# THE PENROSE TRANSFORM FOR COMPLEX PROJECTIVE SPACE

MICHAEL EASTWOOD

Abstract. Various complexes of differential operators are constructed on complex projective space via the Penrose transform, which also computes their cohomology.

## 1. Introduction

Throughout this article $\mathbb{CP}_n$ will denote complex projective space as a homogeneous Riemannian manifold under the natural action of $\mathrm{SU}(n+1)$. The invariant metric is called the Fubini-Study metric. Details may be found in [4]. Let $\mathbb{F}_{1,2}(\mathbb{C}^{n+1})$ denote the complex flag manifold

$$\{(L,P) \text{ s.t. } 0 \subset L \subset P \subset \mathbb{C}^{n+1},\ \dim L = 1,\ \dim P = 2\}$$

and define a mapping

$$\tau : \mathbb{F}_{1,2}(\mathbb{C}^{n+1}) \to \mathbb{CP}_n \quad \text{by} \quad (L,P) \xrightarrow{\tau} L^\perp \cap P, \tag{1}$$

where $L^\perp$ is the orthogonal complement of $L \subset \mathbb{C}^{n+1}$ with respect to a fixed Hermitian inner product on $\mathbb{C}^{n+1}$. Notice that $\tau$ is a submersion and, although it is not itself holomorphic, its fibres are clearly holomorphic since over $\ell \in \mathbb{CP}_n$ the fibre consists of pairs $(\ell^\perp \cap P, P)$ for those planes $P$ containing $\ell$. As such, $\tau^{-1}(\ell) \cong \mathbb{P}(\mathbb{C}^{n+1}/\ell)$ and is intrinsically $\mathbb{CP}_{n-1}$ as a complex manifold.

The classical case is when $n = 2$, for then $\mathbb{F}_{1,2}(\mathbb{C}^3)$ is the twistor space of $\mathbb{CP}_2$. It is a special case of the general construction [2] associating a complex manifold to any anti-self-dual 4-dimensional conformal manifold. In fact, with its usual orientation $\mathbb{CP}_2$ is a self-dual manifold but we shall adopt the reverse orientation as in [1, 11]. Equivalently, it is the orientation that makes the Kähler form $J$ on $\mathbb{CP}_2$ anti-self-dual. The Penrose transform for $\mathbb{CP}_2$ realises the analytic cohomology $H^r(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}(V))$ for any holomorphic homogeneous vector bundle $V$ on $\mathbb{F}_{1,2}(\mathbb{C}^3)$ as the cohomology of an appropriate elliptic complex on $\mathbb{CP}_2$. On the other hand, for irreducible bundles $V$ the Bott-Borel-Weil Theorem realises these cohomology spaces for as irreducible representations of $\mathrm{SL}(4,\mathbb{C})$. So the Penrose transform both constructs natural elliptic complexes on $\mathbb{CP}_2$ and also identifies the cohomology of these complexes. Here are some examples (see [1, 8, 10]).

*Example.* That $H^r(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}) = \mathbb{C}$ for $r = 0$ and vanishes for $r \geq 1$ implies

$$0 \to \mathbb{R} \to \Gamma(\mathbb{CP}_2, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^2_+) \to 0$$

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is exact. Here the mappings $d$ are induced by the exterior derivative and $\Lambda^p$ denotes the bundle of $p$-forms with $\Lambda^2_+$ the self-dual 2-forms. Alternatively, we may use complex notation whence

$$
\begin{array}{ccccccc}
0\to\mathbb{C}\to\Gamma(\mathbb{CP}_2,\Lambda^{0,0}) & \xrightarrow{\overline{\partial}} & \Gamma(\mathbb{CP}_2,\Lambda^{0,1}) & & & & \\
 & \searrow^{\partial} & \oplus & \searrow^{\partial} & & & \\
 & & \Gamma(\mathbb{CP}_2,\Lambda^{1,0}) & \xrightarrow{\overline{\partial}} & \Gamma(\mathbb{CP}_2,\Lambda^{1,1}_\perp)\to 0 & &
\end{array}
\tag{2}
$$

is exact. Here $\Lambda^{p,q}_\perp$ denotes the bundles of forms of type $(p,q)$ and the subscript $\perp$ denotes those that are orthogonal to $J$. With our choice of orientation

$$
\mathbb{C}\otimes_{\mathbb{R}}\Lambda^2=\underbrace{\Lambda^{1,0}\oplus\Lambda^{0,1}\oplus\mathbb{C}J}_{\textstyle\| \atop \textstyle\mathbb{C}\otimes_{\mathbb{R}}\Lambda^2_-}\;\oplus\;\underbrace{\Lambda^{1,1}_\perp}_{\textstyle\| \atop \textstyle\mathbb{C}\otimes_{\mathbb{R}}\Lambda^2_+}
$$

as detailed in [1].

