Global regularity for solutions to Monge-Ampère type equations

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A thesis submitted for the degree of Doctor of Philosophy of the Australian National University
Dedicated to my husband and my parents for their love and unconditional support
Declaration

The work in this thesis is my own except where otherwise stated.

Elina Andriyanova
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Abstract

The optimal transportation problem was formulated by Monge in 1781: given two domains $\Omega, \Omega^* \subset \mathbb{R}^n$ and two mass distributions $f \in L^1(\Omega)$ and $g \in L^1(\Omega^*)$ of the same mass, find the optimal volume-preserving map $T$ between them, where optimality is measured against a cost functional

$$C(s) = \int_{\Omega} f(x)c(x, s(x)).$$

Optimal transportation has undergone a rapid and important development since the pioneering work of Brenier, who discovered that when the cost is the distance squared, optimal maps for the problem are gradients of convex functions. Later Caffarelli proved that in the case of convex target domain $\Omega^*$, potential functions to optimal transportation problem are weak solutions (in the Aleksandrov sense) to the standard Monge-Ampère equation. Following this result and its subsequent extensions, the theory of optimal transportation has flourished, with generalizations to other cost functions and corresponding Monge-Ampère type equations, applications in many other areas of mathematics such as geometric analysis, functional inequalities, fluid mechanics, dynamical systems, and other more concrete applications such as irrigation and cosmology.

In this thesis we are concerned with the global regularity problem for the optimal transportation and the corresponding Monge-Ampère type equation. The manuscript consists from 4 chapters.

Chapter 1 is an introduction, where collected some background information on the topic and main results been made so far.

In Chapter 2 we consider the global regularity of potential functions in optimal transportation with quadratic cost and provide a global $C^{1,\alpha}$ regularity result, which extends the Caffarelli’s regularity theory on a class of nonconvex domains.

In Chapter 3 we deal with degenerate Monge-Ampère type equations with the general cost function $c$. There will be obtained global $C^2$ a priori estimates for generalized solutions of the corresponding Dirichlet problem as well as the
existence and uniqueness of solutions via the classical continuity method.

Finally, Chapter 4 is devoted to singular Monge-Ampère equations. There will be shown a symmetry of smooth convex solutions to singular Monge-Ampère equations in the entire space $\mathbb{R}^n$ as well as different applications of this result to Monge-Ampère equations, Hessian equations and special Lagrangian equations. In addition, we will construct an example which clearly shows that solutions to singular Monge-Ampère equations in a ball may not have a decomposition of a smooth function and a smooth convex cone.
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Chapter 1

Introduction

1.1 Optimal transportation problem

The optimal transportation problem was born in 1781, with a very famous paper - Mémoire sur la théorie des déblais et des remblais - by Gaspard Monge: to find an optimal mapping from one distribution to another such that a cost functional is minimized among all measure preserving mappings. Since then, it has become a classical subject in the probability theory, economics and optimization. The first significant result on this topic was obtained by Kantorovich [97], [98] in 1940’s who introduced a duality theory in optimal transportation and formulated a relaxed version of Monge’s problem. However, the original Monge’s problem was solved only in 1979 by Sudakov [150] using probabilistic techniques.

Later the problem gained extreme popularity, because many researchers coming from different areas in mathematics understood that this topic was strongly linked to their subjects. Indeed, in 1987 Brenier [17] discovered that when the cost is the distance squared, optimal maps for the problem are gradients of convex functions. Following this result and its subsequent extensions, the theory of optimal transportation has flourished, with generalizations to other cost functions, more general spaces such as Riemannian manifolds, applications in many other areas of mathematics such as geometry, probability, meteorology, fluid mechanics, dynamical systems, light reflection.

The optimal transportation problem can be introduced as follows. Given two bounded domains $\Omega, \Omega^*$ in Euclidean space $\mathbb{R}^n$ and two mass distributions $f \in L^1(\Omega)$ and $g \in L^1(\Omega^*)$ with an obvious mass balance condition

$$\int_{\Omega} f(x)dx = \int_{\Omega^*} g(y)dy. \quad (1.1)$$
Let \( s : \Omega \to \Omega^* \) be a mapping between them and \( c(x, y) \) be a continuous cost function for transporting per unit mass from \( x \in \Omega \) to \( y \in \Omega^* \).

We say a Borel measurable mapping \( s : \Omega \to \Omega^* \) is called measure preserving if

\[
\int_{s^{-1}(E)} f(x) \, dx = \int_E g(y) \, dy \quad \text{for all Borel subsets } E \subset \Omega^*,
\]

or equivalently

\[
\int_{\Omega} h(s(x)) f(x) \, dx = \int_{\Omega^*} h(y) g(y) \, dy \quad \text{for all } h \in C^0(\Omega^*).
\]

Denote the set of all measure preserving mappings by \( \mathcal{S} = \mathcal{S}(\Omega, \Omega^*) \). The problem aims to find the optimal measure preserving map \( T \in \mathcal{S} \), where optimality is measured against a cost functional

\[
\mathcal{C}(s) = \int_{\Omega} f(x) c(x, s(x)) \, dx.
\]

In other words, the optimal mapping \( T \) minimizes the total cost of redistributing the mass distribution \( f \in L^1(\Omega) \) to \( g \in L^1(\Omega^*) \) among all measure preserving mappings \( s \in \mathcal{S} \) such that

\[
\mathcal{C}(T) = \inf_{s \in \mathcal{S}} \mathcal{C}(s).
\]

The above problem can be generalized by considering nonuniform mass distributions and more general cost functions. Let \( \mu \) and \( \nu \) be two Borel measures satisfying mass balance condition \( \mu(\Omega) = \nu(\Omega^*) \). Then condition (1.2) is equivalent to

\[
\mu(s^{-1}(E)) = \nu(E) \quad \text{for all Borel subsets } E \subset \Omega^*.
\]

One may ask whether there exists a mapping \( T \in \mathcal{S} \) minimizing the cost functional

\[
\mathcal{C}(s) = \int_{\Omega} c(x, s(x)) \, d\mu(x),
\]

such that

\[
\mathcal{C}(T) = \inf_{s \in \mathcal{S}} \mathcal{C}(s).
\]

This is called the generalized Monge’s problem.

As one can see below, Kantorovich’s problem is just the same as Monge’s problem, except that Monge additionally required no mass to be split i.e., each location \( x \) is associated with a unique destination \( y \). This makes Kantorovich’s problem much more easier to handle. To study the existence of optimal mappings he introduced the dual functional

\[
I(\varphi, \psi) = \int_{\Omega} \varphi(x) f(x) \, dx + \int_{\Omega^*} \psi(y) g(y) \, dy,
\]

(1.3)
1.1. OPTIMAL TRANSPORTATION PROBLEM

over the set
\[ K = \{ (\varphi, \psi) \in C^0(\mathbb{R}^n) \times C^0(\mathbb{R}^n) \mid \varphi(x) + \psi(y) \leq c(x, y), \ \forall x \in \Omega, y \in \Omega^* \} . \]

Kantorovich’s dual problem is to find a pair \((u, v) \in K\) such that
\[ I(u, v) = \sup_{(\varphi, \psi) \in K} I(\varphi, \psi) . \]

The fundamental relation between the cost functional \( C \) and its dual functional \( I \) is the following
\[ \inf_{s \in S} C(s) = \sup_{(\varphi, \psi) \in K} I(\varphi, \psi) . \] (1.4)

It is readily checked that the maximizer to the right hand side always exists and is unique up to a constant, as shown below.

1.1.1 Existence and uniqueness results

In this section we shall restrict our attention on absolutely continuous mass distributions with bounded support and show the existence and uniqueness of optimal mappings. The existence theorem and the sketch of the proof will be introduced below, however for more detailed proof the readers are referred to Brenier [18], Caffarelli [26], Gangbo and McCann [62] and Urbas [170].

**Theorem 1.1.** Suppose that cost function \( c \) is \( C^1 \) smooth on \( \mathbb{R}^n \times \mathbb{R}^n \) and satisfies the following condition:

(A1) For any \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\) and \((p, q) \in \mathbb{R}^n \times \mathbb{R}^n\), there exists unique \( Y = Y(x, p) \) and \( X = X(y, q) \) such that \( D_x c(x, Y) = p \) and \( D_y c(X, y) = q \).

Then there exists an essentially one-to-one mapping \( T \in S(\Omega, \Omega^*) \) such that \( C(T) = \inf_{s \in S} C(s) \). Moreover, the optimal mapping \( T \) is essentially unique, i.e. if \( C(s) \leq C(T) \) for some \( s \in S \), then \( s(x) = T(x) \) for a.e. \( x \in \text{spt} f \).

Here \( T \) is essentially one-to-one mapping means that there exists \( T^* \in S(\Omega^*, \Omega) \) such that
\[ T^*(T(x)) = x \text{ for a.e. } x \in \text{spt} f; \]
\[ T(T^*(y)) = y \text{ for a.e. } y \in \text{spt} g . \]

The idea of the proof is to consider Kantorovich’s dual functional and use it’s maximizers to find the optimal transport to the original problem.
**Step 1.** \( I \) has a maximizing pair \((u, v) \in \mathcal{K}\).

Consider the \( c \)-transform

\[
\begin{align*}
u^*(x) &= \inf_{y \in \Omega^*} \{c(x, y) - v(y)\}, \quad (1.5) \\
v^*(y) &= \inf_{x \in \Omega} \{c(x, y) - u^*(x)\}.
\end{align*}
\]

It is easy to see that \( I(u, v) \) does not decrease if \( u \) is replaced by \( u^* \). Indeed, for any \((x, y) \in \Omega \times \Omega^*
\]

\[
u^*(x) \leq c(x, y) - v(y).
\]

Then by continuity of \( c \) and \( v \) for every \( x \in \Omega \) there is \( y \in \bar{\Omega}^* \) such that

\[
u^*(x) = c(x, y) - v(y) \geq u(x), \quad (u, v) \in \mathcal{K}.
\]

Therefore, \( I(u^*, v) \geq I(u, v) \). Similarly, \( I(u^*, v^*) \geq I(u^*, v) \). Thus, we may replace \((u, v)\) by \((u^*, v^*)\).

It also follows that we may add any constant \( c \) to \( u \) if we subtract it from \( v \). So if \((u, v) \in \mathcal{K}\) then \((u + c, v - c) \in \mathcal{K}\) and \( I(u, v) = I(u + c, u - c) \).

Choose a sequence \( \{(u_k, v_k)\} \in \mathcal{K} \) such that

\[
I(u_k, v_k) \to \sup_{(\varphi, \psi) \in \mathcal{K}} I(\varphi, \psi).
\]

It is easy to show that \((u_k, v_k)\) are Lipschitz pairs, uniformly bounded with respect to \( k \) and therefore there is a subsequence converging uniformly to a bounded Lipschitz pair \((u, v)\) where \( u \) and \( v \) are unique up to a constant and satisfy (1.5). The maximizing functions \( u \) and \( v \) are called potential functions to optimal transportation problem.

**Step 2.** A mapping \( T : \Omega \to \Omega^* \) is uniquely determined a.e. in \( \Omega \).

Since \( c \) and \( u \) are continuous for any \( x_0 \in \Omega \) there is a \( y_0 \in \bar{\Omega}^* \) such that

\[
\begin{align*}
u(x) &= c(x_0, y_0) - v(y_0), \quad (1.6) \\
u(x) \leq c(x, y_0) - v(y_0), \quad \forall x \in \Omega. \quad (1.7)
\end{align*}
\]

By Rademacher’s theorem [51], Lipschitz continuous potentials \( u \) and \( v \) are differentiable a.e. and its gradients are Borel measurable. Then, if \( x_0 \) is a point of differentiability of \( u \), we have

\[
Du(x_0) = D_x c(x_0, y_0).
\]

Using assumption (A1), we obtain a mapping \( T \), \( T(x_0) = y_0 \), which is uniquely determined by the equation

\[
Du(x) = D_x c(x, T(x)) \quad \text{for a.e. } x \in \Omega. \quad (1.8)
\]
Step 3. $T$ is a measure preserving mapping. For a function $h \in C(\mathbb{R}^n)$ and $\varepsilon < 1$ we define
\[ v_\varepsilon(y) = v(y) + \varepsilon h(y), \]
and
\[ u_\varepsilon(x) = \inf_{y \in \Omega^*} \{ c(x, y) - v(y) - \varepsilon h(y) \}, \]
where $u$ and $v$ are potential functions. By the Step 2, for almost every $x \in \Omega$ there exists a unique point $y_0 = T(x)$ for which infimum in (1.5) is attained. Then one can prove (see [18], [26], [62], [170]) that
\[ u_\varepsilon(x) = u(x) - \varepsilon h(T(x)) + o(\varepsilon). \]
Since $(u, v)$ maximizes the dual functional $I$, we obtain
\[ -\int_\Omega h(T(x)) f(x)dx + \int_{\Omega^*} h(y) g(y)dy = \lim_{\varepsilon \to 0} \frac{I(u_\varepsilon, v_\varepsilon) - I(u, v)}{\varepsilon} = 0. \]
Thus, the mapping $T$ is measure preserving, i.e. $T \in S(\Omega, \Omega^*)$.

Step 4. The duality (1.4) holds. Indeed, for any measure preserving mapping $s$ and any pair $(\varphi, \psi) \in \mathcal{K}$, we get
\[
\begin{align*}
\int_\Omega \varphi(x) f(x)dx + \int_{\Omega^*} \psi(y) g(y)dy &= \int_\Omega (\varphi(x) + \psi(s(x))) f(x)dx \\
&\leq \int_\Omega c(x, s(x)) f(x)dx \\
&= C(s).
\end{align*}
\] (1.9)
Therefore, for any $(\varphi, \psi) \in \mathcal{K}$, $s \in \mathcal{S}$ we have $I(\varphi, \psi) \leq C(s)$ and from Step 2 and Step 3 we have $I(u, v) = C(T)$.

Step 5. The minimizer $T$ of the cost functional $C$ is essentially unique. Suppose there is another minimizer $\bar{T}$ of the cost functional $C$. Then
\[
\int_\Omega \varphi(x) f(x)dx + \int_{\Omega^*} \psi(y) g(y)dy = \int_\Omega c(x, \bar{T}) f(x)dx.
\]
Similarly to the Step 2, we get
\[ Du(x) = D_x c(x, T(x)) \text{ for a.e. } x \in \Omega. \] (1.10)
Then, assumption (A1) implies $\bar{T}(x) = T(x)$ for a.e. $x \in \Omega$.

Note that the existence and uniqueness of optimal mappings in the original papers [18], [26], [62], [170] were proved only for convex or concave cost function in the form $c(x, y) = c(|x-y|)$. However it was observed by Ma, Trudinger and Wang in [122] that the result also holds for general cost functions satisfying condition (A1). It is also possible to obtain the existence and uniqueness results for more general mass distributions but this requires additional technical hypotheses.
1.1.2 Regularity results

Before we review the regularity of the optimal mappings let us show the link between optimal transportation and Monge-Ampère equations.

Suppose first that \( c(x, y) = x \cdot y \) (or equivalently \( c(x, y) = \frac{1}{2}|x - y|^2 \)). Then (1.10) takes form

\[
Du(x) = T(x) \quad \text{for a.e. } x \in \Omega.
\]

By (1.5) \( u \) is concave and twice differentiable almost everywhere. Then, differentiating the above equation, we get

\[
detD^2u(x) = \frac{f(x)}{g(Du(x))},
\]

or, more generally,

\[
detD^2u(x) = F(x, Du(x)).
\]

This equation is called the standard Monge-Ampère equation. The natural boundary condition for this equation is

\[
Du(\Omega) = \Omega^*.
\]

It is easy to see that to study the regularity of the optimal maps with the quadratic cost it is enough to study the regularity of the solutions to the standard Monge-Ampère equation.

The study of the Monge-Ampère equation is an old and important subject pioneered by Aleksandrov [3] and Pogorelov [125], [126]. To see how the general theory of Monge-Ampère equations gives some insight on the smoothness of the optimal mappings there are series of papers by Caffarelli [21], [22], [23], [24], [25] and Urbas [166], [167], [169], [171]. Note that the regularity results were obtained by Caffarelli and Urbas independently and by completely different techniques.

Since Caffarelli’s regularity theory is used in Chapter 2 of the thesis, we will briefly describe it below.

**Definition 1.2.** We say \( u \) is a generalized solution (in the sense of Aleksandrov) of the Monge-Ampère equation (1.12) if it satisfies

\[
|\partial u(E)| = \int_E F(x, Du(x))dx \quad \text{for any set } E \subset \Omega,
\]

where \( \partial u \) is the standard sub-gradient map of the convex function \( u \).
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The first step in the Caffarelli’s approach is to study the generalized solution of the following equation:

\[ \lambda \leq \det D^2 u \leq \Lambda, \quad (1.14) \]

where \( \lambda, \Lambda \) are positive constants.

Let us cut a section \( S \) of the graph of \( u \) such that \( S = \{ x \in \Omega | u(x) \leq l(x) \} \), where \( l \) is a linear function. Subtracting the linear function from \( u \) we may suppose that \( u \leq 0 \) inside the convex set \( S \) and study the regularity of \( u \) inside the set \( S \).

Using famous John’s lemma we may also assume that \( S \) is normalized, that is \( B_1 \subset S \subset nB_1 \) (see [118]), where \( B_1 \) is a unit ball in \( \mathbb{R}^n \). Then, for some universal constant \( C \), Caffarelli proved the following Aleksandrov type estimates:

\[ |u(x)|^n \leq C \text{dist}(x, \partial S), \quad (1.15) \]

\[ a \leq \sup |u| \leq b. \]

The following result proved in [23] implies interior strict convexity of potential functions.

**Lemma 1.3.** Let \( u \) be a globally Lipschitz generalized solution of (1.14) in a bounded convex set \( \Omega \) and let \( K \) is the set where \( u \) attains its minimum value. If \( K \) contains more than one point then \( K \) cannot have extreme points in \( \Omega \).

Indeed, since the theorem also holds for functions of the form \( u - l(x) \), where \( l \) is any affine function, it implies that \( u \) is strictly convex inside \( \Omega \) in the sense that every tangent plane has only one contact point with the graph of \( u \).

Now we assume that \( u = 0 \) on \( \partial \Omega \). Let \( x_0 \) be the point such that \( u(x_0) = \inf_\Omega u \) and \( L \) be a line passing through \( (x_0, u(x_0)) \) and \( (x, 0) \) for any \( x \in \partial \Omega \). If \( L \cap \{ u = u(x_0)/2 \} \) touches the graph of \( u \) then the graph of \( u \) contains a line. However, Lemma 1.3 implies that it is impossible. Therefore, there is a universal constant \( C \) such that the distance from \( L \cap \{ u = u(x_0)/2 \} \) to the graph of \( u \) is greater than \( C \). Iterating this process (using \( \{ u = u(x_0)/4 \} \), \( \{ u = u(x_0)/8 \} \),...), one can see that the graph of \( u \) becomes flatter at \( x_0 \) in every direction and \( u \in C^{1, \alpha} \) at \( x_0 \) (see [22]). It is easy to see that after subtraction a linear function from \( u \), any point in \( \Omega \) can be chosen as a minimum of \( u \) in \( \Omega \). Therefore, \( u \) is \( C^{1, \alpha} \). The following theorem summarizes the results.

**Theorem 1.4.** Let \( u \) be a strictly convex generalized solution of

\[ \det D^2 u = f \]

in a bounded set \( \Omega, B_1 \subset \Omega \subset nB_1 \). Then
(i) if $C_1 \leq f \leq C_2$, then $u \in C^{1,\alpha}$ for some $\alpha \in (0,1)$.

(ii) Given any $\alpha > 0$ there exists $\varepsilon > 0$ such that $|f - 1| \leq \varepsilon$, then $u \in C^{1,\alpha}$.

(iii) Given any $p > 1$ there exists $\varepsilon > 0$ such that $|f - 1| \leq \varepsilon$, then $u \in W^{2,p}$. If $f$ is continuous, then $u \in W^{2,p}$ for all $p > 1$.

(iv) If $f \in C^\alpha$, then $u \in C^{2,\alpha}$.

Note that higher regularity follows from the classical theory of elliptic regularity. In particular, if $f \in C^\infty$, then $u \in C^\infty$.

**Remark 1.5.** It is worth noting that the results above are optimal in the following sense. Counterexamples constructed by Wang in [176] show that:

- if $f$ is not continuous, $u$ may fail to be of class $W^{2,p}$;
- if $f$ is not strictly positive, $u$ may fail to be $C^{1,\alpha}$-smooth for any $\alpha > 0$, even though $f$ is continuous.

As a further extension of these techniques Caffarelli obtained a global $C^{1,\alpha}$ regularity result in [24], [25]. The idea of his proof is as follows. Define $\tilde{u}$ to be an extension of $u$ outside of $\Omega$:

$$\tilde{u} = \sup\{L : L \text{ is linear}, L|_\Omega \leq u, L(z) = u(z) \text{ for some } z \in \Omega\}.$$ 

We will further write $u$ instead of $\tilde{u}$ for simplicity. Then $u$ is a globally Lipschitz convex solution of

$$\lambda \chi_\Omega \leq \det D^2 u \leq \Lambda \chi_\Omega.$$ 

Then Caffarelli proved the existence of centered sections, namely:

**Lemma 1.6.** Let $u$ be a global convex function. Assume that

(1) $u(0) = 0$, $u \geq 0$;

(2) $u$ is finite in a neighbourhood of zero;

(3) the graph of $u$ contains no complete lines.

Then for any $h > 0$, there exists a slope $p$ such that the center of mass of the section

$$S_h = \{x : u(x) \leq x \cdot p + h\}$$

is defined and equals to zero.
Here we write $S_h(x)$ to denote the section of $u$ with the height $h$ centered at $x$. Now, by the John’s lemma we can find an affine transformation $A$ such that 

$$B_1 \subset S^*_h = A(S_h) \subset nB_1.$$ 

If we define a new function 

$$u^*(x) = \frac{(u - L)(A^{-1}x)}{h},$$

then $u^*$ solves 

$$\begin{cases} 
\det D^2 u^* = h^{-n} \left( \det A \right)^{-1} f(A^{-1}x) \text{ in } S^*, \\
u^* = 0 \text{ on } \partial S^*.
\end{cases} \quad (1.16)$$

This process is called the normalization, while $(u^*, S^*)$ is a normalized pair.

For such a pair Caffarelli proved Aleksandrov type estimate (1.15) under the following doubling condition:

$$|\partial u(S)| \leq C |\partial u(\frac{1}{2}S \cap \Omega)|,$$

where $\frac{1}{2}S$ means contraction of $S$ around the center of mass. Note that if $\Omega$ is convex and $f$ is bounded from above and below, the doubling condition always holds (see Lemma 2.3 in [25]).

The key result needed for the global $C^{1,\alpha}$ regularity in [25] is the geometric decay of centered sections.

**Theorem 1.7.** Let $x_1, x_2$ be two points in $\Omega$, take $0 < t < \tilde{t} < 1$, and let $S_h(x_1)$ denotes, as above, a section of $u$ with height $h$ and center of mass at $x_1$. Assume that $x_2$ belongs to $tS_h(x_1)$. Then there exists $s_0 > 0$ such that for any $s < s_0$

$$S_{sh}(x_2) \subset \tilde{t}S_h(x_1). \quad (1.17)$$

The proof of this theorem is by contradiction. Assume that there are sequences of $\{u_k\}, \{x^k_1\}, \{x^k_2\}$, and $\{h_k\}$ such that $x^k_2$ belongs to $tS_{h_k}(x^k_1)$ but $S_{1/t h_k}(x^k_2)$ is not contained in $(1 - \frac{1}{2})S_{h_k}(x^k_1)$. Now we normalize pair $(u_k, S_{h_k}(x^k_1))$ to $(u^*_k, S^*_{h_k})$. By convexity of $u^*_k$ we may take a limit as $k \to \infty$, and the limiting function $u_\infty$ will be a generalized solution of (1.16). By the geometric construction, one can see that the graph of $u_\infty$ contains a flat part in $\overline{S}_\infty$. Then on a carefully chosen section $\tilde{S}$, one can show that on one hand the point where $\inf_{\tilde{S}} u$ is achieved is very close to $\partial \tilde{S}$. But on the other hand, by Aleksandrov type estimate $\inf_{\tilde{S}} u$
cannot be achieved near the boundary of $\tilde{S}$. Then contradiction in Theorem 1.7 follows from this consideration.

Once Theorem 1.7 is proved the following quantitative strict convexity estimate follows from (1.17):

$$ u(z) \geq u(x) + Du(x) \cdot (z - x) + C|z - x|^\beta \quad \text{for any } x, z \in \Omega, $$

for some $\beta > 1$. It is easy to check that the above estimate implies $u^* \in C^{1,\alpha}$ on $\overline{\Omega}$, where $u^*$ is the standard Legendre transform of $u$. Since $u^*$ is indeed the potential function of the optimal transport problem from $\Omega^*$ to $\Omega$, the role of $u$ and $u^*$ can be switched and we get the following result [24]:

**Theorem 1.8.** Suppose $\Omega, \Omega^*$ are convex and $f, g$ have positive upper and lower bounds. If $u$ is the generalized solution of (1.11), (1.13), then $u \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha > 0$.

In a subsequent paper [25] Caffarelli also proved a higher global regularity result:

**Theorem 1.9.** Suppose $\Omega, \Omega^*$ are $C^2$-smooth uniformly convex domains in $\mathbb{R}^n$ and $f \in C^\alpha(\overline{\Omega})$, $g \in C^\alpha(\overline{\Omega}^*)$ are positive. Then $u \in C^{2,\alpha}(\overline{\Omega})$.

It should be noted that the similar result, under slightly stronger regularity assumptions, was obtained by Delanoë in [48] for $n = 2$, and by Urbas in [169] for $n \geq 2$, using completely different techniques. We present the result from [169] below:

**Theorem 1.10.** Suppose $\Omega, \Omega^*$ are $C^{2,1}$-smooth uniformly convex domains in $\mathbb{R}^n$ and $f \in C^{1,1}(\overline{\Omega})$, $g \in C^{1,1}(\overline{\Omega}^*)$ are positive. Then the problem (1.11), (1.13) has a convex solution $u \in C^{2,\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1)$ and $u$ is unique up to constants. Then $u \in C^{2,\alpha}(\overline{\Omega})$.

To proceed to the regularity of potential functions for general cost functions $c(x, \cdot)$ let us introduce a convexity notion relative to the cost function and the Monge-Ampère equation.

A $c$-support function of $\varphi$ at $x_0$ is a function of the form $h(x) = c(x, y_0) + a$, where $a$ is a constant and $y_0 \in \mathbb{R}^n$, such that

$$ \varphi(x_0) = c(x_0, y_0) + a, \quad (1.18) $$

$$ \varphi(x) \geq c(x, y_0) + a, \quad \forall x \in \Omega. $$
1.1. OPTIMAL TRANSPORTATION PROBLEM

Definition 1.11. A function $\varphi$ is called $c$-convex if for any $x_0 \in \Omega$ there exists a $c$-support of $\varphi$ at $x_0$. If every $c$-support touches the graph of $\varphi$ at a single point, then $\varphi$ is called strictly $c$-convex.

This convexity notion becomes the standard one if the cost function is $c(x,y) = x \cdot y$.

Similarly there can be introduced the notion of $c$-concavity.

Definition 1.12. A set $\ell_c \subset \mathbb{R}^n$ is a $c$-segment in $\Omega^*$ with respect to a point $x_0$ if $D_x c(x_0, \ell_c)$ is a line segment in $\mathbb{R}^n$.

Definition 1.13. $\Omega^*$ is $c$-convex relative to $\Omega$ if for any two points $y_0, y_1 \in \Omega^*$ and any $x \in \Omega$, the $c$-segment relative to $x$ connecting $y_0$ and $y_1$ lies in $\Omega^*$ or equivalently, if for all $x_0 \in \Omega$, $D_x c(x_0, \Omega^*)$ is convex in $\mathbb{R}^n$.

Definition 1.14. We say $\Omega^*$ is uniformly $c$-convex with respect to $\Omega$ if for all $x_0 \in \Omega$, $D_x c(x_0, \Omega^*)$ is uniformly convex in $\mathbb{R}^n$.

