Multidimensional photonics in synthetic lattices

Kai Wang

A thesis submitted for the degree of
Doctor of Philosophy of
The Australian National University

November, 2019

© Copyright by Kai Wang 2019
All Rights Reserved
Declaration

This thesis is an account of research undertaken between November 2015 and August 2019 at the Nonlinear Physics Centre within in the Research School of Physics (previously named Research School of Physics and Engineering until July 2019) at The Australian National University while enrolled in the Doctor of Philosophy degree.

The research was carried out under the supervision of Assoc. Prof. Andrey A. Sukhorukov, Prof. Dragomir N. Neshev, and Dr. Alexander S. Solntsev.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Kai Wang
November, 2019
List of publications

Papers in peer-reviewed journals

(Papers containing results included in this thesis are listed in bold)


*Equal contributions*


Unpublished papers

(Papers containing results included in this thesis are listed in bold)


Conference proceedings


(12) K. Wang, L. J. Maczewsky, A. A. Dovgiy, A. E. Miroshnichenko, A. Moroz, D. N. Christodoulides, A. Szameit, and A. A. Sukhorukov, "High-dimensional synthetic lattice with enhanced defect sensitivity in planar photonic structures," in Advanced Photonics


(22) S. J. U. White, K. Wang, T. T. Tran, M. Kianinia, J. Titchener, M. Gräfe, S. Fischbach, S. Rodt, J. Song, S. Reitzenstein, I. Aharonovich, A. A. Sukhorukov, A. Szameit, and A. S.


**Selected conference talks**


Acknowledgment

First of all, I gratefully thank my PhD supervisory panel. I deeply appreciate the countless help and strong support from my primary supervisor Assoc. Prof. Andrey A. Sukhorukov, who is a faithful example of theoretical physicists in the solidity of knowledge, accuracy in communication and uncompromising pursuit of useful research. Coming from an experimental background, I benefited from both our similarities and complementarities. I am also highly grateful to Prof. Dragomir N. Neshev for his invaluable efforts towards a relaxed atmosphere and respectful culture in the experimental group, and for his remarkable social skills which I have truly benefited from. I would also like to thank Dr. Alexander S. Solntsev, for being not only a great advisor but also a kind listener, and for our resonating discussions.

I am deeply indebted to Prof. Yuri S. Kivshar, the head of an amazing department, for his powerful leadership and kind help in various aspects, including but not limited to, countless free coffee provided in the department. I would also like to thank Mrs. Kathy Hicks for her exceptional administrative support.

I am highly grateful to Prof. Alexander Szameit, supervisor of my Master thesis, for becoming my main collaborator during PhD and hosting me in two extended visits to his current research group in Germany for a total length of over four months. I also deeply appreciate Prof. Shanhui Fan, Prof. Demetrios Christodoulides, and Assoc. Prof. Mercedeh Khajavikhan for cordially inviting me to their groups in the US for in total over three months and for openly sharing their insights into research frontiers.

Furthermore, I would like to thank all other colleagues and external collaborators who worked together on various projects during my PhD. This includes but is not restricted to James G. Titchener, Shaun Lung, Hung-Pin Chung, Lei Xu, Bryn A. Bell, Matthew Parry, Sergey V. Suchkov, Alexander A. Dovgiy, Steffen Weimann, Lukas J. Maczewsky, Matthias Heinrich, Nicolas Pedersen, Yu Shi, Falk Eilenberger, Ivan I. Kravchenko, Khosro Z. Kamali, Sergey Kruk, Yen-Hung Chen, Mohsen Rahmani, and Benjamin Eggleton. Moreover, despite impossible to mention all names here, I am also grateful to those who did not directly work with me but interacted with or helped me during my PhD, such as helps in the lab, discussions, and encouragements.

Besides, I gratefully thank the financial support from different grants and scholarships, including various grants from the Australian Research Council, the Australia-Germany Joint Research Cooperation Scheme, Robert and Helen Crompton Award, SPIE Optics and Photon-
ics Education Scholarship, and ANU Vice Chancellor's Travel Grant.

Last yet not least, I would like to thank my friends and family for their support – my parents in particular, who deeply love and strongly support me as they always do.
Abstract

Geometrical dimensionality is known to have a great impact on the fundamental characteristics of physical systems, from the stability of planetary orbits to the occurrence of Anderson localization. Importantly, accessing higher dimensions is central to the fundamental enhancement of performance in many applications, such as deep neural network and scalable quantum computation. Discrete photonic state is a potential candidate for information processing due to its well-maintained coherence at the room temperature. The manipulation of these states is typically based on periodic photonic structures interacting with light analogous to electrons in electronic lattices. However, the direct use of such usually planar structures restricts the physical behaviors of photonic states to be within low-dimensional regimes. Recently, synthetic lattices appear as an emerging and rapidly growing topic in photonics, where artificial engineering of discrete potentials are employed to lift up limitations imposed by the geometry. Yet it remains unknown in many aspects how such synthetic lattices can be utilized to facilitate multidimensional photonics.

This thesis focuses on the theoretical and experimental study of multidimensional photonics that uses synthetic lattices for both classical and quantum light. More specifically, we focus on three key aspects in multidimensional photonics, i.e. forming multidimensional lattices, harnessing multidimensional properties and manipulating multidimensional states. We test the feasibility of our concepts on a variety of experimental platforms, including nanosstructured metasurfaces, integrated waveguides, and nonlinear fibers.

First, we introduce and experimentally demonstrate a controllable all-optical approach to synthetic frequency lattices in a nonlinear waveguide, which is suitable to implement long-range interactions and artificial gauge fields. This platform, based on coherent frequency conversion mediated by parametric nonlinearity, enables flexible implementation of synthetic lattices without complex electro-optic modulation. We employ such lattices to generalize the discrete Talbot effect to new instances, synthesize dimensions, and perform measurements of frequency combs.

Secondly, we formulate and experimentally realize a new paradigm that utilizes a tailored isospectral transformation to map arbitrary dimensional networks to planar photonic structures while maintaining excitation dynamics. This facilitates our theoretical and experimental exploration of bound states at the edge of the continuum associated with a sharp transition in higher-dimensional defect localization, paving the way for versatile potential applications from quantum simulation to advanced sensing with the fundamental enhancement
from higher dimensional physics.

Thirdly, we focus on the manipulation and measurement of multidimensional photonic states spanned by entangled photons of quantum light. Specifically, we introduce a new concept with advanced imaging to map multiphoton quantum states to an optimized set of correlation measurements. Such a transformation is achieved by metagrating lattices interleaved on a nanostructured metasurface. We present the experimental observation of multiphoton interference on metasurfaces and reconstruction of single- and multi-photon quantum states using such flat imaging optics.

Finally, we introduce the concept of inline detection and reconstruction of multiphoton quantum states in quantum-photonic circuits. While conventional quantum state measurements were at the output of devices, we make judicious use of integrated detectors for measuring quantum states inside circuits without changing the measured quantum state except for a small overall loss. We perform proof-of-principle experiments and utilize this scheme in the monitoring of coherence in parity-time symmetric couplers.
Contents

Declaration i
List of publications iii
Acknowledgment ix
Abstract xi

1 Plenty of room in higher dimensions: An introduction 1
  1.1 Multidimensional networks in photonics 3
    1.1.1 Tight-binding photonic lattices and dimensionality 4
    1.1.2 Multidimensional photonic states 6
  1.2 Facilitating multidimensional photonics with synthetic lattices 9
    1.2.1 Forming multidimensional lattices 10
    1.2.2 Harnessing multidimensional properties 13
    1.2.3 Manipulating multidimensional photonic states 14
  1.3 Scope and outline of the thesis 18

2 Multidimensional synthetic frequency lattices in nonlinear waveguides 21
  2.1 Introduction 21
  2.2 Frequency lattice with long-range coupling in an optical fiber 23
  2.3 Discrete Talbot effect with arbitrary periodicity 28
  2.4 Synthetic multidimensional space with nontrivial gauge field 31
  2.5 Single-shot measurement of frequency combs with synthetic lattices 38
  2.6 Generalization to quadratic nonlinearity 44
  2.7 Summary 45

3 Realizing higher dimensional dynamics in planar photonics 47
  3.1 Introduction 47
  3.2 Observing multidimensional excitation dynamics in 1D lattices 50
  3.3 Sharp localization transition in four and higher dimensions 58
  3.4 Freeform engineering of local density of states for lattice design 67
  3.5 Non-Hermitian generalization 71
  3.6 Summary 74

4 Imaging-based multiphoton state reconstruction with metasurfaces 77
  4.1 Introduction 77
  4.2 Optimal reconstruction of multiphoton states with correlation measurements 79
### CONTENTS

4.3 Metasurface for quantum state measurement ........................................... 84  
4.4 Multiphoton interference and state reconstruction with metasurfaces ........... 90  
4.5 Summary ................................................................................................. 95  

5 Inline detection of multiphoton states in quantum circuits 99  
5.1 Introduction ........................................................................................... 99  
5.2 Multiphoton state monitoring with integrated detectors ............................ 100  
5.3 Reconstruction of spatially-encoded photonic states ................................. 105  
5.4 Measurement of coherence in parity-time symmetric couplers ................. 106  
5.5 Summary ................................................................................................. 111  

6 Conclusion and outlook 113  

Appendix 117  
A.1 Theoretical details ................................................................................... 117  
  A.1.1 Coupled mode theory of weakly guided waveguides ............................ 117  
  A.1.2 Simulation of synthetic lattice and comparison with experiment ........... 119  
  A.1.3 Lanczos algorithm ............................................................................... 126  
A.2 Experimental methods ............................................................................... 127  
  A.2.1 Waveguide fabrication and measurement ................................. 127  
  A.2.2 Metasurface fabrication and characterization ................................. 131  
  A.2.3 Two-photon state generation and characterization ............................ 132  

Bibliography 139
I too, have lived many years for ideals which I would never be able to explain to myself; but I do something entirely good. I live on trees.

Italo Calvino, *The Baron in the Trees*
It has been six decades since Richard P. Feynman delivered his classic talk "There's Plenty of Room at the Bottom" [1] in December 1959. It was in this talk that he pointed out a pathway towards technologies that enable the observation and manipulation of matter inside a "staggeringly small world" [1]. This pathway, without necessarily relying on discoveries in the fundamental laws of physics, has been proven surprisingly powerful. Nowadays, the advances in nanotechnology have gone way beyond his prognostications, the influence of which spans all aspects of human life.

Among the so many drastic changes these technologies have enabled, an especially influential one to daily life lies in a so-called third industrial revolution characterized by digital information processing. Central to such advances is the semiconductor technology that has substantially minimized the size of computational units, confirming the predictions that wires in computers "should be 10 or 100 atoms in diameter, and the circuits should be a few thousand angstroms across" [1]. Lately, chips incorporating integrated electronic circuits composed of fin field-effect transistors (FinFETs) started using a 7-nm process, substantially exceeding Feynman’s predictions. However, the hardware technology facilitating such rapid growth may face saturation shortly. Moore’s law predicting the exponential growth of the number of transistors per unit area is expected to be no longer fully valid if there is no drastic change in the working principle of transistors. Meanwhile, fabrication becomes far less efficient as the size continues to go down. For instance, nowadays it remains under hot debate in semiconductor industry if it is truly beneficial to move towards 5-nm processes. As such, to further boost the information processing capacity, it becomes more important than ever to make better use of fundamental physics as a driving force to technological progress.

Geometrical dimensionality is known to greatly impact on the characteristics of physical systems [2, 3]. For example, the occurrence of Anderson localization [2, 4–6] and the stability of planetary orbits [3] are functions of the number of the involved spatial dimensions. While experiments are fundamentally limited by the three-dimensional space, a possibility to effectively go beyond three dimensions is of both fundamental and practical interests. Such a capability can facilitate the experimental observation of higher dimensional physics, and importantly, it can play a key role in the fundamental enhancement of performances in various devices. An example of early efforts trying to access higher dimensionality is the hypercubic architecture in computers [7] aiming at parallel computing that reduces the travel time of data and hence enhances speed. Moreover, in quantum simulation there have been tremen-
dous efforts to synthesize dimensions by making use of extra degrees of freedom \[8\], trying to benefit from the many fundamental aspects associated with increased dimensionality.

Importantly, note that when Feynman gave this talk there was even no artificial source of coherent light, as the first functioning laser was only demonstrated in the following year, 1960 \[9\]. While Feynman's prognostications were mostly focused on electronics and such systems indeed have played a key role in the information technology until today, the use of solid-state systems also has shortcomings. For example, a key problem with electronics is that it is difficult for electrons in solid-state systems to maintain coherent at easily accessible temperature, thus their manifestations in the wave nature usually appear suppressed. While tremendous efforts have been devoted to engineered quantum solid-state systems, such as cold atoms \[8\] and superconducting qubits \[10\], they are typically associated with rather complex arrangements and low temperature. By contrast, a coherent superposition with laser light is even possible in classical linear optics at the room temperature. Moreover, research efforts in the past decades in quantum optics have also boosted technologies on harnessing the particle nature of light. Therefore, light can be a promising candidate for novel types of information processing platform in both classical and quantum regimes. Hence, light science can be also promising in harnessing multidimensional physics for fundamental and practical applications, where the rapidly growing research on multidimensional photonics is of great hope.

Nowadays, advances in fabrication have enabled the fine structuring of optical materials for light-matter interaction in photonic periodic structures analogous to electrons in atomic potentials. One of the main focuses in the research field of photonics lies in the study of light interacting with such lattice-like structures in both classical and quantum regimes. This is also being combined with many more recent topics such as deep learning and virtual reality, promising unprecedented new advances in various areas. Notably, the use of photonic lattices does not necessarily need to faithfully replicate structures that are already known to exist in nature. Rather, artificially engineered lattices beyond direct analogs of natural lattices, known as synthetic lattices, are of even more interest. Particularly, such novel structures may overcome limitations imposed by geometry, thereby constituting a versatile tool to facilitate multidimensional photonics.

In this vein, instead of debating on if there is still plenty of room at the bottom, here we attach more importance to the fact that there is plenty of room in higher dimensions. Indeed, multidimensional physics offers us many new opportunities, which can be explored with the current technological advances in artificially engineer photonic structures. Therefore, in this thesis, we will focus on tackling key challenges in multidimensional photonics using synthetic lattices.
1.1 Multidimensional networks in photonics

Many discrete photonic systems can be described by a network model with nodes and connections. In such networks, despite direct imagination should fail beyond three-dimensional space, the network geometry in higher dimensions can be mathematically derived. To illustrate this, we show a simple example as sketched in Figures 1.1(a–d) for multidimensional networks from one to four dimensions. Consider the simplest case with two nodes (denoted by the yellow balls) in one-dimensional (1D) space [Figure 1.1(a)], there can be only one connection between the two nodes (gray line). Its two dimensional (2D) generalization is a square network with $2^2 = 4$ nodes [Figure 1.1(b)]. In this 2D square network, each node is connected with two nearest neighbors, giving rise to in total 4 connections. Then, if we consider three-dimensional (3D) space, the generalization becomes a cubic network with $2^3 = 8$ nodes [Figure 1.1(c)]. Now each node connects to 3 nearest neighbors and hence there are in total 12 such connections.

From this analysis, we can mathematically derive the cases in even higher, $N_d$-dimensions, which is basically a network structure with $2^{N_d}$ nodes and $N_d2^{N_d-1}$ nearest-neighbor connections. For example, despite we are not able to directly visualize the case in four-dimensional (4D) space, we can construct an effective 3D illustration. This is shown in Figure 1.1(d), where we sketch a 4D hypercubic network using two 3D cubic networks and interconnect the nodes of the two cubes. By doing so, the network in Figure 1.1(d) fulfills our mathematical derivation for $N_d = 4$ (16 nodes and 32 connections). Despite in 4D space the nodes can be arranged in a way such that all these connections are equally short-range (i.e. only connecting nearest neighbors), the network geometry of our illustration in Figure 1.1(d) is effectively in 4D as the structure ensures an equivalence in node number and their connectivity. Therefore, for a given number of nodes in a network, geometrical dimensionality is directly determined by the connectivity of these nodes.

![Figure 1.1](image.png)

**Figure 1.1:** Multidimensional hypercubic networks in (a) one-, (b) two-, (c) three-, and (d) four- geometrical dimensions.

As such, geometrical dimensionality naturally occurs in various physical systems and its
impact can play an important role in their manifestations. For example, the extremely powerful capability of the human brain as a processor is associated with neural networks with long-range connections. It has been studied that such connectivity can effectively correspond to 11 geometrical dimensions. Such multidimensional nature does not only enable a faster communication of information but also fundamentally exhibit a better capability to deal with tasks due to the inherent complexity of these networks. This has inspired many techniques creating artificial intelligence, from the hypercubic computer architecture decades ago to the currently extensively studied topic of deep neural networks.

While in many different aspects photonic systems can be modeled as multidimensional networks, two most common ones are multidimensional photonic lattices described by a tight-binding model and multidimensional photonic states. Thereby, in the upcoming two subsections, we will discuss some basics on these two systems.

1.1.1 Tight-binding photonic lattices and dimensionality

Discrete lattice dynamics play an important role in various branches of physics, from condensed-matter physics to topological photonics. Photonic systems constitute a versatile platform to implement Hamiltonian systems described by the tight-binding model, using various systems such as arrays of evanescently coupled waveguides, metamaterials and coupled resonators.

In Figure 1.2 we sketch a 1D exemplary tight-binding lattice with five lattice sites denoted by the blue balls numbered by \( m = 1, 2, \ldots, 5 \). In this model, we can treat each site as a potential with an eigen-frequency of \( \beta_m \), while there can also be a coherent exchange of energy between lattice sites. The latter can be described by a coupling constant \( C_{n,m} \) denoting the rate of coupling (hopping) from site \( n \) to site \( m \). Some couplings are short-range, or equivalently, local, which are only between nearest neighbors, i.e. \( |m-n| = 1 \). For \( |m-n| > 1 \) such interactions are termed long-range, or equivalently nonlocal couplings. For example, in Figure 1.2 the couplings denoted by arrows between site 1 and 2 are local couplings, whereas those between site 1 and 3 are nonlocal ones. The time dynamics of such a lattice is governed...
by Schrödinger’s equation in the discrete form

\[ i \frac{d}{dt} a_m(t) = \beta_m a_m(t) + \sum_n C_{n,m} a_n(t), \]  

(1.1)

where \( t \) represents time, \( a_m(t) \) denotes the time-dependent probability amplitude in lattice site \( m \) and \( i \) represents the imaginary unit. The integer \( m \) runs over the sites of the lattice and \( n - m \) runs over the orders of coupling.

It can be seen from this example that the structure of a tight-binding lattice is inherently a network. Hence, dimensionality is also relevant in such lattices, where light can be coherently manipulated. Higher-dimensional properties may be manifested in such photonic lattices if they can be implemented effectively multidimensional. In general, for such networks higher dimensionality means more connectivity. However, in the commonly-employed implementations of photonic lattices, it is highly challenging to effectively arrange higher dimensional networks. Now we outline the reasons by briefly introducing the main approaches of experimental realizations.

![Diagram of photonic lattices](image)

**Figure 1.3:** Example implementations of photonic lattices, including (a) Coupled waveguides and (b) Coupled-resonator optical waveguides (CROW) adapted from Reference [17].

While there can be many different realizations of spatial photonic lattices, a most widely
used platform lies in coupled waveguides [14, 18–23]. Short-time waveguides can be formed by structured light illumination in photorefractive crystals [18]. Permanent structures can be fabricated as weakly guiding structures with methods such as femtosecond laser writing [19, 20, 22] and diffusion methods [21], or as high-index-contrast waveguides by electron-beam lithography [23] and other lithography approaches. Typically, these waveguide lattices can be described by a coupled-mode equation (see Appendix Section A.1.1 for details), which is formally equivalent as tight-binding Schrödinger’s equation. An example is given in Figure 1.3(a). In such waveguide lattices, usually, time is mapped to spatial propagation $z$, which is convenient to observe the initial evolution of an input photonic state. Each lattice site is represented by one waveguide, where its site potential in the tight-binding lattice is usually controlled by varying the refractive index or geometry of the waveguide. The coupling is usually based on the evanescent field of modes propagating in each waveguide, and therefore the coupling rate exponentially decays in space. Thus it is challenging to realize long-range coupling on this platform.

Another important type of spatial photonic lattices is coupled resonators [16, 17, 24], for which we show an example in Figure 1.3(b) adapted from Reference [17]. Coupled resonators typically operate in real-time, the temporal-spectral behavior of which can be described by a temporal coupled-mode theory [25]. On this platform, each resonator usually represents a lattice site. The site potential is essentially the resonance frequency of each resonator, which can be finely tuned by dynamic elements such as thermal tuners [24]. One way to introduce coupling between lattice sites is to employ the mode overlaps of resonators [24], a similar working principle as the coupled waveguides discussed above. Therefore such resonators are also difficult to exhibit long-range coupling. Another common approach to induce coupling is via waveguides interconnecting different resonators, known as coupled-resonator optical waveguides (CROWs) [16, 17]. Since these waveguides are based on planar photonic structures where waveguides cannot jump one over another, therefore it also appears highly difficult to go beyond 2D space on this platform.

Overall, it is highly challenging for both the waveguide lattices and coupled resonators to be effectively multidimensional. Hence it becomes vital to develop new lattices beyond natural analogs, which we will introduce in the upcoming Section 1.2.

1.1.2 Multidimensional photonic states

Geometrical dimensionality is also inherent to the classical and quantum light encoded in various degrees of freedom, known as multidimensional photonic states. Note that there can be different meanings when a photonic state is called multidimensional. Quite commonly, this refers to the dimensionality of the Hilbert space of the state vector. Indeed, boosting the Hilbert dimensionality has stimulated great research interest at a hope to enhance the information capacity encoded in photons. For example, this is one of the main motivations in developing quantum frequency combs [26, 27] that can have many modes even for single-photon states [27]. However, here we focus on the geometrical dimensionality of the space
that the state inhabits, which is a fundamental factor in the scalability of photonic states. For example, an important aspect of the strength of quantum computing lies in the exponential growth of the state space with particle number, which is inherently due to an enlarged geometrical dimensionality. We would also emphasize that a large Hilbert space can be combined with the multiple geometrical dimensions to further boost the capacity of photonic states.

Figure 1.4: Basic concepts in multidimentional quantum states. (a) Bloch sphere describing the amplitude, phase and coherence of a two-mode state, where the red arrow denotes a pure state and the blue arrow represents a mixed state. (b) Scaling up the space of the state representation to higher dimensions using multiple photons in the representation of horizontal (H) and vertical (V) polarization.

First, we briefly introduce some basics on photonic states. Consider classical coherent light that has a polarization degree of freedom, which can be represented in the bases of horizontal (|H\rangle) and vertical (|V\rangle) polarization components. If we don’t consider all other degrees of freedom, we can write the state vector |\psi\rangle as a coherent superposition of these two components

$$|\psi\rangle = c_H |H\rangle + c_V |V\rangle,$$

(1.2)

where $c_H$ and $c_V$ are the complex amplitudes of the horizontal and vertical polarization components, respectively, and if the state is normalized to unity intensity we have $|c_H|^2 + |c_V|^2 = 1$. If we don’t consider the global phase factor of $|\psi\rangle$, all such states can be represented as vectors on a sphere, known as the Bloch sphere (or Poincaré sphere specifically for the polarization of light). An exemplary Bloch sphere is sketched in Figure 1.4(a), where the red vector denotes such a $|\psi\rangle$ on the sphere. For $|\psi\rangle$ normalized to unity modulus, the coordinates in the Bloch space are given by $S_n = \langle \psi | \hat{\sigma}_n | \psi \rangle$, where $\hat{\sigma}_1 = \hat{\sigma}_z$, $\hat{\sigma}_2 = \hat{\sigma}_x$ and $\hat{\sigma}_3 = \hat{\sigma}_y$ are the Pauli matrices. If a state is on the equator of the Bloch sphere, there is no phase difference between $c_H$ and $c_V$, giving rise to linear polarization. The north and south poles correspond to states with $\pm \pi/2$ phase differences between $c_H$ and $c_V$ that are equal in modulus, giving rise to circular polarization states. The rest area on the sphere represents the more general, elliptical polarization states. In practice, light may not be fully polarized – there exists partially polarized or unpolarized light. Then a state vector $|\psi\rangle$ is no longer sufficient to describe this, since the amplitudes and phases inside parameters of $|\psi\rangle$ can only be definite values. A statistical
description of more general states is the density matrix, which is defined by

\[ \hat{\rho} = \sum_m b_m |\psi_m\rangle \langle \psi_m|, \]  

(1.3)

which is an incoherent superposition of different states \(|\psi_m\rangle\) weighted by a distribution in the set \(\{b_m\}\). Here \(m\) is an index running over all contributed states in the incoherent superposition. For the density matrix \(\hat{\rho}\), its projection to the Bloch space can also be calculated by

\[ S_n = \text{Tr}(\hat{\rho} \hat{\sigma}_n). \]

A not-perfectly polarized state is located inside the Bloch sphere, see the blue arrow for example in Figure 1.4(a).

While it is possible to use different degrees of freedom to implement multidimensional states in photonics, such as polarization and orbital angular momentum (OAM), an especially nontrivial and scalable type of such states are spanned by the multi-particle nature of quantum light. Quantum properties of multiple entangled photons underpin a broad range of applications [26, 28–30] encompassing enhanced sensing, imaging, secure communications, and information processing. Therefore here we introduce the multidimensional states whose geometrical dimensionality is determined by the photon number.

To clarify, for multiphoton cases considered in this thesis, a basic assumption is a known photon number \(N\), or equivalently, a Fock state \(|N\rangle\) in photon number space [31]. This is a widely used assumption in many applications of quantum optics. Note that this consideration does not lose generality, as in linear and static photonic systems there is no crosstalk between different Fock states, allowing each sub-ensemble with a certain photon number to be treated individually.

Now we illustrate such multidimensional states with the example of multiphoton polarization states sketched in Figure 1.4(b). The single-photon case has a similar description as the classical case given above, which inherently inhabits in a one-dimensional space [Figure 1.4(b, left)]. Then, we consider a two-photon polarization state, the bases of which are in a two-dimensional space as sketched in Figure 1.4(b, middle). In this case, the state vector \(|\psi^{(2)}\rangle\) is spanned by a bases of polarization behaviors of two photons, i.e.

\[ |\psi^{(2)}\rangle = c_{HH} |HH\rangle + c_{HV} |HV\rangle + c_{VH} |VH\rangle + c_{VV} |VV\rangle. \]  

(1.4)

If we consider a three-photon polarization state, as illustrated in Figure 1.4(b, right), the state can be decomposed to 8 bases in the three-dimensional cubic geometry.

Importantly, separable photon behaviors, in general, do not fully characterize the degrees of freedom inside a multiphoton state. Consider the product of single-photon polarization states for \(N\) photons, i.e.

\[ |\psi_{\text{sep}}^{(N)}\rangle = |\psi_{\text{1}}^{(1)}\rangle \otimes |\psi_{\text{2}}^{(1)}\rangle \otimes \cdots \otimes |\psi_{\text{N}}^{(1)}\rangle, \]  

(1.5)

where \(\otimes\) denotes the Kronecker product. Since in each \(|\psi_{\text{n}}^{(1)}\rangle\) there can be two complex free parameters (if we don't consider normalization), then there are maximally \(2N\) complex free parameters in this so-called separable state \(|\psi_{\text{sep}}^{(N)}\rangle\). In contrast, the growth of geometrical di-
mensionality indicates an exponential increase of complex-valued free parameters $2^N$ with photon number $N$. As the case given in Equation (1.5) fully describes the free parameters in the state with local variables of each photon, therefore to obtain the extra degrees of freedom it is necessary to have nonlocal behaviors among the multiple photons. This is a core idea inside quantum entanglement in multiphoton states, where entanglement is the enabling factor to the free choice of all parameters in the multidimensional space of the state bases.

A density matrix description of multiphoton states can equally be established following the principle given in Equation (1.3). Note that for a multiphoton density matrix, both local and nonlocal types of coherence are included, where the latter is directly relevant to the degree of entanglement [32]. While in practice it is highly important to not only determine the wavefunction of multiphoton states but also the density matrix in order to gain information about coherence and entanglement, yet a full measurement of such multiphoton states can be highly challenging. This is due to the requirement to implement multiphoton interference in this process. Therefore, one key purpose in the manipulation of such multiphoton states lies in measuring its density matrix, which constitutes an important motivation of this thesis. We will discuss more details on the possible platforms to facilitate this in the upcoming Section 1.2.

### 1.2 Facilitating multidimensional photonics with synthetic lattices

In the above section, we have outlined some main implementations of photonic lattices. As seen, most photonic lattices are limited by the geometry and thus difficult to exhibit multidimensional properties. Importantly, despite photonic lattices are inspired by the atomic potentials, there is way more flexibility enabled by the designer’s engineering beyond direct analogs. This gives rise to the concept of *synthetic lattice*, referring to a lattice that has no direct counterpart in natural materials. In photonics, this includes both lattices that use a synthetic degree of freedom other than space and those lattices arranged in space with certain engineering. A key advantage of using synthetic lattices lies in the great possibility to observe effects that are difficult to be achieved in natural lattices. Notably, many synthetic lattices are not restricted by geometry, giving rise to tremendous opportunities to facilitate multidimensional photonics. Such synthetic lattices may also provide solutions to challenges in the manipulation and measurement of multidimensional photonic states.

As this thesis aims at using synthetic lattices to solve key problems in multidimensional photonics, here we summarize three main aspects in which synthetic lattices are expected to be useful in accessing higher dimensions:

- **Forming multidimensional lattices.** This aspect focuses on the implementation of photonic lattices that are mathematically equivalent to multidimensional lattices.

- **Harnessing multidimensional properties.** This aspect does not rely on faithfully replicating the multidimensional lattice itself; rather, it aims at having certain multidimentionality.
sional properties manifested in low-dimensional lattices.

- **Manipulating multidimensional photonic states.** This aspect focuses on the manipulation of multidimensional photonic states of classical and quantum light in discrete representations, regardless of whether the lattice by itself is multidimensional or not.

Now we follow the order of the above three aspects and provide some introductory discussions.

### 1.2.1 Forming multidimensional lattices

Firstly we discuss the goal on the development of methods and experimental platforms that can effectively obtain multidimensional lattices. Here the general purpose is to exactly replicate the lattices sites and their interactions in limited dimensions up to three-dimensional space.

As a most widely used approach, synthetic space can be formed by using extra degrees of freedom than the spatial lattice itself. We illustrate this approach with the example from its first proposal in photonics \[33, 34\]. As shown in Figure 1.5(a, top), a one-dimensional spatial lattice can be implemented by a line of photonic ring resonators along the spatial dimension denoted by \( \hat{z} \). To realize higher dimensions beyond the one-dimensional spatial lattice, frequency eigenmodes of each resonator is utilized as a second *synthetic dimension*. This is shown as lattice sites separated by \( \Omega \) in Figure 1.5(a, bottom) along the frequency dimension denoted by \( f \). This approach dates back to early proposals in quantum simulation using cold atoms \[8\]. Many theoretical proposals make use of this approach to facilitate higher-dimensional topological physics \[35, 36\]. Such proposals in photonics include four-dimensional quantum Hall effect in three-dimensional lattices \[34\], photonic Weyl point in two-dimensional lattices \[37, 38\], three-dimensional photonic topological insulator using two-dimensional lattices \[39\], and parity-time symmetric lasing \[40\].

![Figure 1.5: Reported methods to implement multidimensional structures in photonics. (a) Implementation of higher dimension with an extra degree of freedom than the spatial lattice, adapted from Reference \[33\]. (b) Higher dimensional structure by longer coupling ranges, adapted from Reference \[41\].](image)
Experimentally, this approach has enabled the realization of photonic topological insulator in synthetic dimension \[42\] using spatial eigenmodes as an extra dimension. Other recent efforts are mostly focused on experiments with dynamically modulated fiber resonators \[43, 44\]. However, the approach of using extra degrees of freedom is limited by the possible degrees of freedom the system can have, therefore it is not well scalable to arbitrary dimensions.

To develop methods with better scalability, another approach has been developed by introducing more connectivity between the lattice sites. This inevitably involves interactions or couplings that are nonlocal (long-range), i.e. beyond nearest neighbors. We illustrate the working principle of this approach in Figure 1.5(b) with an example adapted from Reference \[41\]. In Figure 1.5(b, top) a one-dimensional lattice with both local (blue lines) and non-local (red curves) interactions are considered. The connectivity in this one-dimensional lattice is equivalent to the lattice geometry in two-dimensional space as sketched in Figure 1.5(b, bottom-left), which is inherently a square lattice wrapped into a tube in three dimensions \[see Figure 1.5(b, bottom-right)\]. This approach maps higher orders of coupling to higher dimensions, where the reachable dimensionality is no longer limited by the number of degrees of freedom one can find; rather it is restricted by the capability to realize higher-order couplings.

Early theoretical proposals of this approach can be found in superconducting qubits with a relatively large range of connectivity \[10\] and optical solitons \[45\]. In photonics, it is much easier to create nonlocal interactions in the eigenmode space, since a long-range interaction in the latter can be achieved without long-range interactions in its complementary space. This was recently proposed in the context of photonic synthetic dimension \[41\].