The aim of this article is to extend the Penrose transform to $\mathbb{CP}_n$ where (1) is viewed as the twistor fibration. We shall find, for example, an exact sequence

$$
\begin{array}{ccccccc}
0\to\mathbb{C}\to\Gamma(\mathbb{CP}_3,\Lambda^{0,0}) & \xrightarrow{\overline{\partial}} & \Gamma(\mathbb{CP}_3,\Lambda^{0,1}) & \xrightarrow{\overline{\partial}} & \Gamma(\mathbb{CP}_3,\Lambda^{0,2}) & & \\
 & \searrow^{\partial} & \oplus & \searrow^{\partial} & \oplus & \searrow^{\partial} & \\
 & & \Gamma(\mathbb{CP}_3,\Lambda^{1,0}) & \xrightarrow{\overline{\partial}} & \Gamma(\mathbb{CP}_3,\Lambda^{1,1}_\perp) & \xrightarrow{\overline{\partial}} & \Gamma(\mathbb{CP}_3,\Lambda^{1,2}_\perp)\to 0
\end{array}
\tag{3}
$$

as the counterpart to (2) on $\mathbb{CP}_3$. Notice that the bundles occurring in this complex are irreducible on $\mathbb{CP}_3$ as an Hermitian manifold.

*Example.* The complex generated on $\mathbb{CP}_2$ by the holomorphic tangent bundle $\Theta$ on the twistor space $\mathbb{F}_{1,2}(\mathbb{C}^3)$ is

$$
0\to\Lambda^1\xrightarrow{\nabla}\square\!\square_\circ\Lambda^1\xrightarrow{\nabla^{(2)}}\boxplus_{\circ+}\Lambda^1\to 0. \tag{4}
$$

Here, the first differential operator in local cöordinates is

$$
\phi_a\mapsto[\nabla_a\phi_b+\nabla_b\phi_a]_\circ=\nabla_a\phi_b+\nabla_b\phi_a-\tfrac{2}{n}g_{ab}\nabla^c\phi_c \tag{5}
$$

where the subscript $\circ$ denotes the trace-free part with respect to the Fubini-Study metric $g_{ab}$ and $\nabla_a$ is the Levi-Civita connection for this metric. We are using Young diagrams [9] to specify the symmetries of tensor bundles. For example, the bundle $\square\!\square_\circ\Lambda^1$ consists of symmetric trace-free tensors. The second operator in (4) is

$$
\psi_{ab}\mapsto[\nabla_{(a}\nabla_{c)}\psi_{bd}-\nabla_{(b}\nabla_{c)}\psi_{ad}-\nabla_{(a}\nabla_{d)}\psi_{bc}+\nabla_{(b}\nabla_{d)}\psi_{ac}]_{\circ+}
$$

where $+$ denotes the self-dual part with respect to the skew indices $ab$. The bundle $\boxplus_{\circ+}\Lambda^1$ is precisely the bundle of covariant tensors with the resulting symmetries. It is an irreducible SO(4)-bundle of rank 5. As observed in [6], it is straightforward to write the complexification of (4) in terms of irreducible Hermitian bundles:–

$$
\begin{array}{ccccc}
0\longrightarrow\Lambda^{0,1} & \xrightarrow{\overline{\partial}} & \square\!\square\Lambda^{0,1} & & \\
\searrow\quad\oplus & \searrow^{\partial} & \oplus & \searrow^{\partial^{(2)}} & \\
\Lambda^{1,0} & \xrightarrow{\overline{\partial}} & \Lambda^{1,1}_\perp & \xrightarrow{\partial\overline{\partial}} & \\
 & \searrow^{\partial} & \oplus & & \\
 & & \square\!\square\Lambda^{1,0} & \xrightarrow{\overline{\partial}^{(2)}} & \square\!\square\Lambda^{0,1}\otimes_\perp\square\!\square\Lambda^{1,0}\to 0.
\end{array}
\tag{6}
$$

The Penrose transform also computes the global cohomology of these complexes. Since $H^0(\mathbb{F}_{1,2}(\mathbb{C}^3), \Theta) = \mathfrak{sl}(3, \mathbb{C})$ (arising from the infinitesimal action of $\mathrm{SL}(3, \mathbb{C})$ on $\mathbb{F}_{1,2}(\mathbb{C})$ as a complex homogeneous space) and all higher cohomology vanishes, we conclude that

$$0 \to \mathfrak{su}(3) \to \Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_2, \square\!\square_\circ \Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{CP}_2, \boxplus_{\circ +} \Lambda^1) \to 0$$

is exact. It is the anti-self-dual deformation complex and we conclude that $\mathbb{CP}_2$ is rigid as an anti-self-dual conformal manifold, also that the only conformal motions are infinitesimal isometries (this latter conclusion is well-known by other methods).

*Remark.* The Penrose transform for $\mathbb{CP}_2$ may often be understood, as typified in the previous example, in terms of its being an anti-self-dual conformal manifold. Though we shall be able to construct the transform just as well for $\tau : \mathbb{F}_{1,2}(\mathbb{C}^{n+1}) \to \mathbb{CP}_n$, it is far from clear whether it can be interpreted via some intrinsic geometric structure on $\mathbb{CP}_n$. The complexes of differential operators that arise are surely worthy of further study and interpretation.

