Pivotal categories, matrix units, and towers of biadjunctions

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October 2017

A thesis submitted for the degree of Bachelor of Philosophy (Honours) of the Australian National University

Declaration

The work in this thesis is my own except where otherwise stated.

Sam Quinn

Acknowledgements

First and foremost, I would like to thank my supervisors, both official and unofficial, Scott Morrison, Tony Licata, and Corey Jones. Thanks for picking such a fascinating topic for me to work on this year, and for all your advice and assistance in producing this thesis.

I'd also like to thank all the excellent researchers at the MSI who have taught me so much, including Amnon Neeman, David Smyth, Pierre Portal, Joan Licata, Vigleik Angeltveit and Jim Borger. I would particularly like thank to Amnon and David for running terrific reading courses this year.

Thanks also to my fellow honours students – good friends who have made the year really enjoyable.

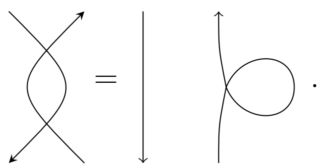
Finally, I want to thank my family for all their support over the years, and for putting up with me over the last few months. I couldn't have done it without you. Particular thanks to my Dad for attempting to proofread this document – I hope the sentences are the right way around.

Abstract

In this thesis we study a tower of biadjunctions coming from a pivotal tensor category with a self dual object. In order to do this, we present some relevant parts of the standard theory of monoidal categories, tensor categories, and pivotal tensor categories. We recall a method for constructing matrix units for the algebras $\mathrm{End}(X^{\otimes n})$ for any object X in a semisimple linear monoidal category.

Using these matrix units, we then prove our main result, Theorem 4.2.14. In a linear monoidal category, endomorphism algebras for tensor powers of a distinguished object X can be used to build a tower of algebras. We prove that when the category is a pivotal tensor category and the object X is self dual, the induction and restriction functors associated to this tower form biadjoint pairs.

Inspired by [Kho14], we use the data of these biadjunctions to construct a graphical category $\mathcal{G}_X$. The morphisms in this category are various planar diagrams, modulo some local relations. For instance, one such relation is



The construction in [Kho14] has been a rich source of interesting mathematics. The hope is that our category might prove to be similarly interesting.

Contents

Chapter 0

Background and motivation

0.1 What is categorification?

The term 'categorification' was first introduced by Crane and Frenkel in their joint work [CF94]. Categorification aims to take familiar concepts, phrased in the language of sets and functions, and shift everything one rung up the categorical ladder. The hope is that by passing from a simpler object to a more complex one, we gain some additional structure or insight that can be used to study the original object. These newly constructed objects are often interesting in their own right as well.

Before we go into any further details, it is easier to first discuss the opposite process of decategorification. If $\mathcal{C}$ is a category, then we can 'decategorify' $\mathcal{C}$ by looking at the set of isomorphism classes of its objects. As an example, let's think about the category $\mathbf{FinVec}_k$ of finite dimensional k-vector spaces. Since two finite dimensional vector spaces are isomorphic if and only if they have the same dimension, there is one isomorphism class of objects for each natural number $n \in \mathbb{N}$.

This might not seem particularly interesting, and at this stage it's not. However, $\mathbb{N}$ is far more than just a set – to start with there are algebraic operations and a natural ordering. The interesting thing is that this additional structure is reflected in the category $\mathbf{FinVec}_k$. Given two finite dimensional vector spaces V and W, we can form their direct sum $V \oplus W$, and their tensor product $V \otimes W$. If V is m-dimensional and W is n-dimensional, then $V \oplus W$ and $V \otimes W$ have dimensions $m+n$ and $m \cdot n$ respectively. In this way, the addition an multiplication in $\mathbb{N}$ are reflected in $\mathbf{FinVec}_k$. One can also show that the familiar rules for arithmetic in $\mathbb{N}$ have categorical analogues in $\mathbf{FinVec}_k$. For example, the isomorphism

$$(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$$

is analogous to the distributive rule $(\ell + m) \cdot n = (\ell \cdot n) + (m \cdot n)$.

We can push this even further though! We could note that the dimension of V is less than or equal to the size of W if and only if there is a monomorphism (injection) from V to W. Thus, the natural order on $\mathbb{N}$ can be expressed in $\mathbf{FinVec}_k$ too. Again, we can show that basic rules of this order, like $\ell \leq m \implies \ell + n \leq m + n$, are reflected in the category.

Decategorification is relatively simple to perform, but we inevitably lose information in the process. Categorification is an attempt to reverse this process and recover the lost information. When attempting to categorify a set-theoretic object, we must produce a categorical object which decategorifies to the original object. Since this process involves the creation of entirely new objects, it is inherently harder than decategorification. The relationship between the two is akin to the relationship between differentiation and integration – the former is a systematic process, while the latter is more of an art, requiring insight into particular situations.

From our above discussion, we could say that the category $\mathbf{FinVec}_k$ is a categorification of the natural numbers $\mathbb{N}$. This is really a toy example though, and it is reasonable to ask whether there are deeper, more interesting examples. The answer is a resounding yes. To list just two:

- Khovanov homology [Kho00] is a categorification of the Jones polynomial. It still assigns an invariant to each knot, but is also functorial in that it assigns an invariant to each cobordsim between knots. This makes Khovanov homology a richer and more interesting invariant than the Jones polynomial. It is known that Khovanov homology detects the unknot, whereas this is still an open question for the Jones polynomial.

- Khovanov and Lauda [KL09], and independently Rouquier [Rou08], categorified the quantum Kac-Moody algebras. This leads to the construction of canonical bases for these algebras that have convenient positive integrality properties.

There are many further examples – see the discussion in [Tub13] or [CY98].

Often, categorification is achieved by means of the following analogy between set theory and category theory:

set-theoretic concept	category-theoretic concept
sets	categories
elements of sets	objects of categories
functions between sets	functors between categories
equalities between elements	isomorphisms between objects
equalities between functions	natural isomorphisms between functors

.

We attempt to replace the set-theoretic concepts on the left with the corresponding category-theoretic concept on the right. We note that equalities are replaced with isomorphisms rather than strict equalities between objects or functors. This is because in practice requiring equality between objects or functors turns out to be far too rigid a requirement. For example, in the above discussion, the algebraic structure of $\mathbb{N}$ was reflected in $\mathbf{FinVec}_k$ by isomorphisms of vector spaces, not equalities.

To demonstrate this idea we'll categorify the definition of a monoid to arrive at the definition of a monoidal category, a structure of central importance in everything that follows.

Recall that a monoid is a set M with a binary operation $M \times M \to M$ written $(a, b) \mapsto a \cdot b$. We require that this operation is associative, i.e.,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{1}$$

for all $a, b, c \in M$. There is also a distinguished element, denoted 1, satisfying the equations

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a \tag{2}$$

for all $a \in M$.

The philosophy of categorification dictates that we should replace the set M with a category $\mathcal{C}$, the function $M \times M \to M$ with a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and the distinguished element 1 with a distinguished object $\mathbf{1}$. We'll denote this functor by $\otimes$ (read 'tensor') and write the image of (X, Y) as $X \otimes Y$.

Since we ask that (1) and (2) hold for all elements of M, these are really equations between functions. This means that we should replace them with natural isomorphisms (written in components)

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z),$$

and

$$r_X : X \otimes \mathbf{1} \xrightarrow{\cong} X \quad \text{and} \quad \ell_X : \mathbf{1} \otimes X \xrightarrow{\cong} X.$$

This gets us most of the way to the definition of a monoidal category.

Definition 0.1.1. A monoidal category is a sextuple $(\mathcal{C}, \otimes, \mathbf{1}, a, \ell, r)$. Here $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor called the 'tensor product' functor, and $\mathbf{1}$ is a distinguished object of $\mathcal{C}$ called the 'tensor unit'. The remaining pieces of data are three natural isomorphisms – the associator $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, the left unitor $\ell : \mathbf{1} \otimes - \to -$, and the right unitor $r : - \otimes \mathbf{1} \to -$. In components, these are isomorphisms

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

for all $X, Y, Z \in \mathcal{C}$, as well as

$$\ell_X : \mathbf{1} \otimes X \xrightarrow{\cong} X \quad \text{and} \quad r_X : X \otimes \mathbf{1} \xrightarrow{\cong} X$$

for all $X \in \mathcal{C}$. We require that this data satisfies the following two axioms.

1. The pentagon axiom. The diagram

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"'] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

commutes for all $W, X, Y, Z \in \mathcal{C}$.

2. The triangle axiom. The diagram

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \ell_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

commutes for all $X, Y \in \mathcal{C}$.

These two axioms are somewhat mysterious – there are no analogous axioms in the definition of a monoid. The need to add these axioms arises when we relax the equalities in the monoid definition to mere isomorphisms in the categorified definition.

In a monoid, equation (1) guarantees that all the ways of bracketing an expression $a_1 \cdot a_2 \cdot \ldots \cdot a_n$ of arbitrary length give the same result. In a monoidal category, the associativity isomorphisms ensure that all the ways of bracketing an expression of the form $X_1 \otimes X_2 \otimes \ldots \otimes X_n$ are isomorphic, however there could conceivably be several different isomorphisms between two different bracketed expressions.

Similar issues arise with the isomorphisms coming from the left and right unitor. We would really like to not worry too much about the bracketing of expressions or the presence of the tensor unit, but in order to safely do this we need canonical isomorphisms between these various expressions. For expressions involving 4 objects, the pentagon axiom says that the two possible isomorphisms are the same, but we could still run in to issues for expression involving 5 or more objects.

Happily, there is a solution – the famous Coherence Theorem of Mac Lane.

Theorem 0.1.2 (Mac Lane's Coherence Theorem)**.** *Let $X_1, \ldots, X_n$ be objects in a monoidal category $\mathcal{C}$. Let P_1 and P_2 be any two bracketed tensor products of $X_1, \ldots, X_n$ (in this order) with arbitrary insertions of the tensor unit $\mathbf{1}$. Let $f, g : P_1 \to P_2$ be any two isomorphisms obtained by composing the associativity and unit isomorphisms and their inverses, possibly tensored with identity morphisms. Then $f = g$.*

Proof. See either [EGNO15] or [ML98] for the proof of this. □

This means that any two bracketed tensor products of $X_1, \ldots, X_n$ with the tensor unit inserted arbitrarily are canonically isomorphic. Because of this, we can safely identify all these various expressions and ignore bracketings and the tensor unit in most of our work. We will do this from now on, suppressing the various isomorphisms unless confusion is likely.

The preceding discussion illustrates the most subtle aspect of this approach to categorification. We don't just have to replace equalities with isomorphisms, but we have to add additional 'coherence laws' amongst the various isomorphisms. These don't directly arise from the structure of the object being categorified, so finding the correct coherence laws can be a difficult task. For further discussion of this aspect of categorification, see [BD98].

0.2 String diagrams

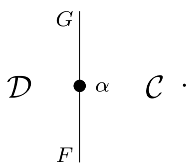
Many successful attempts at categorification (for example, see [Kho14, LS13, KL09]) have proceeded by constructing interesting graphical categories. By this, we mean a category whose hom-spaces consist of various types of diagrams, with some set of local relations that allow us to manipulate diagrams. In order to describe this approach in more detail, we first need to introduce string diagrams for categories, functors and natural transformations. The treatment of this is based on [Kho10].

Suppose we have categories $\mathcal{C}$ and $\mathcal{D}$, functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, and a natural transformation $\alpha : F \to G$. Normally, we might represent this situation with the diagrams

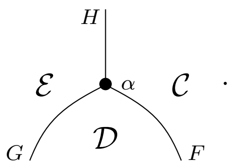
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{D} \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathcal{D} \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{C} \arrow[l, bend right, "G"'] \arrow[l, bend left, "F"] \end{tikzcd} .$$

Here we think of categories as labelling points, the functors labelling 1-dimensional strings between these point, and natural transformations labelling 2-dimensional regions between strings. We will find it more convenient to work with diagrams of the second type, with functors pointing from right to left. This ensures that the algebraic and diagrammatic notations for composition of functors match up.

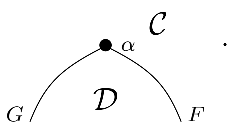
We can now consider taking the so called 'Poincaré dual' of this diagram. To do this, we instead label regions with categories, strings dividing regions by functors between categories, and points on strings by natural transformations. The right hand diagram above becomes



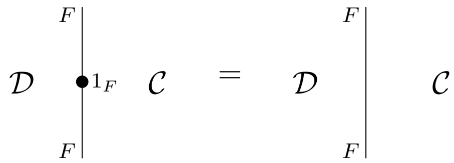
Similar considerations lead us to depict a natural transformation between composite functors by multiple labelled strands meeting in a point. For example, if we have functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ and $H : \mathcal{C} \to \mathcal{E}$, then a natural transformation $\alpha : GF \to H$ can be depicted by



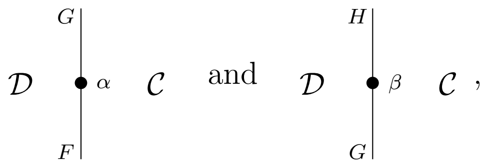
There are some common conventions regarding identity functors and identity natural transformations. Strands labelled by the identity functor are generally omitted from our diagrams. For example, if in the above $\mathcal{C} = \mathcal{E}$ and H is the identity functor on $\mathcal{C}$, then the situation is depicted by



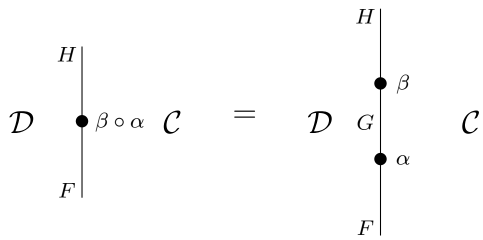
In a similar way, nodes labelled by identity natural transformations are not drawn. For example, If 1_F is the identity natural transformation on a functor $F : \mathcal{C} \to \mathcal{D}$, then



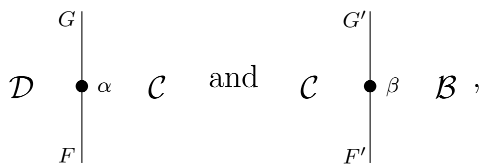
The two different ways of composing natural transformations also have convenient graphical descriptions. If there are natural transformations represented diagrammatically by



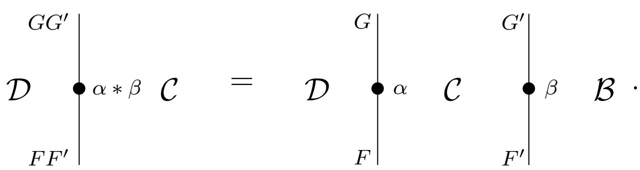
then the composite natural transformation $\beta \circ \alpha : F \to H$ (the vertical composite) is represented by stacking the diagrams:



Similarly, if we have natural transformations represented diagrammatically by



then the composite natural transformation $\alpha * \beta : FF' \to GG'$ (the horizontal composite) is represented by juxtaposing the diagrams:



We start to see the real power of these diagrammatic representations in the context of adjoint functors. Recall [ML98] that a functor $F : \mathcal{D} \to \mathcal{C}$ is left adjoint to a functor $G : \mathcal{C} \to \mathcal{D}$ if there are natural transformations

$$\eta : 1_{\mathcal{D}} \to GF$$

and

$$\varepsilon : FG \to 1_{\mathcal{C}},$$

such that the two composite natural transformations

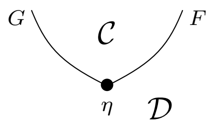
$$F \xrightarrow{1_F * \eta} FGF \xrightarrow{\varepsilon * 1_F} F \tag{3}$$

and

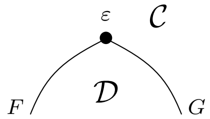
$$G \xrightarrow{\eta * 1_G} GFG \xrightarrow{1_G * \varepsilon} G \tag{4}$$

are equal to 1_F and 1_G respectively.

We can represent η and ε by the diagrams



and



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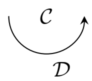
Rather than labelling our strands in this situation, we might assign them a positive orientation near F and a negative orientation near G instead. For example, with this convention the identity natural transformation on F is depicted by an arrow pointing up the page, with the region on the right labelled by $\mathcal{D}$ and the region on the left labelled by $\mathcal{C}$. Similarly, we can represent the unit and counit for the adjunction by an oriented cap and an oriented cup:



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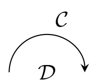

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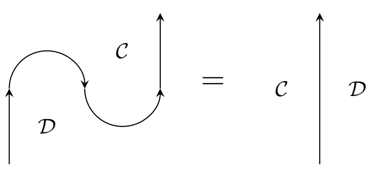
Then equations (3) and (4) become the diagrammatic identities:



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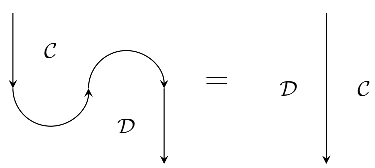
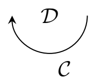


.

These can be interpreted as two of the four possible isotopies of oriented arcs in the plane (with labelled regions). To have complete isotopy invariance we need to assume that F is also a right adjoint of G (or equivalently, that G is a left adjoint of F). If this is the case, then there are natural transformations represented by the two other oriented caps and cups:



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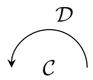

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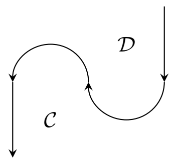
satisfying the diagrammatic identities



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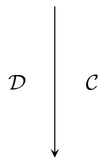

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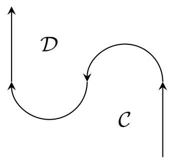

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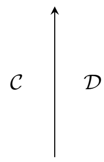

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These give us the other two isotopies of oriented arcs in the plane.

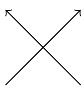
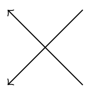
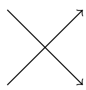
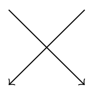
0.3 An approach to graphical categorification

We can now begin to describe how biadjoint functors can be used to produce graphical categorifications. Suppose we have a (possibly infinite) collection of categories $\mathcal{C}_1, \mathcal{C}_2, \ldots$, and biadjoint functor pairs (F_n, G_n) going between consecutive categories:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F_1"] & \mathcal{C}_2 \arrow[l, bend left, "G_1"] \arrow[r, bend left, "F_2"] & \mathcal{C}_3 \arrow[l, bend left, "G_2"] \arrow[r, bend left, "F_3"] & \cdots \arrow[l, bend left, "G_3"] \end{tikzcd}$$

We refer to this general set-up as a 'tower of biadjunctions'.

Since the functor F_n goes from $\mathcal{C}_n$ to $\mathcal{C}_{n+1}$, it make sense to assign the F_n in each pair a positive orientation and the G_n in each pair a negative orientation. If we fix biadjointness natural transformations for each pair, then we can use the graphical calculus described above to build various oriented planar diagrams using the four possible (F_n, G_n)-caps and cups for each biadjoint pair, plus the identity strands on the functors. Within these diagrams, strands and loops will by labelled by functors, and regions will be labelled by categories. Since our categories are indexed by positive integers, we can simplify the notation by labelling regions with positive integers only. To draw more intricate pictures, we might choose to represent additional natural transformations by the various crossings

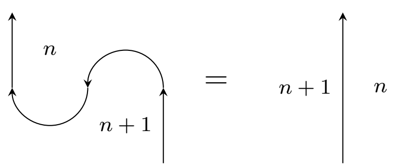
 ,  ,  and  .

Here the different natural transformations correspond to different ways of labelling the strings and regions in these pictures. With these, we can then draw oriented planar diagrams where the strands are now allowed to cross each other.

Note that once we label the rightmost region of one of our diagrams by a positive integer n, the labelling of all the other regions and all the strands is uniquely determined. Reading from right to left, numbers should increase by 1 across an upwards oriented strand and decrease by 1 across a downwards oriented strand. An upwards oriented strand with n on the right and $n+1$ on the left must be labelled by $F_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$, and a downwards oriented strand with $n+1$ on the right and n on the left must be labelled by $G_n : \mathcal{C}_{n+1} \to \mathcal{C}_n$. As such, we only need to label the rightmost region of our diagrams and will do so from now on.

Define a category $\mathcal{S}$ as follows. The objects of $\mathcal{S}$ are compositions of the assorted functors $F_1, G_1, F_2, G_2, \ldots$. The morphisms between two such composites are those natural transformations that can be built (by the two types of composition) using the fixed biadjointness natural transformations, the natural transformations representing the crossings, and the identity natural transformations for the functors. The discussion above describes a graphical calculus for the morphisms in this category.