Let’s now differentiate (1.10),

$$D^2 u(x) = D^2_x c(x, T(x)) + D^2_{x,y} c(x, T(x)) \cdot DT(x).$$

Then potential function $u$ satisfies

$$\det(D^2 u - D^2_c) = |\det D^2_{x,y} c| \frac{f}{g \circ T}, \quad \text{for } x \in \Omega, \quad (1.19)$$

or, more generally

$$\det(D^2 u - A(x, Du)) = F(x, u, Du), \quad (1.20)$$

where $A(x, Du) = D^2_x c(x, T(x))$. This equation is called Monge-Ampère type equation. The natural boundary condition for this equation is

$$T(\Omega) = \Omega^*. \quad (1.21)$$

The problem (1.20),(1.21) is called the second boundary value problem for Monge-Ampère type equation.

As we observed in the previous section, the optimal mapping $T$ is completely determined by the potential functions $u, v$. Therefore, study of the regularity of optimal maps with general cost functions satisfying (A1) is equivalent to study of the regularity of solutions to Monge-Ampère type equation.

The first breakthrough was made by Ma, Trudinger, Wang in [122] where they proved the following interior regularity result:
CHAPTER 1. INTRODUCTION

**Theorem 1.15.** Suppose \( f \in C^2(\Omega) \), \( g \in C^2(\Omega) \), \( f, g \) have positive upper and lower bounds, and the mass balance condition (1.1) holds. Suppose the cost function \( c \) is \( C^4 \)-smooth and satisfies the following assumptions:

1. **(A1)** For any \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\) and \((p, q) \in \mathbb{R}^n \times \mathbb{R}^n\), there exist unique \( Y = Y(x, p) \) and \( X = X(y, q) \) such that \( D_x c(x, Y) = p \) and \( D_y c(X, y) = q \);

2. **(A2)** For any \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\), \( \det D_{x,y}^2 c \neq 0 \);

3. **(A3)** There exists a constant \( c_0 > 0 \) such that for any \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\), and vectors \( \xi, \eta \in \mathbb{R}^n \) with \( \xi \perp \eta \),

\[
\sum_{i,j,k,p,q,r,s} (c^{p,q}c_{ij,p,q,rs} - c_{ij,rs}) c^{r,k} c^{s,l}(x, y) \xi^i \xi^j \eta^k \eta^l \geq c_0 |\xi|^2 |\eta|^2, \tag{1.22}
\]

where the subscripts before (resp. after) comma mean partial derivatives in \( x \)-variables (resp. \( y \)-variables), and \( \{c^{i,j}\} \) stands for the inverse of \( \{c_{i,j}\} \);

Then, if the domain \( \Omega^* \) is \( c \)-convex relative to \( \Omega \), the potential function \( u \) is \( C^3 \)-smooth in \( \Omega \).

**Remark 1.16.** As we observed in the previous section condition **(A1)** is required to prove the existence of optimal transport mappings. Condition **(A2)** implies that for positive densities \( f, g \) the elliptic equation (1.19) is non-degenerate. It is a natural condition for regularity, because, as one can see in the next section, for degenerate Monge-Ampère equations we cannot expect regularity better than \( C^{1,1} \) in general.

**Remark 1.17.** As shown in [122], \( c \)-convexity of domains \( \Omega, \Omega^* \) is also an optimal condition for regularity. Namely, if \( \Omega^* \) is not \( c \)-convex, then there exist smooth, positive functions \( f, g \) satisfying (1.1) such that the optimal mapping is not continuous. For more details, see [122, Section 7.3].

We present here a few examples of cost functions \( c \) satisfying conditions **(A1)**-** (A3):**

1) \( c(x, y) = -\frac{1}{m}|x - y|^m \), \( m \neq 0 \), \( m \in (-2, 1) \);
2) \( c(x, y) = \sqrt{1 \pm |x - y|^2} \);
3) \( c(x, y) = -\log |x - y| \) on \( \mathbb{R}^n \times \mathbb{R}^n \) with \( x \neq y \);
4) \( c(x, y) = -\log |x - y| \) restricted to \( S^n \) (corresponds to the reflector antenna problem) satisfies condition **(A3)** on \( S^{n-1} \times S^{n-1} \) with \( x \neq y \) (see [164]).

For other examples we refer to [53].
As a further extension of Theorem 1.15 Trudinger and Wang [163] obtained the global regularity of potentials under a condition on the cost function weaker than \((A3)\) but with stronger condition on domains \(\Omega, \Omega^*\).

**Theorem 1.18.** Let \(C^4\)-smooth domains \(\Omega, \Omega^*\) are uniformly \(c\)-convex with respect to each other, \(f\) be a positive function in \(C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)\), \(c\) be a cost function satisfying hypotheses \((A1),(A2)\), and

\((A3w)\) for any \((x,y) \in \mathbb{R}^n \times \mathbb{R}^n\), and vectors \(\xi, \eta \in \mathbb{R}^n\) with \(\xi \perp \eta\),

\[
\sum_{i,j,k,l,p,q,r,s} (c^{p,q}c_{ij,p}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}(x,y) \xi^i \xi^j \eta^k \eta^l \geq 0.
\]

Then any elliptic solution \(u \in C^3(\Omega)\) of the second boundary value problem (1.20), (1.21) satisfies the a priori estimate

\[
|D^2u| \leq C,
\]

where \(C\) depends on \(c, f, \Omega, \Omega^*\) and \(\sup_{\Omega}|u|\).

If in addition

\[
F = \frac{f}{g \circ T},
\]

and \(f, g\) satisfy mass balance condition (1.1), then there exists a unique (up to additive constants) elliptic solution \(u \in C^3(\bar{\Omega})\).

The necessity of the condition \((A3w)\) was shown by Loeper in [120]. Namely, if there exist \(x_0 \in \Omega, y_0 \in \Omega^*\) and nonzero vectors \(\xi, \eta \in \mathbb{R}^n\) with \(\xi \perp \eta\) such that,

\[
\sum_{i,j,k,l,p,q,r,s} (c^{p,q}c_{ij,p}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}(x_0, y_0) \xi^i \xi^j \eta^k \eta^l < 0, \quad (1.23)
\]

then there exist smooth, positive functions \(f\) and \(g\) satisfying (1.1) such that the optimal mapping \(T\) is not continuous.

In the same paper Loeper also showed the following geometric properties of conditions \((A3)\) and \((A3w)\):

Fix \(x_0 \in \Omega\) and let \(y_0, y_1\) be two distinct points in \(\bar{\Omega}^*\). Let \(y_0 y_1\) be the \(c\)-segment relative to \(x_0\), connecting \(y_0\) and \(y_1\), namely,

\[
\{y_t : c_x(x_0, y_t) = p_t, \ t \in [0,1]\},
\]

where \(p_t = tp_1 + (1-t)p_0\) and \(p_0 = D_xc(x_0, y_0), p_1 = D_xc(x_0, y_1)\).

Consider functions

\[
h_i(\cdot) = c(\cdot, y_i) - a_i, \quad i = 0, 1,
\]

\[
h_t(\cdot) = c(\cdot, y_t) - a_t,
\]
where \( t \in (0, 1) \) and \( a_0, a_1, a_t \) are constants such that \( h_0(x_0) = h_1(x_0) = h_t(x_0) \).
If condition (A3) holds, then for each \( t \in (0, 1) \),
\[
h_t(x) > \min\{h_0(x), h_1(x)\}. \tag{1.24}
\]
This geometric property obtained by Loeper in [120] is called Loeper’s maximum principle. Relying on this maximum principle Loeper also proved the global \( C^{1,\alpha} \) regularity of potentials (see [120]).

A further development of Loeper’s maximum principle was given by Kim and McCann in [99] and Trudinger and Wang in [164]. Namely, Kim and McCann gave an elementary and direct geometrical proof of a more general version of the Loeper’s maximum principle; whereas, Trudinger and Wang found a monotonicity formula under (A3) or (A3w), by which they proved the strict \( c \)-convexity and \( C^1 \) regularity for potential functions under the (A3) condition. Later Chen and Wang [41] introduced another proof of strict convexity and \( C^{1,\alpha} \) regularity of potentials under relaxed condition (A3w), where they extended Caffarelli’s ideas for quadratic cost functions [25] to more general cost functions.

Following paper [122], the regularity of optimal transports has been intensively investigated. An independent result was obtain by Liu [115], who proved the global \( C^{1,\alpha} \) regularity with optimal constant \( \alpha \in (0, 1) \) for degenerate Monge-Ampère equation \((f \geq 0)\) under condition (A3). The interior \( C^{2,\alpha} \) regularity was given by Liu, Trudinger and Wang in [116]. In a subsequent paper [117] they also obtained local \( W^{2,p} \) regularity of potentials. Combining all these results we may state the following theorem:

**Theorem 1.19.** Denote
\[
F = |\det D^2_{x,y}c| f \bigg/ \frac{g \circ T}{T}.
\]
Assume the cost function satisfies (A1),(A2),(A3) conditions, \( \Omega^* \) is \( c \)-convex relative to \( \Omega \). We have the following regularity results for potential function \( u \):

(i) If \( F \geq 0, F \in L^p(\Omega) \) for some \( p \in \left( \frac{n+1}{2}, \infty \right) \), then \( u \in C^{1,\alpha}(\Omega) \) for some \( 0 < \alpha < 1 \). (The optimal \( \alpha \) was found in [115].)

(ii) If \( C_2 \geq f, g \geq C_1 > 0 \), then for all \( x, y \in \Omega_\delta = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \delta \} \),
the estimate follows
\[
|D^2 u(x) - D^2 u(y)| \leq C \left[ d + \int_0^d \frac{\omega_F(r)}{r} \, dr + d \int_0^1 \frac{\omega_F(r)}{r^2} \, dr \right],
\]
where \( d = |x-y| \), \( \omega_F(r) = \sup_{|x-y|<r} \{|F(x) - F(y)|\} \) denotes the oscillation of \( F \), \( C > 0 \) depends only on \( n, \delta, C_0, C_1, |u|_{C^1(\Omega)} \). Hence:
1.1. OPTIMAL TRANSPORTATION PROBLEM

(a) If $F$ is Dini continuous, then $u \in C^2$;

(b) If $F \in C^\alpha$, then $u \in C^{2,\alpha}$;

(iii) If $f \in C^0(\Omega), g \in C^0(\Omega^*)$ and $C_2 \geq f, g \geq C_1 > 0$, then $u \in W^{2,p}(\Omega')$ for any $1 \leq p < \infty$ and $\Omega' \subset \subset \Omega$.

If (A3) condition is weakened to (A3w), the Hölder continuity and the injectivity of optimal transports were established by Figalli, Kim and McCann [59], and also by Chen and Wang [41]. They proved:

**Theorem 1.20.** Assume that $\Omega, \Omega^*$ are bounded domains in $\mathbb{R}^n$, $\Omega \subset \tilde{\Omega}$ for some $\tilde{\Omega}$, and $\tilde{\Omega}, \Omega^*$ are c-convex to each other. Assume the cost function satisfies the (A1), (A2), (A3w) conditions. If $C_2 \geq f, g \geq C_1 > 0$, then the optimal transport is injective in $\Omega$ and the potential function $u \in C^{1,\alpha}(\Omega)$ for some $0 < \alpha < 1$.

So far, the regularity of optimal transport mappings was obtained if the cost function satisfies conditions (A1), (A2), and (A3) or (A3w). In Chapter 2, we provide a different proof of the global $C^{1,\alpha}$ regularity of potential functions with quadratic cost based on Caffarelli’s regularity theory [24], [25]. Instead of deducing the $C^{1,\alpha}$ regularity of $u$ from the strict convexity of its Legendre transform, we prove the $C^{1,\alpha}$ regularity of $u$ directly. Moreover, this method works for a more general class of domains.
1.2 Dirichlet problem to Monge-Ampère type equations

In the previous sections we described the existence, uniqueness and some regularity results for Monge-Ampère type equations and the corresponding second boundary value problem. Now we move to study the Dirichlet problem for Monge-Ampère type equations, namely:

\[
\det(D^2 u - A(x, Du)) = f \quad \text{in } \Omega, \tag{1.25}
\]

\[
u = \varphi \quad \text{on } \partial \Omega. \tag{1.26}
\]

This problem has attracted a lot of interest, especially in the case of quadratic cost function \(c\) (i.e. standard Monge-Ampère equations). In 1971 Pogorelov [126], [127], [128] demonstrated how to obtain solutions which are smooth in the interior of \(\Omega\) (see also Cheng and Yau [42]). Later Lions [114] gave an independent proof of the existence of interior smooth solutions to more general equations using the penalty method.

The global \(C^2\) a priori estimates were first obtained by Ivochkina in [87]. Few years later the existence of globally smooth solutions on strictly convex domains was proved independently by Caffarelli, Nirenberg and Spruck [30] and Krylov [101]:

**Theorem 1.21.** Suppose \(\Omega\) is a \(C^4\) strictly convex domain, \(\varphi \in C^4(\partial \Omega), F \geq \lambda > 0\) is \(C^2(\Omega)\). Then any convex solution \(u \in C^3(\Omega)\) of the Dirichlet problem (1.12),(1.26) satisfies the estimate:

\[
\|u\|_{C^{2,\alpha}(\Omega)} \leq C,
\]

where \(C\) depends only on \(\partial \Omega, n, \lambda, |\varphi|_{C^4}, |u|_{C^1}, |F|_{C^2}\).

In subsequent works by Guan and Spruck [68] and Guan [67] this result was extended on non-convex but still smooth domains, where the strict convexity assumption on the domain \(\Omega\) was replaced by the existence of a strictly convex subsolution.

For generalized solutions the earliest regularity results for the problem (1.28) dealt with the two variable case only (see Pogorelov [125], Nikolaev and Shefel’ [123], Heinz [79], Sabitov [137], Schulz [145]). Namely, if \(u\) is a solution of the Dirichlet problem (1.12),(1.26) in a bounded domain \(\Omega \subset \mathbb{R}^2\), and \(F\) has positive upper and lower bounds, then \(u \in C^{1,\alpha}(\Omega) \cap W^{2,p}_{loc}(\Omega)\). If, in addition, \(F\) is globally \(C^\alpha\)-smooth, then \(u \in C^{2,\alpha}_{loc}(\Omega)\).
1.2. DIRICHLET PROBLEM TO MONGE-AMPÈRE TYPE EQUATIONS

In higher dimensions though, the analogous assertions are generally false. Indeed, consider the following convex function

\[ w(x) = (1 + x_1^2) \left( \sum_{k=2}^{n} x_k^2 \right)^{1-\frac{1}{n}}. \]  

(1.27)

It is easy to see that \( w \) is a generalized solution of (1.12) in a ball \( B_\varepsilon \) with small \( \varepsilon > 0 \), where the right hand side \( F \) is positive and analytic. However, \( w \) is not even \( C^2 \) in \( B_\varepsilon \cap \{x' = 0\} \), where \( x' = (x_2,...,x_n) \). The above example, which is called Pogorelov’s example (see [130]), explains the necessity of some additional hypotheses in the regularity treatment, such as the strict convexity of the solution, or sufficient initial interior regularity, or sufficient regularity of \( \partial \Omega \) and \( \varphi \).

The interior regularity were first obtained by Pogorelov in [131] and Trudinger in [152]. These results were later extended, improved and generalized by Urbas [167] in the following form:

**Theorem 1.22.** (i) Let \( \Omega \) be a \( C^{1,\alpha} \) bounded convex domain in \( \mathbb{R}^n \) and \( \varphi \in C^{1,\alpha}(\bar{\Omega}) \), where \( \alpha > 1 - 2/n \), \( F \) is a \( C^{1,1} \) positive function. If \( u \in C(\bar{\Omega}) \) is a generalized solution of the Dirichlet problem (1.12),(1.26) then \( u \in C^{3,\beta}(\Omega) \) for all \( \beta \in (0,1) \), and for any \( \Omega' \subseteq \Omega \), we have the estimate

\[ ||u||_{C^{3,\beta}(\Omega')} \leq C, \]

where \( C \) depends only on \( n, \alpha, \beta, \Omega, \Omega' \), \( \sup_{\Omega} |u| \), \( ||\varphi||_{C^{1,\alpha}(\bar{\Omega})} \), \( F \) and its first and second derivatives, and the modulus of continuity of \( u \) on \( \partial \Omega \).

(ii) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( F \) is a \( C^{0,1} \) positive function, and \( u \) is a generalized solution of (1.12). If

a) either \( u \in C^{1,\alpha}(\bar{\Omega}) \) for some \( \alpha > 1 - 1/n \),

or

b) \( u \in W^{2,p}(\Omega) \) for some \( p > n(n-1)/2, n \geq 3 \),

then \( u \in C^{2,\beta} \) for all \( \beta \in (0,1) \) and

\[ ||u||_{C^{2,\beta}(\Omega')} \leq C, \]

where \( C \) depends only on \( n, \alpha \) or \( p, \beta, \Omega, \Omega' \), \( ||u||_{C^{1,\alpha}(\bar{\Omega})} \) or \( ||u||_{W^{2,p}(\Omega)} \), \( F \) and its first derivatives.

The interior \( C^{2,\alpha} \) and \( W^{2,p} \) regularity of solutions to Monge-Ampère equations were significantly developed since then by the Caffarelli’s interior regularity
theory described in the previous section. However, the existence and boundary regularity theory for the Dirichlet problem (1.12), (1.26) were completed several years later. The existence of a unique generalized solution to the Dirichlet problem (1.12), (1.26) was proved by Gutiérrez in [73] for a strictly convex domain Ω. Following this result Hartenstine [78] found a necessary and sufficient condition for the existence of generalized solution for nonstrictly convex domains:

**Theorem 1.23.** Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $\mu$ be a finite Borel measure on $\Omega$. The Dirichlet problem

$$\begin{align*}
det D^2 u &= \mu \text{ in } \Omega, \\
u &= \varphi \text{ on } \partial \Omega,
\end{align*}$$

has a unique Aleksandrov solution $u \in C(\bar{\Omega})$ if and only if $\varphi \in C(\partial \Omega)$ can be extended to a function $\tilde{\varphi} \in C(\bar{\Omega})$ that is convex in $\Omega$.

Few years later the regularity theory for the problem was finally completed by Trudinger and Wang [161] and Savin [141], who obtained global and pointwise $C^{2, \alpha}$ estimates respectively, under optimal assumptions on $F$, $\varphi$ and $\partial \Omega$, and Savin [142], who obtained global $W^{2,p}$ estimates also under optimal assumptions on $\varphi$ and $\partial \Omega$.

It is also worth to note that the corresponding linearized Monge-Ampère equations have been intensively studied during last decades initiated by the fundamental paper by Caffarelli and Gutiérrez [29]. Another contribution to the regularity theory of linearized equations comes from [74], [75], [76], where the sharp interior $C^{1, \alpha}$ and $W^{2,p}$ estimates were obtained, while for the corresponding global regularity we refer the reader to [107], [106].

Let us now consider the case of general cost function $c$, i.e. the Dirichlet problem (1.25), (1.26). In addition to the papers mentioned in the previous section ([122], [116], [117]), which treated interior regularity of solutions, it is worth to note a paper by Jiang, Trudinger and Yang [91], where global $C^2$ a priori estimates were established under hypothesis of the existence of a subsolution $\bar{u} \in C^2(\bar{\Omega})$ of (1.25), (1.26). In a subsequent paper by Jiang and Trudinger [89] this hypothesis was replaced by the existence of a subsolution $\bar{u}$ in a neighbourhood of the boundary, whose boundary is the prescribed boundary function. Then the classical solvability of (1.25), (1.26) with the solution $u \in C^3(\bar{\Omega})$ follows by the continuity method under sharp conditions on the cost function (A1), (A2), (A3w) (see [91]).
1.2. DIRICHLET PROBLEM TO MONGE-AMPÈRE TYPE EQUATIONS

The higher regularity of the problem was established by Huang, Jiang and Liu [85], where results of Trudinger and Wang in [161] were extended for the case of general cost functions. Namely, they obtained the global $C^{2,\alpha}$ regularity of the generalized (or classical) solution of the problem (1.25), (1.26) under conditions (A1)-(A3) and some mild assumptions on the boundary data.

1.2.1 Dirichlet problem to degenerate Monge-Ampère type equations

Let us turn our attention to the degenerate form of the Monge-Ampère type equation (1.25), that is $f$ is assumed to be only nonnegative. In this case, the equation becomes more complicated and a lot of attention has been directed to obtaining $C^2$ a priori estimates of the problem. First significant results were made for the Dirichlet problem of the homogeneous Monge-Ampère equation

\[ \det D^2 u = 0 \text{ in } \Omega, \]
\[ u = \varphi \text{ on } \partial\Omega. \]

It is well known that the unique generalized solution $u$ to the problem (1.29), (1.30) exists and can explicitly be written as:

\[ u(x) = \sup \{ v(x) | v \text{ is affine, } v \leq \varphi \text{ on } \partial\Omega \}. \]

If $\varphi$ is smooth enough and $\Omega$ is uniformly convex, the global $C^2$ estimates were proved by Caffarelli, Nirenberg and Spruck [33], following the interior regularity result by Trudinger and Urbas [157].

**Theorem 1.24.** Let $\Omega$ be a uniformly convex domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ is in $C^{3,1}$, and $\varphi \in C^{3,1}(\partial\Omega)$. Then the convex solution $u$ to (1.29), (1.30) is of class $C^{1,1}(\bar{\Omega})$.

Note that assumptions in the theorem cannot be relaxed, which is shown by the following example.

**Example 1.25.** Consider $\Omega \subset \mathbb{R}^2$ is a unit disc centred at the origin and

\[ w(x, y) = (1 + y)^{2-\varepsilon}, \quad \varepsilon > 0. \]

It is easy to see that $w$ solves the problem (1.29), (1.30) with $\varphi \in C^{3,1-2\varepsilon}(\partial\Omega)$ while $u \in C^{1,1-\varepsilon}(\bar{\Omega})$. The same function shows also that if $\Omega$ is flattened at some point, then $\varphi$ may be very smooth, while $u$ is not $C^{1,1}$ in $\bar{\Omega}$. 
CHAPTER 1. INTRODUCTION

The corresponding Dirichlet problem for the inhomogeneous degenerate equation

$$\det D^2 u = f \geq 0 \text{ in } \Omega,$$

was treated few years later by Caffarelli, Kohn, Nirenberg and Spruck [31] and Krylov [104], where the $C^{1,1}$ regularity was established under assumption $f^{1/n} \in C^{1,1}$. Although this assumption seems natural (because $(\det D^2 u)^{1/n}$ is concave) it is nevertheless quite restrictive and could be relaxed. It was done by Guan in [69], where he obtained the global $C^2$ a priori estimate under the condition that $f^{1/(n-1)} \in C^{1,1}$. More precisely, his proof needs $f$ to be only pseudosubharmonic in $\Omega$, that is

$$\inf \{ \Delta f^{1/(n-1)}(x) \} \geq -\frac{A}{n-1} \text{ in } \Omega,$$

and $f^{1/(n-1)}$ to be Lipschitz continuous near $\partial \Omega$, that is

$$|\nabla f^{1/(n-1)}| \leq \frac{A}{n-1} \text{ for } x \in \Omega, \text{ dist}(x, \partial \Omega) \leq \delta.$$

An example given by Wang in [176] shows that for any $\varepsilon > 0$, there is a $C^{1,1}$ nonnegative function $g$, such that $f = g^{n-1-\varepsilon}$, but problem (1.31), (1.30) with the boundary condition $\varphi \equiv 0$ has no $C^{1,1}$ convex solution. Therefore, the $(n-1)$-power growth in the condition is optimal and cannot be reduced.

Although, the author of [69] obtained the regularity under the sharp conditions on the inhomogeneous term he had to suppose that either $\varphi \equiv 0$ or $f$ is positive near the boundary, to derive an a priori estimate for second derivatives on the boundary. This difficulty has been overcome in the subsequent work by Guan, Trudinger and Wang [71], where authors obtained the following result:

**Theorem 1.26.** Let $\Omega$ be a uniformly convex domain in $\mathbb{R}^n$ with boundary $\partial \Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\overline{\Omega})$ and let $f$ be a non-negative function in $\Omega$ such that $f^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$. Then there exists a unique convex solution $u \in C^{1,1}(\overline{\Omega})$ of the Dirichlet problem (1.31)-(1.30). Consequently, any generalized solution in $C^0(\overline{\Omega})$ must belong to $C^{1,1}(\overline{\Omega})$. Moreover, the solution $u$ satisfies an estimate

$$\|u\|_{C^{1,1}(\overline{\Omega})} \leq C,$$

where $C$ is a constant depending on $\Omega$ and the norms of the functions $\varphi$ and $\tilde{f} = f^{1/(n-1)}$ in the spaces $C^{3,1}(\overline{\Omega})$ and $C^{1,1}(\overline{\Omega})$ respectively.

The significance of the theorem is not so much that it is an improvement of earlier results but that it is optimal. As it was already shown assumptions $f^{1/(n-1)} \in$
1.2. DIRICHLET PROBLEM TO MONGE-AMPERE TYPE EQUATIONS

$C^{1,1}$, $\varphi \in C^{3,1}$ are sharp. It is also worth noting that for degenerate Monge-Ampère equations $C^{1,1}$ regularity is the best that can be expected. This is readily seen from the following example (see [33]):

**Example 1.27.** Consider the convex function

$$w(x) = \left[ \max \left\{ \left( x_1^2 - \frac{1}{2}\right)^+, \left( x_2^2 - \frac{1}{2}\right)^+ \right\} \right]^2.$$ 

It is not hard to verify that $w$ is a solution to (1.29) in the unit disc

$$D = \{ x_1^2 + x_2^2 < 1 \},$$

and $w$ is analytic on the boundary $\partial D$. But $w$ is only of class $C^{1,1}$ in $D$.

However, higher regularity of solutions to degenerate Monge-Ampère equations can be obtained under additional conditions on the inhomogeneous term. In particular, the smooth solution were obtained in two dimensions by Guan and Sawyer [72] and in higher dimension by Rios, Sawyer and Wheeden [135], [136] in case when $f$ is equivalent to some exponential function $|x|^m$, $m \geq 1$.

In the case when $f$ vanishes near the boundary, the regularity of solutions near the boundary has been obtained recently by Hong, Huang and Wang [84] in two dimensions, and by Savin [143] in higher dimensions. Another higher regularity result was obtained by Li and Wang in [119]. Namely,

Let $u$ be a convex solution to (1.29) in a bounded domain $\Omega$. For any $x_0 \in \Omega$, let $l_{x_0}$ be a tangent plane of $u$ at $x_0$. Then the set

$$C_{x_0} := \{ x \in \Omega | u(x) - l_{x_0}(x) = 0 \}$$

is convex and its extreme points are boundary points of $\Omega$. If $C_{x_0}$ is a line segment at $x_0 \in \Omega$ and the boundary function $\varphi$ is uniformly convex at the two endpoints of $C_{x_0}$, then $u$ is smooth at $x_0$.

As one can see the Dirichlet problem to the Monge-Ampère type equations has received considerable study in recent decades. However a satisfactory regularity theory for degenerate Monge-Ampère type equations (1.25) is still lacking. In Chapter 3 we will present the global $C^2$ a priori estimates for the Dirichlet problem

$$\det(D^2u(x) - A(x, Du)) = f(x, Du) \geq 0 \text{ in } \Omega,$$

$$u = \varphi \text{ on } \partial \Omega.$$ 

We will also show that these estimates imply existence of the unique solution $u \in C^{1,1}(\overline{\Omega})$ by the classical continuity method. This result is optimal and generalize the Theorem 1.26 for the case of general cost functions satisfying conditions (A1), (A2), (A3w).
1.3 Monge-Ampère equations in the entire space

So far we have only considered optimal transport problems and Monge-Ampère equations in bounded domains. The next obvious step is to consider Monge-Ampère equations in the entire Euclidean space

$$\det D^2u = 1 \text{ in } \mathbb{R}^n. \quad (1.32)$$

The first result on this topic was obtained in two dimensions by Jörgens in [94]. Using complex analysis methods he proved that any classical solution of equation (1.32) for \( n = 2 \) must be a quadratic polynomial, i.e. unimodular affine equivalent to \( \frac{1}{2}|x|^2 \). A simpler proof of the same result was later introduced by Nitsche [124].