*Remark.* In order to keep the notation to a minimum, the rest of this article will be confined to the case $\tau : \mathbb{F}_{1,2}(\mathbb{C}^4) \to \mathbb{CP}_3$. This already captures the difficulties in encountered for $\mathbb{CP}_n$ in general.

## 2. A spectral sequence

We shall end up following the Penrose transform for $\mathbb{CP}_2$ constructed in [5] but we begin with a general technique for which we only need the following. Let $Z$ be a complex manifold and $M$ a smooth manifold. Suppose

$$\tau : Z \to M \tag{7}$$

is a smooth submersion with compact complex fibres. The mapping (1) is the example we would like to understand. The classical example, however, is the twistor fibration $\tau : \mathbb{CP}_3 \to S^4$, viewed in [1] as a quaternionic version of the Hopf fibration.

**Theorem 2.1.** *There is a smooth complex vector bundle $\Lambda^{1,0}_\mu$ on $Z$ that is holomorphic along the fibres of $\tau$. Let us denote by $\Lambda^{p,0}_\mu$ the p-fold exterior power of $\Lambda^{1,0}_\mu$. Let us suppose that all $\tau^q_* \Lambda^{p,0}_\mu$ are smooth vector bundles on $M$ where $\tau^q_*$ denotes the $q^{\text{th}}$ direct image with respect to the holomorphic structure along the fibres of $\tau$. Then, there is a spectral sequence*

$$E_1^{p,q} = \Gamma(M, \tau^q_* \Lambda^{p,0}_\mu) \Longrightarrow H^{p+q}(Z, \mathcal{O}) \tag{8}$$

*whose differentials are linear differential operators on $M$.*

*Proof.* To say that $\tau$ is a submersion is to say that $d\tau : \tau^* \Lambda^1_M \to \Lambda^1_Z$ is injective. Let $\Lambda^1_\tau$ denote the cokernel of this homomorphism. It is the dual bundle to the bundle of vertical vector fields. To say that the fibres of $\tau$ are holomorphic is to say that complex multiplication preserves the vertical vector fields. It defines a surjection of vector bundles $\Lambda^{0,1}_Z \to \Lambda^{0,1}_\tau$ and thus a complex vector bundle as its kernel. The exact sequence

$$0 \to \Lambda^{1,0}_\mu \to \Lambda^{0,1}_Z \to \Lambda^{0,1}_\tau \to 0 \tag{9}$$

defines the bundle $\Lambda^{1,0}_\mu$. The composition

$$\Lambda^{0,0}_Z \xrightarrow{\overline{\partial}} \Lambda^{0,1}_Z \to \Lambda^{0,1}_\tau$$

coincides with the $\overline{\partial}$-operator intrinsic to the fibres of $\tau$, which we shall write as $\overline{\partial}_\tau$. Commutativity of the diagram

$$\begin{array}{ccc} \Lambda^{0,1}_Z & \xrightarrow{\overline{\partial}} & \Lambda^{0,2}_Z \\ \downarrow & & \downarrow \\ \Lambda^{0,1}_\tau & \xrightarrow{\overline{\partial}_\tau} & \Lambda^{0,2}_\tau \end{array}$$

shows that $\overline{\partial} : \Lambda^{0,1}_Z \to \Lambda^{0,2}_Z$ induces a differential operator $\overline{\partial}_\tau : \Lambda^{1,0}_\mu \to \Lambda^{0,1}_\tau \otimes \Lambda^{1,0}_\mu$, which defines the holomorphic structure along the fibres claimed in the statement of the theorem. The spectral sequence is simply that of a filtered complex, where the assumption that all $\tau^q_*\Lambda^{p,0}$ are smooth vector bundles is saying that the dimensions of these finite-dimensional cohomology groups along the (compact) fibres does not jump (and generically this is true). □

*Remark.* We shall be interested (8) for the fibration (1). In this case, because it is also true for the fibration itself, all bundles are manifestly homogeneous under the action of $\mathrm{SU}(n+1)$. In particular, there can be no rank jumping and $\tau^q_*\Lambda^{p,0}_\mu$ are not only smooth vector bundles but homogeneous to boot. Their computation is a matter of representation theory, which we now pursue.

