Geometry & Cluster Algebras: Finite Type Classification & Double Bruhat Cells

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For my grandparents

Declaration

The work in this thesis is my own except where otherwise stated.

Zhenyi Wang

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I would like to pay my respect and acknowledge the traditional custodians of the land on which this thesis was written, and also pay respect to Elders both past and present.

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Abstract

In 2000, Fomin and Zelevinsky introduced a new language called cluster algebras for describing rings with certain combinatorial structures. Cluster algebras enjoy a variety of nice properties such as well-established collection of classification results and interesting geometric properties with the upper cluster algebra. We approach cluster algebras first from the perspective of operations on quivers, then reacquaint ourselves with a more general definition. We then present the classification of cluster algebras of finite type and explore cluster algebra structures on the ring of regular functions of double Bruhat cells.

Preliminaries

0.1 Some algebraic geometry

Our purpose in this section is to set up enough algebraic geometry for use. We will only mention projective varieties here whilst assuming knowledge of affine varieties. We will follow Hartshorne' *Algebraic Geometry* [Har77].

Definition 0.1.1. Projective space over an algebraically closed field k is defined as

$$\mathbb{P}^n_k = \mathrm{Proj} k[t_0, \ \dots \ t_n] \tag{1}$$

Alternatively, we may think of $\mathbb{P}^n_k$ as the parameter space of lines through the origin in k^{n+1}. In other words:

Definition 0.1.2. $\mathbb{P}^k_n = \{\text{Lines through the origin in } k^{n+1}\} = {}^{k^{n+1} \setminus \{0\}}\!\big/_{k^\times}$

We denote the equivalence class of a point $(x_0, \ \dots \ x_n)$ to be $(x_0 : \ \dots \ : x_n)$.

To give $\mathbb{P}^n_k$ the structure of a variety, we begin by observing:
Let $0 \leqslant i \leqslant n$. Let $U_i := \{(x_0 : \ \dots \ x_n) \in \mathbb{P}^n_k \ ; \ x_i \neq 0\} \subset \mathbb{P}^k_n$.
We observe that the union over all U_i is indeed $\mathbb{P}^n_k$. There are bijections:
$U_i \xrightarrow{\sim} \mathbb{A}^n_k$, given by $(x_0 : \ \dots \ x_n) \mapsto (\frac{x_0}{x_i}, \ \dots \ \frac{\hat{x_i}}{x_i}, \ \dots \ \frac{x_n}{x_i})$. (The $\hat{}$ notation means to omit that element).

This map induces an isomorphism of varieties to $(\mathbb{A}^n_k, \ \mathcal{O}_{\mathbb{A}^n_k})$. We now denote this affine variety $(U_i, \ \mathcal{O}_{U_i})$. We may now give projective space a topology using this connection to affine space.

Definition 0.1.3 (Zariski Topology)**.** A set $U \subset \mathbb{P}^n_k$ is open if $U \cap U_i$ is open in U_i for all i. This defines a topology on $\mathbb{P}^n_k$ as long as $\forall i \ \neq j$, the set $U_i \cap U_j$ is open is in U_i and U_j. Indeed, $U_i \cap U_j = D(\frac{x_j}{x_i}) \subset U_i$ (flip the quotient for U_j).

One may check this defines a topology but could $\not\Rightarrow$ should. We now see that projective space has the structure of a topological space but not yet the structure of a variety. To do this, we will show that $\mathbb{P}^n_k$ is a ringed space with a sheaf of functions. We will give projective space a structure similar to coordinate rings of an affine variety, we will call this ring the **homogeneous coordinate ring**.

Recall from commutative algebra that a homogeneous polynomial $f \in k[x_0, \dots x_n]$ of degree d is one such that $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots x_n)\ \forall \lambda \in k \setminus \{0\}$ s.t. $f(\lambda x_0, \dots, \lambda x_n) = 0 \iff f(x_0, \dots, x_n) = 0$.

Remark 0.1.1. The zero polynomial is homogeneous of every degree. This lets us write $k[x_0, \dots, x_n]_d$ for the submodule inside $k[x_0, \dots x_n]$ consisting of all homogeneous polynomials of degree d.

Remark 0.1.2. $k[x_0, \dots, x_n] = \bigoplus\limits_{d \geqslant 0} k[x_0, \dots, x_n]_d$
This gives $k[x_0, \dots, x_n]$ a natural grading.

Definition 0.1.4. Let P be a point in $\mathbb{P}^n_k$.
Let $\mathcal{O}_{\mathbb{P}^n_k, P} := \left\{ \frac{f}{g} \in k(x_0, \dots x_n)_d;\ g(P) \neq 0 \right\}$ be the local ring corresponding to P.

Definition 0.1.5. $\mathcal{O}_{\mathbb{P}^n_k}(U) := \bigcap\limits_{P \in U} O_{\mathbb{P}^n_k, P}$.

Now we want to verify that this satisfies the sheaf axioms:

Lemma 0.1.1. $\mathcal{O}_{\mathbb{P}^n_k}$ *defines a sheaf of rings on* $\mathbb{P}^n_k$.

Proof. It is clear by construction that $\mathcal{O}_{\mathbb{P}^n_k}$ is a presheaf. We check the sheaf conditions:

- Suppose $U = \bigcup\limits_k O_k$ an open cover. Let $s \in \mathcal{O}_{\mathbb{P}^n_k}(U)$ such that the restriction $Res^U_{O_k}(s) = 0\ \forall k$. Thus $s|_{O_k} = \frac{f}{g}|_{O_k}(P) = 0\ \forall p \in O_k$. This implies that $s = 0$ on U.

- Using the open cover as above. Suppose that $\exists s_k \in \mathcal{O}(O_k) \forall k$ such that $s_i|_{(O_i \cap O_j)} = s_j|_{(O_i \cap O_j)}$ for all i, j.
Then $\exists f, g \in k[x_0, \dots x_n]_d$ such that $\frac{f}{g}|_{O_i \cap O_j}(P) = \frac{f}{g}|_{O_i \cap O_j}(P)\ \forall P \in O_i \cap O_j$
This suggests we may simply pick $s = \frac{f}{g}$.

$\square$

Definition 0.1.6. A vanishing locus of homogeneous polynomials $f_1, \ldots f_m$, $V_+(f_1, \ldots f_m) = \{(x_0 : \ldots : x_n); \forall j : f_j(x_0, \ldots, x_n) = 0\}$ is closed.

Definition 0.1.7. A projective variety is a closed set of $\mathbb{P}^n_k$ in the Zariski topology.

Remark 0.1.3. The fact that $\mathbb{P}(\wedge^d V) \simeq \mathbb{P}^N_k$ follows after applying the Veronese then Segre embedding. We will not deal with it here; the proof can be found in Chapter 9 of Young Tableaux [Ful97].

0.2 The Grassmannian and Plucker embedding

We will approach Plucker Coordinates from the perspective of linear algebra. We will follow the approach used by Hassett's *Introduction to Algebraic Geometry* [Has07] but supplemented by the classic text on the subject - *Young Tableaux* by Fulton. [Ful97]

0.2.1 The Grassmannian as homogenous space

We start by fixing a basis of $\mathbb{C}^n$. How can we describe a k-dimensional subspace Σ inside $\mathbb{C}^n$? Well, we can pick a representative in matrix form as the column space of a $n \times k$ matrix of rank k. This choice is not however unique since if we performed row operations, we still obtain same subspace Σ. We call this matrix B and it looks like:

$$B^t = \begin{bmatrix} 1 & 0 & \cdots & \cdots & b_{1k+1} & \cdots & b_{1n} \\ 0 & \ddots & \cdots & \cdots & b_{2k+1} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & \cdots & \cdots & 1 & b_{nk+1} & \cdots & b_{nn} \end{bmatrix}$$

Or more generally, we can multiply a choice of representative B by an invertible $n \times n$ matrix A and the columns of AB will still span the same subspace. One way of thinking of this is as a group action of the general linear matrices taking representatives to representatives.

Let $X = Gr(k, n)$. Given a subspace Σ and basis $\{v_1, \ldots, v_k\}$ we define the exterior product map $\pi : V \to \bigwedge^k V$ by sending $\{v_1, \ldots, v_k\} \mapsto v_1 \wedge \cdots \wedge v_k$. However depending on our choice of basis, different elements in $\bigwedge^k V$ represent

Σ up to a non-zero scalar.

That is, given some other choice of basis $\{w_1, \ldots, w_k\}$, we can write each $w_i = \sum_{i \in [k]} c_{i,j} v_i$. This defines some $(c_{i,j}) \in GL_k$ such that $v_1 \wedge \cdots \wedge v_k = det(c_{i,j}) w_1 \wedge \cdots \wedge w_k$.

This means, setting $\Pi : \bigwedge^k V \to \mathbb{P} \bigwedge^k V$, the composition $\Pi \circ \pi(\Sigma) := [v_1 \wedge \cdots \wedge v_k]$ gives a unique identification of Σ. We call this composition the Plucker embedding and we denote it by Φ. Our discussion informs us that Φ is well defined.

Now consider $GL_n \curvearrowright X$ by $g < v_1, \ldots, v_>) :=< gv_1, \ldots, gv_k >$. GL_n also acts on $\bigwedge^k V$ (and therefore $\mathbb{P} \bigwedge^k V$) by $g \cdot v_1 \wedge \cdots \wedge v_k = gv_1 \wedge \cdots \wedge gv_k$. It is easily verified that π and Π are GL_n equivariant maps.

Proposition 0.2.1. *The action of $GL_n \curvearrowright X$ is transitive.*

Proof. Pick the canonical basis of V given by $e_1, \ldots, e_n$. Let $E =< e_1, \ldots, e_d >$. For any two $dim(k)$ subspaces $U, W \subset V$, and picking bases $u_1, \ldots, u_k$ and $w_1, \ldots, w_k$ respectively, we will find an element of GL_n such that $g \cdot U = W$. Since each $u_j = \sum_{i \in [n]} a_{i,j} e_i$ and $w_l = \sum_{i \in [n]} b_{i,l} e_i$, we have elements $A, B \in GL_n$ such that $U = A \cdot E$ and $W = B \cdot E$. Then $W = BA^{-} \cdot U$. $\square$

Since the action is transitive, we can identify the Grassmannian as the projective variety GL_n/P where P is a parabolic subgroup stabilising X. The standard option is to choose

$$P = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1k} & x_{1k+1} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k1} & \cdots & x_{kk} & x_{kk+1} & \cdots & x_{kn} \\ & & & \vdots & \ddots & \vdots \\ & 0 & & x_{nk+1} & \cdots & x_{nn} \end{pmatrix} \right\}$$

This establishes the structure of an algebraic variety on the Grassmannian. Next we will embed this variety inside projective space and give a homogeneous coordinate ring with respect to this embedding.

0.2.2 Plucker coordinates

Definition 0.2.1. Let V be a $\mathbb{C}$-vector space. An element $x \in \bigwedge^m V$ is completely decomposible if there exists $\{v_i\}_{i\in[m]} \subset V$ such that $x = v_1 \wedge \cdots \wedge v_m$. x is partially decomposible if $x = v \wedge \tau$ for $v \in V$ and $\tau \in \bigwedge^{k-1} V$.

Proposition 0.2.2. *Let $x \in \bigwedge^k V$. Then the following are true:*

- *x is partially decomposible $\implies$ $x \wedge x = 0$.*
- *x is partially decomposible $\iff$ $\phi(x) : V \to \bigwedge^{k+1} V$ given by $v \mapsto v \wedge x$ has non-trivial kernel.*
- *$\{v_1, \ldots, v_m\}$ is a basis for $ker(\phi(x))$ $\implies$ $x = v_1 \wedge \cdots \wedge v_m \wedge \tau$, $\tau \in \bigwedge^{k-m} V$.*
- *x is completely decomposible $\iff$ $dim(ker(\phi(x))) = k$.*

Remark 0.2.1. One way of thinking of $\phi(x)$ is as a function that 'measures' the failure of x to be completely decomposible.

Proof. [Has07] Page 196 □

From the above proposition, we are able to prove that the Plucker embedding Φ is injective.

Proposition 0.2.3. *The Plucker embedding $\Phi : Gr(k, n) \to \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is injective.*

Proof. Let Σ be a k-dimensional subspace with a basis $\{v_1, \ldots, v_k\}$. The previous proposition suggests that span$\{v_1, \ldots, v_k\} = ker(\phi(v_1 \wedge \cdots \wedge v_k))$. Suppose $v_1 \wedge \cdots \wedge v_k = w_1 \wedge \cdots \wedge w_k$, $w_i \in \mathbb{C}^n$. Then $span\,\{v_1, \ldots, v_k\} = ker(\phi(v_1 \wedge \cdots \wedge v_k)) = ker(\phi(w_1 \wedge \cdots \wedge w_k)) = span\,\{w_1, \ldots, w_k)\}$. Thus $w_1, \ldots, w_k$ is a basis for Σ and so we are done. □

Pick the standard basis of V $\{e_1, \ldots, e_n\}$. Denote by $I_{k,n} := \{\mathbf{i} = (i_1, \ldots, i_k) |\ 1 \le i_1 < \cdots < i_k \le n\}$, the set of all strictly increasing sequences of length k in $[n]$. For each $\mathbf{i} \in I_{k,n}$, we define an element $e_{\mathbf{i}} = e_{i_1} \wedge \cdots \wedge e_{i_k}$. The canonical basis of $\bigwedge^k V$ can then be written as $\{e_{\mathbf{i}}\}$.

Definition 0.2.2. The Kronecker delta is:

$$\delta_{\mathbf{i},\mathbf{j}} = \begin{cases} 1 & \mathbf{i} = \mathbf{j} \\ 0 & \mathbf{i} \neq \mathbf{j} \end{cases} \tag{2}$$

Definition 0.2.3. Consider functions $\Delta_{\mathbf{i}} : \bigwedge^k V \to \mathbb{C}$ given by $\Delta_{\mathbf{i}}(e_{\mathbf{j}}) = \delta_{\mathbf{i},\mathbf{j}}$. We call $\Delta_{\mathbf{i}}$ Plucker coordinates.

Remark 0.2.2. By construction, Plucker coordinates form the standard basis on $(\bigwedge^k V)^*$.

Remark 0.2.3. Given some $\Sigma =< v_1, \dots, v_k >$, its image in $\bigwedge^k V$ under π is $v_1 \wedge \cdots \wedge v_k = \sum_{\mathbf{i}} a_{\mathbf{i}} e_{\mathbf{i}}$. By definition of the basis for the dual, we can write each $a_{\mathbf{i}} = \Delta_{\mathbf{i}}(v_1 \wedge \cdots \wedge v_k) = \Delta_{\mathbf{i}}(\pi(\Sigma))$. Naively, through pulling back Plucker coordinates along π, we want to view them as functions on Σ. However this function may not be well defined as it depends on the choice of basis for Σ. In fact the discrepancy is exactly the determinant of some invertible $n \times n$ matrix. As before, we can deal with this problem by considering the image of Σ in $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$.

0.2.3 The homogeneous coordinate ring of the Grassmannian

Lemma 0.2.1 ([Ful97] Section 8.2 Lemma 1)**.** *Let M and $N \in \mathcal{M}_{p\times p}(\mathbb{C})$ and $k \in [p]$. Then $det(M)det(N) = \sum det(M')det(N')$ for all pairs (M', N') obtained from M and N by exchanging a fixed set of k columns of N with any k columns of M, preserving the order of the columns.*

Theorem 0.2.1. *The Plucker embedding is a bijection from $Gr(k,n)$ to the subvariety of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ defined by the quadratic ideal Q (called the Plucker ideal) generated by:*

$(v_1 \wedge \cdots \wedge v_k) \cdot (w_1 \wedge \cdots \wedge w_k) - \sum_{i_1 < \cdots < i_d} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_d \wedge \cdots \wedge v_k) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_d} \wedge w_{d+1} \wedge \cdots \wedge w_k)$
where $v_i, w_i \in \mathbb{C}^n$. That is $\mathbb{C}[Gr(k,n)] \cong {Sym^{\cdot}(\bigwedge^k \mathbb{C}^n)}/{Q}$.

Proof. [[Ful97], with embellishments] Firstly Sylvester's Lemma tells us that any $\Sigma \in Gr(k,n)$ satisfies the quadratic equations in Q. Now let M and N be the minors of A with columns labelled $i_1, \dots, i_k$ and $j_1, \dots, j_k$ respectively.
In the other direction, pick a point $\mathbf{x}$ with Plucker coordinates $x_{i_1,\dots,i_k}$ of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ satisfying the equations in Q. Abusing notation, we can pick $\mathbf{x}$ such that some $x_{i_1,\dots,i_k} = 1$. We can associate a $n \times k$ matrix $A = (a_{t,s})$ given by:

$$a_{t,s} = x_{i_1,\dots,i_{s-1},t,i_{s+1},\dots,i_k} \text{ where } s \in [k],\ t \in [n]$$

Now we want to show that the image of A is some k-dimensional subspace, say, $\Omega \in Gr(k,n)$, with the property that under the Plucker embedding Ω corresponds to the point of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ with Plucker coordinates in $x_{j_1,\ldots,j_k}$.

We will do this by backwards induction. Let $I = (i_1, \ldots, i_k)$ and consider all $J = (j_1, \ldots, j_k)$ where $1 \leq j_1 < \cdots < j_k \leq n$. We will consider the submatrix of A with rows in J, say A_J. If $J = I$, then A_J is the identity matrix and $\det A_J = 1$. Thus A has rank k and therefore the image is also k dimensional. When I and J have $k-1$ entries in common, say by replacing some i_s by t, then A_J is the matrix which is the identity matrix in every row except the s^th row where the entry on the diagonal is $a_{s,t}$. This is exactly what we expect from the quadratic equation.

Now suppose I and J have l entries in common for some $l \in [k]$. Assume that any $x_{j'_1,\ldots,j'_k}$ is the Plucker coordinate of the corresponding minor of A if I and J' have $> l$ entries in common. Suppose some J and J' differ at some j_r. Thus $j_r \notin I$. We will assume that j_r occupies the first position of J that is reorder so that $j_r = j_1$. Then applying the quadratic equation to swap the single index j_r, we get $x_I \cdot x_J = \sum_h x_{i_1,\ldots,j_r,\ldots,i_k} \cdot x_{i_{r_h},j_2,\ldots,j_k}$. Now note that in each summand, the first term is exactly I with one index swapped and the second term has $r+1$ indices in common with I. Then the right hand side, by the inductive hypothesis, corresponds to relevant minors of A. Then using Sylvester's Lemma, x_J must correspond to the minor of A with rows in J. So we are done.

The above argument shows that the Plucker embedding is surjective. We have already shown that Plucker embedding is injective. Thus we are finished. $\square$

0.2.4 Flag varieties

0.3 Some representation theory

We will follow Fulton and Harriss' *Representation Theory: A First Course.* [Ful04] Let's begin by recalling some facts from Lie theory. The information here will be useful in Chapter 2 and Chapter 3.

Definition 0.3.1. A Lie algebra g is simple if it has no non-trivial ideals.

Definition 0.3.2. A Lie algebra $\mathfrak{g}$ is semisimple if it is a direct sum of simple Lie algebras.

Definition 0.3.3. A Lie group G is called semisimple if its Lie algebra $\mathfrak{g}$ is semisimple.

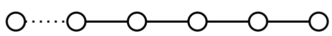
Let $\mathfrak{g}$ be a simple Lie algebra. Let Φ be its root system. Let $\mathfrak{t}$ be a maximal commutative subalgebra (with respect to inclusion). Then for any $\alpha \in \mathfrak{h}^*$ we have an associated root space $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \ [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$. We call α a root if $\mathfrak{g}_\alpha$ is non-trivial. Let $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ denote the corresponding Cartan subalgebra. We can take a root space decomposition of $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha)$.

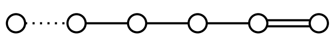
Also recall that we can define an adjoint action of a Lie algebra on itself by: $ad_X : \mathfrak{g} \to \mathfrak{g}$ where $ad_X(Y) = [X, Y] \ \forall Y \in \mathfrak{g} \ \forall X \in \mathfrak{g}$. The Killing form is the bilinear symmetric form (-,-) on $\mathfrak{g}$ given by (X,Y)= $tr(ad_X ad_Y)$. We then called a coroot of some root $\alpha \in \mathfrak{h}^*$ the element $H_\alpha \in \mathfrak{h}$ satisfying $\alpha(H) = (H_\alpha, H) \ \forall H \in \mathfrak{h}$.

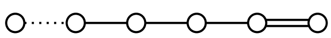
We can restrict (-,-) to the real vector space $V =< H_\alpha >_{\alpha\in\Phi}$ where it is positive definite. In particular the dual V^* spanned by $\alpha \in \Phi$ is a real vector space with inner product (-,-). We can then define the Cartan matrix of $\mathfrak{g}$ by setting $A = (a_{ij}) = (2\dfrac{(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)})$.

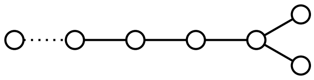
The Cartan matrix determines a Dynkin diagram $D = (V, E)$ where $V = [dim(\mathfrak{h})]$ and two vertices i, j are connected by an edge with multiplicity $a_{ij}a_{ji}$. An edge is oriented if $a_{ij}a_{ji} > 1$ and towards j when $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$. Dynkin diagrams classify Weyl groups and by extension, Lie algebras.

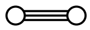
An important result in representation theory is that all finite dimensional simple Lie algebras over $\mathbb{C}$ can be classified by the following Dynkin diagrams :

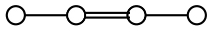
A_n 

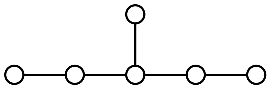
B_n 

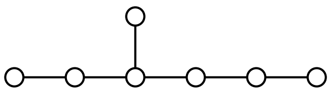
C_n 

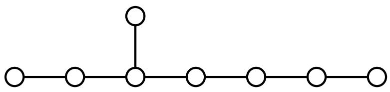
D_n 

G_2 

F_4 

E_6 

E_7 

E_8 

As we shall see, the above diagrams also play an important role in classifying certain cluster algebras.

Definition 0.3.4. A generalised Cartan matrix $A = (a_{ij})$ is an $n \times n$ integer matrix satisfying:

1. $a_{ii} = 2$
2. $a_{ij} \leq 0$ if $i \neq j$
3. $a_{ij} = 0$ iff $a_{ji} = 0$
4. $A = DS$ where D is diagonal and S is symmetric.

We say A is of finite type if all principal minors of A are positive.
We say A is decomposable if $\exists I \subset [n]$ such that $a_{ij} = 0$ for $i \in I,\ j \notin I$. A is indecomposable if it is not decomposable.

Here are some other results about $\mathfrak{g}$ which we shall use throughout this document. Let $\mathfrak{b}_+$ (respectively $\mathfrak{b}_-$) be the subalgebra of $\mathfrak{g}$ spanned by the Cartan $\mathfrak{h}$ and positive (resp. negative) root spaces. Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the maximal nilpotent subalgebra $[b_+, b_+]$ (resp. $[b_-, b_-]$). Let B_+, B_-, N_+, N_- be the images of these algebras under the exponential map. In the group G of $\mathfrak{g}$, we can construct a dense open (Zariski) subset $G^0 = N_- H N_+$.

Every element $x \in G^0$ admits a factorisation $x = x_- x_0 x_+$ where $x_- \in N_-$, $x_0 \in H$, and $x_+ \in N_+$. Note that we are picking representatives here. Fix a fundamental weight w_i. On G^0, we define a function $\Delta_i(x) = x_0^{w_i}$.

We can pick some global section which we will also call Δ_i whose restriction to G^0 coincides with the construction in the above paragraph. When $G = SL_n$, (and also GL_n although GL_n is not semisimple) this function is precisely the minor of x (as a matrix) for the first i rows and columns.

Definition 0.3.5. $\forall u, v \in W$, we define a regular function on G to be:

$$\Delta_{uw_i, vw_i}(x) := \Delta_i(u^{-1} x v)$$

Now this is not obviously well defined as we are really picking representatives of elements of the Weyl group. First we construct maps $\phi_i : SL_2 \to G$

$$\phi_i(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}) = x_i^+(t)$$

$$\phi_i(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}) = x_i^-(t)$$

where the $x_i^\pm$ are the one parameter subgroups $exp(te_i)$, where e_i are the Chevalley generators.

