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Interior $C^{2,\alpha}$ Regularity for Potential Functions in Optimal Transportation

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In this paper we study the continuity of second derivatives of solutions to the Monge–Ampère equation arising in optimal transportation. We obtain Hölder and more general continuity estimates for second derivatives, when the inhomogeneous term is Hölder and Dini continuous, together with corresponding regularity results for potentials.

Keywords Monge–Ampère equations; Optimal transportation; Second derivative estimates.

Mathematics Subject Classification Primary 35J60; Secondary 35B45.

1. Introduction

In this paper we study the continuity of second derivatives of solutions to the Monge–Ampère equation

$$\det\{D^2u(x) - A(x, Du)\} = f(x) \quad \text{in } \Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $A = \{A_{ij}\}$ is an $n \times n$ symmetric matrix defined in $\Omega \times \mathbb{R}^n$.

Equation (1.1) arises in optimal transportation [14, 20] and also in reflector and refractor shape design problems [8, 22]. In these applications, u is the potential function, the matrix A and the right hand side f are given by

$$A(x, Du) = D_x^2 c(x, T_u(x)), \quad (1.2)$$

$$f = |\det\{D_{xy}^2 c\}| \frac{\rho}{\rho^* \circ T_u}, \quad (1.3)$$

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where $c(\cdot, \cdot)$ is the cost function, $T_u : x \rightarrow y$ is the optimal mapping determined by

$$Du(x) = D_x c(x, y), \quad (1.4)$$

and ρ, ρ^* are mass distributions respectively in the initial domain Ω and the target domain Ω^* .

When $\rho, \rho^* > 0$, $\rho \in C^{1,1}(\Omega)$, $\rho^* \in C^{1,1}(\Omega^*)$, and Ω^* is c^* -convex with respect to Ω , the C^3 regularity of potentials has been established in [14], under suitable assumptions on c given below. When the smoothness assumption is removed and the densities are just assumed to be bounded, from above and away from zero, corresponding C^1 regularity was obtained in [3] for the reflector problem and in [19] for the general optimal transportation problem. We remark that in [19], only the boundedness of $\rho, 1/\rho^*$ from above is needed. The $C^{1,\alpha}$ estimate was obtained in [12, 13], with the optimal exponent $\alpha = \frac{1}{2n-1}$, for all dimensions n , obtained in [12]. In this paper we prove the Hölder continuity of the second derivatives D^2u for Hölder continuous densities and the continuity of D^2u for Dini continuous densities. As in [14], our approach is through *a priori* estimates for elliptic solutions of (1.1). Here we say a solution u is elliptic if the matrix $\{D^2u - A\}$ is positive definite. Our estimates correspond to previous ones on the standard Monge–Ampère equation

$$\det\{D^2u\} = f \quad (1.5)$$

obtained by Caffarelli [2] for Hölder continuous f and by Wang [21] for Dini continuous f , (see also [10] for a detailed proof).

As in [14] we assume that the cost function $c \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies the following conditions:

- (A1) For any $x, p \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $D_x c(x, y) = p$; and for any $y, q \in \mathbb{R}^n$, there is a unique $x \in \mathbb{R}^n$ such that $D_y c(x, y) = q$.
- (A2) For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\det\{D_{xy}^2 c(x, y)\} \neq 0$,
- (A3) For any $x, p \in \mathbb{R}^n$, and any $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$A_{ij,kl}(x, p) \xi_i \xi_j \eta_k \eta_l > 0, \quad (1.6)$$

where $A_{ij,kl} = D_{p_k p_l}^2 A_{ij}$ and A is given by (1.2).

In the above conditions we assume that the cost function c is defined in the whole space $\mathbb{R}^n \times \mathbb{R}^n$. This is for simplicity. In many cases the cost function is defined in a proper subset $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^n$. In this case we replace (A1) by assuming that the mapping $D_x c(x, \cdot)$ is one-to-one on $\mathcal{U}_x$ for each x , and the mapping $D_y c(\cdot, y)$ is one-to-one on $\mathcal{U}_y$ for each y , where $\mathcal{U}_x = \{y \in \mathbb{R}^n \mid (x, y) \in \mathcal{U}\}$ and $\mathcal{U}_y = \{x \in \mathbb{R}^n \mid (x, y) \in \mathcal{U}\}$. Accordingly in (A2) and (A3) we assume that $(x, y) \in \mathcal{U}$, $p \in D_x c(\mathcal{U})$.

Denote

$$\omega_f(r) = \sup\{|f(x) - f(y)| : |x - y| < r\} \quad (1.7)$$

the oscillation of f . We say f is Dini continuous if

$$\int_0^1 \frac{\omega_f(r)}{r} dr < \infty. \quad (1.8)$$

The main estimates of this paper are as follows.

Theorem 1.1. *Assume the cost function c satisfies (A1)–(A3) and f satisfies*

$$C_1 \leq f \leq C_2 \quad (1.9)$$

for some positive constants $C_1, C_2 > 0$. Let $u \in C^2(\Omega)$ be an elliptic solution of (1.1). Then for all $x, y \in \Omega_\delta$, we have the estimate

$$|D^2u(x) - D^2u(y)| \leq C \left[d + \int_0^d \frac{\omega_f(r)}{r} + d \int_d^1 \frac{\omega_f(r)}{r^2} \right], \quad (1.10)$$

where $d = |x - y|$, $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$, $C > 0$ depends only on $n, \delta, C_1, C_2, A, \sup |Du|$, and the modulus of continuity of Du . It follows that:

- (i) *If f is Dini continuous, then the modulus of continuity of D^2u can be estimated by (1.10) above;*
- (ii) *If $f \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$, then*

$$\|u\|_{C^{2,\alpha}(\Omega_\delta)} \leq C \left[1 + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)} \right]; \quad (1.11)$$

- (iii) *If $f \in C^{0,1}(\Omega)$, then*

$$|D^2u(x) - D^2u(y)| \leq Cd[1 + \|f\|_{C^{0,1}} |\log d|] \quad \forall x, y \in \Omega_\delta. \quad (1.12)$$

From the estimates in Theorem 1.1, we conclude the corresponding regularity results for potentials. A function $u \in C^0(\Omega)$ is a potential for the cost function c and densities ρ, ρ^* if it is the first component of a pair $(u, v) \in C^0(\Omega) \times C^0(\Omega^*)$ minimizing the functional

$$I(u, v) = \int_\Omega u\rho + \int_{\Omega^*} v\rho^*, \quad (1.13)$$

over the set of all $(u, v) \in C^0(\Omega) \times C^0(\Omega^*)$, satisfying

$$u(x) + v(y) \geq c(x, y), \quad \forall x \in \Omega, y \in \Omega^*. \quad (1.14)$$

The function v is called the dual potential. The existence of potentials, (unique up to additive constants), follows from condition (A1). Moreover they are semi-convex, almost everywhere elliptic solutions of equation (1.1) and its dual, (where x is replaced by y), respectively; (see [14]).