Following [5], it is useful to consider the complexification of the submersion (1). For simplicity, we shall write out the details in case $n = 3$ and abbreviate the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^4)$ as $\mathbb{F}$. A suitable complexification of $\mathbb{CP}_3$ is given by

$$\mathbb{M} \equiv \{(\ell, H) \in \mathbb{CP}_3 \times \mathbb{CP}^*_3 \text{ s.t. } \ell \not\in H\},$$

where $\mathbb{CP}^*_3$ is regarded as the space of hyperplanes $H$ in $\mathbb{CP}_3$ and the totally real embedding $\mathbb{CP}_3 \hookrightarrow \mathbb{M}$ is given by $\ell \mapsto (\ell, \ell^\perp)$. Just as $\mathbb{CP}_3$ is a homogeneous space for $\mathrm{SU}(4)$, so $\mathbb{M}$ is a homogeneous space for $\mathrm{SL}(4,\mathbb{C})$. Specifically,

$$\mathbb{M} = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}\right\} \text{ with basepoint } \left(\begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ * \\ * \\ * \end{bmatrix}\right). \tag{10}$$

The submersion itself complexifies to a double fibration

$$\begin{array}{ccc} & \mathbb{G} & \\ {\scriptstyle\mu}\swarrow & & \searrow{\scriptstyle\nu} \\ \mathbb{F} & & \mathbb{M}. \end{array} \tag{11}$$

Here,

$$\mathbb{G} = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\right\} \qquad \mathbb{F} = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\right\}, \tag{12}$$

the mapping $\nu : \mathbb{G} \to \mathbb{M}$ is the natural one, and $\mu : \mathbb{G} \to \mathbb{F}$ is induced by the homomorphism

$$\mathrm{SL}(4,\mathbb{C}) \ni A \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{SL}(4,\mathbb{C}). \tag{13}$$

Alternatively, a different choice of basepoint would lead to our writing $\mathbb{F}$ as

$$\mathrm{SL}(4,\mathbb{C}) / \left\{ \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\} \tag{14}$$

with both $\mu$ and $\nu$ the natural projections. But (12) is preferred so as to coincide with the conventions established in [3] for writing flag manifolds. Moreover, the irreducible homogeneous vector bundles on $\mathbb{F}$ may then be written as

$$\overset{a}{\times}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet}, \text{ for integers } a, b, c \text{ with } c \geq 0 \text{ (see [3] for details).}$$

The irreducible homogeneous vector bundles on $\mathbb{G}$ are in 1–1 correspondence with the finite-dimensional irreducible representations of the stabiliser subgroup

$$\left\{ \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\}$$

from (12). But these are carried by the smaller subgroup

$$\left\{ \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\},$$

which, as the appropriate Levi factor, also carries the representations inducing the irreducible homogeneous vector bundles on $\mathbb{F}$. We shall, therefore, use the same notation for the homogeneous bundles on $\mathbb{G}$ but add $\mathbb{G}$ as a subscript when needed:–

$$\overset{p}{\times}\!\!-\!\!\overset{q}{\times}\!\!-\!\!\overset{r}{\bullet}_{\mathbb{G}}, \text{ for integers } p, q, r \text{ with } r \geq 0.$$

The price to pay for the conjugation (13) used in defining $\mu$ is that the corresponding simple reflection must be invoked in the Weyl group to pull back homogeneous vector bundles from $\mathbb{F}$ to $\mathbb{G}$:–

$$\mu^* \overset{a}{\times}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet}_{\mathbb{F}} = \overset{-a}{\times}\!\!-\!\!\overset{a+b}{\times}\!\!-\!\!\overset{c}{\bullet}_{\mathbb{G}} \text{ (see [3] for details).} \tag{15}$$

Finally, we need a notation for the homogeneous vector bundles on $\mathbb{M}$ and its real slice $\mathbb{CP}_3$. From (10) it is clear that we may use the usual notation [3]

$$\overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}_{\mathbb{M}}, \text{ for integers } a, b, c \text{ with } b \geq 0, c \geq 0$$

and that every SU(4)-homogeneous bundle on the smooth manifold $\mathbb{CP}_3$ extends uniquely to a $\mathrm{SL}(4,\mathbb{C})$-homogeneous bundle on its complexification $\mathbb{M}$. If we restrict the action of $\mathrm{SL}(4,\mathbb{C})$ on the double fibration (11) to the real form $\mathrm{SU}(4)$, then we obtain a real splitting $\sigma$ of the holomorphic submersion $\mu$ and a diagram

$$\begin{array}{ccccc} & & \mathbb{G} & & \\ & {}^{\mu}\swarrow\!\nearrow_{\sigma} & & \searrow^{\nu} & \\ \mathbb{F} & \xrightarrow{\tau} & \mathbb{CP}_3 & \hookrightarrow & \mathbb{M}. \end{array} \tag{16}$$

so that $\tau = \nu \circ \sigma$. In this way we may view $\mathbb{F}$ as a submanifold of $\mathbb{G}$ and then the fibres of $\tau$, with their complex structure, coincide with the fibres of $\nu$. The canonical homomorphism of vector bundles

$$\Lambda^{1,0}_{\mathbb{G}}|_{\mathbb{F}} = \sigma^*\Lambda^{1,0}_{\mathbb{G}} \hookrightarrow \sigma^*(\mathbb{C}\otimes_{\mathbb{R}}\Lambda^1_{\mathbb{G}}) \xrightarrow{d\sigma} \mathbb{C}\otimes_{\mathbb{R}}\Lambda^1_{\mathbb{F}} \twoheadrightarrow \Lambda^{0,1}_{\mathbb{F}}$$

