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The local index formula in semifinite Von Neumann algebras I: Spectral flow

Alan L. Carey[a], John Phillips[b,*], Adam Rennie[c], Fyodor A. Sukochev[d]

[a] *Mathematical Sciences Institute, Australian National University, Canberra, ACT. 0200, Australia*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada V8W 3P4*
[c] *School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW, 2308 Australia*
[d] *School of Informatics and Engineering, Flinders University, Bedford Park S.A., 5042 Australia*

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Abstract

We generalise the local index formula of Connes and Moscovici to the case of spectral triples for a $*$-subalgebra $\mathcal{A}$ of a general semifinite von Neumann algebra. In this setting it gives a formula for spectral flow along a path joining an unbounded self-adjoint Breuer–Fredholm operator, affiliated to the von Neumann algebra, to a unitarily equivalent operator. Our proof is novel even in the setting of the original theorem and relies on the introduction of a function valued cocycle which is 'almost' a (b, B)-cocycle in the cyclic cohomology of $\mathcal{A}$.

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* Corresponding author. Fax: +250 721 8962.
E-mail addresses: acarey@maths.anu.edu.au (A.L. Carey), phillips@math.uvic.ca (J. Phillips), adam.rennie@newcastle.edu.au (A. Rennie), sukochev@infoeng.flinders.edu.au (F.A. Sukochev).

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1. Introduction

The odd local index theorem of Connes and Moscovici [CoM] may be thought of as a far reaching generalisation of the classical index theorem for Toeplitz operators. It is thus a natural question to ask whether the index theorem of Coburn et al. [CDSS], Curto et al. [CMX] proved in the setting of semifinite von Neumann algebras and giving a topological formula for the Breuer–Fredholm index of Wiener–Hopf operators with almost periodic symbol is the prototype for a von Neumann algebra version of the local index theorem. This question was answered in the affirmative by our noncommutative geometry calculation of the index of Toeplitz operators with noncommutative symbol [CPS2,L,PR]. In both cases there is a clear interpretation of the index as computing spectral flow along a certain path of unbounded self-adjoint Breuer–Fredholm operators. (This follows from recent work of some of us in [CP1,CP2] interpreting the Breuer–Fredholm index of the [CDSS] Wiener–Hopf operators as 'type II' spectral flow.) This motivated the present general study of the local index formula of Connes and Moscovici in the setting of semifinite von Neumann algebras via a computation of spectral flow along a path of self-adjoint unbounded Breuer–Fredholm operators.

This line of reasoning touches on a more general program outlined by [BeF] for developing a theory of 'von Neumann spectral triples'. In addition to [CDSS], examples which suggest that this has interest include differential operators with almost periodic symbol [Sh], the L^2-index theorem (see [M] and references therein), foliations [Co2,BeF,Pr] as well as the example of spectral flow cited above [CP1,CP2].

The starting point is a Hilbert space $\mathcal{H}$ on which there is an unbounded densely defined self-adjoint Breuer–Fredholm operator $\mathcal{D}$. In this setting Carey–Phillips introduced an integral formula for the spectral flow along the linear path joining $\mathcal{D}$ to a unitarily equivalent operator $u\mathcal{D}u^*$, [CP1,CP2], which we present in Eq. (2) in Section 3. The natural framework for this formula is that of odd spectral triples (generalised to the von Neumann setting as in [CP1,BeF,CPS1,CPS2]) so that u is an element of a $*$-subalgebra $\mathcal{A}$ of a semifinite von Neumann algebra $\mathcal{N}$ acting nondegenerately on $\mathcal{H}$.

Given the analytic formula of [CP1,CP2] for the spectral flow, our task is to show that there is an equality with a cohomological formula. We derive the cohomological formula in several steps.

The first step, described in Section 5.3, exploits the fact that the analytic formula gives spectral flow as the integral of an exact one form, [CP1,CP2]. The exactness of the one form allows us to change the path of integration to obtain a new formula which is amenable to perturbation theory methods. In Section 7, we employ a perturbation expansion of the resolvent to write spectral flow in terms of a 'function-valued cochain' in the (b, B) bicomplex of cyclic cohomology. Our function-valued cochain is reminiscent of, though distinct from, Higson's 'improper cocycle' [H]. Our cochain is a cocycle modulo functions holomorphic in a half-plane. We refer to this cochain as the resolvent cocycle and it should be thought of as a substitute for the JLO cocycle (which is the starting point for the argument of Connes–Moscovici).

The resolvent cocycle can be further expanded employing the quantised pseudodifferential calculus of Connes–Moscovici, [CoM]. This is done in Section 8 using material developed in Section 6. The end result is an expression for the spectral flow in terms of a sum of generalised zeta functions of the form

$$\zeta_b(z) = \tau(b(1+\mathcal{D}^2)^{-z}), \quad b \in \mathcal{N}.$$

This sum of zeta functions is meromorphic in a half-plane, with (at worst) only a single simple pole in this half-plane. The residue at this pole is precisely the spectral flow.

Under the assumption that the individual zeta functions in this sum analytically continue to a deleted neighbourhood of the critical point, we may take residues of the individual terms at the critical point. The resulting formula, when $\mathcal{N} = \mathcal{B}(\mathcal{H})$, is essentially that which is obtained by pairing the (odd, renormalised) cyclic cocycle obtained by Connes and Moscovici, [Co3,CoM], with the Chern character of the unitary u^*. The only difference between the two formulae is that we do not assume $\mathcal{D}$ is invertible and hence use inverse powers of $(1+\mathcal{D}^2)^{1/2}$, whereas Connes–Moscovici assume that $\mathcal{D}$ is invertible and use inverse powers of $|\mathcal{D}|$.

The novel aspects of our approach are:

- Our result calculates spectral flow in semifinite von Neumann spectral triples generalising part of the type I theory of [CoM]. Specifically, our formula for spectral flow is given in terms of a cyclic cocycle, which is the generalisation to semifinite von Neumann algebras of the residue cocycle of [CoM]. This provides an extension of [CPS2,L,PR].
- Only the final step of our proof requires the analytic continuation property of the generalised zeta functions. Indeed, we express spectral flow as the *residue* of a sum of zeta functions without invoking *any* analytic continuation hypothesis.
- Assuming the individual zeta functions in the above sum have analytic continuations to a deleted neighbourhood of the critical point allows us to write spectral flow as a sum of residues of zeta functions. The residues of these zeta functions then assemble to form a (b, B) cocycle for the algebra $\mathcal{A}$ of the spectral triple.
  For examples with 'dimension' less than 2, these last two statements are true without *any* analytic continuation property.
- We make no assumptions on the decay of our zeta functions along vertical lines in the complex plane thus reducing the side conditions that need to be checked when applying the local index formula of [CoM].
- Our proof that the residue cocycle [CoM] is indeed a (b, B)-cocycle is quite simple even in the general semifinite case by virtue of using our resolvent cocycle.
- Except for the need to verify a number of estimates, the strategy of our proof is straightforward, and applies to both the type I and type II cases.
- The idea of this proof in the odd case can be adapted to handle the even case of the local index formula. The starting point for the even case is a generalised McKean–Singer formula and the argument is presented in Part II.
- We remark that there is an unrenormalised version of the residue cocycle in [CoM] containing an infinite number of terms in the case that one of the terms in the expansion has an essential singularity, whereas their renormalised version always has a bounded number of terms. The unrenormalised version presents an issue of con-

vergence which is difficult to address. Since we do not pass through an intermediate step where the cocycle contains a potentially infinite number of terms, we are free to allow essential singularities from the outset.

In this paper we do not address the relationship of the residue cocycle to the Chern character as is done in [CoM] leaving that to another place as it is not a trivial step. Our residue cocycle necessarily involves zeta functions of $(1+D^2)^{-1/2}$ because $|D|$ may have zero in its continuous spectrum. This means the usual transgression arguments do not immediately work. It also means that the formula obtained when $\mathcal{N}=\mathcal{B}(\mathcal{H})$ is a modification of a consequence (Corollary II of [CoM]) of the 'renormalised' version of the Connes–Moscovici Local Index Theorem.

The plan of the paper is that we group preliminary material including notation and definitions in Section 2. Section 3 describes integral formulae for Spectral Flow, and the relation to cyclic cohomology. The statement of the main result is in Section 4, and at this point we also provide an outline of the proof for the reader's convenience.

Section 5 establishes the key estimates needed for the rest of the proof. Section 6 summarises what we need to know about the quantised pseudodifferential calculus. In Section 7 we derive our resolvent cocycle. These earlier ingredients come together in Section 8 to prove the main theorem.

2. Definitions and background

2.1. Semifinite spectral triples

We have adopted a notational convention correlated to the context in which we are working. A calligraphic $\mathcal{D}$ will always denote an unbounded self-adjoint operator forming part of a semifinite spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$. A roman D will denote a self-adjoint operator on a Hilbert space, usually with some side conditions. Later, starting from a semifinite spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ in Section 5.3, we construct from the space $\mathcal{H}$ on which a semifinite von Neumann algebra $\mathcal{N}$ acts, a new Hilbert space $\tilde{\mathcal{H}}$, an algebra $\tilde{\mathcal{N}}$ and an operator $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{H}}$ affiliated with $\tilde{\mathcal{N}}$. While this is an odd spectral triple (for an algebra containing $\mathcal{A}$), for us it is just a computational device inspired by ideas of [G]. We begin with some semifinite versions of standard definitions and results. Let $\mathcal{K}_{\mathcal{N}}$ be the τ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E\in\mathcal{N}$ with $\tau(E)<\infty$). Here τ is a fixed faithful semifinite trace on the von Neumann algebra $\mathcal{N}$.

Definition 2.1. A semifinite spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A}\subset\mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that

(1) $[\mathcal{D},a]$ is densely defined and extends to a bounded operator for all $a\in\mathcal{A}$,
(2) $(\lambda-\mathcal{D})^{-1}\in\mathcal{K}_{\mathcal{N}}$ for all $\lambda\notin\mathbf{R}$.

Note. In this paper, for simplicity of exposition, we will deal only with unital algebras $\mathcal{A} \subset \mathcal{N}$ where the identity of $\mathcal{A}$ is that of $\mathcal{N}$. Henceforth we omit the term semifinite as it is implied by the use of a faithful normal semifinite trace τ on $\mathcal{N}$ in all of the subsequent text. In this paper (Part I) we will only deal with odd spectral triples, [Co2], since spectral flow is the pairing of K-homology with K_1.

Definition 2.2. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geqslant 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geqslant 1$.

Note. The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations, commutators and topology. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{N}$ which need not be (weakly) dense in $\mathcal{N}$. More precisely, $\operatorname{dom}\delta = \{T \in \mathcal{N} : \delta(T) \text{ is bounded}\}$. We also note that if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{N}$. In fact, writing $|\mathcal{D}|_1 = (1+\mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have $\operatorname{dom}(\delta^n) = \operatorname{dom}(\delta_1^n)$ for all n.

Proof. Let $f(\mathcal{D}) = (1+\mathcal{D}^2)^{1/2} - |\mathcal{D}|$, so, as noted above, $f(\mathcal{D})$ extends to a bounded operator in $\mathcal{N}$. Since

$$\delta_1(T) - \delta(T) = [f(\mathcal{D}), T]$$

is always bounded, $\operatorname{dom}\delta = \operatorname{dom}\delta_1$. Now $\delta\delta_1 = \delta_1\delta$, so

$$\begin{aligned}\delta_1^2(T) - \delta^2(T) &= \delta_1(\delta_1(T)) - \delta_1(\delta(T)) + \delta_1(\delta(T)) - \delta(\delta(T))\\ &= [f(\mathcal{D}), \delta_1(T)] + [f(\mathcal{D}), \delta(T)].\end{aligned}$$

Both terms on the right-hand side are bounded, so $\operatorname{dom}\delta^2 = \operatorname{dom}\delta_1^2$. The proof proceeds by induction.

Thus the condition defining QC^∞ can be replaced by

$$a, [\mathcal{D}, a] \in \bigcap_{n \geqslant 0} \operatorname{dom}\delta_1^n \quad \forall a \in \mathcal{A}.$$

This is important as we *do not assume at any point that* $|\mathcal{D}|$ *is invertible*.

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, we may endow the algebra $\mathcal{A}$ with the topology determined by the seminorms

$$a \longrightarrow \| \delta^k(a) \| + \| \delta^k([\mathcal{D}, a]) \|, \quad k = 0, 1, 2, \ldots$$

We call this topology the δ-topology and observe that by [R, Lemma 16] we may, without loss of generality, suppose that $\mathcal{A}$ is complete in the δ-topology by completing if necessary. This completion is Fréchet and stable under the holomorphic functional calculus, so we have a sensible spectral theory and $K_*(\mathcal{A}) \cong K_*(\bar{\mathcal{A}})$ via inclusion, where $\bar{\mathcal{A}}$ is the C^*-completion of $\mathcal{A}$.

Next we observe that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$.

Proof. Observe that $\mathcal{D}$ is affiliated with $\mathcal{N}$, and so commutes with all projections in the commutant of $\mathcal{N}$, and the commutant of $\mathcal{N}$ preserves the domain of $\mathcal{D}$. Thus if $[\mathcal{D}, T]$ is bounded, it too commutes with all projections in the commutant of $\mathcal{N}$, and these projections preserve the domain of $\mathcal{D}$, and so $[\mathcal{D}, T] \in \mathcal{N}$. $\square$

Similar comments apply to $[|\mathcal{D}|, T]$, $[(1 + \mathcal{D}^2)^{1/2}, T]$ and the more exotic combinations such as $[\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1/2}$ which we will encounter later.

Recall from [FK] that if $S \in \mathcal{N}$, the *t-th generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{||SE|| \mid E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leqslant t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\| T \|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\| \cdot \|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)\, dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ need not be complete in this norm but it is complete in the norm $||.||_1 + ||.||_\infty$. (where $||.||_\infty$ is the uniform norm).

2.2. *Dimension spectrum*

Many naturally occurring spectral triples satisfy a summability condition, and such conditions allow one to define interesting cocycles. In particular, one can obtain representatives of the Chern character. The (finite) summability conditions give a half-plane where the function

$$z \mapsto \tau((1 + \mathcal{D}^2)^{-z}) \tag{1}$$

is well-defined and holomorphic. In [Co3,CoM], a stronger condition was imposed in order to prove the Local Index Theorem. This condition not only specifies a half-plane where the function in (1) is holomorphic, but also that this function analytically continues to $\mathbf{C}$ minus some discrete set. We clarify this in the following definitions.

Definition 2.3. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple. The algebra $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{N}$ is the algebra of polynomials generated by $\delta^n(a)$ and $\delta^n([\mathcal{D}, a])$ for $a \in \mathcal{A}$ and $n \geqslant 0$. A QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has *discrete dimension spectrum* $Sd \subseteq \mathbf{C}$ if Sd is a discrete set and for all $b \in \mathcal{B}(\mathcal{A})$ the function $\tau(b(1+\mathcal{D}^2)^{-z})$ is defined and holomorphic for $Re(z)$ large, and analytically continues to $\mathbf{C} \setminus Sd$. We say the dimension spectrum is *simple* if this zeta function has poles of order at most one for all $b \in \mathcal{B}(\mathcal{A})$, *finite* if there is a $k \in \mathbf{N}$ such that the function has poles of order at most k for all $b \in \mathcal{B}(\mathcal{A})$ and *infinite*, if it is not finite.

Connes and Moscovici impose the discrete dimension spectrum assumption to prove their version of the local index theorem. In this paper we employ a weaker condition (explained in the next section) that is implied by the discrete dimension spectrum assumption.

3. Spectral flow

To place our results in their proper setting we need some background from [Ph,Ph1,PR]. Let $\pi : \mathcal{N} \to \mathcal{N}/\mathcal{K}_\mathcal{N}$ be the canonical mapping. A Breuer–Fredholm operator is one that maps to an invertible operator under π, [PR]. In the Appendix to [PR], the theory of Breuer–Fredholm operators for the case where $\mathcal{N}$ is not a factor is developed in analogy with the factor case of Breuer, [B1,B2]. In Part II of this work, we develop this theory even further. As usual D is an unbounded densely defined self-adjoint Breuer–Fredholm operator on $\mathcal{H}$ (meaning $D(1+D^2)^{-1/2}$ is bounded and Breuer–Fredholm in $\mathcal{N}$) with $(1+D^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$. For a unitary $u \in \mathcal{N}$ such that $[D, u]$ is a bounded operator, the path

$$D_t^u := (1-t)\,D + tuDu^*$$

of unbounded self-adjoint Breuer–Fredholm operators is continuous in the sense that

$$F_t^u := D_t^u \left(1 + (D_t^u)^2\right)^{-\frac{1}{2}}$$

is a norm continuous path of self-adjoint Breuer–Fredholm operators in $\mathcal{N}$ [CP1]. Recall that the Breuer–Fredholm index of a Breuer–Fredholm operator F is defined by

$$ind(F) = \tau(Q_{\ker F}) - \tau(Q_{\operatorname{coker} F}),$$

where $Q_{\ker F}$ and $Q_{\operatorname{coker} F}$ are the projections onto the kernel and cokernel of F.

Definition. If $\{F_t\}$ is a continuous path of self-adjoint Breuer–Fredholm operators in $\mathcal{N}$, then the definition of the *spectral flow* of the path, $sf(\{F_t\})$ is based on the following sequence of observations in [P1]:

(1) The function $t \mapsto sign(F_t)$ is typically discontinuous as is the projection-valued mapping $t \mapsto P_t = \frac{1}{2}(sign(F_t) + 1)$.
(2) However, $t \mapsto \pi(P_t)$ is continuous.
(3) If P and Q are projections in $\mathcal{N}$ and $||\pi(P) - \pi(Q)|| < 1$ then $PQ : Q\mathcal{H} \to P\mathcal{H}$ is a Breuer–Fredholm operator and so $ind(PQ) \in \mathbf{R}$ is well-defined. This requires Section 3 of Part II.
(4) If we partition the parameter interval of $\{F_t\}$ so that the $\pi(P_t)$ do not vary much in norm on each subinterval of the partition then

$$sf(\{F_t\}) := \sum_{i=1}^{n} ind(P_{t_{i-1}} P_{t_i})$$

is a well-defined and (path-) homotopy-invariant number which agrees with the usual notion of spectral flow in the type I_∞ case.
(5) For D and u as above, we define the *spectral flow* of the path $D_t^u := (1-t)\, D + tuDu^*$ to be the spectral flow of the path F_t where $F_t = D_t^u \left(1 + (D_t^u)^2\right)^{-\frac{1}{2}}$. We denote this by

$$sf(D, uDu^*) = sf(\{F_t\}),$$

and observe that this is an integer in the $\mathcal{N} = \mathcal{B}(\mathcal{H})$ case and a real number in the general semifinite case.

Special cases of spectral flow in a semifinite von Neumann algebra were discussed in [M,P1,P2].

Let P denote the projection onto the nonnegative spectral subspace of D. The spectral flow along $\{D_t^u\}$ is equal to $sf(\{F_t\})$ and by [CP1] this is the Breuer–Fredholm index of $PuPu^*$. (Note that sign $F_1^u = 2uPu^* - 1$ and that for this special path we have $P - uPu^*$ is compact so $PuPu^*$ is certainly Breuer–Fredholm from $uPu^*\mathcal{H} \to P\mathcal{H}$.) Now, [PR, Appendix B], we have $\mathrm{ind}(PuPu^*) = \mathrm{ind}(PuP)$.

3.1. Spectral flow formulae

We now introduce the spectral flow formula of Carey and Phillips, [CP1,CP2] from which our other formulae follow. This formula starts with a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and computes the spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$, where $u \in \mathcal{A}$ is unitary with $[\mathcal{D}, u]$ bounded, in the case where $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is of dimension $p \geqslant 1$. Thus for any

$n > p$ we have by Theorem 9.3 of [CP2]:

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{n/2}} \int_0^1 \tau(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-n/2})\, dt, \tag{2}$$

with $C_{n/2} = \int_{-\infty}^{\infty} (1 + x^2)^{-n/2}\, dx$. This real number $sf(\mathcal{D}, u\mathcal{D}u^*)$ recovers the pairing of the K-homology class $[\mathcal{D}]$ of $\mathcal{A}$ with the $K_1(\mathcal{A})$ class $[u]$ (see below). There is a geometric way to view this formula. It is shown in [CP2] that the functional $X \mapsto \tau(X(1 + (\mathcal{D} + X)^2)^{-n/2})$ on $\mathcal{N}_{\mathrm{sa}}$ determines an exact one-form on an affine space modelled on $\mathcal{N}_{\mathrm{sa}}$. Thus (2) represents (cf. [Si]) the integral of this one-form along the path $\{\mathcal{D}_t = (1-t)\mathcal{D} + tu\mathcal{D}u^*\}$ provided one appreciates that $\dot{\mathcal{D}}_t = u[\mathcal{D}, u^*]$ is a tangent vector to this path. Moreover this formula is scale invariant. By this we mean that if we replace $\mathcal{D}$ by $\varepsilon\mathcal{D}$, for $\varepsilon > 0$, on the right-hand side of (2), then the left-hand side is unchanged, since spectral flow is invariant with respect to change of scale. We will use this fact later on at several points.

3.2. Relation to cyclic cohomology

One can also interpret spectral flow (in the type I case) as the pairing between an odd K-theory class represented by a unitary u, and an odd K-homology class represented by $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, [Co2, Chapter III,IV]. This point of view also makes sense in the general semifinite setting, though one must suitably interpret K-homology, [CPRS1,CP2]. A central feature of [Co2] is the translation of the K-theory pairing to cyclic theory in order to obtain index theorems. One associates to a suitable representative of a K-theory class, respectively a K-homology class, a class in periodic cyclic homology, respectively a class in periodic cyclic cohomology, called a Chern character in both cases. The principal result is then

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\frac{1}{\sqrt{2\pi i}} \langle [Ch_*(u)], [Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle, \tag{3}$$

where $[u] \in K_1(\mathcal{A})$ is a K-theory class with representative u and $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the K-homology class of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

On the right-hand side, $Ch_*(u)$ is the Chern character of u, and $[Ch_*(u)]$ its periodic cyclic homology class. Similarly $[Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the periodic cyclic cohomology class of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. *The analogue of Eq.* (3), *for a suitable cocycle associated to* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, *in the general semifinite case is part of our main result.*

We will not discuss periodic cyclic cohomology Chern characters in great detail here, leaving that to another place. We do, however, require some of the basic definitions and results of cyclic theory, as well as one result on the Chern character of a unitary. We will use the normalised (b, B)-bicomplex (see [Co2,Lo]).