The most widely proposed approach to the eigenmode-based synthetic lattice is based on the use of frequency modes of resonators \[33, 34, 37–40, 43, 44, 46, 47\]. This is a temporal-domain approach, yet it is different than those synthetic lattices directly arranged in time using pulses \[48, 49\]. Rather, it utilizes each frequency eigenmode to denote a lattice site. Since these eigenmodes in general have no crosstalk in linear and static systems, a dynamic modulation to these resonators is typically employed to introduce couplings. To illustrate the working principle of this approach, we adapt a sketch from References \[43, 44\] as shown in Figure 1.6(a). When a ring resonator undergoes a dynamic modulation, for example, a periodic time-dependence in the refractive index, the frequency eigenmodes of the resonator (denoted by rings with different colors) when it is static exhibit couplings (denoted as \(J\)) if the modulation frequency is set to equal the frequency differences \(\Omega_R\) between the eigenmodes. Frequency is highly promising to act as lattice sites, yet so far reported experiments are mostly based on electro-optic modulation (EOM) \[43, 44\] that is rather slow and complex. Therefore, it remains highly important to develop new concepts and platforms to enable better performance and hence more applications of frequency-based synthetic lattices.

Another representative approach uses spatial eigenmodes. Since there can be also multiple spatial modes associated with spatial lattices, hence it is also possible to make use of such spatial eigenmodes as a lattice space. Here we illustrate this scheme with the experiment of
Figure 1.6: Examples of experimental implementations of synthetic lattices in eigenmode space that are possible to have nonlocal coupling. (a) Frequency synthetic lattice using dynamically modulated ring resonators, adapted from References [43, 44]. (b) Synthetic lattice via engineered spatial lattices used in the eigenmode space, adapted from References [35, 42].
Reference [42] as sketched in Figure 1.6(b). In this example, an array of waveguides are used as a one-dimensional lattice, where the propagation direction along $z$ is treated as time. The spatial eigenmodes of such a lattice can also be utilized as a synthetic lattice with each mode as a lattice site. If the structures have no variation in $z$ [Figure 1.6(b, left-top)], then the spatial eigenmodes have no coupling (hopping). Such couplings can be introduced by a modulation along the propagation direction [here along $z$, see Figure 1.6(b, left-bottom)]. With a properly chosen modulation, hoppings will then be achieved between the eigenmodes, as shown in Figure 1.6(b, right). This approach is suitable for synthetic dimensions spanned by a second degree of freedom in addition to the space, yet the modulation along the propagation direction of waveguides is not arbitrary due to the limitation by the geometry and exponential dependence of coupling with waveguide separation. Therefore, there are practical limitations with this platform to obtain well-defined nonlocal couplings, restricting its scalability.

Besides, there also exist other approaches, such as using modes with different orbital angular momenta (OAM) [50–52] and real space in cavities [53]. Overall, these methods are in bulk optics and require rather stringent conditions such as having the orbital angular momenta (OAM) modes in the same frequency.

So far, in photonics, the connectivity-based synthetic dimension is only experimental demonstrated within highly limited cases [54] using EOM in frequency modes. Therefore, it is highly important to develop new concepts and controllable platforms that can be used to form such multidimensional lattices. This serves the main motivation for the results presented in Chapter 2.

### 1.2.2 Harnessing multidimensional properties

The methods for forming multidimensional lattices reply on a site-to-site equivalence, which usually requires substantial experimental complexity to fully replicate the entire lattice. Moreover, since there is an exponential growth of lattice sites with an increased dimensionality, even just accessing the size of the system can become extremely challenging. Nevertheless, in many fundamental physics effects and applications, it may not be necessary to replicate the entire structure in order to obtain the nontrivial feature associated with higher geometrical dimensions. In fact, many works have already exploited tailored mathematical mappings to allow certain physical property of multidimensional lattices to manifest itself in low-dimensional structures [35].

So far the most widely used method to harness multidimensional properties lies in adding a parameter space via the time-dependence of low-dimensional systems, which can be mapped to higher-dimensional static lattices [35, 36]. One example of such a scheme is shown in Figure 1.7 adapted from Reference [22] where a waveguide lattice is modulated along propagation direction $z$ to obtain the parameter space. More specifically, this can be explained by adiabatically varying the Hamiltonian, realizing topological pumpings [55]. Higher-dimensional photonics based on added parameter space has been explored in a va-
riety of aspects such as the evolution of topological edge states [20], topological phase transition[56], and quantum Hall effect [22].

![Figure 1.7: Example realization of harnessing multidimensional properties via specially modulated coupled waveguides along the propagation to form extra parameter space, taken from Reference [22].](image)

However, it remains unknown if it is possible to use mathematical mapping to explore even higher dimensional physics, as the latest advance only reached up to four dimensions [22]. Besides, note that these mapping methods are all based on time(propagation)-dependent Hamiltonians requiring certain modulation, which adds more complexity to its realizations. In particular, if the platform is based on waveguides, not all $z$ modulation is achievable owing to the limitation of geometry. Hence it remains an open question if multidimensional characteristics can be accessed and harnessed by mappings without time-dependence. These facts serve as key motivations of the works presented in Chapter 3, where we will lift up both limitations in an entirely new paradigm for harnessing properties of arbitrarily many-dimensional networks using one-dimensional static lattices.

### 1.2.3 Manipulating multidimensional photonic states

Synthetic lattices may also play an important role in developing new methods for manipulating multidimensional states. As mentioned in the last section, a key goal of photon state manipulation is to judiciously project or transform the state to best match the detection device, which is used for the purpose of measuring multidimensional photonic states. This is especially challenging in multidimensional states realized by photon number since it unavoidably needs interference between multiple photons in order to extract the nonlocal parameters in the state. This has been realized in tunable quantum circuits [57] and in a static synthetic waveguide lattice [58]. However, the measurement of multiphoton states remains a very important question as reported experiments usually lack robustness. Lately, there have been
new advances in photonic structures that allow further miniaturization and better stability, and single-photon detectors are also possible to be made integrated [59, 60]. Thereby it is of great importance to develop new concepts for manipulating multidimensional states, in particular, multiphoton states, combining these new advances in synthetic lattices. This serves as a key motivation of the works presented in Chapters 4 and 5 of this thesis.

Now we give two specific examples of where synthetic lattices can be useful for manipulating multidimensional photonic states.

**Metagratings: Synthetic lattices on metasurfaces**

Gratings have been well known in optics, where recent technological advances in metamaterial enable the so-called *metagratings* that allow each unit element in the grating to be considerably smaller than the wavelength of light [61–63]. This is especially useful in the manipulation of the vectorial nature of light, such as its polarization. A widely used theoretical aspect in their designs is the use of geometric phase [64, 65]. The realization of metagratings is possible with both plasmonic [61] and dielectric materials [63]. Recent efforts in metagratings have also been devoted to freeform engineering of individual element geometry [62].

Chapter 4 will be focused on new possibilities in multiphoton state manipulation enabled by such metagratings on a metasurface. More specifically, we concentrate on nanostructured polarization metagratings. Polarization has been widely used to encode information in light as an intrinsic degree of light originated from the spin angular momentum of paraxial light. The vectorial nature of polarization requires its manipulation to be based on certain anisotropic properties such as an effective birefringence. A unit element in the metagrating employed in Chapter 4 of the thesis is sketched in Figure 1.8(a). It is based on silicon pillars on a glass substrate, with a typical lattice parameter $l$ is smaller than the wavelength of light. The height $h$, cross-section geometrical parameters $d_x$, $d_y$ of the silicon pillars are typically around a few hundred nanometers. The pillar is oriented at an angle $\theta$, which can be designed in the metagrating. Such a unit element can be modeled as a birefringent transformation onto the polarization of light illuminated onto the area of this site [66]. In general, there are two such transformations – one for transmitted light and the other for reflected light. The phase retardance is also complex-valued. Nevertheless, since in these dielectric structures the transmission can be very high for both polarization components [66], it is common to mainly work with the transmitted light and treat the phases as real-valued while multiplying an overall transmission factor.

By arranging such elements with different orientations and phase retardance into a periodic lattice [see Figure 1.8(b) for an exemplary sketch of one period], one obtains a nanostructured polarization metagrating. If an input beam (double bold arrows) illuminates the grating, there can be both transmitted and reflected diffraction orders. Here we mainly use the polarization-dependent transmitted orders as most light goes there due to the highly transparent dielectric material. Their behavior can be well understood by performing a discrete
Figure 1.8: Metagratings as synthetic lattices on all-dielectric metasurfaces. (a) Sketch of one unit element of the metagrating. (b) Sketch of a metagrating consisting of 10 elements. (c) Metasurface consisting of many metagratings interleaved on it.
Fourier transform to the spatially varying transformations, obtaining each transformation associated with its allowed diffraction channels. Such metagratings can be interleaved into metasurfaces to perform certain designed manipulation of light, which is illustrated in Figure 1.8(c).

Quantum-photonic circuits based on synthetic lattices

Another important possibility to use synthetic lattice is in quantum-photonic circuits. While metasurface and metamaterial mentioned above can have many degrees of freedom in space for the manipulation of light, quantum-photonic circuits are more suitable for integrated processing of information encoded in the spatial modes of light.

The most widely used type of quantum-photonic circuits is composed of Mach-Zehnder-like interferometers each realizing a $2 \times 2$ transformation of spatial modes [67–70]. This can be realized in integrated waveguides with mode-splitting sections and phase shifters, which we provide an example in Figure 1.9(a) adapted from Reference [70] (for more details see [71]). While this approach has good compatibility with silicon photonics [70] and is suitable for large-scale fabrication with tunability, it is also rather space-consuming as most of the waveguiding sections do not have coupling to other waveguides.

![Figure 1.9: Quantum-photonic circuits using two types of implementations. (a) Mach-Zehnder-like interferometer sections in waveguides, taken from Reference [70]. (b) Gradually coupled waveguides, taken from Reference [58].](image)

A more compact solution is by using synthetic lattices, which is shown by an example taken from Reference [58] in Figure 1.9(b). In this case, the waveguides form a specially designed lattice and are very close by, hence among most of its propagation length all waveguides do couple with each other. This provides not only a compact but also potentially more efficient solution. Judicious use of such type of quantum circuits typically needs one to break away from the convention of treating the manipulation of quantum states as a series of gates. In Chapter 5, we will employ entirely new concepts to make better use such synthetic-lattice-based quantum circuits for the manipulation of multiphoton quantum states.
1.3 Scope and outline of the thesis

This thesis aims at addressing key challenges with a scope of both fundamental and practical interests on how one can make use of artificially engineered synthetic lattices to facilitate multidimensional photonics in both classical and quantum regimes. Specifically, efforts are devoted to three main aspects in the field of multidimensional photonics in discrete lattices, including forming multidimensional lattices (Chapter 2), harnessing multidimensional properties (Chapter 3) and the manipulation of multidimensional photonic states (Chapters 4 and 5).

Here we provide a more detailed outline of the following chapters:

Chapter 2 targets on developing new approaches to arranging multidimensional lattices. We introduce and experimentally demonstrate a new concept to form multidimensional synthetic frequency space. We show a controllable and all-optical platform that can mediate the interaction of propagation frequency component in a nonlinear waveguide, forming a synthetic frequency lattice with long-range and complex couplings. This enables the experimental observation of both fundamental and practical effects in the frequency space, including spectral Talbot effects with arbitrary periodicity, synthetic dimension with nontrivial gauge field, and one-shot measurement of the amplitude, phase, and coherence of frequency combs. Consequently, this new approach forms an extremely powerful tool of nonreciprocal frequency reshaping with effective multidimensional lattices.

Then, Chapter 3 focuses on harnessing multidimensional characteristics without entirely replicating the multidimensional lattice. We introduce and experimentally demonstrate an entirely new paradigm to access higher dimensions using a tailored isospectral mapping that enables certain excitation dynamics in arbitrary-dimensional networks to be exactly reproduced in practical one-dimensional lattices. This approach does not require nonlocal coupling or dynamic modulation, hence is highly suitable for planar photonic structures. With the unprecedented strength of accessing multidimensional dynamics enabled by this method, we discover a sharp localization transition in four- and higher dimensional lattices and provide direct experimental evidence.

Then, in Chapters 4 and 5 we aim at the manipulation of multidimensional photonic states. More specifically, we focus on quantum-photonic states that are inherently multidimensional due to the multiparticle nature. Our manipulation is devoted to facilitating the measurement of such multiphoton states by judiciously tailored transformations realized by synthetic lattices.

Chapter 4 is based on metagratings interleaved on a monolithic metasurface. We introduce and experimentally show a new concept of using advanced imaging to map out entangled multiphoton quantum states with metasurfaces. Our results serve as one of the first examples of combining quantum light with metasurfaces, where we illustrate the manifestation of multiphoton interference on the metasurface for the first time. We achieve the reconstruc-
tion of both single- and two-photon polarization states using our specially designed metasurface and pave the way to using such synthetic lattices as imaging optics for multidimensional quantum states.

Chapter 5 is based on synthetic-lattice-based quantum-photonic circuits. We introduce and experimentally prove the principle of a new concept in quantum state measurement based on inline detection, making it possible to pinpoint issues of large quantum networks in real-time. With the inline detection approach, we also show its application in measuring the coherence of light propagating in parity-time symmetric waveguide couplers.

Finally, in Chapter 6 we draw the main conclusions for this thesis and provide a discussion on potential future works.
2 Multidimensional synthetic frequency lattices in non-linear waveguides

As outlined in Chapter 1, the use of discrete frequencies to encode information promises a possibility to go beyond the limits that our three-dimensional space imposes onto the lattice network. Since different frequencies cannot interact in the most commonly used linear and time-independent systems, temporal modulation is necessary in order to reshape discrete frequencies. While conventionally this was done with rather complex electro-optic modulation (EOM), nonlinear wave mixing can be a simple and all-optical solution to this problem.

In this chapter, we focus on the aspect of forming synthetic lattices in multidimensional photonics. Specifically, we will introduce and experimentally demonstrate a synthetic frequency lattice in a single nonlinear waveguide.

2.1 Introduction

Systems utilizing solely the physical space are limited by three geometrical dimensions, whereas many theories predict exotic effects in higher dimensions. Therefore, the synthesis of artificially created dimensions stimulates great research interest, giving rise to the rapidly growing field of research namely synthetic dimension [35, 36]. Recently, a particularly promising scheme to synthesize arbitrary dimensions was proposed in photonics using dynamically modulated resonators [41], employing multiple orders of hopping to realize synthetic space [41, 75, 76]. Unlike other approaches (for example, Reference [33]), in this method, the dimensionality is determined by the number of coupling orders and is thus scalable to higher dimensions.

As mentioned in Chapter 1, the key to realizing such synthetic space lies in the implementation of not only local (nearest-neighbor) but also nonlocal (long-range) coupling in photonic lattices. Longer-range coupling is relevant in some electronic systems, and the addition of controlled nonlocal coupling provides more degrees of freedom to tailor the dispersion relation or band-structure of the lattice. In photonics, it is well known that in spatial lattices, such as an array of evanescently coupled waveguides, the interactions are usually dominated by nearest-neighbor (local) coupling, though the use of a two-dimensional geom-

\*Results presented in this chapter are partially included in the published References [72–74], where I am a lead author and my contribution is in theoretical aspects.
etry (e.g. zig-zag arrangement of lattice sites) can introduce next-nearest-neighbor coupling into a 1D lattice \[77, 78\]. However, this approach is highly difficult to scale to longer ranges, and the coupling profile is usually limited by the geometry of the lattice configuration.

Spatial photonic lattices also usually have real-valued coupling coefficients. Complex-valued coupling coefficients cause waves propagating through the lattice to accumulate a momentum-direction-dependent phase which breaks time-reversal symmetry (TRS), i.e. moving from lattice site A to B imparts a different phase-shift to moving from B to A. Importantly, a complex coupling is directly relevant to an artificially-induced gauge field. With the artificially engineered gauge fields and resulted potentials, photons can also exhibit certain properties that were only possible with charged particles, giving rise to many applications, such as topological photonics \[16, 17, 19, 33, 42, 73, 79, 80\] and nonreciprocal light guiding \[81, 82\]. Nevertheless, the use of solely the spatial degree of freedom practically limits the realization of these versatile effects.

Synthetic lattices utilizing degrees of freedom other than space potentially allow greater flexibility to realize these novel features and explore their effects. As mentioned in Chapter 1, one important type of synthetic lattice uses the eigenspace to implement lattice sites, such as frequency space. Since a long-range coupling in frequency space, in general, does not require long time-scale interactions, it becomes especially suitable to be used as a synthetic lattice with long-range coupling. This was most widely realized using electro-optical modulation (EOM) \[43, 44\] to introduce coupling between resonator modes with different frequencies, yet it is usually rather complex in setup and difficult to be integrated to small devices. Besides, it was usually considered necessary to use resonator modes for such applications, yet previous experiments have demonstrated in space a possibility to use propagation waves to emulate discrete modes \[83\].

In photonics, it is known that discrete spectral components of optical waves can couple to each other driven by processes other than EOM, such as nonlinear frequency conversion \[84\] and photon-phonon interactions \[85, 86\]. Discrete spectral components have also been widely investigated in the context of parametric amplification, for instance for frequency comb generation \[87–89\]. In contrast, to emulate lattice dynamics one needs to realize conservative dynamics with no overall amplification. Therefore, it is highly interesting to establish an all-optical platform for synthetic frequency lattices based on optical nonlinearity. This will ideally enable flexible manipulation of many frequency components in a single-port nonlinear waveguide.

Importantly, the development of synthetic frequency lattice based on coherent nonlinear conversion processes is also well-suited for weak quantum light, since non-amplification nonlinear processes such as sum-frequency-generation and four-wave-mixing Bragg scattering can be implemented in a way avoiding introducing quantum noise. Recently, there has been significant progress in quantum optics in the spectral domain \[26, 27\]. Frequency domain Hong-Ou-Mandel interference has proven stable and easy to implement \[90\], and frequency-encoded photonic states are now poised to enable scalable quantum information
processing [91]. Having access to fast and simple means to reshape a quantum state in the frequency domain would be indispensable.

In this chapter, we suggest and develop experimentally a tight-binding synthetic lattice, sites of which are represented by discrete frequency channels that are coupled together by nonlinear frequency conversion. In such a system, controllable long-range and complex coupling are made possible by shaping the spectrum of the optical pump. Our scheme promises flexibility to implement different Hamiltonians with nontrivial band structures, particularly those breaking locality of coupling and TRS, opening up opportunities to experimentally investigate new physical effects in discrete lattices. With the strength of this synthetic lattice, we demonstrate discrete Talbot effect with novel instances. Moreover, we make use of the correspondence between higher-order coupling and higher dimension to synthesize dimensions with nontrivial gauge field. Our approach also promises practical applications, as an example, we also experimentally demonstrate the measurement of the amplitude, phase, and coherence of frequency combs using our synthetic frequency lattice.

Outline of this chapter

The subsequent sections of this chapter are structured as follows: Firstly, in Section 2.2, we introduce the working principle of building up synthetic frequency space in a nonlinear waveguide and its experimental realization. Then, as a direct benefit and example, in Section 2.3 we experimentally demonstrate a spectral version of discrete Talbot effect that is no longer restricted by the limited periodicity due to local coupling. Then, in Section 2.4 we make use of the synthetic frequency lattice to synthesize multidimensional lattices with nontrivial gauge field. In Section 2.5 we bring the synthetic lattice to a practical context of frequency comb measurement, allowing the reconstruction of not only amplitudes but also phase and coherence encoded in discrete frequencies. In Section 2.6 we provide a generalization to the sum-frequency-generation in second-order nonlinearity for a potential to realize lattices with two sub-lattices. In Section 2.7 we will summarize this chapter.

2.2 Frequency lattice with long-range coupling in an optical fiber

Now we show the working principle of our synthetic frequency lattices. We consider a co-propagating signal and pump in a Kerr nonlinear waveguide [see Figure 2.1(a)], whose spectra consist of discrete components separated in angular frequency by $\Omega$ [see Figure 2.1(b)]. In the waveguide, the signal can be up- or down-shifted by a multiple of $\Omega$ in a coherent conversion process known as four-wave-mixing Bragg scattering (FWMBS) [92], depicted in Figures 2.1(a,b). Consequently, the nonlinearity-mediated interactions between different frequency components form a tight-binding lattice, as sketched in Figure 2.1(c) where each lattice site is a frequency component. Note that FWMBS is in principle noiseless and has attracted attention in quantum optics, for manipulating the frequency of single photons [93–
Chapter 2: Multidimensional Synthetic Frequency Lattices in Nonlinear Waveguides

Figure 2.1: Concept of synthetic frequency space in a nonlinear waveguide. (a) Four-wave-mixing Bragg scattering in a $\chi^{(3)}$ waveguide. Pairs of pumps (red lines) can up- or down-shift the signal (orange lines) in frequency. (b) With multiple pumps present (left), the evolution of the signal (right) is governed by multiple hopping coefficients (connecting curves) across the lattice, depending on the amplitudes of pairs of pumps. (c) The discrete signal frequencies effectively form a tight-binding lattice.

97, as well as in classical communications and all-optical switching [98]. We assume that the signal and pump move at the same group-velocity and group-velocity dispersion can be ignored, achieved by placing them equidistant in frequency to either side of a zero dispersion wavelength. This results in a broad phase-matched bandwidth for FWMBS. Other nonlinear processes such as parametric amplification of the signal by the pumps are phase-mismatched to avoid introducing gain and noise into the signal.

Theoretical formalism of the synthetic frequency lattice mediated by Kerr nonlinearity

Now we provide theoretical details on how FWMBS can be used to implement frequency lattices. The signal-envelope dynamics along propagation distance $z$ can be described by a set of amplitudes $a_m$ at frequency components $\Omega m$, which evolve according to coupled-mode equations. We begin by assuming that the signal and pump wavefunctions, $a(t, z)$ and $A(t, z)$ respectively, consist of a lattice of discrete frequencies with complex amplitudes $a_m(z)$ and
\begin{equation}
A_m(z) = \sum_{m=\pm \infty}^+ a_m(z) e^{-i\Omega m t},
\end{equation}

where \( \Omega \) is the channel spacing in angular frequency and \( z \) the position along the fiber. The wavefunctions evolve with \( z \) according to the nonlinear Schrödinger equations (NLSEs), with effective nonlinearity \( \gamma \),

\begin{equation}
\frac{\partial}{\partial z} a(t, z) = 2i \gamma |A(t, z)|^2 a(t, z),
\end{equation}

\begin{equation}
\frac{\partial}{\partial z} A(t, z) = i \gamma |A(t, z)|^2 A(t, z),
\end{equation}

where the signal and pump move at the same group-velocity and dispersion can be ignored - in practice this is achieved by placing them equidistant in frequency to either side of a zero dispersion wavelength. Since all frequencies involved move at the same group velocity, a moving reference frame has been used to remove the usual time-derivative terms in the NLSEs that correspond to translation due to group-velocity. Peschel et al. [99] give a theoretical treatment that includes differing group-velocities and group-velocity dispersion. The signal is also weak enough that the pump evolution is independent of it; hence the pump experiences a nonlinear self-phase-shift, but its intensity as a function of time is unchanged:

\begin{equation}
|A(t, z)|^2 = |A(t, 0)|^2 = \sum_{l=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A_l(0) A^*_l+n(0) e^{i\Omega nt}.
\end{equation}

Inserting this expression into the signal’s NLSE, we obtain

\begin{equation}
\frac{\partial}{\partial z} a(t, z) = 2i \gamma \sum_{l=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A_l(0) A^*_l+n(0) e^{i\Omega nt} a(t, z).
\end{equation}

Then, we expand \( a(t, z) \) into frequency components and group the frequencies together, and get equations of motion for the individual frequency components \( a_m \)

\begin{equation}
\frac{\partial}{\partial z} a_m(z) = 2i \gamma \sum_{l=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A_l(0) A^*_l+n(0) a_{m-n},
\end{equation}

i.e. each pair of pump frequency components, \( A_m \) and \( A_l \), creates coupling between signal channels separated by \( m - l \). It can be seen that while the coupling coefficients in Equation (2.4) are varying with time, they are not dependent on \( z \), and this results in constant but potentially complex-valued coefficients in the frequency domain as seen in Equation (2.5).

The equations of motion can also be written in terms of \( n \)-th order coupling coefficients.
by grouping together pairs of pumps which are separated by $n$:

$$\frac{\partial}{\partial z} a_m = i \sum_{n=-\infty}^{+\infty} C_n a_{m+n} = i \sum_{n=1}^{+\infty} \left( C_n a_{m+n} + C_n^* a_{m-n} \right) + i C_0 a_m,$$  \hspace{1cm} (2.6)

where we have the coupling coefficient

$$C_n = 2\gamma \sum_{l=-\infty}^{+\infty} A_l(0) A_l^*(0).$$ \hspace{1cm} (2.7)

Note that the coupling coefficients are symmetric, $C_n \equiv C_{-n}^*$, and accordingly the evolution is conservative, $\partial_z \sum_m |a_m|^2 \equiv 0$. The term $C_0$ does not result in hopping but rather contains the cross phase-shifts from individual pump frequencies to the signal. This does not affect the dynamics and can be removed via a change of reference frame, $a_m \rightarrow a_m \exp(i C_0 z)$, which maps $C_0 \rightarrow 0$. We assume such a transformation in our analysis. Therefore, we obtain a differential equation analogous to that describing a tight-binding lattice

$$\frac{da_m}{dz} = i \sum_{n=1}^{+\infty} \left( C_n a_{m+n} + C_n^* a_{m-n} \right).$$ \hspace{1cm} (2.8)

This provides an effective tight-binding potential which is periodic in frequency every $\Omega$, and accordingly a dispersion relation or band-structure can be used to describe the evolution, where time plays the role of wave vector. We use a notation $k = \Omega t$ to emphasize that this quantity plays the role of wave-vector, rather than having the state evolving with time. Then, a signal spectrum $a_m(z) = \exp\{i[k m + \beta(k)z]\}$ is a steady state solution with eigenvalue:

$$\beta(k) = \sum_{n=1}^{+\infty} \left( C_n e^{ink} + C_n^* e^{-ink} \right)$$

$$= 2\gamma \sum_{n \neq 0} A_l(0) A_l^*(0) e^{ink}.$$ \hspace{1cm} (2.9)

It is surprising to see time $t$ playing the role of wave-number in the band-structure, but this follows from treating the signal frequency as its position in the lattice, since time is the reciprocal space of frequency. In general the pumps have differing phases and $\beta(-k) \neq \beta(k)$, which breaks TRS. Note that we are treating steady-state fields in this system and the Schrödinger equation is written in terms of propagation $z$, hence the time mentioned in this context is essentially propagation distance. Therefore, the TRS here is actually a reversal symmetry of $z$ by assuming $z$ does not have a back-scattering direction, as is defined in many photonic lattice systems such as coupled waveguides [19].

**Experimental realization of synthetic frequency lattices in a nonlinear fiber**

Now we show our experimental realization of such a synthetic frequency lattice in a nonlinear fiber. The experimental setup is shown in Figure 2.2(a). To produce multiple pump and sig-
nal frequencies which are mutually phase-stable, a single mode-locked laser with bandwidth \( \sim 25 \) nm is filtered into the required frequencies using a spectral wave shaper (SPS, Finisar WaveShaper 4000S). The pump frequencies are amplified and their power is controlled by a variable attenuator. A second SPS is used to recombine the signal and pumps while removing spontaneous emission from the amplifier, as well as to impart phase-shifts to the different frequencies as required. FWMBS occurs in a 750 m length of highly nonlinear fiber (HNLF). The HNLF has a zero dispersion wavelength at 1551 nm, so FWMBS is well phase-matched when the signal and pumps are evenly spaced to either side of this wavelength \([100]\). Hence the pumps were placed around 1540 nm, with a \( \Omega = 2\pi \times 100 \) GHz frequency separation (\( \sim 0.8 \) nm) between the channels, and the input signal was placed at 1562 nm. The phase-matched bandwidth for the signal is estimated to be greater than \( 2\pi \times 2 \) THz, limited by the third order dispersion of the HNLF.

![Figure 2.2:](image)

**Figure 2.2:** (a) Experimental setup. MLL: mode-locked laser; SPS: spectral pulse shaper; EDFA: erbium-doped fiber amplifier; Att.: variable attenuator; DL: tunable delay line; HNLF: highly nonlinear fiber; OSA: optical spectrum analyzer. (b) Measurement result with only two pumps. The dynamics of the evolution are apparent as the total average pump power \( P \) is varied. (c) Calculated band-structure.

First, we realize spectral discrete diffraction in a synthetic lattice with nearest-neighbor coupling, using two equal-amplitude pumps at neighboring spectral positions, \( A_1 = A_2 \). The measured spectra for this initial configuration are presented in Figure 2.2(b), with the vertical axis showing increasing pump power. We note that increasing the power of all pumps uniformly increases all of the \( C_n \) uniformly, and so is equivalent to varying the time of the evolution, allowing the dynamics to be observed without the need to change the fiber length. The result shows the expected discrete diffraction pattern. In Appendix Section A.1.2, we provide simulation results alongside experimental measurements showing good agreement with theory [101], and the fidelity between the measured and ideal spectrum remains \( > 95\% \) over
the range of the evolution. Figure 2.2(c) shows the calculated band-structure for this system, which has the usual cosine-shape.

![Figure 2.3](image)

**Figure 2.3**: Measurement results with the addition of a third pump, $A_4$, creating 2nd and 3rd order hopping $|C_3| \approx |C_2| \approx 0.15|C_1|$. The phase of the new pump is set to (a) 0, (b) $\pi/2$, (c) $\pi$, (d) $3\pi/2$. Insets to the bottom right show calculated band-structures.

Then we implement a lattice with nonlocal coupling by introducing a third pump frequency, with amplitude $A_4$. We choose a lower power for the third component, $|A_4| \approx 0.15|A_{1,2}|$, creating 2nd and 3rd order hopping coefficients $C_3 \approx C_2 \approx 0.15C_1$. The spectral evolution becomes highly dependent on the phase of this third pump, despite its lower power, as shown in Figures 2.3(a-d). For phases of 0 or $\pi$ relative to the other pumps, the evolution remains symmetric, but either signal intensity can be diverted into the propagating lobes to the left and right of the diagram [Figure 2.3(a)], or it can remain more localized in the central three frequency channels [Figure 2.3(c)]. For phases of $\pi/2$ or for $3\pi/2$ radians [Figures 2.3(b,d)], the evolution becomes asymmetric; this asymmetry is also reflected in the calculated band-structure, and is connected to the breaking of TRS.

### 2.3 Discrete Talbot effect with arbitrary periodicity

As an example, in this section we demonstrate a spectral analog of the discrete Talbot effect, a self-repetitive imaging effect observed in diffractive systems with periodic input states [102, 103]. In the discrete Talbot effect, a periodic input pattern is recovered after a certain period of evolution. Whereas a discrete Talbot effect was previously demonstrated using optical waveguide arrays [103], it was fundamentally limited only to certain periodicities due to local coupling. We show how to overcome this restriction by engineering nonlocal coupling, enabling self-imaging of other spectral patterns.
We find two ways in which novel instances can be found which have not been possible in spatial lattices. Firstly, displaced images can be formed such that they appear to be propagating in frequency with a direction, arising from the breaking of TRS by complex-valued hopping. Secondly, whereas previously it was thought that an image would only occur when the input was periodic every $N = 1, 2, 3, 4,$ or $6$ lattice sites [103], we show that altering the band-structure with nonlocal coupling can lead to the Talbot effect occurring with other periodicities, and experimentally demonstrate the $N = 5$ case.

A periodic input signal which repeats every $N$ lattice sites contains only a discrete set of $k$ components, $k_m = 2\pi m/N$, with $m$ an integer such that $-\pi < k_m \leq \pi$. The corresponding eigenvalues are labeled $\beta_m = \beta(k_m)$. If the ratios of the separations between $\beta_m$ are rational numbers, i.e.

$$\frac{\beta_i - \beta_0}{\beta_j - \beta_0} = \frac{p_i}{p_j},$$

where $\{p_i\}$ is a set of coprime integers, then, after some period of evolution the separate $k$ components will come back into phase and produce an image of the input signal. As shown in [103], for the case of nearest-neighbor hopping, this will only occur if $N$ is in the set $\{1, 2, 3, 4, 6\}$. Initially, we realize a spectral version of discrete Talbot effect for $N = 2, 3, 4$. Here, only two pump frequencies are used, and their separation has been decreased to $\Omega = 2\pi \times 50$ GHz ($\sim 0.4$ nm), to allow more periods to fit into the useful signal bandwidth.

![Figure 2.4: Experimental demonstration of spectral discrete Talbot effect, for input signals with periodicity (a) $N = 2$, (b) $N = 3$, and (c) $N = 4$. Horizontal solid (dashed) lines mark the positions of real (displaced) images of the input. The band-structures to the right of each measurement are marked with the positions of the non-zero $k$ components in each case (orange dots).](image-url)
Figure 2.5: (a) Talbot effect combined with image shift for \( N = 3 \). A \( \pi/2 \) phase-shift between the pumps creates an asymmetric propagation in which regular displaced images appear. (b) Talbot effect for \( N = 5 \), which required first and second order coupling realized with three pump frequencies.

