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The challenge of the chiral Potts model

R J Baxter

Centre for Mathematics and its Applications, The Australian National University, Canberra, ACT 0200, Australia

Abstract. The chiral Potts model continues to pose particular challenges in statistical mechanics: it is "exactly solvable" in the sense that it satisfies the Yang-Baxter relation, but actually obtaining the solution is not easy. Its free energy was calculated in 1988 and the order parameter was conjectured in full generality a year later. However, a derivation of that conjecture had to wait until 2005. Here we discuss that derivation.

1. Introduction

In 1970 I was in England, where my wife and I stayed for five months with my parents in Essex. It was largely holiday, as we were on our way back to Australia after two years in Boston, where I had been introduced to the six-vertex models and the Bethe ansatz by Elliott Lieb.

However, I did visit Cyril Domb's group at King's College, London, and it was there that I first interacted with Tony Guttmann, who was also visiting the department: he was an invaluable aid to navigating the labyrinthine corridors and staircases that linked the department's quarters in Surrey Street with the main part of the College.

Tony's natural enthusiasm for statistical mechanics must have been infectious, for it was at this time that I realised that the transfer matrices of the six-vertex model commuted - a vital first step in the subsequent solution of the eight-vertex model.

This led to the solution of a number of other two-dimensional lattice models. One that has proved particularly challenging is the chiral Potts model. Here I wish to discuss some of the insights that led to the recent derivation of its order parameters.

The chiral Potts model is a two-dimensional classical lattice model in statistical mechanics, where spins live on sites of a lattice and each spin takes N values $0, 1, \ldots, N-1$, and adjacent spins interact with Boltzmann weight functions $W, \overline{W}$. We consider only the case when the model is "solvable", by which we mean that $W, \overline{W}$ satisfy the star-triangle ("Yang-Baxter") relations [1]. The free energy of the infinite lattice was first obtained in 1988 by using the invariance properties of the free energy and its derivatives [2]. Then in 1990 the functional transfer matrix relations of Bazhanov and Stroganov [3] were used to calculate the free energy more explicitly as a double integral [4, 5, 6]. The model has a critical temperature, below which the system exhibits ferromagnetic order.

The next step was to calculate the order parameters $\mathcal{M}_1, \ldots, \mathcal{M}_{N-1}$ (defined below). These depend on a constant k which decreases from one to zero as the temperature increases from zero to criticality. In 1989 Albertini *et al* [7] made the elegant conjecture, based on the available series expansions, that

$$\mathcal{M}_r \;=\; k^{r(N-r)/N^2} \quad , \quad 0 \leq r \leq N \quad . \tag{1.1}$$

It might have been expected that a proof of such a simple formula would not have been long in coming, but in fact it proved to be a remarkably difficult problem. Order parameters (spontaneous magnetizations) are notoriously more difficult to calculate than free energies. For the Ising model (to which the chiral Potts model reduces when $N = 2$), the free energy was calculated by Onsager in 1944 [8], but it was five years later when at a conference in Florence he announced his result for the spontaneous magnetization, and not till 1952 that the first published proof was given by Yang [9, 10].

Similarly, the free energy of the eight-vertex model was calculated in 1971 [11]. The spontaneous magnetization and polarization were conjectured in 1973 and 1974, respectively [12, 13], but it was not till 1982 that a proof of the first of these conjectures were published [14]. A proof of the second had to wait until 1993 [15]!

By then three separate methods had been used. The Onsager-Yang calculation was based on the particular free-fermion/spinor/pfaffian/Clifford algebra structure of the Ising model [16]. As far as the auther is aware, this has never been extended to the other models: it would be very significant if it could be.

The eight-vertex and subsequent hard-hexagon calculation was made using the corner transfer matrix method, which had been discovered in 1976 [17]. This worked readily for the magnetization (a single-site correlation), but not for the polarization (a single-edge correlation). This problem was remedied by the "broken rapidity line" technique discovered by Jimbo *et al* [15].

For all the two-dimensional solvable models, the Boltzmann weight functions $W, \overline{W}$ depend on parameters p and q. These parameters are known as *rapidities* and are associated with lines (the dotted lines of Figure 1) that run through the midpoints of the edges of the lattice. In general these are complex numbers, or sets of related complex numbers. In all of the models we have mentioned, with the notable exception of the $N > 2$ chiral Potts model, these parameters can be chosen so that $W, \overline{W}$ depend only on the *rapidity difference* (spectral parameter) $p - q$.

This property seems to be an essential element in the corner transfer matrix method: the star-triangle relation ensures that the corner transfer matrices factor, but the difference property is then needed to show that the factors commute with one another and are exponentials in the rapidities. The difference property is *not* possessed by the $N > 3$ chiral Potts model and one is unable to proceed. At first the author thought this would prove to be merely a technical complication and embarked on a low-temperature numerical calculation [18] in the hope this would reveal the kind of simplifications that happen with the other models. This hope was not realised.

I then looked at the technique of Jimbo *et al* and in 1998 applied it to the chiral Potts model. One could write down functional relations satisfied by the generalized order parameter ratio function $G_{pq}(r)$, and for $N = 2$ these were sufficient (together with an assumed but very plausible analyticity property) to solve the problem. However, for $N > 2$ there was still a difficulty. Then p, q are points on an algebraic curve of genus > 1 and there is no obvious uniformizing substitution. The functional relations themselves do not define $G_{pq}(r)$: one needs some additional analyticity information, and that seems hard to come by.

The calculation of the free energy of the chiral Potts model [5, 6, 19] proceeds in two stages. First one considers a related "$\tau_2(t_q)$" model [20]. This is intimately connected with the superintegrable case of the chiral Potts model [21]. It is much simpler than the chiral Potts model in that its Boltzmann weights depend on the horizontal rapidity q only via a single parameter t_q, and are linear in t_q. Its row-to-row transfer matrix is the product of two chiral Potts transfer matrices, one with horizontal rapidity q, the other with a related rapidity $r = VRq$ defined by eqn. (2.7) of section 2.