We introduce the following linear spaces. Let $C_m = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ where $\bar{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathbf{C}I$ with I being the identity element of $\mathcal{A}$ and (assuming with no loss of generality that $\mathcal{A}$ is complete in the δ-topology) we employ the projective tensor product. Let $C^m = Hom(C_m, \mathbf{C})$ be the linear space of continuous multilinear functionals on C_m. We may define the (b, B) bicomplex using these spaces (as opposed to $C_m = \mathcal{A}^{\otimes m+1}$ etc.) and the resulting cohomology will be the same. This follows because the bicomplex defined using $\mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ is quasi-isomorphic to that defined using $\mathcal{A} \otimes \mathcal{A}^{\otimes m}$ [Lo].

A normalised (b, B)-*cochain*, ϕ is a finite collection of continuous multilinear functionals on $\mathcal{A}$,

$$\phi = \{\phi_m\}_{m=1,2,\ldots,M} \text{ with } \phi_m \in C^m.$$

It is a (normalised) (b, B)-*cocycle* if, for all m, $b\phi_m + B\phi_{m+2} = 0$ where $b : C^m \to C^{m+1}$, $B : C^m \to C^{m-1}$ are the coboundary operators given by

$$\begin{aligned}
&(B\phi_m)(a_0, a_1, \ldots, a_{m-1}) \\
&\quad = \sum_{j=0}^{m-1} (-1)^{(m-1)j} \phi_m(1, a_j, a_{j+1}, \ldots, a_{m-1}, a_0, \ldots, a_{j-1}), \\
&\quad (b\phi_{m-2})(a_0, a_1, \ldots, a_{m-1}) \\
&\quad = \sum_{j=0}^{m-2} (-1)^j \phi_{m-2}(a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_{m-1}) \\
&\qquad + (-1)^{m-1} \phi_{m-2}(a_{m-1}a_0, a_1, \ldots, a_{m-2}).
\end{aligned}$$

We write $(b + B)\phi = 0$ for brevity. Thought of as functionals on $\mathcal{A}^{\otimes m+1}$ a normalised cocycle will satisfy $\phi(a_0, a_1, \ldots, a_n) = 0$ whenever any $a_j = 1$ for $j \geqslant 1$. An *odd* (*even*) cochain has $\{\phi_m\} = 0$ for m even (odd).

Similarly, a (b^T, B^T)-*chain*, c is a (possibly infinite) collection $c = \{c_m\}_{m=1,2,\ldots}$ with $c_m \in C_m$. The (b, B)-chain $\{c_m\}$ is a (b^T, B^T)-*cycle* if $b^T c_{m+2} + B^T c_m = 0$ for all m. More briefly, we write $(b^T + B^T)c = 0$. Here b^T, B^T are the boundary operators of cyclic homology, and are the transpose of the coboundary operators b, B in the following sense.

The pairing between a (b, B)-cochain $\phi = \{\phi_m\}_{m=1}^M$ and a (b^T, B^T)-chain $c = \{c_m\}$ is given by

$$\langle \phi, c\rangle = \sum_{m=1}^{M} \phi_m(c_m).$$

This pairing satisfies

$$\langle (b + B)\phi, c\rangle = \langle \phi, (b^T + B^T)c\rangle.$$

We use this fact in Section 8 in the following way. We call $c = (c_m)_{m\ odd}$ an odd normalised (b^T, B^T)-*boundary* if there is some even chain $e = \{e_m\}_{m\ even}$ with $c_m = b^T e_{m+1} + B^T e_{m-1}$ for all m. If we pair a normalised (b, B)-cocycle ϕ with a normalised (b^T, B^T)-boundary c we find

$$\langle \phi, c\rangle = \langle \phi, (b^T + B^T)e\rangle = \langle (b + B)\phi, e\rangle = 0.$$

There is an analogous definition in the case of even chains $c = (c_m)_{m\ even}$. All of the cocycles we consider in this paper are in fact defined as functionals on $\oplus_m \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$. Henceforth we will drop the superscript on b^T, B^T and just write b, B for both boundary and coboundary operators as the meaning will be clear from the context.

The Chern character $Ch_*(u)$ of a unitary $u \in \mathcal{A}$ is the following (infinite) collection of odd chains $Ch_{2j+1}(u)$ satisfying $bCh_{2j+3}(u) + BCh_{2j+1}(u) = 0$,

$$Ch_{2j+1}(u) = (-1)^j j! u^* \otimes u \otimes u^* \otimes \cdots \otimes u \quad (2j + 2 \quad \text{entries}).$$

It is well known to experts that

$$Ch_*(u^*) + Ch_*(u) \tag{4}$$

is homologous to zero in the normalised (entire) (b, B) chain complex however an accessible argument eluded us so we present one here. A similar statement holds in the periodic theory.

Lemma 3.1. *For u unitary in $\mathcal{A}$, $Ch_*(u^*)+Ch_*(u)$ is a boundary in the odd normalised entire cyclic homology of $\mathcal{A}$.*

Proof. That $Ch_*(u)$ defines an entire cycle is presented in [G]. We work with elements of $\mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes n}$ written as $n + 1$-tuples in the normalised version of the (b, B) complex. So in the normalised complex with $n = 2m + 1$, $m = 1, 2, \ldots$ let

$$w_{2m+1} = (u^{-1}, u, \ldots, u^{-1}, u) \in \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes 2m+1}$$

and then

$$\begin{aligned} Bw_{2m+1} &= B(u^{-1}, u, \ldots, u^{-1}, u) \\ &= (m + 1)(1, u^{-1}, u, \ldots, u^{-1}, u) - (m + 1)(1, u, u^{-1}, \ldots, u, u^{-1}) \end{aligned}$$

and $bw_{2m+1} \in \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes 2m}$

$$bw_{2m+1} = b(u^{-1}, u, \ldots, u^{-1}, u) = (1, u^{-1}, u, \ldots, u^{-1}, u) - (1, u, u^{-1}, \ldots, u, u^{-1}).$$

Thus $Bw_{2m+1} - (m+1)bw_{2m+3} = 0$ and fixes the normalisation up to a constant: $Ch_*(u) = (c_{2m+1}w_{2m+1})$ where

$$w_{2m+1} = (u^{-1}, u, \ldots, u^{-1}, u) \quad (2m+2 \text{ entries}), \;\; c_{2m+1} = (-1)^m m!.$$

Now, in the normalised complex:

$$b(1, u^{-1}, u, \ldots, u^{-1}, u) = (u^{-1}, u, \ldots, u^{-1}, u) + (u, u^{-1}, \ldots, u, u^{-1})$$

and

$$B(1, u^{-1}, u, \ldots, u^{-1}, u) = 0.$$

Hence let $z = (z_{2m+2})$ where $m = 0, 1, 2, \ldots$ and

$$z_{2m+2} = c_{2m+1}(1, u^{-1}, u, \ldots, u^{-1}, u) \quad (2m+3 \quad \text{entries}).$$

Then

$$\begin{aligned} Ch_{2m+1}(u^*) + Ch_{2m+1}(u) \;=\; & c_{2m+1}b(1, u^{-1}, u, \ldots, u^{-1}, u) \\ & + c_{2m-1}B(1, u^{-1}, u, \ldots, u^{-1}, u) \end{aligned}$$

so that $Ch_*(u) + Ch_*(u^{-1}) = (b+B)z$. $\square$

4. The main result and outline of the proof

4.1. Statement of the main result

We introduce some notation in order to be able to state the main theorem.

First, we require multi-indices $(k_1, \ldots, k_m)$, $k_i \in \{0, 1, 2, \ldots\}$, whose length m will always be clear from the context. We write $|k| = k_1 + \cdots + k_m$, and define $\alpha(k)$ by

$$\alpha(k) = \frac{1}{k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}.$$

The numbers $\sigma_{n,j}$ are defined by the equality

$$\prod_{j=0}^{n-1}(z + j + 1/2) = \sum_{j=0}^{n} z^j \sigma_{n,j}.$$

These are just the elementary symmetric functions of $1/2, 3/2, \ldots, n - 1/2$.

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple and $T \in \mathcal{N}$, we write $T^{(n)}$ to denote the iterated commutator $[\mathcal{D}^2, [\mathcal{D}^2, [\cdots, [\mathcal{D}^2, T]\cdots]]]$ where we have n commutators with $\mathcal{D}^2$. It follows from the remarks after Definition 2.2 that operators of the form $T_1^{(n_1)} \cdots T_k^{(n_k)}(1+\mathcal{D}^2)^{-(n_1+\cdots+n_k)/2}$ are in $\mathcal{N}$ when $T_i = [\mathcal{D}, a_i]$, or $= a_i$ for $a_i \in \mathcal{A}$.

Definition 4.1. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, we call

$$p = \inf\{k \in \mathbf{R} : \tau((1+\mathcal{D}^2)^{-k/2}) < \infty\}$$

the *spectral dimension* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has *isolated spectral dimension* if for b of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}$$

the zeta functions

$$\zeta_b(z-(1-p)/2) = \tau(b(1+\mathcal{D}^2)^{-z+(1-p)/2})$$

have analytic continuations to a deleted neighbourhood of $z = (1-p)/2$.

Remark. Observe that we allow the possibility that the analytic continuations of these zeta functions may have an essential singularity at $z = (1-p)/2$. All that is necessary for us is that the residues at this point exist.

Now we define, for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ having isolated spectral dimension and

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}$$

$$\tau_j(b) = res_{z=(1-p)/2}(z-(1-p)/2)^j \zeta_b(z-(1-p)/2).$$

The hypothesis of isolated spectral dimension is clearly necessary here in order to define the residues.

With these preliminaries we can state the main result of the paper.

Theorem 4.2 (*Semifinite odd local index theorem*). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $p \geqslant 1$. Let $N = [p/2]+1$ where $[\cdot]$ denotes the integer part, and let $u \in \mathcal{A}$ be unitary. Then*

(1)

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{\sqrt{2\pi i}} res_{r=(1-p)/2}\left(\sum_{m=1,odd}^{2N-1} \phi_m^r(Ch_m(u))\right),$$

and if $a_0, \ldots, a_m \in \mathcal{A}$, $l = \{a+iv : v \in \mathbf{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda-(1+s^2+\mathcal{D}^2))^{-1}$ *and* $r > 0$ *we define* $\phi_m^r(a_0, a_1, \ldots, a_m)$ *to be*

$$\frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m \tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} a_0 R_s(\lambda)[\mathcal{D}, a_1]R_s(\lambda)\cdots[\mathcal{D}, a_m]R_s(\lambda)\, d\lambda\right) ds.$$

In particular the sum on the right-hand side of (1) *analytically continues to a deleted neighbourhood of* $r = (1-p)/2$ *with* at worst *a simple pole at* $r = (1-p)/2$. *Moreover, the complex function-valued cochain* $(\phi_m^r)_{m=1,odd}^{2N-1}$ *is a* (b, B) *cocycle for* $\mathcal{A}$ *modulo functions holomorphic in a half-plane containing* $r = (1-p)/2$.

(2) *The spectral flow* $sf(\mathcal{D}, u^*\mathcal{D}u)$ *is also the residue of a sum of zeta functions*:

$$\frac{1}{\sqrt{2\pi i}} res_{r=(1-p)/2}\Bigg(\sum_{m=1,odd}^{2N-1}\ \sum_{|k|=0}^{2N-1-m}\ \sum_{j=0}^{|k|+(m-1)/2} (-1)^{|k|+m}\alpha(k)\Gamma((m+1)/2)$$
$$\times\sigma_{|k|+(m-1)/2,j}(r-(1-p)/2)^j \tau\Big(u^*[\mathcal{D}, u]^{(k_1)}[\mathcal{D}, u^*]^{(k_2)}\cdots[\mathcal{D}, u]^{(k_m)}$$
$$\times(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-p)/2}\Big)\Bigg).$$

In particular the sum of zeta functions analytically continues to a deleted neighbourhood of $r = (1-p)/2$ *and has* at worst *a simple pole at* $r = (1-p)/2$.

(3) *If* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *also has isolated spectral dimension then*

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{\sqrt{2\pi i}}\sum_m \phi_m(Ch_m(u)),$$

where for $a_0, \ldots, a_m \in \mathcal{A}$

$$\phi_m(a_0, \ldots, a_m) = res_{r=(1-p)/2}\phi_m^r(a_0, \ldots, a_m) = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|}\alpha(k)$$
$$\times\sum_{j=0}^{|k|+(m-1)/2}\sigma_{(|k|+(m-1)/2),j}\tau_j\left(a_0[\mathcal{D}, a_1]^{(k_1)}\cdots[\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}\right),$$

and $(\phi_m)_{m=1,odd}^{2N-1}$ *is a* (b, B) *cocycle for* $\mathcal{A}$. *When* $[p] = 2n$ *is even, the term with* $m = 2N-1$ *is zero, and for* $m = 1, 3, \ldots, 2N-3$, *all the top terms with* $|k| = 2N-1-m$ *are zero.*

Remark. Since ϕ_m is a multilinear functional, it is well-defined on elements of $\mathcal{A}^{\otimes m+1}$ such as $Ch_m(u)$.

Corollary 4.3. *For* $1 \leqslant p < 2$, *the statements in* (3) *of Theorem* 4.2 *are true without the assumption of isolated dimension spectrum.*

4.2. Outline of the proof of Theorem 4.2

The basic strategy we employ is very simple, but as the technical details may obscure this, we offer an outline of the proof here using the notation of Theorem 4.2. There are two basic parts of the proof.

1. The Carey–Phillips spectral flow formula is manipulated into a form that allows us to use perturbation theory in the form of a resolvent expansion. The resulting formula suggests the definition of a substitute, for finitely summable spectral triples, of the JLO cocycle of entire cyclic cohomology. Our substitute we term the 'resolvent cocycle'. It is a function-valued (b, B)-cocycle, modulo functions holomorphic in a certain half-plane.

2. The pseudodifferential calculus of [CoM] then enables us to write the spectral flow as a sum of zeta functions, modulo functions holomorphic in a certain half-plane. If we impose the isolated spectral dimension assumption we can analytically continue these zeta functions and take residues at a predetermined critical point. We then see that spectral flow is obtained by pairing $Ch_*(u)$ with a variant of the Connes–Moscovici residue cocycle.

We now expand on these two basic parts.

In Sections 5.1 and 5.2, we prove all the basic norm and trace estimates we will require for the resolvent expansion. These technical details may be skipped on a first reading.

The proof proper begins in Section 5.3. To successfully apply perturbation techniques to the Carey–Phillips spectral flow formula, Eq. (2), we require 'more room to manoeuvre'. Three basic steps are involved in this section. First, we 'double-up' the data $(\mathcal{H}, \mathcal{D})$ from our spectral triple and unitary u by tensoring on two copies of $\mathbf{C}^2$ to $\mathcal{H}$ to obtain an unbounded self-adjoint operator $\tilde{\mathcal{D}}$ affiliated with $\tilde{\mathcal{N}} := M_2 \otimes M_2 \otimes \mathcal{N}$ and a self-adjoint unitary q (determined by u) in $M_2 \otimes M_2 \otimes \mathcal{A}$. This may be viewed as employing a formal (Clifford) Bott periodicity and replaces the trace τ by a supertrace $S\tau$.

Second, with the additional freedom, we are now able use an idea of [G] to define a two-parameter family of perturbations $\tilde{\mathcal{D}}_{r,s}$, $r \in [0, 1]$ and $s \in [0, \infty)$. We observe that, as the spectral flow is computed by integrating an *exact* one-form on an affine space of perturbations of $\tilde{\mathcal{D}}$, Lemma 5.6, we may compute spectral flow from $\mathcal{D}$ to $u^*\mathcal{D}u$ along different paths joining the endpoints; initially it is given by integrating with respect to r when $s = 0$.

The third step chooses a path that expresses spectral flow in terms of an integral over the s variable with $r = 0$ where the perturbation in the spectral flow formula of equation (2), instead of being of first order in $\tilde{\mathcal{D}}$, is now zeroth order. Thus we obtain a new formula for spectral flow

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{C_{p/2+r}} \int_0^\infty S\tau \left(q(1 + \tilde{\mathcal{D}}^2 + s\{\tilde{\mathcal{D}}, q\} + s^2)^{-p/2-r} \right) ds,$$

where $\{\cdot,\cdot\}$ denotes the anticommutator. Crucially, the anticommutator $\{\tilde{\mathcal{D}}, q\}$ is bounded, and we are now in a position to employ perturbation theory in the form of the resolvent expansion.

Section 6 reviews the pseudodifferential calculus of [CoM], and the 'Taylor expansion' in the form introduced by Higson, [H]. We then prove several technical results, Lemmas 6.10–6.12, that allow us to easily apply the pseudodifferential calculus in our setting. Again, this section may be omitted on a first reading.

In Section 7.1 we write

$$(1 + \tilde{\mathcal{D}}^2 + s^2 + s\{\tilde{\mathcal{D}}, q\})^{-p/2-r} = \frac{1}{2\pi i} \int_l \lambda^{-p/2-r} (\lambda - (1 + \tilde{\mathcal{D}}^2 + s^2 + s\{\tilde{\mathcal{D}}, q\}))^{-1} \, d\lambda,$$

where the vertical line l lies between 0 and $spec(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}}, q\})$ for all $s \in [0, \infty)$. We then apply the resolvent expansion (writing $R_s(\lambda) = (\lambda - (1 + \tilde{\mathcal{D}}^2 + s^2))^{-1}$)

$$(\lambda - (1 + \tilde{\mathcal{D}}^2 + s^2 + s\{\tilde{\mathcal{D}}, q\}))^{-1} = \sum_{m=0}^{2N-1} \left(R_s(\lambda) s\{\tilde{\mathcal{D}}, q\} \right)^m R_s(\lambda) + Remainder.$$

The estimates in Lemmas 7.1 and 7.2 allow us to show in Lemma 7.4 that *modulo functions of r holomorphic in a half-plane containing* $r = (1 - p)/2$

$$\begin{aligned} &sf(\mathcal{D}, u^*\mathcal{D}u) C_{p/2+r} \\ &\quad = \frac{1}{2\pi i} \sum_{m=1,odd}^{2N-1} \int_0^\infty s^m S\tau \left(q \int_l \lambda^{-p/2-r} (R_s(\lambda)\{\tilde{\mathcal{D}}, q\})^m R_s(\lambda) \, d\lambda \right) ds. \end{aligned} \quad (5)$$

The even terms in the above expansion are seen to vanish by elementary Clifford-type manipulations. The 'constant'

$$C_{p/2+r} = \frac{\Gamma(r - (1 - p)/2)\Gamma(1/2)}{\Gamma(p/2 + r)}$$

has simple poles at $r = (1 - p)/2 - k$, $k = 0, 1, 2, \ldots,$ with residue equal to 1 at $r = (1 - p)/2$. Therefore, since the error terms in Eq. (5) are holomorphic at $r = (1 - p)/2$, we may take residues at $r = (1 - p)/2$ of the analytic continuations of both sides of (5) even though the individual terms in this expansion need not analytically continue.

Section 7.2 begins by performing the 'super' part of the trace to obtain a formula for the spectral flow in terms of the original spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and the unitary u. The general structure of this formula suggests the definition of a function-valued (b, B)-cochain on the algebra $\mathcal{A}$. We call this the resolvent cocycle, and using techniques inspired by Higson, [H], we show that this is a cocycle *modulo functions of* r *holomorphic in a half-plane containing* $(1-p)/2$. This 'almost cocycle' property is proved in Lemma 7.10, and this proves (1) of Theorem 4.2.

Section 8 returns to our spectral flow computations. Section 8.1 applies the pseudodifferential calculus, in the form derived in Lemma 6.11, to each term of the resolvent expansion. This moves all the resolvents to the right, allowing us to use Cauchy's formula to perform the complex line integral. In Section 8.2 we perform the remaining integral over $s \in [0, \infty)$, and so obtain our penultimate formula:

$$\begin{aligned} &sf(\mathcal{D}, u^*\mathcal{D}u)C_{p/2+r} \\ &\quad = \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} C_{k,m,r} S\tau\left(q\{\tilde{\mathcal{D}}, q\}^{(k_1)} \cdots \{\tilde{\mathcal{D}}, q\}^{(k_m)}(1+\tilde{\mathcal{D}}^2)^{-(p-1)/2-|k|-m/2-r}\right), \end{aligned} \tag{6}$$

where equality is again *modulo functions of* r *holomorphic in a half-plane containing* $(1-p)/2$.

This is a remarkable formula. So far we have not invoked the isolated spectral dimension hypothesis, yet the sum of zeta functions in Eq. (6) clearly has a simple pole at $r = (1-p)/2$, with residue equal to the spectral flow. This proves part (2) of Theorem 4.2.

In Section 8.3 we finally assume that the individual zeta functions possess analytic continuations to a deleted neighbourhood of $r = (1-p)/2$ so we can take residues of the zeta functions in Theorem 8.3 to obtain our version of the residue cocycle. We can then prove part (3) of Theorem 4.2. The cocycle property for the residue cocycle follows from the 'almost' cocycle property of the resolvent cocycle upon taking residues. We conclude with the simple proof of Corollary 4.3.

5. Norm and trace estimates

Throughout this section, let $D : \mathrm{dom}\, D \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$.

5.1. Norm estimates

In a number of estimates, we will also consider a bounded self-adjoint operator A. The operators A that are of interest satisfy $s^2 + sA + D^2 \geqslant 0$ for all real $s \geqslant 0$. However it is also convenient at times to assume that $\| A \|$ is relatively small: $\| A \| < \sqrt{2}$, for example. This can be achieved by scaling A: see *Observation 2* of Section 7.