The simplest nontrivial case is \( N = 2 \) [see Figure 2.4(a)]; here, the state can be described by two \( k \) components, \( k = 0, \pi \). At around 0.1 mW pump power a phase-difference of \( \pi \) has accumulated between these \( k \) components, and the input sites are depleted of signal. At around 0.2 mW, a phase-difference of \( 2\pi \) has accumulated, creating an image, then this pattern repeats. For \( N = 3 \) [see Figure 2.4(b)], a higher power of around 0.25 mW is required to see an image, because the allowed \( k \) components are more closely spaced in \( \beta(k) \) and take longer to accumulate a \( 2\pi \) phase difference. The imperfections in this image could be explained by errors in setting the amplitudes and phases of the input state, or a consequence of dispersion in the nonlinear fiber. Since pulsed pumps are used, cross phase modulation from the individual pumps causes broadening of the signal frequency channels at higher powers. Similarly, for \( N = 4 \), as shown in Figure 2.4(c), an image can be seen at a pump power around 0.4 mW, and there is a clear displaced image at 0.2 mW.

Displaced images of the input occur when the relative phase-shifts between \( k \) components are equal to an integer multiple of \( k \). Here, a displacement of one lattice site corresponds to multiplication by a factor \( \exp(ik) \) in reciprocal space. For the cases \( N = 2, 4 \) these displaced images can be seen in Figure 2.4, occurring halfway between the real images. However for \( N = 3 \), it is necessary to break TRS to obtain a displaced image. Figure 2.5(a) shows the \( N = 3 \) case, but with a \( \pi/2 \) phase-shift applied between the two pump frequencies. This creates an asymmetric band-structure where the \( k \) components for \( N = 3 \) lie along a straight line. The components of the state accumulate phase differences proportional to \( k \), and hence can form an image displaced by one site at 0.15 mW pump power, then by two sites at 0.3 mW.
Previously, the discrete Talbot effect has not been demonstrated with $N = 5$ or $N \geq 7$, because when there is only nearest-neighbor hopping these cases cannot satisfy the requirement of Equation (2.10). Here, we show that by inducing nonlocal hopping, this restriction can be lifted. We find that for $N = 5$, the Talbot effect can be achieved by using three equally-spaced pumps, $A_1 = A_3 = 2A_2$. This induces nearest-neighbor and next-nearest-neighbor hopping at equal rates, such that $\beta(k) \propto [\cos(k) + \cos(2k)]$, and

$$
\beta_m = (\beta_0/4) \{-1, -1, 4, -1, -1\},
$$

so an image can be formed. The experimental result is shown in Figure 2.5(b), and an image of the input signal is clearly seen at 0.17 mW of pump power. This approach could be extended to larger $N$, with long-range hopping enabling flexible band-structure engineering such that an image appears.

### 2.4 Synthetic multidimensional space with nontrivial gauge field

Now we show how the synthetic frequency lattice can be used to synthesize higher-dimensional lattices. On our experimental platform via nonlinear frequency conversion, we provide an experimental demonstration of using coupling orders to achieve scalable synthetic dimension. In particular, we implement nontrivial gauge fields (i.e. the existence of nonzero magnetic flux) and show experimental quantum walk with the evolution of coherent light analogous to the superposition principle in quantum states [104, 105].

#### Construction of synthetic dimension in a nonlinear waveguide

Now we show how one can map a one-dimension lattice with both local and nonlocal coupling to a higher-dimensional lattice with local-only hopping. We follow an analogous approach as the theoretical proposal in the context of dynamically modulated resonators [46]. Recall the example shown above in Figures 2.1(b,c), where three equally-separated pumps (separated by $\Omega$) can introduce coupling of first and second orders for signal frequencies separated by $\Omega$. For the convenience of discussion we show the synthetic lattice again in Figure 2.6(a). We can write the Hamiltonian in terms of the creation and annihilation operators $\hat{a}_m^\dagger$ and $\hat{a}_m$ for those discrete signal frequency components with site index $m$

$$
H = \sum_m \sum_{\{n\}} C_n \hat{a}_m^\dagger \hat{a}_{m+n} + \text{H.c.},
$$

where a set of positive integers $\{n\}$ indicates the orders of coupling and $C_n$ being the corresponding coupling constant of each order. Here H.c. denotes Hermitian conjugate, and $m$ is an integer running through all phase-matched frequencies.

With two orders of coupling, a two-dimensional square lattice can be constructed, as shown in Figure 2.6(b). The general idea is to map an order of coupling to a certain basis
Figure 2.6: Concept of using all-optical spectral photonic lattices to synthesize multidimensional lattices for (a) An exemplary synthetic frequency lattice for the pump configuration in Figures 2.1(b) where first order (green arrows) and second order (blue arrows) couplings are present. (b) Two-dimensional equivalent square lattice constructed using the synthetic lattice in (a), where the orange arrow denotes the wrapping vector that wraps the lattice connected by the dashed lines.

For the example shown in Figures 2.6(b), which is a two-dimensional space of square lattice, there are two basis vectors, $k_x$ and $k_y$. Hence we assign the coupling order $n = 1$ to $k_x$, and $n = 2$ to $k_y$. Then, we obtain the Hamiltonian of the two-dimensional equivalent lattice

$$H_{sq} = \sum_m C_1 \hat{a}_{r_m}^\dagger \hat{a}_{r_m+k_x} + C_2 \hat{a}_{r_m}^\dagger \hat{a}_{r_m+k_y} + H.c.,$$

where $r_m$ is a vector indicating the spatial coordinate of the $m$-th site in this two-dimensional space. Despite in a two-dimensional space, $k_x$ and $k_y$ are orthogonal, the algebraic property in the one-dimensional lattice still holds, where the range of two first order coupling is the same as a second order coupling has. This is manifested as a periodic boundary condition in the two-dimensional synthetic space as shown in Figures 2.6(b), which is described by the vector $p$ (denoted by the orange arrow):

$$p = 2k_x - k_y.$$  

Consequently, the two-dimensional equivalent lattice is actually wrapped into a tube connected by the dashed lines in Figures 2.6(b).

Now we move to a more complex case with three orders of coupling. While it is possible to map such a configuration to a cubic lattice in three-dimensional space, here we explore another interesting type of lattice the basis vectors of which are not orthogonal, i.e. a triangular lattice in two-dimensional space that was not considered before in such synthetic space. The synthetic frequency space we employ is sketched in Figure 2.7(a), where 1st, 3rd and 4th orders of coupling are present. In the two-dimensional space of triangular lattice, as shown in Figure 2.7(b), the basis vectors are $k_1 = [1, 0]^T$ and $k_2 = [1/2, \sqrt{3}/2]^T$ (with $^T$ denoting the
2.4 SYNTHETIC MULTIDIMENSIONAL SPACE WITH NONTRIVIAL GAUGE FIELD

Figure 2.7: Synthesis of a two-dimensional triangular lattice using three orders of coupling. (a) A lattice with three coupling orders 1 (gray arrows), 3 (blue arrows) and 4 (orange arrows). (b) The corresponding synthetic triangular lattice in two dimensional space, where the green arrow denotes the wrapping vector. (c) 3D illustration of the lattice in (b).

transpose of a vector), which are not orthogonal with each other. We assign 1st order coupling to \( k_1 \), and 4th order coupling to \( k_2 \). Then, it can be easily shown that the 3rd order coupling is automatically assigned to \( k_2 - k_1 \). This constructs the two-dimensional equivalent triangular lattice as sketched in Figure 2.7(b), with the Hamiltonian written by

\[
H_{tr} = \sum_m C_1 \hat{a}_{\mathbf{r}_m}^\dagger \hat{a}_{\mathbf{r}_m + k_1} + C_2 \hat{a}_{\mathbf{r}_m}^\dagger \hat{a}_{\mathbf{r}_m + (k_2 - k_1)} + C_4 \hat{a}_{\mathbf{r}_m}^\dagger \hat{a}_{\mathbf{r}_m + k_2} + \text{H.c.} \tag{2.15}
\]

Similarly as in the case of the square lattice, there is also a periodic boundary condition here. This can be given by the vector

\[
\mathbf{q} = 4k_1 - k_2, \tag{2.16}
\]

which is shown as a green arrow in Figure 2.7(b). Hence the triangular lattice is also wrapped and connected by the dashed lines in Figure 2.7(b). To more intuitively illustrate this, we sketch the eventually wrapped tube in 3D in Figure 2.7(c).

Multidimensional quantum walk in synthetic space

Now we show an experimental realization of quantum walk in the multidimensional synthetic lattice space as discussed above on Figure 2.7. We employ three pumps to achieve the frequency lattice as illustrated in Figure 2.7(a) with equal 1st, 3rd and 4th order coupling \( C_1 = C_3 = C_4 \). There is no phase difference between the pumps, and therefore all coupling are real-valued. With a single-frequency excitation, we observe a quantum walk in this frequency space, which is shown in Figure 2.8(a).

We map this experimentally realized synthetic lattice with multiple orders of coupling to
Figure 2.8: Quantum walk in synthetic dimension. (a) Original frequency quantum walk implemented on our nonlinear spectral lattice platform. (b,d,f) Mapping of three pump powers $P$ in (a) to a two-dimensional triangular lattice with $P = 0.10, 0.19, 0.28$ mW, respectively. (c,e,g) Corresponding theoretical plots in comparison with (b,d,f), respectively.
the triangular lattice as described in Figure 2.7(b). The mapped quantum walk is shown in Figures 2.8(b,d,f) at three representative average pump powers $P = 0.10$, 0.19, 0.28 mW, respectively. Recall that here pump power acts as time in the quantum walk. In these figures, the site of excitation is marked by a green arrow. We find that this approach indeed can enable the experimental observation of quantum walks in higher synthetic dimensions. Our experimental results also agree resonantly well with the corresponding theoretical plots in Figures 2.8(c,e,g), which are calculated by coupled mode equations describing the quantum walk.

**Artificial gauge field with nonzero flux**

Finally we explore how the complex-valued nature of the coupling constants in the synthetic frequency lattice can be made judicious use in artificially creating gauge fields. Different from the one-dimensional cases presented in sections above, in higher-dimensional lattices an important aspect of the gauge potential lies in inducing magnetic flux as one traces out a closed loop in the lattice. Therefore, it is important not only to implement complex coupling constants but also, in a nontrivial manner, introduce nonzero flux in the lattice. Note that a nontrivial flux may be locally nonzero in each small cell but zero in larger cells, such as the example of the Haldane model [106] that is topologically nontrivial.

To do so, we reexamine the way we used to implement three orders of coupling for achieving the lattice in Figure 2.7(a), where the minimum number of pumps is three. This is illustrated in Figure 2.9(a), where three pumps with complex amplitudes $A_1$, $A_2$, $A_3$ are used. In Figure 2.9(b) we show a simplified synthetic lattice implemented by the pump configuration in Figure 2.9(a), visualizing five sites (1 to 5) as an illustration. Since we aim at analyzing the phases of the coupling here, we use a one-way arrow to show each order of coupling, where the other direction is simply the complex conjugated case due to Hermiticity. According to Equation (2.7), we can easily find the phases of each order of the coupling along the direction of the arrows in Figure 2.9(b): $\varphi_{-1} = \arg(A_2) - \arg(A_1)$ (gray arrow), $\varphi_{-3} = \arg(A_3) - \arg(A_2)$ (blue arrow), and $\varphi_4 = \arg(A_1) - \arg(A_3)$ (orange arrow). This lattice is mapped to the triangular lattice using the approach described above, which is shown in Figure 2.9(c) for the exemplary five sites. We find that the flux (if calculated clockwise) in each triangular cell

$$
\Phi_{1-5-2} = \varphi_4 + \varphi_{-3} + \varphi_{-1} = 0, \\
\Phi_{1-4-5} = -\varphi_4 - \varphi_{-3} - \varphi_{-1} = 0.
$$

This clearly indicates that it is impossible to achieve a nonzero flux in any of the triangular cells of this two-dimensional lattice, as we only use the minimum necessary number of three pumps.

To overcome this issue we add an extra pump, indicated as the green arrow in Figure 2.9(d) with amplitude $A_3$. In this case, the synthetic frequency space is shown in Figure 2.9(e). Since
we are still aiming for three orders of coupling, firstly we need to ensure

\[ C_2 = A_1 A_3^* + A_3 A_5^* = 0. \]  

(2.18)

Such a canceled 2nd order coupling is denoted by the green dashed arrows in Figure 2.9(e). As a sufficient condition to ensure Equation (2.18) is fulfilled, in our experiment we make \(|A_1| = |A_5|\), \(\arg(A_1 A_3^*) = \pi/4\), and \(\arg(A_3 A_5^*) = -3\pi/4\). Then, if we calculate the phases of the rest orders of coupling, \(\phi_{-3}\) (blue arrow) and \(\phi_{4}\) (orange arrow) remain the same as the case analyzed above with three pumps, since each of the two orders is induced by one similar pair of pumps as the three-pump case. The only different phase appears at the first order coupling, where we have

\[ C_1 = A_1 A_2^* + A_2 A_3^*. \]  

(2.19)

Despite we have fixed \(\arg(A_1 A_3^*) = \pi/4\) in our experiment, since we can freely choose the amplitudes of all these three involved pumps and the phase of \(A_2\), in principle the phase \(\phi_1\) of \(C_1\) can be arbitrarily chosen. Therefore, we are no longer subject to the limitation given in Equations (2.17), and hence we can now freely engineer a nonzero flux \(\Phi_{1-5-2}\). Note that the following condition still holds

\[ \Phi_{1-4-5} \equiv -\Phi_{1-5-2}, \]  

(2.20)

which leads to a zero flux if one encircles a pair of neighboring cells. Nevertheless, locally non-zero flux in each cell can already be interesting.

Now we show a representative set of experimental results that includes such a nontrivial gauge potential. Here we intentionally make \(|C_4|\) slightly larger than \(|C_1| = |C_3|\) in order to
Figure 2.10: Quantum walks in synthetic dimension with same coupling amplitudes and artificial gauge fields with different signs. (a,c,e) Experimental results for an effective complex coupling phase $\pi/2$ along the direction of the yellow arrow, for pump average powers $P = 0.10, 0.19, 0.28$ mW, respectively. (b,d,f) The corresponding cases of (a,c,e) with an opposite direction of the $\pi/2$ phase.
more clearly observe the characteristics associated with the artificial gauge potential. Specifically, our chosen pump profiles with four frequencies theoretically match the coupling constants of $C_1 : C_3 : C_4 = 3 : -3 : 5 \exp(i\alpha)$. This arrangement effectively corresponds to a phase of $-\alpha$ along the $k_2$. In two sets of experiments, we realize quantum walks with a single-site excitation in the synthetic triangular lattice with $\alpha = \pi/2$ and $\alpha = -\pi/2$, respectively. In Figures 2.10(a,c,e) we show the case with $\alpha = \pi/2$, where we use a yellow arrow to indicate the direction along which there is a positive $\pi/2$ phase in the coupling if the hopping along all other directions are gauged to have no phase. In the gradually increasing pump powers shown in Figures 2.10(a,c,e), we find that the evolution of the single-site excitation does exhibit asymmetric behavior along the direction of the effective gauge field. This can be better seen if compared to the case $\alpha = -\pi/2$, which is shown in Figures 2.10(b,d,f) for the same pump powers as in (a,c,e), respectively. In particular, if one compares Figure 2.10(e) with (f), the quantum walks at this point look like arrows pointing towards two opposite directions. A comparison of the experimental results presented in Figure 2.10 with theory can be found in Appendix Section A.1.2.

To briefly summarize this section, we have successfully made use of our synthetic frequency lattice to form multidimensional lattices. We observe quantum walks in two-dimensional triangular lattices and implemented gauge fields with nonzero flux. We would like to emphasize that the periodic boundary condition, which is necessary for the mapping, can be neglected, if one uses combinations of large and incommensurate coupling orders. On the other hand, the periodic boundary conditions also provide a unique feature to explore the property of some natural structures, e.g. carbon nanotubes. For the application of such synthetic lattices, it is also interesting to explore topological properties by introducing edges or parity-time symmetry to such synthetic structures via engineered dispersion.

### 2.5 Single-shot measurement of frequency combs with synthetic lattices

The synthetic frequency lattice introduced in this chapter allows complex spectral manipulation of light within a single spatial mode, such as a single optical fiber or waveguide. High bandwidth telecommunications rely on fast and complex routing of light between different spectral channels, as well as the efficient extraction of information about the spectra. Thus, the ability to all-optically manipulate these types of information carrying spectral lattices has many potential applications. In this section, we will show a such an example.

The full characterization of ultra-short optical signals, including both their phase and coherence properties, is crucial for the development and understanding of novel engineered light sources, such as optical frequency combs [107–109] and frequency encoded quantum states [27, 91]. Furthermore, full optical signal characterization is important for the transmission of timing information over fiber networks [110] and wavelength division multiplexing transmission formats where the relative phases between individual carriers are impor-
tant [111]. The most commonly used methods for measuring optical pulses, frequency-resolved optical gating (FROG) [112] and Spectral Phase Interferometry for Direct Electric-field Reconstruction (SPIDER) [107, 113], require complex multimode optical setups in order to recover full amplitude and phase information about optical pulses.

Techniques with the capability to recover spectral phase information with just a single spatial mode are being actively developed. This includes ultra-fast signal conversion methods like stretch transform spectroscopy [114], which could allow real-time measurement of pulse to pulse variations in the spectral phase profile, without need to use an interferometric setup. The technique of electro-optic spectral shear interferometry (EOSI) [113, 115] can also be implemented for determining the spectral phase and amplitude of frequency combs using a single mode setup, by employing an electro-optic modulator (EOM) [116]. The electro-optic phase modulation enables particularly effective characterization of weak coherent frequency combs, even in quantum regime at the single- and few-photon levels [27, 90]. However, EOSI necessarily requires a sequence of different EOM shears to be applied to the signal to fully determine its state, which significantly limits the data acquisition speed.

In this section, we propose and experimentally demonstrate a method for recovering the full spectral amplitude, phase and coherence information of an optical frequency comb from a simple measurement of the spectral intensities after an input signal is specially transformed in a nonlinear fiber. The presented method can be much faster than FROG, SPIDER or EOM based approaches, since it only needs a single measurement and minimal post-processing, while allowing for larger free spectral range compared to the EOM approach. More specifically, we reveal that signal propagation and reshaping in nonlinearly induced spectral lattices can be utilized to allow full reconstruction of an input spectrum from a simple spectral intensity measurement at the output. Specifically, the spectral amplitude, phase and relative coherence of the signal can be recovered.

### Working principle of frequency state reconstruction in synthetic lattices

![Figure 2.11](image.png)

*Figure 2.11:* Conceptual sketch of using synthetic lattices to measure frequency combs. An input signal composed of a number of different frequency channels undergoes a transformation in a spectral lattice with first and second order coupling between frequency channels. The output spectra observed output spectra can then uniquely determine the input signals spectral coherence function for well-chosen transformations.
We consider an input signal, the spectrum of which consists of a number of discrete frequency components that are separated by a constant spectral spacing $\Omega$, as shown in Figure 2.11. After propagating through the frequency lattice, which introduces coupling between the frequency lines, the input signal is spread across an increased number of spectral lattice sites. In the regime of coherent frequency conversion, the output spectral intensity pattern is dependent on the relative phases and coherence of the input signal. Importantly, we find that, with certain well chosen parameters of the lattice coupling, it is sufficient to measure just the output spectral intensities in order to perform robust reconstruction of the amplitude, phase and coherence of the input spectral lines.

The output state after propagation through the fiber length $L$ is related to the input state via a linear transformation which can be expressed through the transfer matrix $T$,

$$a_n(z = L) = \sum_l T_{n,n'}(L)a_{n'}(z = 0), \quad (2.21)$$

where transfer matrix can be obtained with the Hamiltonian given in Equation (2.12)

$$T(z) = \exp(iHz). \quad (2.22)$$

In general, noise and fluctuations may affect the coherence of comb spectrum. Such partially coherent signals can be characterized by the mutual coherence or complex visibility\[31\] between different pairs of spectral lines,

$$V_{n,m}(z) = \langle \hat{a}_n(z)\hat{a}^*_m(z) \rangle, \quad (2.23)$$

where $\hat{a}_n(z)$ are the fluctuating complex field amplitudes of the spectral comb lines and angled brackets denote averaging. The partially coherent field can be represented as an incoherent mixture of coherent fields $a^{(p)}_m$ indexed by $p$, such that

$$V_{n,m}(z) = \sum_p a^{(p)}_n(z)\{a^{(p)}_m(z)\}^* \quad (2.24)$$

We substitute Equation (2.21) into Equation (2.24) and obtain

$$V_{n,m}(z) = \sum_p \sum_{n',m'} T_{n,n'}(z)a^{(p)}_{n'}(0)\{T_{m,m'}(z)a^{(p)}_m(0)\}^* = \sum_{n',m'} T_{n,n'}(z)T^*_{m,m'}(z)V_{n',m'}(z). \quad (2.25)$$

The output comb intensities, which can be directly measured by a spectrometer, are then found as

$$I_n(z = L) \equiv V_{n,n}(L) = \sum_{n',m'} T_{n,n'}(L)T^*_{n,m'}(L)V_{n',m'}(z = 0). \quad (2.26)$$

Therefore the input complex visibility can be retrieved, provided that Equation (2.26) can be inverted to find $V_{n',m'}(z = 0)$ in terms of the output spectral intensities $I_n(z = L)$. The robustness of the inversion process can be characterized by the condition number. It quantifies the amplification of errors and noise in measurement of the output intensities through the
calculation of the input complex visibility function \[117\]. A mathematical description of the condition number will be provided in Chapter 4. In principle, a high condition number implies that small errors in the measurement data would be highly amplified after the inversion, meaning the inverted result is highly unreliable. Thus for an accurate reconstruction, we require a low condition number.

**Experimental measurement of frequency combs with a synthetic lattice**

Now we show an experimental realization of the concept with the synthetic-lattice setup as introduced in this chapter [see Figure 2.2(a)]. We experimentally demonstrate the reconstruction of a signal consisting of up to four frequency channels, but in principle the method is applicable to more complex signals.

![Figure 2.12](image)

**Figure 2.12:** (a) Dispersion of the spectral lattice (red curve) and contributions from the first order coupling \(C_1\) (dashed curve) and second order coupling \(C_2\) (dash-dotted curve), where \(C_1 = C_2\). (b) The condition number of the transformation as a function of first and second order coupling rates. Red dashed line shows the subset of the parameter space investigated experimentally. (c) Measured output spectra for a frequency space quantum walk for input state \(|0\rangle + i|1\rangle\). (d) (left hand axis) Calculated condition number from experimental data in (e). (right hand axis) Fidelity of the reconstruction of the state \(|0\rangle + i|1\rangle\) as a function of pump power.

We choose the pump spectra to consist of 3 different frequency components at 1540 nm, each separated by \(\Omega = 2\pi \times 100 \text{ GHz}\). By having 3 such equally spaced pump spectra the signal can be up- or down-shifted by \(\Omega\) or \(2\Omega\) via the FWMBS. This realizes a spectral lattice with first and second order coupling [see Figure 2.11]. The total dispersion of the resulting spectral
lattice is shown in Figure 2.12(a), with the contributions of both the first and second order coupling shown individually as dashed and dash-dotted lines, respectively.

We first numerically investigate the inversion problem for the case where the input signal is limited to two input frequency lattice sites. We study the effect of varying the coupling constants $C_1$ and $C_2$, while assuming that other coupling are zero ($C_{m>2} \equiv 0$), and the fiber length is held constant at $L = 1$. For each set of values of $C_1$ and $C_2$ we calculated the condition number. The plot of the condition number vs the coupling rates is shown in Figure 2.12(b), indicating that inversion will be practical given both $C_1$ and $C_2$ are chosen in an optimal region. Importantly, the condition number goes to infinity (a large value in the numerical test) if $C_2 = 0$. That confirms it is impossible to reconstruct the input state if we only have the first order coupling.

To confirm this result we take a representative experimental measurement with $C_1 = C_2$ as indicated by the red dashed line in Figure 2.12(b). The transfer function of the system for a given set of pump powers was calculated by coupling a basis set of four different input signals into the fiber. The basis set used was $\{|\omega_0\rangle, |\omega_1\rangle, |\omega_0\rangle + i|\omega_1\rangle, |\omega_0\rangle + i|\omega_1\rangle\}$, where $|\omega_0\rangle$ corresponds to the frequency lattice site 1561.7 nm and $|\omega_1\rangle$ corresponds to the frequency lattice site 1562.1 nm. For example output states resulting from coupling the basis state $|\omega_0\rangle + i|\omega_1\rangle$ into the fiber is shown in Figure 2.12(c), where the x-axis shows the pump power.

Using the experimentally measured output states of the 4 input basis states the transfer function of the fiber was determined (for the range of different pump powers). With such obtained transfer function we then calculated the condition number of the inverse problem as a function of pump power, as shown in Figure 2.12(d). We solve the inverse problem using the data from the basis state $|\omega_0\rangle + i|\omega_1\rangle$ as the input, then calculated the fidelity of the reconstructed state with the expected state, $|\omega_0\rangle + i|\omega_1\rangle$. This fidelity is shown by the right hand y-axis on Figure 2.12(d). The decrease in fidelity with increasing pump power indicates that the assumption that the fiber system preforms linear transformation on the input signal breaks down, as other competing effects in the nonlinear frequency conversion begin to emerge at higher pump powers. Figure 2.12(d) can be used to determine the pump power which minimizes the condition number and maximizes the reconstruction fidelity in order to allow the most accurate reconstruction of arbitrary input states.

A pump power of 135 $\mu$W was chosen to enable the optimal reconstruction accuracy of input signals. We then tested our reconstruction method experimentally by coupling a sequence of random input signals into the fiber. The general form of the input signals was $a_n = \cos(\theta/2)|\omega_0\rangle + e^{i\phi}\cos(\theta/2)|\omega_1\rangle$, where $\theta$ and $\phi$ are randomly chosen, and implemented using a waveshaper. Then the output spectra at the end of the fiber is measured and used to reconstruct the input signal. For example the real and parts of a reconstructed input signals complex visibility function is shown in Figure 2.13(a) and 2.13(b) respectively. Similar reconstructions were carried out for many random input signals parametrized by $\theta$ and $\phi$ as shown in Figure 2.13(c). The reconstructed signals’ fidelity to the target signal that the waveshaper was configured to produce is typically above 96% as shown in Figure 2.13(c).
Figure 2.13: (a) Real and (b) imaginary parts of an example reconstructed 2-channel spectral coherence function. Fidelity to the target state is 97.8%. (c) Reconstruction fidelity (shown by color) of random states in the form $\cos(\theta/2)\ket{\omega_0} + e^{i\phi} \cos(\theta/2)\ket{\omega_1}$, where $\phi$ and $\theta$ are plotted on the $x$ and $y$ axes respectively. The state from (a) and (b) is indicated by a red arrow in (c).

Figure 2.14: (a) Frequency quantum walk output spectra as a function of the pump's intensity for the input state $\ket{\omega_1} + i \ket{\omega_2} - \ket{\omega_3} - i \ket{\omega_4}$. (b) Real and (c) imaginary parts of the states spectral coherence function, reconstructed by measuring the output spectra for 360 $\mu W$ pump power [red dotted line in (a)]. Fidelity of the reconstruction to the intended state is 93.9%. (d) Frequency quantum walk output spectra for the input state $i \ket{\omega_1} + \ket{\omega_2} + i \ket{\omega_3} + \ket{\omega_4}$. (e) Real and (f) imaginary parts of the input states spectral coherence function, again reconstructed by measuring the output spectra for 360 $\mu W$ pump power. Fidelity of the reconstruction to the intended state is 96.7%.
Reconstruction of more complex input signals was also carried out. An input signal consisting of four lattice sites was recovered. This signal consisted of lattice sites centered at frequencies 1561.6 nm, 1562.0 nm, 1562.4 nm and 1.562.8 nm, which will be denoted by the kets $|\omega_1\rangle$, $|\omega_2\rangle$, $|\omega_3\rangle$ and $|\omega_4\rangle$ respectively. First the transfer function of the fiber and pump system was determined by measuring the output spectra of a set of 16 basis states. Then this was used to reconstruct a number of random input states. The output spectra (as a function of pump power) of two such random input states are shown in Figures 2.14(a) and 2.14(b). The pump power providing optimum condition number for inversion was determined to be 360 $\mu$W, as indicated by the dashed vertical lines in Figures 2.14(a) and 2.14(b). These two random input states were reconstructed using their output spectra at 360 $\mu$W pump power. The real and imaginary parts of the resulting complex visibility functions are shown in Figures 2.14(b) and 2.14(c) and Figures 2.14(e) and 2.14(f) respectively. The fidelities of the reconstructed states to the intended states is 93.9% and 96.7% respectively.

To briefly summarize this section, we have experimentally investigated the use of spectral lattices for recovering complex spectral signals. We have shown that the full complex visibility of input spectral signals can be reconstructed just by measuring its output spectra after propagating through a spectral lattice.

### 2.6 Generalization to quadratic nonlinearity

In this section, we briefly discuss a potential of generalization our synthetic frequency lattice to the wave-mixing in second-order nonlinear waveguide. In general, sum-frequency-generation (SFG) with undepleted pump can also be described as a linear coupling between the signal and sum-frequency waves \[118, 119\].

In Figure 2.15(a) we sketch the concept of using SFG to achieve a synthetic frequency space. Similar as the results presented above using FWMBS, the system here also has only one waveguide where we can launch in shaped pump frequencies, such as the three-pump case shown in Figure 2.15(b) where pumps are shown as arrows. Importantly, here we have both signal spectra [shaded in yellow in Figure 2.15(b)] and sum-frequency spectra (shaded in green). Inherently these two regions in the frequency space can form two sub-lattices, as sketched in Figure 2.15(c). Recall that in FWMBS a pair of pumps induce one order of coupling. By contrast, here in $\chi^{(2)}$, each pump gives rise to an order of coupling. If we set the coupling induced by the pump denoted in red color as 0-th order coupling, then, the other two pumps correspond to 1st and 3rd order coupling [see Figures 2.15(b,c)]. Such a lattice with two sub-lattices and three orders of coupling can be mapped to a carbon nanotube-like lattice as illustrated in Figure 2.15(d).
2.7 Summary

To summarize this chapter, we have introduced and experimentally demonstrated an entirely new class of synthetic frequency lattices in nonlinear waveguides. Unlike many other photonic lattices that are implemented in space with structured matter, we use a single-port system that has no spatial engineering to form synthetic lattices in frequency space. The system is all-optical, with the spectral lattice induced through optical nonlinearity and controlled via a set of pump lasers, which can be altered quickly to provide different functionalities. Furthermore, manipulation of light within the spectral dimensions (without resorting to spatial multiplexing) makes such devices very compact. While many efforts in light-matter interaction were devoted to enhancing the interaction time via high quality-factor resonances, here we make judicious use of propagation waves in long nonlinear fibers, where we successfully test the feasibility of this approach. This also lifts up the limitation imposed by the free spectral range (FSR) of most dynamically modulated resonating systems, in particular, integrated resonators. After the publication of our results, EOM-based propagation-wave synthetic lattice was also reported [79], yet the unit coupling range $\Omega$ remain limited to a very small value to allow a sufficient number of spectral lines fulfilling the rather stringent phase-matching condition. Most recently, the use of thin-film lithium niobate modulators has drastically enhanced the efficiency and phase-matching bandwidth of EOM, as reported in a recent preprint [54], which we believe can also boost the capability of our all-optical approach if these waveguiding
and resonating structures can be incorporated to our system. Importantly, the synthetic frequency lattice in both third and second-order nonlinear waveguides are non-reciprocal due to phase-matching condition, which is also free of dynamic reciprocity [120] as we discussed in Reference [72] using sum-frequency generation as an example.

We anticipate this platform will enable the development of novel information processing techniques benefiting from its multidimensional nature in various means. Such a controllable lattice in the frequency space with artificial gauge field may be useful in various contexts of topological photonics, for example in solving the long-existing bottlenecks restricting the performance of optical devices associated with dispersion using topological protection in multidimensional space. The coherent nature of this platform promises great suitability for quantum photonics; for instance, the rapid development in quantum frequency combs may benefit from our platform in the manipulation and measurement of these high-dimensional frequency states.
3 Realizing higher dimensional dynamics in planar photonics

In Chapter 2, a key concept lies in artificially creating space and dimensions for the unconventional implementation and manipulation of photonic states. The approach taken in Chapter 2, along with many reported works, relies on the one-to-one reproduction of lattice sites and therefore leads to an exponential growth of space size with dimensionality.

In this chapter*, we introduce a new concept in synthetic dimension by utilizing the isospectral property of discrete systems, enabling the implementation of arbitrary local density of states for however many-dimensional lattice systems.

3.1 Introduction

"Can one hear the shape of a drum?" was the title of an article [122] by Mark Kac published in 1966, an intriguing question phrased earlier by American mathematician Lipman Bers. Central to this title is the question if it is possible to derive the shape of an object from its eigenfrequencies. This seemingly easy question was only shown conclusively for two-dimensional objects in early 1990’s by Gordon, Webb, and Wolpert [123] with a negative answer that there do exist two different shapes yielding the same set of eigenfrequencies. The fact of systems sharing the same eigenspectrum is called isospectrality.

