EQUILIBRIA UNDER KNIGHTIAN PRICE UNCERTAINTY

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We study economies with Knightian uncertainty about state prices. We introduce an equilibrium concept with sublinear prices and prove that equilibria exist under weak conditions. In general, such equilibria lead to inefficient allocations; they coincide with Arrow–Debreu equilibria if and only if the values of net trades are ambiguity-free in mean. In economies without aggregate uncertainty, inefficiencies are generic. Equilibrium allocations under price uncertainty are efficient in a constrained sense that we call uncertainty–neutral efficient. Arrow–Debreu equilibria turn out to be non-robust with respect to the introduction of Knightian uncertainty.

KEYWORDS: Knightian uncertainty, ambiguity, general equilibrium.

1. INTRODUCTION

WE STUDY economies with Knightian uncertainty about state prices. Knightian uncertainty describes a situation in which the probability distribution of relevant outcomes is not exactly known. In such a situation, it is natural to work with a nonadditive notion of expectation derived from a set of probability distributions. We introduce a corresponding nonlinear concept of equilibrium, Knight–Walras equilibrium, in which the forward price of a contingent consumption plan is given by the maximal expected value of the net consumption value.

In our economy, agents trade contingent plans on a forward market at time 0, as in Debreu’s original model of trade under uncertainty. The market is complete in the sense that all contingent plans are available. However, Knightian uncertainty induces an imperfection to the price formation of the market, resulting in sublinear prices.

Knightian uncertainty is described by a set of priors that is common knowledge; the set describes the imprecise, but objective probabilistic information about the outcome distribution over future states, as in Ellsberg’s thought experiments. Following Walras and Debreu, we do not explicitly model the price formation process; rather, we model its outcome by a sublinear expectation. The invisible hand of the market uses the maximal expected value over the set of priors to price contingent claims. In other words, the cautious market maker has imprecise probabilistic information about the states of the world and computes the maximal expected present value over a set of models to hedge Knightian uncertainty. Such a cautious approach is common practice in today’s financial
markets in which traders use different models in stress testing to price contingent claims. Our approach thus takes such a robustness concern of the market maker into account.

We establish the existence of Knight–Walras equilibrium for general preferences including the well-studied classes of smooth ambiguity preferences (Klibanoff, Marinacci, and Mukerji (2005)) and variational preferences (Maccheroni, Marinacci, and Rustichini (2006)). The proof uses an adaptation of Debreu’s game-theoretic approach that has an interesting economic interpretation. Debreu worked with a Walrasian auctioneer who maximizes the expected value of aggregate excess demand. In our proof, we introduce an additional Knightian price player who chooses the prior that maximizes the expected value of aggregate net demand. Under Knightian uncertainty, we can thus view Adam Smith’s “invisible hand of the market” as consisting of two auctioneers who determine the state price and the relevant prior for the market. The Knightian price player in the social market game plays the role of a cautious regulator who asks for consumption plans that are robust with respect to Knightian uncertainty.

Given that we have a nonlinear price system, whether or not agents can generate arbitrage gains might come into question; any reasonable notion of equilibrium should exclude such arbitrage, of course. In our context, there is no financial market, so the arbitrage notion of a costless portfolio with positive gains does not apply here. We consider two natural notions of arbitrage for our sublinear prices. Following Aliprantis, Florenzano, and Tourky (2005), an arbitrage is a positive consumption plan with a price of zero. We show that this kind of arbitrage is precluded in Knight–Walras equilibrium. Alternatively, in our sublinear context, one could think of making gains by splitting a consumption bundle into two or more plans. The convexity of our price functional excludes such arbitrage opportunities as well.

In case of pure risk, that is, when the set of probability distributions consists of a singleton, the new equilibrium notion coincides with the classic notion of an Arrow–Debreu equilibrium under risk. We thus study the differences that Knightian uncertainty about state prices create in comparison with the Arrow–Debreu equilibrium concept. We show that Arrow–Debreu and Knight–Walras equilibria coincide if and only if the values of the individual net demands are ambiguity-free in mean, that is, when there is no ambiguity about the mean value of net demands.

We then ask how restrictive this latter condition is. To this end, we study the well-known class of economies without aggregate uncertainty and uncertainty-averse agents who share a common subjective belief at certainty. This general notion of uncertainty-averse preferences was introduced by Rigotti, Shannon, and Strzalecki (2008) and covers the well-studied class of pessimistic multiple prior agents (Gilboa and Schmeidler (1989)), smooth ambiguity models (Klibanoff, Marinacci, and Mukerji (2005)), and multiplier preferences (Hansen and Sargent (2001)). Without aggregate uncertainty, Rigotti, Shannon, and Strzalecki (2008) demonstrated that an interior allocation is efficient if and only if each agent is fully insured in equilibrium, a finding that extends previous results from Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), and Dana (2002) for specific classes of preferences. We prove that for generic endowments, Arrow–Debreu equilibria are not Knight–Walras equilibria. Agent’s net demand

\footnote{De Castro and Chateauneuf (2011) extended these results to the case of no aggregate ambiguity. Strzalecki and Werner (2011) introduced the notion of conditional subjective beliefs to study efficient allocations in general. In particular, efficient allocations are measurable with respect to aggregate endowment if agents share a common conditional belief. A further discussion of efficient allocations on the interim stage can be found in Kajii and Ui (2006) and Martins-da Rocha (2010).}
is rarely ambiguity-free in mean when individual endowments are subject to Knightian uncertainty.

Based on the generic non-equivalence of equilibria under Knightian price uncertainty to Arrow–Debreu equilibrium, we show that Knight–Walras equilibria under no aggregate uncertainty are generically inefficient. Efficiency is thus the exception under Knightian price uncertainty. We introduce a notion of constrained efficiency that we call uncertainty-neutral efficiency. An allocation is uncertainty-neutral efficient if it is impossible to improve it by trading in an ambiguity-free way. The fictitious social planner is thus restricted to redistributions with ambiguity-free net values. Knight–Walras equilibrium allocations are uncertainty-neutral efficient.

We subsequently continue to explore the nature of Knight–Walras equilibria in economies without aggregate uncertainty. It turns out that Arrow–Debreu equilibria are not robust with respect to the introduction of Knightian uncertainty in prices. Even with a small amount of Knightian uncertainty, there is no trade in Knight–Walras equilibria, which stands in sharp contrast to the full insurance allocation of Arrow–Debreu equilibria. The equilibrium correspondence is thus not continuous when moving from risk to uncertainty. This result might be seen as an equilibrium extension of the seminal result of Dow and Werlang (1992) who showed that ambiguity-averse agents stay away from the asset market for a whole range of prices.

At the end of this Introduction, we discuss nonlinearities in price systems that have appeared in other economic environments. Our equilibrium and efficiency results apply to these models insofar as the sublinear structure of prices is shared.

While we favor the interpretation that the sublinearity of prices is a result of Knightian uncertainty, our model encompasses and generalizes various other models of market imperfections that have been discussed in the literature. Sublinear (forward) prices appear in incomplete financial markets, in insurance markets, or in markets with transaction costs.

Jouini and Kallal (1995), Luttmer (1996), and Araujo, Chateauneuf, and Faro (2018) discussed transaction cost models; financial markets with bid–ask spreads can be characterized by a set of measures under which the expected payoff of securities remains in the bid–ask interval. This set of so-called martingale measures is a polytope within the set of all probability measures. Transaction costs thus naturally lead to a special form of Knightian expectation given by a finite set of expectations. In Appendix B, we explain how the transaction cost economy can be embedded into a Knightian economy and we prove an equivalence result between the two equilibrium notions. The results of our paper thus hold true for markets under uncertainty in which agents finance their desired consumption bundles by trading in a nominal financial market with transaction cost.

Economies with incomplete financial markets are also closely related because the so-called super-replication functional is a Knightian expectation. Araujo, Chateauneuf, and Faro (2012) discussed sublinear functionals that satisfy similar axioms as do our Knightian expectation. They proved that the sublinear price functional is equal to the superhedging price of an exogenous incomplete and arbitrage-free financial market if and only if the subspace of claims whose expectation does not depend on a specific prior coincides with the subspace of undominated claims under the Knightian expectation. In incomplete financial markets, the hedgeable claims are undominated, and claims that do not belong to the marketed subspace are dominated. In turn, when beginning with a sublinear pricing functional, this latter condition is sufficient to construct an incomplete financial market whose superhedging price functional is equal to the given sublinear price functional. Since we do not impose such a condition, our setup is more general than the setup created by incomplete financial markets.
Castagnoli, Maccheroni, and Marinacci (2002) discussed sublinear prices in insurance markets. In particular, they characterized insurance prices that can be written as the sum of a fair premium—that is, the usual linear expected value of the potential damage—and an ambiguity premium that depends on the size of the set of priors.


The remainder of the paper is organized as follows. Section 2 introduces the concept of Knight–Walras equilibria. Existence is established in Section 3, and Section 4 analyzes the relation to Arrow–Debreu equilibria. Subsequently, we study efficient allocations under Knightian uncertainty and discuss uncertainty-neutral efficiency in Section 5. Section 6 investigates the sensitivity of Arrow–Debreu equilibria with respect to Knightian uncertainty before we present our conclusions in Section 7. The Appendix collects proofs and provides a detailed discussion of financial markets with transaction costs.

2. KNIGHT–WALRAS EQUILIBRIUM

2.1. Expectations and Forward Prices

We consider a static economy under uncertainty with a finite state space \( \Omega \). Let \( X = \mathbb{R}^\Omega \) be the commodity space of contingent plans for our economy. We first describe the sublinear form of the price functional used in our approach. In a general equilibrium model à la Arrow and Debreu, a price functional is a linear mapping \( \Psi : X \to \mathbb{R} \). In financial economics, it is common to write such a price functional in the form of an expectation

\[
\Psi(x) = E^P[\psi x]
\]

for some state price or stochastic discount factor \( \psi \) and a probability measure \( P \) on \( \Omega \); compare Cochrane (2005), Chapter 1 or Föllmer and Schied (2011), Chapter 3.4. It is (often implicitly) assumed that the probability measure \( P \) is objective, or at least common knowledge for all agents. We generalize such an approach to account for Knightian uncertainty in the price formation process by using sublinear expectations. We discuss our approach and its relations to other institutional frictions in more detail in Section 2.3.

We call \( \mathbb{E} : X \to \mathbb{R} \) a (Knightian) expectation if it satisfies the following properties:

1. \( \mathbb{E} \) preserves constants: \( \mathbb{E}[m] = m \) for all \( m \in \mathbb{R} \),
2. \( \mathbb{E} \) is monotone: \( \mathbb{E}[x] \leq \mathbb{E}[y] \) for all \( x, y \in X \) with \( x \leq y \),
3. \( \mathbb{E} \) is sub-additive: \( \mathbb{E}[x + y] \leq \mathbb{E}[x] + \mathbb{E}[y] \) for all \( x, y \in X \),
4. \( \mathbb{E} \) is homogeneous: \( \mathbb{E}[\lambda x] = \lambda \mathbb{E}[x] \) for \( \lambda > 0 \) and \( x \in X \),
5. \( \mathbb{E} \) is relevant: \( \mathbb{E}[-x] < 0 \) for all \( x \in X_+ \setminus \{0\} \).

