An ultimate state bound for a class of linear systems with delay

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Abstract

A new method is given to estimate an ultimate state bound on a time-varying linear system with delay and bounded disturbances by using some results on Metzler matrices. The effectiveness of the obtained results is illustrated by a numerical example.

Key words: State bounding; Time-varying linear systems; Bounded disturbances; Metzler matrix.

1 Introduction

In the last decade, the problem of bounding the state of delay systems has attracted much interest. Some important results about reachable set bounding and state convergence for linear time-delay systems are proposed (Fridman & Shaked, 2003; Kim, 2008; Kwon, Lee & Park, 2011; Nam & Pathirana, 2011; Oucheriah, 2006; Zuo, Ho & Wang, 2010). The most commonly used methods in the existing results are based on Lyapunov-Krasovskii functionals or the Lyapunov-Razumikhin technique. Recently, a new method for time-varying systems with delay and bounded disturbances was presented in Hien & Trinh (2014). The main purpose of this paper is to give some new methods for estimating an ultimate state bound for a class of systems with delay.

The system under consideration is given by

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + D(t)\boldsymbol{x}(t-\tau(t)) + B(t)\omega(t) \qquad (1)$$

where $\boldsymbol{x}(t) = [x_1(t)\cdots x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector, $\omega(t) \in \mathbb{R}^m$ is the external disturbance vector, $A(t) = [a_{ij}(t)] \in \mathbb{R}^{n\times n}$, $D(t) = [d_{ij}(t)] \in \mathbb{R}^{n\times n}$ and $B(t) = [b_{ij}(t)] \in \mathbb{R}^{n\times m}$ are system matrices. The disturbance $\omega(t) = [\omega_1(t) \ \cdots \omega_m(t)]^{\mathrm{T}}$ is unknown but is assumed to be bounded by a given constant $\bar{\omega} = [\bar{\omega}_1 \ \cdots \bar{\omega}_m]^{\mathrm{T}}$ with $\bar{\omega}_i > 0$, $i \in [1,m]$; i.e., for all $t \geq 0$, $|\omega_i(t)| \leq \bar{\omega}_i$, $i \in [1,m]$. The time-varying delay $\tau(t)$ satisfies $0 \leq \tau(t) \leq \bar{\tau}$, for all $t \geq 0$. $\boldsymbol{x}(t) = \phi(t) = [\phi_1(t) \ \cdots \ \phi_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$, $-\bar{\tau} \leq t \leq 0$ is the initial state and $\phi(t)$, $-\bar{\tau} \leq t \leq 0$ is bounded. Let $a_{ij}(t)$, $d_{ij}(t)$, $i,j \in [1,n]$ and $b_{ij}(t)$, $i \in [1,n]$, $j \in [1,m]$, $\phi_i(t)$, $i \in [1,n]$ and $\tau(t)$ be continuous functions. It is assumed that $A(t)$, $B(t)$ and $D(t)$ satisfy the following bounds,

$$\lim_{t\to\infty} \sup |a_{ij}(t)| \leq \bar{a}_{ij}, \quad i,j \in [1,n] \text{ and } i \neq j, \qquad (2)$$

$$\lim_{t\to\infty} \sup a_{ii}(t) \leq \bar{a}_{ii}, \quad i \in [1,n], \qquad (3)$$

$$\lim_{t\to\infty} \sup |d_{ij}(t)| \leq \bar{d}_{ij}, \quad i,j \in [1,n], \qquad (4)$$

$$\lim_{t\to\infty} \sup |b_{ij}(t)| \leq \bar{b}_{ij}, \quad i \in [1,n], \ j \in [1,m]. \qquad (5)$$

We let $\bar{A} = [\bar{a}_{ij}] \in \mathbb{R}^{n\times n}$, $\bar{D} = [\bar{d}_{ij}] \in \mathbb{R}^{n\times n}$, $\bar{B} = [\bar{b}_{ij}] \in \mathbb{R}^{n\times m}$ and $\bar{\omega} = [\bar{\omega}_1 \cdots \bar{\omega}_m]^{\mathrm{T}} \in \mathbb{R}^m$.