Importantly, the scope inside the "hearing the shape of a drum" problem is way beyond a mathematical curiosity. In fact, tremendous fundamental research and practical applications are based on measurements of the spectral response of physical systems, since such detection is way more convenient than performing fast time-resolving measurements. Owing to the complementarity between frequency (energy) and time duration, tracing out a considerable length of time gives rise to a good resolution in frequency space, promising a wide range of applications. For example, material scientists use spectroscopy to understand the structure of materials and synthesize new materials maintaining the same spectrum. In a broader context, these applications are commonly based on a model of complex networks [124], a universal description utilized in almost all aspects of science and technology, from optical [14] and cold-atom [125] lattices to biological macromolecules [126] and neuromorphic circuits [127].

*Results presented in this chapter are partially included in Reference [121], where I am a lead author and my contribution is in theoretical aspects.
It is generally recognized that the network dimensionality can strongly affect the excitation dynamics, such as in the case of Anderson localization [2, 4–6] where diffusion can be absent in a disordered medium. As mentioned and outlined in Chapter 2, the concept of synthetic dimension [8, 35, 36] is a powerful tool to overcome the limitation imposed by the three-dimensional space. Common approaches for synthesizing dimensions typically rely on either introducing additional parameter dependencies [22] to the Hamiltonian, or using artificially engineered degrees of freedom, such as discrete frequencies, to form a synthetic space [33, 36, 42, 44, 48, 49, 128–131], with latest experiments demonstrating up to four dimensions [22]. The works presented in Chapter 2 are also examples of this approach. Generally, the aim is to achieve a perfect one-to-one reproduction of the entire multidimensional space. However, this inevitably comes at the steep cost of exponentially increasing complexity when attempting to access higher effective dimensions. Is there a new approach to synthetic dimension that can break away from this convention?

Isospectrality may serve as a potential answer to this question. Recall the example of “drums that sound the same” and note that, isospectrality can be interdimensional, e.g. interdimensional isospectrality has been investigated inspired by graph networks [132]. Besides, isospectrality associated with fundamental physics theories such as supersymmetry (SUSY) may also be brought to interdimensional applications. Despite being a controversial path to grand unification theory, the isospectrality nature of SUSY partners has inspired various optical applications [23, 133–136]. However, none of these reported methods is an exact solution to synthetic dimension, as the equivalence of eigenvalues, in general, does not automatically lead to shared eigenstates. In other words, the dynamics in one lattice site can correspond to collective behavior of many sites in its isospectral counterpart. Therefore, to use isospectral transformations for synthetic dimension, it is not sufficient to solely reproduce the spectra; rather, we have to ensure certain correspondence of the bases in which we excite and observe the dynamics.

If the above challenge is mapped back to the drum problem, then there is a surprising conclusion – two drums sharing the same eigenfrequencies can sound differently. This is due to two other important factors: the way of drumming, e.g. the shape of the drumstick and how it hits the drum, and the way of hearing, e.g. listening from which angle. These two factors determine the weight of each excited eigenfrequency that eventually goes into the listener’s ears. In the most ideal case, one would like to hear the same sound with the same way of drumming and same way of listening. Therefore, in order to establish the effectively same lattice-site behavior between a multidimensional lattice and a one-dimensional lattice, it is not sufficient to merely preserve the optical eigenvalue spectrum, as supersymmetry (SUSY) and related transformations [132–136] are known to do. Instead, one has to exactly replicate the actual local density of states, which is inherently the weights these frequencies are excited.

In this chapter, we introduce a new paradigm for the selective realization of a wide range of useful physical effects associated with high-dimensional networks and their nontrivial excitation dynamics based on isospectrality. Different from reported works, this approach allows
3.1 INTRODUCTION

Specially tailored isospectral transformation

\[ \mathbf{V} \]

Multidimensional network

1D lattice with only nearest neighbor interactions

Figure 3.1: Conceptual sketch of the concept of using isospectral transformation to reproduce the excitation dynamics of an anchor site in multidimensional networks in a one-dimensional lattice with judiciously tailored nearest-neighbor coupling. The multidimensional network (top left panel) is illustrated by a 3D projection of a 7D hypercube, the 1D equivalent lattice of which is shown in the top right panel.

a multidimensional lattice to be mapped to 1D with not only the same spectrum but also the same local density of states for a chosen anchor site. Consequently, the anchor sites exhibit exactly the same excitation dynamics. This mapping approach enables that arbitrary Hermitian multi-dimensional lattices with any local or non-local coupling distribution can be mapped to a 1D lattice with judiciously tailored nearest-neighbor coupling and detunings.

In order to illustrate this, consider a lattice with arbitrary multidimensional geometry. Figure 3.1 (top left) illustrates this for the example of a seven-dimensional hypercube. Using a specially tailored Lanczos transformation [137, 138], the hypercubic network is mapped to a one-dimensional lattice with nearest-neighbor coupling, as shown in Figure 3.1 (top right). The dynamics of an excitation at the anchor site in the multidimensional lattice is directly mapped onto the dynamics of the first site in the 1D lattice (sites marked in orange). Notably, this holds true for arbitrary defect equally added to the anchor sites, e.g. detuning and/or lossy type, time-dependent and/or nonlinear.

An advantage of this approach is the possibility to directly benefit from fundamental properties in higher dimensional space manifested in a lattice site of an input-output optical system. It is a well-known fact that excitation dynamics, in particular, the emergence of defect-localized states, crucially depends on the lattice dimensionality, as dictated by the fundamental distinctions in the density of states at the band-edges [12]. As it turns out, one of the key features of our technique is that any detuning-type defect at the anchor site identically impacts the excitation dynamics in the multi-dimensional structure and the synthetic 1D equivalent lattice alike. In this vein, in this chapter we theoretically predict and experimentally observe the emergence of a sharp localization transition at a critical defect strength. This behavior is mediated by a new type of bound state at the edge of the continuum (BSEC)
that occurs in four or higher-dimensional hypercubic lattices, which constitute a fundamentally nontrivial addition to the previously identified families of bound states in the continuum (BIC) [139]. The resulting extreme defect sensitivity readily lends itself to multifarious practical applications and for example, may serve as an entirely new approach to a design of integrated optical circuits.

Our method provides an entirely new approach for the design of synthetic lattices, with no dependence on platform. Beyond obtaining inspirations from higher dimensional networks, our approach also allows for freeform engineering of the local density of states for a certain site in one-dimensional lattices with nearest-neighbor coupling. Besides, since Lanczos transformation does, in general, also apply to non-Hermitian Hamiltonians, our method can also be generalized to non-Hermitian cases.

Outline of this chapter

The subsequent sections of this chapter are structured as follows: Firstly, in Section 3.2, general principles of our tailored Lanczos mapping approach for the inter-dimensional reproduction of excitation dynamics will be introduced and discussed with specific examples. Then, as a direct benefit and application of such a method, in Section 3.3 we discuss an ultra-sensitive type of defect state in four and higher dimensional lattices originating from bound states at the edge of the continuum and provide direct experimental evidence. Then, in Section 3.4 we further generalize the problem beyond the dimensionality context and introduce the freeform engineering of local density of states. Then, in Section 3.5 we provide a non-Hermitian generalization of this method. Finally, we will summarize this chapter in Section 3.6.

3.2 Observing multidimensional excitation dynamics in 1D lattices

As mentioned in Section 3.1, recent theoretical works employed Householder and SUSY-related transformations to map the optical spectrum of multi-dimensional arrangements to 1D lattices [132, 136]. However, in those approaches, each site of the 1D lattice inevitably ends up representing a complex superposition of multiple original lattice sites. In this section, we introduce a general mapping procedure to achieve a one-to-one correspondence between the respective anchor sites and provide an experimental demonstration.

The most widely used lattice is based on one-dimensional tight-binding model with only nearest-neighbor coupling. While it is true there is a limitation imposed to the extent that one can freely engineer the dynamics of each site, yet such short-range interaction actually suffices to freely engineer the dynamics of one site. Now we formulate this mathematically, where we start from an \( M \times M \) Hermitian Hamiltonian \( H \) (in arbitrary dimension). We can perform an eigenvalue decomposition for \( H \)

\[
RD = HR, \tag{3.1}
\]
where $R$ is an $M \times M$ matrix, the columns (denoted by $|r_m\rangle$) of which are the eigenvectors of $H$, and $D$ is a diagonal matrix containing the $M$ eigenvalues of $H$ (denoted by $\lambda_m$). For a Hermitian Hamiltonian, the eigenvectors are orthonormal, i.e. $\langle r_m | r_n \rangle = \delta_{m,n}$, and therefore $R$ is a unitary matrix where we have $R^{-1} = R^\dagger$ ($^\dagger$ denotes conjugate transpose). The Hermiticity also leads to real eigenvalues, i.e. $\lambda_m \in \mathbb{R}$. If we multiply $R^\dagger$ from the left to both sides of Equation (3.1), we find

$$D = R^\dagger H R. \quad (3.2)$$

Now if there exists another Hermitian Hamiltonian $H'$ that is isospectral with $H$, then we have to ensure

$$R^\dagger H R = R^\dagger H' R', \quad (3.3)$$

where $R'$ is a matrix with its columns being the eigenvectors of $H'$. If we multiply $R$ from left and $R^\dagger$ from right to both sides of Equation (3.3), we find

$$H = R R'^\dagger H' R R'^\dagger. \quad (3.4)$$

Since both $R$ and $R'$ are unitary matrices, we can define a new unitary matrix $V = R' R^\dagger$, and hence we have

$$H = VH V^\dagger. \quad (3.5)$$

Therefore, to obtain $H'$ that is isospectral to $H$, we have to find a unitary matrix $V$ fulfilling Equation (3.5). Note that despite there can be in total up to $M$ eigenvectors, the set of eigenvalues $\{\lambda_m\}$ may contain only $N < M$ different elements if there exists degeneracy, i.e. multiple eigenvectors correspond to the same eigenvalue.

Now we focus on a three-diagonal Hamiltonian $H' = H^{(3)}$ that restricts the isospectral lattice to a one-dimensional one with only nearest-neighbor coupling, and hence Equation (3.5) becomes

$$H = VH^{(3)} V^\dagger. \quad (3.6)$$

The unitary nature of $V$ requires the column vectors $|v_m\rangle$ of $V$ being orthonormal, i.e. $\langle v_m | v_n \rangle = \delta_{m,n}$. For $H^{(3)}$, if we denote its diagonal elements by $H_{m,m} = \epsilon_m$ and off-diagonal elements $H_{m+1,m} = C_m$ and $H_{m,m+1} = C_m^*$ ($^*$ represents complex conjugate), then the following decomposition is equivalent to Equation (3.6):

$$H = \sum_{m=1}^{M} \epsilon_m |v_m\rangle \langle v_m| + \sum_{m=1}^{M-1} \left( C_m |v_{m+1}\rangle \langle v_m| + C_m^* |v_m\rangle \langle v_{m+1}| \right). \quad (3.7)$$

To obtain elements in $H^{(3)}$, one can recursively multiply $H$ onto $|v_m\rangle$ for $m = 1, 2, \ldots, M$. As a first step, for $m = 1$,

$$H |v_1\rangle = \epsilon_1 |v_1\rangle + C_1 |v_2\rangle. \quad (3.8)$$

We can obtain $\epsilon_1$ by multiplying $\langle v_1 |$ from left to both sides of Equation (3.8) and using the orthonormal property of $|v_m\rangle$,

$$\epsilon_1 = \langle v_1 | H |v_1\rangle. \quad (3.9)$$
This indicates that one can choose an arbitrary unit vector $|v_1⟩ \in \mathbb{C}^M$, since $⟨v_1|H|v_1⟩ \in \mathbb{R}$ can be fulfilled for arbitrary $|v_1⟩ \in \mathbb{C}^M$ and Hermitian $H$. Different from the previous Lanczos methods, here we specially choose $|v_1⟩$ that matches the initial state of excitation $|Ψ(t = 0)⟩$ for $H$, i.e.

$$|v_1⟩ = |Ψ(t = 0)⟩.$$ (3.10)

Then, with $ε_1$ known, we can get the modulus of $C_1$ using Equation (3.8),

$$|C_1| = |(H − ε_1)|v_1⟩|,$$ (3.11)

where we use the property that $⟨v_2|v_2⟩ = 1$. The argument $φ_1$ of $C_1$ is actually another degree of freedom that one can choose, i.e. $C_1 = |C_1|\exp(iφ_1)$. Hence $|v_2⟩$ can be obtained by

$$|v_2⟩ = \frac{1}{C_1}(H − ε_1)|v_1⟩.$$ (3.12)

For $m > 1$, there is

$$H|v_m⟩ = ε_m |v_m⟩ + C_m |v_{m+1}⟩ + C_{m−1}^* |v_{m−1}⟩.$$ (3.13)

Hence, for $m = 2, 3, \ldots$ one can recursively calculate

$$ε_m = ⟨v_m|H|v_m⟩,$$ (3.14a)

$$C_m = [(H − ε_m)|v_m⟩ − C_{m−1}^* |v_{m−1}⟩]|e^{iφ_m},$$ (3.14b)

$$|v_{m+1}⟩ = \frac{1}{C_m}[(H − ε_m)|v_m⟩ − C_{m−1}^* |v_{m−1}⟩],$$ (3.14c)

in which the phase $φ_m$ is a parameter that one can freely choose. To better illustrate the method, we sketch the recursive steps in Figure 3.2, where yellow and blue grids denote zero and non-zero elements in the Hamiltonians, respectively.

The above analysis outlines both the key working principle inside the Lanczos algorithm [137, 138, 140, 141] for the tri-diagonalization of matrices and the new approach we take in making use of such a transformation. For a detailed description of the algorithm used for this chapter, see Appendix Section A.1.3. While this transformation is widely used for converting matrices to a three-diagonal form, in particular as an intermediate step to the diagonalization, the potential of all degrees of freedom behind the transformation is not fully utilized. More specifically, unlike the diagonalization of a Hamiltonian, the tri-diagonalization does not yield a unique solution. In fact, the first unit vector $|v_1⟩$ for the recursive process to start with is at a free choice. Importantly, it is the free choice of $|v_1⟩$ given in Equation (3.10) that enables the synthesis of arbitrary excitation dynamics in the anchor site. If $|v_1⟩$ is chosen to match the excitation of an anchor site, i.e.

$$|Ψ(t = 0)⟩ = |m_a⟩,$$ (3.15)

where $|m_a⟩$ denotes the citation of of the anchor site $m_a$, then the wavefunction $Φ_1(t)$ in the first site of the one-dimensional lattice governed by $H^{(3)}$ is exactly the same as $Ψ_{m_a}(t)$ in the
anchor site of the multidimensional structure governed by $H$:

$$\Phi_1(t) \equiv \Psi_{m_0}(t).$$

(3.16)

Nevertheless, importantly, this approach is not limited to reproducing the excitation dynamics of a local site. As a more general condition of the to-be-replicated excitation, the initial condition in the multi-dimensional lattice can be arbitrary, e.g. an initial wavefunction $|\Psi(t = 0)\rangle$ that coherently excites many sites. Yet the dynamics from the anchor site of the mapped 1D lattice is a projection of the wavefunction $|\Psi(t)\rangle$ in the multi-dimensional lattice to its initial excitation, i.e.

$$\Phi_1(t) \equiv \langle \Psi(t = 0)|\Psi(t)\rangle.$$

(3.17)

As one may find from the discussion above, the argument of each off-diagonal element can also be freely chosen. Nevertheless, the choice of arguments in the off-diagonal elements does not influence the excitation dynamics of the anchor site, as a single-site excitation traces out the whole reciprocal space and the phase in coupling coefficients could only shift the dispersion. Therefore, in the following discussions, the off-diagonal elements are all treated as real-valued in the construction, which is also the most practical case in most lattice systems.

**Mapping example of a ring lattice**

Now we illustrate our approach with an example. Figure 3.3(a) schematically shows a ring lattice in two-dimensional space with 11 identical sites and homogeneous nearest-neighbor coupling, the Hamiltonian of which is plotted in Figure 3.3(c). An anchor site is chosen as numbered 1 and marked in red in Figure 3.3(a). Upon mapping this arrangement to one-
Figure 3.3: Example of mapping a (a) two-dimensional ring lattice to (b) a one-dimensional lattice. (c,d) The Hamiltonian of (c) the original lattice $H$ and (d) the mapped lattice $H^{[3]}$. (e,f) The dynamics of all sites when the anchor (first) site is excited, in (e) the original lattice and (f) the mapped lattice. (g) Elements $V_{p,q}$ of the matrix $V$ that describes the mapping, $H^{[3]} = V^\dagger HV$. 
3.2 OBSERVING MULTIDIMENSIONAL EXCITATION DYNAMICS IN 1D LATTICES

dimensional space, a lattice as sketched in Figure 3.3(b) can be obtained. As seen from the 1D mapped Hamiltonian shown in Figure 3.3(d) directly obtained from the Lanczos algorithm, the coupling between sites 6 and 7 is zero. This is why the 1D lattice is effectively truncated at 6 sites as shown in Figure 3.3(b). Such a truncation occurs because some of the modes on a ring lattice have degenerate eigenvalues, and the mapped 1D equivalent lattice does not have degeneracy. Therefore it is possible to end up with less number of sites in the mapped equivalent 1D lattice. A faithful reproduction of the excitation dynamics therefore does not always require the entirety of the degrees of freedom inherent in the original system. Note that a previous theoretical approach for inter-dimensional mapping presented in [132] only preserved the optical spectrum, whereas it is a distinguishing feature of our approach that the very dynamics in the anchor (first) site [Figure 3.3(e)] is faithfully reproduced in the first site of the mapped 1D equivalent lattice [Figure 3.3(f)].

We also show the matrix $V$ with a color density plot in Figure 3.3(g). As can be seen, in the first column of $V$ only the first element $V_{1,1} = 1$ is non-zero, which exactly matches the excitation of the site 1 in the two-dimensional ring structure.

**Mapping example of structures with long-range interactions**

Note that our mapping works for lattices with any local or non-local coupling distributions. By non-local coupling we refer to connections beyond nearest-neighbours. Mathematically, when the Hamiltonian is arranged in the conventional two-dimensional matrix form, the non-local coupling would manifest as extra non-zero off-diagonal elements. Our mapping procedure is applicable to arbitrary (Hermitian) Hamiltonians, and therefore is fully applicable in presence of nonlocal coupling. To illustrate this, here we provide an example for a three-dimensional cubic structure as sketched in Figure 3.4(a) with inhomogeneous local coupling denoted by different colors of the connecting lines, the Hamiltonian of which is plotted in Figure 3.4(b). The 1D mapping of this 3D structure is sketched in Figure 3.4(c). The 1D equivalent lattice has 7 sites, where we show the coupling constants $C_m$ (left) and on-site potential $\epsilon_m$ (right) in Figure 3.4(d). By contrast, if we add higher order coupling, e.g. diagonal coupling with the value $C_d = 0.5$, as sketched in Figures 3.4(e,f), the 1D equivalent lattice becomes a 6-site one with detuned (non-zero) $\epsilon_m$ for $m \neq 1$ [see Figure 3.4(g,h)]. The detunings are illustrated by the size of balls denoting the sites in Figure 3.4(g), and in Figure 3.4(h) we plot the 1D coupling (left) and detuning (right) profiles.

**Mapping of infinite lattices**

The mapping scheme introduced in this chapter is not only applicable to finite lattices, but also to infinite ones which are then mapped to *semi-infinite* 1D lattices. In practical realizations, the mapped 1D semi-infinite lattice can be truncated at a certain number of sites which contribute to the anchor site dynamics within a finite experimental observation time (i.e. observation distance in a waveguide system).
As an important example and basis of what the later results are based on, here the mapping of infinite hyper-cubic lattices will be presented. As sketched in Figure 3.5(a), here infinite lattices with square (2D), cubic (3D) and hyper-cubic (4D and higher dimension) will be considered as the object to be mapped. The nearest-neighbor coupling are homogeneous in these lattices, denoted by $\bar{C}$. The coupling for 1D arrays mapped from these high-dimensional periodic lattices are plotted in Figures 3.5(b–g) for dimensionalities $N_d = 1, \ldots, 6$.

Note that the case of $N_d = 1$ corresponds to mapping a site from within an infinite 1D chain to an edge site of a semi-infinite lattice. Interestingly, this is achieved through the modification of only the first coupling coefficient [Figure 3.5(b)]. For the higher-dimensional lattices, all coupling coefficients are judiciously engineered in the 1D lattice. However, in each case, the normalized 1D coupling approach unity at large $m$, as the respective lattices and their multidimensional counterparts have been engineered to exhibit photonic bands in the same interval of $-2\bar{C} < \beta < 2\bar{C}$.

**Experimental demonstration for mapping up to 7D hyper-cubic lattices**

Now an experimental realization will be shown using the one-dimensional lattices synthesizing multidimensional hypercubic lattices as presented in Figure 3.5. Note that the theory is in general applicable to arbitrary experimental systems that can be described by a tight-binding lattice. In this chapter, the proof-of-principle experiments are based on femtosecond laser-written photonic lattices [142]. For details on the experimental fabrication, characterization
Figure 3.5: (a) Sketch of 1D, 2D and 3D lattices, each with an anchor site marked in red. (b) Normalized coupling coefficients ($C_m/\bar{C}$) for the mapped 1D structure from $N_d$-dimensional lattices as indicated by labels, shown for the first 16 sites.
and measurement of such waveguides, see Appendix Section A.2.1. Importantly, on such an experimental platform, the propagation direction $z$ corresponds to the time coordinate $t$, and the strength of coupling between two adjacent waveguides can be continuously tuned by adjusting their transverse separation. The light intensity evolution inside waveguides can be experimentally observed via fluorescence microscopy [142].

Consider periodic lattices of increasing dimensionality $N_d$ as illustrated in Figure 3.5(a). For $N_d > 3$, their higher dimensional hypercubic generalizations are analyzed with the nearest-neighbor coupling normalized as

$$\tilde{C}_{N_d} = \tilde{C} / N_d.$$  (3.18)

This ensures that the respective systems exhibit photonic bands in the same interval $-2\tilde{C} < \beta < 2\tilde{C}$ in order to allow for a direct comparison of effects associated with the multidimensional diffraction relation

$$\beta = 2\tilde{C}_{N_d} \sum_{n=1}^{N_d} \cos(k_n),$$  (3.19)

where $k = (k_1, k_2, \ldots, k_{N_d})$ is the wave vector of the Bloch waves [12].

In the calculation the parameters of the mapped 1D lattices, we take $\epsilon_m \equiv 0$. As was discussed above, the coupling of the 1D equivalent systems exhibit characteristic distributions for each dimension $N_d$, and universally converge to $C_m / \tilde{C} \rightarrow 1$ at $m \gg 1$ according to the width of the photonic band. After implementing the 1D equivalent lattices by femtosecond laser direct inscription, they are used to shed light on unique features of higher-dimensional quantum walks by observing the respective escape dynamics of single-site excitations. Such evolution is theoretically predicted to follow

$$|\Phi_1(z)|^2 = [\mathcal{J}_0(2\tilde{C}_{N_d} z)]^{2N_d},$$  (3.20)

where $\mathcal{J}_0$ is the 0th order Bessel function. Figures 3.6(c) and (d) compare theoretical predictions according to the expression given in Equation (3.20) and experimental observations for three representative cases ($N_d = 1, 3, 7$), which coincide with an excellent fidelity of $F > 98\%$. The intriguing impact of dimensionality on the quantum walk becomes apparent: Despite the increasing number of potential coupling destinations in higher-dimensional space, light initially tends to escape slower. Moreover, the observed propagation patterns in the 1D equivalent lattices show that the partial revivals known from 1D lattices are entirely suppressed in higher dimensions.

### 3.3 Sharp localization transition in four and higher dimensions

The conventional wisdom on bound states is associated with discrete eigenvalues lying outside the continuum of energy band. While there exist various approaches to specially engineer the potential achieving the so-called bound states in the continuum, the latter typically
Figure 3.6: (a) Schematic of the experimental realization with appropriate inhomogeneous spacings between adjacent waveguides. (b) Fluorescence images of the propagation dynamics in the 1D mapped lattices corresponding to the dimensionalities $N_d = 1$ and 7. (c) theory and (d) experimental observations in the first waveguides of the mapped 1D lattices with $C = 1 \text{ cm}^{-1}$.
requires rather complex structures with special considerations. Hence it remains highly inter-
ingesting if it is possible to push a bound state to be as close as possible to the continuum, ide-
ally sitting right at the band edge. This is especially interesting for the dynamic switch-
ing of systems between leaky modes (spatially non-localized, in the continuum) and a bound
state (spatially compact, out of the continuum), where the required change to the potential
to achieve the switching can go infinitely close to zero if the bound state is at the edge of the
continuum.

With the mapping approach discussed in this chapter, it becomes possible to investigate
the excitation dynamics in arbitrary dimensionality. Hence in this section we introduce the
new concept of the bound state at the edge of the continuum that appears for defect localiza-
tion in higher dimensions. More specifically, we provide details on the theoretical analysis of
the defect state in multi-dimensional space and describe the origin of the bound state at the
edge of continuum. We show its fundamentally extreme sensitive nature with direct evidence
from waveguide experiments.

**Perturbative analysis of defect localization in higher dimensions**

Now an perturbation-based theoretical analysis will be performed to investigate the local-
ization behavior of single impurity scattering in four- and higher dimensional hyper-cubic
lattices. Note that in this section, we use the propagation distance \( z \) as time in the dynamics
analysis to match the experimental platform of waveguides.

For the infinite \( N_d \)-dimensional lattice as sketched in Figure 3.5(a), from a mathemat-
ical point of view, the dynamics resulting from a single-site excitation represents temporal Green’s
function \( G_{N_d}(z) = \Phi_1(z)/\Phi_1(z = 0) \), which effectively characterizes the response of the entire
lattice. Temporal-domain Green’s function for a non-detuned site of excitation can be given
by

\[
G_{N_d}(z) = \left[ \mathcal{F}_d(z) \right]^{N_d}.
\]

By performing a Fourier transform of \( G_{N_d}(z) \), we obtain its frequency-domain (correspond-
ing to wavenumber-space for waveguides lattices) form:

\[
\tilde{G}_{N_d}(\beta) = -i \int_0^\infty G_{N_d}(z) e^{i\beta z} \, dz,
\]

which provides additional insights into the role of dimensionality. In particular, one can num-
merically evaluate frequency-space Green’s function given in Equation (3.22) and compared
the results with the analytical expressions in 1D and 2D:

\[
\tilde{G}_1(\beta) = \left( \beta^2 - 4\tilde{C}^2 \right)^{-1/2},
\]

\[
\tilde{G}_2(\beta) = 2(\pi \beta)^{-1} \mathcal{K} \left( 4\tilde{C}^2 \beta^{-2} \right),
\]

where \( \mathcal{K} \) is the complete elliptic integral of the first kind.
Figure 3.7: Real and imaginary parts of spectral Green's function at the excitation site for lattices of different number of dimensions $N_d = 1, 2, 3, 4, 5$. 
Figures 3.7 shows the theoretically calculated real and imaginary parts of $\tilde{G}_{N_d}$ for $N_d = 1, 2, 3, 4$ and 5. The imaginary part $\text{Im}[\tilde{G}_{N_d}(\beta)]$ determines the local density of states and is therefore responsible for the rate of escape from the excited site mediated by the different spectral components within the band. The evident lower rate of escape observed in the mapped versions of higher-dimensional lattices is consistent with the calculated behavior of Green's function within the transmission band ($|\beta| < 2$, up to the vertical dashed lines in Figure 3.7). In this vein, the mapping approach can also serve as versatile tool for engineering the band structure and tailoring the associated lattice response towards practical applications.

Then, consider the fundamental effect of localization arising from a detuning $\epsilon$ of the anchor with respect to the surrounding homogeneous $N_d$-dimensional lattice. Importantly, knowledge of Green's function in a defect-free lattice allows us to predict the localization behavior upon introduction of a defect \[12\] as

$$\epsilon = \frac{1}{\tilde{G}_{N_d}(\beta_d)},$$

where $\beta_d$ is the frequency of a localized defect mode. It follows that real-valued detunings $\epsilon$ can support localization in the band-gap, where $\text{Im} \tilde{G}_{N_d}(\beta) = 0$. The real part of Green's function accordingly reflects this tendency of light to remain localized in the excited anchor site. This can be determined by the excitation efficiency for the defect-localized mode, defined as the fraction of population remaining at the input site after a sufficiently long evolution. In the theoretical limit of $z \to +\infty$, it is expressed as

$$\eta = \lim_{z \to +\infty} \left| \frac{\Phi_1(z)}{\Phi_1(z = 0)} \right|^2 = \left( \frac{\partial [1/\tilde{G}_{N_d}(\beta_d)]}{\partial \beta_d} \right)^{-2} = \left( \frac{\partial \epsilon}{\partial \beta_d} \right)^{-2}. \tag{3.26}$$

Fundamentally enhanced defect sensitivity with bound state at the edge of the continuum

Figure 3.8 shows the calculated excitation efficiency over the defect strength for the dimensionalities from 1 to 5. In 1D and 2D, localized modes appear for any defect strength, while for $N_d \geq 3$ localization occurs only above a critical defect magnitude, $\epsilon \geq \epsilon_{cr}$, in agreement with previous studies \[12\]. Remarkably, a distinctive effect for high-dimensional lattices with $N_d \geq 4$ is identified: Here, the excitation efficiency instantly attains a finite non-zero value at the critical defect strength. Specifically, we see in Figure 3.8 that $\eta(\epsilon_{cr}) \approx 0.13$ for 4D and $\approx 0.49$ for 5D lattices.

In order to give a more intuitive perspective, in Figure 3.9(a) the defect mode propagation constant $\beta_d$ of the defect mode versus different values of defect strength $\epsilon$ is plotted. Notably, all defect states emerge from the band edge (indicated by the dashed line). Whereas in 1D and 2D, localized modes appear for any defect strength, localization for $N_d \geq 3$ only occurs above a critical defect magnitude, $\epsilon \geq \epsilon_{cr}$. This is in agreement with previous studies of $N_d = 1, 2, 3$ lattices \[12\]. Moreover, the magnitude of this threshold $\epsilon_{cr}$ systematically increases with the dimensionality $N_d$. Figure 3.9(b–d) shows the anchor-site intensities of defect-localized
Figure 3.8: Theoretically calculated excitation efficiency for infinite propagation distance. Shading outlines the region under the curves for $N_d = 4, 5$, where a distinct localization threshold occurs.

Figure 3.9: (a) Theoretical plot of the defect state propagation constant $\beta_d$ versus the defect strength $\epsilon$. (b) Color plot illustrating the dependence of the population in the first nine waveguides (horizontal axis) of the mapped structure on the propagation constant $\beta_d$ (vertical axis) of the defect state for $N_d = 1, 3$ and 5.
modes vs. the propagation constant $\beta_d$ for $N_d = 1, 3, 5$. In all cases, localized modes appear for $\beta_d \geq 2$. The conventional localization transition for $N_d = 1, 3$ produces weakly localized defect modes that initially extend to infinity at the band-edge and gradually become narrower as $\beta$ is pushed further into the band-gap region. In contrast, a remarkably different behaviour occurs for $N_d \geq 4$: Here, the mode is already strongly localized when it emerges from the band edge ($\beta_d = 2$), as we illustrate in the figure for $N_d = 5$. This latter observation is direct evidence of the presence of a bound state at the edge of the continuum (BSEC), which appears exactly at the critical defect strength $\epsilon_{ct}$ for $N_d \geq 4$. Whereas bound states in the continuum (BIC) have been predicted for 1D waveguide arrays with modulated parameters [143], we here show how bound state at the edge of the continuum give rise to an entirely new regime of enhanced defect sensitivity.

**Propagation-limited localization**

In many practical cases, in particular waveguide systems that map propagation time to length, the observation is limited by the maximum propagation length in the experiment. While our theoretical analysis predicts the sharp transition in the defect localization in higher-dimensional hypercubic lattices at large propagation lengths, here we discuss the defect localization transitions at the intermediate propagation distances.