Now to pick our representatives in a consistent way, we set a simple reflection s_i of W to be:

$$\bar{s_i} = \phi_i(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) \tilde{s_i} = \phi_i(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

We will let $\bar{s_i}$ be the generators of the first copy of W in $W \times W$ and $\tilde{s_i}$ generators for the second. We will pick our representatives u and v in this way. Thus the generalised minor does not depend on u or v; only the weights.

Given a pair of permutations $u, v \in S_n$, we can consider cosets of the form $G^{(u,v)} = B_+ u B_+ \cap B_- v B_-$, where $B_\pm$ are the upper/lower triangular matrices. We will sketch some important properties of $G^{u,v}$ following Part 4 of [FZ98]. In particular, here we will sketch the proof that $G^{u,v}$ contains a dense open torus. The proofs of these results are quite technical; our brief description here omits a considerable amount of detail but some exposition on this subject is necessary to establish some groundwork for Chapter 3.

First, let us pick generators for G. Let $E_{i,j}$ denote the $n \times n$ matrix with 1 at the (i, j) entry and 0 elsewhere. For $i = 1, \ldots, n-1$, $t \in \mathbb{C} \setminus \{0\}$ define:

$$x_i(t) = I + tE_{i,i+1}$$
$$x_{\bar{i}}(t) = I + tE_{i+1,i}$$

For $i \in [n]$, $t \in \mathbb{C} \setminus \{0\}$, define:

$$x_{\hat{i}}(t) = I + (t-1)E_{i,i}$$

Let $S := \left\{1, \ldots, n-1, \hat{1}, \ldots, \hat{n}, \bar{1}, \ldots, \overline{n-1}\right\}$ be the set of all indices from the above construction. These x_s, $s \in S$, generate GL_n.

We can think of each x_s as a one parameter subgroup from $\mathbb{C} \setminus 0 \to GL_n$.

Definition 0.3.6. Let $\mathbf{i} = (i_1, \ldots, i_k) \in S^k$. We define a map $x_{\mathbf{i}} : \mathbb{C} \setminus \{0\}^k \to GL_n(\mathbb{C})$

$$x_{\mathbf{i}} := x_{i_1}(t_1) \cdots x_{i_k}(t_k)$$

Let $u \in S_n$ and let $l(u)$ denote the length of a reduced word for u (Definition 3.2.2).

Theorem 0.3.1 ([FZ98] Theorem 4.4)**.** *Let $u, v \in S_n$. Let $\boldsymbol{i}$ be a reduced word for (u, v) (See Definition 3.2.3). Then $x_{\mathbf{i}}$ (Definition 0.3.6) is a biregular isomorphism between $\mathbb{C} \setminus \{0\}^k$ and a dense open subset of $G^{u,v}$ where $k = n + l(u) + l(v)$.*

Theorem 0.3.2. *][FZ98] Theorem 1.1] $G^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $n + l(u) + l(v)$.*

If we let the diagonal matrices D act freely on $G^{u,v}$ by left multiplication, we can identify $L^{u,v}$ as the orbit space $G^{u,v}/D$. $L^{u,v}$ has dimension $l(u) + l(v)$.

Chapter 1

An Introduction to Mutation

Cluster algebras are certain rings with a combinatorial structure on the set of generators. This structure is called 'mutation'. In this first chapter, we will endeavour to understand cluster algebras through the mutation of quivers. In fact, not all cluster algebras arise in this way; this is something we will discuss in Chapter 2. For the moment, we will discuss compatible operations of mutation on quivers, graphs, and rings. We will look at various examples; the important example of the Grassmannian will be discussed towards the end of this chapter.

Summary of Main Results

A list of important examples:

- The cluster algebra of an orientation of the Dynkin diagram A_2. (Example 1.1.8)
- The cluster algebra of the quiver $\check{A}_2$ which is related to the Fibonacci sequence. (Example 1.1.9)
- The cluster algebra $\mathcal{A}_{n-3}$ coming from flips of a regular n-polygon. (Example 1.2)
- Finally, the main theorem of this chapter: the homogeneous coordinate ring of the Grassmannian of planes in n-space under the Plucker embedding $\mathbb{C}[Gr(2,n)]$ localised at certain coordinates is a cluster algebra. (Theorem 1.3.1)

1.1 Mutation, mutation, mutation

We first introduce a certain operation on certain types of graphs.

Definition 1.1.1. A quiver Q is a directed graph. It consists of vertices Q_0, arrows Q_1, and maps $s : Q_1 \to Q_0$ called source and $t : Q_1 \to Q_0$ called target. Given vertices $i, j \in Q_0$ a loop is some arrow such that $i \to i$ and a two cycle is a pair of arrows $i \to j$ and $j \to i$.

All quivers we shall consider will be **without loops or two cycles**. One interesting operation we may perform on quivers without loops or two cycles is **quiver mutation**. To perform mutation at a chosen vertex k of a quiver, first reverse the direction of all arrows leading to and from the vertex, then for all $i, j \in Q_0$ s.t. $i \to k \to j$ is a path **after reversing arrows**, add a new arrow $j \to i$, as well as removing all pairs of two cycles formed by this process. We call this mutation at k and write μ_k.

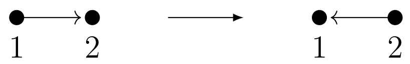
Lemma 1.1.1. *Mutation (Quiver) is an involution.*

Proof. Let Q be a quiver and pick a vertex of Q, say, k. We want to show that $\mu_k\mu_k(Q) = Q$. It is clear that reversing an arrow $i \to k$ (similarly $k \to j$) twice results in $i \to k$ again. So we should check all paths $i \to k \to j$. Mutating at k once gives us $j \to k \to i$ and an extra arrow $i \to j$. Mutating again at k will reverse this path back to $i \to k \to j$ and add a new arrow $j \to i$. This creates a two cycle between i and j which we erase. This takes us back to Q. □

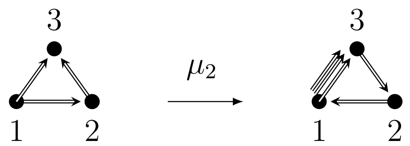
Example 1.1.1. *The simplest quiver consists of a single vertex with no edges. There is only one mutation class. We call this quiver A_1, after the Dynkin diagram.*

$$\bullet$$

Example 1.1.2. *Now consider a quiver with vertices $\{1, 2\}$ and a single arrow $1 \to 2$. When we mutate this quiver at the first vertex, we see that all that occurs is the arrow reverses direction. Similarly for the second vertex. The diagram* *below illustrates this process. We call this quiver A_2 as it is an orientation of the Dynkin diagram A_2.*



Example 1.1.3. *Here is an example of a quiver which is not* ***mutation finite*** *meaning there are infinitely many non-identical quivers the initial quiver may mutate into. The quiver has vertices 1,2,3 and edges* $1 \rightrightarrows 2$, $2 \rightrightarrows 3$, $1 \rightrightarrows 3$. *(This is because there is always a way to mutate to stack more arrows.)*



We may also assign to each quiver Q a signed adjacency matrix.

Definition 1.1.2. Let B be a matrix given by

$$B_{i,j} = \begin{cases} \#(i \to j) & i \to j \\ \#(j \to i) & j \to i \\ 0 & \text{no edges between i and j} \end{cases}.$$

We call B the **signed adjacency matrix** of Q.

Remark 1.1.1. For the above definition, note since Q is assumed to have no loops or two cycles (meaning the matrix is well defined), B is always a skew-symmetric matrix.

As promised, we can also mutate matrices.

Definition 1.1.3. We construct matrix mutation as follows: Let B be a signed adjacency matrix attached to a quiver Q. We construct a new matrix $\mu_k(B)$ by **mutation in the direction k**:

$$\mu_k(B)_{i,j} = \begin{cases} -B_{i,j} & \text{if k} \in \{i, j\} \\ B_{i,j} + \frac{1}{2}(B_{i,k}|B_{k,j}| + |B_{i,k}|B_{k,j}) & \textit{otherwise} \end{cases} \tag{1.1}$$

Example 1.1.4. *Consider A_2 once again (Example 1.1.8).* $B_{A_2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *Now suppose if we applied quiver mutation to A_2 at the first vertex. Then* $B_{\mu_1(A_2)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. *In addition, we see that applying matrix mutation yields* $\mu_1(B_{A_2}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. *This is indeed no coincidence.*

Lemma 1.1.2. *Let Q be a quiver. Let B be its signed adjacency matrix. Then* $B_{\mu(Q)} = \mu(B_Q)$.

Proof. We mutate in direction k. We consider $\mu(B_Q)_{i,j}$.
First suppose that k $\in \{i, j\}$. Then $\mu(B_Q)_{i,j} = -(B_Q)_{i,j}$ by definition of matrix mutation. By definition of quiver mutation, we must reverse the arrows at k, thus we also have that $B_{\mu(Q)_{i,j}} = -(B_Q)_{i,j}$.
Now suppose k $\notin \{i, j\}$. If $B_{i,k}$ or $B_{k,j}$ are zero, then $\mu(B_Q)_{i,j} = (B_Q)_{i,j}$. As this corresponds to no arrows leading to/from k to/from i or j, this means quiver mutation leaves arrows to/from i and j fixed. Thus $B_{\mu(Q)_{i,j}} = (B_Q)_{i,j}$.
If both are non-zero, there exist either a sub-quiver of form $\cdot \to k \to \cdot$ or $\cdot \to k \leftarrow \cdot$. In such cases, $\mu(B_Q)_{i,j} = B_{i,j} + \frac{1}{2}(B_{i,k}|B_{k,j}| + |B_{i,k}|B_{k,j})$ using the matrix mutation formula.
Quiver mutation adds an additional arrow $i \to j$ for every sub-quiver of form $i \to k \to j$ and $j \to i$ for every sub-quiver of form $j \to k \to i$. In such cases the matrix relation $\frac{1}{2}(B_{i,k}|B_{k,j}| + |B_{i,k}|B_{k,j})$ shortens to $B_{i,k}B_{k,j}$. This exactly counts the number of sub-quivers of the above form. That is, if there are n arrows from i to k and m arrows from k to j, then the number of sub-quivers of form $i \to k \to j$ is nm. For the second case of $\cdot \to k \leftarrow \cdot$, no additional arrows are formed. Therefore, the signed adjacency matrix $B_{\mu(Q)_{i,j}}$ should just be $B_{i,j} \pm B_{i,k}B_{k,j}$ (the plus/minus depends on direction of arrows).
Thus $B_{\mu(Q)_{i,j}} = B_{i,j} + \frac{1}{2}(B_{i,k}|B_{k,j}| + |B_{i,k}|B_{k,j})$. □

Remark 1.1.2. As quiver mutation was an involution, by 1.1.2, we see that matrix mutation too, is an involution.

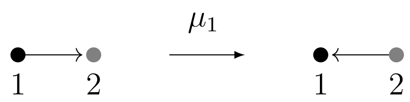
Thus we have shown that our operation on quivers agree with our operation on matrices. From here, we will use mutations on matrices and quivers interchangeably. We can now define a compatible notion of mutation on rings. However, we will restrict our quivers to the following case:

Definition 1.1.4. An **ice quiver** is a pair (Q, P) where Q is a quiver without loops or two cycles and P is a partition of $Q_0 = Q_m \sqcup Q_f$. We call the former set of vertices **mutable** and the latter **frozen** with the convention that there are no edges between frozen vertices. All quivers from henceforth will be ice quivers unless otherwise noted, so we will use the notation Q to refer to ice quivers.

Remark 1.1.3. We redefine **quiver mutation for ice quivers** to be exactly the same as ordinary quiver mutation with the exceptions:

- Mutation only occurs at mutable vertices
- We add an additional step to mutation: remove any edges formed between frozen vertices in the ordinary quiver mutation process

Example 1.1.5. *Consider A_1 with an extra frozen vertex. We call this ice quiver A_1'. We may only mutate at the single mutable vertex (in grey).* ***(A note on convention: all frozen vertices will be drawn in grey)***



Let m ⩾ n be positive integers. Consider a tuple of formal variables $(x_1, \dots, x_n, \dots, x_m)$. Set $F = \mathbb{Z}[x_{n+1}^{\pm 1}, \dots, x_m^{\pm 1}](x_1, \dots, x_n)$. We call F the **ambient field** and $\mathbb{Z}[x_{n+1}^{\pm 1}, \dots, x_m^{\pm 1}]$ the **ground ring** .

Now we will be defining a construction known as a 'seed' in two ways: one in terms of quivers and the other in terms of matrices. We will prove that these two definitions are in fact the same thus justifying us calling them both 'seeds'.

Definition 1.1.5. A **seed** is a pair ($\mathbf{x}$,Q) where

- $(x_1, \dots, x_n) \subset \mathbf{x} = (x_1, \dots, x_m)$ generates F freely over the ground ring;
- Q is a quiver with vertices $1, \dots, m$ where $1, \dots, n$ are called mutable and $n+1, \dots, m$ are called stable.

We call $\mathbf{x}$ **a cluster** with **cluster variables** $\{x_1, \ldots, x_n\}$ and **stable variables** $\{x^{\pm}_{n+1}, \ldots, x^{\pm}_m\}$, where $x^{\pm}_k$ refers to $x^{\pm 1}_k$. The choice of notation here is to be consistent with notation for more general cluster algebras.

Remark 1.1.4. We should think: mutable vertices $\mapsto$ cluster variables and frozen vertices $\mapsto$ stable variables.

Definition 1.1.6. Let $(\mathbf{x}, Q)$ be a seed in F, and let $k \in [n]$. Mutation (in direction k, where k is mutable) μ_k produces a new seed $(\mathbf{x'}, Q)$ defined by:

$$x'_k x_k = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha)=k}} x_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha)=k}} x_{s(\alpha)} \tag{1.2}$$

Also for $i \neq k$, $x'_i = x_i$.

Note that the terms $x_{t(\alpha)}$ have exponents equal to the number of arrows between k and t. This means we can read off the exponents of the quiver exchange relation from the signed adjacency matrix suggesting a way to define the exchange relation using matrices.

Indeed this was precisely the content of Lemma 1.1.2:

Definition 1.1.7. A **seed** is a pair $(\mathbf{x}, B)$ where:

- $(x_1, \ldots, x_n) \subset \mathbf{x} = (x_1, \ldots, x_m)$ generates F freely over the ground ring;
- B is an $n \times m$ matrix where the principal $n \times n$ submatrix is skew-symmetric.

Definition 1.1.8. Let $(\mathbf{x}, B)$ be a seed in F, and let $k \in [n]$. Mutation (in direction k) μ_k produces a new seed $(\mathbf{x'}, B)$ defined by:

$$x'_k x_k = \prod_{\substack{b_{ik}>0 \\ 1 \leqslant i \leqslant m}} x_i^{b_{ik}} + \prod_{\substack{b_{ik}<0 \\ 1 \leqslant i \leqslant m}} x_i^{-b_{ik}} \tag{1.3}$$

Again, for $i \neq k$, $x'_i = x_i$.

Thus we have defined a notion of mutation on certain elements inside the ambient field F. One important object we shall consider is the equivalence class of all seeds obtained from some initial seed through mutation.

Definition 1.1.9. We say two seeds $(\mathbf{x}, B)$ and $(\tilde{\mathbf{x}}, \tilde{B})$ are mutation equivalent if there exists a sequence of mutations $\mu_1, \ldots, \mu_d$, $d \in \mathbb{N}$ s.t. $(\tilde{\mathbf{x}}, \tilde{B}) = \mu_1(\ldots \mu_d((\mathbf{x}), B))$.

Lemma 1.1.3. *Mutation (matrix) equivalence is an equivalence relations on seeds.*

Proof.
- Reflexivity: We could simply not make any mutation. Or, using the fact that mutation is an involution, mutating in any direction twice will return the same seed. Thus $(\mathbf{x}, B) \sim (\mathbf{x}, B)$
- Symmetry: $(\mathbf{x}, B) \sim (\mathbf{y}, C)$ means there exists a sequence of mutations $\mu_1, \ldots, \mu_d$, $d \in \mathbb{N}$ s.t. $(\mathbf{y}, C) = \mu_1(\ldots \mu_d((\mathbf{x}), B))$. Again, since mutation is an involution, we can apply the same sequence backwards to go from $(\mathbf{y}, C)$ to $(\mathbf{x}, B)$.
- Transitivity: $(\mathbf{x}, B) \sim (\mathbf{y}, C) \sim (\mathbf{z}, D)$ means there are two sequences of mutations $\mu_1, \ldots, \mu_s$, $s \in \mathbb{N}$ and $\mu_1, \ldots, \mu_t$, $t \in \mathbb{N}$ connecting the seeds. Composing the mutations will produce the desired result.

$\square$

Let $\chi = \bigcup_{(\mathbf{x},B)\sim(\mathbf{y},C)} \mathbf{y}$ be the union of all clusters whose seeds are mutation equivalent to some initial seed $(\mathbf{x}, B)$.

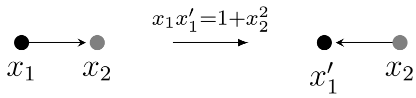
Definition 1.1.10. A **cluster algebra** A is the sub-algebra of F generated by χ over the ground ring. We will write $A = A(\chi) = A(Q) = A(B)$ where Q and B are the relevant quivers and matrices of the seeds inside χ.

Example 1.1.6. *The simplest possible cluster algebra is the one coming from the graph A_1. We start off with the initial seed (x, A_1). Mutation takes this seed to $(2x^{-1}, A_1)$.*

$$\underset{x}{\bullet} \xrightarrow{xx'=2} \underset{x'}{\bullet}$$

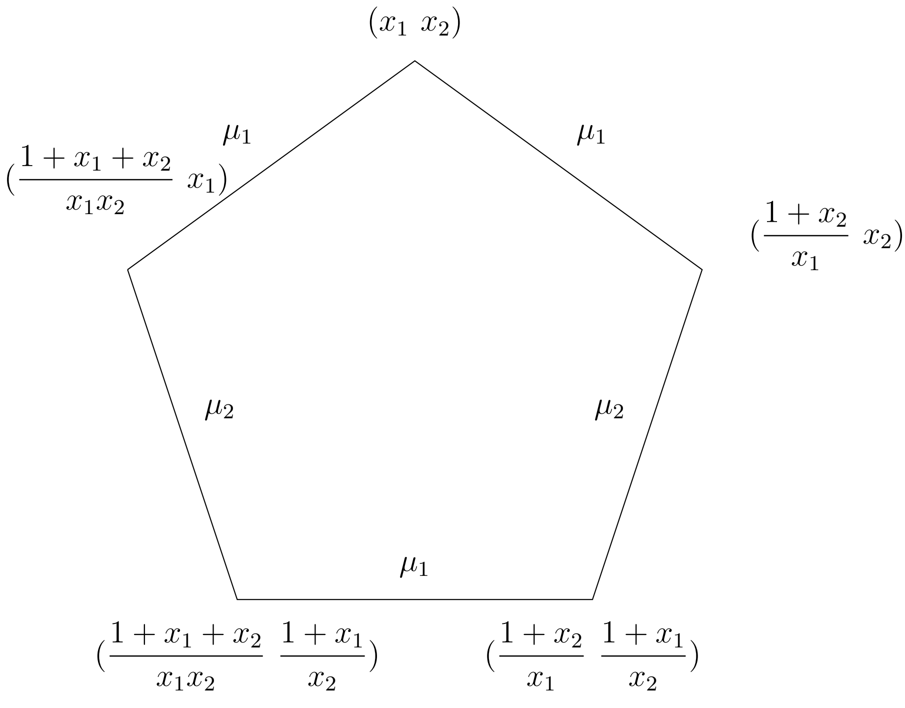
Thus $\chi = \{x, 2x^{-1}\}$ So $A(A_1) = \mathbb{Z}[x, 2x^{-1}]$. Over a field (or any ring where 2 is a unit), say $\mathbb{Q}$, this is the Laurent polynomial ring in one variable.

Example 1.1.7. *We can consider a similar cluster coming from the quiver A_1', which is A_1 with an additional frozen vertex.*



Once again we have a single exchange relation given by: $x_1x_1' = 1 + x_2$*. We get the ring* $A = \mathbb{Z}[x_1, x_1', x_2^{\pm}]/ < x_1x_1' = 1 + x_2 > \cong \mathbb{Z}[x_1, x_1', \frac{1}{x_1x_1'-1}]$*. So over some algebraically closed field* k*,* $k \otimes A \cong \mathcal{O}(D(x_1x_1' - 1))$*, the ring of regular functions of everything avoiding the hyperbola* $x_1x_1' - 1$ *in* $\mathbb{A}^2_k$*.*

Example 1.1.8. *Now we may examine our initial example of* A_2 *with all three notions of mutation. We start with the seed* $((x_1, x_2), A_2)$ *or equivalently* $((x_1, x_2), \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$*.*

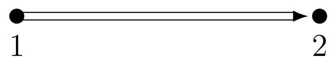


We do assume that $(x_1, x_2) = (x_2, x_1)$*. In Chapter 2, we will prove that the order does not actually matter (Lemma 2.2.4). The set of seeds*

$\chi = \left\{ x_1, x_2, \dfrac{1+x_1}{x_2}, \dfrac{1+x_2}{x_1}, \dfrac{1+x_1+x_2}{x_1x_2} \right\}$. *The cluster algebra* $\mathbb{Z}[\chi] \subset F$ *arising from* A_2 *has five distinct seeds. This will be an important example for the future.*

One thing to observe in the above case is that while only two different quivers are obtained by mutation, the clusters produced by mutation are different, meaning the seeds are different. In all examples we've seen so far, it appears as though this process of mutation always generates finitely many distinct clusters. Interestingly, this is not always true.

Example 1.1.9. *Consider the quiver consisting of two mutable vertices* $\{1, 2\}$ *with two edges* $1 \to 2$. *We call this quiver* $\check{A}_2$. *Note that in representation theory this quiver is more commonly called* $\hat{A}_1$. *We can immediately see that quiver mutation produces simply reverses the two arrows. However, there are infinitely many clusters!*



The two exchange relations for an initial cluster (x_0, x_1) *are:* $\mu_1 : x_0x'_0 = 1+x_1^2$ *and* $\mu_2 : x_1x'_1 = 1 + x_0^2$. *Now we'll prove that there are infinitely many clusters.*

Interestingly however, this algebra actually has a finite presentation given by $\mathbb{Z}[x_1, x'_1, x_2, x'_2] \big/ < x_1x'_1 = 1 + x_2^2,\ x_2x'_2 = 1 + x_1^2 >$. *In particular, this is a Noetherian ring. In fact* $\check{A}_2$ *is an example of an 'acyclic' cluster algebra which is something we will discuss in Chapter 3 (Definition 3.2.4).*

Definition 1.1.11. Cassini's Relation states that if (a_n) denotes the Fibonacci sequence, then the following relation holds:

$$a_{n+1}a_{n-1} = (-1)^n + a_n^2$$

Remark 1.1.5. Observe that Cassini's Relation generate the Fibonacci Sequence.

Proposition 1.1.1. *The cluster variables of* $A(\check{A}_2)$ *surject onto* $\{a_1\} \sqcup (a_{2n})$.

Proof. We observe that for even n Cassini's Relation is precisely the exchange relation on $\check{A}_2$. Therefore we will abuse notation and refer to the Cassini Relation by μ_i for $i \in \{1, 2\}$. Also note that there are only two ways we can mutate in the $\check{A}_2$ case (using that mutation is an involution): first perform μ_1 then μ_2 and repeat, or perform μ_2 then μ_1 and repeat.

We start by picking an initial cluster (x_0, x_1). We will mutate by first performing μ_2 and then μ_1. We will prove that the subset S of all seeds in $\chi := \chi_{\check{A}_2}$ obtainable from (x_0, x_1) by such mutations are in bijection with $\{a_1\} \sqcup (a_{2n})$. Here is a picture of how we will prove the statement:

$$\begin{tikzcd}
x_0\ x_1 \arrow[dr, "\mu_2"] \arrow[rrr] & & & 1\ \ 1 \arrow[dr, "\mu_2"] & \\
& x_0\ x_2 \arrow[dl, "\mu_1"] & & & 1\ \ 2 \arrow[dl, "\mu_1"] \\
x_3\ x_2 \arrow[dr, "\mu_2"] & & & 5\ \ 2 \arrow[dr, "\mu_2"] & \\
& x_3\ x_4 & & & 5\ \ 13
\end{tikzcd}$$

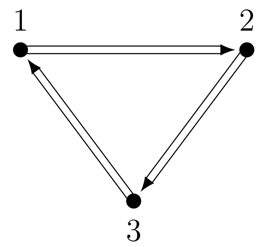
We send $x_0 \mapsto a_1 = 1$, $x_1 \mapsto a_2 = 1$. Then $x_2 := x_1' \mapsto a_4 = 5$, $x_3 := x_0' \mapsto$

$a_6 = 13$ and so on. This induces a map $\phi : S = (x_0, x_1, x_2, \dots) \to (a_n)$ since the exchange relations on x_i hold for their images in (a_n). This map is given by $x_0 \mapsto a_1$ and for $i > 1$, $x_i \mapsto a_{2i}$. This map is injective and the image in (a_n) is precisely $\{a_1\} \sqcup (a_{2n})$.

