Infinitely extended complex KdV equation and its solutions : solitons and rogue waves

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Abstract. We present an infinitely-extended KdV equation that contains an infinite number of arbitrary real coefficients controlling higher-order terms in the extended evolution equation. The higher-order terms are chosen in a way that maintains the integrability of the whole equation. Another significant step in this work is that this extended equation admits complex-valued solutions. This generalization allows us to consider both solitons and rogue waves in the form of rational solutions of this equation. Special choices of the arbitrary coefficients lead to particular cases - the basic KdV and its higher-order versions. Using the extended KdV, instead of the basic one, may improve the accuracy of the description of rogue waves in shallow water.

keywords: solitons ; Rogue waves ; complex Korteweg de Vries equation.

1. Introduction

The Korteweg - de Vries (KdV) equation became the first known integrable evolution equation for modelling nonlinear dispersive waves [1]. Originally, it was derived for modelling shallow water waves [2, 3]. The observation of a soliton by John Scott Russell demonstrated the existence of such waves in water canals [4]. More recently, it was found that the KdV equation is applicable to plasma waves [5, 6] and atto-second optical pulses that contain a few optical cycles, or even a fraction of one cycle [7, 8]. In each case, soliton solutions are of primary interest as waves that keep undistorted profiles over large distances of propagation [9]. However, solitons are not the only structures that exist in the physical systems listed above. Rogue waves represent a different form of nonlinear structure that has received much attention in recent years. In particular, they do exist in shallow water [10]. In fact, they are understood much less than deep water rogue waves. One of the possible ways to model shallow water rogue waves is to use the KdV equation with complex functions [11, 12], rather than the real ones used in most previous publications. This has been done in our recent work [13].

Historically, the second nonlinear wave evolution equation found to be integrable was the Nonlinear Schrödinger Equation (NLSE) [14]. Solitons, breathers and rogue waves described by this equation are all known in analytical form. These solutions equally well model waves in optical fibres [15] and water waves in the deep ocean [16]. It has been extended to include the Hirota equation, the Lakshmanan - Porsezian - Daniel (LPD) equation and other higher-order equations, keeping integrability intact, and thus allowing one to find exact solutions for the relevant physical problems [17]. These equations provide the opportunity to increase the accuracy of modelling of nonlinear waves with the inclusion of higher-order dispersion or nonlinear terms. These higher-order equations are important for modelling oceanic waves [18], [19]. They also cover a wider range of physical phenomena, such as the Heisenberg ferromagnetic spin chain [20], or effects like modulation instability [21, 22] that do not exist in the case of the ordinary KdV equation.

In a similar fashion, expanding the KdV equation with the addition of higher-order terms may improve the accuracy of modeling of known phenomena as well as predicting new effects. For example, this could describe water waves that are shorter but steeper [23]. It also can have better accuracy, compared to the KdV, in modeling shallow water waves facing an obstacle [24] or where the surface tension is not negligible [25]. However, it has been shown that the basic KdV equation describes half-cycle optical solitons in quadratic nonlinear media [9]. As another example, the 5-th order KdV equation has been found relevant to soliton collisions for weakly nonlinear long waves [26]. In these cases, only real solutions have been considered, despite the fact that allowing the solutions to encompass the complex space can significantly increase the variety of possible scenarios in wave evolution [27, 28].

Shallow water rogue waves can have considerable impact on the marine ecosystem and have caused significant coastal erosion [29]. A rogue wave hitting a crowded beach is of particular concern. For example, this occurred [30] in 1938 at Bondi beach in Sydney, Australia, where 'three tremendous waves rolled onto the beach', one after the other. This caused panic and swept many out into the sea. In the end, 5 people were killed and 250 rescued. It would be very useful to gain understanding of such phenomena, with an eye to possible early detection and warning. It is plausible that such waves can be modelled by the KdV equation and its expanded versions. In this work, we extend complex rogue wave ideas to the KdV expansion.