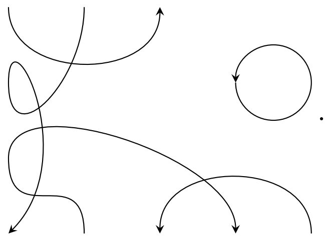
Let $\mathcal{S}_n$ be the full subcategory of $\mathcal{S}$ whose objects are those composites starting at $\mathcal{C}_n$. For instance, $G_4F_4F_3$ is an object in $\mathcal{S}_3$, but F_2 is not. Then $\mathcal{S}$ decomposes as $\bigoplus_{n=1}^{\infty} \mathcal{S}_n$. In many situations, we are interested in understanding the uniform behaviour of the category $\mathcal{S}$. More precisely, we want to understand which relationships between the objects and morphisms of this category are independent of the piece $\mathcal{S}_n$ they live in. For example, since all the functors come in biadjoint pairs, the graphical relation



holds for any positive integer n. This means that there is some uniform equality of morphisms in $\mathcal{S}$ that does not depend on n.

One way that we can explore the uniform behaviour of $\mathcal{S}$ is by finding a single category governing the behaviour of $\mathcal{S}$. By this, we mean a category $\mathcal{C}$ with some collection of functors $\mathcal{F}_\alpha : \mathcal{C} \to \mathcal{S}$. For this to be useful in studying $\mathcal{S}$, we also require that everything in $\mathcal{S}$ is in the image of one of these functors.

There is a clear candidate for such a category. Ignoring the labels in our pictures leaves oriented diagrams in the plane where strands are allowed to cross each other. For example, one such diagram looks like



We will define things more formally later on, but for now consider a category $\mathcal{C}$ whose hom-spaces consist of diagrams like that depicted above. There are obvious functors $\mathcal{F}_n : \mathcal{C} \to \mathcal{S}_n$ for each n, each of which takes such a diagram, labels the rightmost region by n, and interprets the resulting diagram as a morphism in $\mathcal{S}_n$.

As yet, we haven't allowed any local relations (not even isotopy of arcs) in $\mathcal{C}$. This makes the hom-spaces in $\mathcal{C}$ incredibly large, which in turn makes $\mathcal{C}$ a difficult category to work with. To fix this we want to add various local relations in our category, but how do we know which relations to impose?

Above, we saw that certain diagrammatic identities hold in $\mathcal{S}$ regardless of the labellings of the diagrams. This means that we can forget the labels on such an identity, and use the resulting equality of diagrams as a local relation in $\mathcal{C}$. The fact that the relation holds for any labelling means that the functors F_n are still well defined (this is something like quotienting by elements in the kernel of a group homomorphism). Thus, the goal is to find various relations in $\mathcal{S}$ that hold regardless of labellings, and then use these to define local relations in $\mathcal{C}$.

This explains our initial interest in working with biadjoint functors. The fact that all the functors came in biadjoint pairs means that the four basic isotopies of oriented arcs in the plane hold in $\mathcal{S}$ for any labelling by positive integers. Thus, we end up allowing local isotopy of arcs in $\mathcal{C}$. Such isotopies are very convenient in a graphical category, so it is desirable to work with biadjoint functor pairs.

In several papers [Kho14, LS13, KL09], constructing a category with such a collection of functors is a source of interesting mathematics. For example, in [Kho14] Khovanov studies the tower of group algebras for the symmetric groups:

$$k[S_1] \subset k[S_2] \subset k[S_3] \subset \cdots$$

There are functors associated to this tower

$$\mathrm{Ind}_n^{n+1} : k[S_n]\text{-}\mathbf{mod} \longrightarrow k[S_{n+1}]\text{-}\mathbf{mod}$$

and

$$\mathrm{Res}_n^{n+1} : k[S_{n+1}]\text{-}\mathbf{mod} \longrightarrow k[S_n]\text{-}\mathbf{mod},$$

called induction and restriction respectively. It is a standard result from the representation theory of finite groups that these functors form biadjoint pairs $(\mathrm{Ind}_n^{n+1}, \mathrm{Res}_n^{n+1})$. Following the procedure outlined above, Khovanov produces a graphical monoidal category which conjecturally categorifies the Heisenberg algebra.

In this thesis we will examine another tower of algebras which have not been studied in this context before. Inspired by the tower of Temperley-Lieb algebras, we will show that the induction and restriction functors associated to certain towers of endomorphism algebras in pivotal tensor categories come in biadjoint pairs. We can then begin to build a graphical monoidal category and a collection of functors using the techniques described above.

0.4 The structure of this thesis

In Chapter 1 we will introduce the Temperley-Lieb and Temperley-Lieb-Jones categories. The latter will be used as an extended example throughout the thesis to supplement the more abstract material developed. The material in this chapter is well known, our main sources being [Wan10] and [Che14].

In Chapter 2 we will work though a construction of matrix units for certain endomorphism algebras in semisimple linear monoidal categories. This culminates in the proof of Theorem 2.4.7 These matrix units will underpin the work of Chapter 4. The matrix unit construction we describe is well known to experts and dates back to early work on monoidal categories. However, we have not found a good source for this material in the literature. We hope that the work described here can begin to fill this gap.

In Chapter 3 we will take a brief detour and discuss the representation theory of the Temperley-Lieb algebras using the matrix units from Chapter 2. In Propositions 3.1.1, 3.1.2 and 3.1.3 we give a classification of the irreducible representations. Then we introduce the induction and restriction functors for the tower of Temperley-Lieb algebras, and in Proposition 3.3.1 and Corollary 3.4.2 describe how induced or restricted representations decompose into irreducibles. The material in this chapter is not strictly necessary for the remainder of the thesis, but the theory is interesting on its own and illustrates a useful application of the material of Chapter 2. Again, this material is well known. Our main sources here are [RSA14] and [GdlHJ89].

In Chapter 4 we will prove Theorem 4.2.14, the central result of this thesis. In a linear monoidal category, the endomorphism algebras for tensor powers of a distinguished object X form a tower of algebras. When the category is a pivotal tensor category and X is self dual, we prove that the induction and restriction functors associated to this tower form biadjoint pairs. Although experts in the field may be aware of this theorem, it does not appear anywhere in the literature. The background material in this chapter is primarily sourced from [EGNO15] and [TV17].

In Chapter 5 we define a graphical category $\mathcal{G}_X$ using this biadjunction. In Theorem 5.2.2 we prove that there is a collection of functors out of $\mathcal{G}_X$ into another category of interest. To do this, we check that various relations hold between certain natural transformations. The work in this chapter is original.

Chapter 1

The Temperley-Lieb categories

1.1 Quantum numbers

Let $\mathbb{F}$ be a field, and consider the field $\mathbb{F}(q)$ of rational polynomials in a formal variable q. Within this field, the n-th quantum number $[n]_q$ is defined by

$$\begin{aligned}[n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}} \\ &= q^{n-1} + q^{n-3} + \ldots + q^{-(n-3)} + q^{-(n-1)} \quad \text{(when } n \neq 0\text{)}.\end{aligned}$$

It is common to suppress the subscript q in $[n]_q$ and just write $[n]$ when this is unlikely to cause confusion. There are some very basic properties of the quantum numbers that will be used extensively in what follows.

Lemma 1.1.1. *For all $n \in \mathbb{N}$, the following are true:*

(i) $[0] = 0$*;*

(ii) $[1] = 1$*; and*

(iii) $[2]\,[n] = [n+1] + [n-1]$.

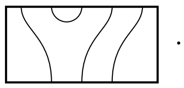
Proof. The first two properties follow trivially from the definition. The last property follows from the simple calculation

$$\begin{aligned}[2]\,[n] &= (q + q^{-1}) \cdot \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q^{n+1} - q^{-n+1} + q^{n-1} - q^{-n-1}}{q - q^{-1}} \\ &= \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + \frac{q^{n-1} - q^{-(n-1)}}{q + q^{-1}} = [n+1] + [n-1]\,.\end{aligned}$$

□

1.2 The Temperley-Lieb category

Definition 1.2.1. Let m and n be natural numbers. A simple (m, n) Temperley-Lieb diagram is a rectangle with m points on its bottom edge and n points on its top edge. The interior of the diagram consists of uncrossing strands pairing the points on the boundary. For example, the following is a simple $(3, 5)$ Temperley-Lieb diagram:



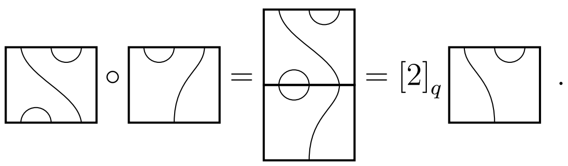
We think of two simple diagrams as equivalent if one can be deformed into the other via a planar isotopy that fixes the boundary points. That is, two diagrams are equivalent if they induce the same pairing on the $n+m$ boundary points.

From now on, we identify all equivalent simple Temeperley-Lieb diagrams. Technically, this means that we're working with equivalence classes of diagrams, but in practice there is little risk in thinking only in terms of representatives of these equivalence classes. Having made these identifications, there are then only finitely many simple (m, n) Temperley-Lieb diagrams for any choices of m and n.

Definition 1.2.2. The Temperley-Lieb category $\mathcal{TL}$ has as objects the natural numbers $\mathbb{N}$. The hom-space $\mathrm{Hom}_{\mathcal{TL}(q)}(m, n)$ is defined to be the $\mathbb{C}(q)$-vector space with basis the simple (m, n) Temperley-Lieb diagrams. Composition

$$\circ : \mathrm{Hom}_{\mathcal{TL}}(m, n) \times \mathrm{Hom}_{\mathcal{TL}}(\ell, m) \to \mathrm{Hom}_{\mathcal{TL}}(\ell, n)$$

is given for simple diagrams by stacking the left hand diagram on top of the right hand diagram and removing any closed loop formed for a factor of $[2]_q$. For example,



This is extended bilinearly to define the composition of two arbitrary morphisms.

The phrase 'Temperley-Lieb diagram' will usually refer to a formal linear combination of simple Temperley-Lieb diagrams. When we want to talk about simple diagrams specifically we will make sure to include the adjective 'simple'.

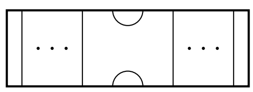
Proposition 1.2.3. *The Temperley-Lieb category $\mathcal{TL}$ is strict monoidal category. The tensor product $\otimes$ of objects n and n' is the sum $n + n'$, and the tensor product of diagrams $f : m \to n$ and $g : m' \to n'$ is the diagram $f \otimes g : m + m' \to n + n'$ obtained by juxtaposing simple diagrams and extending bilinearly.*

Proof. This is all straightforward to check. The tensor product of two integers is their sum, so the tensor product is strictly associative with tensor identity 0. The associator, left unitor and right unitor have all components the identity, so the pentagon and triangle axioms are satisfied trivially. □

Definition 1.2.4. For $n \in \mathbb{N}$, the n-th Temperley-Lieb algebra, TL_n, is defined to be the vector space $\mathrm{Hom}_{\mathcal{TL}}(n, n)$. The algebra multiplication is given by composition of diagrams.

Notation 1.2.5. Given this algebra structure on $\mathcal{TL}(n \to n)$, we will frequently be writing the composition of two morphisms $f, g \in \mathcal{TL}(n \to n)$ as fg instead of $f \circ g$. As such, it makes sense to extend this convention to composition of morphisms in any category. We will do this without further comment for the remainder of this thesis.

Within the Temperley-Lieb algebras there are some important distinguished elements. In TL_n there are the $n-1$ Jones projections $e_1, \ldots, e_{n-1}$ defined by

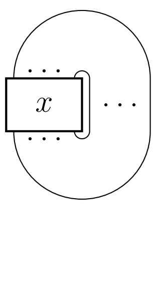
$e_i =$  ,

with the cap-cup pair starting at the i-th position. It is a standard fact [RSA14] that TL_n is generated as a unital algebra by the Jones projections. In fact, TL_n can be (and frequently is) described abstractly as the unital algebra with generators $e_1, \ldots, e_{n-1}$, subject to the relations

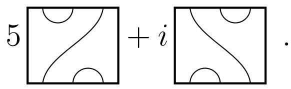
$$e_i^2 = [2]_q \, e_i, \quad e_i e_{i\pm1} e_i = e_i, \quad \text{and} \quad e_i e_j = e_j e_i \text{ if } |i - j| > 1.$$

These relations are obviously satisfied by the Jones projections, but demonstrating that no other relations are required is more involved. We will not make use of this presentation here, preferring to work with the more intuitive diagrammatic description.

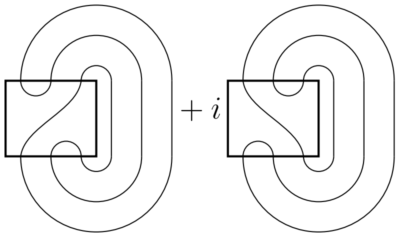
For each natural number n there is useful map $\mathrm{tr}_n : TL_n \to \mathbb{C}(q)$ defined diagrammatically by

$x \mapsto$  .

The way this diagram (and other similar diagrams) should be interpreted is by 'pasting' simple Temperley-Lieb diagrams into the box and then extending linearly. As an example, suppose $n = 3$ and x is the element



Then

$\mathrm{tr}_n(x) = 5$  $= (5\,[2]_q + i\,[2]_q).$

We should point out that we really get some multiple of the empty diagram in TL_0. However, TL_0 is 1 dimensional and is canonically isomorphic to $\mathbb{C}(q)$ by extracting this scalar, so we can think of this as a map $TL_n \to \mathbb{C}(q)$.

1.3 Jones-Wenzl idempotents

Having introduced the Temperley-Lieb algebras, we are now in a position to discuss an important collection of central idempotents – the Jones-Wenzl idempotents (JWI). These were first studied by Jones in [Jon83], however what follows more closely resembles the treatment in [Wen87]. These idempotents will be of critical importance in a lot of what follows. For example, they will be used to define important families of representations and to construct matrix units for the various Temperley-Lieb algebras.

Proposition 1.3.1 (Uniqueness of JWI)**.** *In the algebra TL_n there is at most one element $f^{(n)}$ characterised by the following properties:*

(1) $f^{(n)} \neq 0$*;*

(2) $f^{(n)} f^{(n)} = f^{(n)}$*; and*

(3) $e_i f^{(n)} = f^{(n)} e_i = 0$ *for each* $i \in \{1, \ldots, n-1\}$*.*

When it exists, this element is known as the n-th Jones-Wenzl idempotent.

Proof. Suppose $f^{(n)}$ satisfies these three properties, and write $f^{(n)} = a1 + x$ where x is some linear combination of the standard basis vectors that are not the identity. Then we can compute

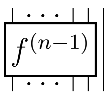
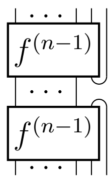
$$f^{(n)} = f^{(n)} f^{(n)} = f^{(n)}(a1 + x) = af^{(n)} + f^{(n)}x.$$

Now, if y is any standard basis vector that is not the identity, then there is a cup appearing somewhere at the top of y. If this cup starts at the i-th position, then we have $e_i y = [2]\, y$, hence $y = \frac{1}{[2]} e_i y$. This means that $f^{(n)} y = 0$ by the third property above, and as a result of this we have $f^{(n)} x = 0$. This yields $f^{(n)} = a f^{(n)}$, hence $a = 1$. Now, if $\widetilde{f^{(n)}}$ also satisfies the above properties we can write $f^{(n)} = 1 + x$ and $\widetilde{f^{(n)}} = 1 + \widetilde{x}$, where $\widetilde{x}$ is also some linear combination of the standard basis vectors except the identity. Taking the product of these two elements and expanding from both sides gives

$$\begin{aligned}\widetilde{f^{(n)}} = \widetilde{f^{(n)}} + x\widetilde{f^{(n)}} &= (1+x)\widetilde{f^{(n)}} \\ &= f^{(n)}\widetilde{f^{(n)}} \\ &= f^{(n)}(1+\widetilde{x}) = f^{(n)} + f^{(n)}\widetilde{x} = f^{(n)},\end{aligned}$$

where $x\widetilde{f^{(n)}} = 0 = f^{(n)}\widetilde{x}$ follows from the same logic as $f^{(n)}x = 0$ above. □

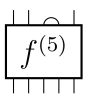
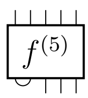
Proposition 1.3.2 (Existence and basic properties of the JWI)**.** *Let* $f^{(0)}$ *be the empty diagram in* TL_0*, let* $f^{(1)}$ *be the strand in* TL_1*, and recursively define* $f^{(n)}$ *by Wenzl's recursion formula [Wen87]:*

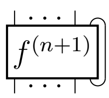
$f^{(n)} =$  $- \dfrac{[n-1]}{[n]}$  . (1.1)

Then for all n, the following are true:

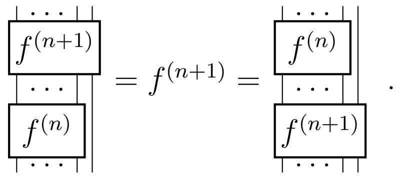
(i) $f^{(n)} f^{(n)} = f^{(n)}$*;*

(ii) capping any two adjacent strands on top of $f^{(n)}$ *or cupping any two adjacent strands on the bottom of* $f^{(n)}$ *gives zero, e.g.,*

 $=$  $= 0$*;*

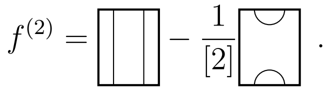
(iii)  $= \frac{[n+2]}{[n+1]} f^{(n)}$*;*

(iv) $\mathrm{tr}(f^{(n)}) = [n+1]$*; and*

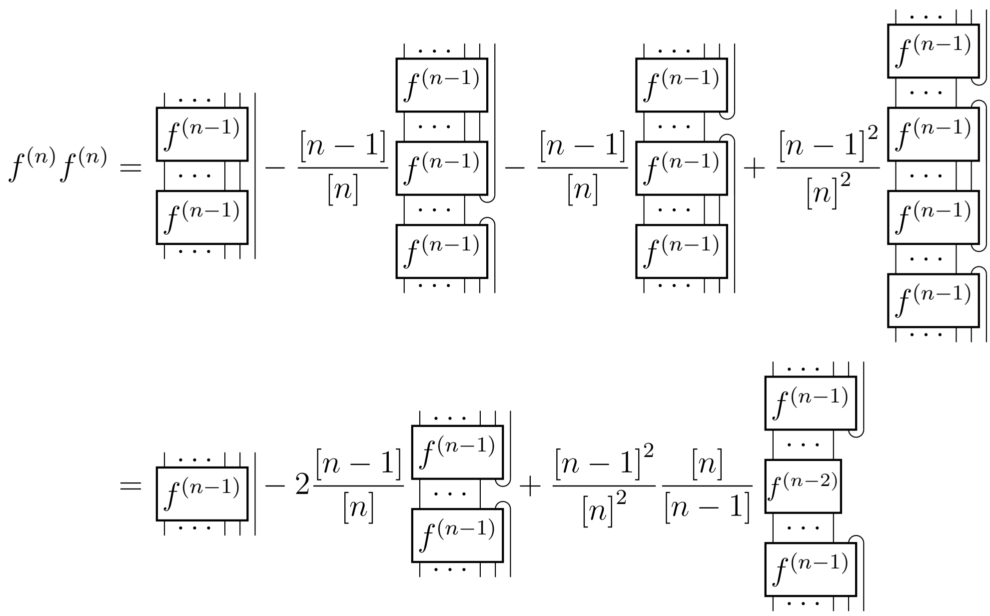
(v) 

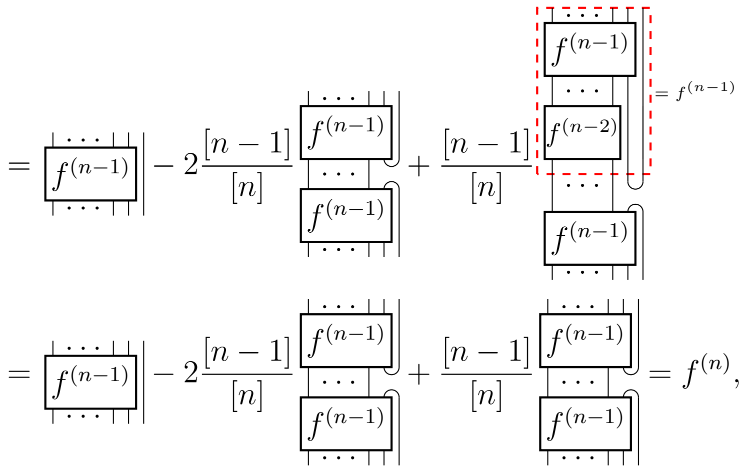
Note that we can express $e_i f^{(n)}$ as the composition of a capped $f^{(n)}$ with an appropriate morphism in $\mathcal{TL}(n-2 \to n)$, so (ii) yields $e_i f^{(n)} = 0$ for each i. Similarly, $f^{(n)} e_i = 0$ for each i. Additionally, since $[n+1] \neq 0$, (iv) shows that $f^{(n)} \neq 0$, so by Proposition 1.3.1 this recursively defined $f^{(n)}$ is the unique n-th Jones-Wenzl idempotent.