Had we originally proceeded by first performing μ_1 then μ_2, we would have obtained an identical map from $S' := (x_1, x_0, x_{-1}, x_{-2}, \dots) \to (a_n)$ which also restricts to a bijection with $\{a_1\} \sqcup (a_{2n})$. Since $\chi = S \cup S'$, this gives us a two-to-one correspondence from $\chi \mapsto \{a_1\} \sqcup (a_{2n})$ except at a_1. $\square$

Remark 1.1.6. In particular, the above proposition implies that $\check{A}_2$ has infinitely many distinct clusters.

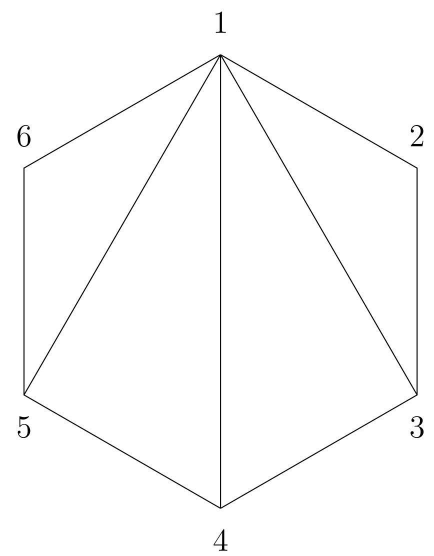
Example 1.1.10. *While the previous cluster algebra $A(\check{A}_2)$ had infinitely many distinct clusters, it did have a finite presentation. However there are some 'bad'* *cluster algebras. Here is an important example of a quiver which is not mutation finite: the Markov quiver, which is linked to solutions of the Markov Equation. (In a similar way to how $\check{A}_2$ is linked to a subsequence of the Fibonacci sequence.) However the algebra coming from the below quiver is **not** a finitely generated algebra. [Lam16]*



Remark 1.1.7. From the previous examples, we should take away the lesson that having finitely many distinct quivers coming from mutation is a necessary but not sufficient condition for a cluster algebra to have finitely many distinct clusters.

1.2 Triangulations of a polygon

We now consider a comprehensive and important example of a cluster algebra. Let R_n be a regular n-gon. A **triangulation** T of R_n is a maximal collection of non-crossing chords. For instance, below is a triangulation of a hexagon:

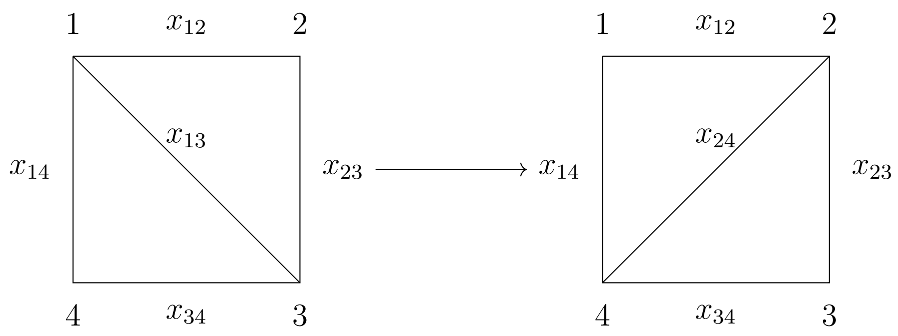


Now let us label the vertices of R_n from 1 to n clockwise. For a given triangulation T, we may further label each side $[i, j]$, where $i, j \in [n]$ by formal variables x_{ij}. These formal variables generate a field of rational functions $F = K(x_{ij} \, |\forall i, \, j \in [n])$. We will take $K = \mathbb{C}$ unless noted otherwise.

Remark 1.2.1. For what follows, we will define a mutation through certain combinatorial moves on polygons. It turns out that these moves actually define two different cluster algebras, one of which we will deal with here, and the other we will see in Chapter 2 (Example 2.1.9).

Triangulations of a polygon, it turns out, also possesses a notion of mutation using Whitehead moves or diagonal flips. This is perhaps best explained pictorially as below:

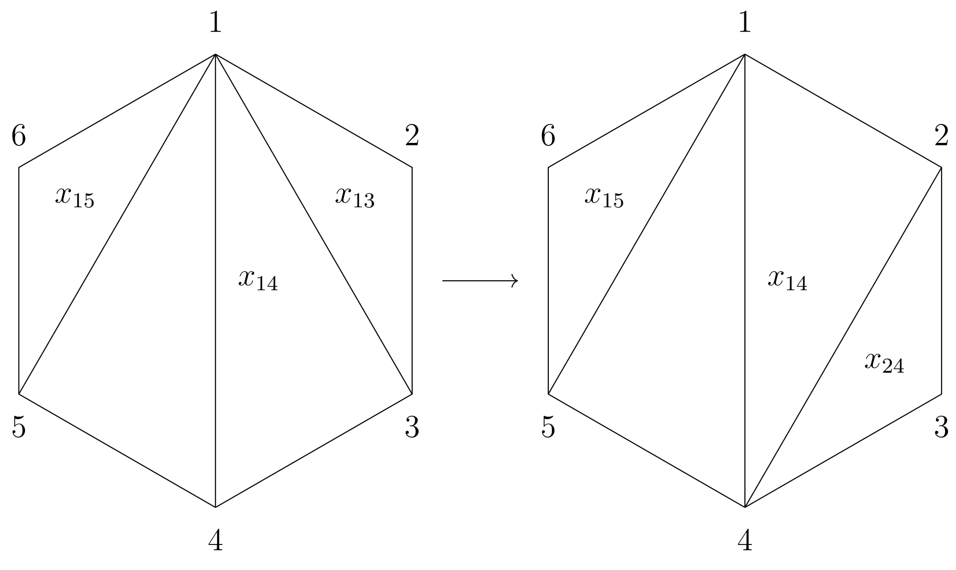
Example 1.2.1.



If a, b, c, d are vertices of some quadrilateral with a diagonal α then associated to the process of diagonal flips are exchange relations: $\alpha\alpha' = ac + bd$ which historically stem from Ptolemy's Rule. In the above picture, this corresponds to

$$x_{13}x_{24} = x_{12}x_{34} + x_{23}x_{14}$$

Example 1.2.2. *Below is an example of mutating a pentagon in the quadrilateral with vertices* $1, 2, 3, 4$ *(omitting side labels).*

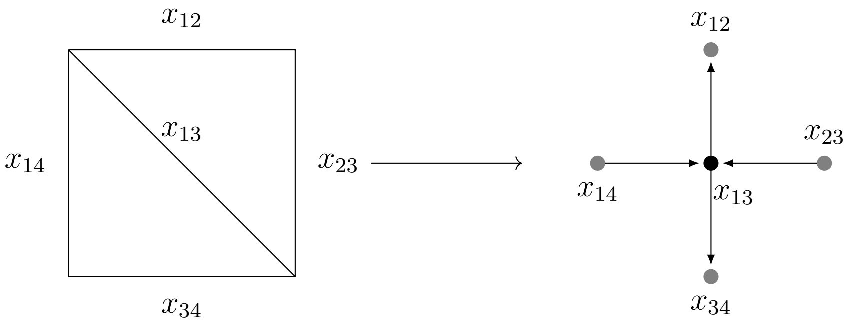


The exchange relation is: $x_{13}x_{24} = x_{12}x_{34} + x_{23}x_{14}$.

For each triangulation T of R_n, we will define an associated quiver. Note that there are n sides and n-3 chords.

To define the associated quiver:
At the midpoint of each chord, place a mutable vertex. At the midpoint of each side, place a frozen vertex. The collection of all these vertices form Q_0. Within each triangle of T, place an additional triangle linking the vertices of Q_0, where edges are oriented clockwise. These edges form Q_1. This forms a quiver Q inside the triangulation.

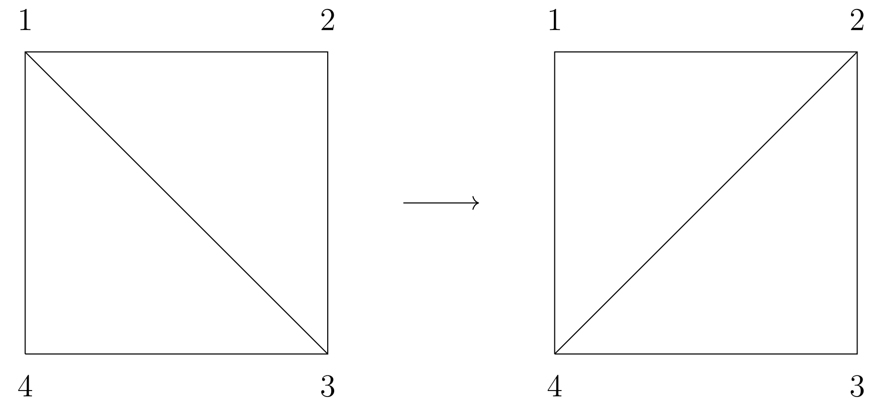
Example 1.2.3. *Here is an example of associating a quiver and cluster to a square:*

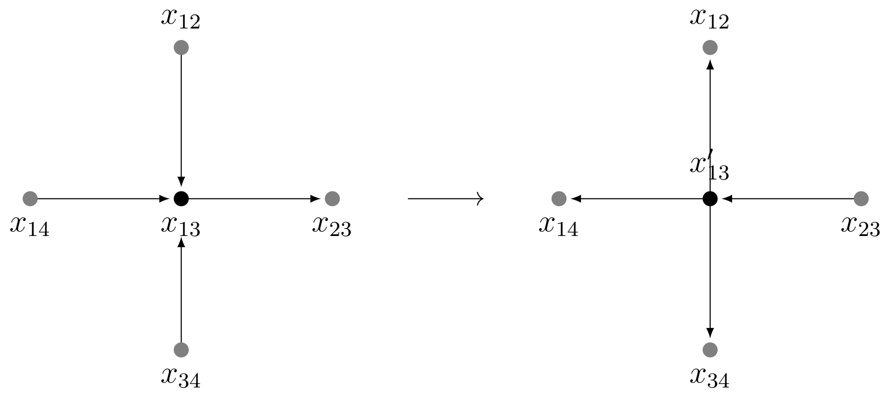


Now let us consider the particular triangulation T_0 where all the chords are of the form x_{1i} for $i \in [n]$. Let Q be the quiver associated to this triangulation and set $\mathbf{x} = (x_{13}, \ldots, x_{1n-1}, x_{12}, \ldots, x_{n1})$ to be the labelling of T_0 where the first $n-3$ variables (corresponding to chords) are cluster variables and the subsequent variables (corresponding to sides) are stable variables. We therefore have a seed $(\mathbf{x}, Q)$ which certainly gives us a cluster algebra but we have yet to show that quiver mutation coincide with diagonal flips.

Lemma 1.2.1. *Let T be a triangulation of R_n and let T' be another triangulation of R_n obtained from T by flipping a single diagonal x_{ij}. Then $Q'_T = \mu_{ij}(Q_T)$.*

Proof. As diagonal flips are local moves, we can prove this statement by looking at a quadrilateral. The proof is best explained with a picture: Idea is to manually compute the quiver associated to a quadrilateral before and after a flip and show that it is the same quiver as from quiver mutation.





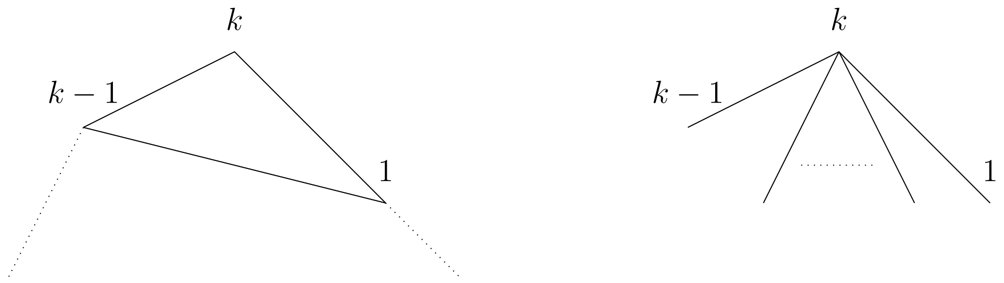
□

One last thing to check is that flipping actually cycles through all triangulations of an n-gon.

Lemma 1.2.2. *Let T_0 be the triangulation of a regular n-gon R_n where the chords are exactly x_{1k}; $3 \leqslant k \leqslant n-1$. Let T be any other triangulation of R_n. Then there exists a sequence of flips connecting T_0 to T.*

Proof. By Strong Induction The statement is clearly true for n $\leqslant 4$. We consider a regular labelled k-gon R_k. Let T be any triangulation of R_k. The inductive hypothesis implies that each triangulation T_s, where $s \in [\binom{k}{k-1}] = [k]$, of each regular (k-1)-gon inside T is mutation equivalent to T_0 (Here we mean T_0 to be the triangulation consisting of $[1, k]$ for R_{k-1}). We will also use the same fact for

R_d where $d \leq k-1$.
We say that two triangles are **adjacent** in R_k if they share a side. We say that d triangles U_i are adjacent if $\forall U_i\ \exists U_j$ such that U_i and U_j are adjacent. this guarantees that d adjacent triangles form a subtriangulation for for $d+2$ gon. First we look at the vertex labelled k and observe there are two possible situations.



In the first case, we pick the unique triangle of which n is a vertex. We call this triangle U_1. Then $T \setminus U_1$ is some R_{k-1} with 1 as a vertex and by the inductive hypothesis $T \sim T_0$.

In the second case, let d be the number of chords radiating out from k. This gives us d adjacent triangles. We consider the $d+2$-gon formed by these adjacent triangles. We see from the diagram that 1 is a vertex. If $d+3 \leq k-1$, by the inductive hypothesis, this is mutation equivalent to T_0 for R_{d+2}. In particular, in this T_0, $1, k-1, k$ forms a triangle. We call this triangle U_1. Then $T \setminus U_1$ is some R_{k-1} with 1 as a vertex. Thus by the inductive hypothesis, $T \sim T_0$.
If $d = k-3$, then the triangulation T is exactly the one where all chords are given by $[k,i]\ \forall i \in \mathbb{Z}/k$. Then we can just relabel the vertices of R_k by $i \mapsto i+1$. Now $T = T_0$. □

As cluster mutation is determined entirely by the quiver, we can now confidently say that flips on triangulations of an n-gon generate a cluster algebra and it is exactly the one given by $(\mathbf{x}, Q)$. We call this algebra $\mathcal{A}_{n-3}$.

Example 1.2.4. *See Example 1.2.1 and Example 1.2.3 for the relevant pictures. First observe that there is a single quadrilateral and therefore there are just two different triangulations.*

The corresponding cluster algebra is:
$\mathbb{Z}[x_{13}, x_{24}, x_{12}^{\pm}, x_{23}^{\pm}, x_{34}^{\pm}, x_{14}^{\pm}] \big/ < x_{13}x_{24} = x_{12}x_{34} + x_{14}x_{23} >$

1.3 The Grassmannian of planes in n-space

[See Preliminaries for background information on Grassmannians.] In this section we will give the homogeneous coordinate ring of the Grassmannian of planes in 2-space, $Gr(2, n)$, the structure of a cluster algebra. We will then discuss how to generalise this to the case of $Gr(k, n)$ for $k > 2$.

We will show how the cluster algebra coming from flips of polygons is isomorphic to the homogeneous coordinate ring of $Gr(2, n)$. Consider the coordinate ring of the $Gr(2, n)$ given by $\mathbb{C}[\Delta^{ij} \ : \ i, j \in [n]] \big/ Q$ where Q is the Plücker Ideal.

Strictly speaking, the cluster algebra $\mathcal{A}_{n-3}$ are the $\mathbb{Z}$-forms on the localisation of $\mathbb{C}[Gr(2, n)]$ at homogeneous coordinates Δ^{ii+1}. For now, we will deal with this technicality by an extension of scalars. Later, we will develop the language to establish a cluster algebra coming from flips of a n-gon which does not correspond to quivers and quiver mutation. This will then allows us to give $\mathbb{C}[Gr(2, n)]$ the structure of a cluster algebra properly.

Theorem 1.3.1. $\mathbb{C}[\Delta^{ii+1^{\pm 1}}] \otimes_{\mathbb{C}[\Delta^{ii+1}]} \mathbb{C}[Gr(2, n)] \big/ Q \cong \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}_{n-3}$

Proof. Define a map: $\phi : \mathbb{C}[\Delta^{ii+1^{\pm 1}}] \otimes_{\mathbb{C}[\Delta^{ii+1}]} \mathbb{C}[Gr(2, n)] \to \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}_{n-3}$ with $1 \otimes \Delta^{ij} \mapsto 1 \otimes x_{ij}$ and $\Delta^{ii+1^{-1}} \otimes 1 \mapsto 1 \otimes x_{ii+1}^{-1}$. This map is well-defined since any Δ^{ii+1} commutes across the tensor.

The map gives a bijection from set of all Plücker Coordinates to cluster variables of $\mathcal{A}_{n-3}$ i.e. a bijection between generators of the rings. This implies that ϕ is surjective. To prove the theorem, we just need to show that ϕ the kernel is Q. We observe that the only relations on the right hand side are the exchange relations (also invertibility ones for stable variables). This implies that kernel is generated by all the exchange relations in $\mathbb{C} \otimes \mathcal{A}_{n-3}$.

Recall that the Plücker Relations in $\dot{Sym}(\wedge^2 E)$, for some n-dim vector space E is given by:

$$\Delta^{ik}\Delta^{jl} = \Delta^{ij}\Delta^{kl} + \Delta^{il}\Delta^{jk} \tag{1.4}$$

We observe that every exchange relation on the cluster algebra is of this form. So the kernel is contained in Q. To show that the kernel is Q, we just need to prove that every Plucker relation is an exchange relation. This means we should look at the image of Equation (1.4) in $\mathcal{A}_{n-3}$.

We have $x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}$ (omitting the tensor with $\mathbb{C}$). We are done as long as this is a valid relation. We should check that this relation is a flip relation. Without loss of generality we may assume $i < j < k < l$.

Let T be any triangulation where $[i\ j\ k\ l]$ defines a quadrilateral with a diagonal $[i\ k]$ or $[j\ l]$ running through it. Applying Lemma 1.2.2, $T \sim T_0$.

Thus the image of a Pl$\ddot{u}$cker Relation is a valid exchange relation in A_{n-3}. Thus implies that the kernel of ϕ is Q. Thus ϕ is an isomorphism of rings. $\square$

Remark 1.3.1. The above theorem identifies the cluster algebra coming from flips of polygons, $\mathcal{A}_{n-3}$, with the $\mathbb{Z}$-form on $\mathbb{C}[Gr(2,n)]_{(\Delta^{ii+1})}$. Homogeneous coordinate rings, unlike in the affine case, depend on the choice of embedding. It might therefore be appealing to instead associate $\mathcal{A}_{n-3}$ with the ring of regular functions on the affine cone of $Gr(2,n)$ inside $\mathbb{A}^n_{\mathbb{C}}$.

With more complicated combinatorics we are also able to describe cluster structures on the homogeneous coordinate ring of $Gr(k,n)$ for $k > 2$. Details of this lie within [Sco03].

Chapter 2

Cluster Algebras

Summary of Main Results

In Chapter 1, our narrative was focused on describing cluster algebras coming from quiver mutation and how this relates to the homogeneous coordinate ring of the Grassmannian. Now we will revisit some of the definitions and techniques from Chapter 1 in greater generality. In the most general case, a cluster algebra over a certain ground ring R is an R-algebra embedded in some ambient field $\mathcal{F}$ where both R and F are associated to some semi-field G. One of the first things we will do in Chapter 2 is to collect different definitions of cluster algebras in the literature and show that they are in fact the same.

- What is a cluster algebras in general? (Definition 2.1.8)
- Which cluster algebras come from quivers? (Theorem 2.1.1)
- Which cluster algebras should we expect to have interesting geometry? (Definition 2.1.10)
- Which cluster algebras have finitely many clusters? (Theorem 2.2.1)

We will also introduce two important operations on cluster algebras (of geometric type) called quotient and restriction. We will prove nice properties of these operations and explain how they can be used to simplify the study of cluster algebras. We will apply these two operations to the question: What can I do to check if a cluster algebra of geometric type A is of finite type? It turns out certain quotients of A classify a family of cluster algebras 'similar' to A (Lemma 2.2.5, Lemma 2.2.2)). Furthermore, if we wanted to see if A is of infinite type, if suffices to check if any restrictions of A are of infinite type (Prop. 2.2.4).

2.1 General cluster algebras

We will also introduce two operations on cluster algebras called restriction and quotient which will allow us to develop some foundational results about cluster algebras.

2.1.1 General cluster algebras and cluster algebras coming from ice quivers

We start by constructing an ambient field inside which we will define a cluster algebra.

Definition 2.1.1. A semi-field $(G, \times, \oplus)$ is an abelian multiplicative group with an associative and commutative addition $\oplus$ which distributes with respect to $\times$.

Example 2.1.1. *Let $\mathbb{P}$ be a free abelian group with finitely many generators p_j. We define addition as:*

$$\prod_j p_j^{a_j} \oplus \prod_j p_j^{b_j} = \prod_j p_j^{min(a_j,b_j)} \tag{2.1}$$

We require that the exponents be positive integers. We will be most frequently working with cluster algebras over this particular semi-field but the definition of cluster algebras applies for semi-fields in general. We call this semi-field $\mathbb{P}$ the tropical semi-field.

The functor $GrpRng:\ Groups \to Rings$ sending $G \mapsto \mathbb{Z}G$ is left adjoint to the functor $GrpUnts:\ Rings \to Groups$ sending $R \mapsto R^{\times}$. In particular, this means $GrpRng$ sends coproducts to coproducts.

Lemma 2.1.1. *Let $\mathbb{P}$ be a finitely generated free Abelian group with n generators p_j. Then the group ring $\mathbb{Z}\mathbb{P} \cong \mathbb{Z}[p_1^{\pm}, \dots, p_n^{\pm}]$.*

Proof. We may write $\mathbb{P} = \prod_{i=1}^n < p_i >$ as a product of infinite cyclic groups. Then $\mathbb{Z}\mathbb{P} = \mathbb{Z}[\prod_{i=1}^n < p_i >] \cong \bigotimes_{i=1}^n \mathbb{Z}[< p_i >]$ using the fact that $GrpRng$ is left adjoint to $GrpRng$ thus sending coproducts to coproducts.
Since each $\mathbb{Z}[< p_i >] \cong \mathbb{Z}[\mathbb{Z}]$, let us compute the latter ring. By definition, $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}\{x^r :\ r \in \mathbb{Z}\} \cong \mathbb{Z}[x^{\pm 1}]$. Thus $\bigotimes_{i=1}^n \mathbb{Z}[< p_i >] \cong \bigotimes_{i=1}^n \mathbb{Z}[p_i^{\pm 1}] \cong \mathbb{Z}[p_1^{\pm 1}, \dots, p_n^{\pm 1}]$. $\square$

Remark 2.1.1. As $\mathbb{ZP}$ is a Laurent polynomial ring in n variables over $\mathbb{Z}$, it is automatically a Noetherian integral domain.

Remark 2.1.2. Noetherian rings are nice rings to consider from the perspective of algebraic geometry. This partially motivates why we will be calling cluster algebras over such rings 'of geometric type'. We will see this in more detail later in this chapter.