The wavefunction dynamics representing localization at the anchor site can be calculated theoretically using Green's function approach as follows. Firstly, we consider $G_{N_d}(z)$ as Green's function in a defect-free case, such that the wavefunction in the anchor site is $\Phi_1 = G_{N_d}(z)$ for a unit-amplitude excitation at the anchor site. The presence of a defect $\epsilon$ at the anchor site acts as an effective distributed source along the waveguide, which can be mathematically expressed through an integral relation for the wavefunction:

$$\Phi_1(z) = G_{N_d}(z) + i \epsilon \int_0^z dz' \Phi_1(z') G_{N_d}(z - z').$$

(3.27)

We emphasize that this is an exact relation describing the wavefunction dynamics in presence of the defect. We can then perform differentiation over $z$ for both sides of Equation (3.27) noting the initial condition $G_{N_d}(z = 0) = 1$,

$$\frac{d\Phi_1(z)}{dz} = G_{N_d}'(z) + i \epsilon \Phi_1(z) + i \epsilon \int_0^z dz' \Phi_1(z') G_{N_d}'(z - z'),$$

(3.28)

where $G_{N_d}'(z) = dG_{N_d}(z)/dz$. As from the above analysis, the analytical expression of $G_{N_d}(z)$ was given in Equation (3.21), from which $G_{N_d}'(z)$ can be analytically given by

$$G_{N_d}'(z) = -2 N_d \tilde{C}_{N_d} \left[ \mathcal{J}_0(2 \tilde{C}_{N_d} z) \right]^{N_d-1} \mathcal{J}_1(2 \tilde{C}_{N_d} z).$$

(3.29)

We calculate the wavefunction numerically by discretising the propagation length $z$ into small steps $\delta z$. Thereby, we determine the fraction of input power remaining at the defect for a
Figure 3.10: Theoretical analysis of the influence of finite propagation length onto multidimensional defect localization. (a–f) Color density plot of the (a–c) defect localization and (d–f) defect sensitivity versus the defect strength (horizontal axis) and normalized propagation length $zC$ (vertical axis) for $Nd=1, 3$ and $5$, as indicated by labels. (g–i) Plots of the defect localization (left) and defect sensitivity (right) versus defect strength for the propagation lengths $zC = 10, 50$ and $100$, respectively.
particular propagation distance as $|\Phi_1(z)|^2$. We also determine the defect sensitivity defined as $\frac{\partial |\Phi_1(z)|^2}{\partial \epsilon}$.

In Figure 3.10 we show the theoretically calculated defect localization (a–c) and defect sensitivity (d–f) for different propagation lengths $z \bar{C}$ normalized to the coupling constant $\bar{C} = \bar{C}_{N_d} N_d$, at different defect strengths $\epsilon$ and different dimensionality $N_d = 1, 3, 5$. In these density plots we mark three propagation distances with dashed lines, $z \bar{C} = 10$ (white dashed lines), $z \bar{C} = 50$ (yellow dashed lines), and $z \bar{C} = 100$ (red dashed lines). The corresponding cross-section plots at these three distances are additionally shown with curves in Figures 3.10 (g–i). Figure 3.10(g) corresponds to the case of $z \bar{C} = 10$, where one can already see a larger defect sensitivity (right) at $N_d = 5$ compared to $N_d = 1, 3$. This will be demonstrated in our upcoming experiment. Importantly, the defect sensitivity increases with the propagation length $z$. As plotted in Figure 3.10 (h,i), for $z \bar{C} = 50$ and $z \bar{C} = 100$, the enhancement in defect sensitivity is already quite substantial. As $z \bar{C}$ increases, the we approach the regime of very sharp transition at a critical defect strength for $N_d = 5$, in agreement with the theoretical limit at the infinite propagation length presented in Figure 3.8.

**Experimental evidence of sensitive bound state at the edge of the continuum**

The key aspect in allowing this characteristic of a high-dimensional system to be mapped to a 1D structure is that the judiciously engineered coupling distribution of the latter likewise supports a bound state at the edge of the continuum (BSEC) exactly for the critical defect strength.

Now we establish how we observe the exotic defect localization behavior associated with bound state at the edge of the continuum using 1D equivalent lattices. Specifically, we aim to provide a theoretical analysis of why a defect state can be maintained by this Lanczos transformation approach. For the multidimensional Hermitian Hamiltonian $H$ and an anchor site defect $\epsilon$ at site $m_d$, the corresponding dynamics are governed by a discrete Schrödinger equation,

$$\frac{i}{\hbar} \frac{d\Psi_m}{dz} = \sum_{m'} H_{m,m'} \Psi_{m'} + \epsilon \delta_{m,m_d} \Psi_m,$$  \hfill (3.30)

where $z$ is the normalized propagation distance coordinate acting as time, the index $m$ denotes the lattice sites, $\Psi_m$ are the complex wave function amplitudes, and $\delta_{m,m_d}$ is a Kronecker delta function. Recall that the Lanczos method helps identify a unitary transformation $V$ that relates $H$ to a three-diagonal Hamiltonian $H^{(3)} = V^\dagger HV$ corresponding to a 1D lattice with nearest-neighbor coupling. Considering single-site excitations at anchor site $m_d$ as the initial step for a recursive Lanczos algorithm, the evolution Equation (3.30) can then be transformed to

$$\frac{i}{\hbar} \frac{d\Phi_m}{dz} = \epsilon_m \Phi_m + C_{m-1} \Phi_{m-1} + C_m \Phi_{m+1} + \epsilon \delta_{m,1} \Phi_m,$$  \hfill (3.31)
where for \( m \geq 1, \epsilon_m \) and \( C_m \) are the respective on- and off-diagonal elements of \( H^{(3)} \). The dynamics of an excitation at the anchor site in the multidimensional lattice (considering \( \Psi_{m \neq m_d}(t = 0) = 0 \)) is directly mapped onto the dynamics of the first site in the 1D lattice:

\[
\Phi_1(z) \equiv \Psi_{m_d}(z).
\]

Note that this holds true for detunings \( \epsilon \) of any strength, which can even be time-dependent and/or nonlinear according to an arbitrary function \( \epsilon(z, \Psi_{m_d}) \).

Since the associated sharp transition from zero to strong localization had not been observed before, we experimentally explored the phenomenon in the respective 1D equivalent lattices, where the anchor site's detuning was implemented by choosing a different inscription speed.

The excitation efficiencies were estimated by averaging the final centimetre of the recorded fluorescence images [Figure 3.11(a)], as indicated by orange frames, and calculating the fraction of light remaining in the anchor site. As an example, for \( N_d = 5 \) the light remains much more strongly trapped for an above-threshold detuning [Figure 3.11(a)]. Figure 3.11(b) shows an overview of the localization transition for different mapped dimensionalities. Note that, while the finite sample length of 10 cm in our experiments necessarily smooths out the abrupt jump predicted for infinite propagation distance, the systematically increased slope clearly showcases the dramatically enhanced sensitivity in higher dimensions. Mapped equivalent lattices provide the means to harness this extreme sensitivity, as well as many other unique features of higher-dimensional systems including nonlinear localization [45], in a single site of an appropriately engineered 1D system.

### 3.4 Freeform engineering of local density of states for lattice design

So far we based our analysis and results in this chapter on synthetic dimension, where inherently the judicious engineering of a one-dimensional lattice is from the inspiration from another higher dimensional lattice. This enables some exotic physical properties of the high dimensional lattices to manifest itself in an anchor site of excitation in the isospectral one-dimensional lattice. In a most general picture, since the dynamical behavior of a lattice site in Hermitian lattices is fully determined by its local density of states, it is highly interesting to freely synthesize local density of states exhibited by an anchor site by judiciously modulating the coupling and on-site potential profiles. This will facilitate the design of arbitrary excitation dynamics associated with the anchor site, without a need to get inspiration from any other systems. Therefore, in this section, we generalize the Lanczos method introduced in this chapter to the functionality of synthesizing arbitrary local density of states for Hermitian Hamiltonians.

The general concept is sketched in Figure 3.12. As sketched in Figure 3.12(left), one can consider an arbitrary distribution of local density of states which includes a set of \( M \) eigen-
Figure 3.11: (a) Fluorescence images of the mapped $N_d = 5$ dimensional lattices with a detuned anchor site, from left to right: below the localization threshold, for a detuning of 0.61 cm$^{-1}$ and 1.02 cm$^{-1}$, light is able to escape; approximately at the localization threshold, for a detuning 1.69 cm$^{-1}$; above the localization threshold, for a detuning and 1.95 cm$^{-1}$, the initial excitation remains largely localized. (b) Experimentally observed localization at the defect (symbols) compared with the theoretical predictions (lines) in the planar lattices with different mapped dimensionality $N_d = 1, 3, 5$, defined as the average relative intensity in the last 1 cm of the first waveguide.
values and their corresponding local density of states. With our tailored Lanczos method, we construct an \( M \)-site one-dimensional lattice [illustrated in Figure 3.12(right)] that has only nearest-neighbor coupling with its first site exhibiting the given distribution of local density of states. Note that it is not possible to have degeneracy in such a 1D lattice with nearest-neighbor-only coupling, therefore the number of sites is exactly the number of eigenvalues.

![Conceptual diagram of synthesizing arbitrary local density of states via tailored Lanczos method.](image)

Figure 3.12: Conceptual diagram of synthesizing arbitrary local density of states via tailored Lanczos method.

To start with, we discuss the possible range of values that can be taken by the local density of states of Hermitian lattices. For a \( M \times M \) Hermitian Hamiltonian \( H \), recall its eigenvalue decomposition \( H = \mathbf{R}\sigma^\dagger \) that can be written in the form

\[
H = \sum_{m=1}^{M} \lambda_m |r_m\rangle \langle r_m|, \quad (3.33)
\]

where \( \lambda_m \) and \( |r_m\rangle \) are the \( m \)-th eigenvalue and corresponding eigenvector in the \( m \)-th column of \( \mathbf{R} \), respectively. Likewise, the unitary evolution operator \( \mathbf{U}(t) \) can be expanded to the same basis,

\[
\mathbf{U}(t) = e^{-iHt} = \sum_{m=1}^{M} e^{-i\lambda_m t} |r_m\rangle \langle r_m| . \quad (3.34)
\]

The time dynamics \( |\Psi_{n,n'}(t)\rangle \) for the excitation of site \( n \) and observation in site \( n' \) can be written as

\[
\Psi_{n,n'}(t) = \langle n'|\mathbf{U}|n \rangle . \quad (3.35)
\]

Inserting Equation (3.34) to Equation (3.35) we find

\[
\Psi_{n,n'}(t) = \sum_{m=1}^{M} \langle n'|r_m\rangle \langle r_m|n \rangle e^{-i\lambda_m t}. \quad (3.36)
\]

Since the density of states \( \rho_m \) is defined by the weight that each eigenstate is excited, therefore it can be given by

\[
\rho_m(n, n') = \langle n'|r_m\rangle \langle r_m|n \rangle = |n\rangle \langle r_m|n\rangle. \quad (3.37)
\]

If one defines the operator \( \mathbf{A} = |n\rangle \langle n| \), then \( \rho_m \) is basically the expectation value of an eigen-
state $|r_m\rangle$ under the operator $A$. Since here we focus on the case where the observation and excitation are in the same site, i.e. $n = n'$. Then we have $\rho_m = \langle r_m|n\rangle \langle n|r_m\rangle$, hence $\rho_m \geq 0$. Additionally, the sum of all density of states needs to be unity $\sum_{m=1}^{M} \rho_m = 1$.

The task here can be outlined as follows: Knowing a set of $\{\lambda_m\}$ and $\{\rho_m\}$ (with $\rho_m \geq 0$), to construct a three-diagonal Hermitian Hamiltonian $H^{(3)}$ for its excitation from $n$-th site and observation from $n$-site to exhibit these parameters. If the anchor site is chosen as the first site, then $n = 1$. Hence,

$$\rho_m = \langle r_m|1\rangle \langle 1|r_m\rangle.$$

(3.38)

Since $\rho_m \geq 0$, we have

$$\langle r_m|1\rangle = \langle 1|r_m\rangle = \sqrt{\rho_m}.$$

(3.39)

Recall the steps employed in Section 3.2, here the to-be-mapped Hamiltonian is actually a diagonal Hamiltonian $D$ consisting of the set of eigenvalues. Hence the mapping can be described by

$$D = VH^{(3)}V^\dagger,$$

(3.40)

which is, interestingly, analogues to the well-known eigenvalue decomposition of $H^{(3)}$:

$$H^{(3)} = RDR^\dagger,$$

(3.41)

with the columns of matrix $R$ being the eigenvectors $|r_m\rangle$ of $H^{(3)}$. Obviously, it suffices to fulfill Equation (3.40) with

$$V = R^\dagger.$$

(3.42)

In this case, the columns of $V$, termed $|v_m\rangle$, is directly related to $|r_m\rangle$ by

$$\langle m|v_n\rangle = \langle r_m|n\rangle.$$

(3.43)

From Equation (3.39), one can obtain all elements in $|v_1\rangle$:

$$\langle m|v_1\rangle = \sqrt{\rho_m}.$$

(3.44)

Now it is clear that with the $|v_1\rangle$ one can simply run the Lanczos algorithm given in Section 3.2 to obtain $H^{(3)}$ from $D$.

While the Lanczos algorithm was typically used for three-diagonalizing matrices with usually with more off-diagonal elements than three-diagonal matrices, yet it is in principle possible to use it to transform a diagonal matrix to three-three diagonal form. The synthesis of arbitrary local density of state is exactly such an example.

The freeform engineering of local density of states may serve as an entirely new approach in the design of lattice-site properties in synthetic lattices. It may also be used to derive the coupling and potential profiles of one-dimensional synthetic lattice from the spectral response of the first site.
3.5 Non-Hermitian generalization

All results above are based on Hermitian Hamiltonians. In experiments, non-Hermiticity serves as a convenient and powerful description of lattice systems, since most lattices, in particular those in real-time systems, are not conservative. The key difference if the Hamiltonian becomes non-Hermitian lies in (i) the eigenstates are no longer orthogonal, and (ii) the eigenvalues could be complex-valued (the second aspect is optional). The formalism hence is based on bi-orthogonality using both left and right eigenvectors, where there also exists a non-Hermitian Lanczos algorithm for its recursive calculation \cite{144}. Here we theoretically formulate a Lanczos-based approach for mapping an anchor-site excitation dynamics from a non-Hermitian Hamiltonian to practical one-dimensional lattices.

We can perform an eigenvalue decomposition of a general, non-Hermitian Hamiltonian $H$, which has both left- and right- eigenvectors

$$HR = RD,$$  \hspace{1cm} (3.45a)

$$L^\dagger H = DL^\dagger,$$  \hspace{1cm} (3.45b)

where the columns of $L$ (termed $|l_m\rangle$) and $R$ (termed $|r_m\rangle$) are the left- and right- eigenvectors, respectively. Similarly as in the Hermitian case discussed above, $D$ is a diagonal matrix, the diagonal elements $\lambda_m \in \mathbb{C}$ are the eigenvalues. In the special case that $H$ is Hermitian, $H = H^\dagger$, we have $L = R$. This gives rise to the analysis in Equations (3.1) and (3.2) that only utilizes $R$. By contrast, for non-Hermitian $H$, we have $L \neq R$ and hence both left- and right- eigenvectors must be taken into account. Note that $L$ and $R$ cannot be both arbitrary matrices; in fact, they have to fulfill the biorthogonal relation

$$\langle l_m | r_n \rangle = \langle l_m | r_m \rangle \delta_{m,n},$$  \hspace{1cm} (3.46)

where $\langle l_m | r_m \rangle$ may not necessarily be unity. Equivalently, $L^\dagger R$ is a diagonal matrix with $\langle l_m | r_m \rangle$ being its diagonal elements. Therefore, $H$ can be decomposed into a basis only taking into account its left- and right- eigenvectors with the same column index $m$,

$$H = \sum_{m=1}^{M} \lambda_m \frac{|r_m\rangle \langle l_m|}{\langle l_m | r_m \rangle},$$  \hspace{1cm} (3.47)

The evolution operator $T$ is no longer a unitary,

$$T = e^{-iHt} = \sum_{m=1}^{M} e^{-i\lambda_m t} \frac{|r_m\rangle \langle l_m|}{\langle l_m | r_m \rangle}.$$  \hspace{1cm} (3.48)

If one excites the $n$-th site and observe in the $n'$-th site, the dynamics can be described by the
wavefunction $\Psi_{n,n'}(t)$:

$$
\Psi_{n,n'}(t) = \langle n' | T | n \rangle = \sum_{m=1}^{M} \frac{\langle l_m | n' \rangle \langle n | r_m \rangle}{\langle l_m | r_m \rangle} e^{-i\lambda_m t} ,
$$

(3.49)

where the coefficient, or weight, of each $e^{-i\lambda t}$ is inherently temporal-domain Green’s function

$$
G_m(A) = \frac{\langle l_m | A | r_m \rangle}{\langle l_m | r_m \rangle} ,
$$

(3.50)

where $A = | n' \rangle \langle n |$. Note that $G_m(A)$ is actually in the format of the well-known two-state-vector formalism [145, 146], which is central to the definition of the quantum weak value [147, 148]. Quantities under such a formalism is known to be in general complex-valued enabled by the non-orthogonality between $| l_m \rangle$ and $| r_m \rangle$. In fact, $G_m(A)$ may not even converge approaching a non-Hermitian degeneracy where $| l_m \rangle$ and $| r_m \rangle$ coalesce as the denominator in Equation (3.50) approaches zero. This is analogous to the so-called quantum weak value amplification of quantum weak measurements with specially chosen projection bases [149]. Here interestingly the amplification effect is actually an inherent property to local Green’s function of non-Hermitian Hamiltonian.

**Brevity consideration and its impact**

Now we discuss the brevity consideration in the construction of three-diagonal isospectral Hamiltonians. The most widely realized and practical case of non-Hermiticity is associated with only gain or loss localized at each lattice site, i.e. the non-Hermiticity only lies in the imaginary parts of the diagonal elements of $H'$. Similarly as above, we target real-valued off-diagonal elements. This can be described by a symmetric nature of the Hamiltonian

$$
H = H^T ,
$$

(3.51)

where $^T$ denotes the transpose of a matrix. Now we check how such a special case affects on the eigenvalue decomposition of Hamiltonian $H$. We can perform eigenvalue decomposition to $H^T$:

$$
H^T = \sum_{m=1}^{M} \lambda_m | l_m^* \rangle \langle r_m^* | ,
$$

(3.52)

where $^*$ denotes complex conjugate. Comparing Equation (3.52) with Equation (3.47), we find

$$
| l_m^* \rangle = | r_m \rangle
$$

(3.53)

can serve as a sufficient condition to ensure Equation (3.51) satisfied. Hence we find the special type of non-Hermiticity can be achieved by making the left- and right- eigenvectors being complex conjugate with each other. This is the case we will discuss in the following.
Mapping the excitation dynamics for non-Hermitian Hamiltonian

Now we generalize the Lanczos mapping approach discussed in Section 3.2 to non-Hermitian Hamiltonians that we consider here. To establish an isospectrality between $H$ and $H^{(3)}$ in the non-Hermitian case, we need to search for matrices $V$ and $W$ such that

$$H = VH^{(3)}W^\dagger,$$  \hfill (3.54)

where $VW^\dagger = I$. We can use $|w_m\rangle$ and $|v_m\rangle$ to represent the columns of $W$ and $V$, respectively. With the three-diagonal nature of $H^{(3)}$, we can write the transformation into the form of

$$H = \sum_{m=1}^{M} \epsilon_m |v_m\rangle \langle w_m| + \sum_{m=1}^{M-1} (C_m |v_{m+1}\rangle \langle w_m| + C_m |v_m\rangle \langle w_{m+1}|),$$  \hfill (3.55)

in which $\epsilon_m \in \mathbb{C}$ represents diagonal elements and $C_m \in \mathbb{R}$ denotes the off-diagonal elements of $H^{(3)}$.

Now we establish the recursive steps to obtain $H^{(3)}$ from a given $H$. For the first step, $m = 1$, if we multiply $\langle w_1|$ from the left and $|v_1\rangle$ from right, for both sides of Equation (3.55), we have

$$\epsilon_1 = \langle w_1|H|v_1\rangle.$$  \hfill (3.56)

Similarly as in the Hermitian case, $|v_1\rangle = |n\rangle$ can be freely chosen to match the anchor site $n$. With the brevity consideration above, $|w_1\rangle = |v_1^*\rangle$ is then determined. If we only multiply $|v_1\rangle$ from right for both sides of Equation (3.55), we have

$$H|v_1\rangle = \epsilon_1 |v_1\rangle + C_1 |v_2\rangle,$$  \hfill (3.57)

and a similar procedure can be done by multiplying $\langle w_1|$ from left for both sides of Equation (3.55),

$$\langle w_1|H = \epsilon_1 \langle w_1| + C_1 \langle w_2|.$$  \hfill (3.58)

Then, with Equations (3.57) and (3.58) we have

$$|C_1|^2 = \left| \langle w_1|(H - \epsilon_1)^2 |v_1\rangle \right|.$$  \hfill (3.59)

As we assume $C_m \geq 0$, we can obtain $C_1$ by

$$C_1 = \left| \langle w_1|(H - \epsilon_1)^2 |v_1\rangle \right|^{\frac{1}{2}}.$$  \hfill (3.60)

Hence $|v_2\rangle$ and $|w_2\rangle$ can be obtained

$$|v_2\rangle = \frac{1}{C_1} (H - \epsilon_1) |v_1\rangle,$$

$$|w_2\rangle = \frac{1}{C_1} (H - \epsilon_1)^\dagger |w_1\rangle.$$  \hfill (3.61)
For \( m = 2, 3, \cdots \), from Equation (3.55) we have

\[
\langle w_m | H | v_m \rangle = \epsilon_m, \tag{3.62a}
\]

\[
H | v_m \rangle = \epsilon_m | v_m \rangle + C_m | v_{m+1} \rangle + C_{m-1} | v_{m-1} \rangle, \tag{3.62b}
\]

\[
\langle w_m | H = \epsilon_m \langle w_m | + C_m \langle w_{m+1} | + C_{m-1} \langle w_{m-1} |. \tag{3.62c}
\]

Hence, we can recursively continue the calculation by

\[
\epsilon_m = \langle w_m | H | v_m \rangle, \tag{3.63a}
\]

\[
C_m = \left[ \langle w_m | (H - \epsilon_m)^2 | v_m \rangle - C_{m-1}^2 \right]^{1/2}, \tag{3.63b}
\]

\[
| v_{m+1} \rangle = \frac{1}{C_m} \left[ (H - \epsilon_m) | v_m \rangle - C_{m-1} | v_{m-1} \rangle \right], \tag{3.63c}
\]

\[
| w_{m+1} \rangle = \frac{1}{C_m} \left[ (H - \epsilon_m)^\dagger | w_m \rangle - C_{m-1} | w_{m-1} \rangle \right]. \tag{3.63d}
\]

### 3.6 Summary

In summary, we propose and demonstrate a general approach for mapping the excitation dynamics of arbitrary high-dimensional lattices onto one-dimensional equivalent lattices with strictly nearest-neighbor interactions. By judiciously engineering the coupling and on-site energy profile, our approach goes beyond globally matching the eigenenergies of the multi-dimensional system, and faithfully reproduces the excitation and localization dynamics at the anchor site. This innovative concept not only allows us to observe the dynamical features of quantum walks in record-high lattice dimensions but also opens up new opportunities to harness otherwise inaccessible high-dimensional features for practical applications. As an important usage case, we demonstrate a sharp localization transition at surface defects in four- or higher-dimensional lattices, facilitated by a new type of bound state at the edge of the continuum in a mapped 1D lattice. We have provided theoretical analysis on the freeform engineering of local density of states and a non-Hermitian generalization of the mapping.

Importantly, despite we focus on the replication of the local density of states in one anchor site or other choices of one certain excitation, the strength of this approach can be further enhanced with higher-order couplings beyond nearest neighbors. For example, with a tailorable second-order coupling, one should be able to freely engineer the dynamics in two sites. While networks used in many cutting-edge information tasks are extremely complex with many nodes, usually the input and output can be represented within a certain number of ports. In this vein, our approach can also be a powerful tool in designing complex networks with greatly simplified arrangement while maintaining the input-output functionality.

Since the method presented here does not require dynamic modulations, the static 1D equivalent lattices are compatible with a broad range of existing technological platforms, including, but not limited to, the planar laser-written photonic circuits used in the proof-of-
concept experiments. The platform-independent nature of our approach will boost the development of tailored one-dimensional lattices in a variety of systems and enable previously inaccessible exotic high-dimensional effects to find their way to applications in many different fields such as topological photonics, quantum simulation, and optical information processing. In this vein, 1D mapped equivalent lattices hold great promise for the exploration of high-dimensional physics under linear, nonlinear and even non-classical conditions. Long-standing questions such as the dimensional scaling behavior of Anderson-localization, the stability of solitons, and entanglement dynamics can be now explored in tabletop experiments.
4 Imaging-based multiphoton state reconstruction with metasurfaces

In Chapters 2 we discussed new concepts in forming multidimensional lattices, and in Chapter 3 we introduced a new approach to access multidimensional characteristics. As mentioned in Chapter 1, multidimensional photonic state is another important ingredient of multidimensional photonics, where the manipulation and measurement of such states are especially challenging in the quantum regime. From Chapter 4, we will move to the manipulation of multiphoton-spanned multidimensional states that aims at state measurement.

Synthetic lattices in photonics provide new opportunities to drastically enhance imaging capabilities. In this chapter*, we make judicious use of flat optics incorporating synthetic lattices to enable the interference and measurement of multiphoton quantum states by means of imaging.

4.1 Introduction

Multidimensional properties can be manifested in the multiparticle nature of light, constituting many exotic behaviors known under the regime of quantum optics [28–30, 152]. As discussed in Chapter 1, with quantum entanglement, a variety of applications from secure commutation to imaging can be drastically enhanced with non-classical light. Importantly, the Hilbert space scales exponentially with the number of photons due to the inherent multidimensional nature, constituting a key factor of the strength of many quantum information processing techniques, such as quantum computing. Therefore multiphoton quantum state constitutes another important aspect in multidimensional photonics in addition to the multidimensional lattices as we discussed in Chapters 2 and 3.

While many research efforts have been devoted to the manipulation of multiphoton states, a particularly useful yet challenging task lies in transforming the state in a way such that it facilitates robust measurement of the quantum state. Such manipulation is necessary due to the fact that photon detectors can only measure counts or correlation events, which in general does not directly access the quantities inside the representation of the state. Conventionally, this was done by the so-called quantum state tomography [153] associated with

*Results presented in this chapter are partially included in the published References [150, 151], where I am the lead author and my contribution is in all aspects including experiment and theory.
projective measurements using reconfigurable bulk optical elements. The general idea is to perform a tomographically complete set of measurements by varying the projection. Then, the quantum state can be mathematically reconstructed with the knowledge of the chosen set of projections and measurement readouts. While the most conventional way of its implementation is based on bulk optical elements and suffers from substantial error, the development in integrated photonic circuits have enabled its realization in waveguides with tuning elements \[57\]. However, since the latter approach still requires dynamic reconfiguration, the measurement remains time-consuming and sensitive to environmental noises.

Recent efforts have brought reconstruction-based quantum measurements to a static device that maps out the state with a judiciously designed transformation \[58, 104, 154, 155\]. In such an approach, there are usually a larger number of ports at the output facing the detectors than the input that matches the space of the measured state. Inherently, this means the measured state is combined with extra inputs from the vacuum, which unavoidably involves nonclassical multiphoton interference. Conventionally, this is performed through a sequence of beam-splitting optical elements, each realizing quantum interference \[154, 156, 157\]. Such sequential implementations present a fundamental limit for miniaturization while being inherently sensitive to fluctuations or misalignment between different elements, especially for higher photon-number states. Recent advances in nanotechnology have enabled the integration of beam-splitters and couplers on tailored plasmonic structures \[158, 159\], yet material losses and complex photon-plasmon coupling interfaces restrict the platform scalability.

Imaging is among the most basic functionalities of optical systems. The manipulation of quantum light with imaging systems can potentially be an efficient and intuitive way of measuring quantum states of light encoded in large dimensionality. While this is extremely challenging with the conventional optical elements, the field of nanostructured metasurfaces offers the possibility of replacing traditionally bulky imaging systems with flat optics devices \[160\] achieving high transmission based on all-dielectric platforms \[66, 161–165\]. The metasurfaces provide the freedom to tailor the light interference by coherently selecting and mixing different components on a sub-wavelength scale, enabling polarization-spatial conversion \[61, 64, 66, 165–168\] and spin-orbital transformation \[169\]. Such capabilities motivated multiple applications for the regime of classical light, yet the metasurfaces have a potential to emerge as essential components for quantum photonics \[170, 171\].

In this chapter, we make use of synthetic lattices interleaved on such flat optics to realize imaging-based multiphoton state manipulation and measurement. The concept is sketched in Figure 4.1. Multiphoton states are combined with vacuum to facilitate a judiciously designed transformation, where the correlation measurements from the output ports can uniquely and robustly determine the multiphoton state at the input in a reconstruction procedure. Our specially designed synthetic lattices on an imaging flat optics facilitate the realization of the transformation. This not only promises great miniaturization and stability but also paves the way towards flat optical components for quantum photonics.

More specifically, we realize several multiphoton interferences in a single flat all-dielectric
metasurface. The parallel quantum state transformations are encoded in multiple interleaved 
metagratings, taking advantage of the transverse spatial coherence of the photon wavefunc-
tions extending across the beam cross-section. In the classical context, the interleaving ap-
proach was effectively used for polarization-sensitive beam splitting [61, 64, 167, 168], yet it 
requires nontrivial development for the application to multiphoton states.

Outline of this chapter

The subsequent sections of this chapter are structured as follows: Section 4.2 sets the stage 
for this chapter by discussing possible cases associated with the quantum state and detect-
sion scheme, giving a quantitative measure for the condition to perform robust multiphoton 
state reconstruction with the transformation-based general approach. Then, in Section 4.3, 
the implementation of such transformation with metasurfaces will be discussed, showing the 
principle of the designed imaging flat optics. Then, Section 4.4 will include experimental re-
sults of using metasurfaces to achieve quantum interference and state reconstruction for up 
to two-photon states. Finally, in Section 4.5 we will summarize this chapter.

4.2 Optimal reconstruction of multiphoton states with correlation 
measurements

To start with, in this section we theoretically summarize some basic principles of using cor-
relation measurements to reconstruct multiphoton states. We discuss the reduced density
matrix, which acts as the object to be measured, under different situations of detection. We also provide a quantitative measure with the condition number.

\[
\begin{align*}
(a) & \quad \text{Mapping in the measurement (M=6)} \\
\text{1-photon} & \quad \text{HH} \quad \text{HV} \\
\text{2-photon} & \quad \text{VH} \quad \text{VV} \\
\text{N-photon} & \quad \vdots \\
(b) & \quad \text{Photon counts (1D space)} \\
& \quad \text{2-fold correlation (2D space)} \\
& \quad \text{N-fold correlation (N-D space)}
\end{align*}
\]

**Figure 4.2:** Conceptual sketch of using N-fold correlations to measure N-photon states. (a) Example multiphoton states in polarization for \( N = 1, 2 \). (b) Corresponding one- and two-photon correlation measurements, where a circle denote a possible channel to yield measurement events.

In Figure 4.2 we illustrate the general idea of multiphoton measurements using correlations. As discussed in Chapter 1, an \( N \)-photon state inherently inhabits in a \( N \)-dimensional state space, such as the example of \( N \)-photon polarization state as sketched in Figure 4.2(a). The transformation of the device actually provides a mapping from the space where the state inhabits [Figure 4.2(a)] to a measurement space. The latter is mostly formed by the so-called correlation measurements, which is sketched in Figure 4.2(b) for the example case with \( M = 6 \) ports and hence 6 detectors. For single-photon states, \( N = 1 \), there can be simply photon counting measurements, where the six detection channels can be treated a one-dimensional space [Figure 4.2(b)]. Then, for two-photon states, \( N = 2 \), two-fold correlation becomes possible in which one counts the events with two photons arriving simultaneously. Thereby the space of the detection is effectively two-dimensional tracing out all two-fold combinations of detection channels, as shown in Figure 4.2(b) by the 2D array of circles. In general, for \( N \)-photon state measurement, the detection space of correlation is \( N \)-dimensional.

**Measured object: Reduced \( N \)-photon density matrices**

Now we define the target of the measurement. We consider quantum states with a fixed photon number \( N \), which is a widely-used approach in photon detection [58, 104, 153, 155]. As outlined in Chapter 1, such a state can be well described by a \( N \)-photon density matrix in relevant certain representations, such as spatial modes, polarization and frequency modes. A reduced density matrix in one representation traces out all other degrees of freedom \( \rho_N = \sum \alpha_i |\psi_i\rangle \langle \psi_i| \), leaving only the space where the representation inhabits.
It is important to understand how many real-valued free parameters there are in the measured $N$-photon density matrix. In the most general case, a $d$-mode system, for an $N$-photon state $\hat{\rho}_N$ is a $d^N \times d^N$ Hermitian matrix. Such a $\hat{\rho}_N$ contains $d^{2N}$ real-valued unknowns (if $\hat{\rho}_N$ is normalized, then $d^{2N} - 1$ unknowns). Importantly, detection scheme influences the independence of parameters in the $N$-photon density matrices. There exist the so-called distinguishable and indistinguishable detection schemes, depending on whether or not the detectors can tell which photon is which among the $N$ photons from some extra degree of freedom. For example, for the two-photon polarization state sketched in Figure 4.2(a) in a two-dimensional space with the geometry of a square, this consideration corresponds to if the detection can distinguish HV and VH. If not, then the $4 \times 4$ density matrix will have a symmetry reducing the number of free parameter [58].