Lemma 5.1. *Let D be an unbounded self-adjoint operator.*

(a) *For $\lambda = a + iv \in \mathbf{C}$, $0 < a < 1/2$, $s \geqslant 0$ we have the estimate*

$$\| (\lambda - (1 + D^2 + s^2))^{-1} \| \leqslant (v^2 + (1 + s^2 - a)^2)^{-1/2} \leqslant \frac{1}{1 - a}.$$

(b) *If A is bounded, self-adjoint and $s^2 + sA + D^2 \geqslant 0$ we have*

$$\| (\lambda - (1 + D^2 + s^2 + sA))^{-1} \| \leqslant (v^2 + (1 - a)^2)^{-1/2} \leqslant \frac{1}{1 - a}.$$

(c) *If A is bounded, self-adjoint and $c = \| A \| < \sqrt{2}$ we have*

$$\| (\lambda - (1 + D^2 + s^2 + sA))^{-1} \| \leqslant (v^2 + (1 + s^2 - a - sc)^2)^{-1/2} \leqslant \frac{1}{1/2 - a}.$$

Proof. Part (a) is an application of the functional calculus. To see (b) note that the spectrum of $(1 + D^2 + s^2 + sA)$ is contained in $[1, \infty)$, so the distance from λ to the spectrum is at least $|\lambda - 1|$. For (c) note that the minimum value in the spectrum of $(1 + D^2 + s^2 + sA)$ is at least $(1 + s^2 - sc) > 1/2$, and so the distance from λ to the spectrum is at least $|\lambda - (1 + s^2 - sc)|$. $\square$

5.2. Trace estimates

For the duration of this section we suppose that the operator D satisfies the summability condition

$$(1 + D^2)^{-n/2} \in \mathcal{L}^1(\mathcal{N}) \quad \forall n > p \geqslant 1. \tag{7}$$

For instance, if $\mathcal{D}$ comes from a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ then this is exactly the condition that $\mathcal{D}$ have dimension p. With hypothesis (7) we will obtain some trace norm estimates for various operators depending on s. The following lemma is the key technical estimate, and the main result of this subsection. Henceforth $1 >> \varepsilon > 0$.

Remark. In the following lemma, if instead of assuming $\| A \| < \sqrt{2}$, we suppose that $D^2 + s^2 + sA \geqslant 0$, then the estimate on the RHS becomes $C_{p+\varepsilon}(1/2 + (1/2)s^2)^{-Re(r)+\varepsilon}$ for $s \geqslant 2 \| A \|$. Since one can show that $(1 + D^2 + s^2 + sA)^{-p/2-r}$ is trace-class for all s, the integrability conclusion still holds also.

Lemma 5.2. *Let A be bounded self-adjoint operator with $\| A \| < \sqrt{2}$. Let $p \geqslant 1$ and let $(1+D^2)^{-1/2}$ be $(p+\varepsilon)$-summable for every $\varepsilon > 0$. Then for each $\varepsilon > 0$ and $r \in \mathbf{C}$ with $Re(r) > 0$, the trace norm of $(1+D^2+s^2+sA)^{-p/2-r}$ satisfies*

$$\| (1+D^2+s^2+sA)^{-p/2-r} \|_1 \leqslant C_{p+\varepsilon}(1/2+s^2-s \| A \|)^{-Re(r)+\varepsilon},$$

where $C_{p+\varepsilon} = \| (1/2+D^2)^{-(p/2+\varepsilon)} \|_1$. So, assuming that $Re(r) > 1/2+\varepsilon$, then $\| (1+D^2+s^2+sA)^{-p/2-r} \|_1$ is integrable in s on $\mathbf{R}$.

Proof. Throughout the proof we suppose, without loss of generality, that r is real and positive. This is possible because for complex r with $Re(r) > 0$ we have

$$\begin{aligned} &\| (1+D^2+s^2+sA)^{-p/2-r} \|_1 \\ &\quad \leqslant \| (1+D^2+s^2+sA)^{-iIm(r)} \| \| (1+D^2+s^2+sA)^{-p/2-Re(r)} \|_1 \\ &\quad \leqslant \| (1+D^2+s^2+sA)^{-p/2-Re(r)} \|_1 \,. \end{aligned}$$

We first consider the case where $A=0$. Since $(1/2+D^2)^{-1} \leqslant 2(1+D^2)^{-1}$ the constant $C_{p+\varepsilon}$ is finite. Now, for positive real numbers X, Y, a, b one easily sees that:

$$(X+Y)^{-a} \leqslant X^{-a} \quad \text{and} \quad (X+Y)^{-b} \leqslant Y^{-b}$$

and so,

$$(X+Y)^{-a-b} \leqslant X^{-a} Y^{-b}.$$

Therefore, by the functional calculus:

$$\begin{aligned} (1+D^2+s^2)^{-p/2-r} &= (1/2+D^2+1/2+s^2)^{-(p/2+\varepsilon)-(r-\varepsilon)} \\ &\leqslant (1/2+D^2)^{-(p/2+\varepsilon)}(1/2+s^2)^{-(r-\varepsilon)}. \end{aligned}$$

Taking the trace norm of this inequality, gives the lemma when $A=0$.

To obtain the general case where $\| A \| < \sqrt{2}$, we observe that

$$0 < 1+s^2+D^2-s \| A \| \leqslant 1+s^2+D^2+sA \leqslant 1+s^2+D^2+s \| A \| \,.$$

Consequently,

$$(1+s^2+D^2+sA)^{-1} \leqslant (1+s^2+D^2-s \| A \|)^{-1}.$$

Now Corollary 4, Appendix B of [CP1] tells us that

$$\tau((1+s^2+D^2+sA)^{-p/2-r}) \leqslant \tau((1+s^2+D^2-s \| A \|)^{-p/2-r}).$$

By the same argument as the case $A=0$ (noting that $1/2+s^2-s\parallel A\parallel>0$) we get:

$$(1+s^2+D^2-s\parallel A\parallel)^{-p/2-r}\leqslant(1/2+D^2)^{-(p/2+\varepsilon)}(1/2+s^2-s\parallel A\parallel)^{-(r-\varepsilon)}).$$

Taking the trace of this inequality and combining it with the immediately previous inequality, yields the proof of the lemma. □

Lemma 5.3. *Let* $0<a=Re(\lambda)<1/2$, $\lambda=a+iv$. *Let* $p\geqslant 1$ *and let* $(1+D^2)^{-1/2}$ *be* $(p+\varepsilon)$*-summable for every* $\varepsilon>0$. *Then for each* $\varepsilon>0$ *and* $N>(p+\varepsilon)/2$, *we have the trace-norm estimate*:

$$\parallel(\lambda-(1+D^2+s^2))^{-N}\parallel_1\ \leqslant C'_{p+\varepsilon}((1/2+s^2-a)^2+v^2)^{-N/2+(p+\varepsilon)/4},$$

where $C'_{p+\varepsilon}=\parallel(1/2+D^2)^{-(p+\varepsilon)/2}\parallel_1$.

Proof. This is similar to Lemma 5.2. Now,

$$\begin{aligned}\left|\left[\lambda-(1+D^2+s^2)\right]^{-1}\right| &= \left[(1/2+D^2+1/2+s^2-a)^2+v^2\right]^{-1/2}\\ &\leqslant \left[(1/2+D^2)^2+(1/2+s^2-a)^2+v^2\right]^{-1/2}.\end{aligned}$$

By the argument of the previous lemma, we have

$$\begin{aligned}&\left|\left[\lambda-(1+D^2+s^2)\right]^{-N}\right|\\ &\quad\leqslant\left[(1/2+D^2)^2+(1/2+s^2-a)^2+v^2\right]^{-N/2}\\ &\quad=\left[(1/2+D^2)^2+(1/2+s^2-a)^2+v^2\right]^{-(p+\varepsilon)/4-(N/2-(p+\varepsilon)/4)}\\ &\quad\leqslant(1/2+D^2)^{-(p+\varepsilon)/2}((1/2+s^2-a)^2+v^2)^{-N/2+(p+\varepsilon)/4}.\end{aligned}$$

Taking the trace of this inequality yields the proof of the lemma. □

We finish this subsection with an integral estimate which we will use several times.

Lemma 5.4. *Let* $0<a<1/2$ *and* $0\leqslant c\leqslant\sqrt{2}$ *and* $A=0$ *or 1. Let J,K, and M be nonnegative constants. Then the integral*

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty s^J\sqrt{a^2+v^2}^{-M}\sqrt{(s^2+1/2-a)^2+v^2}^{-K}\\ &\qquad\times\sqrt{(s^2+1-a-sc)^2+v^2}^{-A}\,dv\,ds \qquad (8)\end{aligned}$$

converges provided $J-2K-2A<-1$ *and* $J-2K-2A+1-2M<-2$.

Proof. We begin with the case $A = 0$. The integrand is positive and continuous (and hence measurable), so by Tonelli's theorem we may evaluate the s integral first. As $1/2 - a > 0$,

$$(s^2 + 1/2 - a)^2 \geqslant s^4 + (1/2 - a)^2$$

and

$$\int_0^\infty s^J ((s^2 + 1/2 - a)^2 + v^2)^{-K/2}\, ds \leqslant \int_0^\infty s^J (s^4 + (1/2 - a)^2 + v^2)^{-K/2}\, ds.$$

Now write $b^2 = (1/2 - a)^2 + v^2$ and set $t = sb^{-1/2}$. Performing the substitution gives

$$\begin{aligned}\int_0^\infty s^J (s^4 + b^2)^{-K/2}\, ds &= b^{-K} \int_0^\infty s^J (s^4/b^2 + 1)^{-K/2}\, ds \\ &= b^{-K+J/2+1/2} \int_0^\infty t^J (t^4 + 1)^{-K/2}\, dt.\end{aligned}$$

This integral converges provided $J - 2K < -1$. Thus

$$\begin{aligned}&\int_0^\infty s^J \int_{-\infty}^\infty \sqrt{a^2 + v^2}^{-M} \sqrt{(s^2 + 1/2 - a)^2 + v^2}^{-K}\, dv\, ds \\ &\quad \leqslant C \int_{-\infty}^\infty ((1/2 - a)^2 + v^2)^{-K/2+J/4+1/4} (a^2 + v^2)^{-M/2}\, dv,\end{aligned}$$

and this is finite provided $-2K + J + 1 - 2M < -2$. When $A = 1$ we observe that

$$1 + s^2 - a - sc = (s - c/2)^2 + (1 - c^2/4 - a)$$

so

$$\begin{aligned}(1 + s^2 - a - sc)^2 &= ((s - c/2)^2 + (1 - c^2/4 - a))^2 \\ &\geqslant (s - c/2)^4 + (1 - c^2/4 - a)^2 \\ &\geqslant (s - c/2)^4 + (1/2 - a)^2.\end{aligned}$$

We also have for $s \geqslant c/2$

$$(s^2 + 1/2 - a)^2 \geqslant (s - c/2)^4 + (1/2 - a)^2.$$

For $s \leqslant c/2$ we have

$$(s^2 + 1/2 - a)^2 \geqslant (1/2 - a)^2.$$

The integral in Eq. (8) is then bounded by

$$\int_{c/2}^{\infty} s^J \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-M} \sqrt{(s-c/2)^4+(1/2-a)^2+v^2}^{-K-1} dv\, ds$$
$$+ \int_0^{c/2} (c/2)^J \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-M} \sqrt{(1/2-a)^2+v^2}^{-K-1} dv\, ds.$$

We look at the integral over s from $c/2$ to ∞. Write $b^2 = (1/2-a)^2+v^2$, so we are considering

$$\int_{c/2}^{\infty} s^J \sqrt{(s-c/2)^4+b^2}^{-K-1} ds = \int_0^{\infty} (s+c/2)^J \sqrt{s^4+b^2}^{-K-1} ds.$$

Now setting $t = sb^{-1/2}$ and making the change of variables this last integral becomes

$$\int_0^{\infty} b^{-K+(J-1)/2}(t+cb^{-1/2}/2)^J(1+t^4)^{-(K+1)/2}$$
$$\leqslant b^{-K-1/2+J/2} \int_0^{\infty} (t+c/2\sqrt{1-2a})^J(1+t^4)^{-(K+1)/2}\, dt,$$

and this converges when $J-2K-2<-1$. Now we have the v-integral, which is bounded by

$$C \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-M} \sqrt{(1/2-a)^2+v^2}^{-K-1/2+J/2} dv$$

and this is finite when $J-2K-1-2M<-2$. The s-integral from 0 to $c/2$ is finite, and the corresponding v integral is given by

$$\int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-M} \sqrt{(1/2-a)^2+v^2}^{-K-1} dv$$

and this is finite for $-M-K-1<-1$ or $M+K>0$. $\square$

5.3. Application: rewriting the formula for spectral flow

In this section we begin with the spectral flow formula (2) of the last section, for a finitely summable odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and rewrite it in a different way so as to be able to exploit resolvent expansions.

Our method borrows an idea from [G] however, whereas Getzler's approach is via the superconnection formalism, we will adopt a more concrete functional analytic approach suggested by [CP0]. See also Section 9 of [CP2].

Definition 5.5. Form the Hilbert space $\tilde{\mathcal{H}} = \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathcal{H}$ acted on by the von Neumann algebra, $\tilde{\mathcal{N}} = M_2 \otimes M_2 \otimes \mathcal{N}$. Introduce the two dimensional Clifford algebra in the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(With $\tilde{\mathcal{D}}$ defined below this gives an odd spectral triple for the algebra $1_2 \otimes M_2 \otimes \mathcal{A}$ but we shall *never* use this fact). Define the grading on $\tilde{\mathcal{H}}$ by $\Gamma = \sigma_2 \otimes \sigma_3 \otimes 1 \in \tilde{\mathcal{N}}$.

Let $u \in \mathcal{A}$ be unitary and introduce the following even operators (i.e., they commute with Γ):

$$\tilde{\mathcal{D}} = \sigma_2 \otimes 1_2 \otimes \mathcal{D}, \quad q = \sigma_3 \otimes \begin{pmatrix} 0 & -iu^{-1} \\ iu & 0 \end{pmatrix},$$

$$\mathcal{D}_r = (1-r)\tilde{\mathcal{D}} - rq\tilde{\mathcal{D}}q, \quad \mathcal{D}_{r,s} = \mathcal{D}_r + sq$$

for $r \in [0, 1], s \in [0, \infty)$. Clearly, the unbounded operators are affiliated with $\tilde{\mathcal{N}}$. Notice that

$$\mathcal{D}_r \equiv \mathcal{D}_{r,0} = \sigma_2 \otimes \begin{pmatrix} \mathcal{D} + ru^{-1}[\mathcal{D}, u] & 0 \\ 0 & \mathcal{D} + ru[\mathcal{D}, u^{-1}] \end{pmatrix}$$

So

$$\dot{\mathcal{D}}_r = \sigma_2 \otimes \begin{pmatrix} u^{-1}[\mathcal{D}, u] & 0 \\ 0 & u[\mathcal{D}, u^{-1}] \end{pmatrix}.$$

The graded trace on $\tilde{\mathcal{N}}$ we write as $S\tau(a) = \frac{1}{2}\tau(\Gamma a)$ for a trace-class in $\tilde{\mathcal{N}}$, and so for example,

$$\begin{aligned} &S\tau(\dot{\mathcal{D}}_r(1+\mathcal{D}_r^2)^{-\frac{n}{2}}) \\ &= \frac{1}{2}\int_0^1 \tau\left(1_2 \otimes \begin{pmatrix} u^*[\mathcal{D}, u](1+\mathcal{D}_r^2)^{-n/2} & 0 \\ 0 & -u[\mathcal{D}, u^*](1+\mathcal{D}_r^2)^{-n/2} \end{pmatrix}\right) dr \\ &= \tau\Big(u^{-1}[\mathcal{D}, u](1+(\mathcal{D}+ru^{-1}[\mathcal{D}, u])^2)^{-\frac{n}{2}} \\ &\quad -u[\mathcal{D}, u^{-1}](1+(\mathcal{D}+ru[\mathcal{D}, u^{-1}])^2)^{-\frac{n}{2}}\Big). \end{aligned}$$

Next we calculate

$$\mathcal{D}_{r,s}^2 = \mathcal{D}_r^2 + s(1-2r)\sigma_1 \otimes \begin{pmatrix} 0 & [\mathcal{D}, u^{-1}] \\ -[\mathcal{D}, u] & 0 \end{pmatrix} + s^2$$

which depends on the relation

$$\tilde{\mathcal{D}}q + q\tilde{\mathcal{D}} = \sigma_1 \otimes \begin{pmatrix} 0 & [\mathcal{D}, u^{-1}] \\ -[\mathcal{D}, u] & 0 \end{pmatrix}.$$

Our immediate aim now is to rewrite the spectral flow in terms of these new operators.

Lemma 5.6. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with dimension $p \geqslant 1$. Consider the affine space Φ of perturbations of $\tilde{\mathcal{D}}$ given by*

$$\Phi = \{\tilde{\mathcal{D}} + X \quad | \ X \in \tilde{\mathcal{N}}_{\text{sa}} \text{ and } [X, \Gamma] = 0\}.$$

Then for $n > p$ the linear functional on the tangent space to Φ at $\tilde{\mathcal{D}}_1 \in \Phi$

$$\alpha_{\tilde{\mathcal{D}}_1}(X) = S\tau(X(1 + \tilde{\mathcal{D}}_1^2)^{-n/2})$$

is well defined and makes $\tilde{\mathcal{D}}_1 \mapsto \alpha_{\tilde{\mathcal{D}}_1}$ an exact one-form (i.e. an exact section of the cotangent bundle to Φ).

Proof. We would like to apply Theorem 9.3 of [CP2] to the pair $(\tilde{\mathcal{N}}, \tilde{\mathcal{D}})$; however, that theorem deals with the ungraded case (no Γ and just the ordinary trace τ) and this lemma is firmly in the graded setup. In order to avoid reworking the whole theory to fit the graded setting, we resort to the following trick. We modify our unbounded operator, setting $\hat{\mathcal{D}} := \Gamma\tilde{\mathcal{D}}$, and apply Theorem 9.3 of [CP2] to the pair $(\tilde{\mathcal{N}}, \hat{D})$. Since Γ is self-adjoint and commutes with all the operators in Φ, we see that $\Gamma\Phi$ is a real affine space and

$$\Gamma\Phi = \{\hat{\mathcal{D}} + X \quad | \ X \in \tilde{\mathcal{N}}_{\text{sa}} \text{ and } [X, \Gamma] = 0\} \subseteq \hat{\mathcal{D}} + \tilde{\mathcal{N}}_{\text{sa}}.$$

Thus, if $\tilde{\mathcal{D}} + X(s)$ is a piecewise C^1 continuous path in Φ, then $\hat{\mathcal{D}} + \Gamma X(s)$ is a piecewise C^1 continuous path in $\Gamma\Phi \subseteq \hat{\mathcal{D}} + \tilde{\mathcal{N}}_{\text{sa}}$, and we have by Theorem 9.3 of [CP2] that the integral:

$$\int_0^1 \tau\left(\frac{d}{ds}(\Gamma X(s))\left(1 + (\hat{\mathcal{D}} + \Gamma X(s))^2\right)^{-n/2}\right) ds$$

depends only on the endpoints $\Gamma X(0)$ and $\Gamma X(1)$ and hence only on $X(0)$ and $X(1)$. But, since

$$(\hat{\mathcal{D}} + \Gamma X(s))^2 = (\Gamma[\tilde{\mathcal{D}} + X(s)])^2 = (\tilde{\mathcal{D}} + X(s))^2$$

we see that our integral becomes:

$$\int_\tau \left(\Gamma X'(s)\left(1+(\tilde{\mathcal{D}}+X(s))^2\right)^{-n/2}\right)ds$$
$$=2\int_0^1 S\tau\left(X'(s)\left(1+(\tilde{\mathcal{D}}+X(s))^2\right)^{-n/2}\right)ds,$$

and its value depends only on the endpoints $X(0)$ and $X(1)$. Since integrals of our one form are independent of path, our one form is exact by Lemma 7.3 of [CP2]. In fact, it is this independence of path that we use below. $\square$

Lemma 5.7. *For $(\mathcal{A},\mathcal{H},\mathcal{D})$ a spectral triple with dimension $p\geqslant 1$ we have for $n>p$*

$$\frac{1}{C_{n/2}}\int_0^1 S\tau(\dot{\mathcal{D}}_r(1+\mathcal{D}_r^2)^{-n/2})\,dr = 2sf(\mathcal{D},u^*\mathcal{D}u).$$

Proof. Using the above computations we have

$$\int_0^1 S\tau(\dot{\mathcal{D}}_r(1+\mathcal{D}_r^2)^{-n/2})\,dr$$
$$=\int_0^1 \tau\Big(u^*[\mathcal{D},u](1+(\mathcal{D}+ru^*[\mathcal{D},u])^2)^{-n/2}$$
$$-u[\mathcal{D},u^*](1+(\mathcal{D}+ru[\mathcal{D},u^*])^2)^{-n/2}\Big)\,dr$$
$$=2C_{n/2}sf(\mathcal{D},u^*\mathcal{D}u).$$

By formula (2) of Section 4.1. $\square$

Remark. Observe that this is the spectral flow from $\mathcal{D}$ to $u^*\mathcal{D}u$, which is $-Index(PuP)$ where P is the spectral projection of $\mathcal{D}$ corresponding to the half-line $[0,\infty)$. This in turn is $-sf(\mathcal{D},u\mathcal{D}u^*)$. The following estimate enables us to exploit exactness of our one form by changing the path of integration.