Proposition 2.1.1. *Let G be any semifield. Then the group ring $\mathbb{Z}G$ is an integral domain.*

Proof. We first show (G,×) is torsion-free. Suppose $g \in G$ is torsion. Then $\exists m \in \mathbb{N}$ s.t. $g^m =$ e. Let $x = g^{m-1} \oplus g^{m-2} \oplus \cdots \oplus 1 \in G$. Then observe that $gx = x$ which implies that $g = e$ by the uniqueness of the identity. Therefore G is torsion free.
This establishes G as an Abelian torsion-free group. Now we will prove that the group ring is a domain by reducing to the finitely generated case.
Suppose $a, b \in \mathbb{Z}G$ such that $ab = 0$. By definition, we can find group elements $\{g_{i_j}\}_{j\in[n]}$, $\{g_{s_k}\}_{k\in[m]}$ such that $a = \sum_j a_i g_{i_j}$ and $b = \sum_k b_k g_{s_k}$ for $a_i, b_k \in \mathbb{Z}$. We can therefore find a finitely generated subgroup $H :=< g_i >_{i\in[h]} \subset < g_{i_j}, g_{s_k} > \subset G$ such that $a, b \in H$.
This establishes H as a finitely generated torsion-free Abelian group which means it must be a finitely generated free Abelian group. By Prop 2.1.1, $\mathbb{Z}H$ is a Laurent polynomial ring in h variables over $\mathbb{Z}$, and therefore an integral domain. Thus $a = 0$ or $b = 0$. Hence $\mathbb{Z}G$ must be an integral domain.

□

Definition 2.1.2. Let G be any semifield. Let $t_1, \ldots, t_n$ be formal variables for n a positive integer. The field of fractions $\mathcal{F} = \mathbb{Z}G(t_1, \ldots, t_n)$ is called the **ambient field**. We note that this is indeed well defined since $\mathbb{Z}G$ is an integral domain.

In this section, we will define a cluster algebra to be a particular sub-algebra inside $\mathcal{F}$. In Chapter 1, we first constructed a quiver, and from the quiver, we obtained a seed. We defined a cluster algebra through mutations of the seeds. The process we are about to embark on is analogous to the more concrete construction we saw in Chapter 1.

Definition 2.1.3. An $n \times n$ matrix $B = (b_{ij})$ is **sign-skew-symmetric** if $\forall i, j \in [n]$ $b_{ii} = 0$ and $b_{ij} = b_{ji} = 0$ or $b_{ij}b_{ji} < 0$.

Example 2.1.2. *The following matrix is sign-skew-symmetric:* $\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$

The following matrix is not sign-skew-symmetric: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Definition 2.1.4. A **seed** $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{F}$ is:

- $\mathbf{x} := (x_1, \ldots, x_n)$ where $\mathbf{x}$ is a transcendence basis of $\mathcal{F}$ over $\mathbb{ZP}$. We call $\mathbf{x}$ a **cluster** and each x_i **cluster variable**.
- $\mathbf{p} := (p_i^{\pm})_{x_i \in \mathbf{x}} = (p_{x_1}^+, \ldots, p_{x_n}^+, p_{x_1}^-, \ldots, p_{x_n}^-)$ where each $p_i^{\pm} \in \mathbb{P}$ such that $p_i^+ \oplus p_i^- = 1\ \forall x_i \in \mathbf{x}$. We call such $\mathbf{p}$ **coefficients**.
- B is a $n \times n$ sign-skew-symmetric matrix with integer entries (b_{ij}) indexed by $x_i, x_j \in \mathbf{x}$.

Example 2.1.3. *We take $\mathbb{P}$ to be the free abelian group on a single letter a. $F := \mathbb{ZP}(x_1, x_2)$. Setting $B = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$, $\boldsymbol{p} = (1, a, a, 1)$, noting that $1 \oplus a = 1$, and $\boldsymbol{x} = (x_1, x_2)$. These three objects define a seed $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$.*

Now that we have defined a seed, we can now define mutation on seeds. We will continue to use the definition of matrix mutation from Chapter 1 (Definition 1.1.8).

Definition 2.1.5 (Seed mutation). Let Σ be a seed in $\mathcal{F}$ as defined above and consider $x_s \in \mathbf{x}$. We define a new seed Σ' as follows:

- $\mathbf{x}' := \mathbf{x} \setminus x_s \cup x_s'$ where $x_s' \in \mathcal{F}$ is the element satisfying the relation:

$$x_s x_s' = p_s^+ \prod_{\substack{x_i \in \mathbf{x} \\ b_{is} > 0}} x^{b_{is}} + p_s^- \prod_{\substack{x_i \in \mathbf{x} \\ b_{is} > 0}} x^{-b_{is}} \tag{2.2}$$

- B' is obtained by applying matrix mutation at s as defined in Definition 1.1.8.
- $\mathbf{p}'$ is determined by:
  - $(p_k^{\pm})' = p_k^{\mp}$
  - $$\frac{p_j^{+'}}{p_j^{-'}} = \begin{cases} \dfrac{p_j^+}{p_j^-}(p_k^+)^{b_{jk}} & b_{jk} \geq 0 \\ \dfrac{p_j^+}{p_j^-}(p_k^-)^{b_{jk}} & b_{jk} \leq 0 \end{cases}$$

We denote the above operation by μ_{x_s}.

Example 2.1.4. *Using the seed from the previous example (2.1.3), let us compute two initial mutations* μ_{x_1} *and* μ_{x_2}.
The two exchange relations on cluster variables are:
$x_1x_1' = x_2^3 + a$ *and* $x_2x_2' = a + x_1$.
$\mu_{x_1}(B) = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$ *and* $\mu_{x_2}(B) = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
$\mu_{x_1}(\boldsymbol{p}) = (1, 1, a, 1)$ *and* $\mu_{x_2}(\boldsymbol{p}) = (a, 1, 1, a)$.

Remark 2.1.3. Mutation defined in this manner is an involution due to Lemma 1.1.1 and Lemma 1.1.2.

The fact that mutation is actually an involution allows us to define an equivalence relation on seeds. If Σ and Ω are two seeds in $\mathcal{F}$, then $\Sigma \sim \Omega$ if there is a series of mutations $\mu_{i_1}, \ldots, \mu_{i_k}$ such that $\Sigma = \mu_{i_1} \cdots \mu_{i_k}(\Omega)$. We proved this was indeed an equivalence relation in Lemma 1.1.3. From a single seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ we are able to define a subalgebra of $\mathcal{F}$ generated by $\chi = \bigcup_{\Sigma \sim \Omega = (\mathbf{y}, \mathbf{q}, C)} \mathbf{y}$. First however, we will define a ring over which this algebra will lie.

Definition 2.1.6. Let $\mathcal{P} = \bigcup_{\Sigma \sim \Omega} \mathbf{p}_\Sigma$ the set of all co-efficients which are mutation equivalent. The **ground ring** is the sub-algebra inside $\mathcal{F}$ generated by $\mathcal{P}$ as a set i.e. $\mathbb{Z}[\mathcal{P}] \subset \mathcal{F}$.

Example 2.1.5. *Let G be the tropical semifield $\mathbb{P}$ (Example 2.1.1) with N generators p_j. Define $\mathcal{P} = (p_1^{\pm 1}, \ldots, p_N^{\pm 1})$. Thus the ground ring $\mathbb{Z}[\mathcal{P}] = \mathbb{Z}[p_1^{\pm 1}, \ldots, p_N^{\pm 1}]$. By Prop 2.1.1, $\mathcal{F} = \mathbb{Z}[p_1^{\pm 1}, \ldots, p_N^{\pm 1}](t_1, \ldots, t_n)$.*
This choice of $\mathcal{P}$ and $\mathcal{F}$ coincides with our choice of ground ring and ambient field from Chapter 1 1.1. This similarity motivates the below definition.

Definition 2.1.7. A seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{F}$ is **of quiver type** if B is skew-symmetric, $\mathbb{P}$ is tropical with k generators p_j, with ground ring $\mathbb{Z}[p_1^{\pm 1}, \ldots, p_k^{\pm 1}]$.

Example 2.1.6. *The previous example (2.1.3) is therefore not a seed of quiver type as B is not skew symmetric (but sign-skew-symmetric).*

Definition 2.1.8. Let Σ be a seed. The **cluster algebra** $A(\Sigma)$ is the sub-algebra of $\mathcal{F}$ generated by χ over $\mathbb{Z}[\mathcal{P}]$. That is, $A(\Sigma) := \mathbb{Z}[\mathcal{P}][\chi]$.

Finally, we have finished constructing the definition for cluster algebras in full generality. This gives us two different definitions of cluster algebras. One in terms of sign-skew-symmetrisable matrices and semifields, and the other in terms of quivers. A natural question to ask is: in which cases do these definitions coincide? The goal for the rest of this section is give a precise answer to the preceding question.

We say a **cluster algebra** $A(\Sigma)$ **is of quiver type** if Σ is of quiver type (Definition 2.1.7). Now, as promised at the start of Chapter 2, we will show that a cluster algebra of quiver type does indeed have an associated quiver.

Theorem 2.1.1 (Classification of cluster algebras of quiver type)**.** *Fix $\mathcal{F}$. Let $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$ be a seed of quiver type as above. Then there exists an ice quiver Q and $\Omega = (Q, \boldsymbol{y})$ (Recall Definition 1.1.5) such that $A(\Sigma) \cong A(\Omega)$.*

Remark 2.1.4. This result gives us a sufficient condition for when a cluster algebra 'comes' from a quiver.
Throughout Chapter 1, we showed how given an ice quiver, Q, we could define a cluster algebra $A(Q)$ from mutations of Q. This means we now have both a necessary and sufficient condition for when a cluster algebra 'comes' from a quiver.

Proof. To construct Ω: Let the set of generators for $\mathbb{P}$ be denoted by $\mathbf{c} = (p_j)_j$. Define $\mathbf{y} := \mathbf{x} \cup \mathbf{c}$.
Since B is skew-symmetric, we can view it as a signed-adjacency matrix for some quiver Γ with vertices in $\mathbf{x}$. We now extend Γ to some quiver Q with vertices in $\mathbf{y}$ by defining additional edges between $\mathbf{c}$ and $\mathbf{x}$.
Now each $p_{x_i}^{+} = \prod_j p_j^{e_j}$ for some choice of exponents $e_j \in \mathbb{N}$, for all $x_i \in \mathbf{x}$. Fixing some x_i, we add e_j arrows $i \to j$. Similarly, writing $p_{x_i}^{-} = \prod_k p_k^{f_k}$, we add f_k arrows $k \to i$. We pick Q to be the above extension of Γ.
We check this produces a genuine ice quiver. We just need to check there are no two cycles. Fixing an i and j, suppose $p_j^{e_j} | p_{x_i}^{+}$ with $e_j > 0$. The normalisation condition $p_{x_i}^{+} \oplus p_{x_i}^{-} = 1$ under tropical addition implies that if $p_j^{f_j} | p_{x_i}^{-}$ then $\min(e_j, f_j) = 0$. This implies $f_j = 0$. Converting to the language of quivers using our algorithm above, this says if there is at least one arrow $i \to j$, there cannot also be an arrow $j \to i$.
We define $\Omega := (Q, \mathbf{y})$. We immediately observe that, by construction, exchange relations on the clusters $\mathbf{x}$ and $\mathbf{y}$ coincide. This also means we have a bijection on sets of clusters $\phi : \chi(A(\Sigma)) \to \chi(A(\Sigma))$ induced by $\mathbf{x} \mapsto \mathbf{y}$. It then follows since

cluster algebras are generated by all the clusters, as $\mathbb{Z}$-algebras, $A(\Sigma) \cong A(\Omega)$

$\square$

2.1.2 Cluster algebras of geometric type and weighted ice quivers

In the previous section we saw a general definition of cluster algebras as well as connected our familiar construction from Chapter 1 with the more general case. Now we will consider an important class of cluster algebras called cluster algebras of geometric type. The methods used here will play an important role in establishing finiteness results later on.

Definition 2.1.9. A $n \times n$ integer matrix B is **skew-symmetrisable** if there exists some diagonal $n \times n$ matrix D with positive integer entries such that DB is skew symmetric.

Example 2.1.7. *The matrix* $\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$ *is skew-symmetrisable by the diagonal matrix* $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

Definition 2.1.10. A seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{F}$ is of **geometric type** if B is skew-symmetrisable, $\mathbb{P}$ is tropical, and all k generators p_j of $\mathbb{P}$ belong to $\mathcal{P}$.

This means $\mathbb{Z}[P] \cong \mathbb{Z}[p_1, \ldots, p_k]$.

When Σ is of geometric type, we will write Σ as a pair $(\tilde{\mathbf{x}}, B)$ where $\tilde{\mathbf{x}} = \mathbf{x} \sqcup \{p_1, \ldots, p_k\}$. When written in this form, the convention is that the first n variables of $\tilde{\mathbf{x}}$ are cluster variables, and the subsequent are stable. We label $\mathbf{c} := \{p_1, \ldots, p_k\}$

A cluster algebra is of geometric type if a seed Σ is of geometric type.

Definition 2.1.11. A **weighted quiver** is a quiver $Q = (V, E)$ with:

- A map of sets $\phi : E \to R = \mathbb{N}[x] \big/ < x^2 - n, \ \forall n \in \mathbb{N} >$
- A set of labels $L = \{\phi(s)\}_{s \in E}$
- For any pair of vertices i, j we allow at most one edge between them.

In other words, a weighted quiver is a simply laced quiver where every edge has a labelled weight which is either a positive integer or the square root of a positive integer.

Definition 2.1.12. A **weighted ice quiver** is a weighted quiver $\Gamma = (V, E, \phi)$ satisfying:

- $V = M \sqcup F$ where vertices in M are called mutable and vertices in F are called frozen
- There are no loops or two cycles in E.
- There are no edges between frozen vertices.

Definition 2.1.13. Let B be a $n \times m$ matrix with positive integer entries where $m \geq n$. The **principal part** of B is the $n \times n$ submatrix consisting of the first n columns.

Definition 2.1.14. Let $B = (b_{ik})$ be a $n \times m$ matrix with positive integer entries with a principal $n \times n$ skew-symmetrisable submatrix. We construct a weighted ice quiver $\Gamma(B) = (V, E, \phi)$ by:

1. $V := [n] \sqcup \{n+1, \ldots, m\}$
2. An arrow $(i \to k) \in E$ iff $b_{ik} > 0$
3. Set $\phi(i \to j) = \sqrt{|b_{ij} b_{ji}|}$

Definition 2.1.15 (Generalised quiver mutation)**.** Let $\Gamma(B)$ be as above. We denote by μ_k mutation at a (mutable) vertex k. This produces a new quiver $\Gamma' := \mu_k(\Gamma(B)) = (V', E', \phi')$. It operates as follows:

- V' = V
- We produce E and ϕ' as follows:
  1. Reverse all arrows into/out of k. (weights remain unchanged)
  2. For every path $i \to k \to j$, add an arrow $j \to i$ with label:

$$\phi'(j \to i) := \begin{cases} \phi(i \to k)\phi(k \to j) - \phi(i \to j) & i \to j \in E \\ \phi(i \to k)\phi(k \to j) - \phi(j \to i) & j \to i \in E \\ \phi(i \to k)\phi(k \to j) & \text{if no edge between i and j} \end{cases}$$

3. Everything else remain unchanged.

Remark 2.1.5. When a labelled edge is a positive integer, we may simply draw that many arrows. This means that when the principal part of B is skew-symmetric, this coincides with usual quiver mutation.

Remark 2.1.6. For simplicity, from here on we will use w_{ij} to denote $\phi(i \to j)$.

Ideally, we want to define mutation from such weighted quivers. However it is not generally possible, given some $\Gamma(B)$ to recover the original matrix B except when B is skew-symmetric. In this case, it is easy to show that $\Gamma(B)$ is actually an ice quiver and B its signed adjacency matrix. Then we proceed as in Chapter 1.

Nonetheless, such weighted quivers are an important combinatorial gadget, particularly in classifying cluster algebras.

2.1.3 Geometric Realisation

The name 'of geometric type' seems to suggest a way to connect such cluster algebras with algebraic geometry. Naively, any cluster algebra A is of course a ring, and so $SpecA$ an affine scheme. Furthermore, as a sub-algebra in a field, A must also be an integral domain. When A is of geometric type, this means that A is Noetherian, and thus $SpecA$ is a Noetherian scheme.

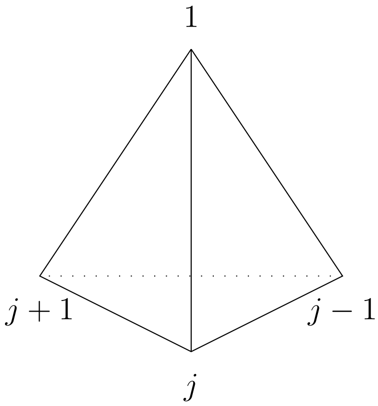
In this section we will give examples of how to associate a cluster algebra to the (homogeneous) coordinate rings of a given variety.

Example 2.1.8. $\mathbb{C}[SL_2(\mathbb{C})]$ *Let* $G = SL_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$ *be the group of* 2×2 *matrices with determinant 1. Viewing* G *as an algebraic variety, we have the coordinate ring*
$\mathbb{C}[SL_2(\mathbb{C})] = \mathbb{C}[a,b,c,d]/_{< ad - bc = 1 >}$

We construct a cluster algebra (of geometric type) where $\boldsymbol{x} = a$, $\boldsymbol{p} = bc$, *with the single exchange relation* $aa' = 1 + bc$. *Thus* $a' = d$. *Further, we set* $\mathcal{P} = \{b, c, bc\}$ *and so the ground ring* $\mathbb{Z}[\mathcal{P}] \cong \mathbb{Z}[b,c]$ *and the ambient field* $\mathbb{Z}[b^{\pm 1}, c^{\pm 1}](a)$. *From this we obtain a cluster algebra* $A = \mathbb{Z}[a,d,b,c]/_{< ad - bc = 1 >}$.

Example 2.1.9. *In Chapter 1, we showed how flips of an n-gon were related to the mutations of a certain quiver. From these mutations, we were able to define a cluster algebra (of quiver type) which is the localisation of $\mathbb{C}[Gr(2,n)]$ at certain coordinates. Now that we have the vocabulary of a cluster algebra of geometric type, we can give $\mathbb{C}[Gr(2,n)]$ the structure of a cluster algebra.*

First we need to define an initial seed Σ where the variables come from $\mathbb{C}[Gr(2,n)]$. Let $\boldsymbol{x} = (\Delta_{13}, \ldots, \Delta_{1n-1})$. This corresponds to the triangulation T_0 from Chapter 1. We know that the flip relation for any Δ_{1j} will be of the form: $\Delta_{1j}\Delta_{j-1j+1} = \Delta_{1j-1}\Delta_{jj+1} + \Delta_{1j+1}\Delta_{j-1j}$ for $3 \leq j \leq n-1$.



This suggests to us how we should set up $\boldsymbol{p}$. For any $j \neq 3, n-1$, we set $p_j^+ = \Delta_{jj+1}$ and $p_j^- = \Delta_{j-1j}$. For $j = 3$, $p_3^+ = \Delta_{12}\Delta_{34}$ and $p_3^- = \Delta_{23}$. For $j = n-1$, $p_{n-1}^+ = \Delta_{n-2n-1}\Delta_{1n}$ and $p_{n-1}^- = \Delta_{n-1n}$. Thus we see that $\mathcal{P} = \Delta_{ii+1}$ for $i \in \mathbb{Z}_n$.

This also determines the matrix B. From the above data, we obtain a cluster algebra $A(\Sigma)$. $A(\Sigma)$ has a presentation

$$\mathbb{Z}[\Delta_{13}, \ldots, \Delta_{1n-1}, \Delta_{12}, \Delta_{23}, \ldots, \Delta_{1n}] \big/ < \Delta_{1j}\Delta_{j-1j+1} = \Delta_{1j-1}\Delta_{jj+1} + \Delta_{1j+1}\Delta_{j-1j} >$$

Recall from Chapter 1 that we were able to obtain all Plucker coordinates and relations through mutation (Lemma 1.2.2). Therefore $A(\Sigma)$ can be pre-

sented as $Sym^{\bullet}(\bigwedge^2 \mathbb{C}^n)/Q$ *where* Q *is the Plucker ideal. This establishes* $A(\Sigma) \cong \mathbb{C}[Gr(2,n)]$. *We will henceforth call this cluster algebra* $\mathbb{C}[Gr(2,n)]$.

Here is a nice table that summarises all the different types of cluster algebras we saw in this section:

Cluster Algebra	Matrix	Ambient Field	Diagram
General	sign-skew-symmetrisable	$\mathbb{Z}G(\mathbf{x})$, G any semifield	n/a
Geometric Type	skew-symmetrisable	$\mathbb{ZP}(\mathbf{x})$, $\mathbb{P}$ tropical	weighted ice quiver
Quiver type	skew symmetric	$\mathbb{Z}[\mathbf{c}^{\pm}](\mathbf{x})$	ice quivers

2.2 Cluster Algebras of Finite Type

As we have already seen, cluster algebras can be quite complicated objects to study. It would be useful if we could deduce properties of a larger cluster algebra by looking at certain subalgebras. The purpose of this section is to show how we can reduce the complexity of a particular problem: does a cluster algebra A have finitely many clusters? To do this, we will introduce two operations on cluster algebras called restriction and quotient.

2.2.1 Restriction and the Exchange Graph

Definition 2.2.1. The rank of a cluster algebra is the number of cluster variables in a cluster.

Definition 2.2.2. Let $B = (b_{kj})$ be a $n \times m$ ($n \leq m$) matrix where the principal $n \times n$ part is skew-symmeterisable. Then the **restriction of** B at n, denoted by $Res_n(B) := (b_{kj})$ for $k \in [n-1]$ and $j \in [m]$. In words, the restriction of B at n is the submatrix given by the first $n-1$ rows.

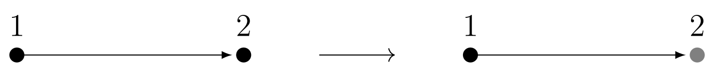
Definition 2.2.3. Let A be a cluster algebra of geometric type (See Definition 2.1.10) with cluster $\mathbf{x} =$
$(x_1, \dots, x_n, y, x_{n+1}, \dots, x_m)$ of rank n+1 and exchange matrix B_A. The **restriction of** A at y is a cluster algebra Res_yA with the cluster $\mathbf{x} = (x_1, \dots, x_n, y, x_{n+1}, \dots, x_m)$ of rank n and exchange matrix $B_{Res_A} := Res_y(B_A)$.

In words, the restriction of A at y, called Res_yA, is the cluster algebra with the same cluster as A obtained by declaring y as a stable variable.

Remark 2.2.1. We also obtain an operation on the weighted ice quiver of B_A, $\Gamma(B)$. We 'redraw' the vertex y as a frozen vertex and erase all edges between frozen vertices formed by this redrawing. The resulting quiver is an 'induced subgraph' of the original quiver.

Definition 2.2.4. Let Q be an ice quiver with vertices V and directed edges E. An **induced subgraph** D of Q consists of vertices $V' \subset V$ and $i \to j \in E'$ if and only if $i \to j \in E$ and $i, j \in V'$ with the one exception that i and j are both frozen.

Example 2.2.1. *Let A be a cluster algebra coming from A_2 (Example 1.1.8). There are no stable variables. $A \cong \mathbb{Z}[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}]$. We will restrict A at x_2: that is, we take a cluster (x_1, x_2) and declare x_2 to be stable. The cluster algebra $A' \cong \mathbb{Z}[x_1, x_1', x_2^{\pm}]/ < x_1x_1' - 1 - x_2 = 0 > \cong \mathbb{Z}[x_1, x_1']_{(x_1x_1'-1)}$. This is the same as the cluster algebra coming from A_1' (Example 1.1.5). The picture on quivers looks as follows:*



Proposition 2.2.1. *The restriction of a cluster algebra of quiver type A is the extension by scalars $A' \cong \mathbb{Z}[y^{\pm 1}] \otimes_{\mathbb{Z}[y]} A$.*

Proof. Through a slight abuse of notation, when we write x we mean some $x_i \in \mathbf{x} \setminus \{y\}$. We may construct a map from $A' \to \mathbb{Z}[y^{\pm 1}] \otimes_{\mathbb{Z}[y]} A$ by sending every A element $x \mapsto 1 \otimes x$ and $y^{-1} \mapsto y^{-1} \otimes 1$. Notice that $y \otimes 1 = 1 \otimes y$. We provide an inverse which will show that this map is an isomorphism. Any element f on the right hand side may be uniquely written as $\sum_i a_i y^{-1^{e_i}} \otimes \prod_j x_j^{f_{i,j}}$ where $x_j \in \mathbf{x} \setminus \{y\}$ and $e_i, f_{i,j} \in \mathbb{Z}$. In particular, y does not divide the right hand side product. The fact that y commutes across the tensor assure a unique expression. This means we may send any element $f \mapsto \sum_i a_i y^{-1^{e_i}} \prod_j x_j^{f_{i,j}} \in A'$ and this will be a well defined map. It is now easy to see that this provides an inverse. $\square$

Remark 2.2.2. The above proposition can be generalised to cluster algebras of geometric type with minor alterations.