Proof. The proof proceeds by induction on n. Using the formula we find that



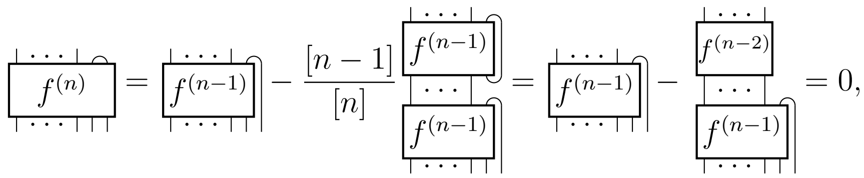
Then it is straightforward to check that (i)-(v) hold in the cases $n = 0$ and $n = 1$, noting that (ii) holds vacuously in both cases. Now, assume $n \geq 2$ and that (i)-(v) hold for all $k < n$ (we refer to these as IH(i)-(v) for the rest of the proof). To prove (i) at the level n we calculate





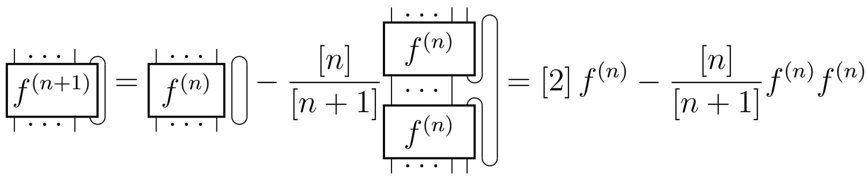
where the second equality uses IH(i) and IH(iii), and the third equality uses IH(v).

For (ii), note that the formula for $f^{(n)}$ immediately shows that capping (respectively cupping) $f^{(n)}$ anywhere but on the far right gives zero (since $f^{(n-1)}$ is both uncappable and uncuppable by hypothesis). When the cap is on the far right we get



using IH(iii) and the boxed part of the previous calculation. A similar calculation shows that cupping $f^{(n)}$ at the far right also gives zero.

To prove (iii) we make use of the fact (just proved) that $f^{(n)} = f^{(n)} f^{(n)}$. We have

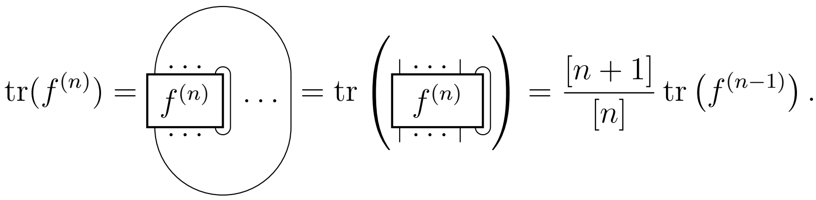


$$= \frac{[2]\,[n+1] - [n]}{[n+1]} f^{(n)}$$

$$= \frac{[n+2]}{[n+1]} f^{(n)},$$

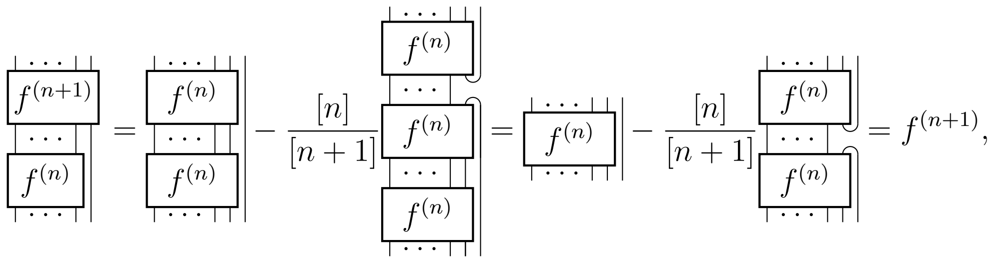
as desired.

For (iv) we use IH(iii) to write



Then the inductive hypothesis yields $\mathrm{tr}(f^{(n)}) = \frac{[n+1]}{[n]}[n] = [n+1]$.

Finally, for (v) we again use $f^{(n)} = f^{(n)}f^{(n)}$ to calculate



and a similar calculation proves the other equality. □

1.4 The generic Temperley-Lieb-Jones category

We now wish to promote the Jones-Wenzl idempotents to be objects in a new category. This category will have a nicer structure and more interesting objects than the Temperley-Lieb category.

Definition 1.4.1. If $\mathcal{C}$ is a category, then the Karoubi envelope, or idempotent completion, of $\mathcal{C}$ is a new category which we denote by $\mathrm{Kar}(\mathcal{C})$. The objects of this category are pairs (X, p) where X is an object of $\mathcal{C}$ and p is a morphism $X \to X$ with $p \circ p = p$ (an idempotent). A morphism $(X, p) \to (X', p')$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $f \circ p = f = p' \circ f$. Composition of morphisms uses the composition in $\mathcal{C}$, and it is easy to check that composing in this way always gives another morphism in $\mathrm{Kar}(\mathcal{C})$.

There is a functor $\mathcal{F} : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ that sends an object X to the pair (X, id_X) and a map $f : X \to Y$ to itself, but now thought of as a map $(X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y)$. This functor is both full and faithful, so we can think of $\mathcal{C}$ as embedded in $\mathrm{Kar}(\mathcal{C})$. For this reason, there is essentially no loss of information when passing to the Karoubi envelope.

It is common to drop the domain X from the notation (X, p) and simply write p for an object in the Karoubi envelope. The domain will generally be clear from the context, so there is little danger in doing this.

Since $\mathcal{TL}$ is strict monoidal category, it isn't too hard to lift its monoidal structure to the Karoubi envelope $\mathrm{Kar}(\mathcal{TL})$. If $p : n \to n$ and $p' : n' \to n'$ are idempotents in $\mathcal{TL}$, then so is the tensor product $p \otimes p : n + n' \to n + n'$. Thus, we define the tensor product of (n, p) and (n', p') to be $(n + n', p \otimes p')$. The empty diagram is an idempotent, so we have a tensor unit in $\mathrm{Kar}(\mathcal{C})$. This defines the structure of a strict monoidal category since $\mathcal{TL}$ is a strict monoidal category.

Definition 1.4.2. Let $\mathcal{C}$ be an **Ab**-enriched category (meaning that the hom-spaces are abelian groups and that composition is bilinear). Then the additive envelope of $\mathcal{C}$ is an additive category which we denote by $\mathrm{Mat}(\mathcal{C})$. The objects of this category are formal finite direct sums $\bigoplus_i X_i$ of objects from $\mathcal{C}$. A morphism $\bigoplus_{i=1}^m X_i \to \bigoplus_{j=1}^n Y_j$ is an $n \times m$ matrix of morphisms in $\mathcal{C}$, where the (j, i)-th entry is a morphism $f_{ji} : X_i \to Y_j$. The identity map on $\bigoplus_{i=1}^n X_i$ is the diagonal matrix with entries id_{X_i}. Composition of maps is given by matrix multiplication. In order for this to make sense, we need to be able to add morphisms, so this justifies the **Ab**-enriched assumption.

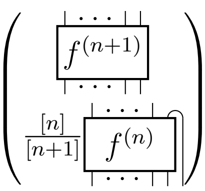
Once again, we can lift the strict monoidal structure of $\mathrm{Kar}(\mathcal{TL})$ to a strict monoidal structure on $\mathrm{Mat}(\mathrm{Kar}(\mathcal{TL}))$. The tensor product is defined by extending the tensor product of $\mathrm{Kar}(\mathcal{TL})$ bilinearly across $\bigoplus$, and the tensor product of two matrices of morphisms is given by the Kronecker product of matrices. Again, this gives a strict monoidal category since $\mathrm{Kar}(\mathcal{TL})$ is a strict monoidal category.

Definition 1.4.3. The generic Temperley-Lieb-Jones category is the full subcategory of $\mathrm{Mat}(\mathrm{Kar}(\mathcal{TL}))$ whose objects are those that can be created from the Jones-Wenzl idempotents by finitely many applications of $\oplus$ and $\otimes$. We use the notation $\mathcal{TLJ}$ for this category.

To demonstrate how things work in this category, we will prove the following useful fact.

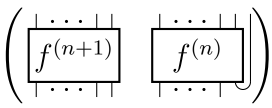
Proposition 1.4.4. *In $\mathcal{TLJ}$, $f^{(n)} \otimes f^{(1)} \cong f^{(n+1)} \oplus f^{(n-1)}$ for any $n \geq 1$.*

Proof. Consider the map $f^{(n)} \otimes f^{(1)} \to f^{(n+1)} \oplus f^{(n-1)}$ given by the matrix



,

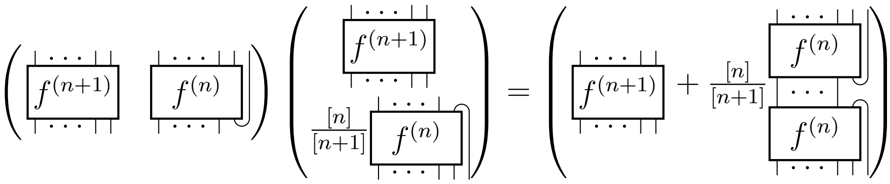
and the map $f^{(n+1)} \oplus f^{(n-1)} \to f^{(n)} \otimes f^{(1)}$ given by the matrix

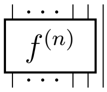


Then the composition

$$f^{(n)} \otimes f^{(1)} \longrightarrow f^{(n+1)} \oplus f^{(n-1)} \longrightarrow f^{(n)} \otimes f^{(1)}$$

is given by the 1×1 matrix



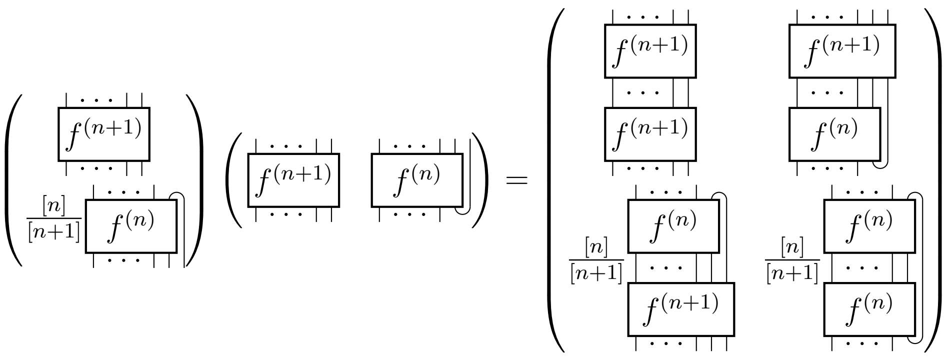
By Wenzl's recursion formula, the entry of this matrix is 

, which is the identity morphism on $f^{(n)} \otimes f^{(1)}$.

The other composite

$$f^{(n+1)} \oplus f^{(n-1)} \longrightarrow f^{(n)} \otimes f^{(1)} \longrightarrow f^{(n+1)} \oplus f^{(n-1)}$$

is given by the 2×2 matrix



The top left entry simplifies to $f^{(n+1)}$. The off-diagonal entries become zero because after applying Proposition 1.3.2(v) $f^{(n+1)}$ is either being capped or cupped. The bottom right entry simplifies to $f^{(n-1)}$ by property Proposition 1.3.2(iii). This means that the matrix is the identity on $f^{(n+1)} \oplus f^{(n-1)}$, completing the proof. $\square$

Chapter 2

Matrix units in a semisimple linear monoidal category

Consider the algebra $M_n(k)$ of $n \times n$ matrices over a field k. There is a particularly convenient basis for this algebra given by the matrices $e_{i,j}$ (for $1 \leq i, j \leq n$) with a 1 in the (i, j)-position and zeros everywhere else. For any $1 \leq i, j, k, \ell \leq n$, these matrices satisfy the useful equation

$$e_{i,j} e_{k,\ell} = \delta_{j,k} e_{i,\ell}, \tag{2.1}$$

where δ is the Kronecker delta. We call this basis the basis of matrix units for $M_n(k)$.

Now, consider the multimatrix algebra $M = \bigoplus_\alpha M_{n_\alpha}(k)$ obtained by taking a finite direct sum of matrix algebras. Within each summand we have a basis of matrix units as above – for each α there is an index set I_α of size n_α, and elements $e_{i,j} \in M$ for $i, j \in I_\alpha$ satisfying equation (2.1). By definition, matrices in different summands multiply to zero. Thus, taking $I = \cup_\alpha I_\alpha$ and setting $e_{i,j} = 0$ when i and j come from different indexing sets, we obtain the equation

$$e_{i,j} e_{k,\ell} = \delta_{j,k} e_{i,\ell}, \tag{2.2}$$

which now holds for all $i, j, k, \ell \in I$. If A is an algebra that is isomorphic to a multimatrix algebra, then there will always be elements satisfying equation (2.2). We call such a collection 'matrix units for A'.

In this chapter we will define the concept of a semisimple linear category. In Lemma 2.1.10 we show that the endomorphism spaces in such a category are multimatrix algebras. After this, the rest of the chapter is devoted to describing an explicit construction that yields a basis of matrix units for the algebras $\mathrm{End}(X^{\otimes n})$ in a semisimple monoidal category. This is finally achieved in Theorem 2.4.7.

2.1 Categorical preliminaries

Definition 2.1.1. An additive category is a category $\mathcal{C}$ satisfying the following three axioms:

(1) Every hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an (additive) abelian group such that the composition of morphisms is biadditive with respect to these group structures.

(2) There is a zero object $0 \in \mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$, the trivial group.

(3) For any objects $X_1, X_2 \in \mathcal{C}$ there is another object $Y \in \mathcal{C}$ with morphisms $p_1 : Y \to X_1$, $p_2 : Y \to X_2$,$i_1 : X_1 \to Y$ and $i_2 : X_2 \to Y$ such that $p_1 i_1 = \mathrm{id}_{X_1}$, $p_2 i_2 = \mathrm{id}_{X_2}$ and $i_1 p_1 + i_2 p_2 = \mathrm{id}_Y$.

It is straightforward to show that the object Y in (3) is unique up to unique isomorphism (see [ML98] for the details). We generally denote this object by $X_1 \oplus X_2$ and call it the direct sum or biproduct of X_1 and X_2.

The existence of biproducts for pairs of objects in $\mathcal{C}$ implies the existence of arbitrary finite direct sums of objects in $\mathcal{C}$. For objects $X_1, \ldots, X_n$, their direct sum $\bigoplus_{j=1}^n X_i$ is characterised by the existence of morphisms $i_j : X_j \to \bigoplus_{j=1}^n X_i$ and $p_j : \bigoplus_{j=1}^n X_i \to X_j$ satisfying the obvious generalisations of the equations in (3). For the full details, we again refer the reader to [ML98]. We also often write mZ to mean the direct sum of m copies of the object Z.

The standard example of an additive category is the category **Ab** of abelian groups. The zero object is the trivial group and the biproduct of X_1 and X_2 is the direct sum of abelian groups. In this situation, the maps p_1 and p_2 are projection onto X_1 and X_2 respectively, while the maps i_1 and i_2 are the inclusions of X_1 and X_2 into $X_1 \oplus X_2$. Because of this, we often refer to these maps in an arbitrary additive category as projections and inclusions.

Lemma 2.1.2. *Let $i_j : X_j \to \bigoplus_{j=1}^n X_i$ and $p_j : \bigoplus_{j=1}^n X_i \to X_j$ be the inclusion and projection maps for the direct sum $Y = \bigoplus_{j=1}^n X_i$. Then $p_j i_k = 0$ when $j \neq k$.*

Proof. We prove this for the case $n = 2$. We have

$$p_1 i_2 = p_1(i_1 p_1 + i_2 p_2) i_2 = \mathrm{id}_{X_1} p_1 i_2 + p_1 i_2 \,\mathrm{id}_{X_2} = p_1 i_2 + p_1 i_2,$$

hence $i_1 p_2 = 0$. Symmetrically, $p_2 i_1 = 0$. The general case follows using the same idea and induction. $\square$

Definition 2.1.3. Let k be an arbitrary field. We say that an additive category $\mathcal{C}$ is k-linear if for any objects $X, Y \in \mathcal{C}$, the abelian groups $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are actually finite dimensional vector spaces over k, such that composition of morphisms is k-bilinear.

We now state and prove a collection of lemmas describing isomorphisms between various hom-spaces in k-linear categories. We describe these isomorphisms explicitly since we will make use of them in some proofs later in this chapter.

Lemma 2.1.4. *Let $\mathcal{C}$ be a k-linear category. If $f : X \to Y$ is an isomorphism, then the map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ given by $g \mapsto g \circ f$ is an isomorphism of vector spaces.*

Proof. Composition in $\mathcal{C}$ is bilinear, so this is a linear map. If $f^{-1} : Y \to X$ is the inverse of f, then the same construction yields another linear map that is an inverse to this map. $\square$

Lemma 2.1.5. *Suppose $\mathcal{C}$ is a k-linear category and that $\bigoplus_{j=1}^{n} Z_j$ is a direct sum of finitely many objects of $\mathcal{C}$. Then there are vector space isomorphisms*

$$\mathrm{Hom}(\textstyle\bigoplus_{j=1}^{n} Z_j, X) \cong \bigoplus_{j=1}^{n} \mathrm{Hom}(Z_j, X)$$

and

$$\mathrm{Hom}(X, \textstyle\bigoplus_{j=1}^{n} Z_j) \cong \bigoplus_{j=1}^{n} \mathrm{Hom}(X, Z_j),$$

where the direct sum on the right hand side is the usual direct sum of vector spaces.

Proof. Consider the map $\mathrm{Hom}(\bigoplus_{j=1}^{n} Z_j, X) \to \bigoplus_{j=1}^{n} \mathrm{Hom}(Z_j, X)$ defined by sending a morphism $f : \bigoplus_{j=1}^{n} Z_j \to X$ to the n-tuple $(fi_1, \ldots, fi_n)$, where the i_k are the inclusions $i_k : Z_k \to \bigoplus_{j=1}^{n} Z_j$. We can also construct a map $\bigoplus_{j=1}^{n} \mathrm{Hom}(Z_j, X) \to \mathrm{Hom}(\bigoplus_{j=1}^{n} Z_j, X)$ in the other direction by sending a n-tuple of morphisms $(g_1, \ldots, g_n)$ to $\sum_{j=1}^{n} g_j p_j$, where the p_j are the projections for the direct sum. Both maps are clearly linear transformations since composition of morphisms is bilinear. The composite

$$\mathrm{Hom}(\textstyle\bigoplus_{j=1}^{n} Z_j, X) \longrightarrow \bigoplus_{j=1}^{n} \mathrm{Hom}(Z_j, X) \longrightarrow \mathrm{Hom}(\bigoplus_{j=1}^{n} Z_j, X)$$

sends a morphism f to $\sum_{j=1}^{n} f i_j p_j = f$, and the composite

$$\textstyle\bigoplus_{j=1}^{n} \mathrm{Hom}(Z_j, X) \longrightarrow \mathrm{Hom}(\bigoplus_{j=1}^{n} Z_j, X) \longrightarrow \bigoplus_{j=1}^{n} \mathrm{Hom}(Z_j, X)$$

sends an n-tuple of morphisms $(g_1, \ldots, g_n)$ to

$$\left(\sum_{j=1}^{n} g_j p_j i_1, \ldots, \sum_{j=1}^{n} g_j p_j i_n \right) = (g_1, \ldots, g_n) \, .$$

For this last equality we make use of Lemma 2.1.2. It follows from these calculations that the maps are mutual inverses, completing the proof of the first statement. The second statement is proved in a very similar way. In this case though, the maps are

$$\mathrm{Hom}(X, \textstyle\bigoplus_{j=1}^n Z_j) \to \bigoplus_{j=1}^n \mathrm{Hom}(X, Z_j), \quad f \mapsto (p_1 f, \ldots, p_n f)$$

and

$$\textstyle\bigoplus_{j=1}^n \mathrm{Hom}(X, Z_j) \to \mathrm{Hom}(X, \bigoplus_{j=1}^n Z_j), \quad (g_1, \ldots, g_n) \mapsto \sum_{j=1}^n i_j g_j.$$

□

Remark 2.1.6. It follows from Lemma 2.1.5, that

$$\mathrm{Hom}\left(\bigoplus_{j=1}^m X_j, \bigoplus_{k=1}^n Y_k\right) \cong \bigoplus_{j=1}^m \bigoplus_{k=1}^n \mathrm{Hom}(X_j, Y_k).$$

This implies that each morphism $f : \bigoplus_{j=1}^m X_j \to \bigoplus_{k=1}^n Y_k$ can be thought of as an $n \times m$ matrix of components $f_{kj} = p_k f i_j : X_j \to Y_k$. Composition of morphisms is then given by the usual multiplication of these matrices of components. See [ML98] for further discussion.