Theorem 1.2. *Assume the cost function c satisfies (A1)–(A3) and ρ, ρ^* are Dini continuous, and uniformly bounded and positive, in Ω, Ω^* respectively. Then if the target domain Ω^* is c^* -convex, with respect to Ω , any potential function $u \in C^2(\Omega)$ and is an*

elliptic solution of (1.1), satisfying the estimates (1.10)–(1.12), with f given by (1.3), where C depends on $n, \delta, C_1, C_2, \Omega, \Omega^*$ and c . Consequently if the densities ρ, ρ^* are Hölder continuous and Ω, Ω^* are c, c^* -convex with respect to each other, then the optimal mapping T_u is a $C^{1,\alpha}$ diffeomorphism from Ω to Ω^* for some $\alpha > 0$.

The c^* -convexity of Ω^* , (as introduced in [14]), is necessary and also permits the removal of the dependence on Du in the constant in (1.10), [19]. Moreover it was shown by approximation in [14] that if Ω^* is not c^* -convex with respect to Ω , there exist smooth, positive mass distributions such that the potential function is not C^1 smooth.

After the normalization in Section 3, the positivity in assumption (A3) is no longer under control and we need to consider its weak form, called (A3w) in [18]. We say the cost function c satisfies (A3w) if (1.6) is replaced by

$$A_{ij,kl}(x, p)\xi_i\xi_j\eta_k\eta_l \geq 0, \quad (1.15)$$

for all $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$. By a related approximation argument, in [13] Loeper also proved (A3w) is necessary for regularity, namely if c does not satisfy (A3w), there exist smooth, positive mass distributions such that the potential function is not C^1 smooth.

This paper is arranged as follows. In Section 2 we establish the interior second derivative estimates for elliptic solutions of (1.1), assuming the cost function satisfies (A3w), which is crucial for the proof of Theorem 1.1. Our proof uses Pogorelov's approach for the standard Monge–Ampère equation, as for the global estimate in [18], but the details are much more complicated. In Section 3 we include various preliminary results, developed in [12, 19], which are used in Section 4 to prove Theorem 1.1. Examples of cost functions and associated matrices satisfying conditions (A3) and (A3w) are presented in [14, 18].

2. Interior Second Derivative Estimate

In this section, we establish interior second derivative estimates for solutions u to the equation

$$\det w_{ij} = 1 \quad \text{in } \Omega, \quad (2.1)$$

where

$$w_{ij} = u_{ij} - A_{ij}(\cdot, Du),$$

and $A = \{A_{ij}\}$ is a given C^2 symmetric matrix valued function on $\Omega \times \mathbb{R}^n$. The interior second derivative estimate for the standard Monge–Ampère equation (1.5) was proved by Pogorelov [15], but the corresponding estimate for equation (2.1) is in general not true, as demonstrated by the Lewy–Heinz example in [17]. In this section we prove the estimate under appropriate conditions.

We assume the matrix A satisfies the weak A3 condition

$$D_{p_k p_l}^2 A_{ij}(x, p)\xi_i\xi_j\eta_k\eta_l \geq 0 \quad (2.2)$$

at all $x \in \Omega$, $p = Du(x)$, $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$, and $u \in C^{0,1}(\overline{\Omega})$ satisfies

$$u = 0 \quad \text{on } \partial\Omega. \quad (2.3)$$

We also assume that $A(x, 0) = 0$,

$$D_{p_k} A_{ij}(x, Du) u_k \xi_i \xi_j \geq A_{ij}(x, Du) \xi_i \xi_j - \kappa_0 u_\xi^2, \quad (2.4)$$

for some constant $\kappa_0 \geq 0$, and that there exists a function φ such that

$$[D_{ij}\varphi(x) - D_{p_k} A_{ij}(x, Du(x)) D_k \varphi(x)] \xi_i \xi_j \geq |\xi|^2 \quad (2.5)$$

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

Theorem 2.1. *Let $u \in C^4(\Omega)$ be an elliptic solution of (2.1). Then under the above assumptions, we have the estimate*

$$(-u)^\tau |D^2 u| \leq C, \quad (2.6)$$

where the constants C, τ depend on $n, \kappa_0, \|A\|_{C^2}, \|\varphi\|_{C^2}$ and $\sup_\Omega |Du|$.

Proof. Our approach is similar to that of Pogorelov but the details are more complicated and various modifications are necessary.

First we note that it follows from the strong comparison principle that $u < 0$ in Ω . Let

$$\rho = \eta^\tau w_{\xi\xi} e^{\frac{\beta}{2}|Du|^2 + \kappa\varphi} \quad (2.7)$$

where ξ is a unit vector, $w_{\xi\xi} = w_{ij} \xi_i \xi_j$, $\eta = -u$, and β, κ, τ are positive constants to be determined.

Suppose ρ attains its maximum at $\bar{x} \in \Omega$ and $\xi = (1, 0, \dots, 0)$. We may assume that the matrix $\{w_{ij}\}$ is diagonal at $\bar{x}$. Then the function

$$v = \tau \log \eta + \log w_{11} + \frac{1}{2} \beta |Du|^2 + \kappa \varphi \quad (2.8)$$

also attains its maximum at $\bar{x}$. Hence $Dv(\bar{x}) = 0$, $D^2 v(\bar{x}) \leq 0$, and

$$L(v)(\bar{x}) \leq 0, \quad (2.9)$$

where L is the linear operator defined by

$$L := \sum_{i,j=1}^n w^{ij}(\bar{x}) [D_{ij} - D_{p_k} A_{ij}(\bar{x}, Du(\bar{x})) D_k], \quad (2.10)$$

and $\{w^{ij}\}$ is the inverse matrix of $\{w_{ij}\}$.

We have, at $\bar{x}$,

$$D_{iv} = \frac{\tau D_i \eta}{\eta} + \frac{D_i w_{11}}{w_{11}} + \beta D_k u D_{ki} u + \kappa D_i \varphi,$$

$$D_{ii}v = \frac{\tau D_{ii}\eta}{\eta} - \frac{\tau(D_i\eta)^2}{\eta^2} + \frac{D_{ii}w_{11}}{w_{11}} - \frac{(D_iw_{11})^2}{w_{11}^2} + \beta \sum_{k=1}^n ((D_{ik}u)^2 + D_k u D_{ik}u) + \kappa D_{ii}\varphi. \quad (2.11)$$

Hence

$$Lv(\bar{x}) = \frac{\tau}{\eta} L\eta - \frac{\tau}{\eta^2} \sum_{i=1}^n \frac{(D_i\eta)^2}{w_{ii}} + \frac{1}{w_{11}} Lw_{11} - \frac{1}{w_{11}^2} \sum_{i=1}^n \frac{(D_iw_{11})^2}{w_{ii}} + \beta \sum_{k=1}^n D_k u Lu_k + \beta \sum_{i,k=1}^n \frac{1}{w_{ii}} (D_{ik}u)^2 + \kappa L\varphi. \quad (2.12)$$

In the following we estimate each term on the right hand side of (2.12). First by (2.5) we have

$$L\varphi(\bar{x}) \geq \sum w^{ii}(\bar{x}). \quad (2.13)$$

To estimate Lw_{11} and Lu_k , we differentiate equation (2.1) to get

$$w^{ii}[D_{ii}u_\xi - D_\xi A_{ii} - D_{p_l} A_{ii} D_l u_\xi] = 0, \quad (2.14)$$

$$w^{ii}[D_{ii}u_{\xi\xi} - D_{\xi\xi} A_{ii} - D_{p_k p_l} A_{ii} u_{k\xi} u_{l\xi} - D_{p_k} A_{ii} D_k u_{\xi\xi} - 2D_{\xi p_k} A_{ii} u_{k\xi}] - w^{ii} w^{jj} (D_\xi w_{ij})^2 = 0 \quad (2.15)$$

at $\bar{x}$, where ξ is a unit vector.