Higher-order KdV equations have been considered earlier in a number of publications [31, 33, 34, 35, 36]. They have been called 'hierarchies', as each separate one can be obtained from a lower order one using certain transformations [32]. Our approach is different in that we take them all together. Firstly, we consider a single infinitely extended equation that includes, as a particular case, the basic KdV equation. Secondly, we include complex functions as potential solutions to this single equation. The infinite extension contains higher-order operators with arbitrary real parameters, and this significantly increases the range of physical problems where this full equation can be applied. Below, we consider, separately, the cases of focusing and defocusing KdV extensions. These differ in signs of nonlinear terms, although a simple change of the sign of the function $u \to -u$ converts one case to another.

2. Infinitely extended KdV equation

Recently, work was presented showing the plausibility of employing complex functions to describe rogue waves occurring in shallow water [13]. This involved only the 'basic' KdV equation. To extend this, we start with the well known 'defocussing' KdV equation,

$$u_x + \alpha_1 \left[u_{ttt} - 6u(t,x)u_t \right] = 0, \tag{1}$$

where t is the transverse variable and x is the variable along the wave propagation direction. In optics, x is the propagation distance, while t is a retarded time, as used in the optical few-cycle KdV example of [9, 37, 39]. In water and other fluid systems, x is usually time while t is a transverse spatial variable. This is an evolution equation which means that waves evolve in the x-direction.

It is easy to show that if $u(x,t)$ is a solution of the 'defocusing' KdV, then $u(x,t) = -u(x,t)$ is a solution of 'focusing' KdV. Thus, the 'focusing' KdV solutions can be found from the 'defocusing' ones simply by changing the sign of the dependent variable.

We can write the infinitely extended equation (IEE) in the form similar to the one used in [17]:

$$u_x(t,x) + \sum_{m=1}^{\infty} \alpha_m K_m(u) = 0, \tag{2}$$

where $K_m(u)$ is the operator of order m. It include the function u and its derivatives up to order m. The lowest order operator here is:

$$K_1 = u_{ttt} - 6u(t,x)u_t.$$

When all coefficients except α_1 are zero, Eq.(2) reduces to the basic KdV equation (1). The next level operator, $K_2(u)$, is

$$K_2 = -\frac{15}{2}u_t u^2 + \frac{5}{2}u_{ttt}u + 5u_t u_{tt} - \frac{1}{4}u_{5t}$$

while the operator $K_3(u)$ is given by

$$K_3 = \frac{1}{16}\Big\{70u_{ttt}u^2 - 140u_t u^3 + +14\left[20u_t u_{tt} - u_{5t}\right]u + 70(u_t)^3 - 70u_{tt}u_{ttt} - 42u_t\, u_{4t} + u_{7t}\Big\} \quad (3)$$

Here, each evolution equation, $u_x(t,x) + \alpha_n K_n(u) = 0$, obtained from eq.(2) when all coefficients except one are zero can be written as

$$\frac{\partial u}{\partial x} + \alpha_n \frac{\partial F_n(u)}{\partial t} = 0,$$

where $F_n(u)$ is a polynomial in u. So

$$\frac{\partial F_n(u)}{\partial t} = K_n(u).$$

Hence, F_1 and F_2 are:

$$F_1 = u_{tt} - 3u^2, \quad (4)$$

$$F_2 = (-\frac{1}{4})(u_{4t} - 5u_t^2 - 10uu_{tt} + 10u^3). \quad (5)$$

The coefficient α_1 in (1) is an arbitrary real parameter that commonly is taken to be 1. Eq.(1) can be called KdV-3 as it has the third-order derivative in the operator within the rectangular brackets. The set of operators can be continued indefinitely, thus making Eq.(2) infinitely extended.