Definition 2.1.7. Let $\mathcal{C}$ be a k-linear category. We say an object $X \in \mathcal{C}$ is simple if $\mathrm{Hom}(X, X) \cong k$, i.e., every morphism $X \to X$ is some scalar multiple of id_X. Two simple objects X and Y are said to be disjoint if $\mathrm{Hom}(X, Y) \cong 0$. A collection of disjoint simple objects of $\mathcal{C}$ is a collection of simple objects that are pairwise disjoint.

Definition 2.1.8. A k-linear category $\mathcal{C}$ is said to be semisimple if there is a collection S of disjoint simple objects, such that every object of $\mathcal{C}$ is isomorphic to a finite direct sum $\bigoplus_{i=1}^n m_i Z_i$ for distinct objects $Z_i \in S$. When the category is monoidal, we require the tensor unit $\mathbf{1}$ belong to S. By an abuse of language, we sometimes refer to the collection S as *the* simple objects of $\mathcal{C}$.

Note that we can determine the coefficients m_i of this decomposition more explicitly. For any j, we have

$$\begin{aligned} \dim \mathrm{Hom}\left(Z_j, \bigoplus_{i=1}^n m_i Z_i\right) &= \dim \bigoplus_{i=1}^n m_i \,\mathrm{Hom}(Z_j, Z_i) \\ &= \dim m_j \,\mathrm{Hom}(Z_j, Z_j) = m_j, \end{aligned}$$

since we're working with disjoint simple objects.

The standard example of a semisimple category is the category $\mathbf{Rep}(G)$ of complex representations of a finite group G. It is a standard fact from representation theory that every representation of such a group can be written as a direct sum of irreducible representations, and that these irreducible representations make up a collection of disjoint simple objects.

Lemma 2.1.9. *Let Z be a simple object in a semisimple k-linear category $\mathcal{C}$. Then $\mathrm{Hom}(mZ, nZ)$ is isomorphic to $M_{n\times m}$, the space of $n \times m$ matrices over k.*

Proof. Recall the discussion of Remark 2.1.6. A morphism $f : mZ \to nZ$ is determined by the $n \times m$ matrix of its component maps $Z \to Z$. Since $\mathrm{Hom}(Z, Z) \cong k$, the result follows. □

Lemma 2.1.10. *Let X be an object in a semisimple k-linear category $\mathcal{C}$. The endomorphism algebra $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ of X is a multimatrix algebra.*

Proof. We can write $X \cong \bigoplus_i m_i Z_i$ for finitely many disjoint simple objects Z_i of $\mathcal{C}$. Then using Lemma 2.1.5, we have isomorphisms

$$\mathrm{End}(X) \cong \mathrm{Hom}\left(\bigoplus_i m_i Z_i, \bigoplus_j m_j Z_j\right) \cong \bigoplus_{i,j} \mathrm{Hom}(m_i Z_i, m_j Z_j).$$

Since the simple objects are disjoint, the only summands that do not vanish are those where $i = j$. Thus, we have

$$\mathrm{End}(X) \cong \bigoplus_i \mathrm{Hom}(m_i Z_i, m_i Z_i) \cong \bigoplus_i M_{m_i}(k)$$

by the previous lemma. This shows that $\mathrm{End}(X)$ isomorphic to a finite direct sum of matrix algebras, as claimed. □

Definition 2.1.11. A k-linear monoidal category is a category that is both k-linear and monoidal, and such that $\otimes$ is bilinear on morphisms.

This compatibility between the monoidal and k-linear structures yields the following important fact.

Lemma 2.1.12. *In a k-linear monoidal category, $\otimes$ distributes over $\oplus$.*

Proof. We prove the isomorphism $(X_1 \oplus X_2) \otimes Y \cong (X_1 \otimes Y) \oplus (X_2 \otimes Y)$. The generalisation to arbitrary finite direct sums in either entry will be clear.

Let i_1, i_2, p_1 and p_2 be the inclusions and projections for the direct sum. Define

$$i'_j = i_j \otimes \mathrm{id}_Y : X_j \otimes Y \to (X_1 \oplus X_2) \otimes Y$$

and

$$p'_j = p_j \otimes \mathrm{id}_Y : (X_1 \oplus X_2) \otimes Y \to X_j \otimes Y$$

for $j = 1, 2$. Then $p'_j i'_j = \mathrm{id}_{X_j} \otimes \mathrm{id}_Y = \mathrm{id}_{X_j \otimes Y}$ and

$$i'_1 p'_1 + i'_2 p'_2 = \mathrm{id}_{X_1 \oplus X_2} \otimes \mathrm{id}_Y = \mathrm{id}_{(X_1 \oplus X_2) \otimes Y}\,.$$

The isomorphism follows from the uniqueness of direct sums. □

Theorem 2.1.13. *The generic Temperley-Lieb-Jones category is a semisimple $\mathbb{C}(q)$-linear monoidal category – the Jones-Wenzl idempotents form a collection of disjoint simple objects and every object is a direct sum of Jones-Wenzl idempotents.*

Proof. We have already seen that this category is monoidal. It is additive since the additive envelope construction produces an additive category. Clearly $\mathcal{TLJ}$ is $\mathbb{C}(q)$-linear with $\otimes$ bilinear on morphisms. The only thing we really need to check is semisimplicity. The proof of this fact is reasonably involved. It makes use of some complicated skein theory for Temperley-Lieb diagrams. As such, we refer the reader to the excellent treatment in [Che14] for a full proof of this. □

We finish this section with an important lemma.

Lemma 2.1.14. *Let $\mathcal{C}$ be a semisimple k-linear category with disjoint simple objects Z_i indexed by a set I. Then for any objects X and Y of $\mathcal{C}$, the map*

$$\bigoplus_{i \in I} \mathrm{Hom}(Z_i, Y) \otimes_k \mathrm{Hom}(X, Z_i) \to \mathrm{Hom}(X, Y)$$

induced by composition is an isomorphism.

Proof. There are finite subsets $J \subset I$ and $L \subset I$ such that $X \cong \bigoplus_{j \in J} m_j Z_j$ and $Y \cong \bigoplus_{\ell \in L} n_\ell Z_\ell$. Then for each $i \in I$ the summand

$$\mathrm{Hom}(Z_i, Y) \otimes \mathrm{Hom}(X, Z_i)$$

on the left vanishes unless $i \in J \cap L$. If $i \in J \cap L$, we have

$$\mathrm{Hom}(X, Z_i) \cong \bigoplus_{j \in J} \mathrm{Hom}(m_j Z_j, Z_i) \cong \mathrm{Hom}(m_i Z_i, Z_i) \cong M_{1 \times m_i}(k)$$

and similarly,

$$\mathrm{Hom}(Z_i, Y) \cong M_{n_i \times 1}(k).$$

For the space on the right hand side, we have

$$\begin{aligned}\mathrm{Hom}(X, Y) &\cong \bigoplus_{j \in J} \bigoplus_{\ell \in L} \mathrm{Hom}(m_j Z_j, n_\ell Z_\ell) \\ &\cong \bigoplus_{i \in J \cap L} \mathrm{Hom}(m_i Z_i, n_i Z_i) \cong \bigoplus_{i \in J \cap L} M_{n_i \times m_i}(k).\end{aligned}$$

This means that corresponding to the map in the statement, we have a map

$$\bigoplus_{i \in J \cap L} M_{n_i \times 1}(k) \otimes_k M_{1 \times m_i} \to \bigoplus_{i \in J \cap L} M_{n_i \times m_i}(k),$$

which is the direct sum of maps $M_{n_i \times 1}(k) \otimes_k M_{1 \times m_i} \to M_{n_i \times m_i}(k)$ given by composition of matrices. One can check that these maps are injective, and a dimension comparison shows that they are isomorphisms. This completes the proof. □

2.2 Dual bases in a semisimple category

Definition 2.2.1. Let V and W be vector spaces over a field k. We say that a bilinear pairing $\langle \cdot, \cdot \rangle : V \times W \to k$ is nondegenerate if the maps

$$v \mapsto \langle v, w \rangle \quad \text{and} \quad w \mapsto \langle v, w \rangle$$

are injective for all $v \in V, w \in W$.

Suppose that V and W are finite dimensional. If $v_1, \dots, v_n$ is a basis for V, then a dual basis with respect to the pairing $\langle \cdot, \cdot \rangle$ is a basis $w_1, \dots, w_n$ for W such that $\langle v_i, w_j \rangle = \delta_{ij}$. Symmetrically, we also say that the basis $v_1, \dots, v_n$ is dual to the basis $w_1, \dots, w_n$. We note that the existence of a dual basis implies that V and W have the same dimension. In the presence of a nondegenerate pairing, dual bases always exist. This is a very useful fact.

The construction is fairly straightforward. We begin by considering the map $\Phi : W \to V^*$ defined by $w \mapsto \langle \cdot, w \rangle$. Since the pairing is bilinear, this is a linear map, and by nondegeneracy it must be injective. This shows that $\dim W \leq \dim V^* = \dim V$, and by considering a similarly defined map $V \to W^*$ we can conclude that $\dim W = \dim V$ and that Φ is an isomorphism. Then we can consider the dual basis $v^1, \dots, v^n$ in V^*, characterised by the equations $v^i(v_j) = \delta_{ij}$. Pulling this basis back along the isomorphism yields a basis $w_1, \ldots, w_n$ for W with

$$\langle v_i, w_j \rangle = \Phi(w_j)(v_i) = v^j(v_i) = \delta_{ij},$$

as desired. If instead we started with a basis for W, then a very similar argument is used to construct a dual basis of V.

Proposition 2.2.2. *Suppose that $\mathcal{C}$ is a semisimple, k-linear, additive category. If X is a simple object, then for any object Y, the natural bilinear pairing*

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}(Y, X) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, X) \cong k$$

given by composition of morphisms is nondegenerate.

Proof. We can write $Y \cong \bigoplus_i m_i Z_i$ for finitely many simple objects Z_i. Then

$$\mathrm{Hom}(Y, X) \cong \bigoplus_i \mathrm{Hom}(m_i Z_i, X) \quad \text{and} \quad \mathrm{Hom}(X, Y) \cong \bigoplus_i \mathrm{Hom}(X, m_i Z_i).$$

If X is not isomorphic to any of the Z_i, then both spaces are zero and the pairing is trivially nondegenerate. If X is isomorphic to one of the Z_i, then $\mathrm{Hom}(Y, X) \cong k^{m_i} \cong \mathrm{Hom}(X, Y)$. Then under these isomorphisms, the pairing $\langle \cdot, \cdot \rangle$ corresponds to the usual inner product on k^{m_i}. This is nondegenerate, so the result follows. □

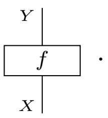
Thus, if X is simple and we're given a basis for either $\mathrm{Hom}(X, Y)$ or $\mathrm{Hom}(Y, X)$, then we can produce a dual basis with respect to the natural pairing. This will prove to be very important when we construct matrix units in the coming sections.

2.3 String diagrams for monoidal categories

A 2-category axiomatises the structure present in the category of all categories. Roughly speaking, a 2-category consists of a collection of objects (or 0-morphisms), a collection of 1-morphisms between these objects, and a collection of 2-morphisms between these 1-morphisms. There is one way to compose 1-morphisms and there are two ways to compose 2-morphisms – normally referred to as 'horizontal' and 'vertical' composition. These obey an interchange law like that for composition of natural transformations. The canonical example is the 2-category of categories, functors and natural transformations.

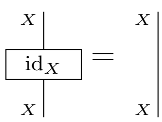
Our graphical calculus for the 2-category of categories works in general for any 2-category. We label regions by objects, strands by 1-morphisms and dots on strands by 2-morphisms. Now, in much the same way that a monoid can be thought of as a category with a single object, a monoidal category $\mathcal{C}$ can be thought of as a 2-category with a single object. The 1-morphisms are the objects of $\mathcal{C}$, and these are composed by taking the tensor product of objects. The 2-morphisms are the morphisms of $\mathcal{C}$, which are composed vertically using the normal composition in $\mathcal{C}$, and which are composed horizontally using the tensor product on morphisms in $\mathcal{C}$. The interchange law comes from the functoriality of $\otimes$.

This means that we have a natural graphical calculus for morphisms in a monoidal category $\mathcal{C}$. A morphism $f : X \to Y$ is depicted diagrammatically by

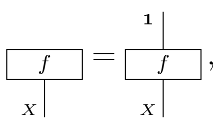


In the work to come, the string diagrams we draw will be fairly complex. As a result, we'll have many labels floating around at once. To alleviate the confusion this could cause, we have made a small concession here and enlarged our dots on strands to boxes on strands, in which we place the appropriate label.

Recall our conventions from Chapter 0 about identities – strands labelled by the identity functor are not drawn, and the identity natural transformation on a functor F is depicted as a strand with no box, labelled by F. In the monoidal category setting, these correspond to omitting strands labelled by the tensor unit $\mathbf{1}$, and depicting id_X as strand labelled by X. For example

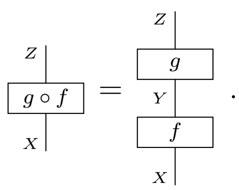


and

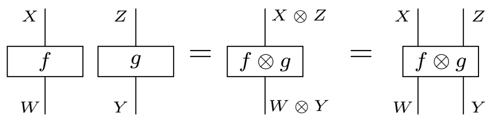


for a morphism $f : X \to \mathbf{1}$.

If $f : X \to Y$ and $g : Y \to Z$ are morphisms, then the composition $g \circ f$ is given by stacking diagrams:



If $f : W \to X$ and $g : Y \to Z$, then their tensor product is the horizontal composition and is therefore depicted by



Here the equation

$$f \otimes g = (f \circ \mathrm{id}_W) \otimes (\mathrm{id}_Z \circ g) = (\mathrm{id}_X \circ f) \otimes (g \circ \mathrm{id}_Y)$$

allows us to push boxes past each other vertically, as if they were beads on a string. For a complete treatment of the details of this graphical calculus see the comprehensive survey [Sel11].

2.4 Matrix units for $\mathrm{End}(X^{\otimes n})$

Definition 2.4.1. Let $\mathcal{C}$ be a semisimple k-linear monoidal category. Let's fix our favourite (i.e., any) object X in this category. The principal graph $\Gamma(\mathcal{C}, X)$ for the pair $(\mathcal{C}, X)$ is a directed graph defined as follows. The vertices are labelled by the disjoint simple objects of $\mathcal{C}$ and for each pair of simple objects Y and Z, there are

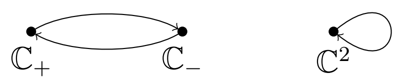
$$N_{Y,X}^{Z} := \dim \mathrm{Hom}(Y \otimes X, Z)$$

directed edges from the vertex labelled by Y to the vertex labelled by Z.

We illustrate this idea with a simple example. Recall that the category $\mathbf{Rep}(S_3)$ has 3 simple objects, the trivial representation $\mathbb{C}_+$, the sign representation $\mathbb{C}_-$, and the two dimensional irreducible representation which is usually denoted $\mathbb{C}^2$. Some character theory (see [EGH+11]) can be used to show that

$$\begin{aligned}
\mathbb{C}_+ \otimes \mathbb{C}_- &\cong \mathbb{C}_-, \\
\mathbb{C}_- \otimes \mathbb{C}_- &\cong \mathbb{C}_+, \\
\mathbb{C}^2 \otimes \mathbb{C}_- &\cong \mathbb{C}^2.
\end{aligned}$$

By Schur's lemma (again, see [EGH+11]), this means that the principal graph for the pair $(\mathbf{Rep}(S_3), \mathbb{C}_-)$ is



For the remainder of this section we will fix bases for the various hom-spaces $\mathrm{Hom}(Y \otimes X, Z)$ as Y and Z range over all the simple objects of $\mathcal{C}$. We will use the notation $\{\gamma_{Y,Z,i}\}_{i \in I_{Y,Z}}$ for these sets, where $I_{Y,Z}$ is some index set of cardinality $N_{X,Y}^{Z}$. Note that the act of choosing actual basis elements means that the matrix unit construction that follows is not canonical.

Definition 2.4.2. A walk a on the principal graph $\Gamma(\mathcal{C}, X)$ consists of a sequence of simple objects (vertices) $A_0, A_1, \ldots, A_n$ and a sequence of basis elements (edges) $\gamma_{A_0,A_1,a_1}, \gamma_{A_1,A_2,a_2}, \ldots, \gamma_{A_{n-1},A_n,a_n}$ connecting them (here $a_k \in I_{A_{k-1},A_k}$). We will use the notation

$$a = (A_0, \gamma_{A_0,A_1,a_1}, A_1, \gamma_{A_1,A_2,a_2}, A_2, \ldots, A_{n-1}, \gamma_{A_{n-1},A_n,a_n}, A_n),$$

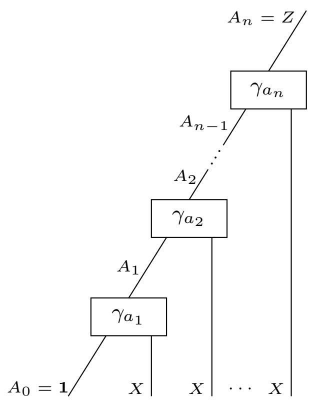
with the various edges interspersed between the vertices. Note that we use an upper case letter for the sequence of vertices and the associated lower case letter to name the walk and index the edges. We might also suppress the object labels for the specified basis elements. When this is the case, γ_{a_k} means the morphism γ_{A_{k-1},A_k,a_k}.

The length of such a walk a is the number of edges n, the starting vertex is A_0, and the ending vertex is A_n.

We will now use length n walks on the principal graph starting at $\mathbf{1}$ to construct matrix units for the algebra $\mathrm{End}(X^{\otimes n})$. Let $P_{Z,n}$ denote the set of all such walks that end at the simple object Z. Then we can define a map $t_{Z,n} : P_{Z,n} \to \mathrm{Hom}(X^{\otimes n}, Z)$ by assigning to a walk

$$a = (\mathbf{1} = A_0, \gamma_{A_0,A_1,a_1}, A_1, \gamma_{A_1,A_2,a_2}, A_2, \ldots, A_{n-1}, \gamma_{A_{n-1},A_n,a_n}, A_n = Z)$$

the morphism represented by the string diagram



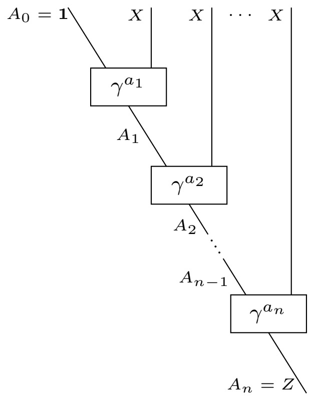
Remark 2.4.3. The morphism depicted actually belongs to $\mathrm{Hom}(\mathbf{1} \otimes X^{\otimes n}, Z)$. However, this space is canonically isomorphic to $\mathrm{Hom}(X^{\otimes n}, Z)$, so this is not an issue.

If we now define $P_n := \bigcup_Z P_{Z,n}$ by taking the union over all simple objects Z, then P_n consists of all length n walks on the principal graph starting at $\mathbf{1}$. Then by combining the various maps $t_{Z,n} : P_{Z,n} \to \mathrm{Hom}(X^{\otimes n}, Z)$ together, we get another map

$$t : \bigcup_{n\geq 1} P_n \longrightarrow \bigcup_{n\geq 1} \bigcup_Z \mathrm{Hom}(X^{\otimes n}, Z),$$

where the union is again taken over all simple objects Z.

As it stands, we have a collection of bases $\{\gamma_{Y,Z,i}\}_{i\in I_{Y,Z}}$ for the spaces $\mathrm{Hom}(Y \otimes X, Z)$. Since Z is always simple, the results of Section 2.2 allow us to pick dual bases for the spaces $\mathrm{Hom}(Z, Y \otimes X)$, which we denote by $\{\gamma^i_{Y,Z}\}_{i\in I_{Y,Z}}$. Having done this, we can define another map $\hat{t}_{Z,n} : P_{Z,n} \to \mathrm{Hom}(Z, X^{\otimes n})$ by instead assigning the morphism represented by



to a. Then in exactly the same way as before, we get a map

$$\hat{t} : \bigcup_{n\geq 1} P_n \longrightarrow \bigcup_{n\geq 1} \bigcup_Z \mathrm{Hom}(Z, X^{\otimes n}),$$

where the union is again over all simple objects Z.