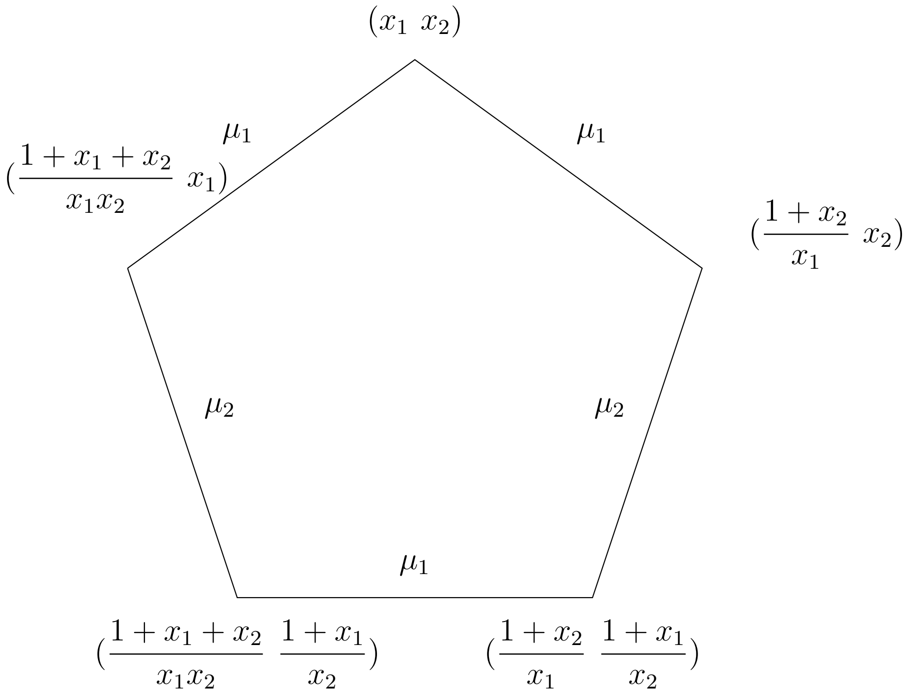
Example 2.2.2. *In Example 2.2.1, observe that A is actually a sub-algebra of A'. To see this, we just need to check that $\frac{1+x_1+x_2}{x_1x_2} \in A'$. We can rewrite this element as $x_2^{-1}(1+\frac{1+x_2}{x_1})$ which is indeed in A'. Using the map above, we can easily verify that $\mathbb{Z} \otimes_{\mathbb{Z}[y]} A \cong A'$.*

Restriction affects something called the "exchange graph" of a cluster graph. Denote by T_A the set of all seeds of some cluster algebra A (observe that this is equivalent to picking any seed and considering the set of all seeds mutation equivalent to it). Note that T_A and χ are in bijection (Lemma 1.1.3). Thus, for brevity, when we draw exchange graphs, we will often label the vertices by clusters as opposed to the whole seed.

Definition 2.2.5. Let A be a cluster algebra of geometric type with a cluster $\mathbf{x} = (x_1, \ldots, x_n, \ldots, x_m)$. The **exchange graph** $\Xi(A)$ is a graph with vertices $V = T_A$ and two seeds Σ_1 and Σ_2 are connected by an edge if and only if $\exists k \in [n]$ such that $\mu_k(\Sigma_1) = \Sigma_2$.

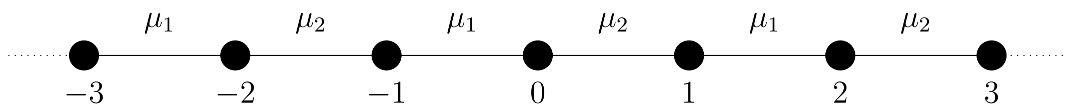
Remark 2.2.3. Notice now that we have two graphs. Firstly, we have the ice quiver (or weighted ice quiver) of a cluster algebra A. Secondly, we have the exchange graph whose vertices can be thought of as seeds (or equivalently clusters). These two graphs are related -this is something we will see later in this chapter. (Remark 2.2.13)

Example 2.2.3. *We have already seen the exchange graph of A_2, it is a pentagon:*



The exchange graph of A_2 coincides with the exchange graph of the cluster algebra coming from flips of a polygon (or as we have seen, the $\mathbb{Z}$-forms on $\mathbb{C}[Gr(2,n)]$ in Example 2.1.9).

Example 2.2.4. *The exchange graph of $\breve{A}_2$ (Example 1.1.9) is a line indexed by $\mathbb{Z}$:*



Example 2.2.5. *Consider an orientation Q of A_3 given by $1 \to 2 \to 3$ with 1 a frozen vertex and 2, 3 mutable. Let us restrict this quiver at 2 giving us the picture:*

$$1 \to 2 \to 3 \quad \longrightarrow \quad 1 \qquad 2 \to 3$$

The resulting quiver is no longer connected. Note that $\Xi(Q)$ is a pentagon whilst

$\Xi(Res_2(Q))$ is a copy of A_2 (the Dynkin diagram). This is a subgraph of $\Xi(Q)$. This is no coincidence.

Proposition 2.2.2. *Let A be a cluster algebra of geometric type with y a cluster variable of some cluster $\boldsymbol{x}$ and let $Res_y(A) = A'$. Then $\Xi(A')$ is a subgraph of $\Xi(A)$.*

Proof. First observe that any cluster of A' is also a cluster of A. This induces an inclusion of seeds from $T'_A \to T_A$ sending each cluster $\mathbf{x} \mapsto \mathbf{x}$ and exchange matrix $B_{A'} \mapsto B_A$. This map is well-defined on exchange matrices since if $Res_y(B_A) = Res_y(C_A)$ for $(\mathbf{x}, B), (\mathbf{x}, C) \in T_A$ then $B = C$.

Since T'_A and T_A are the vertices of $\Xi(A')$ and $\Xi(A)$ respectively, we have a map on vertices.

Similarly, if $\mu_k := \Sigma = (\mathbf{x}, B) \to \Omega$ is an edge in $\Xi(A')$, this corresponds to mutation at some $x_k \in \mathbf{x}$, where $x_k \neq y$. This is still a valid mutation in A. Therefore μ_k is an edge in $\Xi(A)$. Thus there is also an inclusion of edges. Thus $\Xi(A')$ is a subgraph of $\Xi(A)$. $\square$

Example 2.2.6. *$\mathcal{A}_{n-3}$ and $\mathbb{C}[Gr(2,n)]$ One observation we will make is that $\mathcal{A}_{n-3}$ (1.2) and $\mathbb{C}[Gr(2,n)]$ (Example 2.1.9) have the same exchange graph. $\mathcal{A}_{n-3}$ is precisely $\mathbb{C}[Gr(2,n)]$ restricted at the stable variables.*

2.2.2 Quotient

In this section we will introduce the quotient of a cluster algebra. We will learn the definition and compute some examples; relevant results however are in the next section.

Definition 2.2.6. Let A be a cluster algebra of geometric type with a cluster $\mathbf{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m, y)$ of rank n. Then a **quotient of A** at y, called $Quot_y A$ is a cluster algebra of rank n with clusters $\mathbf{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$ obtained by sending $y \mapsto 1$. $Quot_y A \cong {}^{A}\!/_{< y - 1 >}$.

We have the following short exact sequence of $\mathbb{ZP}$-modules (noting that the map $< y-1 > \to A$ depends on the exchange relations in A):

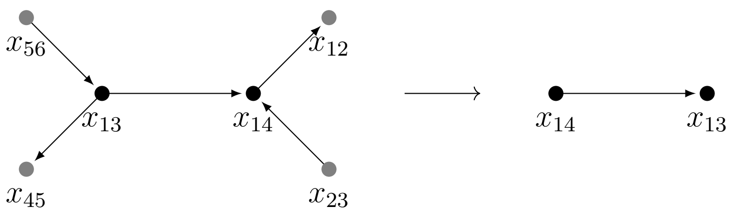
$$0 \to < y-1 > \to A \to Quot_y A \to 0$$

Remark 2.2.4. The exchange matrix $B_{Quot_y A}$ of $Quot_y A$ is the submatrix of B_A corresponding to the first m columns. We can check that if we define a seed of geometric type $\Sigma = (\mathbf{x}, B_{Quot_y A})$, then the cluster algebra $A(\Sigma) = Quot_y A$.
The corresponding quiver operation (when A is of quiver type) is to erase the frozen vertex corresponding to y. We call this "freezing".

Remark 2.2.5. We can also define the quotient at multiple variables $y_1, \ldots, y_k$ in the natural way since the order in which we do the quotient does not affect the seeds of the cluster algebra $Quot_{y_1,\ldots,y_k} A$.

Example 2.2.7. *Consider A_1 and A_1' from Chapter 1. (Example 1.1.5) It is clear that freezing the single stable vertex in A_1' returns precisely A_1. The map on algebras takes $\mathbb{Q} \otimes \mathbb{Z}[x, x', x_2^{\pm}] / < xx' = 1 + x_2 > \twoheadrightarrow \mathbb{Q} \otimes \mathbb{Z}[x, 2x^{-1}]$ by sending $x_2 \mapsto 1$.*

Example 2.2.8. *Let A_{n-3} be an orientation of the Dynkin diagram A_{n-3}. The cluster algebra $A(A_{n-3})$ can be obtained from $\mathcal{A}_{n-3}$ by quotienting out all the stable variables of $\mathcal{A}_{n-3}$ i.e. x_{ii+1} $\forall i \in \mathbb{Z}/n$. Since we showed that $\mathbb{C}[Gr(2,n)]$ and $\mathcal{A}_{n-3}$ differ only by an extension of scalars (Example 2.1.9, 1.2, Example 2.2.6), we can think of $\mathbb{C}[Gr(2,n)]$ as being the same 'type' as $\mathcal{A}_{n-3}$. Below is an example of this quotient on $\mathcal{A}_{n-3}$ (corresponding to $Gr(2,5)$).*



Example 2.2.9. *Consider the quiver with vertices $\{1, 2, 3\}$ where 2 is frozen with edges $1 \to 2$ and $2 \to 3$. If we take the quotient at 2, then the remaining graph is $A_1 \times A_1$.*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} \arrow[r, "Quot_2"] & \underset{1}{\bullet} & & \underset{3}{\bullet} \end{tikzcd}$$

This example shows that quotients may break connectedness.

Remark 2.2.6. An important quotient is the one obtained by freezing **all** the stable variables. Let A be a cluster algebra and let $\mathbf{x} = x \sqcup \mathbf{c}$ a cluster. One important type of quotient we will consider is $\mathbf{Quot(A)} := Quot_{\mathbf{c}}(A)$.

2.2.3 Cluster Algebras of Finite Type

Definition 2.2.7. A cluster algebra is **of finite type** if there are only finitely many distinct clusters.

Theorem 2.2.1. *[FZ03] Let A be a cluster algebra with a seed $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$. Then A is of finite type if and only if the weighted ice quiver $\Gamma(B)$ (See Definition 2.1.14) is mutation equivalent to some Q which is an orientation of a disjoint union of Dynkin diagrams (see 0.3).*

When $\Gamma(B)$ is mutation equivalent to some Q which is an orientation of a disjoint union of Dynkin diagrams Π_i, we say that A is of **type** $\prod_i \Pi_i$.

Remark 2.2.7. In Chapter 1, we saw how mutating seeds often resulted in the same quivers re-occuring despite different clusters generated with each mutation. The example of $\check{A}_2$ informed us that we cannot use finiteness of quiver mutation to conclude that the cluster algebra is of finite type. However, quiver mutation plays an important role in classification of cluster algebras of finite type.

Quotients and Finite Type

Lemma 2.2.1. *Let A be a cluster algebra of geometric type with a cluster $\boldsymbol{x} = x \sqcup \boldsymbol{c}$. Then the set of clusters of A and $Quot_{s \subset cA}$ are in bijection. That is, $\chi_A \leftrightarrow \chi_{Quot_{s\subset c}(A)}$ (See Remark 2.2.6 for notation).*

Proof. Let $\mathbf{x} = x \sqcup \mathbf{c} \in \chi_A$ be a cluster. Consider a set morphism $\pi : \chi_A \to \chi_{Quot(A)}$ by $\mathbf{x} = x \sqcup \mathbf{c} \mapsto x \sqcup \mathbf{c} \setminus \mathbf{s}$. It is easy to show this is a bijection by showing $x \sqcup \mathbf{c} \setminus \mathbf{s} \mapsto x \sqcup \mathbf{c}$ provides an inverse. □

Remark 2.2.8. Since quotients preserve cluster variables, we deduce that quotients also preserve the rank of a cluster algebra as well as the exchange graph. That is, $\Xi(A) = \Xi(Quot_{\mathbf{s}\subset\mathbf{c}})$.

In words, the content of this lemma implies that the exchange graph of a cluster algebra is not affected by quotients. In particular, the following result is immediate:

Lemma 2.2.2. *Let A be a cluster algebra of geometric type. Then A is of finite type if and only if $Quot(A)$ is of finite type.*

In other words, without any loss of generality, if we are interested in the exchange graph of a cluster algebra, say to check finiteness, it suffices to consider the quotient.

Example 2.2.10 ($Gr(2, n)$)**.** *We can now show $\mathbb{C}[Gr(2, n)]$ is of finite type. Let $R = \mathbb{C}[\Delta_{ii+1}]$. Since $R\otimes\mathbb{C}[Gr(2, n)] \cong \mathbb{C}\otimes_{\mathbb{Z}}\mathcal{A}_{n-3}$, we can just consider $\mathcal{A}_{n-3}$. (Nothing that R are coefficients.) By construction, $Quot(\mathcal{A}_{n-3})$ is a choice of orientation of the Dynkin diagram A_{n-3}. By [FZ2, Section 7], $Quot(\mathcal{A}_{n-3})$ is of finite type, and by the previous lemma, so is $\mathcal{A}_{n-3}$ and by extension, $\mathbb{C}[Gr(2, n)]$. Thus we say it is of type A.*

Lemma 2.2.3. *Let F and F' be two ambient fields over a semifield G. Let $\phi : F \to F'$ be a $\mathbb{Z}\mathbb{G}$-algebra isomorphism. Let $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$ be a seed in F. Then there exists a cluster algebra $A' \subset F'$ such that $A' \cong A$ via ϕ and $\chi_A \leftrightarrow \chi_{A'}$.*

Proof. We construct a seed $\Sigma' = (\mathbf{x}', \mathbf{p}', B')$ in F'.
Set $\mathbf{x}' := (\phi(x_i))_{x_i\in\mathbf{x}}$. Set $\mathbf{p}' = \mathbf{p}$. Note that $\phi(\mathbf{p}) = \mathbf{p}$.

Now let $x_k \in \mathbf{x}$ with an associated exchange relation $x_k x'_k = p_k^+ \prod_{b_{ik}>0} x_i^{b_i k} + p_k^- \prod_{b_{ik}<0} x_i^{|b_{ik}|}$. The image $\phi(x_k)\phi(x'_k) = p_k^+ \prod_{b_{ik}>0} \phi(x_i)^{b_{ik}} + p_k^- \prod_{b_{ik}<0} \phi(x_i)^{|b_{ik}|}$. Since B' comes from the exchange relation, we should set $B' = B$. Thus $\Sigma' = (\phi(\mathbf{x}), \mathbf{p}, B)$.

By construction, it is now a simple matter to check $\phi(\chi_A) = \chi_{A(\Sigma')}$. This implies that $\phi_A : A \to A'$ is an isomorphism. $\square$

Definition 2.2.8. Two cluster algebras $X \subset F_X$ and $Y \subset F_Y$ over a semifield G are **strongly isomorphic** if there exists a $\mathbb{Z}G$-algebra isomorphism $F_X \to F_Y$ sending some seed Σ to an isomorphic seed in Y.

Lemma 2.2.4. *Let F be an ambient field G. Let $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$ be a seed in F. Let $\Sigma_\sigma = (\sigma(\boldsymbol{x}), \sigma(\boldsymbol{p}), \sigma(B))$, $\sigma \in S_n$ acting by permuting entries of $\boldsymbol{x}$ and $\boldsymbol{p}$ and simultaneously permuting rows and columns of B, be another seed in F. Then $A(\Sigma) = A(\Sigma_\sigma)$.*

Proof. It suffices to prove that $\chi_{A(\Sigma)} = \chi_{A\Sigma_\sigma}$ and $\mathcal{P}_{A(\Sigma)} = \mathcal{P}_{A(\Sigma)}$. These two statements have identical proofs so we will do the first one. Let $s \in \chi_{A(\Sigma)}$. Thus $s \in \mathbf{s} \in \Omega \sim \Sigma$. This implies that $s \in \sigma(\mathbf{s}) \in \Omega_\sigma \sim \Sigma_\sigma$. Thus $s \in \chi_{A\Sigma_\sigma}$. □

Proposition 2.2.3. *Let G be a semifield. Let X and Y be strongly isomorphic cluster algebras over G. Then $X \cong Y$ as rings.*
Furthermore, if $A(\Sigma)$ be the cluster algebra coming from some seed $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$, then A is determined by B and $\mathbf{p}$ up to strong isomorphism. We write $A = A(B, \boldsymbol{p})$.

Proof. Lemma 2.2.3 and Lemma 2.2.4 implies that $Y \cong \phi(X) \cong X$.
For the second statement, it suffices to prove that given any two seeds $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ and $\Sigma' = (\mathbf{y}, \mathbf{p}, B)$, $A(\Sigma)$ and $A(\Sigma')$ are strongly isomorphic.

Since $\mathbf{x}$ and $\mathbf{y}$ are different transcendence bases for F, let us define a $\mathbb{Z}G$-automorphism $\rho : F \to F$ by sending $x_i \mapsto y_i$. Then it follows from Lemma 2.2.3 that $A(\Sigma') = \rho(A(\Sigma)) \cong A(\Sigma)$. □

Definition 2.2.9. Fix a semifield $\mathbb{G}$. A series $\mathcal{A}(B, -)$ of cluster algebras is a **family** of cluster algebras with a fixed $n \times n$ sign-skew-symmetrisable matrix B with integer entries and we allow $\mathbf{p}$ to vary.

Remark 2.2.9. Observe that the definition of a family of cluster algebras is well defined since any two cluster algebras with seeds sharing B and $\mathbf{p}$ are automatically strongly isomorphic.

Lemma 2.2.5. *Let A be a cluster algebra of geometric type. Then up to strong isomorphism, $\forall R, S \in A(B, -)$, $Quot_{\boldsymbol{c}}R \cong Quot_{\boldsymbol{c}}S$. We denote this cluster algebra $Quot(A(B, -))$.*

Remark 2.2.10. This lemma implies that series of cluster algebras of quiver type are 'determined' by the choice of some ice quiver of full rank.

Proof. Let $R, S \in A(B, -)$. Since the choice of cluster variables depends on the matrix B, and R, S, belong to the same series, this implies that (up to strong isomorphism) R and S share the same cluster variables. We may then pick clusters from R and S given by $\mathbf{x}_R = x \cup \mathbf{c}_R$ and $\mathbf{x}_S = x \cup \mathbf{c}_S$. From here it is clear that freezing all stable variables from R and S would result in strongly isomorphic cluster algebras. □

Remark 2.2.11. **To briefly summarise this section**: Lemma 2.2.2 and Theorem 2.2.1 tells us in order to show that a cluster algebra of geometric type A is of finite type, it suffices to check whether $\Gamma(Quot(A))$ is mutation equivalent to some orientation of a disjoint union of Dynkin diagrams. In particular, cluster algebras of a given Dynkin type Π all belong to a family of cluster algebras $\mathcal{A}(B, -)$. Lemma 2.2.5 then tells us every family of cluster algebras of geometric type correspond to the quotient which freezes all stable variables. For convention, when it comes to proving things of finite type, we will assume we have already taken the quotient.

Restriction and Finite Type

Let $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ be of geometric type. One common (implicit) technique used throughout cluster algebra literature (for instance in Sections 7-9 of [FZ03]) is the idea that we can look at certain subgraphs of $\Gamma(B)$ to deduce whether $A(\Sigma)$ is of infinite type.

Example 2.2.11. *Simply being a subgraph is insufficient. For instance, consider the following two quivers:*

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 4 \arrow[r] & 3 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 4 \arrow[r] & 3 \arrow[ul] \end{tikzcd}$$

The cluster algebra associated to the one on the right is of finite type. But the cluster algebra coming from the subgraph on the left is of infinite type! (Example 2.2.14).

In fact, we should look at induced subgraphs (Definition 2.2.4). To prove this, we will use restriction (it is easy to check that restriction gives all possible induced subgraphs).

Remark 2.2.12. Recall that in our discussion of restriction, for some cluster algebra of geometric type A, we showed that $\Xi(Res_y(A))$ is a subgraph of $\Xi(A)$. This gives us a sufficient condition for a cluster algebra to be of infinite type.

Lemma 2.2.6. *A cluster algebra of geometric type A is of infinite type if and only if $\Xi(A)$ is infinite.*

Proof. Immediate as the vertices of $\Xi(A)$ are exactly the seeds of A. □

Proposition 2.2.4. *Let A be a cluster algebra of geometric type. Let $y_1, \dots, y_k$ be cluster variables such that $A' = Res_{y_1,\dots,y_k}(A)$ is of infinite type. Then A is of infinite type.*

In particular, if A is of quiver type, then if $Q(A)$ has an induced subgraph whose cluster algebra is of infinite type, then A is of infinite type.

Proof. If A' is of infinite type then $\Xi(A')$ is infinite. Then $\Xi(A') \subset \Xi(A)$ implies $\Xi(A)$ is infinite.

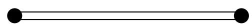
If A is of quiver type, let $T \subset Q(A)$ an induced subgraph with vertices $V' \subset V$ where $A' := A(T)$ is of infinite type. Since A' is the restriction of A at V', A is also of infinite type. □

Remark 2.2.13. More generally, it is easy to see how the previous two propositions imply that a cluster algebra (of quiver type) is of infinite type if a sub-cluster algebra is of infinite type. Equivalently, any sub-cluster algebra of a cluster algebra of finite type is also of finite type.

For cluster algebras of quiver type $A(Q)$, it suffices for us to check if some induced subgraph $T \subset Q$ has an associated cluster algebra $A(T)$ which is of infinite type.

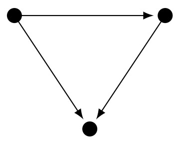
Now let us consider some induced subgraphs of quivers for cluster algebras of quiver type:

Example 2.2.12 (Double lines).



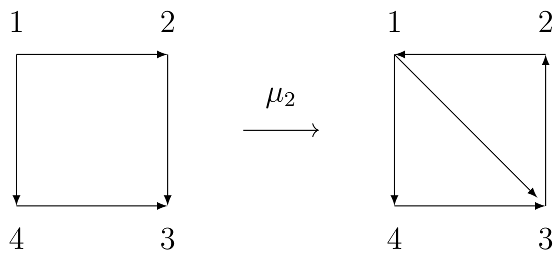
We proved that $A(\check{A}_2)$ is of infinite type in Example 1.1.9. Our work done throughout the previous section allows us to conclude that if a quiver Q has an induced subgraph isomorphic to $A(\check{A}_2)$ then $A(Q)$ is of infinite type.

Example 2.2.13 (Bad triangles).

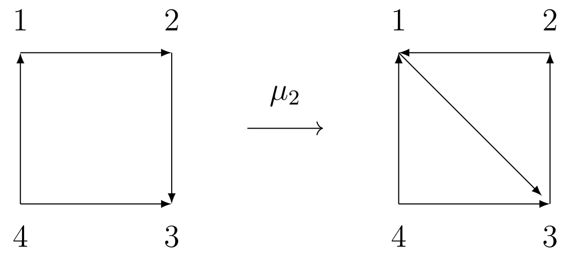


Bad triangles are mutation equivalent (mutate at the vertex with a path through it) to something with a $\check{A}_2$ subgraph and therefore are mutation infinite.