Letting $\xi = e_k$, the k th unit vector, we obtain from (2.14),

$$\begin{aligned} Lu_k &= \sum_i w^{ii} [D_{ii}u_k - D_{p_l} A_{ii} D_l u_k] \\ &= \sum_i w^{ii} D_k A_{ii}. \end{aligned} \quad (2.16)$$

Hence

$$\sum_k D_k u Lu_k \geq -C \sum_i w^{ii},$$

where the constant C depends on n , $\|u\|_{C^1}$ and $\|A\|_{C^2}$.

To estimate Lw_{11} , we first calculate Lu_{11} . Choosing $\xi = e_1$ in (2.15), we get

$$\begin{aligned} Lu_{11} &= \sum_i w^{ii} [D_{ii}u_{11} - D_{p_k} A_{ii} D_k u_{11}] \\ &= w^{ii} w^{jj} (D_1 w_{ij})^2 + w^{ii} [D_{11} A_{ii} + 2D_{1p_k} A_{ii} (w_{1k} + A_{1k})] \\ &\quad + w^{ii} A_{ii,kl} u_{k1} u_{l1}, \end{aligned}$$

where $A_{ij,kl} = D_{p_k p_l} A_{ij}$. For the second term above, we have the following estimate by $\|A\|_{C^2} \leq C$,

$$w^{ii} [D_{11} A_{ii} + 2D_{1p_k} A_{ii} (w_{1k} + A_{1k})] \geq -C \sum_{i,j} w^{ii} (1 + w_{jj}).$$

For the third term, by (A3w) and noticing that $\{w_{ij}\}$ is diagonal, we have

$$\begin{aligned} w^{ii} A_{ii,kl} u_{k1} u_{l1} &= w^{ii} A_{ii,kl} (w_{k1} + A_{k1}) (w_{l1} + A_{l1}) \\ &= w^{ii} [A_{ii,11} w_{11}^2 + 2A_{ii,1k} w_{11} A_{k1} + A_{ii,kl} A_{k1} A_{l1}] \\ &\geq -C w^{ii} (1 + w_{jj}). \end{aligned}$$

Hence

$$Lu_{11} \geq w^{ii} w^{jj} (D_1 w_{ij})^2 - C w^{ii} (1 + w_{jj}).$$

where again the constant C depends on n and $\|A\|_{C^2}$. Since L is linear, from (2.10) and (2.16) we then obtain

$$\begin{aligned} Lw_{11} &= Lu_{11} - LA_{11} \\ &\geq Lu_{11} - C w^{ii} (1 + w_{jj}) \\ &\geq w^{ii} w^{jj} (D_1 w_{ij})^2 - C w^{ii} (1 + w_{jj}). \end{aligned} \tag{2.17}$$

By (2.4),

$$\begin{aligned} L\eta &= \sum_i w^{ii} [-u_{ii} + D_{p_k} A_{ii} u_k] \\ &= \sum_i w^{ii} [-w_{ii} - A_{ii} + D_{p_k} A_{ii} u_k] \\ &\geq -n - \kappa_0 \sum_i w^{ii} u_i^2. \end{aligned}$$

Obviously

$$\begin{aligned} \sum_{i,k} \frac{1}{w_{ii}} (D_{ik} u)^2 &= \sum_i \frac{1}{w_{ii}} (w_{ii} + A_{ii})^2 + \sum_{k \neq i} \frac{1}{w_{ii}} A_{ik}^2 \\ &\geq w_{ii} - C w^{ii}. \end{aligned}$$

Hence from (2.12), when $|\eta| \leq \max_{\Omega} u \leq 1$ and $C \geq 1 + \kappa_0$, we obtain

$$\begin{aligned} Lv &\geq -\frac{\tau n}{\eta} - \frac{\tau C}{\eta^2} \sum_i \frac{(D_i \eta)^2}{w_{ii}} + \frac{1}{w_{11}} \sum_{i,j} \frac{(D_1 w_{ij})^2}{w_{ii} w_{jj}} \\ &\quad + \kappa w^{ii} + \beta w_{ii} - \frac{1}{w_{11}^2} \sum_i \frac{(D_i w_{11})^2}{w_{ii}} \\ &\quad - \frac{C}{w_{11}} w^{ii} (1 + w_{jj}) - \beta C w^{ii}. \end{aligned} \tag{2.18}$$

Next we estimate $\frac{\tau}{\eta^2} \sum_{i=1}^n \frac{(D_i \eta)^2}{w_{ii}}$. Observe that $\frac{(D_1 \eta)^2}{w_{11}}$ is under control. It suffices to consider $\sum_{i=2}^n \frac{(D_i \eta)^2}{w_{ii}}$. From (2.11),

$$\begin{aligned} \frac{D_i \eta}{\eta} &= -\frac{1}{\tau} \left(\frac{D_i w_{11}}{w_{11}} + \beta D_k u [w_{ki} - A_{ki}] + \kappa D_i \varphi \right) \\ \left(\frac{D_i \eta}{\eta} \right)^2 &= \frac{1}{\tau^2} \left(\frac{D_i w_{11}}{w_{11}} + \beta D_i u w_{ii} - \beta D_k u A_{ki} + \kappa D_i \varphi \right)^2 \\ &\leq \frac{4}{\tau^2} \left[\left(\frac{D_i w_{11}}{w_{11}} \right)^2 + \beta^2 (D_i u)^2 w_{ii}^2 + \beta^2 (D_k u A_{ki})^2 + \kappa^2 (D_i \varphi)^2 \right] \\ &\leq \frac{4}{\tau^2} \left(\frac{D_i w_{11}}{w_{11}} \right)^2 + \frac{C}{\tau^2} (\beta^2 w_{ii}^2 + \beta^2 + \kappa^2). \end{aligned}$$

Hence,

$$\begin{aligned} \tau C \sum_{i=2}^n \frac{1}{w_{ii}} \left(\frac{D_i \eta}{\eta} \right)^2 &\leq \frac{4C}{\tau} \sum_{i=2}^n \frac{1}{w_{ii}} \left(\frac{D_i w_{11}}{w_{11}} \right)^2 \\ &\quad + \frac{\beta^2 C}{\tau} w_{ii} + \frac{(\beta^2 + \kappa^2)C}{\tau} w^{ii}. \end{aligned}$$