We note that a simple recursive formula is evident from this diagrammatic description of t and $\hat{t}$. For any walk $a \in P_n$, let a' denote the walk in P_{n-1} obtained by deleting the last vertex and edge from a. Then

$$t(a) = \gamma_{A_{n-1},A_n,a_n} \circ (t(a') \otimes \mathrm{id}_X) \tag{2.3}$$

and

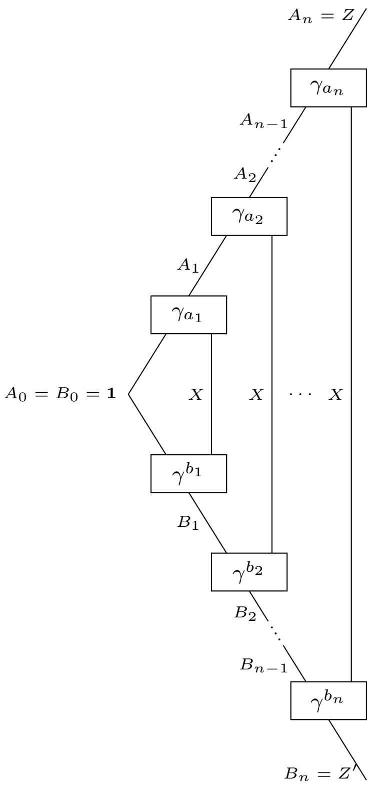
$$\hat{t}(a) = (\hat{t}(a') \otimes \mathrm{id}_X) \circ \gamma^{a_n}_{A_{n-1},A_n}. \tag{2.4}$$

In fact, we could have defined these maps recursively by these formulae. However, the structure would be much less clear had we done this.

Lemma 2.4.4. *If $a \in P_{Z,n}$ and $b \in P_{Z',n}$, then*

$$t(a)\hat{t}(b) = \begin{cases} \mathrm{id}_{Z'} = \mathrm{id}_Z & \text{if } a = b, \\ 0 \in \mathrm{Hom}(Z', Z) & \text{otherwise.} \end{cases}$$

Proof. Consider the following string diagram representing $t(a)\hat{t}(b)$:



If any of the vertices A_k and B_k are different from each other, then A_k and B_k are disjoint simple objects, so the piece of the diagram representing a morphism $B_k \to A_k$ vanishes. This means that the whole diagram is zero unless $A_k = B_k$ for all k. If this is the case, then we work from the innermost composite $(\gamma_{a_1} \circ \gamma^{b_1})$ on the left outwards. That the whole diagram is the identity on $A_n = Z = Z' = B_n$ if and only if $a_k = b_k$ for each k follows from the fact that the bases were chosen to be dual to each other. □

Lemma 2.4.5. *For any simple object Z of $\mathcal{C}$, the set $\{t(a) \mid a \in P_{Z,n}\}$ is a basis for* $\mathrm{Hom}(X^{\otimes n}, Z)$. *Similarly, the set $\{\hat{t}(a) \mid a \in P_{Z,n}\}$ is a basis for* $\mathrm{Hom}(Z, X^{\otimes n})$.

Proof. We prove the first statement only. The proof of the second statement is similar. Linear independence is straightforward to establish. If we have coefficients r_a such that

$$\sum_{a \in P_{Z,n}} r_a t(a) = 0,$$

then multiplying on the right by $\hat{t}(b)$ for any $b \in P_{Z,n}$ yields

$$0 = \sum_{a \in P_{Z,n}} r_a t(a)\hat{t}(b) = r_b \,\mathrm{id}_Z,$$

by the previous lemma. Thus $r_b = 0$, proving linear independence.

Showing that the set also spans is more involved. To do this, we proceed by induction on n. If $n = 1$, then each $t(a)$ is a different distinguished basis element of $\mathrm{Hom}(\mathbf{1} \otimes X, Z)$. These certainly span, so the base case is immediate. Now, suppose that the statement holds for walks of length $n-1$. Length n walks from $\mathbf{1}$ to Z are in bijection with pairs consisting of an edge from Z_i to Z and a length $n-1$ walk from $\mathbf{1}$ to Z_i, where the Z_i run over all simple objects. Thus,

$$|P_{Z,n}| = \sum_{i \in I} N_{Z_i,X}^{Z} \cdot |P_{Z_i,n-1}| = \sum_{i \in I} N_{Z_i,X}^{Z} \cdot \dim \mathrm{Hom}(X^{\otimes n-1}, Z_i),$$

where the last equality comes from the induction hypothesis.

On the other hand, using semisimplicity, we can write $X^{\otimes n-1} \cong \bigoplus_{i \in I} m_i Z_i$ for $m_i = \dim \mathrm{Hom}(X^{\otimes n-1}, Z_i)$. Then $X^{\otimes n} \cong \bigoplus_{i \in I} m_i(Z_i \otimes X)$, which means that

$$\begin{aligned} \dim \mathrm{Hom}(X^{\otimes n}, Z) &= \sum_{i \in I} m_i \cdot \dim \mathrm{Hom}(Z_i \otimes X, Z) \\ &= \sum_{i \in I} \dim \mathrm{Hom}(X^{\otimes n-1}, Z_i) \cdot N_{Z_i,X}^{Z}. \end{aligned}$$

This completes the proof. □

Definition 2.4.6. We say that two length n walks a and b on the principal graph $\Gamma(\mathcal{C}, X)$ are compatible if they have the same end vertex $A_n = B_n$.

If $a, b \in P_n$ are compatible paths, set $e_{a,b} = \hat{t}(a)t(b)$. Then the previous lemma shows that the various $e_{a,b}$ satisfy the equations for matrix units. If, on the other hand, a and b are not compatible, then we extend our definition by setting $e_{a,b} = 0$. Then the equations

$$e_{a,b}e_{c,d} = \delta_{b,c}e_{a,d}$$

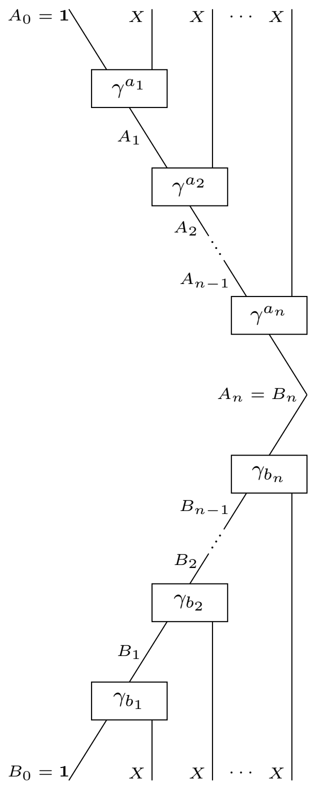
hold for any paths $a, b, c, d \in P_n$. Then we can finally prove the following.

Theorem 2.4.7. *The $e_{a,b}$ for pairs of compatible walks $a, b \in P_n$ form a basis of matrix units for* $\mathrm{End}(X^{\otimes n})$.

Proof. Most of the work has been done already – we just need to apply Lemma 2.1.14. We have bases for each $\mathrm{Hom}(X^{\otimes n}, Z)$ and $\mathrm{Hom}(Z, X^{\otimes n})$, hence we have bases for the tensor products appearing on the left hand side of Lemma 2.1.14. These bases look like $\hat{t}(a) \otimes t(b)$ for $a, b \in P_{Z,n}$, and composing yields the result. □

For compatible walks a and b, the matrix unit $e_{a,b}$ is depicted diagrammatically by

$e_{a,b} =$



. (2.5)

Often the basis of matrix units is referred to as the 'tree basis' because of this diagram's close resemblance to a tree.

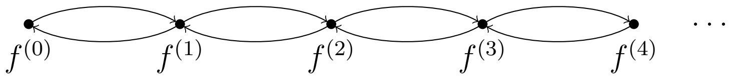
2.5 Example – generic Temperley-Lieb-Jones

We'll now work through the construction given above in detail for the generic $\mathcal{TLJ}$ category. Here, the simple object we use is $f^{(1)}$. Since this is just a single strand, the n-th tensor power is simply the identity in the algebra TL_n. This means that the space $\mathrm{End}(f^{(1)\otimes n})$ is actually the n-th Temperley-Lieb algebra TL_n. As such, what follows will describe explicit matrix units for the generic Temperley-Lieb algebras.

As mentioned before, in the generic TLJ category the simple objects are the Jones-Wenzl idempotents. Recalling that $f^{(n)} \otimes f^{(1)} \cong f^{(n+1)} \oplus f^{(n-1)}$, we find that

$$\operatorname{Hom}(f^{(n)} \otimes f^{(1)}, f^{(m)}) \cong \operatorname{Hom}(f^{(n+1)} \oplus f^{(n-1)}, f^{(m)}) \cong \begin{cases} \mathbb{C}(q) & \text{if } n = m \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

This means that the principal graph $\Gamma(\mathcal{TLJ}, f^{(1)})$ has a vertex for each natural number, and exactly one directed edge going each way between numerically adjacent vertices:



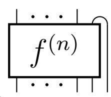
This makes the matrix unit construction reasonably simple to work through in this case. We begin by recalling that an element of $\operatorname{Hom}(f^{(n)} \otimes f^{(1)}, f^{(n+1)})$ is an element $x \in TL_{n+1}$ such that

$$x(f^{(n)} \otimes f^{(1)}) = x = f^{(n+1)}x.$$

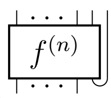
An obvious TL_{n+1}-diagram satisfying these equations is $f^{(n+1)}$, and since the hom-space is 1-dimensional this actually gives us a basis:

$$\operatorname{Hom}(f^{(n)} \otimes f^{(1)}, f^{(n+1)}) = \mathbb{C}(q) \left\{ f^{(n+1)} \right\}.$$

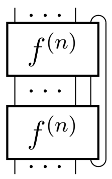
Similarly, there are obvious diagrams with

$\operatorname{Hom}(f^{(n)} \otimes f^{(1)}, f^{(n-1)}) = \mathbb{C}(q) \Bigg\{$  $\Bigg\}$,

$$\operatorname{Hom}(f^{(n+1)}, f^{(n)} \otimes f^{(1)}) = \mathbb{C}(q) \left\{ f^{(n+1)} \right\},$$

$\operatorname{Hom}(f^{(n-1)}, f^{(n)} \otimes f^{(1)}) = \mathbb{C}(q) \Bigg\{$  $\Bigg\}$.

The chosen basis elements for $\operatorname{Hom}(f^{(n)} \otimes f^{(1)}, f^{(n+1)})$ and $\operatorname{Hom}(f^{(n+1)}, f^{(n)} \otimes f^{(1)})$ are already dual to each other. The basis elements for $\operatorname{Hom}(f^{(n)} \otimes f^{(1)}, f^{(n-1)})$ and $\operatorname{Hom}(f^{(n-1)}, f^{(n)} \otimes f^{(1)})$ don't quite work though – they're off by a constant since



$$= \frac{[n+1]}{[n]} f^{(n-1)}.$$

To solve this, we just scale each of these basis elements by $\sqrt{\frac{[n]}{[n+1]}}$. Having done that, these bases are then fixed for the remainder of the construction.

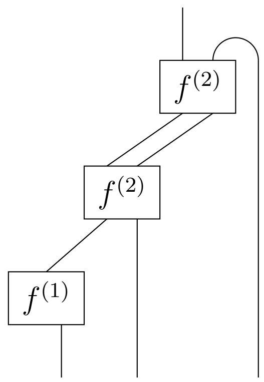
Normally, a walk on the principal graph consists of a sequence of vertices (simple objects) and specified edges (basis vectors) connecting them. Since all the hom-spaces are 1-dimensional in this example, once we have fixed bases there is no danger in thinking of a walk a on the principal graph as simply a sequence of vertices $a = (A_0, A_1, \ldots, A_n)$. Then the sets $P_{f^{(m)},n}$ used above can be identified with the sets $P_{m,n}$ of length n walks in $\mathbb{N}$ starting at 0 and ending at m. Note that these sets $P_{m,n}$ are empty when m and n have different parity, or when $m > n$.

The map t then has a reasonably simple description. If we have a walk

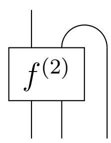
$$a = (0 = A_0, A_1, \ldots, A_n = m),$$

then we build $t(a)$ as follows. We follow the walk through $\mathbb{N}$, adding a diagram each time we move from one vertex to another. Every time a diagram is placed, all but the rightmost strands attach to the top of the previous diagram, and the rightmost strand is pulled down to the bottom. If the walk increases as it goes from A_k to A_{k+1}, then the diagram we use should be the chosen basis element of $\mathrm{Hom}(f^{(A_k)} \otimes f^{(1)}, f^{(A_k+1)})$, and if the walk decreases as it goes from A_k to A_{k+1}, then the diagram we use should be the chosen basis element of $\mathrm{Hom}(f^{(A_k)} \otimes f^{(1)}, f^{(A_k-1)})$.

For example, when $n = 3$, one possible walk is $a = (0, 1, 2, 1)$. Following this prescription, we produce the diagram

$\sqrt{\frac{[2]}{[3]}} \cdot$  .

The basic properties of Jones-Wenzl idempotents mean that this simplifies to

$\sqrt{\frac{[2]}{[3]}} \cdot$  .

The map $\hat{t}$ has a similar description, except we work from the top down and use the dual bases instead.

Chapter 3

Representations of Temperley-Lieb algebras

In Lemma 2.1.10 of Chapter 2, we showed that the endomorphism algebras in a semisimple k-linear monoidal category are isomorphic to multimatrix algebras over k. Algebras of this type are more commonly referred to as semisimple algebras. We recall from basic representation theory [EGH$^+$11] that every representation of a semisimple algebra decomposes as a direct sum of irreducible subrepresentations. Thus, we can say a great deal about the representation theory of a semisimple algebra if we can just understand its irreducible representations.

In Propositions 3.1.1, 3.1.2 and 3.1.3 we give a classification of the irreducible representations of $\mathrm{End}(X^{\otimes n})$ using the matrix units constructed in the previous chapter. In particular, this classifies the irreducible representations of the Temperley-Lieb algebras.

Before proceeding, we remind the reader that (left) modules over an algebra are the same thing as representations of that algebra. Because of this, we use the terms 'module' and 'representation' interchangeably in what follows. Furthermore, all modules are left modules unless otherwise specified.

3.1 Standard representations for $\mathrm{End}(X^{\otimes n})$

The tree basis construction of the previous chapter provides a convenient way to explore the representation theory of $\mathrm{End}(X^{\otimes n})$. For any simple object Z, define

$$W_Z^n := \mathrm{Hom}(Z, X^{\otimes n}).$$

Then W_Z^n is a representation of $\mathrm{End}(X^{\otimes n})$, with the action of $\mathrm{End}(X^{\otimes n})$ given by composition on the left. Lemma 2.4.5 shows that these representations have as a basis the $\hat{t}(a)$ for $a \in P_{Z,n}$. This will prove to be a very convenient basis to work with in what follows.

Amazingly, the representations just constructed are exactly the irreducible representations of $\mathrm{End}(X^{\otimes n})$. This is the content of the next three propositions.

Proposition 3.1.1. *If W_Z^n is nonzero, then it is an irreducible representation.*

Proof. Let $W \subset W_Z^n$ be a nonzero subrepresentation. Take any nonzero $x \in W$ and write $x = \sum_{a \in P_{Z,n}} r_a \hat{t}(a)$. At least one of these coefficients, say r_b, must be nonzero. Then for any $c \in P_{Z,n}$ we can act by $r_b^{-1} e_{c,b}$ to give

$$\frac{1}{r_b} e_{c,b} x = \sum_{a \in P_{Z,n}} \frac{r_a}{r_b} \hat{t}(c) t(b) \hat{t}(a) = \hat{t}(c) \,\mathrm{id}_Z = \hat{t}(c).$$

As a result, we have $W = W_Z^n$, showing that W_Z^n is irreducible. □

Proposition 3.1.2. *If Z and Z' are disjoint simple objects, then the representations W_Z^n and $W_{Z'}^n$ are not isomorphic as* $\mathrm{End}(X^{\otimes n})$ *representations.*

Proof. Suppose $f : W_Z^n \to W_{Z'}^n$ is an intertwining map. Take any $a \in P_{Z,n}$. Then $\hat{t}(a)$ is a basis vector for W_Z^n. Write $f(\hat{t}(a)) = \sum_{b \in P_{Z',n}} r_b \hat{t}(b)$. Then we can compute

$$f(\hat{t}(a)) = f(e_{a,a} \hat{t}(a)) = e_{a,a} \cdot \sum_{b \in P_{Z',n}} r_b \hat{t}(b) = 0,$$

since a is not in $P_{Z',n}$. Since the $\hat{t}(a)$ give a basis for W_Z^n, this completes the proof. □

Proposition 3.1.3. *Every irreducible representation is isomorphic to one of these irreducible representations.*

Proof. It is a standard result [EGH$^+$11] about representations of finite dimensional semisimple algebras that the sum of the squared dimensions of the irreducible representations gives the dimension of the algebra. Here this sum is taken over all simple objects Z and is

$$\sum_Z (\dim W_Z^n)^2 = \sum_Z |P_{Z,n}|^2.$$

The tree basis constructed for $\mathrm{End}(X^{\otimes n})$ in Chapter 2 has exactly as many elements as the right hand sum, so the proposition follows from this. □

3.2 Dimension counting

In general, computing the size of the sets $P_{Z,n}$ (or equivalently, the dimensions of the irreducible representations) is difficult. In the Temperley-Lieb case, however, things are simpler. Although it still takes a bit of work to achieve, we can give explicit formulae for these dimensions.

We can apply the construction of Section 3.1 to the $\mathcal{TLJ}$ category with $X = f^{(1)}$. The simple objects of this category are the Jones-Wenzl idempotents, so we find that the representations are

$$W_m^n := W_{f^{(m)}}^n = \mathrm{Hom}(f^{(m)}, f^{(1)^{\otimes n}}).$$

Since $f^{(1)}$ is just the strand, its n-th tensor power is the identity in TL_n. Thus, we can characterise W_m^n as those (m,n)-Temperley-Lieb diagrams x with $x = xf^{(m)}$. We will make use of this characterisation in the next section. As we mentioned before, when n and m have different parity, or when $m > n$, the set $P_{m,n}$ is empty. This means that the irreducible representations are the W_m^n for $m \in \{0, \dots, n\}$ with the same parity as n.

Proposition 3.2.1. *For $m \in \{0, \dots, n\}$ with the same parity as n, we have*

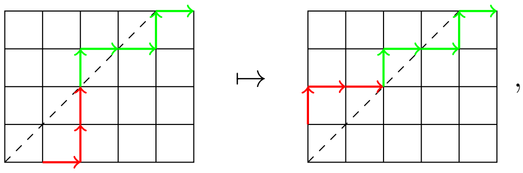
$$\dim W_m^n = \binom{n}{\frac{n-m}{2}} - \binom{n}{\frac{n-m}{2} - 1}.$$

Proof. We can compute the dimension of W_m^n by counting the number of elements of $P_{m,n}$. In order to do this, we'll exhibit a bijection between another set that is easier to count. Given a length n walk in $\mathbb{N}$ starting at 0 and ending at m, we can construct a monotonic increasing walk in $\mathbb{Z}^2$. The walk starts at $(0,0)$ and moves one unit to the right every time the original walk moves to the right, or it moves one unit up every time the original walk moves to the left. The original walk in $\mathbb{N}$ moves to the left at most as many times as it moves to the right. As a result, the constructed walk in $\mathbb{Z}^2$ cannot go above (but may touch) the main diagonal in $\mathbb{Z}^2$. The walk in $\mathbb{N}$ must reach m after n steps. Thus, if (x,y) is the final vertex of the constructed walk, we must have $x + y = n$ and $x - y = m$. Solving these equations, we find that $x = \frac{n+m}{2}$ and $y = \frac{n-m}{2}$. This means that the constructed walk is a monotonic increasing walk in $\mathbb{Z}^2$ from $(0,0)$ to $(\frac{n+m}{2}, \frac{n-m}{2})$ that does not go above the diagonal. It is straightforward to see that these sets are actually in bijection, since every such walk can be constructed this way.