2 Preliminaries

In the following, for any vectors $\boldsymbol{x} = [x_1 \ \cdots \ x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and $\boldsymbol{y} = [y_1 \ \cdots \ y_n]^{\mathrm{T}} \in \mathbb{R}^n$, the notation $x \geq y$ means that for each $i \in [1,n]$, $x_i - y_i \geq 0$ and the notation $\boldsymbol{x} > \boldsymbol{y}$ means that for each $i \in [1,n]$, $x_i - y_i > 0$. For a vector $\boldsymbol{y} = [y_1 \ \cdots y_n]^{\mathrm{T}} \in \mathbb{R}^n$, let $|y|$ denote $|\boldsymbol{y}| = [|y_1| \ \cdots |y_n|]^{\mathrm{T}}$, let $||\boldsymbol{y}||_\infty$ denote $||\boldsymbol{y}||_\infty = \max_{i\in[1,n]}\{|y_i|\}$. Let 0_n denote the n-dimensional null vector. For a given matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$, let $||M||_\infty$ denote $||M||_\infty = \max_{i\in[1,n]} \sum_{j=1}^{n} |m_{ij}|$, let $|M|$ denote $|M| = \tilde{M}$, where $\tilde{M} = [\tilde{m}_{ij}] \in \mathbb{R}^{n\times n}$ and $\tilde{m}_{ij} = |m_{i,j}|$, $i,j \in [1,n]$.

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Definition 1 *A matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ is said to be a Metzler matrix, if for all $i, j \in [1, n]$, $i \neq j$, $m_{ij} \geq 0$.*

Lemma 1 *(Berman & Plemmons, 1994) Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then, the following statements are equivalent,*

(a) The matrix M is Hurwitz;

(b) For all $i, j \in [1, n]$, $n_{ij} \leq 0$ holds, where $N = [n_{ij}] \in \mathbb{R}^{n\times n}$ and $N = M^{-1}$.

(c) There exists a vector $\boldsymbol{x} \in \mathbb{R}^n$ with $\boldsymbol{x} > 0_n$ such that $M\boldsymbol{x} < 0_n$.

Lemma 2 *Let $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n}$ be two Metzler matrices. If M is Hurwitz, then it follows that there exists a scalar $\bar{\lambda} \in \mathbb{R}$ with $\bar{\lambda} > 0$ such that $(M + \bar{\lambda}N)$ is Metzler and Hurwitz. Also, it follows that for all $\lambda \in [0, \bar{\lambda}]$, the matrix $(M + \lambda N)$ is Metzler and Hurwitz.*

Proof. These results can be obtained directly from the Perron-Frobenius Theorem. □

Lemma 3 *Let $\boldsymbol{y} \in \mathbb{R}^n$ be a vector with $\boldsymbol{y} \geq 0_n$ and $M \in \mathbb{R}^{n\times n}$ be a Metzler and Hurwitz matrix. It follows that for a given positive scalar $\delta \in \mathbb{R}$, there exists a positive scalar $\tilde{\lambda} \in \mathbb{R}$ such that*

$$0_n \leq \tilde{\zeta} \leq \zeta + 0.5\delta\boldsymbol{z} \quad \forall\lambda \in [0, \tilde{\lambda}] \tag{6}$$

holds, where $\tilde{M} = M + \lambda Q$, $\tilde{\boldsymbol{y}} = \boldsymbol{y} + \lambda\boldsymbol{z}$, $\zeta = -M^{-1}\boldsymbol{y}$, $\tilde{\zeta} = -\tilde{M}^{-1}\tilde{\boldsymbol{y}}$, $Q = [q_{ij}] \in \mathbb{R}^{n\times n}$ with $q_{ij} = 1$, $i, j \in [1, n]$ and $\boldsymbol{z} = [z_1 \cdots z_n]^{\mathrm{T}} \in \mathbb{R}^n$ with $z_i = 1$, $i \in [1, n]$.

Proof. Note that M is a Metzler and Hurwitz matrix. Then, using Lemma 2, we obtain that there exits a scalar $\bar{\lambda} > 0$ such that for all $\lambda \in [0, \bar{\lambda}]$, $(M + \lambda Q)$ is a Metzler and Hurwitz matrix. Hence, using Lemma 1, $\boldsymbol{y} \geq 0_n$ and $\boldsymbol{z} > 0_n$, we have

$$0_n \leq \tilde{\zeta} \quad \forall\lambda \in [0, \bar{\lambda}], \tag{7}$$

where $\tilde{\zeta} = -\tilde{M}^{-1}\tilde{\boldsymbol{y}}$, $\tilde{M} = M + \lambda Q$ and $\tilde{\boldsymbol{y}} = \boldsymbol{y} + \lambda\boldsymbol{z}$. Then, using $\zeta = -M^{-1}\boldsymbol{y}$, we can obtain that for all $\lambda \in [0, \bar{\lambda}]$, $M(\zeta - \tilde{\zeta}) + \lambda Q(\zeta - \tilde{\zeta}) = \lambda(\boldsymbol{z} + Q\zeta)$, $(\zeta - \tilde{\zeta}) + \lambda M^{-1}Q(\zeta - \tilde{\zeta}) = \lambda M^{-1}(\boldsymbol{z} + Q\zeta)$ and

