Dirac Operators on Orientifolds

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Declaration of Authorship

Except where otherwise stated, the material in this thesis is my own work.

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Abstract

Motivated by Wigner's theorem, a canonical construction is described that produces an Atiyah-Singer Dirac operator [63, §II.6] with both unitary and anti-unitary symmetries. This Dirac operator includes the Dirac operator for KR-theory [2] as a special case, filling a long-standing gap in the literature. The conditions under which this construction can be made are investigated, and the obstruction is identified as a class within a generalisation of equivariant Čech cohomology. An associated geometric K-homology theory [16] is constructed, along with a homomorphism into an appropriate generalisation of analytic K-homology. More broadly, this thesis demonstrates that difficulties surrounding the interaction of K-orientiation and anti-linear symmetry can be naturally resolved by building on Wigner's theory of corepresentations. Potential applications include the classification of D-brane charges in orientifold string theories [87, §5.2], the construction of index invariants for topological insulators [36], and the formulation of a Baum-Connes conjecture [13] for discrete groups with a distinguished order-2 subgroup.

Contents

Notations and Terminology

1. Rather than constantly introducing new names for the many group actions which occur throughout the text, group actions will often be denoted by apposition, so long as this does not create ambiguity. For example, if X is a topological Γ-space for some group Γ which also acts on $\mathbb{C}$ by conjugation, then an action of Γ on $f \in C(X, \mathbb{C})$ may be defined by writing
$$(\gamma f)(x) := \gamma f(\gamma^{-1}x).$$

2. The symbol κ will be reused often. It represents standard conjugation actions on a variety of objects. For example: complex conjugation on $\mathbb{C}$, conjugation on $U(1)$ under the embedding $U(1) \subset \mathbb{C}$, elementwise conjugation on the standard matrix representation of $GL(n, \mathbb{C})$, and conjugation on the $U(1)$ component of $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$.

3. The symbol ι will be used to represent the negation action $x \mapsto -x$ in a number of contexts.

Introduction

The topic of this thesis is anti-linear symmetry in index theory. In its most basic form, an index is an integer associated to an elliptic differential operator on a manifold by taking the difference in dimension between its kernel and cokernel. The key property of the index is that it is stable under continuous perturbation of the underlying operator. Index theory studies the consequences of this property. The primary example of such a consequence is the Atiyah-Singer index theorem [7, 8], which computes the index of an elliptic operator topologically. Far-reaching results in geometry have been obtained by treating the indices of canonical elliptic operators as topological invariants [63, §IV]. The most important of these operators is the Atiyah-Singer Dirac operator [63, §II.6]. This first order differential operator can be constructed on any manifold that satisfies a topological condition known as spin-orientibility. The Dirac operator is at the heart of index theory, in the sense that all classical index theoretic problems reduce to problems regarding Dirac operators [16, 15, 17].

One of the early generalisations of index theory was equivariant index theory [8, 22]. In this setting, a compact Lie group acts on the underlying manifold, and vector bundles are equipped with a linear lifting of the action. Elliptic operators between such vector bundles are required to be equivariant with respect to the group action. This implies that the kernel and cokernel of the operator are representations. The difference between the characters of the resulting representations defines an element in the representation ring of the group. In this way, equivariant index theory intertwines the global geometry of manifolds with the representation theory of groups.

The motivation for investigating anti-linear symmetry is provided by Wigner's Theorem [83, pp. 233-236]. Wigner's Theorem is derived from the basic postulates of quantum mechanics [82, pp. 91-96] . It states that the symmetries of a quantum mechanical system are implemented by operators which are either unitary or anti-unitary. Both types of symmetry, and various combinations of the two, arise in simple systems. In particular, time reversal symmetry is implemented by anti-unitary operators [83, §26].

In view of Wigner's Theorem, it is natural to define an equivariant index theory which accomodates both unitary and anti-unitary symmetries. This means that the action of a group element on the base manifold lifts to either a unitary or an anti-unitary map on a complex vector bundle. The kernel and cokernel of an equivariant elliptic operator are then group representations consisting of both unitary and anti-unitary operators, and the index is taken to be the formal difference of the equivalence classes of these representations. The set of equivalence classes of unitary/anti-unitary representations may be viewed as a subset of the classes of unitary representations for the subgroup of elements which act by unitary operators. This subset is determined by its invariance under a conjugation map constructed from the anti-unitary part of the action. Thus, an index theory with anti-unitary symmetries intertwines the conjugate structure of unitary/anti-unitary representations with the global geometry of manifolds.

Having defined such an index theory, a central task is to determine when a Dirac operator exists and how it can be constructed. At this point, one encounters a problem which has remained unresolved for a some time: such a Dirac operator would include the Real Dirac operator associated to KR-theory as a special case. Atiyah's KR-theory considers *Real bundles*[1], which are complex vector bundles equipped with an anti-linear involution that covers an involution on the base space [2]. In this context, the base space, equipped with its involution, is refered to as a *Real space*. A Real Dirac operator on a Real space must act between Real bundles and be equivariant with respect to their anti-linear involutions. Although KR-theory was introduced in the 1960's, the question of whether a given Real space can be equipped with a Real Dirac operator has not been answered.

The main contribution of this thesis is to construct the Dirac operator for index theory with both unitary and anti-unitary symmetries. This brings classical index theory into line with Wigner's Theorem, and fills the gap in the literature regarding the existence of a Dirac operator for KR-theory. As in the equivariant setting, the geometric data used to construct Dirac operators can be formed into classes for a geometric K-homology theory [18]. This theory will be described, along with a map into an obvious generalisation of analytic K-homology [46, 58]. Some initial steps toward the formulation of a Baum-Connes conjecture for orientifold groups will also be taken.

To emphasise the connection with current research in theoretical physics, the language of orientifolds will be used. The term *orientifold* originates in string theory. In the present context, it will refer to a manifold equipped with an action of a group Γ which, in turn, is equipped with a homomorphism $\epsilon : \Gamma \to \mathbb{Z}_2$. This small amount of extra structure is used

[1]Note that the R in "Real" is capitalised when used in this sense.

to define unitary/anti-unitary actions of Γ on complex vector bundles over the orientifold. An element $\gamma \in \Gamma$ acts via a unitary map if $\gamma \in \Gamma^+ := \ker(\epsilon)$, or an anti-unitary map if $\gamma \in \Gamma^- := \Gamma \setminus \Gamma^+$. These vector bundles will be described as *orientifold bundles*. Note that the set of orientifold bundles over an orientifold depends on the embedding $\Gamma^+ \hookrightarrow \Gamma$. More generally, the term *orientifold* will be used as an adjective to describe objects carrying, or compatible with, unitary/anti-unitary actions. For example, the Dirac operator mentioned above acts between orientifold bundles in an equivariant manner and will be described as the *orientifold Dirac operator*.

The construction of the orientifold Dirac operator and geometric orientifold K-homology depend on an understanding of the global topology of complex vector bundles with anti-unitary symmetries. In the equivariant setting, the obstruction to the existence of a Dirac operator can be identified as an equivariant cohomology class. The main obstacle to understanding the conditions under which an orientifold Dirac operator exists is the failure of equivariant transition cocycles and cohomology to accomodate anti-linear symmetries. This obstacle will be overcome by introducing a new type of transition cocycle which generalises Wigner's notion of a corepresentation [83, pp. 334-335] [51, pp. 169-172] in the same way that an equivariant transition cocycle generalises a representation. In fact, this generalisation extends beyond what is neccesary for applications to orientifolds, yielding the notion of a *semi-equivariant* transition cocycle. A compatible semi-equivariant Čech cohomology theory will also be defined, and an analogue of a theorem due to Plymen [69, p. 312] will allow the topological obstruction to the existence of an orientifold Dirac operator to be identified as a semi-equivariant cohomology class.

Literature Review

When examining the relevant literature, it is helpful to divide the category of orientifolds in two basic ways. First, the action of Γ on the base manifold X may be trivial or non-trivial. Second, Γ^- may or may not contain an involution. When the orientifold group $\mathrm{id} : \mathbb{Z}_2 \to \mathbb{Z}_2$ acts trivially on X, each orientifold bundle corresponds to a real vector bundle by taking fixed points. Thus, the associated K-theory is KO-theory. In KO-theory there is an 8-fold periodicity theorem: $KO^p(X) \simeq KO^p(X \times \mathbb{R}^8)$ [63, p. 63]. And, more generally, there is a Thom isomorphism $KO^p(X) \simeq KO^p(V)$ for rank $8k$ real vector bundles $V \to X$, whenever V carries a Spin-structure [63, p. 387]. The condition that V carries a Spin-structure is equivalent to the vanishing of the second Stiefel-Whitney class in cohomology with $\mathbb{Z}_2$-valued coefficients [63, p. 82]. A Spin-structure on TX can be used to construct a Dirac operator [63,

p. 112]. Thus, in this special case, the conditions for the existence of an orientifold Dirac operator are clearly understood.

The situation in which the orientifold group $\mathrm{id} : \mathbb{Z}_2 \to \mathbb{Z}_2$ acts by a non-trivial involution σ on X corresponds to KR-theory [2]. Orientifold bundles are then Real vector bundles. An initial motivation for the development of KR-theory was the observation that the symbol of the complexification of a real elliptic operator defines a class in KR-theory, not KO-theory. This fact was noticed by Atiyah and Singer, whose index theorem for families of real elliptic operators [10, p. 142] is proved using KR-theory, and holds equally well for Real elliptic operators [10, Remark p. 143]. Atiyah's KR-theory is bigraded, but periodicity theorems ultimately reduce the possible KR-groups $KR^{p,q}(X, \sigma)$ to one of the eight cases where $0 \leq p \leq 7$ and $q = 0$. This reduction can be approached in two related ways. The first way is to use the $(1, 1)$- and 8-fold periodicity theorems

$$KR^{p.q}(X, \sigma) \simeq KR^{p,q}(X \times \mathbb{C}, \sigma \times \kappa) \qquad KR^{p.q}(X) \simeq KR^{p,q}(X \times \mathbb{R}^8, \sigma \times \mathrm{id}),$$

where κ is conjugation. These theorems can be proved using elementary means [2, p. 373, 379], or the elliptic operators method [4, p. 126, 130]. A second approach is to directly prove the (p, q)-periodicity theorem,

$$KR^{p,q}(X, \sigma) \simeq KR^{p,q}(X \times \mathbb{R}^{r,s}, \sigma \times \iota^{r,s})$$

where $\mathbb{R}^{r,s} := \mathbb{R}^r \oplus \mathbb{R}^s$, $\iota^{r,s}(x, y) := (x, -y)$ and $r = s \bmod 8$, by combining observations regarding Real Clifford modules [2, pp. 380-384] [6] with the elliptic operators method [4, p. 131]. As in KO-theory, the above periodicity theorems have corresponding Thom isomorphisms which one expects to be closely related to the construction of Dirac operators. The $(1, 1)$-Thom isomorphism, $KR^{p.q}(X) \simeq KR^{p,q}(E)$ for a Real bundle E, holds for any Real bundle with no additional assumptions [2, p. 374]. However, the (p, q)-Thom isomorphism and the 8-fold Thom isomorphism each require an additional orientation hypothesis. It was noted in [2, pp. 383-384] that the (p, q)-Thom isomorphism holds whenever a $\mathrm{Spin}^c(p, q)$-structure exists. The existence of a $\mathrm{Spin}^c(p, q)$-structure is the hypothesis for Kasparov's (p, q)-Thom isomorphism in KKR-theory [58, pp. 549-550], and equivalent to the notion of KR-orientation defined in [74, pp. 108-115]. While this condition is sufficient to state the Thom isomorphism, it leaves open the question of whether a given Real space carries a $\mathrm{Spin}^c(p, q)$-structure. Several authors have put forward approaches to this problem. These approaches broadly follow the strategy of Plymen, who identified the obstruction to a Spin^c-structure as a Dixmier-Douady class [69, p. 312]. One formulation of the KR-orientiation condition was given by Moutuou in the setting of twisted groupoid KR-theory [66, p. 219].

His approach identifies the obstruction to the existence of a $\mathrm{Spin}^c(p, q)$-structure as a class in a Čech cohomology theory for Real groupoids [66, Ch. 3] [67]. Another approach, by Hekmati et al., proposed that the obstruction to the existence of a $\mathrm{Spin}^c(p, q)$-structure should be a class in a Real $\mathbb{Z}_2$-equivariant sheaf cohomology [45, p. 31].

There is an equivariant version of KR-theory which corresponds to an action of the orientifold group $\mathbb{Z}_2 \ltimes_\theta G$ on X, where $\epsilon(z, g) := z$ and θ is an action of $\mathbb{Z}_2$ on G. Like KR-theory, this theory also has $(1, 1)$, 8-fold, and (p, q)-periodicity theorems. The proofs of these theorems use the elliptic operators method [4, p. 126, 130, 131]. This method will be adapted to prove Thom isomorphisms for the K-theory of orientifold bundles in Chapter 4. The framework of [66] covers equivariant KR-theory also.

When Γ^- does not contain an involution, the set of orientifold bundles differs from the set of Real equivariant bundles. In particular, if $q : H \to \mathbb{Z}_2$ is the orientifold group defined by $H = \{\pm 1, \pm i\}$ and $q(h) = h^2$, then the set of orientifold bundles contains several subsets of bundles that are of independent interest. Each of these is determined by specifying the manner in which the element $-1 \in H$ should act on a bundle. When $-1 \in H$ is specified to act by $-\mathrm{id}$, the resulting orientifold bundles are symplectic analogues of Real vector bundles. The associated K-theory is sometimes denoted KH [35, 44]. Further examples can be obtained by choosing a sign ± 1 for each connected component of the fixed point set of an orientifold. One can then consider orientifold bundles such that $-1 \in H$ acts over each component by either $+\mathrm{id}$ or $-\mathrm{id}$, according its sign. These examples arise in orientifold string theories, where the connected components of the fixed point set are known as O-planes. The associated K-*theory with sign choice*, denoted $K_\pm$, was studied in [34]. The full set of orientifold bundles for the orientifold group (H, q) contains both the set of symplectic orientifold bundles, and the set of orientifold bundles with sign-choice. Although these subsets will not be considered specifically, some of the methods used here to study the larger class of orientifold bundles apply to the study of these subsets after making suitable refinements. Beyond the orientifold group (H, q), there are many other possible orientifold groups (Γ, ϵ) such that Γ^- does not contain an involution. Even for a fixed Γ, these can yield different sets of orientifold bundles. This is demonstrated by the case of a point orientifold, over which orientifold bundles are unitary/anti-unitary representations. In general, the set of such representations depends on the specific embedding $\Gamma^+ \hookrightarrow \Gamma$ defined by ϵ.

In this thesis, attention will be focused on obtaining an orientation condition for the 8-fold Thom isomorphism in orientifold K-theory using an elementary method. Even when restricted to the setting of KR-theory, this method differs from previous approaches. Although the general approach of Plymen is used, the obstruction class is identified in a new semi-equivariant cohomology theory. This method gives a single notion of orientation for the various cases of orientifold K-theory, including cases in which Γ^- contains no involution. It is also conceptually clear, computable, and leads to a method for constructing Dirac operators on orientifolds.

Part of the reason for the renewed interest in KR-theory and its variants lies in applications to physics. The two main areas of potential application are string theory and the classification of topological insulators. The connection between the present investigation and string theory begins with the classification of D-brane charges using K-theory, as described in [65, 87]. Results in index theory allow one to pass from K-theory to an analytic K-homology theory in which classes are represented by elliptic operators. Each class in this K-homology theory may be represented by a Dirac operator. By replacing these Dirac operators with the geometric data used to construct them, it is possible to define a K-homology theory in entirely geometric terms [16, 15, 17]. This characterisation of D-brane charge is of interest, as the geometric data associated to such a K-homology class has physical interpretations [79, §4]. In order to generalise these ideas to orientifold string theories, it is first neccesary to identify an appropriate variant of K-theory, and then construct the corresponding Dirac operator and geometric K-homology theory. Three types of orientifold string theories are listed in [87, p. 26-27], along with the corresponding K-theories that classifying the associated D-brane charges. In the first of these, D-brane charges are classified by KR-theory. The geometric orientifold K-homology defined in Chapter 6 applies to this situation. The other two possibilities involve K-theory with sign choice, as has been studied by Doran et al. [34] using methods from non-commutative geometry. As discussed above, K-theory with sign choice forms a subgroup of the orientifold K-theory considered in this thesis. One further generalisation that is important in string theory is twisted K-theory. Twisted K-theory is closely related to K-theoretic orientiation conditions. The paper of Hekmati et al. proposes the construction of a twisted geometric KR-homology theory, and discusses its applications in string theory [45, §8]. Although twisted orientifold K-theory is not investigated here, the identification of the obstruction to a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure in Section 3.1 provides the key element required to construct such a theory. Twisted geometric K-homology is a topic which is under active development [12, 45].

In recent years, there there has been much interest in the classification of topological insulators. These classification attempts lead naturally to the consideration of topological invariants which respect anti-linear symmetries [54, 38, 39, 40, 71]. Contact with Clifford algebras and K-theory has been made through the work of Kitaev [61]. Another framework for studying topological insulators, using twisted K-theories, has been described by Freed and Moore [36]. Orientifold K-theory, as considered in this thesis, is a primary example within their framework. Thus, it appears that there is potential for index invariants derived from the orientifold Dirac operator to be applied to the classification of topological insulators.

To finish this review of the relevant literature, some general references will now be collected. As mentioned, the motivation for this thesis comes from Wigner's theorem. The derivation of Wigner's theorem can be found in [82, pp. 91-96]. The English translation of Wigner's book contains a discussion from which the theorem can be drawn [83, pp. 233-236]. It also contains a discussion of time reversal symmetry in quantum mechanics [83, Ch. 26], and an analysis of unitary/anti-unitary representations using the theory of corepresentations [83, pp. 334-335]. Another useful exposition of corepresentations, which separates their mathematical and physical aspects, can be found in [51, §II.7]. Two further papers by Wigner that deal with anti-unitary operators are [84] and [85].

The results and constructions in this thesis draw on a large body of standard material from index theory. As a general reference for the representation theory of Clifford algebras, and other topics in index theory, [63] has been used. The results in Chapter 3 concerning decomposition of $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures and connections for $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures generalise standard results in the Spin^c setting that can be found in [37, pp. 48-49, 57-60] and [63, §D]. The analytic orientifold K-homology defined in this thesis is a straightforward generalisation of Kasparovs KKR-theory [58]. General references for analytic K-homology and KK-theory include [46, 21, 52]. Geometric orientifold K-homology is based on the geometric K-homology defined by Baum and Douglas [16, p. 117] [15, p. 1]. An equivariant version of this theory was described in [18], and the map from geometric to analytic orientifold K-homology, defined in Section 6.3, is analogous that described in [18, 17]. The discussion of assembly for orientifold groups in Chapter 7 is based around [11, p. 41] with modifications to adapt it to the orientifold setting. This paper describes one variant of the Baum-Connes conjecture [13], see also [14, pp. 241-291] [18, pp. 21-22].

Overview of Chapters

Chapter 1, develops tools that are used in later chapters to study Spin^c-structures for orientifolds, which will be refered to as $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. In particular, this chapter defines *semi-equivariant principal bundles*, *semi-equivariant transition cocycles*, and *semi-equivariant cohomology theory*. The essential difference between these objects and their analogues in the equivariant setting is that the structure group/coefficient group itself carries an action of the equivariance group. In the case of a semi-equivariant principal bundle, this action controls the commutation relation between the left and right actions on the total space. After defining semi-equivariant principal bundles in Section 1.2, Section 1.3 defines semi-equivariant transition cocycles and their equivalences. These cocycles can be thought of as a cross between the corepresentations of Wigner and the usual notion of a transition cocycle. Emulating proofs that apply in the non-equivariant setting, it is proved that isomorphism classes of semi-equivariant principal bundles are in bijective correspondence with equivalence classes of semi-equivariant transition cocycles. In particular, this means that the action on a semi-equivariant principal bundle can be reconstructed from its semi-equivariant cocycle. A corresponding semi-equivariant Čech cohomology theory is developed in Section 1.4. The semi-equivariant cohomology can be used to classify semi-equivariant transition cocycles with abelian structure groups. Then, viewing the set of transition cocycles as a non-abelian cohomology group, Section 1.5 shows that a central short exact sequence of structure groups induces a connecting map from the semi-equivariant transition cocycles into the semi-equivariant cohomology. This is the main result of the chapter. Combined with earlier results it identifies obstructions to certain liftings of structure groups for semi-equivariant principal bundles. These obstructions can be considered as semi-equivariant Dixmier-Douady invariants. The semi-equivariant associated bundle construction and some related results are treated in Section 1.6. Finally, Section 1.7 defines *semi-equivariant connection* 1*-forms*, and an averaging result, which will be used in the construction of the orientifold Dirac operator, is proved.

Chapter 2, defines the main objects of study and describes their basic properties. First, in Section 2.1, *orientifold groups* are defined and some terminology for different types of actions by an orientifold group is introduced. After this, Section 2.2 discusses unitary/anti-unitary representations, which will be refered to as *orientifold representations*. The classification of orientifold representations in terms of irreducible corepresentations is reviewed. This classification is due to Wigner [83, §26] [51, §II.7]. Next, in Section 2.4, *orientifold bundles* are introduced. A semi-equivariant averaging procedure is used to produce equivariant Hermitian metrics on orientifold bundles, and the frame bundle of an orientifold bundle is then shown to be a semi-equivariant principal bundle. These observations are key to the correct definition of a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. Finally, Section 2.5 defines various operations on orientifold bundles that will be required when considering orientifold K-theory.

Chapter 3, uses the results of the previous chapters to construct Dirac operators on orientifolds. In Section 3.1, $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures are defined as a lifting of an equivariant $SO(n)$-frame bundle to a Γ-semi-equivariant principal $(\mathrm{Spin}^c(n), \kappa_\epsilon)$-bundle, where κ_ϵ is a Γ-action induced by conjugation. These lifts are then classified using the results of Chapter 1. In particular, a *semi-equivariant third integral Stiefel-Whitney class* $W_3^{(\Gamma,\epsilon)}$ is identified as the obstruction to the existence of a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. An important corollary is also proved that reduces the problem of finding a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure to that of finding a certain semi-equivariant principal $(U(1), \kappa_\epsilon)$-bundle. The subsection finishes with the construction of a canonical orientifold Spin^c-structure on the sphere. Next, in Section 3.2, orientifold-Spin^c-structures are used to define the *orientifold spinor bundle* and *reduced orientifold spinor bundle* via the semi-equivariant associated bundle construction. The *orientifold Clifford bundle* is also defined, and some relationships between the three bundles are examined. This requires the introduction of complexified real Clifford algebras equipped with orientifold actions, and a similar complexification of some relevant results from the representation theory of real Clifford algebras. In Section 3.3, the results of Section 1.7 are used to equip the orientifold spinor bundles with equivariant connections that are compatible with Clifford multiplication on sections. Finally, Section 3.4 defines the orientifold Dirac operator and examines its basic properties. An existence theorem for Orientifold Dirac operators can then be stated. The existence theorem for the Real Dirac operators is obtained as a special case. This completes the major aim of the thesis.

Chapter 4, deals with orientifold K-theory. Orientifold K-theory is defined in Section 4.1, along with various Bott and Thom classes that are used later in the chapter. Similar to KR-theory, orientifold K-theory is a bigraded cohomology theory. In Section 4.2, the principal symbol of an elliptic orientifold operator is examined. It is shown to satisfy an equivariance condition generalising that satisfied by the symbol of a complexified real operator. This condition implies that the principal symbol of an elliptic orientifold operator defines a class in $K_{(\Gamma,\epsilon)}(TX, \iota_\epsilon d\sigma)$, where ι is the Γ-action which acts by negation when $\epsilon(\gamma) = -1$, and by id when $\epsilon(\gamma) = 1$. In Section 4.3, basic facts regarding the indicies of elliptic operators and families of elliptic operators are reviewed. These are noted to generalise to the setting of orientifolds. In particular, the index map associated to a family of elliptic orientifold operators is defined. Such maps are the essential ingredient in the proofs of Bott Periodicity and the Thom isomorphisms. Using the computations of [4], the index map associated to the orientifold Doubeault operator on complex projective space is evaluated on the $(1,1)$-Bott class. The pairing between the reduced orientifold Dirac operator on an $8k$-dimensional sphere and the corresponding 8-fold Bott class is also computed. In Section 4.4, the strategy of [4] is used to prove a sufficient condition for equivariant Bott periodicity. Finally, in Section 4.5 equivariant Bott periodicity is proved for orientifold K-theory. The $(1,1)$ and 8-fold Bott periodicity theorems are obtained as special cases of this result. Together $(1,1)$ and 8-fold Bott periodicity imply that, up to isomorphism, there are only eight orientifold K-groups. By combining equivariant periodicity with a semi-equivariant associated bundle construction, the $(1,1)$ and 8-fold Thom isomorphisms are proved. In particular, equivariant periodicity is combined with results on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures to prove the Thom isomorphism for $8k$-dimensional $(\mathrm{Spin}^c, \kappa_\epsilon)$-oriented real equivariant vector bundles.

Chapter 5, defines orientifold KK-theory by introducing orientifold actions into Kasparov's KK-theory. Kasparov's KK-theory is based on the idea of considering C^*-algebras as abstract topological spaces. From this point of view, a class in orientifold KK-theory can be considered as an abstract family of elliptic operators which is equivariant with respect to an orientifold action. In Section 5.1, the K-theory of orientifold C^*-algebras is defined and the connection between commutative orientifold C^*-algebras and orientifolds is indicated. Then, in Section 5.2, operators on orientifold Hilbert modules are introduced. With these definitions in hand, Section 5.3 defines the orientifold KK-theory groups. Orientifold Dirac operators are shown to define classes in orientifold KK-theory in Section 5.4.

Chapter 6, generalises the geometric K-homology of Baum and Douglas [16] to the orientifold setting. The first step, made in Section 6.1, is to prove several small results dealing with operations on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. These depend, in an essential way, on the classification results for $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures proved in Section 3.1. Each class in the geometric K-homology of an orientifold X is represented by a continuous equivariant map $f : M \to X$ from an orientifold M that is equipped with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure and an orientifold bundle. These structures are precisely the data required to form an orientifold Dirac operator on M with coefficients in an orientifold bundle. In Section 6.2, the operations defined in Section 6.1 are used to define equivalence relations on the set of all such representatives, and the resulting classes form the geometric orientifold K-homology. By constructing the Dirac operator associated to a geometric K-homology class, it is possible to define a homomorphism from the geometric to the analytic orientifold K-homology. This is done in Section 6.3.

Chapter 7, constructs geometric K-homology groups and analytic K-theory groups associated to a finite orientifold group. A correspondence between them is defined based on the assembly map in the equivariant setting. In Section 7.1, a group C^*-algebra with an orientifold action is associated to an orientifold group (Γ, ϵ). The analytic K-theory of this

C^*-algebra is then defined by equipping Kasparov modules with an anti-linear operator associated to a choice of element $\zeta \in \Gamma^-$, and imposing an equivalence relation to eliminate the ambiguity introduced by this choice. This definition is related to the notion of *relative conjugation*, which is used to reduce the theory of unitary/anti-unitary representations to that of unitary representations. In Section 7.2, the geometric K-homology of an orientifold group is defined. In Section 7.3, a correspondence between the geometric K-homology and analytic K-theory of an orientifold group is defined by constructing K-theory classes from orientifold Dirac operators. The section finishes with some speculation on the possibility of a Baum-Connes conjecture for infinite discrete orientifold groups.

Chapter 1

Semi-equivariance

The main obstacle to the construction of a Real Dirac operator is that the frame bundle of a Real vector bundle is not an equivariant principal bundle in the usual sense. This is due to the fact that the action of $\mathbb{Z}_2$ on a Real vector bundle is anti-linear. Whereas the total space of a $\mathbb{Z}_2$-equivariant principal $\mathrm{GL}(\mathbb{C}, n)$-bundle carries an action of $\mathbb{Z}_2 \times \mathrm{GL}(\mathbb{C}, n)$, a Real bundle has a frame bundle with a total space that carries an action of the semi-direct product $\mathbb{Z}_2 \ltimes_\kappa \mathrm{GL}(\mathbb{C}, n)$, where κ is the automorphism of $\mathrm{GL}(\mathbb{C}, n)$ given by elementwise conjugation on the standard matrix representation. This is the basic example of a *semi-equivariant* principal bundle. More generally, a semi-equivariant principal bundle has a total space that carries a smooth action of $\Gamma \ltimes_\theta G$, for some equivariance group Γ and some structure group G equipped with an action θ of Γ by automorphisms.

The construction of a $\mathbb{Z}_2$-equivariant Dirac operator depends on the existence of a lifting from the $\mathbb{Z}_2$-equivariant $\mathrm{SO}(n)$-frame bundle of the tangent space to a $\mathbb{Z}_2$-equivariant principal $\mathrm{Spin}^c(n)$-bundle. In an analogous manner, the construction of a Real Dirac operator depends on the existence of a lifting from the $\mathbb{Z}_2$-semi-equivariant $(\mathrm{SO}(n), \mathrm{id})$-frame bundle of the tangent space to a $\mathbb{Z}_2$-semi-equivariant principal $(\mathrm{Spin}^c(n), \kappa)$-bundle. Here id is the trivial $\mathbb{Z}_2$-action on $\mathrm{SO}(n)$, and κ is the $\mathbb{Z}_2$-action induced by conjugation on the $\mathrm{U}(1)$ component of $\mathrm{Spin}^c(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$. In order to find such liftings, the global topology of the space and its interaction with the group action must be considered. In the equivariant setting, this can be approached by encoding the global topology and action into an equivariant transition cocycle. The lifting problem for these transition cocycles is then connected to equivariant Čech cohomology via equivariant Dixmier-Douady theory. This method classifies the possible equivariant liftings and shows how they can be constructed. It also identifies the obstruction to the existence of liftings as a class in equivariant cohomology.

To apply this method to the classification of semi-equivariant liftings, it is necessary to generalise the notions of transition cocycle, Čech cohomology and Dixmier-Douady invariant to the semi-equivaraint setting, where structure groups and coefficient groups are equipped with an action of the equivariance group. As will be discussed in Section 2.4, the frame bundle of an orientifold bundle is semi-equivariant. However, the frame bundles of orientifold bundles form only a small subset of the possible semi-equivariant principal bundles. Thus, most of the results that follow will be more general than is necessary for applications to orientifolds. Although semi-equivariant principal bundles are occasionally mentioned in the literature under various names, and it is well-known that the frame bundle of a Real bundle is semi-equivariant [80, §I.8] [64], it appears that the semi-equivariant generalisations developed here have been somewhat overlooked. Some related constructions can be found in the work of Freed and Moore on topological phases of matter [36, §7], and the work of Karoubi and Weibel on twistings of K-theory [55].

1.1 Semi-direct Products

Before examining semi-equivariant principal bundles, the notion of a semi-direct product is breifly reviewed. Semi-direct products are basic to the notion of semi-equivariance, and are useful for working with orientifold groups, which will be introduced in Section 2.1.

Definition 1.1. Let Γ be a Lie group. A *(smooth)* Γ*-group* (G, θ) is a Lie group equipped with a smooth action

$$\theta : \Gamma \to \mathrm{Aut}(G).$$

A *homomorphism* $\varphi : G \to H$ of Γ-groups is a homomorphism of Lie groups such that, for $\gamma \in \Gamma$ and $g \in G$,

$$\varphi(\gamma g) = \gamma\varphi(g). \tag{1.1}$$

Definition 1.2. Let (G, θ) be a Γ-group. The *(outer) semi-direct product* $\Gamma \ltimes_\theta G$ is the Lie group consisting of elements $(\gamma, g) \in \Gamma \times G$ with multiplication defined, for $\gamma_i \in \Gamma$ and $g_i \in G$, by

$$(\gamma_1, g_1)(\gamma_2, g_2) := (\gamma_1\gamma_2, g_1(\gamma_1 g_2)).$$

One situation in which semi-direct product groups arise is when G and Γ both act on an object X and satsify the relation $\gamma(gx) = (\gamma g)(\gamma x)$, for some action θ of Γ on G. In this case, the two actions combine to form a single action of the group $\Gamma \ltimes_\theta G$ by $(\gamma, g)x := g(\gamma x)$.

Example 1.3. The standard U(1)-action on $\mathbb{C}$ and the $\mathbb{Z}_2$-action on $\mathbb{C}$ by conjugation, combine into a $\mathbb{Z}_2 \ltimes_\kappa U(1)$-action on $\mathbb{C}$, where κ is the $\mathbb{Z}_2$-action on U(1) by conjugation.

1.2 Semi-equivariant Principal Bundles

The structure group of a semi-equivariant principal bundle is a Γ-group (G, θ). The action θ determines the commutation relation between the left action of Γ and right action of G on the total space of the principal bundle. These actions combine into an action of the semi-direct product $\Gamma \ltimes_\theta G$. In the following definitions, let (G, θ) be a smooth Γ-group and X be a manifold equipped with a smooth Γ-action.

Definition 1.4. A *(smooth)* Γ*-semi-equivariant principal* (G, θ)*-bundle* over X is a smooth principal G-bundle $\pi : P \to X$ equipped with a smooth left action of Γ such that, for $\gamma \in \Gamma$, $p \in P$ and $g \in G$,

$$\pi(\gamma p) = \gamma\pi(p) \qquad \gamma(pg) = (\gamma p)(\gamma g).$$

Definition 1.5. An *isomorphism* $\varphi : P \to Q$ of Γ-semi-equivariant principal (G, θ)-bundles is a diffeomorphism such that, for $\gamma \in \Gamma$, $p \in P$ and $g \in G$,

$$\pi_P = \pi_Q \circ \varphi \qquad \varphi(pg) = \varphi(p)g \qquad \varphi(\gamma p) = \gamma\varphi(p).$$

Next, let $\lambda : (G, \theta) \to (H, \vartheta)$ be a homomorphism of Γ-groups, and Q be a Γ-semi-equivariant principal (H, ϑ)-bundle.

Definition 1.6. A *lifting* of Q by λ is a pair (P, φ), where P is a Γ-semi-equivariant principal (G, θ)-bundle and $\varphi : P \to Q$ is a smooth map such that, for $\gamma \in \Gamma$, $p \in P$ and $g \in G$,

$$\pi_P = \pi_Q \circ \varphi \qquad \varphi(pg) = \varphi(p)\lambda(g) \qquad \varphi(\gamma p) = \gamma\varphi(p).$$

Definition 1.7. Two liftings (P_1, φ_1) and (P_2, φ_2) of Q by λ are *equivalent* if there is an isomorphism $\psi : P_1 \to P_2$ such that $\varphi_2 \circ \psi = \varphi_1$.

The set of smooth Γ-semi-equivariant principal (G, θ)-bundles will be denoted $PB_\Gamma(X, (G, \theta))$, and the isomorphisms classes will be denoted $PB_\Gamma^{\cong}(X, (G, \theta))$.

1.3 Semi-equivariant Transition Cocycles

Transition cocycles are used to extract global topological information from a principal bundle into a form which is more easily analysed. A transition cocycle over an open cover $\mathcal{U} := \{U_a\}$ with values in a Lie group G is a collection of smooth maps $\phi_a : U_a \to G$. Maps on overlapping open sets are required to satisfy a *cocycle condition*. This condition ensures that the cocycle can be used to glue together the patches $U_a \times G$ into a principal G-bundle.

In the equivariant setting, a transition cocycle consists of maps $\phi_a(\gamma, \cdot) : U_a \to G$ for each $U_a \in \mathcal{U}$ and $\gamma \in \Gamma$. The equivariant cocycle condition then ensures that the elements $\phi_a(1, \cdot)$ can be used construct the total space of a principal G-bundle, and that the elements $\phi_a(\gamma, \cdot)$ can be used to construct a Γ-action. The derivation of the equivariant cocycle condition uses the fact that the left and right actions on an equivariant principal bundle form an action of $\Gamma \times G$, and thus commute.

Semi-equivariant transition cocycles can be defined in a similar fashion to equivariant transition cocycles. However, the left and right actions on a Γ-semi-equivariant principal (G, θ)-bundle form an action of $\Gamma \ltimes_\theta G$. Thus, the commutation relation between the left and right actions is controlled by θ, and the action θ appears in the semi-equivariant cocycle condition. When this cocycle condition is satisfied, the elements $\phi_a(1, \cdot)$ in a cocycle can be used to construct the total space of a semi-equivariant principal bundle, and the elements $\phi_a(\gamma, \cdot)$ can be used to construct a semi-equivariant Γ-action. The main result of this section is that the set of isomorphism classes of smooth semi-equivariant principal bundles is in bijective correspondence with the set of equivalence classes of smooth semi-equivariant transition cocycles. Throughout this section, let X be a Γ-space, (G, θ) be a Γ-group and $\mathcal{U} := \{U_a\}$ be an open cover of X. The cover $\mathcal{U}$ is not required to be invariant.

Definition 1.8. A *(smooth)* Γ*-semi-equivariant* (G, θ)*-valued transition cocycle* over $\mathcal{U}$ is a collection of smooth maps

$$\phi := \left\{ \phi_{ba}(\gamma, \cdot) : U_a \cap \gamma^{-1} U_b \to G \mid U_a \cap \gamma^{-1} U_b \neq \emptyset \right\},$$

satisfying

$$\phi_{aa}(1, x_0) = 1 \qquad \phi_{ca}(\gamma'\gamma, x) = \phi_{cb}(\gamma', \gamma x)(\gamma' \phi_{ba}(\gamma, x)), \tag{1.2}$$

for $x_0 \in U_a$, $\gamma', \gamma \in \Gamma$ and $x \in U_a \cap \gamma^{-1} U_b \cap (\gamma'\gamma)^{-1} U_c$.

Note that the conditions (1.2) define a non-equivariant cocycle when restricted to $\gamma = 1$, and an equivariant cocycle when $\theta = \mathrm{id}$.

Definition 1.9. An *equivalence* of Γ-semi-equivariant (G, θ)-valued transition cocycles ϕ^1 and ϕ^2 with cover $\mathcal{U}$ is a collection of smooth maps

$$\mu := \{\mu_a : U_a \to G\}$$

such that

$$\mu_b(\gamma x)\phi^1_{ba}(\gamma, x) = \phi^2_{ba}(\gamma, x)(\gamma \mu_a(x)),$$

for $\gamma \in \Gamma$ and $x \in U_a \cap \gamma^{-1} U_b$.

Next, let $\lambda : (G, \theta) \to (H, \vartheta)$ be a homomorphism of Γ-groups, and ϕ be a Γ-semi-equivariant (H, ϑ)-valued transition cocycle over $\mathcal{U}$.

Definition 1.10. A *lifting* of ϕ by λ is a Γ-semi-equivariant (G, θ)-valued transition cocycle ψ such that $\lambda \circ \psi_{ba} = \phi_{ba}$.

Definition 1.11. Two liftings ψ^1 and ψ^2 of ϕ by λ are *equivalent* if there exists an equivalence μ between ψ^1 and ψ^2.

The set of smooth Γ-semi-equivariant (G, θ)-valued transition cocycles over $\mathcal{U}$ will be denoted $TC_\Gamma(\mathcal{U}, X, (G, \theta))$. The set of equivalence classes of smooth Γ-semi-equivariant (G, θ)-valued transition cocycles over $\mathcal{U}$ will be denoted by $TC_\Gamma^{\cong}(\mathcal{U}, X, (G, \theta))$.

The first step toward a correspondence between principal bundle and cocycles, is to show how a semi-equivariant transition cocycle can be constructed from a semi-equivariant principal bundle. Implicit in the proof of this result is the derivation of the semi-equivariant cocycle property.

Proposition 1.12. *Let $P \in PB_\Gamma(X, (G, \theta))$ and $s := \{s_a : U_a \to P|_{U_a}\}$ be a choice of smooth local sections over the cover $\mathcal{U}$. The collection of maps*

$$\phi^s := \left\{\phi_{ba}(\gamma, \cdot) : U_a \cap \gamma^{-1} U_b \to G \mid U_a \cap \gamma^{-1} U_b \neq \emptyset\right\}$$

defined by

$$\gamma s_a(x) = s_b(\gamma x)\phi_{ba}(\gamma, x). \tag{1.3}$$

is a smooth Γ-semi-equivariant (G, θ)-valued transition cocycle.

Proof. The given condition implies the following three identities

$$\gamma'\gamma s_a(x) = s_c(\gamma'\gamma x)\phi^s_{ca}(\gamma'\gamma, x) \quad \gamma' s_b(\gamma x) = s_c(\gamma'\gamma x)\phi^s_{cb}(\gamma', \gamma x) \quad \gamma s_a(x) = s_b(\gamma x)\phi^s_{ba}(\gamma, x),$$

which, together, imply

$$\begin{aligned} s_c(\gamma'\gamma x)\phi^s_{ca}(\gamma'\gamma, x) &= \gamma'\gamma s_a(x) \\ &= \gamma'(s_b(\gamma x)\phi^s_{ba}(\gamma, x)) \\ &= (\gamma' s_b(\gamma x))(\gamma'\phi^s_{ba}(\gamma, x)) \\ &= s_c(\gamma'\gamma x)\phi^s_{cb}(\gamma', \gamma x)(\gamma'\phi^s_{ba}(\gamma, x)). \end{aligned}$$

Thus ϕ^s satisfies the cocycle property $\phi^s_{ca}(\gamma'\gamma, x) = \phi^s_{cb}(\gamma', \gamma x)(\gamma'\phi^s_{ba}(\gamma, x))$. □

Note that (1.3) is the defining relation for a non-equivariant transition cocycle when restricted to $\gamma = 1$. If $\theta = \mathrm{id}$, then (1.3) is the defining relation for an equivariant transition cocycle.

The map from semi-equivariant principal bundles to semi-equivariant transition cocycles, defined by Proposition 1.12, depends on a choice of local sections. However, if one passes to isomorphism classes of principal bundles and equivalence classes of transition cocycles this dependence disappears. The next proposition shows that cocycles associated to isomorphic principal bundles by Proposition 1.12 are always equivalent, regardless of which sections are chosen.

Proposition 1.13. *Let* $P_i \in \mathrm{PB}_\Gamma(X, (G, \theta))$, *and* $\phi^i \in \mathrm{TC}_\Gamma(\mathcal{U}, X, (G, \theta))$ *be the cocycles associated to local sections* $s^i := \{s^i_a : U_a \to P_i|_{U_a}\}$ *as in Proposition 1.12. If* $\varphi : P_1 \to P_2$ *is an isomorphism, then the collection of maps*

$$\mu := \{\mu_a : U_a \to G\} \tag{1.4}$$

defined by

$$\varphi(s^1_a(x)) := s^2_a(x)\mu_a(x) \tag{1.5}$$

is an equivalence between ϕ^1 *and* ϕ^2.

Proof. The properties of semi-equivariant principal bundle isomorphisms and the defining property (1.5) imply that

$$\begin{aligned}
\varphi(\gamma s^1_a(x)) &= \gamma\varphi(s^1_a(x)) \\
\varphi(s^1_b(\gamma x)\phi^1_{ba}(\gamma, x)) &= \gamma(s^2_a(x)\mu_a(x)) \\
\varphi(s^1_b(\gamma x))\phi^1_{ba}(\gamma, x) &= (\gamma s^2_a(x))(\gamma\mu_a(x)) \\
s^2_b(\gamma x)\mu_b(\gamma x)\phi^1_{ba}(\gamma, x) &= s^2_b(\gamma x)\phi^2_{ba}(\gamma, x)(\gamma\mu_a(x)).
\end{aligned}$$

Thus,

$$\mu_b(\gamma x)\phi^1_{ba}(\gamma, x) = \phi^2_{ba}(\gamma, x)(\gamma\mu_a(x)),$$

and μ is an equivalence between ϕ^1 and ϕ^2 for any choice of sections s^i. □

Corollary 1.14. *The map of Proposition 1.12 induces a well-defined map*

$$\begin{aligned}
\mathrm{PB}^{\cong}_\Gamma(X, (G, \theta)) &\to \mathrm{TC}^{\cong}_\Gamma(\mathcal{U}, X, (G, \theta)) \\
[P] &\mapsto [\phi^s],
\end{aligned}$$

where s *is any collection of smooth local sections of* P.

The correspondence between semi-equivariant cocycles and principal bundles has now been shown in one direction. Next, an inverse map reconstructing a semi-equivariant principal bundle from a semi-equivariant transition cocycle is defined.

Proposition 1.15. *Let* $\phi \in TC_\Gamma(\mathcal{U}, X, (G, \theta))$. *The bundle* P^ϕ *defined by*

$$\pi : (\bigsqcup_{a \in A} U_a \times G / \sim) \to X,$$

where

1. $(a, x, g) \sim (b, x, \phi_{ba}(1, x)g)$ *defines the equivalence relation* $\sim$
2. $\pi[a, x, g] := x$ *is the projection map*
3. $[a, x, g]g' := [a, x, gg']$ *defines the right-action of* G
4. $\gamma[a, x, g] := [b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)]$ *defines the left action of* Γ,

is a smooth Γ*-semi-equivariant principal* (G, θ)*-bundle.*

Proof. The elements $\{\phi_{ba}(1, \cdot)\}$ satisfy

$$\phi_{ca}(1, x) = \phi_{cb}(1, x)\phi_{ba}(1, x)$$

and so form a G-valued cocycle in the usual sense. Therefore, the usual proof that P^ϕ is a principal G-bundle applies. The Γ-action is well-defined on equivalence classes as

$$\begin{aligned}
\gamma[b, x, \phi_{ba}(1, x)g] &= [c, \gamma x, \phi_{cb}(\gamma, x)\gamma(\phi_{ba}(1, x)g)] \\
&= [c, \gamma x, \phi_{cb}(\gamma, x)(\gamma\phi_{ba}(1, x))(\gamma g)] \\
&= [c, \gamma x, \phi_{ca}(\gamma, x)(\gamma g)] \\
&= \eta_\gamma[a, x, g].
\end{aligned}$$

The semi-equivariance property $\gamma(pg) = (\gamma p)(\gamma g)$ is satisfied as

$$\begin{aligned}
\gamma([a, x, g]g') &= \gamma([a, x, gg']) \\
&= [b, \gamma x, \phi_{ba}(\gamma, x)(\gamma gg')] \\
&= [b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)(\gamma g')] \\
&= (\gamma[a, x, g])(\gamma g')
\end{aligned}$$

Thus, P^ϕ is a Γ-semi-equivariant principal (G, θ)-bundle. □

This reconstruction map is also well-defined at the level of isomorphism and equivalence classes.

Proposition 1.16. *Let $\phi^i \in TC_\Gamma(\mathcal{U}, X, (G,\theta))$ and $P_i \in PB_\Gamma(X,(G,\theta))$ be the associated principal bundles, constructed using Proposition 1.15. If $\mu := \{\mu_a : U_a \to G\}$ is an equivalence between ϕ^1 and ϕ^2 then*

$$\begin{aligned} \varphi : P_1 &\to P_2 \\ [a,x,g] &\mapsto [a,x,\mu_a(x)g]. \end{aligned}$$

is an isomorphism.

Proof. That φ is a well-defined isomorphism of principal G-bundles follows immediately from the proof in the non-equivarant case. Compatibility with the Γ-action is satisfied as

$$\begin{aligned} \gamma\varphi([a,x,g]) &= \gamma[a,x,\mu_a(x)g] \\ &= [b,\gamma x,\phi'_{ba}(\gamma,x)\gamma(\mu_a(x)g)] \\ &= [b,\gamma x,\phi'_{ba}(\gamma,x)(\gamma\mu_a(x))(\gamma g)] \\ &= [b,\gamma x,\mu_b(\gamma x)\phi_{ba}(\gamma,x)(\gamma g)] \\ &= \varphi([b,\gamma x,\phi_{ba}(\gamma,x)(\gamma g)]) \\ &= \varphi(\gamma[a,x,g]). \end{aligned}$$

Thus, φ is an isomorphism of Γ-semi-equivariant principal (G,θ)-bundles. ☐

Corollary 1.17. *The map of Proposition 1.15 induces a well-defined map*

$$TC_\Gamma^{\cong}(\mathcal{U}, X, (G,\theta)) \to PB_\Gamma^{\cong}(X,(G,\theta)) \tag{1.6}$$

$$[\phi] \mapsto [P^\phi]. \tag{1.7}$$

Finally, one shows that the two maps defined above are inverse to one another.

Proposition 1.18. *The maps*

$$\begin{aligned} TC_\Gamma^{\cong}(\mathcal{U}, X, (G,\theta)) &\to PB_\Gamma^{\cong}(X,(G,\theta)) \\ [\phi] &\mapsto [P^\phi] \end{aligned} \quad \text{and} \quad \begin{aligned} PB_\Gamma^{\cong}(X,(G,\theta)) &\to TC_\Gamma^{\cong}(\mathcal{U}, X, (G,\theta)) \\ [P] &\mapsto [\phi^s] \end{aligned}$$

are inverse to one another.