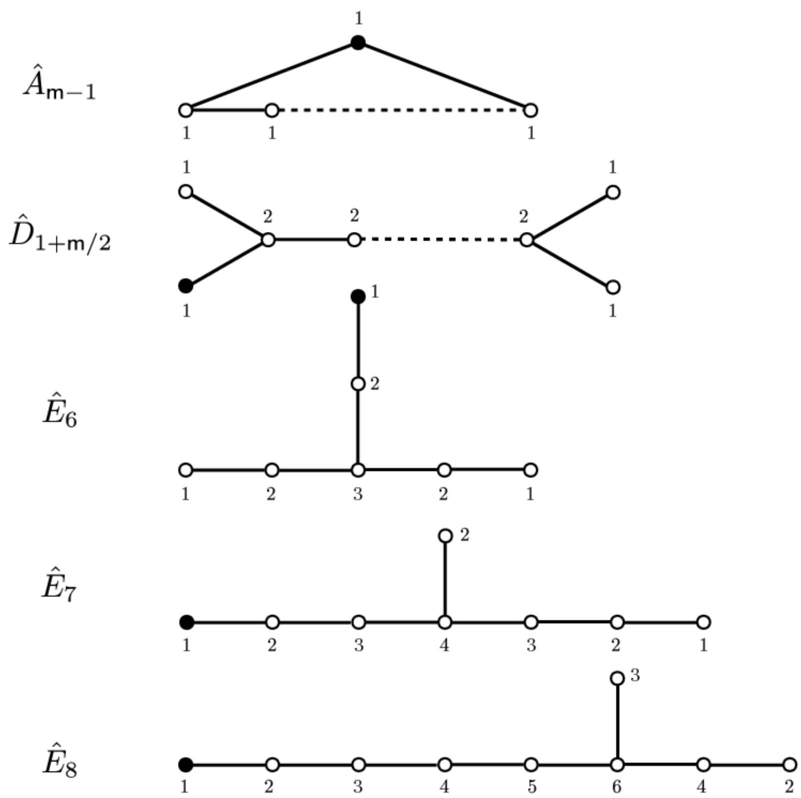
Example 2.2.14 (Any non-cyclic n-gon with no chords). *We can mutate any non-cylic n-gon without chords into something with a bad triangle. Let us see this for the square:*
Case 1:



Case 2:

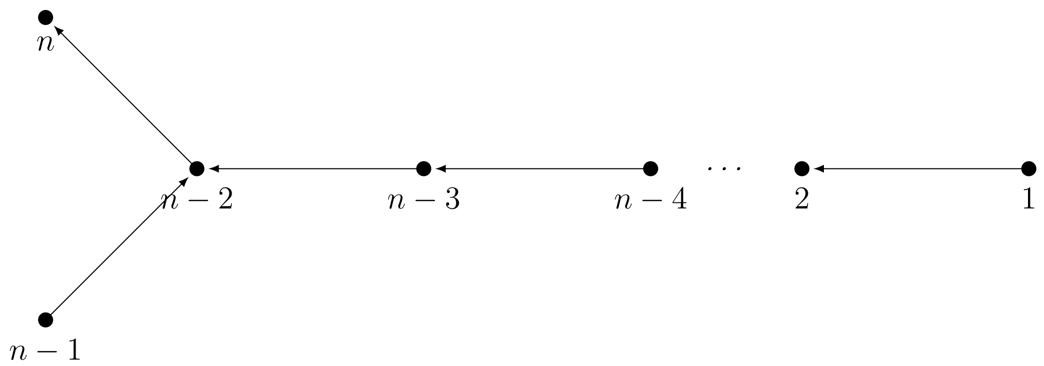


Example 2.2.15 (Extended Dynkin diagrams)**.** *Extended Dynkin diagrams classify affine Lie algebras. These diagrams are also of infinite type. Below we have a picture sourced from the internet (ignore the labelling) (https://inspirehep.net/record/1243914/plots):*

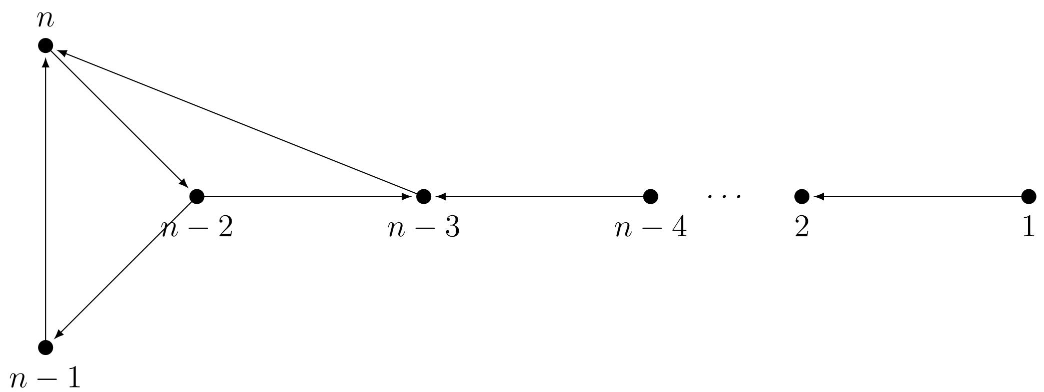


There is one exception which is when the chosen orientation of $A_n^{(1)}$ is cyclic; any oriented n-gon is mutation equivalent to D_n. This has a nice enough proof:

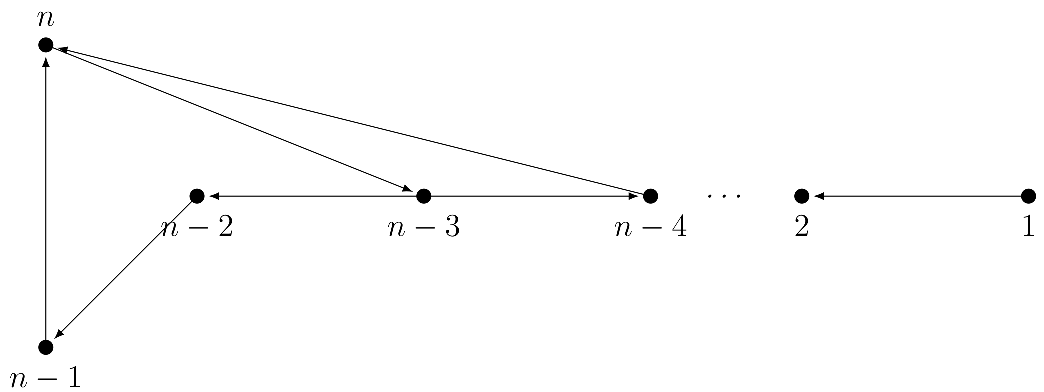
Pick the following orientation and labelling of D_n:



Now mutating at $n-2$ gives the following picture:



Now mutate at $n-3$.



Observe that mutating at $n-3$ creates a cyclic square with vertices $n-3, n-2, n-1, n$ joined along $n \to n-3$ to a cyclic triangle with vertices $n-4, n-3, n$. Next we mutate at $n-4$ for a pentagon and proceed inductively all the way to 1 -this will create a cyclic n-gon.

Chapter 3

Reduced Words and Double Bruhat Cells

For this section, background knowledge comes from Fulton and Harris [Ful04]. For construction of quivers associated to reduced words, we follow the algorithm laid out in [SSVZ99]. Chapter 2 of Cluster Algebras and Poisson Geometry [Gek10] also provides an account of this process.

An Outline of Main Results

The main purpose of Chapter 3 is the construction of a cluster algebra of geometric type $A(Q)$ which is associated to certain elements of a Coxeter group W. When W is a Weyl group, this cluster algebra is isomorphic to the coordinate ring of a certain affine variety (Theorem 3.3.1) called a reduced double Bruhat cell $L^{u,v}$. We will also prove certain combinatorial properties of the quiver Q (Prop 3.2.2) as well as compute the classification of all cluster algebras associated to S_2 and S_3 (Example 3.2.5). Another highlight is an important counterexample in S_4 (Example 3.2.6) which shows how deceptively 'nice' quivers can have complicated cluster structures. Finally we show how different reduced words for some element of a Coxeter group corresponds to mutation of the quiver (Theorem 3.4.1).

3.1 Double Bruhat Cells

Here is a simple question from linear algebra: when can we write some invertible $n \times n$ matrix M as the product of a lower and upper triangular matrix, L and U

respectively? We can phrase this more precisely. Let $G = SL_n(\mathbb{C})$ and consider subgroups of G given by the upper triangular matrices $B_+ \subset G$ and lower triangular matrices $B_- \subset G$. Then a matrix $M \in G$ can be LU factorised if $M \in B_-B_+$. A matrix $M \in G$ can be both LU and UL factorised if $M \in B_-B_+ \cap B_+B_-$. We can give more detail if we establish a little more machinery.

Let G a complex semisimple Lie group over an algebraically closed field k whose Weyl group W is simply laced (Definition 3.2.1). Let Π be the Dynkin diagram of W. Then the **Bruhat Decomposition** $G = \bigcup_{w \in W} B_+wB_+ = \bigcup_{w \in W} B_-wB_-$ decomposes G into a disjoint union of double cosets. Observe these give different partitions of G. Each $B_\pm wB_\pm$ is called a **Bruhat cell**. We will also consider subgroups $N_+ \subset B_+ \subset G$ and $N_- \subset B_- \subset G$. These $N_\pm$ are the **unipotent** subgroup of G. In the SL_n case these are the upper (resp. lower) triangular matrices with 1 on the diagonal.

The Bruhat decomposition tells us that any matrix $M \in G$ can be written as a product U_1wU_2, where $U_1, U_2 \in B_+$ and $w \in S_n$, the symmetric group on n letters. This tells us that any matrix in G can be written as a product of an upper triangular followed by some permutation of columns of upper triangular matrices. We also know that $w_0B_+ = B_-w_0$, where w_0 is the longest element of S_n. This means the Bruhat cell $B_+w_0B_+$ is exactly matrices that can be LU factorised after applying reversing all columns. Intuitively, Bruhat cells give an indication of the failure of a matrix to be LU factorised. Once again we can give a more detailed account if we introduce a few more definitions.

Definition 3.1.1. Given a pair $(u, v) \in W^2$, we define the **double Bruhat cell** as $G^{(u,v)} = B_+uB_+ \cap B_-vB_-$.

Definition 3.1.2. Given a pair $(u, v) \in W^2$, the **reduced double Bruhat cell** is $L^{(u,v)} = N_+uN_+ \cap B_-vB_-$.

Some foundational properties of $G^{u,v}$ were previous described in the Preliminaries 0.3.5. Here, we will establish that the ring of regular functions $\mathcal{O}(L^{u,v})$ is generated by generalised minors, following Part 4 of [FZ98]. The proof of this result is quite technical; our brief description here omits a considerable amount of detail but some exposition on this subject is necessary to establish groundwork for the rest of this chapter.

First we describe regular functions on the Bruhat cell B_+wB_+ (and thus also B_-wB_-).

Let $x \in SL_n$. Let $\Delta_{I,J}(x)$ be the minor with rows in I and columns in J for $I, J \subset [n]$ with the same cardinality. These are in fact regular functions on SL_n. Now if $w \in S_n$, denote by $w[i]$ the action of w on $\{1, \ldots, i\}$. The double Bruhat cell is identified inside SL_n by specifying certain vanishing conditions for $\Delta_{I,J}$.

Proposition 3.1.1 (Proposition 4.1 [FZ98]). $\mathbb{C}[B_+wB_+] \cong$ $\mathbb{C}[\Delta_{I,J}]_{(\Delta_{w[i],[i]})_{i\in[n-1]}} \big/ < \Delta_{w([i-1]\cup\{j\}),[i]} >$ *for* (i, j) *such that* $1 \leq i < j \leq n$ *and* $w(i) < w(j)$.

Now applying the transpose map $x \to x^t$ sends a minor $\Delta_{I,J}$ to $\Delta_{J,I}$ which gives a description of $\mathbb{C}[B_-wB_-]$. Furthermore, adding the condition that x is a permutation of some unipotent matrix i.e. $\Delta_{w[i],[i]} = 1$, we can also describe $\mathbb{C}[N_+wN_+]$. Since double Bruhat cells are an intersection of some B_+uB_+ and B_-vB_- we have a description of the coordinate ring for the double Bruhat cell. This then gives us a description of $\mathcal{O}(L^{u,v}) \cong \mathbb{C}[N_+uN_+ \cap B_-uB_-]$.

We will construct a cluster algebra A from some Coxeter group and later establish that the ring of regular functions on the reduced double Bruhat cell $L^{u,v}$ is in fact isomorphic to A. In general, $G^{u,v}$ also has a cluster algebra structure but we will not deal with it here (See [BFZ03], Section 2).

3.2 Cluster algebras from Coxeter groups

Definition 3.2.1. Let Π be a graph without loops or multiple edges. A group $W := W(\Pi)$ is a simply laced **Coxeter group** if W has a presentation consisting of s_i, $i \in V(\Pi)$ with the relations $s_is_j = s_js_i$ iff $i, j \notin E(\Pi)$, $s_is_js_i = s_js_is_j$ iff $i, j \in E(\Pi)$ and $s_i^2 = 1$.
We write $W = \langle s_i | s_is_j = s_js_i \text{ for } i, j \in E(\Pi),\ s_is_js_i = s_js_is_j \text{ for } i, j \in E(\Pi),\ s_i^2 = 1\rangle$

Example 3.2.1. *Let* Π *be the Dynkin diagram* A_n. *Let* $W = S_n$, *the symmetric group on n letters. Any element of the symmetric group can be written as a product of transpositions so we can give* S_4 *a presentation where the generators are the transpositions* $s_i = (i,\ i + 1)$. *To check that* S_n *is a simply laced Coxeter group:*

1. *We know that a pair s_i, s_j commute if and only if they are disjoint. Thus $s_i s_j = s_j s_i$ iff $|i-j| \geq 2$.*

2. *We can also compute $s_i^2 = 1$*

3. *s_i also satisfies the braid relation.*
   Observe that $(i,\ i+1)(i+1,\ i+2)(i,\ i+1) = (i,\ i+2)$ whilst $(i+1,\ i+2)(i,\ i+1)(i+1,\ i+2) = (i,\ i+2)$. This shows $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$. For the only if direction, suppose $|i-j| > 1$ then $(i,\ i+1)(j,\ j+1)(i,\ i+1) = (j,\ j+1)$. Similarly, $(j,\ j+1)(i,\ i+1)(j,\ j+1) = (i,\ i+1)$ but $s_i \neq s_j$. So we are done.

Definition 3.2.2 (Word decomposition)**.** Given $w \in W$ we call any expression of $w = s_{i_1} \cdots s_{i_k}$ (where s_{i_j} is a generator from the above presentation) a **word decomposition** of w. We say that such a decomposition into products of generators is reduced when k is minimal.

Now to establish some convention, we will write a permutation $\sigma \in S_n$ in product notation that is: $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$.

Example 3.2.2. *We consider 1342$\in S_4$. To compute a word decomposition, we start with the word 1234 and consecutively apply transpositions on the word until we get 1324.*
$1234 \xrightarrow{s_3} 1243 \xrightarrow{s_2} 1423 \xrightarrow{s_3} 1432 \xrightarrow{s_2} 1342$
A word decomposition of 1342 is then given by $s_3 s_2 s_3 s_2$. Notice however this is not a reduced decomposition. If we apply the braid relation we can get $s_2 s_3 s_2 s_2 = s_2 s_3$.

Example 3.2.3. *Consider $W = S_3$. The longest element $w_0 = 321$ has a reduced word decomposition $s_1 s_2 s_1$. To see this, we start from the word 123 and apply the transpositions $s_1 s_2 s_1$. This gives us:*
$123 \underset{s_1}{\longrightarrow} 213 \underset{s_2}{\longrightarrow} 231 \underset{s_1}{\longrightarrow} 321$
This decomposition is reduced in that there are no further relations we can apply which would reduce its length to less than three. Equivalently, we could have also obtained a reduced word decomposition $s_2 s_1 s_2$ as:
$123 \underset{s_2}{\longrightarrow} 132 \underset{s_1}{\longrightarrow} 312 \underset{s_2}{\longrightarrow} 321.$
Observe how the two reduced words differ by a braid move.

Let F be the free group generated by s_i. **Denote the relations of the above Coxeter presentation by** $r_1,\ r_2,\ r_3$ **respectively in the order they appear**. We denote r_j^i the operation of applying the relation r_j at position i in some word $w = w_1w_2 \ldots w_n$. If it is not possible to apply the move, then r_j^i does nothing. Each r_j^i then gives a map of sets $\mathrm{F} \to \mathrm{F}$ where given $w \in \mathrm{F}$, $r_j^i(w)$ is the word one obtains after applying the relation r_j to w at the i^{th} position.

Proposition 3.2.1. *Let $w \in W$. Suppose $w = s_{i_1} \cdots s_{i_k}$. Let r_1 denote the two-move and r_2 denote the braid move. Then $\exists r_{j_1}, \ldots, r_{j_m}$ for $m \in \mathbb{N}$ and $j_k \in \{1, 2\}$ such that $r_{j_1} \cdots r_{j_m}(s_{i_1}, \ldots, s_{i_k})$ is reduced.*

Remark 3.2.1. This is a famous result due to Tits. See Chapter 1 of [Bjo00] for the proof.

Let $s, t \in \mathbb{N}$. Let's pick reduced word decompositions for $u, v \in W$ respectively $\mathbf{j} = s_{j_1} \cdots s_{j_s}$ and $\mathbf{k} = s_{k_1} \cdots s_{k_t}$. Henceforth we will no longer be writing reduced words in terms of products of elementary reflections of the Coxeter group but instead simplify notation by **writing the products in terms of tuples of indices**, that is $\mathbf{j} = (j_1, \ldots, j_s)$ and $\mathbf{k} = (k_1, \ldots, k_t)$. Sometimes we will omit the commas for brevity. Now we will consider the group $W^2 = W \times W$ which is also a simply laced Coxeter group with graph $\Pi \times Pi$. We will now describe reduced word decompositions on $W \times W$. We will construct an ice quiver associated to a reduced word of W^2. This construction is adapted from [SSVZ99].

Definition 3.2.3. Given a pair $(u, v) \in W^2$, a **reduced word** decomposition is a pairing $\mathbf{i} = (i_1, \ldots, i_n) := (\mathbf{j}, -\mathbf{k}) = (j_1, \ldots, j_s, -k_1, \ldots, -k_t)$.

Given $\mathbf{i} = (i_1, \ldots, i_n)$ a reduced word for $(u, v) \in W$, we say that $l \in [n]$ is **bounded** if $\exists k < l$ such that $|i_k| = |i_l|$ and **unbounded** otherwise. We denote by l^- the largest such $k \in 0, \ldots, l-1$. Here we use the convention that if l is **unbounded** then $l^- = 0$.

Keeping the above notation, define a map of sets $\theta : \{1, \ldots, n\} \to \{+, -\}$ where $\theta(i) = sign(i_k)$.

We now construct an ice quiver $Q = (V, E)$ depending on a pair of elements $(u, v) \in W$ as well as a reduced word $\mathbf{i}$ in the sense of Definition 3.2.3.

- We set $V = \{1, \ldots, m\}$ where the labels of each vertex are made as follows: Moving from left to right, the i^{th} vertex receives the label i. A vertex is mutable if it is bounded, and frozen otherwise.

- To construct E, we say that a pair $k, l \in E$ where $k < l$ iff one of the following three conditions are satisfied:

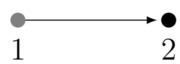
  1. $k = l^-$
  2. $k^- < l^- < k$, $\{|i_k|, |i_l|\} \in \Pi$,and $\theta(k) = \theta(l^-)$
  3. $l^- < k^- < k$, $\{|i_k|, |i_l|\} \in \Pi$,and $\theta(k) \neq \theta(k^-)$

  We draw edges of the first type horizontally, and edges of latter two types diagonally.
  Given a horizontal edge k, l, we orient $k \to l$ iff $\theta(k) = +$. Given a diagonal edge k, l, we orient $k \to l$ iff $\theta(k) = -$.

We denote this quiver by $Q(\mathbf{i})$. We denote the cluster algebra coming from mutations of this quiver $A(\mathbf{i})$.

Example 3.2.4 (Classification of all quivers and cluster algebras coming from S_2^2)**.** *Let $W = S_2$. All possible pairs are given by a = (e, e), b = (e, (1, 2)), c = ((1, 2), e), d = ((1, 2), (1, 2)). All corresponding reduced words are given by* $i_a = ()$, $i_b = (-1)$, $i_c = (1)$, $i_d = (1, -1)$. *Let us compute the quiver coming from each word.*

- $Q(i_a)$. *In this case, we just have the empty quiver. A cluster algebra coming from this quiver is just some choice of ring R.*
- $Q(i_b)$. *There is a single vertex and no edges. This is the Dynkin diagram A_1. A cluster algebra coming from this quiver is of the form $R[x, x^-1]$.*
- $Q(i_c)$. *Again we have A_1.*
- $Q(i_d)$. *There are two vertices. To compute the edges, we note that $1^- = 0$, $2^- = 1$. This implies 1 is stable and 2 is mutable. We also observe there is a single edge $1 \to 2$. The picture is thus:*

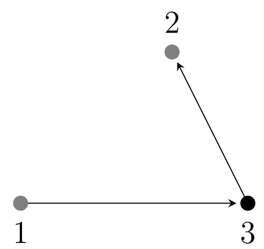


  A cluster algebra coming from this quiver is of the form $R[x_1, x_1', x_2^{\pm}]/ < x_1 x_1' = 1 + x_2 >$. We know this algebra as A_1'.

These are all the quivers coming from a pair of elements in S_2^2 using our algorithm.

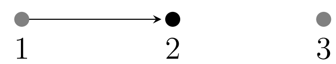
Example 3.2.5 (Classification of all quivers and cluster algebras coming from S_3^2)**.** *We break down into cases. There are 36 elements of S_3^2 but we only need to consider 30 cases as it is clear that $\forall x \in S_3$ (e,-x) and (x,e) will generate the same quiver. We will omit some calculations.*

- *The quivers coming from the reduced words 1, 2, 12, 21 will not have any edges. So we have either A_1 or A_1^2. Thus their cluster algebras will be isomorphic to some Laurent polynomial ring (in either one or two variables).*
- *$Q(121)$. There are three vertices where 1 and 2 are stable. We have edges $1 \to 3$, $3 \to 2$.*

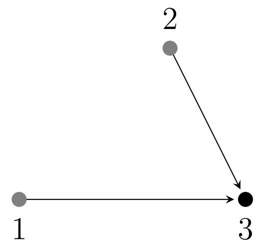


The corresponding cluster algebra has the form $R[x_1^{\pm}, x_2^{\pm}, x_3, x_3']/ < x_3x_3' = x_1 + x_2 >$.

- *$Q(1-1)$. There are two vertices where 1 is stable with an edge $1 \to 2$. This is the quiver A_1'.*
- *$Q(1-2)$. There are two vertices with no edges. This is A_1^2.*
- *$Q(1-1-2)$. We have three vertices where 1 and 3 are stable. We have a single edge $1 \to 2$. We call this quiver $A_1' \times A1$.*

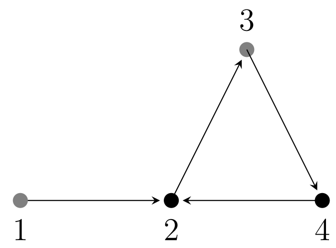


- $Q(1-2-1)$. *We have three vertices where 1 and 2 are stable. We have edges* $1 \to 3$, $2 \to 3$.



The corresponding cluster algebra has the form $R[x_1^{\pm}, x_2^{\pm}, x_3, x_3']/ < x_3x_3' = 1 + x_2x_1 >$.

- $Q(1-1-2-1)$. *There are 4 vertices where 1 and 3 are stable. We have edges* $1 \to 2$, $2 \to 3$, $4 \to 2$, $3 \to 4$.



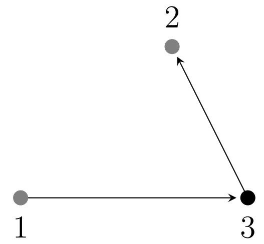
The cluster algebra is of type A_2.

- $Q(2-1)$. *This is isomorphic to* A_1^2.

- $Q(2-2)$. *This is isomorphic to* A_1'.

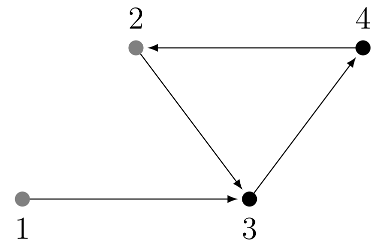
- $Q(2-1-2)$. *This is isomorphic to* $A_1' \times A_1$.