In the sequel, we denote by \( \Delta \) the set of all probability measures on \( \Omega \). It is well known that \( \mathbb{E} \) is uniquely characterized by a convex and compact set \( \mathbb{P} \subset \Delta \) of probability mea-

\[\text{Note that the two approaches are equivalent as, for any linear price functional } \Psi \text{ and prior } P, \text{ there exists a state price } \psi \text{ such that } \Psi(x) = E^P[\psi x] \text{ holds true. Conversely, the price functional given by } x \mapsto E^P[\psi x] \text{ is linear, of course. See also Example 1.}\]

\[\text{See Lemma 3.5 in Gilboa and Schmeidler (1989), Peng (2004), Artzner, Delbaen, Eber, and Heath (1999), or Föllmer and Schied (2011), Corollary 4.18.}\]
su res on $\Omega$ such that
\[ E[x] = \max_{P \in P} E^P[x] \] (1)
for all $x \in X$; $E^P$ denotes the usual linear expectation here. Relevance implies that the representing set $P$ in (1) consists of measures with full support in the sense that, for every $P \in P$, we have $P(\omega) > 0$ for every $\omega \in \Omega$.

The sublinear expectation $E$ leads naturally to a concept of (forward) price for contingent plans. Let $\psi : \Omega \to \mathbb{R}_+$ be a positive state price. The forward price for a contingent plan $x \in X$ is
\[ \Psi(x) = E[\psi x], \]
in analogy to the usual forward (or risk-adjusted) price under risk. We call $\Psi : X \to \mathbb{R}$ coherent price system.

2.2. The Economy With Sublinear Forward Prices

We introduce now an economy with sublinear forward prices. Uncertainty is described by the state space $\Omega$ and the Knightian expectation $E$, respectively the representing set of priors $P$. There is one physical commodity for consumption; an extension to finitely many commodities is straightforward.

DEFINITION 1: A Knightian economy (on $\Omega$) is a triple
\[ \mathcal{E} = (I, (e_i, U_i)_{i=1,\ldots,I}, E), \]
where $I \geq 1$ denotes the number of agents, $e_i \in X_+ = \{c \in X : c(\omega) \geq 0 \text{ for all } \omega \in \Omega\}$ is the endowment of agent $i$, $U_i : X_+ \to \mathbb{R}$ agent $i$’s utility function, and $E$ is a Knightian expectation.

As we fix the agents throughout the paper, we will sometimes use the shorthand notation $E_P$ to emphasize the dependence of the economy on the Knightian expectation $E$ that is generated by $P$.

DEFINITION 2: We call a pair $(\psi, c)$ of a state price $\psi : \Omega \to \mathbb{R}_+$ and an allocation $c = (c_i)_{i=1,\ldots,I} \in X_+^I$ a Knight–Walras equilibrium of the economy $\mathcal{E} = (I, (e_i, U_i)_{i=1,\ldots,I}, E)$ if
1. the allocation $c$ is feasible, that is, $\sum_{i=1}^I (c_i - e_i) \leq 0$,
2. for each agent $i$, $c_i$ is optimal in the Knight–Walras budget set
\[ \mathbb{B}(\psi, e_i) = \{c \in X_+ : E[\psi(c - e_i)] \leq 0\}, \] (2)
that is, $c_i \in \mathbb{B}(\psi, e_i)$ and, for all $d \in X_+$ with $U_i(d) > U_i(c_i)$, we have $d \notin \mathbb{B}(\psi, e_i)$.

We discuss some immediate properties of the new concept.

EXAMPLE 1: 1. When $P = \{P\}$ is a singleton, the budget constraint is linear. In this case, Knight–Walras and Arrow–Debreu equilibria coincide. In particular, equilibrium allocations are efficient.

Note that in expected utility economies, the probability measure $P$ plays a minor role in equilibrium. As Harrison and Kreps (1979) have pointed out, the role of $P$ consists of
determining the null sets and the commodity space of the economy. Indeed, the probability $P$ and the state price $\psi$ determine a linear mapping $x \mapsto E^P[\psi x]$ for $x \in X$; it is thus common in general equilibrium theory to look only at linear functionals $\Psi$ of the form $\Psi(x) = \sum_{\omega \in \Omega} q(\omega) x(\omega)$ for some $q$. As long as $P$ has full support, the two approaches are equivalent with $P(\omega) \psi(\omega) = q(\omega)$.

2. At the other extreme, when $P = \Delta$ consists of all probability measures, and the state price $\psi$ is strictly positive, the budget sets consist of all plans $c$ with $c \leq e_i$ in all states. We are economically in the situation where all spot markets at time 1 operate separately and there is no possibility to transfer wealth over states. As a consequence, with strictly monotone utility functions, no trade is an equilibrium for every strictly positive state price $\psi$. Equilibrium allocations are inefficient, in general, and equilibrium prices are indeterminate.

2.3. Discussion of Sublinear Prices

In our economy, agents trade contingent plans on a forward market at time 0, as in Debreu’s original model of trade under uncertainty. The market is complete in the sense that all contingent plans are available. However, Knightian uncertainty induces an imperfection to the price formation of the market, resulting in sublinear prices.

Knightian uncertainty is described by a set of priors that is common knowledge; the set describes the imprecise, but objective probabilistic information about the outcome distribution over future states, as in Ellsberg’s thought experiments. Following Walras and Debreu, we do not explicitly model the price formation process; rather, we model its outcome by a sublinear expectation. The invisible hand of the market uses the maximal expected value over the set of priors to price contingent claims. In other words, the cautious market maker has imprecise probabilistic information about the states of the world and computes the maximal expected present value over a set of models to hedge Knightian uncertainty. Such a cautious approach is common practice in today’s financial markets in which traders use different models in stress testing to price contingent claims (compare Artzner et al. (1999)). Our approach thus takes such a robustness concern of the market maker into account.

The price $\psi(\omega)$ is the spot price of the real consumption good in state $\omega$ at time 1. For simplicity, we consider the case of one physical good, but the extension to any finite number is straightforward; with $d$ physical goods, $\psi(\omega)$ and $c(\omega)$ would take values in $\mathbb{R}^d$ and $\langle \psi, c \rangle$ would be the scalar product. The price $\psi$ is used by the market to value contingent claims; the actual trade is then carried out via the contracts made at time 0 and markets do not re-open at time 1.

The Knight–Walras budget set $\mathbb{B}(\psi, e_i)$ in (2) is the intersection of budget sets under linear prices of the form $E^P[\psi \cdot]$, that is,

$$
\mathbb{B}(\psi, e_i) = \bigcap_{P \in \mathcal{P}} \mathbb{B}^P(\psi, e_i),
$$

where $\mathbb{B}^P(\psi, e_i) = \{c \in X_+: E^P[\psi(c - e_i)] \leq 0\}$ denotes the budget set in an Arrow–Debreu economy under $P = \{P\}$. Hence, agents in the Knightian economy $\mathcal{E}^P$ only consider those consumption bundles that are robustly affordable against the price uncertainty $\mathcal{P}$.

While we favor the interpretation that the sublinearity of prices is a result of Knightian uncertainty, our model encompasses and generalizes various other models of market
imperfections that have been discussed in the literature. Sublinear (forward) prices appear in incomplete financial markets, in insurance markets, or in markets with transaction costs.

Jouini and Kallal (1995), Luttmer (1996), and Araujo, Chateauneuf, and Faro (2018) discussed transaction cost models. Financial markets with bid–ask spreads can be characterized by a set of measures under which the expected payoff of securities remains in the bid–ask interval. This set of so-called martingale measures is a polytope within the set of all probability measures. Transaction costs thus lead naturally to a special form of Knightian expectation given by a finite set of expectations. In Appendix B, we show how the transaction cost economy can be embedded into a Knightian economy and we prove an equivalence result between the two equilibrium notions. In particular, all results that follow hold true for markets under uncertainty in which agents finance their desired consumption bundles by trading in a nominal financial market with transaction cost.

Economies with incomplete financial markets are also closely related because the so-called super-replication functional is also a Knightian expectation. Araujo, Chateauneuf, and Faro (2012) discussed sublinear functionals that satisfy similar axioms as our Knightian expectation. They showed that the sublinear price functional is equal to the super-hedging price of an exogenous incomplete and arbitrage-free financial market if and only if the subspace of claims whose expectation does not depend on a specific prior $P \in \mathcal{P}$ coincides with the subspace of undominated claims under $\mathbb{E}$. In their setup, a claim $x$ is called undominated if there is no claim $y > x$ with the same price. In incomplete financial markets, exactly the hedgeable claims are undominated; claims that do not belong to the marketed subspace are dominated. In turn, if one starts with a sublinear pricing functional, this latter condition is sufficient to construct an incomplete financial market whose superhedging price functional is equal to the given sublinear price functional. Since we do not impose such a condition, our setup is more general than the setup created by incomplete financial markets. Moreover, as Araujo, Chateauneuf, and Faro (2012) considered the properties of superhedging prices in financial markets, they worked directly with the set of martingale measures, or set $\psi = 1$. In the case where $\psi$ is simultaneously a density for all $P \in \mathcal{P}$, and the above condition on the equality of mean–ambiguity-free and undominated claims are equal, the results of Araujo, Chateauneuf, and Faro (2012) would thus allow to construct an incomplete financial market with our sublinear prices.

Castagnoli, Maccheroni, and Marinacci (2002) discussed sublinear prices in insurance markets. In particular, they characterized insurance prices that can be written as the sum of a fair premium, that is, the usual linear expected value of the potential damage, and an ambiguity premium of the form $\text{Amb}_P(x) = \sup_{P Q \in \mathcal{P}} |E_P[x] - E_Q[x]|$. Their characterization thus shares a certain conceptual analogy to our above interpretation of the sublinear price functional. Knightian uncertainty leads to sublinear insurance prices.

The papers cited above discuss properties related to sublinear functionals, but do not study equilibrium. Our paper completes this gap in the literature.

2.4. On Utility Functions Under Uncertainty

For our analysis, we will use two assumptions on preferences and endowments. First, we list general properties that are well known from general equilibrium theory.

**Assumption 1:** Each agent’s endowment $e_i$ is strictly positive. Each utility function $U_i: \mathbb{R}_+ \to \mathbb{R}$ is

- continuous,
monotone, that is, if \( c \geq c' \), then \( U_i(c) \geq U_i(c') \),

semi-strictly quasi-concave, that is, for all \( c, c' \in \mathbb{X}_+ \) with \( U(c) > U(c') \), we have, for all \( \lambda \in (0, 1) \),

\[
U(\lambda c + (1 - \lambda)c') > U(c'),
\]

and non-satiated, that is, for \( c \in \mathbb{X}_+ \), there exists \( c' \in \mathbb{X}_+ \) with \( U_i(c') > U_i(c) \).

Every concave utility function is semi-strictly quasi-concave. Semi-strict quasi-concavity and non-satiation imply local non-satiation because for \( c \in \mathbb{X}_+ \) and \( \epsilon > 0 \), non-satiation allows to choose \( c' \in \mathbb{X}_+ \) with \( U_i(c') > U_i(c) \). We can then find \( \lambda \in (0, 1) \) such that \( c'' = \lambda c + (1 - \lambda)c' \) satisfies \( \|c - c''\| < \epsilon \); by semi-strict quasi-concavity, \( U_i(c'') > U_i(c) \). Thus, \( U_i \) is locally non-satiated.

There are utility functions that are monotone, semi-strictly quasi-concave, non-satiated, but not strictly monotone. An example are multiple priors utilities as, in its simplest form,

\[
U_i(x) = \min_{\omega \in \Omega} x(\omega).
\]

Subjective reactions to the imprecise probabilistic information \( \mathbb{P} \) in the spirit of Gajdos, Hayashi, Tallon, and Vergnaud (2008) can be described by preferences of the form

\[
U_i(c) = \min_{P \in \Phi_1 i(\mathbb{P})} E^P [u_i(c)]
\]

for a selection \( \Phi_1 i(\mathbb{P}) \subset \mathbb{P} \). Note that a singleton \( \Phi_1 i(\mathbb{P}) = \{P_i\} \) leads to ambiguity-neutral subjective expected utility agents.