For a finite lattice, the partition function Z of the $\tau_2(t_q)$ model is therefore a polynomial in t_p. The free energy is the logarithm of $Z^{1/M}$, where M is the number of sites of the lattice, evaluated in the thermodynamic limit when the lattice becomes infinitely big. This limiting function of course may have singularities in the complex t_q plane. *A priori*, one might expect it to have N branch cuts, each running though one of the N roots of unity. However, one can argue that in fact it only has one such cut. As a result the free energy (i.e. the maximum eigenvalue of the transfer matrix) can be calculated by a Wiener-Hopf factorization.

The second stage is to factor this free energy to obtain that of the chiral Potts model.

It was not until 2004 that I realised that:

(1) If one takes p, q to be related by eqn. (4.1) below, then $G_{pq}(r)$ can be expressed in terms of partition functions that involve p, q only via the Boltzmann weights of the $\tau_2(t_{p'})$ model, with $p' = R^{-1}p$.

(2) It is *not* necessary to obtain $G_{pq}(r)$ for arbitrary p and q. To verify the conjecture (1.1) it is sufficient to obtain it under the restriction (4.1).

I indicate the working in the following sections: a fuller account is given in Ref. [22]. The calculation of $G_{pq}(r)$ for general p, q remains an unsolved problem: still interesting, but not necessary for the derivation of the order parameters $\mathcal{M}_r$.

2. Chiral Potts model

We use the notation of [1, 4, 23]. Let k, k' be two real variables in the range $(0, 1)$, satisfying

$$k^2 + k'^2 = 1 \ . \tag{2.1}$$

Consider four parameters x_p, y_p, μ_p, t_p satisfying the relations

$$k x_p^N = 1 - k'/\mu_p^N \ \ , \ \ k y_p^N = 1 - k'\mu_p^N \ \ , \ \ t_p = x_p y_p \ . \tag{2.2}$$

Let p denote the set $\{x_p, y_p, \mu_p, t_p\}$. Similarly, let q denote the set $\{x_q, y_q, \mu_q, t_q\}$. We call p and q "rapidity" variables. Each has one free parameter and is a point on an algebraic curve.

Define Boltzmann weight functions $W_{pq}(n), \overline{W}_{pq}(n)$ by

$$W_{pq}(n) \ = \ (\mu_p/\mu_q)^n \prod_{j=1}^{n} \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q} \ , \tag{2.3a}$$

$$\overline{W}_{pq}(n) \ = \ (\mu_p \mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p} \ , \tag{2.3b}$$

where

$$\omega \ = \ \mathrm{e}^{2\pi \mathrm{i}/N} \ .$$

They satisfy the periodicity conditions

$$W_{pq}(n+N) = W_{pq}(n) \ \ , \ \ \overline{W}_{pq}(n+N) = \overline{W}_{pq}(n) \ .$$

Now consider the square lattice $\mathcal{L}$, drawn diagonally as in Figure 1, with a total of M sites. On each site i place a spin σ_i, which can take any one of the N values $0, 1, \ldots, N-1$.

The solid lines in Figure 1 are the edges of $\mathcal{L}$. Through each such edge there pass two dotted or broken lines - a vertical line denoted v and a horizontal line denoted h (or p or q). These v, h, p, q are rapidity variables, as defined above. We refer to each dotted line as a "rapidity line".

With each SW - NE edge (i, j) (with i below j) associate an edge weight $W_{vh}(\sigma_i - \sigma_j)$. Similarly, with each SW - NE edge (j, k) (j below k), associate an edge weight $\overline{W}_{vh}(\sigma_j - \sigma_k)$. (Replace h by p or q for the broken left and right half-lines.) Then the partition function is

$$Z \ = \ \sum_{\sigma} \prod W_{vh}(\sigma_i - \sigma_j) \prod \overline{W}_{vh}(\sigma_j - \sigma_k) \ , \tag{2.4}$$

the products being over all edges of each type, and the sum over all N^M values of the M spins. We expect the partition function per site

$$\kappa \ = \ Z^{1/M}$$

to tend to a unique limit as the lattice becomes large in both directions.

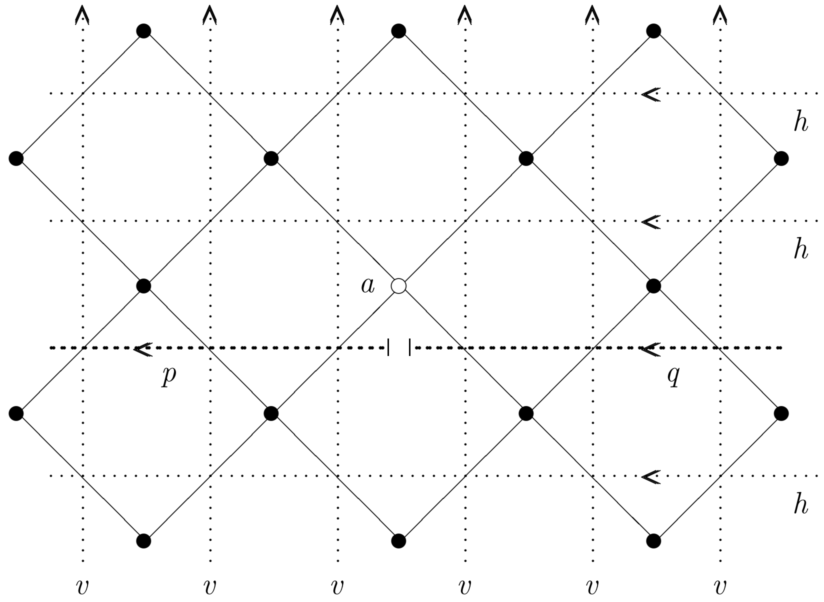

Figure 1. The square lattice (solid lines, drawn diagonally), and the associated rapidity lines (broken or dotted).