$$(1 - \lambda\alpha)\,||\zeta - \tilde{\zeta}||_\infty \leq \lambda\beta \quad \forall\lambda \in [0, \bar{\lambda}], \tag{8}$$

where $\alpha = ||M^{-1}||_\infty \cdot ||Q||_\infty$ and $\beta = ||M^{-1}||_\infty \cdot (||\boldsymbol{z}||_\infty + ||Q\zeta||_\infty)$. Using the definitions of M, Q and $\boldsymbol{z}$, we obtain $\alpha > 0$ and $\beta > 0$. Let $\tilde{\lambda}$ denote $\tilde{\lambda} = \min\{\bar{\lambda}, \gamma\}$, where $\gamma = \delta(2\beta + \alpha\delta)^{-1}$. Then, it follows that for all $\lambda \in [0, \tilde{\lambda}]$, $(1 - \lambda\alpha) > 0$ and $\lambda\beta(1 - \lambda\alpha)^{-1} \leq 0.5\delta$. Hence, using (8), we obtain that for all $\lambda \in [0, \tilde{\lambda}]$, $||\tilde{\zeta} - \zeta||_\infty \leq 0.5\delta$ and

$$\tilde{\zeta} - \zeta \leq 0.5\delta\boldsymbol{z} \quad \forall\lambda \in [0, \tilde{\lambda}]. \tag{9}$$

Using (7) and (9), we obtain (6). This completes the proof. □

Lemma 4 *Let $f(t) \in \mathbb{R}$ denote a given function. If for all $t \geq t_0$, $\dot{f}(t) \leq \alpha f(t)$ holds, then it follows that for all $t \geq t_0$, $f(t) \leq f(t_0)e^{\alpha(t-t_0)}$, where α is a real constant. Especially, it is satisfied that $f(t) < 0$, $\forall t \geq t_0$, provided that $f(t_0) < 0$.*

Proof. It follows from $\dot{f}(t) \leq \alpha f(t)$, $t \geq t_0$ that for all $t \geq t_0$, $\int_{t_0}^{t} e^{-\alpha\tau}[\dot{f}(\tau) - \alpha f(\tau)]d\tau \leq 0$. Then, we can obtain the conclusions of this lemma. □

3 An ultimate state bound for a class of time-varying linear systems

Theorem 1 *Suppose that for the system (1)-(5), the matrix $M = (\bar{A} + \bar{D})$ is Hurwitz. Then the trajectories of the system (1) satisfy*

$$\lim_{t\to\infty} \sup |\boldsymbol{x}(t)| \leq \zeta, \tag{10}$$

where $\zeta = -M^{-1}\bar{B}\bar{\omega}$.

Proof. It follows that if for any positive scalar $\delta \in \mathbb{R}$, there exists a positive scalar $t_\delta \in \mathbb{R}$ such that

$$|\boldsymbol{x}(t)| \leq \zeta + \delta\boldsymbol{z} \quad \forall t \geq t_\delta, \tag{11}$$

then (10) is satisfied, where $\boldsymbol{z}$ is defined as in Lemma 3. Let $\delta \in \mathbb{R}$ denote an arbitrary positive scalar. Then, we will prove that there exists a positive scalar $t_\delta \in \mathbb{R}$ satisfying (11).

Note that M is a Metzler and Hurwitz matrix. Hence, using Lemma 2 and Lemma 3, we obtain that there is a positive scalar $\lambda \in \mathbb{R}$ such that

$$0_n \leq \tilde{\zeta} \leq \zeta + 0.5\delta\boldsymbol{z} \tag{12}$$

holds and the matrix $\tilde{M}$ is Metzler and Hurwitz, where $\tilde{M} = \bar{A} + \bar{D} + \lambda Q$, $\zeta = M^{-1}\bar{B}\bar{\omega}$, $\tilde{\zeta} = \tilde{M}^{-1}(\bar{B}\bar{\omega} + \lambda\boldsymbol{z})$, $\boldsymbol{z}$ and Q are defined as in Lemma 3. Then, using Lemma 2, we obtain that there exists a positive scalar $\gamma \in \mathbb{R}$ such that the matrix $G \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz, where $G = \tilde{A} + e^{\gamma\bar{\tau}}\tilde{D} + \gamma I_n$, $\tilde{A} = [\tilde{a}_{ij}] \in \mathbb{R}^{n\times n}$, $\tilde{D} = [\tilde{d}_{ij}] \in \mathbb{R}^{n\times n}$, $\tilde{a}_{ij} = \bar{a}_{ij} + 0.5\lambda$, $i, j \in [1, n]$ and $\tilde{d}_{ij} = \bar{d}_{ij} + 0.5\lambda$, $i, j \in [1, n]$.