2. The smooth model of Klibanoff, Marinacci, and Mukerji (2005) has

\[
U_i(c) = \int_{\mathbb{P}} \phi_i(E^P [u_i(c)]) \mu_i(dP)
\]

for a continuous, monotone, strictly concave ambiguity index \( \phi_i : \mathbb{R} \rightarrow \mathbb{R} \) and a second-order prior \( \mu_i \), a measure with support included in \( \mathbb{P} \).

3. Dana and Riedel (2013) introduced anchored preferences of the form

\[
U_i(c) = \min_{P \in \mathbb{P}} E^P [u_i(c) - u(e_i)].
\]

These preferences have recently been axiomatized by Faro (2015). With multiple prior utilities, they belong to the larger class of variational preferences (Maccheroni, Marinacci,
and Rustichini (2006)) of the form

\[ U_i(c) = \inf_{P \in \mathcal{P}} E^P[u_i(c)] + \gamma(P) \]

for a suitable penalty function \( \gamma : \mathbb{P} \rightarrow \mathbb{R}_+ \cup \{\infty\} \).

The above preferences share some common features that are characteristic for uncertainty-averse preferences. We gather them in the following assumption. In order to do so, the concept of subjective beliefs introduced by Rigotti, Shannon, and Strzalecki (2008) is useful. The set of subjective beliefs \( \pi_i(c) \) of agent \( i \) at \( c \in \mathbb{X}_+ \) is given by

\[ \pi_i(c) = \{ Q \in \Delta : E^Q[y] \geq E^Q[c] \text{ for all } y \text{ with } u_i(y) \geq u_i(c) \} \]  

The set \( \pi_i(c) \) consists of the normalized supports of the upper contour sets of \( U_i \) at the consumption plan \( c \); it contains all beliefs for which the agent is unwilling to trade net consumption plans with zero expected net payoff.

**Assumption 2:**
- The utility functions \( U_i \) are concave and strictly monotone.
- Each \( U_i \) is translation invariant at certainty: For all \( h \in \mathbb{X} \) and all constant bundles \( c, c' > 0 \), if \( U_i(c + \lambda h) \geq U_i(c) \) for some \( \lambda > 0 \), then there exists \( \lambda' > 0 \) such that \( U_i(c' + \lambda' h) \geq U_i(c') \). We denote the subjective belief of agent \( i \) at any constant bundle \( m > 0 \) by \( \pi_i = \pi_i(m) \).
- Preferences are consistent with the set of priors \( \mathbb{P} \), that is, we have \( \pi_i \subset \mathbb{P} \), and agents share some common subjective belief at certainty: \( \bigcap_{i=1}^{I} \pi_i \neq \emptyset \).

Concavity is slightly more restrictive than semi-strict quasi-concavity, but satisfied by most models. In the same spirit, strict monotonicity is a slightly more restrictive condition than mere monotonicity, but will be satisfied in most applications. Translation invariance at certainty was introduced in Rigotti, Shannon, and Strzalecki (2008). It ensures that subjective beliefs are constant across constant bundles, and thus independent of the particular constant \( m \) (Proposition 8 in their paper).

Translation invariance at certainty is satisfied by the common utility functions that model uncertainty aversion, including the ambiguity-neutral expected utility case, the smooth ambiguity model, multiple priors, and variational preferences that we listed above. The second part of the assumption ensures that the subjective beliefs at constant bundles are consistent with the set of priors \( \mathbb{P} \) that describes the Knightian uncertainty of the economy. As we assume that \( \mathbb{P} \) is common knowledge, it is a natural assumption that the subjective beliefs at constant consumption plans are consistent with the given imprecise probabilistic information and that the agents share some belief about possible priors. According to Rigotti, Shannon, and Strzalecki (2008), efficient allocations under no aggregate uncertainty are full insurance allocations if and only if agents share some common subjective belief at certainty.

### 3. Existence of Knight–Walras Equilibria

In this section, we establish existence of a Knight–Walras equilibrium. If agents have single-valued demand, one can modify a standard proof, as, for example, in Hildenbrand and Kirman (1988), to prove existence. Under Knightian uncertainty, natural examples arise where demand can be set-valued. A point in case are ambiguity-averse, yet risk-neutral agents with multiple prior expected utilities. If we include this general case, one
needs to work more. We think that the proof, beyond the natural interest in generality, provides additional insights into the working of markets under Knightian uncertainty, as we explain below.

**THEOREM 1:** Under Assumption 1, Knight–Walras equilibria \((\psi, c)\) exist.

The basic idea of the proof is as follows. Debreu (1952) introduced a price player who maximizes the expected value of aggregate excess demand over state prices. Let us call this type of player a Walrasian price player. The consumers maximize their utility given the budget constraint. The equilibrium of this (extended) game exists by a straightforward extension of Nash’s existence proof for noncooperative games, and it is the desired Arrow–Debreu equilibrium. Our method to prove existence follows this game-theoretic approach. Due to Knightian uncertainty, we have to introduce a second, Knightian, price player. This player maximizes the expected value of aggregate excess demand over the priors \(P \in \mathbb{P}\), taking the state price as given. The Walrasian price player in the Knight–Walras equilibrium acts in the same way as in the Arrow–Debreu equilibrium. The equilibrium of this game with two auctioneers is a Knight–Walras equilibrium.

Note that even though our market features sublinear prices, a weak form of Walras’s law holds true. As semi-strict quasi-concavity and non-satiation imply local non-satiation, the budget constraint is binding for each agent \(i\) at an optimal consumption plan \(c_i\); by sublinearity, net aggregate demand \(\zeta = \sum_{i=1}^I c_i - e_i\), satisfies \(\Psi(\zeta) \leq \sum_{i=1}^I \Psi(c_i - e_i) = 0\).

Given that we have a nonlinear price system, one might ask if agents can generate arbitrage gains; any reasonable notion of equilibrium should exclude such arbitrage, of course. In our context, there is no financial market, so the arbitrage notion of a costless portfolio with positive gains does not apply here. We consider two natural notions of arbitrage for our sublinear prices. Following Aliprantis, Florenzano, and Tourky (2005), an arbitrage is a consumption plan \(c \in \mathbb{X}_+ \setminus \{0\}\) with \(\Psi(c) = 0\). Alternatively, in our sublinear context, one could think of making gains by splitting a consumption bundle into two or more plans. Note that the gain from selling a plan \(c\) is the negative of “buying” the plan \(-c\), that is, \(-\Psi(-c)\). The following proposition shows that neither form of arbitrage is possible in Knight–Walras equilibrium.

**PROPOSITION 1:** Let \((\psi, (c_i)_{i=1,...,I})\) be a Knight–Walras equilibrium and let \(\Psi(x) = \mathbb{E}[\psi x]\). Under Assumption 2, the following absence of arbitrage conditions hold true:

1. We have \(\Psi(c) > 0\) for all \(c \in \mathbb{X}_+ \setminus \{0\}\).
2. Let \(x = y + z\) for \(x, y, z \in \mathbb{X}\). Buying (selling) \(x\) and selling (buying) \(y\) and \(z\) separately yields no profits. We have

\[
\Psi(x) \geq -\left(\Psi(-y) + \Psi(-z)\right) \quad \text{and} \quad \Psi(y) + \Psi(z) \geq -\Psi(-x).
\]

4. **(NON-)EQUIVALENCE TO ARROW–DEBREU EQUILIBRIUM**

If the expectation \(\mathbb{E}\) is linear, Knight–Walras equilibria are Arrow–Debreu equilibria; by the first welfare theorem, equilibrium allocations are thus efficient. It seems thus natural to ask whether such a result extends to general Knightian economies.

In a first step, we show that Knight–Walras equilibria are Arrow–Debreu equilibria if and only if the expected net consumption values of all agents do not depend on the specific prior in the representing set \(\mathbb{P}\) in the sense of the following definition. We then show for the particular transparent example of no aggregate uncertainty that this property is generically not satisfied in Knight–Walras equilibrium.
DEFINITION 3: We call $\xi \in \mathbb{X}$ ambiguity-free in mean (with respect to $\mathbb{P}$) if $\xi$ has the same expectation for all $P \in \mathbb{P}$, that is, there is a constant $k \in \mathbb{R}$ with $E^P[\xi] = k$ for all $P \in \mathbb{P}$. We denote the set of plans that are ambiguity-free in mean by $L$ or $L^P$.

Note that $\xi$ is ambiguity-free in mean if and only if we have $E[-\xi] = -E[\xi]$. We will use this fact sometimes below.4

We can now clarify when Arrow–Debreu equilibria of a particular linear economy $\mathcal{E}^{(P)}$ are also Knight–Walras equilibria.

THEOREM 2: Fix a prior $P \in \mathbb{P}$. Let $(\psi, (c_i))$ be an Arrow–Debreu equilibrium for the (linear) economy $\mathcal{E}^{(P)}$. The following assertions are equivalent:
1. $(\psi, (c_i))$ is a Knight–Walras equilibrium for $\mathcal{E}^P$.
2. The value of net demands $\xi_i = \psi(c_i - e_i)$ are ambiguity-free in mean with respect to $\mathbb{P}$ for all agents $i$.

Let us consider the particularly transparent case of no aggregate uncertainty. We shall show that generically in endowments, Arrow–Debreu equilibria are not Knight–Walras equilibria.

THEOREM 3: Assume that $\mathbb{E}$ is not linear. Under no aggregate uncertainty and Assumptions 1 and 2, generically in endowments, Arrow–Debreu equilibria of $\mathcal{E}^{(P)}$ for some $P \in \mathbb{P}$ are not Knight–Walras equilibria of $\mathcal{E}^P$.

More precisely: let $M = \{(e_i)_{i=1, \ldots, I} : \sum e_i = 1\}$ be the set of economies without aggregate uncertainty normalized to 1. Let $N$ be the subset of elements $(e_i)$ of $M$ such that there exist $P \in \mathbb{P}$ and an Arrow–Debreu equilibrium $(\psi, (c_i))$ of the economy $\mathcal{E}^{(P)}$ that is also a Knight–Walras equilibrium of the economy $\mathcal{E}^P$. $N$ is a Lebesgue null subset of the $(I - 1) \cdot \#\Omega$–dimensional manifold $M$.

A key step in the proof of the above theorem is the insight that the subspace of mean–ambiguity-free contingent plans $L$ has a strictly smaller dimension than the full space $\mathbb{R}^\Omega$ under Knightian uncertainty. Under our assumptions, Arrow–Debreu equilibrium allocations are full insurance allocations; after changing the measure, the first-order conditions allow to identify the state price with the constant 1. Since $L$ contains the constant functions, one can show that an Arrow–Debreu equilibrium is a Knight–Walras equilibrium only if the endowments are in $L$.

LEMMA 1: 1. The set $L$ of plans $\xi \in \mathbb{X}$ that are ambiguity-free in mean forms a subspace of $\mathbb{X}$. $L$ includes all constant functions. If $\#\mathbb{P} > 1$, $L$ has a strictly smaller dimension than $\mathbb{X}$.
2. For $\eta \in \mathbb{X}$ and $\xi \in L$, we have $E[\eta + \xi] = E[\eta] + E[\xi]$.