induces an isomorphism between $\Lambda^{1,0}_\mu$ on $\mathbb{G}$ defined by the exact sequence

$$0 \to \mu^*\Lambda^{1,0}_{\mathbb{F}} \xrightarrow{d\mu} \Lambda^{1,0}_{\mathbb{G}} \to \Lambda^{1,0}_\mu \to 0 \tag{17}$$

and its restriction to $\mathbb{F}$ defined by (9). Furthermore, the holomorphic structure on $\Lambda^{1,0}_\mu$ along the fibres of $\tau$ inherited from (9) coincides with its evident holomorphic structure on $\mathbb{G}$ defined by (17) (notice from (16) that the fibres of $\tau$ agree with the fibres of $\nu$ over $\mathbb{CP}_3 \hookrightarrow \mathbb{M}$). We reach the standard conclusion that the spectral sequence (8) may be obtained by restricting the terms in the spectral sequence

$$E_1^{p,q} = \Gamma(\mathbb{M}, \nu^q_*\Lambda^{p,0}_\mu)$$

derived in [3] to $\mathbb{CP}_3$ replacing holomorphic sections on the complexification $\mathbb{M}$ by smooth sections on the real slice $\mathbb{CP}_3$. The point of this manœuvre is that the direct image bundles $\nu^q_*\Lambda^p_\mu$ are easily computed by representation theory as detailed in [3]. Firstly, we need to identify $\Lambda^{1,0}_\mu$ as a homogeneous vector bundle on $\mathbb{G}$. According to (12) and (14) this bundle is induced by the co-Adjoint representation of

$$\left\{\begin{bmatrix} * & 0 & 0 & 0\\ 0 & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & * & * \end{bmatrix} \in \mathrm{SL}(4,\mathbb{C})\right\} \quad\text{on}\quad \left[\frac{\left\{\begin{bmatrix} * & 0 & * & *\\ * & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & * & * \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{C})\right\}}{\left\{\begin{bmatrix} * & 0 & 0 & 0\\ 0 & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & * & * \end{bmatrix} \in \mathfrak{sl}(4,\mathbb{C})\right\}}\right]^*$$

Therefore,

$$\begin{array}{rcl} \Lambda^{1,0}_\mu & = & \overset{1\quad 0\quad 1}{\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}_{\mathbb{G}} \oplus \overset{-2\quad 1\quad 0}{\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}_{\mathbb{G}} \\ \Lambda^{2,0}_\mu & = & \overset{2\quad 1\quad 0}{\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}_{\mathbb{G}} \oplus \overset{-1\quad 1\quad 1}{\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}_{\mathbb{G}} \\ \Lambda^{3,0}_\mu & = & \overset{0\quad 2\quad 0}{\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}_{\mathbb{G}} \end{array} \tag{18}$$

and elementary application of the formulæ in [3] yields

$$
\begin{array}{l}
q\uparrow \\
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
| & & & & & & \\
\Gamma(\mathbb{CP}_3, \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) & \to & \Gamma\left(\mathbb{CP}_3, \begin{array}{c}\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} \\ \oplus \\ \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\end{array}\right) & \to & \Gamma\left(\mathbb{CP}_3, \begin{array}{c}\overset{2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} \\ \oplus \\ \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\end{array}\right) & \to & \Gamma(\mathbb{CP}_3, \overset{0}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}) \quad \underset{p}{\to}
\end{array}
\end{array}
$$

for the spectral sequence (8) applied to the submersion (1). Convergence to

$$
H^r(\mathbb{F}, \mathcal{O}) = H^r(\mathbb{F}, \overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}) = \begin{cases} \mathbb{C} & \text{if } r = 0 \\ 0 & \text{if } r \geq 1, \end{cases}
$$

implies that this spectral sequence collapses to an exact sequence, which is readily identified as (3).

## 3. Further examples

As a natural higher dimensional counterpart to (6) let us now consider the Penrose transform of $H^r(\mathbb{F}_{1,2}(\mathbb{C}^4), \Theta)$. One immediate issue that must be dealt with is that the holomorphic tangent bundle $\Theta$ is reducible but not decomposable. Specifically [3], there is a short exact sequence

$$
0 \to \begin{array}{c} \overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}_{\mathbb{F}} \\ \oplus \\ \overset{-1}{\times}\!-\!\overset{1}{\times}\!-\!\overset{1}{\bullet}_{\mathbb{F}} \end{array} \to \Theta \to \overset{1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}_{\mathbb{F}} \to 0 \tag{19}
$$

that does not split as $\mathrm{SL}(4, \mathbb{C})$-homogeneous bundles. But we can form the Penrose transform of each irreducible subfactor with the following results.