The set of the operators for 'focusing' KdV, $u_x + \alpha_n W_n = 0$, can be found by setting $u(x,t) = -u(x,t)$, thus obtaining:

$$W_1 = u_{ttt} + 6u(t,x)u_t, \quad (6)$$

$$W_2 = -\frac{15}{2}u_t u^2 - \frac{5}{2}u_{ttt}u - 5u_t u_{tt} - \frac{1}{4}u_{5t}$$

and

$$W_3 = \frac{1}{16}\Big\{70u_{ttt}u^2 + 140u_t u^3 + 14\left[20u_t u_{tt} + u_{5t}\right]u + 70(u_t)^3 + 70u_{tt}u_{ttt} + 42u_t\, u_{4t} + u_{7t}\Big\}. \quad (7)$$

As stated above, the parameters α_m are arbitrary real numbers that provide infinite variability for this equation. When all coefficients are zero except one, we obtain particular cases. For example, KdV-5 is an equation with 5-th order highest derivative in t:

$$u_x + \alpha_2 K_2 = 0. \quad (8)$$

Similarly,

$$u_x + \alpha_3 K_3 = 0. \tag{9}$$

can be called KdV-7, as its highest derivative is of seventh order in t. For each individual example, the equation with K_n can be called KdV-(2n+1), as its highest derivative is of order $2n+1$.

The set of operators K_m provided above ensures the integrability of Eq.(2). This means that solutions of Eq.(2) can be written in analytic forms. We demonstrate this by providing a few solutions to Eq.(2). We start with the most common – the soliton solution.

3. Soliton solutions

For KdV-3, the well-known soliton solution is:

$$u_1 = -2k^2 sech^2 \left[k\left(t - 4\alpha_1 k^2 x\right)\right], \tag{10}$$

where k is an arbitrary real constant that defines the amplitude and the velocity of the soliton.

We can write down the soliton solution in general form for the whole extended equation (2):

$$u(t,x) = -2k^2 sech^2 \left[k\left(t + J(x)\right)\right], \tag{11}$$

where

$$J(x) = 4x \sum_{n=1}^{\infty} (-1)^n \alpha_n k^{2n}.$$

This can be written as

$$u(t,x) = -2\left[log(f(t,x))\right]_{tt}$$

where $f(t,x) = 1 + \exp\left[2k(t + J(x))\right]$. This solution is valid for any real values of the coefficients α_n which are present in the equation (2), even if there are infinitely many. Clearly, if some of these coefficients are zero, the solution is simplified, with a reduced number of the terms in the sum. For the case where only α_1 and α_2 are non-zero, this solution has been given in [38].

For example, if all of them are zero except for α_n, the solution becomes:

$$u_n = -2k^2 sech^2 \left\{k\left[t + 4(-1)^n \alpha_n k^{2n} x\right]\right\}, \tag{12}$$

where $n = 1, 2, 3, \cdots$. This is the soliton solution for KdV-(2n+1). By having a closer look at this form of the solution, we realize that for each KdV-(2n+1) (and setting $\alpha_n = 1$ and $k = 1$), we get the same solution, apart from the sign on the x term. This does not occur in the forms of the rational solutions, since we will obtain different shapes and solutions for each order of the KdV.

Furthermore, the soliton solutions given above in this section are real. However, in this paper we allow for complex solutions. Here we can add a parameter, $i\,c$, with

c being an arbitrary real number, to the argument of the `sech` function in each of the above solutions. Hence

$$u_n = -2k^2 sech^2 \left\{ k \left[t + 4(-1)^n \alpha_n k^{2n} x \right] + i\, c \right\} \tag{13}$$

is a valid solution of the equation of order n. As an illustrative example, Fig. 1 shows the profiles of the real and imaginary parts, respectively, of the complex $n = 2$ soliton defined by Eq. (13).

We now take $k = 1$ for convenience and expand this. Thus:

$$u_n = \frac{4}{B}[-1 - \cos(2c)\cosh(A) + i\, \sin(2c)\sinh(A)], \tag{14}$$

where $A = 2(t + 4(-1)^n \alpha_n x)$ and $B = [\cos(2c) + cosh(A)]^2$.