4The concept has appeared before in De Castro and Chateauneuf (2011) and Beißner and Riedel (2018). For the notion of unambiguous events, see also Epstein and Zhang (2001). A stronger notion would require that the probability distribution of a plan is the same under all priors in $\mathbb{P}$; Ghirardato, Maccheroni, and Marinacci (2004) called such plans “crisp acts.”
5. EFFICIENCY

The previous section shows that Knight–Walras and Arrow–Debreu equilibria rarely coincide under no aggregate uncertainty. The question thus arises if the first welfare theorem holds true. To tackle this question, we discuss the welfare properties of Knight–Walras equilibria in the light of recent results on efficient allocations under Knightian uncertainty.

5.1. Pareto Efficiency and Knight–Walras Equilibria

Rigotti, Shannon, and Strzalecki (2008) used the concept of subjective beliefs that we introduced in Assumption 2 to characterize efficient allocations in the economy $\mathcal{E}^\mathcal{P}$. They showed that an interior allocation $(c_1, \ldots, c_I) \in (\mathbb{X}^+)^I$ is efficient if and only if the agents share a common subjective belief, that is, $\bigcap_{i=1}^I \pi_i(c_i) \neq \emptyset$ with $\pi_i(c_i)$ being the set of subjective beliefs of agent $i$ at consumption plan $c_i$, defined in (4).

Without aggregate uncertainty and when agents have multiple prior or Choquet expected utility preferences, the above condition entails that an interior allocation is efficient if and only if it fully insures each agent (Billot et al. (2000), Chateauneuf, Dana, and Tallon (2000)). De Castro and Chateauneuf (2011) extended these results to the case of no aggregate ambiguity in mean, that is, $\sum e_i \in \mathbb{L}$. Strzalecki and Werner (2011) introduced the notion of conditional subjective beliefs to study efficient allocations in general. In particular, efficient allocations are measurable with respect to aggregate endowment if agents share a common conditional belief.\footnote{A further discussion of efficient allocations on the interim stage can be found in Kajii and Ui (2006) and Martins-da Rocha (2010).}

In analogy to subjective beliefs, we now introduce the concept of effective pricing measures; for a coherent price system $\Psi : \mathbb{X} \to \mathbb{R}$, we call

$$\varphi_\Psi(\xi) = \{ Q \in \Delta : E^\mathcal{P}[\xi] \geq E^\mathcal{P}[\eta] \text{ for all } \eta \text{ with } \Psi(\eta) \leq \Psi(\xi) \}$$

the set of effective pricing measures at $\xi \in \mathbb{X}$.

**PROPOSITION 2:** For any $\xi \in \mathbb{X}$, we have

$$\varphi_\Psi(\xi) = \{ Q \in \Delta : Q = \lambda\psi \cdot P \text{ for some } \lambda > 0 \text{ and some } P \in \text{argmax}_{P \in \mathcal{P}} E^\mathcal{P}_\psi[\psi \xi] \}.$$

The concepts of subjective beliefs and effective pricing measures allow to characterize Knight–Walras equilibria in a compact way.

**THEOREM 4:** Let $(c_i)$ be a feasible interior allocation in $\mathcal{E}^\mathcal{P}$.

Under Assumptions 1 and 2, $(\psi, (c_i))$ is a Knight–Walras equilibrium for $\mathcal{E}^\mathcal{P}$ if and only if

$$\pi_i(c_i) \cap \varphi_\Psi(c_i - e_i) \neq \emptyset \text{ for all } i = 1, \ldots, I.$$

With the help of the previous theorem, we show that Knight–Walras equilibria usually fail to be efficient when there is no aggregate uncertainty.
Theorem 5: Next to Assumptions 1 and 2, assume that the utility functions $U_i$ are differentiable at certainty. Under no aggregate uncertainty, generically in endowments, Knight–Walras equilibrium allocations of $E^P$ are inefficient.

More precisely: let $M = \{(e_i)_{i=1,...,I} \in \mathbb{X}_{++}^I : \sum e_i = 1\}$ be the set of economies without aggregate uncertainty normalized to 1. Let $N_e$ be the subset of elements $(e_i)$ of $M$ for which there exists a Knight–Walras equilibrium $(\psi, (c_i))$ such that $(c_i)$ is efficient. $N_e$ is a Lebesgue null subset of the $(I-1) \cdot \#\Omega$-dimensional manifold $M$.

By the above theorem, Knight–Walras equilibria have inefficient allocations for large classes of economies. In particular, for the widely used classes of smooth ambiguity preferences (Klibanoff, Marinacci, and Mukerji (2005)) and of multiplier preferences (Hansen and Sargent (2001)), Knight–Walras equilibria of economies without aggregate uncertainty are generically inefficient.

5.2. Uncertainty-Neutral Efficiency

In general, Knight–Walras equilibria are inefficient. We introduce now a concept of constrained efficiency for our Knightian framework. If the Walrasian auctioneer aims for robust rules, he might consider only values of net trades that are independent of the specific priors in $\mathbb{P}$.

We might also consider a situation of cooperative negotiation among the agents. In a framework of Knightian uncertainty described by the set of priors $\mathbb{P}$, different priors may matter for different agents. For multiple prior agents, for example, different priors are usually relevant for buyers and sellers of a contingent claim.

The preceding reasoning suggests the following concept of constrained efficiency.

Definition 4: Let $E = (I, (e_i, U_i)_{i=1,...,I}, \mathbb{E})$ be a Knightian economy. Let $c = (c_i)_{i=1,...,I}$ be a feasible allocation. Let $\psi$ be a state-price density. We call the allocation $c$ uncertainty-neutral efficient (given $\psi$ and $\mathbb{P}$) if there is no other feasible allocation $d = (d_i)_{i=1,...,I}$ with

$$\eta_i = \psi(d_i - e_i) \in \mathbb{L}^P$$

and $U_i(d_i) > U_i(c_i)$ for all $i = 1, \ldots, I$.

Our notion of uncertainty-neutral efficiency shares some similarities with other notions of constrained efficiency, but is slightly stronger. In analogy to constrained efficiency notions in other contexts, we require the social planner to use the market as given by the state price $\psi$ and the sublinear expectation $\mathbb{E}$ in order to implement potential improvements for the agents. The social planner is thus constrained to use the market maker’s prices.

Our notion of constrained inefficiency is slightly stronger than the one used for incomplete financial markets. Suppose that $\mathbb{L}$ is the marketed subspace of an incomplete financial market. In this case, a feasible allocation $c = (c_i)_{i=1,...,I}$ is $\mathbb{L}$-constrained efficient if the net consumption bundle is in the marketed subspace, $(c_i - e_i) \in \mathbb{L}$ for all $i = 1, \ldots, I$, and there is no other feasible allocation $d = (d_i)_{i=1,...,I}$ with $(d_i - e_i) \in \mathbb{L}$ and $U_i(d_i) > U_i(c_i)$ for all $i = 1, \ldots, I$; compare Magill and Quinzii (2002). We do not impose the first condition that the net consumption plan be in $\mathbb{L}$, and thus uncertainty-neutral efficiency with respect to $(\psi, \mathbb{P})$ is a stronger condition than $\mathbb{L}$-constrained efficiency.

Knight–Walras equilibria satisfy our robust version of efficiency.
THEOREM 6: Let \((\psi, c)\) be a Knight–Walras equilibrium of the Knightian economy \(E = (I, (e_i, U_i)_{i=1,\ldots,i}, \mathbb{E})\). Then \(c\) is uncertainty-neutral efficient (given \(\psi\) and \(\mathbb{E}\)).

6. SENSITIVITY OF ARROW–DEBREU EQUILIBRIA WITH RESPECT TO KNIGHTIAN PRICE UNCERTAINTY

In this section, we explore first the robustness of Arrow–Debreu equilibria with respect to the introduction of a small amount of Knightian uncertainty when agents have multiple prior utilities. With no aggregate uncertainty, equilibria change in a discontinuous way with small uncertainty perturbations; whereas agents attain full insurance under pure risk, no trade (and thus no insurance) occurs in equilibrium with a tiny amount of Knightian uncertainty.\(^6\) We then take the opposite view and consider growing uncertainty. When uncertainty is sufficiently large, no trade is again the unique equilibrium.

For the next example and Theorem 7 below, we fix continuously differentiable, strictly concave, and strictly increasing Bernoulli utility functions \(u_i : \mathbb{R}_+ \to \mathbb{R}\) and write, for a given set of priors \(\mathbb{P}\),

\[
U^\mathbb{P}_i(c) = \min_{P \in \mathbb{P}} \mathbb{E}^P[u_i(c)]
\]

for the associated multiple-prior utility function.

We illustrate by a simple example that the introduction of a tiny amount of uncertainty can change the equilibrium allocation drastically.

EXAMPLE 3: Let \(\Omega = \{1, 2\}\). Let the set of priors be \(\mathbb{P}_\epsilon = \{p \in \Delta : p_1 \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]\}\) for some \(\epsilon \in [0, \frac{1}{2}]\).

For \(\epsilon > 0\), a consumption plan is ambiguity-free in mean if and only if it is full insurance; we have \(\mathbb{L}^\mathbb{P} = \{c \in \mathbb{X} : c(1) = c(2)\}\).

Let there be no aggregate ambiguity; without loss of generality, \(e = 1\) in both states. Let there be two agents \(I = 2\) (with multiple-prior utilities as stated above) and uncertain endowments, for example, \(e_1 = (1/3, 2/3)\) and \(e_2 = (2/3, 1/3)\).

In a Knight–Walras equilibrium, the state price has to be strictly positive because of strictly monotone utility functions \(U^\mathbb{P}_i\). Since we have two agents, the budget constraint implies that

\[
0 = \mathbb{E}[\psi(c_1 - e_1)] = \mathbb{E}[\psi(c_2 - e_2)]
\]

or

\[
0 = \mathbb{E}[\psi(c_1 - e_1)] = \mathbb{E}[-\psi(c_1 - e_1)].
\]

Hence, \(\psi(c_1 - e_1)\) is mean–ambiguity-free, thus constantly equal to zero here. Since \(\psi\) is strictly positive, \(c_1 = e_1\) follows. There is no trade in Knight–Walras equilibrium for every \(\epsilon > 0\). In sharp contrast, agents achieve full insurance in every Arrow–Debreu equilibrium of any linear economy \(E^{\mathbb{P}}\).

\(^6\)Dana (2004) discussed the dependence of Arrow–Debreu equilibria on aggregate endowment when agents have common convex capacities. As equilibria are indeterminate at no aggregate uncertainty, but unique when aggregate endowment varies with states, agents’ welfare may change substantially with respect to small changes in aggregate endowment. The equilibrium correspondence is continuous, though.
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FIGURE 1.—An Edgeworth box for Example 3.

The example and Figure 1 use the fact that we are in a simple world with two states and two agents. In general, the situation will be more involved. Nevertheless, the discontinuity when passing from a risk economy $\mathcal{E}^{(P)}$ to a Knightian economy $\mathcal{E}^{P}$ remains.

Let us now consider economies of the form

$$\mathcal{E}^{P} = (I, (e_i, U_i^P)_{i=1,...,I}, E)$$

with strictly positive initial endowment allocation $e = (e_1, \ldots, e_I) \in X_{++}^I$. Here, $E$ denotes the Knightian expectation induced by the set of priors $P$. Let $\mathbb{K}(\Delta)$ be the set of closed and convex subsets of $\text{int}(\Delta)$ equipped with the usual Hausdorff metric $d_H$. Define the Knight–Walras (KW) equilibrium correspondence $\mathbb{KW} : \mathbb{K}(\Delta) \Rightarrow X_+ \times X_{++}^I$ via

$$\mathbb{KW}(P) = \{ (\psi, c) \in X_+ \times X_{++}^I : (\psi, c) \text{ is a KW equilibrium in } \mathcal{E}^{P} \}.$$ 

According to Theorem 1, the set of KW equilibria $\mathbb{KW}(P)$ in the economy is nonempty.