In higher dimension the extension of this result was given few years later by Calabi [38] for \( n \leq 5 \) and by Pogorelov [129] for all dimensions \( n \geq 2 \). Another more analytical proof was introduced by Cheng and Yau [43]. Moreover, it was proved by Trudinger and Wang [159] that the problem

$$\begin{cases}
\det D^2u = 1 \text{ in } \Omega, \\
\lim_{|x| \to \partial \Omega} u = \infty,
\end{cases}$$

admits a \( C^2 \) convex solution \( u \) only if \( \Omega = \mathbb{R}^n \). These results for classical solutions were extended to generalized solutions by Caffarelli [28].

Few years later Caffarelli and Li [37] established Jörgens-Calabi-Pogorelov type theorem in the following form:

**Theorem 1.28.** Assume that \( u \) is a convex viscosity solution of

$$\det D^2u = f \text{ in } \mathbb{R}^n.$$ 

Let \( f \in C^0(\mathbb{R}^n) \) satisfy conditions:

$$0 < \inf f \leq \sup f < \infty,$$

$$\text{support } (f - 1) \text{ is bounded}.$$ 

Then

(i) For \( n \geq 3 \), there exist some \( c \in \mathbb{R} \), \( b \in \mathbb{R}^n \), and \( A \in \mathcal{A} \), such that \( E(x) := u(x) - \left( \frac{1}{2}x'Ax + b \cdot x + c \right) \) satisfies

$$\limsup_{|x| \to \infty} |x|^{n-2}|E(x)| < \infty. \quad (1.33)$$

Moreover, \( u \) is \( C^\infty \) in the complement of the support of \( (f - 1) \) and

$$\limsup_{|x| \to \infty} |x|^{n-2+k}|D^k E(x)| < \infty \quad \forall k \geq 1. \quad (1.34)$$
(ii) For $n = 2$, there exist some $c \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $A \in \mathcal{A}$ such that $E(x) := u(x) - \left( \frac{1}{2} x'Ax + b \cdot x + d \log \sqrt{x'Ax + c} \right)$ satisfies

$$
\limsup_{|x| \to \infty} |x||E(x)| < \infty,
$$

where $d = \frac{1}{2\pi} \int_{\mathbb{R}^2} (f - 1)$. Moreover, $u$ is $C^\infty$ in the complement of the support of $(f - 1)$ and

$$
\limsup_{|x| \to \infty} |x|^{k+1} |D^k E(x)| < \infty \quad \forall k \geq 1.
$$

Here

$$\mathcal{A} = \{ A : A \text{ is real } n \times n \text{ symmetric positive definite matrix with } \det(A) = 1 \}.$$

Therefore, the behaviour of solutions to Monge-Ampère equations in the entire space is well understood at the moment. Let’s now consider an exterior problem for the Monge-Ampère equation

$$\det D^2 u = 1 \text{ in } \mathbb{R}^n \setminus O. \quad (1.37)$$

The first result for the problem was obtained in two dimensions due to Jörgens [95]. He showed that in $\mathbb{R}^2 \setminus \{0\}$ every smooth locally convex solution of (1.37) is unimodular affine equivalent to

$$
\int_0^{|x|} (\tau^2 + c)^\frac{1}{2} d\tau.
$$

Later the same result was proved in higher dimensions for every generalized solution of (1.37) by Jin and Xiong [92].

Since the Dirichlet problem on exterior domains is closely related to asymptotic behaviour of solutions defined on entire $\mathbb{R}^n$, the Theorem 1.28 also implies the following corollary which will be used in Chapter 4 of this thesis.

**Corollary 1.29.** Let $O$ be a bounded open convex subset of $\mathbb{R}^n$, and let $u \in C^0(\mathbb{R}^n \setminus O)$ be a locally convex viscosity solution of

$$\det D^2 u = 1 \text{ in } \mathbb{R}^n \setminus O. \quad (1.38)$$

Then $u \in C^\infty(\mathbb{R}^n \setminus O)$, and we have:

(i) For $n \geq 3$, there exist some $c \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $A \in \mathcal{A}$ such that (1.33) and (1.34) hold.
(ii) For \( n = 2 \), there exist some \( c, d \in \mathbb{R}, b \in \mathbb{R}^2 \), and \( A \in \mathcal{A} \) such that (1.35) and (1.36) hold. Moreover, if \( O = \emptyset \), then \( d = 0 \).

The original Jörgens-Calabi-Pogorelov theorem is a simple consequence of this result.

The corollary also enabled authors to establish an existence theorem for the exterior Dirichlet problem

\[
\det D^2 u = f \text{ in } \mathbb{R}^n \setminus \overline{O}, \\
u = \varphi \text{ on } \partial O,
\]

with prescribed asymptotic behaviour at infinity. Indeed, the second main result in [37] tells us

for any given \( b \in \mathbb{R} \) and \( A \in \mathcal{A} \) there exist \( c^* \), such that for every \( c > c^* \) there exists a unique solution \( u \in C^\infty(\mathbb{R}^n \setminus O) \cap C^0(\mathbb{R}^n \setminus O) \) of (1.39) that satisfies (1.33).

The similar exterior Dirichlet problem in two dimensions was also studied by Ferrer, Martínez and Milán in [52] and Bao and Li in [11] with different asymptotic behaviours at infinity.

Recently, Bao, Li and Zhang [10] extended Caffarelli-Li’s results in [37] to the case \( f(x) = 1 + O(|x|^{-\beta}) \) at infinity for some \( \beta > 2 \); whereas, Ju and Bao [96] proved the existence of a locally convex viscosity solution of (1.39) which is asymptotic to a radial function.

### 1.3.1 Entire solutions to more general equations

#### Hessian equation.

A natural extension of the standard Monge-Ampère equation is the following \( k \)-Hessian equation

\[
\sigma_k(\lambda(D^2 u)) = f,
\]

where

\[
\sigma_k(\lambda(D^2 u)) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k}.
\]

The Hessian equation is an important class of fully nonlinear elliptic equations. For \( k = 1 \) equation (1.40) is the Poisson equation \( \Delta u = f \), while for \( k = n \), (1.40) is the Monge-Ampère equation \( \det D^2 u = f \).
1.3. MONGE-AMPRÈ EQUATIONS IN THE ENTIRE SPACE

There are many significant works concerning the existence of solutions to (1.40) in bounded domains (see, [32], [44], [39], [66], [88], [154], [155], [158], [168]). In particular, Caffarelli, Nirenberg and Spruck in [32] established the existence of a classical solution of the Dirichlet problem of Hessian equations. Trudinger in [155] demonstrated the existence and uniqueness of weak solutions and Urbas in [168] proved the existence of viscosity solutions.

As for the entire Dirichlet problem to the Hessian equation, there are no results similar to Jörgens-Calabi-Pogorelov theorem established for the Monge-Ampère equation. Moreover, it was recently shown by Warren [175] that:

For \( n \geq 2k - 1 \), there exist nonpolynomial elliptic entire solutions to the equation

\[
\sigma_k(\lambda(D^2u)) = 1 \text{ on } \mathbb{R}^n. \tag{1.41}
\]

In particular, when \( n = 3 \) function

\[
u(x, y, z) = (x^2 + y^2)e^z + \frac{1}{4}e^{-z} - e^z \tag{1.42}
\]

solves

\[
\sigma_2(\lambda(D^2u)) = 1.
\]

Although, examples in [175] clearly show that not every elliptic solution of the entire Dirichlet problem of Hessian equation is quadratic, the same question for convex solutions remains open. So far, the entire solutions of Hessian equations are not fully understood.

Nevertheless, there are many papers on exterior Dirichlet problems. To the best of our knowledge, the first result on the existence of solutions of (1.41) with prescribed asymptotic behaviour in exterior domains was obtained by Dai [46] for the constant boundary data and by Dai and Bao [47] for a general boundary function. Recently, these results were extended for the case of a general nonhomogeneous term \( f \) in the following theorem.

**Theorem 1.30.** Let \( D \) be a smooth bounded strictly convex open subset in \( \mathbb{R}^n \), \( n \geq 3 \), and let \( \varphi \in C^2(\partial D) \). Suppose that \( f \geq 0 \) on \( \mathbb{R}^n \setminus D \), and there exists a constant \( \beta > 2 \) such that \( \limsup_{|x| \to \infty} |x|^\beta |f(x) - 1| < \infty \). Then for any given \( b \in \mathbb{R}^n \) and any real \( n \times n \) symmetric positive definite matrix \( A \), with \( \sigma_k(\lambda(A)) = 1 \), \( 2 \leq k \leq n \), there exists a constant \( c^* \), such that for every \( c > c^* \) there exists a unique viscosity solution \( u \in C^0(\mathbb{R}^n \setminus D) \) of

\[
\sigma_k(\lambda(D^2u)) = f \text{ in } \mathbb{R}^n \setminus \overline{D},
\]

\[
u = \varphi \text{ on } \partial D,
\]
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with prescribed asymptotic behavior

\[
\limsup_{|x| \to \infty} |x|^{\min(k/H_k-\beta)-2} \left| u(x) - \left( \frac{1}{2} x' Ax + b \cdot x + c \right) \right| < \infty, \quad \text{if } \beta \neq \frac{k}{H_k},
\]

or

\[
\limsup_{|x| \to \infty} |x|^{k/H_k-2} (\ln |x|)^{-1} \left| u(x) - \left( \frac{1}{2} x' Ax + b \cdot x + c \right) \right| < \infty, \quad \text{if } \beta = \frac{k}{H_k},
\]

where

\[ H_k = \{ \lambda_i(A) \frac{\partial}{\partial \lambda_i} \sigma_k(\lambda(A)) | i = 1, ..., n \}. \]

In all papers concerning the existence of solutions with prescribed asymptotic behavior there exists a constant \( c^* \), such that for any \( c \geq c^* \) the exterior Dirichlet problem of Hessian equation (or Monge-Ampère equation) is solvable. So it is reasonable to say that for \( c \) small enough the problem doesn’t have any admissible solutions. However, to the author’s knowledge the exact value for \( c^* \) has not been obtained. The only known result on this problem obtained by Wang and Bao [178] is as follows:

**Theorem 1.31.** Let \( n \geq 3, \ 2 \leq k \leq m \leq n \). Then there exists a unique \( m \)-convex radially symmetric function \( u \in C^1(\mathbb{R}^n \setminus B_1) \cap C^2(\mathbb{R}^n \setminus \overline{B}_1) \) satisfying

\[
\sigma_k(\lambda(D^2u)) = 1 \ \text{in} \ \mathbb{R}^n \setminus B_1, \quad u = b \ \text{on} \ \partial B_1,
\]

and

\[
\limsup_{x \to \infty} \left( |x|^{n-2} \left| u(x) - \left( \frac{c^*}{2} |x|^2 + c \right) \right| \right) < \infty, \quad (1.43)
\]

if and only if \( c \in [\mu(-1), +\infty) \) for \( m = k \), and \( c \in [\mu(-1), \mu(m-k)] \) for \( m > k \),

where \( c^* = (\frac{1}{C_n})^{1/k} \), \( C_n = \frac{n!}{k!(n-k)!} \) and

\[
\mu(\alpha) = b - \frac{c^*}{2} + c^* \int_{1}^{\infty} s \left[ \left( 1 + \frac{\alpha}{s^n} \right)^{1/k} - 1 \right] \text{d}s.
\]

**Lagrangian equation.**

Another extension of the Monge-Ampère equation is the special Lagrangian equation

\[
\sum_{i=1}^{n} \lambda_i(D^2u) = \Theta. \quad (1.44)
\]

When \( n = 2 \), the equation takes the algebraic form \( \cos \Theta \Delta u + \sin \Theta \det D^2u = \sin \Theta \); while for \( n = 3 \), and \( |\Theta| = \pi \) or \( \pi/2 \), the equation is equivalent to \( \Delta u = \det D^2u \) or \( \sigma_2(D^2u) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 1 \), respectively.
Entire solutions of the equation (1.44) have been well-studied by many authors. The first known result is by Borishenko [16], who showed that any convex solution to (1.44) with a linear growth and $\Theta = k\pi$ is linear. Later Yuan [179], [180] proved that any convex solution to (1.44) is a quadratic polynomial for all $n \geq 2$ and $|\Theta| \geq (n-2)\pi/2$; the case $n = 2$ was considered earlier by Fu [60]. For $\Theta = k\pi$ in case $n = 3$, it was proved in [12] that any strictly convex solution with quadratic growth must be quadratic. Under the assumption that the Hessian is bounded and $\lambda_i \lambda_j \geq 3/2$, it was also shown in [165] that any global solution to (1.44) is quadratic.

The Lagrangian angle $(n - 2)\pi/2$ is called a critical phase, because the level set $\{ \lambda \in \mathbb{R}^n | \lambda \text{ satisfying (1.44)} \}$ is convex only for $|\Theta| \geq (n-2)\pi/2$. The example by Warren (1.42) ($n = 3$) as well as a simple solution $\sin x_1 e^{x_2}$ ($n = 2$), show that when $\Theta = (n - 2)\pi/2$ entire solutions of (1.44) may not be even polynomial.

The corresponding exterior problems were studied in low dimensions by Bers [15] (for $n = 2$) and Simon [149] (for $3 \leq n \leq 7$), where authors asserted that all solutions approach to linear functions asymptotically near infinity. The general result for the exterior problem was recently obtained by Li, Li and Yuan [112]:

**Theorem 1.32.** Let $u$ be a smooth solution to (1.44) in $\mathbb{R}^n \setminus \bar{\Omega}$, where $|\Theta| \geq (n-2)\pi/2$, $\Omega$ is a bounded domain, and $\lambda_i(D^2 u)$ denote the eigenvalues of the Hessian $D^2 u$. Then there exists a unique quadratic polynomial $Q(x)$ such that when $n \geq 3$

$$u(x) = Q(x) + O_k(|x|^{2-n}) \text{ as } |x| \to \infty,$$

for all $k \in \mathcal{N}$, and when $n = 2$,

$$u(x) = Q(x) + \frac{d}{2} \log x^T (I + (D^2 Q)^2) x + O_k(|x|^{2-n}) \text{ as } |x| \to \infty,$$

for all $k \in \mathcal{N}$, where

$$d = \frac{1}{2\pi} \left( \int_{\partial \Omega} \cos \Theta u_\nu + \sin \Theta u_1 (u_{22}, -u_{12}) \cdot \nu ds - \sin \Theta |\Omega| \right),$$

where $\nu$ is the unit outward normal of the boundary $\partial \Omega$, and the notation $\varphi(x) = O_k(|x|^m)$ means that $|D^k \varphi(x)| = O(|x|^{m-k})$.

### 1.3.2 An open question in a theory of singular Monge-Ampère equations

After the existence, uniqueness (in a sense) and asymptotic behaviour of solutions to singular Monge-Ampère equations have been established, it is natural to ask:
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Question 1.33. Does any solution \( u \) of

\[
det D^2 u = 1 + \delta_0 \quad \text{in} \quad \Omega,
\]

(1.45)

has the following decomposition

\[
u = h + g,
\]

where \( h \) is a \( C^{2,\alpha} \) smooth function and the graph of \( g \) is a convex cone with uniformly convex, smooth \( \partial\{\nabla g\} \)?

Here \( \delta_0 \) is a delta measure centered at 0. Question 1.33 is the main problem studied in Chapter 4. In two dimensional case the positive answer was given for a bounded domain \( \Omega \) by Galves, Jimenez and Mira [61] in the following form:

Assume that \( \Omega := \{(x, y) \in \mathbb{R}^2 | 0 < x^2 + y^2 < \rho^2 \} \) is a punctured disc centered at the origin, \( f > 0 \) is \( C^k \)-smooth function (analytic, respectively), \( u \in C^{k+1,\alpha}_{loc}(\Omega) \) is a solution to

\[
det D^2 u = f \quad \text{in} \quad \Omega,
\]

with an isolated singularity at \((0,0)\). Then the limit gradient of \( u \) is a regular, strictly convex Jordan curve in \( \mathbb{R}^2 \), which is \( C^{k,\alpha} \), \( \forall \alpha \in (0,1) \) (analytic, respectively).

For the case \( n \geq 3 \), the partial answer was given by Savin [140], who established the \( C^{1,1} \) regularity of the tangent cone of a strictly convex solution to (1.45) in \( B_1 \). Indeed, consider the following obstacle problem: given a finite measure \( \mu_0 \) in \( \Omega \) we define \( \mathcal{D}_{\mu,0} \) the class of nonnegative supersolutions

\[
\mathcal{D}_{\mu,0} = \{ \varphi : \Omega \to [0, \infty), \varphi \text{ convex}, \varphi = 1 \text{ on } \partial \Omega, M\varphi \leq \mu_0 \}.
\]

Then, we study the minimization problem

\[
v = \inf_{\varphi \in \mathcal{D}_{\mu,0}} \varphi.
\]

(1.46)

Define

\[
\mathcal{D}_{1,\delta} := \{ v : \Omega, v = \delta \text{ on } \partial \Omega, B_1 \subset \{v = 0\} \subset B_{k(n)}, v \text{ solves (1.46) with } d\mu_0 = dx \}.
\]

Let \( u \) be the Legendre transform of \( v \in \mathcal{D}_{1,\delta} \). Then \( u \) solves (1.45) in \( B_1 \setminus \{0\} \) and the set \( \{v = 0\} \) is the image of the subgradients of the tangent cone of \( u \) at 0. Denote a convex set \( L \) to be 1 level set of the tangent cone of \( u \). Then the set \( L \) is convex dual of \( \{v = 0\} \) and we obtain the regularity of the tangent cone from the following theorem (see [140]):
Theorem 1.34. There exist positive constants $r$ small and $R$ large depending only on $n$ and $\delta$ such that:

If $v \in D_{1,\delta}$ then at each point of $\partial \{v = 0\}$ there exist a tangent ball of radius $r$ contained in $\{v = 0\}$ and a tangent ball of radius $R$ which contains $\{v = 0\}$.

Thus, the mentioned papers considered Question 1.33 either for $n = 2$ or for a strictly convex solution. In Chapter 4 we will present a radial symmetry result for solutions of (1.45) in $\mathbb{R}^n \setminus \{0\}$ and a counterexample, which give answers to the Question 1.33 for both, bounded and unbounded domains $\Omega$, for $n \geq 3$. Finally, Chapter 4 provides different applications of the symmetry result to Monge-Ampère equations as well as more general equations, such us Hessian equations and special Lagrangian equations with the aid of Corollary 1.29 and Theorems 1.30, 1.32.
1.4 Thesis outline

This thesis consists from 4 chapters.

Chapter 1 is an introduction, which includes the history of the optimal transportation problem and Monge-Ampère equations, and a comprehensive overview of main results been made so far.

Chapter 2 concerns global regularity of potential functions in optimal transportation with quadratic cost. First, there is a proof of global $C^1$ regularity on any Lipschitz domain $\Omega$ under some natural conditions. Then using $C^1$ regularity result we obtain the $C^{1,\alpha}$ regularity of potential functions on a domain $\Omega$ obtained by removing finitely many disjoint convex subsets from a convex domain.

Chapter 3 is devoted to degenerate Monge-Ampère type equations which belongs to a wider class of equations than ones considered in Chapter 2. There will be presented the global $C^2$ a priori estimates for the corresponding Dirichlet problem. We also show that these estimates imply existence of the unique solution $u \in C^{1,1}(\overline{\Omega})$ by the classical continuity method.

In the last Chapter we deal with singular Monge-Ampère equations on bounded and unbounded domains. First, we construct an example which shows that solutions to singular Monge-Ampère equations in a ball for $n \geq 3$ may not have a decomposition of a smooth function and a smooth convex cone. Then we study the symmetry of smooth convex solutions to singular Monge-Ampère type equations in the entire space $\mathbb{R}^n$. Moreover, we consider different application of this result to Monge-Ampère equations with constant nonhomogeneous term as well as more general equations, such us Hessian equations and special Lagrangian equations.
Chapter 2

Regularity of potential functions in optimal transportation with quadratic cost

2.1 Main result

In this Chapter we study the global $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost. Let $\Omega, \Omega^*$ be the source and target domains associated with densities $C^{-1} \leq f, g \leq C$ respectively, where $C$ is a positive constant. The optimal transport problem with quadratic cost is about finding a map $T : \Omega \rightarrow \Omega^*$ among all measure preserving maps minimizing the transportation cost

$$\int_{\Omega} |x - Tx|^2 dx.$$ Here the term “measure preserving” means that $\int_{T^{-1}(B)} f(x)dx = \int_B g(y)dy$ for any Borel set $B \subset \Omega^*$.

As it was mentioned in Chapter 1 the potential function $u$ satisfies $T(x) = Du(x)$ a.e. in $\Omega$ and if the target domain $\Omega^*$ is convex $u$ satisfies $\frac{1}{C}|A \cap \Omega| \leq |\partial u(A)| \leq C|A \cap \Omega|$ for any Borel set $A \subset \Omega$. Moreover, if we extend $u$ to $\mathbb{R}^n$ as following

$$\tilde{u} := \sup \{ L : L \text{ is linear, } L|_\Omega \leq u, L(z) = u(z) \text{ for some } z \in \Omega \},$$

then $\tilde{u}$ is a globally Lipschitz convex solution of

$$C^{-1} \chi_\Omega \leq \det \tilde{u}_{ij} \leq C \chi_\Omega.$$
We will still use $u$ to denote this extended function.

The main result in this Chapter is the following theorem.

**Theorem 2.1.** Let $\Omega$ and $\Omega^*$ be two bounded domains in $\mathbb{R}^n$, $n \geq 2$, and $f$ and $g$ be densities of two positive probability measures defined respectively in $\Omega$ and $\Omega^*$, satisfying $C^{-1} \leq f, g \leq C$ for a positive constant $C$. Assume that $\Omega^*$ is convex, $\Omega$ is Lipschitz. Then,

(i) If for any given $x \in \Omega$, there exists a small ball $B_{r_x}(x)$ such that for any convex set $\omega \subset B_{r_x}(x)$ centered in $\Omega$ we have $\int_{\omega} f \leq C \int_{\frac{1}{2}\omega} f$ for some constant $C$ independent of $\omega$, then the potential function $u \in C^1(\overline{\Omega})$. Here $f$ is defined to be 0 outside $\Omega$.

(ii) If $\Omega$ is a domain obtained by removing finitely many disjoint convex subsets from a convex set, then the potential function $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

**Remark 2.2.** a) It is easy to see that in Theorem 2.1 (i) we allow $\Omega$ to be any polytope (not necessarily convex). We also note that the $C^1$ regularity always holds in dimension two without any condition on $\Omega$. This is a classical result of Aleksandrov, see also [53].

b) One may want to prove higher regularity when the densities are smooth, however, in view of the following simple example we see that this is impossible. Let the dimension $n = 2$. Let $\Omega := B_2 - B_1$ associated with uniform probability density, and let $\Omega^* := B_{\sqrt{3}}$ associated with uniform probability density. Then by symmetry it is easy to compute that the optimal transport map is

$$T(x) = \sqrt{|x|^2 - 1} \frac{x}{|x|},$$

which is only $C^{1,\frac{1}{2}}$ on $\partial B_1 \subset \partial \Omega$.

The results of this chapter were published in the joint paper [6] with Dr. Shijing Chen.

### 2.2 Preliminaries

In the following we will use $S_h(x_0)$ to denote a section of $u$ with height $h$, namely,

$$S_h(x_0) := \{x | u < p \cdot (x - x_0) + u(x_0) + h\},$$
where \( p \) is chosen such that \( x_0 \) is the center of mass of \( S_h(x_0) \). We say a point \( x_0 \in \Omega \) localized (with respect to \( u \)), if for any sequence \( h_k \to 0 \) and \( x_k \to x_0 \) satisfying \( x_0 \in S_{h_k}(x_k) \) we have that \( S_{h_k}(x_k) \) shrinks to the point \( x_0 \in \Omega \).

Now, we record a fundamental property of convex sets which is called John’s Lemma. We refer to Liu and Wang [118] for a simplified proof.

**Lemma 2.3.** Let \( U \subset \mathbb{R}^n \) be a bounded, convex domain with its centre of mass at the origin. There exists an ellipsoid \( E \), also centered at the origin, such that \( E \subset U \subset nE \).

By John’s Lemma we can show the following property of convex functions.

**Lemma 2.4.** Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a convex function. Let \( L \) be a supporting function of \( u \). Then any extreme point of \( \{ u = L \} \) is localized.

**Proof.** Suppose to the contrary that there exists an extreme point \( x_0 \) of \( \{ u = L \} \) which is not localized. Then there exists a sequence \( x_k \to x_0, h_k \to 0 \) such that \( x_0 \in S_{h_k}(x_k) \), and that \( S_{h_k}(x_k) \) contains a segment of length greater or equal to some positive constant \( \delta \). Since \( S_{h_k}(x_k) \) is convex and centered at \( x_k \), by John’s Lemma there exists a unit vector \( \xi_k \) such that \( I_k, \) the segment connecting \( x_k - \frac{\delta}{2n} \xi_k \) and \( x_k + \frac{\delta}{2n} \xi_k \), is contained in \( S_{h_k}(x_k) \).

Denote by \( L_k \) the defining function of \( S_{h_k}(x_k) \), namely \( S_{h_k}(x_k) = \{ u \leq L_k \} \).

Then, it is easy to see that \( DL_k \) is bounded, hence by passing to a subsequence \( L_k \to L_\infty \) for some linear function \( L_\infty \). Also by passing to a subsequence we may assume \( \xi_k \to \xi_\infty \) for some unit vector \( \xi_\infty \). Then \( u \) is linear on \( I_\infty \), which is the segment connecting \( x_0 - \frac{\delta}{2n} \xi_\infty \) and \( x_0 + \frac{\delta}{2n} \xi_\infty \). Hence \( I_\infty \subset \{ u = L \} \), which contradicts to the assumption that \( x_0 \) is an extreme point of \( \{ u = L \} \).

The following property of sections of convex functions was proved by Caffarelli [23]. Here we provide a different proof by using a well known fact that if a continuous map from a ball to itself fixes the boundary, then it must be surjective. We learned this method from Wang, see [148, Section 4].

**Lemma 2.5.** Let \( u : \mathbb{R}^n \to [0, \infty] \) be a convex function. Assume that:
1. \( u(0) = 0, u \geq 0 \).
2. \( u \) is finite in a neighbourhood of \( 0 \).
(3) The graph of \( u \) contains no complete lines. Then for \( h > 0 \), there exists a slope \( p \) such that the centre of mass of the section

\[
S_{h,p} := \{ x \mid u \leq x \cdot p + h \}
\]

is defined and equal to 0.

Proof. Let

\[
\begin{align*}
  u_k(x) &= u(x) \quad \text{in } B_k, \\
  u_k &= \infty \quad \text{in } \mathbb{R}^n \setminus B_k.
\end{align*}
\]  

(2.1)

We only need to show the existence of section

\[
S^k := \{ x \mid u_k \leq x \cdot p_k + h \}
\]

centered at 0 with bounded \( p_k \). Then, \( S_{h,p} = \lim_{k \to \infty} S^k \) is the desired section in the lemma.

Take a large ball \( B_r \). For any \( p \in B_r \), denote by \( z_p \) the centre of mass of the section

\[
S_p := \{ x \mid u_k(x) \leq x \cdot p + h \}.
\]

Then we obtain a mapping \( M_1 : p \to z_p \) from \( B_r \) to \( \mathbb{R}^n \). If \( p \in \partial B_r \), it is easy to see that \( p \cdot z_p > 0 \) provided \( r \) is sufficiently large.

If there is no \( p \in B_r \) such that \( z_p = 0 \), then we can define a mapping \( M_2 : z_p \to t_p z_p \), where \( t_p > 0 \) is a constant such that \( t_p z_p \in \partial B_r \). We then obtain a continuous mapping \( M = M_2 \circ M_1 \) from \( B_r \) to \( \partial B_r \) with the property that

\[
p \cdot M(p) > 0 \quad \text{on } \partial B_r.
\]  

(2.2)

To get a contradiction, we extend the mapping \( M \) to \( B_{2r} \) as follows. For any point \( p \in \partial B_{2r} \), denote \( p_1 = p, p_0 = \frac{1}{2} p \in \partial B_r \), and \( p_t = (1-t)p_0 + p_1 \). We extend the mapping \( M \) to \( B_{2r} \) by letting

\[
M(p_t) = (1-t)M(p_0) + tp_1.
\]

Then by (2.2) \( M(p) \neq 0 \) on \( B_{2r} \) and \( M \) is the identity mapping on \( \partial B_{2r} \). This is a contradiction.