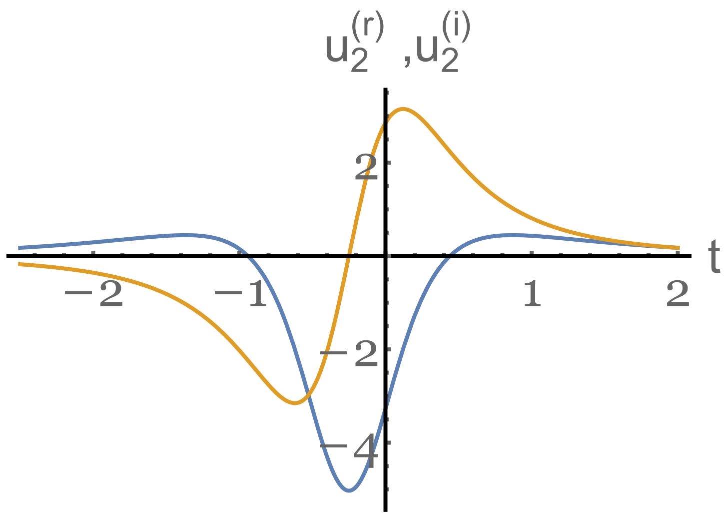

Figure 1. Real part, $u_2^{(r)}$ (blue, reaching a minimum of approx.-5), and imaginary part, $u_2^{(i)}$ of $n = 2$ soliton of eq.13. Here $\alpha_2 = 1, x = 1, c = 2.5, k = 0.5$.

This form can also be seen in the next section.

4. First order rational solutions.

The first order complex rational solution of KdV-3 has been given in [13]:

$$u_1 = \frac{8}{(2t - 12k\alpha_1 x + ic)^2} - k. \tag{15}$$

where k and c are arbitrary real constants, apart from the restriction that $c \neq 0$.

Now, this form of solution can be extended for other equations in the set. When only one α_n is non-zero, we obtain solutions of higher-order KdV equations. For example, for the KdV-5 case, the first order rational solution of Eq. (8), can be written as

$$u_2 = \frac{8}{(2t + 15k^2\alpha_2 x + ic)^2} - k. \tag{16}$$

Next, the first order rational solution of KdV-7, i.e. Eq.(9) can be written as:

$$u_3 = \frac{8}{\left(2t - k^3\alpha_3\frac{35x}{2} + ic\right)^2} - k. \tag{17}$$

The first-order rational solution for KdV-9 is:

$$u_4 = \frac{8}{\left(2t + k^4\alpha_4\frac{315x}{16} + ic\right)^2} - k. \tag{18}$$

We continue in this way, finding the coefficients needed.

Further particular cases up to and including u_7, i.e. up to the equation $KdV - 15$, reveal the general solution:

$$u_n = \frac{8}{(2t + p_n x + ic)^2} - k, \tag{19}$$

where

$$p_n = \alpha_n \frac{(-k)^n}{2^{n+1}}\,(4n^2 - 1)\, b_n,$$

with $b_n = \{16, 8, 8, 10, 14, 21, 33, \cdots\}$ for $n = 1, 2, 3, \cdots, 7, \cdots$ and k is an arbitrary real parameter. Taking into account this form of b_n, we can write the expression for p_n explicitly:

$$p_n = 8\alpha_n \left(-\frac{k}{2}\right)^n \frac{(2n+1)!!}{n!}, \tag{20}$$

where the double factorial is defined as $(2n + 1)!! = (2n + 1)(2n - 1)(2n - 3) \cdots 1$.

In fact, we can solve the infinitely extended equation (2) with

$$u_\infty = \frac{8}{(2t + px + ic)^2} - k$$

where $p = \sum_{n=1}^{\infty} p_n$, or

$$p = 8\sum_{n=1}^{\infty} \alpha_n (-\frac{k}{2})^n \frac{(2n+1)!!}{n!}.$$

Each of the above solutions is a rational soliton with a fixed velocity v. It has a ridge of height 9 while the background is $|u| = 1$. In order to avoid large numbers in the solution, we replace the independent variables with $X = 12x$ and $T = 2t$. Then, we can write the solution in log-form:

$$u_1(x,t) + k = \frac{8}{R_1^2} = -2\,\frac{\partial^2}{\partial t^2}\,(log\, R_1)\,, \tag{21}$$

where $R_1 = 2t - 12\alpha_1\, k\, x + ic$. Hence, $R_1 = T - \alpha_1\, k\, X + ic$. For the arbitrary n,

$$u_n(x,t) + k = \frac{8}{R_n^2} = -2\,\frac{\partial^2}{\partial t^2}\,(log\, R_n)\,, \tag{22}$$

where $R_n = 2t + p_n x + ic$. Hence, $R_n = T + p_n X/12 + ic$, with p_n given by Eq.(20).