**Theorem 7:** Assume that aggregate endowment is constant, $\bar{e} = \sum_{i=1}^I e_i \in \mathbb{R}_{++}$. Assume that agents have multiple-prior utilities of the form (5) for continuously differentiable, strictly concave, and strictly increasing Bernoulli utility functions $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$.

Let $P : [0, 1] \rightarrow \mathbb{K}(\Delta)$ be a $d_H$-continuous function with $P(0) = \{P_0\}$ for some $P_0 \in \text{rint}(\Delta)$. For $0 < \epsilon \leq 1$, assume $P_0 \in \text{rint} P(\epsilon)$ and $(e_i) \notin (L_{P(\epsilon)})^I$. Then the Knight–Walras equilibrium correspondence

$$\epsilon \mapsto \mathbb{KW}(P(\epsilon))$$

is neither upper hemi- nor lower hemi-continuous in zero.

The previous result shows that in an economy without aggregate uncertainty and multiple prior agents, a small amount of Knightian uncertainty has a large impact on equilibria and allocations may react very abruptly on a small change in Knightian uncertainty. The focus of the theorem is thus on the discontinuity of the equilibrium map.

As the introduction of Knightian uncertainty leads to no trade in the special setup of no aggregate uncertainty and multiple prior utilities, we might ask if we can expect a no-trade theorem to hold true in general. If aggregate endowment $\bar{e} = \sum e_i$ is not constant and if
we allow for general preferences, agents do usually trade in Knight–Walras equilibria. However, no trade is indeed the only equilibrium allocation if Knightian uncertainty is large enough. We thus generalize part 2 of our initial Example 1.

**Theorem 8:** Suppose that Assumption 1 holds true and assume that at least one agent has a strictly monotone utility function. If Knightian uncertainty is sufficiently large, every Knight–Walras equilibrium is a no-trade equilibrium: There exists $P \in K(\Delta)$ such that, for every $P' \in K(\Delta)$ with $P' \supseteq P$, we have

$$KW(P') = X_{P'} \times \{e\}$$

for

$$X_{P'} = \{ \psi \in X_{+++} | \pi_i(e_i) \cap P' \neq \emptyset \text{ for } i = 1, \ldots, I \}.$$ (6)

The theorems of this section might transport the impression that Knight–Walras equilibria tend to be no-trade equilibria. In general, however, this is not true. Note that we used the fact that aggregate endowment is constant and that the set of priors has nonempty (relative) interior in Theorem 7. For general aggregate endowments and sets of priors, trade might occur in Knight–Walras equilibrium when a small amount of ambiguity is introduced. Theorem 8, in contrast, holds true for general aggregate endowments; a sufficiently large amount of Knightian price uncertainty drives down the set of potential equilibrium trades to such an extent that no trade is the only possible equilibrium.

7. **Conclusion**

We introduce an equilibrium concept for markets under Knightian uncertainty with sublinear prices derived from a set of priors. We established existence of such equilibrium points and studied its efficiency properties. While one cannot expect fully efficient allocations, in general, the allocation of a Knight–Walras equilibrium satisfies a restricted efficiency criterion: if the authority is restricted to ambiguity-neutral trades, it cannot improve upon a Knight–Walras equilibrium allocation.

The introduction of Knightian friction on the price side rather than the utility side can have strong effects. In a world without aggregate uncertainty, no-trade equilibria result even with a tiny amount of uncertainty. The abrupt change of equilibria with respect to Knightian uncertainty has potentially strong implications for consumption-based asset pricing results that rely on the assumption of probabilistically sophisticated agents and markets.

In general, we conjecture that one can decompose the state space into a “risk” part and an “uncertainty” part. In the risk part of the economy, one would expect trade to occur, whereas we expect rather few trades in the uncertainty part. This question as well as the extension of our model to dynamic and continuous-time models remain to be explored.

**Appendix A: Existence**

The proof of Theorem 1 follows Debreu’s game-theoretic approach. We will prove existence first in the compactified or truncated economy in order to ensure a compact-valued demand correspondence. The budget set $B(\psi, e_i)$, defined in (2), is in general not compact within $X$, so we truncate $B$ by introducing $\overline{B}(\psi, e_i) = B(\psi, e_i) \cap [0, 2\bar{c}]$, where
The aggregate endowment and the compact order interval \([0, 2\bar{e}]\) denote the corresponding truncated economy \(\mathcal{E}^p\) is given by
\[
\mathcal{E} = (I, (e_i, \bar{U}_i)_{i=1,\ldots,I}, \mathbb{E}),
\]
where \(\bar{U}_i : [0, 2\bar{e}] \to \mathbb{R}\) is the restriction of \(U_i\) to the truncated consumption set \([0, 2\bar{e}]\).

To prepare the proof of the existence of Knight–Walras equilibria, we begin with an investigation of the truncated Knight–Walras budget correspondence \(\mathcal{B}\). To prove the continuity of our budget correspondence, we follow the lines of Debreu (1982). We let

\[
\Delta = \left\{ \psi \in \mathbb{X}_+ : \sum_{\omega \in \Omega} \psi(\omega) = 1 \right\}.
\]

**Lemma 2:** Under Assumption 1, we have the following properties for the budget sets in the truncated economy:

1. The budget sets \(\mathcal{B}(\psi, e_i)\) are nonempty, compact, and convex for all \(\psi \in \Delta\).
2. The correspondence \(\psi \mapsto \mathcal{B}(\psi, e_i)\) is homogeneous of degree zero.
3. The correspondence \(\psi \mapsto \mathcal{B}(\psi, e_i)\) is continuous at any \(\psi \in \Delta\).

**Proof of Lemma 2:** 1. Since \(0, e_i \in \mathcal{B}(\psi, e_i)\) and \(\mathbb{E}\) is compact, the truncated budget set \(\mathcal{B}\) is nonempty. The untruncated budget set \(\mathcal{B}(\psi, e_i)\) is the intersection of budget sets under linear prices of the form \(E^p[\psi \cdot]\), that is,
\[
\mathcal{B}(\psi, e_i) = \bigcap_{P \in \mathcal{P}} \mathcal{B}^P(\psi, e_i),
\]
where \(\mathcal{B}^P(\psi, e_i) = \{ c \in \mathbb{X}_+ : E^p[\psi(c - e_i)] \leq 0 \}\) denotes the closed and convex budget in an Arrow–Debreu economy under \(\mathcal{P} = \{P\}\). The arbitrary intersection of convex (closed) sets is again convex (closed) and so is \(\mathcal{B}(\psi, e_i)\). Consequently, \(\mathcal{B}(\psi, e_i)\) is nonempty, compact, and convex.

2. By definition, the Knightian expectation \(\mathbb{E}\) is positively homogeneous. The result then follows by the same arguments as in the case with linear price systems.

3. The order interval \([0, 2\bar{e}]\) is a compact, convex, nonempty set in \(\mathbb{X} = \mathbb{R}^{\Omega}\). We prove the continuity of \(\mathcal{B} : \Delta \Rightarrow [0, 2\bar{e}]\).

To establish upper semi-continuity, it suffices to show the closed graph property, since \(\mathcal{B}\) is compact-valued by part 1. The graph of the budget correspondence \(\text{gr}(\mathcal{B}) = \left\{ (\psi, x) \in \Delta \times [0, 2\bar{e}] : x \in \mathcal{B}(\psi, x) \right\}\) is closed since \(\psi \mapsto \max_{P \in \mathcal{P}} E^p[\psi x]\) is continuous for all \(x \in \mathbb{X}\), by an application of Berge’s maximum theorem.

Now let us consider lower semi-continuity. Let \(\psi_n \to \psi\) and \(x \in \mathcal{B}(\psi, e_i)\). Let us denote by \(\Psi_n\) the price system induced by a normalized \(\psi_n \in \Delta\). We consider two cases.

Case 1: If \(\Psi_n(x - e_i) < 0\), then by continuity, for some \(\tilde{n} \in \mathbb{N}\), we have \(\Psi_n(x - e_i) < 0\) for \(n \geq \tilde{n}\). We define the following converging sequence:
\[
\begin{align*}
x_n = \begin{cases} 
x'_n \in \mathcal{B}(\psi_n, e_i) \text{ arbitrary} & \text{if } n \leq \tilde{n}, \\
x & \text{if } n > \tilde{n}.
\end{cases}
\end{align*}
\]

Then \(x_n \to x\) and \(x_n \in \mathcal{B}(\psi_n, e_i)\).
Case 2: We now consider the case $\Psi(x - e_i) = 0$. Note that $\Psi(x' - e_i) < 0$ for $x' = \frac{e_i}{2}$: since $E$ is relevant and endowments are strictly positive by Assumption 1, we get

$$\Psi(x' - e_i) = \frac{1}{2} \Psi(-e_i) = \frac{1}{2} E[-\psi e_i] < 0.$$ 

For $n$ large,

$$\Psi_n = \{ y \in X : \Psi_n(y - e_i) = 0 \} \cap \{ y \in X : \exists \lambda \in \mathbb{R} : y = \lambda x + (1 - \lambda)x' \}$$

is nonempty. Since $\Psi_n$ is the closed subset of a line, $\bar{x}_n = \arg \min_{y \in \Psi_n} \|y - x\|$ is unique. Now, set

$$x_n = \begin{cases} \bar{x}_n & \text{if } \bar{x}_n \in [x', x], \\ x & \text{else.} \end{cases}$$

By construction, we have $x_n \in \overline{B}(\psi_n, e_i)$ and $x_n \to x$ in $X$. 

**Q.E.D.**

**PROOF OF THEOREM 1:** We show first existence of an equilibrium in the truncated economy $\overline{E} = (I, (e_i, U_i)_{i=1,...,I}, \overline{E})$ and verify later that this candidate is also an equilibrium in the original economy $E^P$.

The existence proof of an equilibrium in $\overline{E}$ is divided into six steps.

1. **Continuity of the Budget Correspondence:** By Assumption 1, each initial endowment $e_i$ is strictly positive. The continuity of the correspondence $B : \Delta \to [0, 2\bar{e}]$ follows from Lemma 2.3.

2. **Properties of the Demand Correspondence:** Consider the (truncated) demand correspondence $\overline{X}_i : \Delta \to [0, 2\bar{e}]$. By step 1, $B(\cdot, e_i) : \Delta \to [0, 2\bar{e}]$ is continuous, hence by Berge’s maximum theorem the demand

$$\overline{X}_i(\psi) = \arg \max_{x \in B(\psi, e_i)} U_i(x)$$

is upper hemi-continuous, compact, and nonempty-valued, since $U_i$ is continuous on $\text{gr}(B)$. By quasi-concavity of $U_i$, $\overline{X}_i(\psi)$ is convex-valued.

3. **Walrasian Auctioneer:** Define the Walrasian price adjustment correspondence $W : [0, 2\bar{e}]^I \times P \Rightarrow \Delta$ via

$$W(x_1, \ldots, x_I, P) = \arg \max_{\psi \in \Delta} E^P \left[ \psi \sum_{i=1}^I (x_i - e_i) \right].$$

As $W$ consists of the maximizers of a linear functional over a compact set, the correspondence is upper hemi-continuous by Berge’s maximum theorem, and it attains convex, compact, and nonempty values.