In Table 4.1, using the analysis presented in Reference [58] on reduced N-photon density matrices including relevant calculations, we summarize the number of independent unknowns with $d$ modes (e.g. $d = 2$ for polarization states, which we also show as a specific example in the table) under both distinguishable and indistinguishable detection cases.

| Table 4.1: Free parameters in $N$-photon density matrix |
|-----------------|-----------------|-----------------|
|                | Distinguishable detection | Indistinguishable detection |
| $\hat{\rho}_N(d)$ | $d^{2N}$ | $\frac{(N+d^2-1)!}{N!(d-1)!}$ |
| $\hat{\rho}_N(d = 2)$ | $2^{2N}$ | $\frac{(N+3)!}{N!3!}$ |

Measurement readouts: $N$-fold correlation

Then we discuss how many correlation measurement readouts one can get in a system with $M$ output ports. Detection scheme also affects this. To illustrate this, we show an example set of two-fold correlation measurements in Figure 4.3(a) in four frames of measurement (i.e. measured four times). The green and blue stars denote the pair of photons. If the detection is designed in a way that these two photons are distinguishable, for example the photon denoted by the blue and green stars differ in wavelength and the detectors can tell which is which. Then the two-fold correlation measurements by summing up all the four frames are as shown in Figure 4.3(b), where each orange triangle denotes an event. By contrast, if the detectors cannot tell which photon is which, then the correlation measurement space reduces to the case shown in Figure 4.3(c), where the 2D square matrix is reduced to a triangular matrix. The correlation events in frame 1 and frame 3 in Figure 4.3(a), for instance, are now treated as the same event in Figure 4.3(c).

Furthermore, there is another factor influencing this. There are detectors that can only recognize on-off events, known as click or bucket detectors (for details on the terminology of detectors, see Reference [172]), without a knowledge of how many photons are detected in one detection event. In contrast, another type of detector is the so-called photon-number-resolvable detector that can detect the number of photons. In Figures 4.3(b,c), we use orange circles to denote those correlations that have local multiphoton events, which are only de-
Figure 4.3: Example sketch of two-fold correlation measurements with different detection schemes. (a) An example measurement with four frames. (b) The two-fold correlation statistics after summing up the detection in (a) for distinguishable detection scheme. (c) Two-fold correlation statistics of (a) for indistinguishable detection scheme.

tectable by photon-number-resolvable detectors.

In Table 4.2, we show the number of distinct \( N \)-fold correlations one can get out of \( M \) multiplexed number-resolvable or click detectors for both distinguishable and indistinguishable detection schemes. To enable reconstruction of the quantum state, the number of correlation measurements has to be no less than the number of unknowns in the reduced density matrix.

<table>
<thead>
<tr>
<th></th>
<th>Distinguishable detection</th>
<th>Indistinguishable detection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photon-number-resolvable detectors</td>
<td>( M^N )</td>
<td>( \frac{M+N-1}{N(M+1)} )</td>
</tr>
<tr>
<td>Click detectors</td>
<td>( \frac{M!}{N!} )</td>
<td>( \frac{M!}{M-N} )</td>
</tr>
</tbody>
</table>

Quantifying the robustness of reconstruction by the condition number

In the above discussion of this section, we have made clear the object to be measured, and the measurement readouts we can have in correlation measurements. Now we discuss the optimization of the mapping from the space of the measured state to the measurement space. In general, the transformation aims at an eventual reconstruction, which is inherently an inverse problem. This determines that some transformations are well-conditioned while some are not. It is thus vital to have a measure on such a condition.
Firstly we discuss the condition number, which is the measure we use throughout this thesis on such reconstruction problems. In general, a measurement process can be described by

\[ \vec{O} = J \vec{S}, \quad (4.1) \]

where a to-be-measured vector \( \vec{S} \) that contains all unknowns of a state is transformed to another vector \( \vec{O} \) that contains the observables by an instrument matrix \( J \). The instrument matrix is basically a description of the measurement device. Specifically for the measurement of the density matrix of a quantum system, \( \vec{S} \) is a vector that contains all independent elements in the measured density matrix. Then \( J \) is inherently the transformation implemented by the measurement setup. Hence \( \vec{O} \) is a vector containing the counts or correlations.

From the measurement readout \( \vec{O} \), one may use the inversion (or pseudo-inversion) \( J^{-1} \) to retrieve the reconstructed \( \vec{S} \) (denoted as \( \vec{S}' \)), i.e.

\[ \vec{S}' = J^{-1} \vec{O}, \quad (4.2) \]

where it is vital to know how well \( J \) can be reversed. This essentially depends on to what extent an error in \( \vec{O} \) gets amplified in such a construction process as described by Equation (4.2).

Now consider a vector denoting such errors \( \Delta \vec{O} \) from the measurement, which will then propagate to another vector \( \Delta \vec{S} \) impacting on the reconstructed \( \vec{S}' \). These error vectors are also linked by the instrument matrix \( J \):

\[ \Delta \vec{S} = J^{-1} \Delta \vec{O}. \quad (4.3) \]

If we calculate the norm, denoted by \( || \cdot || \), of all quantities in Equation (4.3), according to the Cauchy's Inequality, \( ||\Delta \vec{S}|| \) has an upper bound

\[ ||\Delta \vec{S}|| \leq ||J^{-1}|| ||\Delta \vec{O}||. \quad (4.4) \]

For the relative error \( ||\Delta \vec{S}||/||\vec{S}'|| \) with respect to the reconstructed vector, using Equation (4.2) and again the Cauchy's Inequality we can find its upper bound

\[ \frac{||\Delta \vec{S}||}{||\vec{S}'||} \leq ||J|| ||J^{-1}|| \frac{||\Delta \vec{O}||}{||\vec{O}||}, \quad (4.5) \]

where \( ||J|| \cdot ||J^{-1}|| \) can act as a upper bound of the amplification of the relative error in such a reconstruction process. In other words, it quantifies the robustness of the system against error propagation in the reconstruction. Therefore such a quantity is defined as the condition number \( \kappa \)

\[ \kappa(J) = ||J|| \cdot ||J^{-1}||. \quad (4.6) \]

An infinite \( \kappa \) means that, for a given measurement \( \vec{O} \), one cannot derive a unique \( \vec{S}' \). Hence we can tell that the reconstruction is ill-conditioned. By contrast, the closer a finite \( \kappa \) gets to its minimum possible value, the more robust this system is against error amplification in the reconstruction.
In this thesis, we employ Euclidean norm. While other norms have been used in the literature, such as Hilbert-Schmidt norm in Reference [173] formulating optimal frames for polarization state reconstruction, for all definitions the minimum condition number universally corresponds to the optimal regime of reconstruction with minimum errors.

**Optimal frame in transformation-based state reconstruction**

Now we briefly discuss how the transformation can be chosen to be optimally conditioned, i.e. yielding the minimum possible condition number. This is described in the so-called quantum frame theory [173]. For the transformation, for single-photon states, we could treat the measurement at each output port as one projective measurement with a projection basis. Then, the choice of the set of projection bases in an equivalent description of tailoring the transformation.

The general idea is to ensure the projection bases spread evenly in the Bloch space. Choices consisting of pairwise mutually orthogonal polarization projections can be obtained with platonic polyhedra in spherical \( t \)-design, with \( M = 6, 8, 12, 20, \cdots \) [173]. For example, in Figures 4.4(a–d) we show such platonic solids and the projective bases (arrows) in a sphere for \( M = 6, 8, 12, 20 \), respectively. Note that the set of bases have a rotational degree of freedom without changing the optimal-frame nature. A good condition number for single-photon states in principle also means good condition for its multiphoton counterparts, since the latter transformation is essentially obtained by tensor products of the single-photon case. Yet this may involve certain considerations on detector types, the impact of which will be discussed in Chapter 5.

![Figure 4.4: (a–d) Example sets of optimal-frame projections for \( M = 6, 8, 12, 20 \), respectively. Arrow denotes a projection basis.](image)

**4.3 Metasurface for quantum state measurement**

Now we present the design of our flat-optical element to perform the transformation that can enable multiphoton quantum-polarization state reconstruction using correlation measurements.
Firstly we outline our concept, which is sketched in Figure 4.5. We develop a metasurface incorporating a set of $M/2$ interleaved metagratings, each composed of nano-pillars with specially varying dimensions and orientation according to the principle of geometric phase [64] to split specific elliptical polarization states [165], which would not be possible with conventional gratings. This performs quantum projections in a multiphoton Hilbert space to $M$ imaging spots, each corresponding to a different elliptical polarization state [Figure 4.5(a)], which is essential to minimize the error amplification in quantum state reconstruction [173]. Then, by directly measuring all possible $N$-photon correlations from the $M$ output beams, it becomes possible to reconstruct the initial $N$-photon density matrix providing full information on the multiphoton quantum entanglement. For example, in Figure 4.5(b) we show a sketch of three gratings (top) which realize an optimal set of projective bases shown as vectors on the Poincaré sphere (middle) for $M = 6$.

The photon correlations between $M$ output ports can be obtained with simple polarization-insensitive click single-photon detectors. The metasurface can be potentially combined with single-photon sensitive electron-multiplying CCD (EMCCD) cameras [174, 175] to determine the spatial correlations by processing multiple time-frame images of quantum states. The $N$-fold correlation data, stored in an array with $N$ dimensions, are obtained by averaging the coincidence events over multiple time frames. For example, in Figure 4.5(c) we sketch a case with $N = 2$ and $M = 6$. In each frame, two photons arrive at different combinations of spots. After summing up the coincidence events over multiple time frames, we obtain a correlation in two-dimensional space. Following the general measurement theory of Reference [58], we establish that for an indistinguishable detection of $N$-photon polarization states (i.e. the detectors cannot distinguish which is which of the $N$ photons), the required number of output ports to perform the reconstruction scales linearly with the photon number as $M \geq N + 3$, see Figure 4.5(b, bottom). For instance, with $M = 6$ up to $N = 3$ photon states can be measured.

**Working principle and design of individual metagratings**

Now we provide details on the working principle of each of the metagratings. As mentioned above, the metasurface is assembled from $M/2$ metagratings, with each one diffracting a pair of specially chosen polarization states to two directions symmetric with respect to the transmissive axis. The novel aspect of our geometric-phase metagrating lies in the feature that we extend the capability to project our states to any pairs of elliptical polarization states. This is a crucial property for quantum-polarization measurements – unlike classical polarization measurement, for multiphoton quantum states more diffracted polarization components are needed for on-off click detectors. This has been considered difficult to realize with non-chiral birefringent structures, until recently a generalized geometric phase was implemented in Reference [165] for polarization sensitive hologram. Here we provide the theory for designing metagratings that can achieve decomposition to elliptical polarization components.

We assume that each element in the metagrating (i.e. each nano-pillar) behaves as a bire-
Figure 4.5: Concept of quantum state imaging via nanostructured flat optics. (a) Sketch of using a metasurface to image an input $N$-photon polarization state into an $M$-spot image. Top-right inset shows an SEM image of the fabricated all-dielectric metasurface. (b) An example of correlation measurement with $N=2$ and $M=6$, with several time-frame measurements combined into a two-dimensional correlation image.
fringent crystal with its fast and slow axes in the plane of the metasurface, depending on the orientation angle $\theta$ of its fast axis. We denote the phase picked up along the fast and slow axes as $\phi_1$ and $\phi_2$, respectively ($\phi_1 < \phi_2$). A single-photon pure state can be described by the state vector $\psi_{\text{in}} = [c_H, c_V]^T$, where $c_H$ and $c_V$ are the wavefunctions of photons polarized in horizontal (x) and vertical (y) directions, respectively. Here $^T$ represents the matrix transpose.

The transformation that each grating element does can be expressed by a $2 \times 2$ matrix $U$

$$U = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \quad (4.7)$$

The eigenstates of the static Hamiltonian that generates $U$ are always pairs of orthogonally-polarized linear polarization states. The angle $\theta$ determines the orientation of the two eigenstates. Such a fact means that $U$ has two basic properties: (1) it transforms an orthogonal pair of polarization states to another orthogonal pair, which is governed by its Hermiticity; (2) such a transformation is not chiral, as the eigenstates are linear polarizations. With such properties, it can be easily found that for any pair of elliptical polarizations, changing their handedness can be realized with any $\theta$, as long as one chooses proper difference between $\phi_1$ and $\phi_2$. The phase we utilize is directly related to the Pancharatnam-Berry phase [65]. The working principle is conceptually shown in Figure 4.6(a). The nano-pillars in a grating exhibit both spatially varying orientation $\theta$ and cross-section geometrical parameters that can change $\phi_1$ and $\phi_2$. They are tailored in a way that each nano-pillar changes the handedness of a pair of elliptical polarization states $|\psi\rangle$ and $|\tilde{\psi}\rangle$, changing them to $|\psi'\rangle$ and $|\tilde{\psi}'\rangle$, respectively. In such a transformation, each state acquires a linear phase gradient in space with inverse signs for $|\psi\rangle$ and $|\tilde{\psi}\rangle$.

For a more detailed description, we firstly write down $|\psi\rangle$ and $|\tilde{\psi}\rangle$:

$$|\psi\rangle = \begin{bmatrix} \cos \alpha, \exp(i\beta) \sin \alpha \end{bmatrix}^T,$$

$$|\tilde{\psi}\rangle = \begin{bmatrix} -\sin \alpha, \exp(i\beta) \cos \alpha \end{bmatrix}^T. \quad (4.8)$$

Note that $\langle \tilde{\psi}|\psi\rangle = 0$, showing the orthogonality of the two polarization states. As described above, we target changing the handedness of $|\psi\rangle$ and $|\tilde{\psi}\rangle$. Hence we are looking for the different $U(\theta)$ that can achieve such a transformation:

$$\exp(i\gamma)|\psi'\rangle = U(\theta)|\psi\rangle, \quad (4.9)$$

where $|\psi'\rangle = \begin{bmatrix} \cos \alpha, \exp(-i\beta) \sin \alpha \end{bmatrix}^T$. This can be seen from Figure 4.6(b) where $|\psi\rangle$ and $|\psi'\rangle$ are represented as two arrows on the Poincaré sphere, and different angles $\theta$ for different nano-pillars lead to different paths in converting the state $|\psi\rangle$ to $|\psi'\rangle$ with a reversed handedness. Similarly, after reversing the handedness of the orthogonal counterpart $|\tilde{\psi}\rangle$, it becomes $|\tilde{\psi}'\rangle = \begin{bmatrix} -\sin \alpha, \exp(-i\beta) \cos \alpha \end{bmatrix}^T$. So firstly we look for the solution of Equation (4.9). The general condition is

$$|\langle \psi'|U(\theta)|\psi\rangle| = 1. \quad (4.10)$$
Figure 4.6: Working principle of a metagrating that spatially diffracts a pair of elliptical polarization components to two directions, where a metagrating with a super-cell of 8 nano-pillars is shown: (a) Conceptual 3D structure of the metagrating with periodic boundaries extending to the left and right sides, with the phases picked up by $|\psi\rangle$ and $|\tilde{\psi}\rangle$ shown as dots for each nano-pillar; (b) The paths that $|\psi\rangle$ goes on the Poincaré sphere, with each nano-pillar changing the handedness of $|\psi\rangle$ by going along paths at different angles; (c) Similar paths for $|\tilde{\psi}\rangle$. 
We numerically solve Equation (4.9) at different angles $\theta$ to obtain $\phi_1(\theta)$ and $\phi_2(\theta)$. Note that the solution of $\phi_1(\theta)$ and $\phi_2(\theta)$ is not unique, which determines the weights of geometric and dynamic phases. Here we conveniently take $\phi_1 = -\phi_2$ and hence simplified the calculation. Then, the phase picked up when converting $|\psi\rangle$ to $|\psi'\rangle$ can be easily calculated by

$$
\gamma(\theta) = \arg \langle \psi' | U(\theta) | \psi \rangle. \quad (4.11)
$$

When designing the metagrating, we introduce a spatially varying $\theta(x_n)$ for $x_n = (km + n)d$ where $k$ is an integer, $m$ is the number of nano-pillars (essentially the periodicity) and $d$ is the lattice constant for the nano-pillars. We design the metagrating such that

$$
\gamma(n) = -\frac{2\pi}{m} n + c_1, \quad (4.12)
$$

where $C_1$ is a constant. It can be demonstrated that for the orthogonal counterpart $|\tilde{\psi}\rangle$, when its handedness is reversed and as it becomes $|\tilde{\psi}'\rangle$ [see Figure 4.6(c)], the phase $\gamma'$ it picks up is

$$
\gamma'(n) = \frac{2\pi}{m} n + c_1, \quad (4.13)
$$

which has an inverse slope as $|\psi\rangle$. Therefore $|\psi\rangle$ and $|\tilde{\psi}\rangle$ get spitted spatially with inverse transverse state vectors. If the sampling rate is high enough (i.e. high enough number of nano-pillars in one supercell of the metagrating) and the phase slopes as shown in Figure 4.6(a) are really linear, the diffraction efficiency can be close to unity. Note that the typically used geometric-phase grating that splits two circular polarizations is a special case of the scheme given here.

**Interleaving the metagratings onto a monolithic metasurface**

The metasurface consists of multiple metagratings, interleaved to generate the functionality of spreading out $M$ output polarization components in parallel. Here we briefly show how we interleave these metagratings and estimate the maximum number of different gratings one can incorporate in a metasurface.

Along the grating direction $x$ (see Figure 4.7), the number of output ports which can be formed without overlapping can be estimated as follows. First, a grating period should consist of at least $Q_g$ nano-pillars, to perform reliable splitting of elliptical polarizations. Such a grating imposes the transverse wavevector component of $k_{g,max} = 2\pi/(Q_g \Delta r)$, where $\Delta r$ is the spacing between neighboring nano-pillars (or, lattice constant). Second, the width of diffraction peaks from the gratings is defined by the metasurface size as $\Delta k = 2\pi/L_x$. Then, the maximum number of gratings along the $x$ direction is $M/2 < k_{g,max} / \Delta k = L_x / (Q_g \Delta r)$. In our metasurface, $L_x = 2\text{mm}$, the minimum $Q_g = 8$ (see Figure 4.7), $\Delta r = 800\text{nm}$, and the maximum number of gratings is $M/2 < 312$.

In the vertical ($y$) direction as shown in Figure 4.7, it is preferential to interleave the gratings to (i) lower down the diffracted transverse wave vector along $y$ to enhance the diffraction
efficiency and (ii) minimize the dependence of the transfer matrix on the spatial beam profile. For (i), we repeat the same grating $Q_{i1}$ times before putting a different one, as illustrated in Figure 4.7. For (ii), we ensure that the size along $y$ is large enough to repeat the whole group of $Q_{i1} \times M/2$ gratings several times ($Q_{i2}$), as shown in Figure 4.7. Then, the maximum number of gratings along the $y$ direction is $M/2 < L_y/(Q_{i1} Q_{i2} \Delta r)$. Since $Q_g \ll Q_{i1} Q_{i2}$ for a square metasurface with $L_x \approx L_y$, the main limitation comes from the grating interleaving in the vertical direction. In our metasurface design, $L_y = 2$ mm, $Q_{i1} = 100$, $Q_{i2} = 6$, $\Delta r = 800$ nm, such that $M/2 \leq 4$. By increasing the interleaving density to still practically suitable regime with $Q_{i1} = 50$, we could increase the maximum number of gratings interleaved in the $y$-direction to $M/2 \leq 8$, enabling the measurement of up to $N = 13$ photon-number states. By also increasing the metasurface dimension to the experimentally feasible size of $L_y = 5$ mm, up to $M/2 \leq 20$ gratings can be interleaved. There appears an interesting open research question on optimizing the metasurface design for encoding most efficiently a large number of gratings [176] for multiple quantum measurements.

Overall, the number of interleaved gratings $M/2$ scales linearly with the metasurface dimensions, which in combination with linear scaling vs. the number of photons makes the platform suitable for characterization of multiphoton states.

4.4 Multiphoton interference and state reconstruction with metasurfaces

We fabricate silicon-on-glass metasurfaces with $M = 6$ and $M = 8$ using standard semiconductor fabrication technology (see Appendix Section A.2.2 for details). The experimentally determined polarization projective bases obtained through classical characterization are plotted on the Poincaré sphere in Figure 4.9(a) for a metasurface with $M = 6$ that is used later for quantum experiments. The transfer matrix measurements confirm that the polarization
projective bases are close to the optimal frame. The condition number, a measure of error amplification in the reconstruction is 2.08, close to the fundamental theoretical minimum of $\sqrt{3} \approx 1.73$. The reconstruction is immune to fabrication imperfections, as their effect is fully taken into consideration by performing an experimental metasurface characterization with classical light after the fabrication (see Appendix Section A.2.2).

Reconstruction of heralded single-photon states

![Diagram](image)

Figure 4.8: (a) Setup for heralded single-photon measurements, where APD stands for avalanche photodiode that is used as our single-photon detector. The two APDs are connected to time-to-digital conversion electronics to record the correlation of events from the two detectors (not shown in the figure). (b) A representative readout from the single-photon measurement, where the error bars are based on the shot noise in photon detection. (c) the reconstructed single-photon density matrix $\hat{\rho}$ using the measurement data in (b).

First, we show that our metasurface enables accurate reconstruction of the quantum-polarization state of single photons. A heralded photon source [177] is used at a wavelength of 1570.6 nm based on spontaneous parametric down conversion (SPDC) in a nonlinear waveguide. The schematic of the setup is shown in Figure 4.8(a). After the pump-filtered horizontally-polarized (H) and vertically-polarized (V) photons are separated by a polarizing beam splitter (PBS), the H photons are sent to a single-photon detector (APD 1) as the master arm in the photon heralding. The V photons go through a quarter wave-plate (QWP) and then are sent to the metasurface, and photons at the six diffracted spots are collected by the fiber couplers. Each fiber can be connected to another single-photon detector (APD 2), which acts as the slave detector. To efficiently collect the photons in the quantum measurements, we
use cylindrical lenses in between the metasurface and the fiber couplers to focus the vertically diffracted photons (not shown in the figure). Since the SPDC process in our experiment has very low probably to generate photon states with higher than 2 photons, if both APD 1 and APD 2 click we can be sure that this event on APD 2 is a single-photon event, which is the so-called heralded single-photon. In Figure 4.8(b), we show a representative readout of such correlation counts between APD 1 and APD 2, where APD 2 is switched between different output ports (1–6). With such counts, we are able to reconstruct both the real and imaginary parts of the single-photon density matrix $\hat{\rho}$, as shown in Figure 4.8(c). The reconstructed state is consistent with the theoretical predictions.

![Figure 4.9: Experimental measurement of heralded single-photon states with the metasurface. (a) Classically characterized projective bases of the metasurface for ports numbered 1 to 6. (b) Accumulated single-photon counts in each of $M=6$ output ports vs. the angle of a quarter-wave plate realizing a photon state transformation before the metasurface. Experimental data are shown with dots, with error bars indicating shot noise. Solid lines represent theoretical predictions based on classically measured metasurface transfer matrix. (c) Comparison between the prepared (solid line) and reconstructed (dots) states based on the measurements presented in (b), plotted on a Poincaré sphere.](image)

We further prepare different polarization states by varying the angle of the QWP. By measuring the correlations with the master detector, we reconstruct the quantum-polarization state from the photon counts at the six ports. The results are shown in Figure 4.9(b), where the curves are theoretical predictions and dots are experimental measurements. We observe that the measurement errors are dominated by the single-photon detection shot noise, which
is proportional to the square root of the photon counts, as indicated by the error bars. We use the measured photon counts to reconstruct the input single-photon states by performing a maximum-likelihood estimation \[153\] and plot them on a Poincaré sphere in Figure 4.9(c). The reconstructed states present a high average fidelity of 99.35% with respect to the prepared states.

Two-photon interference on the metasurface

Next, we realize two-photon interference, the setup of which is conceptually sketched in Figure 4.10(a). A detailed setup is shown in Appendix Section A.2.2. The SPDC source generates a photon pair with horizontal (H) and vertical (V) polarizations, with their path length difference controllable by a delay-line.

We measure the effect of delay on the two-photon interference, analogous to the Hong-Ou-Mandel (HOM) experiment \[178\]. In such a nontrivially generalized two-photon interference, we expect a dip or peak depending on the $2 \times 2$ transfer matrix $T_{ab} \propto [u_a, u_b]^\dagger$ from the two-dimensional polarization state vector to a chosen pair of ports, where $\dagger$ denotes transpose conjugate, and $u_a$, $u_b$ are the projective bases of ports $a$ and $b$, respectively. We note that $T_{ab}$ corresponds to an effective Hermitian Hamiltonian resulting in a conventional HOM dip only if $u_a$ and $u_b$ are orthogonal, while otherwise a HOM peak can appear analogous to a lossy beam-splitter \[159\]. Here we set the angle of the QWP at $\theta = 0^\circ$, which means that the photon pairs are in a state $\hat{\rho}(\theta = 0^\circ)$, where one photon is H- and another is V-polarized. As reflected in the Poincaré plot of Figure 4.10(b, right), where the red arrows denote projective bases of the two ports ($u_1$, $u_5$) and blue arrows represent the polarization of the photon pairs – one photon in H- and the other in V-polarization, we see that the state vector $u_1$ points to the opposite direction of $u_5$. We find that in this case photons with cross-polarized entanglement in H-V basis will give rise to a dip in the interference pattern with the variation of path length difference, see Figure 4.10(b, left). Such a behavior is directly caused by the coalescence nature of bosons.

The situation is quite different if we measure such an interference between ports $a = 1$ and $b = 5$, since $u_1$ and $u_5$ are far from being orthogonal. This can be seen from the red arrows in the Poincaré sphere of Figure 4.10(c, right), where the angle between the two vectors representing $u_1$ and $u_5$ is much smaller than $\pi$. For entangled photons with H and V polarization in a pair, interference under the transfer matrix $T_{15}$ leads to a peak instead of a dip when varying the path difference in the delay-line. Indeed, in Figure 4.10(c, left) we observe a peak, which is related to the anti-coalescence of bosons in transformations induced by non-Hermitian Hamiltonians, a nontrivial generalization of the HOM interference analogous to Reference \[159\].
Figure 4.10: Experimental two-photon interference with the metasurface. (a) Schematic setup. (b),(c) Quantum correlations between ports (b) 1 and 6 with close-to-orthogonal bases and (c) 1 and 5 with non-orthogonal bases, shown with dots and error bars indicating shot noise. Solid curves represent theoretical predication. Red arrows in the Poincaré spheres denote projective bases of different ports. Blue arrows indicate the polarization state of entangled photons, with one photon in H- and the other in V-polarization.
Two-photon state reconstruction

As a following step, we measure all 15 two-fold nonlocal correlations between the $M=6$ outputs from the metasurface for a given input state where the time delay is fixed to zero. This provides us full information to accurately reconstruct the input two-photon density matrix. We use two single-photon detectors to map out all possible output combinations, while this could be potentially accomplished even simpler with an EMCCD camera.

We show representative results for two different states $\hat{\rho}(\theta = 0^\circ)$ and $\hat{\rho}(\theta = 37.5^\circ)$ in Figures 4.11(a,b) and 4.11(c,d), respectively. Note that $\hat{\rho}(\theta = 0^\circ)$ is a state where photon pairs have cross-polarized entanglement beyond the classical limit, yet it is not fully pure, providing a suitable test case for reconstruction of general mixed states. In Figure 4.11(a) we show the measured two-fold correlations for the input state $\hat{\rho}(\theta = 0^\circ)$, and the reconstructed density matrix is shown in Figure 4.11(b). The fact that only the bunched four central elements are non-zero confirms cross-polarized property of our photon pairs in H-V basis. Moreover, the non-zero $|H\,V\,V\,H\rangle$ element implies the presence of two-photon entanglement. It is smaller compared to the diagonal element $|H\,V\,H\,V\rangle$, indicating that the polarization state is not fully pure. While $\hat{\rho}(\theta = 0^\circ)$ only has non-zero elements in the real part of the density matrix, we also show the measurement and reconstruction of $\hat{\rho}(\theta = 37.5^\circ)$ that contains nontrivial imaginary elements in Figures 4.11(c,d). In both cases, we achieve a very good agreement between the predicted and reconstructed density matrices as evidenced by high fidelity exceeding 95%. The correlation counts are obtained by a Gaussian fitting to the correlation histogram to remove the background, which is less than 10% of the signal for all measurements shown in Figure 4.11(c).

4.5 Summary

To summarize, the results presented in this chapter illustrate the manifestation of multiphoton quantum interference on metasurfaces. We formulate a concept of parallel quantum state transformation with metasurfaces, enabling single- and multiphoton state measurements solely based on the interaction of light with sub-wavelength thin nanostructures and nonlocal correlation measurements without a requirement of photon-number-resolvable detectors. This presents the ultimate miniaturization and stability combined with high accuracy and robustness, as we demonstrate experimentally via reconstruction of one- and two-photon quantum-polarization states including the amplitude, phase, coherence, and quantum entanglement.

The use of metamaterial to facilitate quantum photonics is an emerging and rapidly growing field of research. This work, together with Reference [179], opens a pathway of metasurfaces for quantum photonics. This is characterized by judiciously engineering the spatially-varying sub-wavelength structures in the matter for nontrivial manipulation of quantum light, which can potentially have extremely large degrees of freedom compared to systems
Figure 4.11: Experimental two-photon state reconstruction with the metasurface. (a),(c) Representative two-fold correlation measurements and (b),(d) the corresponding reconstructed density matrices $\hat{\rho}$ labeled 'Measured' alongside with the theoretically predicted states labeled 'Predicted' for QWP orientations (a),(b) $\theta = 0^\circ$ and (c),(d) $\theta = 37.5^\circ$. 
solely using integrated waveguides.

Our approach is particularly suitable for imaging-based measurements of multiphoton polarization states, where the metasurface can act as a quantum lens to transform the photons to a suitable format for the camera to recognize and retrieve more information. Despite the demonstration in this chapter is based on several diffracted spots, such patterns can be designed with further flexibility tailored for computer vision. The work presented in this chapter utilizes interleaved metagratings to perform the measurement. Currently, we are also researching on designing a single metagrating for the same functionality by using general positive operator-valued measures (POVMs), where we optimize a single metagrating by assuming each diffracted spot being an indefinite projection. This approach can also achieve robust reconstruction while maintaining a good beam quality and enhancing the diffraction efficiency, which we recently reported in the conference paper Reference [180]. Furthermore, there is a potential to capture other degrees of freedom associated with spatially varying polarization states for the manipulation and measurement of high-dimensional quantum states of light, with applications including free-space communications and quantum imaging.
5 Inline detection of multiphoton states in quantum circuits

In Chapter 4, where the reconstruction of classical and quantum photonic states were discussed, all these measurements are based on a detection at the output of the device. While photons are believed to be an ideal carrier of quantum information in large networks, the principle of conventional quantum state tomography at the output actually makes it difficult to pinpoint issues inside the network in real-time. With recent advances in integrated single-photon detectors, is it possible to judiciously utilize these integrated detection to facilitate a full reconstruction of the state density matrix without non-trivially changing the measured quantum state?

This chapter∗ will give a positive answer to this question by introducing and experimentally proving the principle of the new concept of inline detection and reconstruction of multiphoton states.

5.1 Introduction

The conventional wisdom of quantum state measurement is associated with detection at the output, from the most traditional quantum state tomography in bulk optics [153] to the metasurface-imaging approach as we presented in Chapter 4. In a large network of quantum circuits where photons are transported, however, the capability to measure the state within the circuits could enable a direct monitoring of their operation and pinpointing possible issues in real-time.

Latest advances in nanofabrication enable the integration of multiple single-photon detectors based on superconducting nanowires [59, 60], opening new possibilities for photon monitoring inside photonic circuits [182]. However, approaches for judiciously utilizing these integrated detectors to achieve the functionality of inline measurement and robustly reconstruct quantum states remain largely untapped. This requires measurements of the quantum features of multiphoton states encoded in their density matrices, while ideally keeping the transmitted states undisturbed apart from weak overall loss.

Such an inline detection capability would be highly desirable similar to the classical

∗Results presented in this chapter are partially included in the published References [151, 181], where I am the lead author and my contribution is in all aspects including experiment and theory.
analogs [166], yet it presents a challenging problem since traditional approaches for quantum measurements are not suitable. In particular, direct measurement methods [183, 184] are difficult to employ due to complex measurement operators. On the other hand, the conventional state tomography [153], a most widely used reconstruction method, requires reconfigurable elements to apply modified projective measurements in different time windows. Recently, the reconstruction was achieved in static optical circuits [58, 104, 155] or metasurfaces as we presented in Chapter 4, yet it comes at a cost of spreading out quantum states to a larger number of outputs, which is incompatible with inline detection principle.

Most linear quantum-photonic circuits for information processing are based on gates that form Mach-Zehnder-like interferometers [68], in order to achieve certain transformations from the input to the output ports. Gradually coupled waveguides provide possibilities for more compact quantum-photonic circuits. However, so far works on such linear platforms were mostly devoted to reproducing or mimicking the functionality of those input-output transformation schemes, e.g. the design in Reference [58].

In this chapter, we introduce a new conceptual approach for practical inline measurement of multiphoton states using integrated detectors which is suitable for multiport systems. We aim to provide a new model that makes judicious use of the intermediate stages of the gradually coupled waveguides for quantum interferometric measurement, which could inspire more designs in using gradually coupled waveguides for highly compact quantum photonic circuits. We show that the detector positions on top of waveguides can be chosen such that the state is transmitted undisturbed apart from an overall loss, while the density matrix of the input multiphoton quantum state can be robustly reconstructed. Therefore the method is well suited for monitoring operation inside complex quantum circuits.