Solutions presented in this section are rational solitons. They have long tails and would not be classified as 'rogue waves' in common parlance. The latter should have a localized bump that 'appears from nowhere' [43]. Below, we present the higher-order rational solutions that do show the rogue wave features.

5. Second order rational (rogue wave) solutions

Rational solutions to many integrable evolution equations comprise hierarchies that start with the lowest order and continue to infinitely high order. The infinitely-extended KdV equation and its particular cases have the same property. All of them have higher-order rational solutions. In contrast to solutions of the previous section, they can describe rogue waves. Higher-order rational solution of the 'defocusing' equation (2) with a single nonzero coefficient α_n can be written in general form:

$$u_n^r(x,t) = -2\,\frac{\partial^2}{\partial t^2}\,(log\, G_n) - 1, \tag{23}$$

where

$$G_n = (T-X)^3 + 11X - 3T - 3i(T-X)^2 + d, \tag{24}$$

with $X = 12\alpha_n\, x\, r_n v_n$ and $T = 2\, t\, v_n$, as above, and with $v_n = 1/\sqrt{n}$. In all of this section, $d = d_r + i\, d_i$ is a non-zero complex constant with $d_i < 0$. The denominator of u_n can be zero at a value of x that depends on $\sqrt{d_i}$. If $d_i > 0$, this occurs at real x (giving a singular solution), but if $d_i < 0$ then no such real value of x exists, and the solution is finite everywhere.

The definition of the variables X and T thus depends on the particular value of n chosen. Namely, when $n = 1$, i.e for KdV-3 equation, we have $r_1 = 1$. Next, when $n = 2$, i.e. for KdV-5, we have $r_2 = -\frac{5}{4}$. Further, for $n = 3$, i.e. in the case of KdV-7, we have $r_3 = \frac{35}{24}$. When $n = 4$, i.e. for KdV-9, we have $r_4 = -\frac{105}{64}$. The highest order for which we explicitly provide this coefficient is $n = 5$. Namely, for KdV-11, we have $r_5 = \frac{231}{128}$. In fact, these coefficients can be written for arbitrary n in the form

$$r_n = (-1)^{n+1}\frac{2^{1-n}}{3n!}(2n+1)!!\ .$$

Let us consider a few examples. The complex polynomial G_n in (24) for KdV-3 is:

$$G_1 = 8(t - 6\alpha_1 x)^3 - 12i(t - 6\alpha_1 x)^2 + 132\alpha_1 x - 6t + d.$$

Then, if $d_r = 0$, the maximum of the corresponding solution $|u_1|$ is $|u_1(0,0)| = -u_1(0,0) = 1 + \frac{72}{d_i^2} - \frac{48}{d_i}$. So, if $d_i = -3$, this is 25. The polynomial G_n for KdV-5 is:

$$G_2 = \frac{1}{8}[2\sqrt{2}(15\alpha_2 x + 2t)^3 - 12i(15\alpha_2 x + 2t)^2 - 660\sqrt{2}\alpha_2 x - 24\sqrt{2}t + 8d].$$

Then, if $d_r = 0$, the maximum of the corresponding solution $|u_2|$ is $|u_2(0,0)| = -u_2(0,0) = 1 + \frac{36}{d_i^2} - \frac{24}{d_i}$. So, if $d_i = -3$, this is 13. Finally, the polynomial G_n for KdV-7 is:

$$G_3 = \frac{(4t - 35\alpha_3 x)^3}{24\sqrt{3}} - \frac{i}{4}(4t - 35\alpha_3 x)^2 + \frac{385\alpha_3 x}{2\sqrt{3}} - 2\sqrt{3}t + d. \tag{25}$$