**Proposition 3.1.** *There is an exact sequence*

$$
0 \to \Gamma(\mathbb{CP}_3, \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}) \to \Gamma\left(\mathbb{CP}_3, \begin{array}{c}\overset{-1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet} \\ \oplus \\ \overset{-4}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\end{array}\right) \to \Gamma\left(\mathbb{CP}_3, \begin{array}{c}\overset{0}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet} \\ \oplus \\ \overset{-3}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{1}{\bullet}\end{array}\right) \to \Gamma(\mathbb{CP}_3, \overset{-2}{\times}\!-\!\overset{3}{\bullet}\!-\!\overset{0}{\bullet}) \to 0.
$$

*Proof.* As usual [3], the spectral sequence (8) may be generalised to incorporate a holomorphic vector bundle on $\mathbb{F}$ in the coefficients and the discussion of §2 is easily modified to calculate the appropriate direct images. Specifically, the exact sequence we are aiming for arises as the Penrose of the vector bundle $\overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}_{\mathbb{F}}$. As a singular bundle, all its holomorphic cohomology vanishes. To incorporate it into the Penrose transform, we firstly use (15) to conclude that

$$
\mu^* \overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}_{\mathbb{F}} = \overset{-2}{\times}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet}_{\mathbb{G}}.
$$

The modified spectral sequence is therefore

$$
E_1^{p,q} = \Gamma(\mathbb{CP}_3, \tau_*^q \Lambda_\mu^{p,0}(\overset{-2}{\times}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet})) \Longrightarrow H^{p+q}(\mathbb{F}, \overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}) = 0.
$$

From (18) we see that on $\mathbb{G}$

$$\Lambda^{\bullet,0}_\mu(\overset{-2\;1\;0}{\times\!-\!\times\!-\!\bullet}) = \overset{-2\;1\;0}{\times\!-\!\times\!-\!\bullet}_{\mathbb{G}} \to \begin{matrix}\overset{-1\;1\;1}{\times\!-\!\bullet\!-\!\bullet}_{\mathbb{G}}\\ \oplus \\ \overset{-4\;2\;0}{\times\!-\!\bullet\!-\!\bullet}_{\mathbb{G}}\end{matrix} \to \begin{matrix}\overset{0\;2\;0}{\times\!-\!\bullet\!-\!\bullet}_{\mathbb{G}}\\ \oplus \\ \overset{-3\;2\;1}{\times\!-\!\bullet\!-\!\bullet}_{\mathbb{G}}\end{matrix} \to \overset{-2\;3\;0}{\times\!-\!\bullet\!-\!\bullet}_{\mathbb{G}},$$

and the required conclusion follows by taking direct images as in [3]. □

**Proposition 3.2.** *There is an exact sequence*

$$0 \to \Gamma(\mathbb{CP}_3, \overset{1\;0\;1}{\times\!-\!\bullet\!-\!\bullet}) \to \Gamma\left(\mathbb{CP}_3, \begin{matrix}\overset{2\;0\;2}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{2\;1\;0}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{-1\;1\;1}{\times\!-\!\bullet\!-\!\bullet}\end{matrix}\right) \to \Gamma\left(\mathbb{CP}_3, \begin{matrix}\overset{3\;1\;1}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{0\;1\;2}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{0\;2\;0}{\times\!-\!\bullet\!-\!\bullet}\end{matrix}\right) \to \Gamma(\mathbb{CP}_3, \overset{1\;2\;1}{\times\!-\!\bullet\!-\!\bullet}) \to 0.$$

*Proof.* Noting that

$$\mu^*\overset{-1\;1\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}} = \overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{G}},$$

the required conclusion is the Penrose transform of the singular bundle $\overset{-1\;1\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}}$. The extra bundles arise via various tensor decompositions, such as

$$\Lambda^{1,0}_\mu \otimes \mu^*\overset{-1\;1\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}} = \begin{matrix}\overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{G}}\\ \oplus \\ \overset{-2\;1\;0}{\times\!-\!\times\!-\!\bullet}_{\mathbb{G}}\end{matrix} \otimes \overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet} = \begin{matrix}\overset{2\;0\;2}{\times\!-\!\times\!-\!\bullet}\\ \oplus \\ \overset{2\;1\;0}{\times\!-\!\times\!-\!\bullet}\\ \oplus \\ \overset{-1\;1\;1}{\times\!-\!\times\!-\!\bullet}\end{matrix}.$$

Again, there are only $0^{\text{th}}$ direct images in the spectral sequence. □

The remaining irreducible bundle from (19) is $\overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}}$ and its Penrose transform is as follows.