Hence for each \( k > 0 \), there exists a \( p_k \in \mathbb{R}^n \) such that

\[
S^k := \{ x \mid u_k \leq x \cdot p_k + h \}
\]
is centered at 0. Moreover, $|p_k| \leq C$ for some constant independent of $k$. Indeed, we can argue as follows. By rotating the coordinates we may assume $p_k = (a, 0, \cdots, 0)$ with $a > 0$. Denote
\[
\alpha^+ = \sup \{x_1 | (x_1, 0, \cdots, 0) \in S^k\},
\]
\[
\alpha^- = -\inf \{x_1 | (x_1, 0, \cdots, 0) \in S^k\}.
\]
Then $\frac{\alpha^+}{a} \to \infty$ as $a \to \infty$. Since $S_k$ is centered at 0, $a$ can not be too large.

The following Aleksandrov type estimates were proved by Caffarelli, see for instance [25], [73].

**Lemma 2.6.** Let $u$ be a convex solution of
\[
\det D^2u = d\mu
\]
in the convex domain $S$ and $u = 0$ on $\partial S$. Assume $S$ is normalised, namely, $B_1 \subset S \subset nB_1$. Assume $d\mu(S) \leq \theta d\mu(\frac{1}{2}S)$ for some constant $\theta$, where $\frac{1}{2}S$ is a dilation of $S$ with respect to the center of mass. Then,
a) there is a constant $C$ depending only on $n$ such that
\[
|u(x)|^n \leq C d\mu(S) d(x, \partial S);
\]
b) there is a constant $C$ depending only on $\theta$ and $n$ such that
\[
\frac{1}{C} |\inf_S u|^n \leq d\mu(S) \leq C |\inf_S u|^n.
\]

**Proof.** a) Fix an arbitrary point $x_0 \in S$ and denote $v$ to be the convex function whose graph is the cone with vertex $(x_0, u(x_0))$ and base $S$ such that $v = 0$ on $\partial S$. Since $u$ is convex and $v \geq u$ in $S$ by maximum principle
\[
\partial v(S) \subset \partial u(S).
\]

**Claim.** The set $\partial v(S)$ is convex.

Suppose $p \in \partial v(x_1)$ for some $x_1 \in S$, $x_1 \neq x_0$. Then $v(x_1) + p \cdot (x - x_1)$ is a supporting hyperplane of $v$ at $x_0$ and $p \in \partial v(x_0)$. Therefore, $\partial v(S) = \partial v(x_0)$ and since $\partial v(x_0)$ is convex, the claim is proved.

Next, it is easy to see that convexity of the section $S$ implies that there exists $p \in \partial v(S)$ such that
\[
|p| = \frac{|u(x_0)|}{\text{dist}(x_0, \partial S)}.
\]
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Now since the ball $B$ with radius $|u(x_0)|/C(n)$ and center at the origin is contained in $\partial v(S)$, the convex hull of $B$ and $p$ is also contained in $\partial v(S)$. Therefore,
\[ d\mu(S) \geq C|u(x_0)|^{n-1}|p_0|, \]
and a) is proved.

b) The estimate for $d\mu(S)$ from below readily follows from a). The estimate for $d\mu(S)$ from above can be easily obtained by the doubling condition. Indeed, since $u$ is convex and vanish on the boundary
\[ d\mu\left(\frac{1}{2}S\right) \leq C|\inf_{\frac{1}{2}S} u|^{n}. \]
Then doubling condition implies the estimate
\[ d\mu(S) \leq \theta d\mu\left(\frac{1}{2}S\right) \leq C|\inf_{S} u|^{n}, \]
which finishes the proof of the lemma. \qed

2.3 Global $C^1$ regularity of potential functions

In this section, we prove Theorem 2.1 (i). First we have

**Lemma 2.7.** Suppose $u$ is a globally Lipschitz convex function. Assume that $u$ is $C^1$ at all of the extreme points of a convex set $K = \{u = L\}$, where $L$ is a linear function satisfying $u \geq L$ and $u(y) = L(y)$ for some $y \in \mathbb{R}^n$. Then $u$ is $C^1$ on $K$.

**Proof.** By subtracting $L$ we may assume $K = \{u = 0\}$. If $K$ is a bounded convex set, then for any $x \in K$ we have
\[ x = \sum_{i=1}^{k} \lambda_i x_i, \]
where $x_i, i = 1, \cdots, k$ are extreme points of $K$, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since $u$ is $C^1$ at $x_i, i = 1, \cdots, k$, we have $0 \leq u(z) = o(z - x_i), i = 1, \cdots, k$. Now, by convexity we have
\[ 0 \leq u(z) = u \left( \sum_{i=1}^{k} \lambda_i(z - x + x_i) \right) \leq \sum_{i=1}^{k} \lambda_i u(z - x + x_i) \]
\[ = \sum_{i=1}^{k} \lambda_i o(z - x) = o(z - x). \]
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Hence, $u$ is $C^1$ at $x$.

If $K$ is unbounded, it is well known that $K = \text{covext}[K] + \text{rc}[K]$, where $\text{covext}[K]$ is the convex hull of the extreme points of $K$, and $\text{rc}[K] := \lim_{\lambda \to 0} \lambda K$ is the recession cone of $K$. Hence we need only to show that $u$ is $C^1$ at points represented by $x = x_0 + q$, where $x_0$ is an extreme point of $K$ and $q \in \text{rc}[K]$. For any $M \geq 0$, by using the facts that $u$ is Lipschitz and $\xi := x_0 + Mq \in K$ we have that $u(z - x + \xi) \leq C|z - x|$. By convexity we have

$$u(z) = u\left(\frac{M-1}{M}(z-x+x_0) + \frac{1}{M}(z-x+\xi)\right) \leq \frac{M-1}{M}o(|z-x|) + \frac{C}{M}|z-x|.$$  

By letting $M \to \infty$ we have $0 \leq u(z) \leq o(|z-x|).$ Hence $u$ is $C^1$ at $x$. \hfill \Box

Since $u$ is convex, for any unit vector $\gamma$, the lateral derivatives

$$\partial^+_{\gamma} u(x) := \lim_{t \searrow 0} \frac{1}{t} [u(x + t\gamma) - u(x)],$$

$$\partial^-_{\gamma} u(x) := \lim_{t \searrow 0} \frac{1}{t} [u(x) - u(x - t\gamma)]$$

exist. To prove that $u \in C^1(\overline{\Omega})$, it suffices to prove that

$$\partial^+_{\gamma} u(x_0) = \partial^-_{\gamma} u(x_0)$$  \hfill (2.5)

at any point $x_0 \in \partial \Omega$ and any unit vector $\gamma$. By the convexity, it suffices to prove it for $\xi = \xi_k$ for all $k = 1, 2, \ldots, n$, where $\xi_k, k = 1, \ldots, n$ are any $n$ linearly independent unit vectors.

Proof of Theorem 2.1 (i).

By Lemma 2.7 and Lemma 2.4 we only need to show that $u$ is $C^1$ at localized points. Assume to the contrary that $u$ is not $C^1$ at $x_0 \in \partial \Omega$. Let us assume that

$$x_0 = 0, \quad u(0) = 0, \quad u \geq 0;$$  \hfill (2.6)

$$\partial^+_{\gamma} u(0) > \partial^-_{\gamma} u(0) = 0.$$  \hfill (2.7)

Since $\partial \Omega$ is Lipschitz, we may also assume that $-te_1 \in \Omega$ for $t \in (0, 1)$, where $e_1$ is the first coordinate direction.

Now we consider a section $S_h(x')$, where $x' = (-a', 0, \ldots, 0)$ for some small constant $0 < a' < \frac{r_0}{2}$, where $r_0 := r_{x_0}$ is the radius in the condition of Theorem
2.1 (i). Note that by John’s lemma, there exists an ellipsoid $E$ with center $x'$ such that $E \subset S_h(x') \subset nE$. Since $u$ is Lipschitz and $\partial_1^+ u(0) > 0$, we have that

$$C^{-1} \varepsilon \leq u(\varepsilon e_1) \leq C \varepsilon$$

for any small $\varepsilon > 0$, where $C$ is a positive constant. Since $\partial_1^- u(0) = 0$, we have $u(-Ma'e_1) = o(a')$, where $M = 2n$. Hence, we can choose small $\varepsilon$ and $a'$ so that the following properties hold:

1) $o(a') = u(-Ma'e_1) \leq C^{-1} \varepsilon << a'$,
2) $\varepsilon e_1$ is on the boundary of some section $S_h(x')$,
3) $S_h(x') \subset B_{r_0}(0)$.

The existence of such section $S_h(x')$ in 2) follows from the property that centered section, say $S_h(x)$, varies continuously with respect to the height $h$, see [34, Lemma A.8]. 3) follows from the assumption that $x_0 = 0$ is localized.

Let $L$ be the defining linear function of $S_h(x')$, by the property 1) it is easy to see that $L$ is increasing in $e_1$ direction, hence

$$(L - u)(0) \geq (L - u)(x) = h. \quad (\text{2.8})$$

Since $\int_{S_h(x')} f \leq C \int_{\frac{1}{2} S_h(x')} f$, we have that

$$(L - u)(0) \leq C \left( \frac{\varepsilon}{a'} \right)^{\frac{h}{2}} h \quad (\text{2.9})$$

contradicting to (2.8) since $a' >> \varepsilon$. Here we followed the argument of Caffarelli [25]. Indeed, let $A$ be an affine transform normalizing $S_h(x')$, then

$$v := \frac{(u - L)(A^{-1} x)}{h}$$
satisfies
\[
\det D^2 v = \frac{f(A^{-1}x)}{h^n} \quad \text{in } A(S_h(x')),
\]
and
\[
v = 0 \text{ on } \partial S_h(x').
\]
Hence, by applying Lemma 2.6 to \(v\) and translating back to \(u\) we get (2.9).

Hence \(u\) must be \(C^1\) at any localized point \(x_0\). Therefore \(u \in C^1(\mathbb{R}^n)\).

**Remark 2.8.** The proof of Theorem 2.1 shares some similarities with the proof of \(C^1\) regularity for the obstacle problem in [140] (see Proposition 2.8 in that paper).

### 2.4 Global \(C^{1,\alpha}\) regularity of potential functions

In this section, we prove Theorem 2.1 (ii). First we point out that to prove \(u \in C^{1,\alpha}(\Omega)\), it suffices to prove that there exist positive constants \(C > 0\), \(\alpha \in (0,1)\) and \(r > 0\) such that for any point \(x_0 \in \Omega\),

\[
u(x) - \ell_{x_0}(x) \leq C|x - x_0|^{1+\alpha}
\]

for any \(x \in B_r(x_0) \cap \bar{\Omega}\). From (2.10) one can prove that \(u \in C^{1,\alpha}(\bar{\Omega})\), using the convexity of \(u\). In the following we will show that a relaxed version of (2.10) is enough to show \(u \in C^{1,\alpha}(\bar{\Omega})\), and it has the advantage to avoiding some annoying limiting picture.

By the assumption of Theorem 2.1 (ii) we write \(\Omega = U - \sum_{i=1}^{k} C_i\), where \(U\) is an open convex set, and \(C_i, i = 1, \cdots, k\) are closed disjoint convex subsets of \(U\). Given any \(x \in \bar{\Omega}\), we introduce the following function

\[
\rho_x(t) := \sup \{u(z) - u(x) - Du(x) \cdot (z - x) | |z - x| = t, x + s \frac{z - x}{|z - x|} \in \bar{\Omega} \text{ for any } s \in [0, r_0]\},
\]

where \(r_0\) is a fixed small positive constant depending on \(\Omega\), and its smallness will be clear in the proof of Lemma 2.9. Indeed we need to take \(r_0\) small enough so that \(B_{r_0}(x) \cap \partial U\) can be represented as the graph of some Lipschitz function for any \(x \in \partial U\) with the Lipschitz constant independent of \(x\), and that

\[
r_0 << \min \{\text{dist}(\partial U, \partial C_i), i = 1, \cdots, k, \text{dist}(\partial C_j, \partial C_l), 1 \leq j \neq l \leq k\}.
\]
Lemma 2.9. Suppose that there exist \( r > 0, \delta \in (0, 1) \) such that for any \( x \in \bar{\Omega} \) we have
\[
\rho_x \left( \frac{1}{2} t \right) \leq \frac{1}{2} (1 - \delta) \rho_x (t)
\] (2.12)
whenever \( t \leq r \). Then \( u \in C^{1,\alpha}(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \).

Proof. For \( t = \frac{1}{2^k} r \), we have
\[
\rho_x (t) \leq \frac{(1 - \delta)^k}{2^k} \rho_x (r) \leq \frac{r}{r} (1 - \delta) \frac{\log \frac{r}{r}}{\log 2} \rho_x (r) \leq Ct^{1+\alpha},
\] (2.13)
where \( C \) depends on \( r, \delta \) and \( \rho_x (r) \), and \( \alpha = -\frac{\log (1-\delta)}{\log 2} \).

Suppose \( x, y \in \bar{\Omega} \) and \( |x - y| << r << r_0 \). We need to consider two cases:

a) \( x, y \) are close to \( \partial U \),

b) \( x, y \) are close to \( \partial C_i \) for some \( 1 \leq i \leq k \).

We will deal with case a) first, and case b) follows from a similar argument.

Without loss of generality we may assume that \( B_{3r_1} \subset U \) for some small fixed \( r_1 \), that \( r_0 << r_1 \), and that \( \text{dist}(\partial B_{3r_1}, \partial U) >> r_1 \). Denote by \( C_{x,r_1} \) the convex hull of \( x \) and \( B_{r_1} \). By convexity \( C_{x,3r_1} \subset U \). Then we prove the following claim.

Claim 1. For any \( z \in B_{r/2} (x) \cap C_{x,2r_1} \), we have
\[
|Du(x) - Du(z)| \leq C|x - z|^\alpha.
\] (2.14)

Proof of claim 1. Observe that \( \text{dist}(z, \partial C_{x,3r_1}) \geq \frac{1}{C}|x - z| \) for some large constant \( C \). Hence \( B_{1} \frac{|x - z|}{C} (z) \subset B_r \cap C_{x,3r_1} \). Now, for any \( \tilde{z} \in \partial B_{1} \frac{|x - z|}{C} (z) \), by (2.13) we have that
\[
u(\tilde{z}) \leq u(x) + Du(x) \cdot (\tilde{z} - x) + C|\tilde{z} - x|^{1+\alpha}.
\] (2.14)

By convexity we also have
\[
u(\tilde{z}) \geq u(z) + Du(z) \cdot (\tilde{z} - z),
\] (2.15)
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and

\[ u(z) \geq u(x) + Du(x) \cdot (z-x). \quad (2.16) \]

By (2.14), (2.15) and (2.16) we have

\[ (Du(z) - Du(x)) \cdot (\tilde{z} - z) \leq C|\tilde{z} - x|^{1+\alpha}. \quad (2.17) \]

Note that $|\tilde{z} - z| \approx |\tilde{z} - x| \approx |z - x|$ provided $\tilde{z} \in \partial B_{\tilde{z} - x}^{1/2}(z)$, and $C$ is sufficiently large. Since (2.17) holds for any $\tilde{z} \in \partial B_{\tilde{z} - x}^{1/2}(z)$, it follows that

\[ |Du(x) - Du(z)| \leq C|x - z|^\alpha. \]

Now, suppose $|x - y| << r$. If either $y \in C_{x,2r_1}$ or $x \in C_{y,2r_1}$ holds, then by claim 1 we have $|Du(x) - Du(y)| \leq C|x - y|^\alpha$. Otherwise one may find a point $z \in C_{x,r_1} \cap C_{y,r_1}$ such that $|z - x| \approx |z - y| \approx |x - y|$. Then by applying the estimate in claim 1 we have

\[ |Du(x) - Du(y)| \leq C(|x - z|^\alpha + |y - z|^\alpha) \leq C|x - y|^\alpha. \]

We can prove case b) by a similar argument. Indeed, for any fixed $x \in \partial C_1$, $\partial C_1 \cap B_r(x)$ can be represented as the graph of some Lipschitz function provided $r << r_0$. Then by the assumption that $C_i$ are disjoint, it is easy to find a small ball $B_{3r_1} \subset \Omega$ such that $C_{z,3r_1} \subset \Omega$ for any $z \in B_r(x) \cap \overline{\Omega}$. Then by a similar argument to the proof of case a) we can show that $|Du(x) - Du(y)| \leq C|x - y|^\alpha$, provided $|x - y| << r$.

\[ \square \]

The following lemma shows that the centered sections are well localized provided the heights are sufficiently small.

Lemma 2.10. There exists a height $h_0 > 0$ such that for any $x \in \overline{\Omega}$, $S_h(x)$ intersects at most one of $\partial U, \partial C_i, i = 1, \ldots, m$, provided $h \leq h_0$.

Proof. Suppose to the contrary, there exists a sequence $x_k \in \overline{\Omega}$, $h_k \to 0$, such that $S_{h_k}(x_k)$ intersects at least two of $\partial U, \partial C_i, i = 1, \ldots, m$. Passing to a subsequence we may assume $x_k \to y \in \overline{\Omega}$. Since $u$ is strictly convex in the interior of $\Omega$, we have either $y \in \partial U$ or $y \in \partial C_i$ for some $i$. Denote $L_k$ the defining function of $S_{h_k}(x_k)$, namely, $S_{h_k}(x_k) = \{u \leq L_k\}$. Then, passing to a subsequence we may
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assume $L_k \to L$ for some affine function $L$, and $S_{h_k}(x_k) \to S \subset \{ u \leq L \}$. It follows from the properties of $S_{h_k}(x_k)$ that:

(i) $S$ is centered at $y$.
(ii) $S$ intersects at least two of $\partial U, \partial C_i, i = 1, \ldots, m$.
(iii) $L(y) = \lim_{k \to \infty} L_k(x_k) = \lim_{k \to \infty} u(x_k) + h_k = u(y)$.

By (i) and (iii) we have that $S \subset \{ u = L \}$. Then by (ii) we see that $S$ passes through the interior of $\Omega$, which contradicts to the fact that $u$ is strictly convex in the interior of $\Omega$.

Proof of Theorem 2.1 (ii).

Step 1. The main observation in this step is that if (2.12) is violated for small $\delta$, then $u$ is close to a linear function on a segment connecting $x$ and some point $z_\delta \in \bar{\Omega}$. Hence, if (2.12) is violated for arbitrary $r, \delta$, then one can find a sequence of points $x_k$, such that $u$ is more and more linear around $x_k$ in some direction as $k \to \infty$. The “almost linearity” will be clear if we perform blow-up and affine transform on $u$ restricted to some carefully chosen section around $x_k$ properly, and a line segment will appear on the graph of the limiting function. The detailed argument goes as follows.

To prove $\rho_x(t) \leq Ct^{1+\alpha}$ for any $x \in \bar{\Omega}$ and any $t \leq r$, by Lemma 2.9 we assume to the contrary that there exists a sequence $t_k \leq 1/k$, $\delta_k = 1/k$, and $x_k \in \bar{\Omega}$ such that

$$\rho_{x_k}(\frac{1}{2}t_k) \geq \frac{1}{2}(1 - \frac{1}{k})\rho_{x_k}(t_k).$$

Suppose the supremum in (2.11) (when $x = x_k, t = \frac{1}{2}t_k$) is attained at $z_k, x_k \in \bar{\Omega}$, by the definition of function $\rho_x$ we see that $z_kx_k \subset \bar{\Omega}$, where $z_k, x_k$ denote the segment connecting $z_k$ and $x_k$. By passing to a subsequence, we may assume $x_k \to x_\infty \in \partial \Omega$.

Choosing sections. For each $k$, let $S_{h_k}(x_k)$ be a section of $u$ with center $x_k$, where $h_k$ is chosen such that $z_k \in \partial S_{h_k}(x_k)$. Similar to the proof of Theorem 2.1(i), the existence of such kind of section follows from the property that centered section, say $S_h(x)$, various continuously with respect to the height $h$, see [34, Lemma A.8] for a proof. It is easy to see that $h_k \to 0$.

Normalization. Let $L_k$ be the defining function of $S_{h_k}(x_k)$. We normalise the section $S_{h_k}(x_k)$ by a linear transformation $T_k$, and denote $S_k = T_k(S_{h_k}(x_k))$. 

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Note that $T_k(x_k) = 0$, and $B_1 \subset S_k \subset nB_1$. Also we denote

$$u_k = \frac{(u - L_k)(T_k^{-1}x)}{h_k}.$$

Then $u_k$ solves

$$\begin{cases}
\det(D^2u_k) = f_k & \text{in } S_k, \\
u_k = 0 & \text{on } \partial S_k,
\end{cases}$$

where

$$f_k = h_k^{-n}(\det T_k)^{-1} \frac{f(T_k^{-1}x)}{g(Du(T_k^{-1}x))}.$$

After a rotation of coordinates, we may assume $T_k(z_k)$ is on the $x_1$-axis.

**Linearity estimate.** Let

$$v_k(x) := u(x) - Du(x_k) \cdot (x - x_k) - u(x_k),$$

from (2.18) we have that $v_k(x_k + z_k) \geq \frac{1}{2}(1 - 1/k)v_k(z_k)$. Let

$$\tilde{L}_k(x) := L_k(x) - Du(x_k) \cdot (x - x_k) - u(x_k).$$

Then we have that $S_{h_k}(x_k) = \{v_k \leq \tilde{L}_k\}$. Since $S_{h_k}(x_k)$ is centered at $x_k$, $z_k \in \partial S_{h_k}(x_k)$, $v_k \geq 0$ and $\tilde{L}_k(x_k) = h_k$, by John’s Lemma we have that $0 \leq \tilde{L}_k(z_k) \leq 2nh_k$. Now,

$$(v_k - \tilde{L}_k)(\frac{x_k + z_k}{2}) - \frac{1}{2}(1 - \frac{1}{k}) \left( (v_k - \tilde{L}_k)(x_k) + (v_k - \tilde{L}_k)(z_k) \right)$$

$$\geq -\frac{1}{2k} \left( \tilde{L}_k(x_k) + \tilde{L}_k(z_k) \right)$$

$$\geq -\frac{3n}{2k} h_k.$$

Since $v_k - \tilde{L}_k = u - L_k$, from the above estimate and the definition of $u_k$ we have

$$u_k(\frac{1}{2}T_k z_k) \geq \frac{1}{2}(1 - \frac{1}{k})(u_k(0) + u_k(T_k z_k)) - \frac{3n}{2k}. \quad (2.20)$$

**Limiting problem.** Now, by convexity we may take limits

$$S_k \to S_\infty, \quad u_k \to u_\infty.$$

Let $f_\infty$ be the weak limit of $f_k$. Then $u_\infty$ satisfies $\det(D^2u_\infty) = f_\infty$ in Aleksandrov sense. Let $z_\infty := \lim_{k \to \infty} T_k(z_k)$. By (2.20) we have

$$u_\infty = L \text{ on the segment connecting } 0 \text{ and } z_\infty, \quad (2.21)$$
where $L$ is a supporting function of $u_\infty$ at 0.

**Step 2.** In this step, we need to consider two situations:

a) $x_\infty \in \partial C_i$ for some $1 \leq i \leq k$;

b) $x_\infty \in \partial U$.

In each case, a contradiction is made at some carefully chosen extreme point (denote by $y$) of $\{u_\infty = L\}$. Heuristically, we can choose a section of $u_\infty$ (denote by $S$) around $y$ such that $y$ is much more closer to $\partial S$ in one direction than in the opposite direction. Hence, on one hand the Aleksandrov type estimate Lemma 2.6 (a) shows that $h$, the height of section $S$, should not be too small. On the other hand, Lemma 2.6 (b) shows that $h$ is very small, which is a contradiction.

We deal with case a) first.

**Proof of case a).**

Note that since $x_\infty \in \partial C_i$ for some $1 \leq i \leq k$ and $h_k \to 0$ as $k \to \infty$, by Lemma 2.10 we have that the support of $f_k$ can be represented by $S_k - A_k$ when $k$ is large, where $A_k$ is an open convex subset of $S_k$. Let the convex set $A_\infty$ be the limit of $A_k$. Then $S_\infty - A_\infty$ is the support of $f$. Since the centre of mass of $S_\infty$ is 0 and $0 \in S_\infty - A_\infty$, we have that the volume of $S_\infty - A_\infty$ is positive. Hence, it is easy to see that there exists a constant $C$ such that

$$C^{-1}\chi_{S_\infty - A_\infty} \leq f_\infty \leq C\chi_{S_\infty - A_\infty}.$$

Since $\overline{z_\infty x_\infty} \subset \bar{\Omega}$, we have $\overline{0z_\infty} \cap A_\infty = \emptyset$. Hence we need only to deal with the following two sub-cases:

sub-case 1, $\{u_\infty = L\}$ contains an interior point of $S_\infty - A_\infty$.

sub-case 2, $\{u_\infty = L\} \cap S_\infty \subset A_\infty$.

For sub-case 1, take $x_0 \in (S_\infty - A_\infty) \cap \{u = L\}$. Take $\delta$ sufficiently small such that $B_\delta(x_0) \subset S_\infty - A_\infty$.

**Choosing extreme point.** Let $y \in \{u = L\}$ be the point such that

1) $u_\infty(y) = \inf_{\{u = L\}} u_\infty$.

2) $y$ is an extreme point of the convex set $\{u_\infty = L\} \cap \{u_\infty = u(y)\}$.

It is easy to see that $y$ is an extreme point of $\{u_\infty = L\}$.

**Cutting a suitable section.** By rotating the coordinates we may assume that $\{u_\infty = L\} \subset \{x_1 \leq b\}$ for some constant $b > 0$, and that $\{u_\infty = L\} \cap \{x_1 = b\} = \{y\}$. Then we consider section $S = \{u_\infty < L + \varepsilon(x_1 - b + a)\}$, where we fix $a$ sufficiently small and then take $\varepsilon << a$ so that $S \subset S_\infty$, and that $a \gg d := \max\{x_1(x_1, 0, \cdots, 0) \in S\} - b$. 

Using Aleksandrov estimates to make contradiction. On one hand by Aleksandrov estimate we have

\[ |S|^2 > C \frac{a}{d} \varepsilon^n. \]  \hspace{1cm} (2.22)

On the other hand, we consider another section \( \tilde{S} = \{ u_\infty < L + C \varepsilon \} \). Since \( u \) is Lipschitz, it is easy to see that \( S \subset \tilde{S} \) provided \( C \) (independent of \( \varepsilon \)) is sufficiently large. By convexity we have

\[ |B_\varepsilon(x_0) \cap \tilde{S}| \geq C|\tilde{S}|, \quad \text{for some } C > 0. \]

We claim

\[ |S|^2 \leq C \varepsilon^n, \]  \hspace{1cm} (2.23)

where the constant \( C \) is independent of \( d \). The claim follows from the following argument. Let

\[ v = u_\infty - L - C \varepsilon, \quad G := \tilde{S} \cap B_\varepsilon(x_0). \]

By John’s lemma, there exists an affine transformation \( A \) with \( \det A = 1 \) such that

\[ B_\varepsilon \subset A(G) \subset n^2 B_\varepsilon. \]
for some $\bar{r}$. Now $\bar{v} = v(A^{-1}x)$ satisfies
\[ \det D^2\bar{v} = f_\infty(A^{-1}x) \geq C^{-1} \text{ in } A(G), \]
and $|v| \leq C\varepsilon$ in $A(G)$. Then we have
\[ C^{-1}|G| \leq \int_{\frac{1}{2}G} f_\infty = |\partial \bar{v}(A(\frac{1}{2}G))| \leq C\varepsilon^n. \tag{2.24} \]
(2.23) follows from (2.24) and the fact that $|\bar{S}| \approx |G| \approx \bar{r}^n$. Since $d << a$, it is easy to see that (2.23) contradicts to (2.22).

For sub-case 2, we need to choose the extreme point more carefully.