Lemma 5.8. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a spectral triple of dimension $p\geqslant 1$. Then if $n>p$, we have*

$$\lim_{s_0\to\infty}\int_0^1 S\tau\left(\frac{d\mathcal{D}_{r,s_0}}{dr}(1+\mathcal{D}_{r,s_0}^2)^{-n/2}\right)dr=0.$$

Proof. First observe that from the definitions we have

$$S\tau\left(\frac{d\mathcal{D}_{r,s_0}}{dr}(1+\mathcal{D}_{r,s_0}^2)^{-n/2}\right)=S\tau\left(\frac{d\mathcal{D}_{r,0}}{dr}(1+\mathcal{D}_{r,s_0}^2)^{-n/2}\right).$$

Next $\mathcal{D}_{1/2} = (1/2)(\tilde{\mathcal{D}} - q\tilde{\mathcal{D}}q)$, so that $\mathcal{D}_{1/2}$ anticommutes with q. Hence $(\mathcal{D}_{1/2} + s_0 q)^2 = \mathcal{D}_{1/2}^2 + s_0^2$. So

$$S\tau(\dot{\mathcal{D}}_{1/2}(1+\mathcal{D}_{1/2,s_0}^2)^{-n/2}) = S\tau(\dot{\mathcal{D}}_{1/2}(1+\mathcal{D}_{1/2}^2 + s_0^2)^{-n/2}).$$

By Lemma 5.2 with $r = (n-p)/2$, we have the following estimate (for a positive constant $C_{p+\varepsilon}$)

$$\| (1+\mathcal{D}_{1/2}^2 + s_0^2)^{-n/2} \|_1 \leqslant C_{p+\varepsilon}(1/2 + s_0^2)^{-n/2+p/2+\varepsilon}.$$

Since this exponent is negative (if we choose ε sufficiently small so that $n > p + 2\varepsilon$), we see that as $s_0 \to \infty$,

$$\begin{aligned} S\tau(\dot{\mathcal{D}}_{1/2}(1+\mathcal{D}_{1/2,s_0}^2)^{-n/2}) &\leqslant S\tau\left(\left|\dot{\mathcal{D}}_{1/2}(1+\mathcal{D}_{1/2,s_0}^2)^{-n/2}\right|\right) \\ &\leqslant C_{p+\varepsilon}(1/2 + s_0^2)^{-n/2+p/2+\varepsilon} \to 0. \end{aligned}$$

Set $A_r = \mathcal{D}_r - \mathcal{D}_{1/2} = (1/2 - r)(\tilde{\mathcal{D}} + q\tilde{\mathcal{D}}q)$, and observe that there is a positive constant c such that $\| A_r \| \leqslant c$ for all $r \in [0,1]$. Then

$$\mathcal{D}_{r,s_0} = \mathcal{D}_{1/2,s_0} + \mathcal{D}_{r,s_0} - \mathcal{D}_{1/2,s_0} = \mathcal{D}_{1/2,s_0} + A_r.$$

Using [CP1, Corollary 8, p. 710] we obtain

$$\| (1+\mathcal{D}_{r,s_0}^2)^{-n/2} \|_1 \leqslant f(c)^{n/2} \| (1+\mathcal{D}_{1/2,s_0}^2)^{-n/2} \|_1,$$

where $f(c) = 1 + c^2/2 + (c/2)\sqrt{c^2+4}$. Hence

$$\int_0^1 S\tau\left(\frac{d\mathcal{D}_{r,s_0}}{dr}(1+\mathcal{D}_{r,s_0}^2)^{-n/2}\right) dr \to 0 \quad \text{as} \quad s_0 \to \infty.$$

Lemma 5.9. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple of dimension $p \geqslant 1$. For $n > p$ and $s_0 > 0$ we have*

$$\int_0^{s_0} S\tau\left(\frac{d\mathcal{D}_{1,s}}{ds}(1+\mathcal{D}_{1,s}^2)^{-n/2}\right) ds = -\int_0^{s_0} S\tau\left(\frac{d\mathcal{D}_{0,s}}{ds}(1+\mathcal{D}_{0,s}^2)^{-n/2}\right) ds.$$

Proof. First observe that $\mathcal{D}_{1,s} = -q\tilde{\mathcal{D}}q + sq$ and $\mathcal{D}_{0,s} = \tilde{\mathcal{D}} + sq$ so that

$$\frac{d\mathcal{D}_{1,s}}{ds} = q = \frac{d\mathcal{D}_{0,s}}{ds}.$$

Set $\rho = \sigma_2 \otimes 1_2 \otimes 1$, so that $\rho q \rho = -q$, $\rho^2 = 1$ and $\rho\Gamma\rho = \Gamma$. Then one easily calculates that:

$$\rho q \mathcal{D}_{1,s} = -(\tilde{\mathcal{D}} + sq)\rho q = -\mathcal{D}_{0,s}\rho q.$$

So that:

$$\rho q \mathcal{D}_{1,s}^2 = \mathcal{D}_{0,s}^2 \rho q,$$

and hence for any Borel function, *f*:

$$\rho q f(\mathcal{D}_{1,s}^2) = f(\mathcal{D}_{0,s}^2)\rho q.$$

Then

$$\begin{aligned}
S\tau\left(\frac{d\mathcal{D}_{1,s}}{ds}(1+\mathcal{D}_{1,s}^2)^{-n/2}\right) &= \frac{1}{2}\tau(\rho^2\Gamma q(1+\mathcal{D}_{1,s}^2)^{-n/2}) \\
&= \frac{1}{2}\tau(\Gamma\rho q(1+\mathcal{D}_{1,s}^2)^{-n/2}\rho) \\
&= \frac{1}{2}\tau(\Gamma(1+\mathcal{D}_{0,s}^2)^{-n/2}\rho q\rho) \\
&= -\frac{1}{2}\tau(\Gamma(1+\mathcal{D}_{0,s}^2)^{-n/2}q) \\
&= -\frac{1}{2}\tau(\Gamma q(1+\mathcal{D}_{0,s}^2)^{-n/2}) \\
&= -S\tau\left(\frac{d\mathcal{D}_{0,s}}{ds}(1+\mathcal{D}_{0,s}^2)^{-n/2}\right).
\end{aligned}$$

This completes the proof. □

Lemma 5.10. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple of dimension $p \geqslant 1$, then for $n > p$ we have*

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{C_{n/2}}\int_0^\infty S\tau\left(\frac{d\mathcal{D}_{0,s}}{ds}(1+\mathcal{D}_{0,s}^2)^{-n/2}\right)ds.$$

Proof. The exactness of our one form gives us

$$\frac{1}{C_{n/2}}\int_0^{s_0} S\tau\left(\frac{d\mathcal{D}_{1,s}}{ds}(1+\mathcal{D}_{1,s}^2)^{-n/2}\right)ds - \frac{1}{C_{n/2}}\int_0^{s_0} S\tau\left(\frac{d\mathcal{D}_{0,s}}{ds}(1+\mathcal{D}_{0,s}^2)^{-n/2}\right)ds$$
$$+\frac{1}{C_{n/2}}\int_0^{1} S\tau\left(\frac{d\mathcal{D}_{r,0}}{dr}(1+\mathcal{D}_{r,0}^2)^{-n/2}\right)dr = \frac{1}{C_{n/2}}\int_0^{1} S\tau\left(\frac{d\mathcal{D}_{r,s_0}}{dr}(1+\mathcal{D}_{r,s_0}^2)^{-n/2}\right)dr.$$

Rearranging, using Lemma 5.9 to combine the first two integrals and then Lemma 5.7 to substitute $2sf(\mathcal{D}, u^*\mathcal{D}u)$ for the third integral gives

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{C_{n/2}}\int_0^{s_0} S\tau\left(\frac{d\mathcal{D}_{0,s}}{ds}(1+\mathcal{D}_{0,s}^2)^{-n/2}\right)ds$$
$$+\frac{1}{2C_{n/2}}\int_0^{1} S\tau\left(\frac{d\mathcal{D}_{r,s_0}}{dr}(1+\mathcal{D}_{r,s_0}^2)^{-n/2}\right)dr.$$

We can now take the limit as $s_0 \to \infty$ of the right-hand side using Lemma 5.8 to show that the improper integral converges to the spectral flow as claimed. □

The conclusion we draw from this is a shift of the path of integration of the integral calculating spectral flow. This leads us to a new formula which, on setting, in Lemma 5.10, $n = p + 2r$ we may write as

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{C_{p/2+r}}\int_0^{\infty} S\tau\left(q(1+\tilde{\mathcal{D}}^2 + s\{\tilde{\mathcal{D}}, q\} + s^2)^{-p/2-r}\right)ds, \tag{9}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator. This formula will be the starting point for writing the spectral flow in terms of (b, B) cocycles.

Before discussing the pseudodifferential calculus, we wish to point out that while the improper integral in Eq. (9) converges for $r > 0$, it converges absolutely only when $r > 1/2$. We omit the argument as it will not affect the subsequent discussion.

6. Summary of the pseudodifferential calculus

6.1. Basic definitions and results

In this section we introduce the terminology and basic results of the Connes–Moscovici pseudodifferential calculus. We will describe our version of the asymptotic expansions of this calculus in Section 6.2 and then use them in Sections 7 and 8. This calculus works in great generality, only needing an unbounded self-adjoint operator D. Just as we did in the remarks following Definition 2.2, we set

$$|D|_1 = (1+D^2)^{1/2}, \qquad \delta_1(T) = [|D|_1, T], \quad T \in \operatorname{dom}\delta.$$

We follow the discussion of the pseudodifferential calculus in [Co3], using $|D|_1$ and δ_1, instead of $|D|$ and δ. In order to ensure that the calculus works in this modified setting we flesh out explanations in [Co3] and record some elementary properties which are trivial to prove, but are often used without comment. The most important results are Proposition 6.5 and its Corollary 6.8. Those familiar with [Co3] or [CoM] can skip to Section 6.2.

So let $D : \mathrm{dom}\, D \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$. For all $k \geqslant 0$, we set

$$\mathcal{H}_k = \mathrm{dom}(1 + D^2)^{k/2} = \mathrm{dom}|D|^k \subseteq \mathcal{H}$$

and $\mathcal{H}_\infty = \cap_{k\geqslant 0} \mathcal{H}_k$. Recall that the graph norm topology makes $\mathcal{H}_k$ into a Hilbert space with norm $\|\cdot\|_k$ given by

$$\| \xi \|_k^2 = \| \xi \|^2 + \| (1 + D^2)^{k/2} \xi \|^2,$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$.

We assume that all of our operators T, in particular D, are affiliated to $\mathcal{N}$ and as in [Co3] that $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$. In this way, all computations involving bounded or unbounded operators make sense on the dense subspace $\mathcal{H}_\infty$.

Definition 6.1. For $r \in \mathbf{R}$, let op^r be the linear space of operators affiliated to $\mathcal{N}$ and mapping $\mathcal{H}_\infty \to \mathcal{H}_\infty$ which are continuous in the norms $(\mathcal{H}_\infty, \|\cdot\|_k) \to (\mathcal{H}_\infty, \|\cdot\|_{k-r})$ for all k such that $k - r \geqslant 0$.

Example. The operators $|D|^r$ and $(1 + D^2)^{r/2}$ are in op^r.

Lemma 6.2 (*Compare Lemma 1.1 of [Co3]*). *Let* $b \in \cap_{n\geqslant 0} \mathrm{dom}\, \delta_1^n$. *With* $\sigma_1(b) = |D|_1 b |D|_1^{-1}$ *and* $\varepsilon_1(b) = \delta_1(b)|D|_1^{-1}$ *we have*

(1) $\sigma_1 = Id + \varepsilon_1$,
(2) $\varepsilon_1^n(b) = \delta_1^n(b)|D|_1^{-n} \in \mathcal{N} \quad \forall n$,
(3) $\sigma_1^n(b) = (Id + \varepsilon_1)^n(b) = \sum_{k=0}^n \binom{n}{k} \delta_1^k(b)|D|_1^{-k} \in \mathcal{N} \quad \forall n$.

Proof. The first statement is straightforward. The second follows because δ_1 is a derivation with $\delta_1(|D|_1) = 0$. The third is just the binomial theorem applied to (1). $\square$

Similarly, if $b \in op^0$, $\sigma_1^{-n}(b) := |D|_1^{-n} b |D|_1^n \in \mathcal{N}$ for all n and

$$|D|_1^{-n} b |D|_1^n = \sum_{k=0}^n \binom{n}{k} |D|_1^{-k} \delta_1^k(b).$$

Corollary 6.3. *If $b \in \cap_{n\geqslant 0}$ dom δ_1^n then $b \in op^0$.*

Proof. For k an integer and $\xi \in \mathcal{H}_\infty$,

$$\| b\xi \|_k^2 = \| b\xi \|^2 + \| |D|_1^k b\xi \|^2 = \| b\xi \|^2 + \| \sigma_1^k(b)|D|_1^k \xi \|^2 \leqslant C(\| \xi \|^2 + \| |D|_1^k \xi \|^2).$$

The case of general $k \in \mathbf{R}$ now follows by interpolation. $\square$

Observe that by the above Lemma, if $b \in op^0$ then $b - \sigma_1(b) = -\varepsilon_1(b) = -\delta_1(b)|D|_1^{-1} \in op^{-1}$. Thus if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, and $b = a$ or $[\mathcal{D}, a]$ for $a \in \mathcal{A}$, then (with $\mathcal{D}$ playing the role of D) $b \in op^0$ and $b - |\mathcal{D}|_1 b |\mathcal{D}|_1^{-1} \in op^{-1}$.

Example. In even the most elementary case $\mathcal{A} = C^\infty(S^1)$, $\mathcal{H} = L^2(S^1)$, $a = M_z$, the operator of multiplication by z, and $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$ one can easily see that $a \in \cap_{n\geqslant 0}$ dom δ_1^n but that $[\mathcal{D}^2, a]$ is not bounded. In general, $[\mathcal{D}^2, a]$ is about the same size as $|\mathcal{D}|$.

Definition 6.4. We define the commuting operators L_1, R_1 on the space of operators on $\mathcal{H}_\infty$ by

$$L_1(T) = (1 + D^2)^{-1/2}[D^2, T] = |D|_1^{-1}[|D|_1^2, T],$$

$$R_1(T) = [D^2, T](1 + D^2)^{-1/2} = [|D|_1^2, T]|D|_1^{-1}.$$

Proposition 6.5 (*Compare Lemma 2 [Co3]*). *For all $b \in op^0$ the following are equivalent*:

(1) $b \in \bigcap_{n\geqslant 0}$ dom δ_1^n,
(2) $b \in \bigcap_{k,l\geqslant 0}$ dom $L_1^k \circ R_1^l$.

Proof. First observe that L_1, R_1 and δ_1 are mutually commuting as maps on the space of operators on $\mathcal{H}_\infty$, which is dense in $\mathcal{H}$. To see that (1) implies (2), let $b \in \cap_{n\geqslant 0}$ dom δ_1^n and observe $L_1(b) = \delta_1(b) + \sigma_1^{-1}(\delta_1(b)) \in op^0$ by Corollary 6.3. Similarly $R_1(b) = \delta_1(b) + \sigma_1(\delta_1(b)) \in op^0$, thus $b \in$ dom $R_1 \cap$ dom L_1. Now σ_1 and δ_1 commute and leave $\cap_{n\geqslant 0}$ dom δ_1^n invariant. Since $L_1^m = \sum_{k=0}^m \binom{m}{k} \sigma_1^{-k}\delta_1^m$, and we easily see that b is in the domain of all powers of L_1. A little more work shows that $b \in \cap_{k,l\geqslant 0}$ dom $L_1^k \circ R_1^l$.

As noted in [Co3], the implication (2) $\Rightarrow$ (1) is more subtle. We first want to see that $b \in \cap_{k,l\geqslant 0}$ dom $L_1^k \circ R_1^l$ implies that $b \in$ dom δ_1. Using

$$|D|_1 = \frac{1}{\pi}\int_0^\infty (1 + D^2)(1 + D^2 + u)^{-1} u^{-1/2}\, du$$

we show by a similar calculation to Lemma 2 of [Co3]:

$$[|D|_1, b] = \frac{1}{\pi}\int_0^\infty \Big(R_1(b)|D|_1(1+D^2+u)^{-2}u^{1/2} \\ +h_u(D)L_1^2(b)(1+D^2+u)^{-1}u^{1/2}\Big)\,du,$$

where $h_u(D) = \big(u(1+D^2+u)^{-1}-1\big)$. We show that the integrals of both terms are bounded. For the first, $R_1(b)$ is bounded so we compute

$$\begin{aligned}&\frac{1}{\pi}\int_0^\infty |D|_1(1+D^2+u)^{-2}u^{1/2}\,du\\ &= |D|_1\frac{1}{\pi}\int_0^\infty (1+D^2+u)^{-2}u^{1/2}\,du\\ &\leqslant |D|_1\frac{1}{\pi}\int_0^\infty (1+D^2+u)^{-1}u^{-1/2}\,du = |D|_1|D|_1^{-1} = 1.\end{aligned}$$

Here we used the formula $x^{-1/2} = \frac{1}{\pi}\int_0^\infty (x+u)^{-1}u^{-1/2}\,du$. To see that the second term is bounded we observe that $\| h_u(D) \| \leqslant 1$, $L_1^2(b)$ is bounded by hypothesis, and the integral $\int_0^\infty (1+D^2+u)^{-1}u^{1/2}\,du$ converges absolutely by estimating $\int_0^1$ and $\int_1^\infty$ separately.

The conclusion is that $b \in \cap_{l,k}\,\mathrm{dom}\,L_1^k \circ R_1^l$ implies $b \in \mathrm{dom}\,\delta_1$. As $L_1(b) \in \cap_{l,k}\,\mathrm{dom}\,L_1^k \circ R_1^l$, we have $L_1(b) \in \mathrm{dom}\,\delta_1$. Since $\delta_1(L_1(b)) = L_1(\delta_1(b))$, we have $\delta_1(b) \in \mathrm{dom}\,L_1$. Similarly $\delta_1(b) \in \cap_{k,l}\,\mathrm{dom}\,L_1^k \circ R_1^l$ and so by the above $\delta_1(b) \in \mathrm{dom}\,\delta_1$; i.e. $b \in \mathrm{dom}\,\delta_1^2$. We continue by induction. $\square$

Definition 6.6. For $r \in \mathbf{R}$

$$OP^r = |D|_1^r\left(\bigcap_{n\geqslant 0}\mathrm{dom}\,\delta_1^n\right) \subseteq op^r \cdot op^0 \subseteq op^r.$$

If $T \in OP^r$ we say that the order of T is (at most) r. The definition is actually symmetric, since for r an integer (at least) we have by Lemma 6.2

$$OP^r = |D|_1^r\left(\bigcap \mathrm{dom}\,\delta_1^n\right) = |D|_1^r\left(\bigcap \mathrm{dom}\,\delta_1^n\right)|D|_1^{-r}|D|_1^r \subseteq \left(\bigcap \mathrm{dom}\,\delta_1^n\right)|D|_1^r.$$

From this we easily see that $OP^r \cdot OP^s \subseteq OP^{r+s}$. Finally, we note that if $b \in OP^r$ for $r \geqslant 0$, then since $b = |D|_1^r a$ for some $a \in OP^0$, we get $[|D|_1, b] = |D|_1^r[|D|_1, a] = |D|_1^r\delta_1(a)$, so $[|D|_1, b] \in OP^r$.

Remarks. An operator $T \in OP^r$ if and only if $|D|_1^{-r}T \in \cap_{n \geq 0} \operatorname{dom} \delta_1^n$. Observe that operators of order at most zero are bounded. If $|D|_1^{-1}$ is p-summable and T has order $-n$ then, T is p/n-summable.

Important Observations:

(1) If f is a bounded Borel function then $f(D) \in \mathcal{N}$ and $\delta_1(f(D)) = 0$, implies $f(D) \in OP^0$.
(2) If g is an unbounded Borel function such that $1/g$ is bounded on $spec(D)$ and both $g(D)|D|_1^{-1}$ and $g(D)^{-1}|D|_1$ are bounded, then for each r, $OP^r = |g(D)|^r OP^0$. This follows since OP^0 is an algebra and both $|g(D)|^r|D|_1^{-r}$ and $g(D)^{-r}|D|_1^r$ are in OP^0. We note that if $|D|$ is *not* invertible then we get strict containment $|D|^r OP^0 \subset |D|_1^r OP^0$. These observations prove the next lemma.

Lemma 6.7. *If $\mu \in \mathbf{C}$ is in the resolvent set of D^2 then*

$$OP^r = |(\mu - D^2)^{1/2}|^r \left(\bigcap_{n \geq 0} \operatorname{dom} \delta_1^n \right).$$

Corollary 6.8. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple, and suppose $a \in \mathcal{A}$. Then for $n \geq 0$, $a^{(n)}$ and $[\mathcal{D}, a]^{(n)}$ are in OP^n.*

Proof. Writing $T^{(n)} = [D^2, [D^2, [\cdots [D^2, T] \cdots]]]$ for the nth iterated commutator with D^2, we have already observed that $L_1^2(b) = |D|_1^{-2}[D^2, [D^2, b]] = |D|_1^{-2} b^{(2)}$. Similarly, $L_1^n(b) = |D|_1^{-n} b^{(n)}$. So if $b \in \cap_{k \geq 0} \operatorname{dom} \delta_1^k$, then $L_1^n(b)$ is also in $\cap_{k \geq 0} \operatorname{dom} \delta_1^k$ and hence $b^{(n)} = |D|_1^n L_1^n(b) \in OP^n$. $\square$

6.2. The pseudodifferential expansion

Next we describe the asymptotic expansions introduced by Connes and Moscovici in [Co3,CoM]. Their principal result is that if $T \in OP^k$ for k integral, then for any $z \in \mathbf{C}$

$$\begin{aligned}(1 + D^2)^z T &= T(1 + D^2)^z + zT^{(1)}(1 + D^2)^{z-1} + \frac{z(z-1)}{2} T^{(2)}(1 + D^2)^{z-2} \\ &\quad + \cdots + \frac{z(z-1)\cdots(z-n+1)}{n!} T^{(n)}(1 + D^2)^{z-n} + P,\end{aligned}$$

where $P \in OP^{k-(n+1)+2Re(z)}$. This result is proved in both of the papers [Co3,CoM], but subsequently a simpler proof has been given by Higson [H].

Because of Higson's idea, we will not need the full force of this expansion, but only a simple algebraic version of it (Lemmas 6.9 and 6.11). We briefly sketch the idea behind Higson's proof in the way in which we will use it. So we suppose that we have a QC^∞ (odd) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with dimension $p \geqslant 1$. We use $\mathcal{D}$ to define the pseudodifferential calculus on $\mathcal{H}$ as in the previous section. Let $Q = (1 + s^2 + D^2)$ where D is $\tilde{\mathcal{D}}$ as defined in Section 5.3 and where $s \in [0, \infty)$. For $Re(z) > p/2$ we write Q^{-z} using Cauchy's formula

$$Q^{-z} = \frac{1}{2\pi i} \int_l \lambda^{-z} (\lambda - Q)^{-1} d\lambda,$$

where l is a vertical line $\lambda = a + iv$ parametrized by $v \in \mathbf{R}$ with $0 < a < 1/2$ fixed. One checks that the integral indeed converges in operator norm (to an element of $\mathcal{N}$) and by using the spectral theorem for Q (in terms of its spectral resolution) it converges to Q^{-z} (principal branch). Computing commutators of Q^{-z} with an operator $T \in OP^m$ then reduces to an iterative calculation of commutators with $(\lambda - Q)^{-1}$. The exact result we need is the following.