In order to count the elements of this new set, we begin by counting monotonic increasing walks in $\mathbb{Z}^2$ from (a,b) to (c,d). (From now on, 'walk' should be read as 'monotonic increasing walk'). If $a > c$ or $b > d$, then there are no walks, so we assume that $a \leq c$ and $b \leq d$. Then a walk from (a, b) to (c, d) consists of a sequence of $c - a + d - b$ edges, where $c - a$ are horizontal and $d - b$ are vertical. To describe a walk of this type, we only need to specify which edges are horizontal, so these walks are counted by

$$N_{(a,b)}^{(c,d)} := \binom{c - a + d - b}{d - b}.$$

Now we make the additional assumptions that $a > b$ and $c > d$. Suppose we have a walk that touches the diagonal at some point. If (i, i) is the coordinate where the walk first touches the diagonal, then we can construct a new monotonic walk from (b, a) to (c, d) by leaving the walk unchanged past (i, i), but reflecting the segment before (i, i) across the main diagonal. For example, for the pictured walk from $(1, 0)$ to $(5, 4)$ we do the following



producing a walk from $(0, 1)$ to $(5, 4)$. Since $a > b$, the point (b, a) is above the main diagonal, and since $c > d$, the point (c, d) is below the main diagonal. This means that a walk from (b, a) to (c, d) must cross the main diagonal. Thus, the same process produces an inverse map, establishing a bijection between the set of walks from (a, b) to (c, d) that touch the diagonal, and the set of all walks from (b, a) to (c, d). The latter set is counted by $N_{(b,a)}^{(c,d)}$, so we conclude that the number of walks from (a, b) to (c, d) that do not touch the diagonal is

$$N_{(a,b)}^{(c,d)} - N_{(b,a)}^{(c,d)} = \binom{c - a + d - b}{d - b} - \binom{c - a + d - b}{d - a}.$$

Returning to the original problem, we actually want to count walks from $(0, 0)$ to $(\frac{n+m}{2}, \frac{n-m}{2})$ which do not go above the main diagonal (but can touch the main diagonal). Shifting such a walk 1 unit to the right, we obtain a bijection between these walks and those walks from $(1, 0)$ to $(\frac{n+m}{2} + 1, \frac{n-m}{2})$ that do not touch the diagonal. Then the previous formula shows that the total number of these walks is

$$\binom{\frac{n+m}{2} + \frac{n-m}{2}}{\frac{n-m}{2}} - \binom{\frac{n+m}{2} + \frac{n-m}{2}}{\frac{n-m}{2} - 1} = \binom{n}{\frac{n-m}{2}} - \binom{n}{\frac{n-m}{2} - 1}.$$

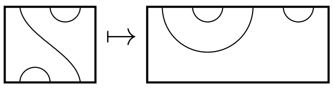
This proves the given formula for $\dim W_m^n$. $\square$

We can use these results to calculate the dimension of TL_n:

Proposition 3.2.2. *The generic Temperley-Lieb algebras have dimensions given by the equation*

$$\dim TL_n = \dim W_0^{2n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Proof. Since $f^{(0)}$ is the empty diagram, we find that W_0^{2n} is exactly the space $\mathrm{Hom}_{\mathcal{TL}}(0, 2n)$. Then there is a straightforward bijection between the standard basis of TL_n and simple diagrams in $\mathrm{Hom}_{\mathcal{TL}}(0, 2n)$. Given a basis element of TL_n take the bottom n dots and pull them up to the right as pictured in the following example



For the inverse map, take a simple diagram in $\mathrm{Hom}_{\mathcal{TL}}(0, 2n)$ and pull the n rightmost dots on the top down to the right. □

Remark 3.2.3. The numbers $C_n = \binom{2n}{n} - \binom{2n}{n-1}$ are known as the Catalan numbers and appear all over the place in combinatorics. For example, an exercise in [Sta99] asks the reader to show that the Catalan numbers count the number of elements of 66 different families of objects.

Remark 3.2.4. We should also note that there are far more direct ways to calculate the dimension of TL_n (see for example [RSA14]). The approach described here is just the quickest in the framework that's already been set up.

3.3 Restriction

There is a natural inclusion of TL_n into TL_{n+1} given by adding a single strand on the right of a simple diagram and extending linearly. This means that we can view any TL_{n+1}-module as a TL_n-module, yielding a functor

$$\mathrm{Res}_n^{n+1} : TL_{n+1}\textbf{-mod} \to TL_n\textbf{-mod},$$

which is the identity on TL_{n+1}-module maps.

Consider an irreducible representation W_m^n of TL_n. We can restrict to TL_{n-1}, obtaining the TL_{n-1}-representation $\mathrm{Res}_{n-1}^n(W_m^n)$. It is natural to try to understand how this restricted representation decomposes as a direct sum of irreducible TL_{n-1}-representations. Pleasantly, there is a very simple formula answering this question.

Proposition 3.3.1. *Let W^n_m be an irreducible TL_n-representation for $n \geq 1$. Then if $m \geq 1$,*

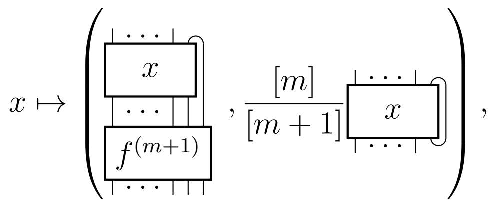
$$\mathrm{Res}^n_{n-1}(W^n_m) = W^{n-1}_{m+1} \oplus W^{n-1}_{m-1},$$

and if $m = 0$,

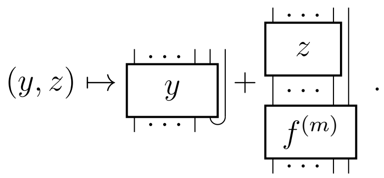
$$\mathrm{Res}^n_{n-1}(W^n_0) = W^{n-1}_1.$$

Proof. When $m = 0$ it is easy to verify that

are mutually inverse TL_{n-1}-module maps, so $\mathrm{Res}^n_{n-1}(W^n_0) \cong W^{n-1}_1$. For $m \geq 1$, define a map $W^n_m \to W^{n-1}_{m+1} \oplus W^{n-1}_{m-1}$ by



noting that the right hand entry is an element of W^{n-1}_{m-1} since $x = xf^{(m-1)}$ (here we are including $f^{(m-1)}$ into TL_m in the usual way). Because the $n-1$ leftmost strands on top of x are left untouched, it is immediate that this is a map of TL_{n-1}-modules. Next, define a map $W^{n-1}_{m+1} \oplus W^{n-1}_{m-1} \to W^n_m$ by

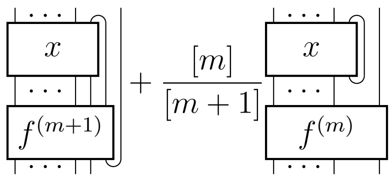


Once again this is clearly a TL_{n-1}-module map, and the output lies in W^n_m since $y = yf^{(m-1)}$.

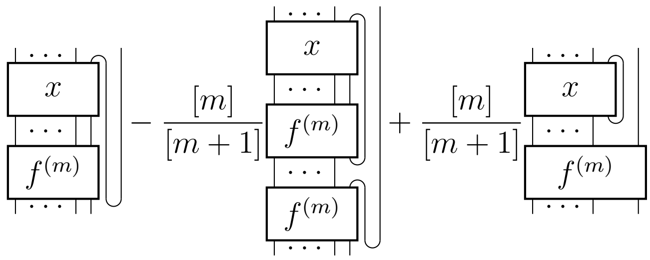
The claim now is that these two maps are mutual inverses. To verify this, we observe that the composition

$$W^n_m \longrightarrow W^{n-1}_{m+1} \oplus W^{n-1}_{m-1} \longrightarrow W^n_m$$

takes an element $x \in W^n_m$ to



Applying Wenzl's recursion formula (1.1) to the $f^{(m+1)}$ in the first term yields

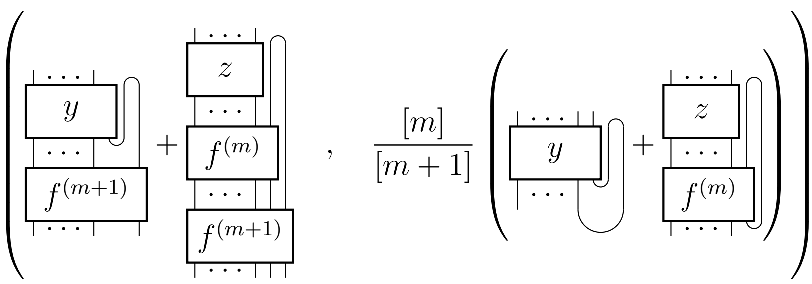


which evaluates to x since $xf^{(m)} = x$.

For the other composition

$$W^{n-1}_{m+1} \oplus W^{n-1}_{m-1} \longrightarrow W^n_m \longrightarrow W^{n-1}_{m+1} \oplus W^{n-1}_{m-1}$$

a pair $(y, z) \in W^{n-1}_{m+1} \oplus W^{n-1}_{m-1}$ is sent to



Then since $y = yf^{(m+1)}$, the first summand in the left term simplifies to y and the first summand in the right term vanishes (JWI are uncuppable). In the second summand of the left term the $f^{(m)}$ pulls back into the $f^{(m+1)}$, which is then being capped, so the summand vanishes. Finally, the second summand in the right term evaluates to z using $zf^{(m-1)} = z$ and the partial trace relation for the JWI. Thus, the map is the identity on $W^{n-1}_{m+1} \oplus W^{n-1}_{m-1}$, so the proof is complete. □

3.4 Induction

Having discussed the process of restricting a representation of TL_{n+1} to a representation of TL_n, it is only natural to ask whether the reverse is possible. That is, given a representation of TL_n, is there a way to produce a representation of TL_{n+1}? The answer is yes, but the story is more complicated. This process of taking a representation of a smaller object and producing a representation of a bigger object is known as induction, and in our context is achieved using the tensor product of bimodules.

Induction works as follows. Note that we can view the algebra TL_{n+1} as a (TL_{n+1}, TL_n)-bimodule, with the action on the right coming from the inclusion of TL_n into TL_{n+1}. Thus, if V is a TL_n-module, then the tensor product

$$TL_{n+1} \otimes_{TL_n} V$$

is a TL_{n+1}-module. To reduce notational clutter, we can abbreviate the tensor product symbol $\otimes_{TL_n}$ to $\otimes_n$. If $L : V \to W$ is a map of TL_n-modules, then

$$\mathrm{id}_{TL_{n+1}} \otimes_n L : TL_{n+1} \otimes_n V \to TL_{n+1} \otimes W$$

is a map of TL_{n+1}-modules, so we have a functor

$$\mathrm{Ind}_n^{n+1} : TL_n\text{-}\mathbf{mod} \to TL_{n+1}\text{-}\mathbf{mod}.$$

Whereas the dimension of a restricted module was immediately obvious, the dimension of an induced module is much less clear. Given a basis for W, it is not at all obvious what a basis for $\mathrm{Ind}_n^{n+1}(W)$ would be. We know that the dimension is bounded above by $(\dim TL_{n+1}) \cdot (\dim W)$, but we will soon see that this number is far too large. Thankfully, we can characterise induction for the standard modules in terms of restriction for other standard modules. This is the content of the following proposition.

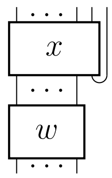
Proposition 3.4.1. *For an irreducible TL_n-representation W_m^n, we have*

$$\mathrm{Ind}_n^{n+1}(W_m^n) = \mathrm{Res}_{n+1}^{n+2}(W_m^{n+2}).$$

In particular,

$$\dim \mathrm{Ind}_n^{n+1}(W_m^n) = \binom{n+2}{\frac{n-m}{2}+1} - \binom{n+2}{\frac{n-m}{2}}.$$

Proof. Define a map $TL_{n+1} \otimes_n W_m^n \to W_m^{n+2}$ by

$x \otimes_n w \mapsto$ 

and extending linearly. To ensure that this is well defined we need to check that $x \otimes_n yw$ and $xy \otimes_n w$ map to the same thing when $y \in TL_n$, but this is immediate from the way that TL_n includes into TL_{n+1}. That this is a map of TL_{n+1}-modules is also immediate – TL_{n+1} acts on the first tensor factor on the left, which corresponds to acting on the leftmost $n+1$ strands of the right hand diagram. To establish that this map is an isomorphism we will proceed as follows:

(1) Construct a spanning set for $TL_{n+1} \otimes_n W_m^n$ containing at most $\dim W_m^{n+2}$ vectors.

(2) Show that each basis vector of W_m^{n+2} is hit by one of the elements in this set under the map.

(3) Part (1) shows that $\dim \operatorname{Ind}_n^{n+1}(W_m^n) \leq \dim W_m^{n+2}$ and (2) allows us to deduce the reverse inequality. Then it follows that the spanning set is a basis and the surjective map is an isomorphism, completing the proof.

For (1), we fix some arbitrary walk $d \in P_{m,n}$ and consider the set

$$S = \{e_{a,(d,A_{n+1})} \otimes_n t(d) \mid a \in P_{m\pm1,n+1}\} \subset TL_{n+1} \otimes_n W_m^n.$$

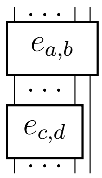
Here (d, A_{n+1}) denotes the walk constructed by appending the vertex A_{n+1} to the end of d. In order for this to actually define a walk we need $A_{n+1} = m\pm1$, hence the condition on a in the above definition.

By moving elements of TL_n through the tensor product, we could theoretically transform one element of this set into another, so it's hard to say exactly how many distinct elements S contains. However, we can say that there are at most $|P_{m-1,n+1}| + |P_{m+1,n+1}| = |P_{m,n+2}| = \dim W_m^{n+2}$ elements. This first equality comes from the simple bijection between $P_{m\pm1,n+1}$ and $P_{m,n+2}$ obtained by deleting the last entry from a walk in $P_{m,n+2}$. Alternatively, we could also see this by some simple manipulation of binomial coefficients.

We know that $TL_{n+1} \otimes_n W_m^n$ is spanned by elements of the form $e_{a,b} \otimes_n t(c)$, for pairs of compatible walks $a, b \in P_{n+1}$ and $c \in P_{m,n}$. Therefore, to show that S spans, it suffices to show that every element of this kind is either zero, or actually lies in S. To do this, note that since $c, d \in P_{m,n}$, we can write $\hat{t}(c) = e_{c,d}\hat{t}(d)$. Then we can calculate

$$e_{a,b} \otimes_n \hat{t}(c) = e_{a,b} \otimes_n e_{c,d}\hat{t}(d) = e_{a,b}e_{c,d} \otimes_n \hat{t}(d).$$

In the left hand factor, the $e_{c,d}$ is being included into TL_{n+1}, so this factor looks like



.

It is straightforward to check that this element is equal to $\delta_{b',c}\, e_{a,(d,B_{n+1})}$ (recall that b' is the walk obtained by deleting the last vertex of b) – either by drawing the tree basis out explicitly, or by using the recursive formulas (2.3) and (2.4). Thus, we have

$$e_{a,b} \otimes_n \hat{t}(c) = \delta_{b',c}\, e_{a,(d,B_{n+1})} \otimes_n \hat{t}(d).$$

Since a and b are compatible, we have $A_{n+1} = B_{n+1}$. As a result, the right hand side is either zero, or a member of S, as claimed. This completes the proof. □

Applying Proposition 3.3.1 in conjunction with Proposition 3.4.1 immediately yields the following corollary.

Corollary 3.4.2. *Let W_m^n be a standard TL_n-module for $n \geq 0$. Then if $m \geq 1$,*

$$\mathrm{Ind}_n^{n+1}(W_m^n) = W_{m+1}^{n+1} \oplus W_{m-1}^{n+1},$$

and if $m = 0$,

$$\mathrm{Ind}_n^{n+1}(W_0^n) = W_1^{n+1}.$$

□

Chapter 4

The biadjunction

In this chapter we prove the main result of this thesis, Theorem 4.2.14. To prove this theorem, we need to establish two seperate adjunctions. We treat one of these in a very general context in Section 4.1. The adjunction proved here is very straightforward. In Section 4.2 we develop some general theory about pivotal tensor categories, including an expansion of our graphical calculus for monoidal categories. This provides us with enough background to state and prove the other adjunction of Theorem 4.2.14. This part is harder, and the proof takes the entirety of Section 4.3.

4.1 An easy adjunction

The concepts of induction and restriction make sense in a more general context than that discussed so far. Whenever there is an inclusion of unital algebras $A \subset B$ we can restrict a B-module to obtain an A-module as in the Temperley-Lieb case. In the other direction, we can take an A-module and produce a B-module using the bimodule construction explained previously. This gives functors

$$\mathrm{Ind}_A^B : A\text{-}\mathbf{mod} \to B\text{-}\mathbf{mod}$$

and

$$\mathrm{Res}_A^B : B\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}.$$

Conveniently, Ind_A^B is left adjoint to Res_A^B even at this level of generality. We will demonstrate this by explicitly describing a unit

$$\bar{\eta} : 1_{A\text{-}\mathbf{mod}} \to \mathrm{Res}_A^B \mathrm{Ind}_A^B$$

and a counit

$$\bar{\varepsilon} : \mathrm{Ind}_A^B \mathrm{Res}_A^B \to 1_{B\text{-}\mathbf{mod}}$$

for the adjunction. Knowing about these natural transformations will be important later on.

If V is an A-module, then define $\bar{\eta}_V : V \to B \otimes_A V$ by $v \mapsto 1 \otimes_A v$. This is obviously a map of A-modules since any element of A can be pulled through the tensor product. On the other hand, if W is a B-module, then define a B-module map $\bar{\varepsilon}_W : B \otimes_A W \to W$ by $b \otimes_A w \mapsto bw$ and extending linearly. (That this is well defined comes from the definition of a B-module where we require that $(ba)w = b(aw)$.) Checking that these maps assemble into natural transformations is straightforward and is omitted.

We must verify that these maps actually give rise to an adjunction. The first map to consider is the composition

$$\mathrm{Ind}_A^B(V) \xrightarrow{\mathrm{Ind}_A^B(\bar{\eta}_V)} \mathrm{Ind}_A^B \mathrm{Res}_A^B \mathrm{Ind}_A^B(V) \xrightarrow{\bar{\varepsilon}_{\mathrm{Ind}_A^B(V)}} \mathrm{Ind}_A^B(V)$$

for any A-module V. This is the map

$$B \otimes_A V \xrightarrow{\mathrm{id}_B \otimes_A \bar{\eta}_V} B \otimes_A (B \otimes_A V) \xrightarrow{\bar{\varepsilon}_{B \otimes_A V}} B \otimes_A V$$

taking $b \otimes_A v$ to $b \otimes_A (1 \otimes_A v)$ and this to $b(1 \otimes_A v) = b \otimes_A v$. This is simply the identity on $B \otimes_A V$, so everything works for this map.

The second map to consider is the composition

$$\mathrm{Res}_A^B(W) \xrightarrow{\bar{\eta}_{\mathrm{Res}_A^B(W)}} \mathrm{Res}_A^B \mathrm{Ind}_A^B \mathrm{Res}_A^B(W) \xrightarrow{\mathrm{Res}_A^B(\bar{\varepsilon}_W)} \mathrm{Res}_A^B(W)$$

for any B-module W. This is the map

$$W \xrightarrow{\bar{\eta}_W} B \otimes_A W \xrightarrow{\overline{\varepsilon_W}} W$$

sending w to $1 \otimes_A w$, which is sent to $1w = w$. Thus, both maps behave as desired, establishing the adjunction.

4.2 Pivotal tensor categories with a self dual object

Let $\mathcal{C}$ be a k-linear monoidal category. Given any object X in this category, we can consider the k-algebra $R_n := \mathrm{End}(X^{\otimes n})$ of endomorphisms of the n-th tensor power of X. There is an obvious injective map $R_n \to R_{n+1}$ given by tensoring a morphism $f : X^{\otimes n} \to X^{\otimes n}$ with the identity morphism $\mathrm{id}_X : X \to X$. When $\mathcal{C}$ is the generic $\mathcal{TLJ}$ category and X is the strand $f^{(1)}$, we find that $R_n = TL_n$ and the map $TL_n \to TL_{n+1}$ is the familiar inclusion. Consequently, this generalises a situation we have studied before.

Ideally, the induction and restriction functors associated with this tower of algebras would form biadjoint pairs. It turns out that if we assume that our category has a little bit of additional structure and pick a sufficiently nice object X, then this does actually hold.

Definition 4.2.1. Let $\mathcal{C}$ be a monoidal category and X be an object of $\mathcal{C}$. An object X^* of $\mathcal{C}$ is a left dual of X if there are morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ (called evaluation and coevaluation) such that the two composites

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X, \tag{4.1}$$

and

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^* \tag{4.2}$$

are the identity on their respective domains. Recall that here we are suppressing the associativity and unit isomorphisms.

As a simple example, consider the category $\mathbf{FinVec}_k$ of finite dimensional vector spaces over k. The dual of a vector space V is the dual vector space V^* (as you would hope). The evaluation map $V^* \otimes V \to k$ comes from the canonical pairing between V^* and V. The coevaluation map $k \to V \otimes V^*$ is defined by picking a basis $\{v_i\}$ for V and sending $1 \in k$ to $\sum_i v_i \otimes v^i$, where $\{v^i\}$ is the dual basis to $\{v_i\}$. Checking that these maps satisfy the required equations is a straightforward exercise.

There is also an analogous definition of an object having a right dual.

Definition 4.2.2. Let $\mathcal{C}$ be a monoidal category and X be an object of $\mathcal{C}$. An object ${}^*X$ of $\mathcal{C}$ is a right dual of X if there are morphisms $\widetilde{\mathrm{ev}}_X : X \otimes {}^*X \to \mathbf{1}$ and $\widetilde{\mathrm{coev}}_X : \mathbf{1} \to {}^*X \otimes X$ such that the two composites

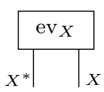
$$X \xrightarrow{\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X} X \otimes {}^*X \otimes X \xrightarrow{\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X} X \tag{4.3}$$

and

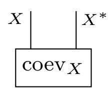
$${}^*X \xrightarrow{\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{{}^*X}} {}^*X \otimes X \otimes {}^*X \xrightarrow{\mathrm{id}_{{}^*X} \otimes \widetilde{\mathrm{ev}}_X} {}^*X \tag{4.4}$$

are the identity of their respective domains.