**Proposition 3.3.** *There is a exact sequence of* $\mathrm{SL}(4,\mathbb{C})$*-modules*

$$\begin{matrix} 0 \to \mathfrak{sl}(4,\mathbb{C}) \\ \downarrow \\ \Gamma(\mathbb{CP}_3, \overset{-1\;1\;1}{\times\!-\!\bullet\!-\!\bullet})\end{matrix} \to \Gamma\left(\mathbb{CP}_3, \begin{matrix}\overset{0\;1\;2}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{0\;2\;0}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{-3\;2\;1}{\times\!-\!\bullet\!-\!\bullet}\end{matrix}\right) \to \Gamma\left(\mathbb{CP}_3, \begin{matrix}\overset{1\;2\;1}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{-2\;2\;2}{\times\!-\!\bullet\!-\!\bullet}\\ \oplus \\ \overset{-2\;3\;0}{\times\!-\!\bullet\!-\!\bullet}\end{matrix}\right) \to \Gamma(\mathbb{CP}_3, \overset{-1\;3\;1}{\times\!-\!\bullet\!-\!\bullet}) \to 0.$$

*Proof.* Apply the standard machinery [3] to compute

$$E_1^{p,q} = \Gamma(\mathbb{CP}_3, \tau_*^q\Lambda^{p,0}_\mu(\mu^*\overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}})) = \Gamma(\mathbb{CP}_3, \tau_*^q\Lambda^{p,0}_\mu(\overset{-1\;1\;1}{\times\!-\!\times\!-\!\bullet})) \Longrightarrow H^{p+q}(\mathbb{F}, \overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}}).$$

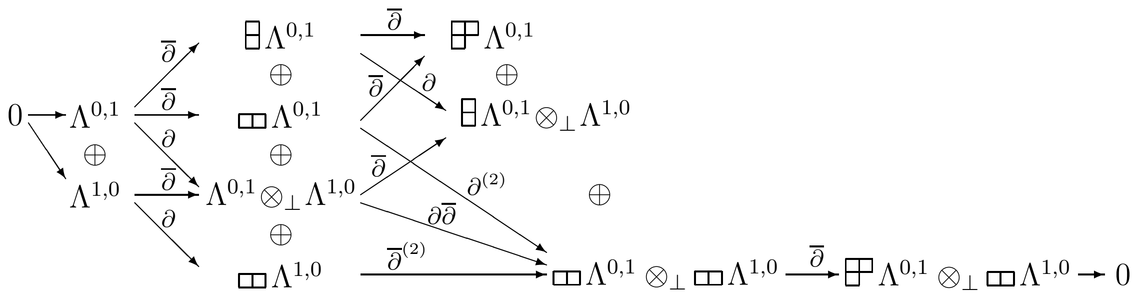
By the Bott-Borel-Weil Theorem, in the conventions of [3],

$$H^r(\mathbb{F}, \overset{1\;0\;1}{\times\!-\!\times\!-\!\bullet}_{\mathbb{F}}) = \begin{cases} \overset{1\;0\;1}{\bullet\!-\!\bullet\!-\!\bullet} = \mathfrak{sl}(4,\mathbb{C}) & \text{if } r=0 \\ 0 & \text{if } r \geq 1 \end{cases}$$

and the required exact sequence emerges. □

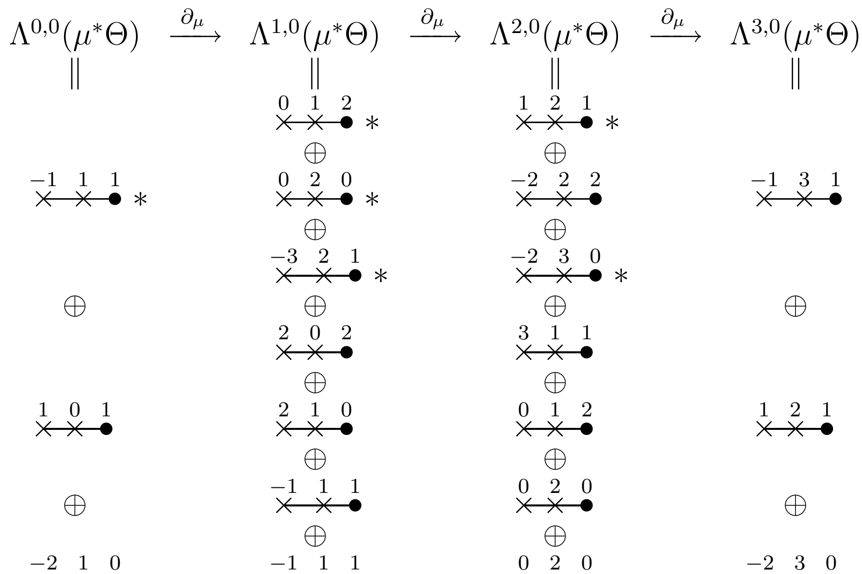
Propositions 3.1, 3.2, and 3.3 may be combined to give the following result for the Penrose transform of $H^r(\mathbb{F}_{1,2}(\mathbb{C}^4))$ in terms of differential operators on $\mathbb{CP}_3$.