Lemma 6.9. *Let m, n, k be non-negative integers and $T \in OP^m$. Then*

$$\begin{aligned}(\lambda - Q)^{-n} T &= T(\lambda - Q)^{-n} + nT^{(1)}(\lambda - Q)^{-(n+1)} + \frac{n(n+1)}{2} T^{(2)}(\lambda - Q)^{-(n+2)} \\ &\quad + \cdots + \binom{n+k-1}{k} T^{(k)}(\lambda - Q)^{-(n+k)} + P(\lambda) \\ &= \sum_{j=0}^{k} \binom{n+j-1}{j} T^{(j)}(\lambda - Q)^{-(n+j)} + P(\lambda),\end{aligned}$$

where the remainder $P(\lambda)$ has order $-(2n + k - m + 1)$ and is given by

$$P(\lambda) = \sum_{j=1}^{n} \binom{j+k-1}{k} (\lambda - Q)^{j-n-1} T^{(k+1)} (\lambda - Q)^{-j-k}.$$

Proof. The proof is inductive using the resolvent formula

$$(\lambda - Q)^{-1} T = T(\lambda - Q)^{-1} + (\lambda - Q)^{-1} T^{(1)} (\lambda - Q)^{-1}.$$

For example with $k = 0$

$$\begin{aligned}(\lambda - Q)^{-n} T &= (\lambda - Q)^{-(n-1)} \left(T(\lambda - Q)^{-1} + (\lambda - Q)^{-1} T^{(1)} (\lambda - Q)^{-1} \right) \\ &= \cdots\end{aligned}$$

$$= T(\lambda - Q)^{-n} + \Big[(\lambda - Q)^{-1}T^{(1)}(\lambda - Q)^{-n}$$
$$+ \cdots + (\lambda - Q)^{-(n-1)}T^{(1)}(\lambda - Q)^{-2} + (\lambda - Q)^{-n}T^{(1)}(\lambda - Q)^{-1}\Big],$$

the term in square brackets being the remainder, which clearly has order $-2(n+1)+m+1 = -(2n+0-m+1)$. For $k=1$, we move $T^{(1)}$ to the left in each term of the remainder using the resolvent formula again. We arrive at

$$(\lambda - Q)^{-n}T = T(\lambda - Q)^{-n} + nT^{(1)}(\lambda - Q)^{-(n+1)}$$
$$+ \sum_{j=1}^{n} j(\lambda - Q)^{-(n+1-j)}T^{(2)}(\lambda - Q)^{-(j+1)},$$

and the new remainder has order $(m+2) - 2(n+2) = -(2n+1-m+1)$. While it is clear how to proceed, in order to keep track of the constants in the remainder terms one must use the formula:

$$\sum_{j=1}^{n} \binom{j+k-1}{k} = \binom{n+k}{k+1}$$

which is easily proved by induction on n with k held fixed. □

Corollary. *Let n, M be positive integers and $A \in OP^k$. Let $R = (\lambda - Q)^{-1}$. Then,*

$$R^n A R^{-n} = \sum_{j=0}^{M} \binom{n+j-1}{j} A^{(j)} R^j + P$$

where

$$P = \sum_{j=1}^{n} \binom{j+M-1}{M} R^{n+1-j} A^{(M+1)} R^{M+j-n}$$

and P has order $k - M - 1$.

Lemma 6.10. *Let k, n be non-negative integers, $s \geqslant 0$, and suppose $\lambda \in \mathbf{C}$, $a = Re(\lambda)$ with $0 < a < 1/2$. Then for $A \in OP^k$ and with $R_s(\lambda) = (\lambda - (1 + D^2 + s^2))^{-1}$ we have*

$$\| R_s(\lambda)^{n/2+k/2} A R_s(\lambda)^{-n/2} \| \leqslant C_{n,k} \quad \text{and} \quad \| R_s(\lambda)^{-n/2} A R_s(\lambda)^{n/2+k/2} \| \leqslant C_{n,k},$$

where $C_{n,k}$ is constant independent of s and λ (square roots use the principal branch of log).

Proof. Since $(OP^k)^* = OP^k$ and $(R_s(\lambda))^* = R_s(\bar{\lambda})$ we only need to prove the first inequality. We begin by using the numerical inequality $(a^2+b^2)^{1/2} \leqslant |a|+|b|$ repeatedly to see that

$$|R_s(\lambda)|^{-1/2} = (v^2 + ((1-a) + D^2 + s^2)^2)^{1/4} \leqslant \cdots \leqslant (|v|^{1/2} + (1-a)^{1/2} + |D| + s).$$

We then use the companion inequality $|a| + |b| \leqslant \sqrt{2}(a^2 + b^2)^{1/2}$ and the fact that $1 < \sqrt{2}$ repeatedly to see that

$$\begin{aligned}(|v|^{1/2} + (1-a)^{1/2} + |D| + s) &\leqslant \cdots \leqslant 4(v^2 + ((1-a) + D^2 + s^2)^2)^{1/4} \\ &= 4|R_s(\lambda)|^{-1/2}.\end{aligned}$$

Hence, as nonnegative bounded operators in OP^0 we have for $j \geqslant 0$,

$$(|v|^{1/2} + (1-a)^{1/2} + |D| + s)^{-j} |R_s(\lambda)|^{-j/2} \leqslant 1,$$

$$|R_s(\lambda)|^{j/2}((1-a)^{1/2} + |D|)^j \leqslant |R_s(\lambda)|^{j/2}(|v|^{1/2} + (1-a)^{1/2} + |D| + s)^j \leqslant 4^j.$$

Now, $A = ((1-a)^{1/2} + |D|)^k B$ for $B \in OP^0$ by Lemma 6.7. Letting $|D|_a = (1-a)^{1/2} + |D|$, making the substitution $A = |D|_a^k B$, and using the identity (valid for any constant, c):

$$B(c + |D|)^n = \sum_{j=0}^{n} \binom{n}{j} (c + |D|)^j (-\delta)^{n-j}(B)$$

we have,

$$\begin{aligned}
&\| R_s(\lambda)^{k/2+n/2} A R_s(\lambda)^{-n/2} \| \\
&\quad = \| \, |D|_a^k R_s(\lambda)^{k/2+n/2} B R_s(\lambda)^{-n/2} \| \\
&\quad \leqslant \| \, |D|_a^k R_s(\lambda)^{k/2+n/2} B (|v|^{1/2} + (1-a)^{1/2} + |D| + s)^n \| \\
&\quad = \left\| |D|_a^k R_s(\lambda)^{k/2+n/2} \sum_{j=0}^{n} \binom{n}{j} (|v|^{1/2} + (1-a)^{1/2} + |D| + s)^j (-\delta)^{n-j}(B) \right\| \\
&\quad \leqslant \sum_{j=0}^{n} \binom{n}{j} \| \, |D|_a^k R_s(\lambda)^{k/2} \| \cdot \| R_s(\lambda)^{n/2} (|v|^{1/2} + (1-a)^{1/2} + |D| + s)^j \| \cdot \| \delta^{n-j}(B) \| \\
&\quad \leqslant \sum_{j=0}^{n} \binom{n}{j} 4^k \| R_s(\lambda)^{n/2-j/2} \| \cdot \| R_s(\lambda)^{j/2}
\end{aligned}$$

$$\times(|v|^{1/2} + (1-a)^{1/2} + |D| + s)^j \parallel \cdot \parallel \delta^{n-j}(B) \parallel$$

$$\leqslant \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{1-a}\right)^{(n-j)/2} 4^{k+j} \parallel \delta^{n-j}(B) \parallel . \qquad \square$$

Remarks. We are thinking of the operator X in the following lemmas as $\{\tilde{\mathcal{D}}, q\}$ in Section 5.3 so that indeed $s^2 + s\{\tilde{\mathcal{D}}, q\} + \tilde{\mathcal{D}}^2 = (\tilde{\mathcal{D}} + sq)^2 \geqslant 0$. With this hypothesis on X and with $0 < a = Re(\lambda) < 1/2$ we see that the spectrum of $(1 + s^2 + sX + \tilde{\mathcal{D}}^2)$ is bounded away from the line $Re(\lambda) = a$ by at least $(1-a)$ independent of s and λ. This hypothesis is crucial in Lemma 6.12. Lemma 6.9 and the next lemma form the algebraic heart of our pseudodifferential expansion.

Lemma 6.11. *Let $A_i \in OP^{n_i}$ for $i = 1, \ldots, m$ and let $0 < a = Re(\lambda) < 1/2$ as above. We consider the operator*

$$R_s(\lambda)A_1 R_s(\lambda)A_2 R_s(\lambda) \cdots R_s(\lambda)A_m \tilde{R}_s(\lambda),$$

where $R_s(\lambda) = (\lambda - (1 + s^2 + D^2))^{-1}$ and $\tilde{R}_s(\lambda) = (\lambda - (1 + s^2 + sX + D^2))^{-1}$ where X is self-adjoint, bounded and $s^2 + sX + D^2 \geqslant 0$. Then for all $M \geqslant 0$

$$R_s(\lambda)A_1 R_s(\lambda)A_2 \cdots A_m \tilde{R}_s(\lambda) = \sum_{|k|=0}^{M} C(k) A_1^{(k_1)} \cdots A_m^{(k_m)} R_s(\lambda)^{m+|k|} \tilde{R}_s(\lambda) + P_{M,m},$$

where $P_{M,m}$ is of order (at most) $-2m - M - 3 + |n|$, and k and n are multi-indices with $|k| = k_1 + \cdots + k_m$ and $|n| = n_1 + \cdots + n_m$. The constant $C(k)$ is given by

$$C(k) = \frac{(|k|+m)!}{k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m)} = (|k|+m)!\alpha(k).$$

Proof. In order to lighten the notation, we abbreviate $R := R_s(\lambda)$ and $\tilde{R} := \tilde{R}_s(\lambda)$. Then we write:

$$RA_1RA_2 \cdots A_m\tilde{R} = RA_1R^{-1}R^2A_2R^{-2} \cdots R^mA_mR^{-m}R^m\tilde{R} \tag{10}$$

so that we have a product of m operators $R^jA_jR^{-j}$ and an additional factor of $R^m\tilde{R}$.
Now by the corollary to Lemma 6.9, if i is a positive integer

$$R^iA_iR^{-i} = \sum_{j=0}^{M} \binom{i+j-1}{j} A_i^{(j)} R^j + P,$$

where P is order $n_i - M - 1$. So if we expand the term RA_1R^{-1} on the right-hand side of Eq. (10) we obtain

$$
\begin{aligned}
&RA_1R^{-1}R^2A_2R^{-2}\cdots R^mA_mR^{-m}R^m\tilde{R}\\
&\quad=\sum_{k_1=0}^{M}A_1^{(k_1)}R^{2+k_1}A_2R^{-2}\cdots A_mR^{-m}R^m\tilde{R}+P_1\\
&\quad=\sum_{k_1=0}^{M}A_1^{(k_1)}R^{2+k_1}A_2R^{-2-k_1}\cdots R^{m+k_1}A_mR^{-m-k_1}R^{m+k_1}\tilde{R}+P_1.
\end{aligned}
$$

Here P_1 is of order $|n|-2m-M-3$. In fact,

$$P_1 = RA_1^{(M+1)}R^{M+1}RA_2R\cdots RA_m\tilde{R}.$$

Now, expanding this new second term, $R^{2+k_1}A_2R^{-(2+k_1)}$, gives

$$\sum_{k_1=0}^{M}\sum_{k_2=0}^{M}C_1(k)A_1^{(k_1)}A_2^{(k_2)}R^{3+k_1+k_2}A_3R^{-3-k_1-k_2}\ldots R^{-m-k_1-k_2}R^{m+k_1+k_2}\tilde{R}+P_1+P_2,$$

where $C_1(k)=\binom{1+k_1+k_2}{k_2}$ and P_2 is of order at most

$$\max(|n|-M-2m-3-k_1)=|n|-M-2m-3.$$

In fact,

$$P_2=\sum_{k_1=0}^{M}\sum_{j=1}^{2+k_1}\binom{j+M-1}{M}A_1^{(k_1)}R^{3+k_1-j}A_2^{(M+1)}R^{M+j}RA_3R\cdots RA_m\tilde{R}.$$

Repeating this argument shows that

$$
\begin{aligned}
&RA_1RA_2\cdots A_m\tilde{R}\\
&\quad=\sum_{k_1,k_2,\ldots,k_m=0}^{M}C(k)A_1^{(k_1)}\cdots A_m^{(k_m)}R^{m+|k|}\tilde{R}+\ \text{order }(|n|-M-2m-3),
\end{aligned}
$$

where

$$C(k)=\binom{k_1}{k_1}\binom{1+k_1+k_2}{k_2}\ldots\binom{m-1+k_1+\cdots+k_m}{k_m}.$$

What remains is to eliminate excess terms in the previous sum and determine the coefficient $C(k)$. First, if $k_1 + \cdots + k_m > M$ then the order of $A_1^{(k_1)} \cdots A_m^{(k_m)} R^{m+|k|} \tilde{R}$ is $|n| - 2 - 2m - |k| < |n| - 2 - 2m - M$ which in turn is $\leqslant |n| - 3 - 2m - M$ (since only integers are considered here.) Thus

$$RA_1RA_2 \cdots A_m \tilde{R} = \sum_{|k|=0}^{M} C(k) A_1^{(k_1)} \cdots A_m^{(k_m)} R^{m+|k|} \tilde{R} + P_{M,m},$$

where $P_{M,m}$ has order $(|n| - M - 2m - 3)$. Finally $C(k)$ is

$$\begin{aligned}
&\frac{(1+k_1+k_2)!}{k_2!(1+k_1)!} \frac{(2+k_1+k_2+k_3)!}{k_3!(2+k_1+k_2)!} \cdots \frac{(m-1+k_1+\cdots+k_m)!}{k_m!(m-1+k_1+\cdots+k_{m-1})!} \\
&\quad = \frac{k_1!}{k_1!} \frac{(1+k_1+k_2)!}{k_2!(1+k_1)!} \frac{(2+k_1+k_2+k_3)!}{k_3!(2+k_1+k_2)!} \cdots \frac{(m-1+k_1+\cdots+k_m)!}{k_m!(m-1+k_1+\cdots+k_{m-1})!} \\
&\quad = \frac{(m-1+k_1+\cdots+k_m)!}{k_1!\cdots k_m!(1+k_1)(2+k_1+k_2)\cdots(m-1+k_1+\cdots+k_{m-1})} \\
&\quad = \frac{(m+|k|)!}{k_1!\cdots k_m!(1+k_1)(2+k_1+k_2)\cdots(m+k_1+\cdots+k_m)}. \qquad \square
\end{aligned}$$

Lemma 6.12. *With the assumptions and notation of the last Lemma including the assumption that $A_i \in OP^{n_i}$ for each i, there is a positive constant C such that*

$$\| \, (\lambda - (1 + D^2 + s^2))^{m+M/2+1/2-|n|/2} P_{M,m} \, \| \leqslant C$$

independent of s and λ (though it depends on M and m and the A_i). If the final $\tilde{R}_s(\lambda)$ is actually an $R_s(\lambda)$ then we can replace the $\frac{1}{2}$ in the exponent with $\frac{3}{2}$: this is important in the proof of Proposition 8.1.

Proof. The remainder $P_{M,m}$ in the previous lemma obtained after applying the pseudodifferential expansion has terms of two kinds. The first kind we consider are the bookkeeping terms at the end of the proof of the last lemma. They are of the form

$$P = A_1^{(k_1)} \cdots A_m^{(k_m)} R^{m+|k|} \tilde{R}$$

with $|k| > M$. Then $R^{-(m+M/2+1/2)+|n|/2} P$ is uniformly bounded by an application of Lemma 6.10.

The other terms are the ones $P_1, P_2, \ldots, P_m$ obtained in the proof of the last lemma. Recall:

$$P_1 = RA_1^{(M+1)} R^{M+1} RA_2 R \cdots RA_m \tilde{R}$$

while a typical summand of P_2 is an integer multiple of:

$$A_1^{(k_1)} R^{3+k_1-j} A_2^{(M+1)} R^{M+j} R A_3 R \cdots R A_m \tilde{R}$$

where $1 \leqslant j \leqslant 2 + k_1$ and $0 \leqslant k_1 \leqslant M$.

Similarly, for $1 \leqslant i \leqslant m$, a typical summand of P_i is an integer multiple of:

$$A_1^{(k_1)} A_2^{(k_2)} \cdots A_{i-1}^{(k_{i-1})} R^{i+1+k_1+k_2+\cdots+k_{i-1}-j} A_i^{(M+1)} R^{M+j} R A_{i+1} R \cdots R A_m \tilde{R}$$

where $1 \leqslant j \leqslant i + k_1 + k_2 + \cdots + k_{i-1}$ and $0 \leqslant k_1, k_2, \ldots, k_{i-1} \leqslant M$.

We work with the typical summand of P_i above, and let $B = A_1^{(k_1)} A_2^{(k_2)} \cdots A_{i-1}^{(k_{i-1})}$ which has order $(k_1 + k_2 + \cdots + k_{i-1}) + (n_1 + n_2 + \cdots + n_{i-1}) = |k| + |n|_{i-1}$, where we have used the notation $|n|_j = n_1 + n_2 + \cdots + n_j$. We will also use the notation $|n|^{j+1} = n_{j+1} + \cdots + n_m$ so that $|n| = |n|_j + |n|^{j+1}$. We need to show that

$$R^{-(m+M/2+1/2)+|n|/2} B R^{i+1+|k|-j} A_i^{(M+1)} R^{M+j} R A_{i+1} R \cdots R A_m \tilde{R}$$

is bounded independent of s and λ. So, we calculate

$$\begin{aligned}
&R^{-(m+M/2+1/2)+|n|/2} B R^{i+1+|k|-j} A_i^{(M+1)} R^{M+j} R A_{i+1} R \cdots R A_m \tilde{R}\\
&\quad = R^{-(m+M/2+1/2-|n|/2)} B R^{(|k|+|n|_{i-1})/2} R^{(m+M/2+1/2-|n|/2)}\\
&\qquad \times R^{|k|/2} R^{-(m+M/2+j-i-1/2)+|n|^i/2} A_i^{(M+1)} R^{M+j} R A_{i+1} R \cdots R A_m \tilde{R}\\
&\quad = \left(R^{-(m+M/2+1/2-|n|/2)} B R^{(|k|+|n|_{i-1})/2} R^{(m+M/2+1/2-|n|/2)}\right) R^{|k|/2}\\
&\qquad \times \left(R^{-(M/2+m+j-i-1/2)+|n|^i/2} A_i^{(M+1)} R^{[(M/2+m+j-i-1/2)-|n|^i/2]+(n_i+M+1)/2}\right)\\
&\qquad \times \left(R^{-(m-i-1)+|n|^{i+1}/2} A_{i+1} R^{(m-i-1-|n|^{i+1}/2)+n_{i+1}/2}\right) \cdots\\
&\qquad \times \left(R^{-1+(n_{m-1}+n_m)/2} A_{m-1} R^{1-(n_{m-1}+n_m)/2+n_{m-1}/2}\right) \left(R^{n_m/2} A_m\right) \tilde{R}.
\end{aligned}$$

Then each bracketed term in the last expression is bounded independent of s and λ by an application of Lemma 6.10, while

$$\| R^{|k|/2} \| \leqslant \left(\frac{1}{1-a}\right)^{|k|/2} \quad \text{and} \quad \| \tilde{R} \| \leqslant \frac{1}{1-a}$$

by Lemma 5.1 parts (a) and (b) and the condition on the operator B. We remark that the cases when $|k| = 0$ are included in this calculation. $\square$

7. The resolvent cocycle

The standing assumptions for the computations in this section and Section 8 are that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ semifinite spectral triple. We denote the dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ by p, and we suppose that $p \geqslant 1$. For $r > 0$, $(1 + \mathcal{D}^2)^{-p/2-r}$ is trace class, and if $u \in \mathcal{A}$ is unitary, the spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$ is given by

$$\frac{1}{C_{p/2+r}} \int_0^1 \tau\left(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-p/2-r}\right) dt,$$

where the constant is

$$C_{p/2+r} = \int_{-\infty}^{\infty} (1 + x^2)^{-p/2-r}\, dx = \frac{\Gamma((p-1)/2 + r)\Gamma(1/2)}{\Gamma(p/2 + r)}.$$

We saw at the end of Section 5 that we may replace this formula with

$$C_{p/2+r} sf(\mathcal{D}, u^*\mathcal{D}u) = \int_0^\infty S\tau(q(1 + \tilde{\mathcal{D}}^2 + s^2 + s\{\tilde{\mathcal{D}}, q\})^{-p/2-r})\, ds, \tag{11}$$

for any $r > 0$ where we remind the reader that we are now computing $sf(\mathcal{D}, u^*\mathcal{D}u) = -sf(\mathcal{D}, u\mathcal{D}u^*)$. It is important to observe that the left-hand side of Eq. (11) provides a meromorphic continuation of the function of r defined by the integral on the right-hand side.

This meromorphic continuation has simple poles at $r = (1-p)/2-k$, $k = 0, 1, 2, \ldots$. The residue of the left-hand side at $r = (1 - p)/2$ is precisely $sf(\mathcal{D}, u^*\mathcal{D}u)$ so we may write

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{2\pi i} \int_\gamma C_{p/2+z} sf(\mathcal{D}, u^*\mathcal{D}u)\, dz,$$

where $\gamma = \{z = (1 - p)/2 + \varepsilon e^{i\theta},\ 0 \leqslant \theta \leqslant 2\pi\}$ for a suitably small ε. Our aim now is to compute these residues from the integral formula. We note however that our previous estimates (Sections 5.1, 5.2) do not allow us to use the formula in Eq. (11) for the analytic continuation of the right-hand side to a deleted neighbourhood of the critical point $r = (1 - p)/2$.

In order to obtain such a formula we need to perform further manipulations on this integral. These involve expanding the integrand using the Cauchy formula, the resolvent expansion, and the pseudodifferential calculus. Then we make some estimates to show that we can discard remainders in these expansions and that the terms left over do indeed analytically continue to a deleted neighbourhood of the critical point

$r = (1-p)/2$. These continuations exist by virtue of the isolated spectral dimension hypothesis. We note some facts here.

Observation 1. Let $N = [p/2] + 1$ be the least integer strictly greater than $p/2 \geqslant 1/2$. Observe that if p is an odd integer, $N = (p+1)/2$, while if p is an even integer, $N = p/2 + 1$. More generally there is always a $\delta > 0$ so that $N \geqslant (p+\delta)/2$.

Observation 2. Let l be the vertical line $\{\lambda : Re(\lambda) = a\}$ where $0 < a < 1/2$ is fixed. We have already observed in a previous remark that the line l is uniformly bounded away from the spectrum of $(1 + \tilde{\mathcal{D}}^2 + s\{\tilde{\mathcal{D}}, q\} + s^2)$ by $(1-a)$ as s varies since $(1 + \tilde{\mathcal{D}}^2 + s\{\tilde{\mathcal{D}}, q\} + s^2) = 1 + (\tilde{\mathcal{D}} + sq)^2$. This implies, in particular, that $\| \tilde{R}_s(\lambda) \| \leqslant \sqrt{v^2 + (1-a)^2}^{-1} \leqslant 1/(1-a)$ for all s and λ. In some of our estimates it also helps to have $\| \{\tilde{\mathcal{D}}, q\} \| < \sqrt{2}$. This can be achieved by scaling $\tilde{\mathcal{D}}$; since both our spectral flow formula and our final cocycle formulas are invariant under this multiplicative change of scale we are free to rescale for each individual q.