Proof. Let $P \in PB_\Gamma(X,(G,\theta))$, $\phi := \phi^s$ and $P' := P^\phi$ for some collection of local sections $s := \{s_a : U_a \to P|_{U_a}\}$. The sections $\{s_a\}$ define a trivialization $\{t_a\}$ of P by

$$\begin{aligned} t_a : P|_{U_a} &\to U_a \times G \\ s_a(x) &\mapsto (a,x,1) \end{aligned}$$

and a collection of maps $\{T_a : P|_{U_a} \to G\}$ by $t_a(p) =: (a, x, T_a(p))$ where $x = \pi_P(p)$. Note that $T_a(pg) = T_a(p)g$. Define

$$\begin{aligned} \varphi : P &\to P' \\ p &\mapsto [t_a(p)]. \end{aligned}$$

That φ is a well-defined isomorphism of principal G-bundles follows from the proof in the non-equivariant case. To check that φ is compatible with the Γ-actions first note that

$$\begin{aligned} t_b \circ \eta_\gamma \circ t_a^{-1}(a, x, g) &= t_b(\gamma(s_a(x)g)) \\ &= t_b((\gamma s_a(x))(\gamma g)) \\ &= t_b(s_b(\gamma x)\phi_{ba}(\gamma, x)(\gamma g)) \\ &= (b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)) \end{aligned}$$

where η is the Γ-action on P. Thus,

$$\begin{aligned} \gamma\varphi(p) &= \gamma[t_a(p)] \\ &= \gamma[a, x, T_a(p)] \\ &= [b, \gamma x, \phi_{ba}(\gamma, x)T_a(p)] \\ &= [t_b \circ \eta_\gamma \circ t_a^{-1}(a, x, T_a(p))] \\ &= [t_b(\gamma p)] \\ &= \varphi(\gamma p). \end{aligned}$$

Therefore, φ is an isomorphism of Γ-semi-equivariant principal (G, θ)-bundles and

$$P \mapsto \phi^s \mapsto P^{\phi^s}$$

is the identity map at the level of isomorphism classes. □

The main theorem of this section has now been proved.

Theorem 1.19. *There is a bijective correspondence*

$$\mathrm{PB}_\Gamma^\cong(X, (G, \theta)) \leftrightarrow \mathrm{TC}_\Gamma^\cong(\mathcal{U}, X, (G, \theta))$$

between semi-equivariant cocycles and principal bundles.

It will be shown, in Proposition 2.29, that the frame bundle of a complex vector bundle with anti-linear symmetries is semi-equivariant. Together with Theorem 1.19, this allows the global topology of bundles with anti-linear symmetries to be analysed using semi-equivariant cocycles.

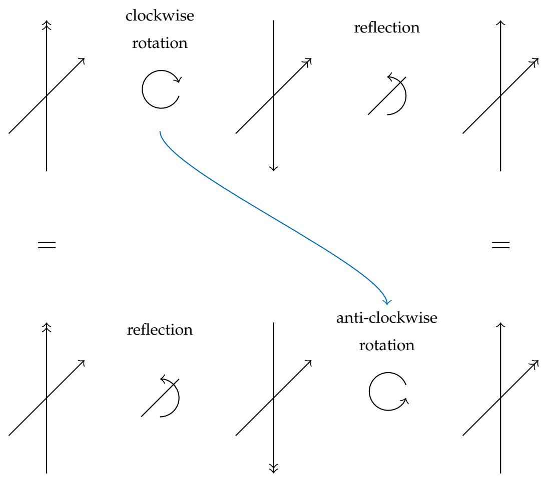


Figure 1.1: This figure corresponds to $\mathbb{C}$ equipped with conjugation as a $\mathbb{Z}_2$-action and $U(1)$ acting by rotations, as in Example 1.3. The blue line represents the conjugation automorphism on $U(1)$. This conjugation is required in order to obtain the same final result when the two actions are applied in reversed order.

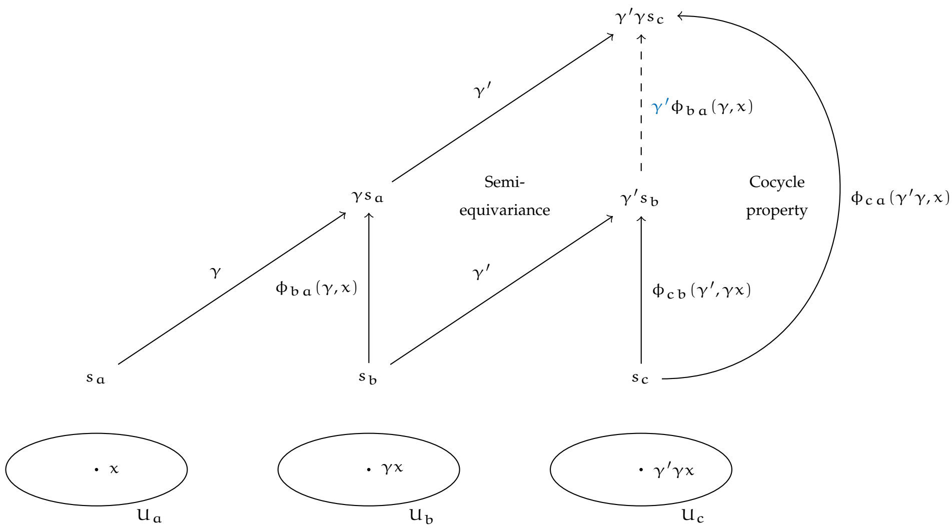


Figure 1.2: This diagram represents the derivation of the semi-equivariant cocycle property, as in Proposition 1.12. Each node of the diagram represents a local section of a principal bundle. The diagonal arrows represent applications of the Γ-action, while the vertical arrows represent the action of a cocycle ϕ via the right action of the structure group. With the exception of the dashed line, all of the arrows follow from the definitions. The dashed line follows by the semi-equivariance property of the principal bundle, the blue γ' is acting on the element $\phi_{ba}(\gamma, x)$ of the structure group.

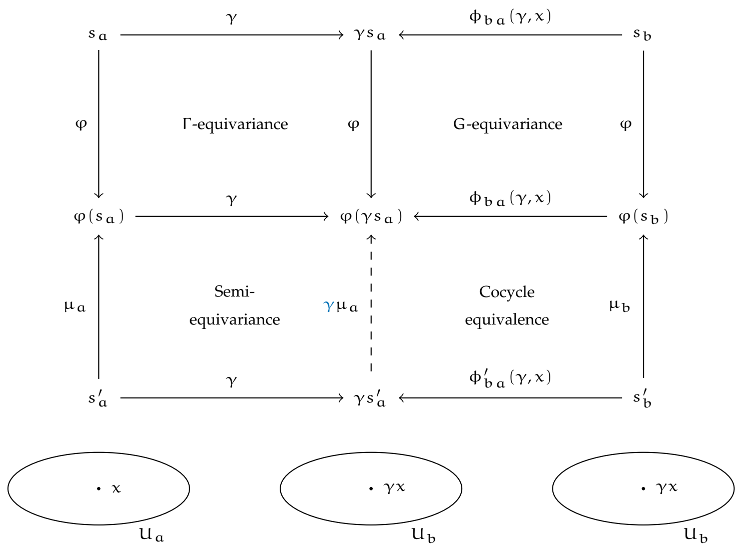


Figure 1.3: This diagram represents the derivation of the equivalence property for semi-equivariant cocycles, see Definition 1.9. Here φ is a semi-equivariant principal bundle isomorphism. Each node of the diagram represents a local section of a principal bundle. The arrows running downward are applications of a principal bundle isomorphism φ. The arrows running left to right are applications of the Γ-action. The arrows running right to left are right actions by the cocycle ϕ. Those running upward are right actions of the cocycle equivalence μ. With the exception of the dashed arrow, all of the arrows follow from definitions. The commutation of the top two squares follows from the properties of principal bundle isomorphisms. The dashed arrow is follows from the semi-equivariance property of the principal bundle. This twists the equivalence μ_a by the action of Γ on the structure group, which is marked in blue. The lower right square is the semi-equivariant cocycle equivalence condition.

1.4 Semi-equivariant Cohomology

In order to study liftings of semi-equivariant principal bundles, a cohomology theory is needed. The existing notions of equivariant cohomology are inappropriate for this task, and a new cohomology theory must be constructed. In this section, a Γ-semi-equivariant Čech cohomology theory is developed with an abelian Γ-group (G, θ) as its coefficient group. The theory makes use of a simplicial space which encodes the group structure of Γ, and the action of Γ on the manifold X. In addition to these actions, the effect of the action θ must be incorporated. This is achieved by twisting the coboundary map using θ. There are a few details to check, but everything works as one would wish. This semi-equivariant cohomology theory generalises an equivariant cohomology theory outlined by Brylinski [26, §A]. Another helpful reference is [41, §3.3]. One feature of the presentation here is that it avoids the use of hypercohomology. The second dimension of the bicomplex appearing in [26, §A] is an artifact of the choice to separate the cocycle into two parts, one encoding the transition functions for the total space and one encoding the action. Although this is ultimately a notational matter, the reduced book-keeping is helpful when checking higher cocycle conditions.

The construction of semi-equivariant Čech cohomology begins with the definition of a simplicial space. The coboundary map on the underlying chain complex of the cohomology theory will be constructed using the face maps of this space.

Definition 1.20. Let X be a manifold equipped with a smooth action of Γ. The *simplicial space* associated to X is defined by

$$X^\bullet := \{\Gamma^p \times X\}_{p \geq 0} .$$

The simplicial space carries *face* and *degeneracy* maps

$$d_i^p : X^p \to X^{p-1} \qquad\qquad e_i^p : X^p \to X^{p+1}$$

defined by

$$d_i^p(\gamma_1, \dots, \gamma_p, x) := \begin{cases} (\gamma_2, \dots, \gamma_p, x) & \text{for } i = 0 \\ (\gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_p, x) & \text{for } 1 \leq i \leq p-1 \\ (\gamma_1, \dots, \gamma_{p-1}, \gamma_p x) & \text{for } i = p \end{cases} \tag{1.8}$$

$$e_i^p(\gamma_1, \dots, \gamma_p, x) := (\gamma_1, \dots, \gamma_i, 1, \gamma_{i+1}, \dots, \gamma_p, x) \quad \text{for } 0 \leq i \leq p+1$$

Notice that in (1.8) the face map d_0^p discards the element γ_1, this element will be used to define the simplicial twisting maps, in Definition 1.22.

Proposition 1.21. *The face and degeneracy maps satisfy the simplicial identities*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i \quad \textit{for } i < j \\ e_i \circ e_j &= e_{j+1} \circ e_i \quad \textit{for } i \le j \end{aligned} \qquad d_i \circ e_j = \begin{cases} e_{j-1} \circ d_i & \textit{for } i < j \\ \mathrm{id} & \textit{for } i = j, j+1 \\ e_j \circ d_{i-1} & \textit{for } i > j+1 \end{cases} \tag{1.9}$$

Corresponding to the face maps d_i^p, twisting maps $\theta_i : X^p \times G \to G$ can be defined. These maps encode the action θ of Γ on G and will be used to twist the coboundary map. They are the basic ingredient needed for generalisation to the semi-equivariant setting. Note that it is only the twisting map θ_0 that has any effect. The rest of the twisting maps are included for notational convenience when dealing with simplical identities.

Definition 1.22. The *simplicial twisting maps* $\theta_i : X^p \times G \to G$ are given by

$$\theta_i^{(\gamma_1,\dots,\gamma_p,x)} := \begin{cases} \theta_{\gamma_1} & \text{for } i = 0 \\ \mathrm{id} & \text{for } 1 \le i \le p-1 \\ \mathrm{id} & \text{for } i = p \end{cases}$$

The twisting maps also satisfy simplicial identities which help to ensure that the coboundary map in semi-equivariant cohomology squares to zero.

Proposition 1.23. *The simplicial twisting maps satisfy the identities*

$$\theta_j^{x^{p+1}} \circ \theta_i^{d_j(x^{p+1})} = \theta_i^{x^{p+1}} \circ \theta_{j-1}^{d_i(x^{p+1})} \qquad \textit{for } i < j$$

$$\theta_i^{e_j(x^p)} = \begin{cases} \theta_i^{x^p} & \textit{for } i < j \\ \mathrm{id} & \textit{for } i = j, j+1 \\ \theta_{i-1}^{x^p} & \textit{for } i > j+1, \end{cases}$$

where $x^p \in X^p$.

Proof. The identities are trivial for most combinations of i and j. The remaining cases can be checked individually. In particular, the first identity reduces to

$$\begin{aligned} \mathrm{id} \circ \theta_{\gamma_1\gamma_2} &= \theta_{\gamma_1} \circ \theta_{\gamma_2} && \text{for } i = 0, j = 1 \\ \mathrm{id} \circ \theta_{\gamma_1} &= \theta_{\gamma_1} \circ \mathrm{id} && \text{for } i = 0, j \ge 2 \\ \mathrm{id} &= \mathrm{id} && \text{otherwise.} \end{aligned}$$

□

To construct a Čech-type theory, a simplicial cover $\mathcal{U}^\bullet$ of $X^\bullet$ is needed. Such a cover can be constructed from an appropriate cover $\mathcal{U} := \{U_a \mid a \in A\}$ of X. First, the indexing set of the simplicial cover is defined. This indexing set has a simplicial structure defined by face and degeneracy maps, which will again be denoted by d_i^p and e_i^p.

Definition 1.24. Define the indexing set for $\mathcal{U}^\bullet$ by

$$A^\bullet := \{A^p\}_{p\geq 0}$$

where $A^p := \{(a_0, \dots, a_p) \mid a_i \in A\}$. Elements of A^p will be denoted by a^p. This set carries face and degeneracy maps

$$d_i^p : A^p \to A^{p-1} \qquad\qquad e_i^p : A^p \to A^{p+1}$$

defined by

$$\begin{aligned} d_i^p(a_0, \dots, a_p) &:= (a_0, \dots, \hat{a}_i, \dots, a_p) \\ e_i^p(a_0, \dots, a_p) &:= (a_0, \dots, a_i, a_i, a_{i+1}, \dots, a_p), \end{aligned}$$

where $\hat{a}_i$ denotes the removal of the element a_i.

Proposition 1.25. *The face and degeneracy maps of the indexing set* $A^\bullet$ *satisfy*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i \quad \textit{for } i < j \\ e_i \circ e_j &= e_{j+1} \circ e_i \quad \textit{for } i \leq j \end{aligned} \qquad d_i \circ e_j = \begin{cases} e_{j-1} \circ d_i & \textit{for } i < j \\ \mathrm{id} & \textit{for } i = j, j+1 \\ e_j \circ d_{i-1} & \textit{for } i > j+1. \end{cases}$$

Before defining the simplicial cover itself, observe that the elements of the simplicial space define sequences of points in X.

Definition 1.26. Let $x^p = (\gamma_1, \dots, \gamma_p, x) \in X^p$. The associated sequence $\{\boldsymbol{x}_i^p\}$ is defined by

$$\boldsymbol{x}_i^p := \gamma_{p-i} \cdots \gamma_p x \in X.$$

Simplicial covers generalise the nerves of covers. The definition will be made using the definitions of the sequences $\boldsymbol{x}_i^p$ and indexing set $A^\bullet$.

Definition 1.27. The *simplicial cover*

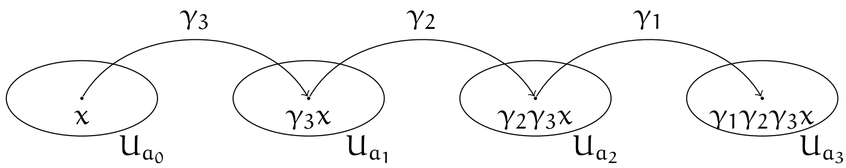
$$\mathcal{U}^\bullet := \{\mathcal{U}^p\}_{p\geq 0}$$

associated to $\mathcal{U}$ is a sequence of covers $\mathcal{U}^p$ of X^p each indexed by A^p. A set

$$U_{(a_0,\dots,a_p)} \in \mathcal{U}^p$$

consists of all points in X^p such that $\boldsymbol{x}_i^p \in U_{a_i}$ for $0 \leq i \leq p$.

For example, $(\gamma_1, \gamma_2, \gamma_3, x) \in U_{(a_0,a_1,a_2,a_3)}$ can be visualised as a path



.

Note that a refinement of $\mathcal{U}$ induces a refinement of $\mathcal{U}^\bullet$. Also, the face maps of the simplicial cover are compatible with those of the simplicial space. This is necessary to ensure that the coboundary map is well-defined.

Proposition 1.28. *The pullback maps of the simplicial space are compatible with those on the indexing set of the cover in the sense that* $d_i(U_{a^p}) \subseteq U_{d_i(a^p)}$.

Semi-equivariant Čech cohomology is based on a single cochain complex. A p-cochain for this cohomology theory consists of a smooth function on each set in the pth level of the simplicial cover.

Definition 1.29. The group of p-*cochains* is defined by

$$K^p_\Gamma(\mathcal{U}, X, (G, \theta)) := \prod_{a^p \in A^p} C^\infty(U_{a^p}, G),$$

with the group operation $(\phi'\phi)_{a^p} := \phi'_{a^p}\phi_{a^p}$.

These cochains can be pulled back by the face maps. In the semi-equivariant setting, the pullback maps are composed with the twisting maps. This modifies the pullback by d_0.

Definition 1.30. The *twisted pullback maps*

$$\partial^p_i : K^p_\Gamma(\mathcal{U}, X, (G, \theta)) \to K^{p+1}_\Gamma(\mathcal{U}, X, (G, \theta))$$

are defined by

$$(\partial^p_i \phi)_{a^{p+1}}(x^{p+1}) := \theta_i^{x^{p+1}} \circ \phi_{d^p_i(a^{p+1})} \circ d^p_i(x^{p+1})$$

Note that the property $d_i(U_{a^p}) \subseteq U_{d_i(a^p)}$ of the cover ensures that $\partial_i(\phi)$ is a well-defined element of $K^{p+1}_\Gamma(\mathcal{U}, X, (G, \theta))$.

Proposition 1.31. *The twisted pullback maps are group homomorphisms.*

Proof. Using the fact that θ_γ is an automorphism for all $\gamma \in \Gamma$,

$$\begin{aligned}
&(\partial_i(\phi'\phi))_{a^{p+1}}(x^{p+1}) \\
&\quad = \theta_i^{x^{p+1}} \circ (\phi'\phi)_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\
&\quad = \theta_i^{x^{p+1}} \Big((\phi'_{d_i(a^{p+1})} \circ d_i(x^{p+1}))(\phi_{d_i(a^{p+1})} \circ d_i(x^{p+1})) \Big) \\
&\quad = \Big(\theta_i^{x^{p+1}} \circ \phi'_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \Big) \Big(\theta_i^{x^{p+1}} \circ \phi_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \Big) \\
&\quad = \Big((\partial_i\phi')_{a^{p+1}}(x^{p+1}) \Big) \Big((\partial_i\phi)_{a^{p+1}}(x^{p+1}) \Big)
\end{aligned}$$

□

The simplicial identities of the face maps for the simplicial space, the simplicial cover and the twisting maps combine to produce a simplicial identity for the twisted pullback maps.

Proposition 1.32. *For* $i < j$ *the twisted pullback maps satisfy the identity*

$$\partial_j \circ \partial_i = \partial_i \circ \partial_{j-1}.$$

Proof. Using the corresponding simplicial identities between face maps on the simplicial complex, those on the simplicial cover, and those between the simplical twisting maps one can directly compute

$$\begin{aligned}
(\partial_j(\partial_i\phi))_{a^{p+2}}(x^{p+2}) &= \theta_j^{x^{p+2}} \circ (\partial_i\phi)_{d_j(a^{p+2})} \circ d_j(x^{p+2}) \\
&= \theta_j^{x^{p+2}} \circ \theta_i^{d_j(x^{p+2})} \circ \phi_{d_i \circ d_j(a^{p+2})} \circ d_i \circ d_j(x^{p+2}) \\
&= \theta_i^{x^{p+2}} \circ \theta_{j-1}^{d_i(x^{p+2})} \circ \phi_{d_{j-1} \circ d_i(a^{p+2})} \circ d_{j-1} \circ d_i(x^{p+2}) \\
&= \theta_i^{x^{p+2}} \circ (\partial_{j-1}\phi)_{d_i(a^{p+2})} \circ d_i(x^{p+2}) \\
&= (\partial_i(\partial_{j-1}\phi))_{a^{p+2}}(x^{p+2}).
\end{aligned}$$

□

Finally, the coboundary maps are defined.

Definition 1.33. The *coboundary maps*

$$\partial^p : K_\Gamma^p(\mathcal{U}, X, (G, \theta)) \to K_\Gamma^{p+1}(\mathcal{U}, X, (G, \theta))$$

are defined by

$$\partial^p := \sum_{0 \le i \le p} (-1)^i \partial_i^p.$$

Using the simplicial identity for the twisted pullback maps, the square of the coboundary map is shown to be zero.

Proposition 1.34. *The coboundary map satisfies $\partial\partial = 0$.*

Proof. First note, using Proposition 1.32, that

$$\sum_{i<j,j\leq p+2} (-1)^{i+j}\partial_j\partial_i = \sum_{i<j,j\leq p+2} (-1)^{i+j}\partial_i\partial_{j-1} = \sum_{i\leq j,j\leq p+1} (-1)^{i+j}\partial_i\partial_j = \sum_{j\leq i,i\leq p+1} (-1)^{i+j}\partial_j\partial_i.$$

Therefore,

$$\begin{aligned}\partial\partial &= \sum_{0\leq j\leq p+2} (-1)^j\partial_j(\sum_{0\leq i\leq p+1} (-1)^i\partial_i) \\ &= \sum_{0\leq j\leq p+2}\sum_{0\leq i\leq p+1} (-1)^{i+j}\partial_j\partial_i \\ &= \sum_{j\leq i,i\leq p+1} (-1)^{i+j}\partial_j\partial_i + \sum_{i<j,j\leq p+2} (-1)^{i+j}\partial_j\partial_i \\ &= 0.\end{aligned}$$

□

When (G, θ) is abelian, Proposition 1.34 allows the cohomology groups

$$H^p_\Gamma(\mathcal{U}, X, (G, \theta))$$

of the complex $(K^\bullet_\Gamma(\mathcal{U}, X, (G, \theta)), \partial)$ to be defined. The restriction to abelian Γ-groups is neccesary to ensure that the coboundary maps ∂^p are group homomorphisms. In order to obtain a cohomology theory which is independent of the cover $\mathcal{U}$, the direct limit of these cohomology groups will be taken with respect to refinements of the cover. A refinement of $\mathcal{U}$ consists of another cover $\mathcal{V}$ indexed by some set B, and a refining map $r : B \to A$ such that $V_b \subset U_{r(b)}$ for all $b \in B$. Such a refinement induces a refinement of the associated simplicial covers, and restriction homomorphisms $r_* : K^p_\Gamma(\mathcal{U}, X, (G, \theta)) \to K^p_\Gamma(\mathcal{V}, X, (G, \theta))$ defined by

$$(r_*\phi)_{(b_0,\ldots,b_p)} := \phi_{(r(b_0),\ldots,r(b_p))}|_{V_{(b_0,\ldots,b_p)}}.$$

These restriction homomorphisms, in turn, induce maps

$$H^p_\Gamma(\mathcal{U}, X, (G, \theta)) \to H^p_\Gamma(\mathcal{V}, X, (G, \theta))$$

on the cohomology of the complexes. In order for the direct limit of cohomology groups to be well-defined, the maps induced on cohomology by two different refining maps need to be equal. This is true in the equivariant setting, and in the semi-equivariant setting it just needs to be checked that the twisting of the coboundary map using θ doesn't cause any problems.

Lemma 1.35. *Let $(\mathcal{V}, r)$ and $(\mathcal{V}, s)$ be refinements of $\mathcal{U}$ with refining maps $r, s : B \to A$. The maps induced on semi-equivariant cohomology by r and s are identical.*

Proof. By analogy with the proof in the non-equivariant case (see for example [72, pp. 78-79]), a cochain homotopy

$$\begin{array}{ccccc} & & K^p_\Gamma(\mathcal{U}, X, (G,\theta)) & \xrightarrow{\partial^p} & K^{p+1}_\Gamma(\mathcal{U}, X, (G,\theta)) \\ & \swarrow^{h^p} & r_* \downarrow\downarrow s_* & \swarrow^{h^{p+1}} & \\ K^{p-1}_\Gamma(\mathcal{V}, X, (G,\theta)) & \xrightarrow{\partial^{p-1}} & K^p_\Gamma(\mathcal{V}, X, (G,\theta)). & & \end{array}$$

is defined by

$$(h^p \phi)_{(b_0,\dots,b_{p-1})} = \sum_{k=0}^{p-1} (-1)^k \phi_{(r(b_0),\dots,r(b_k),s(b_k),\dots,s(b_{p-1}))} \circ e_k,$$

where e_k is the kth degeneracy map. Just as in the non-equivariant case, expanding the expression

$$(h^{p+1}\partial^p \phi)_{(b_0,\dots,b_p)} - (\partial^{p-1} h^p \phi)_{(b_0,\dots,b_p)} \in K^p_\Gamma(\mathcal{V}, X, (G,\theta))$$

results in a large amount of cancelation. The remaining expression is

$$(\partial^p_0 \phi)_{(r(b_0),s(b_0),\dots,s(b_p))} \circ e_0 - (\partial^p_{p+1} \phi)_{(r(b_0),\dots,r(b_p),s(b_p))} \circ e_p.$$

The twisted coboundary maps ∂^0_0 and ∂^p_{p+1} involve the Γ-actions θ on G and σ on X, respectively. However, in the above expression, the degeneracy maps e_0 and e_p ensure that θ and σ only ever act via the identity element of Γ. Thus, the above expression simplifies to

$$\phi_{(s(b_0),\dots,s(b_p))} - \phi_{(r(b_0),\dots,r(b_p))} = (s_*\phi)_{(b_0,\dots,b_p)} - (r_*\phi)_{(b_0,\dots,b_p)}.$$

Therefore, if $\phi \in H^p_\Gamma(\mathcal{V}, X, (G,\theta))$ is a cocycle, then

$$(s_*\phi) - (r_*\phi) = h^{p+1} \circ \partial^p(\phi) - \partial^{p-1} \circ h^p(\phi) = \partial^{p-1} \circ h^p(\phi),$$

which is a coboundary. Thus, r_* and s_* induce the same cohomology groups. □

It is now possible to define the semi-equivariant cohomology groups.

Definition 1.36. The *(smooth) Γ-semi-equivariant Čech cohomology groups* with coefficients in an abelian Γ-group (G,θ) are defined by

$$H^p_\Gamma(X, (G,\theta)) := \varinjlim H^p_\Gamma(\mathcal{U}, X, (G,\theta)),$$

where $H^p_\Gamma(\mathcal{U}, X, (G,\theta))$ are the cohomology groups of the complex $(K^\bullet_\Gamma(\mathcal{U}, X, (G,\theta)), \partial)$, and the direct limit is taken with respect to refinements of $\mathcal{U}$.

Remark 1. Semi-equivariant Čech cohomology $H^\bullet_\Gamma(X,(G,\theta))$ is closely related to several other cohomology theories:

1. If Γ is the trivial group, then $H^\bullet_\Gamma(X,(G,\theta))$ is Čech cohomology $\check{H}^\bullet(X,G)$.

2. If θ is the trivial action, then $H^\bullet_\Gamma(X,(G,\theta))$ is equivariant Čech cohomology $\check{H}^\bullet_\Gamma(X,G)$. When X is a compact manifold acted upon by a finite group, the equivariant Čech cohomology can be related to Grothendieck's equivariant sheaf cohomology [43, §5.5] or Borel cohomology [26, §A], [41, §3.3].

   Note that there is a restriction homomorphism

   $$H^p_\Gamma(X,(G,\theta)) \to H^p_{\Gamma_G}(X,(G,\theta)) \simeq \check{H}^p_{\Gamma_G}(X,G),$$

   where $\Gamma_G \subseteq \Gamma$ is the stabiliser subgroup that acts trivially on G. In this way, the semi-equivariant cohomology can be regarded as a restriction of equivariant cohomology.

3. If X is a point, then $H^\bullet_\Gamma(X,(G,\theta))$ is the group cohomology $H^\bullet(\Gamma,G_\theta)$ of Γ with coefficients in the Γ-module G_θ defined by G and θ [19, p. 35]. With this in mind, semi-equivariant cohomology can be viewed as a cross between group cohomology and equivariant cohomology. In applications to orientifolds, the group Γ is equipped with a homomorphism into $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}_2$. In this case, $H^p_\Gamma(X,(G,\theta))$ incorporates aspects of equivariant Čech cohomology and Galois cohomology for the field extension $\mathbb{C}/\mathbb{R}$.

Semi-equivariant cohomology is functorial with respect to homomorphisms of abelian Γ-groups.

Proposition 1.37. *A homomorphism $\alpha : A \to B$ of abelian Γ-groups induces a morphism of complexes*

$$\alpha^\bullet : (K^\bullet_\Gamma(\mathcal{U},X,A),\partial) \to (K^\bullet_\Gamma(\mathcal{U},X,B),\partial)$$

defined by $(\alpha^p\phi)_{a^p} := \alpha \circ \phi_{a^p}$.

Proof. Let θ be the Γ-action on A and ϑ be the Γ-action on B. As α is a homomorphism of Γ-groups $\alpha^p \circ \theta_i^{x^p} = \vartheta_i^{x^p} \circ \alpha^p$ for all $x^p \in X^p$ and $0 \le i \le p$. Thus,

$$\begin{aligned}
(\alpha^{p+1}(\partial_i\phi))_{a^{p+1}}(x^{p+1}) &= \alpha \circ (\partial_i\phi)_{a^{p+1}}(x^{p+1}) \\
&= \alpha \circ \theta_i^{x^{p+1}} \circ \phi_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\
&= \vartheta_i^{x^{p+1}} \circ \alpha \circ \phi_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\
&= \vartheta_i^{x^{p+1}} \circ (\alpha^p\phi)_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\
&= (\partial_i(\alpha^p\phi))_{a^{p+1}}(x^{p+1}).
\end{aligned}$$

Therefore, $\alpha^{p+1} \circ \partial = \partial \circ \alpha^p$ and α^p defines a morphism of complexes. □

Given a short exact sequence of abelian Γ-groups, connecting maps for a long exact sequence can be constructed.

Theorem 1.38. *A short exact sequence of abelian* Γ*-groups*

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$$

induces a long exact sequence

$$\dots \xrightarrow{\Delta^{p-1}} H^p_\Gamma(X, A) \xrightarrow{\alpha^p} H^p_\Gamma(X, B) \xrightarrow{\beta^p} H^p_\Gamma(X, C) \xrightarrow{\Delta^p} H^{p+1}_\Gamma(X, A) \xrightarrow{\alpha^{p+1}} \dots$$

where $\Delta^p(\phi) := [\partial(\psi)]$ *for any element* $\psi \in K^p_\Gamma(B)$ *such that* $\beta^p(\psi) = \phi$.

Proof. The proposition follows by standard diagram chasing arguments applied to the exact sequence of complexes

$$1 \to (K^\bullet_\Gamma(X, A), \partial) \xrightarrow{\alpha^\bullet} (K^\bullet_\Gamma(X, B), \partial) \xrightarrow{\beta^\bullet} (K^\bullet_\Gamma(X, C), \partial) \to 1.$$

For an example, see the proof of [72, Theorem 4.30]. □

1.5 Semi-equivariant Dixmier-Douady Classes

In order to apply semi-equivariant cohomology to the classification of semi-equivariant liftings, its relationship with semi-equivariant principal bundles must be clarified. By Theorem 1.19, this reduces to the problem of relating semi-equivariant transition cocycles and semi-equivariant cohomology classes. In this section, semi-equivariant transition cocycles will be interpreted as degree-1 cocycles which can take values in a non-abelian coefficient group. An analogue of Theorem 1.38 will be proved that constructs a connecting map from the transition cocycles into degree-2 cohomology. The theorem can be used to classify certain liftings of semi-equivariant principal bundles between non-abelian structure groups. This method has its origins in the work of Dixmier-Douady on continuous trace C^*-algebras [33]. See also [27, § 4] and [72, § 4.3].

To begin, note that the p-cochains of Definition 1.29 and the twisted pullback maps of Definition 1.30 are well-defined for non-abelian Γ-groups. Thus, it is possible to make the following definitions.

Definition 1.39.

$$TC^0_\Gamma(\mathcal{U}, X, (G, \theta)) := \left\{ \mu \in K^0_\Gamma(\mathcal{U}, X, (G, \theta)) \mid (\partial_1 \mu)^{-1}(\partial_0 \mu) = 1 \right\} \quad (1.10)$$

$$TC^1_\Gamma(\mathcal{U}, X, (G, \theta)) := \left\{ \phi \in K^1_\Gamma(\mathcal{U}, X, (G, \theta)) \mid (\partial_1 \phi)^{-1}(\partial_2 \phi)(\partial_0 \phi) = 1 \right\} / \sim \quad (1.11)$$

where $\phi^1 \sim \phi^2$ if and only if there exists a $\mu \in K^0_\Gamma(\mathcal{U}, X, (G, \theta))$ such that $(\partial_1\mu)\phi^1 = \phi^2(\partial_0\mu)$.

The set $TC^1_\Gamma(\mathcal{U}, X, (G, \theta))$ is just $TC^{\simeq}_\Gamma(\mathcal{U}, X, (G, \theta))$ with the transition cocycle condition and equivalence condition expressed in terms of twisted pullback maps. Note that the particular order of the terms $\partial_i\mu$ in (1.10) and $\partial_i\phi$ in (1.11) is important as the elements μ and ϕ take values in G, which is not necessarily abelian. When G is abelian, these terms may be rearranged to give the corresponding cocycle properties in semi-equivariant cohomology. An abelian structure group also ensures that pointwise multiplication is a well-defined group structure on TC^0_Γ and TC^1_Γ, which, in general, are only pointed sets.

Theorem 1.40. *When* G *is abelian*

$$TC^0_\Gamma(\mathcal{U}, X, (G, \theta)) \simeq H^0_\Gamma(\mathcal{U}, X, (G, \theta)) \tag{1.12}$$

$$TC^1_\Gamma(\mathcal{U}, X, (G, \theta)) \simeq H^1_\Gamma(\mathcal{U}, X, (G, \theta)). \tag{1.13}$$

Proof. When G is abelian, the defining condition on $TC^0_\Gamma(\mathcal{U}, X, (G, \theta))$ and the 0-cocycle condition on cohomology are equivalent as

$$0 = -(\partial_1\mu) + (\partial_0\mu) = (\partial_0\mu) - (\partial_1\mu) = \partial\mu.$$

This proves (1.12). Similarly, the defining condition on $TC^1_\Gamma(\mathcal{U}, X, (G, \theta))$ and the 1-cocycle condition on cohomology are equivalent as

$$0 = -(\partial_1\phi) + (\partial_2\phi) + (\partial_0\phi) = (\partial_0\phi) - (\partial_1\phi) + (\partial_2\phi) = \partial\phi,$$

and the equivalence relations on $TC^1_\Gamma(\mathcal{U}, X, (G, \theta))$ and $H^0_\Gamma(\mathcal{U}, X, (G, \theta))$ are the same as

$$\begin{aligned}(\partial_1\mu) + \phi^1 &= \phi^2 + (\partial_0\mu)\\ \phi^1 - \phi^2 &= (\partial_0\mu) - (\partial_1\mu)\\ \phi^1 - \phi^2 &= \partial\mu.\end{aligned}$$

These two facts imply (1.13). □

Together, Theorem 1.38 and Theorem 1.40 enable liftings of semi-equivariant principal bundles between abelian structure groups to be classified. However, the construction of a Dirac operator involves the construction of liftings between non-abelian groups. The next theorem is a generalisation of Theorem 1.38 that can be used to classify certain liftings between non-abelian structure groups.

Theorem 1.41. *A short exact sequence of* Γ*-groups*

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1,$$

where $\alpha(A)$ *is central in* B, *induces an exact sequence of pointed sets*

$$0 \to H^0_\Gamma(X,A) \xrightarrow{\alpha^0} TC^0_\Gamma(X,B) \xrightarrow{\beta^0} TC^0_\Gamma(X,C) \xrightarrow{\Delta^0} H^1_\Gamma(X,A) \xrightarrow{\alpha^1} TC^1_\Gamma(X,B) \xrightarrow{\beta^1} TC^1_\Gamma(X,C) \xrightarrow{\Delta^1} H^2_\Gamma(X,A),$$

where

1. $\Delta^0([\mu]) := [(\partial_1\eta)^{-1}(\partial_0\eta)]$ *for any element* $\eta \in K^0_\Gamma(X,B)$ *such that* $\beta^0(\eta) = \mu$,
2. $\Delta^1([\phi]) := [(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi)]$ *for any element* $\psi \in K^1_\Gamma(X,B)$ *such that* $\beta^1(\psi) = \phi$.

Proof. The diagram chasing arguments used in the proof of Theorem 1.38 do not apply directly. However, they can be imitated while carefully working around any lack of commutivity in the groups B and C. Note that Proposition 1.37 and Proposition 1.32 continue to hold when the structure groups involved are non-abelian. Thus, the twisted pullback maps ∂_i commute with the maps α^i and β^i induced by α and β, and also satisfy the simplicial identity $\partial_j \circ \partial_i = \partial_i \circ \partial_{j-1}$ for $i < j$.

First, the map Δ^0 will be considered. Let $\nu := (\partial_1\eta)^{-1}(\partial_0\eta) \in K^1_\Gamma(X,B)$. The cochain η is a lifting by β of μ so $\beta(\nu) = 1$. Thus, ν takes values in $\ker(\beta) \simeq A$ and defines an element of $K^1_\Gamma(X,A)$. The simplicial identity can be used to show that the cochain ν satisfies the cocycle property,

$$\begin{aligned}
(\partial_1\nu)^{-1}(\partial_0\nu) &= \Big[(\partial_1\partial_1\nu)^{-1}(\partial_1\partial_0\nu)\Big]^{-1}\Big[(\partial_0\partial_1\nu)^{-1}(\partial_0\partial_0\nu)\Big] \\
&= (\partial_1\partial_0\nu)^{-1}(\partial_1\partial_1\nu)(\partial_0\partial_1\nu)^{-1}(\partial_0\partial_0\nu) \\
&= (\partial_1\partial_0\nu)^{-1}(\partial_1\partial_1\nu)(\partial_0\partial_1\nu)^{-1}(\partial_0\partial_0\nu) \\
&= (\partial_0\partial_0\nu)^{-1}(\partial_1\partial_1\nu)(\partial_1\partial_1\nu)^{-1}(\partial_0\partial_0\nu) \\
&= 1.
\end{aligned}$$

Therefore, $\Delta^0([\mu]) := [\nu] \in H^1_\Gamma(X,A)$. Next, it needs to be shown that $\Delta^0([\mu]) := [(\partial_1\eta)^{-1}(\partial_0\eta)]$ is independent of the choice of η. Let $\eta' \in K^0_\Gamma(X,B)$ be another element such that $\beta(\eta') = \mu$. Set $\omega := \eta'\eta^{-1}$ and $\nu' := (\partial_1\eta')^{-1}(\partial_0\eta') \in K^1_\Gamma(X,B)$. Then $\beta(\omega) = \beta(\eta'\eta^{-1}) = \mu\mu^{-1} = 1$. Thus, ω defines an element of $K^0_\Gamma(X,A)$ and $\partial\omega \in K^1_\Gamma(X,A)$ is a coboundary. Using the fact that ν and $\partial\omega$ take values in the abelian group A,

$$\begin{aligned}
(\partial\omega)\nu &= (\partial\omega)(\partial_1\eta)^{-1}(\partial_0\eta) \\
&= (\partial_1\eta)^{-1}(\partial\omega)(\partial_0\eta) \\
&= (\partial_1\eta)^{-1}(\partial_1\eta)(\partial_1\eta')^{-1}(\partial_0\eta')(\partial_0\eta)^{-1}(\partial_0\eta) \\
&= (\partial_1\eta')^{-1}(\partial_0\eta') \\
&= \nu'.
\end{aligned}$$

Therefore, $[\nu] = [\nu'] \in H^1_\Gamma(X, A)$.

In order to examine the map Δ^1, let $\nu := (\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi) \in K^2_\Gamma(X, B)$. The cochain $\psi \in K^1_\Gamma(X, B)$ is a β-lifting of the cocycle $\phi \in TC^1_\Gamma(X, C)$ so $\beta(\nu) = 1$. Therefore, ν defines an element of $K^2_\Gamma(X, A)$. Using the simplicial identity, and the fact that ν takes values in the centre of B, it can be shown that ν satisfies the 2-cocycle propery. First, compute

$$
\begin{aligned}
(\partial_1\nu)(\partial_3\nu) &= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)(\partial_1\partial_0\psi)(\partial_3\nu) \\
&= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)(\partial_3\nu)(\partial_1\partial_0\psi) \\
&= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)\Big[(\partial_3\partial_1\psi)^{-1}(\partial_3\partial_2\psi)(\partial_3\partial_0\psi)\Big](\partial_1\partial_0\psi) \\
&= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)\Big[(\partial_1\partial_2\psi)^{-1}(\partial_3\partial_2\psi)(\partial_3\partial_0\psi)\Big](\partial_1\partial_0\psi) \\
&= (\partial_1\partial_1\psi)^{-1}(\partial_3\partial_2\psi)(\partial_3\partial_0\psi)(\partial_1\partial_0\psi) \\
&= (\partial_1\partial_1\psi)^{-1}(\partial_3\partial_2\psi)\Big[(\partial_2\partial_0\psi)(\partial_2\partial_0\psi)^{-1}\Big](\partial_3\partial_0\psi)(\partial_1\partial_0\psi) \\
&= (\partial_2\partial_1\psi)^{-1}(\partial_2\partial_2\psi)\Big[(\partial_2\partial_0\psi)(\partial_0\partial_1\psi)^{-1}\Big](\partial_0\partial_2\psi)(\partial_0\partial_0\psi) \\
&= \Big[(\partial_2\partial_1\psi)^{-1}(\partial_2\partial_2\psi)(\partial_2\partial_0\psi)\Big]\Big[(\partial_0\partial_1\psi)^{-1}(\partial_0\partial_2\psi)(\partial_0\partial_0\psi)\Big] \\
&= (\partial_2\nu)(\partial_0\nu).
\end{aligned}
$$

Then

$$
\begin{aligned}
(\partial\nu) &= (\partial_0\nu)(\partial_1\nu)^{-1}(\partial_2\nu)(\partial_3\nu)^{-1} \\
&= (\partial_0\nu)(\partial_2\nu)(\partial_3\nu)^{-1}(\partial_1\nu)^{-1} \\
&= (\partial_0\nu)(\partial_2\nu)\Big[(\partial_1\nu)(\partial_3\nu)\Big]^{-1} \\
&= (\partial_0\nu)(\partial_2\nu)\Big[(\partial_0\nu)(\partial_2\nu)\Big]^{-1} \\
&= 1,
\end{aligned}
$$

and so $[\nu] \in H^2_\Gamma(X, A)$.

Next, it needs to be shown that Δ^1 is well-defined. Specifically, that

$$\Delta^1([\phi]) := [(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi)]$$

is independent of the choice of ψ, and depends only on the class of ϕ in $TC^1_\Gamma(X, C)$. To prove the first statement, let $\psi' \in K^1_\Gamma(X, B)$ be another β-lifting of ϕ and $\nu' := (\partial_1\psi')^{-1}(\partial_2\psi')(\partial_0\psi')$ be the corresponding element of $H^2_\Gamma(X, A)$. If $\omega := \psi'\psi^{-1}$ then $\beta(\omega) = \beta(\psi'\psi^{-1}) = \phi\phi^{-1} = 1$. Thus, $\omega \in K^1_\Gamma(X, A)$ and $\partial\omega \in K^2_\Gamma(X, A)$ is a coboundary . Next, using the fact that ω takes values in the center of B,

$$\begin{aligned}
(\partial\omega)\nu &= (\partial_0\omega)(\partial_1\omega)^{-1}(\partial_2\omega)(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi) \\
&= (\partial_1\psi)^{-1}(\partial_1\omega)^{-1}(\partial_2\omega)(\partial_2\psi)(\partial_0\omega)(\partial_0\psi) \\
&= (\partial_1\psi)^{-1}(\partial_1\psi'\psi^{-1})^{-1}(\partial_2\psi'\psi^{-1})(\partial_2\psi)(\partial_0\psi'\psi^{-1})(\partial_0\psi) \\
&= (\partial_1\psi)^{-1}(\partial_1\psi)(\partial_1\psi')^{-1}(\partial_2\psi')(\partial_2\psi)^{-1}(\partial_2\psi)(\partial_0\psi')(\partial_0\psi)^{-1}(\partial_0\psi) \\
&= (\partial_1\psi')^{-1}(\partial_2\psi')(\partial_0\psi') \\
&= \nu'.
\end{aligned}$$

Therefore, $[\nu] = [\nu'] \in H^2_\Gamma(X, A)$.

In order to prove that $\Delta^1([\phi])$ depends only on the class of ϕ, suppose that ϕ is a coboundary i.e. that $\phi = (\partial_1\tilde{\phi})^{-1}(\partial_0\tilde{\phi})$ for some $\tilde{\phi} \in K^0_\Gamma(X, C)$. By surjectivity of β, there exists an element $\tilde{\psi}$ such that $\beta(\tilde{\psi}) = \tilde{\phi}$. Then $\psi := (\partial_1\tilde{\psi})^{-1}(\partial_0\tilde{\psi})$ is a lifting by β of ϕ as

$$\begin{aligned}
\beta(\psi) &= \beta\left[(\partial_1\tilde{\psi})^{-1}(\partial_0\tilde{\psi})\right] \\
&= (\beta\partial_1\tilde{\psi})^{-1}(\beta\partial_0\tilde{\psi}) \\
&= (\partial_1\beta\tilde{\psi})^{-1}(\partial_0\beta\tilde{\psi}) \\
&= (\partial_1\tilde{\phi})^{-1}(\partial_0\tilde{\phi}) \\
&= \phi.
\end{aligned}$$

So, again applying the simplicial identity,

$$\begin{aligned}
\Delta^1([\phi]) &= [(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi)] \\
&= [(\partial_1\partial_0\tilde{\psi})^{-1}(\partial_1\partial_1\tilde{\psi})(\partial_2\partial_1\tilde{\psi})^{-1}(\partial_2\partial_0\tilde{\psi})(\partial_0\partial_1\tilde{\psi})^{-1}(\partial_0\partial_0\tilde{\psi})] \\
&= [(\partial_0\partial_0\tilde{\psi})^{-1}(\partial_1\partial_1\tilde{\psi})(\partial_1\partial_1\tilde{\psi})^{-1}(\partial_0\partial_1\tilde{\psi})(\partial_0\partial_1\tilde{\psi})^{-1}(\partial_0\partial_0\tilde{\psi})] \\
&= 1.
\end{aligned}$$

Thus, $\Delta^1([\phi])$ depends only on the class of ϕ in $TC^1_\Gamma(X, C)$. □

1.6 Semi-equivariance and Associated Bundles

This section collects results regarding vector bundles constructed from Γ-semi-equivariant principal (G, θ)-bundles. The construction of these associated bundles differs slightly from the corresponding equivariant construction. When forming an equivariant vector bundle from an equivariant principal bundle, the only requirement on the model fibre is that it carries carries an action of the structure group G. However, when forming a vector bundle from a semi-equivariant principal bundle, it is neccesary to use a model fibre that carries both an action of the structure group G and an action of the equivariance group Γ. As on the semi-equivariant principal bundle, these two actions are required to combine into an action of the semi-direct product group $\Gamma \ltimes_\theta G$. Although the action of the equivariance group G on the model fibre is required to be linear, the action of the equivariance group Γ is not. This makes it possible to construct associated bundles with Γ-actions that are not linear. In particular, it is possible to construct complex vector bundles equipped with linear/anti-linear actions as semi-equivariant associated bundles.

Definition 1.42. Let P be a Γ-semi-equivariant principal (G, θ)-bundle. A *semi-equivariant fibre* for P is a vector space V equipped with a linear action of G and an action of Γ by diffeomorphisms, such that

$$\gamma(gv) = (\gamma g)(\gamma v).$$

Definition 1.43. Let P be a Γ-semi-equivariant principal (G, θ)-bundle, and V be a semi-equivariant fibre for P. The *semi-equivariant associated bundle* is the vector bundle

$$P \times_{(G,\theta)} V := P \times V/\sim$$

where $(p, v) \sim (pg^{-1}, gv)$. This bundle carries an action of Γ defined by

$$\gamma(p, v) := (\gamma p, \gamma v).$$

Note that the Γ-action on $P \times_{(G,\theta)} V$ is well-defined because

$$\gamma[pg^{-1}, gv] = [\gamma(pg^{-1}), \gamma(gv)] = [(\gamma p)(\gamma g)^{-1}, (\gamma g)(\gamma v)] = [\gamma p, \gamma v] = \gamma[p, v].$$

Sections of associated bundles are often represented as equivariant maps from the principal bundle into the model fibre. It is sometimes useful to express the action of Γ on a section in this way.