**Theorem 3.4.** *There is a complex of differential operators on* $\mathbb{CP}_3$




*whose global* $0^{\text{th}}$ *cohomology is* $\mathfrak{sl}(4,\mathbb{C})$ *and is otherwise exact. The operators in this complex are the natural ones induced by the Fubini-Study connection.*

*Proof.* To compute the Penrose transform of $H^r(\mathbb{F},\Theta)$ we need to identify the complex $\Lambda^{\bullet,0}(\mu^*\Theta)$ and compute direct images $\tau_*^q\Lambda^{p,0}(\mu^*\Theta)$ down on $\mathbb{CP}_3$. From (19) and (18) we obtain




Notice that all the bundles marked $*$ are repeated in the next column. A more detailed analysis shows that, when restricted to these bundles and an appropriate target bundle, the differential operator $\partial_\mu$ is simply the identity mapping. Also, notice that all the bundles have only $0^{\text{th}}$ direct images down on $\mathbb{CP}_3$. Diagram chasing, either on $\mathbb{G}$ or down on $\mathbb{CP}_3$, now shows that all the bundles labelled $*$ can be eliminated from the resulting complex at the expense of introducing second order operators. (This construction is similar to that of the Bernstein-Gelfand-Gelfand operators in [3] but there appears to be no precise link.) The complex in the statement of the theorem is obtained by using the more traditional notation for irreducible Hermitian bundles (the differential operators in this complex are the only possible Hermitian-invariant ones). Its cohomology realises $H^r(\mathbb{F},\Theta)$. But from (19) and the algorithms of [3] it follows that $H^0(\mathbb{F},\Theta) = \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$ and all higher cohomology vanishes. $\square$

The complex in Theorem 3.4 is somewhat ugly compared to the pleasing complex (6) on $\mathbb{CP}_2$. There does appear to be a more satisfying analogue as follows.

$$\begin{array}{ccccccccc}
0\to & \Lambda^{0,1} & \longrightarrow & \square\!\square\,\Lambda^{0,1} & \longrightarrow & \boxplus\,\Lambda^{0,1} & & & \\
 & \oplus & & \oplus & & \oplus & & & \\
 & \Lambda^{1,0} & \longrightarrow & \Lambda^{0,1}\otimes_\perp\Lambda^{1,0} & \longrightarrow & {\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}\,\Lambda^{0,1}\otimes_\perp\Lambda^{1,0} & \longrightarrow & \boxplus\,\Lambda^{0,1}\otimes_\perp\Lambda^{1,0} & \\
 & & & \oplus & & \oplus & & \oplus & \\
 & & & \square\!\square\,\Lambda^{1,0} & \longrightarrow & \square\!\square\,\Lambda^{0,1}\otimes_\perp\square\!\square\,\Lambda^{1,0} & \longrightarrow & {\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}\,\Lambda^{0,1}\otimes_\perp\square\!\square\,\Lambda^{1,0} & \to 0.
\end{array}$$

Though the details are not yet worked out, it appears that this complex may also be obtained from the Penrose transform of a suitable homogeneous bundle $V$ on $\mathbb{F}$. This bundle $V$ arises as extension

$$0\to\begin{array}{c}\overset{2}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}_{\mathbb{F}}\\ \oplus\\ \overset{-1}{\times}\!-\!\overset{1}{\times}\!-\!\overset{1}{\bullet}_{\mathbb{F}}\end{array}\to V\to\begin{array}{c}\overset{-2}{\times}\!-\!\overset{3}{\times}\!-\!\overset{0}{\bullet}_{\mathbb{F}}\\ \oplus\\ \overset{1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}_{\mathbb{F}}\end{array}\to 0.$$

If true, this would gives rise to some $1^{\text{st}}$ cohomology too since

$$H^1(\mathbb{F},V)=H^1(\mathbb{F},\overset{-2}{\times}\!-\!\overset{3}{\times}\!-\!\overset{0}{\bullet})=\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}.$$

For the moment, this remains a conjecture. The geometric significance of the vector bundle $V$, if any, also remains unclear. Presumably, this more satisfactory complex on $\mathbb{CP}_3$ is elliptic.

We conclude with some final geometric observations, which also provided the main motivation for this study. The complex (6) has a simple real form (4). Whilst this is not true for the complex just suggested on $\mathbb{CP}_3$ as a satisfactory analogue, it is still true that the first operator is simply (5) but acting on complex-valued 1-forms. This is the conformal Killing operator. The Killing operator is similar but does not remove the trace:–

$$\Lambda^1\to\square\!\square\,\Lambda^1\quad\text{by}\quad\phi_a\longmapsto\nabla_a\phi_b+\nabla_b\phi_a.$$

In [6] these observations were used establish results concerning real integral geometry on $\mathbb{CP}_2$. Unfortunately, as we have seen, the Penrose transform for $\mathbb{CP}_n$ when $n\geq 3$ yields much more complicated results. Fortunately, for the purposes of real integral geometry on $\mathbb{CP}_n$, other methods found in joint work with Hubert Goldschmidt [7] have circumvented this approach.

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Department of Mathematics, University of Adelaide, South Australia 5005
*E-mail address*: meastwoo@member.ams.org