Observation 3. In Section 5.3 we began with a QC^∞ semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and looked at various operators affiliated with $\tilde{\mathcal{N}} = M_2 \otimes M_2 \otimes \mathcal{N}$. In particular, we considered

$$\tilde{\mathcal{D}} = \sigma_2 \otimes 1_2 \otimes \mathcal{D}, \quad q = \sigma_3 \otimes \begin{pmatrix} 0 & -iu^{-1} \\ iu & 0 \end{pmatrix} \quad \text{and}$$

$$\{\tilde{\mathcal{D}}, q\} = \sigma_1 \otimes \begin{pmatrix} 0 & [\mathcal{D}, u^{-1}] \\ -[\mathcal{D}, u] & 0 \end{pmatrix}.$$

Now clearly, $|\tilde{\mathcal{D}}|_1 = 1_2 \otimes 1_2 \otimes |\mathcal{D}|_1$ while both q and $\{\tilde{\mathcal{D}}, q\}$ are in the algebra $M_2 \otimes M_2 \otimes (OP^0_{\mathcal{D}})$. Letting $\tilde{\delta}_1$ and δ_1 be the derivations induced by $|\tilde{\mathcal{D}}|_1$ and $|\mathcal{D}|_1$, we see that $\tilde{\delta}_1 = Id \otimes Id \otimes \delta_1$, so

$$OP^0_{\tilde{\mathcal{D}}} = M_2 \otimes M_2 \otimes (OP^0_{\mathcal{D}}).$$

That is, both q and $\{\tilde{\mathcal{D}}, q\}$ are in OP^0 relative to the operator $\tilde{\mathcal{D}}$ which satisfies the same summability condition as $\mathcal{D}$.

7.1. Resolvent expansion of the spectral flow

We require two estimates to guarantee that various operators which arise from the Cauchy formula and the resolvent expansion are trace class. We present these as separate lemmas as we will use them repeatedly. The techniques we use in these lemmas are indicative of the methods employed in the remainder of the proof.

Lemma 7.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ semifinite spectral triple of dimension $p \geqslant 1$. Let m be a nonnegative integer, and for $j = 0, \ldots, m$ let $A_j \in OP^0$. Let l be the vertical* *line* $v \mapsto \lambda = a + iv$ *for* $v \in \mathbf{R}$ *and* $0 < a < 1/2$, $R_s(\lambda) = (\lambda - (1 + s^2 + \tilde{\mathcal{D}}^2))^{-1}$ *and* $\tilde{R}_s(\lambda) = (\lambda - (1 + s^2 + \tilde{\mathcal{D}}^2 + s\{\tilde{\mathcal{D}}, q\}))^{-1}$. *Then for* $r \in \mathbf{C}$ *and* $Re(r) > 0$ *the operator*

$$B(s) = \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 R_s(\lambda) A_2 \cdots R_s(\lambda) A_m \tilde{R}_s(\lambda)\, d\lambda,$$

is trace class for $m > p/2$ *and the function* $s^m \parallel B(s) \parallel_1$ *is integrable on* $[0, \infty)$ *when*

$$p + \varepsilon < 1 + m \quad \text{and} \quad 1 + \varepsilon < m + 2Re(r).$$

Proof. By Lemma 6.11 we have

$$\begin{aligned} B(s) &= \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 \cdots R_s(\lambda) A_m \tilde{R}_s(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} \sum_{|n|=0}^{M} C(n) A_0 A_1^{(n_1)} \cdots A_m^{(n_m)} R_s(\lambda)^{m+|n|} \tilde{R}_s(\lambda)\, d\lambda \\ &\quad + \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} P \tilde{R}_s(\lambda)\, d\lambda, \end{aligned}$$

where P is of order $-M - 1 - 2m$ and $|n| = n_1 + \cdots + n_m$. Now by Lemma 6.10,

$$A_0 A_1^{(n_1)} \cdots A_m^{(n_m)} R_s(\lambda)^{|n|/2}$$

is bounded independently of s, λ. Thus, provided that $m > p/2$ (so that $R_s(\lambda)^{m+|n|/2}$ is trace class); using Lemma 6.12 to estimate the remainder term; and noting that since r is complex and $\lambda \in l$,

$$|\lambda^{-p/2-r}| = e^{Re[(-p/2-r)Log(\lambda)]} \leqslant |\lambda|^{-p/2-Re(r)} e^{|Im(r)|\pi/2} = C_r |\lambda|^{-p/2-Re(r)},$$

we find that $\|B(s)\|_1$ is at most:

$$\begin{aligned} &\sum_{|n|=0}^{M} \frac{C'(n)}{2\pi}\int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{\,-p/2-Re(r)} \parallel R_s(\lambda)^{m+|n|/2} \parallel_1 \parallel \tilde{R}_s(\lambda) \parallel dv \\ &\quad + \frac{C}{2\pi}\int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{\,-p/2-Re(r)} \parallel R_s(\lambda)^{m+M/2+1/2} \parallel_1 \parallel \tilde{R}_s(\lambda) \parallel dv \\ &\leqslant \sum_{|n|=0}^{M} C''(n) \Bigg(\int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{\,-p/2-Re(r)} \sqrt{(1/2+s^2-a)^2+v^2}^{\,-m-|n|/2+(p+\varepsilon)/2} \end{aligned}$$

$$\times \sqrt{(1+s^2-a-sc)^2+v^2}^{\,-1} dv\Big)$$

$$+C' \int_{-\infty}^{\infty} \left(\sqrt{a^2+v^2}^{\,-p/2-Re(r)} \sqrt{(1/2+s^2-a)^2+v^2}^{\,-M/2-1/2-m+(p+\varepsilon)/2}\right.$$

$$\left. \times \sqrt{(1+s^2-a-sc)^2+v^2}^{\,-1} dv\right). \tag{12}$$

The integrals over v are finite for $p/2+Re(r)+m+|n|/2 > (p+\varepsilon)/2$, which (with r, p, m as above) is always true, and $p/2+Re(r)+m+M/2+1/2 > (p+\varepsilon)/2$, which is also always true. Now multiply Eq. (12) by s^m and integrate over $[0, \infty)$. An application of Lemma 5.4 shows that the integral arising from the remainder converges if

$$M > -m-2+p+\varepsilon \quad \text{and} \quad M > -m-2Re(r)+\varepsilon.$$

Since M is positive, the second holds trivially while the first will hold if $M \geqslant p/2-2+\varepsilon$. The other terms are all finite provided

$$m+1 > p+\varepsilon \quad \text{and} \quad m+2Re(r) > 1+\varepsilon.$$

This completes the proof. $\square$

Lemma 7.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ semifinite spectral triple of dimension $p \geqslant 1$. Let m be a nonnegative integer, and for $j = 0, \ldots, m$ let $A_j \in OP^{k_j}$, $k_j \geqslant 0$. Let l be the vertical line described above, $R_s(\lambda) = (\lambda - (1+s^2+\tilde{\mathcal{D}}^2))^{-1}$. Then for $Re(r) > 0$ the operator*

$$B(s) = \frac{1}{2\pi i} \int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 R_s(\lambda) A_2 \cdots R_s(\lambda) A_m R_s(\lambda)\, d\lambda,$$

is trace class for $Re(r)+m-|k|/2 > 0$ and the function $s^\alpha \times \| B(s) \|_1$ is integrable on $[0, \infty)$ when

$$1+\alpha+|k|-2m < 2(Re(r)-\varepsilon).$$

Remark. Observe that the estimate in Lemma 7.2 is vastly superior to that in Lemma 7.1. This is because the resolvents are all of the same kind, and we may employ Cauchy's formula to do integrals, whereas in Lemma 7.1 we are forced to employ trace estimates under the integral.

Proof of Lemma 7.2. We employ Lemma 6.11 again to find

$$\begin{aligned}B(s) &= \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 R_s(\lambda) A_2 \cdots R_s(\lambda) A_m R_s(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} \sum_{|n|=0}^{M} C(n) A_0 A_1^{(n_1)} \cdots A_m^{(n_m)} R_s(\lambda)^{m+1+|n|}\, d\lambda \\ &\quad + \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} P\, d\lambda,\end{aligned}$$

where the remainder P is of order $-M-3-2m+|k|$. We will ignore the remainder for a moment, and observe that the integrals over λ in all the other terms may be performed using Cauchy's formula

$$f^{(b)}(z) = \frac{b!}{2\pi i}\int_C \frac{f(\lambda)}{(\lambda-z)^{b+1}}\, d\lambda$$

and the derivative formula

$$\frac{d^b}{d\lambda^b}\lambda^{-p/2-r} = (-1)^b \frac{\Gamma(p/2+b+r)}{\Gamma(p/2+r)} \lambda^{-p/2-r-b}.$$

There are two subtle points here. One, is pulling the unbounded operator $A_0 A_1^{(n_1)} \cdots A_m^{(n_m)}$ out of the integral, and the second is the application of Cauchy's Theorem in the operator setting. The second point is handled as in the previous application of Cauchy's Theorem in the introduction to Section 6.2. The first difficulty is overcome by noting that

$$\begin{aligned}&\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} R_s(\lambda)^{m+1+|n|}\, d\lambda \\ &\quad = (-1)^{m+|n|}\frac{\Gamma(m+|n|+p/2+r)}{\Gamma(p/2+r)(m+|n|)!}(1+\tilde{\mathcal{D}}^2+s^2)^{-p/2-r-m-|n|}\end{aligned}$$

and that both the integrand and the RHS map our core $\mathcal{H}_\infty$ into itself, so that evaluating our integrals on vectors in $\mathcal{H}_\infty$, we see that we can "push" $A_0 A_1^{(n_1)} \cdots A_m^{(n_m)}$ (which is defined on all of $\mathcal{H}_\infty$) through the integral sign. This yields (modulo the remainder term)

$$\begin{aligned}B(s) &= \sum_{|n|=0}^{M} (-1)^{m+|n|}\frac{\Gamma(m+|n|+p/2+r)}{\Gamma(p/2+r)(m+|n|)!} C(n) A_0 A_1^{(n_1)} \cdots A_m^{(n_m)} \\ &\quad \times (1+\tilde{\mathcal{D}}^2+s^2)^{-p/2-r-m-|n|}. \qquad (13)\end{aligned}$$

Now, there is $B \in OP^0$ with $A_0 A_1^{(n_1)} \cdots A_m^{(n_m)} = B(1+\tilde{\mathcal{D}}^2)^{+|n|/2+|k|/2}$ so that,

$$A_0 \cdots A_m^{(n_m)}(1+\tilde{\mathcal{D}}^2+s^2)^{-|k|/2-|n|/2} = B(1+\tilde{\mathcal{D}}^2)^{+|n|/2+|k|/2}(1+\tilde{\mathcal{D}}^2+s^2)^{-|k|/2-|n|/2}$$

is uniformly bounded independent of s. So (again modulo the remainder term)

$$\| B(s) \|_1 \leqslant \sum_{|n|=0}^{M} C_n' \| (1+\tilde{\mathcal{D}}^2+s^2)^{-p/2-r-m-|n|/2+|k|/2} \|_1, \tag{14}$$

and this is finite when $Re(r)+m+|n|/2-|k|/2>0$. For the worst case ($|n|=0$) we obtain the condition in the statement of the lemma.

For the remainder term we have, using Lemmas 6.12 and 5.3,

$$\begin{aligned}\left\| \int_l \lambda^{-p/2-r} P \, d\lambda \right\|_1 &\leqslant C_r \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-p/2-Re(r)} \| R_s(\lambda)^{M/2+3/2+m-|k|/2} \|_1 \, dv \\ &\leqslant C_r \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-p/2-Re(r)} \\ &\quad \times \sqrt{(1/2+s^2-a)^2+v^2}^{-M/2-3/2-m+|k|/2+(p+\varepsilon)/2} dv. \end{aligned} \tag{15}$$

Thus the remainder term is trace class for $M > \varepsilon + |k| - 1 - 2Re(r) - 2m$, which can always be arranged.

Finally, adding Eqs. (14) and (15), multiplying by s^α, and integrating over $[0,\infty)$ gives (using Lemma 5.2)

$$\begin{aligned}&\int_0^\infty s^\alpha \| B(s) \|_1 \, ds \\ &\leqslant \sum_{|n|=0}^{M} C_n' \int_0^\infty s^\alpha (1/2+s^2)^{\varepsilon-Re(r)-m-|n|/2+|k|/2} ds \\ &\quad + C_r \int_0^\infty s^\alpha \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{-p/2-Re(r)} \\ &\quad \times \sqrt{(1/2+s^2-a)^2+v^2}^{-M/2-3/2-m+|k|/2+(p+\varepsilon)/2} dv \, ds.\end{aligned}$$

Inspection shows that the non-remainder terms are finite for

$$\alpha + 2\varepsilon - 2Re(r) - 2m - |n| + |k| < -1,$$

which gives the final statement of the lemma upon considering the worst case, $|n| = 0$. Using Lemma 5.4, the remainder term is finite for

$$M > \alpha - 2 - 2m + |k| + p + \varepsilon,$$

which may always be arranged. □

Lemma 7.3. *With the notation as set out at the beginning of this subsection and with* $R_s(\lambda) = (\lambda - 1 - \tilde{\mathcal{D}}^2 - s^2)^{-1}$, $\tilde{R}_s(\lambda) = (\lambda - (1 + \tilde{\mathcal{D}}^2 + s^2 + s\{\tilde{\mathcal{D}}, q\}))^{-1}$ *we have for* $Re(r) > 0$ *and any positive integer* $M > p - 1$:

$$\begin{aligned} &S\tau(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r}) \\ &= \sum_{m=1,odd}^{M} s^m S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\,d\lambda\right) \\ &\quad + s^{M+1} S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^{M+1} \tilde{R}_s(\lambda)\,d\lambda\right). \end{aligned}$$

Proof. We will use Cauchy's formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\lambda)}{(\lambda - z)}\,d\lambda$$

and the resolvent expansion (easily proved by induction on M)

$$\tilde{R}_s(\lambda) = \sum_{m=0}^{M}\left(R_s(\lambda)s\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda) + \left(R_s(\lambda)s\{\tilde{\mathcal{D}},q\}\right)^{M+1}\tilde{R}_s(\lambda),$$

valid for $\lambda \in l$ to expand $q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r}$. Since q and $\{\tilde{\mathcal{D}},q\}$ are in OP^0 by Observation 3 we can employ the previous lemmas to first use Cauchy's formula to obtain

$$\begin{aligned} &S\tau(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r}) \\ &= S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q(\lambda - (1+\tilde{\mathcal{D}}^2+s\{\tilde{\mathcal{D}},q\}+s^2))^{-1}\,d\lambda\right) \end{aligned}$$

and then apply the resolvent expansion to arrive at

$$S\tau\left(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r}\right)$$

$$= S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r}\sum_{m=1,odd}^{M} s^m q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\,d\lambda\right)$$
$$+s^{M+1}S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^{M+1}\tilde{R}_s(\lambda)\,d\lambda\right).$$

Separating out the remainder term is valid, because $M+1>p>p/2$ ensures, by Lemma 7.1, that it is trace-class.

We have retained only odd terms in the resolvent expansion, because the even terms vanish under the supertrace. For if we consider a single term in the sum with k $\{\tilde{\mathcal{D}},q\}$'s, we find (since $\rho^2=1$ and ρ commutes with $\tilde{\mathcal{D}}$ and Γ, but anticommutes with q)

$$S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r}\rho^2 q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^k R_s(\lambda)\,d\lambda\right)$$
$$= (-1)^{k+1}S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r}\rho q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^k R_s(\lambda)\rho\,d\lambda\right)$$
$$= (-1)^{k+1}S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^k R_s(\lambda)\,d\lambda\right).$$

So if k is even we get zero. This argument does not apply to the remainder term

$$s^{M+1}S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^{M+1}\tilde{R}_s(\lambda)\,d\lambda\right).$$

as ρ neither commutes nor anticommutes with $\tilde{R}_s(\lambda)$.

Now, we may apply the conclusion of Lemma 7.2 to the nonremainder terms to see that each integral in the sum is trace-class. Hence we may move the trace through the sum and obtain the final statement of the Lemma. We note that we cannot push the trace through the integrals in the nonremainder terms as these integrands are not trace-class: we did *not* apply the actual pseudodifferential expansion in the proof of Lemma 7.2 to each of these terms to obtain trace-class integrands. $\square$

The main result of this subsection is the following lemma.

Lemma 7.4. *Let $N=[p/2]+1$ be the least positive integer strictly greater than $p/2$. Then there is a δ', $0<\delta'<1$ such that*

$$sf(\mathcal{D},u^*\mathcal{D}u)C_{p/2+r}$$

$$= \int_0^\infty S\tau(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r})\,ds$$
$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m S\tau\left(\int_l \lambda^{-p/2-r} q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\,d\lambda\right) ds + holo,$$

where holo is a function of r holomorphic for $Re(r) > -p/2+\delta'/2$.

Proof. Writing $A = \{\tilde{\mathcal{D}}, q\}$, we have by Hölder's inequality

$$\| (R_s(\lambda)A)^{2N} \|_1 \leqslant (\| R_s(\lambda)A \|_{2N})^{2N} \leqslant \| A \|^{2N} \| R_s(\lambda) \|_{2N}^{2N} = \| A \|^{2N} \| R_s(\lambda)^{2N} \|_1 .$$

Consequently, we can estimate the super-trace of the remainder term from Lemma 7.3 using Lemmas 5.3 and 5.4:

$$\int_0^\infty s^{2N} \left\| \int_l \lambda^{-p/2-r} q \left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^{2N} \tilde{R}_s(\lambda)\,d\lambda \right\|_1 ds$$
$$\leqslant C \int_0^\infty s^{2N} \int_{-\infty}^\infty \sqrt{a^2+v^2}^{\,-p/2-Re(r)} \| R_s(\lambda)^{2N} \|_1 \| \tilde{R}_s(\lambda) \| \, dv\,ds.$$

Then Lemmas 5.3 and 5.1 give the bound

$$\leqslant C' \int_0^\infty \int_{-\infty}^\infty s^{2N} \frac{(a^2+v^2)^{-p/4-Re(r)/2}((1/2+s^2-a)^2+v^2)^{(p+\varepsilon)/4-N}}{((1+s^2-a-sc)^2+v^2)^{1/2}}\,dv\,ds,$$

where $c = \| \{\tilde{\mathcal{D}}, q\} \|$. We now apply Lemma 5.4 with $J = 2N$, $M = p/2+r$, $A = 1$ and $K = 2N-(p+\varepsilon)/2$. A simple check shows that the preceding integral is finite for $Re(r) > -N+(1+\varepsilon)/2$. Letting $N = p/2+\delta/2$ where $1 \geqslant \delta > 0$ by Observation 1 of this Section, we see that the integral is finite for $Re(r) > -p/2+(1-\delta+\varepsilon)/2$. Choose $\varepsilon < \delta$ and then $\delta' = (1-\delta+\varepsilon)$.

To see that the remainder term:

$$F(r) = \frac{1}{2\pi i}\int_0^\infty s^{2N} \int_l \lambda^{-p/2-r} S\tau\left(q(R_s(\lambda)\{\tilde{\mathcal{D}},q\})^{2N}\tilde{R}_s(\lambda)\right) d\lambda\,ds$$

is defined and holomorphic for r in the half-plane, $Re(r) > -p/2+\delta'/2$, we first fix r. Since $\lambda \in l$, $|\lambda^{-p/2-r}| \leqslant C_r|\lambda|^{-p/2-Re(r)}$, so the integral converges by Lemma 5.4. We will show that $F'(r)$ is given by the above integral with $\lambda^{-p/2-r}$ replaced by $-\lambda^{-p/2-r}Log(\lambda)$. To see that this new integral converges, we observe that since r is fixed, there exists a fixed $\eta > 0$ with $Re(r)-\eta > -p/2+\delta'/2$, and so

$$|\lambda^{-p/2-r}Log(\lambda)| \leqslant C|\lambda|^{-p/2-Re(r)+\eta}|\lambda|^{-\eta}|Log(\lambda)| \leqslant C'|\lambda|^{-p/2-(Re(r)+\eta)}.$$

So, that integral converges by Lemma 5.4. Now, the difference quotient $1/h(F(r+h)-F(r))$ is the above integral, with $\lambda^{-p/2-r}$ replaced by $\lambda^{-p/2-r}(1/h)(\lambda^h-1)$. We use Taylor's theorem applied to $f(h)=\lambda^{-h}$ about 0 (λ fixed) to estimate the difference $(1/h)(\lambda^h-1)-(-Log\,\lambda)$. So, let C be the circle $z=\eta e^{i\theta}$ for $0\leqslant\theta\leqslant 2\pi$, and consider $h<\eta/2$. By ([A, pp. 125–126]):

$$\frac{\lambda^h-1}{h}-(-Log\lambda)=f_2(h)h \ \ where \ \ f_2(h)=\frac{1}{2\pi i}\int_C\frac{\lambda^{-z}dz}{z^2(z-h)}.$$

Moreover,

$$|f_2(h)|\leqslant\frac{\max\{|\lambda^{-z}|:z\in C\}}{\eta\eta/2}\leqslant\frac{2|\lambda|^{\eta}e^{\eta\pi/2}}{\eta^2}=(const)|\lambda|^{\eta}.$$

Finally,

$$\left|\lambda^{-p/2-r}\frac{\lambda^{-\eta}-1}{h}-\lambda^{-p/2-r}(-Log\lambda)\right|\leqslant|\lambda^{-p/2-r}|\cdot|f_2(h)h|\leqslant C_2|h||\lambda|^{-p/2-Re(r)}.$$

Hence the quotient $1/h(F(r+h)-F(r))$ differs from the formal derivative by $|h|$ times an absolutely convergent integral with nonnegative integrand. $\square$

7.2. The resolvent cocycle

At this point it is interesting to perform the 'super' bit of the trace, so that we have an expression which only depends on our original spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$. The computation begins by recalling the definition of q and $\{\tilde{\mathcal{D}},q\}$. Then,

$$\begin{aligned}&q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\\&\quad=i(-1)^{(m-1)/2}\sigma_3\sigma_1^m\\&\qquad\otimes\begin{pmatrix}u^*R[\mathcal{D},u]R[\mathcal{D},u^*]\cdots[\mathcal{D},u]R & 0\\ 0 & uR[\mathcal{D},u^*]R[\mathcal{D},u]\cdots[\mathcal{D},u^*]R\end{pmatrix}.\end{aligned}$$

On the right-hand side, by an abuse of notation, we have written $R=(\lambda-(1+\mathcal{D}^2+s^2))^{-1}$. The grading operator is $\Gamma=\sigma_2\otimes\sigma_3\otimes 1$. Since $\sigma_2\sigma_3\sigma_1^m=i1_2$ for m odd, we have (writing Tr_4 for the operator-valued trace which maps $\tilde{\mathcal{N}}=M_2\otimes M_2\otimes\mathcal{N}\to\mathcal{N}$)

$$\begin{aligned}&Tr_4(\Gamma qR_s(\lambda)\{\tilde{\mathcal{D}},q\}R_s(\lambda)\cdots\{\tilde{\mathcal{D}},q\}R_s(\lambda))\\&\quad=2(-1)^{(m+1)/2}(u^*R[\mathcal{D},u]R[\mathcal{D},u^*]\cdots[\mathcal{D},u]R-uR[\mathcal{D},u^*]R[\mathcal{D},u]\cdots[\mathcal{D},u^*]R).\end{aligned}$$

Consequently, there is a δ' with $0 < \delta' < 1$ such that for $r > 0$

$$\int_0^\infty S\tau(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r}\,ds$$
$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m S\tau\left(q\int_l \lambda^{-p/2-r}\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\,d\lambda\right)ds$$
$$+holo$$
$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m \frac{1}{2}\tau\left(Tr_4\Gamma q\left(\int_l \lambda^{-p/2-r}\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\,d\lambda\right)\right)ds$$
$$+holo$$
$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m \frac{1}{2}\tau\left(\int_l \lambda^{-p/2-r}Tr_4\left(\Gamma q\left(R_s(\lambda)\{\tilde{\mathcal{D}},q\}\right)^m R_s(\lambda)\,d\lambda\right)\right)ds$$
$$+holo$$
$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}(-1)^{(m+1)/2}\int_0^\infty s^m \times \tau\left(\int_l \lambda^{-p/2-r}\left(u^*R[\mathcal{D},u]R[\mathcal{D},u^*]\right.\right.$$
$$\cdots[\mathcal{D},u]R - uR[\mathcal{D},u^*]R[\mathcal{D},u]$$
$$\left.\left.\cdots[\mathcal{D},u^*]R\right)d\lambda\right)ds + holo,$$

where *holo* is a function of r holomorphic for $Re(r) > -p/2+\delta'/2$.