Let a be a spin on a site near the centre of the lattice, as in the figure, and r be any integer. Then the thermodynamic average of ω^{ra} is

$$\tilde{F}_{pq}(r) \;=\; \langle \omega^{ra} \rangle \;=\; Z^{-1} \sum_{\sigma} \omega^{ra} \prod W_{vh}(\sigma_i - \sigma_j) \prod \overline{W}_{vh}(\sigma_j - \sigma_k) \;\; . \tag{2.5}$$

We expect this to also tend to a limit as the lattice becomes large.

We could allow each vertical (horizontal) rapidity line α to have a different rapidity v_α (h_β). If an edge of $\mathcal{L}$ lies on lines with rapidities v_α, h_β, then the Boltzmann weight function of that edge is to be taken as $W_{vh}(n)$ or $\overline{W}_{vh}(n)$, with $v = v_\alpha$ and $h = h_\beta$.

The weight functions $W_{pq}(n)$, $\overline{W}_{pq}(n)$ satisfy the star- triangle relation [1]. For this reason we are free to move the rapidity lines around in the plane, in particular to interchange two vertical or two horizontal rapidity lines [24]. So long as no rapidity line crosses the site with spin a while making such rearrangements, the average $\langle \omega^{ra} \rangle$ is *unchanged* by the rearrangement.[1]

All of the v, h rapidity lines shown in Figure 1 are "full", in the sense that they extend without break from one boundary to another. We can move any such line away from the central site to infinity, where we do not expect it to contribute to $\langle \omega^{ra} \rangle$. Hence in the infinite lattice limit $\tilde{F}_{pq}(r) = \langle \omega^{ra} \rangle$ must be *independent* of *all* the full-line v and h rapidities.

The horizontal rapidity line immediately below a has different rapidity variables p, q on the left and the right of the break below a. This means that we cannot use the star-triangle relation to move it away from a.

It follows that $\tilde{F}_{pq}(r)$ will in general depend on p and q, as well as on the " universal" constants k or k'. We are particularly interested in the case when $q = p$. Then the p, q line is not broken, it can be removed to infinity, so

$$\mathcal{M}_r \;=\; \tilde{F}_{pp}(r) \;=\; \langle \omega^{ra} \rangle \;=\; \text{independent of } \; p \;\; . \tag{2.6}$$

[1] Subject to boundary conditions: here we are primarily interested in the infinite lattice, where we expect the boundary conditions to have no effect on the rearrangements we consider.

These are the desired order parameters of the chiral Potts model, studied by Albertini *et al.* By using this "broken rapidity line" approach, I was finally ably to verify their conjecture (1.1) in 2005 [25, 22]. Here I shall present some of the observations that enabled me to do this.

Automorphisms

There are various automorphisms that change $x_p, y_p\mu_p, t_p$ while leaving the relations (2.2) still satisfied. Four that we shall use are R, S, M, V, defined by:

$$\begin{aligned} \{x_{Rp}, y_{Rp}, \mu_{Rp}, t_{Rp}\} &= \{y_p, \omega x_p, 1/\mu_p, \omega t_p\} \ , \\ \{x_{Sp}, y_{Sp}, \mu_{Sp}, t_{Sp}\} &= \{1/y_p, 1/x_p, \omega^{-1/2} y_p/(x_p\mu_p), 1/t_p\} \ , \\ \{x_{Mp}, y_{Mp}, \mu_{Mp}, t_{Mp}\} &= \{x_p, y_p, \omega\mu_p, t_p\} \ , \\ \{x_{Vp}, y_{Vp}, \mu_{Vp}, t_{Vp}\} &= \{x_p, \omega y_p, \mu_p, \omega t_p\} \ . \end{aligned} \tag{2.7}$$

The central sheet $\mathcal{D}$ and its neighbours.

We shall find it natural, at least for the special case discussed below, to regard t_p as the independent variable, and x_p, y_p, μ_p to be defined it terms of it by (2.2). They are not single-valued functions of t_p: to make them single-valued we must introduce N branch cuts $B_0, B_1, \ldots, B_{N-1}$ in the complex t_p-plane as indicated in Figure 2. They are about the points $1, \omega, \ldots, \omega^{N-1}$, respectively,

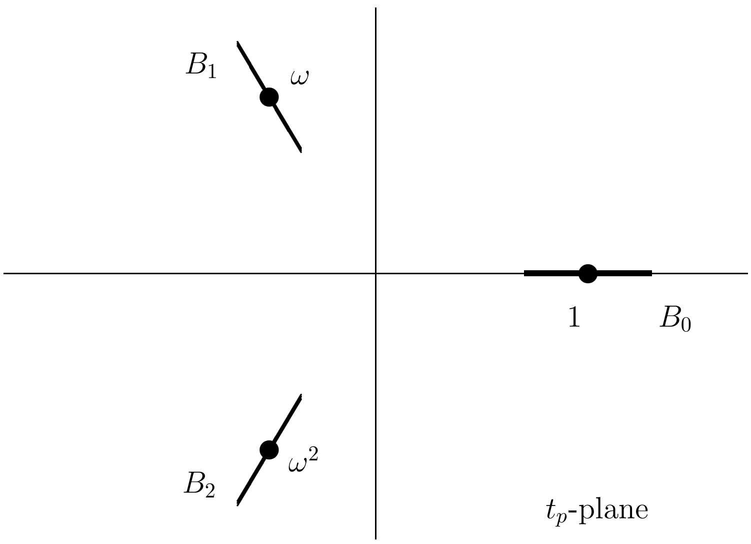

Figure 2. The cut t_p-plane for $N = 3$.

Since the Boltzmann weights are rational functions of x_p, y_p, we expect $G_{pq}(r)$, considered as a function of t_p or t_q, to also have these N branch cuts.