- $Q(2-2-1)$. *There are three vertices where 1 and 3 are frozen. We have edges* $1 \to 2$ *and* $2 \to 3$.

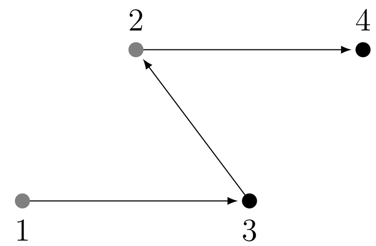


This is the same cluster algebras as $A(Q(121))$.

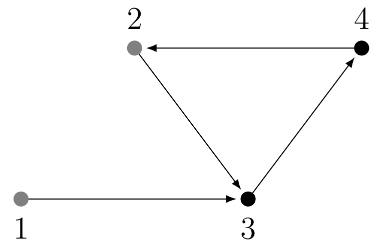
- $Q(2-1-2-1)$. *There are four vertices where 1 and 2 are stable. We have edges* $1 \to 3$, $2 \to 3$, $3 \to 4$, $4 \to 2$.



- $Q(12-1)$. *This has three vertices with 1 and 2 frozen. We have edges* $1 \to 3$ *and* $3 \to 2$. *This is the same quiver as* $Q(121)$.
- $Q(12-2)$. *This is isomorphic to* $A_1' \times A_1$.
- $Q(12-1-2)$. *1 and 2 are stable vertices. We have edges* $1 \to 3$, $3 \to 2$, $2 \to 4$.

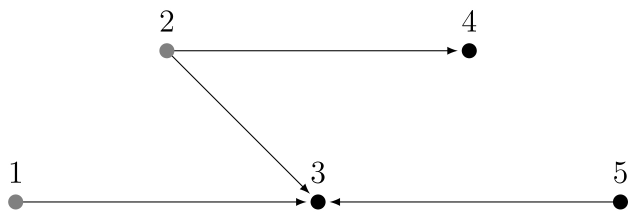


- $(12-2-1)$. *1 and 2 are stable vertices. We have edges* $1 \to 4$, $4 \to 2$, $2 \to 3$, $3 \to 4$.

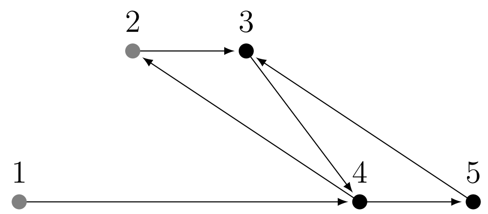


This is therefore isomorphic to $Q(2-1-1-2)$.

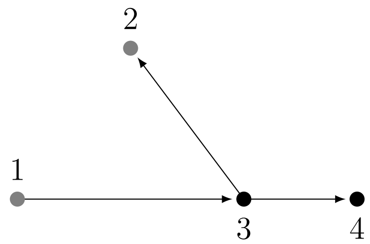
- $Q(12-1-2-1)$. *1 and 2 are stable vertices with arrows* $1 \to 3$, $5 \to 3$, $2 \to 3$, $2 \to 4$.



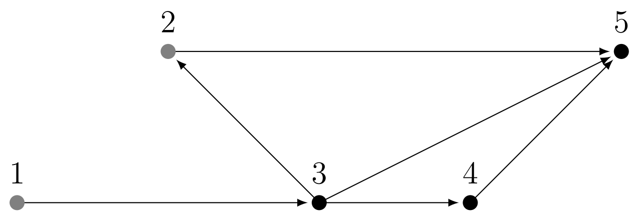
- $Q(21-1)$. *This is isomorphic to* $A_1' \times A_1$.
- $Q(21-2)$. *1 and 2 are stable with arrows* $1 \to 3$ *and* $3 \to 2$. *This is the same as* $Q(121)$.
- $Q(21-1-2)$. *1 and 2 are stable with arrows* $1 \to 4$, $4 \to 2$, $2 \to 3$, $3 \to 4$. *This is isomorphic to* $Q(21-1-2)$.
- $Q(21-2-1)$. *1 and 2 are stable with arrows* $1 \to 3$, $3 \to 2$, $2 \to 4$. *This is the same as* $Q(12-1-2)$.
- $Q(21-1-2-1)$. *1 and 2 are stable with arrows* $1 \to 4$, $4 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 5$, $5 \to 3$.



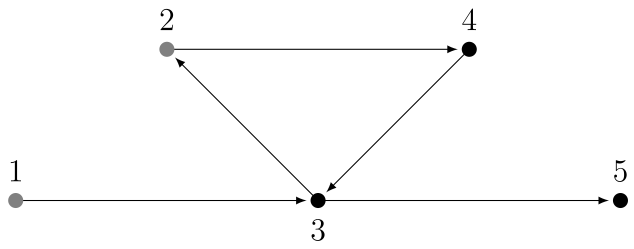
- $Q(121-1)$. *1 and 2 are stable with arrows* $1 \to 3$, $3 \to 4$, $3 \to 2$.



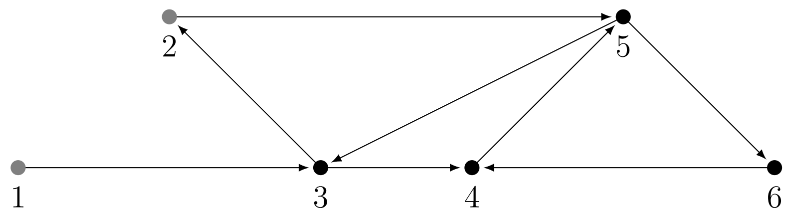
- $Q(121-2)$. *1 and 2 are stable with arrows* $1 \to 3$, $3 \to 2$, $2 \to 4$, $4 \to 3$. *This quiver is isomorphic to* $Q(21-1-2)$.
- $Q(121-1-2)$. *1 and 2 are stable with arrows* $1 \to 3$, $3 \to 4$, $4 \to 5$, $5 \to 3$, $3 \to 2$, $2 \to 5$.



- $Q(121-2-1)$. *1 and 2 are stable with arrows* $1 \to 3$, $3 \to 2$, $2 \to 4$, $4 \to 3$, $3 \to 5$.

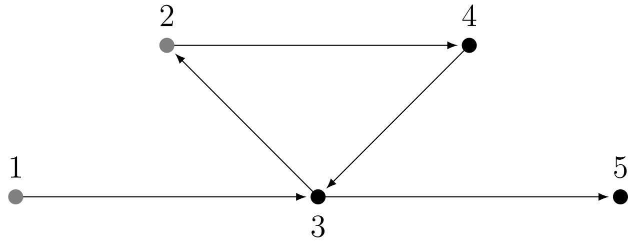


- $Q(121-1-2-1)$.

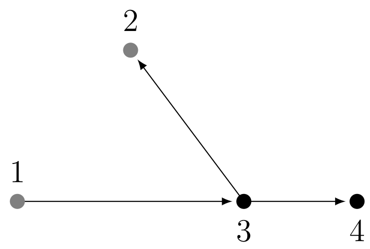


Some isomorphism classes of cluster algebras from above quivers (omitting the easy cases):

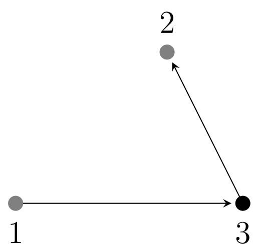
- $A(Q(21-1-2-1)) \cong A(Q(121-1-2)) \cong A(Q(121-2-1))$ *of type A.* *(A_3)*



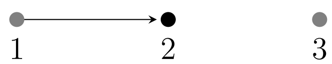
- $A(Q(12-2-1)) \cong A(Q(21-1-2)) \cong A(Q(121-2)) \cong A(Q(1-1-2-1)) \cong A(Q(2-1-2-1))$ *of type A.* *(A_2)*



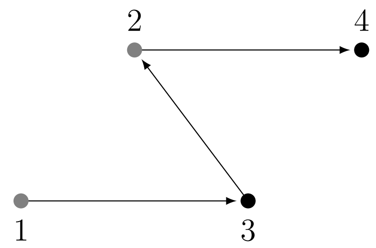
- $A(Q(12-1)) \cong A(Q(121)) \cong A(Q(21-2)) \cong A(Q(1-2-1)) \cong A(Q(2-2-1))$ *of type A.* *(A_1)*



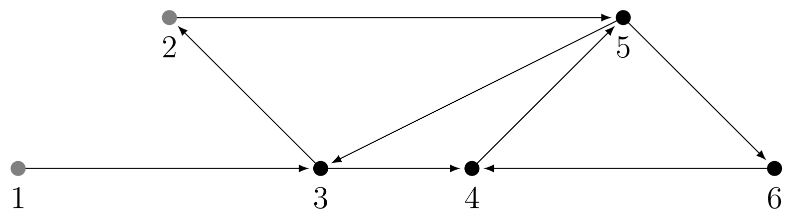
- $A(Q(21-1)) \cong A(Q(1-1-2)) \cong A(Q(2-1-2))$ *of type A.* *(A_1)*



- $A(Q(12-1-2)) \cong A(Q(21-2-1))$ *of type A. (A_1^2)*



- $A(Q(121-1-2-1))$ *of type D. (D_4)*



Mutating at 5 *gives an orientation of* D_4 *with two frozen vertices.*

Remark 3.2.2. We claimed that the above examples were a classification of cluster algebras from reduced words of $S_3 \times S_3$. Yet given a pair $(u, v) \in S_3^2$ there are different reduced words for (u, v) linked by braid moves. In other words, it is possible that different choices of reduced words for some $w \in W \times W$ lead to non-isomorphic cluster algebras.

It turns out (Theorem 3.4.1) braid moves of some reduced word $\mathbf{i}$ are equivalent to mutations on the quiver $Q(\mathbf{i})$ and thus the cluster algebra $A(\mathbf{i})$ does not depend on the choice of reduced word.

Remark 3.2.3. In particular, we see that all cluster algebras coming from reduced words of S_3^2 are of finite type (Theorem 2.2.1).

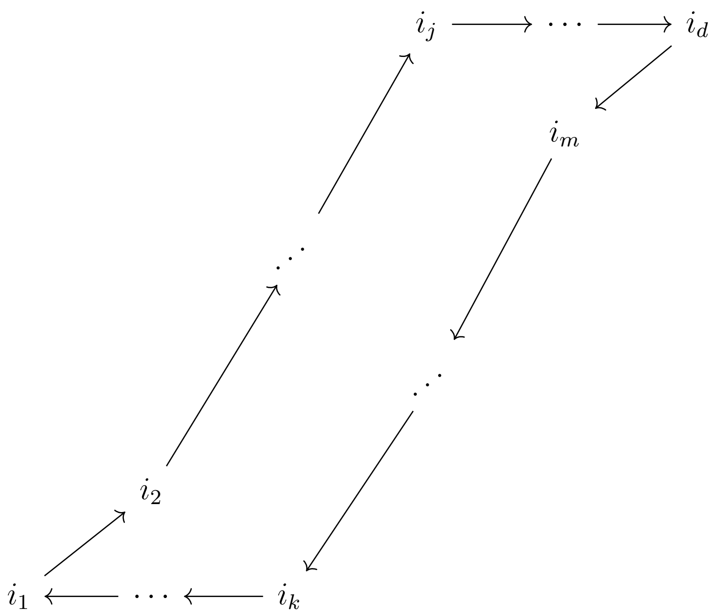
A **triangle** is a graph consisting of three vertices $\{i, j, k\}$ and edges $\{i, j\}$ $\{j, k\}$ $\{i, k\}$. An **n-gon** is a graph consisting of n vertices $\{1, \ldots, n\}$ and edges between $i, i+1$ for $i \in \mathbb{Z}/n$. A quiver Q is called an n-gon if as a graph, Q is an n-gon. We say that an **n-gon of a quiver** Q is an induced sub-quiver which is an n-gon.

Lemma 3.2.1. *Let W be a Weyl group whose Dynkin diagram Π is a tree. Let $\boldsymbol{i} \in W \times W$ a reduced word. Consider a quiver $Q(\boldsymbol{i})$ with an n-gon as an* ***induced*** *subgraph (recall Definition 2.2.4) with vertices $\{i_1, \ldots, i_k\}$. Then the number of distinct values any $\boldsymbol{i}_{ij}$ can take is 2, i.e. $\#\left\{|\boldsymbol{i}_{i_j}|\right\}_j = 2$.*

Remark 3.2.4. The content of the above lemma says that if there is an n-gon inside $Q(\mathbf{i})$ then all n vertices has to fit on two 'levels'.

Proof. First observe that if there are two vertices i_j, i_k with $i_j < i_k$ such that $|\mathbf{i}_{i_j}| = |\mathbf{i}_{i_k}|$ then there are additional (ordered) vertices $\{l_s\}_{s=1}^m$ all connected in a path: $i_j \leftrightarrow l_1 \leftrightarrow \cdots \leftrightarrow l_m \leftrightarrow i_k$. In words, this argument lets us eliminate the case where two 'intermediate' vertices (meaning they lie between the minimal and maximal 'levels') share a 'level'.

Now suppose $\#\left\{|\mathbf{i}_{i_j}|\right\}_j \geq 2$. The previous paragraph tells us that for all i_j, i_k where $|\mathbf{i}_{i_j}|, |\mathbf{i}_{i_k}| \neq \min\left\{|\mathbf{i}_{i_j}|\right\}_j$ and $|\mathbf{i}_{i_j}|, |\mathbf{i}_{i_k}| \neq \max\left\{|\mathbf{i}_{i_j}|\right\}_j$, then $|\mathbf{i}_{i_j}| \neq |\mathbf{i}_{i_k}|$.



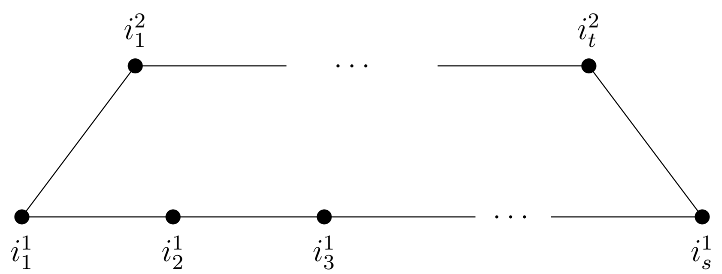
This then implies that if $\#\left\{|\mathbf{i}_{i_j}|\right\}_j > 2$ then Π contains a cycle with vertices $|\mathbf{i}_{i_1}|, \ldots, |\mathbf{i}_{i_j}|$ and is therefore not a tree. Thus $\#\left\{|\mathbf{i}_{i_j}|\right\}_j = 2$. $\square$

Remark 3.2.5. This lemma tells us that if there are any n-gons of $Q(\mathbf{i})$, it must lie on exactly two 'levels'. Next we will prove that any n-gon of $Q(\mathbf{i})$ is in fact an oriented cycle. The lemma greatly reduces the complexity and all that remains is an inductive argument which proceeds by checking every possible case.

Proposition 3.2.2. *Let W be a Weyl group whose Dynkin diagram Π is a tree. Let $\boldsymbol{i} \in W \times W$ a reduced word. Then every n-gon of $Q(\boldsymbol{i})$ is an oriented cycle.*

Proof. We will use induction on the number of vertices. Suppose for all $k < n$, every k-gon of $Q(\mathbf{i})$ is an oriented cycle.

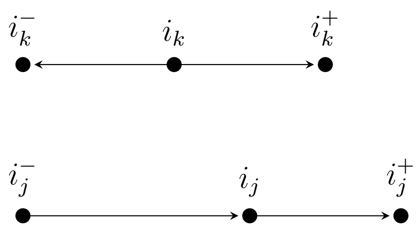
Consider an n-gon with vertices $i_1, \ldots, i_n$. By the previous lemma, we know that $\#\left\{|\mathbf{i}_{i_j}|\right\}_j = 2$. Then our n-gon will look like the below diagram (choosing an appropriate labelling):



Note that the labelling implies that $s + t = n$.

Suppose $\theta(i_1^1) = +$ and assume that $i_1^1 < i_1^2$.
If $\theta(i_1^2) = +$ then we want to show $\exists i_2^1 \to i_1^2$. We have that $i_1^2 \to i_2^2$ horizontally.
Note that $i_2^1- = i_1^1$ and $i_2^2- = i_1^2$. Now $i_1^2 \to i_1^1 \implies i_1^{2-} < i_1^1$.
If $i_1^2 < i_2^1$, then $i_1^{2-} < i_2^{1-} < i_1^2$ and $\theta(i_1^2) = + = \theta(i_2^{1-})$. Thus $i_2^1 \to i_1^2$. On the other hand, if $i_2^1 < i_1^2$, we come to a similar conclusion. Then using the inductive hypothesis we are done.

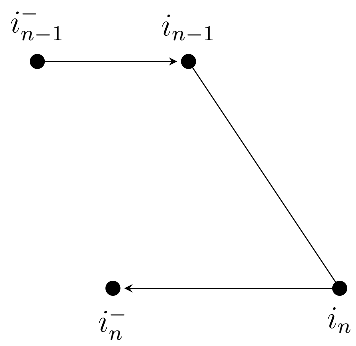
Else $\theta(i_1^2) = -$, in which case we proceed to $\theta(i_2^1)$, and so on, alternating between the two strands. We consider a case of some i_k where either $\theta(i_k) = -$ and i_k is on the top row or $\theta(i_k) = +$ and i_k is on the bottom row. (Note that it does not actually matter whether i_k is on the top or bottom.) We will consider the case of the below picture:



We can assume that $i_j > i_k$ (by virtue of how we picked i_k). We want to show $\exists i_j \to i_k$. We have that $\theta(i_j^-) = +$ and $\theta(i_k^-) = -$. If $i_j^- > i_k$, we check if $i_j^{--} > i_k$ and so on some m times until we find some $i_j^{-m} < i_k$ at which point we set $i_j := i_j^{-(m-1)}$.

If $i_k^- < i_j^- < i_k$. We also have that $\theta(i_j^-) = - = \theta(i_k)$ and so we are done. Else if $i_j^- < i_k^- < i_k$, we have that $\theta(i_k) \neq \theta(i_k^-)$. Thus we have an arrow $i_j \to i_k$ and we are done by induction.

Else there are no such i_k for $k < n-1$ and so we proceed until i_{n-1}. We can assume that $i_{n-1} < i_n$. We can assume that say $\theta(i_n^-) = -$ and $\theta(i_{n-1}^-) = +$. Also observe there is already an edge between i_{n-1} and i_n which implies that $i_n^- < i_{n-1}$. We must show that $i_{n-1} \to i_n$ which will then complete the proof.



If $i_n^- < i_{n-1}^- < i_{n-1}$, then $\theta(i_{n-1}) = -$ and $i_{n-1} \to i_n$. Otherwise $i_{n-1}^- < i_n^- < i_{n-1}$ with $\theta(i_{n-1}) = -$. Thus $i_{n-1} \to i_n$. This shows the n-gon is an oriented n-cycle and so we are done once we check the initial case for $n = 3$ i.e triangles are oriented.

Suppose for a contradiction, we can find a triangle with vertices i, j, k such that they do not form an oriented cycle. We can assume that $i < j < k$ as well as $i \to k \to j$ and $i \to j$.

- Suppose $i \to k$ is horizontal. This means $\theta(i) = +$ and $k^- = i$.
  If $i \to j$ is diagonal, then it forces $\theta(i) = -$, which is impossible. So $i \to j$ is horizontal. But then $i < j < k$ implies $k^- = j$ contradicting our assumption that $i \to k$ being horizontal.

- Suppose $i \to k$ is diagonal. This means $\theta(i) = -$.
  If $i \to j$ is horizontal, then it forces $\theta(i) = +$, which is impossible. So $i \to j$ must be diagonal. Thus $|i_i| \neq |i_j|$ and $|i_i| \neq |i_k|$ with at least two of $|i_i|, |i_j|, |i_k|$ vertices in Π with edges between them. We now consider two further subcases.
  Suppose $k \to j$ is horizontal. Then $j^- = k$. However a diagonal arrow $i \to j \implies j^- < i$. We now have $j^- < i < j < k$. This is absurd. Thus $k \to j$ must be diagonal. This means $|i_k| \neq |i_j|$. So in fact $|i_i|, |i_j|, |i_k| \in V(\Pi)$ with edges connecting all three vertices.

The above proves that Π must contain a triangle thus it is not a tree. So we are done.

□

A natural question to ask is: are the cluster algebras associated to reduced words of finite type? In our classification of S_2 and S_3, we might find this plausible. However there is a counter-example in S_4 where the quiver coming from $\mathbf{i} = (w_0, -w_0)$ contains a subgraph of infinite type. We will now compute the case of this example explicitly.

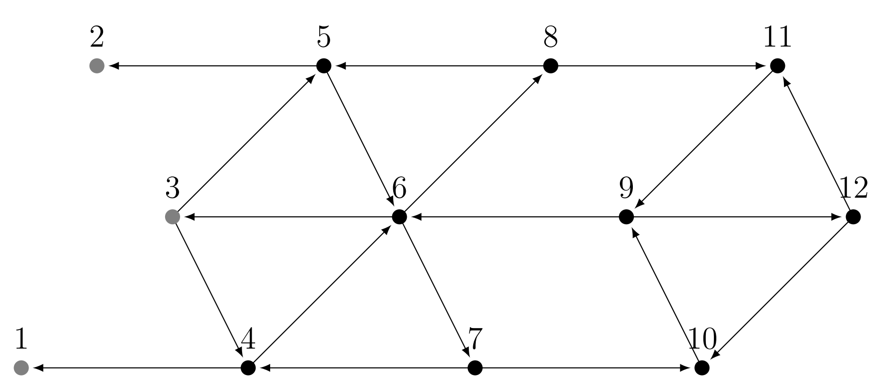
First we will introduce the idea of a 2-finite weighted quiver. A weighted ice quiver Q is **2-finite** if $\forall k \in V(Q)$, $\forall i \to j \in E(\mu_k(Q))$, the weight $w_{ij} \leq \sqrt{3}$.

Proposition 3.2.3. *Let A be a cluster algebra of geometric type with a seed $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$. Then A is of finite type if and only if the weighted ice quiver $\Gamma(B)$ is 2-finite.*

[FZ03], Section 7-9. □

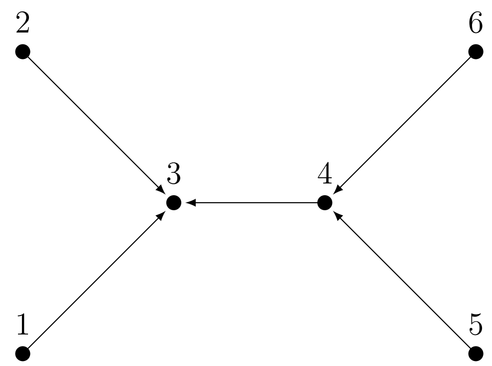
Remark 3.2.6. To prove this proposition, it suffices to show that $\Gamma(B)$ is 2-finite if and only if it is mutation equivalent (using weighted ice quiver mutation from Definition 2.1.15) to an orientation of a Dynkin Diagram (Preliminaries 0.3) after which we apply Theorem 2.2.1. The proof of this proposition is therefore straightforward but quite long as it involves checking all possible cases.