We choose the constant τ sufficiently large, say,

$$\tau = \beta^2 + \kappa^2 + 8C.$$

Then we have the estimate

$$\tau C \sum_{i=2}^n \frac{1}{w_{ii}} \left(\frac{D_i \eta}{\eta} \right)^2 \leq \frac{1}{2w_{11}^2} \sum_{i=2}^n \frac{(D_i w_{11})^2}{w_{ii}} + C(w_{ii} + w^{ii}). \quad (2.19)$$

Therefore by (2.18),

$$\begin{aligned} Lv &\geq -\frac{\tau n}{\eta} - \frac{1}{2w_{11}^2} \sum_{i=2}^n \frac{(D_i w_{11})^2}{w_{ii}} + \frac{1}{w_{11}} \sum_{i,j} \frac{(D_1 w_{ij})^2}{w_{ii} w_{jj}} \\ &\quad + (\kappa - C\beta) w^{ii} + (\beta - C) w_{ii} - \frac{1}{w_{11}^2} \sum_i \frac{(D_i w_{11})^2}{w_{ii}}. \end{aligned} \quad (2.20)$$

We have

$$\begin{aligned} &\frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2 - \frac{1}{w_{11}^2} \sum_{i=1}^n \frac{(D_i w_{11})^2}{w_{ii}} - \frac{1}{2w_{11}^2} \sum_{i=2}^n \frac{(D_i w_{11})^2}{w_{ii}} \\ &= \frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2 - \frac{1}{w_{11}^2} w^{11} (D_1 w_{11})^2 - \frac{3}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 \\ &\geq \frac{1}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 + \frac{2}{w_{11}^2} \sum_{i=2}^n w^{ii} [(D_1 w_{11})^2 - (D_i w_{11})^2] \\ &\geq -C w^{ii} / w_{11}^2, \end{aligned}$$

where the last inequality follows from $w_{ij} = u_{ij} - A_{ij}$ and the following estimate

$$\begin{aligned}
 & \frac{2}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_1 w_{1i} - D_i w_{11}) (D_1 w_{1i} + D_i w_{11}) \\
 &= \frac{2}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i A_{11} - D_1 A_{1i}) (2D_i w_{11} + D_i A_{11} - D_1 A_{1i}) \\
 &\geq -\frac{1}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 - \frac{6}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i A_{11} - D_1 A_{1i})^2 \\
 &\geq -\frac{1}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 - C w^{ii} / w_{11}^2.
 \end{aligned}$$

Therefore from (2.20) we obtain

$$Lv \geq (\kappa - C\beta)w^{ii} + (\beta - C)w_{ii} - \frac{\tau n}{\eta}. \quad (2.21)$$

Choose $\beta > C + 1$ and then $\kappa > C\beta$. We then obtain

$$\eta \sum_i w_{ii}(\bar{x}) \leq C.$$

Consequently $\sup_{\Omega} \rho \leq C$. $\square$

Remark 2.1. In the proof of Theorem 2.1, the assumption (2.4) is used to obtain the estimate of $L\eta$ in (2.12). It can be removed if one assumes that the matrix function A satisfies (2.2) without the orthogonality restriction, i.e.,

$$D_{p_k p_l}^2 A_{ij}(x, p) \xi_i \xi_j \eta_k \eta_l \geq 0, \quad \text{for all } \xi, \quad \eta \in \mathbb{R}^n. \quad (2.22)$$

Indeed, using the Taylor formula and the assumption $A(x, 0) = 0$, for some $\theta \in (0, 1)$ we have

$$\begin{aligned}
 \sum_k A_{ii, p_k}(x, Du) u_k - A_{ii}(x, Du) &= \sum_k A_{ii, p_k}(x, Du) u_k - \sum_k A_{ii, p_k}(x, \theta Du) u_k \\
 &= \frac{1}{2} (1 - \theta) \sum_{k, l} A_{ii, kl}(x, \bar{\theta} Du) u_k u_l \\
 &\geq 0,
 \end{aligned}$$

where $\bar{\theta} \in (\theta, 1)$, and the last inequality is from (2.22) by letting $\xi = e_i$, the i th unit vector, and $\eta = Du$, the gradient vector of u . Therefore, we can obtain the estimate

$$\begin{aligned}
 L\eta &= \sum_i w^{ii} [-w_{ii} - A_{ii} + D_{p_k} A_{ii} u_k] \\
 &\geq -n.
 \end{aligned}$$

In particular, we recover the estimates in [15] for the standard Monge–Ampère equation which corresponds to the cost function $c(x, y) = x \cdot y$ that satisfies (2.22).

More generally, the cost function,

$$c(x, y) = x \cdot y + f(x)g(y), \quad (2.23)$$

will satisfy (A1), (A2), (2.22) and (2.4) if f and g are convex functions in $C^2(\mathbb{R}^n)$ satisfying $Df(x) \cdot Dg(y) > -1$, for all $x, y \in \mathbb{R}^n$, [18].

Remark 2.2. If we assume that the oscillation of Du is sufficiently small then (2.4) can also be removed.

From the condition (2.2), with $\theta, \bar{\theta}$ as shown in Remark 2.1

$$\begin{aligned} \sum_k A_{ii,p_k}(x, Du)u_k - A_{ii}(x, Du) &= \frac{1-\theta}{2} \sum_{k,l} A_{ii,kl}(x, \bar{\theta}Du)u_k u_l \\ &\geq \frac{1-\theta}{2} A_{ii,ii}u_i^2 + (1-\theta) \sum_k A_{ii,ik}u_i u_k \\ &\geq -Cu_i^2 - \frac{\varepsilon u_i^2}{\eta} - C_\varepsilon \eta \sum_k u_k^2, \end{aligned}$$

for any $\varepsilon > 0$ small, where the constant C depends on n and $\|A\|_{C^2}$. Then we have

$$\frac{L\eta}{\eta} \geq \frac{-n}{\eta} - \frac{(\varepsilon + C\eta)}{\eta^2} \sum_i \frac{\eta_i^2}{w_{ii}} - C_\varepsilon \sum_{k,i} u_k^2 w^{ii}.$$

We may choose the cut-off function $\eta = \sigma + \inf u - u$ for some small $\sigma > 0$, so that $C_\varepsilon \sum_k u_k^2$ is sufficiently small. By the above argument we obtain estimate (2.6) without assuming (2.4).

3. Local Properties of Potential Functions

In this section we verify the conditions in Theorem 2.1 for a rescaled solution. First we recall some terminologies related to potential functions in [14, 19]. Note that in [14, 19] we consider c -concave potentials but in this paper we deal with c -convex potentials.

We say a semi-convex function u is c -convex if for any point $x_0 \in \Omega$, there exists a c -support at x_0 in Ω ; a function of the form $\varphi = c(\cdot, y_0) + a_0$ is a c -support of u at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$ and $u(x) \geq \varphi(x)$ for all $x \in \Omega$.