Lemma 1.44. *Sections of* $P \times_{(G,\theta)} V$ *are in bijective correspondence with maps* $\psi : P \to V$ *such that* $\psi(pg) = g^{-1}\psi(p)$*. The* Γ*-action on sections of* $P \times_{(G,\theta)} V$ *corresponds to the* Γ*-action*

$$(\gamma\psi)(p) = \gamma\psi(\gamma^{-1}p)$$

on these maps.

Proof. A map $\psi : P \to V$ with $\psi(pg) = g^{-1}\psi(p)$ corresponds to the section of $P \times_{(G,\theta)} V$ defined by $s(p) := [p, \psi(p)]$. The Γ-action on such a section is

$$(\gamma s)(p) := \gamma s(\gamma^{-1}p) = \gamma[\gamma^{-1}p, \psi(\gamma^{-1}p)] = [\gamma\gamma^{-1}p, \gamma\psi(\gamma^{-1}p)] = [p, \gamma\psi(\gamma^{-1}p)].$$

Thus, the corresponding map on P is $p \mapsto \gamma\psi(\gamma^{-1}p)$. □

Next, trivialisations of semi-equivariant associated bundles, and their interaction with the Γ-action will be considered.

Definition 1.45. The *trivialisation* $t : (P \times_{(G,\theta)} V) \to V$ associated to a local section $s : U \to P$ is defined, for $e \in (P \times_{(G,\theta)} V)_x$, by

$$e = [s(x), t(e)].$$

In Section 4.2, it will be necessary to examine the symbols of pseudo-differential operators that have anti-linear symmetries. The symbol of an equivariant operator satisfies a corresponding equivariance property. When the symmetries are anti-linear, the factor of i in definition of the Fourier transform causes sign changes in the equivariance property for the symbol. The remaining results of this section begin the calculations needed to explicitly identify this phenomena. The results are stated in terms of a collection of data, which will now be described.

For each semi-equivariant associated bundle

$$(B, \tau^B) := (P^B, \eta^B) \times_{(G^B, \theta^B)} (V^B, \rho^B),$$

over a Γ-space (X, σ), define the following collection of data,

1. $\mathcal{U} := \{U_a\}$, an open cover of X
2. $h := \{h_a : U_a \to \mathbb{R}^n\}$, a collection of smooth coordinate charts for $\mathcal{U}$.
3. $h_{ba}(\gamma, \cdot) := h_b \circ \sigma_\gamma \circ h_a^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, and $h_{ab} := h_{ba}^{-1}$
4. $s^B := \left\{s_a^B : U_a \to P^B|_{U_a}\right\}$, collections of smooth local frames
5. $t^B := \left\{t_a^B : B|_{U_a} \to V^B\right\}$, the local trivialisations associated to h and s^B
6. ϕ^B, the semi-equivariant cocycles defined by s^B.

Using this data, the Γ-actions τ^B can be expressed locally, resulting in the following lemma.

Lemma 1.46. *For* $u \in B_x$,

$$t_b^B \circ \tau_\gamma^B(u) = \phi_{ba}^B(\gamma, x)^{-1} \rho_\gamma^B \circ t_a^B(u).$$

Proof. This can be checked directly by using the definition of the trivialisation t^B, the Γ-action in the associated bundle, and the cocycle ϕ^B,

$$\begin{aligned}[s_b^B(\gamma x), t_b^B(\gamma u)] &= \gamma u \\ &= \gamma[s_a^B(x), t_a^B(u)] \\ &= [\gamma s_a^B(x), \gamma t_a^B(u)] \\ &= [s_b^B(\gamma x)\phi_{ba}^B(\gamma, x), \gamma t_a^B(u)] \\ &= [s_b^B(\gamma x), \phi_{ba}^B(\gamma, x)^{-1}(\gamma t_a^B(u))].\end{aligned}$$

□

Using Lemma 1.46, an equivariance condition for locally defined operators between semi-equivariant associated bundles E and F can be computed. While doing this, it is necessary to keep careful track of the way in which sections are trivialised, and the way in which functions in a trivialisation are pulled back again to local sections.

Lemma 1.47. *Let* $D : C^\infty(E) \to C^\infty(F)$ *be an operator defined locally by operators*

$$D_a : C^\infty(h_a(U_a), V^E) \to C^\infty(h_a(U_a), V^F).$$

The operator D *is equivariant if and only if*

$$D_a\psi_a = \theta^F_{\gamma^{-1}} \circ \phi^F_{ba,\gamma} \circ h_a^{-1} \cdot \rho^F_{\gamma^{-1}} \circ D_b(\phi^{E,-1}_{ba,\gamma} \circ h_a^{-1} \circ h_{ab} \cdot \rho^E_\gamma \circ \psi_a \circ h_{ab}) \circ h_{ba}$$

for all $\gamma \in \Gamma$ *and* $U_b, U_a \in \mathcal{U}$ *such that* $U_a \cap \gamma^{-1}U_b \neq \emptyset$.

Proof. Suppose that the function $\psi_a : h_a(U_a) \subset \mathbb{R}^n \to V^E$ is the local representative of the section $\psi|_{U_a}$. Then, by Lemma 1.46, the local representative of $\gamma\psi|_{U_b}$ is

$$\phi^{E,-1}_{ba,\gamma} \circ h_a^{-1} \circ h_{ab} \cdot \rho^E_\gamma \circ \psi_a \circ h_{ab} : h_b(U_b) \subset \mathbb{R}^n \to V^E.$$

Next, if $\psi_b : h_b(U_b) \subset \mathbb{R}^n \to V^F$ is the local representative of the section $D(\gamma\psi)|_{U_b}$, then the local representative of $\gamma^{-1}D(\gamma\psi)|_{U_a}$ is

$$(\phi^F_{ab,\gamma^{-1}} \circ \sigma^{-1}_{\gamma^{-1}} \circ h_a^{-1})^{-1} \cdot \rho^F_{\gamma^{-1}} \circ \psi_b \circ h_{ba} : h_a(U_a) \subset \mathbb{R}^n \to V^F.$$

The semi-equivariant cocycle property implies that $\phi_{ba}(\gamma, x)^{-1} = \gamma\phi_{ab}(\gamma^{-1}, \gamma x)$. So the representative function for $\gamma^{-1}D(\gamma\psi)|_{U_a}$ becomes

$$\theta^F_{\gamma^{-1}} \circ \phi^F_{ba,\gamma} \circ h_a^{-1} \cdot \rho^F_{\gamma^{-1}} \circ \psi_b \circ h_{ba} : h_a(U_a) \subset \mathbb{R}^n \to V^F.$$

Putting these two together gives the local representation of $\gamma(D(\gamma\psi))|_{U_a}$

$$\theta^F_{\gamma^{-1}} \circ \phi^F_{ba,\gamma} \circ h_a^{-1} \cdot \rho^F_{\gamma^{-1}} \circ D_b(\phi^{E,-1}_{ba,\gamma} \circ \sigma_\gamma^{-1} \circ h_b^{-1} \cdot \rho^E_\gamma \circ \psi_a \circ h_{ab}) \circ h_{ba} : h_a(U_a) \subset \mathbb{R}^n \to V^F.$$

The equivariance condition $(\gamma D) = D$ is equivalent to the statment that the above function is equal to $D_a\psi_a$. □

1.7 Semi-equivariant Connections

In the smooth non-equivariant setting, a connection for a principal G-bundle P can be expressed as a $\mathfrak{g}$-valued 1-form on the tangent space TP, where $\mathfrak{g}$ is the Lie algebra of the structure group G [62, Chapter 2], [37, Appendix B]. A Γ-semi-equivariant (G, θ)-principal bundle has a Γ-group (G, θ) as its structure group. The differentials $(\theta_\gamma)_*$ of the Γ-action on G form a Γ-action on the Lie algebra $\mathfrak{g}$. A connection in a semi-equivariant principal bundle must be compatible with this action if it is to produce an equivariant connection in an associated bundle. The definition of a semi-equivariant connection 1-form is given below, along with an averaging proceedure that can be used to construct semi-equivariant connections. In what follows, let $R_g(p) = R^p(g) := pg$ denote the multiplication maps associated to the right action on a principal G-bundle P. Also, let $R_g(h) := hg$ denote the right action of G on itself. Note that $(R^p)_*(A_e)$ defines the vector field induced on P by an element $A \in \mathfrak{g}$, and the adjoint map on $\mathfrak{g}$ may be expressed as $\mathrm{Ad}_{g^{-1}} = (R_g)_*$.

Definition 1.48. Let (P, η) be a smooth Γ-semi-equivariant principal (G, θ)-bundle with Γ-action η, and let $\mathfrak{g}$ be the Lie alegebra of G. A *Γ-semi-equivariant connection* 1*-form* on P is a Lie algebra valued 1-form

$$\omega : TP \to \mathfrak{g}$$

such that for all $\gamma \in \Gamma$, $g \in G$, $A \in \mathfrak{g}$, and $p \in P$,

$$\omega \circ (R^p)_*(A_e) = A \qquad \omega \circ (R_g)_* = (R_g)_* \circ \omega \qquad \omega \circ (\eta_\gamma)_* = (\theta_\gamma)_* \circ \omega.$$

When Γ is finite, a semi-equivariant connection can be constructed from a given connection by a twisted averaging procedure.

Proposition 1.49. *Let Γ be a finite Lie group, and suppose that P is a smooth Γ-semi-equivariant principal (G, θ)-bundle with Γ-action η. If $\omega : TP \to \mathfrak{g}$ is a connection form on P, then*

$$\omega_\Gamma := \sum_{\gamma \in \Gamma} (\theta_\gamma)_* \circ \omega \circ (\eta_{\gamma^{-1}})_*$$

is a Γ-semi-equivariant connection on P.

Proof. First note that, as θ is an automorphism and P is semi-equivariant, identities are induced between the differentials of the various actions. For $\gamma \in \Gamma$, $g, h \in G$, and $p \in P$

$$\gamma(hg) = (\gamma h)(\gamma g) \implies \quad (\theta_\gamma)_* \circ (R_g)_* = (R_{\gamma g})_* \circ (\theta_\gamma)_*$$

$$\gamma(pg) = (\gamma p)(\gamma g) \implies \begin{cases} (\eta_\gamma)_* \circ (R_g)_* = (R_{\gamma g})_* \circ (\eta_\gamma)_* \\ (\eta_\gamma)_* \circ (R^p)_* = (R^{\gamma p})_* \circ (\theta_\gamma)_*. \end{cases}$$

To check that ω_Γ is a connection, first observe that the condition $\omega_\Gamma \circ (R^p)_*(A_e) = A$ holds,

$$\begin{aligned}
(\theta_\gamma)_* \circ \omega \circ (\eta_{\gamma^{-1}})_* \circ (R^p)_*(A_e) &= (\theta_\gamma)_* \circ \omega \circ (R^{\gamma^{-1}p})_* \circ (\theta_{\gamma^{-1}})_*(A_e) \\
&= (\theta_\gamma) \circ \omega \circ (R^{\gamma^{-1}p})_*((\theta_{\gamma^{-1}})_*(A)_e) \\
&= (\theta_\gamma)_* \circ (\theta_{\gamma^{-1}})_*(A) \\
&= A.
\end{aligned}$$

The condition $\omega_\Gamma \circ (R_g)_* = (R_g)_* \circ \omega_\Gamma$ also holds, as

$$\begin{aligned}
(\theta_\gamma)_* \circ \omega \circ (\eta_{\gamma^{-1}})_* \circ (R_g)_* &= (\theta_\gamma)_* \circ \omega \circ (R_{\gamma^{-1}g})_* \circ (\eta_{\gamma^{-1}})_* \\
&= (\theta_\gamma)_* \circ (R_{\gamma^{-1}g})_* \circ \omega \circ (\eta_{\gamma^{-1}})_* \\
&= (R_g)_* \circ (\theta_\gamma)_* \circ \omega \circ (\eta_{\gamma^{-1}})_*.
\end{aligned}$$

Finally, semi-equivariance holds, as

$$\begin{aligned}
\omega_\Gamma \circ (\eta_\gamma)_* &= (\sum_{\gamma_1 \in \Gamma} (\theta_{\gamma_1})_* \circ \omega \circ (\eta_{\gamma_1^{-1}})_*) \circ (\eta_\gamma)_* \\
&= \sum_{\gamma_1 \in \Gamma} (\theta_{\gamma_1})_* \circ \omega \circ (\eta_{\gamma_1^{-1}\gamma})_* \\
&= \sum_{\gamma_2 \in \Gamma} (\theta_{\gamma\gamma_2^{-1}})_* \circ \omega \circ (\eta_{\gamma_2})_* \\
&= (\theta_\gamma)_* \circ (\sum_{\gamma_2 \in \Gamma} (\theta_{\gamma_2^{-1}})_* \circ \omega \circ (\eta_{\gamma_2})_*) \\
&= (\theta_\gamma)_* \circ \omega_\Gamma.
\end{aligned}$$

□

Chapter 2

Orientifolds

This chapter begins with a discussion of orientifold groups and a review of the theory of linear/anti-linear representations, which will be refered to as *orientifold representations*. Orientifold groups are topological groups equipped with a small amount of extra structure that allows them to act in a linear/anti-linear manner. The representation theory of such actions can be reduced to the theory of unitary representations that are invariant under a conjugate structure on the space of equivalence classes of representations. This reduction is achieved by using the notion of a corepresentation, which coincides precisely with that of a semi-equivariant $(U(n), \kappa_\epsilon)$-valued transition cocycle over a point.

After briefly defining *orientifolds*, *orientifold bundles* will be introduced as complex vector bundles equipped with linear/anti-linear actions. On any orientifold bundle, it is possible to construct a Hermitian metric that is compatible with the linear/anti-linear action. Moreover, the frame bundle of an orientifold bundle is a Γ-semi-equivariant principal $(U(n), \kappa_\epsilon)$-bundle. Thus, a neat generalisation is formed, in which an orientifold bundle over a point is an orientifold representation, and the semi-equivariant $(U(n), \kappa_\epsilon)$-valued transition cocycle of its frame bundle is the corresponding corepresentation. From this perspective, the results of Chapter 1 are a part of a generalised theory of corepresentations.

As with equivariant bundles, orientifold bundles admit a number of natural operations which will be used when considering orientifold K-theory and the symbols of orientifold operators. Semi-equivariant cocycles again prove useful, in Section 2.5, for defining and working with these operations.

2.1 Orientifold Groups

Any group Γ which acts by a combination of linear and anti-linear operators must have an index-2 subgroup of elements which act via linear operators, and a complementary subset of elements which act via anti-linear operators. In general, if Γ contains more than one subgroup of index 2, then the set of orientifold representations of Γ depends on which of these groups is chosen as Γ^+. These facts motivate the definition of an orientifold group.

Definition 2.1. An *orientifold group* (Γ, ϵ) is a Lie group equipped with a non-trivial homomorphism $\epsilon : \Gamma \to \mathbb{Z}_2$. For any orientifold group define $\Gamma^+ := \ker(\epsilon)$ and $\Gamma^- := \Gamma \setminus \ker(\epsilon)$.

Definition 2.2. A *homomorphism* $\varphi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ of orientifold groups is a group homomorphism such that $\epsilon \circ \varphi = \epsilon'$.

The next lemma collects some basic facts about orientifold groups.

Lemma 2.3. *If* (Γ, ϵ) *is an orientifold group, then*

1. $\Gamma^+ \subset \Gamma$ *is a normal subgroup*
2. $\Gamma/\Gamma^+ \simeq \mathbb{Z}_2$
3. $1 \to \Gamma^+ \to \Gamma \xrightarrow{\epsilon} \mathbb{Z}_2 \to 1$ *is an extension of topological groups*
4. $\gamma^2 \in \Gamma^+$ *for all* $\gamma \in \Gamma$
5. $\Gamma = \Gamma^+ \sqcup \Gamma^- = \Gamma^+ \sqcup \zeta\Gamma^+$ *for any* $\zeta \in \Gamma^-$.

The simplest non-trivial example of an orientifold group is provided by $\mathrm{id} : \mathbb{Z}_2 \to \mathbb{Z}_2$. Given an orientifold group, its semi-direct product with a Γ-group can yield another orientifold group.

Lemma 2.4. *Let* $\epsilon : \Gamma \to \mathbb{Z}_2$ *be an orientifold group and* (G, θ) *be a* Γ*-group. Then the group extension*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma^+ \ltimes_\theta G & \xrightarrow{i} & \Gamma \ltimes_\theta G & \xrightarrow{\epsilon\circ\pi_1} & \mathbb{Z}_2 & \longrightarrow & 1 \\ & & (\gamma, g) & \longmapsto & (\gamma, g) & \longmapsto & \epsilon(\gamma) & & \end{array}$$

makes $\Gamma \ltimes_\theta G$ *into an orientifold group. The notation* $(\Gamma, \epsilon) \ltimes_\theta G$ *will be used to denote orientifold groups of this form.*

The following example commonly arises when G is a group of linear operators and κ represents conjugation with respect to a fixed basis.

Example 2.5. Let (G, θ) be a $\mathbb{Z}_2$-group with unit e, then $(\mathbb{Z}_2, \mathrm{id}) \ltimes_\theta G$ is an orientifold group

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \xrightarrow{i} & \mathbb{Z}_2 \ltimes_\theta G & \xrightarrow{\mathrm{id}\circ\pi_1} & \mathbb{Z}_2 & \longrightarrow & 1 \\ & & g & \longmapsto & (z, g) & \longmapsto & z. & & \end{array}$$

Note that the element $(-1, e) \in \Gamma^-$ is an involution,

$$(-1, e)^2 = ((-1)^2, e(-1e)) = (-1^2, e^2) = (+1, e).$$

It is also possible to construct examples in which Γ^- does not contain an involution.

Example 2.6. The Weil group [1, §XV] of $\mathbb{R}$ is the subgroup $\mathbb{C}^\times \sqcup \mathbb{C}^\times j \subset \mathbb{H}^\times$ of the multiplicative group of quaternions. It fits into the non-split extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times \sqcup \mathbb{C}^\times j & \longrightarrow & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \longrightarrow & 1 \\ & & & & j & \longmapsto & -1 & & \end{array}$$

of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}_2$, making it into an orientifold group. Note that there is no element $\zeta \in \mathbb{C}^\times j = \Gamma^-$ such that $\zeta^2 = 1$.

Example 2.7. If $H := \{\pm 1, \pm i\}$ is the orientifold group equipped with the homomorphism $q(h) := h^2$, then $\Gamma := (H, q) \ltimes_\theta G$ is the orientifold group

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} \ltimes_\theta G & \xrightarrow{i} & \{\pm 1, \pm i\} \ltimes_\theta G & \xrightarrow{q\circ\pi_1} & \mathbb{Z}_2 & \longrightarrow & 1 \\ & & (h, g) & \longmapsto & (h, g) & \longmapsto & h^2. & & \end{array}$$

If $(h, g) \in \Gamma^-$, then $h = \pm i$ and $(h, g)^2 = (h^2, g(hg)) = (-1, g(hg)) \in \Gamma^+$. Thus, there is no element $\gamma \in \Gamma^-$ such that $\gamma^2 = (1, e)$.

Given an orientifold group (Γ, ϵ), the parity information provided by ϵ can be used when defining actions on various objects. Three different types of actions of an orientifold group will be distinguished. The first type of action uses the parity information assigned to group elements to dictate whether an element acts linearly or anti-linearly. It will be neccesary to define these actions on a variety of $\mathbb{C}$-modules from different categories, including complex vector spaces, complex vector bundles, and algebras over $\mathbb{C}$. Given objects X and Y in an appropriate category, define

$$\mathrm{Hom}^{+1}(X, Y) := \mathrm{Hom}(X, Y) \qquad \mathrm{Hom}^{-1}(X, Y) := \left\{ a_{\bar{Y}} \circ \varphi \mid \varphi \in \mathrm{Hom}(X, \bar{Y}) \right\},$$

where $a_{\bar{Y}} : \bar{Y} \to Y$ is the identity map on the underlying set for Y. The map $a_{\bar{Y}}$ is anti-linear and the elements of $\mathrm{Hom}^{-1}(X, Y)$ can be considered as anti-linear homomorphisms. The conjugation map $Y \mapsto \bar{Y}$ changes the $\mathbb{C}$-module structure of Y to its conjugate $\mathbb{C}$-module structure, and, depending on the category, it may change other structures on Y. For example, the conjugate of a Hilbert space carries a conjugate inner product. Denote the disjoint union of Hom^+ and Hom^- by $\mathrm{Hom}^\pm$. The spaces $\mathrm{End}^\pm$ and $\mathrm{Aut}^\pm$ are defined similarly.

Definition 2.8. Let (Γ, ϵ) be an orientifold group. An *orientifold action* is a homomorphism $\rho : \Gamma \to \mathrm{Aut}^\pm(W)$ such that

$$\rho(\gamma) \in \mathrm{Aut}^{\epsilon(\gamma)}(W).$$

A second type of action uses an involution ρ to define an action of Γ. Typically, an involution of this type represents the change of some structure to a conjugate structure, occuring in parallel with the application of an orientifold action.

Definition 2.9. An *involutive action* of an orientifold group, is an action of the form

$$\rho \circ \epsilon : \Gamma \to \mathbb{Z}_2 \to \mathrm{Aut}(Y), \tag{2.1}$$

where $\rho : \mathbb{Z}_2 \to \mathrm{Aut}(Y)$ is an involution.

Example 2.10. Some examples of involutive actions are

1. $\iota_\epsilon^{p,q} : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$, where $\mathbb{R}^{p,q} := \mathbb{R}^p \oplus \mathbb{R}^q$ and $\iota^{p,q} : (x, y) \mapsto (x, -y)$.

2. $\kappa_\epsilon : GL(n, \mathbb{C}) \to GL(n, \mathbb{C})$, where κ_ϵ is elementwise conjugation on the standard matrix representation of $GL(n, \mathbb{C})$.

3. $d\theta_\epsilon : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra and $\theta : G \to G$ is an involution on its Lie group.

Of course, the parity of the group elements can also be ignored. This type of action occurs on an orientifold and its tangent bundle. In order to differentiate it from the other types of action, it will be refered to as a *basic* action.

2.2 Orientifold Representations

In this section, the theory of unitary/anti-unitary representations is reviewed. These appear under a variety of names in the literature. They will be refered to here as *orientifold representations*. This theory uses the notion of a corepresentation to show that equivalence classes of orientifold representations for Γ correspond to equivalence classes of representations for Γ^+ which are invariant under *relative conjugation* by an element of Γ^-. Most of the results in this thesis are based on some combination of unitary/anti-unitary representation theory and standard constructions from index theory. From this perspective, there are two main things to observe. First, the index of an elliptic orientifold operator is a formal difference of orientifold representations. Thus, understanding the relationship between the orientifold representations for Γ and the representations of Γ^+ helps to identify the extra information captured by the index of an orientifold operator, as compared with that of an operator which only has linear symmetries. The second thing to observe is that the corepresentations of an orientifold group are semi-equivariant transition cocycles over a point. Together with the upcoming results of Section 2.4, this shows that the results of Chapter 1 form a natural extension of the theory of corepresentations.

Definition 2.11. An *orientifold representation* of an orientifold group $\epsilon : \Gamma \to \mathbb{Z}_2$ is a complex vector space equipped with an orientifold action $\rho : \Gamma \to \mathrm{Aut}^{\pm}(V)$.

Definition 2.12. A homomorphism $\varphi : V \to V'$ of orientifold representations is a linear map satisfying $\varphi(\gamma v) = \gamma\varphi(v)$.

An action by finite dimensional linear operators may be encoded into a matrix representation by allowing it to act on a basis for the representation space. In this case, there is a homomorphism from the original linear representation to the resulting matrix representation. For an orientifold action, the same procedure can be performed to associate a matrix to each element of the group. However, in general, there is not a homomorphism between the matrix group and the original group. The resulting collection of matrices is a *corepresentation* of the group. The concept is due to Wigner [83, pp. 334-335] [51, pp. 169-172].

Definition 2.13. A *corepresentation* of an orientifold group (Γ, ϵ) is a map

$$\phi : \Gamma \to GL(n, \mathbb{C})$$

satisfying $\phi(1) = \mathrm{id}$ and

$$\phi(\gamma'\gamma) = \phi(\gamma')(\gamma'\phi(\gamma)).$$

There are a few points to note. If ϵ is non-trivial, then the map ϕ is not a homomorphism unless $\phi(\gamma) \in GL(n, \mathbb{R})$ for all γ. Also, ϕ depends on the homomorphism $\epsilon : \Gamma \to \mathbb{Z}_2$. Finally, notice that these are exactly the defining properties of a Γ-semi-equivariant $(GL(n, \mathbb{C}), \kappa_\epsilon)$-valued transition cocycle over a point, see Definition 1.8. The appropriate notion of equivalence is also slightly different for a corepresentation, as compared to that of a representation.

Definition 2.14. Two corepresentations ϕ and ϕ' are *equivalent* if there exists a $\mu \in GL(n, \mathbb{C})$ such that

$$\phi'(\gamma) = \mu^{-1}\phi(\gamma)(\gamma\mu).$$

An equivalence of two corepresentations corresponds precisely to an equivalence of their associated Γ-semi-equivariant $(\mathrm{GL}(n, \mathbb{C}), \kappa_\epsilon)$-valued transition cocycles over a point, see Definition 1.9.

Observe that if ϕ is a corepresentation of Γ then $\phi^+ := \phi|_{\Gamma^+}$ is a representation. The following result shows that there is a strong relationship between the corepresentations of Γ and the representations of Γ^+.

Theorem 2.15. *Two corepresentations ϕ and ψ are equivalent if and only if the representations ϕ^+ and ψ^+ are equivalent.*

Proof. See [51, pp. 174-175]. □

This theorem implies that the equivalence class of a corepresentation ϕ is determined by the character of ϕ^+. However, not every representation φ of Γ^+ determines a corepresentation of Γ. To understand which representations of Γ^+ do extend to corepresentations, the operation of relative conjugation is defined on representations of Γ^+. The use of relative conjugation, rather than elementwise conjugation, is neccesary to deal with the case in which Γ^- does not contain an involution.

Definition 2.16. Let $\epsilon : \Gamma \to \mathbb{Z}_2$ be an orientifold group and fix an element $\zeta \in \Gamma^-$. If

$$\phi : \Gamma^+ \to \mathrm{GL}(n, \mathbb{C})$$

is a matrix representation of Γ^+, then the *conjugate of ϕ relative to ζ* is the representation

$$({}^{\zeta}\phi)(\gamma) := \overline{\phi(\zeta^{-1}\gamma\zeta)}.$$

Note that $({}^{\zeta^2}\phi)(\gamma) = \phi(\zeta^2)^{-1}\phi(\gamma)\phi(\zeta^2)$. This implies that conjugation relative to a fixed $\zeta \in \Gamma^-$ is not an involution on the set of representations unless $\zeta^2 = 1$. It is, however, an involution on the set of equivalence classes of representations, as $\phi(\zeta^2)$ provides an equivalence between ${}^{\zeta^2}\phi$ and ϕ.

An irreducible representation φ of Γ^+ can be classified into one of three types based on its relationship to ${}^{\zeta}\varphi$ for some fixed $\zeta \in \Gamma^-$. Suppose that an irreducible representation is equivalent to its relative conjugate. Then, there exists a $\mu \in \mathrm{GL}(n, \mathbb{C})$ such that ${}^{\zeta}\varphi = \mu^{-1}\varphi\mu$. One can show, using Schur's lemma, that $\mu\bar{\mu} = \lambda\varphi(\zeta^2)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. The scalar λ is then used to define the *type* of an irreducible representation relative to the element ζ.

Definition 2.17. The *type* of an irreducible representation φ of Γ^+ is said to be

$$\begin{cases} \textit{real} & \text{if } \lambda > 0 \\ \textit{complex} & \text{if } \varphi \text{ is not equivalent to } \zeta\varphi \\ \textit{quaternionic} & \text{if } \lambda < 0. \end{cases}$$

This definition turns out to be independent of the specific element $\zeta \in \Gamma^-$ chosen. It is also possible to determine the type of a representation in a more direct way from its character [31] [51, pp. 168-169]. Because a corepresentation satisfies

$$(\zeta\phi)(\gamma) = \phi(\zeta^{-1})\phi(\gamma)\phi(\zeta),$$

the representations $\zeta\phi^+$ and ϕ^+ of Γ^+ must always be equivalent. Thus, a complex-type representation φ of Γ^+ does not correspond to ϕ^+ for any corepresentation ϕ of Γ. In general, a complex-type representation must be paired with its conjugate to obtain a representation $\varphi \oplus (\zeta\varphi)$ of Γ^+ coming from a corepresentation of Γ.

By considering the types of representations, it is possible to reconstruct a complete set of irreducible corepresentations for Γ from a complete set of representations for Γ^+.

Theorem 2.18. *Let $\epsilon : \Gamma \to \mathbb{Z}_2$ be a finite orientifold group and $\{\varphi_i\}$ be a complete set of irreducible representations for Γ^+. Then the set of irreducible corepresentations of Γ is determined as follows:*

1. *Each φ_i of real-type determines an irreducible corepresentation by*

$$\phi(\gamma) = \varphi_i(\gamma) \qquad\qquad \phi(\zeta) = \lambda_i^{-\frac{1}{2}}\mu_i$$

2. *Each pair $(\varphi_i, \zeta\varphi_i)$ of complex-type irreducible representations determines a single irreducible corepresentation by*

$$\phi(\gamma) = \begin{pmatrix} \varphi_i(\gamma) & 0 \\ 0 & (\zeta\varphi_i)(\gamma) \end{pmatrix} \qquad\qquad \phi(\zeta) = \begin{pmatrix} 0 & \varphi_i(\zeta^2) \\ \mathrm{id} & 0 \end{pmatrix}$$

3. *Each φ_i of quaternionic-type determines an irreducible corepresentation by*

$$\phi(\gamma) = \begin{pmatrix} \varphi_i(\gamma) & 0 \\ 0 & \varphi_i(\gamma) \end{pmatrix} \qquad\qquad \phi(\zeta) = |\lambda_i|^{-\frac{1}{2}} \begin{pmatrix} 0 & \mu_i \\ -\mu_i & 0 \end{pmatrix}$$

where $\mu_i \in GL(n, \mathbb{C})$ satisfies $\zeta\varphi_i = \mu_i^{-1}\varphi\mu_i$, and $\lambda_i \in \mathbb{R}\setminus\{0\}$ satisfies $\mu_i\bar{\mu}_i = \lambda_i\varphi(\zeta^2)$. The set of equivalence classes of irreducible representations determined does not depend on the choice of $\zeta \in \Gamma^-$.

Proof. See [51, pp. 176-181]. ☐

Remark 2. The theory of orientifold representations shows that the index of an elliptic orientifold operator can be considered as a difference of equivalence classes of representations for Γ^+ each of which is invariant under relative conjugation by elements of Γ^-.

2.3 Orientifolds

In order to maintain a clear focus on anti-linearity in index theory, only the simplest definition of an orientifold will be treated. These orientifolds are essentially global quotient orbifolds with a small amount of extra structure. Using the language of Section 2.1, they could be described as manifolds equipped with a basic action of an orientifold group. The origin of the term orientifold is in string theory, where orientifolds are often considered to have a sign choice ± 1 associated to the connected components of their fixed point sets. However, these sign choice structures will not be considered here.

Definition 2.19. An *orientifold* is a compact manifold X equipped with a smooth action

$$\rho : \Gamma \to \mathrm{Diff}(X),$$

where Γ is a finite orientifold group. The category of orientifolds with orientifold group $\epsilon : \Gamma \to \mathbb{Z}_2$ will be denoted $\mathrm{Ori}_{(\Gamma,\epsilon)}$.

Example 2.20. Let Γ be any orientifold group. Then $\mathbb{R}^{p,q} := \mathbb{R}^p \oplus \mathbb{R}^q$ equipped with the involutive action induced by $(x, y) \mapsto (x, -y)$ is an orientifold. This orientifold will be used to form suspensions in orientifold K-theory.

Example 2.21. Let $X \in \mathrm{Ori}_{(\Gamma,\epsilon)}$ with Γ-action σ. The tangent bundle TX equipped with the basic Γ-action $d\sigma$ is again an orientifold. The K-theory of this orientifold will be the target space of the 8-fold Thom isomorphism for orientifold K-theory.

The category of real vector bundles equipped with a basic action of the orientifold group (Γ, ϵ) will be denoted $\mathrm{Vect}_{(\Gamma,\epsilon)}(X, \mathbb{R})$. The isomorphism classes of such bundles will be denoted $\mathrm{Vect}^{\cong}_{(\Gamma,\epsilon)}(X, \mathbb{R})$.

2.4 Orientifold Bundles

Orientifold bundles are the main object of interest in the study of orientifolds. In the language of Section 2.1, they are complex vector bundles carrying orientifold actions that cover the action on the base orientifold.

Definition 2.22. If $\pi : E \to X$ is a complex vector bundle, define $\mathrm{Aut}_{\mathrm{Diff}}(E)$ to be the set of maps $\varphi : E \to E$ such that

1. $\pi \circ \varphi(e) = f \circ \pi(e)$, for some diffeomorphism $f \in \mathrm{Diff}(X)$ and all $e \in E$.

2. $\varphi : E_x \to E_{f(x)}$ is a linear bijection, for all x.

Definition 2.23. An *orientifold bundle* $\pi : E \to X$ is a complex vector bundle equipped with an orientifold action

$$\tau : \Gamma \to \mathrm{Aut}^{\pm}_{\mathrm{Diff}}(E)$$

such that $\pi(\gamma v) = \gamma\pi(v)$.

The category of orientifold bundles over $X \in \mathrm{Ori}_{(\Gamma,\epsilon)}$ will be denoted $\mathrm{Vect}_{(\Gamma,\epsilon)}(X, \mathbb{C})$. The set of isomorphism classes of orientifold bundles will be denoted $\mathrm{Vect}^{\simeq}_{(\Gamma,\epsilon)}(X, \mathbb{C})$.

Example 2.24. An orientifold representation (V, ρ) can be considered as an orientifold bundle over a point. If (X, σ) is an orientifold then an orientifold bundle of the form

$$(X \times V, \sigma \times \rho)$$

will be described as a *trivial* orientifold bundle.

Note that if ϵ is non-trivial, then every orientifold bundle for (Γ, ϵ) carries at least one anti-linear map, and so there is no orientifold bundle (E, τ) such that $\tau_\gamma = \mathrm{id}$ for all $\gamma \in \Gamma$.

Just as in the equivariant setting, it is possible to average an hermetian metric on an orientifold bundle to make it compatible with the orientifold action. The averaging process needs to be twisted with conjugation to account for the anti-linearity of the action, as does the compatibility condition.

Definition 2.25. An *orientifold metric* on an orientifold bundle E is an hermitian metric h on E such that, for all $v_1, v_2 \in E$ and $\gamma \in \Gamma$,

$$h(\gamma v_1, \gamma v_2)_{\gamma x} = \gamma h(v_1, v_2)_x.$$

Proposition 2.26. *Every orientifold vector bundle E over a paracompact orientifold X carries an orientifold metric.*

Proof. It is a standard result that every complex vector bundle over a paracompact space carries an hermitian metric [78, Lemma 2]. Given an hermitian metric h on an orientifold bundle E, define

$$h_\Gamma(u, v)_x = \sum_{\gamma \in \Gamma} \gamma^{-1} h(\gamma u, \gamma v)_{\gamma x}.$$

This metric is an orientifold metric as

$$\begin{aligned} h_\Gamma(\gamma u, \gamma v)_{\gamma x} &= \sum_{\gamma' \in \Gamma} \gamma'^{-1} h(\gamma'\gamma u, \gamma'\gamma v)_{\gamma'\gamma x} \\ &= \sum_{\gamma'' := \gamma'\gamma \in \Gamma} \gamma\gamma''^{-1} h(\gamma'' u, \gamma'' v)_{\gamma'' x} = \gamma \sum_{\gamma'' \in \Gamma} \gamma''^{-1} h(\gamma'' u, \gamma'' v)_{\gamma'' x} = \gamma h_\Gamma(u, v)_x. \end{aligned}$$

□

Using an orientifold metric it is possible to split sequences of orientifold bundles.

Corollary 2.27. *Let* X *be a paracompact orientifold. If*

$$0 \to E' \xrightarrow{\varphi'} E \xrightarrow{\varphi} E''$$

is an exact sequence of orientifold bundles over X*, then* $E \simeq E' \oplus E''$*.*

Proof. By Proposition 2.26, there exists an orientifold metric h on E. It is a standard result [78, Proposition 2] that h determines a projection $p : E \to E$ and a splitting of complex vector bundles $E = \operatorname{im}(p) \oplus \ker(p) \simeq E' \oplus E''$. The projection p is defined fibrewise by

$$\begin{aligned} p_x : E_x &\to E_x \\ v &\mapsto \sum_i \frac{h(v, b_i)_x}{h(b_i, b_i)_x} b_i, \end{aligned}$$

where $\{b_i\}$ is any basis for $\varphi'(E')_x$. Therefore, if $p_x(v) = 0$, then $h(v, b_i)_x = 0$ for all i, and

$$p_{\gamma x}(\gamma v) = \sum_i \frac{h(\gamma v, \gamma b_i)_{\gamma x}}{h(\gamma b_i, \gamma b_i)_{\gamma x}}(\gamma b_i) = \sum_i \frac{\gamma h(v, b_i)_x}{\gamma h(b_i, b_i)_x}(\gamma b_i) = \sum_i \frac{\gamma 0}{\gamma h(b_i, b_i)_x}(\gamma b_i) = 0.$$

Thus, $\ker(p)$ is invariant under the action of Γ, as is the given splitting. □

Next, the frame bundle of an orientifold bundle will be examined.

Definition 2.28. The *frame bundle* $\mathrm{Fr}(E)$ of an orientifold bundle E is the principal $GL(n, \mathbb{C})$-bundle of frames for the total space of E, equipped with a left Γ-action defined on a frame $s = (s_1, \ldots, s_n) \in \mathrm{Fr}(E)_x$ by $(\gamma s)_i = \gamma s_i$.

Although the frame bundle of an orientifold is defined in the same manner as that of an equivariant bundle, the anti-linearity present in the Γ-action gives it different properties. In particular, there is a mild noncommutivity between the left action of Γ and the right action of the structure group $GL(n, \mathbb{C})$. This non-commutivity makes the frame bundle of an orientifold bundle into a semi-equivariant principal bundle.

Proposition 2.29. *Let* E *be an orientifold bundle and consider* $GL(n, \mathbb{C})$ *to be equipped with the involutive action of* (Γ, ϵ) *induced by conjugation. Then,*

$$\mathrm{Fr}(E; GL(n, \mathbb{C})) \in PB_{(\Gamma,\epsilon)}(X, (GL(n, \mathbb{C}), \kappa_\epsilon)).$$

In particular, the left and right actions on the frame bundle satisfy

$$\gamma(sg) = (\gamma s)(\gamma g),$$

for $\gamma \in \Gamma$*,* $s \in \mathrm{Fr}(E)$ *and* $g \in GL(n, \mathbb{C})$*.*

Proof. The action of g on a frame s is given by $(sg)_j = \sum_{1\le i\le n} s_i g_{ij}$. Thus,

$$\gamma(sg)_j = \sum_{1\le i\le n} \gamma(s_i g_{ij}) = \sum_{1\le i\le n} (\gamma s_i)(\gamma g_{ij}) = \sum_{1\le i\le n} (\gamma s)_i(\gamma g)_{ij} = ((\gamma s)(\gamma g))_j.$$

□

Note that, by using an orientifold metric, the structure group can always be reduced to $(U(n), \kappa_\epsilon)$, where κ_ϵ is the action induced on $U(n)$ by its inclusion into $GL(n, \mathbb{C})$.

2.5 Operations on Orientifold Bundles

Some basic operations on orientifold bundles will now be defined. It will be useful to make these definitions in terms of semi-equivariant cocycles. To start with, consider the following operations on Γ-groups.

Definition 2.30. Let $a^k \in GL(\mathbb{C}^{m_k})$, and denote by $[a_{ij}]$ the matrix representation of an element $a \in GL(\mathbb{C}^m)$ with respect to the standard basis of $\mathbb{C}^m$. Define the following operations

1. The *dual* $a^* \in GL(\mathbb{C}^m)$,

$$[(a^*)_{ij}] := ([a_{ij}]^t)^{-1}$$

2. The *direct sum* $a^1 \oplus a^2 \in GL(\mathbb{C}^{m_1+m_2})$,

$$[(a^1 \oplus a^2)_{ij}] := \begin{pmatrix} [a^1_{ij}] & 0 \\ 0 & [a^2_{ij}] \end{pmatrix}.$$

3. The *tensor product* $a^1 \otimes a^2 \in GL(\mathbb{C}^{m_1 m_2})$,

$$[(a^1 \otimes a^2)_{ij}] := \begin{pmatrix} a^1_{11}[a^2_{ij}] & \dots & a^1_{1m}[a^2_{ij}] \\ \vdots & \ddots & \vdots \\ a^1_{m1}[a^2_{ij}] & \dots & a^1_{mm}[a^2_{ij}] \end{pmatrix}.$$

Examining Definition 2.30, it is clear that the dual, direct sum and tensor product on the groups $GL(\mathbb{C}^m)$ are compatible with involutive Γ-actions induced by conjugation.

Lemma 2.31. *The dual, direct sum and tensor product operations are homomorphisms*

$$* : (GL(\mathbb{C}^m), \kappa_\epsilon) \to (GL(\mathbb{C}^m), \kappa_\epsilon)$$
$$\oplus : (GL(\mathbb{C}^{m_1}), \kappa_\epsilon) \times (GL(\mathbb{C}^{m_2}), \kappa_\epsilon) \to (GL(\mathbb{C}^{m_1+m_2}), \kappa_\epsilon)$$
$$\otimes : (GL(\mathbb{C}^{m_1}), \kappa_\epsilon) \times (GL(\mathbb{C}^{m_2}), \kappa_\epsilon) \to (GL(\mathbb{C}^{m_1 m_2}), \kappa_\epsilon)$$

of Γ-groups.

Lemma 2.31, allows the dual, direct sum and tensor product of $(GL(m,\mathbb{C}),\kappa_\epsilon)$-valued cocycles to be defined in the obvious way. Pullbacks of cocycles can also be defined. It is routine to prove that these satisfy the semi-equivariant cocycle condition.

Definition 2.32. Let $\phi^i \in TC_{(\Gamma,\epsilon)}(\mathcal{U}, X, (GL(\mathbb{C}^{m_i}),\kappa_\epsilon))$ and $f : X \to Y$ be a homomorphism orientifolds. Define the following operations on cocycles

Pullback	$(f^*\phi)_{ba}(\gamma, x) := \phi_{ba}(\gamma, f(x))$	$\in TC_{(\Gamma,\epsilon)}(f^*\mathcal{U}, Y, (GL(\mathbb{C}^m),\kappa_\epsilon))$
Dual	$(\phi^*)_{ba}(x,\gamma) := \phi_{ba}(x,\gamma)^*$	$\in TC_{(\Gamma,\epsilon)}(\mathcal{U}, X, (GL(\mathbb{C}^m),\kappa_\epsilon))$
Direct sum	$(\phi^1 \oplus \phi^2)_{ba}(x,\gamma) := \phi^1_{ba}(x,\gamma) \oplus \phi^2_{ba}(x,\gamma)$	$\in TC_{(\Gamma,\epsilon)}(\mathcal{U}, X, (GL(\mathbb{C}^{m_1+m_2}),\kappa_\epsilon))$
Tensor product	$(\phi^1 \otimes \phi^2)_{ba}(x,\gamma) := \phi^1_{ba}(x,\gamma) \otimes \phi^2_{ba}(x,\gamma)$	$\in TC_{(\Gamma,\epsilon)}(\mathcal{U}, X, (GL(\mathbb{C}^{m_1 m_2}),\kappa_\epsilon))$,

where $f^*\mathcal{U} := \left\{f^{-1}(U_a) \mid a \in A\right\}$ is the pullback of the cover $\mathcal{U} := \{U_a \mid a \in A\}$.

The above operations on cocycles induce operations on orientifold bundles via the semi-equivariant associated bundle construction, see Definition 1.43.

Definition 2.33. Let $E_i \in \mathrm{Vect}^{m_i}_{(\Gamma,\epsilon)}(X,\mathbb{C})$. Let ϕ^i denote a semi-equivariant cocycle associated $\mathrm{Fr}(E_i)$ by Proposition 1.12, and P^ϕ denote the semi-equivariant principal bundle constructed from a cocycle ϕ via Proposition 1.15. Define the following operations on orientifold bundles

Pullback	$f^*E := P^{f^*\phi} \times_{(GL(m,\mathbb{C}),\kappa_\epsilon)} (\mathbb{C}^m,\kappa_\epsilon)$	$\in \mathrm{Vect}^m_{(\Gamma,\epsilon)}(X,\mathbb{C})$
Dual	$E^* := P^{\phi^*} \times_{(GL(m,\mathbb{C}),\kappa_\epsilon)} ((\mathbb{C}^m)^*,\kappa_\epsilon)$	$\in \mathrm{Vect}^m_{(\Gamma,\epsilon)}(X,\mathbb{C})$
Direct sum	$E_1 \oplus E_2 := P^{\phi_1\oplus\phi_2} \times_{(GL(m_1+m_2,\mathbb{C}),\kappa_\epsilon)} (\mathbb{C}^{m_1+m_2},\kappa_\epsilon)$	$\in \mathrm{Vect}^{m_1+m_2}_{(\Gamma,\epsilon)}(X,\mathbb{C})$
Tensor product	$E_1 \otimes E_2 := P^{\phi_1\otimes\phi_2} \times_{(GL(m_1 m_2,\mathbb{C}),\kappa_\epsilon)} (\mathbb{C}^{m_1 m_2},\kappa_\epsilon)$	$\in \mathrm{Vect}^{m_1 m_2}_{(\Gamma,\epsilon)}(X,\mathbb{C})$,

where $\kappa_\epsilon : (\mathbb{C}^m)^* \to (\mathbb{C}^m)^*$ is the action defined by $(\gamma\lambda)(z) := \gamma\lambda(\gamma^{-1}z)$.

As in the non-equivariant setting, it is possible to construct the bundle of homomorphisms between two orientifold bundles using their tensor products and duals. This will be of interest when investigating the symbols of orientifold operators.

Proposition 2.34. *Let* $E_i \in \mathrm{Vect}^{m_i}_{(\Gamma,\epsilon)}(X,\mathbb{C})$. *The homomorphisms* $\varphi \in \mathrm{Hom}(E_1,E_2)$ *are in bijective correspondence with equivariant sections of the orientifold bundle* $E_2 \otimes E_1^*$.

In order to define the Thom homomorphism in Chapter 4, it is neccesary to define the external tensor product for orientifold bundles. This is a notion of tensor product between vector bundles over different base spaces.

Definition 2.35. Let $E_i \in \mathrm{Vect}^{m_i}_{(\Gamma,\epsilon)}(X_i, \mathbb{C})$, and $\pi_i : X_1 \times X_2 \to X_i$ be the coordinate projection for $i \in \{1,2\}$. The *external tensor product* is defined by

$$E_1 \boxtimes E_2 := \pi_1^* E_1 \otimes \pi_2^* E_2 \in \mathrm{Vect}^{m_1 m_2}_{(\Gamma,\epsilon)}(X_1 \times X_2, \mathbb{C}).$$

The most important application of the external tensor product occurs when when X_1 and X_2 are vector bundles over a common orientifold X. More specifically, when $X_i \in \mathrm{Vect}_{(\Gamma,\epsilon)}(X, \mathbb{F}_i)$ for $\mathbb{F}_i \in \{\mathbb{R}, \mathbb{C}\}$. In this case, a diagonal restriction map Δ_* is defined, which, when composed with the external tensor product, yields an action of $\mathrm{Vect}_{(\Gamma,\epsilon)}(X, \mathbb{C})$ on $\mathrm{Vect}_{(\Gamma,\epsilon)}(V_1, \mathbb{C})$.

Definition 2.36. Let $V_i \in \mathrm{Vect}_{(\Gamma,\epsilon)}(X, \mathbb{F}_i)$ with projections π_i, and $F \in \mathrm{Vect}^{m}_{(\Gamma,\epsilon)}(V_1 \times V_2, \mathbb{C})$. The *diagonal restriction* of F is defined by

$$\Delta_* F := \left\{ F_{(v,w)} \mid \pi_1(v) = \pi_2(w) \right\} \in \mathrm{Vect}^{m}_{(\Gamma,\epsilon)}(V_1 \oplus V_2, \mathbb{C}).$$

Definition 2.37. Let $E_i \in \mathrm{Vect}^{m_i}_{(\Gamma,\epsilon)}(V_i, \mathbb{C})$. Then

$$E_1 E_2 := \Delta_*(E_1 \boxtimes E_2) \in \mathrm{Vect}^{m_1 m_2}_{(\Gamma,\epsilon)}(V_1 \oplus V_2, \mathbb{C}).$$

In particular, if $V_2 = X$ is the vector bundle with zero-dimensional fibres, then $E_1 E_2 \in \mathrm{Vect}^{m_1 m_2}_{(\Gamma,\epsilon)}(V_1, \mathbb{C})$, and this action is a right action of $\mathrm{Vect}^{m_2}_{(\Gamma,\epsilon)}(X, \mathbb{C})$ on the semi-group $\mathrm{Vect}^{m_1}_{(\Gamma,\epsilon)}(V_1, \mathbb{C})$. A left action of $\mathrm{Vect}_{(\Gamma,\epsilon)}(X, \mathbb{C})$ may be defined similarly.