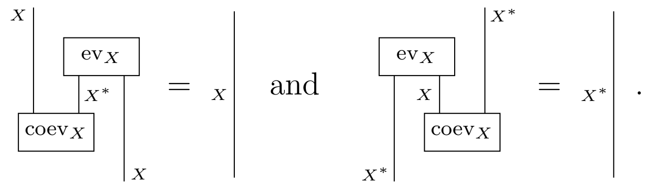
There are graphical interpretations for equations (4.1) through (4.4). If we depict ev_X and coev_X by



and



respectively (recall that we always leave out strands labelled by the tensor unit), then equations (4.1) and (4.2) say that we can pull 'zigzags' in our diagrams straight:



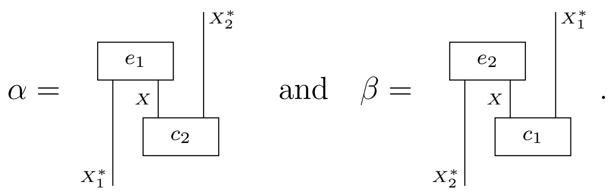
The situation for right duals is very similar. Because of this graphical interpretation, equations (4.1) through (4.4) are generally referred to as the 'zigzag equations'.

For the remainder of this section we will focus on results concerning left duals, remembering that there is always an analogous statement for right duals.

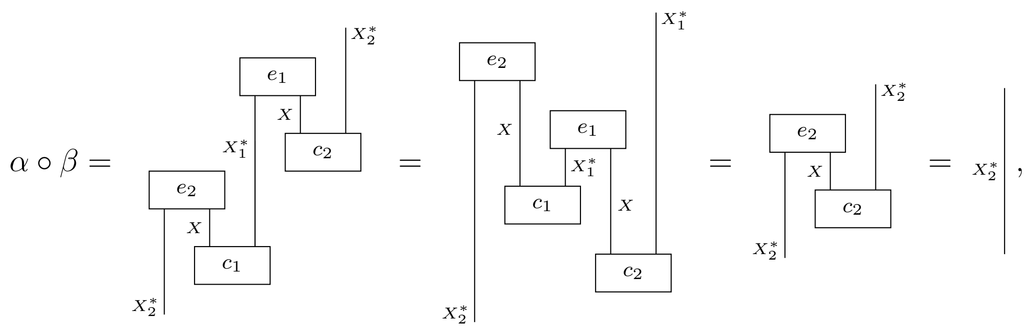
Proposition 4.2.3. *If an object X has a left dual X^*, then this dual is unique up to unique isomorphism. More precisely, if X_1^* is a left dual with evaluation and coevaluation maps e_1 and c_1 respectively, and if X_2^* is another left dual with evaluation and coevaluation maps e_2 and c_2 respectively, then there is a unique isomorphism $\alpha : X_1^* \to X_2^*$ preserving the evaluation and coevaluation maps:*

$$e_2 \circ (\alpha \otimes \mathrm{id}_X) = e_1 \quad \textit{and} \quad (\mathrm{id}_X \otimes \alpha^{-1}) \circ c_2 = c_1.$$

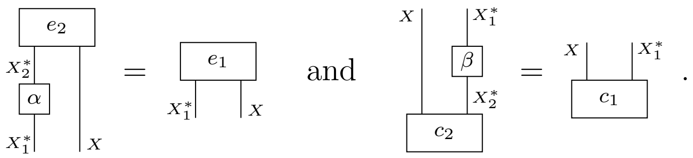
Proof. Define $\alpha : X_1^* \to X_2^*$ and $\beta : X_2^* \to X_1^*$ by the diagrams



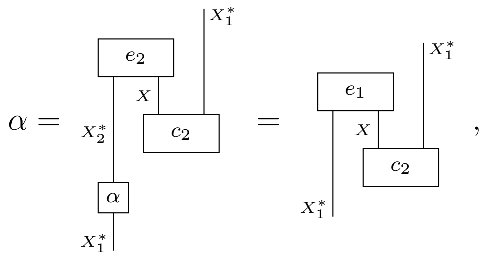
Then



where we use the interchange law and the zigzag equations twice. A similar diagrammatic calculation shows that $\beta \circ \alpha = \mathrm{id}_{X_1^*}$, establishing the isomorphism. Likewise, it is easy to see that



For uniqueness, suppose we have an isomorphism α with inverse β, satisfying these two relations. Then we must have



using the first equation. A similar argument, using the second equation, shows that β must be the isomorphism defined above. This completes the proof. $\square$

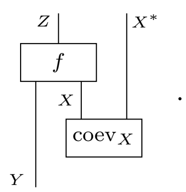
Proposition 4.2.4. *Let $\mathcal{C}$ be a k-linear monoidal category. If an object X of $\mathcal{C}$ has a left dual X^*, then for any objects Y and Z in $\mathcal{C}$ we have vector space isomorphisms*

$$\mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(Y, Z \otimes X^*)$$

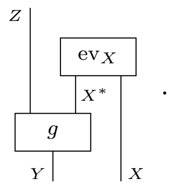
and

$$\mathrm{Hom}(X^* \otimes Y, Z) \cong \mathrm{Hom}(Y, X \otimes Z).$$

Proof. Define a map $\mathrm{Hom}(Y \otimes X, Z) \to \mathrm{Hom}(Y, Z \otimes X^*)$ by sending a morphism $f : Y \otimes X \to Z$ to the morphism represented by the diagram



Similarly, define a map $\mathrm{Hom}(Y, Z \otimes X^*) \to \mathrm{Hom}(Y \otimes X, Z)$ by sending a morphism $g : Y \to Z \otimes X^*$ to the morphism represented by the diagram



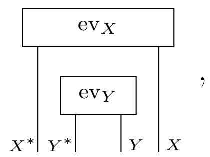
The linearity of both maps follows from the bilinearity of both the composition and the tensor product in a k-linear monoidal category. That these maps are mutual inverses follows from an application of the zigzag equations.

The second isomorphism is much the same. □

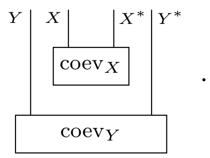
Definition 4.2.5. A monoidal category $\mathcal{C}$ is rigid if every object of $\mathcal{C}$ has both left and right duals. A tensor category is a semisimple k-linear rigid monoidal category.

Proposition 4.2.6. *Let X and Y be objects in a rigid monoidal category. Then $(Y \otimes X)^* \cong X^* \otimes Y^*$.*

Proof. Consider the map $(X^* \otimes Y^*) \otimes (Y \otimes X) \to \mathbf{1}$ defined by the diagram

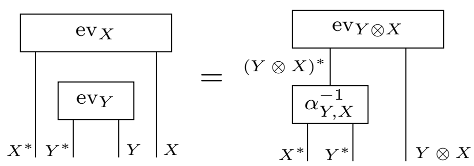


and the map $\mathbf{1} \to (Y \otimes X) \otimes (X^* \otimes Y^*)$ defined by the diagram



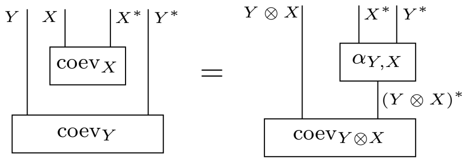
The zigzag identities for ev_X, coev_X, ev_Y and coev_Y show that these morphisms define evaluation and coevaluation maps for a dual of $Y \otimes X$. Thus, by the uniqueness of duals, we must have $(Y \otimes X)^* \cong X^* \otimes Y^*$. □

Remark 4.2.7. The uniqueness statement of Proposition 4.2.3 actually tells us more. Namely, that for any objects X and Y, there is a unique isomorphism $\alpha_{Y,X} : (Y \otimes X)^* \to X^* \otimes Y^*$ such that



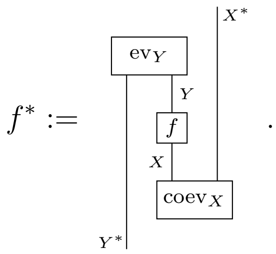
(4.5)

and



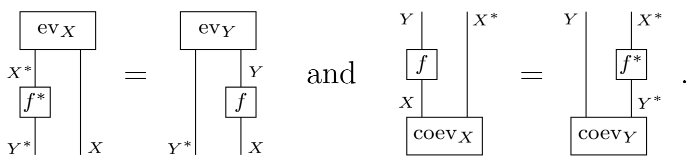
(4.6)

If $f : X \to Y$ is a morphism in a rigid monoidal category, and X^* is a left dual of X, and Y^* is a left dual of Y, then there is a dual morphism $f^* : Y^* \to X^*$ defined diagrammatically by



If $g : Y \to Z$ is another morphism and Z^* is a left dual of Z, then as an easy consequence of the zigzag equations we have $(g \circ f)^* = f^* \circ g^*$. It is also straightforward to see that $\mathrm{id}_X^* = \mathrm{id}_{X^*}$. We would like to say that taking duals is a contravariant functor, but there are some subtleties here. Since duals are only unique up to isomorphism, defining a duality functor requires us to pick specific duals for each object in our category. It is also reasonable to insist that functors between monoidal categories respect the additional monoidal structure in some way. This leads one to define 'monoidal functors', and in this setting we would say that duality is a monoidal functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op,mop}}$. Here $\mathcal{C}^{\mathrm{op,mop}}$ is the opposite and 'monoidal opposite' category of $\mathcal{C}$, obtained by reversing the arrows in $\mathcal{C}$ and switching the order of tensor products. We won't go into any further detail about monoidal functors or this monoidal opposite construction here. The reader is referred to the books [EGNO15, TV17] for further discussion.

Another immediate consequence of the zigzag equations are the following two useful identities:



(4.7)

These equations say that we can pull f 'through' an evaluation or coevaluation map, replacing f with its dual f^*, and the evaluation or coevaluation map for the domain with the same map for the codomain, or vice versa.

Definition 4.2.8. Let $\mathcal{C}$ be a rigid monoidal category. A pivotal structure on $\mathcal{C}$ is a collection of isomorphisms $\phi_X : X \to X^{**}$ for each object X in $\mathcal{C}$. We require these isomorphisms to be natural in X – meaning that if $f : X \to Y$ is a morphism, then the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\ X^{**} \arrow[r, "f^{**}"] & Y^{**} \end{tikzcd} .$$

We also require that the diagram

$$\begin{tikzcd} X \otimes Y \arrow[d, "\phi_{X\otimes Y}"'] \arrow[dr, "\phi_X \otimes \phi_Y"] & \\ (X \otimes Y)^{**} \arrow[r, "\cong"] & X^{**} \otimes Y^{**} \end{tikzcd}$$

commutes for any objects X and Y. Here the isomorphism $(X \otimes Y)^{**} \to X^{**} \otimes Y^{**}$ is the canonical isomorphism constructed as the composite

$$(Y \otimes X)^{**} \xrightarrow{(\alpha^*_{Y,X})^{-1}} (X^* \otimes Y^*)^* \xrightarrow{\alpha_{X^*,Y^*}} Y^{**} \otimes X^{**},$$

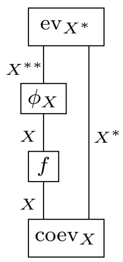
where $\alpha_{Y,X} : (Y \otimes X)^* \to X^* \otimes Y^*$ and α_{X^*,Y^*} are the unique isomorphisms of Remark 4.2.7.

This last requirement may seem somewhat strange, but there is a good reason for it. Both the identity functor and the double dual functor are monoidal functors. The commutativity of the second diagram simply expresses the requirement that ϕ be a monoidal natural isomorphism. Again, the reader is referred to [EGNO15, TV17] for the details.

Within a pivotal tensor category we can define left and right traces for a morphism $f : X \to X$. The left trace $\mathrm{tr}_L(f)$ is the composite

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X \otimes X^* \xrightarrow{\phi_X \otimes \mathrm{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\mathrm{ev}_{X^*}} \mathbf{1},$$

which we often identify with the corresponding scalar in $k \cong \mathrm{End}(\mathbf{1})$. The right trace is defined similarly. In string diagrams, this looks like

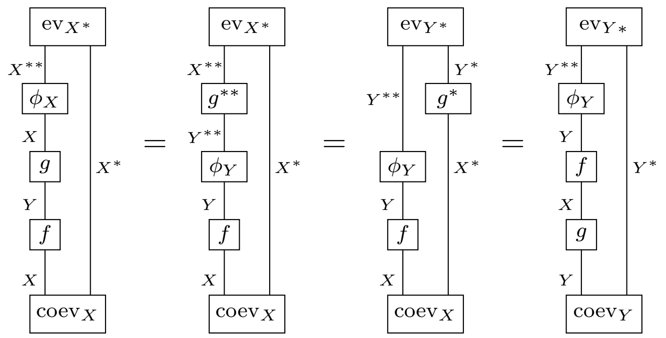
$\mathrm{tr}_L(f) =$  .

Lemma 4.2.9. *For morphisms* $f : X \to Y$ *and* $g : Y \to X$*, we have*

$$\mathrm{tr}_L(gf) = \mathrm{tr}_L(fg),$$

so the trace deserves its name.

Proof. This follows from the naturality of ϕ, the identities in (4.7) and diagrammatic manipulation:

$\mathrm{tr}_L(gf) =$ 

$= \mathrm{tr}_L(fg).$

□

Definition 4.2.10. If X is an object in a pivotal tensor category, then we define the (left) dimension of X, $\dim X$, to be the scalar $\mathrm{tr}_L(\mathrm{id}_X)$.

Lemma 4.2.11. *If* X *is a simple object, then* $\dim X \neq 0$.

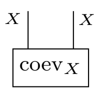
Proof. By proposition 4.2.4, we have

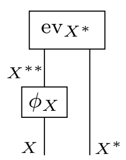
$$\dim \mathrm{Hom}(\mathbf{1}, X \otimes X^*) = \dim \mathrm{Hom}(\mathbf{1} \otimes X, X) = \dim \mathrm{Hom}(X, X) = 1.$$

Similarly,

$$\dim \mathrm{Hom}(X \otimes X^*, \mathbf{1}) = \dim \mathrm{Hom}(X, X^{**}) = \dim \mathrm{Hom}(X, X) = 1.$$

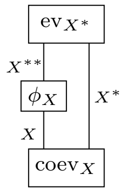
This implies that $X \otimes X^*$ has a single copy of $\mathbf{1}$ in its decomposition as a direct sum of simple objects. Thus, there are maps $p : X \otimes X^* \to \mathbf{1}$ and $i : \mathbf{1} \to X \otimes X^*$ with $p \circ i = \mathrm{id}_{\mathbf{1}}$. In particular, this last equality implies that neither p nor i can be the zero element of their respective hom-spaces. In a similar way, the zigzag equations and the invertiblity of ϕ_X imply that neither $\mathrm{ev}_{X^*} \circ (\phi_X \otimes \mathrm{id}_{X^*})$ nor coev_X are zero. Since the two spaces above are 1-dimensional, it follows that there are nonzero scalars $\lambda_1, \lambda_2 \in k$ with



$= \lambda_1 i$ and


$= \lambda_2 p.$

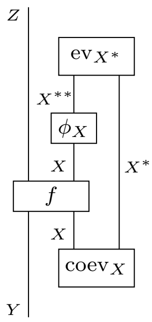
From this we can conclude that

$$\mathrm{tr}_L(\mathrm{id}_X) =$$


$$= \lambda_1\lambda_2 p \circ i = \lambda_1\lambda_2\,\mathrm{id}_{\mathbf{1}}\,.$$

Hence $\dim X = \lambda_1\lambda_2 \neq 0$, as claimed. □

There is also a notion of a (left) partial trace for a morphism $f : Y \otimes X \to Z \otimes X$, denoted $\mathrm{ptr}_L^X(f) : Y \to Z$ and defined by the string diagram

$$\mathrm{ptr}_L^X(f) =$$


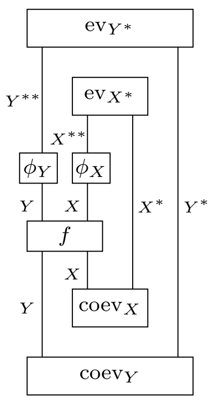
.

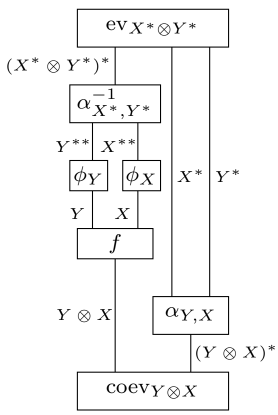
If Y and Z happen to be the same object in the above, then it is geometrically plausible that taking two consecutive partial traces should be the same as taking the (full) trace once. Happily, this turns out to be true.

Lemma 4.2.12. *If f is a morphism $Y \otimes X \to Y \otimes X$ then*

$$\mathrm{ptr}_L^Y(\mathrm{ptr}_L^X(f)) = \mathrm{tr}_L(f).$$

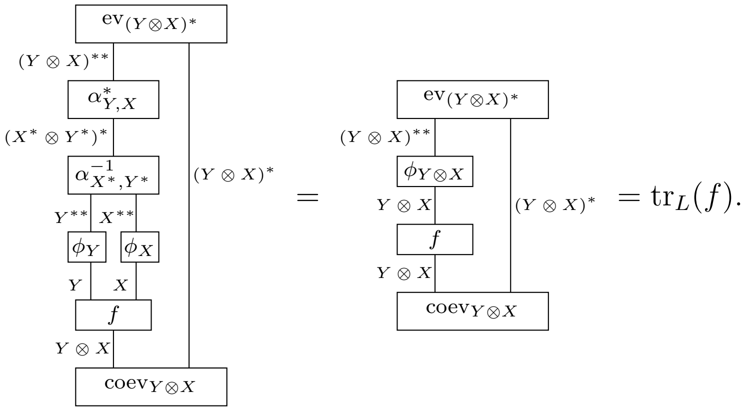
Proof. Using equations (4.5) and (4.6) of Remark 4.2.7, we can calculate

$$\mathrm{ptr}_L^Y(\mathrm{ptr}_L^X(f)) =$$


$=$


.

Pulling the $\alpha_{Y,X}$ through the top evaluation map, this becomes



For the first equality here we have used the commutativity of the second diagram in the definition of the pivotal structure. □

In fact, the analogous statement for endomorphisms of arbitrary tensor products also holds. However, we will only need Lemma 4.2.12 in what follows. See [TV17] for a proof of the general statement.

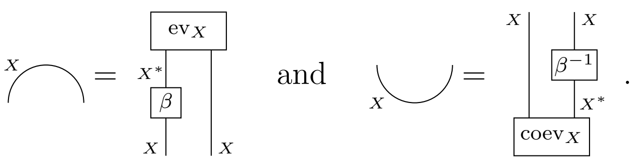
Definition 4.2.13. An object X with left dual X^* in a pivotal category $\mathcal{C}$ is **self dual** if there is an isomorphism $\beta : X \to X^*$. When there is such an isomorphism, we can also produce another isomorphism

$$X \xrightarrow{\beta} X^* \xrightarrow{(\beta^{-1})^*} X^{**},$$

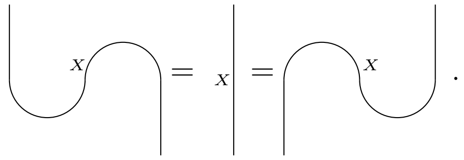
and we require that this is the same as the isomorphism $\phi_X : X \to X^{**}$ coming from the pivotal structure.

This additional requirement is there to ensure that the self duality interacts conveniently with the pivotal structure. [Sel10] outlines several proposed definitions of self duality in various types of monoidal categories. Our requirement is just one of many axioms proposed, but it will be the only one we make use of in what follows.