We say a set of points $\ell \subset \mathbb{R}^n$ is a c -segment with respect to a point $y_0 \in \mathbb{R}^n$ if $D_y c(\ell, y_0)$ is a line segment in $\mathbb{R}^n$; and a set U is c -convex with respect to another set V if the image $D_y c(U, y)$ is convex for each $y \in V$.

Similarly, we can define c^* -support, c^* -segment and c^* -convexity by exchanging variables x and y .

Let $u \in C^2(\Omega)$ be an elliptic solution to (1.1). If Ω is c -convex with respect to $T_u(\Omega)$ and (A1)–(A3w) hold, then u is c -convex [19]. For arbitrary Ω , we may assume that u is a c -convex potential for an associated optimal transportation problem with c^* -convex target. To see this, we fix a point $x_0 \in \Omega$ and set $y_0 = T_u(x_0)$. Then T_u is one-to-one from a sufficiently small neighborhood $\mathcal{N}$ of x_0 to a neighborhood of y_0 . We now take $\Omega^* = B_r(y_0)$ for sufficiently small r and replace

Ω by the subdomain $\Omega_0 = T_u^{-1}(\Omega^*)$, using the C^1 control on u to ensure $\Omega_0 \subset \mathcal{N}$. Now let x_0 be a given point in Ω , (modified as above if necessary) and let $c(x, y_0) + a_0$ be the c -support of u at x_0 , where $y_0 = T(x_0)$. Replacing u by $u - c(x, y_0) - a_0$ and $c(x, y)$ by $c(x, y) - c(x, y_0) - a_0$, we assume that $\phi \equiv 0$ is the c -support of u at x_0 , so that

$$u(0) = 0, \quad Du(0) = 0. \quad (3.1)$$

Henceforth we assume that the cost c satisfies (A1)–(A3). By [19] the potential function u is strictly c -convex at x_0 when the inhomogeneous term f satisfies (1.9). For a positive constant $h > 0$, we denote

$$S_h^0 = \{x \in \Omega : u(x) < h\}$$

the sub-level set of u . By [12], S_h^0 is c -convex with respect to y_0 , namely $D_y c(S_h^0, y_0)$ is a convex set. We also make the coordinate transform

$$\tilde{x} = D_y c(x, y_0) \quad (3.2)$$

so that S_h^0 is convex, for all small $h > 0$ such that $S_h^0 \Subset \Omega$. Under this transform, equation (1.1) becomes

$$\det[D_{\tilde{x}}^2 u - \tilde{A}(\tilde{x}, D_{\tilde{x}} u)] = \tilde{f}(\tilde{x}), \quad (3.3)$$

where $\tilde{f}$ satisfies (1.9), and the matrix $\tilde{A} = \{\tilde{A}_{ij}\}$ satisfies

$$\tilde{A}_{ij,p_k} = 0 \quad (3.4)$$

and (A3) with the same positivity as in (1.6), see the computations (2.9)–(2.12) in [12]. Therefore we may assume that S_h^0 is convex directly, and A satisfies (3.4).

From [9], for any convex set D of positive volume, there is a unique ellipsoid E attaining the minimum volume among all ellipsoids containing the set D , and E is called the *minimum ellipsoid* of D . By a translation and rotation of coordinates, we assume that $E = \{\sum_i \frac{x_i^2}{r_i^2} < 1\}$, where $r_1 \geq \dots \geq r_n$, is the minimum ellipsoid of S_h^0 . We make the linear transform

$$\begin{aligned} v(\bar{x}) &= u(x)/h, \\ \bar{x}_i &= x_i/r_i, \end{aligned} \quad (3.5)$$

such that

$$v = 1 \quad \text{on } \partial U, \quad (3.6)$$

where $U = \{\bar{x} \in \mathbb{R}^n : (r_1 \bar{x}_1, \dots, r_n \bar{x}_n) \in S_h^0\}$ is the normalization of S_h^0 . Then equation (1.1) becomes

$$\det[v_{ij} - \bar{A}_{ij}(\bar{x}, Dv)] = \bar{f}(\bar{x}), \quad (3.7)$$

where

$$\bar{f}(\bar{x}) = \frac{(r_1 \cdots r_n)^2}{h^n} f(x),$$

$$\bar{A}_{ij}(\bar{x}, Dv) = \frac{r_i r_j}{h} A_{ij}(x, Du) \quad \forall i, j.$$

Here and below the repeated indices do not indicate summation, unless otherwise indicated. By (3.4) and the Taylor formula,

$$\bar{A}_{ij}(\bar{x}, Dv) = \frac{1}{2} \sum_{k,l} h \frac{r_i r_j}{r_k r_l} A_{ij,kl}(x, \hat{p}) v_k v_l, \quad (3.8)$$

where $A_{ij,kl} = D_{p_k p_l} A_{ij}$, $\hat{p} = \theta Du$, with $\theta \in (0, 1)$ depending on x .

It was proved in [12] that

$$hr_1^2/r_n^2 \leq C \quad (3.9)$$

for some constant $C > 0$ depending only on n and the cost function c . Hence we have

$$|\bar{A}_{ij,kl}(\bar{x}, p)| \leq C$$

where

$$\bar{A}_{ij,kl} = h \frac{r_i r_j}{r_k r_l} A_{ij,kl}. \quad (3.10)$$

When f satisfies (1.9), we have

$$C^{-1} \leq \frac{(r_1, \dots, r_n)^2}{h^n} \leq C. \quad (3.11)$$

Indeed, the first inequality was established in [12], see the computation (4.4) and (4.5) there. To show the second inequality, we observe by (3.8) and the condition (A3) that the matrix in (3.7),

$$\{v_{ij} - \bar{A}_{ij}\} \leq \left\{ v_{ij} - h A_{ij,ij} v_i v_j - h \frac{r_i}{r_k} A_{ij,jk} v_j v_k \right\}.$$

By the $C^{1,\alpha}$ estimate, we have $h/r_k \rightarrow 0$ as $h \rightarrow 0$. Hence

$$\{v_{ij} - \bar{A}_{ij}\} \leq \{v_{ij} + \varepsilon |Dv|^2 \delta_{ij}\}$$

with $\varepsilon \rightarrow 0$ as $h \rightarrow 0$. We obtain from (3.7),

$$\det\{v_{ij} + \varepsilon |Dv|^2 \delta_{ij}\} \geq C_1 \frac{(r_1, \dots, r_n)^2}{h^n}.$$

When the right hand side is sufficiently large, by (3.6) and since U is normalized, one easily constructs a barrier to show that $\inf v < 0$. But by (3.1) we have $v \geq 0$. Hence (3.11) holds.