Note that the two possibilities, $\mathbb{F}_i = \mathbb{R}$ or $\mathbb{C}$, correspond to an action on bundles defined over an equivariant real bundle or an orientifold bundle, respectively. These actions induce module structures on the orientifold K-theory groups, and the two different cases are used to define the two different types of Bott periodicity which exist in orientifold K-theory.

Another construction which is imporant in K-theory is that of perpendicular bundles.

Definition 2.38. Let E be an orientifold bundle. A *perpendicular bundle* for E is an orientifold bundle F such that $E \oplus F$ is a trivial orientifold bundle.

In the case where Γ is finite and X is compact, the proof that perpendicular bundles always exist extends to orientifolds bundles. This result makes use of the standard orientifold action on the vector space of sections s of an orientifold bundle, which is defined by

$$(\gamma s)(x) := \gamma s(\gamma^{-1} x).$$

Lemma 2.39. *Let X be a compact orientifold with a finite orientifold group (Γ, ϵ), and $E \to X$ be an orientifold bundle. There is a finite dimensional orientifold representation $(V, \rho) \subset C(X, E)$ such* *that the evaluation map*

$$\varphi : X \times V \to E$$
$$(x, s) \mapsto s(x)$$

is a surjective map of orientifold bundles.

Proof. A subspace of $C(X, E)$ with surjective evaluation map is called an *ample* subspace. It is a standard result [3, Lemma 1.4.12] that a finite dimensional ample subspace $V \subset C(X, E)$ may be constructed for any complex vector bundle E. The space

$$V_\Gamma := \bigoplus_{\gamma \in \Gamma} \gamma V \subset C(X, E)$$

is then finite dimensional, ample, and invariant under the action of Γ on sections. Thus, $X \times V$ is an orientifold bundle when equipped with the action $(x, s) \to (\gamma x, \gamma s)$. The evaluation map is equivariant with respect to this action as

$$\varphi(\gamma x, \gamma s) = (\gamma s)(\gamma x) = \gamma s(\gamma^{-1} \gamma x) = \gamma s(x) = \gamma \varphi(x, s).$$

□

Corollary 2.40. *If X is a compact orientifold with a finite orientifold group (Γ, ϵ) and $E \to X$ is an orientifold bundle, then there exists a perpendicular orientifold bundle for E*

Proof. The required bundle is $F = \ker(\varphi)$, where φ is the evaluation map as in Lemma 2.39. □

Chapter 3

The Orientifold Dirac Operator

In this chapter, Dirac operators are constructed for orientifolds. First, $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures are defined. Using results from Chapter 1, these structures are classified, and shown to decompose into $\mathrm{Spin}(n)$ and $(U(1), \kappa_\epsilon)$ components. By applying the semi-equivariant associated bundle construction with a Clifford module as the model fibre, it is possible to construct spinor bundles with orientifold actions. Both a total spinor bundle, with a right action of $(\mathbb{C}l_n, \kappa_\epsilon)$, and a reduced spinor bundle, with the complexification of an irreducible Cl_{8k}-module as a model fibre, are defined. As in the usual setting, the sections of orientifold spinor bundles are acted on by sections of a Clifford bundle. This action is compatible with the orientifold action on the spinor bundle and a canonical orientifold action on the complex Clifford bundle. In order to construct a Dirac operator on an orientifold, it is neccesary to have a connection which is compatible with Clifford multiplication on sections and the orientifold action. Such a connection can be constructed using the results on semi-equivariant connection forms from Section 1.4. After equipping the orientifold spinor bundles with compatible connections, the orientifold Dirac operator and its reduced counterpart will be defined.

3.1 Classification of Orientifold Spin^c-structures

In order to define and classify Spin^c-structures for orientifolds, it is neccesary to consider the interaction of Clifford algebras and the Spin groups with orientifold actions. The idea is to complexify results which apply to real Clifford algebras, whilst keeping track of the associated conjugation maps. These maps can then be used to define involutive actions of orientifold groups. To begin, the definitions of the real Clifford algebra, Spin group, and adjoint map are recalled.

Definition 3.1. The Clifford algebra Cl_n is the algebra generated by the standard basis $\{e_i\}$ of $\mathbb{R}^n$ subject to the relations $e_i^2 = -1$ and $e_i e_j + e_j e_i = 0$.

Note that the set $\{e_{i_1} \cdots e_{i_k} \in Cl_n \mid i_1 < \cdots < i_k\}$ is a basis for Cl_n. The group $Spin(n)$ sits inside Cl_n. Elements of $Spin(n)$ are products of an even number of unit vectors from $\mathbb{R}^n$.

Definition 3.2. The group $Spin(n)$ is defined by

$$Spin(n) := \{x_1 \cdots x_{2k} \mid x_i \in \mathbb{R}^n, \|x_i\| = 1\} \subset Cl_n.$$

If $g \in Spin(n)$ and $x \in \mathbb{R}^n$ one can show that $gxg^{-1} \in \mathbb{R}^n$. The transformation $x \mapsto gxg^{-1}$ defines an element of $SO(n)$, and the resulting assignment $Spin(n) \to SO(n)$ is a double covering.

Definition 3.3. The *adjoint map* $Ad : Spin(n) \to SO(n)$ is defined, for $g \in Spin(n)$, $x \in \mathbb{R}^n$, by

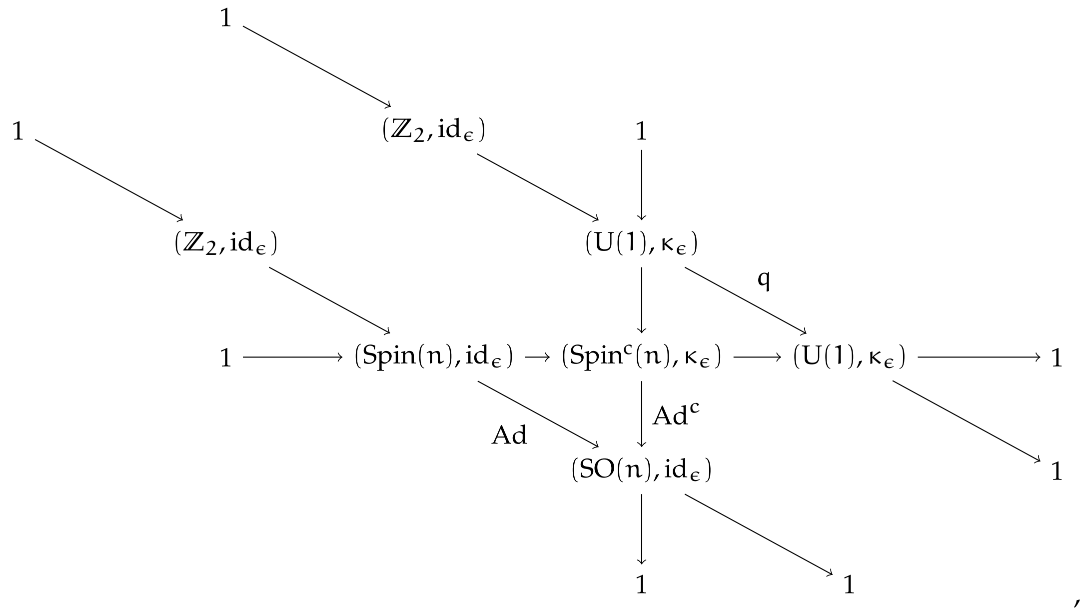
$$Ad_g(x) := gxg^{-1}.$$

For applications to orientifolds, it is neccesary to work with the complexifications of Cl_n and $Spin(n)$. These complexifications are equipped with conjugation maps which induce involutive actions of orientifold groups. The complexified adjoint map is a homomorphism of Γ-groups.

Definition 3.4. Let (Γ, ϵ) be an orientifold group and define the following

1. $(\mathbb{C}l_n, \kappa_\epsilon) := Cl_n \otimes \mathbb{C}$ with the Γ-action $\kappa_\epsilon(\varphi \otimes z) := \varphi \otimes \kappa_\epsilon(z)$
2. $(Spin^c(n), \kappa_\epsilon) := (Spin(n) \times U(1)) / \{\pm(1,1)\}$ with the induced action $\kappa_\epsilon[g,z] := [g, \kappa_\epsilon(z)]$
3. $Ad^c : (Spin^c(n), \kappa_\epsilon) \to (SO(n), id_\epsilon)$ defined by $Ad^c[g,z] := Ad(g)$.

Note that $Ad^c \circ \kappa_\epsilon[g,z] = Ad^c[g,z]$. The properties of Ad^c, and the decomposition of $Spin^c(n)$, produce two central exact sequences of Γ-groups about $Spin^c(n)$. These sequences fit into the following diagram



(3.1)

where q is the square map. The above sequences will be used to classify Spin^c-structures for orientifolds.

Having examined semi-equivariance, orientifolds, and orientifold actions on $\mathrm{Spin}^c(n)$, it is now possible to define a notion of Spin^c-structure which is appropriate for orientifolds.

Definition 3.5. An $(\mathrm{Spin}^c, \kappa_\epsilon)$-*structure* for a real Γ-equivariant vector bundle V over an orientifold is a semi-equivariant lifting $\varphi : P \to \mathrm{Fr}(V)$ by $\mathrm{Ad}^c : (\mathrm{Spin}^c(n), \kappa_\epsilon) \to (\mathrm{SO}(n), \mathrm{id}_\epsilon)$.

If V has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure, then it is said to be $(\mathrm{Spin}^c, \kappa_\epsilon)$-*oriented*. If the tangent bundle TM of an orientifold M is $(\mathrm{Spin}^c, \kappa_\epsilon)$-oriented, then M is said to be $(\mathrm{Spin}^c, \kappa_\epsilon)$-*oriented*.

The $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures associated to a vector bundle V can be classified using the result of Chapter 1. The following theorem is obtained by applying Theorem 1.41 to the central exact sequence running vertically in (3.1).

Theorem 3.6. *The central exact sequence*

$$1 \to (U(1), \kappa_\epsilon) \to (\mathrm{Spin}^c(n), \kappa_\epsilon) \overset{\mathrm{Ad}^c}{\to} (\mathrm{SO}(n), \mathrm{id}_\epsilon) \to 1,$$

induces an exact sequence

$$H^1_\Gamma(X, (U(1), \kappa_\epsilon)) \to TC^1_\Gamma(X, (\mathrm{Spin}^c(n), \kappa_\epsilon)) \overset{\mathrm{Ad}^c}{\to} TC^1_\Gamma(X, (\mathrm{SO}(n), \mathrm{id}_\epsilon)) \overset{\Delta_{sc}}{\to} H^2_\Gamma(X, (U(1), \kappa_\epsilon)).$$

Theorem 3.6 has the following corollaries, which classify $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures in terms of semi-equivariant cohomology with coefficients in $(U(1), \kappa_\epsilon)$.

Corollary 3.7. *A real Γ-equivariant vector bundle V over an orientifold has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure if and only if $\Delta_{sc}(\phi^V) = 1$, where ϕ^V is the transition cocycle for V.*

Corollary 3.8. *A given* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure is unique up to tensoring by semi-equivariant principal* $(\mathrm{U}(1), \kappa_\epsilon)$*-bundles.*

To obtain an obstruction class with integer coefficients, involutive actions can be taken on the groups in the exponential exact sequence. This results in the following proposition.

Lemma 3.9. *The exponential exact sequence*

$$0 \to (\mathbb{Z}, \iota_\epsilon) \to (\mathbb{R}, \iota_\epsilon) \overset{\exp}{\to} (\mathrm{U}(1), \kappa_\epsilon) \to 1 \tag{3.2}$$

induces isomorphisms

$$H^p_\Gamma(X, (\mathrm{U}(1), \kappa_\epsilon)) \overset{\Delta^p_{\exp}}{\to} H^{p+1}_\Gamma(X, (\mathbb{Z}, \iota_\epsilon)),$$

where ι_ϵ *is the involutive orientifold action induced by the map* $t \mapsto -t \in \mathbb{R}$*.*

Proof. By Theorem 1.38, the exact sequence (3.2) induces a long exact sequence

$$H^p_\Gamma(X, (\mathbb{Z}, \iota_\epsilon)) \to H^p_\Gamma(X, (\mathbb{R}, \iota_\epsilon)) \overset{\exp}{\to} H^p_\Gamma(X, (\mathrm{U}(1), \kappa_\epsilon)) \overset{\Delta^p_{\exp}}{\to} H^{p+1}_\Gamma(X, (\mathbb{Z}, \iota_\epsilon)).$$

The existence of a smooth partition of unity on X implies that $H^p_\Gamma(X, (\mathbb{R}, \iota_\epsilon)) = 0$ for all p. Therefore, the maps $\Delta^p_{\exp}$ are isomorphisms. □

Using Proposition 3.2, it is possible to define an analogue of the third integral Stiefel-Whiney class.

Definition 3.10. The *third integral orientifold Stiefel-Whiney class* is defined by

$$W_3^{(\Gamma,\epsilon)}(V) := \Delta_{\exp} \circ \Delta_{sc}(\phi^V) \in H^3_\Gamma(X, (\mathbb{Z}, \iota_\epsilon)),$$

where ϕ^V is the transition cocycle associated to V.

Corollaries 3.7 and 3.8 can then be restated in terms of semi-equivariant cohomology with coefficients in $(\mathbb{Z}, \iota_\epsilon)$.

Corollary 3.11. *A real* Γ*-equivariant bundle* V *is* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-oriented if and only if* $W_3^{(\Gamma,\epsilon)}(V) = 0$*.*

Corollary 3.12. *The* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structures on a* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-oriented real* Γ*-equivariant vector bundle are in bijective correspondence with the elements of* $H^2_\Gamma(X, (\mathbb{Z}, \iota_\epsilon))$*.*

It is possible to further isolate the semi-equivariance in a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure by splitting it via the decomposition

$$(\mathrm{Spin}^c(n), \kappa) \simeq (\mathrm{SO}(n), \mathrm{id}) \times_{\mathbb{Z}_2} (\mathrm{U}(1), \kappa).$$

This decomposition immediately implies that, for any cochain $\phi_{sc} \in K^1_\Gamma(X, (\mathrm{Spin}^c(n), \kappa_\epsilon))$, there exist cochains $\phi_s \in K^1_\Gamma(X, (\mathrm{Spin}(n), \mathrm{id}_\epsilon))$ and $\phi_u \in K^1_\Gamma(X, (U(1), \kappa_\epsilon))$ such that $\phi_{sc} = [\phi_s, \phi_u]$. It also allows the definition of the map

$$\begin{aligned} \mathrm{Ad} \times q : (\mathrm{Spin}^c(n), \kappa_\epsilon) &\to (SO(n), \mathrm{id}_\epsilon) \times (U(1), \kappa_\epsilon) \\ [s, z] &\mapsto (\mathrm{Ad}(s), q(z)). \end{aligned}$$

The next proposition shows that every $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure extends to a lifting of a semi-equivariant principal $(SO(n), \mathrm{id}) \times (U(1), \kappa)$-bundle by $\mathrm{Ad} \times q$.

Proposition 3.13. *If $\varphi_0 : P \to Q$ is a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure, then there exists a lifting*

$$\varphi : P \to Q \times_X L \tag{3.3}$$

by $\mathrm{Ad} \times q$, where L is a Γ-semi-equivariant principal $(U(1), \kappa_\epsilon)$-bundle.

Proof. Let $\phi \in TC^1_\Gamma(X, (SO(n), \mathrm{id}_\epsilon))$ be the cocycle for Q. If Q has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure there is a cocycle $[\phi_s, \phi_u] \in TC^1_\Gamma(X, (\mathrm{Spin}^c(n), \kappa_\epsilon))$ with $\mathrm{Ad}^c([\phi_s, \phi_u]) = \mathrm{Ad}(\phi_s) = \phi$. The cocycle $[\phi_s, \phi_u]$ is a lifting by $\mathrm{Ad} \times q$ of (ϕ, ϕ_u^2). It remains to check that ϕ_u^2 is a cocycle. First, note that $\mathrm{Ad}(\partial\phi_s) = \partial \circ \mathrm{Ad}(\phi_s) = \partial(\phi) = 1$. Thus, $\partial\phi_s$ takes values in $\ker(\mathrm{Ad}) = \mathbb{Z}_2$, and

$$(\partial\phi_s)^{-1}(\partial\phi_u) \in K^2_\Gamma(X, (U(1), \kappa_\epsilon)) \subset K^2_\Gamma(X, (\mathrm{Spin}^c(n), \kappa_\epsilon)).$$

This cochain is a cocycle as

$$(\partial\phi_s)^{-1}(\partial\phi_u) = [1, (\partial\phi_s)^{-1}(\partial\phi_u)] = [\partial\phi_s, \partial\phi_u] = \partial[\phi_s, \phi_u] = 1.$$

The cochain $\phi_u^2 \in K^1_\Gamma(X, (U(1), \kappa_\epsilon))$ is then a cocycle as

$$\partial(\phi_u^2) = (\partial\phi_u)^2 = (\partial\phi_s)^{-2}(\partial\phi_u)^2 = \Big((\partial\phi_s)^{-1}(\partial\phi_u)\Big)^2 = 1.$$

Therefore, the required bundle L can be constructed from ϕ_u^2 using Proposition 1.15. $\square$

Proposition 3.13 can be refined into a statement about cohomology classes. This refinement uses the exact sequences in cohomology obtained by applying Theorem 1.41 to the two exact sequences of Γ-groups running diagonally in diagram (3.1).

Lemma 3.14. *The central exact sequences*

$$1 \to (\mathbb{Z}_2, \mathrm{id}_\epsilon) \to (\mathrm{Spin}(n), \mathrm{id}_\epsilon) \xrightarrow{\mathrm{Ad}} (SO(n), \mathrm{id}_\epsilon) \to 1,$$

$$1 \to (\mathbb{Z}_2, \mathrm{id}_\epsilon) \to (U(1), \kappa_\epsilon) \xrightarrow{q} (U(1), \kappa_\epsilon) \to 1,$$

induce the exact sequences

$$H^1_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)) \to TC^1_\Gamma(X,(\mathrm{Spin}(n),\mathrm{id}_\epsilon)) \overset{\mathrm{Ad}}{\to} TC^1_\Gamma(X,(\mathrm{SO}(n),\mathrm{id}_\epsilon)) \overset{\Delta_s}{\to} H^2_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)),$$

$$H^1_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)) \to H^1_\Gamma(X,(U(1),\kappa_\epsilon)) \overset{q}{\to} H^1_\Gamma(X,(U(1),\kappa_\epsilon)) \overset{\Delta_u}{\to} H^2_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)).$$

Proposition 3.13 and Lemma 3.14 can now be combined to establish an alternative criteria for the existence of a $(\mathrm{Spin}^c,\kappa_\epsilon)$-structure.

Theorem 3.15. *A Γ-equivariant principal $\mathrm{SO}(n)$-bundle Q with cocycle ϕ has a $(\mathrm{Spin}^c,\kappa_\epsilon)$-structure if and only if there exists a cocycle $\psi \in H^1_\Gamma(X,(U(1),\kappa_\epsilon))$ such that*

$$\Delta_s(\phi) = \Delta_u(\psi) \in H^2_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)).$$

Proof. Assume that Q has a $(\mathrm{Spin}^c,\kappa_\epsilon)$-structure. By Proposition 3.13, there exists an cocycle $[\phi_s,\phi_u] \in TC^1_\Gamma(X,(\mathrm{Spin}^c(n),\kappa_\epsilon))$ such that ϕ_u^2 is a cocycle and

$$(\mathrm{Ad}\times q)[\phi_s,\phi_u] = (\phi_s,\phi_u^2).$$

As $[\phi_s,\phi_u]$ is a cocycle, $[\partial\phi_s,\partial\phi_u] = \partial[\phi_s,\phi_u] = 1$. This implies that $\partial\phi_s = \partial\phi_u$. Therefore, applying Lemma 3.14 to ϕ and ϕ_u^2,

$$\Delta_s(\phi) = [\partial\phi_s] = [\partial\phi_u] = \Delta_u(\phi_u^2) \in H^2_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)).$$

Thus, $\psi := \phi_u^2$ is the required cocycle.

Conversely, suppose there exists a cocycle $\psi \in H^1_\Gamma(X,(U(1),\kappa_\epsilon))$ such that

$$\Delta_s(\phi) = \Delta_u(\psi) \in H^2_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)).$$

Then, there are a cochains ϕ_s with $\mathrm{Ad}(\phi_s) = \phi$, and ϕ_u with $\phi_u^2 = \psi$ such that

$$[\partial\phi_s] = [\partial\phi_u] \in K^2_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon)).$$

This implies that $\partial\phi_s = \partial\phi'\partial\phi_u = \partial(\phi'\phi_u)$ for some $\phi' \in K^1_\Gamma(X,(\mathbb{Z}_2,\mathrm{id}_\epsilon))$. Then $\partial[\phi_s,\phi'\phi_u] = [\partial\phi_s,\partial(\phi'\phi_u)] = 1$, and $\mathrm{Ad}^c[\phi_s,\phi'\phi_u] = \mathrm{Ad}(\phi_s) = \phi$. Thus, $[\phi_s,\phi'\phi_u]$ defines a $(\mathrm{Spin}^c,\kappa_\epsilon)$-structure on Q. □

If X is a manifold acted on by a finite group H, and $V \to X$ is a real H-equivariant vector bundle with cocycle $\phi \in TC^1_H(X,\mathrm{SO}(n))$, then the obstruction to the existence of an H-equivariant Spin-structure on V is the second $\mathbb{Z}_2$-valued equivariant Stiefel-Whitney class, which can be defined by $w_2^H(V) := \Delta_{\mathrm{Spin}}(\phi) \in H^2_H(X,\mathbb{Z}_2)$. Here $\Delta_{\mathrm{Spin}}(\phi)$ is the connecting map for the exact sequence

$$H^1_H(X,\mathbb{Z}_2) \longrightarrow TC^1_H(X,\mathrm{Spin}(n)) \xrightarrow{\mathrm{Ad}} TC^1_H(X,\mathrm{SO}(n)) \xrightarrow{\Delta_{\mathrm{Spin}}} H^2_H(X,\mathbb{Z}_2),$$

induced by the central exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \stackrel{\mathrm{Ad}}{\to} \mathrm{SO}(n) \to 1$.

If (Γ, ϵ) is the orientifold group defined by $\Gamma := \mathbb{Z}_2 \times H$ and $\epsilon(z, h) := z$, then X can be made into an orientifold $\tilde{X}$ for (Γ, ϵ) by trivially extending its H-action to the Γ-action $(z, h)x := hx$. Similarly, the H-equivariant vector bundle V can be made into a Γ-equivariant vector bundle $\tilde{V}$ by trivially extending its H-action to the Γ-action $(z, h)v := hv$. The cocycle of $\tilde{V}$ is an element $\tilde{\phi} \in TC^1_\Gamma(X, (\mathrm{SO}(n), \mathrm{id}_\epsilon))$.

In this situation, the quotient map $\pi : \Gamma \to \Gamma/\mathbb{Z}_2 \simeq H$ induces a map $\pi : X^\bullet_\Gamma \to X^\bullet_H$ between the simplicial spaces associated to the groups Γ and H. Because π is a homomorphism and satisfies $\pi(\gamma)x = \gamma x$, it commutes with the face maps on these spaces, and defines a pulback map π^* on cochains. The map π^* also commutes with the coboundary maps, and provides well-defined extension maps

$$\pi^* : TC^p_H(X, G) \to TC^p_\Gamma(\tilde{X}, (G, \mathrm{id}_\epsilon)) \qquad \pi^* : H^p_H(X, G) \to H^p_\Gamma(\tilde{X}, (G, \mathrm{id}_\epsilon)).$$

One then has the following result.

Proposition 3.16. *If $\tilde{V} \to \tilde{X}$ is the trivial extension of a real H-equivariant vector bundle $V \to X$, as described above, then*

1. *the cocycle for $\tilde{V}$ is the pullback of the cocycle for V by the quotient map $\pi : \Gamma \to H$,*

$$\tilde{\phi} = \pi^* \phi \in H^1_\Gamma(\tilde{X}, (\mathrm{SO}(n), \mathrm{id}_\epsilon)).$$

2. *the second $\mathbb{Z}_2$-valued equivariant Stiefel-Whitney class for V satisfies*

$$\pi^* w_2^H(V) = \Delta_s(\pi^* \phi) \in H^2_\Gamma(\tilde{X}, (\mathbb{Z}_2, \mathrm{id}_\epsilon)).$$

3. *$\tilde{V}$ has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure if and only if*

$$\pi^* w_2^H(V) = \Delta_u(\psi) \in H^2_\Gamma(\tilde{X}, (\mathbb{Z}_2, \mathrm{id}_\epsilon)),$$

for some cocycle $\psi \in H^1_\Gamma(\tilde{X}, (U(1), \kappa_\epsilon))$.

Here Δ_s and Δ_u are the connecting maps of Lemma 3.14.

Proof. If $\{s_a\}$ is a collection of local sections for V, then

$$\pi(z, h)x = hx = (z, h)x \qquad \pi(z, h)s_a(x) = hs_a(x) = (z, h)s_a(x),$$

where $(z, h) \in \Gamma = \mathbb{Z}_2 \times H$, $x \in X$. Together with the property (1.3), which defines the cocycles ϕ and $\tilde{\phi}$, this implies

$$s_b(\pi(z, h)x)\phi_{ba}(\pi(z, h), x) = \pi(z, h)s_a(x) = (z, h)s_a(x) = s_b((z, h)x)\tilde{\phi}_{ba}((z, h), x)$$
$$= s_b(\pi(z, h)x)\tilde{\phi}_{ba}((z, h), x).$$

Thus, $\pi^*\phi = \tilde{\phi}$, which proves the the first statement.

The second statement follows from the existence of the commutative diagram

$$\begin{array}{ccccccc}
H^1_H(X,\mathbb{Z}_2) & \longrightarrow & TC^1_H(X,\mathrm{Spin}(n)) & \xrightarrow{\mathrm{Ad}} & TC^1_H(X,SO(n)) & \xrightarrow{\Delta_{\mathrm{Spin}}} & H^2_H(X,\mathbb{Z}_2) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\
H^1_\Gamma(\tilde{X},(\mathbb{Z}_2,\mathrm{id}_\epsilon)) & \longrightarrow & TC^1_\Gamma(\tilde{X},(\mathrm{Spin}(n),\mathrm{id}_\epsilon)) & \xrightarrow{\mathrm{Ad}} & TC^1_\Gamma(\tilde{X},(SO(n),\mathrm{id}_\epsilon)) & \xrightarrow{\Delta_s} & H^2_\Gamma(\tilde{X},(\mathbb{Z}_2,\mathrm{id}_\epsilon)).
\end{array}$$

To see that the right-most cell of this diagram commutes, note that if ψ is a lifting of ϕ, then $\pi^*\psi$ is a lifting of $\pi^*\phi$. The commutation of π^* with the coboundary maps then implies

$$\pi^* w_2^H(V) := \pi^*\Delta_{\mathrm{Spin}}(\phi) = \pi^*\partial(\psi) = \partial(\pi^*\psi) = \Delta_s(\pi^*\phi).$$

The third statement follows from the first and second by applying Theorem 3.15. $\square$

To end this section, two canonical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures will be described. The first of these is the canonical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure associated to a real representation V of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$. When $\dim(V) = 8$, this $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure is used to construct a canonical reduced orientifold spinor bundle over the point orientifold for $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$, which, in turn, is used to construct the 8-fold Bott class over V. The second is a canonical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on the n-sphere. This $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure is used to construct a canonical reduced orientifold spinor bundle on S^{8k}. The reduced orientifold spinor bundle on S^{8k} will be used in the next chapter when describing the compactification of the 8-fold Bott class over a real representation of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$.

Lemma 3.17 (The canonical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure over a point)**.** *Let V be the representation of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$ on $\mathbb{R}^n$ defined by $(\gamma, g) \cdot v := \mathrm{Ad}^c(g)v$. Then*

$$\mathrm{Ad}^c : \mathrm{Spin}^c(n) \to SO(n) \simeq \mathrm{Fr}(V).$$

is a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for the real equivariant vector bundle $V \to \mathrm{pt}$ over the point orientifold for $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$.

Proof. The group $\mathrm{Spin}^c(n)$ forms a principal bundle over a point with the trivial projection $\pi(p) = \mathrm{pt}$, and right $\mathrm{Spin}^c(n)$ action defined by multiplication. The left action of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$ is taken to be

$$(\gamma, g) \cdot p := g\kappa_\gamma(p),$$

for $\gamma \in \Gamma$ and $g, p \in \mathrm{Spin}^c(n)$. The inclusion of the conjugation κ is the only difference from the corresponding construction in the usual equivariant setting. $\square$

Lemma 3.18 (The canonical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on the sphere)**.** *The map*

$$\mathrm{Ad}^c : \mathrm{Spin}^c(n+1) \to \mathrm{SO}(n+1)$$

forms a $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure for the orientifold*

$$S^n \subset \mathbb{R}^{n+1}$$

equipped with the action of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n+1)$ *defined by* $(\gamma, g) \cdot \nu := \mathrm{Ad}^c(g)\nu$*.*

Proof. In what follows, let $\gamma \in \mathbb{Z}_2$, $g, p \in \mathrm{Spin}^c(n+1)$, $h \in \mathrm{Spin}^c(n)$, $q \in \mathrm{SO}(n+1)$, $f \in \mathrm{SO}(n)$. Also, let $\alpha_1 : \mathrm{SO}(n) \to \mathrm{SO}(n+1)$ and $\beta_1 : \mathrm{Spin}^c(n) \to \mathrm{Spin}^c(n+1)$ be the maps induced by the inclusion $\mathbb{C}l_n \to \mathbb{C}l_{n+1}$ defined on standard basis elements by $e_k \mapsto e_{k+1}$. Equip $\mathrm{Spin}^c(n+1)$ with the projection, left action, and right $\mathrm{Spin}^c(n)$-action

$$\pi_{sc}(p) := \mathrm{Ad}^c(p)e_1 \qquad (\gamma, g) \cdot p := g\kappa_\gamma(p) \qquad p \cdot h := p\beta_1(h),$$

respectively. Again, the presence of the conjugation action κ in the left action is the only difference from the corresponding construction in the usual equivariant setting [18, p. 5]. Using the properties of κ, Ad^c and β_1, it is straightforward to check that $\mathrm{Spin}^c(n+1)$ forms a $(\Gamma, \epsilon) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n+1)$-semi-equivariant principal $(\mathrm{Spin}^c(n), \kappa_\epsilon)$-bundle,

$$\begin{aligned}\pi_{sc}((\gamma, g) \cdot p) &= \pi(g(\gamma p)) \\ &= \mathrm{Ad}^c(g(\gamma p))e_1 = \mathrm{Ad}^c(g)\mathrm{Ad}^c(\gamma p)e_1 = \mathrm{Ad}^c(g)\mathrm{Ad}^c(p)e_1 = (\gamma, g)\pi_{sc}(p),\end{aligned}$$

$$\begin{aligned}(\gamma, g) \cdot (p \cdot h) &= (\gamma, g) \cdot (p\beta_1(h)) \\ &= g(\gamma(p\beta_1(h))) = g(\gamma p)(\gamma\beta_1(h)) = g(\gamma p)\beta_1(\gamma h) = ((\gamma, g)p) \cdot (\gamma h).\end{aligned}$$

Next, equip $\mathrm{SO}(n+1)$ with the projection, left action, and right $\mathrm{SO}(n)$-action defined by

$$\pi_{so}(q) := qe_1 \qquad (\gamma, g) \cdot q := \mathrm{Ad}^c(g)q \qquad q \cdot f := q\alpha_1(f),$$

respectively. It can then be checked that $\mathrm{SO}(n+1)$ forms a $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n+1)$-equivariant principal $\mathrm{SO}(n)$-bundle,

$$\begin{aligned}\pi_{so}((\gamma, g) \cdot q) &= \pi_{so}(\mathrm{Ad}^c(g)q) = \mathrm{Ad}^c(g)qe_1 = (\gamma, g)\pi(q), \\ (\gamma, g) \cdot (q \cdot f) &= (\gamma, g) \cdot (q\alpha_1(f)) = \mathrm{Ad}(g)q\alpha_1(f) = ((\gamma, g) \cdot q) \cdot f.\end{aligned}$$

That Ad^c is a semi-equivariant lifting can be checked directly by verifying compatibility with projections, right actions, and left actions,

$$\begin{aligned}\pi_{sc}(p) &= \mathrm{Ad}^c(p)e_1 = \pi_{so} \circ \mathrm{Ad}^c(p), \\ \mathrm{Ad}^c(p \cdot h) &= \mathrm{Ad}^c(p\beta_1(h)) = \mathrm{Ad}^c(p)\mathrm{Ad}^c(\beta_1(h)) = \mathrm{Ad}^c(p)\alpha_1(\mathrm{Ad}^c(h)) = \mathrm{Ad}^c(p) \cdot \mathrm{Ad}^c(h), \\ \mathrm{Ad}^c((\gamma, g) \cdot p) &= \mathrm{Ad}^c(g(\gamma p)) = \mathrm{Ad}^c(g)\mathrm{Ad}^c(\gamma p) = \mathrm{Ad}^c(g)\mathrm{Ad}^c(p) = (\gamma, g) \cdot \mathrm{Ad}^c(p).\end{aligned}$$

It remains to check that $SO(n+1)$ with the given action of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n+1)$ is isomorphic to the equivariant principal $SO(n)$-bundle $\mathrm{Fr}(S^n)$. First, identify the tangent space of the n-sphere with a subbundle of the tangent space to $\mathbb{R}^{n+1}$,

$$TS^n \simeq \left\{(v_1, v_2) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|v_1\| = \|v_2\| = 1, \langle v_1, v_2 \rangle = 0\right\} \subset T\mathbb{R}^{n+1}$$

The standard action of $SO(n+1)$ on $\mathbb{R}^{n+1}$ associates a matrix to each element $q \in SO(n+1)$, which will also be denoted q. The columns q_i of this matrix determine an orthonormal frame

$$F(q) := \{(q_1, q_2), \ldots, (q_1, q_{n+1})\} \in \mathrm{Fr}_{q_1}(TS^n).$$

In this way, $SO(n+1)$ can be identified with $\mathrm{Fr}(TS^n)$. This identification is compatible with projections as

$$\pi_{so}(q) = qe_1 = q_1 = \pi_{TS^n}(F(q)).$$

Compatibility with right actions follows from the fact that

$$(q \cdot f)_j = (q\alpha_1(f))_j = \begin{cases} q_1 & \text{for } j = 1 \\ \sum_{2 \le i \le n+1} q_i f_{(i-1)(j-1)} & \text{for } j \ge 2. \end{cases}$$

Finally, the left action on $\mathrm{Fr}(TS^n)$ can be characterised by observing that a vector $(v_1, v) \in TS^n$ is tangent to the curve $(\cos t)v_1 + (\sin t)v$ at $t = 0$. Acting on this curve by $(\gamma, g) \in (\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n+1)$ produces a new curve $(\cos t)(\mathrm{Ad}^c(g)v_1) + (\sin t)(\mathrm{Ad}^c(g)v)$ which has $(\mathrm{Ad}^c(g)v_1, \mathrm{Ad}^c(g)v)$ as its tangent vector at $t = 0$. Thus,

$$(\gamma, g)F(q) = F(\mathrm{Ad}^c(g)q) = F((\gamma, g)q),$$

and the identification of $SO(n+1)$ and $\mathrm{Fr}(TS^n)$ is compatible with the left actions. □

3.2 Orientifold Spinor Bundles

In this section, orientifold spinor bundles are constructed. This is done by applying the semi-equivariant associated bundle construction, from Definition 1.43, with a Clifford module as the model fibre. In order to do this, the Clifford modules used must be semi-equivariant with respect to the action of $(\mathrm{Spin}^c(n), \kappa_\epsilon)$. The principal bundle used in the construction is the principal bundle P from a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(V)$. The central property of a spinor bundle is that its sections are acted upon by sections of the Clifford bundle $\mathbb{C}l(V)$. This action is sometimes described as *Clifford multiplication*. Clifford multiplication on sections is defined in terms of the action of $\mathbb{C}l_n$ on the model fibre. In order for Clifford multiplication on sections to be well-defined, this fibrewise definition of Clifford multiplication must be compatible with the global topology of the base space. In the orientifold setting, Clifford multiplication is also required to be compatible with an orientifold action on the spinor bundle, and a canonical orientifold action on $\mathbb{C}l(V)$. The $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure used to construct an orientifold spinor bundle ensures that both of these requirements are fulfilled. Thus, the benefit of working on semi-equivariance and $(\mathrm{Spin}^c, \kappa_\epsilon)$-orientiation in earlier chapters is finally observed.

Before defining the orientifold spinor bundles, some results from the representation theory of real Clifford algebas are reviewed. The main Cl_n-modules of interest are Cl_n, considered as a module over itself, and the irreducible Cl_{8k}-modules. Up to equivalence, there is only one irreducible Cl_{8k}-module [63, p. 33]. A representative of this equivalence class will be denoted by Δ. For applications in index theory, it is also important to consider graded modules. The gradings on the orientifold spinor bundles will be derived from special gradings on Cl_n and Δ that are connected with the representation theory of real Clifford algebras. Graded Clifford modules are defined with respect to the standard grading on Cl_n.

Definition 3.19. The *standard grading* on the Clifford algebra Cl_n is the decomposition

$$\mathrm{Cl}_n = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1,$$

defined by the grading operator $\boldsymbol{\alpha} : e_i \mapsto -e_i$.

Definition 3.20. A *graded* Cl_n*-module* is a Cl_n-module V equipped with a decomposition $V = V^0 \oplus V^1$ such that

$$\varphi^i v^j \in V^{i+j}$$

for $i, j \in \mathbb{Z}_2, \varphi \in \mathrm{Cl}_n^i, v^j \in V^j$.

The following two graded Cl_n-modules exist for all n. Example 3.21, will be used to construct the Clifford bundle. Example 3.22, will be used to relate the Clifford bundle to the exterior algebra bundle.

Example 3.21. If Cl_n is considered as a Clifford module over itself, then the standard grading provides Cl_n with a graded Cl_n-module structure.

Example 3.22. The exterior algebra $\Lambda^\bullet(\mathbb{R}^n)$ defines a graded Cl_n-module. To see this, first observe that the map

$$\begin{aligned} \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1 &\to \Lambda^{\mathrm{even}}(\mathbb{R}^n) \oplus \Lambda^{\mathrm{odd}}(\mathbb{R}^n) \\ e_{i_1} \cdots e_{i_k} &\mapsto e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned} \tag{3.4}$$

is an isomorphism of graded vector spaces. Exterior multiplication on $\Lambda(\mathbb{R}^n)$ is not the same as Clifford multiplication on Cl_n. However, the above isomorphism can be used to express Clifford multiplication in terms of the exterior and interior products on $\Lambda(\mathbb{R}^n)$ [63, p.25]. If $\varphi \in \mathrm{Cl}_n$ and $\omega \in \Lambda(\mathbb{R}^n)$, then

$$\varphi\omega := \tilde{\varphi} \wedge \omega - \tilde{\varphi} \lrcorner \omega, \tag{3.5}$$

where $\tilde{\varphi}$ is the image of φ under the map of (3.4). With this multiplication, $\Lambda^\bullet(\mathbb{R}^n)$ is isomorphic to Cl_n as a graded Cl_n-module.

In dimensions 4k, the representation theory of real Clifford algebras provides another natural method to grade Cl_{4k}-modules.

Proposition 3.23. *If* V *is a* Cl_{4k}*-module then multiplication by the* oriented volume element

$$\boldsymbol{\omega} := e_1 \cdots e_{4k} \in \mathrm{Cl}_{4k}$$

is a grading operator, and the associated grading $V^+ \oplus V^-$ *defines a graded* Cl_{4k}*-module.*

Proof. See [63, p. 23]. □

Example 3.24. Considering Cl_{8k} as a right module over itself, Proposition 3.23 implies that right multiplication by $\boldsymbol{\omega}$ determines a graded Cl_{8k}-module structure. The resulting graded Cl_{8k}-module will be denoted $\mathrm{Cl}_{8k}^{\pm} := \mathrm{Cl}_{8k}^{+} \oplus \mathrm{Cl}_{8k}^{-}$.

Example 3.25. An irreducible left Cl_{8k}-module Δ, can be graded using left multiplication by $\boldsymbol{\omega}$. This results in a decomposition $\Delta = \Delta^+ \oplus \Delta^-$, where $\Delta^{\pm}$ are the two inequivalent irreducible Cl_{8k-1}-modules. See [63, p. 35-36].

The graded Cl_{8k}-modules in Examples 3.25 and 3.24 will be used later in this section to define the spinor bundles and their gradings. The two examples can be related using the following proposition.

Proposition 3.26. *For any irreducible* Cl_{8k}*-module* Δ*, there is an isomorphism of graded* $\mathrm{Cl}_{8k}\hat{\otimes}\mathrm{Cl}_{8k}$*-modules,*

$$\Delta\hat{\otimes}\Delta^* \simeq \mathrm{Cl}_{8k},$$

where the action on Cl_{8n} *is defined by* $(\varphi_1, \varphi_2)\varphi := \varphi_1\varphi\varphi_2^*$.

Proof. The proof of this proposition for complex Clifford algebras can be found in [63, p. 38]. The same argument applies, using facts from the representation theory of real Clifford algebras. First, note that $\mathrm{Cl}_{8k}\hat{\otimes}\mathrm{Cl}_{8k} = \mathrm{Cl}_{16k}$ [63, pp. 27-28] and that Cl_{16k} has a single irreducible representation of dimension 2^{8k} [63, p. 33]. Therefore, all representations of $Cl_{8k}\hat{\otimes}Cl_{8k}$ with dimension 2^{8k} are equivalent. The algebra Cl_{8k} has a single irreducible representation of dimension 2^{4k} [63, p. 33]. Thus, $\dim(\Delta) = 2^{4k}$ and $\dim(\Delta\hat{\otimes}\Delta^*) = 2^{8k}$. As $\dim(Cl_{8k}) = 2^{8k}$, this proves that $\Delta\hat{\otimes}\Delta^*$ and Cl_{8k} are isomorphic as $Cl_{8k}\hat{\otimes}Cl_{8k}$ modules. □

Corollary 3.27. $\Delta \otimes (\Delta^+)^* \simeq Cl_{8k}^+$.

Proof. The submodule Cl_{8k}^+ is defined as the $+1$-eigenspace under muliplication by the orientifold volume element $\boldsymbol{\omega}$ for the right Cl_{8k}^+-module structure. By Proposition 3.26,

$$Cl_{8n} \simeq \Delta\hat{\otimes}\Delta^* = (\Delta^+ \otimes (\Delta^+)^*) \oplus (\Delta^- \otimes (\Delta^-)^*) \oplus (\Delta^+ \otimes (\Delta^-)^*) \oplus (\Delta^- \otimes (\Delta^+)^*),$$

so the $+1$-eigenspace is $(\Delta^+ \otimes (\Delta^+)^*) \oplus (\Delta^- \otimes (\Delta^+)^*) = \Delta \otimes (\Delta^+)^*$. □

In Section 3.1, results involving the groups $\mathrm{Spin}^c(n)$ were complexified and equipped with involutive orientifold actions. In a similar manner, it is neccesary to complexify the above definitions and results involving real Clifford modules. In regards to this, it is important to note that the complexification $\Delta \otimes \mathbb{C}$ is an irreducible module for $\mathbb{C}l_{8k}$. This is a non-trivial fact which depends on the representation theory of Clifford algberas. Also, in dimensions $8k$, the complexified volume element $\boldsymbol{\omega} \otimes \mathrm{id}$ is the same as the volume element that is conventionally used to grade complex Clifford modules [63, p. 34].

Definition 3.28. Define the following

1. $(\Lambda_c(\mathbb{R}^n), \kappa_\epsilon) := (\Lambda(\mathbb{R}^n) \otimes \mathbb{C}, \mathrm{id} \otimes \kappa_\epsilon)$ with the even/odd grading.

2. $(\mathbb{C}l_n, \kappa_\epsilon) := (Cl_n \otimes \mathbb{C}, \mathrm{id} \otimes \kappa_\epsilon)$ graded by $\boldsymbol{\alpha} \otimes \mathrm{id}$

3. $(\mathbb{C}l_{8k}^\pm, \kappa_\epsilon) := (Cl_{8k} \otimes \mathbb{C}, \mathrm{id} \otimes \kappa_\epsilon)$ graded by $\boldsymbol{\omega} \otimes \mathrm{id}$

4. $(\Delta_c^\pm, \kappa_\epsilon) := (\Delta \otimes \mathbb{C}, \mathrm{id} \otimes \kappa_\epsilon)$ graded by $\boldsymbol{\omega} \otimes \mathrm{id}$.

Complexifying Example 3.22, Proposition 3.26, and Corollary 3.27 provides corresponding results, in the setting of orientifolds.

Example 3.29. $(\Lambda_c(\mathbb{R}^n), \kappa_\epsilon) \simeq (\mathbb{C}l_n, \kappa_\epsilon)$ as graded Cl_n-modules.

Proposition 3.30. $(\Delta_c, \kappa_\epsilon)\hat{\otimes}(\Delta_c, \kappa_\epsilon) \simeq (\mathbb{C}l_{8k}, \kappa_\epsilon)$ *as* $(\mathbb{C}l_{8k}, \kappa_\epsilon)\hat{\otimes}(\mathbb{C}l_{8k}, \kappa_\epsilon)$*-modules.*

Corollary 3.31. $(\Delta_c, \kappa_\epsilon) \otimes (\Delta_c^+, \kappa_\epsilon)^* \simeq (\mathbb{C}l_{8k}^+, \kappa_\epsilon)$.

Having considered the Clifford modules that will form their model fibres, it is now possible to define the orientifold spinor bundles.

Definition 3.32. Let $P \to \mathrm{Fr}(V)$ be an orientifold-Spin^c-structure, and define the following orientifold bundles:

The *orientifold spinor bundle* $\quad \mathbb{S}\!\!\!/ := P \times_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} (\mathbb{C}l_n, \kappa_\epsilon),$

The *reduced orientifold spinor bundle* $\quad S\!\!\!/ := P \times_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} (\Delta_c, \kappa_\epsilon).$

Note that if one disregards the orientifold action, then an orientifold spinor bundle is a complex spinor bundle in the usual sense. In the case of the reduced orientifold spinor bundle, Δ_c is an irreducible module for $\mathbb{C}l_{8k}$, as mentioned above. This implies that, disregarding the orientifold action, the reduced orientifold spinor bundle is a reduced complex spinor bundle.

Example 3.33 (The canonical reduced orientifold spinor bundle over a point)**.** Using Lemma 3.17 it is possible to construct a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(V)$, for the adjoint representation V of $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$. If $\dim(V) = 8k$, then the irreducible Cl_n-module Δ can be used to construct a canonical reduced spinor bundle $S\!\!\!/ \to \mathrm{pt}$ over the point orientifold.

Example 3.34 (The canonical reduced orientifold spinor bundle over S^{8k})**.** By Lemma 3.18, each sphere S^n has a canonical $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$-equivariant $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. If $\dim(V) = 8k$, then the irreducible Cl_n-module Δ can be used to construct a canonical reduced spinor bundle $S\!\!\!/ \to S^{8k}$ over the 8-dimensional sphere. This construction is an adaptation, to the orientifold setting, of the Real equviariant spinor bundle defined on S^{8k} by Atiyah [4, p. 128].

The space of sections of the orientifold spinor bundle carries an action by sections of an *orientifold Clifford bundle* $\mathbb{C}l(V)$. When a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(V)$ exists, the orientifold Clifford bundle can be expressed as an associated bundle

$$\mathbb{C}l(V) := P \times^{\mathrm{Ad}^c}_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} (\mathbb{C}l_n, \kappa_\epsilon)$$

of P, and this characterisation can be used to define Clifford multiplication on sections of the spinor bundle. In what follows, consider sections of associated bundles to be represented by equivariant maps from the principal bundle P of an underlying $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(V)$ into the semi-equivariant fibre, as in Lemma 1.44.