Outline of this chapter

The subsequent sections of this chapter are organized as follows: in Section 5.2, the concept and method of quantum state reconstruction via inline detection will be introduced. Then, in Section 5.3 proof-of-principle experimental results demonstrating the inline approach in integrated waveguides will be presented. Then, in Section 5.4 we show an application of the inline measurement approach in monitoring the spatial coherence of light in a parity-time symmetric directional coupler. Finally, we will summarize this chapter in Section 5.5.

5.2 Multiphoton state monitoring with integrated detectors

Now we establish theoretically the working principle of our inline approach to multiphoton state measurements. We employ a new regime of correlation measurements among detectors at different propagation stages associated with the same static Hamiltonian, taking the unique advantage of photons in waveguides implementing gradually-coupled transformations. We illustrate it for two waveguides in Figure 5.1(a). The coupled waveguides (coupling
5.2 MULTIPHOTON STATE MONITORING WITH INTEGRATED DETECTORS

Coefficient $C$ have different propagation constants (detuned by $\beta$) and the length $L$ is exactly a revival period. An even number $M$ of weakly coupled single-photon click detectors are placed at $M/2$ cross-sections, starting from $z_1$ with equal distances $2L/M$ between each other. As we demonstrate in the following, the measurement of $N$-fold nonlocal correlations by averaging the coincidence events enables a full reconstruction of the density matrix $\hat{\rho}_N$ for $N$-photon states.

\[
\hat{H} = \beta \hat{a}_1^\dagger \hat{a}_1 - \beta \hat{a}_2^\dagger \hat{a}_2 + C \hat{a}_1^\dagger \hat{a}_2 + C \hat{a}_2^\dagger \hat{a}_1,
\]

where $\hat{a}_q^\dagger$ ($\hat{a}_q$) are the photon creation (annihilation) operators in waveguide number $q$. We use Heisenberg representation $[185, 186]$ and map time $t$ to the propagation distance $-z$.

Figure 5.1: Conceptual sketch of inline detection and reconstruction of $N$-photon state $\hat{\rho}_N$ with two spatial modes. (a) Two waveguides with coupling constant $C$ and detuning $\beta$ in propagation constants. An even number $M$ of single-photon click detectors are positioned at $M/2$ equidistant cross-sections, illustrated for $M = 6$ with D1–D6 labels. (b) Examples of simulated single-photon counts ($N = 1$) and $N$-fold correlations (for $N = 2$) for $z_1 = 0$ and $\beta = C/\sqrt{2}$, which enable full reconstruction of the input density matrix $\hat{\rho}_N$. The quantum state evolution along the waveguides, under the assumption of weakly coupled detectors, is governed by the Hamiltonian

The quantum state evolution along the waveguides, under the assumption of weakly coupled detectors, is governed by the Hamiltonian
Figure 5.2: Robust reconstruction with waveguide detuning. (a,b) The analysis states on a Bloch sphere for (a) identical waveguides ($\beta = 0$) and (b) introduced detuning ($\beta = C/\sqrt{2}$). Curves show the evolution along $z$, and arrows correspond to six detectors as indicated in Figure 5.1(a). The $M = 6$ analysis states consist of 3 pairs of orthogonal ones, with each pair indicated by two arrows pointing in opposite directions. Red and blue colors denote waveguide a and b, respectively. (c,d) Inverse condition number $\kappa^{-1}$ versus (c) normalized detuning $\beta / C$ and (d) number of detectors $M$ for the optimal detuning $\beta = C/\sqrt{2}$.

Hence the operator evolution reads

$$\hat{a}_q(z) = \sum_{q' = 1,2} T_{q,q'}(z) \hat{a}_{q'}(0), \quad (5.2a)$$
$$\hat{a}_q^\dagger(z) = \sum_{q' = 1,2} T_{q,q'}(z) \hat{a}_{q'}^\dagger(0). \quad (5.2b)$$

Here the linear transfer matrix elements of the waveguide coupler are

$$T_{q,q}(z) = \cos(\eta z) + i(-1)^q \beta \eta^{-1} \sin(\eta z), \quad (5.3a)$$
$$T_{q,3-q} = -i C \eta^{-1} \sin(\eta z), \quad (5.3b)$$

where $\eta = (C^2 + \beta^2)^{1/2}$. We note that the input state is restored at the revival length $L = 2\pi/\eta$, since $\hat{a}_q^\dagger(L) = \hat{a}_q^\dagger(0)$. While the inline detectors introduce small loss, due to their symmetric positions the revival effect will remain, and the output $N$-photon density matrix would only exhibit an overall loss $(1 - \mu)^N \hat{\rho}_N$, where $\mu$ is a one-photon transmission coefficient which can be close to unity.

We consider the measurement of states with a fixed photon number $N$, which is a practically important regime [58, 104, 153, 155] as we have discussed in Chapter 4. Note that our
method can be generalized to knowing the maximum photon number \( N_{\text{max}} \) in a measured photon system \([154]\). More specifically, it is possible to firstly use \( N_{\text{max}} \)-fold correlations to reconstruct the photon states in the \( N_{\text{max}} \) sub-Fock-space with the same method as introduced in this thesis. Then one can simply measure the \((N_{\text{max}}-1)\)-correlations and exclude those events caused by \( N_{\text{max}} \)-fold correlations, and reconstruct the density matrix in the \((N_{\text{max}}-1)\) sub-Fock-space. Following this procedure, it is possible to reconstruct the spatial-mode density matrix ranging from \( N_{\text{max}} \) to 1 sub-Fock-space.

We base the analysis on the most common type of click detectors that cannot resolve the number of photons arriving simultaneously at the same detector, and cannot distinguish which photon is which. Nevertheless, our design is also compatible with number-resolvable detectors and distinguishable detection scheme. If either zero or one photons (but not more) are incident on detector \( m \), then the general POVM detection operator \([172]\) can be represented as \( \hat{A}_m = \hat{a}_{q_m}^\dagger (z_m) \hat{a}_{q_m}(z_m) \), where \( q_m \) is the waveguide number and \( z_m \) is the detector coordinate. Accordingly, the photon correlations corresponding to the simultaneous detection of \( N \) photons by a combination of \( N \) detectors with numbers \( m_1, m_2, \ldots, m_N \), such that there is at most one photon at each detector, are:

\[
\Gamma(m_1, m_2, \ldots, m_N) = \text{Tr}(\hat{\rho}_N \hat{A}_{m_1} \hat{A}_{m_2} \cdots \hat{A}_{m_N}).
\] 

For a single photon \((N = 1)\), the measurement corresponds to a direct accumulation of counts at each detector without correlations, and we show an example in Figure 5.1(b, left) for a pure input state \( |\psi_i\rangle = [1,1]^T/\sqrt{2} \) with the density matrix \( \hat{\rho}_1 = |\psi_i\rangle \langle \psi_i| \). We present an example of coincidence counts for a two-photon N00N state \([172]\) \((N = 2)\) in Figure 5.1(b, right), corresponding to the events of clicks at different combinations of two detectors.

We outline the principle of reconstructing the input \( N \)-photon density matrix \( \hat{\rho}_N \) from the correlation measurements. The vectors \( |\psi_{q}\rangle = [T_{q,1}, T_{q,2}]^T \) are the analysis states, which define the measurements at the different detector positions. We visualize their evolution along \( z \) on a Bloch sphere, where the projector \( |\psi_{q}\rangle \langle \psi_{q}| \) is decomposed into the Pauli matrices \( \hat{\sigma}_i \) to generate the coordinates \( S_i = \text{Tr}(\hat{\sigma}_i |\psi_{q}\rangle \langle \psi_{q}|) \). First we present the trajectories for the case of no detuning \((\beta = 0)\) in Figure 5.2(a), where we see that the analysis states simply trace out a circle along the Bloch sphere. Considering for example \( M = 6 \) detectors as sketched in Figure 5.1(a), all the analysis states corresponding to detectors shown by arrows are in one plane. Such a configuration cannot probe states beyond that plane, therefore one cannot perform state reconstruction with a static, non-detuned directional coupler. In contrast, for detuned waveguides \((\beta \neq 0)\) the circular trajectories in the first and second waveguides become non-degenerate, see the red and blue curves in Figure 5.2(b). We indicate the analysis states with arrows for \( M = 6 \), and note that they are spread out to different directions in the sphere and can thus be utilized to probe all information about the states and enable the density matrix reconstruction.

The input state reconstruction is performed as follows. First, we introduce an index \( p = 1, 2, \ldots, P \) with \( P = M!/[N!(M-N)!] \) to enumerate all possible \( N \) combinations of \( M \)
detectors, \((m_1, m_2, \ldots, m_N)_p\). Second, we select \(S = (N + 3)!/(N!3!)\) real-valued parameters \(r_1, r_2, \ldots, r_S\), representing independent real and imaginary parts of the density matrix elements following the procedure in the Supplementary of Reference \([58]\). Here, we consider the indistinguishable detection scheme that does not tell which photon is which, and formulate Equation (5.4) in a matrix form:

\[
\Gamma_p \equiv \Gamma(m_1, m_2, \ldots, m_N)_p = \sum_{s=1}^{S} B_{p,s} r_s,
\]

(5.5)

where the elements of the instrument matrix \(B\) are expressed through the transfer matrices \(T\) at the detector positions. The minimum requirement for the density matrix parameters \(\{r_s\}\) to be reconstructed from the correlations by performing pseudo-inversion of Equation (5.4) is \(P \geq S\), i.e. when the number of detectors

\[
M \geq N + 3.
\]

(5.6)

We now analyze the robustness of reconstruction with respect to possible experimental inaccuracies in the correlation measurements, such as shot noise. This can be quantified by the condition number \(\kappa\) of the matrix \(B\), see References \([58, 173]\) for details. Then, the most accurate reconstruction corresponds to the smallest condition number. We numerically calculate the condition numbers for different combinations of \(M\) and \(N\), considering a symmetric arrangement of detectors as sketched in Figure 5.1(a). We find that accurate reconstruction can be achieved for the number of click detectors \(M > N + 3\). However, the condition number is infinite and the reconstruction cannot be performed for \(M = N + 3\). This happens because the \(M\) analysis states are not fully independent, but actually constitute \(M/2\) pairs of orthogonal states, as illustrated in Figure 5.2(b). There is no contradiction with Equation (5.6), since the latter establishes only an ultimate lower bound for reconstruction.

We determine that a detuning of the waveguide propagation constants \((\beta)\) is essential to perform the reconstruction. We illustrate in Figure 5.2(c) that the variation of \(\beta\) strongly changes the reconstruction condition for a different number of photons \(N = 1, 2, 3\), considering the minimum possible even number of detectors. With no detuning, for \(\beta = 0\), \(\kappa^{-1}\) goes to zero, meaning that the reconstruction is ill-conditioned and cannot be performed in practice. The optimal measurement frames occur at \(\beta/C \simeq 1/\sqrt{2}\). In Figure 5.2(d) we fix \(\beta/C = 1/\sqrt{2}\) and plot \(\kappa^{-1}\) versus the number of detectors \(M\). We find that \(\kappa^{-1} = 1/\sqrt{3}\) for \(N = 1\) and any \(M \geq 6\), in agreement with classical measurement theory \([173]\). Importantly, an optimal condition number can also be preserved for stronger couplings to the detectors, when detection rate is enhanced at a cost of losing more photons in the output. The post-fabricated device can be calibrated and described by a transfer matrix, which thereby accounts for fabrication imperfections and different detection efficiencies and enables accurate reconstructions, provided the condition number is close to optimal, without a need to characterize individual structure parameters.
5.3 Reconstruction of spatially-encoded photonic states

We demonstrate the scheme with proof-of-principle experiments for classical light. Mathematically, this is equivalent to the single-photon problem [105, 186], since monochromatic laser light can be described by a density matrix of the same dimensionality as for a single photon, $2 \times 2$ corresponding to two waveguides.

\[ \rho(z) \]

\[ \rho(z + L) \]

\[ \hat{\rho}(z) \]

\[ \hat{\rho}(z + L) \]

\[ \text{Fluorescence image} \]

\[ \text{Normalized power} \]

\[ \text{Propagation distance (mm)} \]

**Figure 5.3:** A fluorescence image (top), showing light intensity along the coupler. Bottom – the corresponding normalized power evolution in each waveguide (solid curves) compared to theory (dashed curves). Shading indicates a section $(z, z + L)$, where inline measurements are used to reconstruct the full density matrix at the section input, $\hat{\rho}(z)$.

We employ laser-written waveguides in fused silica to perform a reconstruction with a source of coherent laser light. In the fabrication process, we use different laser energies to write the coupled waveguides, and thereby achieve a pre-determined offset $(\pm \beta)$ of the propagation constants for the fundamental modes in the two waveguides [142]. We launch 633 nm laser light into one of the waveguides, and perform inline measurements of the evolution of intensity by observing the fluorescence (around 650 nm wavelength) emitted from the color centers of the glass material under a microscope [142, 187]. More details on the fabrication and measurement of such waveguides can be found in the Appendix Section A.2.1.

We present a characteristic fluorescence image in Figure 5.3 (top), which features a clear beating pattern due to light coupling between the waveguides along the propagation direction $z$. The extracted normalized power in each waveguide is shown in Figure 5.3 (bottom). We observe a good agreement with the theory (dashed line) for the coupled waveguide modes with $C = 0.0885 \text{ mm}^{-1}$ and propagation constants detuning $\beta = 0.0240 \text{ mm}^{-1}$, corresponding to a single-photon condition number $\kappa = 2.8$. The observation of emitted fluorescence is conceptually equivalent to putting many weakly coupled detectors homogeneously along both waveguides.
Figure 5.4: Experimental inline measurement and reconstruction of $N = 1$ states emulated with classical laser light in detuned directional couplers. (a) The reconstructed real and imaginary parts of the density matrix elements for different combinations of the waveguide numbers (WG No.) at $z = 10$ mm, the beginning of the shaded section in (a). (b) Experimentally reconstructed $\hat{\rho}(z)$ from a set of different positions $z$ (crosses) compared with theoretical predictions (green curve) on a Bloch sphere, demonstrating an average fidelity of 99.65%.

We then test our reconstruction method using the fluorescence image of the waveguide coupler. In the 80-mm long section of the fabricated structure shown in Figure 5.3, light periodically couples several times between the waveguides. We truncate a section along $z$ with one revival period $L$ to mimic the inline quantum detection. Then, the state $\hat{\rho}(z)$ can be predicted by the coupled wave equation using the characterized parameters as described above. We consider different $z$ as starting positions, which allows us to effectively change the input state. We employ a maximum-likelihood method to perform a pseudo-inversion of Equation (5.5) and thereby find an input density matrix that best fits the evolution of the fluorescence power in the two waveguides, see an example in Figure 5.4(a). We analyze 34 different input states, truncated from successive $z$ with 1 mm increments. The reconstructed states are plotted on a Bloch sphere with crosses in Figure 5.4(b), and compared to theoretical modelling shown with a green curve. The coordinates are obtained via decomposing the density matrices to the Pauli matrices, i.e. $S_i = \text{Tr}(\sigma_i \hat{\rho})$ with $i$ spanning $x, y, z$. We observe an excellent consistency between direct theoretical modelling and reconstruction from the experimental fluorescence images of the full density matrix, where we reach a very high average fidelity of 99.65%. The high fidelity shows that reliable reconstructions are possible with a reasonable non-zero detuning, where the condition number of reconstruction and shot noise act as fundamental limits.

5.4 Measurement of coherence in parity-time symmetric couplers

With the unique capability to fully reconstruct multiphoton states in quantum circuits such as integrated waveguides that we have developed, it becomes possible to directly monitor the evolution of light in such structures including amplitude, phase and coherence information. Even just for classical light, this is an important functionality as it eliminates the efforts
in implementing interferometry setups at the output waveguides. Thereby, such a scheme enables the experimental study of light propagation in waveguides with unprecedented capability. Now, in this section, we will show an example that uses the inline detection principle for measuring coherence in parity-time (PT) symmetric couplers.

The degree of coherence is of fundamental importance for both classical and quantum states of light and their applications. For instance, classical optical interferometry requires the light field to remain highly coherent. On the other hand, decoherence limits the performance of quantum computers and communications. This motivates strong research efforts in developing approaches for increasing the degree of coherence, which can be quantified by a purity measure of states of light. It is recognized that in linear conservative systems, where dynamics are governed by Hermitian Hamiltonians, the degree of purity is preserved. Thereby, bulk optical setups with specially introduced loss were developed to perform purification and control of coherence was demonstrated through diffraction on metal films involving plasmon coupling.

Parity-time (PT) symmetry gives rise to a class of non-Hermitian Hamiltonian that challenges a conventional wisdom in quantum mechanics, where it was found that non-Hermitian Hamiltonian with such a symmetry can also have real eigenvalues. While there is an important phase transition in a PT system between PT-symmetric phase (real eigenvalue) and broken phase (complex eigenvalue), within the PT-symmetric regime there is an important feature associated with the non-orthogonality of eigenvectors under the most common definition of orthogonality. This fundamentally allows coherence to vary in systems governed by such Hamiltonian. Yet unlike in Hamiltonians with complex eigenvalues, in the PT-symmetric phase the state can still exhibit revivals. Hence the use of PT symmetry in the manipulation of optical coherence and state purification is of fundamental and practical interests. This can be efficiently realized in integrated PT-symmetric photonic structures composed of elements with different loss or gain.

![Figure 5.5:](image)

**Figure 5.5:** (a) Concept of coherence manipulation between the optical modes through propagation in a parity-time symmetric structure composed of coupled conservative and lossy waveguides. (b) Theoretically calculated purity of an initially fully incoherent state vs. the normalized propagation distance $CL$ and the waveguide loss $\gamma/C$. 
While the concept of manipulating coherence with PT Hamiltonian can be applied to a variety of photonic systems with PT symmetry, to be specific, we analyze a structure composed of coupled conservative and lossy waveguides [191]. As a representative example, we consider input light which is fully incoherent between the two input waveguide ports, see Figure 5.5(a). Classical light propagation is governed by a non-Hermitian PT-symmetric Hamiltonian [192] for the density matrix $\hat{\rho}$. We derive the purity expression as

$$
p = \frac{\text{Tr}(\hat{\rho}^2)}{[\text{Tr}(\hat{\rho})]^2} = 1 - \frac{(4 - \frac{\gamma^2}{C^2})^2}{2[4 - \frac{\gamma^2}{C^2} \cos(\xi CL)]^2},
$$

where $\xi = \sqrt{4 - \gamma^2 C^{-2}}$. We plot it in Figure 5.5(b), where the horizontal axis is the propagation distance $L$ normalized to the waveguide coupling $C$, and the vertical axis is the normalized difference in the losses of two waveguides $\gamma$. The value $p = 0.5$ indicates a fully incoherent state, while $p = 1$ is reached in case of perfect coherence. For losses below a PT breaking threshold at the exceptional point indicated by the dashed line in Figure 5.5(b), $\gamma / C < 2$, the purity varies along the propagation direction with a period $L_p = 2\pi \xi^{-1} C^{-1}$, while above the loss threshold the purity gradually increases approaching a perfectly coherent state.

We experimentally realize the manipulation and detection of optical coherence using a specially designed waveguide coupler fabricated with the same direct femtosecond laser writing technique as discussed in the section above, where the whole structure is sketched in Figure 5.6. The first section (shaded in yellow) realizes a quasi-PT coupler acting as the part for the manipulation of coherence, where the second waveguide features a sinusoidal modulation of its position in space which results in radiative losses [193]. The second section (shaded in green) contains straight coupled waveguides with specially detuned propagation constants following the principle of inline detection introduced in this chapter, which are optimized to enable a full reconstruction of the degree of purity and mode coherence by measuring the interference pattern in both waveguides along this section. Note that here it is not necessary to maintain the state after the measurement in the inline measurement section, therefore we design a rather long section that can capture sufficiently fruitful evolution dynamics.

The measured power evolution in both waveguides determined through fluorescence images. We show two example cases of the fluorescence images of the whole coupler structure in Figures 5.7(a,c) for the PT coupler section length of $L = 4$mm and $L = 19$mm, respectively.
We launch fully incoherent state to the PT coupler achieved by a double exposure of two fluorescence measurements, each illuminated from only one of the two waveguides with coherent laser light. In Figure 5.7(a, bottom), after only a very short propagation in the PT coupler section (yellow shaded), the light entering the inline measurement section (green shaded) exhibit almost no oscillation in power except for an overall decay due to the intrinsic loss of the waveguides. In Figures 5.7(b) we show the theoretical input density matrix Figure 5.7(b, left) and the experimentally reconstructed output density matrix after the PT section Figure 5.7(b, right), as the inline detection enables us to achieve this measurement. It can be seen that input density matrix is fully diagonal confirming the fully incoherent nature of the input state. Yet the measured output density matrix also has almost zero off-diagonal elements, indicating there is little purification due to the short propagation distance in the PT section. Now we compare with the case that has longer PT section, \( L = 19 \text{mm} \), as shown in Figure 5.7(c, bottom). In this case, we observe interferometric oscillations inside the green-shaded second section, which clearly indicates that the state is gaining coherence. The reconstructed output density matrix for this case, shown in Figure 5.7(d, right), exhibits non-zero off-diagonal elements while the diagonal elements are also changed due to the transformation of the state by the PT section. This confirms that PT-symmetric Hamiltonian can indeed purify an incoherent photonic state for certain well-chosen propagation lengths.

![Figure 5.7](image)

Figure 5.7: Representative measurements with different PT coupler lengths. (a) (b) A representative measurement with PT coupler length \( L = 4 \text{ mm} \): the fluorescence image (top) and extracted powers along the two waveguides (bottom). (c) (d) A representative measurement with PT coupler length \( L = 19 \text{ mm} \): the fluorescence image (top) and extracted powers along the two waveguides (bottom).

Then, to experimentally study the impact of the propagation length \( L \) on the purification,
we prepare several such structures with different $L$ and perform special algorithmic processing of experimental data to precisely determine the density matrix. Moreover, we have a degree of freedom to vary the input state in the double exposure, where

$$\hat{\rho}_{\text{in}} = c_1 |1\rangle \langle 1| + c_2 |2\rangle \langle 2|.$$  \hfill (5.8)

We change the ratio $\Delta_0 = c_1/c_2$ in the summation of two double-exposed images, which effectively change the coherence of the input state. As a measure of the state evolution, we also define a general $\Delta$ as the ratio between the light powers in waveguide 1 and 2, $\Delta = \rho_{1,1}/\rho_{2,2}$.

As a representative example, in Figure 5.8(a) we show our experimentally reconstructed states after the PT coupler with different propagation lengths $L$ in a Bloch sphere (red dots) with an input $\Delta_0 = 0.5$. The corresponding theoretical prediction is shown as a blue curve. It can be seen that the origin of the coherence variation lies in the cyclic evolution of the state vector inside the Bloch sphere, where the state vector changes its length (i.e., distance to the center of the ball). Such a variation is not possible with Hermitian Hamiltonians. This can be further seen from Figure 5.8(b), where we plot the evolution of purity of state $P$, coherence $\zeta$ and the power ratio in the two waveguides $\Delta$ versus the propagation length $L$. Here scattering symbols denote experimental measurements and curves represent theoretical predictions. Specifically, the coherence measure $\zeta$ is defined by

$$\zeta = \frac{|\rho_{1,2}|}{\sqrt{\rho_{1,1}\rho_{2,2}}},$$  \hfill (5.9)

which is not an invariant measure. Our observation constitutes the first experimental evidence of reversible manipulation of light coherence in PT coupled waveguides.

![Figure 5.8](image)

**Figure 5.8:** (a) Theoretical (blue curve) and experimentally measured (red dots) evolution of normalized state under the PT Hamiltonian for an initial $\Delta_0 = 0.5$ plotted in a Bloch sphere. (b) The theoretically predicted (curves) and experimentally measured (discrete markers) values of $\Delta$, coherence $\zeta$ and purity $p$ under the same Hamiltonian for varying length $L$.

In Figures 5.9(a–f) we show the density plots of all possible combinations of different $L$ and input $\Delta$. More specifically, In Figures 5.9(a,b,c) plots the purity of state $p$, coherence $\zeta$ and power ratio $\Delta$ obtained from our experiments, respectively. The corresponding theoretical predictions are shown in In Figures 5.9(d,e,f), respectively. It can be seen that our
5.5 Summary

To summarize, in this chapter, we proposed a practical and efficient approach for inline detection and measurement of single and multiphoton quantum states in coupled waveguides with integrated photon detectors, suitable for various applications in quantum photonics. We showed proof-of-principle results with laser-written waveguides for classical light emulating single-photon regime. It should also be feasible with various types of other quantum photonic circuits, such as silicon nanowaveguides, where the propagation constant of which can be tailored by geometric parameters and superconducting nanowire detectors can be placed in the same integrated configuration. This promises various applications in multiphoton quantum photonics. We presented the theory and experiments for a two-port system, and this can be scaled to multiple coupled waveguides. Moreover, our approach has a potential for translation from spatial to frequency and time-domain measurements using synthetic lattices in nonlinear waveguides as presented in Chapter 2, where engineered phase-matching condition can act as detunings to the frequency lattice sites therein.

With the strength of integrated state reconstruction enabled by the inline detection scheme, we suggest and demonstrate experimentally light state purification and measurement of coherence in integrated parity-time symmetric photonic structures with specially designed asymmetric loss. This does not only show the strength of our inline detection approach but also presents a new method for coherence manipulation. The latter can have a wide range of applications from classical to quantum optics, including filtering out noise.
added to states and optimizing the visibility of interferometric measurements.

Importantly, our inline detection scheme provides an example of the unique advantage to use photons in integrated quantum information processing compared to other particles. We made use of correlations among detectors placed at different positions along the waveguides, corresponding to various effective propagation times of the same static Hamiltonian governing the photon-state evolution. This is enabled by the fast-propagation nature of photons in waveguides, such that time is mapped to space. Such types of interferometric measurements are difficult to realize for other quantum particles operating in real-time (e.g. superconducting qubits or trapped ions). Our inline detection principle focuses on not disturbing the state apart from a small overall loss by always placing detectors at the same propagation distance in the coupled waveguides with the same coupling to the detector. Nevertheless, there should be plenty of flexibility to engineer the arrangement of these weakly coupled detectors in a judicious way realizing certain quantum measurement operators, such as those described by the quantum weak measurement \[183\]. Above all, we anticipate the judicious engineering of quantum circuits with integrated detectors can be of both great fundamental and application interests in many aspects.
6 Conclusion and outlook

This thesis is devoted to addressing key problems in multidimensional photonics using synthetic lattices, with a scope of both fundamental and practical interests. Based on various platforms, novel concepts and approaches are established and experimentally demonstrated in the three main aspects of this field.

First, we show a new approach to form multidimensional lattices in photonics. This is presented in Chapter 1, where we propose and experimentally realize a new type of frequency lattice using nonlinear frequency conversion in a nonlinear waveguide. This method is designed to operate in an all-optical manner, which does not rely on complex electro-optic modulation. Our synthetic frequency lattices allow the implementation of long-range couplings and artificial gauge fields, both being key enabling factors for accessing many exotic lattice behaviors in higher dimensions. We experimentally realize novel instances of discrete Talbot effect and quantum walks in the synthetic space from a single nonlinear fiber with nontrivial gauge potential. In a more practical aspect, this platform also enables the demonstration of the one-shot measurement of amplitude, phase and coherence encoded in discrete frequencies. This new platform has the potential to bring multidimensional physics effects, such as topological behaviors in higher dimensions, to the temporal-spectral domain to enable a fundamental solution to practical problems. It may also be a powerful tool in the nonreciprocal manipulation of quantum light encoded in the frequency degree of freedom with drastically-enhanced information capacity.

Secondly, we explore the possibility to use judiciously engineered planar photonic structures to go beyond the geometrical limitation and harness multidimensional properties. This is presented in Chapter 3, where we introduce a fundamentally new approach to access higher-dimensional dynamics based on isospectrality transformations. Specifically, we formulate and experimentally demonstrate a way of specially engineering the coupling and potential profiles of nearest-neighbor-coupled one-dimensional lattices, enabling a chosen site to yield arbitrary-dimensional dynamics. This approach does not need any dynamical or propagation modulation and is highly practical to realize in planar photonics. This approach is platform-independent, the impact of which can be beyond photonics, e.g. in quantum-engineered systems such as cold-atoms. Enabled by the unprecedented capability in accessing higher dimensional dynamics, we reveal and experimentally test the unique property of defect localization in larger-than-three dimensions, which is manifested in a sharp localization transition originated from a bound state at the edge of the continuum. We anticipate
this approach may enable the experimental observation of many long-standing fundamental questions such as the dimensional scaling behavior of Anderson-localization, the stability of solitons, and entanglement dynamics in higher dimensions. Such a new paradigm as a design method for planar photonics can also find its way in many applications from thresholded switching to advanced sensing.

Thirdly, we focus on key difficulties in the manipulation of multidimensional photonic states. Specifically, we research on the especially nontrivial type of multiphoton states, with the manipulation aiming at a judicious transformation to facilitate the challenging task of state measurement.

In Chapter 4, for the first time we make use of metasurfaces for multiphoton interference and measurement, opening up a pathway of imaging-based quantum photonic state reconstruction with flat optics. Our results illustrate the manifestation of multiphoton interference on metasurfaces for the first time and test the feasibility of using metasurfaces as a new platform to map out multiphoton quantum states with imaging methods. This presents ultimate miniaturization and stability combined with high accuracy and robustness, as we have demonstrated experimentally via reconstruction of one- and two-photon quantum-polarization states including the amplitude, phase, coherence, and quantum entanglement. We anticipate such a new field of using flat optics for the manipulation of multidimensional quantum-photonic states will boost versatile applications in quantum photonics, including free-space communications and quantum imaging.

In Chapter 5, for the first time, we bring quantum state measurement to the inline regime. More specifically, we propose a practical and efficient approach for inline detection and measurement of single and multiphoton quantum states in quantum circuits with a judicious combination of integrated single-photon detectors. We prove the principle for this approach with integrated waveguide structures, providing a recipe for the judicious use of integrated detectors for the inline monitoring of operation in quantum-photonic circuits for pinpointing issues in real-time. We also apply the inline detection scheme to the measurement of light state purification in integrated parity-time symmetric photonic structures. This further confirms the strength of our inline detection approach.

Above all, in this thesis, we have shown the strength of engineered synthetic lattices in facilitating multidimensional photonics on various experimental platforms including nonlinear waveguide, planar photonics, metasurfaces, and quantum-photonic circuits. To achieve this, we developed new concepts surrounding the construction of lattices, the manifestation of multidimensional features and control of multidimensional states in photonics. Indeed, there is plenty of room in higher dimensions.

As an outlook, we anticipate the developments in multidimensional photonics using synthetic lattices can benefit and inspire versatile research directions in many fields, including but not limited to, topological photonics, quantum photonics, artificial intelligence, optical computing, advanced sensing, and communications. The approaches developed on this topic may also inspire new ideas in fields beyond photonics, such as material science and
electronics. This is illustrated as a diagram in Figure 6.1.

![Diagram](image.png)

**Figure 6.1**: Potential fields that can benefit from the research on multidimensional photonics in synthetic lattices.

On specific ideas and suggestions for future work, we have discussed in the summary sections of each chapter. Additionally, there are also some more suggestions:

- Utilizing the all-optical synthetic frequency space presented in Chapter 2 on topological photonics. Since such a platform provides particular flexibility to the exploration of many fundamental physical effects beyond 3D, one can consider the realization of many topological effects in higher dimensions, for example, higher-order topological states.

- Designing novel types of lasers, passive or active sensors, neural networks and other information-processing devices using the strength enabled by the freeform engineering of local density of states shown in Chapter 3.

- Using analogous types of metasurfaces designed in Chapter 4 for the reconstruction of even larger-dimensional Hilbert space by introducing the spatial variation or orbital angular momentum of entangled photons.

- Performing multiphoton experiments in silicon photonics with superconducting nanowire detectors based on our inline detection scheme shown in Chapter 5.
This page is intentionally left blank.
Appendix

A.1 Theoretical details

A.1.1 Coupled mode theory of weakly guided waveguides

To illustrate the behavior of light in a set of evanescently coupled waveguides, it is convenient to find discrete modes for each channel. The coupled mode theory is a possible approximate approach. The following derivation can also be found in Reference [194].