Choosing extreme point. Let $\tilde{K} \subset \mathbb{R}^n$ be a supporting plane of the convex set $A_\infty$ at 0. If $A_\infty$ is not $C^1$ at 0 we choose $\tilde{K}$ to be the one containing $\bar{z}_\infty$. Let $y'$ be the point where $u_\infty$ attains its minimum on
\[ D := \{u = L\} \cap \tilde{K} \cap S_\infty. \]
It is easy to check that $D$ is a convex set, and the set $D \cap \{x|u(x) = u(y')\}$ is also convex. Let $y$ be an extreme point of $D \cap \{x|u(x) = u(y')\}$. We claim that $y$ is an extreme point of $\{u = L\}$. Indeed, suppose not, then there exists $y_1, y_2 \in \{u = L\} \cap S_\infty \subset \bar{A}_\infty$ such that $y = \frac{y_1 + y_2}{2}$. By the facts that $\tilde{K}$ is a supporting plane of $A_\infty$ and $y \in \bar{A}_\infty$, we have that $y_1, y_2 \in D$. However, since $u(y) = \min\{u(x)|x \in D\}$, we have $y_1, y_2 \in D \cap \{x|u(x) = u(y')\}$ which contradicts to the choice of $y$ as an extreme point of $D \cap \{x|u(x) = u(y')\}$.

Cutting a suitable section. By subtracting $L$ and translating the coordinates we may assume that $y = 0$, that $u_\infty \geq 0$, that $u_\infty(te_1) = 0$ for $t \in (0,1)$, and that $u_\infty(te_1) > 0$ for $t < 0$. Let $0 < \varepsilon << a$ be small positive numbers. Let $S_h(\varepsilon e_1)$ be a section of $u_\infty$ with center $ae_1$, where $h$ is chosen such that $-\varepsilon e_1 \in \partial S_h(\varepsilon e_1)$. Since $y$ is an extreme point of $\{u = L\}$, we have that $S_h(\varepsilon e_1) \subset S_\infty$ provided $h$ is sufficiently small. Note that $h \to 0$ as $\varepsilon \to 0$.

Using Aleksandrov estimates to make contradiction. Since $A_\infty$ is convex, it is easy to see that
\[ \int_{S_h(\varepsilon e_1)} f_\infty \leq C \int_{\frac{1}{2}S_h(\varepsilon e_1)} f_\infty \]
for some constant $C$. Let $L_1$ be the defining function of the section $S_h(\varepsilon e_1)$, it is obviously decreasing in $e_1$ direction. Hence $(L_1 - u_\infty)(0) \geq h$. Then, by Lemma 2.6 we also have
\[ (L_1 - u_\infty)(0) \leq C\left(\frac{\varepsilon}{a}\right)^{\frac{1}{2}}h, \]
which contradicts to the previous estimate.

\[
\text{Proof of case b).}
\]

The proof of case b) follows from a similar argument of the proof of [24, Lemma 4], we sketch the argument here. Note that now \( f_k \) is supported in a convex domain \( D_k \subset \overline{S_k} \). Let

\[
D_\infty := \lim_{k \to \infty} D_k.
\]

We have \( z_\infty \in D_\infty \). Let \( L \) be the supporting function of \( u_\infty \) at 0 such that \( \overline{0z_\infty} \subset \{ u_\infty = L \} \). Similar to the proof of sub-case 1 of case a), let \( y \in \{ u_\infty = L \} \) be the point such that

1) \( u_\infty(y) = \inf_{\{ u_\infty = L \}} u_\infty \).

2) \( y \) is an extreme point of the convex set \( \{ u_\infty = L \} \cap \{ u_\infty = u(y) \} \).

It is easy to see that \( y \) is an extreme point of \( \{ u_\infty = L \} \). Observe that \( y \in D_\infty \), since otherwise \( u_k \) has positive Monge-Ampère measure outside \( D_k \) for large \( k \).

Denote

\[
z = (1 - \sigma)y + \sigma z_\infty
\]

for some small positive \( \sigma \). We may also find a section satisfying

\[
S_h(z) := \{ u_\infty < L \} \subset \subset S_\infty
\]

and \( y + \varepsilon \frac{y - z_\infty}{|y - z_\infty|} \in \partial S_h(z) \) for small \( \varepsilon << \sigma \). Since \( y \in D_\infty \), there exists a sequence \( y_k \in D_k \) such that \( y_k \to y \) as \( k \to \infty \).

Let

\[
\tilde{z}_k := (1 - \sigma)y_k + \sigma T(z_k),
\]

it is easy to see that \( \tilde{z}_k \to z \) as \( k \to \infty \). Recall that \( z_\infty := \lim_{k \to \infty} T(z_k) \) with \( T(z_k) \in D_k \). Let \( \tilde{S}_k := \{ u_k \leq L \} \) be a section of \( u_k \) centered at \( \tilde{z}_k \) with height \( h \). Then passing to a subsequence \( \tilde{S}_k \to S_h(z) \) in Hausdorff distance. In particular, \( \tilde{S}_k \subset \subset S_k \) provided \( k \) is sufficiently large. Then by Lemma 2.6, we have that

\[
Ch \leq (L_k - u_k)(y_k) \leq \left( \frac{\varepsilon}{\sigma} \right)^{1/2} h
\]

for large \( k \), which is a contradiction because \( \varepsilon << \sigma \).

\[
\text{Theorem 2.1 (ii) follows from the above discussions.}
\]
CHAPTER 2. REGULARITY OF POTENTIAL FUNCTIONS
Chapter 3

Degenerate Monge-Ampère type equations

3.1 Main result

In this Chapter we are concerned with evaluation of the global $C^{1,1}$ estimates for the $c$-convex generalized solutions $u$ to the Dirichlet problem for a degenerate Monge-Ampère type equation of the form

\[
\det[D^2u - A(x, Du)] = f(x, Du) \text{ in } \Omega, \tag{3.1}
\]

\[
u = \varphi \text{ on } \partial \Omega, \tag{3.2}
\]

where $\Omega$ is a bounded domain in the Euclidean space $\mathbb{R}^n$, $f \geq 0$ and $\varphi$ are given functions. We also assume that the matrix $A$ and function $f$ are given by

\[
A(x, Du) = D^2_x c(x, T(x)),
\]

\[
f(x, Du) = |\det\{D^2_{xy} c\}| \frac{\rho}{\rho^* \circ T},
\]

where $\rho \geq 0$, $\rho^* > 0$ are mass distributions respectively in the initial domain $\Omega$ and the target domain $\Omega^* \subset T(\Omega)$ satisfying the mass balance condition

\[
\int_{\Omega} \rho(x) dx = \int_{\Omega^*} \rho^*(y) dy,
\]

c$(\cdot, \cdot)$ is a $C^\infty$-smooth cost function on $\mathbb{R}^n \times \mathbb{R}^n$ (that is, the expense of moving a point $x \in \Omega$ to a point $y \in \Omega^*$) and $T : x \to y$ is the optimal mapping determined by

\[
Du(x) = D_x c(x, y). \tag{3.3}
\]
We assume that the cost function \( c \) satisfies the following conditions:

(A1) For any \( x, p \in \mathbb{R}^n \), there exist a unique \( y = y(x, p) \in \mathbb{R}^n \) such that \( D_x c(x, y) = p \) and for any \( y, q \in \mathbb{R}^n \), there exist a unique \( x = x(y, q) \in \mathbb{R}^n \) such that \( D_y c(x, y) = q \).

(A2) For any \( x, p \in \mathbb{R}^n \), \( \det \{ c_{i,j}(x, y) \} \neq 0 \), where subscripts of \( c \) before the comma mean derivatives in \( x \), after the comma mean derivatives in \( y \).

(A3w) For any \( x, p \in \mathbb{R}^n \),

\[
D_{p,p_i} A_{kl}(x, p) \zeta_i \zeta_j \eta_k \eta_l \geq 0, \quad \forall \zeta \perp \eta \in \mathbb{R}^n.
\]

Uniqueness of the optimal mapping \( T \) now is guaranteed by condition (A1).

For later application we notice here that the function \( f \) can be written in the form

\[
f(x, Du) = q(x) \xi(x, Du), \quad q \geq 0, \quad \xi > 0.
\]

(3.4)

The main statement of this Chapter is as follows.

**Theorem 3.1.** Let \( \Omega \) be a uniformly \( c \)-convex domain with respect to \( T_u(\Omega) \) with boundary \( \partial \Omega \in C^{3,1} \), \( \varphi \in C^{3,1}(\overline{\Omega}) \) and let \( c \) be \( C^\infty \)-smooth cost function satisfying conditions (A1)-(A3w). Assume \( f \) is given by (3.4) and \( q^{1/(n-1)} \in C^{1,1}(\overline{\Omega}) \), \( \xi^{1/(n-1)} \in C^{1,1}(\overline{\Omega} \times \mathbb{R}^n) \). Then a generalized solution \( u \) of the Dirichlet problem (3.1),(3.2) satisfies the estimate

\[
||u||_{C^{1,1}(\overline{\Omega})} \leq C,
\]

(3.5)

where \( C \) is a constant depending on \( \Omega, |\varphi|_{3,1}, |q^{1/(n-1)}|_{1,1}, |\log \xi|_{1,1}, \) the cost function \( c \) up to its fifth order derivatives and the lower bound for \( |\det D_{x,y} c| \).

**Remark 3.2.** The uniform \( c \)-convexity of \( \Omega \) with respect to \( T_u(\Omega) \) can be easily verified using a subsolution. Indeed, let \( w \) be a subsolution to (3.1), (3.2). If \( \Omega \) is uniformly \( c \)-convex with respect to \( T_w(\Omega) \), where \( T_w \) is the optimal mapping corresponding to the function \( w \), then it is \( c \)-convex with respect to \( T_u(\Omega) \).

**Remark 3.3.** Note that Theorem 3.1 is optimal. Indeed, an example by Loeper [120] shows that if (A3w) is violated somewhere in \( \Omega \times \Omega \), then there exists a \( c \)-convex solution to equation (3.1) which is not \( C^1 \)-smooth. An example by Wang [176] shows that \( C^{1,1} \)-regularity for the case \( c(x, y) = \frac{1}{2}|x - y|^{1/2} \) is false if the function \( f^{1/(n-1)} \notin C^{1,1}(\Omega) \). Assumption \( \varphi \in C^{3,1}(\overline{\Omega}) \) is also sharp for global regularity [33]. By considering a simple example (see [71]) we also conclude that \( C^{1,1} \)-regularity is the best possible result for the degenerate case (see Chapter 1 for more details).
Corollary 3.4. Let, in addition to conditions from Theorem 3.1, there exists a subsolution \( w \) to the Dirichlet problem (3.1),(3.2) and \( \Omega \) is a uniformly \( c \)-convex domain with respect to \( T_w(\Omega) \). Then there exists a unique \( c \)-convex generalized solution \( u \in C^{1,1}(\Omega) \) of the Dirichlet problem (3.1),(3.2).

In this Chapter we use techniques proposed in [71], where the Dirichlet problem of the standard Monge-Ampère equation was studied. The argument in [71] involves a delicate evaluation of the upper and lower bounds of second derivatives on \( \partial \Omega \). Our equation (3.1), which arises in optimal transportation, contains the inhomogeneous term \( A \) and the right hand side \( f \) depends on \( Du \), which makes the argument more complicated.

The results of this chapter are presented in [7].

### 3.2 Global second-derivative bounds

Let us make the following transforms:

\[
\begin{align*}
  u(x) &\rightarrow [u(x) - u(x_0)] - [c(x, y_0) - c(x_0, y_0)]; \\
  c(x, y) &\rightarrow [c(x, y) - c(x_0, y)] - [c(x, y_0) - c(x_0, y_0)]; \\
  \varphi(x) &\rightarrow [\varphi(x) - u(x_0)] - [c(x, y_0) - c(x_0, y_0)];
\end{align*}
\]

where \( x_0 \in \bar{\Omega} \) and \( y_0 = T(x_0) \). Then the cost function satisfies [117]

\[
c(x, y_0) \equiv 0, \forall x \in \Omega, \quad c(x_0, y) \equiv 0, \forall y \in T(\Omega),
\]

and function \( u \) satisfies

\[
u(x_0) = 0, \quad Du(x_0) = 0, \quad u \geq 0 \text{ in } \Omega;
\]

Lemma 3.5. Let \( u \in C^4(\Omega) \cap C^2(\bar{\Omega}) \) satisfies equation (3.1) in \( \Omega \) with \( f > 0 \) in \( \Omega \), \( \tilde{f} := f^{1/(n-1)} \in C^{1,1}(\bar{\Omega} \times \mathbb{R}^n) \) and the cost function \( c \) satisfies (A1)-(A3w).

Then

\[
\sup_{\Omega} |D^2u| \leq C + \sup_{\partial \Omega} |D^2u|,
\]

where \( C \) is a constant depending only on \( n, \Omega, |\tilde{f}|_{1,1}, |u|_{1,\Omega} \) and \( c \) up to its fourth-order derivatives.

Proof. Our reduction to the boundary estimation follows the approach in [163], [89], [90].
Define a linearized operator
\[ L := \omega^{ij}(D_{ij} - D_{pl}A_{ij}D_{l}). \]
Let \( v \) be an auxiliary function defined by
\[ v(x) = \log(w_{kk}) + \tau |Du|^2 + \kappa \eta, \]
where \( \{w_{ij}\} := \{u_{ij} - A_{ij}\} \) and \( \eta = e^{(\bar{u} - u)} \) is a barrier of a linearized operator of equation (3.1).

**Remark 3.6.** The barrier function \( \eta \) is crucial for the derivation of the global second derivative bounds. Previously, to obtain the second derivative bounds, a kind of global barrier condition was assumed in [163]. Namely, it was assumed that there exists a function \( \varphi \in C^2(\bar{\Omega}) \) satisfying
\[ [D_{ij}\varphi - D_{pk}A_{ij}(:, Du)D_{k}\varphi]\xi_i\xi_j \geq |\xi|^2 \text{ in } \Omega, \]
for all \( \xi \in \mathbb{R}^n \). Later, it was shown in [91] that such a condition can be replaced by the existence of a subsolution. In particular, they used condition (A3w) on the matrix function \( A \) to construct a barrier function \( \eta = e^{(\bar{u} - u)} \) for the linearized operator from a subsolution \( \bar{u} \) and obtained the following barrier inequality
\[ L\eta \geq \delta \sum_{i=1}^{n} w_{ii} - C, \] (3.9)
where \( C \) is a positive constants depending on \( n, \Omega, \) cost function \( c, |u|_{1,\Omega}. \) Subsequently, there was proved the existence of such a function \( \bar{u} \in C^2(\bar{\Omega}) \) (see [89, Lemma 2.1], [89, Lemma 2.2] and [90, Lemma 2.1]), for which the crucial barrier inequality (3.9) holds. This inequality will be used to control the second order global estimates in this section.

Suppose that \( v \) attains its maximum at \( x_0. \) If \( x_0 \in \partial\Omega, \) we are through. So to prove (4.10) we need only to consider the case \( x_0 \in \Omega. \) By our transform (3.7) \( \{w_{ij}\} = \{u_{ij}\} \) at \( x_0 \) and rotating the coordinates we may assume that \( D^2u \) is a diagonal matrix at \( x_0. \) Let \( w_{11} = \max w_{ii}. \) If \( w_{11} \) is bounded we are done, therefore we may suppose that \( w_{11} \) is as large as we want.

Using (3.4) write equation (3.1) in the form
\[ \log \det(D^2w) = \log q(x) + \log \xi(x, Du). \]
Differentiation of this equation yields

\[ w^{ij} (u_{ijk} - A_{ijk} - D_{pk}A_{ij}u_{kl}) = \frac{q_k}{q} + \frac{\xi_k}{\xi} + \frac{\xi_{pm}}{\xi} u_{mk}; \quad (3.10) \]

\[ w^{ij} = \frac{q_{kk}}{q} - \frac{q_k^2}{q^2} + \sum_{p} \frac{\xi_{pkm}}{\xi} + \left( \frac{\xi_{ppm}}{\xi} - \frac{\xi_{pm}^2}{\xi^2} \right) u_{mk}u_{mk} \]

\[ + 2 \left( \frac{\xi_{kpm}}{\xi} - \frac{\xi_{kx}}{\xi} \right) u_{mk} + \frac{\xi_{pm}}{\xi} u_{mkk} + w^i w^{jm} D_k w_{ij} D_k w_{lm}, \]

where \( \{w^{ij}\} \) is the inverse of \( \{w_{ij}\} \), which is also diagonal at \( x_0 \). Note that a repeated index always implies a summation.

Set

\[ \mathcal{L} := L - \frac{\xi_{pm}}{\xi} D_m. \]

Then using (3.11) and condition (A3w) we estimate at \( x_0 \)

\[ \mathcal{L} u_{kk} = w^{ij} (u_{ijk} - D_{pk}A_{ij}u_{kk}) - \frac{\xi_{pm}}{\xi} u_{mkk} \]

\[ \geq \left( \frac{q_{kk}}{q} - \frac{n - 2 q_k^2}{n - 1 q^2} \right) - \frac{1}{n - 1} q_k^2 + \left( \frac{\xi_{kk}}{\xi} - \frac{n - 2 \xi_k^2}{n - 1 \xi^2} \right) - \frac{1}{n - 1} \xi_k^2 \]

\[ + \left( \frac{\xi_{ppm}}{\xi} - \frac{n - 2 \xi_{pm}^2}{n - 1 \xi^2} \right) u_{mk}u_{mk} - \frac{1}{n - 1} \frac{\xi_{pm} \xi_{pm}}{\xi} u_{mk}u_{mk} \]

\[ + 2 \left( \frac{\xi_{kpm}}{\xi} - \frac{\xi_{kx}}{\xi} \right) u_{mk} - \frac{2}{n - 1} \frac{\xi_{pm} \xi_{pm}}{\xi} u_{mkk} \]

\[ - C \sum_{i,j=1}^n (w^{ii} (1 + w_{jj}) + w^{ii}) + w^i w^{jm} D_k w_{ij} D_k w_{lm}. \]

Since \( \tilde{f}(x, Du) = f^{1/(n-1)}(x, Du) \in C^{1,1}(\mathbb{R} \times \mathbb{R}^n) \), we have

\[ \frac{q_k}{q} - \frac{n - 2 q_k^2}{n - 1 q^2} \geq -C q^{-1/(n-1)}, \quad \frac{\xi_{kk}}{\xi} - \frac{n - 2 \xi_k^2}{n - 1 \xi^2} \geq -C \xi^{-1/(n-1)}; \]

\[ \frac{\xi_{ppm}}{\xi} - \frac{n - 2 \xi_{pm}^2}{n - 1 \xi^2} \geq -C \xi^{-1/(n-1)}, \quad \frac{\xi_{kpm}}{\xi} - \frac{n - 2 \xi_{km} \xi_{pm}}{\xi} \geq -C \xi^{-1/(n-1)}; \]

\[ \left| \frac{\partial q^{1/(n-1)}}{\partial x_i} \right| \leq C q^{1/2(n-1)}, \quad \left| \frac{\partial \xi^{1/(n-1)}}{\partial x_j} \right| \leq C \xi^{1/2(n-1)}, \quad \left| \frac{\partial \xi^{1/(n-1)}}{\partial p_m} \right| \leq C \xi^{1/2(n-1)}, \]

where \( i < n; j, m \leq n \) and hence also

\[ -\frac{1}{n - 1} q_k^2 \geq -C q^{-1/(n-1)}, \quad -\frac{1}{n - 1} \xi_k^2 \geq -C \xi^{-1/(n-1)}; \]

\[ -\frac{1}{n - 1} \xi_{pm}^2 \geq -C \xi^{-1/(n-1)}, \quad -\frac{1}{n - 1} \xi_{km} \xi_{pm} \geq -C \xi^{-1/(n-1)}, \]

\[ -\frac{1}{n - 1} \xi_{pm} \xi_{pm} \geq -C \xi^{-1/(n-1)}, \]

\[ -\frac{1}{n - 1} \xi_{km} \xi_{pm} \geq -C \xi^{-1/(n-1)}, \]

\[ -\frac{1}{n - 1} \xi_{km} \xi_{pm} \geq -C \xi^{-1/(n-1)}, \]
where $C$ depends on $|\tilde{f}|_{1,1}$ and $\Omega$. Recalling that
\[
\sum_{i=1}^{n} w_{ii} = \sum_{i=1}^{n} \frac{1}{w_{ii}} \geq \left[ \frac{1}{\prod_{i=2}^{n} w_{ii}} \right]^{1/(n-1)} \geq f^{-1/(n-1)} = (q\xi)^{-1/(n-1)},
\]
and using boundedness of $q, \xi$ from above, we obtain
\[
\mathcal{L}u_{kk} \geq w^{i}w^{jm}D_{k}w_{ij}D_{k}w_{lm} - C \sum_{i,j=1}^{n} (w^{ii}(1 + w_{jj}) + w_{ii}).
\]

Let us now estimate $\mathcal{L}A_{kk}$. We can easily estimate
\[
w^{ij} D_{p_{m}}A_{kk}u_{il}u_{jm} = w^{ij} D_{p_{m}}A_{kk}(w_{il} + A_{il})(w_{jm} + A_{jm}) \leq C \sum_{i,j=1}^{n} (w^{ii}(1 + w_{jj}) + w_{ii}).
\]

Therefore, by (3.10), (3.13) and (3.14) we obtain
\[
\mathcal{L}A_{kk} = w^{ij}(A_{kkj} + 2D_{p_{s}}A_{kkj}u_{sj} + D_{p_{s}}A_{kkj}u_{ij} + D_{p_{s}}A_{kkj}u_{ii}u_{jm}) - w^{ij}(D_{p_{s}}A_{ij}A_{kk} - D_{p_{s}}A_{ij}D_{p_{s}}A_{kk}u_{st}) - \frac{\xi_{pm}}{\xi}(A_{kkm} + D_{p_{s}}A_{kk}u_{sm}) \leq w^{ij}D_{p_{s}}A_{kk}u_{ij} + C \sum_{i,j=1}^{n} \left( w^{ii}(1 + w_{jj}) + w_{ii} + \xi - \frac{1}{\xi^{1/(n-1)}}(1 + w_{jj}) \right) \leq C \sum_{i,j=1}^{n} (w^{ii}(1 + w_{jj}) + w_{ii}) + C \sqrt{\sum_{i=1}^{n} w^{ii} \left( 1 + \sum_{j=1}^{n} w_{jj} \right)}.
\]

Since $\mathcal{L}$ is linear, combining (3.15), (3.16) we then have
\[
\mathcal{L}w_{kk} \geq w^{il}w^{jm}D_{k}w_{ij}D_{k}w_{lm} - C \sum_{i,j=1}^{n} (w^{ii}(1 + w_{jj}) + w_{ii}) - C \sqrt{\sum_{i=1}^{n} w^{ii} \left( 1 + \sum_{j=1}^{n} w_{jj} \right)}.
\]

Since $v$ attains its maximum at $x_{0}$, differentiating function $v$ we get
\[
0 = v_{i}(x_{0}) = \frac{D_{i}w_{kk}}{w_{kk}} + 2\tau u_{il} + \kappa D_{i}\eta,
\]
\[
0 \geq v_{ij}(x_{0}) = \frac{D_{ij}w_{kk}}{w_{kk}} - \frac{D_{i}w_{kk}D_{j}w_{kk}}{w_{kk}^{2}} + 2\tau(u_{il}u_{jl} + u_{il}u_{jl}) + \kappa D_{ij}\eta.
\]

Assume $k = 1$. Then since $w_{11} = \max w_{ii}$ and $\{w_{ij}\}$ is diagonal at $x_{0}$, we have
3.3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

(see (3.12) in [163])

\[
\begin{align*}
\frac{1}{w_{kk}} w^{im} D_k w_{ij} D_k w_{lm} - \frac{1}{w_{kk}^2} w^{ij} D_i w_{kk} D_j w_{kk} &= \frac{1}{w_{11}^2} w^{ii} (D_i w_{11})^2 - \frac{1}{w_{11}^2} w^{ii} (D_i w_{11})^2 \\
&\geq \frac{1}{w_{11}^2} \sum_{i>1} (2w^{ii}(D_i w_{11})^2 - w^{ii}(D_i w_{11})^2) \\
&= \frac{1}{w_{11}^2} \sum_{i>1} w^{ii}(D_i w_{11})^2 \\
&\quad + \frac{2}{w_{11}^2} \sum_{i>1} w^{ii}(D_i w_{11})(D_i w_{11} + D_i w_{11}) \\
&\geq \frac{1}{w_{11}^2} \sum_{i>1} w^{ii}(D_i w_{11})^2 \\
&\quad + \frac{2}{w_{11}^2} \sum_{i>1} w^{ii}(D_i w_{11})(D_i w_{11} + D_i w_{11}) \\
&\geq -C \sum_{i=1}^n w^{ii}.
\end{align*}
\]

Using this inequality and (3.9) we obtain

\[
0 \geq L v(x_0) \geq 2\tau \sum_{i=1}^n w_{ii} + \kappa \sum_{i=1}^n w^{ii} - C \frac{1}{w_{11}} \sum_{i,j=1}^n (\tau w^{ii}(1 + w_{jj}) + w_{ii}) \\
- C \sqrt{\sum_{i=1}^n w^{ii} \left( \tau + \kappa + \frac{1}{w_{11}} \sum_{j=1}^n w_{jj} \right)}.
\]

Hence, since \( w_{11} \) is large enough, we get the estimate,

\[
0 \geq L v(x_0) \geq 2\tau \sum_{i=1}^n w_{ii} + \kappa \sum_{i=1}^n w^{ii} - C \tau \sum_{i=1}^n w^{ii} - Cn - C \sqrt{\sum_{i=1}^n w^{ii} (\tau + \kappa + n)},
\]

which gives us a contradiction for \( \kappa \) sufficiently large.

\[
\square
\]

3.3 Existence and uniqueness of the solution

This section is devoted to the proof of Corollary 3.4.
Assume that \( u_{ij}(x_0) = 0 \) for \( i \neq j, \ i, j = 1, ..., n - 1 \). Then, by our transform (3.7), we can write equation (3.1) at the boundary point \( x_0 \) in the form
\[
\prod_{i=1}^{n-1} u_{ii}(x_0) u_{nn} = \sum_{j=1}^{n-1} \left( \prod_{i \neq j} u_{ii}(x_0) \right) u_{jn}^2(x_0) + f(x_0, Du(x_0)),
\]
so that
\[
0 \leq u_{nn}(x_0) = \sum_{j=1}^{n-1} \frac{u_{jn}^2(x_0)}{\prod_{i=1}^{n-1} u_{ii}(x_0)} + f(x_0, Du(x_0)).
\]
Hence in order to estimate second derivatives on the boundary we need to get:
\[
\begin{align*}
&u_{ii}(x_0) \leq C, \quad i < n; \quad (3.19) \\
&\prod_{i=1}^{n-1} u_{ii}(x_0) \geq Cf(x_0, Du(x_0)); \quad (3.20) \\
&u_{jn}^2(x_0) \leq Cu_{jj}(x_0), \quad j = 1, ..., n - 1; \quad (3.21)
\end{align*}
\]
where all constants are independent on the lower bound of \( f \).

Once the above estimates are obtained, we get the global second derivative estimate
\[
|D^2u| \leq C,
\]
where \( C \) is independent on the lower bound of \( f \). Estimate (3.5) is then established for non-degenerate case \( f > 0 \) on \( \bar{\Omega} \). Next, we need to solve approximating Dirichlet problem
\[
\det[D^2u - A(x, Du)] = tf_\varepsilon := f + \varepsilon \text{ in } \Omega, \\
u = \varphi \text{ on } \partial \Omega,
\]
for \( \varepsilon > 0 \), using continuity method.

The continuity method used in the proof is well-known (for example see [30], [77]), but for the reader’s convenience we will recall it here.

Let \( u^0 \in C^{4,\alpha}(\bar{\Omega}) \) be a uniformly convex function satisfying
\[
\det[D^2u^0 - A(x, Du^0)] = f_\varepsilon \text{ in } \Omega, \\
u^0 = \varphi \text{ on } \partial \Omega.
\]
Set \( f^0 = \det(u_{ij}^0 - A_{ij}) \). Then \( f^0 \geq f_\varepsilon \text{ in } \Omega \).

For each \( t \in [0, 1] \), we want to find a uniformly convex solution \( u^t \in C^{2,\alpha}(\bar{\Omega}) \) of
\[
\det[D^2u^t - A(x, Du^t)] = tf_\varepsilon + (1 - t)f^0 \text{ in } \Omega, \\
u^t = \varphi \text{ on } \partial \Omega.
\]
We set
\[ I = \{ t \in [0, 1] : (3.25) \text{ has a uniformly convex solution } u^t \in C^{2,\alpha}(\overline{\Omega}) \}. \]

It is easy to see that \( 0 \in I \) since \((3.25)\) for \( t = 0 \) has a solution \( u^0 \).