Example 3.2.6. *Consider the word $(-1-3-2-1-3-2132132) \in S_4 \times S_4$. This cluster algebra is isomorphic to the coordinate ring on the reduced double Bruhat cell L^{w_0,w_0} for S_4. This is something we will see in Theorem 3.3.1. Its quiver has the form:*

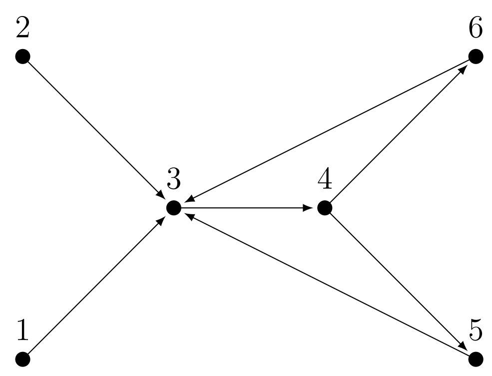


Note that vertices $4, 5, 6, 9, 10, 11$ *form a induced subgraph of type* $D_5^{(1)}$ *(an extended Dynkin diagram). If we show that* $D_5^{(1)}$ *is 2-infinite, then it will follow that* $Q(\boldsymbol{i})$ *is also 2-infinite. (Proposition 3.2.3 implies that* $D_n^{(1)}$ *is 2-infinite since it is clearly not mutation equivalent to a Dynkin diagram but let us do it explicitly here). To show its 2-infinity, it suffices to show there is a 'bad' triangle as an induced subgraph (recall Remark 2.2.13) since any 'bad' triangle is mutation equivalent to something with weight 2.*

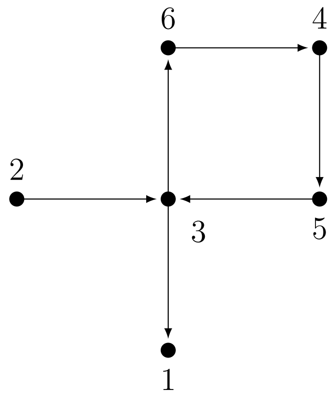
Below we have the induced subgraph of $D_5^{(1)}$ *inside* $Q(\boldsymbol{i})$ *(relabelled).*



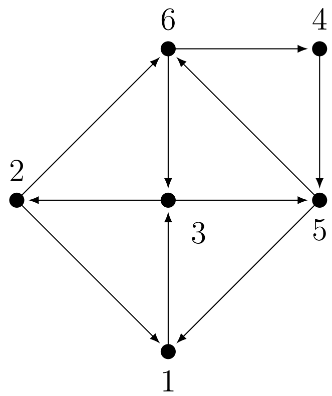
Mutate at 4:



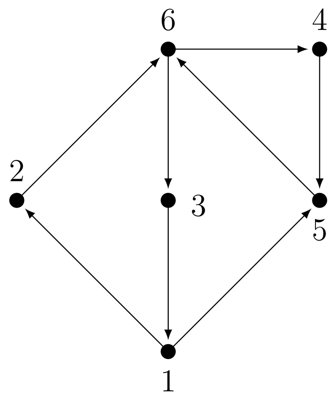
Mutate at 6 then 1:



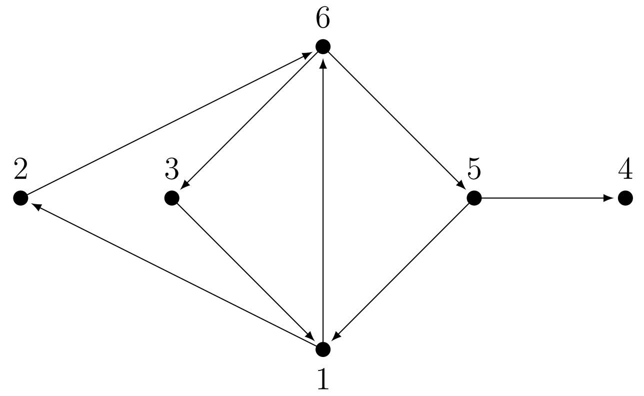
Mutate at 3:



Mutate at 1:



Mutate at 5:



Now observe that the vertices 1,2,6 form a 'bad triangle' which is of infinite type.

Thus $D_4^{(1)}$ *is of infinite type. Thus* $A(\boldsymbol{i})$ *is of infinite type.*

Thus from the above example, we see that not all cluster algebras coming from reduced words of a simply laced Coxeter group are well-behaved cluster algebras of finite type. This counterexample is quite unintuitive since other examples of quivers whose cluster algebras were of infinite type (Example 1.1.9, Example 2.2.14) tended to have either multiple edges or have non-oriented cycles.

There is a property which is similar to being finite type called 'acyclicity'. If A is acyclic, then there is a 'nice' way to present a cluster algebra.

Definition 3.2.4. A seed Σ of quiver type is **acyclic** if the quiver $Q(\Sigma)$ does not have any oriented cycles.

Example 3.2.7. A_2 *and* $\check{A}_2$ *(Example 1.1.9) are both acyclic. In general, any tree is acyclic.*

Definition 3.2.5. A cluster algebra A is **acyclic** if there exists a seed Σ such that Σ is acyclic.

Example 3.2.8. *Any quiver* Q *such that* $A(Q)$ *is of finite type is mutation to some* Q' *which is an orientation of a Dynkin diagram. Thus* Q' *is acyclic. This implies that any cluster algebra of finite type is also acyclic.*

Definition 3.2.6. Let Σ be a seed with cluster $\mathbf{x} = (x_1, \ldots, x_m)$. The **lower bound** of Σ is the ring $\mathcal{L}(\Sigma) = \mathbb{ZP}[x_1, x_1', \ldots, x_m, x_m'] \subset \mathcal{F}$ where $x_i x_i'$ satisfy the exchange relation at i.

Theorem 3.2.1. *Let Σ be a seed of quiver type. Then $\mathcal{L}(\Sigma) = A(\Sigma)$ if and only if Σ is acyclic.*

Proof. [BFZ03] Theorem 1.20 □

Remark 3.2.7. So if a cluster algebra A is acyclic, then we can present the whole ring as long as we have an acyclic seed Σ.

Proposition 3.2.4. *Let Σ be an acyclic seed. Then $A(\Sigma)$ is Noetherian.*

Proof. We know from Theorem 3.2.1 that $A(\Sigma)$ coincides with the lower bound, which is a finitely generated algebra over $\mathbb{ZP}$. Thus it suffices to show that $\mathbb{ZP}[t_1, \dots, t_{2n}]$ is Noetherian. Through the Hilbert Basis Theorem, all we need is for $\mathbb{ZP}$ to be Noetherian. Fortunately, we proved this at the start of Chapter 2 (2.1.1). □

Remark 3.2.8. So acyclicity is a sufficient condition for a cluster algebra to be Noetherian.

It turns out that when W is a finite simply laced Weyl group, many of the cluster algebras $A(\mathbf{i})$ are acyclic, for instance, any word of the form (w, w) for $w \in W$ ([BFZ03] Section 2.5). There is a conjectured counterexample in S_6 but we will not deal with it here ([BFZ03] Remark 2.17). In fact it is not known which $A(\mathbf{i})$ are acyclic in general.

Question: Let W be a simply laced Coxeter group. Let $\mathbf{i}$ be a reduced word for $(u, v) \in W^2$. When is $A(\mathbf{i})$ an acyclic cluster algebra?

3.3 Structure of the ring $\mathcal{O}(L^{u,v})$

Theorem 3.3.1. *Let G be a simply connected semisimple complex algebraic group. Let W be its Weyl group. The cluster algebra coming from a pair $(u, v) \in W$ is isomorphic to the coordinate ring of the reduced double Bruhat cell inside G, $L^{u,v}$. That is: $\mathcal{O}(L^{u,v}) \cong A(\boldsymbol{i}_{(u,v)})$*

Proof. [BFZ03] Theorem 2.10 and Lemma 2.12 □

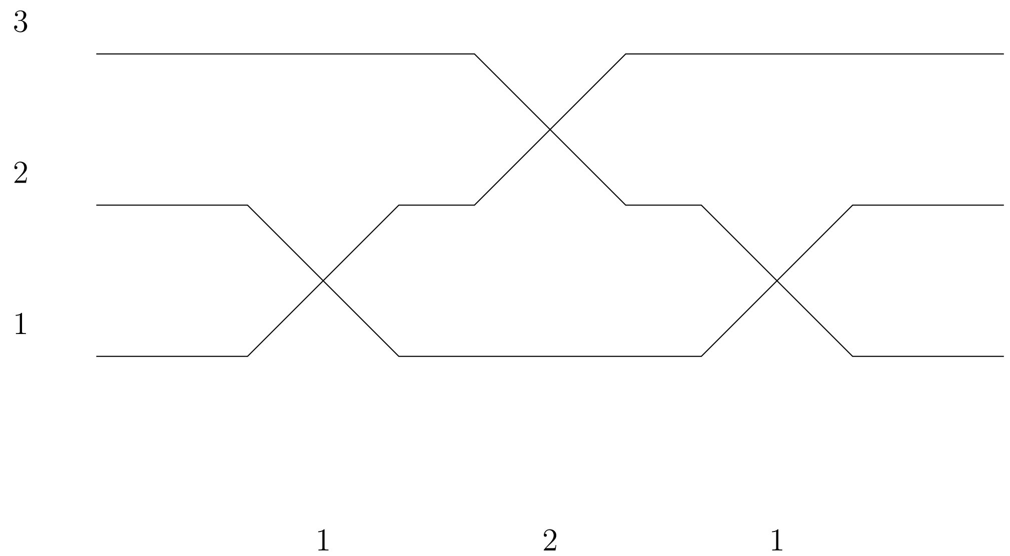
This theorem requires considerable work therefore we will only sketch the case of $G = SL_n(\mathbb{C})$.

First we will introduce a combinatorial gadget called a double pseudoline arrangement.

Given an element $w \in S_n$, we take a reduced word $w = s_{i_1} s_{i_2} \dots s_{i_k}$. We associate to this reduced word a pseudoline arrangement which is constructed with the following algorithm:

1. From left to right, write down the indices $i_1, \dots, i_k$.
2. Label n wires from bottom to top by $1, \dots, n$.
3. Moving from left to right, above each index i_j create a wire crossing in the i_jth level from bottom to top.

Example 3.3.1. *Consider* $121 \in S_3$. *It has the following pseudoline arrangement:*



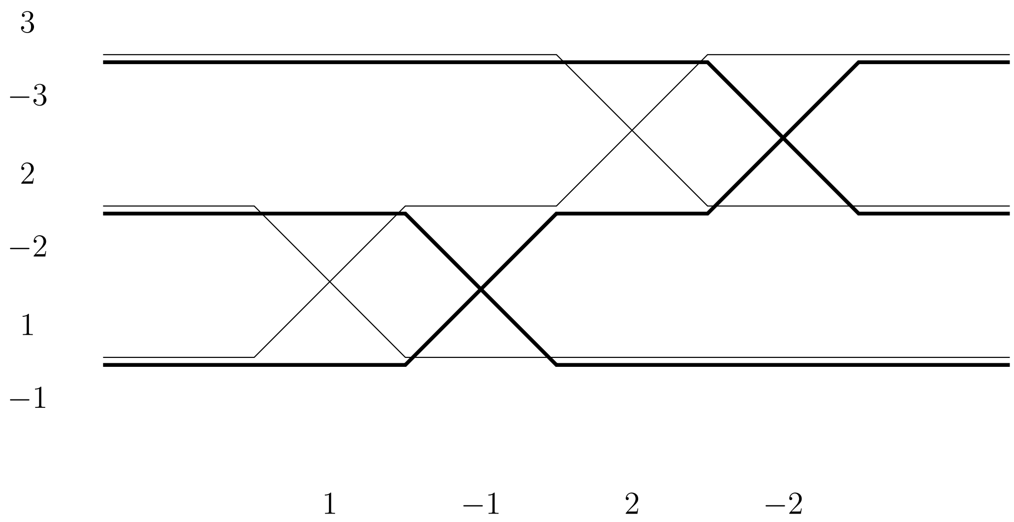
Now consider the case of a pair $(u, v) \in S_n^2$. We can associate a reduced word $\mathbf{i} = (i_1, \dots, i_m)$. Let us denote by U the pseudoline arrangement for u and by V the pseudoline arrangement for v. Then we obtain a pseudoline arrangement for $\mathbf{i}$ by stacking U on top of V in the following way:

1. From left to right, write down the indices $i_1, \ldots, i_m$.

2. Label 2n wires from bottom to top by $-1, 1, -2, 2, \ldots, -n, n$ in order.

3. Moving from left to right, above each positive index i_j create a wire crossing at the i_jth level (in U) and above each negative index i_s create a wire crossing at the i_s level (in V).

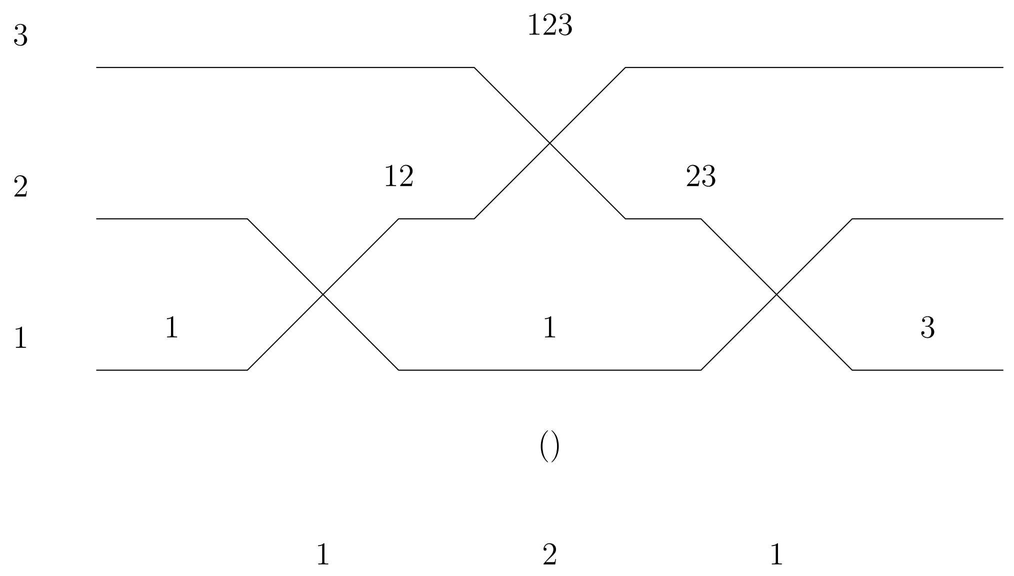
Example 3.3.2. *Consider* $\boldsymbol{i} = (1, -1, 2, -2) \in S_3 \times S_3$. *The double pseudoline arrangement has the following form:*



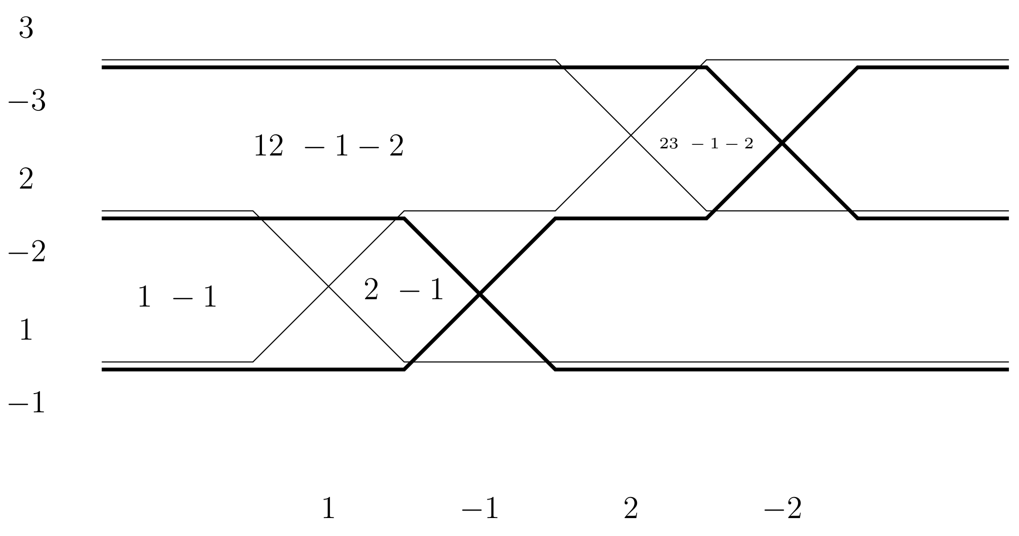
We are interested in regions of a double pseudoline arrangement diagram demarcated by the crossing of two wires in U. By this, we mean that the rightmost point of a region should be a wire crossing in U. Such a region is called a chamber. To each chamber we assign some $C(i)$ for $i \in [m]$.

We label a chamber first by assigning it indices i for each wire i beneath the chamber. Followed by indices $-j$ for each wire $-j$ beneath the chamber. We call the former $I(C)$ and the latter $J(C)$. We then repeat this process for the crossing of two wires in V.

Example 3.3.3. *Let's label all the regions of the pseudoline crossing corresponding to the word (121) in* S_3*.*



Example 3.3.4. *We will label all the chambers of the double pseudoline arrangement of* $\boldsymbol{i} = (1, -1, 2, -2) \in S_3^2$*.*



This lets us define a minor $\Delta_{I(C),J(C)}(X)$ for any $X \in G$. From Proposition 3.1.1, we had that such $\Delta_{I(C),J(C)}$ generated $\mathcal{O}(L^{u,v})$, satisfying certain vanishing conditions.

Consider the cluster algebra $A(\mathbf{i})$. We send variables $x_{i_j} \mapsto \Delta_{I(C(i_j)),J(C(i_j))}$, the corresponding minor. We observe that this is a bijection of sets. This map $\phi : A(\mathbf{i}) \to \mathcal{O}(L^{u,v})$ is an isomorphism. This requires checking a few things, for instance, unbounded indices must map to non-vanishing functions and the exchange relations for the cluster algebra must hold in the image. The complete proof of this result can be found in the proof of Lemma 3.1 of [Zel00].

3.4 Reduced words and quiver mutation

We saw earlier that given a pair $(u, v) \in W$, the choice of reduced word decomposition is not unique and are related by a series of braid moves. How does the choice of word affect the quiver obtained from the above process? To be more precise, given a pair $(u, v) \in W$ and two reduced word decompositions of (u, v), $\mathbf{i}$ and $\mathbf{j}$ respectively, how are $Q(\mathbf{i})$ and $Q(\mathbf{j})$ related?

Let $u, v \in W$. Let $R(u, v)$ be the set of all reduced words of $(u, v) \in W \times W$. There are two moves we may apply on any reduced word $\mathbf{i} \in R(u, v)$.

1. 2-move: Interchange any two consecutive entries $i_{k-1}, i_k \in \mathbf{i} = (i_1, \dots, i_m)$ for $\{i_{k-1}, i_k\} \notin \Pi \times \Pi$. We say a 2-move is trivial if $i_k \neq -i_{k-1}$.
2. 3-move: Replace three consecutive entries i_{k-2}, i_{k-1}, i_k in $\mathbf{i}$ by $i_{k-1}i_{k-2}, i_{k-1}$ if $i_k = i_{k-2}$ and $\{i_{k-1}, i_k\} \in \Pi \times \Pi$.

We can associate transpositions to the above two moves.

- If two reduced words $\mathbf{i}$ and $\mathbf{i'}$ are related by a trivial 2-move at position k then we define a permutation $\sigma_{\mathbf{i'},\mathbf{i}} = (k-1, k)$.
- If two reduced words $\mathbf{i}$ and $\mathbf{i'}$ are related by a non-trivial 2-move at position k then we define a permutation $\sigma_{\mathbf{i'},\mathbf{i}} = (e)$.

- If two reduced words $\mathbf{i}$ and $\mathbf{i'}$ are related by a 3-move at k then we define a permutation $\sigma_{\mathbf{i'},\mathbf{i}} = (k-2, k-1)$.

Now we will relate these moves to mutation on the quiver $Q(\mathbf{i})$:

Theorem 3.4.1 ([SSVZ99]). *Let $u, v \in W$. Let $\boldsymbol{i}$ and $\boldsymbol{i'}$ be two reduced words for (u, v) related by either a non-trivial 2-move or 3-move at k. Then $Q(\boldsymbol{i'}) = \mu_k(Q(\boldsymbol{i}))$ (up to relabelling).*

Proof. In fact, Theorem 3.5 from [SSVZ99], which pre-dates cluster algebras, essentially proves this very proposition. Theorem 3.5 states the following:

Suppose $\mathbf{i}$ and $\mathbf{i'}$ related by either 2 or 3 moves at k. Let $\sigma = \sigma_{\mathbf{i'},\mathbf{i}}$ be the associated transposition. Let $a, b \in [m]$ with $a \neq b$ such that at least one is $\mathbf{i}$-bounded. Then:

$$(a \to b) \in E(Q(\mathbf{i})) \iff (\sigma(a) \to \sigma(b)) \in E(Q(\mathbf{i'}))$$

With two exceptions:

1. If σ is non-trivial then $(a \to k) \in E(Q(\mathbf{i})) \iff (k \to \sigma(a)) \in E(Q(\mathbf{i'}))$. (Similarly for $k \to a$)
2. If σ is non-trivial and $a \to k \to b$ is a path in $Q(\mathbf{i})$ then $(b \to a) \in E(Q(\mathbf{i}))$ if and only if $\{\sigma(a), \sigma(b)\}$ is not an edge in $Q(\mathbf{i'})$.

This implies statement of our proposition; here we rephrase the result in the language of cluster algebras and mutation.

First observe that we have a bijection of vertices between $Q(\mathbf{i'})$ and $\mu_k(Q(\mathbf{i}))$.

Let $i \to j$ be an arrow in $Q(\mathbf{i})$. $i \to j$ remains an arrow in $\mu_k(Q(\mathbf{i}))$ if $i \neq k$, $j \neq k$, and $\{i, k\}$, $\{j, k\}$ are not edges in $Q(\mathbf{i})$. The SSVZ algorithm on the other hand tells us $\sigma(i) = i \to \sigma(j) = j$ is also an arrow in $Q(\mathbf{i})$. Since σ transposes k with an adjacent index, we only need to look locally near k.

If $i \to k$ in $Q(\mathbf{i})$, then mutating at k will leave $k \to i$ in $\mu_k(Q(\mathbf{i}))$. For any path $i \to k \to j$, mutating at k gives a new path $j \to k \to i$ and a new arrow $i \to j$ in $\mu_k(Q(\mathbf{i}))$.

Comparing $\mu_k((Q(\mathbf{i})))$ to $Q(\mathbf{i})$, we now differ by an application of σ to the vertices. $\square$

A famous theorem by Tits (alluded to earlier in this chapter in Proposition 3.2.1) proves that applying such 2-moves and 3-moves on some reduced word $\mathbf{i}$ generates all reduced words inside $R(u,v)$. Therefore, we can think of $R(u,v)$ as a graph where vertices are reduced words $\mathbf{i}$ and two reduced words $\mathbf{i}$ and $\mathbf{j}$ are connected by an edge if $\mathbf{i}$ and $\mathbf{j}$ are related by a non-trivial 2-move or 3-move. Theorem 3.4.1 then allows us to think of $R(u,v)$ as a subgraph of the exchange graph $\Xi(A(u,v))$.

Remark 3.4.1. One interesting observation we can make is that there are always finitely many reduced word expressions. These reduced word expressions give rise to different quivers which are all mutation equivalent (or the same quiver with permuted indices). Since there are only finitely many reduced word expressions for a pair (u,v), this implies that if $\Xi(A(u,v)) = R(u,v)$ then $A(u,v)$ is of finite type. Yet we know there is at least one example (Example 3.2.6) of such cluster algebras which is not of finite type. Thus: **do these additional mutations correspond to any words?** Moreover, we can ask: **Given a mutation, how do we know if it comes from a 2-move or 3-move?**

Bibliography

[BFZ03] A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras III: Upper bounds and double Bruhat cells. *ArXiv Mathematics e-prints*, May 2003.

[Bjo00] Anders Bjorner. *Combinatorics of Coxeter Groups*. Springer, 2000.

[Ful97] W. Fulton. *Young Tableaux*. London Mathematical Society, 1997.

[Ful04] Harris J. Fulton, W. *Representation Theory: A First Course*. Springer, 2004.

[FZ98] S. Fomin and A. Zelevinsky. Double Bruhat cells and total positivity. *ArXiv Mathematics e-prints*, February 1998.

[FZ03] S. Fomin and A. Zelevinsky. Cluster algebras II: Finite type classification. *Inventiones Mathematicae*, 154:63–121, October 2003.

[Gek10] Schapiro M. Vainshtein A. Gekhtman, M. *Cluster Algebras and Poisson Geometry*. American Mathematical Society, 2010.

[Har77] R. Hartshorne. *Algebraic Geometry*. Springer, 1977.

[Has07] B. Hassett. *Introduction to Algebraic Geometry*. Cambridge University Press, 2007.

[Lam16] P. Lampe. Diophantine equations via cluster transformations. *ArXiv e-prints*, February 2016.

[Sco03] J. S. Scott. Grassmannians and Cluster Algebras. *ArXiv Mathematics e-prints*, November 2003.

[SSVZ99] B. Shapiro, M. Shapiro, A. Vainshtein, and A. Zelevinsky. Simply-laced Coxeter groups and groups generated by symplectic transvections. *ArXiv Mathematics e-prints*, June 1999.

[Zel00] A. Zelevinsky. Connected components of real double Bruhat cells. *ArXiv Mathematics e-prints*, March 2000.