4. **Knightian Auctioneer:** Define the Knightian adjustment correspondence $K : [0, 2\bar{e}]^I \times \Delta \Rightarrow P$ via

$$K(x_1, \ldots, x_I, \psi) = \arg \max_{P \in P} E^P \left[ \psi \sum_{i=1}^I (x_i - e_i) \right].$$

Once again, by Berge’s maximum theorem, the correspondence is upper hemi-continuous. Since $E^P$ is linear and $P$ is convex, $K$ is convex, compact, and nonempty-valued.
5. Existence of a Fixed Point: Set \( \bar{X} = (\bar{X}_1, \ldots, \bar{X}_I) \). Putting things together, we have the combined correspondence

\[
[K\bar{X}W] : \mathbb{P} \times [0, 2\bar{e}]^I \times \Delta \Rightarrow \mathbb{P} \times [0, 2\bar{e}]^I \times \Delta
\]
as a product of nonempty, compact, and convex-valued upper hemi-continuous correspondences (see steps 2, 3, and 4). Consequently, a fixed point

\[
(\bar{P}, \bar{x}_1, \ldots, \bar{x}_I, \bar{\psi}) \in [K\bar{X}W](\bar{P}, \bar{x}_1, \ldots, \bar{x}_I, \bar{\psi})
\]
events by an application of Kakutani’s fixed-point theorem.

6. Feasibility: We check the feasibility of the fixed-point allocation \( \bar{x} \).

By the budget constraint, the sublinearity of \( c \mapsto \mathbb{E}[\bar{\psi}c] \) (since \( \bar{\psi} \geq 0 \)), we get for the fixed point \((\bar{P}, \bar{x}_1, \ldots, \bar{x}_I, \bar{\psi})\)

\[
0 \geq \sum_{i=1}^{I} \mathbb{E}[\bar{\psi}(\bar{x}_i - e_i)] \geq \mathbb{E}[\bar{\psi} \sum_{i=1}^{I} (\bar{x}_i - e_i)] = \mathbb{E}^P[\bar{\psi} \sum_{i=1}^{I} (\bar{x}_i - e_i)] \geq \mathbb{E}^P[\psi \sum_{i=1}^{I} (\bar{x}_i - e_i)].
\] (7)
The first inequality follows from the definition of the budget set and \( \bar{x}_i \in \bar{X}_i(\bar{\psi}) \) for all \( i = 1, \ldots, I \). The last inequality holds for all \( \bar{\psi} \in \Delta \) and by the positive homogeneity of linear expectations, it holds even for all \( \bar{\psi} \in \mathbb{X}_+ \). We thus have \( l(\sum_{i=1}^{I} (\bar{x}_i - e_i)) \leq 0 \) for all positive linear forms on \( \mathbb{X} \). This implies \( \sum_{i=1}^{I} (\bar{x}_i - e_i) \leq 0 \).

For the feasibility of the equilibrium allocation, the truncation is irrelevant.

7. Maximaliy in \( \mathcal{E}^P \): Since \( \bar{x}_i \in \bar{X}_i(\bar{\psi}) \), we have

\[
\bar{x}_i \in \arg \max_{x \in \mathbb{B}(\psi, e_i) \cap [0, 2\bar{e}]} \bar{U}_i(x).
\]

We have to show that \( \bar{x}_i \) also maximizes \( U_i \) on \( \mathbb{B}(\psi, e_i) \). Suppose there is an \( x \in \mathbb{B}(\psi, e_i) \) in the original budget set, such that \( U_i(x) > U_i(\bar{x}_i) \). Then we have, for some \( \lambda \in (0, 1) \),

\[
\lambda x + (1 - \lambda)\bar{x}_i \in \bar{X}(\psi, e_i) = \mathbb{B}(\psi, e_i) \cap [0, 2\bar{e}].
\]
The semi-strict quasi-concavity of \( U_i \) yields \( U_i(\lambda x + (1 - \lambda)\bar{x}_i) > U_i(\bar{x}_i) \), a contradiction. Therefore, \( (\bar{x}_1, \ldots, \bar{x}_I, \bar{\psi}) \) is also an equilibrium in the original economy \( \mathcal{E}^P \).

Q.E.D.

APPENDIX B: MARKETS WITH TRANSACTION COSTS

Sublinear prices arise naturally in various economic environments. While we focus on the frictions created by Knightian uncertainty in the price formation process, other institutional frictions can be embedded into our framework as well. In this section, we consider financial markets with transaction costs as a particularly important example.

In addition to our setup consisting of the commodity space \( \mathbb{X} \) and the \( I \) agents with endowments \( e_i \in \mathbb{X}_+ \) and utility functions \( U_i : \mathbb{X}_+ \to \mathbb{R} \), let us consider a financial market with \( m + 1 \) assets. The assets are traded at bid–ask prices \( 0 \leq q^B_j \leq q^A_j, j = 0, \ldots, m \) and pay off \( x_j \in \mathbb{X}_+, j = 0, \ldots, m \). We furthermore assume that “cash” is frictionless and normalize the interest rate to zero: \( q^B_0 = q^A_0 = x_0 = 1 \). We also assume (without loss of
generality) that the market has no redundant assets, that is, that $x_0, \ldots, x_m$ are linearly independent. We call

$$E^{tc} = (I, (e_i, U_i)_{i=1,\ldots,I}, (q^B_j, q^A_j, x_j)_{j=0,\ldots,m})$$

a transaction cost economy with financial market $M = (q^A, q^B, x)$.

We assume that the consumption good is traded for a price $\phi(s) \geq 0$ on a spot market at time 1 in state $s$. Agents have to finance their desired value of excess demand $\phi(c_i - e_i)$ by trading in the financial market. A portfolio is a vector $\theta = (\theta^A, \theta^B) \in \mathbb{R}^{2(m+1)}_+, \theta^A_j$ is the number of assets $j$ bought at time 0, $\theta^B_j$ is the number of assets $j$ sold short at time 0. We say that a portfolio $\theta$ superhedges a contingent plan $\xi \in \mathbb{X}$ if its cost is less than or equal to zero,

$$\gamma(\theta) := \sum_{j=0}^{M} (\theta^A_j q^A_j - \theta^B_j q^B_j) \leq 0,$$

and its value at time 1 in state $s$ suffices to cover $\xi(s)$, that is,

$$V(\theta) := \sum_{j=0}^{M} (\theta^A_j - \theta^B_j) x_j \geq \xi.$$

The budget set of agent $i$ in the transaction cost economy $E^{tc}$ consists of all consumption plans $c_i$ whose value of excess demand can be superhedged:

$$\mathbb{B}^{tc}(\phi, e_i) = \{ c_i \in \mathbb{X}_+ : \text{there is } \theta \in \mathbb{R}^{2(m+1)}_+ \text{ with } \gamma(\theta) \leq 0 \text{ and } V(\theta) \geq \phi(c_i - e_i) \}.$$

We are now able to formulate a corresponding equilibrium concept for the transaction cost economy.

**DEFINITION 5:** An equilibrium of the transaction cost economy $E^{tc}$ consists of a spot price $\phi : \Omega \rightarrow \mathbb{R}_+$, an allocation $c = (c_i)_{i=1,\ldots,I} \in \mathbb{X}^I_+$, and portfolios $\theta_i = (\theta^A_i, \theta^B_i) \in \mathbb{R}^{2(m+1)}_+, i = 1, \ldots, I$ such that

1. the allocation $c$ is feasible, that is, $\sum_{i=1}^{I} (c_i - e_i) \leq 0$,
2. the financial market clears, that is, $\sum_{i=1}^{I} \theta^A_i = \sum_{i=1}^{I} \theta^B_i$ for $j = 0, \ldots, m$,
3. for each agent $i$, $c_i$ is optimal in the budget set $\mathbb{B}^{tc}(\phi, e_i)$, that is, $c_i \in \mathbb{B}^{tc}(\phi, e_i)$ and, for all $d \in \mathbb{X}_+$ with $U_i(d) > U_i(c_i)$, we have $d \notin \mathbb{B}^{tc}(\phi, e_i)$.

Equilibria in financial markets only exist if they do not admit arbitrage. Jouini and Kallal (1995) showed that the financial market $M = (q^A, q^B, x)$ is free of arbitrage if and only if there exist martingale measures with full support. A probability measure $P \in \Delta$ is a martingale measure if we have

$$q^B_j \leq E^P x_j \leq q^A_j$$

for all $j = 1, \ldots, m$. Let $\mathbb{Q}_M$ denote the set of all martingale measures (not necessarily with full support). We assume that $\mathbb{Q}_M$ contains at least one probability measure with full support.

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7Note that we require a double index for the portfolios: $\theta^A_{ij}$ is the number of assets $j$ that agent $i$ buys at time 0.
EQUILIBRIA UNDER KNIGHTIAN PRICE UNCERTAINTY

Araujo, Chateauneuf, and Faro (2018) showed that $\mathcal{Q}_M$ is a polytope, that is, there exist finitely many (extremal) martingale measures $P_1, \ldots, P_k$ such that $\mathcal{Q}_M$ is their convex hull,

$$\mathcal{Q}_M = \text{conv}\{P_1, \ldots, P_k\}.$$

The transaction cost economy thus leads to a Knightian expectation

$$\mathbb{E}_M[x] = \max \{E_{p_l}^x \mid l = 1, \ldots, k\}.$$ 

$\mathbb{E}_M$ is called the super-replication functional of the financial market.

We show now that our results on Knight–Walras equilibria hold also true for transaction cost economies. To this end, we show that each Knight–Walras equilibrium of the economy $\mathcal{E} = (I, (e_i, U_i))_{i=1, \ldots, I}, \mathbb{E}_M$ corresponds to an equilibrium of the transaction cost economy $\mathcal{E}^tc$ and vice versa.

**THEOREM 9:** Let $(\phi, (c_i, \theta_i))_{i=1, \ldots, I}$ be an equilibrium of the transaction cost economy $\mathcal{E}^tc$. Then $(\phi, (c_i))_{i=1, \ldots, I}$ is a Knight–Walras equilibrium of the economy $\mathcal{E} = (I, (e_i, U_i))_{i=1, \ldots, I}, \mathbb{E}_M$.

1. Let $(\phi, (c_i, \theta_i))_{i=1, \ldots, I}$ be a Knight–Walras equilibrium of the economy $\mathcal{E} = (I, (e_i, U_i))_{i=1, \ldots, I}, \mathbb{E}_M$. Then there exist portfolios $\theta_i \in \mathbb{R}^{2(m+1)}_+$ for the agents $i = 1, \ldots, I$ such that $(\psi, (c_i, \theta_i))_{i=1, \ldots, I}$ is an equilibrium of the transaction cost economy $\mathcal{E}^tc$.

**PROOF:** We recall the super-replication duality (see Jouini and Kallal (1995), Theorem 3.2 or Araujo, Chateauneuf, and Faro (2018), Appendix A)

$$\mathbb{E}_M[x] = \min_{\theta \in \mathbb{R}^{2(m+1)}_+: V(\theta) \geq x} \gamma(\theta). \quad (8)$$

The maximal expected payoff over all martingale measures coincides with the cost of the cheapest superhedge of a given claim.