To apply Theorem 2.1 to the function v , we need to verify conditions (2.2)–(2.5) for small h . Clearly the matrix $\bar{A} = \{\bar{A}_{ij}\}$ satisfies (2.2). Replacing v by $v - 1$, we see that (2.3) is also satisfied. To verify (2.4), by the C^1 regularity [19], we have $|Du(x)| \rightarrow 0$ uniformly in S_h^0 as $h \rightarrow 0$. Hence (2.4) follows by Taylor expansion,

$$\sum_k \bar{A}_{ii,p_k}(\bar{x}, Dv)v_k - \bar{A}_{ii}(\bar{x}, Dv) = \frac{1-\theta}{2} \sum_{k,l} \bar{A}_{ii,kl}(\bar{x}, \bar{\theta}Dv)v_kv_l,$$

where $\bar{\theta} \in (\theta, 1)$, and $\theta \in (0, 1)$ is a constant depending on $\bar{x}$. (See Remark 2.1) Meanwhile, from (3.10),

$$\begin{aligned} \sum_{k,l} \bar{A}_{ii,kl}v_kv_l &= \sum_{k,l} h \frac{r_i^2}{r_k r_l} A_{ii,kl}v_kv_l \\ &= h A_{ii,ii}v_i^2 + \sum_{k \neq i} h \frac{r_i}{r_k} A_{ii,ik}v_iv_k + \sum_{k,l \neq i} h \frac{r_i^2}{r_k r_l} A_{ii,kl}v_kv_l \\ &\geq h(A_{ii,ii} - C_\varepsilon A_{ii,ik}^2)v_i^2 + \sum_{k,l \neq i} h \frac{r_i^2}{r_k r_l} (c_0 - \varepsilon)v_kv_l. \end{aligned}$$

By choosing $\varepsilon < c_0$ (the positivity constant in (A3)), we have $\bar{A}_{ii,kl}v_kv_l \geq -Cv_i^2$ and (2.4) is verified. By direct computation, one sees that (2.4) is invariant under rotation of coordinates. Alternatively we may use Remark 2.2, directly.

To verify (2.5), we use the strict convexity of u . Note that by Taylor's formula and (3.4),

$$\bar{A}_{ij,k}(\bar{x}, Dv) = \bar{A}_{ij,kl}(\bar{x}, \hat{p})v_l,$$

where $\hat{p} = \theta Dv$ with $\theta \in (0, 1)$. From (2.4) and (3.10), all the terms $\bar{A}_{ij,k}$ are bounded. By Lemma 3.1 below, the diameter of the set $\{\eta > 0\}$ converges to 0 as $\sigma \rightarrow 0$ if we choose $\eta = \sigma - v$. Hence (2.5) is satisfied by letting $\phi(\bar{x}) = |\bar{x}|^2$. By (3.9), we can also construct an explicit function φ satisfying (2.5) even if $\eta = 1 - v$.

The interior gradient estimate for rescaled solutions follows from the argument in [12]. The next lemma provides quantitative controls on the continuous differentiability and strict convexity of u .

Lemma 3.1. *Let u be a c -convex potential solution of an optimal transportation problem with densities uniformly bounded and positive. Suppose c satisfies (A1)–(A3), and Ω^* is c^* -convex with respect to Ω . Then for any $\delta > 0$, there exist $\alpha, \beta > 0$, depending only on n, δ , the cost function c , the target domain Ω^* , and the constants C_1, C_2 in (1.9), such that for any $x_0 \in \Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$,*

$$C^{-1}|x - x_0|^{1+\beta} \leq u(x) - \varphi(x) \leq C|x - x_0|^{1+\alpha}, \quad (3.12)$$

where φ is the c -support of u at x_0 , C is a constant independent of u .

Proof. The second inequality has been proved in [12, 13]. Here we prove the first inequality.

Let $B = B_r(x_0) \subset \Omega$ and $B^* = T(B_r(x_0)) \subset \Omega^*$. For a sufficient small $r > 0$, B is c -convex with respect to Ω^* [14]. The strict c^* -convexity of u^* implies the existence of an $r^* > 0$ such that $B_{r^*}(y_0) \subset B^*$, where $y_0 = T(x_0)$, and u^* is the dual function of u . By the duality we have

$$\begin{aligned} u(x) &= \sup_{y \in \Omega^*} \{c(x, y) - u^*(y)\}, \\ u^*(y) &= \sup_{x \in \Omega} \{c(x, y) - u(x)\}, \end{aligned} \quad (3.13)$$

namely u and u^* are the c^* and c -transforms of each other.

It is easy to see that for any c -convex functions w_1, w_2 satisfying $w_1 \leq w_2$, their c -transforms, as defined in (3.13), satisfy $w_1^* \geq w_2^*$. This is also true for the c^* -transform.

In the domain B^* , u^* satisfies the equation

$$\det\{D^2 u^* - A^*(\cdot, Du^*)\} = f^*,$$

where $A^* = D_{yy}^2 c$ satisfies the conditions (A1)–(A3), and f^* satisfies (1.9) (for different constants C_1 and C_2). Note that B is c -convex with respect to B^* . Hence from the $C^{1,\alpha}$ regularity in [12, 13], there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that

$$u^*(y) - \psi(y) \leq C|y - y_0|^{1+\alpha} \quad (3.14)$$

where ψ is the c^* -support of u^* at $y_0 = Tx_0$. We may assume $C = 1$ by taking a smaller $\alpha \in (0, 1)$.

As before by replacing u^* by $u^* - c(x_0, \cdot) - b_0$ and $c(x, y)$ by $c(x, y) - c(x_0, y) - b_0$, we may assume that $\psi \equiv 0$ is the c^* -support of u^* at y_0 . By subtracting the function $c(x, y_0)$ we also assume that $\phi \equiv 0$ is the c -support of u at x_0 . In particular we have $c(x, y_0) = c(x_0, y) \equiv 0$.

Now we apply the c^* -transform to both sides of (3.14) and obtain

$$u(x) \geq \sup_{y \in B^*} \{c(x, y) - |y - y_0|^{1+\alpha}\}. \quad (3.15)$$

When x is close to x_0 , the supremum on the right hand side is attained when

$$D_y c(x, y) = (1 + \alpha)|y - y_0|^{\alpha-1}(y - y_0).$$

By the Taylor expansion, $D_y c(x, y) = D_y c(x_0, y) + D_{xy}^2 c(\tilde{x}, y)(x - x_0)$, where $\tilde{x}$ is a point on the segment connecting x_0 and x . Noting that $\partial_{y_i} c(x_0, y) = 0$ for any $1 \leq i \leq n$, we obtain

$$D_{xy}^2 c(\tilde{x}, y)(x - x_0) = (1 + \alpha)|y - y_0|^{\alpha-1}(y - y_0).$$

Hence

$$|y - y_0|^\alpha = \frac{|D_{xy}^2 c(\tilde{x}, y)|}{(1 + \alpha)} |x - x_0|.$$

By making a linear transform of the coordinates, see [12, 19], we may assume that $D_{xy}^2 c(x_0, y_0)$ is the identity matrix. Therefore we have

$$|y - y_0|^\alpha = \frac{1 + o(1)}{(1 + \alpha)} |x - x_0|. \quad (3.16)$$

Recalling that $c(x, y_0) = c(x_0, y) \equiv 0$, by the Taylor's expansion of $c(x, y)$ at (x_0, y_0) ,

$$c(x, y) = D_{xy}^2 c(x_0, y_0)(x - x_0)(y - y_0) + o(|x - x_0||y - y_0|).$$

Hence

$$|c(x, y)| = (1 + o(1))|x - x_0||y - y_0| \simeq |x - x_0|^{1+1/\alpha}. \quad (3.17)$$