Given t_p in the cut plane of Figure 2, choose μ_p^N to be outside the unit circle. Then x_p must lie in one of N disjoint regions centred on the points $1, \omega, \ldots, \omega^{N-1}$. Choose it to be in the region centred on 1. We then say that p lies in the domain $\mathcal{D}$. When this is so (and t_p is not close to a branch cut), then in the limit $k' \to 0$, $\mu_p^N = O(1/k')$ and $x_p \to 1$.

The domain $\mathcal{D}$ has N neighbours $\mathcal{D}_0, \ldots, \mathcal{D}_{N-1}$, corresponding to t_p crossing the N branch cuts $B_0, \ldots, B_{N-1}$, respectively. The automorphism that takes $\mathcal{D}$ to $\mathcal{D}_i$, while leaving t_p unchanged, is

$$A_i \ = \ V^{i-1} R V^{N-i} \ . \tag{2.8}$$

The mappings A_i are involutions: $A_i^2 = 1$.

3. Functional relations

We define the ratio function

$$G_{pq}(r) \;=\; \tilde{F}_{pq}(r)/\tilde{F}_{pq}(r-1) \;\; . \tag{3.1}$$

The functions $\tilde{F}_{pq}(r)$, $G_{pq}(r)$ satisfy two reflection symmetry relations. Also, although we cannot move the break in the (p,q) rapidity line away from the spin a, we can rotate its parts about a and then cross them over. As we show in [23] and [22], this leads to functional relations for $G_{pq}(r)$:

$$\begin{aligned} G_{Rp,Rq}(r) &= 1/G_{pq}(N-r+1) \;\; , \\ G_{p,q}(r) &= 1/G_{RSq,RSp}(N-r+1) \;\; , \\ G_{pq}(r) &= G_{Rq,R^{-1}p}(r) \;\; , \\ G_{pq}(r) &= \frac{x_q\mu_q - \omega^r x_p\mu_p}{y_p\mu_q - \omega^{r-1}y_q\mu_p}\, G_{R^{-1}q,Rp}(r) \\ G_{Mp,q}(r) &= G_{p,M^{-1}q}(r) = G_{pq}(r+1) \;\; , \\ \prod_{r=1}^{N} G_{pq}(r) &= 1 \;\; . \end{aligned} \tag{3.2}$$

Also, from (2.6),

$$\mathcal{M}_r \;=\; G_{pp}(1)\cdots G_{pp}(r) \;\; . \tag{3.3}$$

For the case when $N=2$ we regain the Ising model. As is shown in [23], there is then a uniformizing substitution such that x_p, y_p, μ_p, t_p are all single-valued meromorphic functions of a variable u_p, and $W_{pq}(n), \overline{W}_{pq}(n)$ and hence $G_{pq}(r)$ depend on u_p, u_q only via their difference $u_q - u_p$. In fact all quantities are Jacobi elliptic functions of u_p, u_q with modulus k. One can argue (based on low-temperature series expansions) that $G_{pq}(r)$ is analytic and non-zero in a particular vertical strip in the complex $u_q - u_p$ plane. The relations (3.2) then define $G_{pq}(r)$. They can be solved by Fourier transforms and one readily obtains the famous Onsager result

$$\mathcal{M}_1 \;=\; (1-k'^2)^{1/8} \;\; . \tag{3.4}$$

For $N >$ the problem is much more difficult. There then appears to be no uniformizing substitution and $G_{pq}(r)$ lives on a many-sheeted Riemann surface obtainable from $\mathcal{D}$ by repeated crossings of the branch cuts. One can argue from the physical cases (when the Boltzmann weights are real and positive) that $G_{pq}(r)$ should be analytic and non-zero when p, q both lie in $\mathcal{D}$, but the relations (3.2) only relate these sheets to a small sub-set of all possible sheets. There seems to be a basic lack of information.

4. Solvable special case: $q = Vp$

The author spent much time mulling over this problem, then towards the end of 2004 he realised that the case

$$q \;=\; Vp \tag{4.1}$$

may be much simpler to handle, and still be sufficient to obtain the order parameters $\mathcal{M}_r$.

The reason it is simpler is that one can rotate the left-half line p anti-clockwise below a until it lies immediately below the half-line q, as in Fig. 5 of [22]. One has to reverse the direction of the arrow, which means the rapidity is not p but $p' = R^{-1}p$.

The result is that p enters the sums in (2.4), (2.5) only via the weights of the edges shown in Figure 3. The left-hand spins are the same - the spin a. The right-hand spins are set to the boundary value of zero.

Further, we can sum over the spins between lines p' and q. For instance, summing over the spin g gives a contribution

$$U(b,c,d,e) \;=\; \sum_g W_{vp'}(b-g)\overline{W}_{vp'}(c-g)W_{vq}(g-d)\overline{W}_{vq}(g-e) \;\; .$$

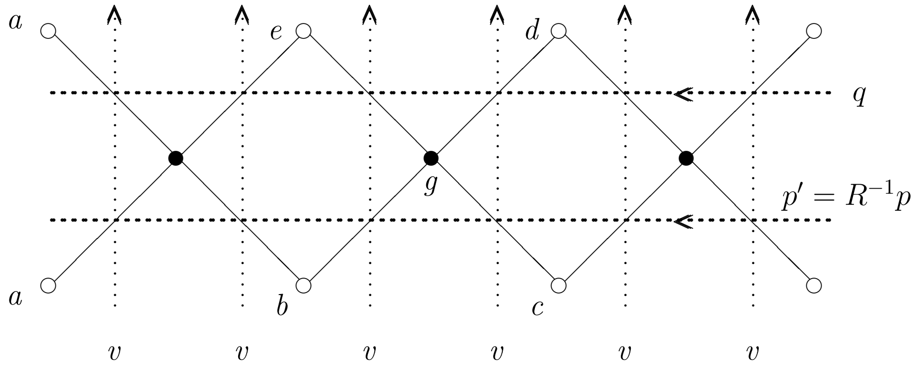

Figure 3. The lattice after rotating the half-line p to a position immediately below q.