It follows from (2)-(5) that there exists a scalar $t_\lambda \in (\bar{\tau}, \infty)$ such that for all $t \geq t_\lambda$,

$$a_{ii}(t) \leq \tilde{a}_{ii}, \quad i \in [1, n], \tag{13}$$

$$|a_{ij}(t)| \leq \tilde{a}_{ij}, \quad i, j \in [1, n] \text{ with } i \neq j, \tag{14}$$

$$|d_{ij}(t)| \leq \tilde{d}_{ij}, \quad i, j \in [1, n], \tag{15}$$

$$|B(t)\omega(t)| \leq \boldsymbol{u}, \tag{16}$$

where $\boldsymbol{u} = [u_1 \cdots u_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{u} = \lambda \boldsymbol{z} + \bar{B}\bar{\omega}$.

Note that G is a Metzler and Hurwitz matrix. Hence, using Lemma 1, we obtain $\tilde{g}_{ij} \leq 0$, $i, j \in [1, n]$ and $|\tilde{G}| = -\tilde{G}$, where $\tilde{G} = [\tilde{g}_{ij}] \in \mathbb{R}^{n \times n}$, $\tilde{G} = G^{-1}$. Then, it follows that

$$|\boldsymbol{x}(t)| < \boldsymbol{y} \quad \forall t \in [(t_\lambda - \bar{\tau}),\ t_\lambda], \tag{17}$$

$$G\boldsymbol{y} \leq 0_n, \tag{18}$$

where $\boldsymbol{y} = -\tilde{G}|G(\bar{\boldsymbol{x}} + \boldsymbol{z})|$, $\bar{\boldsymbol{x}} = [\bar{x}_1 \cdots \bar{x}_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\bar{x}_i = \sup_{t \in [(t_\lambda - \bar{\tau}),\ t_\lambda]} |x_i(t)|$, $i \in [1, n]$.

In the following, we will use the method of contradiction to show that

$$|\boldsymbol{x}(t)| \leq F(t) \quad \forall t \geq t_\lambda, \tag{19}$$

where $F(t) = [f_1(t) \ \cdots \ f_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is defined as follows:

$$F(t) = e^{-\gamma(t - t_\lambda)}\boldsymbol{y} + \tilde{\zeta}. \tag{20}$$

It follows from (12), (17), (18) and (20) that

$$F(t) \geq 0_n \quad \forall t \geq t_\lambda - \bar{\tau}, \tag{21}$$

$$F(t) > |\boldsymbol{x}(t)| \quad \forall t \in [(t_\lambda - \bar{\tau}),\ t_\lambda]. \tag{22}$$

$$\tilde{A}F(t) + \tilde{D}F(t - \bar{\tau}) + \boldsymbol{u} \leq \dot{F}(t) \quad \forall t \geq t_\lambda, \tag{23}$$

$$\tilde{D}F(t - \tau(t)) \leq \tilde{D}F(t - \bar{\tau}) \quad \forall t \geq t_\lambda. \tag{24}$$

Suppose that (19) is not true. Then, it follows from (22) that there exists a scalar $t_J \in (t_\lambda, \infty)$ and an integer $J \in [1, n]$ such that

$$x_J^2(t_J) - f_J^2(t_J) > 0. \tag{25}$$

We now define the functions $U_i(t)$, $i \in [1, n]$ as follows:

$$U_i(t) = x_i^2(t) - f_i^2(t), \quad i \in [1, n]. \tag{26}$$

It follows from (22) and (26) that

$$U_i(t_\lambda) < 0, \quad i \in [1, n]. \tag{27}$$

Using the definitions of the functions $U_i(t)$, $i \in [1, n]$, we obtain that the functions $U_i(t)$, $i \in [1, n]$ are continuous for all $t \geq t_\lambda$. Then, using (25) and (27), we conclude that there exists an integer $L \in [1, n]$ and a scalar $t_0 \in (t_\lambda, t_J)$ such that

$$U_L(t_0) = 0 \tag{28}$$

and for all $t \in [t_\lambda,\ t_0]$,

$$U_i(t) \leq 0, \quad i \in [1, n]. \tag{29}$$

It follows from (1), (13)-(16) and (26) that

$$\dot{U}_L(t) = 2x_L(t)\dot{x}_L(t) - 2f_L(t)\dot{f}_L(t), \tag{30}$$

$$x_L(t)\dot{x}_L(t) \leq \tilde{a}_{LL}x_L^2(t) + |x_L(t)| \sum_{j=1, j \neq L}^{n} \tilde{a}_{Lj}|x_j(t)| + |x_L(t)| \sum_{j=1}^{n} \tilde{d}_{Lj}|x_j(t - \tau(t))| + |x_L(t)|u_L. \tag{31}$$