Plugging (3.16), (3.17) into (3.15), we obtain

$$u(x) \geq C|x - x_0|^{1+1/\alpha}$$

for some constant C depending on α . Therefore the first inequality in (3.12) holds with $\beta = 1/\alpha$. $\square$

For any subset $U' \subset\subset U$, applying Theorem 2.1 to the function v at every point in U' , we obtain $\sup_{U'} |D^2 v| \leq C$, where C depends also on $\text{dist}(U', U)$. Therefore we obtain, by the regularity theory of Evans [4] and Krylov [11], the following *a priori* estimate

Theorem 3.1. *Let v be the function given in (3.5) and U be the normalization of S_h^0 as given in (3.6), and suppose $\bar{f}$ is constant in U . Then for any $t < 1$, we have the estimate*

$$\|v\|_{C^4(U_t)} \leq C, \quad (3.18)$$

where $U_t = \{x \in U : v(x) < t\}$, C is a constant depending on t .

In Theorem 3.1, the domain U is normalized. The estimate (3.18) is still true if the domain U has a good shape, namely if $B_r \subset U \subset B_R$ for some positive constant $R > r > 0$ but in this case the constant C in (3.18) also depends on R/r .

4. Proof of Theorem 1.1

The $C^{2,\alpha}$ regularity for strictly convex solutions to the standard Monge–Ampère equation (1.5) with Hölder continuous f is due to Caffarelli [2]. A detailed proof can be found in [10]. In this paper we adopt a similar argument but we will just prove the second derivatives are bounded. Therefore the equation becomes uniformly elliptic and so the estimate (1.10) follows from [1, 16]. We remark that a different simple proof of the estimate (1.10) for fully nonlinear, concave (or convex), uniformly elliptic equations is given in [23].

Let $u \in C^2(\Omega)$ be a solution of (1.1). Let $x_0 = 0$ be a given point in Ω . By subtracting a function of the form $c(\cdot, y_0) + a_0$, we assume as in Section 3 that $\varphi \equiv 0$ is the c -support of u at x_0 , and (3.4) holds. By making

a coordinate transform (3.2) we may also assume the sub-level set $S_h^0 = \{x \in \Omega : u(x) < h\}$ is convex. By a translation and rotation of coordinates, we assume that $E = \{\sum_i \frac{x_i^2}{r_i^2} < 1\}$, where $r_1 \geq \dots \geq r_n$, is the minimum ellipsoid of S_h^0 . Then to prove the second derivatives are uniformly bounded, it suffices to prove that

$$r_1 \leq c^* r_n \quad (4.1)$$

for some constant c^* under control. In the following we say a convex set has a *good shape* if its minimum ellipsoid satisfies (4.1). The constant c^* is called a *shape constant*.

For clarity, we formulate two lemmas to be used in our proof.

Lemma 4.1. *Let u be a C^2 smooth elliptic solution of*

$$\det[u_{ij} - A_{ij}(x, Du)] = 1, \quad (4.2)$$

in Ω , vanishing on $\partial\Omega$. Suppose u attains its minimum at the origin and $|D^2u(0)| \leq C$. Then the domain Ω is of good shape.

Proof. If Ω does not have a good shape, we make a linear transform $\hat{x} = Tx$ to normalize Ω . Then the ratio of the largest and least eigenvalues of T is arbitrarily large and equation (4.2) transforms to

$$\det\left[v_{ij} - \left(hA_{ij,kl} \frac{r_i r_j}{r_k r_l}\right) v_k v_l\right] = \frac{(r_1, \dots, r_n)^2}{h^n}.$$

By Theorem 3.1, $|D^2v(0)|$ is uniformly bounded. Hence by (3.11), $C^{-1}I \leq D^2v(0) \leq CI$, where I is the unit matrix. Hence $D_x^2u(0) = T'D^2v(0)T$ cannot be uniformly bounded, where T' is the transpose of T . $\square$

Lemma 4.2. *Let $u^{(m)}$, $m = 1, 2$, be two c -convex solutions of (4.2) in $B_1(0)$. Suppose $\|u^{(m)}\|_{C^4} \leq C_0$. Then if $|u^{(1)} - u^{(2)}| \leq \delta$ in $B_1(0)$ for some constant $\delta > 0$, we have, for $1 \leq k \leq 3$,*

$$|D^k(u^{(1)} - u^{(2)})| \leq C\delta \quad \text{in } B_{1/2}(0). \quad (4.3)$$

Proof. Denote $w_{ij}^{(m)} = u_{ij}^{(m)} - A_{ij}(x, Du^{(m)})$. We have

$$\begin{aligned} 0 &= \det w_{ij}^{(2)} - \det w_{ij}^{(1)} \\ &= \int_0^1 \frac{d}{dt} \det[w_{ij}^{(1)} + t(w_{ij}^{(2)} - w_{ij}^{(1)})] dt \\ &= a_{ij}(x) [\{u_{ij}^{(2)} - A_{ij}(x, Du^{(2)})\} - \{u_{ij}^{(1)} - A_{ij}(x, Du^{(1)})\}] \\ &= a_{ij}(x) [\partial_i \partial_j (u^{(2)} - u^{(1)}) - A_{ij,p_k} \partial_k (u^{(2)} - u^{(1)})], \end{aligned} \quad (4.4)$$

where $a_{ij}(x) = \int_0^1 C_t^{ij}(x) dt$, and C_t^{ij} is the cofactor of $M_{t,ij} := w_{ij}^{(1)} + t(w_{ij}^{(2)} - w_{ij}^{(1)})$. By assumptions, $\|w_{ij}^{(m)}\|_{C^2} \leq C_0$ for a different constant C_0 . And the matrices

$M_t = \{M_{t,ij}\}$ are C^2 smooth, positive definite for all $t \in [0, 1]$. Hence from the equation (4.2), the operator

$$L = a_{ij}(x)D_{ij} - a_{ij}(x)A_{ij,p_k}D_k$$

is linear and uniformly elliptic, with C^2 coefficients. By the Schauder estimates of linear elliptic equations [7], we obtain (4.3). $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. As above we assume $\varphi \equiv 0$ is the c -support of u at 0, so that $u(0) = 0$, $Du(0) = 0$, and the sub-level set S_h^0 is convex.

Choose $h_0 > 0$ small such that $S_{h_0}^0$ is compactly contained in $B_1(0)$. By the strict c -convexity of u , we can also choose h_0 small such that when $S_{h_0}^0$ is normalized

$$\int_0^1 \frac{\omega(r)}{r} \leq \varepsilon \quad (4.5)$$

and the right hand side $C\varepsilon$ of (4.11) is small, where $\omega(r) = \omega_{\bar{f}}(r)$.