This leads to the complex solution u_3. Then, if $d_r = 0$, the maximum of the corresponding solution $|u_3|$ is $|u_3(0,0)| = -u_3(0,0) = 1 + \frac{24}{d_i^2} - \frac{16}{d_i}$. So, if $d_i = -3$, this is 9. This latter solution is plotted in Fig.2. So for $d_r = 0$, we have the maximum of $|u_n|$ being $|u_n(0,0)| = 1 + \frac{24}{n\,d_i^2}(3 - 2d_i)$, for n=1,2,3. As n increases, the maximum possible value thus decreases.

It has long 'soliton'-like tails but the most prominent feature of this solution is the rogue wave bump at the origin. This feature is present in all solutions considered in this Section. As we can see from the above, the highest amplitude is in the case of 'standard' KdV-1. As noted above, its value depends on d_r and d_i.

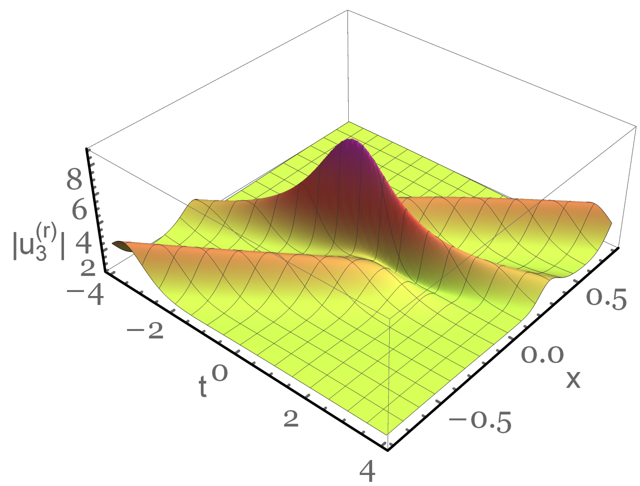

Figure 2. Rogue wave solution of KdV-7, $|u_3^r(x,t)|$ derived from Eqs.(23) and (25). Here $d = 2 - 3i$ and $\alpha_3 = 1$.

6. 'Focussing' form of the KdV equation

For completeness, let us consider an alternative, 'focussing', form of the KdV equation. In contrast to the previous case, all terms in this equation are positive. Thus, the lowest order case of this equation which we call KdV-a-3 is:

$$u_x + 6u(t,x)u_t + u_{ttt} = 0, \tag{26}$$

with the same notation for variables x and t as before. The sign of the nonlinear term $6u(t,x)u_t$ here is reversed, relative to Eq.(1). Similarly to the 'defocussing' case, we can write the set of these equations using operators $W_n(u)$:

$$u_x(t,x) + \sum_{m=1}^{\infty} \alpha_m W_m = 0. \tag{27}$$

Then, recalling Eq.(6), etc, Eq.(26) can be written in the form $u_x + \alpha_1 W_1 = 0$ with $W_1 = 6u(t,x)u_t + u_{ttt}$. The next equation in this set, using the same notation, is: $u_x + \alpha_2 W_2 = 0$ with $W_2 = -\frac{15}{2}u_t u^2 - \frac{5}{2}u_{ttt}u - 5u_t u_{tt} - \frac{1}{4}u_{5t}$. Explicitly, KdV-a-5 is:

$$u_x - \alpha_2(\frac{15}{2}u_t u^2 + \frac{5}{2}u_{ttt}u + 5u_t u_{tt} + \frac{1}{4}u_{5t}) = 0. \tag{28}$$

Finally, the adapted form of KdV-7, i.e. KdV-a-7 is: $u_x + \alpha_3 W_3 = 0$, where

$$W_3 = \frac{1}{16}\Big\{70u_{ttt}u^2 + 140u_t u^3 + 14[20u_t u_{tt} + u_{5t}]u + 70(u_t)^3 + 70u_{tt}u_{ttt} + 42u_t\, u_{4t} + u_{7t}\Big\}. \tag{29}$$

For W_n, the highest derivative is order $2n+1$. For these, when n is odd, all the terms in the equation have positive coefficients. Namely, for $n = 1$, this is KdV-a-3. For $n = 3$, we have KdV-a-7 and $n = 5$, we have KdV-a-11, etc. On the contrary, when n is even, all coefficients in the operators W_n are negative.