Now we claim that \( I \) is open. Set
\[ X = \{ w \in C^{2,\alpha}(\Omega) : \text{on } \partial \Omega \} \]
and
\[ G(w, t) = \det[(w + \varphi)_{ij} - A_{ij}(x, (w + \varphi)_i)] - tf_x - (1 - t)f^0. \]

Then, \( G : X \to C^\alpha(\Omega) \) is a \( C^1 \) map and its Fréchet derivative with respect to \( w \in X \) is given by
\[ G_w(w, t) = a_{ij} \frac{\partial}{\partial x^i} v, \]
where \( \{a_{ij}\} \) is the cofactor matrix of \( \{(w + \varphi)_{ij} - A_{ij}(x, (w + \varphi)_i)\} \). If \((w + \varphi)\) is uniformly convex with respect to the cost function and \( C^{2,\alpha}(\overline{\Omega})\)-smooth, \( G_w(w, t) \) is a uniformly elliptic linear operator with \( C^\alpha(\Omega) \) coefficients. By the Schauder theory, \( G_w(w, t) : X \to C^\alpha(\Omega) \) is an invertible operator. By writing \( u^0 = w^0 + \varphi \) for some \( w^0 \in X \), we have \( G_w(w^0, 0) = 0 \). By the implicit function theorem, for any \( t \) close to 0, there is a unique \( w^t \in X \), close to \( w^0 \) in the \( C^{2,\alpha}\)-norm and satisfying \( G(w, t) = 0 \). Obviously, \( w^t + \varphi \) is uniformly convex for \( t \) close to 0. Then, \( u^t = w^t + \varphi \) is the desired solution of \((3.25)\). Hence, \( t \in I \) for all such \( t \), and therefore \( I \) is open.

Next, we claim that
\[ |u^t|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \quad (3.26) \]
where \( C \) is independent of \( t \). Indeed, the estimate easily follows from the estimate \((3.22)\) and the following theorem (see [77]):

**Theorem 3.7.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a \( C^3 \) boundary \( \partial \Omega \) and \( F \) be a concave \( C^{2} \) function satisfying
\[ \lambda |\xi|^2 \leq F_{r_{ij}}(x, u, Du, D^2 u) \xi_i \xi_j \leq \Lambda |\xi|^2, \]
\[ |F_p|, |F_z|, |F_x(x, u, Du, D^2 u)| \leq \lambda \mu, \]
defined on the space of \( n \times n \) symmetric matrices. Suppose that \( u \) is a \( C^3(\overline{\Omega}) \cap C^4(\Omega) \) solution of
\[ F(x, u, Du, D^2 u) = 0 \text{ in } \Omega, \]
\[ u = \varphi \text{ on } \partial \Omega, \]
for $\varphi \in C^3(\overline{\Omega})$. Then, for some constant $\alpha \in (0,1)$ depending only on $n$, $\lambda$ and $\Lambda$,

$$|D^2u|_{C^\alpha(\overline{\Omega})} \leq C(1+\mu)(1+|u|_{C^2(\overline{\Omega})}) + |\varphi|_{C^0(\overline{\Omega})},$$

where $C$ is a positive constant depending only on $n$, $\lambda$, $\Lambda$ and $\Omega$.

Then, it follows that $I$ is also closed, by the Arzela-Ascoli theorem, and therefore, $I$ is the whole unit interval. The function $u^1$ is then our desired solution of (3.23), (3.24).

Therefore, the full existence result of Corollary 3.4 for $f \geq 0$ can be deduced by sending $\varepsilon$ to zero. The uniqueness assertion follows immediately by the maximum principle.

The following sections will be dedicated to the derivation of estimates (3.19), (3.20), (3.21). The first and second inequalities will be established in Sect. 3.4 while the third one we derive in two steps. The mixed tangential-normal derivative estimate in terms of the largest double tangential derivative will be found in Sect. 3.5. Then using this result we prove the inequality (3.21) by the induction method and affine transformations.

### 3.4 Tangential derivatives

Consider any boundary point $x_0$; by a translation of coordinates we may suppose $x_0 = 0$ and $y_0 = 0$. Without loss of generality we may suppose that $x_0 = 0$ and $y_0 = 0$. Moreover, by making the coordinate transform

$$x \to D_y c(x,0), \quad y \to D_x c(0,y), \quad (3.27)$$

the domain $\Omega$ becomes uniformly convex and equation (3.1) becomes

$$\det[u_{ij} - D_{p_k p_l}\overline{A}_{ij} u_k u_l] = \overline{f}(x, Du); \quad (3.28)$$

where $\overline{A}$ still satisfies condition (A1)-(A3w) and

$$\overline{f}(x, Du) = \frac{1}{|\det D_{xy} c|^2} f(x, Du),$$

(see [115]). For simplicity, we will write $A$, $f$ instead of $\overline{A}$, $\overline{f}$ keeping in mind that $f$ depends on the lower bound of $|\det D_{xy} c|$.

Let the $x_n$-axis be the inner normal of the boundary $\partial \Omega$ at 0. Then in a small neighbourhood $N$ of the origin, $\partial \Omega$ is represented by

$$x_n = \varrho(x') = \frac{1}{2}|x'|^2 + \text{cubic of } x' + O(|x'|^4) \quad (3.29)$$
as \( x' \rightarrow 0 \) where \( x' = (x_1, \ldots, x_{n-1}) \). By Taylor expansion,

\[
    u = \varphi = \frac{1}{2} \sum_{i=1}^{n-1} b_i x_i^2 + R(x') + O(|x'|^4),
\]

(3.30)

where \( b_i = u_{ii}(0) > 0 \) for \( i = 1, \ldots, n-1 \) and \( R(x') \) denotes the cubic term. We may further suppose

\[
    0 < b_1 \leq \ldots \leq b_{n-1}.
\]

On the \( \partial \Omega \) we have \( u - \varphi = 0 \), so also

\[
    (\partial_i + g_i \partial_n)(u - \varphi) = 0,
\]

and

\[
    (\partial_i + g_i \partial_n)(\partial_j + g_j \partial_n)(u - \varphi) = 0, \quad i, j = 1, 2, \ldots, n-1.
\]

In particular, \( \nabla \varphi = 0 \) at the origin, so that

\[
    b_i \leq C, \quad i = 1, \ldots, n-1,
\]

(3.31)

where \( C \) depends on \( \Omega, |\varphi|_{1,1} \) and \( \sup |Du| \). An upper bound for the tangential second derivatives is, thus, established while the lower bound is given by the following lemma.

**Lemma 3.8.** There exists a positive constant \( C \) depending on \( \Omega, |\varphi|_{1,1}, |\tilde{f}|_{1,1}, \) the cost function \( c \) up to its fourth-order derivatives and the lower bound for \( |\det D_x^2 c| \) such that

\[
    \prod_{i=1}^{n-1} b_i \geq C f(0, Du(0)).
\]

(3.32)

**Proof.** Arguments in this paper are built upon an affine transformation \( S : x \rightarrow \tilde{x} \).

Let \( M = 1/b_1 \), define new functions

\[
    v(\tilde{x}) = M^2 u(x), \quad \tilde{c}(\tilde{x}, \tilde{y}) = M^2 c(x, y).
\]

where

\[
    \begin{align*}
    \tilde{x}_i &= M x_i, \quad i = 1, \ldots, n, \\
    \tilde{y}_i &= M^2 y_i / M_i, \quad i = 1, \ldots, n,
    \end{align*}
\]

(3.33)

and

\[
    \begin{align*}
    M_i &= b_i^{1/2} M, \quad i = 1, \ldots, n-1, \\
    M_n &= M = 1/b_1.
    \end{align*}
\]

(3.34)
Define also
\[ g(\tilde{x}, Dv) = \frac{M^{2n}}{(M_1 \cdots M_n)^2} f \left( \frac{\tilde{x}_i}{M_i} \frac{M_i v_i}{M^2} \right). \]

Then function \( v \) satisfies equation
\[ \det \left[ v_{ij} - \frac{1}{M^2} \frac{M_k M_l}{M_i M_j} D_{p_k p_l} A_{ij} v_k v_l \right] = g(\tilde{x}, Dv) \text{ in } \tilde{\Omega}, \quad (3.35) \]
where \( \tilde{\Omega} = S(\Omega) \).

The first step is to check that coefficients
\[ \left[ \frac{1}{M^2} \frac{M_k M_l}{M_i M_j} D_{p_k p_l} A_{ij} \right] \quad (3.36) \]
are bounded. Indeed, by (3.31)
\[ \frac{1}{M^2} \frac{M_k M_l}{M_i M_j} \leq \frac{1}{M^2} \max\{M_{n-1}^2, M_n^2\} = b_1 \max\{b_{n-1}, 1\} \leq C b_1. \quad (3.37) \]

Then, since the cost function \( c \) is smooth enough, coefficients (3.36) are bounded in terms of \( \Omega \), \( |\varphi|_{1,1} \) and the cost function \( c \) up to its fourth-order derivatives.

The second step is to show that \( \partial \tilde{\Omega} \) and \( \tilde{\varphi} \) are \( C^{3,1} \)-smooth independently of the lower bound of \( f \). From (4.13) the boundary \( \partial \tilde{\Omega} \) can be represented in a neighbourhood \( \tilde{N}(=S(N)) \) of the origin by
\[ \tilde{x}_n = \tilde{\varphi}(\tilde{x}') = \frac{1}{2} \tilde{x}_1^2 + \frac{1}{2} \sum_{i=2}^{n-1} \frac{b_1}{b_i} \tilde{x}_i^2 + O(|\tilde{x}'|^3), \quad (3.38) \]
and
\[ v = \tilde{\varphi} = \sum_{i=1}^{n-1} \tilde{x}_i^2 + R(\tilde{x}') + O(|\tilde{x}'|^4), \quad (3.39) \]
where \( R(\tilde{x}') \) is the cubic term.

One can see that all third- and fourth-order derivatives of \( \tilde{\varphi} \) can be easily controlled independently of the lower bound of tangential derivative \( b_1 \) by corresponding derivatives of \( \varphi \). The fourth-order derivatives of \( \tilde{\varphi} \) can also be controlled by derivatives of \( \varphi \), namely,
\[ |D^4 \tilde{\varphi}(\tilde{x})| \leq \frac{b_1^2}{b_i^4} |D^4 \varphi(x)| \leq C, \]
where \( C \) is independent of \( b_1 \) since \( b_1/b_i \leq 1 \). Therefore, in (3.39), the third term on the right hand side
\[ O(|\tilde{x}'|^4) \leq C(|\tilde{x}'|^4) \]
3.4. TANGENTIAL DERIVATIVES

for some constant C depending only on \( \partial \Omega, |\varphi|_{3,1} \) and the cost function \( c \) up to its fourth-order derivatives. Since coefficients of both second- and fourth-order terms in (3.39) are uniformly bounded and \( v \geq 0 \) we conclude that coefficients of the cubic term are uniformly bounded. Hence near the origin both \( \partial \tilde{\Omega} \) and \( \tilde{\varphi} \) are \( C^{3,1} \)-smooth independently of the lower bound of \( b_1 \).

By (3.38) one can see that the boundary of \( \Omega \) expands in directions of \( x_2, \ldots, x_{n-1} \) axes and remains uniformly convex in the direction of \( x_1 \)-axis near the origin. Therefore the domain

\[
\omega := \{ \tilde{x} \in \tilde{\Omega} \mid \tilde{x}_n < 1, |\tilde{x}_i| < 1, \ i = 2, \ldots, n - 1 \} \subset \tilde{N}
\]

is bounded in terms of \( \partial \tilde{\Omega} \) and \( |\varphi|_{3,1} \).

Now by coordinate transforms (3.7), (3.27) and Taylor expansion we may write cost function \( c \) near the origin in the form (see [117])

\[
c(x, y) = x \cdot y + \sum_{s,t,l,m} D_{p,s;rt} A_{ml} x_m x_l y_s y_t + O(|x|^\alpha |y|^\beta),
\]

and hence by (3.37)

\[
\tilde{c}(\tilde{x}, \tilde{y}) = \tilde{x} \cdot \tilde{y} + \sum_{s,t,l,m} \frac{1}{M^2} M_s M_t D_{p,s;rt} A_{ml} \tilde{x}_m \tilde{x}_l \tilde{y}_s \tilde{y}_t + O(b_1 |\tilde{x}|^\alpha |\tilde{y}|^\beta) (3.41)
\]

\[
\leq \tilde{x} \cdot \tilde{y} + C b_1 \sum_{s,t,l,m} D_{p,s;rt} A_{ml} \tilde{x}_m \tilde{x}_l \tilde{y}_s \tilde{y}_t + O(b_1 |\tilde{x}|^\alpha |\tilde{y}|^\beta),
\]

where \( \alpha, \beta \geq 2, \alpha + \beta \geq 5, \gamma \geq 3/2 \). Since \( b_1 \) and \( \omega \) are bounded and \( c \) is smooth enough, function \( v(\tilde{x}) + C \tilde{x}^2 \) is convex for some constant \( C \) depending on \( \omega \), the upper bound of \( b_1 \) and the cost function \( c \) up to its fourth-order derivatives. Therefore, noticing that \( v = \tilde{\varphi} \leq c \) on \( \partial \omega \cap \partial \tilde{\Omega} \), we have

\[
v \leq C \text{ in } \omega,
\]

where \( C \) depends on \( \Omega, |\varphi|_{3,1} \) and the cost function \( c \) up to its fourth-order derivatives.

The last step is to prove that

\[
g(\tilde{x}, Dv) = \frac{f(\tilde{x}_i/M_i, D_i M_i/M^2)}{\Pi_{i=1}^{n-1} b_i} \leq C \text{ in } \omega,
\]

for some constant \( C \) depending on \( \Omega, |f|_{1,1}, |\varphi|_{3,1}, \) cost function \( c \) up to its fourth-order derivatives and the lower bound of \( |\det D^2_{xy} c| \).
Suppose by contradiction that \( \sup g \) is as large as we want. By (3.4) we may write \( g = \tilde{g}(\tilde{x}, \tilde{p}) \tilde{\xi}(\tilde{x}, \tilde{p}) \)

where

\[
\tilde{g}(\tilde{x}) = \frac{q(\tilde{x}/M_i)}{\prod_{i=1}^{n-1} b_i} \quad \text{and} \quad \tilde{\xi}(\tilde{x}, \tilde{p}) = \xi(\tilde{x}/M_i, \tilde{p}_i M_i/M^2).
\]

Noticing that

\[
M_i(\prod_{i=1}^{n-1} b_i)^{1/2(n-1)} \geq 1,
\]

and by (3.12) we can compute

\[
\begin{align*}
\frac{\partial}{\partial \tilde{x}_i} \tilde{g}^{1/2(n-1)} &\leq C \frac{\partial \tilde{g}^{1/(n-1)} / \partial \tilde{x}_i}{\tilde{g}^{1/2(n-1)}} \leq C \frac{\partial q^{1/(n-1)} / \partial \tilde{x}_i q^{-1/2(n-1)}}{M_i(\prod_{i=1}^{n-1} b_i)^{1/2(n-1)}} \leq C, \\
\frac{\partial}{\partial \tilde{x}_i} \tilde{\xi}^{1/2(n-1)} &\leq C \frac{\partial \tilde{\xi}^{1/(n-1)} / \partial \tilde{x}_i}{\tilde{\xi}^{1/2(n-1)}} \leq C \frac{\partial \xi^{1/(n-1)} / \partial \tilde{x}_i}{\xi^{1/2(n-1)}} \leq C \frac{b_1}{b_i^{1/2}}, \\
\frac{\partial}{\partial \tilde{p}_i} \tilde{\xi}^{1/2(n-1)} &\leq C \frac{\partial \tilde{\xi}^{1/(n-1)} / \partial \tilde{p}_i}{\tilde{\xi}^{1/2(n-1)}} \leq C \frac{M_i \partial \xi^{1/(n-1)} / \partial \tilde{p}_i}{M^2 \xi^{1/2(n-1)}} \leq C b_1.
\end{align*}
\]

Hence if \( \sup g \) is large enough then so is \( \inf g \).

Since our aim is to establish tangential derivative estimates in degenerate case we may suppose that \( b_1 \) is as small as we want and by (3.37) coefficients (3.36) become very small. Therefore we may use

\[
\bar{v} = K(|x|^2 - 1) + C
\]

as a barrier function. Indeed, choosing large \( C \) we may assure that \( v \leq \bar{v} \) on \( \partial \omega \).

Since \( \inf_\omega g \) is large we can choose \( K \) so large that \( \bar{v}(0) < 0 \). Notice that for any large \( K \) we can choose \( b_1 \) so small that the matrix

\[
\left\{ \bar{v}_{ij} - \frac{1}{M_i M_j} D_{p^i p^j A_{ij} \bar{v}_k \bar{v}_l} \right\}
\]

is positive definite. Now by comparison principle we get \( \inf_\omega v < 0 \). On the other hand by (3.7) we have \( v \geq 0 \). Therefore (3.43) is proved and (3.32) follows immediately.

\[ \square \]
3.5 Mixed tangential-normal derivatives. Part 1

Lemma 3.9. There exists a constant $\lambda$ such that for $|x| \leq \lambda b_{n-1}$ we have the estimate

$$|u_{\gamma\tau}| \leq C \sqrt{b_{n-1}}, \quad x \in \mathcal{N} \cap \partial \Omega,$$  \hfill (3.45)

where $C$ is a constant depending on $\Omega$, $|\varphi|_{3,1}$, $|\hat{f}|_{1,1}$, $b_{n-1}$, the cost function $c$ up to its fourth-order derivatives and the lower bound of $|\det D_{xy}^2 c|$ and $\gamma$, $\tau$ denote respectively the unit inner normal to $\partial \Omega$ and any unit tangential vector to $\partial \Omega$.

Proof. Again we make an affine transformation $x \to \tilde{x} = S(x)$. Let $M = 1/b_{n-1}$, define new functions

$$v(\tilde{x}) = M^2 u(x), \quad \tilde{c}(\tilde{x}, \tilde{y}) = M^2 c(x, y),$$

where

$$\begin{align*}
\tilde{x}_i &= M x_i, & i &= 1, \ldots, n, \\
\tilde{y}_i &= M^2 y_i/M_i, & i &= 1, \ldots, n,
\end{align*}$$

and

$$\begin{align*}
M_i &= \sqrt{M}, & i &= 1, \ldots, n-1, \\
M_n &= M = 1/b_{n-1}.
\end{align*}$$

Define also

$$g(\tilde{x}, Dv) = \frac{1}{b_{n-1}^2} f \left( \frac{\tilde{x}_i}{M_i}, \frac{M_i v_i}{M^2} \right).$$

Function $v$ then satisfies equation

$$\det \left[ v_{ij} - \frac{1}{M^2 M_i M_j} D_{p_k p_l} A_{ij} v_k v_l \right] = g(\tilde{x}, Dv) \text{ in } \tilde{\Omega},$$

\hfill (3.48)

where $\tilde{\Omega} = S(\Omega)$. As before, coefficients

$$\left[ \frac{1}{M^2 M_i M_j} D_{p_k p_l} A_{ij} \right] \leq C \max\{b_{n-1}, b_{n-1}^3\} \leq C$$

\hfill (3.49)

are bounded. Note that by Taylor expansion, equation (3.48) is equivalent to

$$\det \left[ v_{ij} - \tilde{A}_{ij}(\tilde{x}, Dv) \right] = g(\tilde{x}, Dv),$$

\hfill (3.50)
where
\[ \tilde{A}_{ij}(\tilde{x}, Dv) = \frac{1}{M^2 M_i M_j} D_{pqrs} A_{ij}(x, q)v_{k}v_{l}, \] (3.51)
where \( q = \theta Du \), with \( \theta \in (0, 1) \) depending on \( x \).

In the neighbourhood \( \tilde{N} = S(N) \) of the origin, \( \partial \tilde{\Omega} \) is represented by
\[ \tilde{x}_{n} = \hat{\varphi}(\tilde{x}') = \frac{1}{2} |\tilde{x}'|^2 + O(|\tilde{x}'|^3) \] (3.52)
and on the boundary \( \tilde{\Omega} \cap \tilde{N} \)
\[ v = \hat{\varphi} = \frac{1}{2} \tilde{x}_{n-1}^2 + \frac{1}{2} \sum_{i=1}^{n-2} b_i \tilde{x}_i^2 + R(\tilde{x}') + O(|\tilde{x}'|^4). \] (3.53)

As in the second step of the preceding section, we see that near the origin both \( \partial \tilde{\Omega} \) and \( \hat{\varphi} \) are \( C^{3,1} \)-smooth with \( C^{3,1} \)-norms independent of the lower bound of \( b_1 \). From (3.52) one can see that domain \( \Omega \) after the transformation remains uniformly convex and therefore instead of (3.40) we may choose a small constant \( \varepsilon > 0 \) and \( \omega = \{ \tilde{x} \in \tilde{\Omega} | \tilde{x}_{n} < \varepsilon \} \) which is again bounded independently of \( b_1 \).

Repeating all arguments from the last step of the previous section we get uniform boundedness of \( g \) in \( \omega \). By standard barrier considerations we obtain a normal derivative bound
\[ |v_{\tilde{\gamma}}| \leq C \text{ on } \partial \omega, \] (3.54)
where \( \tilde{\gamma} \) is the unit inner normal to \( \partial \tilde{\Omega} \) and \( C \) depends as usual on \( \Omega, |\varphi|_{3,1} \) and \( |f|_{1,1} \) and the cost function \( c \) up to its fourth-order derivatives. From (3.54) we have
\[ |Dv| \leq C \text{ on } \partial \omega. \] (3.55)

Then, as in the previous section, we notice that function \( v(\tilde{x}) + C \tilde{x}^2 \) is convex in \( \omega \) with \( C \) depending on \( b_{n-1}, \varepsilon \), the cost function \( c \) up to its fourth-order derivatives and therefore the estimate above can be extended to the whole \( \omega \).

Define a linearised operator
\[ L := w^{ij}(D_{ij} - D_{pq} \tilde{A}_{ij} D_{pk}), \]
where \( \{w^{ij}\} \) is the inverse of \( \{w_{ij}\} = \{v_{ij} - \tilde{A}_{ij}\} \). Consider the vector field
\[ V := \partial_{\alpha} + \sum_{\beta < n}(\tilde{x}_{\beta} \partial_{n} - \tilde{x}_{n} \partial_{\beta}). \]
Since \( v - \hat{\varphi} = 0 \) on \( \partial \tilde{\Omega} \) so also
\[ (\partial_{\alpha} + \tilde{\varphi}_{\partial_{n}})(v - \hat{\varphi}) = 0, \quad \alpha < n, \]
and by (3.52)
\[(v - \tilde{\varphi})_\alpha + \tilde{x}_\beta (v - \tilde{\varphi})_\alpha \leq C|\tilde{x}|^2 \text{ on } \partial \Omega,\]
hence
\[V(v - \tilde{\varphi}) \leq C|\tilde{x}|^2 \text{ on } \partial \Omega. \tag{3.56}\]

Set
\[\tilde{q}(\tilde{x}) = \frac{q(\tilde{x}/M_i)}{b^i_{n-1}} \quad \text{and} \quad \tilde{\xi}(\tilde{x}, \tilde{p}) = \xi(\tilde{x}/M_i, \tilde{p}M^2/M_i).\]

Then differentiating equation (3.50) and using (3.4) we get
\[w^{ij}(v_{ij \alpha} - \tilde{A}_{ij \alpha} - D_{\tilde{p}_k} \tilde{A}_{ij} v_{\alpha k}) = (\log \tilde{q})_\alpha + (\log \tilde{\xi})_\alpha + (\log \tilde{\xi})_{\tilde{p}_i} v_{i\alpha},\]
which leads to
\[L v_\alpha := L v_\alpha - \tilde{\xi}_{\tilde{p}_i} v_{i\alpha} = w^{ii} \tilde{A}_{ij \alpha} (\log \tilde{q})_\alpha + (\log \tilde{\xi})_\alpha. \tag{3.57}\]

Now since \(\{w^{ij}\}\) positive definite by (3.13), (3.14) and the above equation we obtain
\[|L V(v - \tilde{\varphi})| \leq \left| L V(v) + C \sum_{i=1}^{n} (w^{ii} + \tilde{\xi})_{\tilde{p}_i} \right| \leq C \left( \sum_{i=1}^{n} w^{ii} + V(\log \tilde{q}) + V(\log \tilde{\xi}) + w^{ij} V(\tilde{A}_{ij}) \right) \leq C \left( \sum_{i=1}^{n} w^{ii} + \tilde{\xi}_{\tilde{p}_i} \right), \tag{3.57}\]

We shall use the function
\[\bar{v} = \left( \frac{1}{2} - \mu \right) |\tilde{x}|^2 + \frac{1}{2} M x_n^2 - \tilde{x}_n\]
as a barrier function in \(\omega\) for suitable constants \(\mu < \frac{1}{2}, M\). We have
\[L \bar{v} \geq (1 - 2\mu) \sum_{i<n} w^{ii} + M w^{nn} - (1 - 2\mu) \sum_{k<n} \tilde{x}_k w^{ij} D_{\tilde{p}_k} \tilde{A}_{ij} - M \tilde{x}_n w^{ij} D_{\tilde{p}_n} \tilde{A}_{ij} - |w^{ij} D_{\tilde{p}_n} \tilde{A}_{ij}| - (1 - 2\mu) \sum_{i<n} \frac{\tilde{\xi}_{\tilde{p}_i}}{\tilde{\xi}} \tilde{x}_i - M \tilde{\xi}_{\tilde{p}_n} \tilde{x}_n + \tilde{\xi}_{\tilde{p}_n} \]
\[\geq \frac{1}{2} (1 - 2\mu) \sum_{i<n} w^{ii} + \frac{M}{2} w^{nn} - |w^{ij} D_{\tilde{p}_n} \tilde{A}_{ij}| - C \tilde{\xi} \frac{1}{2(n-1)^2},\]
provided \(M\) is large enough and \(\varepsilon\) is sufficiently small.
CHAPTER 3. DEGENERATE MONGE-AMPÈRE TYPE EQUATIONS

Now by (3.51) and $C^{1,\alpha}$-smoothness of $u$ (see Remark 3.10) we have
\[
D_p \tilde{A}_{ij}(\tilde{x}, Dv) \leq \frac{1}{4}(1 - 2\mu) \quad \text{in } \tilde{\Omega} \cap \{\tilde{x}_n < \varepsilon\},
\]
provided $\varepsilon$ is small enough. Note that uniform boundedness of $g$ implies that $w^{ii} > 0$. Therefore, by positivity of $\tilde{\xi}$ it follows that for $\varepsilon$ small and $M$ large enough
\[
\mathcal{L} \tilde{v} \geq \frac{1}{4}(1 - 2\mu) \sum_{i<n} w^{ii} + \frac{M}{4} w^{nn}.
\]
Hence, by (3.57) we obtain
\[
\mathcal{L}(a\tilde{v} \pm V(v - \tilde{\phi})) \geq a \frac{1}{4}(1 - 2\mu) \sum_{i<n} w^{ii} + M \frac{w^{nn}}{4} - C \left( \sum_{i=1}^{n} w^{ii} + \sqrt{n \sum_{i=1}^{n} w^{ii}} + 1 \right) \geq 0,
\]
provided $a$ is large enough.

Next we examine $\tilde{v}$ on $\partial \omega$. On $\partial \omega \cap \partial \tilde{\Omega}$ we have by (3.52)
\[
\tilde{v} = -\mu |\tilde{x}'|^2 + O(|\tilde{x}'|^3).
\]
Thus,
\[
a\tilde{v} \pm V(v - \tilde{\phi}) \leq C |\tilde{x}'|^2 - \frac{1}{2} a\mu |\tilde{x}'|^2.
\]
On the remaining boundary $\tilde{\Omega} \cap \{\tilde{x}_n = \varepsilon\}$ we consider 2 cases:

1) $\frac{1}{2} \mu |\tilde{x}'|^2 > M \tilde{x}_n^2$. Then since $\tilde{x}_n \geq \varrho$, we infer
\[
\tilde{v} \leq \left( \frac{1}{2} - \mu \right) |\tilde{x}'|^2 + \frac{1}{2} M \tilde{x}_n^2 - \varrho(\tilde{x}')
\leq - \frac{1}{2} \mu |\tilde{x}'|^2 + \frac{1}{2} M \tilde{x}_n^2 + O(|\tilde{x}'|^3)
\leq - \frac{1}{4} \mu |\tilde{x}'|^2 + O(|\tilde{x}'|^3)
\leq - \frac{1}{8} \mu |\tilde{x}'|^2,
\]
for $\varepsilon$ small enough.