If $a, \sigma_1, \ldots, \sigma_L$ are the spins on the lowest row of Figure 3, and $a, \sigma'_1, \ldots, \sigma'_L$ are those in the upper, then the combined weight of the edges shown in Figure 3 is

$$\prod_{i=1}^{L} U(\sigma_{i-1}, \sigma_i, \sigma'_i, \sigma'_{i-1}) \quad . \tag{4.2}$$

Now $q = VRp'$, which from (2.7) means that

$$x_q = y_{p'} \quad , \quad y_q = \omega^2 x_{p'} \quad , \quad \mu_q = 1/\mu_{p'} \quad . \tag{4.3}$$

This is the equation (3.13) of [4], the q, r therein being our p', q and k, ℓ having the values $0, 2$. From (3.17) therein, $U(b, c, d, e)$ vanishes if $0 \leq \mathrm{mod}(b - e, N) \leq 1$ and $2 \leq \mathrm{mod}(c - d, N) \leq N - 1$. It follows that the spins in the upper row are either equal to the corresponding spins in the lower row, or just one less than them. From (2.29) and (3.39) of [4], it follows that to within "gauge factors" (i.e. factors that cancel out of eqn. 4.2) $U(b, c, d, e)$ depends on p very simply: it is *linear* in t_p.

In fact, these Boltzmann weights $U(b, c, d, e)$ are those of the $\tau_2(t_{p'})$ model [4, 5, 6] mentioned earlier. Just as this model plays a central role in the calculation of the chiral Potts free energy, so it naturally enters this calculation of the order parameters.

In the low-temperature limit, when $k' \to 0$, $\mu_p, \mu_q \sim O(k'^{-1/N})$, $x_p, x_q \to 1$, we can verify that the dominant contribution to the sums in (2.4), (2.5) comes from the case when $\sigma_1, \ldots, \sigma_L, \sigma'_1, \ldots, \sigma'_L$ are all zero. Also, to within factors that cancel out of(4.2) and (2.5),

$$U(b, c, c, b) = 1 - \omega t_{p'} = 1 - t_p \quad . \tag{4.4}$$

It follows that the RHS of (2.5), and therefore of (3.1), is a ratio of two polynomials in t_p, each of degree L, and each equal to $(1 - t_p)^L$ in the limit $k' \to 0$. By continuity (keeping L finite), for small values of k' their L zeros must be close to one. Provided this remains true (which we believe it does) when we take the limit $L \to \infty$, we expect $G_{p,Vp}(r)$ to be an analytic and non-zero function of t_p, except in some region near $t_p = 1$. As k' becomes small, this region must shrink down to the point $t_p = 1$.

Similarly, if we rotate the half line p in Figure 1 clockwise above a, we can move it be immediately above q, with p replaced by Rp, as in Fig. 6 of [22]. The p', q of Figure3 herein are now replaced by q, Rp. This corresponds equation (3.13) of [4] with the q, r therein replaced by q, Rp. From (4.1) it follows that k, ℓ in [4] now have the values $-1, N + 1$. The combined star weights U are now those of the $\tau_N(t_p)$ model. They are polynomials in t_p of degree $N-1$, except for terms which contribute a factor $x_p^{\epsilon(r)}$ to the contribution of (4.2) to $G_{p,Vp}(r)$, where

$$\epsilon(r) \;=\; 1 - N\delta_{r,0} \quad , \tag{4.5}$$

the δ function being interpreted modulo N, so $\epsilon(0) = \epsilon(N) = 1 - N$.

When $k' \to 0$ these polynomials are $(1-\omega t_p)(1-\omega^2 t_p)\cdots(1-\omega^{N-1}t_p)$. In the large-$L$ limit, with k' not too large, we therefore expect $x_p^{\epsilon(r)}G_{p,Vp}(r)$ to have singularities near $t_p = \omega, \ldots, \omega^{N-1}$, but *not* near $t_p = 1$.

If we define

$$g(p;r) \;=\; G_{p,Vp}(r) \quad , \tag{4.6}$$

then this implies that the function $x_p^{\epsilon(r)} g(p;r)$ does *not* have B_0 as a branch cut. This is in agreement with the fourth and sixth functional relations in (3.2). If we set $q = Vp$ therein we obtain

$$x_p^{-\epsilon(r)} g(p;r) \;=\; y_p^{-\epsilon(r)} g(V^{-1}Rp;r) \quad , \tag{4.7}$$

using $V^{-1}R = R^{-1}V$. Here we have used the fourth relation for $r \neq 0$ and the sixth to then determine the behaviour for $r = 0$. (For $r = 0$ the fourth relation merely gives $0 = 0$.) From (2.8) the automorphism $V^{-1}R$ is the automorphism A_0 that takes p across the branch cut B_0, returning t_p to its original value, while interchanging x_p with y_p. Thus (4.7) states that $x_p^{-\epsilon(r)} g(p;r)$ is the same on both sides of the cut, i.e. it does not have the cut B_0.

These are the key analyticity properties that we need to calculate $g(p;r)$ and $\mathcal{M}_r$. We do this in [22, 25], but this meeting is in honour of Tony Guttmann, an expert in series expansion methods, so it seems appropriate to here describe the series expansion checks I made (for $N = 3$) when I first began to suspect these properties.

5. Consequences of this analyticity

The above observations imply that $g(p;r)$, considered as a function of t_p, does *not* have the branch cuts of Figure 2, except for the branch cut on the positive real axis.

This means that $g(p;r)$ is unchanged by taking allowing t_p to cross any of the branch cuts $B_1, \ldots, B_{N-1}$ and then returning it to its original value, i.e. it satisfies the $N-1$ symmetry relations:

$$g(p;r) \;=\; g(A_i\, p;r) \quad \text{for} \quad i = 1, \ldots, N-1 \quad , \tag{5.1}$$

A_i being the automorphism (2.8).