Using (16) and (29), we obtain that for all $t \in [t_\lambda,\ t_0]$,

$$|x_L(t)| \sum_{j=1, j \neq L}^{n} \tilde{a}_{Lj}|x_j(t)| \leq f_L(t)\tilde{A}_L F(t) - \tilde{a}_{LL}f_L^2(t), \tag{32}$$

$$|x_L(t)| \sum_{j=1}^{n} \tilde{d}_{Lj}|x_j(t - \tau(t))| \leq f_L(t)\tilde{D}_L F(t - \bar{\tau}), \tag{33}$$

$$|x_L(t)|u_L \leq f_L(t)u_L, \tag{34}$$

where $\tilde{A}_L$ and $\tilde{D}_L$ denote the L-th row of $\tilde{A}$ and $\tilde{D}$, respectively. From (23) and (30)-(34), it follows that

$$\dot{U}_L(t) \leq 2\tilde{a}_{LL}U_L(t) \quad \forall t \in [t_\lambda,\ t_0]. \tag{35}$$

Then, using (27), (35) and Lemma 4, we have $U_L(t_0) < 0$. This contradicts (28). Thus, we obtain (19). It follows from (12), (19) and (20) that

$$|\boldsymbol{x}(t)| \leq e^{-\gamma(t - t_\lambda)}\boldsymbol{y} + \zeta + 0.5\delta\boldsymbol{z} \quad \forall t \geq t_\lambda. \tag{36}$$

Using the definition of $\boldsymbol{y}$, we have $||\boldsymbol{y}||_\infty < \infty$. Then, it follows from (36) that there exists a scalar $t_\delta \in \mathbb{R}$ satisfying $e^{-\gamma(t - t_\lambda)}\boldsymbol{y} \leq 0.5\delta\boldsymbol{z}$ and (11). Hence, we can obtain (10). This completes the proof. □

4 A numerical example

Consider a system of the form (1) with $\tau(t) = 6|\sin(2\sqrt{t})|$, $\bar{\omega} = 0.5$,

$$A(t) = \begin{bmatrix} -4 - |\sin t|e^{-t} & \cos 2t & \sin^2 t \\ e^{-t}\cos t & -6 + \sin 2t & 2\cos 3t \\ \frac{t\sin t}{1+t} & \frac{1}{\sqrt{1+|\sin t|}} & -5 - |\cos t| \end{bmatrix},$$

$$D(t) = \begin{bmatrix} \sin 3t & -e^{-2t} & 0 \\ e^{-2t}\sin t & 0 & \cos 3t \\ 0 & \cos^2 t & e^{-t}\sin 2t \end{bmatrix}, B(t) = \begin{bmatrix} 0.1e^{\sin t} \\ 0.2\cos 2t \\ 0.1\sin 4t \end{bmatrix},$$

which is proposed in Hien & Trinh (2014).

Using the results in Hien & Trinh (2014), we obtain that

$$\lim_{t\to\infty} \sup |\boldsymbol{x}(t)| \leq [0.68\ 0.68\ 0.68]^{\mathrm{T}}. \tag{37}$$

Using the results in Nam, Pathirana & Trinh (2015), we have

$$\lim_{t\to\infty} \sup |\boldsymbol{x}(t)| \leq [0.3139\ 0.2859\ 0.2339]^{\mathrm{T}}. \tag{38}$$

It follows that

$$\bar{A} = \begin{bmatrix} -4 & 1 & 1 \\ 0 & -5 & 2 \\ 1 & 1 & -5 \end{bmatrix}, \bar{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0.1e \\ 0.2 \\ 0.1 \end{bmatrix}.$$

Then, using the results proposed in this note, we obtain that

$$\lim_{t\to\infty} \sup |\boldsymbol{x}(t)| \leq [0.0752\ 0.0461\ 0.0435]^{\mathrm{T}}. \tag{39}$$

Thus, we can see that for this example, a small ultimate bound can be obtained by using the results of this note. However, for other examples, the results in Hien & Trinh (2014) and Nam, Pathirana & Trinh (2015) may be better than Theorem 1 in the this note.

5 Conclusion

In this note, a new method to estimate a bound on the ultimate state for a class of systems with time-varying delay is presented. The effectiveness of the obtained approach is illustrated by using a numerical example.

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