This last expression suggests how we might define a one-parameter family of functionals which form a cyclic cocycle (up to functionals holomorphic for $Re(r) > -p/2+\delta'/2$) which we term the *resolvent cocycle*. It will eventuate that the Connes–Moscovici residue cocycle may be derived from this resolvent cocycle. We have been using the notation $R_s(\lambda)$ for the $\tilde{\mathcal{D}}$ resolvent and R for the $\mathcal{D}$ resolvent. For the remainder of this section, $R_s(\lambda) = (\lambda-(1+\mathcal{D}^2+s^2))^{-1}$, the $\mathcal{D}$ resolvent.

Definition 7.5. For $m\geqslant 0$, operators $A_0,\ldots,A_m$, $A_i\in OP^{k_i}$, and $2Re(r) > k_0+\cdots+k_m-2m$ define

$$\langle A_0,\ldots,A_m\rangle_{m,s,r} = \tau\left(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r}A_0R_s(\lambda)A_1\cdots A_mR_s(\lambda)\,d\lambda\right).$$

‘Expectations’ like these were first considered by Higson in [H]. The conditions on the orders and on r are sufficient for the trace to be well-defined, by Lemma 7.2.

Lemma 7.6. *For any integers $m \geqslant 0, k \geqslant 1$ and operators $A_0, \ldots, A_m$ with $A_j \in OP^{k_j}$, and $2Re(r) > k + \sum k_j - 2m$, we may choose r with $Re(r)$ sufficiently large such that*

$$\begin{aligned} k &\int_0^\infty s^{k-1} \langle A_0, \ldots, A_m \rangle_{m,s,r}\, ds \\ &= -2 \sum_{j=0}^m \int_0^\infty s^{k+1} \langle A_0, \ldots, A_j, 1, A_{j+1}, \ldots, A_m \rangle_{m+1,s,r}\, ds. \end{aligned}$$

Proof. For $Re(r)$ sufficiently large, determined by Lemma 7.2, we have

$$\begin{aligned} &\frac{d}{ds} s^k \frac{1}{2\pi i} \tau \left(\int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 \cdots A_m R_s(\lambda)\, d\lambda \right) \\ &= k s^{k-1} \frac{1}{2\pi i} \tau \left(\int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 \cdots A_m R_s(\lambda)\, d\lambda \right) \\ &\quad + \sum_{j=0}^m 2 s^{k+1} \frac{1}{2\pi i} \tau \left(\int_l \lambda^{-p/2-r} A_0 R_s(\lambda) A_1 \cdots A_j R_s(\lambda)^2 A_{j+1} \cdots A_m R_s(\lambda)\, d\lambda \right). \end{aligned}$$

The fundamental theorem of calculus completes the argument. □

Lemma 7.7. *For $m \geqslant 0$, $Re(r)$ sufficiently large, $k \geqslant 1$ and $A_j \in OP^{(k_j)}$, $j = 0, \ldots, m$, we have*

$$\int_0^\infty s^k \langle A_0, \ldots, A_m \rangle_{m,s,r}\, ds = \int_0^\infty s^k \langle A_m, A_0, \ldots, A_{m-1} \rangle_{m,s,r}\, ds.$$

Proof. The size of $Re(r)$ is determined by Definition 7.5 via Lemma 7.2. We can repeat Lemma 7.6 until the integrand of

$$\langle A_0, \ldots 1, \ldots, 1, \ldots, A_m \rangle_{m+M,s,r}$$

is trace-class. The cyclicity of the trace allows us to conclude. □

Lemma 7.8. *For operators $A_0, \ldots, A_m$, $A_j \in OP^{k_j}$, $k_j \geqslant 0$, and $Re(r)$ sufficiently large we have*

$$\begin{aligned} &-\langle A_0, \ldots, [\mathcal{D}^2, A_j], \ldots, A_m \rangle_{m,s,r} \\ &\quad = \langle A_0, \ldots, A_{j-1} A_j, \ldots, A_m \rangle_{m-1,s,r} - \langle A_0, \ldots, A_j A_{j+1}, \ldots, A_m \rangle_{m-1,s,r}, \end{aligned}$$

and for $k \geqslant 1$

$$\int_0^\infty s^k \langle \mathcal{D}A_0, A_1, \ldots, A_m\rangle_{m,s,r}\, ds = \int_0^\infty s^k \langle A_0, A_1, \ldots, A_m\mathcal{D}\rangle_{m,s,r}\, ds.$$

Proof. The first identity follows from observing that

$$-[\mathcal{D}^2, A_j] = R_s(\lambda)^{-1}A_j - A_j R_s(\lambda)^{-1},$$

and cancelling neighbouring $R_s(\lambda)$'s. The second follows by applying Lemma 7.7, then commuting $\mathcal{D}$ past the (hidden) $R_s(\lambda)$ and applying Lemma 7.7 again

$$\begin{aligned}
&\int_0^\infty s^k \langle \mathcal{D}A_0, A_1, \ldots, A_m\rangle_{m,s,r}\, ds \\
&\quad = \int_0^\infty s^k \langle A_m, \mathcal{D}A_0, A_1, \ldots, A_{m-1}\rangle_{m,s,r}\, ds \\
&\quad = \int_0^\infty s^k \langle A_m\mathcal{D}, A_0, A_1, \ldots, A_{m-1}\rangle_{m,s,r}\, ds \\
&\quad = \int_0^\infty s^k \langle A_0, A_1, \ldots, A_m\mathcal{D}\rangle_{m,s,r}\, ds \qquad \Box
\end{aligned}$$

Suspecting that the spectral flow is given by pairing a cocycle with the Chern character of a unitary, we remove the normalisation coming from $Ch_m(u)$ from our resolvent formula to define a cocycle. The factor of $\sqrt{2\pi i}$ is for compatability with the Kasparov product, [Co2].

Definition 7.9. Let $\mathcal{C}(m)$ denote the constant $\frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)}$. Then, for $Re(r) > -m/2+1/2$ and $da = [\mathcal{D}, a]$ we define $\phi_m^r : \mathcal{A}^{m+1} \to \mathbf{C}$ by

$$\phi_m^r(a_0, \ldots, a_m) = \mathcal{C}(m) \int_0^\infty s^m \langle a_0, da_1, .., da_m\rangle_{m,s,r}\, ds.$$

By Lemma 7.2 the condition on r ensures that the integral converges. We note that this constant $\mathcal{C}(m)$ is distinct from $C(k)$ which takes a multi-index k as its argument. The next result captures the main new idea of this section.

Proposition 7.10. *For* $p \geqslant 1$ *the collection of functionals* $\phi^r = \{\phi_m^r\}_{m=1}^{2N-1}$, *m odd, is such that*

$$(B\phi_{m+2}^r + b\phi_m^r)(a_0, \ldots, a_{m+1}) = 0 \quad m = 1, 3, \ldots, 2N-3, \quad (B\phi_1^r)(a_0) = 0, \quad (16)$$

where the $a_i \in \mathcal{A}$. Moreover, there is a δ', $0 < \delta' < 1$ such that $b\phi^r_{2N-1}(a_0, \dots, a_{2N})$ is a holomorphic function of r for $Re(r) > -p/2 + \delta'/2$.

Proof. We start with the computation of the coboundaries of the ϕ^r_m. The definition of the operator B and ϕ^r_{m+2} gives

$$\begin{aligned}
&(B\phi^r_{m+2})(a_0, \dots, a_{m+1})\\
&\quad = \sum_{j=0}^{m+1} \phi^r_{m+2}(1, a_j, \dots, a_{m+1}, a_0, \dots, a_{j-1})\\
&\quad = \sum_{j=0}^{m+1} \mathcal{C}(m+2) \int_0^\infty s^{m+2} \langle 1, [\mathcal{D}, a_j], \dots, [\mathcal{D}, a_{j-1}]\rangle_{m+2,s,r}\, ds.
\end{aligned}$$

Using Lemmas 7.7 and 7.6, this is equal to

$$\begin{aligned}
&= \sum_{j=0}^{m+1} \mathcal{C}(m+2) \int_0^\infty s^{m+2} \langle [\mathcal{D}, a_0], \dots, [\mathcal{D}, a_{j-1}], 1, [\mathcal{D}, a_j], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+2,s,r}\, ds\\
&= -\mathcal{C}(m+2)\frac{(m+1)}{2} \int_0^\infty s^m \langle [\mathcal{D}, a_0], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,s,r}\, ds.
\end{aligned}$$

We observe at this point that $\mathcal{C}(m+2)(m+1)/2 = \mathcal{C}(m)$, using the functional equation for the Gamma function.

Next we write $[\mathcal{D}, a_0] = \mathcal{D}a_0 - a_0\mathcal{D}$ and anticommute the second $\mathcal{D}$ through the remaining $[\mathcal{D}, a_j]$ using $\mathcal{D}[\mathcal{D}, a_j] + [\mathcal{D}, a_j]\mathcal{D} = [\mathcal{D}^2, a_j]$. This gives us

$$\begin{aligned}
&(B\phi^r_{m+2})(a_0, \dots, a_{m+1})\\
&\quad = -\mathcal{C}(m) \int_0^\infty s^m \langle \mathcal{D}a_0 - a_0\mathcal{D}, [\mathcal{D}, a_1], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,s,r}\, ds\\
&\quad = -\mathcal{C}(m) \int_0^\infty s^m \Big(\langle \mathcal{D}a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,s,r}\\
&\qquad - \langle a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}, a_{m+1}]\mathcal{D}\rangle_{m+1,s,r}\Big)\, ds\\
&\qquad -\mathcal{C}(m) \int_0^\infty s^m \sum_{j=1}^{m+1} (-1)^j \langle a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}^2, a_j], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,s,r}\, ds,\\
&\quad = -\mathcal{C}(m) \int_0^\infty s^m \sum_{j=1}^{m+1} (-1)^j \langle a_0, [\mathcal{D}, a_1], .., [\mathcal{D}^2, a_j], .., [\mathcal{D}, a_{m+1}]\rangle_{m+1,s,r}\, ds,
\end{aligned} \tag{17}$$

where the last line follows from the second identity of Lemma 7.8. Observe that for ϕ_1^r we have

$$(B\phi_1^r)(a_0) = \frac{\mathcal{C}(1)}{2\pi i}\int_0^\infty s\tau\left(\int_l \lambda^{-p/2-r} R_s(\lambda)[\mathcal{D}, a_0]R_s(\lambda)\,d\lambda\right) ds = 0,$$

by an easy variant of the argument in Lemma 7.8. We now compute the Hochschild coboundary of ϕ_m^r. From the definitions we have

$$\begin{aligned}(b\phi_m^r)(a_0,\dots,a_{m+1}) &= \phi_m^r(a_0a_1, a_2,\dots,a_{m+1})\\ &\quad+\sum_{i=1}^m(-1)^i\phi_m^r(a_0,\dots,a_ia_{i+1},\dots,a_{m+1})\\ &\quad+\phi_m^r(a_{m+1}a_0, a_1,\dots,a_m)\\ &= \mathcal{C}(m)\int_0^\infty s^m\big(\langle a_0a_1,[\mathcal{D},a_2],\dots,[\mathcal{D},a_{m+1}]\rangle_{m,s,r}\\ &\quad+\sum_{i=1}^m(-1)^i\langle a_0,[\mathcal{D},a_1],\dots,a_i[\mathcal{D},a_{i+1}]\\ &\quad+[\mathcal{D},a_i]a_{i+1},\dots,[\mathcal{D},a_{m+1}]\rangle_{m,s,r}\\ &\quad+\langle a_{m+1}a_0,[\mathcal{D},a_1],\dots,[\mathcal{D},a_m]\rangle_{m,s,r}\big)\,ds.\end{aligned}$$

We now reorganise the terms so that we can employ the first identity of Lemma 7.8. So

$$\begin{aligned}&(b\phi_m^r)(a_0,\dots,a_{m+1})\\ &= \mathcal{C}(m)\int_0^\infty s^m\Big(\big(\langle a_0a_1,[\mathcal{D},a_2],\dots,[\mathcal{D},a_{m+1}]\rangle_{m,r,s}\\ &\quad-\langle a_0,a_1[\mathcal{D},a_2],\dots,[\mathcal{D},a_{m+1}]\rangle_{m,r,s}\big)\\ &\quad-\mathcal{C}(m)\int_0^\infty s^m\big(\langle a_0,[\mathcal{D},a_1]a_2,\dots,[\mathcal{D},a_{m+1}]\rangle_{m,r,s}\\ &\quad-\langle a_0,[\mathcal{D},a_1],a_2[\mathcal{D},a_3],\dots,[\mathcal{D},a_{m+1}]\rangle_{m,r,s}\big)\\ &\quad\vdots\\ &\quad-\mathcal{C}(m)\int_0^\infty s^m\big(\langle a_0,[\mathcal{D},a_1],\dots,[\mathcal{D},a_m]a_{m+1}\rangle_{m,r,s}\\ &\quad-\langle a_{m+1}a_0,[\mathcal{D},a_1],\dots,[\mathcal{D},a_m]\rangle_{m,r,s}\big)\Big)\,ds\end{aligned}$$

$$= \sum_{j=1}^{m+1} (-1)^j \mathcal{C}(m) \int_0^\infty s^m \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}^2, a_j], \cdots$$
$$\ldots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,r,s}\, ds, \tag{18}$$

where to get the term with $[\mathcal{D}^2, a_{m+1}]$ we have used Lemma 7.7.

For $m = 1, 3, 5, \ldots, 2N-3$ comparing Eqs. (18) and (17) now shows that

$$(B\phi^r_{m+2} + b\phi^r_m)(a_0, \ldots, a_{m+1}) = 0.$$

So we just need to check the claim that $b\phi^r_{2N-1}$ is holomorphic for $Re(r) > -p/2+\delta'$ for some suitable δ'. By Lemma 7.2

$$\int_0^\infty s^m \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}^2, a_j], \ldots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,r,s}\, ds \tag{19}$$

is finite when

$$Re(r) > \frac{1+m+2\varepsilon+1}{2} - (m+1) = \frac{1-p}{2} + \frac{2\varepsilon - 1 - m + p}{2}.$$

For $m = 2N-1 = 2[p/2]+1$ this reduces to

$$Re(r) > \frac{1-p}{2} + \frac{2\varepsilon - 2[p/2] - 2 + p}{2}.$$

As $p/2-1 < [p/2] \leqslant p/2$, we see that $0 > -2[p/2]-2+p \geqslant -2$, and we may always find an $\varepsilon > 0$ so that $2\varepsilon - 2[p/2] - 2 + p < 0$. The proof that (18) is holomorphic for $m = 2N-1$ is similar to the analyticity proof in Lemma 7.4. □

Thus we have an odd (b, B) cochain $(\phi^r_m)^{2N-1}_{m=1,odd}$ with values in the functions which are holomorphic in a half-plane. Moreover, modulo those functions holomorphic in the half-plane $Re(r) > -p/2-\delta$, our resolvent cochain is a cocycle. This, together with Lemma 7.4, actually proves Part (1) of Theorem 4.2.

8. The residue cocycle

This section proves Theorem 4.2. It is organised into 3 subsections. In the first of these we begin with the resolvent expansion of the spectral flow formula Lemma 7.4. We use the pseudodifferential calculus to derive from the resolvent expansion a new expression stated as Proposition 8.1. This leaves us with a formula for spectral flow that involves an integral over the parameter s. By integrating out the s dependence in the formula of Proposition 8.1 we find in Section 8.2 (Proposition 8.2) a spectral flow formula which involves a sum of zeta functions. One immediately recognises that individual terms in this formula may be obtained from our resolvent cocycle of Section 7 by using the pseudodifferential calculus. Thus, from the resolvent cocycle we derive in the final subsection, in Theorem 8.4, the residue cocycle. Our final formula for the spectral flow follows immediately by evaluating the residue cocycle on $Ch_*(u^*)$.

8.1. Pseudodifferential expansion of the spectral flow

The aim of this section is to establish just one formula which is summarised in the following result. Recall that $N = [p/2] + 1$.

Proposition 8.1. *There is a* δ', $0 < \delta' < 1$ *such that*

$$sf(\mathcal{D}, u^*\mathcal{D}u)C_{p/2+r} = \int_0^\infty S\tau(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r})\,ds$$

$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m S\tau\Bigg(\int_l \lambda^{-p/2-r}\sum_{|k|=0}^{2N-1-m} C(k)q\{\tilde{\mathcal{D}},q\}^{(k_1)}$$

$$\cdots\{\tilde{\mathcal{D}},q\}^{(k_m)}R_s(\lambda)^{m+1+|k|}\,d\lambda\Bigg)\,ds + holo$$

$$= \sum_{m=1,odd}^{2N-1}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|+m}C(k)\frac{\Gamma(p/2+r+m+|k|)}{\Gamma(p/2+r)(|k|+m)!}$$

$$\times\int_0^\infty s^m S\tau\left(q\{\tilde{\mathcal{D}},q\}^{(k_1)}\cdots\{\tilde{\mathcal{D}},q\}^{(k_m)}(1+\tilde{\mathcal{D}}^2+s^2)^{-(p/2+r+m+|k|)}\right)ds + holo,$$

where holo is a function of r holomorphic for $Re(r) > -p/2 + \delta'/2$. *Consequently the sum of functions on the right-hand side has an analytic continuation to a deleted neighbourhood of* $r = (1-p)/2$ *(given by the left-hand side) with at worst a simple pole at* $r = (1-p)/2$.

Proof. The proof starts by applying the pseudodifferential expansion to each of the terms of the resolvent expansion of Lemma 7.4. Thus, modulo the holo of Lemma 7.4:

$$\int_0^\infty S\tau(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r})\,ds$$

$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m S\tau\left\{\int_l \lambda^{-p/2-r}\sum_{|k|=0}^{2N-1-m}C(k)q\{\tilde{\mathcal{D}},q\}^{(k_1)}\cdots\right.$$

$$\cdots \{\tilde{\mathcal{D}}, q\}^{(k_m)} R_s(\lambda)^{m+1+|k|}\, d\lambda \Bigg\}\, ds$$

$$+\frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m S\tau\left(\int_l \lambda^{-p/2-r} P_m(s,\lambda)\, d\lambda\right) ds.$$

Here we are considering multi-indices k with m terms, with $|k| = k_1 + \cdots + k_m$, and using Lemma 6.12 with $M = 2N - 1 - m$ to give us that

$$P_m(s,\lambda)(\lambda - (1 + \tilde{\mathcal{D}}^2 + s^2))^{N+(m+2)/2}$$

is bounded where the bound is uniform in s and λ. We refer to the last expression in this formula as the error term. We want to compute the trace norm of the error term, and show that it is integrable in s. We have by Lemma 5.3

$$\begin{aligned}
&\left\| \int_0^\infty s^m \int_l \lambda^{-p/2-r} P_m\, d\lambda\, ds \right\|_1 \\
&\quad \leqslant C \int_0^\infty s^m \int_{-\infty}^\infty \sqrt{a^2+v^2}^{\,-p/2-r} \, \| \, (\lambda - (1 + \tilde{\mathcal{D}}^2 + s^2))^{-N-(m+2)/2} \, \|_1 \; dv\, ds \\
&\quad \leqslant C' \int_0^\infty s^m \int_{-\infty}^\infty \sqrt{a^2+v^2}^{\,-p/2-r} \\
&\qquad \times \sqrt{(1/2 + s^2 - a)^2 + v^2}^{\,-N-(m+2)/2+(p+\varepsilon)/2} \, dv\, ds.
\end{aligned}$$

We now apply Lemma 5.4. Since $N \geqslant p/2 + \delta/2$ and $0 < \varepsilon < 1$, this tells us that the remainder term is finite for $r > -p/2 + (1-\delta)/2$. That the error term is holomorphic as a function of r is proved exactly as in the proof of analyticity in Lemma 7.4. We now argue as in the proof of Lemma 7.2 using the Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\lambda)}{(\lambda - z)^{n+1}}\, d\lambda$$

and the derivative formula

$$\frac{d^b}{d\lambda^b}\lambda^{-p/2-r} = (-1)^b \frac{\Gamma(p/2+b+r)}{\Gamma(p/2+r)}\lambda^{-p/2-r-b}.$$

Applying the Cauchy formula produces the expression in the statement of the proposition. $\square$

We now have an expression which is a finite sum of terms, and which can be used to compute the residues at $r = (1-p)/2$.

8.2. Eliminating the s dependence in Proposition 8.1

In this section we integrate out the s dependence in the formula of Proposition 8.1 to obtain the following new expression.