For $N = 3$, this can be checked using the series expansions obtained in [26]. We use the hyperelliptic parametrisation introduced in [27, 28, 29]. We define parameters x, z_p, w_p related to one another and to t_p by

$$(k'/k)^2 = 27x \prod_{n=1}^{\infty} \left(\frac{1-x^{3n}}{1-x^n} \right)^{12} \quad . \tag{5.2}$$

$$w = \prod_{n=1}^{\infty} \frac{(1-x^{2n-1}z/w)(1-x^{2n-1}w/z)(1-x^{6n-5}zw)(1-x^{6n-1}z^{-1}w^{-1})}{(1-x^{2n-2}z/w)(1-x^{2n}w/z)(1-x^{6n-2}zw)(1-x^{6n-4}z^{-1}w^{-1})} \tag{5.3}$$

(writing z_p, w_p here simply as z, w), and

$$t_p = \omega \frac{f(\omega z_p)}{f(\omega^2 z_p)} = \frac{f(-\omega/w_p)}{f(-\omega^2/w_p)} = \omega^2 \frac{f(-\omega w_p/z_p)}{f(-\omega^2 w_p/z_p)} \quad , \tag{5.4}$$

where $f(z)$ is the function

$$f(z) \;=\; \prod_{n=1}^{\infty} (1-x^{n-1}z)(1-x^n/z) \quad . \tag{5.5}$$

Note that x, like k', is a constant (not a rapidity variable) and is small at low temperatures. We develop expansions in powers of x. For p in $\mathcal{D}$, the parameters z_p, w_p are of order unity, so to leading order $w_p = z_p + 1$, $x_p = 1$, $y_p = (\omega - \omega^2 z_p)/(1-\omega^2 z_p)$.

The automorphisms R, S, V transform z_p, w_p to

$$z_{Rp} = x z_p \;\; , \;\; z_{Sp} = 1/(x z_p) \;\; , \;\; z_{Vp} = -1/w_p$$

$$w_{Rp} = z_p/w_p \;\; , \;\; w_{Sp} = 1/(x w_p) \;\; , \;\; w_{Vp} = z_p/w_p \;\; , \tag{5.6}$$

so from (2.8), if $p_i = A_i p$ then

$$\begin{array}{lll} z_{p_0} = -1/(x w_p), & z_{p_1} = -x w_p/z_p, & z_{p_2} = z_p \\ w_{p_0} = -1/(x z_p), & w_{p_1} = w_p, & w_{p_2} = x z_p/w_p \quad . \end{array} \tag{5.7}$$

If we write $g(p;r)$ more explicitly as $g(z_p, w_p; r)$, then the relations (5.1) become

$$g(z_p, w_p; r) \;=\; g(-x w_p/z_p, w_p; r) \tag{5.8a}$$

$$g(z_p, w_p; r) \;=\; g(z_p, x z_p/w_p; r) \quad . \tag{5.8b}$$

Using (2.4), (2.5), we can write (3.1) as

$$G_{pq}(r) \;=\; \sum_{j=0}^{2} \omega^{jr} F_{pq}(j) \Big/ \sum_{j=0}^{2} \omega^{j(r-1)} F_{pq}(j) \quad , \tag{5.9}$$

where $F_{pq}(j)$ is the probability that spin a has value j.

We use the series expansions (39) – (52) of [26] for $F_{pq}(1)/F_{pq}(0)$ and $F_{pq}(2)/F_{pq}(0)$ in terms of

$$\alpha = z_q/z_p \;\; , \;\; \beta = w_q/w_p \quad . \tag{5.10}$$

Since $q = Vp$, $z_q = -1/w_p$, $w_q = z_p/w_p$ and we find from (39) of [26] that $u = -\omega\, w_p/z_p$. (Choosing the cube root for u to ensure that $F_{pq}(i)/F_{pq}(0)$ is real when $y_p = y_q = 0$ which is when $z_p = \omega^2$, $w_p = -\omega$: we then regain the physically interesting $q = p$ case of eqn. 2.6.) For p, q in $\mathcal{D}$, the parameters $z_p, w_p, z_q, w_q, \alpha, \beta$ are all of order unity, we can then use the expansion (48) of [26] to obtain

$$F_{pq}(1)/F_{pq}(0) = \omega^2 \psi_1(z_p) \;=\; \omega^2 \psi_2(-w_p) \;\; ,$$

$$F_{pq}(2)/F_{pq}(0) = \omega \psi_2(z_p) \;=\; \omega \psi_1(-w_p) \;\; , \tag{5.11}$$

where

$$\psi_1(z) = -(z+1)x + (z+1)^3 x^2/z - (z^3 + 6z^2 + 16z + 16 + 4z^{-1} + z^{-2})x^3$$
$$+(z^4 + 11z^3 + 41z^2 + 85z + 81 + 25z^{-1} + 7z^{-2} + z^{-3})x^4 + O(x^5) \;\; ,$$

and

$$\psi_2(z) = zx - (2z + 1 + z^{-1})x^2 - (z^2 - 8z - 2 - 3z^{-1} - z^{-2})x^3$$
$$-(2z^3 - 5z^2 + 31z + 6 + 14z^{-1} + 5z^{-2} + z^{-3})x^4 + O(x^5) \;\; .$$

The automorphism (5.8a) interchanges $\mathcal{D}$ with $\mathcal{D}_1$. To leading order in x, the mid-point is when $z_p = \mathrm{i}\, x^{1/2}, w_p = 1$. This is on the boundary of the domain $\mathcal{D}$, in which the series (48) of [26] was obtained, so the series is not necessarily convergent at this point. Nevertheless, if we take $z_p = O(x^{1/2})$ in the above two series, we find the terms originally of order x^j become of order not larger than $x^{(j+1)/2}$. Extrapolating, this suggests that the series do still converge at the midpoint, so we can use them to check whether the symmetry is satisfied.

The first check occurs at order $x^{3/2}$, where both series contain a term

$$\pm\,(xz_p - x^2 w_p/z_p)$$

(using the fact that to leading order $w_p = 1$ at the midpoint). This is indeed symmetric under $z_p \to -xw_p/z_p$. If we subtract this term from the series (using the expansion of w_p in terms of z_p), we can then check the behaviour at order x^2, and similarly then at order $x^{5/2}$. All three checks are satisfied by both series.