2) $\frac{1}{2} \mu |\tilde{x}'|^2 \leq M \tilde{x}_n^2$. Then since $\tilde{x}_n = \varepsilon$,
\[
\tilde{v} \leq \frac{1}{2} |\tilde{x}'|^2 + \frac{1}{2} M \tilde{x}_n^2 - \tilde{x}_n
\leq \left( \frac{M}{\mu} + \frac{1}{2} M \right) \varepsilon^2 - \varepsilon
\leq - \frac{1}{2} \varepsilon,
for $\varepsilon$ small enough.

Thus, we fix $\varepsilon$ to satisfy all these requirements and then choose $a$ so large that

$$
\mathcal{L}(a\tilde{v} \pm V(v - \tilde{\varphi})) \geq 0
$$

and

$$
a\tilde{v} \pm V(v - \tilde{\varphi}) \leq 0 \text{ on } \partial \omega.
$$

Applying the comparison principle we infer that

$$
|V(v - \tilde{\varphi})| \leq a\tilde{v} \text{ in } \omega,
$$

and therefore

$$
|v_i(0) - \tilde{\varphi}_i(0)| \leq a.
$$

Thus we have proved

$$
|v_i| \leq C \text{ on } \partial \tilde{\Omega} \cap \{x_n < \varepsilon \} \text{ for } i = 1, \ldots, n - 1.
$$

Changing back to the original coordinates we obtain

$$
\left| (x) - \tilde{\varphi}(x) \right| \leq CM_2 \sqrt{b_n - 1}, \quad i = 1, \ldots, n - 1.
$$

Remark 3.10. The $C^{1,\alpha}$-smoothness of $u$ can be easily seen using the following interpolation inequality

$$
||u||_{C^{1,\alpha}} \leq C||u||_{1-\alpha}^{1-\alpha}||u||_{C^{1,1}}^{\alpha}.
$$

Indeed, the above inequality can be deduced by the direct computation

$$
||u||_{C^{1,\alpha}} \leq |u|_1 + \sup_{x,y \in \Omega} \frac{|Du(x) - Du(y)|}{|x - y|^\alpha} \\
\leq |u|_1 + \sup_{x,y \in \Omega} \left( \frac{|Du(x) - Du(y)|}{|x - y|} \right)^\alpha \sup_{x,y \in \Omega} |Du(x) - Du(y)|^{1-\alpha} \\
\leq |u|_1^{\alpha} |u|_1^{1-\alpha} + 2^{1-\alpha} |Du|_{1-\alpha}^{\alpha} \\
\leq 2^{1-\alpha} |u|_1^{1-\alpha} (|u|_1^{\alpha} + |Du|_{1-\alpha}^{\alpha}) \\
\leq 4^{1-\alpha} |u|_{1-\alpha}^{\alpha} ||u||_{C^{1,1}}^{\alpha}.
$$

Then for $t \in [0,1]$ we consider the equation

$$
de \det[D^2u_t - A(x, Du_t)] = (1 - t)f(x, Du_t) + t. \quad (3.60)
$$

Obviously, for $t = 1$ the solution $u_1$ to the corresponding Dirichlet problem is $C^{1,\alpha}(\Omega)$ (see [91], [89]) and the inequality (3.58) holds. Therefore, using results of Sect. 3.5 and 3.6 we get

$$
||u_1||_{C^{1,1}} \leq C,
$$

(3.61)
where $C$ is the constant independent of $b_i$, $\varepsilon$ and $t$. Now by the interpolation inequality we get the estimate

$$||u_1||_{C^{1,\alpha}} \leq C,$$

(3.62)

where constant $C$ is independent of $t$. Interpolating that process we see that the same estimate holds for every $t \in [0,1]$.

### 3.6 Mixed tangential-normal derivatives. Part 2

Now we are in a position to establish the final tangential-normal derivative estimate in terms of corresponding tangential derivatives.

**Lemma 3.11.** There exists a constant $C_0$ depending on $\Omega$, $|\varphi|_{3,1}$, $|\tilde{f}|_{1,1}$, $b_i = u_{ii}(0)$, the cost function $c$ up to its fourth-order derivatives and the lower bound of $|\text{det}D^2_{x,y}c|$ such that

$$|u_{in}| \leq C_0 \sqrt{b_i}, \quad i = 1,\ldots,n-1.$$

(3.63)

**Proof.** The proof is by the induction method. Denote by $\tau_i = \tau_i(x)$, $i = 1,\ldots,n-1$ the tangential direction of $\partial\Omega$ at $x \in \mathcal{N} \cap \partial\Omega$, which lies in the two-dimensional plane parallel to the $x_i$- and $x_n$- axis and passes through the point $x$.

1) By Lemma 3.9 for $i = n-1$, there exists a constant $\lambda_{n-1} = \lambda$, such that for $|x| \leq \lambda \sqrt{b_{n-1}}$

$$|u_{\gamma\tau_j}(x)| \leq C \sqrt{b_{n-1}}, \quad j = 1,\ldots,n-1, \quad x \in \mathcal{N} \cap \partial\Omega.$$  

2) Assume that for $i = k+1,\ldots,n-1$ and some $k = 1,\ldots,n-2$ there exists a constant $\lambda_i > 0$, depending on $\Omega$, $|\varphi|_{3,1}$, $|\tilde{f}|_{1,1}$ and the cost function $c$ such that for $|x| \leq \lambda_i \sqrt{b_i}$ we have estimates

$$|u_{\gamma\tau_j}(x)| \leq C \sqrt{b_i}, \quad j = 1,\ldots,i, \quad x \in \mathcal{N} \cap \partial\Omega,$$

(3.64)

where $C$ is a constant depending on $\Omega$, $|\varphi|_{3,1}$, $|\tilde{f}|_{1,1}$ and the cost function $c$ up to its fourth-order derivatives and the lower bound of $|\text{det}D^2_{x,y}c|$.

3) Our aim is to prove that for $i = k$ there exists a constant $\lambda_k$, also depending on $\Omega$, $|\varphi|_{3,1}$, $|\tilde{f}|_{1,1}$ and the cost function $c$, such that for $|x| \leq \lambda_k \sqrt{b_k}$ we have

$$|u_{\gamma\tau_j}(x)| \leq C \sqrt{b_k}, \quad j = 1,\ldots,k, \quad x \in \mathcal{N} \cap \partial\Omega,$$

(3.65)
where $C$ is a constant depending on $\Omega$, $|\varphi|_{3,1}$, $|\tilde{f}|_{1,1}$, $b_i$ and the cost function $c$ up to its fourth-order derivatives and the lower bound of $|\det D^2_{xy}c|$.

We introduce a new transformation $x \to \tilde{x} = S(x)$. Let $M = 1/b_k$, define new functions

$$v(\tilde{x}) = M^2 u(x), \quad \tilde{c}(\tilde{x}, \tilde{y}) = M^2 c(x, y),$$

where

$$\begin{align*}
\tilde{x}_i &= M x_i, \quad i = 1, \ldots, n, \\
\tilde{y}_i &= M^2 y_i/M_i, \quad i = 1, \ldots, n,
\end{align*}$$

(3.66)

and

$$M_i = \begin{cases} \sqrt{M}, & i = 1, \ldots, k, \\ \sqrt{b_i M}, & i = k + 1, \ldots, n - 1, \\ M, & i = n. \end{cases}$$

(3.67)

Define also

$$g(\tilde{x}, Dv) = \frac{M^{2n}}{\prod_{i=1}^{n} M_i^2} f \left( \frac{\tilde{x}_i}{M_i}, \frac{M_i v_i}{M^2} \right).$$

Function $v$ now satisfies equation

$$\det \left[ v_{ij} - \frac{1}{M_k M_l} A_{ijkl} v_i v_j \right] = g(\tilde{x}, Dv) \text{ in } \tilde{\Omega},$$

(3.68)

where $\tilde{\Omega} = S(\Omega)$. We may suppose that $b_k << b_{k+1}$, otherwise (3.65) follows immediately from (3.64). As in the first step of the previous section, coefficients

$$\left[ \frac{1}{M^2 M_i M_j} A_{ijkl} \right] \leq c b_k \max\{b_{n-1}, 1\} \leq C b_k$$

(3.69)

are bounded.

Accordingly to the second step in the neighbourhood $\tilde{N} = S(\mathcal{N})$ of the origin, $\partial \tilde{\Omega}$ is represented by

$$\tilde{x}_n = \tilde{\varphi}(\tilde{x}') = \frac{1}{2} \sum_{i=1}^{k} \tilde{x}_i^2 + \frac{1}{2} \sum_{i=k+1}^{n-1} \frac{b_k}{b_i} \tilde{x}_i^2 + O(|\tilde{x}'|^3)$$

(3.70)

and

$$v = \tilde{\varphi} = \frac{1}{2} \sum_{i=1}^{k-1} \frac{b_i}{b_k} \tilde{x}_i^2 + \frac{1}{2} \sum_{i=k}^{n-1} \tilde{x}_i^2 + R(\tilde{x}') + O(|\tilde{x}'|^4).$$

(3.71)

As above we see that near the origin, both $\partial \tilde{\Omega}$ and $\tilde{\varphi}$ are $C^{3,1}$-smooth with $C^{3,1}$-norms independent of the lower bound of $b_k$. 

From (3.70) one can see that the domain \( \Omega \) after the transformation remains uniformly convex in directions of \( x_{1},...,x_{k} \)-axis and becomes almost flat in directions of \( x_{k+1},...,x_{n} \)-axis near the origin (see Fig. 3.1) since \( b_{k} \ll b_{i} \) for \( i = k + 1,...,n - 1 \) near the origin. Therefore the domain

\[
\omega := \{ \tilde{x} \in \tilde{\Omega} | \tilde{x}_{n} < \beta^{2}, \ |\tilde{x}_{i}| < \beta, \ i = k + 1,...,n - 1 \} \subset \tilde{N}, \tag{3.72}
\]

with a suitable choice of \( \beta \) is bounded independently of \( b_{k} \). Again by the c-convexity of \( v \) (see (3.42)) we have

\[
v \leq C \text{ in } \omega.
\]

Similarly to the last step of Sect. 4 we may write \( g \) in the form

\[
g(\tilde{x}, \tilde{p}) = \tilde{q}(\tilde{x})\tilde{\xi}(\tilde{x}, \tilde{p}),
\]

where

\[
\tilde{q}(\tilde{x}) = \frac{M^{2n}}{\prod_{i=1}^{n} M_{i}^{2}} q(\tilde{x}_{i}/M_{i})
\]

and

\[
\tilde{\xi}(\tilde{x}, \tilde{p}) = \xi(\tilde{x}_{i}/M_{i}, \tilde{p}_{i}M_{i}/M^{2}).
\]

Using similar arguments we have uniform boundedness of \( g \) in \( \omega \) as well as

\[
\frac{\partial}{\partial \tilde{x}_{i}} \tilde{q}^{1/2(n-1)} \leq C; \quad \frac{\partial}{\partial \tilde{x}_{i}} \tilde{\xi}^{1/2(n-1)} \leq C; \quad \frac{\partial}{\partial \tilde{p}_{i}} \tilde{\xi}^{1/2(n-1)} \leq Cb_{k}. \tag{3.73}
\]

The key point in this proof is to find a bound for the normal derivative of \( v \) near the origin. The proof is based on the comparison principle with a suitable barrier function. For any point \( \tilde{x}_{0} \in \partial \omega \cup \partial \tilde{\Omega} \) we choose an orthogonal basis at \( \tilde{x}_{0} \) which we denote \( (e_{1},...,e_{n}) \) where \( e_{n} \) coincides with the inner normal of \( \partial \tilde{\Omega} \) at \( \tilde{x}_{0} \). Let us choose, for simplicity, our coordinate vectors as \( (e_{1},...,e_{n}) \). Subtracting the supporting function \( l \) at the point \( \tilde{x}_{0} \) from \( v(\tilde{x}) \) we may also suppose that the plane \( \{x_{n} = 0\} \) is tangent to the function \( v \) at the point \( \tilde{x}_{0} \). We set

\[
\psi(\tilde{x}) = \frac{1}{2} \sigma |\tilde{x}'|^{2} + \frac{1}{2} K \tilde{x}_{n}^{2} - K^{2} \tilde{x}_{n}. \tag{3.74}
\]

By (3.69) and choosing \( K \) large and \( b_{k} \) small enough we get

\[
\det \left[ \psi_{ij} - \frac{1}{M^{2}} M_{k} M_{l} D_{p_{k}p_{l}} A_{ij} \psi_{k} \psi_{l} \right] \geq \det \left[ \psi_{ij} - cb_{k} \psi_{k} \psi_{l} \right] \geq C \sigma^{n-1} K \geq \sup_{\omega} g. \tag{3.75}
\]
3.6. MIXED TANGENTIAL-NORMAL DERIVATIVES. PART 2

Figure 3.1: Schematic of domain $\omega$ defined by (3.72). Points $S, S^*, \bar{S}, S^0$ denote $S(p), S(p^*), S(\bar{p}), S(p^0)$ respectively.

Our aim is to prove

$$v \leq \frac{1}{2} v \text{ on } \partial \omega.$$  \hfill (3.76)

To this end we divide the boundary $\partial \omega$ by 3 parts as it is shown on Fig. 3.1.

At first, we consider the piece $\partial_1 \omega := \partial \omega \cap \partial \bar{\Omega}$. Let $\tilde{x}^* = (\tilde{x}_1, ..., \tilde{x}_k), \tilde{x}^{**} = (\tilde{x}_{k+1}, ..., \tilde{x}_{n-1})$ For $\tilde{x} \in \partial_1 \omega$ we have by (3.70), (3.71)

$$v(\tilde{x}) \leq \frac{1}{2} \sigma |\tilde{x}'|^2 - \frac{1}{2} K^2 \tilde{x}_n \leq \frac{1}{4} \sigma |\tilde{x}^{**}|^2 - \frac{1}{8} K^2 |\tilde{x}'|^2$$ \hfill (3.77)

provided $\sigma, \beta$ are small and $K$ is large enough. Hence (3.76) holds on $\partial_1 \omega$. On $\partial_2 \omega := \partial \omega \cap \{\tilde{x}_n = \beta^2\}$ we have $v \geq 0$. Again for $\sigma > 0$ small and $K > 1$ large, we have $v \leq -\frac{1}{2} K^2 \beta^2$, so that (3.76) also holds on $\partial_2 \omega$.

Finally we consider the piece

$$\partial_3 \omega := \partial \omega \cap \{\tilde{x}_i = \beta \text{ for some } i = k + 1, ..., n - 1\}.$$  

Firstly, changing back to our coordinates, we get

$$\tilde{\omega} := S^{-1}(\omega) = \{x \in \Omega | x_n < \frac{\beta^2}{M}, |x_i| \leq \frac{\beta}{\sqrt{b_i}M}, i = k + 1, ..., n - 1\},$$

and

$$\partial_3 \tilde{\omega} = \partial \tilde{\omega} \cap \{|x_i| = \frac{\beta}{\sqrt{b_i}M} \text{ for some } i = k + 1, ..., n - 1\}.$$  

Now we prove that

$$u(x) \geq \frac{\beta^2}{8M^2} \text{ on } \partial_3 \tilde{\omega} \cap \{x_n < \frac{\epsilon_0 \beta^2}{M}\}$$ \hfill (3.78)
provided $\epsilon_0$ is small enough, where

$$\partial^*_3 \omega = \partial \omega \cap \{ x_{n-1} \geq \frac{\beta}{\sqrt{b_{n-1}M}} \}.$$  

All other pieces of $\partial_3 \omega$ can be handled in the same way. If $\partial^*_3 \omega \subset \{ x_n \geq \frac{\epsilon_0 \beta^2}{M} \}$, then choosing $K$ large enough, (3.76) holds. So we may suppose that $\partial^*_3 \omega \cap \{ \tilde{x}_n < \frac{\epsilon_0 \beta^2}{M} \} \neq \emptyset$. To prove (3.78) we first fix a point $p = (\hat{p}, \tilde{p}, p_n) \in \partial_3 \omega$, where $\hat{p} = (p_1, ..., p_k) \neq 0$, $\tilde{p} = (p_{k+1}, ..., p_{n-1})$, $p_n < \frac{\epsilon_0 \beta^2}{M}$. For $\delta \geq 0$ sufficiently small we then fix a point $p^* = (0, \tilde{p}, p_n + \delta)$. Let the straight line through $p$ and $p^*$ intersects $\partial \Omega$ at a point $\bar{p} = (\hat{p}, \tilde{p}, \bar{p}_n)$ where $\hat{p} = (\bar{p}_1, ..., \bar{p}_k)$, $\bar{p}_n \leq 2 \frac{\epsilon_0 \beta^2}{M}$ and

$$\frac{1}{2} |p - p^*| \leq |\bar{p} - p^*|. \quad (3.79)$$

Now let $p^0 = (0, \tilde{p}, \rho(0, \tilde{p}))$ be the projection of $p^*$ on $\partial \Omega$. Noticing that $p_{n-1} = \frac{\beta}{\sqrt{b_{n-1}M}}$ and using (4.14) we get

$$u(p^0) = \frac{1}{2} \sum_{i=k+1}^{n-1} b_i p_i^2 + O(|p^0|^3) \geq \frac{3}{8} \frac{\beta^2}{M^2}, \quad (3.80)$$

provided $\beta$ is small enough. Since $\bar{p} \in \partial \Omega$ from (4.13) we see that

$$|\tilde{p}|^2 = \sum_{i=1}^{k} \tilde{p}_i^2 \leq 2(|\bar{p}_n - p_n^0|) + O(|\bar{p}|^3) \quad (3.81)$$

$$\leq 4 \frac{\epsilon_0 \beta^2}{M} + O(|\bar{p}|^3) \leq 5 \frac{\epsilon_0 \beta^2}{M},$$

and

$$u(\bar{p}) = \frac{1}{2} \sum_{i=1}^{k} b_i \tilde{p}_i^2 + \frac{1}{2} \sum_{i=k+1}^{n-1} b_i p_i^2 + O(|\bar{p}|^3) \quad (3.82)$$

$$\leq u(p^0) + C \frac{\epsilon_0 \beta^2}{M} + O(|\bar{p}|^3).$$

Now using $c$-convexity of $u$

$$u(p^*) \geq u(p^0) + c(p^*, y^0) - c(p^0, y^0), \text{ where } y^0 = T(p^0).$$

By Taylor expansion

$$c(p^*, y^0) = c(p^0, y^0) + c_{x_n}(p^0, y^0)|p^*_n - p^0_n| + O(|p^*_n - p^0_n|^2).$$
Therefore since \( p_n^* \leq \bar{p}_n \leq \frac{2\epsilon_0\beta^2}{M} \),

\[
u(p^*) \geq u(p^0) + c_x(p^0, y^0)|p^*_n - p^0_n| + O(|p^*_n - p^0_n|^2) \tag{3.83}
\]

\[\geq u(p^0) - C\epsilon_0\beta^2 - \tilde{C} \left( \frac{\epsilon_0^2\beta^4}{M^2} \right).\]

Hence from (3.82) and (3.83) we can fix \( \beta \) and then choose \( \epsilon_0 \) small so that

\[u(p^*) \geq \frac{8}{9} u(p) \]

and by (3.80)

\[u(p^*) \geq \frac{8}{9} u(p^0) \geq \frac{8}{9} u(p_0) \geq \beta^2 3M^2.\]

As before using an estimate similar to (3.41) we can choose

\[M = \frac{1}{b} k\]

so large that the function \( w := u(x) + |x|^2 \) is convex in \( \varpi \). Then \( w(p) \geq l(p) \) where \( l \) is the linear function connecting points \( w(p^*) \) and \( w(p) \). Then by (3.79) we get

\[w(p) \geq \frac{l(p) - w(p^*)}{|p - p^*|} |p - p^*| + w(p^*)\]

\[\geq 3w(p^*) - 2w(p).\]

Using inequality (3.79) one can easily calculate that for \( \delta \) small enough \( |\tilde{p}|^2 \leq 5|\bar{p}|^2 \). Noticing that \( p_n < \bar{p}_n \leq 2\epsilon_0\beta^2/M \) by (3.81) we get

\[u(p) \geq \frac{3}{4} u(p^*) + 3|p^*|^2 - 2|\bar{p}|^2 - |p|^2\]

\[\geq \frac{\beta^2}{4M^2} - 35\frac{\epsilon_0\beta^2}{M^2} - 12\frac{\epsilon_0^2\beta^4}{M^2}\]

\[\geq \frac{\beta^2}{8M^2},\]

provided \( \epsilon_0 \) is small enough. Thus, (3.78) is proved and hence \( v(\tilde{x}) \geq \frac{1}{8}\beta^2 - C\tilde{x}_n \) on \( \partial \tilde{x} \omega = \partial \omega \cap \{\tilde{x}_n = \beta\} \) where \( C \) depends on \( \beta \) and \( \epsilon_0 \). On the other hand,

\[v(\tilde{x}) \leq C \sigma - K^2 \tilde{x}_n.\]

Therefore \( v \leq \frac{1}{2} v \) on \( \partial \tilde{x} \omega \) if \( \sigma \) is small and \( K \) is large enough. This completes the proof of inequality (3.76).

By (3.75) and the comparison principle, it follows that \( \overline{\nu} \leq v \) on \( \omega \). We therefore obtain

\[|v_\tilde{x}(\tilde{x})| \leq C \text{ for } \tilde{x} \in \partial \omega \cap \partial \tilde{\Omega}. \tag{3.84}\]

and consequently, similarly to (3.55), from the \( c \)-convexity of \( v \), we infer

\[|Dv| \leq C \text{ in } \omega. \tag{3.85}\]
We can now complete the proof of (3.63) by standard arguments. As in the previous section let

\[ L u := Lu - \tilde{\xi} \tilde{\rho}_i u_i = w^{ij}(u_{ij} - D_p \tilde{A}_{ij} u_s) - \tilde{\xi} \tilde{\rho}_i u_i, \]

where \( \{w^{ij}\} \) is the inverse of \( \{v_{ij} - \tilde{A}_{ij}\} \) and

\[ \tilde{A}_{ij}(\tilde{x}, Dv) = \frac{1}{M^2 M_i M_j} D_{pkp} A_{ij}(\tilde{x}, q) v_k v_l, \]

where \( q = \theta Du \), with \( \theta \in (0, 1) \) depending on \( x \). Let \( V = (V_1, ..., V_k) \)

\[ V_i = \partial_i + (\tilde{x}_i \partial_n - \tilde{x}_n \partial_i), \quad i = 1, ..., k. \]

Setting

\[ z(\tilde{x}) = \pm V(v - \tilde{\varphi})(\tilde{x}) + \frac{B}{2} |\tilde{x}|^2, \]

we have

\[ |z(\tilde{x})| \leq C_B (|\tilde{x}|^2 + \tilde{x}_n) \text{ on } \partial \omega. \]  
(3.86)

From (3.73) one can see that

\[ \left| \frac{\partial (\log \tilde{q})}{\partial \tilde{x}_i} \right| \leq C q^{-\frac{1}{2(n-1)}} (\tilde{x}) \leq C g^{-\frac{1}{2(n-1)}} (\tilde{x}), \quad i < n; \]

\[ \left| \frac{\partial (\log \tilde{\xi})}{\partial \tilde{x}_i} \right| = \frac{\tilde{\xi} \tilde{\rho}_i}{\tilde{\xi}} \leq C \xi^{-\frac{1}{2(n-1)}} (\tilde{x}, \tilde{p}) \quad i \leq n; \]

\[ \left| \frac{\partial (\log \tilde{\xi})}{\partial \tilde{p}_i} \right| = \frac{\tilde{\xi} \tilde{\rho}_i}{\tilde{\xi}} \leq C b_k \xi^{-\frac{1}{2(n-1)}} (\tilde{x}, \tilde{p}) \quad i \leq n. \]  
(3.87)

Hence following (3.57) and using (3.69) positivity of \( \xi \) and uniform boundedness of \( g \) we infer

\[ \mathcal{L} z \geq \frac{B}{2} L |\tilde{x}|^2 - \mathcal{L} V(v) - C \sum_{i=1}^{n} (w^{ii} - \frac{\tilde{\xi} \tilde{\rho}_i}{\tilde{\xi}}) \]

\[ \geq \left( B - C \right) \sum_{i=1}^{n} w^{ii} - cBb_k \beta \sum_{i=1}^{n} w^{ii} - C q^{-\frac{1}{2(n-1)}} \]

\[ - C \xi^{-\frac{1}{2(n-1)}} - C B b_k \xi^{-\frac{1}{2(n-1)}} \]

\[ \geq \frac{B}{2} g^{-\frac{1}{n}} - C g^{-\frac{1}{2(n-1)}} - C (1 + B b_k) \xi^{-\frac{1}{2(n-1)}} \]

\[ \geq \frac{B}{4} g^{-\frac{1}{n}}, \]
provided \( \beta, b_k \) are small and \( B \) is large enough. We set

\[
\tilde{z}(\tilde{x}) = a(v - v - \tilde{x}_n) + z(\tilde{x}),
\]

where \( v \) is given by (3.74), and \( a > 1 \) is a large constant to be chosen later. By (3.75) and the concavity of \( F \), we have

\[
L(v - v) \geq F(D^2v) - F(D^2v) \geq 0.
\]

Consequently, we derive

\[
\mathcal{L}\tilde{z} \geq a\mathcal{L}(v - v - \tilde{x}_n) + \mathcal{L}z - cab_k \sum_{i=1}^{n} w^i
\]

\[
- \alpha \sum_{i=1}^{n} \tilde{\xi}_{\tilde{x}_i} - a(K\tilde{x}_n + K^2 + 1) \frac{\tilde{\xi}_{\tilde{x}_n}}{\xi} - a \sum_{i=1}^{n} \tilde{\xi}_{\tilde{x}_i} v_i
\]

\[
\geq \frac{B}{8} g^{-\frac{1}{n}} - Cab_k \xi^{-\frac{1}{n(n-1)}} (\sigma \beta + K^2 + |Dv| + 1) \geq 0,
\]

provided \( B \) is sufficiently large and \( \beta, b_k \) are small. This implies that the function \( \tilde{z} \) attains the maximum on the boundary of \( \omega \).

By (3.76) and (3.86),

\[
\tilde{z}(\tilde{x}) \leq C_B(|\tilde{x}|^2 + \tilde{x}_n) - \frac{1}{2} a(v(\tilde{x}) + \tilde{x}_n).
\]

Using (3.70) and (3.71), we then choose \( a \) large enough so that \( \tilde{z} \leq 0 \) on \( \partial \omega \).

Noticing that \( \tilde{z}(0) = 0 \) we have therefore \( \partial \tilde{z}/\partial \tilde{x}_n(0) \leq 0 \). Namely \( |v_i| \leq C, i = 1, \ldots, k - 1 \). Similarly we have \( |v_{\gamma_\tilde{t}_i}| \leq C \), for \( \tilde{x} \in \partial \omega \cap \partial \tilde{\Omega} \), near the origin, where \( \tilde{\tau}_i \) is a tangential direction of \( \partial \tilde{\Omega} \) at \( \tilde{x} \). Changing back to the \( x \)-coordinates we derive (3.65) and by induction we finally obtain (3.63).
Chapter 4

Singular Monge-Ampère equations on bounded and unbounded domains

4.1 Main result

This Chapter is concerned with the following question:

**Question 4.1.** Does any solution $u$ of

$$\det D^2 u = 1 + \delta_0 \text{ in } \Omega,$$

(4.1)

has the following decomposition

$$u = h + g,$$

where $h$ is a $C^{2,\alpha}$-smooth function and the graph of $g$ is a convex cone with uniformly convex, smooth $\partial \{\nabla g\}$?