6.1. Soliton solutions

Soliton solutions for these equations are similar to those considered in Section 3. Again, we can solve the whole infinitely extended equation (27) for an infinite number of arbitrary real coefficients α_m. Thus, the most general soliton solution for Eq. (27) is:

$$u(t,x) = 2k^2 sech^2\left[k\left(t + 4x\sum_{m=1}^{\infty}(-1)^m\alpha_m k^{2m}\right) + i\,c\right], \tag{30}$$

where k and c are arbitrary real constants This result allows one to find soliton solutions for an elaborate equation that takes into account higher-order dispersion and nonlinear terms. This might be important for more involved cases when the depth of the water layer changes or other complications need to be taken into account.

When only one coefficient, α_n, is non-zero, the general solution, Eq.(30), is simplified. Namely, for KdV-a-3, the soliton solution is:

$$u_1 = 2k^2 sech^2\left[k\left(t - 4\alpha_1 k^2 x\right) + i\,c\right]$$

where k and c are still arbitrary real constants. The only difference from the soliton solution (10) here is its negative amplitude. The soliton solution for higher order equations KdV-a-$(2n+1)$ with $n = 1, 2, 3, \cdots$ can be written in general form:

$$u_n = 2k^2 sech^2\left[k\left(t + 4(-1)^n\alpha_n k^{2n} x\right) + i\,c\right],$$

Again, apart from sign of the amplitude, it is the same as in Eq.(13).

6.2. First order rational solutions

The primary rational solution for the KdV-a-n of order n is:

$$u_n = -k - \frac{8}{(2t + s_n x + ic)^2}, \tag{31}$$

where k, c are arbitrary real numbers and

$$s_n = k^n\,\alpha_n \frac{b_n}{2^{n+1}}\,(4n^2 - 1)\,,$$

with $b_n = \{16, 8, 8, 10, 14, 21, 33, \cdots\}$ for $n = 1, 2, 3, \cdots, 7, \cdots$. In this section, c is an arbitrary real number.

The convenient form of the expression for s_n is:

$$s_n = 8\alpha_n(\frac{k}{2})^n \frac{(2n+1)!!}{n!}, \tag{32}$$

where $(2n+1)!!$ was defined earlier. Eq.(32) differs from p_n of Eq.(20) in that it does not have the $(-1)^n$ part in the coefficient. So, these solutions are similar to solutions in Eq.(19), apart from sign changes.

Finally, we provide the general first order rational solution for the infinitely-extended KdV equation of eq. (27):

$$u_\infty = -k - \frac{8}{(2t + sx + ic)^2}, \tag{33}$$

where $s = \sum_{n=1}^{\infty} s_n$, so

$$s = \sum_{n=1}^{\infty} s_n = 8 \sum_{n=1}^{\infty} \alpha_n \, (\frac{k}{2})^n \frac{(2n+1)!!}{n!}.$$

All of the previous solutions in this section are particular cases of this general expression. The coefficients α_n are real and can be chosen arbitrarily in accordance with particular physical problem at hand.

7. Conclusions

In conclusion, we have presented an infinitely extended KdV equation that contains an infinite number of arbitrary real coefficients controlling higher-order terms in this extended evolution equation. The higher-order terms are chosen in such a way as to maintain the integrability of the whole equation. Another significant step in this work is that this extended equation admits complex-valued solutions. This generalization allows us to expand the set of solutions and include both solitons and rogue waves in the family. The rogue waves take the form of rational solutions of this equation. Special choices of the arbitrary coefficients in the equation lead to particular cases - the basic KdV and a multiplicity of its higher-order elaborations. Using the extended KdV instead of the basic one may well improve the accuracy of the description of rogue waves in shallow water.

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