1. Let $(\phi, (c_i, \theta_i))_{i=1, \ldots, I}$ be an equilibrium of the transaction cost economy $\mathcal{E}^tc$. Let $\mathbb{E}_M$ be the super-replication functional of the financial market $\mathcal{M}$. We have to show that $(\phi, (c_i))_{i=1, \ldots, I}$ is a Knight–Walras equilibrium of the economy $\mathcal{E} = (I, (e_i, U_i))_{i=1, \ldots, I}, \mathbb{E}_M$. Since the consumption market clears, we only have to show that for an agent $i$, $c_i$ is optimal in the budget set of the Knightian economy $\mathcal{E}$. Since the value of net demand $\phi(c_i - e_i)$ can be superhedged at zero cost in the equilibrium of $\mathcal{E}^tc$, the super-replication duality (8) yields

$$\mathbb{E}_M[\phi(c_i - e_i)] \leq 0.$$

The consumption plan $c_i$ thus belongs to the budget set of agent $i$ in the Knightian economy. Now assume that, for some consumption plan $d \in \mathbb{R}_+$, we have $U_i(d) > U_i(c_i)$ and

$$\mathbb{E}_M[\phi(d - e_i)] \leq 0.$$

By invoking the super-replication duality (8) again, we find a portfolio $\theta$ with cost $\gamma(\theta) \leq 0$ and $V(\theta) \geq \phi(d - e_i)$. Thus, $d$ is in the budget set $\mathbb{B}^tc(\phi, e_i)$ of the transaction cost economy, a contradiction to the optimality of $c_i$ in $\mathbb{B}^tc(\phi, e_i)$.

2. Let $(\psi, (c_i, \theta_i))_{i=1, \ldots, I}$ be a Knight–Walras equilibrium of the economy $\mathcal{E} = (I, (e_i, U_i))_{i=1, \ldots, I}, \mathbb{E}_M$. We have to find portfolios $\theta_i \in \mathbb{R}^{2(m+1)}_+$, $i = 1, \ldots, I$ such that $(\psi, (c_i, \theta_i))_{i=1, \ldots, I}$ is an equilibrium of the transaction cost economy $\mathcal{E}^tc$. 

By the super-replication duality (8), for each agent \( i \) there exists a portfolio \( \theta_i \in \mathbb{R}^{2(m+1)}_+ \) with cost \( \gamma(\theta_i) \leq 0 \) and payoff \( V(\theta_i) \geq \psi(c_i - e_i) \). \( c_i \) is thus in the budget set \( \mathbb{B}^{tc}(\psi, e_i) \) of the transaction cost economy. For any \( d \in \mathbb{B}^{tc}(\psi, e_i) \), (8) implies \( \mathbb{E}_i[\psi(d - e_i)] \leq 0 \). As \( (\psi, (c_i)_{i=1,\ldots,l}) \) is a Knight–Walras equilibrium, \( U_i(d) \leq U_i(c_i) \) follows.

It remains to show that the portfolio market clears. Let \( \bar{\theta} = \sum_{i=1}^l \theta_i \). We have

\[
V(\bar{\theta}) = \sum_{i=1}^l V(\theta_i) \geq \sum_{i=1}^l \psi(c_i - e_i) = 0
\]

and

\[
\gamma(\bar{\theta}) = \sum_{i=1}^l \gamma(\theta_i) \leq 0.
\]

Let \( P \in \mathbb{Q}_M \) be a martingale measure with full support. From the previous two inequalities, we conclude

\[
0 \leq E_P[V(\bar{\theta})] = \sum_{i=1}^l \sum_{j=0}^m (\theta_j^A - \theta_j^B)E_P[x_j] \leq \sum_{i=1}^l \sum_{j=0}^m (\theta_j^A q_j^A - \theta_j^B q_j^B) = \gamma(\bar{\theta}) \leq 0.
\]

We thus have \( E_P[V(\bar{\theta})] = 0 \) and since \( P \) has full support, we conclude \( V(\bar{\theta}) = \sum_{i=1}^l \sum_{j=0}^m (\theta_j^A - \theta_j^B) x_j = \sum_{j=0}^m \sum_{i=1}^l (\theta_j^A - \theta_j^B) x_j = 0 \). As the asset payoffs \( x_j, j = 0, \ldots, m \) are linearly independent,

\[
\sum_{i=1}^l (\theta_j^A - \theta_j^B) = 0
\]

follows.

Q.E.D.

APPENDIX C: PROOFS

PROOF OF PROPOSITION 1: 1. Let \( (\psi, (c_i)_{i=1,\ldots,l}) \) be a Knight–Walras equilibrium and let \( \Psi(x) = \mathbb{E}[\psi x] \). Suppose that for some \( c \in \mathbb{X}_+ \setminus \{0\} \), we have \( \Psi(c) \leq 0 \). Then sublinearity of \( \Psi \) and the equilibrium budget constraint imply that \( \bar{c} = c_1 + c \) satisfies \( \Psi(\bar{c} - e_1) \leq \Psi(c) + \Psi(c_1 - e_1) \leq 0 \). Hence, \( \bar{c} \) is in the budget set of agent 1; by strict monotonicity, \( U_1(\bar{c}) > U_1(c_1) \), a contradiction to the equilibrium conditions.

2. For \( x, y, z \in \mathbb{X} \), assume that \( x = y + z \) holds true. Buying the asset \( x \) at cost \( \Psi(x) \) and selling \( y \) and \( z \) separately yields no profits. Let \( P_x \) minimize \( E_P[\psi z] \) over \( \mathbb{P} \). Then

\[
-(\Psi(-y) + \Psi(-z)) = \min_{P \in \mathbb{P}} E_P[\psi y] + \min_{P \in \mathbb{P}} E_P[\psi z] \\
\leq E_{P^x}[\psi y] + E_{P^z}[\psi z] \\
= E_{P^z}[\psi x] \leq \max_{P \in \mathbb{P}} E_P[\psi x] = \Psi(x).
\]

Similarly, selling the asset \( x \) short and buying \( y \) and \( z \) separately yields no profits because of the sublinearity of \( \Psi \). Let \( P_x \) minimize \( E_P[\psi z] \) over \( \mathbb{P} \). Then

\[
\Psi(y) + \Psi(z) = \max_{P \in \mathbb{P}} E_P[\psi y] + \max_{P \in \mathbb{P}} E_P[\psi z] \\
\geq E_{P^x}[\psi y] + E_{P^z}[\psi z]
\]

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PROOF OF THEOREM 2: Let \((\psi, (c_i))\) be an Arrow–Debreu equilibrium for the (linear) economy \(E^{(P)}\). Then markets clear.

Suppose first that the values of net demands \(\xi_i = \psi(c_i - e_i)\) are ambiguity-free in the mean for all agents \(i\). We need to check that \(c_i\) is optimal in agent \(i\)'s budget set for the Knightian economy \(E^P\). By assumption, we have

\[
E^Q[\psi(c_i - e_i)] = k
\]

for all \(Q \in P\) for some constant \(k\). As \(c_i\) is budget-feasible in \(E^{(P)}\) and utility functions are locally non-satiated by Assumption 1, we have \(k = 0\), that is,

\[
E[\psi(c_i - e_i)] = E^P[\psi(c_i - e_i)] = 0.
\]

As \(c_i\) is part of an Arrow–Debreu equilibrium, \(c_i\) is optimal in the linear budget set given by the prior \(P\); this budget set contains the budget of the Knightian economy \(E^P\), defined in (2). Hence, \(c_i\) is optimal for agent \(i\) in the Knightian economy. We conclude that \((\psi, (c_i))\) is a Knight–Walras equilibrium for \(E^P\).

Now suppose that \((\psi, (c_i))\) is a Knight–Walras equilibrium. We need to check that all \(\xi_i\) have expectation zero under all \(P \in P\) for all \(i\).

As utility functions are locally non-satiated, the budget constraint is binding for all agents, \(E[\xi_i] = 0\) for all \(i\). It is enough to show that \(E[\xi_i] = 0\) for all \(i\) (because this entails \(\min_{P \in P} E^P[\xi_i] = \max_{P \in P} E^P[\xi_i] = 0\)). By sublinearity, we have \(E[\xi_i] \geq 0\). Market clearing implies

\[
E[\xi_i] = E\left[\sum_{j \neq i} \xi_j\right] \leq \sum_{j \neq i} E[\xi_j] = 0.
\]

We conclude that \(E[-\xi_i] = 0\) for all \(i\), as desired.  

Q.E.D.

PROOF OF THEOREM 3: Let \((e_i)\) be an allocation in \(N\). Let \((\psi, (c_i))\) be an Arrow–Debreu equilibrium of the economy \((I, (e_i, U_i)_{i=1,...,l}, \{P\})\) which is also a Knight–Walras equilibrium of \(E^P\).

Due to our assumptions, Proposition 9 in Rigotti, Shannon, and Strzalecki (2008) yields that \((c_i)\) is a full insurance allocation. Utility maximization in Arrow–Debreu equilibrium implies that, for some \(\lambda_i > 0\), we have \(\lambda_i \psi \cdot P \in \pi_i\) for all \(i\); in particular, there exists \(Q \in \bigcap_{i=1}^l \pi_i \subset \mathbb{P}\) such that \(\lambda_i \Psi(c_i - e_i) = E^Q[c_i - e_i]\). Therefore, the allocation \((c_i)\) and the price \(\psi = 1\) form an Arrow–Debreu equilibrium in the economy \((I, (e_i, U_i)_{i=1,...,l}, \{Q\})\).

From Theorem 2, we then know that \((c_i - e_i) \in \mathbb{L}\). As \(c_i\) is constant, it belongs to \(\mathbb{L}\); as \(\mathbb{L}\) is a vector space by Lemma 1, we conclude \(e_i = -(c_i - e_i) + c_i \in \mathbb{L}\). As the vector space \(\mathbb{L}\) has strictly smaller dimension than \(\mathbb{X}\), again by Lemma 1, we conclude that \(N\) is a null set in \(M\).

Q.E.D.

PROOF OF LEMMA 1: 1. Let \(\mathbb{L}\) denote the set of all contingent plans which are ambiguity-free in mean. Constant plans are obviously ambiguity-free in mean, hence \(\mathbb{L}\) is not empty. As expectations are linear, the property of being ambiguity-free in mean is preserved by taking sums and scalar products. Hence, \(\mathbb{L}\) is a subspace of \(\mathbb{X}\).
If $|\mathbb{P}| > 1$, we have $P_1, P_2 \in \mathbb{P}$ such that $P_1 - P_2 \neq 0 \in \mathbb{X}$. In abuse of notation, $x \in \mathbb{X}$ is $(P_1, P_2)$-ambiguity-free in the mean, if

\[ \langle P_1, x \rangle = \langle P_2, x \rangle. \]

This equation yields a hyperplane $H = \{ x \in \mathbb{X} : \langle P_1 - P_2, x \rangle = 0 \}$, with $0 \in H$. Consequently, $H$ is subvector space of $\mathbb{X}$ with strictly smaller dimension and contains all plans being $(P_1, P_2)$-ambiguity-free in mean.

The result follows from the first part and $(P_1, P_2) \subseteq \mathbb{P}$ implies $\mathbb{L} \subseteq H$.

2. As $\xi$ is ambiguity-free in mean, we have $E_P^\xi = E[\xi]$ for all $P \in \mathbb{P}$. As $E$ is additive with respect to constants, we obtain

\[
E[X] + E[\xi] = \max_{P \in \mathbb{P}} E_P^X + E[\xi] = \max_{P \in \mathbb{P}} (E_P^X + E_P^\xi) = \max_{P \in \mathbb{P}} E_P^X + E\xi = E[X + \xi].
\]

Q.E.D.