The perceptive reader will remark that (5.11) allows us to work with w_p instead of z_p. Since w_p is unchanged by A_1, the symmetry appears obvious. Indeed it is, but only because a quite remarkable event occurred in deriving these series, namely the z series contains no powers of $z+1$ as denominators, and the w series contains no powers of $w-1$. If one expands w in terms of z (or z in terms of w), then one does find such terms. It is their absence from (5.11) that makes the series obviously convergent near $w=1$ or $z=-1$. I have presented the argument in terms of z_p to make it clear that one does indeed have three non-trivial checks on the symmetry to the available order of the series expansion.

Similarly, (5.8b) interchanges $\mathcal{D}$ with $\mathcal{D}_2$, with mid-point $z_p = -1, w_p = \mathrm{i}\, x^{1/2}$. If one now works with w_p as the variable, one can verify to the same three orders the symmetry $w_p \to xz_p/w_p$.

So our series provide no less than six checks on the symmetries (5.8a), (5.8b). When I first observed this, I could see the resemblance to the properties of the free energy of the $\tau_2(t_q)$ model. One such property is that $\tau_2(t_q)\tau_2(\omega t_q)\cdots\tau_2(\omega^{N-1}t_q)$ is a rational function of x_q^N, so I looked at the series for

$$\begin{aligned}\mathcal{L}(p;r) &= \prod_{j=0}^{N-1} g(V^j\, p;r)\\ &= g(z_p,w_p;r)\, g(-1/w_p,z_p/w_p;r)\, g(-w_p/z_p,-1/z_p;r)\ .\end{aligned} \qquad (5.12)$$

Choosing an arbitrary value for z_p and working to 30 digits of accuracy, I soon found that the series (known to order x^4) fitted with the simple formulae

$$\mathcal{L}(p;0) = 1/x_p^2\ ,\ \ \mathcal{L}(p;1) = k^{1/3}x_p\ ,\ \ \mathcal{L}(p;2) = k^{-1/3}x_p\ . \qquad (5.13)$$

All this strongly suggested that I was on the right track. It did not take long to justify my observations for general N. For instance, if $g(p;r)$ only has the branch cut B_0, and $x_p^{-\epsilon(r)}g(p;r)$ does not have that cut, then $x_p^{-\epsilon(r)}\mathcal{L}(p;r)$ does not have the cut B_0. But this function is unchanged by $p \to Vp$, which rotates the t_p plane through an angle $2\pi/N$. Hence it cannot have any of the cuts $B_0, B_1, \ldots, B_{N-1}$. We do not expect any other singularities (e.g. poles) for p in $\mathcal{D}$, so the function is analytic in the entire t_p plane. It is bounded (the Boltzmann weights $W, \overline{W}$ remain finite and non-zero as $y_p \to \infty$, the ratio μ_p/y_p remaining finite), so from Liouville's theorem it is a constant (independent of p but dependent on r).

We can relate these constants to the desired order parameters $\mathcal{M}_r$ in two ways, and then use these relations to calculate the $\mathcal{M}_r$. When $y_p = y_q = 0$ and $x_p = k^{1/N}$, our special case $q = Vp$ intersects with physically interesting case $q = p$, so from (2.6),

$$x_p^{-\epsilon(r)}\mathcal{L}(p;r) \;=\; k^{-\epsilon(r)/N}\,(\mathcal{M}_r/\mathcal{M}_{r-1})^N\ . \qquad (5.14)$$

When $y_p = y_q = \infty$ (μ_p/y_p remaining finite) and $x_p = k^{-1/N}$ we find not $q = p$ but $q = M^{-1}p$, which is related to $q = p$ by the fifth of the functional relations (3.2), giving

$$x_p^{-\epsilon(r)}\mathcal{L}(p;r) \;=\; k^{\epsilon(r)/N}\,(\mathcal{M}_{r+1}/\mathcal{M}_r)^N \quad . \tag{5.15}$$

The left-hand sides of these last two equations, being constants, are the same in both equations. We can therefore equate the two right-hand sides, for $r = 1,\ldots,N-1$. Using the fact that $\mathcal{M}_0 = \mathcal{M}_N = 1$, we can solve for $\mathcal{M}_1,\ldots,\mathcal{M}_{N-1}$ to obtain

$$\mathcal{M}_r \;=\; k^{r(N-r)/N^2} \;\;\text{for}\;\; r = 0,\ldots,N \quad , \tag{5.16}$$

which verifies the conjecture (1.1) of Albertini *et al* [7]. For $N = 3$ these results do of course agree with my original conjectures (5.13).

In [22] I also show that one can calculate $G_{P,Vp}(r) = g(p;r)$ by a Wiener-Hopf factorization, giving

$$g(p;r) \;=\; k^{(N+1-2r)/N^2}\,\mathcal{S}_p^{\epsilon(r)} \tag{5.17}$$

for $r = 1,\ldots,N$, where

$$\log\mathcal{S}_p \;=\; -\frac{2}{N^2}\log k + \frac{1}{2N\pi}\int_0^{2\pi}\frac{k'\mathrm{e}^{\mathrm{i}\theta}}{1-k'\mathrm{e}^{\mathrm{i}\theta}}\,\log[\Delta(\theta)-t_p]\,\mathrm{d}\theta \quad , \tag{5.18}$$

and

$$\Delta(\theta) \;=\; [(1-2k'\cos\theta+k'^2)/k^2]^{1/N} \quad . \tag{5.19}$$

(This function $\mathcal{S}_p$ should not be confused with the automorphism S defined in (2.7).