Let $E = \{\sum_i \frac{x_i^2}{r_i^2} < 1\}$, where $r_1 \geq \dots \geq r_n$, be the minimum ellipsoid of $S_{h_0}^0$. Make the transform

$$\begin{aligned} v(\bar{x}) &= u(x)/h_0, \\ \bar{x}_i &= x_i/r_i, \end{aligned} \quad (4.6)$$

such that $v = 1$ on ∂U , where $U = \{\bar{x} \in \mathbb{R}^n : (r_1\bar{x}_1, \dots, r_n\bar{x}_n) \in S_h^0\}$. Then v satisfies

$$\det[v_{ij} - \bar{A}_{ij}(\bar{x}, Dv)] = \bar{f}(\bar{x}), \quad (4.7)$$

where $\bar{A}$ and $\bar{f}$ are as in (3.7).

Denote

$$\begin{aligned} v(t) &= \sup_{z \in B_1} \{|\bar{f}(\bar{x}) - \bar{f}(\bar{y})| : \bar{x}, \bar{y} \in S_{t^2,v}^0(z)\}, \\ v_k &= v(2^{-k}). \end{aligned}$$

If $S_{t^2,v}^0$ has a good shape, we have $v(t) \leq \omega(Ct)$.

Let $v^{(k)}$, $k = 0, 1, \dots$, be the elliptic solution of

$$\begin{aligned} \det\{v_{ij}^{(k)} - \bar{A}_{ij}(\bar{x}, Dv^{(k)})\} &= \bar{f}_k \quad \text{in } U_k, \\ v^{(k)} &= v \quad \text{on } \partial U_k, \end{aligned} \quad (4.8)$$

where $U_k = S_{4^{-k},v}^0$ and $\bar{f}_k = \inf_{U_k} \bar{f}$. The classical solvability of the Dirichlet problem (4.8) is obtained by the classical Perron process [7], using its solvability in small balls as in [14]. Since $S_{1,v}^0$ has a good shape, by Theorem 3.1, $\|v^{(0)}\|_{C^4(S_{3/4,v}^0)} \leq C$. Similarly to (4.4), $v - v^{(0)}$ satisfies the equation

$$L[v - v^{(0)}] = \bar{f}(\bar{x}) - \bar{f}_0$$

where

$$L = a_{ij}(\bar{x})D_{ij} - a_{ij}(\bar{x})A_{ij,p_k}D_k$$

Noting that $v = v^{(0)}$ on ∂U_0 , by (2.5) and using the comparison principle in [7], we obtain

$$\sup_{U_0} |v - v^{(0)}| \leq C v_0.$$

In fact, since U_0 has a good shape, we can make a dilation with respect to 0 such that the radius R of outer sphere becomes sufficiently small. Then instead of using (2.5) directly, we can construct auxiliary function $\tilde{v} = v + \frac{\varepsilon}{2}(|x|^2 - R^2)$ for $\varepsilon > 0$ small. By using comparison principle [14] and re-scaling back to U_0 , we also get the above estimate.

Similarly we have $\sup_{U_1} |v - v^{(1)}| \leq C v_1$. Hence we obtain $\sup_{U_1} |v^{(1)} - v^{(0)}| \leq C v_0$.

Since U_0 is normalized, U_1 has a good shape, then so does $S_{4^{-1},v}^0$ i.e., $S_{4^{-1},v^{(1)}}^0$. It follows that $\|v^{(1)}\|_{C^4(S_{3/16,v^{(1)}}^0)} \leq C$. Therefore we obtain

$$|D^k v^{(0)}(\bar{x}) - D^k v^{(1)}(\bar{x})| \leq C v_0 \quad (4.9)$$

for $\bar{x} \in S_{4^{-2},v^{(1)}}^0$, where $1 \leq k \leq 3$. By Lemma 4.1, the estimate also implies that $S_{4^{-2},v^{(1)}}^0$ has a good shape. In return, since $|v - v^{(1)}| \leq C v_1$, we have that $S_{4^{-2},v}^0$ has a good shape.

By induction we assume that $S_{4^{-k-1},v}^0$ has a good shape, with the constant c^* independent of k . Hence $v_k \leq \omega(C2^{-k})$ for some $C > 0$ independent of k (depending on c^*). We apply the same argument to $\hat{v}^{(0)} := 4^k v^{(k)}(\frac{\bar{x}}{2^k})$ and $\hat{v}^{(1)} := 4^k v^{(k+1)}(\frac{\bar{x}}{2^k})$, which satisfy the equation

$$\begin{aligned} \det\{\hat{v}_{ij}^{(l)} - \bar{A}_{ij}(2^{-k}\bar{x}, 2^{-k}D\hat{v}^{(l)})\} &= \bar{f}_{k+l} \quad \text{in } \widehat{S}_{4^{-k-l},v}^0, \\ \hat{v}^{(l)} &= 4^{-l} \quad \text{on } \partial\widehat{S}_{4^{-k-l},v}^0, \end{aligned}$$

where $l = 0, 1$, $\widehat{S}_{4^{-k-l},v}^0 = 2^k S_{4^{-k-l},v}^0$, and number 2^k is the scaling constant. By induction, both domains have good shapes. So we obtain, similar to (4.9),

$$|D^\gamma \hat{v}^{(0)}(\bar{x}) - D^\gamma \hat{v}^{(1)}(\bar{x})| \leq C v_k,$$

for $\bar{x} \in \widehat{S}_{4^{-k-2},v^{(k+1)}}^0$, where $\gamma = 1, 2, 3$. By re-scaling back we obtain, for $\bar{x} \in S_{4^{-k-2},v^{(k+1)}}^0$,

$$|D^2 v^{(k)}(\bar{x}) - D^2 v^{(k+1)}(\bar{x})| \leq C v_k. \quad (4.10)$$

Therefore,

$$\begin{aligned} |D^2 v^{(0)}(0) - D^2 v^{(k+1)}(0)| &\leq C \sum_{i=0}^k v_i \\ &\leq C \int_{2^{-k}}^1 \frac{\omega(r)}{r} dr \leq C\varepsilon, \end{aligned} \quad (4.11)$$

where $C > 0$ is independent of k . Hence $D^2v(0)$ is uniformly bounded. Changing back to the solution u , we obtain $|D^2u(x_0)| \leq C$ for a different constant C depending on h_0 . Therefore equation (1.1) is uniformly elliptic and Theorem 1.1 follows from [23]. $\square$

Remark 4.1. Theorems 1.1 and 1.2 extend to the degenerate (A3w) condition, provided the constant C in Theorem 1.2 also depends on a c -convexity modulus of the solution u in Theorem 1.1 and the potential u in Theorem 1.2 is assumed in advance to be continuously differentiable and strictly c -convex. From recent work of Figalli and Loeper [6] and Figalli et al. [5] these conditions hold if either $n = 2$ or (A3w) is strengthened to (2.22). The necessary modifications to this paper, along with an improved version of Theorem 2.1, will be presented in a sequel.

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