Proposition 3.35. *Sections $\varphi \in \Gamma(\mathbb{C}l(V))$ of the orientifold Clifford bundle act from the left on the sections $\psi \in \Gamma(\mathbb{S}\!\!\!/)$ of the orientifold spinor bundle by*

$$(\varphi\psi)(p) = \varphi(p)\psi(p).$$

This action is well-defined and satisfies $\gamma(\varphi\psi) = (\gamma\varphi)(\gamma\psi)$.

Proof. Multiplication is well-defined, as

$$(\varphi\psi)(pg) = \varphi(pg)\psi(pg) = (g^{-1}\varphi(p)g)(g^{-1}\psi(p)) = g^{-1}\varphi(p)\psi(p) = g^{-1}(\varphi\psi)(p).$$

Compatibility with the orientifold actions is verified using Lemma 1.44,

$$\begin{aligned}
(\gamma(\varphi\psi))(p) &= \gamma(\varphi\psi)(\gamma^{-1}p) \\
&= \gamma(\varphi(\gamma^{-1}p)\psi(\gamma^{-1}p)) \\
&= (\gamma\varphi(\gamma^{-1}p))(\gamma\psi(\gamma^{-1}p)) \\
&= (\gamma\varphi)(p)(\gamma\psi)(p) \\
&= ((\gamma\varphi)(\gamma\psi))(p).
\end{aligned}$$

□

Sections of the orientifold Clifford bundle act on sections of the reduced orientifold spinor bundle in the same way. One can also check that the Clifford multiplication between sections of the orientifold Clifford bundle is well-defined and compatible with the orientifold action.

Because the orientifold spinor bundle has $(\mathbb{C}l_n, \kappa_\epsilon)$ as its model fibre, it carries a right action by elements of $\mathbb{C}l_n$. This right action is sometimes described as a multigrading [46, pp. 379-380].

Proposition 3.36. *An element $\varphi \in \mathbb{C}l_n$ acts from the right on sections $\psi \in \Gamma(\not{S})$ by*

$$(\psi\varphi)(p) = \psi(p)\varphi.$$

For $\gamma \in \Gamma$, this action satisfies $\gamma(\psi\varphi) = (\gamma\psi)(\gamma\varphi)$.

Proof. Consider φ as a constant section of the trivial orientifold bundle $P \times^{\mathrm{id}}_{(G,\theta)} (\mathbb{C}l_n, \kappa_\epsilon)$. The right action is well-defined,

$$(\psi\varphi)(pg) = \psi(pg)\varphi(pg) = g^{-1}\psi(p)\varphi(p) = g^{-1}(\psi\varphi)(p).$$

It is also compatible with the orientifold actions,

$$\begin{aligned}
(\gamma(\psi\varphi))(p) &= \gamma(\psi\varphi)(\gamma^{-1}p) \\
&= \gamma(\psi(\gamma^{-1}p)\varphi(\gamma^{-1}p)) \\
&= (\gamma\psi(\gamma^{-1}p))(\gamma\varphi(\gamma^{-1}p)) \\
&= (\gamma\psi)(p)(\gamma\varphi)(p) \\
&= ((\gamma\psi)(\gamma\varphi))(p).
\end{aligned}$$

□

Similar considerations show that there is also a right action of $\mathbb{C}l_n$ on $\mathbb{C}l(V)$ which is compatible with their orientifold actions.

The relationships between Clifford modules determined by Example 3.22 and Proposition 3.26 induce relationships between the corresponding orientifold spinor bundles.

Lemma 3.37. *Let $V \to X$ be a real equivariant vector bundle over an orientifold X. The complexification of the exterior algebra bundle for V forms an orientifold bundle*

$$\Lambda_c(V) := \mathrm{Fr}(V) \times_{(SO(n),\mathrm{id})} (\Lambda_c(\mathbb{R}^n), \kappa_\epsilon).$$

The isomorphism of graded Clifford modules $\Lambda(\mathbb{R}^n) \simeq \mathbb{C}l_n$, of Example 3.22, induces an isomorphism $\Lambda_c(V) \simeq \mathbb{C}l(V)$ compatible with orientifold actions on $\Lambda_c(V)$ and $\mathbb{C}l(V)$.

Proposition 3.38. *Let $V \to X$ be an $8k$-dimensional real equivariant vector bundle over an orientifold X, and $P \to \mathrm{Fr}(V)$ be a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for V. The following relationships exist between the associated bundles $\mathbb{C}l(V)$, $\not{\mathfrak{S}}$ and $\not{S}$,*

$$\mathbb{C}l(V) \simeq \not{S} \otimes \not{S}^* \qquad\qquad \not{\mathfrak{S}}(V) \simeq \not{S} \boxtimes (\Delta_c^*, \kappa_\epsilon).$$

Proof. Using Proposition 3.26, the Clifford bundle decomposes,

$$\mathbb{C}l(V) = P \times^{\mathrm{Ad}^c}_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} (\mathbb{C}l_{8k}, \kappa_\epsilon) = P \times^{\mathrm{Ad}^c}_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} ((\Delta_c, \kappa_\epsilon) \otimes (\Delta_c^*, \kappa_\epsilon)) = \not{S} \otimes \not{S}^*,$$

Similarly, the spinor bundle decomposes,

$$\not{\mathfrak{S}} = P \times_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} (\mathbb{C}l_{8k}, \kappa_\epsilon) = P \times_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} ((\Delta_c, \kappa_\epsilon) \otimes (\Delta_c^*, \kappa_\epsilon)) = \not{S} \boxtimes (\Delta_c^*, \kappa_\epsilon).$$

$\square$

Corollary 3.39. $\not{S} \otimes (\not{S}^+)^* \simeq \mathbb{C}l^+(V)$.

3.3 Connections in Orientifold Spinor Bundles

In order to define an orientifold Dirac operator, a semi-equivariant connection 1-form is needed for the semi-equivariant principal $(\mathrm{Spin}^c(n), \kappa_\epsilon)$-bundle P of the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to Q$ underlying the orientifold spinor bundle. Such a form can be obtained by using Proposition 3.13 to extend the lifting $\varphi : P \to Q$ to a lifting $P \to Q \times_X L$, where L is a semi-equivariant principal $(U(1), \kappa_\epsilon)$-bundle. A semi-equivariant connection form can then be constructed on $Q \times_X L$ using Proposition 1.49, and lifted to P using the relationship between the Lie algebras $\mathfrak{spin}^c(n)$ and $\mathfrak{so}(n) \oplus \mathfrak{u}(1)$. In the next proposition, q denotes the square map of Diagram (3.1).

Proposition 3.40. *The map*

$$(\mathrm{Ad}^c \times q)_* : \mathfrak{spin}^c(n) = \mathfrak{spin}(n) \oplus \mathfrak{u}(1) \to \mathfrak{so}(n) \oplus \mathfrak{u}(1)$$

is an isomorphism, and satisfies

$$(\mathrm{Ad}^c \times q)_* \circ (\mathrm{id} \times \kappa_\epsilon)_* = (\mathrm{id} \times \kappa_\epsilon)_* \circ (\mathrm{Ad}^c \times q)_*.$$

Proof. That $(\mathrm{Ad}^c \times q)_*$ is an isomorphism is a standard result [37, p. 18-20,29]. The isomorphism can be written down explicitly by making the following identifications

1. $\mathfrak{so}(n)$ can be identified with the real $n \times n$ skew-symmetric matricies. A basis for the skew-symmetric matricies is defined by $\{E_{ij} \mid 1 \le i < j \le n\}$ where E_{ij} is the $n \times n$ matrix with all entries equal to 0 except for the (i, j)th and (j, i)th entry, which are equal to 1 and -1 respectively.

2. $\mathfrak{spin}(n)$ can be identified with the linear subspace $\Lambda^2 \subset \mathrm{Cl}_n$ spanned by the elements $\{e_i e_j \mid 1 \le i < j \le n\}$, see [37, p. 18].

3. $\mathfrak{u}(1)$ can be identified with $\mathbb{R}$.

With these identifications, $(\mathrm{Ad}^c \times q)_*$ is the map

$$\begin{aligned}(\mathrm{Ad}^c \times q)_* : \mathfrak{spin}(n) \oplus \mathfrak{u}(1) &\to \mathfrak{so}(n) \oplus \mathfrak{u}(1)\\ (e_i e_j, t) &\mapsto (2E_{ij}, 2t),\end{aligned}$$

see [37, pp. 19-20,29]. Also, the Γ-actions on $\mathfrak{spin}(n) \oplus \mathfrak{u}(1)$ and $\mathfrak{so}(n) \oplus \mathfrak{u}(1)$ are

$$\begin{aligned}(\mathrm{id} \oplus \kappa_\epsilon)_* : \mathfrak{spin}(n) \oplus \mathfrak{u}(1) &\to \mathfrak{spin}(n) \oplus \mathfrak{u}(1) \qquad & (\mathrm{id} \oplus \kappa_\epsilon)_* : \mathfrak{so}(n) \oplus \mathfrak{u}(1) &\to \mathfrak{so}(n) \oplus \mathfrak{u}(1)\\ (e_i e_j, t) &\mapsto (e_i e_j, \iota_\epsilon(t)) & (E_{ij}, t) &\mapsto (E_{ij}, \iota_\epsilon(t)),\end{aligned}$$

where $\iota_\epsilon : \mathbb{R} \to \mathbb{R}$ is the involutive action induced by $\iota : t \mapsto -t \in \mathbb{R}$. Examining these maps, it is clear that $(\mathrm{Ad}^c \times q)_* \circ (\mathrm{id} \times \kappa_\epsilon)_* = (\mathrm{id} \times \kappa_\epsilon)_* \circ (\mathrm{Ad}^c \times q)_*$. □

Proposition 3.41. *Let $\varphi_Q : P \to Q$ be a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. The semi-equivariant principal bundle P carries a Γ-semi-equivariant connection 1-form.*

Proof. By Proposition 3.13, there exists a lifting

$$\varphi_Q \times \varphi_L : P \to Q \times_X L$$

by $\mathrm{Ad}^c \times q$, where L is a semi-equivariant principal $(U(1), \kappa_\epsilon)$-bundle. The equivariant principal bundle Q has an equivariant connection 1-form $\omega_Q : TQ \to \mathfrak{so}(n)$ determined by an equivariant metric. The semi-equivariant principal bundle L has a semi-equivariant connection 1-form $\omega_L : TL \to \mathfrak{u}(1)$ constructed by applying Proposition 1.49 to any choice of connection 1-form for L. Together, these two connection 1-forms define a semi-equivariant connection 1-form

$$\omega_Q \oplus \omega_L : T(Q \times_X L) \to \mathfrak{so}(n) \oplus \mathfrak{u}(1).$$

Using the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure φ and Proposition 3.40, the connection 1-form $\omega_Q \oplus \omega_L$ can be lifted to a connection 1-form

$$\begin{aligned} \omega : TP &\to \mathfrak{spin}^c(n) \\ v &\mapsto (\mathrm{Ad}^c \times q)_*^{-1} \circ (\omega_Q \oplus \omega_L) \circ (\varphi_Q \times \varphi_L)_*(v). \end{aligned}$$

The semi-equivariance of ω follows from the semi-equivariance of $\omega_Q \oplus \omega_L$, and the equivariance of $(\varphi_Q \times \varphi_L)_*$ and $(\mathrm{Ad}^c \times q)_*$. □

The next proposition shows that the connection 1-form constructed by Proposition 3.41 defines a covariant derivative on the orientifold spinor bundle that is equivariant with respect to the action of Γ. In this proposition, sections will be considered as maps $\psi : P \to \mathbb{C}l_n$ satisfying $\psi(gp) = g^{-1}\psi(p)$, and will be acted on by the Γ-action defined in Lemma 1.44. From the point of view of the exterior covariant derivative, these maps are order zero tensorial forms $\psi \in \Lambda^0(P, \mathbb{C}l_n)$. For the details of tensorial forms and exterior covariant derivatives, see [37, §B.3-4] [62, §II.5].

Proposition 3.42. *Let $\varphi : P \to Q$ be a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. The semi-equivariant connection 1-form ω, defined on P by Proposition 3.41, determines an exterior covariant derivative*

$$d^\omega : \Lambda^0(P, \mathbb{C}l_n) \to \Lambda^1(P, \mathbb{C}l_n)$$

that satisfies the condition

$$d^\omega(\kappa_{\epsilon(\gamma)} \circ \psi \circ \eta_{\gamma^{-1}}) = \kappa_{\epsilon(\gamma)} \circ d^\omega\psi \circ (\eta_{\gamma^{-1}})_*,$$

where $\psi \in \Lambda^0(P, \mathbb{C}l_n)$, η is the Γ-action on P, and κ_ϵ is the conjugation action on $\mathbb{C}l_n$.

Proof. The vertical projection associated to the connection form ω is defined by

$$\pi_V|_p := (R^p)_* \circ \omega : TP_p \to TP_p.$$

Therefore, the exterior covariant derivative can be written as

$$d^\omega\psi(v) = d\psi \circ \pi_H(v) = d\psi(v) - d\psi \circ \pi_V(v) = d\psi(v) - d\psi \circ (R^p)_* \circ \omega(v), \tag{3.6}$$

where $v \in TP_p$, $\psi \in \Lambda^0(P, \mathbb{C}l_n)$, and π_H is the horizontal projection. The first term of the decomposition (3.6) is equivariant, as the properties of the exterior derivative imply that

$$d(\kappa_{\epsilon(\gamma)} \circ \psi \circ \eta_{\gamma^{-1}}) = \kappa_{\epsilon(\gamma)} \circ d\psi \circ (\eta_{\gamma^{-1}})_*.$$

The semi-equivariance of P implies the identity $(\eta_\gamma)_* \circ (R^p)_* = (R^{\gamma p})_* \circ (\theta_\gamma)_*$. Together with the the semi-equivariance of ω, this implies that

$$\begin{aligned} d(\kappa_{\epsilon(\gamma)} \circ \psi \circ \eta_{\gamma^{-1}}) \circ (R^p)_* \circ \omega &= \kappa_{\epsilon(\gamma)} \circ d\psi \circ (\eta_{\gamma^{-1}})_* \circ (R^p)_* \circ \omega \\ &= \kappa_{\epsilon(\gamma)} \circ d\psi \circ (R^{\gamma^{-1}p})_* \circ (\theta_{\gamma^{-1}})_* \circ \omega \\ &= \kappa_{\epsilon(\gamma)} \circ d\psi \circ (R^{\gamma^{-1}p})_* \circ \omega \circ (\eta_{\gamma^{-1}})_*. \end{aligned}$$

Therefore, the second term of the decomposition (3.6) is also equivariant. □

Proposition 3.42 applies equally well to the reduced orientifold spinor bundle if $\mathbb{C}l_n$ is replaced with Δ_c.

As in the non-equivariant case, the exterior covariant derivative is also equivariant with respect to the right action of $\mathbb{C}l_n$ on the orientifold spinor bundle.

Proposition 3.43. *Let $\varphi : P \to Q$ be a* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure. The semi-equivariant connection* 1*-form ω, defined on P by Proposition 3.41, determines an exterior covariant derivative*

$$d^\omega : \Lambda^0(P, \mathbb{C}l_n) \to \Lambda^1(P, \mathbb{C}l_n)$$

that satisfies

$$d^\omega(\psi\varphi) = d^\omega(\psi)\varphi,$$

for $\psi \in \Lambda^0(P, \mathbb{C}l_n)$ and $\varphi \in \mathbb{C}l_n$.

3.4 Dirac Operators on Orientifolds

At this stage, all of the preliminary constructions have been completed. It is now possible to construct the orientifold Dirac operator and reduced orientifold Dirac operator.

Definition 3.44. Let ∇^L denote the connection associated to a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(TM)$ by Proposition 3.41, and μ denote Clifford multiplication by sections of $T^*M \simeq TM \subset \mathbb{C}l(TM)$. Define the following orientifold operators:

The *orientifold Dirac operator*	$\not{\mathfrak{D}} := \mu \circ \nabla^L : \Gamma(\not{\mathfrak{S}}) \to \Gamma(T^*M \otimes \not{\mathfrak{S}}) \to \Gamma(\not{\mathfrak{S}})$,
The *reduced orientifold Dirac operator*	$\not{D} := \mu \circ \nabla^L : \Gamma(\not{S}) \to \Gamma(T^*M \otimes \not{S}) \to \Gamma(\not{S})$.

The orientifold Dirac operator and reduced orientifold Dirac operator are complex Dirac operators, in the usual sense. However, they are equivariant with respect to the orientifold actions on their spinor bundles. Thus, when $\epsilon : \Gamma \to \mathbb{Z}_2$ is non-trivial, they have anti-linear symmetries.

Proposition 3.45. *The orientifold Dirac operator is equivariant with respect to the left action of* Γ *on sections of* $\mathfrak{S}$*,*

$$\mathfrak{D}(\gamma\psi) = \gamma\mathfrak{D}(\psi).$$

Proof. This follows from Propositions 3.35 and 3.42. □

The same arguments show that the reduced orientifold spinor bundle is also Γ-equivariant. In addition to Γ-equivariance, the orientifold Dirac operator is equivariant with respect to the right action of $(\mathbb{C}l_n, \kappa_\epsilon)$ on the orientifold spinor bundle.

Proposition 3.46. *The orientifold Dirac operator is equivariant with respect to the right action of* $\mathbb{C}l_n$ *on sections of* $\mathfrak{S}$*,*

$$\mathfrak{D}(\psi\varphi) = \mathfrak{D}(\psi)\varphi.$$

Proof. This follows from Propositions 3.36 and 3.43. □

Note, in particular, that left and right equivariance together imply that the index of $\mathfrak{D}$ consists of vector spaces which are both Clifford modules and orientifold representations of (Γ, ϵ).

The main aim of this thesis is now complete, and the following theorem has been proved.

Theorem 3.47. *Any orientifold* (X, σ) *with* $W_3^{(\Gamma,\epsilon)}(X, \sigma) = 0$ *carries an orientifold Dirac operator. If* $\dim(X) = 8$*, then* X *also carries a reduced orientifold Dirac operator. In particular, a reduced Real Dirac operator exists on any* 8*-dimensional Real manifold* (X, σ) *such that* $W_3^{(\mathbb{Z}_2,\mathrm{id})}(X, \sigma) = 0$*.*

Chapter 4

The K-theory of Orientifold Bundles

The aim of this chapter is to prove the Bott periodicity and Thom isomorphism theorems in orientifold K-theory, and to provide context for the investigation of orientifold K-homology in later chapters. Orientifold K-theory is a bigraded cohomology theory. Like KR-theory, it has two periodicity theorems. These can be proved by adapting Atiyah's proofs of periodicity for equivariant KR-theory to the setting of orientifolds [4]. Atiyah's proofs construct an inverse to the periodicity homomorphism using index maps associated to families of elliptic operators. In doing so, they tie together many ideas from index theory, and foreshadow constructions that will be described in Chapters 5 and 6 on K-homology. Together, the periodicity theorems imply that, up to isomorphism, an orientifold has eight orientifold K-theory groups. Combining 8-fold periodicity with results on $(\mathrm{Spin}^c, \kappa_\epsilon)$-orientibility from Section 3.1 produces an 8-fold Thom isomorphism in orientifold K-theory.

4.1 Orientifold K-theory

As with equivariant K-theory [75, §3], orientifold K-theory can be defined in terms of complexes of bundles. Using this definition, it is possible to deal more directly with locally compact orientifolds and to characterise the symbol of an elliptic orientifold operator as a class in orientifold K-theory. The set of representative complexes can be reduced so that each class is represented by a length-1 complex [6, §II]. Rather than describing this reduction, the definition of orientifold K-theory below will be made directly in terms of length-1 complexes.

Definition 4.1. Let X be a locally compact Hausdorff orientifold. A length-1 *complex* is a homomorphism $E^0 \xrightarrow{\sigma} E^1$ of orientifold bundles over X. The *support* $\mathrm{supp}(\sigma)$ of such a complex is the set of $x \in X$ such that the restriction $\sigma|_x : E^0|_x \to E^1|_x$ is not an isomorphism.

Definition 4.2. Two complexes $E^0 \xrightarrow{\sigma} E^1$ and $F^0 \xrightarrow{\rho} F^1$ over an orientifold X are *isomorphic* if there exist isomorphisms $\varphi^0 : E^0 \to F^0$ and $\varphi^1 : E^1 \to F^1$ such that the diagram

$$\begin{array}{ccc} E^0 & \xrightarrow{\sigma} & E^1 \\ \varphi^0 \downarrow & & \downarrow \varphi^1 \\ F^0 & \xrightarrow{\rho} & F^1 \end{array}$$

commutes.

Definition 4.3. A *pair* (X, A) of orientifolds consists of a locally compact orientifold X and a closed Γ-invariant subspace $A \subseteq X$. A *homomorphism* $f : (Y, B) \to (X, A)$ between two pairs of orientifolds is a proper homomorphism of orientifolds $f : Y \to X$ such that $f(B) \subseteq A$.

Definition 4.4. Let (X, A) be a pair of orientifolds. The set $L_{(\Gamma,\epsilon)}(X, A)$ consists of isomorphism classes of complexes $E^0 \xrightarrow{\sigma} E^1$ such that $\mathrm{supp}(\sigma)$ is a compact subset of $X \setminus A$.

The operations on orientifold bundles, defined in Section 2.5, induce operations on complexes.

Definition 4.5. Let $(E^0_i \xrightarrow{\sigma_i} E^1_i) \in L_{(\Gamma,\epsilon)}(X, A)$, $(F^0_i \xrightarrow{\rho_i} F^1_i) \in L_{(\Gamma,\epsilon)}(Y, B)$ and $f : (Y, B) \to (X, A)$ be a homomorphism. In addition, let $(G^0_i \xrightarrow{\vartheta_i} G^1_i) \in L_{(\Gamma,\epsilon)}(V_i, C_i)$, where $\pi_i : V_i \to Z$ is either an orientifold bundle or a real equivariant vector bundle over a compact orientifold Z. Define the following operations.

1. *pullback*

$$f^*(E^0 \xrightarrow{\sigma} E^1) := (f^*\sigma : f^*E^0 \to f^*E^1) \in L_{(\Gamma,\epsilon)}(Y, B)$$

2. *direct sum*

$$(E^0_0 \xrightarrow{\sigma_0} E^1_0) \oplus (E^0_1 \xrightarrow{\sigma_1} E^1_1) := (\sigma_0 \oplus \sigma_1 : E^0_0 \oplus E^0_1 \to E^1_0 \oplus E^1_1) \in L_{(\Gamma,\epsilon)}(X, A)$$

3. *external tensor product*

$$\begin{aligned} &(E^0 \xrightarrow{\sigma} E^1) \boxtimes (F^0 \xrightarrow{\rho} F^1) \\ &\quad := \left(\begin{pmatrix} \sigma \boxtimes 1 & -1 \boxtimes \rho^* \\ 1 \boxtimes \rho & \sigma^* \boxtimes 1 \end{pmatrix} : (E^0 \boxtimes F^0) \oplus (E^1 \boxtimes F^1) \to (E^1 \boxtimes F^0) \oplus (E^0 \boxtimes F^1) \right) \\ &\quad \in L_{(\Gamma,\epsilon)}(X \times Y, (A \times Y) \cup (X \times B)). \end{aligned}$$

4. *multiplication*

$$\begin{aligned} (G^0_0 \xrightarrow{\vartheta_0} G^1_0) \cdot (G^0_1 \xrightarrow{\vartheta_1} G^1_1) &:= \Delta_*((G^0_0 \xrightarrow{\vartheta_0} G^1_0) \boxtimes (G^0_1 \xrightarrow{\vartheta_1} G^1_1)) \\ &\in L_{(\Gamma,\epsilon)}(V_0 \oplus V_1, (C_0 \times V_1) \cup (C_1 \times V_0)|_{Z \subset Z \times Z}), \end{aligned}$$

where Δ_* is restriction to the diagonal $Z \subset Z \times Z$.

The orientifold K-groups can be defined by introducing an equivalence relation on the semi-group $(L_{(\Gamma,\epsilon)}(X,A),\oplus)$.

Definition 4.6. Two complexes $(E_0^0 \overset{\sigma_0}{\to} E_0^1), (E_1^0 \overset{\sigma_1}{\to} E_1^1) \in L_{(\Gamma,\epsilon)}(X,A)$ are

1. *homotopic* $(E_0^0 \overset{\sigma_0}{\to} E_0^1) \approx (E_1^0 \overset{\sigma_1}{\to} E_1^1)$ if there exists an element

$$(E^0 \overset{\sigma}{\to} E^1) \in L_{(\Gamma,\epsilon)}(X \times [0,1], A \times [0,1])$$

   such that $\sigma_0 \simeq \sigma|_{X\times\{0\}}$ and $\sigma_1 \simeq \sigma|_{X\times\{1\}}$.

2. *equivalent* $(E_0^0 \overset{\sigma_0}{\to} E_0^1) \sim (E_1^0 \overset{\sigma_1}{\to} E_1^1)$ if there exist isomorphisms $F_0^0 \overset{\rho_0}{\to} F_0^1$ and $F_1^0 \overset{\rho_1}{\to} F_1^1$ of orientifold bundles over X such that

$$(E_0^0 \overset{\sigma_0}{\to} E_0^1) \oplus (F_0^0 \overset{\rho_0}{\to} F_0^1) \approx (E_1^0 \overset{\sigma_1}{\to} E_1^1) \oplus (F_1^0 \overset{\rho_1}{\to} F_1^1).$$

Definition 4.7. The *orientifold* K*-theory* groups are defined by

$$K_{(\Gamma,\epsilon)}(X,A) := L_{(\Gamma,\epsilon)}(X,A)/\sim \qquad K^{p,q}_{(\Gamma,\epsilon)}(X,A) := K_{(\Gamma,\epsilon)}(X \times \mathbb{R}^{p,q}, A),$$

where $\mathbb{R}^{p,q} := \mathbb{R}^p \oplus \mathbb{R}^q$ is equipped with the involutive action $\iota^{p,q} : (x,y) \to (x,-y)$.

For convenience, set the notation

$$K^{p,q}_{(\Gamma,\epsilon)}(X) := \begin{cases} K^{p,q}_{(\Gamma,\epsilon)}(X,\emptyset) & \text{when } X \text{ is compact} \\ K^{p,q}_{(\Gamma,\epsilon)}(X^+,\{\infty\}) & \text{when } X \text{ is locally compact,} \end{cases}$$

where ∞ is the point at infinity in the one-point compactification X^+. Note that when X is compact, any pair of vector bundles E^0 and E^1 defines a class

$$[E^0] - [E^1] := [E^0 \overset{z}{\to} E^1] \in K^{p,q}_{(\Gamma,\epsilon)}(X),$$

where *z* is the zero map.

The operations on complexes, defined in Definition 4.5, induce corresponding operations on classes in orientifold K-theory. In particular, multiplication of complexes induces a $K_{(\Gamma,\epsilon)}(X)$-module structure

$$K_{(\Gamma,\epsilon)}(X) \times K_{(\Gamma,\epsilon)}(B) \to K_{(\Gamma,\epsilon)}(B)$$

on the orientifold K-theory group of a Γ-equivariant vector bundle $B \to X$, which may be real or complex. The different types of Bott periodicity and Thom isomorphisms are all of the form

$$K_{(\Gamma,\epsilon)}(X) \to K_{(\Gamma,\epsilon)}(B)$$
$$x \mapsto \boldsymbol{b}x,$$

where $\boldsymbol{b}x$ is module multiplication of x with a special class in $\boldsymbol{b} \in K_{(\Gamma,\epsilon)}(B)$. Each Bott or Thom map corresponds to a different choice of bundle B and class $\boldsymbol{b}$. The following examples define various classes corresponding to Bott and Thom maps. These classes are described using characterisations that will be useful later in this chapter, when it is proved that the Bott and Thom maps are isomorphisms.

Example 4.8. Let W be an orientifold representation. Then W and its exterior algebra $\Lambda^\bullet W$ can be considered as orientifold bundles over a point, and $\Lambda^\bullet W$ can be pulled back over W,

$$\begin{array}{ccc} \pi^*\Lambda^\bullet W & & \Lambda^\bullet W \\ \downarrow & & \downarrow \\ W & \xrightarrow{\pi} & \text{pt}. \end{array}$$

From this starting point, associated Bott and Thom classes can be defined.

1. the *equivariant Bott class* $\underline{\boldsymbol{\lambda}}^W_{\text{pt}} \in K_{(\Gamma,\epsilon)}(W)$ associated to the orientifold representation W is the class of the complex

$$\begin{aligned} \sigma|_\xi : \pi^*\Lambda^{\text{even}}W|_\xi &\to \pi^*\Lambda^{\text{odd}}W|_\xi \\ \omega &\mapsto \xi \wedge \omega - \xi^* \lrcorner\, \omega, \end{aligned}$$

   where $\xi \in W$ and $\omega \in \pi^*\Lambda^{\text{even}}W|_\xi$. Note that although W is not compact, σ is an isomorphism away from the zero-section $\text{pt} \subset W$, which is compact.

2. the *equivariant Bott class* of a trivial orientifold bundle $X \times W \to X$ over a compact orientifold X is defined by

$$\underline{\boldsymbol{\lambda}}^W_X := (f \times \text{id})^*(\underline{\boldsymbol{\lambda}}^W_{\text{pt}}) \in K_{(\Gamma,\epsilon)}(X \times W),$$

   where $f : X \to \text{pt}$ is the map to the point orientifold.

3. the $(1,1)$-*Bott class* is the equivariant Bott class

$$\underline{\boldsymbol{\lambda}}^{(\mathbb{C}^n,\kappa_\epsilon)}_X \in K_{(\Gamma,\epsilon)}(X \times (\mathbb{C}^n, \kappa_\epsilon)) =: K^{n,n}_{(\Gamma,\epsilon)}(X),$$

   associated to the orientifold representation $(\mathbb{C}^k, \kappa_\epsilon)$.

4. the $(1,1)$-*Thom class* of an orientifold bundle $E \to X$ is

$$\underline{\boldsymbol{\lambda}} := q(\underline{\boldsymbol{\lambda}}^{(\mathbb{C}^n,\kappa_\epsilon)}_{\text{Fr}(E)}) \in K_{(\Gamma,\epsilon)}(E),$$

   where q is the canonical map

$$q : K_{(\Gamma,\epsilon)\ltimes(U(n),\kappa_\epsilon)}(\text{Fr}(E) \times (\mathbb{C}^n, \kappa_\epsilon)) \to K_{(\Gamma,\epsilon)}(\text{Fr}(E) \times_{(U(n),\kappa_\epsilon)} (\mathbb{C}^n, \kappa_\epsilon)) = K_{(\Gamma,\epsilon)}(E).$$

Example 4.9. Let V be a real 8k-dimensional representation of (Γ, ϵ) that factors through the group $\mathbb{Z}_2 \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(8k)$. Then V can be considered as a real equivariant vector bundle over a point, and the reduced orientifold spinor bundle $\$$ over a point can be constructed, see Example 3.33. The spinor bundle $\$$ can be pulled back over V

$$\begin{array}{ccc} \pi^*\$ & & \$ \\ \downarrow & & \downarrow \\ V & \xrightarrow{\pi} & \mathrm{pt} \end{array}$$

and used to define Bott and Thom classes.

1. the *equivariant Bott class* $\underline{\beta}^V_{\mathrm{pt}} \in K_{(\Gamma,\epsilon)}(V)$ associated to the real representation V is the class of the complex

$$\sigma|_\xi : \pi^*\$^+|_\xi \to \pi^*\$^-|_\xi$$
$$\psi \mapsto \xi \cdot \psi,$$

   where $\xi \in V$ and $\psi \in \pi^*\$^+|_\xi$. As in the previous example, this map is an isomorphism away from the compact zero-section $\mathrm{pt} \subset V$.

2. the *equivariant Bott class* of a trivial real equivariant vector bundle $X \times V \to X$ over a compact orientifold X is defined by

$$\underline{\beta}^V_X := (f \times \mathrm{id})^*(\underline{\beta}^V_{\mathrm{pt}}) \in K_{(\Gamma,\epsilon)}(X \times V),$$

   where $f : X \to \mathrm{pt}$ is the map to the point orientifold.

3. the *8-fold Bott class* is the equivariant Bott class

$$\underline{\beta}^{(\mathbb{R}^{8k},\mathrm{id}_\epsilon)}_X \in K_{(\Gamma,\epsilon)}(X \times (\mathbb{R}^{8k}, \mathrm{id}_\epsilon)) =: K^{8k,0}_{(\Gamma,\epsilon)}(X)$$

   associated to the trivial real representation $(\mathbb{R}^{8k}, \mathrm{id}_\epsilon)$.

4. the *8-fold Thom class* of a real equivariant vector bundle $V \to X$ with $\dim(V) = 8k$ and $W_3^{(\Gamma,\epsilon)}(V) = 0$ is

$$\underline{\beta} := q(\underline{\beta}^{(\mathbb{R}^{8k},\mathrm{id}_\epsilon)}_P) \in K_{(\Gamma,\epsilon)}(V),$$

   where $P \to \mathrm{Fr}(V)$ is a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for V, and q is the canonical map

$$q : K_{(\Gamma,\epsilon)\ltimes(\mathrm{Spin}^c(n),\kappa_\epsilon)}(P \times (\mathbb{R}^{8k}, \mathrm{id}_\epsilon)) \to K_{(\Gamma,\epsilon)}(P \times_{(\mathrm{Spin}^c(n),\kappa_\epsilon)} (\mathbb{R}^{8k}, \mathrm{id}_\epsilon)) = K_{(\Gamma,\epsilon)}(V).$$

In order to prove that the Bott and Thom maps are isomorphisms, it will also be neccesary to consider the images of some of the classes from Examples 4.8 and 4.9 under maps on orientifold K-theory induced by compactification.

Example 4.10. There is an inclusion

$$K_{(\Gamma,\epsilon)}(W) \simeq K_{(\Gamma,\epsilon)}(\mathbb{P}(W \oplus \mathbb{C}), \mathbb{P}(W)) \subset K_{(\Gamma,\epsilon)}(\mathbb{P}(W \oplus \mathbb{C})),$$

where $\mathbb{P}(W)$ denotes the projective space of W. The image of $\underline{\boldsymbol{\lambda}}_{\mathrm{pt}}^{W}$ under this inclusion is the class

$$\boldsymbol{\lambda}_{\mathrm{pt}} := \bigoplus_{i \text{ even}} [H]^{i}[\pi^{*}\Lambda^{i}E] - \bigoplus_{i \text{ odd}} [H]^{i}[\pi^{*}\Lambda^{i}E],$$

where H is the dual of the tautological line bundle on $\mathbb{P}(W)$ [3, p. 100].

Example 4.11. The one point compactification of V defines an inclusion

$$K_{(\Gamma,\epsilon)}(V) = K_{(\Gamma,\epsilon)}(S^{8k}, \infty) \subset K_{(\Gamma,\epsilon)}(S^{8k}),$$

and the image of $\underline{\boldsymbol{\beta}}_{\mathrm{pt}}^{V}$ under this inclusion is the class

$$[(\$^{+})^{*}] + [(\$^{+})^{*}_{\infty}],$$

where $\$$ is the canonical reduced orientifold spinor bundle on S^{8k} that was defined in Example 3.34, $\infty \in S^{8k}$ is the fixed point at infinity, and $(\$^{+})^{*}_{\infty}$ is the trivial bundle with fibre $(\$^{+})^{*}_{\infty}$. These two classes will be denoted by $\boldsymbol{\beta} := [(\$^{+})^{*}]$ and $\boldsymbol{\beta}_{\infty} := [(\$^{+})^{*}_{\infty}]$.

4.2 The Symbol Class of an Elliptic Orientifold Operator

Using the characterisation of K-theory in terms of complexes allows K-theory classes to be associated to elliptic operators via their principal symbol. For simplicity, attention will be restricted to first order differential operators $D : \Gamma(E) \to \Gamma(F)$ on an manifold X. Over an open subset $U_a \subset X$, such an operator has the form

$$(D\psi)(x) = \sum_{j}^{n} A_{a}^{j}(x)(\frac{\partial}{\partial x_j}\psi)(x) + B_a(x)\psi(x),$$

where the A_a^j and B_a are matrix valued functions on U_a. Using the Fourier transform, differentiation may be replaced by multiplication, resulting in the pseudodifferential operator

$$D\psi(x) = \int e^{i(x-y)\cdot\xi} p_a(x,\xi)\psi(y)\,dy\,d\xi + B_a(x)\psi(x),$$

where

$$p_a(x,\xi) = i\sum_{j} A_a^j(x)\xi^j. \tag{4.1}$$

The maps p_a transform consistently under coordinate changes, producing a well-defined section

$$\boldsymbol{\sigma}(D) \in \Gamma(E^{*} \otimes F \otimes T^{*}X) \simeq \Gamma(\mathrm{End}(E,F) \otimes T^{*}X).$$

The section $\sigma(D)$ is called the *principal symbol* of D. If $\sigma(D)_x(v) \in \mathrm{End}(E_x, F_x)$ is an isomorphism, for all $x \in X$ and non-zero $v \in T_xX$, then D is said to be *elliptic*.

In the case that D is a G-equivariant operator $D : (E, \eta^E) \to (F, \eta^F)$ over a G-space (X, σ), the principal symbol defines an equivariant section

$$\sigma(D) \in \Gamma\big((F, \eta^F) \otimes (E, \eta^E)^* \otimes (T^*X, d\sigma)\big).$$

However, due to the factor of i in (4.1), the principal symbols of operators with anti-linear symmetries satisfy a slightly different equivariance condition. This fact was noticed by Atiyah and Singer, and lead to the development of KR-theory [2, §5] [10]. The next result identifies the principal symbol of an orientifold operator as an equivariant section of an orientifold bundle.

Proposition 4.12. *The principal symbol* $\sigma(D)$ *of an equivariant first-order pseudodifferential operator* $D : E \to F$ *between orientifold bundles defines an equivariant section of the orientifold bundle*

$$(F, \eta^F) \otimes (E, \eta^E)^* \otimes (T^*X, \iota_\epsilon d\sigma),$$

where ι_ϵ *is the involutive action induced by negation.*

Proof. The equivariance condition for the symbol of a locally defined equivariant operator between E and F can be computed using Lemma 1.47. First the orientifold bundles E and F are expressed as semi-equivariant associated bundles

$$E := \mathrm{Fr}(E) \times_{(GL(m_1,\mathbb{C}),\kappa_\epsilon)} (\mathbb{C}^{m_1}, \kappa_\epsilon) \qquad F := \mathrm{Fr}(F) \times_{(GL(m_1,\mathbb{C}),\kappa_\epsilon)} (\mathbb{C}^{m_2}, \kappa_\epsilon).$$

Then, taking trivialisations and cocycles as in Lemma 1.47, the equivariance condition is

$$\begin{aligned}
&\sum_j^n A_a^j(x)\Big(\frac{\partial}{\partial x_j}\psi_a\Big)(x) + B_a(x)\psi_a(x) \\
&\quad = \kappa_{\epsilon(\gamma^{-1})} \circ \phi^F_{ba,\gamma} \circ h_a^{-1}(x) \\
&\qquad\qquad \cdot \kappa_{\epsilon(\gamma^{-1})}\Big(\sum_j A_b^j \circ h_{ba}(x) \cdot \frac{\partial}{\partial y_j}\big(\phi^{E,-1}_{ba,\gamma} \circ h_a^{-1} \circ h_{ab} \cdot \kappa_{\epsilon(\gamma)} \circ \psi_a \circ h_{ab}\big) \circ h_{ba}(x)\Big) \\
&\qquad + \kappa_{\epsilon(\gamma^{-1})} \circ \phi^F_{ba,\gamma} \circ h_a^{-1}(x) \\
&\qquad\qquad \cdot \kappa_{\epsilon(\gamma^{-1})}\Big(B_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1} \circ h_{ab}(x) \cdot \kappa_{\epsilon(\gamma)} \circ \psi_a(x)\Big).
\end{aligned}$$

Applying the Leibniz Rule and discarding lower-order terms produces the expression

$$\sum_j^n A_a^j(x)\Big(\frac{\partial}{\partial x_j}\psi_a\Big)(x)$$
$$= \kappa_{\epsilon(\gamma^{-1})} \circ \phi^F_{ba,\gamma} \circ h_a^{-1}(x)$$
$$\cdot \kappa_{\epsilon(\gamma^{-1})}\Big(\sum_j A_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1}(x) \cdot \kappa_{\epsilon(\gamma)} \circ \big(\frac{\partial}{\partial y_j}\psi_a \circ h_{ab}\big) \circ h_{ba}(x)\Big)$$
$$= \sum_j \kappa_{\epsilon(\gamma^{-1})}\Big(\phi^F_{ba,\gamma} \circ h_a^{-1}(x) \cdot A_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1}(x)\Big) \cdot \big(\frac{\partial}{\partial y_j}\psi_a \circ h_{ab}\big) \circ h_{ba}(x).$$

If H_{ba} is the Jacobian of h_{ba} this becomes

$$\sum_j \kappa_{\epsilon(\gamma^{-1})}\Big(\phi^F_{ba,\gamma} \circ h_a^{-1}(x) \cdot A_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1}(x)\Big) \cdot H_{ba}(x) \cdot \Big(\frac{\partial}{\partial x_j}\psi_a\Big)(x).$$

Thus, the the matrix coefficients A^j satisfy

$$\sum_j A_a^j(x) = \sum_j \kappa_{\epsilon(\gamma^{-1})}\Big(\phi^F_{ba,\gamma} \circ h_a^{-1}(x) \cdot A_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1}(x)\Big) \cdot H_{ba}(x).$$

However, the principal symbol of the operator consists of the maps $iA_a^j(x)$, which satisfy

$$\sum_j iA_a^j(x) = \sum_j i\kappa_{\epsilon(\gamma^{-1})}\Big(\phi^F_{ba,\gamma} \circ h_a^{-1}(x) \cdot A_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1}(x)\Big) \cdot H_{ba}(x)$$
$$= \sum_j \kappa_{\epsilon(\gamma^{-1})}\Big(\phi^F_{ba,\gamma} \circ h_a^{-1}(x) \cdot iA_b^j \circ h_{ba}(x) \cdot \phi^{E,-1}_{ba,\gamma} \circ h_a^{-1}(x)\Big) \cdot \iota_{\epsilon(\gamma^{-1})}H_{ba}(x).$$

In view of Lemma 1.46, a collection of matrix valued maps satisfying this condition defines an equivariant section of the orientifold bundle $(F, \eta^F) \otimes (E, \eta^E)^* \otimes (TX, \iota_\epsilon d\sigma)^*$. □

Proposition 4.12 implies that the principal symbol of an elliptic operator defines an orientifold K-theory class.

Proposition 4.13. *The principal symbol of a first order elliptic orientifold operator defines an element*

$$[\sigma(D)] \in K_{(\Gamma,\epsilon)}(TX, \iota_\epsilon d\sigma).$$

Proof. By Proposition 4.12, $\sigma(D)$ defines a homomorphism $E_x \to F_x$ for each $\xi \in TX_x$. As D is elliptic, this map is an isomorphism for all non-zero ξ. The zero section of TX is diffeomorphic to X, which is compact. Thus, $\sigma(D)$ is a complex and represents a class in $K_{(\Gamma,\epsilon)}(TX, \iota_\epsilon d\sigma)$. □

Conversely, each class in $K_{(\Gamma,\epsilon)}(TX, \iota_\epsilon d\sigma)$ is of the form $[\sigma(D)]$ for some elliptic orientifold operator D. This operator is clearly not unique. However, the index of any such operator will define the same class in the orientifold K-theory of a point. This will be discussed further in the next section.

Proposition 4.13 generalises the observation that lead to the development of KR-theory. The complexification of a real elliptic operator $D : E \to F$ defines a Real elliptic operator

$$D \otimes \mathrm{id} : (E \otimes \mathbb{C}, \mathrm{id} \otimes \kappa) \to (F \otimes \mathbb{C}, \mathrm{id} \otimes \kappa).$$

By Proposition 4.13, the symbol of this operator forms a class

$$[\sigma(D)] \in K_{(\mathbb{Z}_2,\mathrm{id})}(TX, \iota_\epsilon) = KR(TX, \iota).$$

Thus, if one wishes to retain information about the reality of the operator whilst considering the symbol of its complexification, it is neccesary to deal with KR-theory. This is significant when constructing topological indicies of the type used in the index theorem for families of real elliptic operators [10]. It is important to note that although the index theorem for families of real operators is stated in terms of KO-theory, the proof is given in terms of KR-theory. Thus, the theorem can also be applied to Real operators [10, Remark p. 5]. Similarly, the method is described in [63, III.16] for computing the Clifford index of a real Clifford linear operator, using the families index theorem for real operators, also applies to Real Clifford linear operators. Using this method, it is possible to compute the Clifford index of the orientifold Dirac operator when $(\Gamma, \epsilon) = (\mathbb{Z}_2, \mathrm{id})$. When applied to the real Clifford linear Dirac operator, this Clifford index provides the *Atiyah-Milnor-Singer invariant* of a Spin-manifold. Thus, applying this method to the orientifold Dirac operator for the orientifold group $(\mathbb{Z}_2, \mathrm{id})$ yields an Atiyah-Milnor-Singer invariant for Real spaces X satisfying $W_3^{(\mathbb{Z}_2,\mathrm{id})}(X) = 0$.

4.3 Index Maps in Orientifold K-theory

Recall the following basic facts about bounded linear operators $T : H_1 \to H_2$ between Hilbert spaces. Each operator T has a *kernel*, *image*, and *cokernel* defined respectively by

$$\ker(T) := \{h \in H_1 \mid T(h) = 0\}, \quad \mathrm{im}(T) := \{T(h_1) \in H_2 \mid h_1 \in H_1\}, \quad \mathrm{coker}(T) := H_2 \setminus \overline{\mathrm{im}(T)}.$$

If the kernel and cokernel of T are finite dimensional, then it is said to be a *Fredholm operator*. Each Fredholm operator has a well-defined *index*,

$$\mathrm{ind}(F) := \dim(\ker(F)) - \dim(\mathrm{coker}(F)).$$

By Atkinson's theorem, this is equivalent to the criteria that F be invertible modulo compact operators [63, p.192]. If $F : H_1 \to H_2$ is a Fredholm operator which is equivariant with respect to orientifold actions on H_1 and H_2, then its kernel and cokernel are finite dimensional orientifold representations. The formal difference of these orientifold representations defines a class in the orientifold K-theory of a point. This class is taken to be the orientifold index.

Definition 4.14. The *orientifold index* of a Fredholm orientifold operator $F : H \to H$ is the class

$$\mathrm{ind}(F) := [\ker(F)] - [\mathrm{coker}(F)] \in K_{(\Gamma,\epsilon)}(\mathrm{pt}).$$

Standard results from the theory of elliptic operators show that every elliptic pseudodifferential operator extends to a bounded operator between Hilbert spaces, and has an inverse modulo compact operators called a parametrix. There are many such extensions, however the index does not depend on which extension is chosen [63, III.5,III.7]. Thus, an elliptic operator has a well-defined index. Given an elliptic orientifold operator $D : \Gamma(E) \to \Gamma(F)$, the orientifold actions on E and F can be extended to unitary/anti-unitary orientifold actions on the associated Hilbert spaces. Any extension of D is equivariant with respect to the corresponding unitary/anti-unitary orientifold actions. The orientifold index of D is defined as the orientifold index of any extension.

The key property of the usual index map is stability under continuous deformation. This result is proved for operators which are equivariant with respect to linear actions in [63, III.7, III.9]. The same arguments made there hold for orientifold operators and result in the following theorem for the index of an elliptic orientifold operator.

Theorem 4.15. *The orientifold index* $\mathrm{ind}(D) \in K_{(\Gamma,\epsilon)}(\mathrm{pt})$ *of an elliptic orientifold operator* D *depends only on the orientifold* K*-theory class* $[\boldsymbol{\sigma}(D)] \in K_{(\Gamma,\epsilon)}(TX, \iota_\epsilon d\sigma)$ *of its principal symbol.*

Because the index map is well-defined at the level of the symbol class in K-theory, its interaction with operations in K-theory can be examined. Given an elliptic operator D on X and an orientifold bundle B on X, let D_B be an elliptic operator with principal symbol

$$\boldsymbol{\sigma}(D_B) = \boldsymbol{\sigma}(D) \otimes \mathrm{id}_B \in \Gamma\Big(\big((F,\eta^F) \otimes B\big) \otimes \big((E,\eta^E) \otimes B\big)^* \otimes (TX, \iota_\epsilon d\sigma)^*\Big).$$

Such an operator will be refered to as an operator *with coefficients in* B. By Theorem 4.15, the index class $\mathrm{ind}(D_B) \in K_{(\Gamma,\epsilon)}(\mathrm{pt})$ depends only on $[\boldsymbol{\sigma}(D_B)]$, and not on the specific operator D_B chosen. Using this construction, one can define a map from the orientifold K-theory of X into the orientifold K-theory of a point.