As it was mentioned previously, the two dimensional case was treated by Galves, Jimenez and Mira [61], where the positive answer was given for a bounded domain $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < \rho^2\}$ with an isolated singularity at $(0, 0)$.

The case $n \geq 3$ was studied by Savin in [140], where the $C^{1,1}$ regularity of the tangent cone was obtained for a strictly convex solution to (4.1) in $B_1$.

In this Chapter we obtain answers to Question 4.1 in the following cases:

**Case 1.** $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) and the solution is non-strictly convex.
Then an example introduced in the next section shows that the answer to the question is "no". Indeed, it will be shown that there exists a convex function $\tilde{u}$ satisfying
\[
\det D^2 \tilde{u} = \delta_{p_1} + \delta_{p_2} + C \quad \text{in } B_1,
\]
such that it is only $C^{1,\alpha}$ on the line segment $\overline{p_1p_2}$. Here $\delta_{p_i}$ is a delta measure centered at $p_i$ and $p_1, p_2$ are two points in $B_1 \cap \{x' = 0\}$.

Case 2. $\Omega$ is the entire Euclidean space $\mathbb{R}^n$.

Then the answer follows from results obtained in sections 4.3, 4.4. Namely, Theorem 4.3 and Corollary 1.29 imply that any smooth convex solution of (4.1) in $\mathbb{R}^n$ must be radially symmetric after appropriate coordinate transform. That implies the positive answer to the Question 4.1.

Moreover, in section 4.4 one can find different applications of Theorems 4.3 not only to Monge-Ampère equations with constant nonhomogeneous term but also to more general equations, such as Hessian equations and special Lagrangian equations.

The results of this chapter are presented in [8].

4.2 An example

This section provides an example that shows that the $C^{1,1}$ regularity may fail if the solution of (4.1) is merely convex.

Let $w_1$ be the Pogorelov’s function
\[
w_1(x) = (1 + x_n^2)|x'|^2 - \frac{2}{n},
\]
which vanishes along $x' = (x_1, ..., x_{n-1}) = 0$ and satisfies
\[
\det D^2 w_1 = f \quad \text{in } B_1,
\]
where $f$ is a strictly positive smooth function. Set
\[
w_2(x) = |x - p_1| + |x - p_2|,
\]
where $p_1, p_2 \in B_1 \cap \{x' = 0\}$. Then function $w(x) := w_1(x) + w_2(x)$ is constant on the line segment $\overline{p_1p_2}$ connecting points $p_1$ and $p_2$.

Let $\tilde{u}$ be a solution of
\[
\det D^2 \tilde{u} = \delta_{p_1} + \delta_{p_2} + \inf_{x \in B_1} f \quad \text{in } B_1, \tag{4.2}
\]
\[
\tilde{u} = w \quad \text{on } \partial B_1 \cup \{p_1\} \cup \{p_2\}, \tag{4.3}
\]
where $\delta_{p_i}$ is a delta measure centered at $p_i$.

It is easy to see that $\det D^2 w \geq \det D^2 \bar{u}$, and by comparison principle $\bar{u} \geq w$ in $B_1$. Therefore, $\bar{u}$ is constant on the line segment $\overline{p_1p_2}$ and $\bar{u}$ is only $C^{1,\alpha}$ there. Therefore, the negative answer to the Question 4.1 is obtained.

**Remark 4.2.** The existence of such a solution $\bar{u}$ follows from the following considerations. Suppose $\inf_{x \in B_1} f = 1$ for simplicity. Let $u_1$ be a sub-solution to

$$\det D^2 \bar{u} = \delta_{p_1} + 1 \text{ in } B_1, \quad u = w \text{ on } \partial B_1,$$

and let $u_2$ be a sub-solution to

$$\det D^2 u = \delta_{p_2} + 1 \text{ in } B_1, \quad u = w \text{ on } \partial B_1.$$

Then $u_1 + u_2$ is a sub-solution to

$$\det D^2 u = \delta_{p_1} + \delta_{p_2} + 1 \text{ in } B_1, \quad u = w \text{ on } \partial B_1.$$

Let $U$ be the set of sub-solutions to the problem (4.2)-(4.3). Then $U$ is not empty. Let $\bar{u}(x) = \sup\{u(x)|u \in U\}$. Then $\bar{u}$ is a solution to (4.2)-(4.3).

### 4.3 Symmetry result

Let us now consider Monge-Ampère equation

$$\det D^2 u = f(x, u, Du) \text{ in } \mathbb{R}^n \setminus \{0\}, \quad (4.4)$$

with asymptotic behaviour given by

$$|u(x) - \frac{1}{2}|x|^2| = O(|x|^{-k}),$$
$$|Du(x) - x| = O(|x|^{-k-1}), \quad (4.5)$$

as $|x| \to \infty$, for some $k > 0$.

The main result of this section is as follows.

**Theorem 4.3.** Let $u$ be a $C^2$ solution of (4.4) satisfying asymptotic condition (4.5). Suppose that $f$ is a positive function of class $C^1$, $f$ is symmetric in $(x_1, p_1)$ and satisfies

$$f(x_1, x', z, p_1, ..., p_n) \leq f(\bar{x}_1, x', z, -p_1, p_2, ..., p_n), \quad \text{for } x_1 \geq \bar{x}_1, p_1 \geq 0, \quad (4.6)$$
where \( x' = (x_2, ..., x_n) \).

Then \( u \) is symmetric in \( x_1 \) direction.

The proof of the theorem relies on the moving plane method, which goes back to Aleksandrov [2], Serrin [147] and Gidas, Ni and Nirenberg [64], [65]. We also refer to [13], [14], [109], [108] for other extensions and generalizations of the method.

In all of these papers the maximum principle plays the crucial role, but the papers had to use many forms of maximum principle including the Hopf lemma at the boundary. Various forms of it may be found in [64] and [109], but for the readers convenience we will collect all lemmas used in this Section below.

### 4.3.1 Preliminaries

Let \( u \leq 0 \) be a \( C^2 \) solution of the following differential inequality

\[
Lu := a_{ij}u_{ij} + b_iu_i + cu \geq 0,
\]

in a domain \( \Omega \subset \mathbb{R}^n \). Here \( L \) is a uniformly elliptic operator, i.e., for some constant \( c_0 > 0 \)

\[
a_{ij}\xi_i\xi_j \geq c_0|\xi|^2,
\]

and the coefficients of \( L \) are uniformly bounded in absolute value.

**Maximum Principle 4.4.** If \( u \) vanishes at some point in \( \Omega \) then \( u \equiv 0 \).

**Hopf Lemma 4.5.** Suppose there is a ball contained in \( \Omega \) with a point \( x_0 \in \partial\Omega \cap \partial B \) and suppose \( u \) is differentiable in \( \Omega \cup x_0 \) and \( u(x_0) = 0 \). Then, if \( u \neq 0 \), we have for any outward directional derivative at \( x_0 \),

\[
\frac{\partial u(x_0)}{\partial \nu} > 0.
\]

**Lemma 4.6 ([64]).** Let \( x_0 \in \partial\Omega \) with \( \nu_1(x_0) > 0 \). For some \( \varepsilon > 0 \) assume \( u \) is a \( C^2 \) function satisfying (4.7) in \( \bar{\Omega}_\varepsilon \), where \( \Omega_\varepsilon = \Omega \cap \{|x-x_0| < \varepsilon\} \), \( u < 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial\Omega \cap \{|x-x_0| < \varepsilon\} \). Then \( \exists \delta > 0 \) such that in \( \Omega \cap \{|x-x_0| < \delta\} \),

\[
u_1 > 0.
\]

**Proof.** Since \( u < 0 \) in \( \Omega \), necessarily, \( u_{x_1} \geq 0 \), on \( \partial\Omega \cap \{|x-x_0| < \varepsilon\} = S \), and hence \( u_{x_1} \geq 0 \) on \( S \), for \( \nu_1 > 0 \) everywhere there, which we may assume, by decreasing \( \varepsilon \) if necessary.
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If the lemma were false there would be a sequence of points \(x_j \to x_0\), with \(u_{x_1}(x_j) \leq 0\). For \(j\) large the interval in the \(x_1\) direction going from \(x_j\) to \(\partial \Omega\) hits \(S\) at a point where \(u_{x_1} \geq 0\). Consequently,
\[
    u_{x_1}(x_0) = 0.
\]
On the other hand, \(u\) satisfies (4.7) in \(\Omega_\varepsilon\) and applying Hopf Lemma to the function \(u\) we find \(u_\nu(x_0) > 0\), and so \(u_{x_1}(x_0) > 0\), which contradicts to the previous estimate. The lemma is proved.

4.3.2 Proof of Theorem 4.3

Let \(S_M\) be a section of \(u\) with height \(M\), i.e.
\[
    S_M = \{x \in \mathbb{R}^n | u(x) \leq M\}.
\]
Obviously \(S_M\) is a convex set in \(\mathbb{R}^n\). Denote \(T_\lambda\) to be a hyperplane in \(\mathbb{R}^n\),
\[
    T_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 = \lambda\},
\]
and \(\Sigma(\lambda)\) is a set in \(S_M\),
\[
    \Sigma(\lambda) = \{(x_1, x') \in S_M | x_1 > \lambda\}.
\]

We move the plane \(T_\lambda\) continuously toward the origin, i.e. decrease \(\lambda\) until it touches the boundary \(\partial S_M\). Denote \(\lambda_0\) be the largest value such that \(T_\lambda \cap \bar{S}_M \neq \emptyset\). Set \(\Sigma'(\lambda)\) is the reflection of \(\Sigma(\lambda)\) in the plane \(T_\lambda\). Obviously, \(\Sigma'(\lambda) \subset S_M\) for \(\lambda\) close to \(\lambda_0\).

Then there are two cases when we stop decreasing \(\lambda\):

Case 1. \(T_\lambda\) becomes orthogonal to \(\partial S_M\) at some point.

Case 2. \(\partial \Sigma'(\lambda) \cap (\partial S_M \setminus T_\lambda) \neq \emptyset\).

Let \(\lambda_1\) be the value when \(T_\lambda\) first reaches one of these positions. Obviously, for \(\lambda \in (\lambda_1, \lambda_0)\) we have \(\Sigma'(\lambda) \subset S_M\).

Define functions
\[
    v(x, \lambda) = u(x^\lambda) = u(2\lambda - x_1, x'),
\]
\[
    w(x, \lambda) = v(x, \lambda) - u(x).
\]

Then the proof consists from 3 steps:

In **step 1** we prove for \(\lambda\) close to \(\lambda_0\),
\[
    u(x^\lambda) < u(x) \text{ in } \Sigma(\lambda); \quad (4.8)
\]
\[
    u_1(x) > 0 \text{ on } T_\lambda.
\]
In step 2 we prove (4.8) for all \( \lambda \in (\lambda_1, \lambda_0) \).

In step 3 we show that \( \lambda_1 \to 0 \) as \( M \to \infty \) and \( u \) is symmetric in \( x_1 \) direction.

**Step 1.** Prove \( w(x, \lambda) < 0 \) in \( \Sigma(\lambda) \) for \( \lambda \in (\mu, \lambda_0) \), \( \lambda_0 - \mu > 0 \) small.

Since \( v(x, \lambda) = u(x^\lambda) = u(2\lambda - x_1, x') \) it satisfies

\[
\det D^2 v = f(x^\lambda, v, -v_1, v_2, ..., v_n) \text{ in } \Sigma(\lambda), \ \lambda \in (\mu, \lambda_0).
\]

Then, by (4.6), for \( v_1 \geq 0 \)

\[
\det D^2 v - f(x, v, v_1, ..., v_n) = f(x^\lambda, v, -v_1, v_2, ..., v_n) - f(x, v, v_1, ..., v_n) \geq 0.
\]

Therefore

\[
\det D^2 v - \det D^2 u + f(x, u, u_1, ..., u_n) - f(x, v, v_1, ..., v_n) \geq 0,
\]

and

\[
\bar{L}w := a_{ij} w_{ij} + \bar{b}_i w_i + \bar{c} w \geq 0, \tag{4.9}
\]

provided \( v_1 \geq 0 \), where

\[
\begin{cases}
  a_{ij} = \frac{1}{0} \left[ \text{cofactor of } \{ u_{ij} + t(v_{ij} - u_{ij}) \} \right] dt, \\
  \bar{b}_i = \frac{1}{0} f_{p_i}(x, v + t(u - v), v_i + t(u_i - v_i)) dt, \\
  \bar{c} = \frac{1}{0} f_z(x, v + t(u - v), v_i + t(u_i - v_i)) dt.
\end{cases}
\]

**Claim.** \( w(x, \lambda) \leq 0 \) in \( \Sigma(\lambda) \), \( \lambda \in (\mu, \lambda_0) \).

For any \( \lambda \) close to \( \lambda_0 \) the function \( g(x_1) = e^{-\frac{1}{2\pi}x_1} \) satisfies \( g \geq 1 \) and \( \bar{L}g \leq -1 \) in \( \Sigma(\lambda) \) (see [109]). Obviously we only need to prove that \( \bar{w} := \frac{w}{g} \leq 0 \). If \( v_1 \geq 0 \) the inequality (4.9) implies

\[
L' \bar{w} + \frac{\bar{L}g}{g} \bar{w} = \frac{\bar{L}w}{g} \geq 0 \text{ in } \Sigma(\lambda), \tag{4.10}
\]

where

\[
L' \bar{w} = a_{ij} \bar{w}_{ij} + 2 \frac{a_{1i} g_1 \bar{w}_i}{g} + \bar{b}_i \bar{w}_i.
\]

If the maximum of \( \bar{w} \) is attained on \( \partial \Sigma(\lambda) \) for any \( \lambda \in (\mu, \lambda_0) \), then by the definition of function \( w \) one can see that \( \bar{w}(x, \lambda) \leq 0 \) on \( \partial \Sigma(\lambda) \) and the claim is proved. Therefore, suppose \( \bar{w} \) achieves it's maximum at \((z^0, \lambda)\), for some \( z^0 \in \Sigma(\lambda) \) and \( \lambda \in (\mu, \lambda_0) \). That is \( \bar{w}(z^0, \lambda) > 0 \). Then

\[
\nabla_x \bar{w}(z^0, \lambda) = 0, \quad \{ \bar{w}_{ij}(z^0, \lambda) \} \leq 0.
\]
4.3. SYMMETRY RESULT

From these identities we get

\[ L' \bar{w}(z^0, \bar{\lambda}) \leq 0, \quad \frac{\bar{L} g}{g} \bar{w}(z^0, \bar{\lambda}) < 0. \]

Then at \((z^0, \bar{\lambda})\)

\[ L' \bar{w} + \frac{\bar{L} g}{g} \bar{w} < 0. \quad (4.11) \]

On the other hand, by the definition of \(w\) we know that \(z^0_1 > \bar{\lambda}\). Then \(\bar{w}_\lambda(z^0, \bar{\lambda}) \leq 0\) and \(v_1(z^0, \bar{\lambda}) = -\frac{g}{2} \bar{w}_\lambda(z^0, \bar{\lambda}) \geq 0\). Therefore, (4.10) holds at \((z^0, \bar{\lambda})\) which contradicts to (4.11). The claim is proved.

Now we know that \(w(x, x', \lambda) \leq 0\) in \(\Sigma(\lambda), \lambda \in (\mu, \lambda_0)\);

\(w(x, x', \lambda) = 0\) for \(x_1 = \lambda\).

Therefore, \(w_1(x_1, x', \lambda) = -2u_1(x_1, x') \leq 0\) and \(u_1 \geq 0\) for \(x_1 \geq \mu\).

Derive an elliptic differential inequality for \(u_1 \geq 0\):

\[ \det D^2 v - \det D^2 u + f(x^\lambda, u, -u_1, u_2, \ldots, u_n) - f(x^\lambda, v, -v_1, v_2, \ldots, v_n) \geq 0, \]

and

\[ Lw := a_{ij} w_{ij} + b_i u_i + cw \geq 0, \quad (4.12) \]

where

\[
\begin{align*}
    a_{ij} &= \int_0^1 \text{cofactor of } \{u_{ij} + t(u_{ij} - u_{ij})\} \, dt, \\
    b_i &= \int_0^1 f_{p_i}(x^\lambda, v + t(u - v), -v_1 + t(v_1 - u_1), v_i + t(u_i - v_i)) \, dt, \\
    c &= \int_0^1 f_z(x^\lambda, v + t(u - v), -v_1 + t(v_1 - u_1), v_i + t(u_i - v_i)) \, dt.
\end{align*}
\]

Since \(f > 0\), we have \(w(x_1, x', \lambda) \neq 0\) in \(\Sigma(\lambda), \lambda \in (\mu, \lambda_0)\) for large \(M\). Then applying Maximum principle and Hopf lemma we get for \(\lambda \in [\mu, \lambda_0)\)

\[ w \leq 0 \text{ in } \Sigma(\lambda), \]

\[ u_1 > 0 \text{ on } T_\lambda \cap S_M. \]

**Step 2.** As we have seen above it is enough to prove that for all \(\lambda \in (\lambda_1, \lambda_0)\)

\[ w(x, \lambda) \leq 0 \text{ in } \Sigma(\lambda). \quad (4.13) \]
CHAPTER 4. SINGULAR MONGE-AMPÈRE EQUATIONS

Suppose \( \bar{\mu} \) be the constant such that \([\bar{\mu}, \lambda_0]\) is the maximal interval where (4.13) holds. Repeating arguments from the first step we get

\[
\begin{align*}
  w &< 0 \text{ in } \Sigma(\bar{\mu}), \\
  u_1 &> 0 \text{ on } T_{\bar{\mu}} \cap S_M.
\end{align*}
\]

(4.14)

We only need to prove that \( \bar{\mu} = \lambda_1 \).

Suppose by contradiction that \( \bar{\mu} > \lambda_1 \). Then there exists sequences \( \lambda^k \nearrow \bar{\mu} \) and \( x_k \in \Sigma(\lambda^k) \) such that

\[ w(x_k, \lambda^k) > 0. \]

Without lose of generality, we may assume

\[
\begin{align*}
  w(x_k, \lambda^k) &= \max\{w(x, \lambda^k) | x \in \Sigma(\lambda^k)\}; \\
  x_k &\rightarrow \bar{x} = (\bar{x}_1, \bar{x}') \in \Sigma(\bar{\mu}), \text{ as } k \rightarrow \infty.
\end{align*}
\]

Taking limit, we get

\[ w(\bar{x}, \bar{\mu}) \geq 0. \]

Therefore, by (4.14) we conclude that \( \bar{x} \in \partial \Sigma(\bar{\mu}) \).

If \( \bar{x} \in \partial \Sigma(\bar{\mu}) \setminus T_{\bar{\mu}} \) then \( \bar{x} \) lies in \( S_M \) and

\[ v(\bar{x}, \bar{\mu}) < u(\bar{x}) = M, \quad w(\bar{x}, \bar{\mu}) < 0, \]

which is impossible.

So, we conclude that \( \bar{x} \) is on \( T_{\bar{\mu}} \cap S_M \), i.e. \( \bar{x} = \bar{x} \). For sufficiently large \( k \) the segment \( x_k \rightarrow \bar{x} \) lies in \( S_M \). Since \( u(x_k^\lambda) > u(x_k) \), by the theorem of the mean there exists a point \( z_k \in x_k^\lambda \) such that

\[ u_1(z_k) \leq 0. \]

On the other hand, by (4.14) and Lemma 4.6 we conclude that for some small \( \varepsilon > 0 \)

\[ u_1 > 0 \text{ in } S_M \cap \{x_1 > \bar{\mu} - \varepsilon\}, \]

which contradicts to the previous inequality for sufficiently large \( k \). We finished the second step.

**Step 3.** We first prove that \( \lambda_1 \rightarrow 0 \) as \( M \rightarrow \infty \). By the definition of \( \lambda_1 \) it is enough to prove that for any \( \lambda > 0 \) situations described in Case 1 and Case 2 do not happen, provided \( M \) is sufficiently large.
4.3. SYMMETRY RESULT

Case 1. Let \( \nu(x) \) be the exterior unit normal of \( \partial S_M \) at the boundary point \( x \). By the definition of \( \lambda_1 \) it is easy to see that we only need to prove

\[
\nu_1(x) > 0, \text{ on } \partial S_M \cap T_\varepsilon,
\]

for any \( \varepsilon > 0 \), provided \( M \) is large enough. By Lemma 4.6 it is equivalent to \( u_1(x) > 0 \) on \( \partial S_M \cap T_\varepsilon \). Let \( p \in \partial S_M \cap T_\varepsilon \), then, by (4.5)

\[
u_1(p) \geq \frac{1}{|p|^{k+1}} > 0,
\]

provided \( M \) is large enough. Therefore, for any \( \varepsilon > 0 \), \( T_\varepsilon \) is not orthogonal to \( \partial S_M \) for \( M \) large enough.

Case 2. Let us choose points \( \bar{x} = (\bar{x}_1, 0, ..., 0) \in \partial B_{\sqrt{2M}}(0) \) and \( \tilde{x} = (\tilde{x}_1, 0, ..., 0) \in \partial S_M \) which lie on \( x_1 \) axis, \( x_1 \geq 0 \). Obviously \( \bar{x}_1 = \sqrt{2M} \) and for \( M \) large

\[
\tilde{x}_1 \geq \sqrt{2M} \left(1 - \frac{C}{2M \bar{x}_1^k} - O \left(\frac{1}{M^{2/3/2} \bar{x}_1^k}\right)\right).
\]

Then

\[
|\bar{x} - \tilde{x}| \leq \frac{C}{M^{1/2} \bar{x}_1^k} + O \left(\frac{1}{M^{3/2} \bar{x}_1^k}\right).
\]

Since \( x_1 \) is an arbitrary direction we may conclude that for \( x \in \partial S_M \)

\[
\text{dist}(x, \partial B_{\sqrt{2M}}(0)) \leq C \max_{x \in \partial S_M} \left(\frac{1}{M^{1/2} |x|^k} + \frac{1}{M^{3/2} |x|^{2k}}\right).
\]

In other words, \( S_M \) becomes very close to a ball for \( M \) large. Therefore, for any \( \varepsilon > 0 \), \( \partial \Sigma(\varepsilon) \cap (\partial S_M \setminus T_\varepsilon) = \emptyset \) provided \( M \) is large enough.

Thus, we have proved that \( \lambda_1 \to 0 \) as \( M \to \infty \). We want to show now that \( u \) is symmetric in \( \{x_1 = 0\} \). Then, since \( f \) is symmetric in \( (x_1, p_1) \), by step 2 it follows that

\[
\begin{align*}
u_1 > 0 & \text{ for } x_1 > 0, \\
u_1 < 0 & \text{ for } x_1 < 0.
\end{align*}
\]

Therefore, \( u_1(0) = 0 \). Note that \( w(x, 0) = u(-x_1, x') - u(x_1, x') \leq 0 \).

Suppose that \( w \not\equiv 0 \). Then \( w \) satisfies (4.12) in \( \Sigma(0) \). By maximum principle and Hopf lemma \( w < 0 \) and \( u_1(0, x', 0) = -2u_1(0, x') < 0 \). Hence \( u_1 > 0 \) on \( S_M \cap \{x_1 = 0\} \cap \{|x'| > 0\} \), but \( u_1(0) = 0 \). We get a contradiction. Therefore \( w \equiv 0 \) and \( u \) is symmetric in \( x_1 \) direction.
4.4 Applications of Theorem 4.3

In this section we mention several applications of Theorem 4.3.

Monge-Ampère equation.
Consider equation
\[
\det D^2 u = 1 \text{ in } \mathbb{R}^n \setminus \{0\}.
\]
(4.15)

A well-known result by Caffarelli and Li [37] tells us:

For any convex viscosity solution \( u \) of (4.15) in \( \mathbb{R}^n \setminus \{\bar{O}\} \), where \( O \) is a bounded open convex set of \( \mathbb{R}^n \), there exists \( c, d \in \mathbb{R} \), \( b \in \mathbb{R}^n \), and a positive definite matrix \( A \) with \( \det A = 1 \) such that

(i) for \( n \geq 3 \)
\[
\limsup_{|x| \to \infty} |x|^{n-2} |u(x) - \frac{1}{2} x'Ax + b \cdot x + c| < \infty,
\]
and
\[
\limsup_{|x| \to \infty} |x|^{n-2-k} D^k |u(x) - \frac{1}{2} x'Ax + b \cdot x + c| < \infty, \quad \forall k \geq 1;
\]

(ii) for \( n = 2 \)
\[
\limsup_{|x| \to \infty} |x| |u(x) - \frac{1}{2} x'Ax + b \cdot x + d \log \sqrt{x'Ax} + c| < \infty,
\]
and
\[
\limsup_{|x| \to \infty} |x|^{k+1} D^k |u(x) - \frac{1}{2} x'Ax + b \cdot x + d \log \sqrt{x'Ax} + c| < \infty, \quad \forall k \geq 1.
\]

Therefore, subtracting a linear function from \( u \), rotating and scaling coordinates we can get \( b, c = 0, A = I \). Then, condition (4.5) is satisfied and Theorem 4.3 implies:

Corollary 4.7. Let \( u \) be a smooth convex solution of (4.15) in \( \mathbb{R}^n \setminus \{0\} \), \( n \geq 2 \). Then \( u \) is rotationally symmetric about the origin after a corresponding coordinate transform.

\( k \)-Hessian equation.
It is easy to see that the arguments in the proof of Theorem 4.3 can be extended also for the following \( k \)-Hessian equation
\[
\sigma_k(\lambda(D^2 u)) = 1 \text{ in } \mathbb{R}^n \setminus \{0\},
\]
(4.16)
where
\[ \sigma_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1}...\lambda_{i_k}. \]

During the last few years there were several papers concerning the existence of a solution to equation (4.16) on exterior domains with prescribed asymptotic behaviour (see [47], [46], [110], [151], [19]). Due to these results, the following theorem is well known.

**Theorem 4.8.** Let \( n \geq 3 \), then there exists \( c_0 \) such that for any \( c > c_0 \), there exists a unique \( k \)-convex function \( u \in C^0(\mathbb{R}^n \setminus \{0\}) \) satisfying (4.16) in the viscosity sense and
\[
\limsup_{x \to \infty} \left( |x|^{n-2} \left| u(x) - \left( \frac{c^*}{2} |x|^2 + c \right) \right| \right) < \infty,
\]
where \( c^* = \left( \frac{1}{C^n} \right)^{1/k} \), \( C^n_k = \frac{n!}{k!(n-k)!} \).

Then Theorem 4.3 implies:

**Corollary 4.9.** The solution \( u \) from Theorem 4.8 is rotationally symmetric about the origin after a corresponding coordinate transform.

**Special Lagrangian equation.**

Let us now consider the following equation
\[
\sum_{i=1}^{n} \arctan \lambda_i(D^2u) = \Theta \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.
\]

When \( \Theta = \frac{\pi}{2} \) the equation (4.18) becomes equivalent to Monge-Ampère equation (4.15) (for \( n = 2 \)), or to Hessian equation (4.16) (for \( n = 3 \)). In a recent paper by D. Li, Z. Li and Y. Yuan [112] it was proved:

For any smooth solution of (4.18) with \( \Theta > (n-2)\pi/2 \) there is a unique quadratic polynomial \( Q(x) \) such that

(i) for \( n \geq 3 \)
\[
u(x) = Q(x) + O_k(|x|^{2-n}), \quad \text{as} \quad |x| \to \infty, \quad \forall k \in \mathbb{N},
\]

(ii) for \( n = 2 \)
\[
u(x) = Q(x) + O_k(|x|^{-1}), \quad \text{as} \quad |x| \to \infty, \quad \forall k \in \mathbb{N}.
\]
The proof of Theorem 4.3 can be also applied to equation (4.18) with a minor difference. Indeed, following the arguments of section 4.3 one can see that condition (4.6) in this case can be easily replaced by $F(p_{11},...,p_{nn}) = F(-p_{11}, p_{1\alpha}, p_{\alpha\beta})$, which is satisfied for

$$F(D^2 u) = \sum_{i=1}^{n} \arctan \lambda_i (D^2 u) - \Theta.$$ 

Then subtracting a linear function from $u$ and rotating coordinates we get the following result:

**Corollary 4.10.** Let $u$ be a smooth solution of (4.18) with $\Theta > (n - 2)\pi/2$. Then $u$ is symmetric in each direction.
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