PROOF OF PROPOSITION 2: From Proposition 2 of Rigotti, Shannon, and Strzalecki (2008), we have that a risk-neutral multiple-prior expected utility (see Example 2) with state-dependent utility index $u(\omega, c) = \psi(\omega)u(c)$ satisfies

\[
\pi^{MEU} = \left\{ \frac{q}{\|q\|} : q = \psi P \text{ for some } P \in \arg\min_{P \in \mathbb{P}} E_P^\psi \right\}.
\]

Using $\min(\cdot) = -\max(-\cdot)$ and the definition of $\varphi$, the result follows. Q.E.D.

PROOF OF THEOREM 4: The condition

\[
\pi_i(c_i) \cap \varphi(c_i - e_i) \neq \emptyset
\]

is the necessary and sufficient first-order condition for the utility maximization problem of agent $i$ in our non-differentiable setup. Let us denote the sub-differential of a convex function $f$ at $x \in \mathbb{X}$ by $\partial f(x) = \{ Df(x) \in \mathbb{X} : f(y) - f(x) \geq Df(x)(y - x) \, \forall y \in \mathbb{X} \}$. Clearly, $-U_i$ and $\Psi$ are convex. Optimality of $c_i$ for agent $i$’s problem is then characterized by $0 \in \partial - U_i(c_i) + \partial_d U_i \Psi(c_i - e_i)$, for some $\mu_i \geq 0$, for all $i = 1, \ldots, I$. This yields $\partial - U_i(c_i) \cap \mu_i \partial \Psi(c_i - e_i) \neq \emptyset$. The assumption of an interior allocation makes each consumption $c_i$ not binding to the positivity constraint $c_i \in \mathbb{X}_+$. Hence, $\mu_i > 0$ and the result follows after an appropriate normalization, since subjective beliefs $\pi_i$ and subjective pricing measures $\varphi$ are collinear with the respective to the sub-differentials $\partial - U_i$ and $\partial \Psi$.

Q.E.D.

PROOF OF THEOREM 5: Let $(\psi, c)$ be a Knight–Walras equilibrium and assume that $(c_i)$ is efficient.

Due to our assumptions and Proposition 9 in Rigotti, Shannon, and Strzalecki (2008), $(c_i)$ is a full insurance allocation. As the utility functions are differentiable at certainty, the subjective belief $\pi_i$ is a singleton; as the agents share a common subjective belief, we have $\pi_i = \{ Q \}$ for some $Q \in \mathbb{P}$. By Proposition 2 and Theorem 4, $Q \in \varphi(c_i - e_i)$; in particular,

\[
E_Q^\varphi[c_i - e_i] = 0
\]

for all $i = 1, \ldots, I$. We conclude that $\bar{\psi} = 1$ and $(c_i)$ form an Arrow–Debreu equilibrium in the economy $\mathcal{E}^Q$ because $(c_i)$ is feasible and satisfies the (necessary and sufficient)
first-order condition of utility maximization under the Arrow–Debreu budget constraint. Theorem 3 concludes the proof. Q.E.D.

**Proof of Theorem 6**: Let \((\psi, c)\) be a Knight–Walras equilibrium of the Knightian economy \(E = (I, (e_i, U_i)_{i=1,\ldots,I}, \mathbb{P})\). Suppose there is a feasible allocation \(d = (d_i)_{i=1,\ldots,I}\) with \(U_i(d_i) > U_i(c_i)\) for all \(i = 1,\ldots,I\). From optimality, we have then \(d_i \notin \mathbb{B}(\psi, e_i)\), or \(\mathbb{E}[\eta_i] > 0\). Suppose furthermore \(\eta_i = \psi(d_i - e_i) \in \mathbb{L}^P\). Take any prior \(P \in \mathbb{P}\). As the net excess demand is ambiguity-free in mean, we have

\[
E^P[\eta_i] = \mathbb{E}[\eta_i] > 0.
\]

As the expectation under \(P\) is linear, we obtain by summing up and feasibility of the allocation \(d\)

\[
0 = E^P\left[ \sum_{i=1}^I \psi(d_i - e_i) \right] = \sum_{i=1}^I E^P[\psi(d_i - e_i)] > 0,
\]

a contradiction. Q.E.D.

**Proof of Theorem 7**: For \(\epsilon = 0\), we are in an Arrow–Debreu economy without aggregate uncertainty. As a consequence, if \((\psi, c) \in \mathbb{KW}(\mathbb{P}(0))\), then \(c\) is a full insurance allocation.

For \(0 \leq \epsilon < 1\), define the Knightian expectation \(E_{\epsilon}[X] = E^{P(\epsilon)}[X] = \max_{P \in \mathbb{P}(\epsilon)} E^P[X]\).

Lemma 3 below shows that a mapping \(X : \Omega \rightarrow \mathbb{R}\) is ambiguity-free in mean with respect to \(\mathbb{P}(\epsilon)\) if and only if it is constant. We use this fact to show that \((\psi, c) \in \mathbb{KW}(\mathbb{P}(\epsilon))\) implies \(c = e\). Let \((\psi, c)\) be a Knight–Walras equilibrium for the economy \(E^{P(\epsilon)}\). Let \(\xi_i = \psi(c_i - e_i)\) be the value of net trade for agent \(i\). Then we have \(\sum_{i=1}^I \xi_i = 0\) by market clearing in equilibrium. As the utility functions are strictly monotone, the budget constraint is binding, so \(E_{\epsilon}[\xi_i] = 0\) for all \(i\). From subadditivity, we get \(E_{\epsilon}[-\xi_i] \geq 0\). On the other hand, market clearing, subadditivity, and the binding budget constraint yield

\[
E_{\epsilon}[-\xi_i] = E_{\epsilon}\left[ \sum_{j \neq i} \xi_j \right] \leq \sum_{j \neq i} E_{\epsilon}[\xi_j] = 0.
\]

We conclude that \(\xi_i\) is ambiguity-free in mean, thus constant by Lemma 3. Due to the budget constraint, \(\xi_i = 0\). As state-prices must be strictly positive in equilibrium due to strictly monotone utility functions, we conclude that \(c_i = e_i\).

Now take a sequence \(\epsilon_n \downarrow 0\) and choose \((\psi^n, c^n) \in \mathbb{KW}(\mathbb{P}(\epsilon_n))\) and assume that \(\psi = \lim \psi^n\) and \(c = \lim c^n\) exist. Then we must have \(c^n_i = e_i\) by the preceding argument, hence \(c_i = e_i\) for all agents \(i\). By assumption, the initial endowment is not a full insurance allocation, so the Knight–Walras equilibrium correspondence is not upper hemi-continuous. Now let \((\psi, c) \in \mathbb{KW}(\mathbb{P}(0))\). Then \(c\) is a full insurance allocation. For every sequence \(\epsilon_n \downarrow 0\), it is impossible to find a sequence of equilibria \((\psi^n, c^n) \in \mathbb{KW}(\mathbb{P}(\epsilon_n))\) with \(\lim c^n = c\) because we have \(c^n = e \neq c\) for all \(n\). Hence, the Knight–Walras equilibrium correspondence is not lower hemi-continuous. Q.E.D.

**Lemma 3**: 1. Let \(\mathbb{P} \subset \Delta\) be a set of priors with nonempty relative interior. Then a mapping \(X : \Omega \rightarrow \mathbb{R}\) is ambiguity-free in mean with respect to \(\mathbb{P}\) if and only if it is constant.

2. Let \(\mathbb{P} : [0, 1] \rightarrow \mathbb{K}(\Delta)\) be a continuous function with \(\mathbb{P}(0) = \{P_0\}\) for some \(P_0 \in \text{rint}(\Delta)\). For \(0 < \epsilon < 1\), assume \(P_0 \in \text{rint} \mathbb{P}(\epsilon)\) and \((e_i) \notin (\mathbb{L}^P(\epsilon))'\). For \(\epsilon > 0\), a mapping \(X : \Omega \rightarrow \mathbb{R}\) is \(\mathbb{P}(\epsilon)\)-ambiguity-free in mean if and only if it is constant.
**Proof of Lemma 3:** Due to our assumptions, $\mathbb{P}$ contains a ball (relatively to $\Delta$) around some $P_0 \in \text{rint} \Delta$ of the form

$$B_\eta(P_0) = \{ Q \in \Delta : \|Q - P_0\| < \eta \}$$

for some $\eta > 0$. We use here, without loss of generality, the maximum norm in $\mathbb{R}^\Omega$.

Suppose $E^0[X] = k$ for some $k \in \mathbb{R}$ and all $Q \in \mathbb{P}$. Let $1 = (1, 1, \ldots, 1) \in \mathbb{X}$ denote the vector with all components equal to 1. Let $Z \in \mathbb{X}$ satisfy $Z \cdot 1 = 0$ with $\|Z\| = 1$. Then $P_0 + \tilde{\eta}Z \in B_\eta(P_0) \subset \mathbb{P}$ for all $0 < \tilde{\eta} < \eta$. Hence, we have

$$k = E^{P_0 + \eta Z}[X] = E^{P_0}[X] + \eta(Z, X).$$

As $0 < \tilde{\eta} < \eta$ is arbitrary, we get $(Z, X) = 0$. By linearity, this extends to all $Z$ that are orthogonal to 1. It follows that $X$ is a multiple of 1, hence constant.

The second part of the lemma follows directly from the first part. Q.E.D.

**Proof of Theorem 8:** Suppose that $(\psi, c)$ is a Knight–Walras equilibrium. As the utility function of at least one agent is strictly increasing, $\psi$ is strictly positive (otherwise, there would be infinite demand for consumption in the state that has zero state price).

Let $\xi_i = \psi(c_i - e_i)$. By market clearing, $\sum_{i=1}^I \xi_i = 0$. From the budget constraint, we also have $\mathbb{E}[^]{\xi_i} \leq 0$, or

$$E^P[\xi_i] \leq 0 \quad (9)$$

for all $P \in \mathbb{P}$.

We claim that all $\xi_i$ are ambiguity-free in mean, that is, $\xi_i \in \mathbb{L}^P$. To see this, take, without loss of generality, $i = 1$. We have $-\xi_1 = \sum_{i=2}^I \xi_i$. For all $P \in \mathbb{P}$, we thus get with (9)

$$E^P[-\xi_1] = \sum_{j=2}^I E^P[\xi_j] \leq 0.$$

Since we also have $E^P[\xi_1] \leq 0$ by (9), $E^P[\xi_1] = 0$ for all $P \in \mathbb{P}$ follows.

Now take some $\mathbb{P}$ with nonempty relative interior. By Lemma 3, the space of $\mathbb{P}$-mean–ambiguity-free claims consists of all constant claims. We conclude that $\xi_i = 0$ for all agents $i = 1, \ldots, I$, and as the state price $\psi$ is strictly positive, $c_i = e_i$ follows.

For larger sets of priors $\mathbb{P}' \supseteq \mathbb{P}$, the same conclusion holds true by Lemma 3.

It remains to show $\mathcal{KW}(\mathbb{P}') = \mathbb{X}_{\mathbb{P}'} \times \{e\}$. By Theorem 4, any equilibrium state price $\psi \in \mathbb{X}_{++}$ at $e$ satisfies

$$\pi_i(e_i) \cap \varphi_{\psi}(e_i - e_i) \neq \emptyset, \quad \forall i = 1, \ldots, I.$$

By Proposition 2, the effective pricing measures of $\Psi(\cdot) = \mathbb{E}^P[\psi \cdot]$ are

$$\varphi_{\psi}(e_i - e_i) = \arg \max_{P \in \mathbb{P}} E^P[\psi 0] = \mathbb{P} \quad \text{for all } i = 1, \ldots, I.$$

We thus obtain (6). Q.E.D.
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