Definition 4.16. Let $D : E \to F$ be an elliptic orientifold operator over X. The *index map* associated to D is the $K_{(\Gamma,\epsilon)}(\mathrm{pt})$-module homomorphism defined by

$$\begin{aligned} \mathrm{ind}_D : K_{(\Gamma,\epsilon)}(X) &\to K_{(\Gamma,\epsilon)}(\mathrm{pt}) \\ [B] &\mapsto \mathrm{ind}(D_B). \end{aligned}$$

Two computations of the index map, related to the elements $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$, are particularly important for the proofs of the Thom isomorphisms. The first of these is the evaluation of the index map associated to the Dolbeault operator over $(\mathbb{C}P^n, \kappa_\epsilon)$ on the class $\boldsymbol{\lambda}$. This computation is connected with the $(1,1)$-Thom isomorphism because $(\mathbb{C}P^n, \kappa_\epsilon)$ is the projective compacification of the model fibre $(\mathbb{C}^n, \kappa_\epsilon)$ for an orientifold bundle.

Lemma 4.17. *The index map associated to the Dolbeault operator*

$$\bar{\partial} + \bar{\partial}^* : \Omega^{(0,\mathrm{even})}(\mathbb{C}P^n, \kappa_\epsilon) \to \Omega^{(0,\mathrm{odd})}(\mathbb{C}P^n, \kappa_\epsilon)$$

applied to the compactification of the $(1,1)$*-Bott class* $\underline{\boldsymbol{\lambda}}_{\mathrm{pt}}$ *is equal to the class of the trivial one-dimensional orientifold representation,*

$$\mathrm{ind}_{\bar{\partial}+\bar{\partial}^*}(\boldsymbol{\lambda}) = [\mathbb{C}, \kappa_\epsilon] \in K_{(\mathbb{Z}_2,\mathrm{id})\ltimes(U(n),\kappa_\epsilon)}(\mathrm{pt}).$$

Proof. The orientifold action on the bundles $\Omega^{(0,\mathrm{even})}(\mathbb{C}P^n, \kappa_\epsilon) \otimes \boldsymbol{\lambda}$ and $\Omega^{(0,\mathrm{odd})}(\mathbb{C}P^n, \kappa_\epsilon) \otimes \boldsymbol{\lambda}$ is an involutive action obtained from their Real structure. Thus, it is only neccesary to carry out the calculation in the Real case, and this was done in [4, pp. 122-123, 126-127]. The proof proceeds by using the Hodge decomposition for Kähler manifolds [20, Thm. 7.2, §I.7] to relate $\ker((\bar{\partial} + \bar{\partial}^*)_{\boldsymbol{\lambda}})$ to cohomology with coefficients in $\boldsymbol{\lambda}$. These cohomology groups are then computed using vanishing theorems due to Kodaira [47, Ch. 18]. □

Another important index computation is associated to the element $\boldsymbol{\beta}$, and the reduced orientifold Dirac operator on a sphere of dimension $8k$. A canonical reduced orientifold spinor bundle, and orientifold Dirac operator, always exist on the spheres of dimension $8k$, due to Lemma 3.18. The spheres S^{8k} are relevant to the 8-fold Thom isomorphism as S^{8k} can be regarded as the one-point compactification of the model fibre $\mathbb{R}^{8k}$ for a real equivariant vector bundle of dimension $8k$. As noted in Section 3.2, the restriction to dimension $8k$ is necessary in order to construct the reduced spinor bundle, and is related to the representation theory of real Clifford algebras.

Lemma 4.18. *The index map associated to the positive part*

$$\not{\mathfrak{D}}^+ : \Gamma(\not{\mathfrak{S}}^+) \to \Gamma(\not{\mathfrak{S}}^-)$$

of the orientifold Dirac operator over S^{8k} applied to the compactification of the 8-fold Bott class $\underline{\boldsymbol{\beta}}_{\mathrm{pt}}$ is equal to the class of the trivial one-dimensional orientifold representation,

$$\mathrm{ind}_{\not{\mathfrak{D}}^+}(\boldsymbol{\beta}+\boldsymbol{\beta}_\infty) = (\mathbb{C},\kappa_\epsilon) \in K_{(\mathbb{Z}_2,\mathrm{id})\ltimes_{\kappa_\epsilon}\mathrm{Spin}^c(n)}(\mathrm{pt}).$$

Proof. Several reductions can be made. First note that $\mathrm{ind}_{\not{\mathfrak{D}}^+}(\boldsymbol{\beta}_\infty) = 0$ because $\boldsymbol{\beta}_\infty$ is trivial and $\not{\mathfrak{D}}^+$ is self-adjoint. Next, the decomposition $\not{\mathfrak{S}}(V) \simeq \not{S} \otimes (\Delta_c,\kappa_\epsilon)$ provided by Proposition 3.38 implies that, in dimension $8k$, $\not{\mathfrak{D}} = \not{D}_{(\Delta_c,\kappa_\epsilon)}$. Because $(\Delta_c,\kappa_\epsilon)$ is trivial this implies that $\mathrm{ind}_{\not{\mathfrak{D}}^+} = \mathrm{ind}_{\not{D}^+}$. Furthermore, Corollary 3.39 and Lemma 3.37 imply

$$\not{S} \otimes (\not{S}^+)^* \simeq \mathbb{C}l^+(V) \simeq \Lambda^+(V)\otimes\mathbb{C},$$

so that $\mathrm{ind}(\not{D}^+_{\boldsymbol{\beta}})$ can be identified with the index of

$$(d+d^*)^+ \otimes \mathrm{id} : \Lambda^+(TS^{8k})\otimes\mathbb{C} \to \Lambda^-(TS^{8k})\otimes\mathbb{C},$$

where $\Lambda^\pm$ denotes grading by $\boldsymbol{\omega}$. A section in the kernel or cokernel of $(d+d^*)^+\otimes\mathrm{id}$ is the complexification of section in the kernel of

$$(d+d^*) : \Lambda(TS^{8k}) \to \Lambda(TS^{8k})$$

that is invariant under $\boldsymbol{\omega}$. The kernel of $d+d^*$ can be computed using the self-adjointness of $d+d^*$ followed by the Hodge isomorphism theorem for Riemannian manifolds [20, p. 20] [53, Thm 3.41],

$$\ker(d+d^*) = \ker(d+d^*)^2 \simeq H^\bullet_{dR}(S^{8k}) \simeq \begin{cases} \mathbb{R} \text{ for } p = 0, 8k \\ 0 \text{ otherwise }, \end{cases}$$

where $H^\bullet_{dR}(S^{8k})$ is the de Rham cohomology of S^{8k}. Thus, $\ker(d+d^*)$ is the span of two sections, $\psi^0 \in \Lambda^0(TS^{8k})$ and $\psi^{8k} \in \Lambda^{8k}(TS^{8k})$. The grading operator interchanges $\Lambda^{8k}(TS^{8k})$ and $\Lambda^0(TS^{8k})$. Without loss of generality, assume that $\psi^{8k} = \boldsymbol{\omega}\psi^0$. The subspace of $\ker(d+d^*)$ that is invariant under $\boldsymbol{\omega}$ is then spanned by the single section $\psi = \frac{1}{2}(\psi^0 + \boldsymbol{\omega}\psi^0)$. Complexifying this section and taking into account the anti-linear action on $\not{S}^+\otimes\boldsymbol{\beta}$ proves that

$$\mathrm{ind}_{\not{\mathfrak{D}}^+}(\boldsymbol{\beta}+\boldsymbol{\beta}_\infty) = [\mathbb{C},\kappa_\epsilon] \in K_{(\mathbb{Z}_2,\mathrm{id})\ltimes_{\kappa_\epsilon}\mathrm{Spin}^c(n)}(\mathrm{pt})$$

as required. □

In order to prove Bott periodicity, it will be neccesary to extend the preceeding discussion of the index map to equivariant families of elliptic operators. In the orientifold setting, a family of operators acts between orientifold bundles $\boldsymbol{E}$ and $\boldsymbol{F}$ over an orientifold $\boldsymbol{Y}$. The orientifold $\boldsymbol{Y}$ is required to be a fibre bundle $\pi : \boldsymbol{Y} \to X$ with an equivariant projection map. This makes the bundles $\boldsymbol{E}$ and $\boldsymbol{F}$ into fibre bundles over X, where each fibre $\boldsymbol{E}_x := \boldsymbol{E}|\pi^{-1}(x)$ is a vector bundle over $\boldsymbol{Y}_x := \pi^{-1}(x)$. Similarly, a section $\boldsymbol{\psi}$ of $\boldsymbol{E}$ or $\boldsymbol{F}$ decomposes into a family of sections $\boldsymbol{\psi}_x := \boldsymbol{\psi}|\boldsymbol{Y}_x \in \Gamma(\boldsymbol{E}_x)$. A family of operators $\boldsymbol{D}$ is then an assignment of operators

$$\boldsymbol{D}_x : \Gamma(\boldsymbol{E}_x) \to \Gamma(\boldsymbol{F}_x)$$

to each $x \in X$ in a continuous manner. Such a family of operators acts on a family of sections by $(\boldsymbol{D\psi})_x := \boldsymbol{D}_x\boldsymbol{\psi}_x$. The orientifold actions on $\boldsymbol{E}$ and $\boldsymbol{F}$ induce actions on sections in the usual manner, $(\gamma\boldsymbol{\psi})(y) := \gamma\boldsymbol{\psi}(\gamma^{-1}y)$ for $y \in \boldsymbol{Y}$. Equivariance of the family of operators $\boldsymbol{D}$ is then interpreted to mean that $\boldsymbol{D}(\gamma\boldsymbol{\psi}) = \gamma(\boldsymbol{D\psi})$.

Taking further advantage of the stability properties of the index, it is possible to define an index

$$\operatorname{ind}(\boldsymbol{D}) \in K_{(\Gamma,\epsilon)}(X)$$

associated to an equivariant family $\boldsymbol{D}$ of elliptic operators parameterised by an orientifold X. Naïvely, one can understand the index of a family $\boldsymbol{D}$ by noting that, for each $x \in X$, the kernel $\ker(\boldsymbol{D}_x)$ defines a vector space over the point $x \in X$. The idea is then that, because the family of operators varies continuously, the vector spaces $\ker(\boldsymbol{D}_x)$ might combine to form a vector bundle over X. If the same were true for $\operatorname{coker}(\boldsymbol{D}_x)$, then the resulting pair of vector bundles would define a class in $K(X)$. Equivariance of $\boldsymbol{D}$ would imply that these vector bundles are orientifold bundles, and so define a class $\operatorname{ind}(\boldsymbol{D}) \in K_{(\Gamma,\epsilon)}(X)$. This idea cannot be applied directly because, even when varying x continuously, the dimension of the vector spaces $\ker(\boldsymbol{D}_x)$ and $\operatorname{coker}(\boldsymbol{D}_x)$ can change. However, the K-theory class $[\ker(\boldsymbol{D}_x)] - [\operatorname{coker}(\boldsymbol{D}_x)]$ of the index is more stable than the kernel or cokernel alone. Thus, a procedure exists for modifying the family of operators $\boldsymbol{D}$ to give another equivariant family of operators $\tilde{\boldsymbol{D}}$ such that the dimensions of $\ker(\tilde{\boldsymbol{D}}_x)$ and $\operatorname{coker}(\tilde{\boldsymbol{D}}_x)$ are constant in x. There are some choices involved in the construction of $\tilde{\boldsymbol{D}}$, however it can be shown that the index is independent of these. Thus, a well-defined index $\operatorname{ind}(\tilde{\boldsymbol{D}})$ can be associated to any equivariant family $\boldsymbol{D}$ parameterised by a compact orientifold. In the non-equivariant setting, the following lemma holds [63, Lemma III.8.4, pp. 206-207].

Lemma 4.19. *Let $\boldsymbol{D}$ be a continuous family of elliptic operators parameterised by a compact Hausdorff space X. There exists a finite set of sections $\{\varphi_i \in \Gamma(\boldsymbol{F}) \mid 1 \leq i \leq N\}$ such that the map*

$$\begin{aligned}\tilde{\boldsymbol{D}}_x : \Gamma(\boldsymbol{E}_x) \oplus \mathbb{C}^N &\to \Gamma(\boldsymbol{F}_x)\\ (\psi, z_1, \cdots, z_N) &\mapsto \boldsymbol{D}_x(\psi) + \sum z_j\varphi_j|_x\end{aligned}$$

is surjective for all $x \in X$. *The vector spaces* $\ker(\tilde{\boldsymbol{D}}_x)$ *have constant dimension and combine to form the fibres of a vector bundle* $\ker(\tilde{\boldsymbol{D}})$ *over* X. *The class*

$$[\ker(\tilde{\boldsymbol{D}})] - [\mathbb{C}^N] \in K(X)$$

depends only on the original operator D.

An equivariant version of this result is proved in [76]. The corresponding result for orientifold operators asserts the existence of finite set of sections $\{\varphi_i \in \Gamma(\boldsymbol{F}) \mid 1 \leq i \leq N\}$ which are equivariant with respect to the orientifold action on $\boldsymbol{F}$, and that make the operators

$$\begin{aligned} \tilde{\boldsymbol{D}}_x : \Gamma(\boldsymbol{E}_x) \oplus (\mathbb{C}^N, \kappa_\epsilon) &\to \Gamma(\boldsymbol{F}_x) \\ (\psi, z_1, \cdots, z_N) &\mapsto \boldsymbol{D}_x(\psi) + \sum z_j \varphi_j|_x \end{aligned}$$

surjective. The family $\tilde{\boldsymbol{D}}$ is then equivariant and its index $[\ker(\tilde{\boldsymbol{D}})] - [(\mathbb{C}^N, \kappa_\epsilon)] \in K_{(\Gamma,\epsilon)}(X)$ is well-defined. As in the non-equivariant case, it can be shown that this class is independent of the sections φ_i chosen.

Definition 4.20. Let $\boldsymbol{E}$ and $\boldsymbol{F}$ be orientifold bundles over a family of orientifolds $\boldsymbol{Y} \to X$, where X is a compact. The *index* of an equivariant family of elliptic operators $\boldsymbol{D} : \boldsymbol{E} \to \boldsymbol{F}$ is defined by

$$\operatorname{ind}(\boldsymbol{D}) := [\ker\tilde{\boldsymbol{D}}] - [(\mathbb{C}^N, \kappa_\epsilon)] \in K_{(\Gamma,\epsilon)}(X).$$

Using this index, an index map can be defined by analogy with Definition 4.16. A family of elliptic operators $\boldsymbol{D}$ defines a family of principal symbols $\boldsymbol{\sigma}(\boldsymbol{D}_x)$. If $\boldsymbol{B} \to \boldsymbol{Y}$ is an orientifold bundle, then the restrictions $\boldsymbol{B}_x := \boldsymbol{B}|\boldsymbol{Y}_x$ form a family of vector bundles parameterised by X. The corresponding symbols and bundles can be twisted together to form a new family of symbols $\boldsymbol{\sigma}(\boldsymbol{D}_x) \otimes \mathrm{id}_{\boldsymbol{B}_x}$. These symbols define a new family $\boldsymbol{D}_{\boldsymbol{B}}$ of elliptic operators parameterised by X. If $\boldsymbol{D}$ is equivariant, then $\boldsymbol{D}_{\boldsymbol{B}}$ is also equivariant. This leads to the following definition, which generalises Definition 4.16.

Definition 4.21. The *index map* associated to an equivariant family $\boldsymbol{D}$ of elliptic operators between orientifold bundles parameterised by X is defined by

$$\begin{aligned} \operatorname{ind}_{\boldsymbol{D}} : K_{(\Gamma,\epsilon)}(\boldsymbol{Y}) &\to K_{(\Gamma,\epsilon)}(X) \\ \boldsymbol{B} &\mapsto \operatorname{ind}(\boldsymbol{D}_{\boldsymbol{B}}). \end{aligned}$$

It can be shown that ind_D is a $K_{(\Gamma,\epsilon)}(X)$-module homomorphism so that

$$\operatorname{ind}_{\boldsymbol{D}}(yx) = \operatorname{ind}_{\boldsymbol{D}}(y)x,$$

for all $\boldsymbol{D}$, $x \in K_{(\Gamma,\epsilon)}(\boldsymbol{Y})$ $y \in K_{(\Gamma,\epsilon)}(X)$.

Index maps associated to families of operators will be used to construct inverses to the Bott periodicity maps. In order to do this, it will be neccesary to construct equivariant families of operators parameterised by a given orientifold X. This can be done by taking a product family of operators as follows. Suppose that $D : \Gamma(E) \to \Gamma(F)$ is an elliptic orientifold operator over an orientifold M. Let $\boldsymbol{Y} := X \times M$ be the product orientifold, and denote the component projections by $\pi_X : X \times M \to X$ and $\pi_M : X \times M \to M$. The map π_X makes $\boldsymbol{Y}$ into a family of orientifolds. The orientifold bundles E and F can be pulled back to orientifold bundles $\boldsymbol{E} = \pi_M^* E$ and $\boldsymbol{F} = \pi_M^* F$ over $\boldsymbol{Y}$. A family of operators $\boldsymbol{D}_x : \Gamma(\boldsymbol{E}_x) \to \Gamma(\boldsymbol{F}_x)$ can then be defined by identifying $\boldsymbol{E}_x$ with E, $\boldsymbol{F}_x$ with F, and setting $\boldsymbol{D}_x := D$ for all x. Such a family is always equivariant, regardless of the action on X.

By constructing a product family of operators and taking its index map, it is possible to associate an index map

$$\mathrm{ind}_X^D : K_{(\Gamma,\epsilon)}(X \times M) \to K_{(\Gamma,\epsilon)}(X)$$

to any orientifold operator D over M, and any compact orientifold X. These maps are functorial in X, in the sense that if $f : Y \to X$ is a map of orientifolds then

$$f^* \circ \mathrm{ind}_X^D = \mathrm{ind}_Y^D \circ (f \times \mathrm{id})^*.$$

Applying this construction to the operators $\bar{\partial} + \bar{\partial}^*$ and $\not{\mathfrak{D}}^+$ on complex projective space and the sphere of dimension $8k$ produces maps

$$\mathrm{ind}_X^{\bar{\partial}+\bar{\partial}^*} : K_{(\mathbb{Z}_2,\mathrm{id})\ltimes(U(n),\kappa_\epsilon)}(X \times \mathbb{C}P^n) \to K_{(\mathbb{Z}_2,\mathrm{id})\ltimes(U(n),\kappa_\epsilon)}(X) \tag{4.2}$$

$$\mathrm{ind}_X^{\not{\mathfrak{D}}^+} : K_{(\mathbb{Z}_2,\mathrm{id})\ltimes(\mathrm{Spin}^c(n),\kappa_\epsilon)}(X \times S^{8k}) \to K_{(\mathbb{Z}_2,\mathrm{id})\ltimes(\mathrm{Spin}^c(n),\kappa_\epsilon)}(X). \tag{4.3}$$

The orientifold groups for these index maps are fixed. However, further flexibility can be introduced by noting that a homomorphism $\varphi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ of orientifold groups makes an orientifold (M, τ) for (Γ, ϵ) into an orientifold $(M, \tau \circ \varphi)$ for (Γ', ϵ'), and that if D is an (Γ, ϵ)-equivariant orientifold operator over M, then it is also (Γ', ϵ')-equivariant orientifold operator over $(M, \tau \circ \varphi)$. Thus, given a single (Γ, ϵ)-equivariant orientifold operator D over (M, τ), it is possible to construct index maps

$$\mathrm{ind}_X^D : K_{(\Gamma',\epsilon')}(X \times (M, \tau \circ \varphi)) \to K_{(\Gamma',\epsilon')}(X),$$

for all homomorphisms $\varphi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ and compact orientifolds X acted on by (Γ', ϵ'). Applying this construction to $\bar{\partial} + \bar{\partial}^*$ and $\not{\mathfrak{D}}^+$ produces maps

$$\mathrm{ind}_X^{\bar{\partial}+\bar{\partial}^*} : K_{(\Gamma_1,\epsilon_1)}(X \times \mathbb{C}P^n) \to K_{(\Gamma_1,\epsilon_1)}(X) \tag{4.4}$$

$$\mathrm{ind}_X^{\not{\mathfrak{D}}^+} : K_{(\Gamma_2,\epsilon_2)}(X \times S^{8k}) \to K_{(\Gamma_2,\epsilon_2)}(X), \tag{4.5}$$

for any action of (Γ_1, ϵ_1) on $\mathbb{C}P^n$ that acts through a homomorphism to $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} U(n)$, and any action of (Γ_2, ϵ_2) on S^{8k} that acts through a homomorphism to $(\mathbb{Z}_2, \mathrm{id}) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(n)$.

The index maps (4.4) and (4.5) will be used to construct inverses to the Bott and Thom maps. The next proposition collects together the results and observations from this section that will be required for the proof of equivariant Bott periodicity.

Proposition 4.22. *The homomorphisms*

$$\mathrm{ind}_X^D : K_{(\Gamma,\epsilon)}(X \times M) \to K_{(\Gamma,\epsilon)}(X)$$

satisfy

$$f^* \circ \mathrm{ind}_X = \mathrm{ind}_Y \circ (f \times \mathrm{id})^* \qquad\qquad \mathrm{ind}_X(yx) = \mathrm{ind}_X(y)x,$$

for all elliptic orientifold operators D *on compact orientifolds* M*, compact orientifolds* X*, continuous maps* $f : Y \to X$*, and orientifold* K*-theory classes* $y \in K_{(\Gamma,\epsilon)}(X \times M)$ *and* $x \in K_{(\Gamma,\epsilon)}(X)$*. Furthermore, the maps* (4.4) *and* (4.5) *satisfy*

$$\mathrm{ind}_{\mathrm{pt}}^{\bar\partial+\bar\partial^*}(\boldsymbol{\lambda}) = [\mathbb{C}, \kappa_\epsilon] \in K_{(\Gamma_1,\epsilon_1)}(\mathrm{pt}),$$
$$\mathrm{ind}_{\mathrm{pt}}^{\not{D}^+}(\boldsymbol{\beta} + \boldsymbol{\beta}_\infty) = [\mathbb{C}, \kappa_\epsilon] \in K_{(\Gamma_2,\epsilon_2)}(\mathrm{pt}),$$

where $[\mathbb{C}, \epsilon]$ *is the class of the trivial one-dimensional orientifold bundle.*

4.4 Functoriality and Index Pairings in Orientifold K-theory

In this section, it will be shown that the various properties collected in Proposition 4.22 are enough to prove that the index maps (4.4) and (4.5) provide two-sided inverses to the Bott periodicity maps. The method used closely follows [4]. To deal with the two separate cases at once, it is helpful to abstract the discussion. To this end, define the following objects which will be used throughout this section:

1. A representation V of (Γ, ϵ). This may be either an orientifold representation or a real representation.

2. A distinguished class $\boldsymbol{b}_{\mathrm{pt}} \in K_{(\Gamma,\epsilon)}(\mathrm{pt} \times V)$.

3. For each compact orientifold X, a class defined by

$$\boldsymbol{b}_X := (f \times \mathrm{id})^*(\boldsymbol{b}_{\mathrm{pt}}) \in K_{(\Gamma,\epsilon)}(X \times V),$$

where $f : X \to \mathrm{pt}$.

4. Multiplication maps

$$\mathbb{B}_X : K_{(\Gamma,\epsilon)}(X) \to K_{(\Gamma,\epsilon)}(X \times V)$$
$$x \mapsto \boldsymbol{b}_X x,$$

associated to the classes $\boldsymbol{b}_X$.

5. Homomorphisms

$$\mathbb{A}_X : K_{(\Gamma,\epsilon)}(X \times V) \to K_{(\Gamma,\epsilon)}(X)$$

which satisfy

$$f^* \circ \mathbb{A}_X = \mathbb{A}_Y \circ (f \times \mathrm{id})^* \qquad \mathbb{A}_X(yx) = \mathbb{A}_X(y)x \qquad \mathbb{A}_{\mathrm{pt}}(\boldsymbol{b}_{\mathrm{pt}}) = [\mathbb{C}, \kappa_\epsilon] \in K_{(\Gamma,\epsilon)}(\mathrm{pt}),$$

for all continuous maps $f : Y \to X$, $y \in K_{(\Gamma,\epsilon)}(X \times V)$ and $x \in K_{(\Gamma,\epsilon)}(X)$.

A short series of results will be used to shown that the maps $\mathbb{B}_X$ and $\mathbb{A}_X$ are two-sided inverses to one another. First, the maps $\mathbb{A}_X$ are extended.

Lemma 4.23 (cf. [4, Lemma 1.2])**.** *Let W be an orientifold representation or a real representation of (Γ, ϵ). The homomorphisms $\mathbb{A}_X$ extend to homomorphisms*

$$\mathbb{A}_{X\times W} : K_{(\Gamma,\epsilon)}(X \times W \times V) \to K_{(\Gamma,\epsilon)}(X \times W)$$

satisfying

1. *for all continuous maps $f : Y \to X$*

$$f^* \circ \mathbb{A}_{X\times W} = \mathbb{A}_{Y\times W} \circ (f \times \mathrm{id})^*$$

2. *for all $y \in K_{(\Gamma,\epsilon)}(X \times W \times V)$, $x \in K_{(\Gamma,\epsilon)}(X \times Z)$,*

$$\mathbb{A}_{X\times W\times Z}(yx) = \mathbb{A}_{X\times W}(y)x \in K(X \times W \times Z),$$

where Z is either an orientifold representation or a real representation of (Γ, ϵ).

Proof. Suppose that X is locally compact. For non-compact X, $K_{(\Gamma,\epsilon)}(X) := \ker(i^*)$ where $i : \mathrm{pt} \to X^+$ is the inclusion of the point at infinity. The maps $\mathbb{A}$ can then be extended to include maps $\mathbb{A}_X : K_{(\Gamma,\epsilon)}(X \times V) \to K_{(\Gamma,\epsilon)}(X)$ by observing that the square on the right side of the following diagram commutes due to the functoriality property of $\mathbb{A}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{(\Gamma,\epsilon)}(X \times V) & \longrightarrow & K_{(\Gamma,\epsilon)}(X^+ \times V) & \xrightarrow{(i\times \mathrm{id})^*} & K_{(\Gamma,\epsilon)}(V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbb{A}_X} & & \downarrow{\scriptstyle \mathbb{A}_{X^+}} & & \downarrow{\scriptstyle \mathbb{A}_{\mathrm{pt}}} & & \\ 0 & \longrightarrow & K_{(\Gamma,\epsilon)}(X) & \longrightarrow & K_{(\Gamma,\epsilon)}(X^+) & \xrightarrow{i^*} & K_{(\Gamma,\epsilon)}(\mathrm{pt}) & \longrightarrow & 0 \end{array}$$

Replacing X by $X \times W$ in the above diagram defines the extension

$$\mathbb{A}_{X\times W} : K_{(\Gamma,\epsilon)}(X \times V \times W) \to K_{(\Gamma,\epsilon)}(X \times W).$$

of $\mathbb{A}_X$. This extension of $\mathbb{A}$ inherits the functoriality property. It remains to check that the extension of $\mathbb{A}$ satisfies the stated module homomorphism property.

Let X and Y be compact, and $\pi_X : X \times Y \to X$ $\pi_Y : X \times Y \to Y$ be the coordinate projections. Consider the following diagram of orientifold K-theory groups[1]

$$\begin{CD}
K(X\times V)\otimes K(Y) @>(\pi_X\times \mathrm{id})^*\otimes\pi_Y^*>> K(X\times Y\times V)\otimes K(X\times Y) @>\boxtimes>> K((X\times Y\times V)\times(X\times Y)) @>\Delta_*>> K(X\times Y\times V)\\
@V\mathbb{A}_X\otimes\mathrm{id}VV @V\mathbb{A}_{X\times Y}\otimes\mathrm{id}VV @. @V\mathbb{A}_{X\times Y}VV\\
K(X)\otimes K(Y) @>\pi_X^*\otimes\pi_Y^*>> K(X\times Y)\otimes K(X\times Y) @>\boxtimes>> K((X\times Y)\times(X\times Y)) @>\Delta_*>> K(X\times Y).
\end{CD}$$

The square on the left commutes by the functoriality property of $\mathbb{A}$. The module homomorphism property implies that the square on the right commutes. Simplifying the composition of horizontal maps in the above diagram results in the commutative diagram

$$\begin{CD}
K_{(\Gamma,\epsilon)}(X\times V)\otimes K_{(\Gamma,\epsilon)}(Y) @>\boxtimes>> K_{(\Gamma,\epsilon)}(X\times Y\times V)\\
@V\mathbb{A}_X\otimes\mathrm{id}VV @V\mathbb{A}_{X\times Y}VV\\
K_{(\Gamma,\epsilon)}(X)\otimes K_{(\Gamma,\epsilon)}(Y) @>\boxtimes>> K_{(\Gamma,\epsilon)}(X\times Y).
\end{CD}$$

This diagram induces a corresponding diagram for locally compact X and Y. Then, replacing X with $X \times W$ and Y with $X \times Z$ produces a diagram

$$\begin{CD}
K_{(\Gamma,\epsilon)}(X\times V\times W)\otimes K_{(\Gamma,\epsilon)}(X\times Z) @>\boxtimes>> K_{(\Gamma,\epsilon)}(X\times X\times V\times W\times Z) @>\Delta_*>> K_{(\Gamma,\epsilon)}(X\times V\times W\times Z)\\
@V\mathbb{A}_{X\times W}\otimes\mathrm{id}VV @V\mathbb{A}_{X\times X\times W\times Z}VV @V\mathbb{A}_{X\times W\times Z}VV\\
K_{(\Gamma,\epsilon)}(X\times W)\otimes K_{(\Gamma,\epsilon)}(X\times Z) @>\boxtimes>> K_{(\Gamma,\epsilon)}(X\times X\times W\times Z) @>\Delta_*>> K_{(\Gamma,\epsilon)}(X\times W\times Z),
\end{CD}$$

where Δ is restriction to the diagonal in X. The commutivity of this diagram proves the required module homomorphism property. □

The next lemma will help to show that if $\mathbb{A}$ is a one-sided inverse to $\mathbb{B}$, then it is a two-sided inverse.

Lemma 4.24 (cf. [4, Remark p. 116]). *If* $x, y \in K_{(\Gamma,\epsilon)}(X \times V)$ *then*

$$xy = y\tilde{x} = \tilde{y}x \in K_{(\Gamma,\epsilon)}(X \times V \times V),$$

where $u \mapsto \tilde{u}$ *is the automorphism of* $K_{(\Gamma,\epsilon)}(X \times V)$ *induced by* $(x, v) \mapsto (x, -v)$.

Proof. Define the maps

[1] in this diagram K denotes $K_{(\Gamma,\epsilon)}$. The subscript (Γ, ϵ) has been suppressed to save space.

$$\theta : X \times V \times V \to X \times V \times V \qquad\qquad \vartheta : X \times V \times V \to X \times V \times V$$
$$(x, v, w) \mapsto (x, w, v), \qquad\qquad (x, v, w) \mapsto (x, v, -w).$$

The map θ satisfies $\theta^*(xy) = yx \in K_{(\Gamma,\epsilon)}(X \times V \times V)$. Thus,

$$\tilde{y}x = \theta^* \circ \vartheta^*(xy) \qquad xy = \mathrm{id}(xy) \qquad y\tilde{x} = \vartheta^* \circ \theta^*(xy).$$

The family of maps

$$r_t : X \times V \times V \to X \times V \times V$$
$$(x, u, v) \mapsto (x, u\cos t - v\sin t, v\cos t + u\sin t).$$

is an equivariant homotopy between the maps

$$r_{-\frac{\pi}{2}} = \theta^* \circ \vartheta^* \qquad r_0 = \mathrm{id} \qquad r_{\frac{\pi}{2}} = \vartheta^* \circ \theta^*.$$

Note that this homotopy is still equivariant when V is an orientifold representation because the coefficients $\cos t$ and $\sin t$ are real. The existence of a homotopy implies that the above maps induce the same map on K-theory, proving the lemma. □

Lemmas 4.23 and 4.24 can be now be used to show that $\mathbb{A}$ and $\mathbb{B}$ are two-sided inverses to one another.

Theorem 4.25 (c.f. [4, Prop. 1.5])**.** *The maps*

$$\mathbb{B}_X : K_{(\Gamma,\epsilon)}(X) \to K_{(\Gamma,\epsilon)}(X \times V) \qquad\qquad \mathbb{A}_X : K_{(\Gamma,\epsilon)}(X \times V) \to K_{(\Gamma,\epsilon)}(X)$$

are tw-sided inverses to one another for all X.

Proof. First, note that

$$\mathbb{A}_X(\boldsymbol{b}_X) = \mathbb{A}_X \circ (f \times \mathrm{id})^*(\boldsymbol{b}_{\mathrm{pt}}) = f^* \circ \mathbb{A}_{\mathrm{pt}}(\boldsymbol{b}_{\mathrm{pt}}) = f^* \circ [\mathbb{C}, \kappa_\epsilon]$$

where $f : X \to \mathrm{pt}$. Then, as $\mathbb{A}_X$ is a $K(X)$-module homomorphism,

$$\mathbb{A}_X \circ \mathbb{B}_X(x) = \mathbb{A}_X(\boldsymbol{b}_X x) = \mathbb{A}_X(\boldsymbol{b}_X)x = [\mathbb{C}, \kappa_\epsilon]x = x,$$

for $x \in K(X)$. Therefore, by Lemmas 4.23 and 4.24,

$$\mathbb{B}_X \circ \mathbb{A}_X(y) = \mathbb{A}_X(y)\boldsymbol{b}_X = \mathbb{A}_{X\times V}(y\boldsymbol{b}_X) = \mathbb{A}_{X\times V}(\boldsymbol{b}_X\tilde{y}) = \mathbb{A}_X(\boldsymbol{b}_X)\tilde{y} = [\mathbb{C}, \kappa_\epsilon]\tilde{y} = \tilde{y},$$

for $y \in K(X \times V)$. As $y \mapsto \tilde{y}$ is an automorphism, $\mathbb{B}_X$ and $\mathbb{A}_X$ are isomorphisms that are inverse to one another. □

4.5 Bott Periodicity and Thom isomorphisms

The development in Sections 4.3 and 4.4 followed the methods of [4] very closely. In [4] the elliptic operators method was applied in the Real equivariant setting. However, the small change in perspective, from the Real equivariant setting to the orientifold setting, means that the next theorem can accomodate the case in which Γ^- does not contain an involution, cf. [4, Theorems 5.1, 6.2]. The various Bott periodicity and Thom isomorphism theorems will be proved as consequences of this theorem.

Theorem 4.26 (Equivariant Bott Periodicity for Orientifolds.)**.** *The maps*

$$\begin{aligned} K_{(\Gamma,\epsilon)}(X) &\to K_{(\Gamma,\epsilon)}(X \times W) \\ x &\mapsto \underline{\boldsymbol{\lambda}}_X^W x \end{aligned} \qquad\qquad \begin{aligned} K_{(\Gamma,\epsilon)}(X) &\to K_{(\Gamma,\epsilon)}(X \times Z) \\ x &\mapsto \underline{\boldsymbol{\beta}}_X^Z x, \end{aligned}$$

are isomorphisms, for any orientifold representation W, and real representation Z of dimension $8k$ that acts through a homomorphism to $\mathbb{Z}_2 \ltimes_\kappa \mathrm{Spin}^c(8k)$.

Proof. To prove the first result, Theorem 4.25 is applied with $V = W$, and $\boldsymbol{b} = \underline{\boldsymbol{\lambda}}$. The map $\mathbb{A}$ is taken to be

$$\mathrm{ind}_X^{\bar\partial+\bar\partial^*} \circ i : K_{(\Gamma,\epsilon)}(X \times W) \to K_{(\Gamma,\epsilon)}(X \times \mathbb{P}(W \oplus (\mathbb{C}, \kappa_\epsilon))) \to K_{(\Gamma,\epsilon)}(X),$$

where i is the map induced by the fibrewise projective compactification of the trivial orientifold bundle $X \times (W \oplus (\mathbb{C}, \kappa_\epsilon)) \to X$. Because (Γ, ϵ) acts on W by unitary and anti-unitary operators, $\bar\partial + \bar\partial^*$ is equivariant with respect to the induced (Γ, ϵ)-action on $\mathbb{P}(W \oplus (\mathbb{C}, \kappa_\epsilon))$. Thus, the index maps $\mathrm{ind}_X^{\bar\partial+\bar\partial^*}$ can be constructed. By Proposition 4.22, the map $\mathrm{ind}_X^{\bar\partial+\bar\partial^*} \circ i$ has all of the properties required of $\mathbb{A}$. Thus, by Theorem 4.25 it is a two-sided inverse to the map $x \mapsto \boldsymbol{\lambda} x$, proving that it is an isomorphism.

Similarly, to prove the second result, Theorem 4.25 is applied with $V = Z$, and $\boldsymbol{b} = \underline{\boldsymbol{\beta}}$. The map $\mathbb{A}$ is taken to be

$$\mathrm{ind}_X^{\not{\mathfrak{D}}^+} \circ j : K_{(\Gamma,\epsilon)}(X \times V) \to K_{(\Gamma,\epsilon)}(X \times S(V \oplus \mathbb{R})) \to K_{(\Gamma,\epsilon)}(X),$$

where j is the map induced by the fibrewise one-point compactification of the trivial real equivariant bundle $X \times V \to X$. If (Γ, ϵ)-acts on V via $\mathbb{Z}_2 \ltimes_\kappa \mathrm{Spin}^c$, then the operator $\not{\mathfrak{D}}$ on $S(V \oplus \mathbb{R})$ is equivariant, and so the index map can be constructed. When $\dim V = 8$, Proposition 4.22, shows that the map $\mathrm{ind}_X^{\not{\mathfrak{D}}^+} \circ i$ has all of the properties required of $\mathbb{A}$. Thus, by Theorem 4.25 it is a two-sided inverse to the map $x \mapsto \boldsymbol{\beta} x$, proving that it is an isomorphism. □

The $(1,1)$ and 8-fold periodicity theorems can now be proved as special cases of equivariant periodicity, by applying Theorem 4.26 to the appropriate representations.

Theorem 4.27 ($(1,1)$- and 8-fold Bott Periodicity for Orientifolds)**.** *The maps*

$$\begin{aligned} K_{(\Gamma,\epsilon)}(X) &\to K^{n,n}_{(\Gamma,\epsilon)}(X) \qquad\qquad & K_{(\Gamma,\epsilon)}(X) &\to K^{8k,0}_{(\Gamma,\epsilon)}(X) \\ x &\mapsto \underline{\lambda}_X^{(\mathbb{C}^n,\kappa_\epsilon)} x & x &\mapsto \underline{\beta}_X^{(\mathbb{R}^{8k},\mathrm{id}_\epsilon)} x, \end{aligned}$$

are isomorphisms.

Proof. Apply Theorem 4.26 with $W = (\mathbb{C}^n, \kappa_\epsilon)$ and $Z = (\mathbb{R}, \mathrm{id}_\epsilon)$. ☐

Bott periodicity shows that all of the information in orientifold K-theory is captured by eight orientifold K-theory groups.

Corollary 4.28. *If* $p - q = n \bmod 8$*, then* $K^{p,q}_{(\Gamma,\epsilon)}(X) \simeq K^{n,0}_{(\Gamma,\epsilon)}(X)$.

Combining Theorem 4.26 with the semi-equivariant associated bundle construction produces a $(1,1)$-Thom isomorphism.

Theorem 4.29 (The $(1,1)$-Thom Isomorphism)**.** *If* $\pi : E \to X$ *is an orientifold bundle, then*

$$\begin{aligned} \mathbb{B}_E : K_{(\Gamma,\epsilon)}(X) &\to K_{(\Gamma,\epsilon)}(E) \\ x &\mapsto \underline{\lambda} x, \end{aligned}$$

is an isomorphism.

Proof. Apply Theorem 4.26 with the Bott element $\underline{\lambda}^{(\mathbb{C}^n,\kappa_\epsilon)}_{\mathrm{Fr}(E)}$,

$$\begin{aligned} K_{(\Gamma,\epsilon)}(X) &\simeq K_{(\Gamma,\epsilon)\ltimes(U(n),\kappa_\epsilon)}(\mathrm{Fr}(E)) \\ &\simeq K_{(\Gamma,\epsilon)\ltimes(U(n),\kappa_\epsilon)}(\mathrm{Fr}(E) \times (\mathbb{C}^n,\kappa_\epsilon)) \simeq K_{(\Gamma,\epsilon)}(\mathrm{Fr}(E) \times_{(U(n),\kappa_\epsilon)} (\mathbb{C}^n,\kappa_\epsilon)) \simeq K_{(\Gamma,\epsilon)}(E). \end{aligned}$$

☐

The final theorem of this chapter is the 8-fold Thom isomorphism in orientifold K-theory for a real equivariant vector bundle V. Once again, the idea is to combine a semi-equivariant associated bundle construction with Theorem 4.26. Given any orientifold group (Γ, ϵ), the homomorphism

$$\begin{aligned} (\Gamma,\epsilon) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(8k) &\to \mathbb{Z}_2 \ltimes_\kappa \mathrm{Spin}^c(8k) \\ (\gamma,\varphi) &\mapsto (\epsilon(\gamma),\varphi), \end{aligned}$$

defines the correct representation on the model fibre $(\mathbb{R}^{8k}, \mathrm{id}_\epsilon)$. This means that the total space of the principal bundle carries an action of $(\Gamma, \epsilon) \ltimes_{\kappa_\epsilon} \mathrm{Spin}^c(8k)$, making it a Γ-semi-equivariant principal $(\mathrm{Spin}^c(8k), \kappa_\epsilon)$-bundle. Thus, in order to prove the 8-fold Thom isomorphism V must have a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. In this way, Theorem 4.26, which adapts Atiyah's 8-fold Real equivariant periodicity theorem [4, p. 130] to the orientifold setting, is complemented by the work done earlier on semi-equivariance and $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. It is now possible to prove the 8-fold Thom isomorphism.

Theorem 4.30 (The 8-fold Thom Isomorphism)**.** *If $V \to X$ is an 8-dimensional real equivariant vector bundle over an orientifold X and $W_3^{(\Gamma,\epsilon)}(V) = 0$, then the map*

$$\begin{aligned} \mathbb{B}_E : K_{(\Gamma,\epsilon)}(X) &\to K_{(\Gamma,\epsilon)}(V) \\ x &\mapsto \underline{\boldsymbol{\beta}} x \end{aligned}$$

is an isomorphism.

Proof. Corollary 3.11 implies that V has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(V)$. The isomorphism is then proved by applying Theorem 4.26 with the equivariant Bott element $\underline{\boldsymbol{\beta}}_P^{(\mathbb{R}^{8k}, \mathrm{id}_\epsilon)}$,

$$\begin{aligned} K_{(\Gamma,\epsilon)}(X) &\simeq K_{(\Gamma,\epsilon)\ltimes(\mathrm{Spin}^c(8k),\kappa_\epsilon)}(P) \\ &\simeq K_{(\Gamma,\epsilon)\ltimes(\mathrm{Spin}^c(8k),\kappa_\epsilon)}(P \times (\mathbb{R}^{8k}, \mathrm{id}_\epsilon)) \simeq K_{(\Gamma,\epsilon)}(P \times_{(\mathrm{Spin}^c(8k),\kappa_\epsilon)} (\mathbb{R}^{8k}, \mathrm{id}_\epsilon)) \simeq K_{(\Gamma,\epsilon)}(V). \end{aligned}$$

□

Chapter 5

Analytic K-homology for Orientifolds

Before defining the KK-theory of orientifolds, it is helpful to start with a short discussion of KK-theory in the non-equivariant setting. In the previous chapter, the proof of the Thom isomorphism was based around the observation that elliptic operators are closely related to K-theory via their principal symbols and index maps. On the one hand, elliptic operators are dual to K-theory classes in the sense that an elliptic operator D on X defines an index map from K(X) to the integers

$$\mathrm{ind}_D : K^0(X) \to K^0(\mathrm{pt}) \simeq \mathbb{Z}.$$

This point of view leads to the construction of an analytic K-homology theory, dual to K-theory, in which classes are represented by elliptic operators. On the other hand, it is possible to represent a K-theory class as the index

$$\mathrm{ind}(\boldsymbol{D}) \in K^0(X)$$

of a family $\boldsymbol{D}$ of elliptic operators parameterised by X. This is a manifestation of the Atiyah-Jänich theorem, which states that the space of Fredholm operators is a classifying space for K-theory [3, §A] [50]. These characterisations of K-homology and K-theory can be generalised, from functors on the category of topological spaces to functors from the category of C^*-algebras, by using an abstracted notion of elliptic operator. Abstract elliptic operators were first introduced by Atiyah [5], and were used to define the analytic K-homology of C^*-algebras[1] by Kasparov [56, 57]. Kasparov then combined the C^*-algebraic definitions of K-theory and K-homology into groups $KK(A, B)$ that depend on a pair of C^*-algebras A,B [57, 58, 59] [60, p. 101]. Classes in $KK(A, B)$ are represented by *Kasparov modules*, which

[1]the K-homology groups of a C^*-algebra can also be defined via extensions of C^*-algebras. This viewpoint is due to Brown, Douglas, and Fillmore [25, 23, 24].

can be regarded as abstract families of elliptic operators. The analytic K-homology and the topological K-theory of a compact topological space X can be recovered from the KK-functor as

$$KK(C(X), \mathbb{C}l_j) = K_j(X) \qquad\qquad KK(\mathbb{C}l_j, C(X)) \simeq K^j(X),$$

where $K_j(X)$ is the analytic K-homology of X, and the isomorphism $KK(\mathbb{C}l_j, C(X)) \to K^j(X)$ is a families index map related to the Atiyah-Jänich theorem for the space of Clifford linear Fredholm operators [9] [63, p. 222]. An important feature of KK-theory is the presence of a product structure

$$KK(A, B)\hat{\otimes}KK(B, C) \to KK(A, C),$$

known as the *Kasparov product*. This operation is closely related to the tensor product of families of principal symbols that was used in Section 4.3 to define the index map associated to a family of operators. The proof of the Thom isomorphism and other important theorems from classical index theory can be formulated in terms of this product, and generalised to new contexts.

In this chapter, the aim is to set down the definition of an orientifold KK-theory $KK_{(\Gamma,\epsilon)}(A, B)$ which can accomodate the anti-linear symmetries possessed by the orientifold Dirac operator. Even in the early papers on KK-theory, the equivariant Real case was treated. Given this generalisation, the appropriate definition of $KK_{(\Gamma,\epsilon)}$ is relatively clear. For the present purposes, it will not be neccesary to define a Kasparov product for $KK_{(\Gamma,\epsilon)}$. Aside from Kasparov's original papers, mentioned above, further references on KK-theory include [21, 52, 74]

5.1 The K-theory of Orientifold C^*-algebras

In Section 4.1, the K-theory of an orientifold was defined in terms of orientifold bundles. In the C^*-algebraic setting, orientifolds and orientifold bundles are generalised by *orientifold C^*-algebras* and *finitely generated projective orientifold modules*, respectively. The K-theory of orientifold C^*-algebras can be defined using isomorphism classes of finitely generated projective orientifold modules, in place of isomorphism classes of orientifold bundles. This section begins by defining these generalisations. Afterward, the connection to orientifolds, orientifold bundles, and orientifold K-theory will be described. A general reference for C^*-algebras is [32].

Definition 5.1. A graded (Γ, ϵ)-orientifold C^**-algebra* $(A, \|\cdot\|, *, \alpha, \chi)$ is a complex Banach

$*$-algebra which satisfies the C^*-*identity*

$$\|a^*a\| = \|a\|^2, \tag{5.1}$$

and is equipped with

1. an orientifold action[2] α such that, for all $\gamma \in \Gamma$ and $a \in A$,

$$\gamma(ab) = (\gamma a)(\gamma b) \qquad\qquad \gamma(a^*) = (\gamma a)^*,$$

2. an algebra automorphism χ of A such that, for all $\gamma \in \Gamma$ and $a \in A$,

$$\chi^2 = \mathrm{id} \qquad\qquad \chi(a^*) = \chi(a)^* \qquad\qquad \chi(\gamma a) = \gamma\chi(a). \tag{5.2}$$

Together with the properties of the norm, the C^*-identity (5.1) implies that $\|a^*\| = \|a\|$. The ± 1-eigenspaces of the grading automorphism χ provide a decomposition $A = A^0 \oplus A^1$. An element $a \in A^i$ is called *homogeneous*, and its *degree* is defined as $\deg(a) := i$. Any ungraded orientifold C^*-algebra (A, α) can be made into a graded orientifold C^*-algebra $(A \oplus A, \alpha \oplus \alpha, \mathrm{id} \oplus -\mathrm{id})$, where $(\mathrm{id} \oplus -\mathrm{id})$ is the grading automorphism.

Definition 5.2. A *homomorphism* $\varphi : (A_1, \alpha_1, \chi_1) \to (A_2, \alpha_2, \chi_2)$ of graded (Γ, ϵ)-orientifold C^*-algebras is an algebra homomorphism satisfying

$$\varphi(a^*) = \varphi(a)^* \qquad\qquad \varphi \circ \chi_1 = \chi_2 \circ \varphi \qquad\qquad \varphi(\gamma a) = \gamma\varphi(a).$$

The C^*-identity (5.1) implies that any homomorphism $\varphi : A_1 \to A_2$ between C^*-algebras satisfies $\|\varphi(a)\| \leq \|a\|$, making it continuous.