As is implied by the above equations, $\mathcal{S}_p$ satisfies the product relation

$$\mathcal{S}_p\mathcal{S}_{Vp}\cdots\mathcal{S}_{V^{N-1}p} \;=\; k^{-1/N}x_p \quad . \tag{5.20}$$

Also, if one sets $q = Vp$ in the second of the relations (3.2), uses the identity $RS = MVRSV$ and the fifth relation, one obtains $g(p;r)g(RSVp;N-r) = 1$, from which we can deduce the symmetry

$$\mathcal{S}_p\,\mathcal{S}_{RSVp} = k^{-2/N^2} \quad . \tag{5.21}$$

For $N = 3$ the automorphism $p \to RSVp$ takes z_p, w_p to $-w_p, -z_p$, so this relation can then be written

$$\mathcal{S}(z_p,w_p)\mathcal{S}(-w_p,-z_p) \;=\; k^{-2/9} \quad . \tag{5.22}$$

6. Another interesting case: $q = V^2p$

We now have the solution for $G_{pq}(r)$ for $q = p$ and for $q = Vp$. This suggests looking at one more case: $q = V^2p$, where $y_q = \omega^2 y_p$. Similarly to section 5, we set $g_2(p;r) = G_{pq}(r)$ and

$$L_2(p;r) = \prod_{j=0}^{N-1} g_2(V^jp;r) \quad .$$

For $N = 3$ we have used the series expansions of [26] to obtain for this case

$$F_{pq}(1) = \omega\phi(w_p) \quad , \quad F_{pq}(2) = \omega^2\phi(1/w_p) \quad , \tag{6.1}$$

where

$$\phi(w) = (w-1)x - (2w^2-2w+1)x^2/w + (2w^3+6w^2-6w+1)x^3/w$$

$$- (2w^4 + 8w^3 + 24w^2 - 22w + 5)x^4/w + O(x^5) \quad . \tag{6.2}$$

As in the previous case, the coefficients are Laurent polynomials in w. There is no sign of any singularity near $w_p = 1, t_p = \omega$ so this suggests that $G_{pq}(r)$, considered as a function of t_p, does not have the branch cut B_1.

Indeed, this is a consequence of the third functional relation (3.2). Setting $q = V^2 p$ therein, we obtain

$$g_2(p;r) \;=\; g_2(A_1 p;r) \quad ,$$

which tells us that $g_2(p;r)$ is unchanged by taking t_p across the branch cut B_1 and returning it to its original value. This means that the cut B_1 is unnecessary. However, $g_2(p;r)$ does appear to have the other two cuts B_0 and B_2.

To the available four terms in the series expansion we found

$$L_2(p;1) \;=\; x_p^2 \quad ,$$

and

$$L_2(p;0) \;=\; k^{-1/3} x_p^{-1} h(z_p,w_p)^3 \quad , \quad L_2(p;2) \;=\; k^{1/3} x_p^{-1} h(z_p,w_p)^{-3} \quad , \tag{6.3}$$

where

$$\begin{aligned} h(z,w) = 1 &+ (x^2 - 6x^3 + 35x^4)(w/z^2 + zw - z/w^2 + 3) \\ &+ \; x^4(w^2/z^4 + z^2/w^4 + z^2 w^2 - 3) + O(x^5) \quad . \end{aligned} \tag{6.4}$$

The result for $L_2(p;1)$ looks encouraging, and indeed to the four available terms in the series expansion we also find

$$g_2(p;1) = k^{2/9}\, \mathcal{S}_p\, \mathcal{S}_{Vp} \quad . \tag{6.5}$$

The results for $L_2(p;0)$ and $L_2(p;2)$ are not so encouraging and I have failed to find any obvious result for these or for $g_2(p;0)$, $g_2(p;2)$. In [22] I conjecture that for general N the functions $G_{p,V^i p}(r)$ have a simple form as a product of $\mathcal{S}$ functions provided $i = 0, \ldots, N-1$ and $r = 1, \ldots, N-i$. For other values of i, r they remain a puzzle. (Except when $i = 1$ and $r = N$: this case can be deduced from the sixth relation of eqn 3.2.)

If (6.5) is correct, then we have some information on the function $L_{pq}(r)$ of eqn. 56 of [23]. From this and the first equation of (3.2),

$$L_{pq}(r) = G_{pq}(r) G_{Rq,Rp}(r) = G_{pq}(r)/G_{qp}(N-r+1) \quad . \tag{6.6}$$

Setting $q = Vp$ and using (4.6), we obtain

$$L_{pq}(r) = g(p;r)/g_2(Vp; N-r+1) \quad . \tag{6.7}$$

Taking $r = 0$, it follows from (5.17) and (6.5) that

$$L_{pq}(0) = k^{-4/9}/(\mathcal{S}_p^2\, \mathcal{S}_{Vp}\, \mathcal{S}_{V^2 p}) = k^{-1/9}/(x_p \mathcal{S}_p) \quad . \tag{6.8}$$

The function L_{pq}, for arbitrary p, q, was introduced in [23] partly because its square is a rational function of $x_p, y_p, \mu_p, x_q, y_q, \mu_q$ when $N = 2$, so the hope was that it might be similarly simple for all N. We see that this cannot be so: $\mathcal{S}_p$ is *not* such a function.

7. Summary

I have outlined the recent derivation of the order parameters of the solvable chiral Potts model, a derivation that verifies a long-standing and elegant conjecture [7]. As with all the calculations on solvable models satisfying the star-triangle relations, the trick is to generalize the model to a point where one has a function, here $G_{pq}(r)$, to calculate, rather than a constant, as one can obtain relations and properties that define this function. On the other hand, this is an example where it pays *not* to over-generalize: we can handle the particular function $G_{p,Vp}(r)$, and this is sufficient for the purpose of obtaining the order parameters. The general $G_{pq}(r)$ continues to defy calculation.

Series expansion methods can provide a valuable check on such derivations, which are of their nature believable but hard to make fully mathematically rigorous. One usually tries to present the argument in as logical a manner as possible, but this is usually *not* the manner in which it was originally developed. Here I have indicated the points in the calculation when I found the available checks both reassuring and encouraging.

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