Quite generally, we start from Maxwell’s equations in frequency domain without external source. The vectorial electric field $E(r, \omega)$, dielectric flux density $D(r, \omega)$, magnetic field $H(r, \omega)$ and magnetic flux density $B(r, \omega)$ obey

\begin{align}
\nabla \times E(r, \omega) &= i\omega B(r, \omega) \\
\nabla \times H(r, \omega) &= -i\omega D(r, \omega) \\
\n\nabla \cdot D(r, \omega) &= 0 \\
\n\nabla \cdot B(r, \omega) &= 0.
\end{align}

In this method, the influences of other waveguides are considered as perturbations. For a waveguide with number $n$, we assume there exist perturbed and unperturbed modes, where $E_{n,p}$, $H_{n,p}$ and $E_{n,u}$, $H_{n,u}$ represent the electric and magnetic fields of perturbed and unperturbed modes, respectively. We assume the dielectric medium to be linear, isotropic and non-magnetizable, yet inhomogeneous with the permittivity $\epsilon_n(r, \omega)$ that depends on position $r(x, y, z)$ and frequency $\omega$. For the unperturbed mode, the constitutive equations reads

\begin{align}
D_{n,u}(r, \omega) &= \epsilon_0 \epsilon_n(r, \omega) E_{n,u}(r, \omega) \\
B_{n,u}(r, \omega) &= \mu_0 H_{n,u}(r, \omega),
\end{align}

where the electric constant $\epsilon_0$ and magnetic constant $\mu_0$ are related to the light velocity in vacuum $c$ by $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. Nevertheless, for perturbed mode, Equation (A.2a) should be modified to

\begin{align}
D_{n,p}(r, \omega) &= \epsilon_0 \epsilon_n(r, \omega) E_{n,p}(r, \omega) + P_n(r, \omega),
\end{align}

where $P_n(r, \omega)$ is the polarization caused by the perturbation. Equation (A.2b) remains
Summing all the equations in Equation (A.4) we can obtain

\[
\nabla \cdot \left[ \mathbf{E}_{n,p} \times \mathbf{H}_{n,u}^* + \mathbf{E}_{n,u}^* \times \mathbf{H}_{n,p} \right] = i \omega \mathbf{E}_{n,u}^* \cdot \mathbf{P}_n. \tag{A.5}
\]

By integrating over the transverse plane we have

\[
\int \int \frac{\partial}{\partial z} \left( \mathbf{E}_{n,p} \times \mathbf{H}_{n,u}^* + \mathbf{E}_{n,u}^* \times \mathbf{H}_{n,p} \right) \cdot \mathbf{e}_z \, dx \, dy = i \omega \int \int \mathbf{E}_{n,u}^* \cdot \mathbf{P}_n \, dx \, dy, \tag{A.6}
\]

with the assumption that all fields decay to zero at the boundaries such that the following relations are fulfilled:

\[
\int \int \frac{\partial}{\partial x} \left( \mathbf{E}_{n,p} \times \mathbf{H}_{n,u}^* + \mathbf{E}_{n,u}^* \times \mathbf{H}_{n,p} \right) \cdot \mathbf{e}_x \, dx \, dy = 0 \tag{A.7a}
\]
\[
\int \int \frac{\partial}{\partial y} \left( \mathbf{E}_{n,p} \times \mathbf{H}_{n,u}^* + \mathbf{E}_{n,u}^* \times \mathbf{H}_{n,p} \right) \cdot \mathbf{e}_y \, dx \, dy = 0. \tag{A.7b}
\]

Lately, the influence of nonorthogonal modes in laser-written waveguides were also studied and modeled [195].

We consider only one mode with wave number \( k_n \) in each waveguide, and the guiding is very weak that there is no \( z \) component of the EM filed. If we only consider forward propagation along \( z \) direction, the plane wave ansatz for the unperturbed mode reads

\[
\begin{align*}
\mathbf{E}_{n,u}(\mathbf{r}) &= E_n(x, y) e^{ik_n z} \mathbf{e}_l, \tag{A.8a} \\
\mathbf{H}_{n,u}(\mathbf{r}) &= H_n(x, y) e^{ik_n z} (\mathbf{e}_z \times \mathbf{e}_l), \tag{A.8b}
\end{align*}
\]

where \( E_n(x, y) \) and \( H_n(x, y) \) are the amplitudes of the electric and magnetic fields, respectively, and \( \mathbf{e}_l \) is a unit vector pointing to the direction of the electric field in transverse plane.

In the coupled mode theory, we assume the perturbed field maintains its original mode profile, thus the ansatz for perturbed mode is

\[
\begin{align*}
\mathbf{E}_{n,p}(\mathbf{r}) &= a_n(z) E_n(x, y) \mathbf{e}_l, \tag{A.9a} \\
\mathbf{H}_{n,p}(\mathbf{r}) &= a_n(z) H_n(x, y) (\mathbf{e}_z \times \mathbf{e}_l), \tag{A.9b}
\end{align*}
\]

where \( a_n(z) \) is a \( z \)-dependent envelope multiplied to the original mode profile of the unperturbed mode.
turbed field. From the mode profiles, we can calculate the average field energy by

\[ P_0 = \frac{1}{2} \int \int \Re \left[ E_n(x, y) \times H_n^*(x, y) \right] \text{d}x \text{d}y. \]  

(A.10)

Also for the polarization from perturbation \( P_n(r) \) in linear medium can be written as

\[ P_n(r) = \varepsilon_0 \sum_{m=1}^{M} \Delta \varepsilon_n(r) a_m(z) E_m(x, y) e^{i \beta_m z}, \]  

(A.11)

where \( \Delta \varepsilon_n(r) = \varepsilon(r) - \varepsilon_n(r) \) with \( \varepsilon(r) \) being the permittivity profile of the unperturbed structure, and \( M \) is the total number of waveguides in the lattice.

Now if we insert Equation (A.8), Equation (A.9) and Equation (A.11) into Equation (A.6) and use the relation in Equation (A.10), we get

\[ i \frac{\partial}{\partial z} a_n(z) + k_n a_n(z) + \frac{\varepsilon_0 \omega}{4 P_0} \sum_{m=1}^{M} \int \int \Delta \varepsilon_n(r) a_m(z) E_n(x, y) E_m(x, y) \text{d}x \text{d}y = 0. \]  

(A.12)

Now if we define the coupling constant \( C_{m,n} \) from the waveguide \( m \) to \( n \) by

\[ C_{m,n} = \frac{\varepsilon_0 \omega}{4 P_0} \int \int \Delta \varepsilon_n(r) E_n(x, y) E_m(x, y) \text{d}x \text{d}y, \]  

(A.13)

we can rewrite Equation (A.12) in a more concise manner:

\[ i \frac{\partial}{\partial z} a_n(z) + k_n a_n(z) + \sum_{m=1}^{M} C_{m,n} a_m(z) = 0. \]  

(A.14)

If we just consider the coupling of a waveguide with its next neighbors and use \( \beta_n = k_n + C_{n,n} \) to represent a generalized propagation constant, Equation (A.14) further reduces to the coupled mode equation

\[ i \frac{\partial}{\partial z} a_n(z) + \beta_n a_n(z) + C_{n-1,n} a_{n-1}(z) + C_{n+1,n} a_{n+1}(z) = 0. \]  

(A.15)

### A.1.2 Simulation of synthetic lattice and comparison with experiment

Here, we provide simulations in comparison with representative experimental results presented in Chapter 2. The simulations made use of the fast-fourier transform (FFT) function in Matlab to solve the NLSE for the signal - the input signal and pump spectra were transformed into the time-domain, then the time-dependent cross-phase-shift experienced by the signal was applied. Finally the signal was transformed back into the frequency domain to obtain the output spectrum. This approach assumes that dispersion can be ignored, as described above, but can include the finite bandwidths of the pumps so that some of the resulting broadening of the individual signal channels can be seen. Simulated results without this broadening are also shown.
Figure A.2: (a) Experimental result, (b) simulated result with realistic pump bandwidths, and (c) ideal case for diffraction of a single-site input under the influence of two pumps with a frequency separation of 100GHz ($A_1 = A_2$, giving rise to local coupling $C_1$). (d) Fidelity between measured and ideal spectrum as a function of pump power.

Figure A.2(a) shows the measured result for the simple case of two pumps separated by 100GHz and excitation of a single input site in the signal spectrum. Figures A.2(b,c) shows the corresponding simulations, where the range of pump powers has been tuned to match the experimental result. The two contain very similar diffraction patterns as expected. Figure A.2(d) shows the fidelity between the measured spectrum and an ideal spectrum as a function of power. This ideal spectrum is taken from a simulation with narrow-band pumps, so that there is no broadening of the individual channels. The signal spectrum is divided up into 100GHz broad channels and the total power within each channel is found, then each spectrum is normalised to sum to 1. The fidelity $F$ is calculated according to $F = \sum_n |a_n^{exp}|/|a_n^{ideal}|$ where the absolute values of the experimental amplitudes $|a_n^{exp}|$ are the square-roots of the intensities. This fidelity decreases slightly and fluctuates at higher pump power, because there the experimental spectra are affected by factors such as broadening from the finite-bandwidth pumps, the third-order dispersion of the fibre, and the introduction of Raman scattered noise from the pumps. However the fidelity remains > 95% over the range of pump powers used.

Figure A.3 shows the four cases where a weaker third pump is added to introduce 2nd and 3rd order hopping, with the phase of the third pump shifted by $0$, $\pi/2$, $\pi$, and $3\pi/2$ radians. The top row shows the experimental results and the bottom row the simulations. In each case qualitatively very similar behavior is observed, though at higher pump powers some discrepancies can be seen. Partly this is because of the broadening of individual signal channels due to the finite bandwidths of the pump channels. Imperfections could also result from the small third-order dispersion of the nonlinear fiber, or from phase errors between the three pumps, originating from the original mode-locked laser pulse, the amplifier, or the spectral pulse shapers.

Figure A.4 shows the results for the spectral discrete Talbot effect using only 1st order coupling (two pumps) with a real coupling coefficient, with input signals which are periodic every 2, 3, or 4 lattice sites. The experimental results (top row) show the formation of images as predicted by the simulations (bottom row), although the experimental images show varying degrees of distortion; in particular, some occupied sites have higher intensities than others, and some sites which should not be occupied contain some background. Since these measurements make use of a large number of input sites for the signal, spread over a wide bandwidth,
Figure A.3: Experimental (top row) and simulation (middle and bottom rows) results when a third pump $A_4$ is used to introduce 2nd and 3rd order coupling. $A_1 = A_2, A_4 = 0.15A_1 e^{i\theta}$ and $C_3 = C_2 = 0.15C_1 e^{i\theta}$. The middle row includes the broadening due to finite pump bandwidths, the bottom row is the ideal case. The phase $\theta$ of the third pump is set to (a) 0, (b) $\pi/2$, (c) $\pi$, and (d) $3\pi/2$ radians.
Figure A.4: Experimental (top row) and simulation (middle and bottom rows) results for the spectral discrete Talbot effect with periodicities (a) $N = 2$, (b) $N = 3$, and (c) $N = 4$. Two pump frequencies are used ($A_1 = A_2$, giving rise to local coupling $C_1$). The middle row includes the broadening due to the finite pump bandwidths, the bottom row is the ideal case.
phase-errors between input sites are an additional source of experimental imperfection. The first spectral pulse shaper in the setup was used to attempt to correct the phase of each input site, relative to its neighboring sites, but some residual errors are likely to have been present.

**Figure A.5:** (a) Experimental result, (b) simulated result with finite pump bandwidths, and (c) simulated ideal case for the $N = 3$ discrete Talbot effect with an imaginary coupling coefficient to break time-reversal symmetry $A_1 = iA_2$, $C_1 = -i$. (d) Experimental result, (e) simulated result with finite pump bandwidths, and (f) simulated ideal case for the $N = 5$ discrete Talbot effect, requiring a 1st and 2nd order hopping ($A_1 = 2A_2 = A_3$, $C_1 = C_2$).

Figures A.5(a) and (b) show the experimental and simulated result for the $N = 3$ discrete Talbot effect when a $\pi/2$ phase-shift is applied to one of the pumps, resulting in an imaginary hopping coefficient between neighboring waveguides. The propagation is asymmetric and displaced images appear - good agreement is seen between experiment and simulation.

Figures A.5(c) and (d) show the experimental and simulated result for the $N = 5$ discrete Talbot effect, which requires a 2nd order coupling coefficient introduced by a third pump in order for an image to form. There is good agreement between experiment and simulation, with a clear image of the input state occurring.

**Theoretical comparison for quantum walks in synthetic dimensions with gauge field**

Here we also provide a comparison between experiment and theory for the results presented in Figure 2.10 on the quantum walks in triangular synthetic lattices incorporating artificial gauge fields.

The experimental results shown in Figures 2.10(a,c,e) are replicated in Figures A.6(a,c,e), where on the right side, in Figures A.6(b,d,f) we provide the corresponding theoretical comparisons calculated by the tight-binding evolution using coupled mode theory.
Figure A.6: Comparison between experimental results of quantum walk in triangular synthetic lattice with a gauge field, where $\alpha = \pi/2$. (a,c,e) Experimental results replicated from Figures 2.10(a,c,e), respectively. (b,d,f) Theoretical comparison calculated by coupled mode theory.
Figure A.7: Comparison between experimental results of quantum walk in triangular synthetic lattice with a gauge field, where $\alpha = -\pi/2$. (a,c,e) Experimental results replicated from Figures 2.10(b,d,f), respectively. (b,d,f) Theoretical comparison calculated by coupled mode theory.
Similarly, the experimental results shown in Figures 2.10(b,d,f) are replicated in Figures A.7(a,c,e), respectively, where on the right side, in Figures A.7(b,d,f) we provide the corresponding theoretical comparisons calculated by the tight-binding evolution using coupled mode theory.

### A.1.3 Lanczos algorithm

While the Lanczos transformation is in general applicable to both Hermitian and non-Hermitian Hamiltonians [144], for reasons of brevity we will in the following consider only the Hermitian case. For an \( m \times m \) Hamiltonian \( H \), the algorithm is comprised of the following steps [144]:

1. For a unit vector \( v_1 \) in an \( m \)-dimensional Hilbert space, run the following iterations:
   
   (a) Let \( w'_1 = Hv_1 \).
   
   (b) Calculate \( \epsilon_1 = (w'_1)^\dagger v_1 \equiv (Hv_1)^\dagger v_1 \).
   
   (c) Let \( w_1 = w'_1 - \epsilon_1 v_1 \).

2. For \( j = 2, \ldots, m \):
   
   (a) Let \( C_{j-1} = ||w_{j-1}|| \), where \( ||.|| \) denotes the Euclidean norm.
   
   (b) If \( C_{j-1} \neq 0 \), then let \( v_j = w_{j-1} / C_{j-1} \); otherwise choose a unit vector \( v_j \) such that it is orthogonal to all \( v_1 \) to \( v_{j-1} \).
   
   (c) Let \( w'_j = Hv_j \).
   
   (d) Calculate \( \epsilon_j = (w'_j)^\dagger v_j \equiv (Hv_j)^\dagger v_j \).
   
   (e) Let \( w_j = w'_j - \epsilon_j v_j - C_{j-1} v_{j-1} \).

3. Let \( V \) be a matrix, the columns of which are \( v_1, \ldots, v_m \).

4. Let \( H^{[3]} \) be formed by the parameters \( \epsilon_j \) and \( C_j \) such that:

\[
H^{[3]} = \begin{bmatrix}
\epsilon_1 & C_1 & 0 & 0 & \ldots & 0 \\
C_1 & \epsilon_2 & C_2 & 0 & \ldots & 0 \\
0 & C_2 & \epsilon_3 & C_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \epsilon_m
\end{bmatrix}
\]

Importantly, in our work, we specially tailor the tri-diagonalization by choosing a specific \( v_1 = |m_{d}\rangle \), where \( |m_{d}\rangle \) represents the anchor site in the multi-dimensional lattice. As outlined in Chapter 3, this step is crucial to let the first site of the 1D lattice exhibit the same dynamics as the anchor site \( m_{d} \) in the multi-dimensional lattice. We note that various other tri-diagonalization algorithms, including the Householder transformation previously applied for isospectral mapping [132, 136], do not allow for such an explicit choice.
A.2 Experimental methods

A.2.1 Waveguide fabrication and measurement

In this part, we provide some details on experimental approach of fabricating and characterizing femto-second laser writing waveguides. Due to the similarity in experimental approach taken, this section partially involves material from my Master’s thesis [194].

Fabrication using femto-second laser writing

Optically transparent materials normally do not absorb light in the visible or even near-infrared spectral range. This attributes to the large band-gap between the valence and conduction bands compared to the energy of a single photon. However, when it comes to very intensive light, nonlinear absorption can be initiated through nonlinear ionization (multiphoton absorption and tunneling ionization) and avalanche ionization. Such effects bind the bandgap with high cumulative photon energy and create an electron-ion plasma in the volume where they occur. Permanent structural changes can be induced as the plasma recombines and its energy is dissipated [196]. A change of refractive index is one kind of such structural changes, thus the above effects can be used to modify the refractive index of transparent materials.

Experimentally the high intensity required for the nonlinear absorption effects can be achieved by tightly focusing a pulsed laser beam. The tight focusing and the nonlinear nature of the absorption confines the absorption volume within the focal volume. Therefore one can focus the laser inside the bulk of the material to control the volume of refractive index modification without causing absorption at the surface [197]. Through moving the substrate using a 3D translation stage, one can fabricate a variety of waveguide geometries. Using this technique, waveguides have been achieved in a variety of bulk materials such as fused silica [198] and chalcogenides [199].

As shown in Figure A.8, we write waveguides in glass along the direction that light is guided, denoted by z, while the writing laser propagates along y direction. As seen in Figure A.8, the mode profile is elliptical in shape and typically about 20 µm in width. The induced refractive index modification (in the order of $10^{-4}$ to $10^{-3}$) can be controlled by properly choosing the writing parameters. Pulse duration, pulse energy and writing velocity are among the parameters influencing the amplitude of refractive index changes, while the beam shaping method is a key factor that affects the exact shape of the refractive index gradient.

Now we provide more technical details on the fabrication: A Titanium:sapphire amplifier system (Coherent Mira/RegA) supplied 150fs pulses with an energy of 250 nJ at the carrier wavelength of 800 nm with a repetition rate of 100 kHz. A microscope objective (0.35 NA) was employed to focus these pulses into the bulk of a 100 mm long fused silica sample (Corning 7980), which in turn was translated at speeds of approximately 100 mm min$^{-1}$ by a high-
Figure A.8: Schematic of direct writing waveguide using femto-second laser and a representative image of the mode profile.

precision positioning system (AEROTECH ALS 130). The exact parameters of incripted waveguides do vary from sample to sample, yet here we outline the typical values: With their index contrast on the order of $7 \times 10^{-4}$, the resulting waveguides exhibit mode field diameter of approximately $10.4 \mu m \times 8 \mu m$ at the probe wavelength of 633 nm (Helium-Neon laser). Propagation losses and birefringence were estimated at 0.2 dB cm$^{-1}$ and $10^{-7}$, respectively. Depending on the exact type of glass, the losses induced in the waveguides are different. However, for the same sample, the values of loss that cause exponential decays of the light fields can be considered as equal in all waveguides. Therefore the exact value of this loss does not influence the experimental results.

**Measurement of light evolution in waveguides via fluorescence microscopy**

To observe the evolution of light at each propagation distance we use a fluorescence microscope. The basic idea of this method is to visualize the guided light via fluorescence of color centers [142].

During the writing of waveguide using femtosecond laser, in addition to an increase of refractive index, non-bridging oxygen hole color centers (NBOHC) can be formed [200]. When light is guided in the waveguides fabricated by this method, the intensity of fluorescence generated from the NBOHC can be considered approximately linear to the intensity of light guided in the structure. Therefore one can easily observe the whole spatial guiding profile through imaging at each propagation distance. For similar writing parameters of the femtosecond laser, the life time of formed NBOHC mainly depends on the material of substrate. In our experiments, the fluorescence microscopy is performed within a few days after the waveguide inscription, which is much shorter than the life time of NBOHC.

In Figure A.9, we show the setup of a fluorescence microscope. The laser light from a He-Ne laser ($\lambda = 633$ nm) is launched into the waveguides using a microscope objective, in which
Figure A.9: Schematic of the experimental setup using a fluorescence microscope to measure light evolution in waveguides.

the fluorescence of about $\lambda = 650$ nm is excited. We put a microscope at the top view of the sample together with a CCD camera to image the fluorescence. By translating the microscope along the propagation direction of light in the waveguides we can obtain the fluorescence image of the waveguide structure. In the microscope, a long pass filter is used to filter out the stray light with the same wavelength as the illumination laser, for example the diffused light from the edge of the sample or the stage where the sample is mounted.

Figure A.10: A fluorescence microscopy image of an integrated 50/50 beam splitter when light is guided mainly along $+z$ direction, as an example.

In Figure A.10 we show an example of image obtained from the fluorescence microscopy by putting together a series of images from the microscope at each propagation distance. The structure is an integrated 50/50 beam splitter.

Control of coupling and detuning parameters

The coupling constants between adjacent waveguides and the propagation constant of each waveguide are considered in this thesis. Here we show how we obtain our targeting values in these two parameters in the fabrication.
The basic principle to control the coupling constants is to vary the distances between parallel waveguides. Since the field in each waveguide decays exponentially in space, the coupling constant also exponentially depends on the separation of waveguides. The principle of manipulating the propagation constants is to use different writing velocities in the writing process, such that different energies accumulate in the focal area of the laser beam.

In order to achieve the precise control, we perform extensive parameter scans with waveguide couplers. Here we assume a general waveguide coupler with two propagation constants $\beta_1$ and $\beta_2$ for each channel and a coupling constant $C$ (assume $C_{1,2} = C_{2,1} = C$). When injecting light from one channel, we can obtain its normalized power evolution $P_1(z)$ and $P_2(z)$ of each channel by solving the coupled mode equation

$$
P_1(z) = \cos^2(g z) + \left(\frac{\Delta\beta}{2}\right)^2 \frac{\sin^2(g z)}{g^2},$$
$$P_2(z) = \frac{C^2}{g^2} \sin^2(g z),$$

where $\Delta\beta = |\beta_1 - \beta_2|$ is the relative propagation constant and the parameter $g$ is defined by $g = C^2 + \left(\frac{\Delta\beta}{2}\right)^2$. One can see that, if we know the evolution of power in the two channels of the coupler, we can determine the coupling constant $C$ and the relative propagation constant $\Delta\beta$. Experimentally the evolution of powers can be easily obtained by the image from fluorescence microscopy.

Under a set of certain fixed parameters of the writing laser, we firstly write several non-detuned waveguide couplers ($\Delta\beta = 0$) with different separations $d$ under a certain global writing velocity $v_0$. From Equation (A.16) we can easily obtain the coupling constant in each coupler. A representative result is shown in Figure A.11. One can see that, the relation of coupling constant versus waveguide separation fits very well with an exponential curve, which is consistent with the exponential nature of the field decay.

![Figure A.11: Representative experimental results of the coupling constant $C$ dependence on waveguide separation $d$ from parameter scan.](image-url)

Where $y = 3.82895 e^{-0.23591 x}$.
Detuned couplers ($\Delta \beta \neq 0$) are used to determine the non-trivial relation between the relative propagation constant $\Delta \beta$ with the writing velocity $v$. We fabricate a series of couplers with one channel in each coupler written by a fixed writing velocity $v_0$, while for the other one we vary the writing velocity around $v_0$. Exploiting Equation (A.16) for the power evolutions in different couplers, we can obtain a relation of the relative propagation constant $\Delta \beta$ with the relative writing velocity $\Delta v = v - v_0$ (for an example, see Figure A.12). The relation fits well with an exponential curve.

![Figure A.12: Representative experimental results of relative propagation constant $\Delta \beta$ v.s. the relative writing velocity $\Delta v$ in the parameter scan.](image)

$$y = 0.68875 e^{-0.00526x} - 0.72074$$

A.2.2 Metasurface fabrication and characterization

In this part we provide more details on the fabrication and characterization of the metasurface used in Chapter 4.

Metasurface fabrication

An 792-nm-thick poly-crystalline silicon (Poly-Si) thin film is prepared on a four-inch quartz wafer (thickness 500 $\mu$m) via low-pressure chemical vapor deposition (LPCVD) in horizontal tube furnace. A layer of Polymethyl Methacrylate (950K PMMA A4) covers the sample surface as a positive tone resist via a spinner. Then the sample is exposed to electron beam lithography (EBL, model JEOL 9300FS) to obtain the pattern of the metasurfaces. Since silicon is only semi-conductive, before the EBL a 10 nm layer of chromium (Cr) is evaporated on top of the photo-resist for charge dissipation. After the EBL, wet chemistry removal of the Cr layer and sequential development are performed. Then, another 20-nm-thick Cr layer is evaporated on top of the poly-Si layer as a mask for the etching, and we use anisotropic reactive ion etching (RIE) to remove poly-Si in the regions that are not covered by Cr. Finally, we use wet etching to remove the Cr layer.
Characterization of the metasurface

Before going into quantum experiments, we perform a classical measurement to characterize the actual transfer matrix that is exactly the same for quantum measurements and to verify that the system works properly. The setup for such an measurement is shown in Figure A.13(a). A CW laser at 1570.6 nm wavelength is firstly linearly polarized by a Glan-Taylor prism (polarizer) and then launched into the waveguide which we use for photon pair generation. Starting from the SPDC waveguide, the laser beam goes through practically the same optical elements as the V-polarized photons in the pair of the quantum light, to ensure a similar spatial mode. Then we transform the V polarized component of the laser beam with a quarter-wave-plate (QWP) to prepare many different states and slightly reduce the waist of the laser beam by a lens. The spot of the laser beam illuminated on the metasurface has a diameter of approximately 1 mm, which is smaller than the size of the metasurface (2 mm × 2 mm). We then use an IR detector to read out the power from each fiber port. By doing so, the transfer matrix is fully characterized. Then, we prepare another set of polarization states by the same QWP, and reconstruct the input polarization states using the transfer matrix. The experimental results of powers measured from each port are shown in Figure A.13 (b) with the dots for experimental data and the curves for theoretical predictions by the characterized transfer matrix.

The characterized transfer matrix of the metasurface is

\[
\mathbf{T} = \xi \begin{bmatrix}
1.000 & -0.3227 - 0.7070i \\
1.2022 + 0.2874i & 0.6484 \\
0.1781 + 0.1282i & 0.7935 \\
-0.2692 - 0.8502i & 0.2683 \\
-0.6830 + 0.0063i & 0.8625 \\
0.1971 - 0.5392i & 1.1189
\end{bmatrix},
\]

where \(\xi\) is a scaling factor and each row is essentially a projective basis of one of the 6 spots from the metasurface for classical or single-photon measurements. The two columns denote H and V polarizations, respectively. Note that each row can be gauge transformed up to a phase factor. The \(N\)-photon transfer matrix is basically the Kronecker product of \(\mathbf{T}\) as \(\mathbf{T}^{\otimes N} = \mathbf{T} \otimes \cdots \otimes \mathbf{T}\).

A.2.3 Two-photon state generation and characterization

Here we provide details on the generation and characterization of the two-photon polarization state in the experiment presented in Chapter 4.
Figure A.13: (a) Experimental setup for classical measurements. (b) Powers measured from different ports after the metasurface (dots) compared to theoretical predictions (curves) based on the characterized transfer matrix. The error bars are not plotted as they are smaller than the marker size.
Generation of two-photon cross-polarization in a nonlinear waveguide

We use photon pairs generated via spontaneous parametric down conversion (SPDC) via a nonlinear waveguide as shown in Figure A.14(a). More specifically, we obtain orthogonally polarized photon pairs from a type-II phase-matched periodically-poled lithium niobate (PPLN) waveguide. The PPLN grating period is around 9.4 \( \mu \text{m} \) for a 785.3-nm pumped degenerate spontaneous parametric down-conversion (SPDC) process at 120°C (design value) according to the oeo quasi-phase matched (QPM) condition in Ti-diffused LiNbO\(_3\) single (fundamental)-mode waveguides. The fabrication process for the PPLN waveguides mainly involves Ti strips thermal in-diffusion followed by electric field poling in a LiNbO\(_3\) crystal. The waveguides were fabricated using the titanium thermal diffusion (TTD) method in a 31-mm long, 10-mm wide, and 0.5-mm thick \( z \)-cut LiNbO\(_3\) crystal. Initially, an array of 7 \( \mu \text{m} \)-wide, 90 nm-thick Ti strips was coated on the \(-z\) surface of the crystal along the crystallographic \( x \) axis using the standard lithographic and lift-off process. The titanium diffusion process was then performed in a high-temperature furnace at 1035°C with a constant oxygen flow for 12 hours. Before the crystal poling, optical polish on the \(+z\) surface of the crystal was conducted to remove the domain-inverted layer on that surface formed during the high-temperature TTD process. Considering the possible fabrication errors, a QPM domain structure of multiple grating periods varying from 9.1 to 9.7 \( \mu \text{m} \), stepped by 0.1 \( \mu \text{m} \) (along the crystallographic \( y \) axis), was then implemented in the LiNbO\(_3\) waveguides using the standard electric field poling process. The device fabrication was then accomplished after the end faces of the crystal were optically polished.

On our experimental setup, the PPLN waveguide is heated up in a oven with closed-loop temperature control and is kept at 125°C in order to avoid photorefraction. The waveguide is pumped by a CW laser with a wavelength of 785.3 nm, with about 10 mW power launched into the waveguide by a microscope objective. The PPLN waveguide is designed for type-II phase matching, such that the horizontally (H) and vertically (V) polarized modes at around 1570.6 nm are phase matched. Therefore, we can obtain polarization entangled photon pairs with cross-polarized modes of H and V photons in a pair. The phase-matching is tunable by slightly changing the temperature, and we experimentally adjust it to degenerate frequencies for both photons in a pair. Since the H and V polarized modal profiles of photons in the waveguide are slightly different according to the transverse refractive index profile, we also perform a spatial filtering with a small iris [see Figure A.14(a)] in order to select the well-overlapped and central region out of the transverse cross section of the photon beams, where both modes have approximately homogeneous intensity and polarization.

Characterization of generated two-photon state

We apply the knowledge that type-II SPDC process is designed to be phase-matched and it generates a pair of photons which must be H and V cross-polarized, whereas the \( HH \) or \( VV \) photon pair generation processes are highly phase-mismatched and thus co-polarized pho-
Figure A.14: (a) Schematic setup for photon pair generation and interferometric characterizations, where HWP stands for half-wave-plate, PPLN is periodically poled lithium niobate, PBS is polarizing beam splitter, LP filter is long-pass filter (above 1100 nm can pass), BP filter is bandpass filter (1570 nm central wavelength with 12 nm FWHM) and QWP is quarter-wave-plate. (b) Experimental results of Hong-Ou-Mandel interference: the dots are experimental measurements and the solid curve represents a Gaussian fitting. The dashed line denotes half of the maximum counts given by the fitting, where a dip below this line exceeds the classical limit.
ton pairs are practically never generated. Accordingly, we predict that only the central four elements in the two-photon density matrix (in \{HH, HV, VH, VV\} basis) are non-zero. Moreover, with the symmetry of the density matrix due to the use of the indistinguishable detection scheme (see a related discussion on indistinguishable detection in the supplementary material section 1.2 of Reference [58]), we theoretically predict the reduced density matrix of the photon-pair state to be

\[
\hat{\rho} = \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & \eta & 0 \\
0 & \eta & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

(A.18)

where \(\eta\) is a real-valued number describing the spectral overlap of the photons in a pair (for details see supplementary material section 3 of Reference [58]). The absolute value of \(\eta\) is unity for photons with identical spectra, and accordingly identical temporal profiles. On the other hand, in the HOM measurement the mismatched time delay between the photons results in the spectral phase tilt, and then \(\eta\) approaches zero. Our state measurement experiment is performed when the time delay between the photons is best matched. At this point, it can be demonstrated that \(\eta\) is exactly the depth of the dip in the HOM interference (i.e. the normalized HOM counts value is \(1-\eta\)). The experimentally measured HOM dip of our photon pair source is 58% in depth with respect to the fully mismatched case, thus the theoretically predicted density matrix is

\[
\hat{\rho} = \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0.58 & 0 \\
0 & 0.58 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

(A.19)

We calculate the concurrence [32] of our prepared state, which is a measure between 0 and 1 of the degree of entanglement, as

\[
C_c = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} = 0.58,
\]

(A.20)

where \(\lambda_i\)s are the square roots of eigenvalues (in decreasing order) of \(\hat{\rho} \hat{\rho}_2\) with \(\hat{\rho}_2 = (\hat{\sigma}_y \otimes \hat{\sigma}_y) \hat{\rho}^\dagger (\hat{\sigma}_y \otimes \hat{\sigma}_y)\). We also calculate the purity of state,

\[
p = \Tr(\hat{\rho}^2) / [\Tr(\hat{\rho})]^2 = 66.82%.
\]

(A.21)

For such a four-dimensional state, \(p\) can take the values ranging from 0.25 for fully mixed (incoherent) state to 1 for a pure (coherent) state. Thus the 66.82% purity means that the state we use is a partially coherent state, which is also consistent with the intermediate degree of entanglement according to Equation (A.20). Such a mixed state instead of an ideally pure one (i.e. \(\eta\) does not approach unity at the HOM dip) is likely to be caused by the different spectra of the H and V photons and hence an imperfect spectral overlap even when the time delay between the H and V photons is matched. The joint spectra of the photon pairs are determined
by the phase-matching condition of the SPDC waveguide and are affected by fabrication errors of the waveguide, in particular in the periodic poling.

For reference, if the delay between the photons is not matched temporally (i.e. out of the dip in HOM), then a pair of photons cannot interfere but they are still cross-polarized (H and V). The reduced density matrix in this case is

\[
\hat{\rho}' = \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \tag{A.22}
\]
This page is intentionally left blank.
Bibliography