Graded algebras carry a graded commutator, which will be needed later to define Kasparov modules.

Definition 5.3. The *graded commutator* $[\cdot, \cdot]$ on a graded C^*-algebra A is defined on homogeneous elements $a_k \in A$ by

$$[a_1, a_2] := a_1 a_2 - (-1)^{\deg a_1 \deg a_2} a_2 a_1.$$

Several of the objects examined in previous chapters give rise to orientifold C^*-algebras.

Example 5.4. Each compact orientifold (X, σ), has an associated orientifold C^*-algebra defined by $(C(X), \kappa_\epsilon \circ (\sigma^{-1})^*)$, where

$$f^* := \bar{f} \qquad\qquad \|f\| := \sup_{x \in X} |f(x)|.$$

[2] a linear/anti-linear action, in the sense of Definition 2.8

Example 5.5. Each Clifford algebra $\mathbb{C}l_n$ is a C^*-algebra when equipped with the norm and inner product

$$x^* := \overline{x}^t, \qquad \|x\| := \operatorname{tr}(x^*x),$$

where $\operatorname{tr}(x)$ is the scalar part of x and $x \mapsto x^t$ is the canonical anti-involution on $\mathbb{C}l_n$, see [70, pp. 12-13]. The involutive orientifold action κ_ϵ makes $\mathbb{C}l_n$ into an orientifold C^*-algebra. The grading automorphisms $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$, defined in Section 3.3, produce graded orientifold C^*-algebras $(\mathbb{C}l_n, \|\cdot\|, *, \kappa_\epsilon, \boldsymbol{\alpha})$ and $(\mathbb{C}l_{8k}, \|\cdot\|, *, \kappa_\epsilon, \boldsymbol{\omega})$.

Example 5.6. An orientifold group (Γ, ϵ) defines an orientifold C^*-algebra $(\mathbb{C}\Gamma^+, \|\cdot\|, *, \boldsymbol{\rho})$, where $\mathbb{C}\Gamma^+$ is the group algebra of $\Gamma^+ := \ker(\epsilon)$, $\|\cdot\|$ is the operator norm associated to the regular representation of $\mathbb{C}\Gamma^+$ on $\ell^2(\Gamma^+)$, $f^*(\gamma) = \overline{f(\gamma^{-1})}$, and

$$(\boldsymbol{\rho}_\zeta f)(\gamma) := \zeta f(\zeta^{-1}\gamma\zeta),$$

for all $\zeta \in \Gamma$. The action $\boldsymbol{\rho}$ is related to relative conjugation, see Definition 2.16. This algebra will be used in Chapter 7.

The K-theory of a C^*-algebra A is defined in terms of finitely generated projective (f.g.p.) modules over A. When A is an orientifold C^*-algebra, the appropriate generalisation of an orientifold bundle is an f.g.p. module equipped with a compatible orientifold action. The definition of an f.g.p. orientifold module is based on the definition of an equivariant f.g.p. module [68, § 2] [21, § 11.2].

Definition 5.7. An *f.g.p. orientifold module* (E, λ) over an orientifold C^*-algebra (A, α) is an A-module which can be expressed as a direct summand

$$E \oplus F = A^N$$

in some free A-module A^N, equipped with an action $\lambda : \Gamma \to \mathcal{L}^{\pm}(E)$ such that

$$\lambda(\gamma) \in \mathcal{L}^{\epsilon(\gamma)}(E) \qquad \gamma(xa) = (\gamma x)(\gamma a),$$

for $x \in E$ and $a \in A$. Here $\mathcal{L}^{\pm}(E)$ denotes the space of bounded linear/anti-linear Banach space operators. The topology on E is induced from its embedding as a subspace of A^N. This topology is independent of the particular embedding chosen.

Definition 5.8. A *homomorphism* $\varphi : E \to F$ of f.g.p. orientifold modules over (A, α) is an A-linear map satisfying $\varphi(\gamma e) = \gamma\varphi(e)$.

The set of f.g.p. orientifold modules over (A, α) will be denoted $\mathrm{Mod}^{(\Gamma,\epsilon)}_{\mathrm{fgp}}(A, \alpha)$. The main example of an f.g.p. orientifold module is provided by the space of sections of an orientifold bundle.

Example 5.9. If $E \to X$ is an orientifold bundle, then $\Gamma(E)$ with its standard action is an f.g.p. orientifold module over $(C(X), \kappa_\epsilon \circ (\sigma^{-1})^*)$. To see this, note that multiplication by functions $f \in C(X)$ makes the space of sections $\Gamma(E)$ of an orientifold bundle $E \to X$ into a module over $C(X)$. The existence of a perpendicular bundle, see Proposition 2.40, ensures that $\Gamma(E)$ is finitely generated and projective. The standard action $(\gamma s)(x) := \gamma s(\gamma^{-1}x)$ on $\Gamma(E)$ and the action $\kappa_\epsilon \circ (\sigma^{-1})^*$ on $C(X)$ together satisfy

$$(\gamma(sf))(x) = \gamma(sf)(\gamma^{-1}x) = \gamma(s(\gamma^{-1}x)f(\gamma^{-1}x)) = \gamma s(\gamma^{-1}x)\gamma f(\gamma^{-1}x) = ((\gamma s)(\gamma f))(x).$$

Some basic operations on f.g.p. orientifold modules can be defined as follows.

Definition 5.10. Let $(E_i, \lambda_i) \in \mathrm{Mod}^{(\Gamma,\epsilon)}_{\mathrm{fgp}}(A, \alpha)$, V be a orientifold representation, and $\varphi : (A, \alpha) \to (B, \beta)$ be a homomorphism of orientifold C^*-algebras. Define,

1. the *direct sum* of (E_1, λ_1) and (E_2, λ_2) by

$$(E_1, \lambda_1) \oplus (E_2, \lambda_2) := (E_1 \oplus E_2, \lambda_1 \oplus \lambda_2).$$

2. the *tensor product* of V and E to be the f.g.p. orientifold module

$$V \otimes E \in \mathrm{Mod}^{(\Gamma,\epsilon)}_{\mathrm{fgp}}(A, \alpha)$$

with the left Γ-action and right A-action

$$\gamma(v \otimes e) := (\gamma v) \otimes (\gamma e) \qquad\qquad (v \otimes e)a := v \otimes (ea).$$

3. the *pushforward* of E by φ to be the f.g.p. B-module

$$\varphi_*(E) := E \otimes_\varphi B := E \otimes B / \sim \quad \in \mathrm{Mod}^{(\Gamma,\epsilon)}_{\mathrm{fgp}}(B, \alpha)$$

where $(ea) \otimes b \sim e \otimes (\varphi(a)b)$, with the left Γ-action and right B-action defined by

$$\gamma(e \otimes b) := (\gamma e) \otimes (\gamma b) \qquad\qquad (e \otimes b)b' := e \otimes (bb').$$

Direct sum makes $\mathrm{Mod}^{(\Gamma,\epsilon)}_{\mathrm{fgp}}(A, \alpha)$ into a semi-group. The K-theory of (A, α) is obtained by taking the group completion of this semi-group.

Definition 5.11. The K-theory of an orientifold C^*-algebra $K^{(\Gamma,\epsilon)}(A, \alpha)$ is defined as the group completion of the semi-group $(\mathrm{Mod}^{(\Gamma,\epsilon)}_{\mathrm{fgp}}(A, \alpha), \oplus)$ of isomorphism classes of finitely generated projective orientifold modules over (A, α).

The operations of Definition 5.10 induce maps on orientifold K-theory, these make $K^{(\Gamma,\epsilon)}(A,\alpha)$ into a functor from the category of C^*-algebras to the category of $K_{(\Gamma,\epsilon)}(pt)$-modules.

The theory of f.g.p. modules over C^*-algebras can be viewed as a generalisation of the theory of vector bundles over topological spaces. This interpretation is justified by the Gelfand-Naimark and Serre-Swan theorems, see [42, p. 7, p. 59]. Using these theorems, it is a straightforward matter to reconstruct orientifolds and orientifold bundles from orientifold C^*-algebras and f.g.p. orientifold modules. This correspondence justifies the interpretation of orientifold C^*-algebras and f.g.p. orientifold modules as generalised orientifolds and orientifold bundles. Note that the discussion here is simplified considerably by the restriction to finite orientifold groups, though more general cases could be treated as in the equivariant setting, see [68, § 2]. First, consider the Gelfand-Naimark theorem.

Theorem 5.12 (Gelfand-Naimark)**.** *Let A be a commutative C^*-algebra, and $M \subseteq A^*$ be its space of characters*[3]*. If M is equipped with the restriction of the weak-$*$ topology on A^*, then M is a locally compact topological space and the map*

$$\begin{aligned} A &\to C_0(M) \\ a &\mapsto \Big(\hat{a} : m \mapsto m(a)\Big) \end{aligned}$$

is an isometric $*$-isomorphism.

Given an arbitrary commutative orientifold C^*-algebra (A,α), its space M of characters forms a locally compact Hausdorff space and can be equipped with the Γ-action

$$\sigma_\gamma(m) := \kappa_{\epsilon(\gamma)} \circ m \circ \alpha_{\gamma^{-1}}.$$

This, in turn, defines a corresponding orientifold action $\kappa_\epsilon \circ (\sigma^{-1})^*$ on the space of continuous functions $C_0(M)$. The Gelfand-Naimark isomorphism is compatible with these orientifold actions,

$$\begin{aligned} (\kappa_{\epsilon(\gamma)} \circ \sigma^*_{\gamma^{-1}}(\hat{a}))(m) &= \kappa_{\epsilon(\gamma)} \circ \hat{a} \circ \sigma_{\gamma^{-1}}(m) \\ &= \kappa_{\epsilon(\gamma)} \circ (\sigma_{\gamma^{-1}}(m))(a) \\ &= \kappa_{\epsilon(\gamma)} \circ \kappa_{\epsilon(\gamma^{-1})} \circ m \circ \alpha_\gamma(a) \\ &= m \circ \alpha_\gamma(a) \\ &= (\widehat{\alpha_\gamma(a)})(m), \end{aligned}$$

for $\gamma \in \Gamma, a \in A, m \in M$. Similarly, the Serre-Swan theorem expresses a correspondence between vector bundles and f.g.p. modules, which can be extended to a correspondence between orientifold bundles and f.g.p. orientifold modules.

[3] A *character* of A is a homomorphism from A to $\mathbb{C}$.

Theorem 5.13 (Serre-Swan)**.** *Let* X *be a compact topological space. The section functor*

$$\Gamma : \text{Vect}(X) \to \text{Mod}_{\text{fgp}}(C(X))$$

from the the category of vector bundles over X *to the category of finitely generated projective modules over* $C(X)$ *is an equivalence of categories.*

Given an f.g.p. orientifold module (M, λ) over $(C(X), \kappa_\epsilon \circ (\sigma^{-1})^*)$, the Serre-Swan theorem implies that $M = \Gamma(E)$ for some complex vector bundle E. The fibres of E can be identified with equivalence classes of sections using the maps

$$\begin{aligned} \alpha_x : E_x &\to \Gamma(E)/I_x\Gamma(E) \\ e &\mapsto [s] \end{aligned}$$

where $s \in \Gamma(E)$ is any section such that $s(x) = e$, and $I_x = \{f \in C(X) \mid f(x) = 0\}$. Compatibility with the module action implies that the map

$$\begin{aligned} \lambda_\gamma : \Gamma(E)/I_x\Gamma(E) &\to \Gamma(E)/I_{\gamma x}\Gamma(E) \\ [s] &\mapsto [\lambda_\gamma(s)] \end{aligned}$$

is well-defined. Thus, the action defined by

$$\alpha_{\gamma x}^{-1} \circ \lambda_\gamma \circ \alpha_x : E_x \to E_{\gamma x},$$

makes E into an orientifold bundle.

5.2 Orientifold Hilbert modules and Hilbert Module Operators

In Section 4.3, the index of a family of elliptic operators parameterised by a compact topological space X was defined. Part of the definition involved extending each operator in the family to a Fredholm operator between Hilbert spaces. Taken together these Hilbert spaces form a continuous field of Hilbert spaces $\boldsymbol{H}$ parameterised by X. Such a field of Hilbert spaces can be considered as a right module over the commutative C^*-algebra $C(X)$, with a multiplication defined on a family of sections $\boldsymbol{\psi}$ by

$$(\boldsymbol{\psi} f)_x = f(x)\boldsymbol{\psi}_x,$$

for $\boldsymbol{\psi}_x \in \boldsymbol{H}_x$. The inner products $\langle \cdot, \cdot \rangle_x$ of the Hilbert spaces $\boldsymbol{H}_x$ combine to form a $C(X)$-valued inner product, defined on families of sections by

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}' \rangle(x) = \langle \boldsymbol{\psi}_x, \boldsymbol{\psi}'_x \rangle_x.$$

A family of Hilbert spaces such as this, equipped with its $C(X)$-valued inner product, is one of the prototypical examples of a Hilbert module. The notion of a Hilbert module formalises the definition of a family of Hilbert spaces, and extends it to the non-commutative setting by allowing the algebra $C(X)$ to be replaced by a more general C^*-algebra B. By equipping Hilbert modules with group actions, the definition of an equivariant family of operators can also be formalised and generalised. In the orientifold setting, C^*-algebras B are replaced with orientifold C^*-algebras (B, β), and Hilbert (B, β)-modules are equipped orientifold actions satisfying appropriate compatibility conditions.

Definition 5.14. Let B be a C^*-algebra. A *pre-Hilbert* B*-module* is a complex vector space E equipped with a right-action of B, and a continuous B*-valued inner product*

$$\langle \cdot, \cdot \rangle : E \times E \to B,$$

such that

$$\langle x, \lambda y \rangle = \lambda \langle x, y \rangle \qquad \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle \qquad \langle x, yb \rangle = \langle x, y \rangle b$$
$$\langle x, y \rangle = \langle y, x \rangle^* \qquad \langle x, x \rangle = 0 \iff x = 0 \qquad \langle x, x \rangle \geq 0,$$

where the condition $\langle x, x \rangle \geq 0$ denotes *positivity* in the sense of C^*-algebras, meaning that $x = yy^*$ for some element $y \in B$. The additive structures on E, as a complex vector space and as a B-module, are assumed to coincide. A pre-Hilbert module E carries both a B*-valued norm* and a *scalar-valued norm*, defined respectively by

$$|x| := \langle x, x \rangle^{\frac{1}{2}} \qquad \|x\| := \|\langle x, x \rangle\|_B^{\frac{1}{2}}.$$

If, in addition to the conditions above, E is complete with respect to its scalar-valued norm, then it is refered to as a *Hilbert* B*-module*.

Definition 5.15. A *homomorphism* $\varphi : E_1 \to E_2$ of Hilbert B-modules, is a B-linear map such that $\langle \varphi(x), \varphi(y) \rangle_2 = \langle x, y \rangle_1$, for all $x, y \in E_1$.

An *orientifold Hilbert module* is a Hilbert module, over an orientifold C^*-algebra (B, β), that is equipped with an orientifold action λ. The action λ is required to be compatible with the B-valued inner product and the orientifold action β on B.

Definition 5.16. A *graded orientifold Hilbert module* (E, λ, χ_E) over a graded orientifold C^*-algebra (B, β, χ_B) is an orientifold Hilbert module E equipped with

1. an orientifold action $\lambda : \Gamma \to \mathcal{L}^{\pm}(E)$ such that

$$\gamma(xb) = (\gamma x)(\gamma b) \qquad \langle \gamma x_1, \gamma x_2 \rangle = \gamma \langle x_1, x_2 \rangle,$$

2. a grading operator χ_E such that

$$\chi_E(xb) = \chi_E(x)\chi_B(b) \qquad \langle \chi_E(x), \chi_E(y)\rangle = \chi_B(\langle x, y\rangle) \qquad \gamma\chi_E(x) = \chi_E(\gamma x),$$

The ± 1-eigenspaces of χ_E provide a decomposition $E = E^0 \oplus E^1$. An element $x \in E^i$ is called *homogeneous*, and its *degree* is defined as $\deg(x) := i$.

Definition 5.17. A *homomorphism* $\varphi : (E_1, \lambda_1, \chi_1) \to (E_2, \lambda_2, \chi_2)$ of graded orientifold Hilbert-B-modules, is a homomorphism of Hilbert-B modules such that, for all $\gamma \in \Gamma$, $x \in E_1$,

$$\varphi(\gamma x) = \gamma\varphi(x) \qquad \varphi \circ \chi_1 = \chi_2 \circ \varphi.$$

Example 5.18. Let (X, σ) be an orientifold of dimension $8k$ with $W_3^{(\Gamma,\epsilon)}(X, \sigma) = 0$. Then the L^2-sections of the orientifold spinor bundle form a graded orientifold Hilbert module $(L^2(X, \not{S}), \langle\cdot,\cdot\rangle_{Cl}, \lambda, \boldsymbol{\omega})$ over the graded orientifold C^*-algebra $(\mathbb{C}l_{8k}, \kappa_\epsilon, \boldsymbol{\omega})$ where

1. the right $\mathbb{C}l_{8k}$-action on $L^2(X, \not{S})$ is induced from the right action of $\mathbb{C}l_{8k}$ on $\not{S}$,
2. the $\mathbb{C}l_{8k}$-valued inner product on $L^2(X, \not{S})$ is defined by
$$\langle \psi_1, \psi_2\rangle_{Cl} := \int_X \mathrm{tr}(\psi_1\psi_2^*)dx.$$
3. the orientifold action λ is induced by the orientifold action on $\not{S}$,
4. the grading operator on $L^2(X, \not{S})$, which will again be denoted $\boldsymbol{\omega}$, is induced from the grading operator $\boldsymbol{\omega}$ on $\not{S}$.

One can easily check that the various compatibility conditions between the above actions and maps are satisfied. If (E, τ) is an orientifold bundle, then a graded orientifold Hilbert module $(L^2(X, \not{S} \otimes E), \langle\cdot,\cdot\rangle_{Cl}\langle\cdot,\cdot\rangle_E, \lambda \otimes \lambda^\tau, \omega \otimes \mathrm{id})$ can be defined similarly.

In the next section, it will be neccesary to consider the *pushout* of a Hilbert module.

Definition 5.19. Let (E, λ, χ) be a graded orientifold Hilbert module, $\varphi : B \to C$ be a surjective homomorphism of graded orientifold C^*-algebras, $I_\varphi := \{x \in E : \varphi(< x, x >) = 0\}$ and $q : E \to E/I_\varphi$ be the quotient map. The *pushout* E_φ of E is the completion of the orientifold Hilbert C-module

$$E'_\varphi := E/I_\varphi,$$

where E'_φ is equipped with the C-module structure and C-valued inner product defined respectively by

$$q(x)\varphi(b) := q(xb) \qquad \langle q(x), q(y)\rangle := \varphi(\langle x, y\rangle).$$

To illustrate the basic idea behind Hilbert module operators, consider a continuous family $\boldsymbol{F}$ of Hilbert space operators parameterised by a compact topological space X. Given any function $f \in C(X)$ such a family satisfies

$$(\boldsymbol{F}(\boldsymbol{\psi} f))_x = \boldsymbol{F}_x(f(x)\boldsymbol{\psi}_x) = f(x)(\boldsymbol{F}_x\boldsymbol{\psi}_x) = ((\boldsymbol{F}\boldsymbol{\psi})f)_x,$$

making $\boldsymbol{F}$ $C(X)$-linear. Thus, a family of operators can be considered as a $C(X)$-linear operator between Hilbert $C(X)$-modules. Generalising this construction, a Hilbert B-module operator is a B-linear operator between Hilbert B-modules. The theory of Hilbert module operators is based on the theory of Hilbert space operators. However, the generalisation to families of operators and then further, to B-linear operators, introduces extra subtleties. The first step in developing the theory of Hilbert module operators is to address the issue of adjointable operators. On a Hilbert space every bounded operator has an adjont operator. However, this is not the case for Hilbert module operators.

Definition 5.20. Let E_1 and E_2 be Hilbert B-modules. The space of *adjointable operators* $\mathcal{L}_B(E_1, E_2)$ is the set of maps $T : E_1 \to E_2$ for which there exists a map $T^* : E_2 \to E_1$ such that

$$\langle Tx, y\rangle_2 = \langle x, T^* y\rangle_1.$$

One can show that every adjointable operator is a bounded B-module map, that the adjoint T^* of an adjointable operator is unique, and that $T^{**} = T$. The adjoint map and operator norm make $\mathcal{L}_B(E)$ into a C^*-algebra [81, pp. 240-241].

Proposition 5.21. *If (E, λ, χ_E) is a graded orientifold Hilbert module over a graded orientifold C^*-algebra (B, β, χ_B). Then $\mathcal{L}_B(E, \lambda, \chi_E)$ is a graded orientifold C^*-algebra with*

1. *norm given by the operator norm $\|\cdot\|$,*

2. *$*$-structure given by the adjoint operation $T \mapsto T^*$,*

3. *orientifold action defined by $(\gamma T) := \lambda_\gamma T \lambda_{\gamma^{-1}}$,*

4. *grading operator defined by $\chi(T) := \chi_E T \chi_E^{-1}$.*

A homogeneous element $T \in \mathcal{L}_B^j(E)$ is called even *if $j = 0$,* odd *if $j = 1$, and satisfies $T(E^i) \subseteq E^{i+j}$ for $i, j \in \mathbb{Z}_2$.*

Kasparov's KK-theory is concerned with the indicies of Hilbert module operators. Thus, it is important to determine the set of Hilbert module operators which have a well-defined index. On a Hilbert space, this is the set of Fredholm operators. Recall that the Fredholm operators on a Hilbert space can be characterised, using Atkinson's theorem, as those operators that are invertible modulo compact operators [63, p.192]. Once an appropriate generalisation of compact Hilbert module operator has been made, a similar characterisation can be used to define Fredholm operators between Hilbert modules.

Definition 5.22. The space of *compact operators* $\mathcal{K}_B(E_1, E_2)$ is the subspace of $\mathcal{L}_B(E_1, E_2)$ spanned by operators of the form

$$\theta_{x,y}(z) := x\langle y, z\rangle,$$

where $x \in E_2$ and $y, z \in E_1$.

The compact operators form a two-sided ideal in $\mathcal{L}_B(E_1, E_2)$ [81, pp. 242].

Definition 5.23. An operator $F \in \mathcal{L}_B(E_1, E_2)$ is said to be *Fredholm* if there exists an operator $G \in \mathcal{L}_B(E_2, E_1)$ such that

$$\mathrm{id} - GF \in \mathcal{K}_B(E_1, E_1) \qquad \mathrm{id} - FG \in \mathcal{K}_B(E_2, E_2).$$

Each Fredholm operator $F \in \mathcal{L}_B(E_1, E_2)$ has well-defined index. However, it cannot always be taken directly. This is already the case for a family of Hilbert space Fredholm operators, as was discussed in Section 4.3. The solution to this problem is to perturb F by a compact operator $K \in \mathcal{K}_B(E_1, E_2)$ to another Fredholm operator $\tilde{F} := F + K$ which is *regular*. Regularity ensures that the index $\tilde{F}$ can be taken directly. Such perturbations always exist, and it is possible to show that any two have the same index. Thus, a Fredholm operator F has a well-defined index given by the index of any compact perturbation to a regular operator.

Definition 5.24. An operator $T \in \mathcal{L}_B(E_1, E_2)$ is said to be *regular* if there exists an operator $S \in \mathcal{L}_B(E_2, E_1)$ such that $TST = T$ and $STS = S$.

Definition 5.25. The *index* of F is defined by

$$\mathrm{ind}(F) := [\ker(\tilde{F})] - [\ker(\tilde{F}^*)] \in K^{(\Gamma,\epsilon)}(B),$$

where $\tilde{F}$ is any regular operator such that $\tilde{F} = F + K \in \mathcal{L}_B(E_1, E_2)$ for some $K \in \mathcal{K}_B(E_1, E_2)$.

Further details regarding the indicies of Fredholm operators on Hilbert modules can be found in [42, §4.3] and [81, §17].

5.3 KK-theory for Orientifold C^*-algebras

Using the definitions of the previous section, it is possible to define orientifold Kasparov modules. As mentioned, Hilbert module operators generalise families of Hilbert space operators. Each Kasparov module is a Hilbert module equipped with a Hilbert module operator satisfying certain properties. These properties are analogous to those satisfied by the operatorwise extension of a family of order-zero elliptic operators to a family of Hilbert space operators. Rather than considering operators between a pair of separate Hilbert modules, a Kasparov module organises the pair into a single graded Hilbert module. The operator for the module is then required to be odd, so that it maps between the components of the grading. An orientifold Kasparov module is equipped with an orientifold action, and its operator is required to satisfy an additional equivariance property. Aside from the anti-linearity of the orientifold action, the definition is identical to that used in the usual equivariant setting [21, §20].

Definition 5.26. Let A and B be graded orientifold C^*-algebras. An *orientifold Kasparov* (A, B)*-module* is a triple $\mathcal{E} := ((E, \lambda), \phi, F)$ such that

1. (E, λ) is a countably generated graded orientifold Hilbert B-module
2. $\phi : A \to \mathcal{L}_B(E)$ is a homomorphism of graded orientifold C^*-algebras
3. $F \in \mathcal{L}_B(E)$ is an odd operator such that the operators

$$[F, \phi(a)] \qquad (F^2 - 1)\phi(a) \qquad (F^* - F)\phi(a) \qquad ((\gamma F) - F)\phi(a)$$

are in $\mathcal{K}_B(E)$ for all $a \in A$ and $\gamma \in \Gamma$. Here $[\cdot, \cdot]$ denotes the graded commutator.

The set of orientifold Kasparov (A, B)-modules will be denoted by $\mathbb{E}_{(\Gamma,\epsilon)}(A, B)$.

The conditions imposed on the operator of a Kasparov module have their origin in Atiyah's definition of an abstract elliptic operator [5, §2]. The property $[F, \phi(a)] \in \mathcal{K}_B(E)$ generalises a property satisfied by order-zero pseudodifferential operators. The property $(F^2 - 1)\phi(a) \in \mathcal{K}_B(E)$ ensures that F is Fredholm, and thus has a well-defined index in the K-theory of (B, β). This can be regarded as an abstraction of ellipticity. The properties $(F^* - F)\phi(a) \in \mathcal{K}_B(E)$ and $((\gamma F) - F)\phi(a) \in \mathcal{K}_B(E)$ correspond to self-adjointness and equivariance of the operator. These properties need only hold up to a compact operator because KK-theory is concerned with the indicies of the operators, which are invariant under compact perturbation.

In order to define the orientifold KK-groups, some operations on orientifold Kasparov modules are needed. These are straightforward generalisations of the operations used in non-equivariant KK-theory, see [52, §2.1] [21, §20].

Definition 5.27. Let $\mathcal{E}_i = ((E_i, \lambda_i, \chi_i), \phi_i, F_i) \in \mathbb{E}_{(\Gamma,\epsilon)}(A, B)$, $\psi : A' \to A$ be a homomorphism of graded orientifold C^*-algebras, $\varphi : B \to B'$ be a surjective homomorphism of graded orientifold C^*-algebras, I_φ be $\{x \in E : \varphi(< x, x >) = 0\}$, and $q : E \to E/I_\varphi$ be the quotient map. Define the following operations on graded orientifold Kasparov modules

1. The *direct sum* $\mathcal{E}_1 \oplus \mathcal{E}_2 \in \mathbb{E}_{(\Gamma,\epsilon)}(A, B)$ is defined by

$$(E_1, \phi_1, F_1) \oplus (E_2, \phi_2, F_2) := (E_1 \oplus E_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2).$$

2. The *pullback* of $\mathcal{E}$ by ψ is defined by

$$\psi^*\mathcal{E} := ((E, \lambda, \chi), \phi \circ \psi, F) \in \mathbb{E}_{(\Gamma,\epsilon)}(A', B).$$

3. The *pushout* of $\mathcal{E}$ by φ is defined by

$$\mathcal{E}_\varphi := ((E_\varphi, \lambda_\varphi, \chi_\varphi), \phi_\varphi, F_\varphi) \in \mathbb{E}_{(\Gamma,\epsilon)}(A, B'),$$

where $(E_\varphi, \lambda_\varphi, \chi_\varphi)$ is the pushout of (E, λ, χ) and

$$\phi_\varphi(x) := q(\phi(x)) \qquad\qquad F_\varphi(x) := q(F(x)).$$

The definition of the KK-groups is based on the realisation of K-homology and K-theory classes via the index. For this reason, it is neccesary to identify Kasparov modules with related indicies. This is achieved by placing equivalence relations on $\mathbb{E}_{(\Gamma,\epsilon)}(A, B)$. In particular, due to the homotopy invariance of the index, Kasparov modules which are homotopic in an appropriate sense should belong to the same class. The following equivalence relations are a straightforward generalisation of the equivalence relations used in the non-equivariant setting [52, §2.1] [21, §20].

Definition 5.28. Orientifold Kasparov modules $\mathcal{E}_i = ((E_i, \lambda_i, \chi_i), \phi_i, F_i) \in \mathbb{E}_{(\Gamma,\epsilon)}(A, B)$ are

1. *isomorphic* $\mathcal{E}_1 \simeq \mathcal{E}_2$, if there exists an isomorphism $\varphi : E_1 \to E_2$ of graded orientifold Hilbert B-modules, such that

$$\lambda_2 \circ \varphi = \varphi \circ \lambda_1 \quad \chi_2 \circ \varphi = \varphi \circ \chi_2 \quad (\phi_2(a)) \circ \varphi = \varphi \circ (\phi_1(a)) \quad F_2 \circ \varphi = \varphi \circ F_1$$

2. *homotopic* $\mathcal{E}_1 \sim_h \mathcal{E}_2$, if there exists a triple

$$\mathcal{W} := (E, \phi, F) \in \mathbb{E}_{(\Gamma,\epsilon)}(A, B \otimes (C[0,1], \kappa_\epsilon)),$$

such that $\pi_0\mathcal{W} \simeq \mathcal{E}_1$ and $\pi_1\mathcal{W} \simeq \mathcal{E}_2$ where $\pi_t : B \otimes (C[0,1], \kappa_\epsilon) \to B$ is the evaluation homomorphism at t, and $\pi_t\mathcal{W}$ are the associated pushout modules.

Definition 5.29. The orientifold Kasparov groups are defined by

$$KK_{(\Gamma,\epsilon)}(A,B) := \mathbb{E}_{(\Gamma,\epsilon)}(A,B)/\sim_h$$

with addition given by $[\mathcal{E}_1] + [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2]$.

Definition 5.30. Define the analytic K-homology groups of an orientifold C^*-algebra (A,α), and an orientifold (X,σ) by

$$K^j_{(\Gamma,\epsilon)}(A,\alpha) := KK_{(\Gamma,\epsilon)}((A,\alpha),(\mathbb{C}l_j,\kappa_\epsilon))$$
$$K^{(\Gamma,\epsilon)}_j(X,\sigma) := KK_{(\Gamma,\epsilon)}((C(X),\kappa_\epsilon \circ (\sigma^{-1})^*),(\mathbb{C}l_j,\kappa_\epsilon)).$$

It is often useful to consider a form of homotopy called *operator homotopy*. Operator homotopy varies the operator F continuously, and introduces stabilisation by *degenerate* Kasparov modules, which represent the zero class in $KK_{(\Gamma,\epsilon)}(A,B)$ [21, p. 148]. Operator homotopy implies homotopy in the sense of Definition 5.28.

Definition 5.31. An orientifold Kasparov module $(E,\phi,F) \in \mathbb{E}_{(\Gamma,\epsilon)}(A,B)$ is *degenerate* if

$$[F,\phi(a)] = (F^2-1)\phi(a) = (F^*-F)\phi(a) = (\gamma F - F)\phi(a) = 0, \tag{5.3}$$

for all $\gamma \in \Gamma$, $a \in A$. The set of degenerate Kasparov modules will be denoted by $\mathbb{D}_{(\Gamma,\epsilon)}(A,B)$.

Proposition 5.32. *Every $\mathcal{D} \in \mathbb{D}_{(\Gamma,\epsilon)}(A,B)$ is homotopic to 0.*

Definition 5.33. Define an equivalence relation on $\mathbb{E}_{(\Gamma,\epsilon)}(A,B)$ by letting $\mathcal{E}_1 \sim \mathcal{E}_2$ if there exists a triple $\mathcal{F}_t := (E,\phi,F_t)$, where

1. E is a graded orientifold Hilbert B-module
2. $\phi : A \to \mathcal{L}_B(E)$ is a graded homomorphism of orientifold C^*-algebras
3. F_t is a norm continuous path in $\mathcal{L}_B(E)$ for $t \in [0,1]$,

such that $\mathcal{F}_0 \simeq \mathcal{E}_1$, $\mathcal{F}_1 \simeq \mathcal{E}_2$, and $\mathcal{F}_t \in \mathbb{E}_{(\Gamma,\epsilon)}(A,B)$ for all $t \in [0,1]$. Two Kasparov modules $\mathcal{E}_1,\mathcal{E}_2 \in \mathbb{E}_{(\Gamma,\epsilon)}(A,B)$ are said to be *operator homtopic* $\mathcal{E}_1 \sim_{oh} \mathcal{E}_2$ if there exist degenerate modules $\mathcal{D}_1,\mathcal{D}_2 \in \mathbb{D}_{(\Gamma,\epsilon)}(A,B)$ such that

$$\mathcal{E}_1 \oplus \mathcal{D}_1 \sim \mathcal{E}_2 \oplus \mathcal{D}_2.$$

Because degenerate modules are homotopic to 0, operator homotopy can be considered as form of homotopy in which only the operator varies.

Proposition 5.34. *If $\mathcal{E}_1,\mathcal{E}_2 \in \mathbb{E}_{(\Gamma,\epsilon)}(A,B)$ are operator homotopic then they are homotopic.*

5.4 The K-homology Class of an Orientifold Dirac Operator

As discussed in the previous section, the KK-groups are modelled on order-zero elliptic pseudo-differential operators. It is always possible to normalise a self-adjoint elliptic operator to an order-zero pseudodifferential operator in such a way that its index is preserved [46, §10.6]. This makes it possible to associate an orientifold Kasparov module to the orientifold Dirac operator.

Proposition 5.35. *Let* (X, σ) *be a compact orientifold of dimension* n *such that* $W_3^{(\Gamma,\epsilon)}(X, \sigma) = 0$. *An orientifold Dirac operator*

$$\not{\mathfrak{D}}_E : \Gamma(\not{\mathfrak{S}} \otimes E) \to \Gamma(\not{\mathfrak{S}} \otimes E),$$

with coefficients in an orientifold bundle E*, defines an orientifold Kasparov module*

$$[\not{\mathfrak{D}}_E] := [F, \phi, S_E] \in KK_{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_n, \kappa_\epsilon)),$$

where

1. $S_E := (L^2(X, \not{\mathfrak{S}} \otimes E), \langle \cdot, \cdot \rangle_{\mathbb{C}l} \langle \cdot, \cdot \rangle_E, \lambda, \boldsymbol{\omega})$ *is the Hilbert* $(\mathbb{C}l_n, \kappa_\epsilon, \boldsymbol{\omega})$*-module associated to the orientifold spinor bundle with coefficients in* E*, as in Example 5.18.*
2. ϕ *is the representation of* $(C(X), \kappa_\epsilon \circ (\sigma^{-1})^*)$ *on* S_E *by multiplication operators*
3. F *is the normalisation of the Dirac operator* $\not{\mathfrak{D}}_E$ *defined by*

$$F := \not{\mathfrak{D}}_E (1 + \not{\mathfrak{D}}_E^2)^{\frac{1}{2}} \in \mathcal{L}_{(\mathbb{C}l_n, \kappa_\epsilon)}(S_E).$$

Proof. It is a standard result that the normalisation of a Dirac operator defines a class in the analytic K-homology, see [46, Theorem 10.6.5, pg. 288]. The result applies to the orientifold Dirac operator to produce a non-equivariant Kasparov module. In addition to this, compatibility with an orientifold action is required. As mentioned in Example 5.18, S_E is an orientifold Hilbert $(\mathbb{C}l_n, \kappa_\epsilon)$-module with the orientifold action inhertited from $\not{\mathfrak{S}}_E$. In particular, the actions of Γ and $(\mathbb{C}l_n, \kappa_\epsilon)$ are compatible,

$$\gamma(\psi\varphi) = (\gamma\psi)(\gamma\varphi),$$

for $\gamma \in \Gamma$, $\psi \in S_E$ and $\varphi \in (\mathbb{C}l_n, \kappa_\epsilon)$. The Γ-equivariance of $\not{\mathfrak{D}}$ was proved in Proposition 3.45. It is inhertied by $\not{\mathfrak{D}}_E$ and the normalisations F, making $[\not{\mathfrak{D}}_E]$ an orientifold Kasparov module. □

Chapter 6

Geometric K-homology for Orientifolds

The previous chapter described a realisation of orientifold K-homology in terms of elliptic orientifold operators. In this chapter, a realisation $K^{geo}_{(\Gamma,\epsilon),j}(X)$ of orientifold K-homology will be described in which each class is represented by a continuous equivariant map $f : M \to X$ from an orientifold M equipped with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure and an orientifold bundle E. The groups $K^{geo}_{(\Gamma,\epsilon),j}(X)$ are formed by introducing a direct sum operation, and appropriate equivalence relations, on the set of all such maps. After applying these relations, $K^{geo}_{(\Gamma,\epsilon),j}(X)$ resembles a cross between a bordism theory and K-theory, modulo an extra equivalence relation that captures the structure of the 8-fold Thom isomorphism. Although the definition of geometric orientifold K-homology makes no mention of elliptic operators, its interpretation depends on the realisation of K-homology in terms of elliptic operators, and on the proof of the 8-fold Thom isomorphism in terms of families of elliptic operators, see Chapter 4. This interpretation can be formalised by defining a map from the geometric orientifold K-homology to the analytic orientifold K-homology. Such a map is constructed in Section 6.3. In the usual equivariant setting, this map is an isomorphism [18], though the corresponding proof for orientifolds will not be presented here. Geometric K-homology was first defined by Baum and Douglas [16, 15], see also [17]. An equivariant generalisation is treated in [18].

It should be noted that, rather than being bigraded, the geometric orientifold K-homology groups $K^{geo}_{(\Gamma,\epsilon),j}(X)$ defined here are graded with respect to a single integer $0 \leq j \leq 7$. This simplification is justified by Corollary 4.28, which reduces the number of distinct K-groups to eight. A bigraded approach, using $\mathrm{Spin}^c(p, q)$-structures, has been proposed in the Real setting by Hekmati et al. [45]. Bigraded groups could also be defined using suspension. However, this chapter focuses on capturing the information present at the level of K-theory, and then using the orientifold Dirac operator constructed in Section 3.4 to define a map into analytic K-homology.

6.1 Operations on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures

Several operations on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures will need to be understood in order to define the geometric K-homology of orientifolds. The first step is to prove a *Two-of-Three* Lemma for $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. This lemma induces a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on any real equivariant bundle that fits into a short exact sequence with two other real equivariant bundles that have $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. The Two-of-Three Lemma will be used to define further operations on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. Its proof relies on basic facts regarding the groups $\mathrm{Spin}^c(n)$ and $\mathrm{SO}(n)$, and on results from Section 1.5 regarding the semi-equivariant Dixmier-Douady class.

Lemma 6.1. *There is a commutative diagram of* Γ*-groups*

$$\begin{array}{ccc}
1 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
(\mathrm{U}(1) \times \mathrm{U}(1), \kappa_\epsilon \times \kappa_\epsilon) & \xrightarrow{\nu} & (\mathrm{U}(1), \kappa_\epsilon) \\
\downarrow & & \downarrow \\
(\mathrm{Spin}^c(p) \times \mathrm{Spin}^c(q), \kappa_\epsilon \times \kappa_\epsilon) & \xrightarrow{\beta} & (\mathrm{Spin}^c(p+q), \kappa_\epsilon) \\
\downarrow{\scriptstyle \mathrm{Ad}^c} & & \downarrow{\scriptstyle \mathrm{Ad}^c} \\
\mathrm{SO}(p) \times \mathrm{SO}(q) & \xrightarrow{\alpha} & \mathrm{SO}(p+q) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}$$

where

1. $\nu(z, z') := zz'$ *for* $z, z' \in \mathrm{U}(1)$.

2. $\beta(h, h') := \beta_1(h)\beta_2(h')$ *for* $h \in \mathrm{Spin}^c(p)$ *and* $h' \in \mathrm{Spin}^c(q)$, *where* β_1 *and* β_2 *are the maps defined on the standard basis elements of* $\mathbb{R}^p \subset \mathbb{C}l_p$ *and* $\mathbb{R}^q \subset \mathbb{C}l_q$ *by*

$$\begin{aligned} \beta_1 : \mathbb{C}l_p &\to \mathbb{C}l_{p+q} \\ e_i &\mapsto e_i \end{aligned} \qquad\qquad \begin{aligned} \beta_2 : \mathbb{C}l_q &\to \mathbb{C}l_{p+q} \\ e_i &\mapsto e_{p+i}, \end{aligned}$$

3. $\alpha(g, g') := \alpha_1(g)\alpha_2(g')$ *for* $g \in SO(p)$ *and* $g' \in SO(q)$, *where*

$$\alpha_1 : SO(p) \to SO(p+q) \qquad \qquad \alpha_2 : SO(q) \to SO(p+q)$$
$$[g_{ij}] \mapsto \begin{pmatrix} [g_{ij}] & 0 \\ 0 & 1 \end{pmatrix} \qquad \qquad [g'_{ij}] \mapsto \begin{pmatrix} 1 & 0 \\ 0 & [g'_{ij}] \end{pmatrix}.$$

Here $[a_{ij}]$ *denotes the standard matrix representations of an element* $a \in SO(n)$.

Proof. The only non-trivial part of the lemma is to show that β is a homomorphism, which amounts to showing that $\beta_1(h)\beta_2(h') = \beta_2(h')\beta_1(h)$. Each element of $\beta_1(\mathrm{Spin}(p))$ is the product of an even number of unit vectors $(x, 0) \in \mathbb{R}^p \oplus \mathbb{R}^q \subset \mathbb{C}l_{p+q}$ and each element of $\beta_2(\mathrm{Spin}(q))$ is the product of an even number of unit vectors $(0, y) \in \mathbb{R}^p \oplus \mathbb{R}^q \subset \mathbb{C}l_{p+q}$. Due to the relation $e_ie_j = -e_je_i \in \mathbb{C}l_{p+q}$ for $i \neq j$, such elements satisfy $(x, 0)(0, y) = -(0, y)(x, 0)$. Thus, every element of $\beta_1(\mathrm{Spin}(p))$ commutes with every element of $\beta_2(\mathrm{Spin}(q))$. □

Proposition 6.2. *The diagram of Lemma 6.1 induces a commutative diagram*

$$\begin{array}{ccc}
H^1_{(\Gamma,\epsilon)}(X, (U(1), \kappa_\epsilon) \times (U(1), \kappa_\epsilon)) & \xrightarrow{\nu^1} & H^1_{(\Gamma,\epsilon)}(X, (U(1), \kappa_\epsilon)) \\
\downarrow & & \downarrow \\
TC^1_{(\Gamma,\epsilon)}(X, (\mathrm{Spin}^c(p), \kappa_\epsilon) \times (\mathrm{Spin}^c(q), \kappa_\epsilon)) & \xrightarrow{\beta^1} & TC^1_{(\Gamma,\epsilon)}(X, (\mathrm{Spin}^c(p+q), \kappa_\epsilon)) \\
\downarrow{\scriptstyle \mathrm{Ad}^c \times \mathrm{Ad}^c} & & \downarrow{\scriptstyle \mathrm{Ad}^c} \\
TC^1_{(\Gamma,\epsilon)}(X, (SO(p), \mathrm{id}_\epsilon) \times (SO(q), \mathrm{id}_\epsilon)) & \xrightarrow{\alpha^1} & TC^1_{(\Gamma,\epsilon)}(X, (SO(p+q), \mathrm{id}_\epsilon)) \\
\downarrow{\scriptstyle \Delta_{sc} \times \Delta_{sc}} & & \downarrow{\scriptstyle \Delta_{sc}} \\
H^2_{(\Gamma,\epsilon)}(X, (U(1), \kappa_\epsilon) \times (U(1), \kappa_\epsilon)) & \xrightarrow{\nu^2} & H^2_{(\Gamma,\epsilon)}(X, (U(1), \kappa_\epsilon)).
\end{array}$$

Proof. The above diagram is produced by applying Theorem 1.41 to the two central exact sequences running vertically in the diagram of Lemma 6.1. The commutivity of the diagram in Lemma 6.1 implies the commutivity of the top two cells in the above diagram. To see that the bottom cell commutes, note that if

$$(\psi_1, \psi_2) \in TC^1_{(\Gamma,\epsilon)}(X, (\mathrm{Spin}^c(p), \kappa_\epsilon) \times (\mathrm{Spin}^c(q), \kappa_\epsilon))$$

is a lifting by $\mathrm{Ad}^c \times \mathrm{Ad}^c$ of

$$(\phi_1, \phi_2) \in TC^1_{(\Gamma,\epsilon)}(X, (SO(p), \mathrm{id}_\epsilon) \times (SO(q), \mathrm{id}_\epsilon)),$$

then the commutivity of the middle cell implies that

$$\beta^1(\psi_1, \psi_2) \in TC^1_{(\Gamma,\epsilon)}(X, (\mathrm{Spin}^c(p+q), \kappa_\epsilon))$$

is a lifting by Ad^c of

$$\alpha^1(\phi_1, \phi_2) \in TC^1_{(\Gamma,\epsilon)}(X, (SO(p+q), \mathrm{id}_\epsilon)).$$

Together with the definition of Δ_{sc}, and the fact that β is a homomorphism of Γ-groups, this implies that

$$\begin{aligned}\Delta_{sc} \circ \alpha^1[\phi_1, \phi_2] &= [\partial\beta^1(\psi_1, \psi_2)] \\ &= [\beta^2\partial(\psi_1, \psi_2)] \\ &= \nu^2 \circ (\Delta_{sc} \times \Delta_{sc})(\phi_1, \phi_2).\end{aligned}$$

The last line of the calculation follows from the definition of $\Delta_{sc} \times \Delta_{sc}$, and the fact that $\beta^2 = \nu^2$ as a map from[1]

$$H^2_{(\Gamma,\epsilon)}(X, (U(1), \kappa_\epsilon) \times (U(1), \kappa_\epsilon)) \subset K^2_{(\Gamma,\epsilon)}(X, (\mathrm{Spin}^c(p), \kappa_\epsilon) \times (\mathrm{Spin}^c(q), \kappa_\epsilon)).$$

to

$$H^2_{(\Gamma,\epsilon)}(X, (U(1), \kappa_\epsilon)) \subset K^2_{(\Gamma,\epsilon)}(X, (\mathrm{Spin}^c(p+q), \kappa_\epsilon)).$$

□

Lemma 6.3 (Two-of-Three Lemma). *Let* (Γ, ϵ) *be finite and*

$$0 \to V_1 \to V \to V_2 \to 0$$

be an exact sequence of Γ*-equivariant real vector bundles. Specifying* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structures on two of the bundles in the sequence determines a specific* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure on the remaining bundle.*

Proof. By taking an equivariant metric on V, the sequence of bundles can be split so that $V_1 \oplus V_2 \simeq V$. The existence of a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on implies orientibility, and an orientation on any two of the vector bundles induces an orientation on the third. Thus, it can be assumed that all three bundles are oriented. In this situation, there exist transition cocycles

$$\phi^1 \in TC^1_{(\Gamma,\epsilon)}(X, (SO(p), \mathrm{id}_\epsilon)) \quad \phi^2 \in TC^1_{(\Gamma,\epsilon)}(X, (SO(q), \mathrm{id}_\epsilon)) \quad \phi \in TC^1_{(\Gamma,\epsilon)}(X, (SO(p+q), \mathrm{id}_\epsilon)),$$

for V_1, V_2 and V respectively, such that $\phi^1 \oplus \phi^2$ is equivalent to ϕ. Suppose that two of the vector bundles in the sequence are equipped with specific $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. This is equivalent to specifying a lifting to $(\mathrm{Spin}^c, \kappa_\epsilon)$ for two of the three cocycles ϕ^1, ϕ^2 and ϕ. The commutivity of the bottom cell in the diagram from Proposition 6.2 implies that

$$\Delta_{sc}(\phi) = \Delta_{sc}(\phi^1)\Delta_{sc}(\phi^2).$$

[1]Here $K^2_{(\Gamma,\epsilon)}$ indicates the space of 2-cochains, rather than K-theory.