Proposition 8.2. *There is a δ', $0 < \delta' < 1$ such that*:

$$
\begin{aligned}
&sf(\mathcal{D}, u^*\mathcal{D}u)C_{p/2+r}\\
&\quad = \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} C(k)(-1)^{|k|+m} \frac{\Gamma((m+1)/2)\Gamma(p/2+r+|k|+(m-1)/2)}{2(|k|+m)!\Gamma(p/2+r)}\\
&\qquad \times S\tau\left(q\{\tilde{\mathcal{D}}, q\}^{(k_1)}\cdots\{\tilde{\mathcal{D}}, q\}^{(k_m)}(1+\tilde{\mathcal{D}}^2)^{-(p/2+r+|k|+(m-1)/2)}\right) + holo,
\end{aligned}
$$

where holo is a function of r holomorphic for $Re(r) > -p/2 + \delta'/2$. Consequently the sum of functions on the right-hand side has an analytic continuation to a deleted neighbourhood of $r = (1-p)/2$ (given by the left-hand side) with at worst a simple pole at $r = (1-p)/2$. Moreover, if $[p] = 2n$ is even, each of the top terms with $|k| = 2N - 1 - m$ are holomorphic at $r = (1-p)/2$, including the one term with $m = 2N - 1$.

Remark. We started with a spectral flow formula, Eq. (9), given by an integral defined only for $r > 0$ (holomorphic for $Re(r) > 1/2$ by the comments following Eq. (9)). The above result is telling us that the sum of zeta functions differs from the left-hand side by a function which is holomorphic for $Re(r) > -p/2 + \delta'/2$. Thus this sum of zeta functions has only a single simple pole in this same region $Re(r) > -p/2+\delta'$ and the residue at $r = (1-p)/2$ is the spectral flow with no need to invoke the isolated spectral dimension hypothesis (although of course we cannot conclude that the individual terms are meromorphic in this region). This proves part (2) of Theorem 4.2.

Proof. We first need to interchange the s integral in Proposition 8.1 with the supertrace. To this end we show that for $Re(r) > -m - |k|/2$ the function $A_{k,m} : \mathbf{R}_+ \to \mathcal{L}^1(\tilde{\mathcal{N}})$ given by

$$
A_{k,m}(s) = q\{\tilde{\mathcal{D}}, q\}^{(k_1)}\cdots\{\tilde{\mathcal{D}}, q\}^{(k_m)}(1+\tilde{\mathcal{D}}^2+s^2)^{-p/2-r-m-|k|}
$$

is continuous. Fix s_0 and let $B = q\{\tilde{\mathcal{D}}, q\}^{(k_1)}\cdots\{\tilde{\mathcal{D}}, q\}^{(k_m)}$. Since $B \in OP^{|k|}$, by Lemma 6.7 $B = A(1+\tilde{\mathcal{D}}^2+s_0^2)^{|k|/2}$ where $A \in OP^0$. Then, for $t \geqslant 0$ the resolvent equation gives:

$$
\begin{aligned}
&\| B(1+\tilde{\mathcal{D}}^2+s_0^2)^{-p/2-r-m-|k|} - B(1+\tilde{\mathcal{D}}^2+t^2)^{-p/2-r-m-|k|} \|_1\\
&\quad \leqslant \| A \| \cdot \| (1+\tilde{\mathcal{D}}^2+s_0^2)^{-p/2-r-m-|k|/2} \|_1
\end{aligned}
$$

$$\times|t-s|\cdot \parallel (1+\tilde{\mathcal{D}}^2+t^2)^{-p/2-r-m-|k|} \parallel$$
$$\leqslant \parallel A \parallel \cdot \parallel (1+\tilde{\mathcal{D}}^2+s_0^2)^{-p/2-r-m-|k|/2} \parallel_1 \cdot|t-s|\cdot 1.$$

So, we can employ the Bochner integral to get

$$\int_0^\infty s^m S\tau(A_{k,m}(s))\,ds = S\tau\left(\int_0^\infty s^m A_{k,m}(s)\,ds\right).$$

The s integral on the right-hand side can now be performed using the following Laplace transform computation. This requires an interchange of order of integration, which follows from a small variant of the argument of Lemma 9.1 of [CP2]. So

$$\begin{aligned}
&\int_0^\infty s^m(1+s^2+\tilde{\mathcal{D}}^2)^{-(|k|+m+p/2+r)}\,ds\\
&\quad= \frac{1}{\Gamma(p/2+r+|k|+m)}\int_0^\infty s^m\int_0^\infty u^{|k|+m+p/2+r-1}e^{-(1+\tilde{\mathcal{D}}^2)u}e^{-s^2u}\,du\,ds\\
&\quad= \frac{1}{\Gamma(|k|+m+p/2+r)}\int_0^\infty u^{|k|+m+p/2+r-1}\left(\int_0^\infty s^m e^{-s^2u}\,ds\right)e^{-(1+\tilde{\mathcal{D}}^2)u}\,du\\
&\quad= \frac{\Gamma((m+1)/2)}{2\Gamma(|k|+m+p/2+r)}\int_0^\infty u^{|k|+(m-1)/2+p/2+r-1}e^{-(1+\tilde{\mathcal{D}}^2)u}\,du\\
&\quad= \frac{\Gamma((m+1)/2)\Gamma(|k|+(m-1)/2+p/2+r)}{2\Gamma(|k|+m+p/2+r)}(1+\tilde{\mathcal{D}}^2)^{-(|k|+(m-1)/2+p/2+r)}.
\end{aligned}$$

Observe the factor of two introduced by this computation. Since both the integrand and the final result map the core subspace $\tilde{\mathcal{H}}_\infty$ into itself, we can "push" the operator $q\{\tilde{\mathcal{D}},q\}^{(k_1)}\cdots\{\tilde{\mathcal{D}},q\}^{(k_m)}$ through the integral sign.

Applying the above calculations we now obtain a formula in which the s and λ dependence has been integrated out. This yields

$$\begin{aligned}
&\int_0^\infty S\tau\left(q(1+\tilde{\mathcal{D}}^2+s^2+s\{\tilde{\mathcal{D}},q\})^{-p/2-r}\right)ds\\
&\quad= \sum_{m=1,odd}^{2N-1}\sum_{|k|=0}^{2N-1-m} C(k)(-1)^{|k|+m}\frac{\Gamma((m+1)/2)\Gamma(p/2+r+|k|+(m-1)/2)}{2(|k|+m)!\Gamma(p/2+r)}\\
&\qquad\times S\tau\left(q\{\tilde{\mathcal{D}},q\}^{(k_1)}\cdots\{\tilde{\mathcal{D}},q\}^{(k_m)}(1+\tilde{\mathcal{D}}^2)^{-p/2-r-|k|-(m-1)/2}\right)+holo,
\end{aligned}$$

where *holo* is a function of r holomorphic for $Re(r) > -p/2 + \delta'/2$. We now observe that

$$\left| S\tau \left(q\{\tilde{\mathcal{D}}, q\}^{(k_1)} \cdots \{\tilde{\mathcal{D}}, q\}^{(k_m)} (1+\tilde{\mathcal{D}}^2)^{-p/2-r-|k|-(m-1)/2} \right) \right|$$
$$\leqslant C \parallel (1+\tilde{\mathcal{D}}^2)^{-p/2-r-|k|/2-(m-1)/2} \parallel_1 .$$

This is finite when

$$r > \frac{1}{2} - \frac{m}{2} - \frac{|k|}{2} = \frac{1-p}{2} + \frac{p-m-|k|}{2}.$$

So whenever $m + |k| > p$, we obtain a term holomorphic at $r = (1-p)/2$ which may be discarded. Thus for $[p] = 2n$ we see $[p/2] = n$ and $2N - 1 = 2([p/2]+1) - 1 = 2[p/2] + 1 = 2n + 1 > p$. So for $m = 2N - 1$ (and so $|k| = 0$), $m + |k| > p$ and this term is holomorphic at $r = (1-p)/2$. Similarly when $|k| = 2N - 1 - m$ for any $m = 1, 3, \ldots, 2N-1$, we have $m + |k| = 2N - 1 > p$. $\square$

Observe that at this point we have almost proved Part (2) of Theorem 4.2. The only outstanding item is the precise form of the constants, and these are identified in the next section.

8.3. The residue cocycle

In this subsection we will prove Theorem 4.2. We have now come to the point where we need to assume that our semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension. So this means we may analytically continue our zeta functions to a deleted neighbourhood of the critical point. We denote the analytic continuation of a function analytic in a right half-plane to a deleted neighbourhood of the critical point by putting the function in boldface. Thus define the functionals for each integer $j \geqslant 0$:

$$S\tau_j (q\{\tilde{\mathcal{D}}, q\}^{(k_1)} \cdots \{\tilde{\mathcal{D}}, q\}^{(k_m)} (1+\tilde{\mathcal{D}}^2)^{-(|k|+m/2)})$$
$$= res_{r=(1-p)/2} (r-(1-p)/2)^j \mathbf{S}\tau(\mathbf{q}\{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_1})} \cdots$$
$$\cdots \{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_m})} (\mathbf{1}+\tilde{\mathcal{D}}^{\mathbf{2}})^{-(\mathbf{r}-(\mathbf{1}-\mathbf{p})/\mathbf{2}+|\mathbf{k}|+\mathbf{m}/\mathbf{2})}).$$

Analogously we can define τ_j in terms of the trace τ. Observe that taking residues and performing the ‘super’ bit of the trace commute. Thus we have the Laurent expansion

$$\mathbf{S}\tau(\mathbf{q}\{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_1})} \cdots \{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_m})} (\mathbf{1}+\tilde{\mathcal{D}}^{\mathbf{2}})^{-(\mathbf{r}-(\mathbf{1}-\mathbf{p})/\mathbf{2}+|\mathbf{k}|+\mathbf{m}/\mathbf{2})})$$
$$= \sum_{j \geqslant 0} (r-(1-p)/2)^{-j-1} S\tau_j (q\{\tilde{\mathcal{D}}, q\}^{(k_1)} \cdots$$
$$\cdots \{\tilde{\mathcal{D}}, q\}^{(k_m)} (1+\tilde{\mathcal{D}}^2)^{-(|k|+m/2)}) + holo.$$

Here *holo* is a function of r holomorphic for $Re(r) > (1-p)/2 - \delta$.

Now we start from the formula of Proposition 8.2 and take residues at $r = (1-p)/2$ of both sides. By our assumption that the zeta functions analytically continue to a deleted neighbourhood of this critical point (and using the boldface notational convention as above)

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+m} \frac{\Gamma((m+1)/2)}{2} \alpha(k)$$
$$\times res\left(\frac{\Gamma(|k| + (p+m-1)/2 + z)}{\Gamma(p/2+z)} \mathbf{S}\tau(\mathbf{q}\{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_1})} \right.$$
$$\left. \cdots \{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_m})} (\mathbf{1} + \tilde{\mathcal{D}}^{\mathbf{2}})^{-(\mathbf{z}-(\mathbf{1}-\mathbf{p})/\mathbf{2}+|\mathbf{k}|+\mathbf{m}/\mathbf{2})}) \right).$$

Now we have, writing the integer $|k| + (m-1)/2$ as $h = |k| + (m-1)/2$,

$$\frac{\Gamma(p/2+h+z)}{\Gamma(p/2+z)} = \frac{\Gamma((z-(1-p)/2)+h+1/2)}{\Gamma((z-(1-p)/2)+1/2)}$$
$$= \prod_{j=0}^{h-1} \frac{\Gamma((z-(1-p)/2)+j+1+1/2)}{\Gamma((z-(1-p)/2)+j+1/2)} = \prod_{j=0}^{h-1}((z-(1-p)/2)+j+1/2)$$
$$= \sum_{j=0}^{h}(z-(1-p)/2)^j \sigma_{h,j}.$$

Recall that the $\sigma_{h,j}$'s are the symmetric functions of the half-integers $1/2, 3/2, \ldots, h-1/2$. So finally, we obtain the following formula for $sf(\mathcal{D}, u^*\mathcal{D}u)$:

$$\sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+m} \frac{\Gamma((m+1)/2)}{2} \alpha(k) \sum_{j=0}^{h} \sigma_{h,j}$$
$$\times res\left((z-(1-p)/2)^j \mathbf{S}\tau(\mathbf{q}\{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_1})} \cdots \{\tilde{\mathcal{D}}, \mathbf{q}\}^{(\mathbf{k_m})} (\mathbf{1} + \tilde{\mathcal{D}}^{\mathbf{2}})^{-(\mathbf{z}-(\mathbf{1}-\mathbf{p})/\mathbf{2}+|\mathbf{k}|+\mathbf{m}/\mathbf{2})})\right)$$
$$= \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+m} \frac{\Gamma((m+1)/2)}{2} \alpha(k)$$
$$\times \sum_{j=0}^{h} \sigma_{h,j} S\tau_j(q\{\tilde{\mathcal{D}}, q\}^{(k_1)} \cdots \{\tilde{\mathcal{D}}, q\}^{(k_m)} (1+\tilde{\mathcal{D}}^2)^{-(|k|+m/2)})$$

$$= \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{(m+1)/2+|k|} \frac{\Gamma((m+1)/2)}{2} \alpha(k) \sum_{j=0}^{h} \sigma_{h,j} \tau_j$$
$$\Big((u[\mathcal{D}, u^*]^{(k_1)} \cdots [\mathcal{D}, u^*]^{(k_m)} - u^*[\mathcal{D}, u]^{(k_1)} \cdots [\mathcal{D}, u]^{(k_m)})(1+\mathcal{D}^2)^{-|k|-m/2} \Big).$$

The last line follows from converting the super trace into an ordinary trace. The normalisation of $\frac{1}{2}$ for the super trace has been cancelled by a trace over the 2×2 identity matrix, so we still have a factor of a $\frac{1}{2}$ which arose from the s-integral.

To understand this formula in terms of cyclic (co)homology and Chern characters, we show that our spectral flow formula is obtained by pairing a cyclic cocycle with the Chern character of a unitary. We note that our resolvent cocycle ϕ^r pairs with normalised chains so that by Lemma 3.1,

$$\phi^r(Ch_*(u)) = -\phi^r(Ch_*(u^*))$$

modulo functions holomorphic for $Re(r) > (1-p)/2 - \delta$.

In the next theorem we introduce the functionals that form the residue cocycle and from which we obtain Part (3) of Theorem 4.2.

Theorem 8.3. *Assume that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an odd QC^∞ spectral triple with isolated spectral dimension $p \geqslant 1$. For m odd, define functionals ϕ_m by*

$$\phi_m(a_0, \ldots, a_m) = \sqrt{2\pi i} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|} \alpha(k)$$
$$\times \sum_{j=0}^{h} \sigma_{h,j} \tau_j \left(a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)} (1+\mathcal{D}^2)^{-|k|-m/2} \right),$$

where $h = |k| + (m-1)/2$. Then $\phi = (\phi_m)$ is a (b, B)-cocycle and

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{\sqrt{2\pi i}} \sum_{m=1}^{2N-1} \langle \phi_m, Ch_m(u) \rangle.$$

Proof. The proof is implicit in what has gone before. However we set out the brief direct argument that the ϕ_m define a cocycle as this is by far the most difficult part of [CoM].

We know that the resolvent cocycle is given by

$$\phi_m^r(a_0, \ldots, a_m) = \mathcal{C}(m) \int_0^\infty s^m \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}, a_m] \rangle_{m,s,r}\, ds.$$

We now employ the same arguments as in the proof of Propositions 8.1 to obtain *modulo functions of r holomorphic for Re*$(r) > (1-p)/2 - \delta$

$$\phi_m^r(a_0,\ldots,a_m) = \frac{\mathcal{C}(m)}{2\pi i}\sum_{|k|=0}^{2N-1-m} C(k)\int_0^\infty s^m \tau\left(\int_\ell \lambda^{-p/2-r} A_k R_s(\lambda)^{|k|+m+1}\,d\lambda\right)ds,$$

where $A_k = a_0[\mathcal{D},a_1]^{(k_1)}\ldots[\mathcal{D},a_m]^{(k_m)}$ are fixed (for this discussion) operators of order $|k|$. We find, by the same proof as Proposition 8.2,

$$\begin{aligned}
&\phi_m^r(a_0,\ldots,a_m)\\
&\quad = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m} \frac{2C(k)(-1)^{|k|}\Gamma(p/2+r+|k|+m)}{\Gamma((m+1)/2)\Gamma(p/2+r)(|k|+m)!}\\
&\qquad \times \int_0^\infty s^m\tau(A_k(1+s^2+\mathcal{D}^2)^{-p/2-r-|k|-m})\,ds\\
&\quad = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m} \frac{(-1)^{|k|}\alpha(k)\Gamma(p/2+r+|k|+(m-1)/2)}{\Gamma(p/2+r)}\\
&\qquad \times\tau(A_k(1+\mathcal{D}^2)^{-p/2-r-|k|-(m-1)/2})\\
&\quad = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m} (-1)^{|k|}\alpha(k)\sum_{j=0}^{h}(r-(1-p)/2)^j\\
&\qquad \times\sigma_{h,j}\tau(A_k(1+\mathcal{D}^2)^{-p/2-r-|k|-(m-1)/2}),
\end{aligned}$$

where $h = |k| + (m-1)/2$ and $\alpha(k) = \frac{1}{k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}$.

Taking the residue of the analytic continuations at the critical point

$$\begin{aligned}
&res_{r=(1-p)/2}\boldsymbol{\phi}_{\mathbf{m}}^{\mathbf{r}}(\mathbf{a_0},\ldots,\mathbf{a_m})\\
&\quad = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m}\alpha(k)(-1)^{|k|}\sum_{j=0}^{h}\sigma_{h,j}\tau_j(A_k(1+\mathcal{D}^2)^{-|k|-m/2}).
\end{aligned}$$

That is

$$res_{r=(1-p)/2}\boldsymbol{\phi}_{\mathbf{m}}^{\mathbf{r}}(\mathbf{a_0},\ldots,\mathbf{a_m}) = \phi_m(a_0,\ldots,a_m).$$

The proof that $b\phi_{m-2} + B\phi_m = 0$ now follows from the fact that the ϕ_m^r satisfy this linear relation modulo functions holomorphic for $Re(r) > (1-p)/2 - \delta$ so the analytic continuations of the zeta functions satisfy a similar linear relation and so do the residues, since residues depend linearly on the analytic function. $\square$

Part (3) of Theorem 4.2 follows from the fact that the ϕ_j form a cocycle, and Lemma 3.1. We have (with (y) representing the boundary chain in the proof of Lemma 3.1)

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{2\sqrt{2\pi i}} \sum_{m=1}^{2N-1} \langle \phi_m, 2Ch_m(u) + (b+B)(y)\rangle = \frac{1}{\sqrt{2\pi i}} \sum_{m=1}^{2N-1} \langle \phi_m, Ch_m(u)\rangle$$

which is just a rewriting of the formula of Part (3) of Theorem 4.2.

Notice that we have, by this result, the correct normalisation for the pairing between cyclic cohomology and $K_1(\mathcal{A})$. Observe that since $h = |k| + (m-1)/2$, and $m \leqslant 2N-1$, $|k| \leqslant 2N - m - 1$, no matter what order the singularity of the trace is, we need only consider the first h terms in the principal part of the Laurent expansion, and

$$h \leqslant 2N - m - 1 + (m-1)/2 \leqslant 2N - (m+3)/2 \leqslant 2N - 2 \leqslant 2(p/2+1) - 2 = p.$$

Observe that this is the renormalised version of Connes–Moscovici's formula. This is, essentially, because we started from a scale invariant formula, whereas Connes–Moscovici started from the JLO cocycle. The upshot is that they obtained a formula with infinitely many terms, and then used the easily determined behaviour of the functionals τ_j under change of scale to obtain counterterms. Alternatively, they replace $\zeta \to res_{s=0}\Gamma(h+s+1/2)\zeta(s)$ by the functional

$$\zeta \to res_{s=0} \frac{\Gamma(1/2)\Gamma(h+s+1/2)}{\Gamma(s+1/2)} \zeta(s)$$

thus allowing the removal of the hypothesis of finiteness of the dimension spectrum. It is not clear to us whether one can perform this manoeuvre in the middle of their proof, or whether it is necessary for them to first consider the unrenormalised version.

Proof of Corollary 4.3. To prove that for $1 \leqslant p < 2$ (so $N = 1$) we need make no assumptions about the analytic continuation properties of zeta functions. Consider the expression for spectral flow obtained in Proposition 8.2. Specialising this formula to $N = 1$ and so $k = 0 = j$ we find

$$sf(\mathcal{D}, u^*\mathcal{D}u) C_{p/2+r} = -\frac{1}{2} S\tau \left(q\{\tilde{\mathcal{D}}, q\}(1+\tilde{\mathcal{D}}^2)^{-(p/2+r)} \right) + holo,$$

where *holo* is a function holomorphic at $r = (1-p)/2$. The super part of the trace gives

$$sf(\mathcal{D}, u^*\mathcal{D}u) C_{p/2+r} = (-1/2)\tau((u[\mathcal{D}, u^*] - u^*[\mathcal{D}, u])(1+\mathcal{D}^2)^{-p/2-r}) + holo \quad (20)$$

which, by the proof of Theorem 8.3, is (up to functions holomorphic at $r = (1-p)/2$) $(\sqrt{2\pi i})^{-1}\phi_1^r$ evaluated on $Ch_1(u^*) - Ch_1(u)$. Since $b\phi_1^r(a_0, a_1, a_2)$ is holomorphic at $r = (1-p)/2$ for all $a_0, a_1, a_2 \in \mathcal{A}$, we can write Eq. (20) as

$$sf(\mathcal{D}, u^*\mathcal{D}u)C_{p/2+r} = \frac{1}{\sqrt{2\pi i}}\phi_1^r(Ch(u)) + holo.$$

Taking residues gives

$$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{\sqrt{2\pi i}}\phi_1(Ch(u))$$

and the residue on the right necessarily exists and is given by ϕ_1 by Proposition 8.3.

Remark. In order to see how this last formula fits with Theorem 6.2 of [CPS2], we note that in [CPS2] the spectral dimension is 1 and that p is a variable which we will write here as $p = 1 + z$. Then, (after switching the roles of u and u^* and writing $\mathcal{D}$ in place of D) the formula in Theorem 6.2 of [CPS2] is calculated as

$$1/2res_{z=0}\tau(u^*[\mathcal{D}, u](1 + \mathcal{D}^2)^{-1/2-z/2}).$$

While the formula in this paper is calculated as

$$res_{z=0}\tau(u^*[\mathcal{D}, u](1 + \mathcal{D}^2)^{-1/2-z}).$$

It is a simple general fact about residues that these are identical.

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