Thus, by Corollary 3.7, the remaining transition cocycle must also lift to $(\mathrm{Spin}^c, \kappa_\epsilon)$. Corollary 3.8, and the commutivity of the top cell in the diagram from Proposition 6.2, then imply that the two initial $(\mathrm{Spin}^c, \kappa_\epsilon)$-liftings determine a specific $(\mathrm{Spin}^c, \kappa_\epsilon)$-lifting for the remaining transition cocycle. This lifting determines a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on the remaining vector bundle. □

Lemma 6.3, will allow the *vector bundle modification* and *boundary induction* operations to be defined on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. Vector bundle modification is defined by the next proposition, which has two parts. The first part constructs a family of $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures, each one lying over a fibre in the fibrewise compactification

$$S(V \oplus \mathbb{R}) := P \times_{(\mathrm{Spin}^c(n), \kappa_\epsilon)} S^n$$

of a $(\mathrm{Spin}^c, \kappa_\epsilon)$-vector bundle $V := P \times_{(\mathrm{Spin}^c(n), \kappa_\epsilon)} \mathbb{R}^n \to M$. Here $\mathrm{Spin}^c(n)$ is considered to act on S^n via its inclusion into $\mathrm{Spin}^c(n+1)$, see Lemma 3.18. Considered together, this family of $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures forms a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for the vertical tangent bundle

$$P \times_{(\mathrm{Spin}^c(n), \kappa_\epsilon)} TS^n.$$

The significance of this construction is that a family of vertical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures can be used to define a family of vertical orientifold Dirac operators. When the vector bundle V underlying the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure is trivial, this family of operators is precisely the product family of operators used to construct the inverse to the 8-fold Bott periodicity map, as in Theorem 4.27. When the vector bundle underlying the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure is non-trivial, this family of vertical Dirac operators can be used to construct an inverse for the corresponding 8-fold Thom isomorphism, see Theorem 4.30. The second part of the proposition uses the Two-of-Three Lemma to show that a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on the base orientifold M can be combined with the vertical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure to form a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for the tangent space

$$TS(V \oplus \mathbb{R}) = T(P \times_{(\mathrm{Spin}^c(n), \kappa_\epsilon)} S^n)$$

of the sphere bundle associated to V.

Proposition 6.4 (Vector bundle modification)**.** *Let M be an orientifold, and $V \to M$ be a real equivariant vector bundle with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $\varphi : P \to \mathrm{Fr}(V)$. Then*

1. *there is a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure*

$$\mathrm{id} \times \mathrm{Ad}^c : P \times_{(\mathrm{Spin}^c(n), \kappa)} \mathrm{Spin}^c(n+1) \to P \times_{(\mathrm{Spin}^c(n), \kappa)} \mathrm{SO}(n+1) \tag{6.1}$$

for the vertical tangent bundle $P \times_{\mathrm{Spin}^c(n)} TS^n$ of $S(V \oplus \mathbb{R})$.

2. *a* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure on* TM *determines a* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure for* $TS(V \oplus \mathbb{R})$*.*

Proof. The bundle $P \times_{(\mathrm{Spin}^c(n),\kappa)} \mathrm{Spin}^c(n+1)$ is equipped with the left and right actions

$$\gamma(p, g) = (\gamma p, \gamma g) \qquad\qquad (p, g)h = (p, gh),$$

This should be compared with the semi-equivariant associated bundle construction of Definition 1.43. That (6.1) is a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for the vertical tangent bundle of $S(V \oplus \mathbb{R})$ can be be checked directly, making use of Lemma 3.18.

When TM has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure, the Two-of-Three Lemma can be applied to the decomposition

$$T(P \times_{\mathrm{Spin}^c(n+1)} S^n) = \pi^* TM \oplus (P \times_{\mathrm{Spin}^c(n+1)} TS^n),$$

where π is the projection for the bundle $S(V \oplus \mathbb{R})$. This determines a unique $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for $TS(V \oplus \mathbb{R})$. □

The next two operations are used to define bordism relations between K-cycles. The Two-of-Three Lemma induces $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures on the boundary of any Spin^c-orientifold with boundary.

Proposition 6.5 (Boundary Induction)**.** *The boundary* ∂W *of a* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-orientifold* W *with boundary has a unique* $(\mathrm{Spin}^c, \kappa_\epsilon)$*-structure.*

Proof. There is an exact sequence of equivariant vector bundles

$$0 \to T(\partial W) \to TW|_{\partial W} \to N_{\partial W} \to 0,$$

where $N_{\partial W}$ is the inward pointing normal bundle of ∂W. As W is $(\mathrm{Spin}^c, \kappa_\epsilon)$-oriented, it is oriented. Therefore, $N_{\partial W}$ is trivial. As $N_{\partial W}$ is trivial, it can be equipped with a canonical $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. A $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for $TW|_{\partial W}$ is produced by restricting the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for TW. The Two-of-Three Lemma 6.3 then implies that $T(\partial W)$ also has a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. □

The notion of a $(\mathrm{Spin}^c, \kappa_\epsilon)$ structure involves a choice of orientation on the frame bundle. This determines the bundle of positively oriented orthonormal frames. Given a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $\varphi : P \to \mathrm{Fr}(V)$ for a real equivariant vector bundle V equipped with a choice of orientation, there is a corresponding $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $\varphi^- : P^- \to \mathrm{Fr}^-(V)$ where $\mathrm{Fr}^-(V)$ is the bundle of oppositely oriented orthonormal frames. The next proposition defines the operation which takes a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure to this *opposite* $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. To state the proposition a few basic facts are needed. First note that the bundle of oppositely oriented frames can be written as

$$\mathrm{Fr}^-(V) := \mathrm{Fr}(V) \times_{SO(n)} O^-(n),$$

where $O^-(n)$ is the space of orientiation reversing isometries, represented by the orthogonal matricies with determinant -1. Next, note that $\mathrm{Spin}(n)$ sits inside the larger group

$$\mathrm{Pin}(n) := \{x_1 \cdots x_k \mid x_i \in \mathbb{R}^n, \|x_i\| = 1\} \subset \mathrm{Cl}_n.$$

The adjoint map extends to a double covering $\mathrm{Ad} : \mathrm{Pin}(n) \to O(n)$. Complexifying produces a map

$$\mathrm{Ad}^c : \mathrm{Pin}^c(n) = \mathrm{Pin}(n) \times_{\mathbb{Z}_2} U(1) \to O(n)$$

which extends the adjoint map from $\mathrm{Spin}^c(n)$ [6, p. 9]. Using this extension, it is possible to lift $O^-(n)$ to $\mathrm{Pin}^c(n)$. Define

$$\tilde{O}_c^-(n) := \left\{ r \in \mathrm{Pin}^c(n) \mid \mathrm{Ad}^c(r) \in O^-(n) \right\}.$$

With these preliminaries in place, the opposite $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure can be defined.

Proposition 6.6 (Opposite $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure)**.** *Let $V \to M$ be a real equivariant vector bundle equipped with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $\varphi : P \to \mathrm{Fr}(V)$. Then*

$$\varphi \times \mathrm{Ad}^c : P \times_{(\mathrm{Spin}^c(n), \kappa_\epsilon)} (\tilde{O}_c^-(n), \kappa_\epsilon) \to \mathrm{Fr}(V) \times_{SO(n)} O^-(n)$$

defines a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on the oppositely oriented bundle of frames.

Proof. Let $\gamma \in \Gamma$, $p \in P$, $r \in \tilde{O}_c^-(n)$, and $h \in \mathrm{Spin}^c(n)$. Well-definedness of the various actions is straightforward to check. The bundle $P^- := P \times_{(\mathrm{Spin}^c(n), \kappa_\epsilon)} (\tilde{O}_c^-(n), \kappa_\epsilon)$ is semi-equivariant as

$$\gamma([p, r]h) = \gamma[p, rh] = [\gamma p, \gamma(rh)] = [\gamma p, (\gamma r)(\gamma h)] = [\gamma p, \gamma r](\gamma h) = (\gamma[p, r])(\gamma h).$$

The fact that $\varphi^- := \varphi \times \mathrm{Ad}^c$ is a semi-equivariant lifting can also be checked directly

$$\varphi^-(\gamma[p, r]) = [\varphi(\gamma p), \mathrm{Ad}^c(\gamma r)] = [\gamma\varphi(p), \gamma \mathrm{Ad}^c(r)] = \gamma[\varphi(p), \mathrm{Ad}^c(r)] = \gamma(\varphi^-[p, r])$$
$$\varphi^-([p, r]h) = [\varphi(p), \mathrm{Ad}^c(rh)] = [\varphi(p), \mathrm{Ad}^c(r)\mathrm{Ad}^c(h)] = (\varphi^-[p, r])\mathrm{Ad}^c(h).$$

Thus, $\varphi^- : P^- \to \mathrm{Fr}^-(V)$ forms a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure for V, under the opposite choice of orientation. $\square$

If M is an orientifold equipped with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure, the same manifold with the opposite $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure will be denoted by $-M$.

6.2 The Geometric Orientifold K-homology Groups

Geometric K-homology shares characteristics of bordism and K-theory. Classes in the geometric K-homology of a topological space X are represented by K-cycles. The definition of a K-cycle starts with a manifold M mapped into X. The manifold M is equipped with all of the data needed to construct a spinor bundle with coefficients in a vector bundle E. A similar definition of K-cycle is used in geometric orientifold K-homology, except that the manifolds are replaced with orientifolds and each orientifold M is equipped with the structures needed to construct an orientifold Dirac operator with coefficients in an orientifold bundle. As in bordism, K-cycles can be added using a disjoint union operation.

Definition 6.7. A K-*cycle* for an orientifold X is a triple (M, E, f), where M is a smooth orientifold without boundary equipped with a specific $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure, E is an orientifold bundle over M, and $f : M \to X$ is a continuous equivariant map.

Definition 6.8. The *disjoint union* of two K-cycles is defined by

$$(M_1, E_1, f_1) \sqcup (M_2, E_2, f_2) := (M_1 \sqcup M_2, E_1 \sqcup E_2, f_1 \sqcup f_2).$$

A bordism-type equivalence relation will be defined between those cycles that arise as the boundary of a K-*cycle with boundary*, in the sense of the following definition.

Definition 6.9. A K-*cycle with boundary* is a triple (W, E, f), where W is a $(\mathrm{Spin}^c, \kappa_\epsilon)$-orientifold with boundary, $E \to W$ is an orientifold bundle, and $f : W \to X$ is a continuous Γ-equivariant map. A *boundary* K-*cycle* is a K-cycle of the form $(\partial W, E|_{\partial W}, f|_{\partial W})$, where (W, E, f) is a K-cycle with boundary.

Three equivalence relations on the set of K-cycles will now be introduced. The first of these relations relates the bordism-type disjoint union operation with the K-theoretic relation of vector bundle direct sum. The second relation uses boundary K-cycles to specify a notion of bordism which respects the structures carried by a K-cycle. The third relation expresses the 8-fold Thom isomorphism in terms of K-cycles.

Definition 6.10. Define the following equivalence relations on the set of triples $\{(M, E, f)\}$,

disjoint union/direct sum	$(M \sqcup M, E_1 \sqcup E_2, f \sqcup f) \sim_u (M, E_1 \oplus E_2, f)$
bordism	$(M_1, F_1, f_1) \sim_b (M_2, F_2, f_2)$
vector bundle modification	$(M, E, f) \sim_v (S(V \oplus \mathbb{R}), \boldsymbol{\beta} \otimes \pi^* E, f \circ \pi)$

for $(M_1 \sqcup -M_2, F_1 \sqcup F_2, f_1 \sqcup f_2)$ a boundary K-cycle, $\pi : V \to M$ an equivariant 8k-dimensional real vector bundle equipped with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure, and $\boldsymbol{\beta}$ the 8-fold Thom class of V.

The three equivalence relations of Definition 6.10 combine into a single equivalence relation which is used to define the the geometric orientifold K-homology groups.

Definition 6.11. Two K-cycles are said to be *equivalent* $(M_1, E_1, f_1) \sim (M_2, E_2, f_2)$ if they can be connected by any finite sequence of the equivalence relations $\sim_u$, $\sim_b$, and $\sim_v$.

Definition 6.12. The *geometric orientifold* K*-homology groups* are defined by

$$K^{\text{geo}}_{(\Gamma,\epsilon),j}(X) := \widetilde{K}^{\text{geo}}_{(\Gamma,\epsilon),j}(X)/\sim,$$

where $0 \leq j \leq 7$ and $\widetilde{K}^{\text{geo}}_{(\Gamma,\epsilon),j}(X)$ is the set of K-cycles (M, E, f) such that the dimension of each connected component of M is equal to j modulo 8.

6.3 Relationship to Analytic Orientifold K-homology

As discussed in the introduction to this chapter, geometric orientifold K-homology can be interpreted by constructing orientifold Dirac operators from K-cycles. In Section 3.4, $(\text{Spin}^c, \kappa_\epsilon)$-structures were used to construct orientifold Dirac operators. In Section 4.3, an elliptic orientifold operator and an orientifold bundle E were used to construct orientifold operators with coefficients in E. Applying these constructions using the $(\text{Spin}^c, \kappa_\epsilon)$-orientifold M and the orientifold bundle E from a K-cycle (M, E, f), results in a Dirac operator $\not{\mathfrak{D}}_E$ on M with coefficients in E. Proposition 5.35 shows that the normalisation of such an operator defines a class in the analytic orientifold K-homology of the C^*-algebra $(C(M), \kappa_\epsilon \circ (\sigma_M^{-1})^*)$. Using the map f, this class can be pushed forward to a class in the analytic orientifold K-homology of $(C(X), \kappa_\epsilon \circ (\sigma_X^{-1})^*)$.

Theorem 6.13. *The map*

$$\begin{aligned} \mu : K^{\text{geo}}_{(\Gamma,\epsilon),j}(X, \sigma) &\to KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma_X^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon)) \\ [M, E, f] &\mapsto f_*[\not{\mathfrak{D}}_E] \end{aligned}$$

from geometric to analytic orientifold K*-homology is a well-defined homomorphism.*

Proof. As discussed above, the results of previous chapters show that if (M, E, f) is a specific K-cycle representing a class in $K_j^{(\Gamma,\epsilon)}(X, \sigma)$, then $\mu(M, E, f) := f_*[\not{\mathfrak{D}}_E]$ represents a class in $KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon))$. However, it remains to check that μ is well-defined with respect to the equivalence relations on geometric orientifold K-homology. These will be considered one at a time.

It is straightforward to show that μ is well-defined with respect to the disjoint union/direct sum relation,

$$\mu(M \sqcup M, E_1 \sqcup E_2, f \sqcup f) \sim \mu(M, E_1 \oplus E_2, f) \in KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon)).$$

Next, suppose that (M_1, E_1, f_1) and (M_2, E_2, f_2) are K-cycles, and (W, E, f) is a K-cycle with boundary such that

$$(M_1, E_1, f_1) \sqcup (-M_2, E_2, f_2) \simeq (\partial W, E|_{\partial W}, f|_{\partial W}).$$

Define $\mathcal{E}_1 := \mu(M_1, E_1, f_1), \mathcal{E}_2 := \mu(-M_2, E_2, f_2) \in KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon))$. Because W is a perfectly normal topological space, and M_1 and M_2 are closed subsets, it is always possible to find a continuous map $\theta' : W \to [0,1]$ such that $\theta'^{-1}(0) = M_1$ and $\theta'^{-1}(0) = M_2$, [86, p. 103,105]. Averaging this map over the group action produces an equivariant map

$$\theta(w) := |\Gamma|^{-1} \sum_{\xi \in \Gamma} \theta'(\xi w)$$

such that $\theta^{-1}(0) = M_1$ and $\theta^{-1}(1) = M_2$. The K-cycle (W, E, f) then determines a class

$$\mathcal{W} := [F, \phi \circ f^*, L^2(W, \not{S}_E)] \in KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon) \otimes (C[0,1], \kappa_\epsilon)),$$

where the action of $\mathbb{C}l_j \otimes C[0,1]$ on $L^2(W, \not{S}_E)$ is defined by

$$(\psi(\varphi \otimes f))(w) := f \circ \theta(w)\psi(w)\varphi,$$

and the $\mathbb{C}l_j \otimes C[0,1]$-valued inner product on $L^2(W, \not{S}_E)$ is

$$\langle \psi_1, \psi_2 \rangle(t) = \int_{\theta^{-1}(t)} tr(\psi_1^* \psi_2) dx.$$

The evaluation map $\pi_t : \mathbb{C}l_j \otimes C[0,1] \to \mathbb{C}l_j$ is surjective, allowing the pushout modules $\pi_0 \mathcal{W}$ and $\pi_1 \mathcal{W}$ to be formed. Because the $(Spin^c, \kappa_\epsilon)$-structures on M_1 and M_2 are induced from the boundary of W, the Dirac operator for W restricts to the Dirac operators for M_1 and M_2. This implies that the pushout operators for $\pi_0 \mathcal{W}$ and $\pi_1 \mathcal{W}$ are the same as those for $\mathcal{E}_1, \mathcal{E}_2$ respectively. Thus, $\pi_0 \mathcal{W} \simeq \mathcal{E}_1, \pi_1 \mathcal{W} \simeq \mathcal{E}_2$, and $\mathcal{W}$ defines a homotopy equivalence

$$\mathcal{E}_1 \sim_h \mathcal{E}_2 \in KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon)).$$

To prove that μ is well-defined with respect to vector bundle modification, the method described in [17, Prop. 3.6] can be applied. The equivalence

$$\mu(M, E, f) \sim \mu(S(V \oplus \mathbb{R}), \boldsymbol{\beta} \otimes \pi^* E, f \circ \pi) \in KK^{(\Gamma,\epsilon)}((C(X), \kappa_\epsilon \circ (\sigma^{-1})^*), (\mathbb{C}l_j, \kappa_\epsilon))$$

can be shown by decomposing the Kasparov module $\mu(S(V \oplus \mathbb{R}), \boldsymbol{\beta} \otimes \pi^* E, f \circ \pi)$ into two components, one trivial component and one isomorphic to $\mu(M, E, f)$. This depends on the calculation of the index pairing between $\boldsymbol{\beta}$ and the canonical Dirac operator on the $8k$-dimensional sphere, see Lemma 4.18.

Let $V \to M$ be an equivariant $8k$-dimensional real vector bundle equipped with a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure $P \to \mathrm{Fr}(V)$, $\pi_P : P \to M$ be the projection associated to P, and $\pi : S(V \oplus \mathbb{R}) \to M$ be the sphere bundle of V. As discussed in the proof of Propostion 6.4, the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures on $\pi^* TM$ and S^n induce a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure on $S(V \oplus \mathbb{R})$ via the decomposition

$$TS(V \oplus \mathbb{R}) = \pi^* TM \oplus (P \times_{\mathrm{Spin}^c(n+1)} TS^n).$$

The orientifold spinor bundle associated to this $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure is

$$\mathfrak{S}^{S(V \oplus \mathbb{R})} = \pi^* \mathfrak{S}^M \hat{\otimes} (P \times_{\mathrm{Spin}^c(n+1)} \mathfrak{S}^S),$$

where $\mathfrak{S}^M$ is the orientifold spinor bundle on M and $\mathfrak{S}^S$ is the canonical orientifold spinor bundle on S^{8k}. After twisting by $\boldsymbol{\beta} \otimes \pi^* E$ this becomes

$$\mathfrak{S}^{S(V \oplus \mathbb{R})}_{\boldsymbol{\beta} \otimes \pi^* E} = \pi^* \mathfrak{S}^M_E \hat{\otimes} (P \times_{\mathrm{Spin}^c(n+1)} \mathfrak{S}^S_{\boldsymbol{\beta}}),$$

The Hilbert module of the corresponding class in analytic K-homology is therefore

$$L^2(P \times_{\mathrm{Spin}^c(n+1)} S^{8k}, \pi^* \mathfrak{S}^M_E \hat{\otimes} (P \times_{\mathrm{Spin}^c(n+1)} \mathfrak{S}^S_{\boldsymbol{\beta}})).$$

Rather than forming associated bundles as a quotient by $\mathrm{Spin}^c(n+1)$, the above space can be considered as a space of $\mathrm{Spin}^c(n+1)$-equivariant sections and decomposed as the graded Hilbert space tensor product

$$\begin{aligned} \left[L^2(P \times S^{8k}, \pi^* \mathfrak{S}^M_E \hat{\otimes} (P \times \mathfrak{S}^S_{\boldsymbol{\beta}}))\right]^{\mathrm{Spin}^c(n+1)} &= \left[L^2(P \times S^{8k}, \pi_1^* \pi_P^* \mathfrak{S}^M_E \hat{\otimes} \pi_2^* \mathfrak{S}^S_{\boldsymbol{\beta}})\right]^{\mathrm{Spin}^c(n+1)} \\ &= \left[L^2(P, \pi_P^* \mathfrak{S}^M_E) \hat{\otimes} L^2(S^{8k}, \mathfrak{S}^S_{\boldsymbol{\beta}})\right]^{\mathrm{Spin}^c(n+1)}, \end{aligned}$$

where $\pi_1 : P \times S^{8k} \to P$ and $\pi_2 : P \times S^{8k} \to S^{8k}$ are the component projections. The associated Dirac operator has a corresponding decomposition of the form

$$D := \mathfrak{D}^P_E \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} \mathfrak{D}^S_{\boldsymbol{\beta}},$$

where $\mathfrak{D}^S_{\boldsymbol{\beta}}$ is the $\mathrm{Spin}^c(n+1)$-equivariant orientifold Dirac operator on S^{8k} with coefficients in $\boldsymbol{\beta}$, $\mathfrak{D}^M_E$ is the orientifold Dirac operator on M with coefficients in E, and $\mathfrak{D}^P_E$ is a $\mathrm{Spin}^c(n+1)$-equivariant lifting of $\mathfrak{D}^M_E$ to P. Such liftings can be constructed by patching together local liftings using a partition of unity and then averaging the resulting operator over the action of $\mathrm{Spin}^c(n+1)$ to obtain an equivariant operator, see [17, p. 8].

The decomposition $L^2(S^{8k}, \not{S}^S_\beta) = \ker \not{D}^S_\beta \oplus (\ker \not{D}^S_\beta)^\perp$ can be used to further decompose the Hilbert module,

$$\begin{aligned}\left[L^2(P, \pi_P^* \not{S}^M_E) \hat{\otimes} L^2(S^{8k}, \not{S}^S_\beta)\right]^{\mathrm{Spin}^c(n+1)} \\ = \left[L^2(P, \pi_P^* \not{S}^M_E) \hat{\otimes} \ker \not{D}^S_\beta\right]^{\mathrm{Spin}^c(n+1)} \oplus \left[L^2(P, \pi_P^* \not{S}^M_E) \hat{\otimes} (\ker \not{D}^S_\beta)^\perp\right]^{\mathrm{Spin}^c(n+1)}.\end{aligned}$$

This results in a corresponding decomposition of the Kasparov module into two Kasparov modules for the operator $D := \not{D}^P_E \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} \not{D}^S_\beta$. However, by Lemma 4.18, $\ker \not{D}^S_\beta = (\mathbb{C}, \kappa_\epsilon)$ and the index of $\not{D}^S_\beta$ on $\ker(\not{D}^S_\beta)^\perp$ is $\{0\}$. Thus, the Kasparov module simplifies to

$$\left[F^P, \phi^P, L^2(P, \pi_P^* \not{S}^M_E)\right]^{\mathrm{Spin}^c(n+1)}.$$

where F^P is the normalisation of $\not{D}^P_E$. Pushing forward the associated Kasparov module via π_P recovers the Kasparov module $\mu(M, E, f)$. □

In the usual equivariant setting, the homomorphism corresponding to μ is an isomorphism. The proof proceeds by showing that the map is a natural transformation between generalised cohomology theories and is an isomorphism on the one-point space. A general result from algebraic topology then ensures that the map is an isomorphism [77, §4.6]. The main difficulty is to show that geometric K-homology satisfies the Eilenberg-Steenrod axioms for a generalised cohomology theory. In [18, 17] the isomorphism was proved by constructing an intermediate generalised homology theory based on framed bordism. Another approach is to replace the bordism and vector bundle modification relations with a single *normal bordism* relation. A result due to Jakob [49, 48] then shows that the geometric K-homology is a generalised homology theory. Variations on this technique have been applied to prove isomorphisms between geometric and analytic K-homology in a variety of settings [73, §4.5] [28, §3.3.2] [29, 30] [12, §5]. It seems likely that this normal bordism approach could also be adapted to prove that the map μ of Theorem 6.13 is an isomorphism.

Chapter 7

The K-homology of Orientifold Groups

This chapter makes some notes regarding the possibility of an assembly map for orientifold groups. Using the orientifold Dirac operator, a correspondence

$$\mu : K_j^{geo}(\Gamma, \epsilon) \to KK^{\rho}(\mathbb{C}l_j, C_r^*\Gamma^+)$$

is sketched between a geometric K-homology group associated to a finite orientifold group and an analytic K-theory group which is modelled after $KK(\mathbb{C}l_j, C_r^*G)$. This construction is based on the description of the assembly map in [11, pp. 41-44] and on the theory of unitary/anti-unitary representations, which was reviewed in Section 2.2. See also [51, II.7].

Because representations of orientifold groups involve anti-linear operators, it is not possible to directly define an analogue of C_r^*G for orientifold groups. However, the theory of unitary/anti-unitary representations, outlined in Section 2.2, indicates a way around this problem. Rather than trying to define an algebra $C_r^*\Gamma$, one considers the group algebra $C_r^*\Gamma^+$ of those elements which act linearly, and equips it with an orientifold action $\boldsymbol{\rho}$ of Γ that corresponds to relative conjugation, see Definition 2.16. The analytic K-theory of this algebra is then defined by requiring each Kasparov module over $C_r^*\Gamma^+$ to carry a specific choice of $\zeta \in \Gamma^-$, and an operator R which satisfies the two conditions

$$R^2x = x\delta_{\zeta^2} \qquad\qquad R(xf) = R(x)\boldsymbol{\rho}_{\zeta}(f).$$

If $\zeta^2 = 1$, R corresponds to the anti-linear map carried by a Kasparov module in KKR-theory [58, p. 518]. However, in general there may be no element ζ with this property. For this reason, any choice of $\zeta \in \Gamma^-$ is permitted. The resulting ambiguity can then be removed by introducing an equivalence relation $\sim_{\rho}$ on the set of Kasparov modules. The groups

$KK^{\rho}(\mathbb{C}l_j, C^*_r\Gamma^+)$ are formed as the set of Kasparov modules (E, ϕ, F, ζ, R) modulo $\sim_\rho$ and the appropriate homotopy equivalence relation.

The geometric K-homology groups $K^{geo}_j(\Gamma, \epsilon)$ of an orientifold group are defined in almost exactly the same way as the groups $K^{geo}_{(\Gamma,\epsilon),j}(X, \sigma)$, except that all reference to the orientifold X is removed. Thus, instead of considering K-cycles (M, E, f) where $f : M \to X$ is a continuous equivariant map, K-cycles for $K^{geo}_j(\Gamma, \epsilon)$ are just pairs (M, E).

Once the groups $K^{geo}_j(\Gamma, \epsilon)$ have been defined, it can be shown that K-cycles (M, E) define elements of $[\not{\mathfrak{D}}_E] \in KK^{\rho}(\mathbb{C}l_j, C^*_r\Gamma^+)$ via the orientifold Dirac operators $\not{\mathfrak{D}}_E$. The linear symmetries of the orientifold spinor bundle $\not{\mathfrak{S}}$ make its L^2-sections into a Kasparov module over $C^*_r\Gamma^+$, and the anti-linear symmetries of $\not{\mathfrak{S}}$ provide an anti-linear operator R for any choice of $\zeta \in \Gamma^-$. The equivariance of the orientifold Dirac operator ensures that it is compatible with any possible R.

There are two motivations for making these construction. The first, is to highlight the extra information, appearing as the anti-linear operator R, that is captured by the orientifold Dirac operator. The second motivation is to suggest a generalisation which could be used to investigate infinite discrete orientifold groups. Although only finite orientifold groups have been treated in this thesis, it seems likely that the constructions described here could be generalised along the same lines as in the usual equivariant case.

7.1 Analytic K-theory for Orientifold Groups

Recall from Example 5.6 that a finite orientifold group (Γ, ϵ) defines an orientifold C^*-algebra.

Definition 7.1. If (Γ, ϵ) is a finite orientifold group, define the orientifold C^*-algebra

$$(C^*_r\Gamma^+, \rho) := (\mathbb{C}\Gamma^+, *, \|\cdot\|, \rho)$$

where

1. $\mathbb{C}\Gamma^+$ is the algebra of complex valued functions on Γ^+ with product, $*$-structure, and norm defined respectively by

$$(f * g)(\gamma) := \sum_{\xi\in\Gamma^+} f(\xi)g(\xi^{-1}\gamma) \qquad f^*(\gamma) := \overline{f(\gamma^{-1})} \qquad \|f\| := \|\pi(f)\|_2,$$

where $\|\cdot\|_2$ is the ℓ^2-norm, and π is the regular representation of $\mathbb{C}\Gamma^+$ on ℓ^2 defined by

$$(\pi(f)\nu)(\gamma) := \sum_{\xi\in\Gamma^+} f(\xi)\nu(\xi^{-1}\gamma)$$

for $f \in \mathbb{C}\Gamma^+$ and $\nu \in \ell^2(\Gamma^+)$.

2. $\boldsymbol{\rho}$ is the orientifold action on $\mathbb{C}\Gamma^+$ defined by

$$(\boldsymbol{\rho}_\zeta f)(\gamma) := \zeta f(\zeta^{-1}\gamma\zeta),$$

for all $f \in \mathbb{C}\Gamma^+$ and $\zeta \in \Gamma$.

The appropriate type of Kasparov module to take over this algebra is one equipped with a choice of $\zeta \in \Gamma^-$, and an anti-linear map R. The map R must be compatible with the action of $\boldsymbol{\rho}_\zeta$ and interrelated with the right $C^*_r\Gamma^+$-module action.

Definition 7.2. Let $\mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)$ be the set of all tuples $((E,\zeta,R,\chi),\phi,F)$ where

1. (E,ζ,R,χ) is a countably generated graded Hilbert $C^*_r\Gamma^+$-module equipped with a choice of element $\zeta \in \Gamma^-$, and an anti-linear map $R : E \to E$ which satisfies

$$R^2x = x\delta_{\zeta^2} \qquad R(xf) = (Rx)(\boldsymbol{\rho}_\zeta f) \qquad \langle Rx, Ry\rangle = \boldsymbol{\rho}_\zeta\langle x,y\rangle \qquad R\chi = \chi R$$

   for $x, y \in E$, $f \in C^*_r(\Gamma^+)$.

2. $\phi : \mathbb{C}l_j \to \mathcal{L}_{C^*_r\Gamma^+}(E)$ is a homomorphism of graded C^*-algebras such that

$$\phi \circ \kappa(a) = R\phi(a)R^{-1},$$

   for $a \in \mathbb{C}l_j$.

3. $F \in \mathcal{L}_{C^*_r\Gamma^+}(E)$ is an odd operator such that the operators

$$[F,\phi(a)] \qquad (F^2-1)\phi(a) \qquad (F^*-F)\phi(a) \qquad (RFR^{-1}-F)\phi(a)$$

   are in $\mathcal{K}_{C^*_r\Gamma^+}(E)$ for all $a \in A$ and $\gamma \in \Gamma$, where $[\cdot,\cdot]$ denotes the graded commutator.

For each element $((E,\zeta,R),\phi,F) \in \mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)$ and each $\xi \in \Gamma^+$, there exists another element $((E,\xi\zeta,SR),\phi,F) \in \mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)$ which should correspond to the same Kasparov module. Deeming these to be equivalent eliminates dependence on the choice of $\zeta \in \Gamma^-$.

Proposition 7.3. *If* $((E,\zeta,R),\phi,F) \in \mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)$, $\xi \in \Gamma^+$ *and* $Sx = x\delta_\xi$, *then*

$$((E,\xi\zeta,SR),\phi,F) \in \mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+).$$

This operation determines an equivalence relation

$$((E,R,\zeta),\phi,F) \sim_\rho ((E,\xi\zeta,SR),\phi,F).$$

Proof. First, note that the right action of $C^*_r\Gamma^+$ is defined by the action of the elements $\delta_\gamma \in C^*_r\Gamma^+$. Let λ_γ denote the operator corresponding to the action of δ_γ, but acting from the left on E. The condition $R(\psi f) = (R\psi)(\boldsymbol{\rho}_\zeta f)$ can be rewritten as

$$R(\sum_{\gamma\in\Gamma^+} f(\gamma)\lambda_\gamma)\psi = (\sum_{\gamma\in\Gamma^+} \bar{f}(\zeta^{-1}\gamma\zeta)\lambda_\gamma)R\psi.$$

This implies the corresponding condition for SR,

$$\begin{aligned}
(SR)(\psi f) &= SR(\sum_{\gamma\in\Gamma^+} f(\gamma)\lambda_\gamma\psi) \\
&= S\sum_{\gamma\in\Gamma^+} \bar{f}(\zeta^{-1}\gamma\zeta)\lambda_\gamma R\psi \\
&= S\sum_{\gamma\in\Gamma^+} \bar{f}(\zeta^{-1}\xi^{-1}\gamma\xi\zeta)\lambda_{\xi^{-1}\gamma\xi} R\psi \\
&= S\sum_{\gamma\in\Gamma^+} \bar{f}(\zeta^{-1}\xi^{-1}\gamma\xi\zeta)\lambda_{\xi^{-1}}\lambda_\gamma\lambda_\xi R\psi \\
&= SS^{-1}\sum_{\gamma\in\Gamma^+} \bar{f}((\xi\zeta)^{-1}\gamma\xi\zeta)\lambda_\gamma SR\psi \\
&= (SR\psi)\boldsymbol{\rho}_{\xi\zeta}(f).
\end{aligned}$$

The condition $R^2\psi = \psi\delta_{\zeta^2}$ can be used to prove $(SR)^2\psi = \psi\delta_{(\xi\zeta)^2}$. First, note that

$$RS = R\sum_{\gamma\in\Gamma^+} \delta_\xi(\gamma)\lambda_\gamma = \sum_{\gamma\in\Gamma^+} \bar{\delta}_\xi(\zeta^{-1}\gamma\zeta)\lambda_\gamma R = \sum_{\gamma\in\Gamma^+} \bar{\delta}_\xi(\gamma)\lambda_{\zeta\gamma\zeta^{-1}} R = \lambda_{\zeta\xi\zeta^{-1}} R.$$

Then,

$$SRSR\psi = S\lambda_{\zeta\xi\zeta^{-1}}RR\psi = \lambda_{\xi\zeta\xi\zeta^{-1}}R^2\psi = \lambda_{\xi\zeta\xi\zeta^{-1}}\lambda_{\zeta^2}\psi = \lambda_{\xi\zeta\xi\zeta}\psi = \lambda_{(\xi\zeta)^2}\psi = \psi\delta_{(\xi\zeta)^2}.$$

Because the inner product and operator F are compatible with R and are $C^*_r\Gamma^+$-linear, they are both compatible with the action of SR. The group properties of Γ^+ ensure that $\sim_\rho$ is an equivalence relation. ☐

Definition 7.4. Define two elements of $\mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)$ to be equivalent $x \sim y$ if they can be connected by any finite sequence of the equivalence relations $\sim_\rho$ and homotopy equivalence within $\mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)$, see Definition 5.28.

Definition 7.5. Define the analytic K-*theory* of an orientifold group (Γ, ϵ) by

$$KK^\rho(\mathbb{C}l_j, C^*_r\Gamma^+) := \mathbb{E}^\rho(\mathbb{C}l_j, C^*_r\Gamma^+)/\sim \tag{7.1}$$

with addition given by $[\mathcal{E}_1] + [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2]$.

7.2 Geometric K-homology for Orientifold Groups

The geometric K-homology $K_j^{\text{geo}}(\Gamma, \epsilon)$ of an orientifold group is defined in the same way as the geometric K-homology $K_{(\Gamma,\epsilon),j}^{\text{geo}}(X)$ of an orientifold, except that the orientifold X is disregarded. Thus, rather than considering triples (M, E, f) as in Definition 6.7, a K-cycle for $K_j^{\text{geo}}(\Gamma, \epsilon)$ is a pair (M, E). The set of K-cycles for $K_j^{\text{geo}}(\Gamma, \epsilon)$ is equipped with the equivalence relations *disjoint union/direct sum*, *bordism* and *vector bundle modification*. These relations are defined in the same manner as the corresponding equivalence relations on K-cycles (M, E, f) for $K_{(\Gamma,\epsilon),j}^{\text{geo}}(X)$ except that the maps f are omitted, see Definition 6.10.

Definition 7.6. A K-*cycle* for an orientifold X is a triple (M, E), where M is a smooth compact orientifold equipped with a specific $(\text{Spin}^c, \kappa_\epsilon)$-structure and an orientifold bundle E.

Definition 7.7. The *disjoint union* of two K-cycles is defined by

$$(M_1, E_1) \sqcup (M_2, E_2) := (M_1 \sqcup M_2, E_1 \sqcup E_2) \tag{7.2}$$

Definition 7.8. A K-*cycle with boundary* is a triple (W, E), where W is a $(\text{Spin}^c, \kappa_\epsilon)$-orientifold with boundary, and $E \to W$ is an orientifold bundle. A *boundary* K-*cycle* is a K-cycle of the form $(\partial W, E|_{\partial W})$, where (W, E) is a K-cycle with boundary.

Definition 7.9. Define the following equivalence relations on the set of triples $\{(M, E)\}$:

disjoint union/direct sum	$(M \sqcup M, E_1 \sqcup E_2) \sim_u (M, E_1 \oplus E_2)$.
bordism	$(M_1, F_1) \sim_b (M_2, F_2)$,
vector bundle modification	$(M, E) \sim_v (S(V \oplus \mathbb{R}), \boldsymbol{\beta} \otimes \pi^* E)$,

where $(M_1 \sqcup -M_2, F_1 \sqcup F_2)$ is a boundary K-cycle, and $\pi : V \to M$ is an equivariant $8k$-dimensional real vector bundle equipped with a $(\text{Spin}^c, \kappa_\epsilon)$-structure.

Definition 7.10. Two K-cycles are said to be *equivalent* $(M_1, E_1) \sim (M_2, E_2)$ if they can be connected by any finite sequence of the equivalence relations $\sim_u$, $\sim_b$, and $\sim_v$.

Definition 7.11. The *geometric* K-*homology* of an orientifold group (Γ, ϵ) is defined by

$$K_j^{\text{geo}}(\Gamma, \epsilon) := \widetilde{K}_j^{\text{geo}}(\Gamma, \epsilon) / \sim,$$

where $0 \le j \le 7$ and $\widetilde{K}_j^{\text{geo}}(\Gamma, \epsilon)$ is the set of K-cycles (M, E) such that the dimension of each connected component of M is equal to j modulo 8.

7.3 Assembly and Orientifold Groups

In Section 6.3, a map was defined from the geometric K-homology to the analytic K-homology of an orientifold, using orientifold Dirac operators. The next proposition also uses the orientifold Dirac operator to define an analytic class. However, the perspective is changed and the orientifold Dirac operator is viewed as a family of operators over a group algebra rather than a single equivariant operator. In this way, a class in the analytic K-theory group $KK^{\rho}(\mathbb{C}l_j, C^*_r\Gamma^+)$ is associated to each K-cycle. Note that in the following proposition only finite orientifold groups are considered. See [11, p. 41-44] for the case of an infinite discrete group in the equivariant setting.

Proposition 7.12. *Each* K*-cycle* (M, E) *representing a class in* $K_j^{geo}(\Gamma, \epsilon)$ *defines class*

$$\mu(M, E) := [(S_E, \zeta, R, \omega), \phi, F] \in KK^{\rho}(\mathbb{C}l_j, C^*_r\Gamma^+)$$

where

1. $S_E := L^2(M, \not{S}_E)$ *is the underlying Hilbert module equipped with the right* $C^*_r\Gamma^+$*-action*

$$(\psi f) := \sum_{\xi \in \Gamma^+} f(\xi)(\lambda_\xi \psi),$$

where λ *is the orientifold action on the orientfold spinor bundle and* $f \in C^*_r\Gamma^+$*. The* $C^*_r\Gamma^+$*-valued inner product on* S_E *is defined by*

$$\langle \psi_1, \psi_2 \rangle(\gamma) := \int_M \langle \psi_1(m), (\lambda_\gamma \psi_2)(m) \rangle \, dm,$$

2. ζ *is an arbitrary element of* Γ^- *and* $R : S_E \to S_E$ *is the anti-linear operator*

$$R\psi := \lambda_\zeta \psi,$$

3. ϕ *is the representation of* $\mathbb{C}l_n$ *on* S_E *by right multiplication operators,*

4. F *is the normalisation of* $\not{D}_E$ *is defined by*

$$F := \not{D}_E(I + \not{D}_E^2)^{\frac{1}{2}} \in \mathcal{L}_{\mathbb{C}l_n}(S_E).$$

This class is independent of the choice of ζ*.*

Proof. The condition $R^2 x = x\delta^2$ follows immediately, as the right action of δ^2 is defined in terms of the left action on $\not{S}_E$.

The invariance of property of the orientifold metric makes the $C_r^*\Gamma^+$-valued inner product compatible with R and the action of $C_r^*\Gamma^+$,

$$\begin{aligned}(\zeta\langle\psi_1,\psi_2\rangle)(\gamma) = \zeta\langle\psi_1,\psi_2\rangle(\zeta^{-1}\gamma\zeta) &= \zeta\int_M \langle\psi_1(x),(\lambda_{\zeta^{-1}\gamma\zeta}\psi_2)(x)\rangle dm \\ &= \int_M \zeta\langle\psi_1(x),\zeta^{-1}\gamma\zeta\psi_2(\zeta^{-1}\gamma^{-1}\zeta x)\rangle dm \\ &= \int_M \zeta\langle\psi_1(\zeta^{-1}x),\zeta^{-1}\gamma\zeta\psi_2(\zeta^{-1}\gamma^{-1}x)\rangle dm \\ &= \int_M \langle\zeta\psi_1(\zeta^{-1}x),\gamma\zeta\psi_2(\zeta^{-1}\gamma^{-1}x)\rangle dm \\ &= \langle\lambda_\zeta\psi_1,\lambda_\zeta\psi_2\rangle(\gamma). \\ &= \langle R\psi_1,R\psi_2\rangle(\gamma).\end{aligned}$$

The property $R(\psi f) = (R\psi)(\zeta f)$ holds as

$$\begin{aligned}R(\psi f) = \lambda_\zeta(\psi f) = \lambda_\zeta\sum_{\gamma\in\Gamma} f(\gamma)\lambda_\gamma\psi &= \sum_{\gamma\in\Gamma}(\zeta f(\gamma))\lambda_{\zeta\gamma}\psi \\ = \sum_{\gamma\in\Gamma}(\zeta f(\gamma))\lambda_{\zeta\gamma\zeta^{-1}\zeta}\psi = \sum_{\gamma\in\Gamma}\zeta f(\zeta^{-1}\gamma\zeta)\lambda_{\gamma\zeta}\psi &= \sum_{\gamma\in\Gamma}(\zeta f)(\gamma)\lambda_\gamma\lambda_\zeta\psi = (R\psi)(\zeta f).\end{aligned}$$

The properties $[F,\phi(a)] \in \mathcal{K}_{C_r^*\Gamma^+}$, $(F^2 - \mathrm{id})\phi(a) \in \mathcal{K}_{C_r^*\Gamma^+}$, $(F^* - F)\phi(a) \in \mathcal{K}_{C_r^*\Gamma^+}$ follow from the properties of $\mathfrak{D}_E$ as usual. The property $(RFR^{-1},F)\phi(a) \in \mathcal{K}_{C_r^*\Gamma^+}$ follows from the Γ-equivariance of F for any choice of ζ.

If $\mu(M,E)$ is defined using a different element $\zeta' \in \Gamma^-$, then $\zeta' = \xi\zeta$ for some $\xi \in \Gamma^+$ and the resulting class is $[(S_E,\xi\zeta,SR,\boldsymbol{\omega}),\phi,F]$, where $S = \lambda_\xi$. This class is equivalent to $[(S_E,\zeta,R,\boldsymbol{\omega}),\phi,F]$ under $\sim_\rho$. □

The above correspondence for finite orientifold groups is inspired by the assembly map of the Baum-Connes conjecture. It would be interesting to know if the correspondence could be generalised to provide an assembly map for infinite orientifold groups. To do so, it would be neccesary to extend the constructions of this thesis to deal with infinite discrete orientifold groups and open orientifolds. An assembly map would then be obtained by composing μ with the index map

$$\mathrm{ind}_{(\Gamma,\epsilon)} : K_j^{\mathrm{geo}}(\Gamma,\epsilon) \xrightarrow{\mu} KK^\rho(\mathbb{C}l_j, C_r^*\Gamma^+) \xrightarrow{\mathrm{ind}} K_j^\rho(C_r^*\Gamma^+),$$

where $K_j^\rho(C_r^*\Gamma^+)$ is the K-theory group formed from f.g.p. projective modules equipped with an anti-linear operator R associated to an element $\zeta \in \Gamma^-$, and quotient by an equivalence relation similar to the relation $\sim_\rho$ on $KK^\rho(\mathbb{C}l_j, C_r^*\Gamma^+)$.

Conclusion

The aim of this thesis has been to solidify the understanding of anti-linear symmetry in index theory. In particular, to identify the conditions under which an orientifold Dirac operator can be constructed. In order to do this, the notion of a semi-equivariant transition cocycle was introduced. Semi-equivariant transition cocycles generalise both equivariant transition cocycles and Wigner's corepresentations. A corresponding semi-equivariant cohomology theory was constructed, and analogs of standard results allowed the obstruction to the existence of an orientifold Dirac operator to be identified as a semi-equivariant cohomology class $W_3^{(\Gamma,\epsilon)}$. This class generalises the third integral Stiefel-Whitney class that obstructs Spin^c-structures. Using the decomposition $\mathrm{Spin}^c(n) = \mathrm{Spin}^c(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$, it was possible to show that the existence of complementary semi-equivariant cochains for the structure groups $(\mathrm{Spin}(n), \mathrm{id}_\epsilon)$ and $(\mathrm{U}(1), \kappa_\epsilon)$ is equivalent to the existence of a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. A twisted averaging procedure over the $\mathrm{U}(1)$-component of this splitting allowed the construction of semi-equivariant connections for $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures. The orientifold spinor bundles were then constructed as semi-equivariant associated bundles, using a $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure and a semi-equivariant fibre. The total spinor bundle was constructed using the fibre $(\mathbb{C}l_n, \kappa_\epsilon)$, and, in dimensions $8k$, the complexification $(\Delta \otimes \mathbb{C}, \mathrm{id} \otimes \kappa_\epsilon)$ of the irreducible Spin-representation Δ was used to construct the reduced spinor bundle. Sections of these spinor bundles carry a multiplication by 1-forms that is compatible with the orientifold action. The total and reduced orientifold Dirac operators were obtained by composing multiplication by 1-forms with the connections induced from the semi-equivariant connection on the $(\mathrm{Spin}^c, \kappa_\epsilon)$-structure. The total and reduced orientifold Dirac operators were shown to be equivariant with respect to the linear/anti-linear actions on the spinor bundles. The construction of the orientifold Dirac operator, and the identification of the condition $W_3^{(\Gamma,\epsilon)}(X) = 0$ for its existence, completed the main aim of the thesis. In particular, the Real Dirac operator was found to exist on all Real spaces X such that $W_3^{(\mathbb{Z}_2,\mathrm{id})}(X) = 0$.

Having constructed the orientifold Dirac operator, attention turned to investigating its place in K-theory and K-homology. Atiyah's proof of equivariant Bott periodicity was adapted to the setting of orientifold K-theory. The $(1,1)$ and 8-fold Bott periodicity theorems for orientifold K-theory were obtained as a special cases of an equivariant periodicity theorem. Together, these showed that every orientifold K-group $K^{p,q}_{(\Gamma,\epsilon)}(X)$ is isomorphic to one of the 8 groups $K^{p,0}_{(\Gamma,\epsilon)}(X)$ for $0 \leq p \leq 7$. By combining equivariant periodicity with results on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures, it was possible to prove an 8-fold Thom isomorphism theorem $K_{(\Gamma,\epsilon)}(X) \simeq K_{(\Gamma,\epsilon)}(V)$ for real equivariant vector bundles V such that $W_3^{(\Gamma,\epsilon)}(V) = 0$. A straightforward generalisation of analytic K-homology was made, based on Kasparovs KK-theory. The orientifold Dirac operator was shown to define a class in the resulting analytic orientifold K-homology theory. A geometric orientifold K-homology theory was also defined. In this theory, cycles are represented by orientifolds equipped with $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures and orientifold bundles. A two-of-three lemma was proved using earlier results on semi-equivariant cocycles and cohomology. This allowed operations on $(\mathrm{Spin}^c, \kappa_\epsilon)$-structures and equivalence relations for geometric K-homology to be defined. The interpretation of geometric orientifold K-cycles via orientifold Dirac operators was formalised by constructing a map from geometric to analytic orientifold K-homology. Finally, some speculations were made regarding the possibility of using orientifold Dirac operators to construct assembly maps.

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