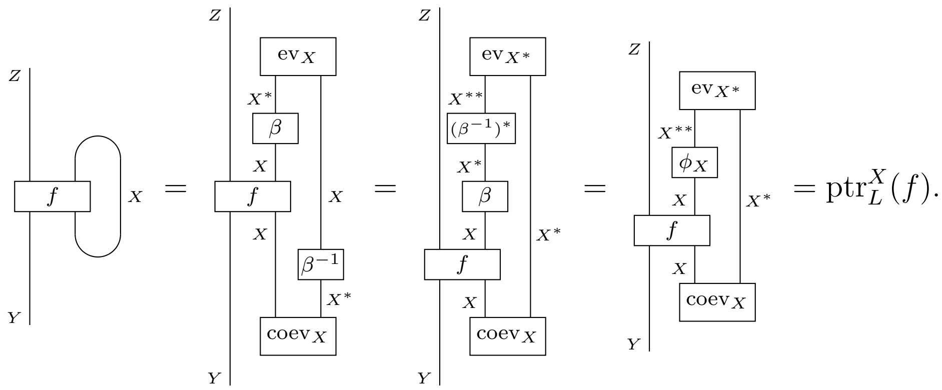
If X is a self dual object with self duality isomorphism $\beta : X \to X^*$, then we can further refine our graphical calculus by defining an unoriented cap and an unoriented cup, both labelled by X. Let



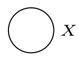
It is then straightforward to check that the zigzag equations for this pair hold:



We also have the geometrically plausible identity

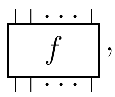


The proof of this fact makes use of the compatibility condition in the definition of a self dual object. A special case of this identity is the equation



$$= \dim X.$$

Now suppose we have an endomorphism $f \in R_n = \mathrm{End}(X^{\otimes n})$. Then we might depict f by the diagram



where we suppress the labels on the strands since they are all labelled by the same object X. This exactly resembles the Temperley-Lieb diagrams we have worked with previously. Morally, the identities for the cap and cup above say that we can manipulate string diagrams of this type exactly as if they were Temperley-Lieb diagrams, the only difference being that now the value of the circle is $\dim X$ rather than $[2]_q$. We will perform these manipulations without further comment for the remainder of the work.

We now come to the central result:

Theorem 4.2.14. *Let $\mathcal{C}$ be a pivotal tensor category and let X be a self dual object in $\mathcal{C}$. Then for any n, the induction and restriction functors*

$$\mathrm{Ind}_n^{n+1} : R_n\text{-}\mathbf{mod} \to R_{n+1}\text{-}\mathbf{mod}$$

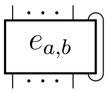
and

$$\mathrm{Res}_n^{n+1} : R_{n+1}\textbf{-mod} \to R_n\textbf{-mod}$$

form a biadjoint pair.

One adjunction is easy and was proved in Section 4.1. The other adjunction is harder and is covered in the next section. A key part of the proof is the following lemma.

Lemma 4.2.15. *The following formulae hold for any walks $a, b \in P_n$:*

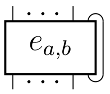


$$= \frac{\dim B_n}{\dim B_{n-1}} \delta(A_{n-1}, B_{n-1}) \delta_{a_n, b_n} e_{a', b'}. \tag{4.8}$$

Recall that a' denotes the walk in P_{n-1} obtained by deleting the last vertex and edge from a, and that δ is the Kronecker delta.

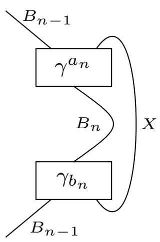
Proof. If a and b are not compatible paths, then $e_{a,b} = 0$ by definition, so the left hand side is zero. If the right hand side were not zero in this case, then that would imply that $A_{n-1} = B_{n-1}$ and $a_n = b_n$. But then $\gamma_{b_n} : B_{n-1} \otimes X \to B_n$ and $A_n \to A_{n-1} \otimes X$ are dual. This means that we must have $A_n = B_n$, which is a contradiction. Thus, the right hand side vanishes also.

Having resolved that case, we now assume that $A_n = B_n$. We want to understand the diagram



.

To do this, refer to the diagrammatic description of $e_{a,b}$ in equation (2.5). Taking the partial trace of the diagram depicted there, we find that within this diagram there is a morphism $B_{n-1} \to A_{n-1}$. If these objects are distinct, then their simplicity implies that the whole morphism vanishes. This accounts for the $\delta(A_{n-1}, B_{n-1})$ factor above. If we assume that these objects coincide, then the innermost piece in this diagram looks like



$$= \mathrm{ptr}_L^X(\gamma^{a_n} \gamma_{b^n}).$$

Since B_{n-1} is simple, this is some multiple S of the identity morphism on B_{n-1}. To find this coefficient, we take traces and make use of Lemmas 4.2.9 and 4.2.12 to calculate

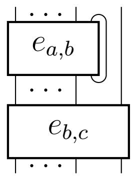
$$\begin{aligned} S \cdot \mathrm{tr}_L(\mathrm{id}_{B_{n-1}}) = \mathrm{ptr}_L^{B_{n-1}}(\mathrm{ptr}_L^X(\gamma^{a_n}\gamma_{b_n})) &= \mathrm{tr}_L(\gamma^{a_n}\gamma_{b_n}) \\ &= \mathrm{tr}_L(\gamma_{b_n}\gamma^{a_n}) \\ &= \delta_{a_n,b_n}\,\mathrm{tr}_L(\mathrm{id}_{B_n}). \end{aligned}$$

Rearranging (and using $\mathrm{tr}_L(\mathrm{id}_{B_{n-1}}) = \dim B_{n-1} \neq 0$) yields

$$S = \delta_{a_n,b_n}\frac{\dim B_n}{\dim B_{n-1}}.$$

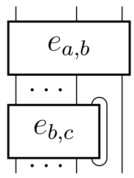
completing the proof. Replacing the piece pictured above with $S \cdot \mathrm{id}_{B_{n-1}}$ yields the result. □

Corollary 4.2.16. *The following formulae hold for any walks $a, b, c \in P_n$:*



$$= \frac{\dim B_n}{\dim B_{n-1}}\delta(A_{n-1}, B_{n-1})\delta_{a_n,b_n}e_{a,c} \tag{4.9}$$

and



$$= \frac{\dim B_n}{\dim B_{n-1}}\delta(B_{n-1}, C_{n-1})\delta_{b_n,c_n}e_{a,c}. \tag{4.10}$$

Proof. Draw these diagrams in the tree basis and apply the Lemma 4.2.15. □

4.3 Proof of Theorem 4.2.14

We will now establish that Res_n^{n+1} is left adjoint to Ind_n^{n+1} by constructing explicit natural transformations

$$\eta : 1_{R_{n+1}\textbf{-mod}} \to \mathrm{Ind}_n^{n+1}\mathrm{Res}_n^{n+1}$$

and

$$\varepsilon : \mathrm{Res}_n^{n+1}\mathrm{Ind}_n^{n+1} \to 1_{R_n\textbf{-mod}}$$

for the adjunction.

For any R_{n+1}-module V, define $\eta_V : V \to R_{n+1} \otimes_n V$ by

$$v \mapsto \sum_{a,b} r(b) e_{a,b} \otimes_n (e_{b,a} v),$$

where the $e_{a,b}$ are the matrix units in R_{n+1}, the sum is taken over all pairs of compatible paths in P_{n+1}, and $r(b)$ is the coefficient

$$r(b) := \frac{\dim B_{n-1}}{\dim B_n} \frac{1}{|P_{B_{n-1},n-1}|}.$$

We note that this coefficient only depends on the vertices B_{n-1} and B_n of the walk.

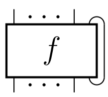
We need to check that this map intertwines the action of R_{n+1}. This follows from the fact that $\sum_{a,b} r(b) e_{a,b} \otimes_n (e_{b,a} f) = \sum_{a,b} r(b)(f e_{a,b}) \otimes_n e_{b,a}$ for any $f \in R_{n+1}$. To prove this, it suffices to prove it for each matrix unit in R_{n+1}. This is easy – if $e_{c,d}$ is a matrix unit, then

$$\begin{aligned} \sum_{a,b} r(b) e_{a,b} \otimes_n (e_{b,a} e_{c,d}) &= \sum_{a,b} r(b) e_{a,b} \otimes_n \delta_{a,c} e_{b,d} \\ &= \sum_{b} r(b) e_{c,b} \otimes_n e_{b,d}, \end{aligned}$$

while

$$\begin{aligned} \sum_{a,b} r(b)(e_{c,d} e_{a,b}) \otimes_n e_{b,a} &= \sum_{a,b} \delta_{d,a} r(b) e_{c,b} \otimes_n e_{b,a} \\ &= \sum_{b} r(b) e_{c,b} \otimes_n e_{b,d}. \end{aligned}$$

Now, for any R_n-module W, define $\varepsilon_W : R_{n+1} \otimes_n W \to W$ by

$f \otimes_n w \mapsto$  $\cdot\, w$

and extending linearly. Here we understand that every strand is labelled by X, and use the self duality of X to get the cap and cup.

There are two things to check for this map, namely that it is well-defined and that it is a map of R_n-modules. For well-definedness we need to check that if $g \in R_n$ then $fg \otimes_n w$ and $f \otimes_n gw$ map to the same thing. This is immediate from the definition of the map, the way that R_n includes into

R_{n+1} and the isotopy invariance of string diagrams. For similar reasons we can conclude that this defines an R_n-module map.

Checking that η and ε both define natural maps is a straightforward exercise. To show that these natural transformations constitute the data of an adjunction we need to check that for a R_n-module W and a R_{n+1}-module V, the composites

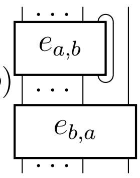
$$\mathrm{Res}_n^{n+1}(V) \xrightarrow{\mathrm{Res}_n^{n+1}(\eta_V)} \mathrm{Res}_n^{n+1}\mathrm{Ind}_n^{n+1}\mathrm{Res}_n^{n+1}(V) \xrightarrow{\varepsilon_{\mathrm{Res}_n^{n+1}(V)}} \mathrm{Res}_n^{n+1}(V)$$

and

$$\mathrm{Ind}_n^{n+1}(W) \xrightarrow{\eta_{\mathrm{Ind}_n^{n+1}(W)}} \mathrm{Ind}_n^{n+1}\mathrm{Res}_n^{n+1}\mathrm{Ind}_n^{n+1}(W) \xrightarrow{\mathrm{Ind}_n^{n+1}(\varepsilon_w)} \mathrm{Ind}_n^{n+1}(W)$$

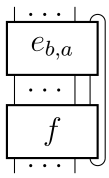
are the identity on their respective domains. Writing things out, these are the maps

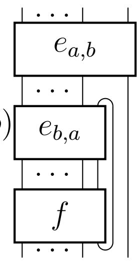
$$V \xrightarrow{\eta_V} R_{n+1}\otimes_n V \xrightarrow{\varepsilon_V} V$$

$$v \mapsto \sum_{a,b} r(b) e_{a,b}\otimes_n e_{b,a}v \mapsto \sum_{a,b} r(b)$$

$$\cdot\, v$$

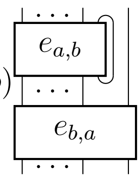
and

$$R_{n+1}\otimes_n W \xrightarrow{\eta_{R_{n+1}\otimes_n W}} R_{n+1}\otimes_n (R_{n+1}\otimes_n W) \xrightarrow{\mathrm{id}_{R_{n+1}}\otimes_n \varepsilon_W} R_{n+1}\otimes_n W$$

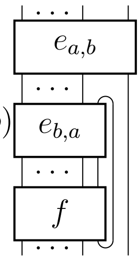
$$f\otimes_n w \mapsto \sum_{a,b} r(b) e_{a,b}\otimes_n (e_{b,a}f\otimes_n w) \mapsto \sum_{a,b} r(b) e_{a,b}\otimes_n$$

$$\cdot\, w$$

$$= \sum_{a,b} r(b)$$

$$\otimes_n w$$

Thus, to establish the adjunction it suffices to prove that

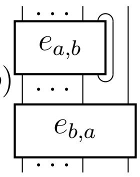
$$\sum_{a,b} r(b)$$

$$= 1 \tag{4.11}$$

and

$$\sum_{a,b} r(b)$$  $$= f \tag{4.12}$$

for any $f \in R_{n+1}$.

Applying (4.9) to the left hand side of (4.11) we get

$$\sum_{a,b} r(b)$$ 

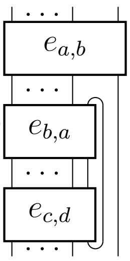
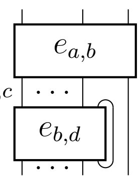
$$= \sum_{a,b} r(b) \frac{\dim B_n}{\dim B_{n-1}} \delta(A_{n-1}, B_{n-1}) \delta_{a_n,b_n} e_{a,a}$$

$$= \sum_{a,b} \frac{1}{|P_{B_{n-1},n-1}|} \delta(A_{n-1}, B_{n-1}) \delta_{a_n,b_n} e_{a,a}$$

$$= \sum_{a} \frac{1}{|P_{A_{n-1},n-1}|} \sum_{\substack{b \\ B_{n-1}=A_{n-1}, b_n=a_n}} e_{a,a}.$$

Since b and a are compatible, they end at the same point. Thus, the walks b in the second sum are those with $A_n = B_n$, $A_{n-1} = B_{n-1}$ and $a_n = b_n$. The number of such walks is exactly $|P_{A_{n-1},n-1}|$, so the sum reduces to $\sum_a e_{a,a}$. This is the sum of the diagonal matrix units and is consequently equal to the identity.

This shows that equation (4.11) holds, and it remains to verify equation (4.12). Clearly it suffices to just check (4.12) for each matrix unit $e_{c,d} \in R_{n+1}$. Using (4.10) and the same logic used in the last calculation, we can calculate

$$\sum_{a,b} r(b)$$  $$= \sum_{a,b} r(b) \delta_{a,c}$$ 

$$= \sum_{a,b} r(b) \delta_{a,c} \frac{\dim B_n}{\dim B_{n-1}} \delta(B_{n-1}, D_{n-1}) \delta_{b_n,d_n} e_{a,d}$$

$$= \sum_{b} \frac{1}{|P_{B_{n-1},n-1}|} \delta(B_{n-1}, D_{n-1}) \delta_{b_n,d_n} e_{c,d}$$

$$= \left(\sum_{\substack{b \\ B_{n-1}=D_{n-1}, b_n=d_n}} \frac{1}{|P_{B_{n-1},n-1}|} \right) e_{c,d}$$
$$= e_{c,d},$$

as desired. Thus, we have established the adjunction. This completes the proof of Theorem 4.2.14.

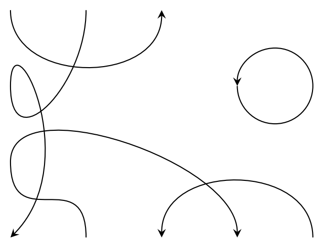
Chapter 5

Towards a graphical category

With the biadjunction in hand, we can now start to build a graphical category as outlined in Chapter 0. The definition we are about to give closely mimics those in [Kho14, LS13], the main difference being the relations we choose to impose.

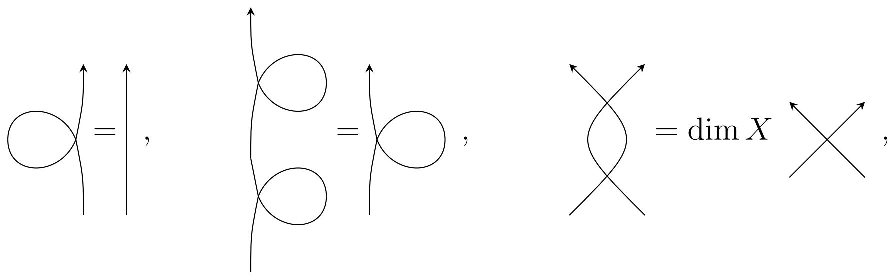
For the duration of this chapter we fix a pivotal tensor category $\mathcal{C}$ with a distinguished self dual object X.

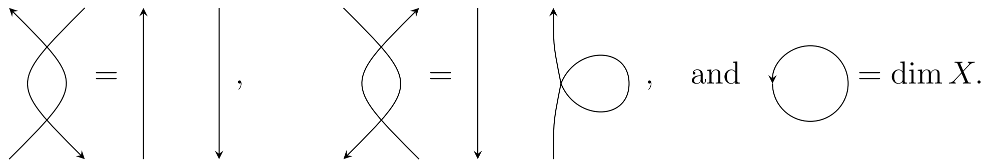
Definition 5.0.1. Define a k-linear strict monoidal category $\mathcal{G}_X$ as follows. The objects of $\mathcal{G}_X$ are words in the symbols $\{+,-\}$. For two words w and w', the space of morphisms $\mathrm{Hom}_{\mathcal{G}_X}(w,w')$ is the k-vector space with a basis consisting of suitable planar diagrams, modulo local isotopy of arcs, and modulo some other local relations. The diagrams consist of oriented compact one-manifolds immersed in the strip $\mathbb{R}\times[0,1]$. If the length of w is m and the length of w' is n, then we require that the endpoints of the one-manifold are located at $\{1,\dots,n\}\times\{1\}$ and $\{1,\dots,m\}\times\{0\}$. Finally, we require that the orientation of the one-manifold at the endpoints must agree with the corresponding signs in the words and that no triple intersections occur. For instance, the diagram



from the introduction depicts a morphism in $\mathrm{Hom}_{\mathcal{G}_X}(-+--+,--+)$. We can compose morphisms by stacking basis diagrams and extending bilinearly. The tensor product of words w and w' is their concatenation ww', hence the tensor unit is the empty word. The tensor product of morphisms is realised by juxtaposing basis diagrams and again extending bilinearly.

The local relations are





The right hand side of this last relation should be interpreted as $\dim X$ times the empty diagram.

5.1 Some set-up

Before continuing, it will be helpful to recap the results of Chapter 4. In our pivotal tensor category $\mathcal{C}$ with a self dual object X, we constructed algebras $R_n = \mathrm{End}(X^{\otimes n})$. These include naturally into each other, producing a tower

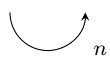
$$R_1 \subset R_2 \subset R_3 \subset \cdots$$

We showed that for each $n \geq 1$, the functors Ind_n^{n+1} and Res_n^{n+1} for this tower form a biadjoint pair. Since induction is the process of 'lifting' an R_n-module to an R_{n+1}-module, it makes sense to assign Ind_n^{n+1} a positive orientation, and consequently we assign Res_n^{n+1} a negative orientation.

Then the work of Chapter 4 produced four natural transformations for this biadjunction. We will list them, show their diagrammatic depiction and describe their component maps. From Section 4.1, we have

$$\bar{\eta}_n : 1_{R_n\textbf{-mod}} \to \mathrm{Res}_n^{n+1} \mathrm{Ind}_n^{n+1} .$$

This is represented by the diagram

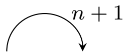


For an R_n-module V, the component $\bar{\eta}_{n,W} : W \to R_{n+1} \otimes_n W$ is the R_n-module map $w \mapsto 1 \otimes_n w$.

Also from Section 4.1, we have

$$\bar{\varepsilon}_n : \operatorname{Ind}_n^{n+1} \operatorname{Res}_n^{n+1} \to 1_{R_{n+1}\text{-}\mathbf{mod}}.$$

This is represented by the diagram

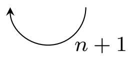


For an R_{n+1}-module V, the component $\bar{\varepsilon}_{n,V} : R_{n+1} \otimes_n V \to V$ is the R_{n+1}-module map $f \otimes_n v \mapsto f \cdot v$.

From Section 4.3, we have

$$\eta_n : 1_{R_{n+1}\text{-}\mathbf{mod}} \to \operatorname{Ind}_n^{n+1} \operatorname{Res}_n^{n+1}.$$

This is represented by the diagram



For an R_{n+1}-module V, the component $\eta_{n,W} : W \to R_{n+1} \otimes_n W$ is the R_{n+1}-module map

$$v \mapsto \sum_{a,b} r(b) e_{a,b} \otimes_n (e_{b,a} v),$$

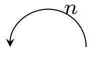
where the $e_{a,b}$ are the matrix units in R_{n+1}, the sum is taken over all pairs of compatible paths in P_{n+1}, and $r(b)$ is the coefficient

$$r(b) := \frac{\dim B_{n-1}}{\dim B_n} \frac{1}{|P_{B_{n-1},n-1}|}.$$

Also from Section 4.3, we have

$$\varepsilon_n : \operatorname{Res}_n^{n+1} \operatorname{Ind}_n^{n+1} \to 1_{R_n\text{-}\mathbf{mod}}.$$

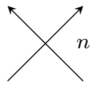
This is represented by the diagram



For an R_n-module W, the component $\varepsilon_{n,W} : R_{n+1} \otimes_n W \to W$ is the R_n-module map

$$f \otimes_n w \mapsto \boxed{f} \cdot w.$$

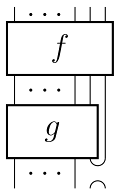
Next we pick a natural transformation (really a family of natural transformations) to be represented by the crossings



To be consistent with rest of a graphical calculus, this should represent a natural transformation from $\mathrm{Ind}_{n+1}^{n+2}\mathrm{Ind}_{n}^{n+1}$ to itself. That is, for any R_n-module V, we should define an R_{n+2}-module map

$$R_{n+2} \otimes_{n+1} R_{n+1} \otimes_n V \longrightarrow R_{n+2} \otimes_{n+1} R_{n+1} \otimes_n V.$$

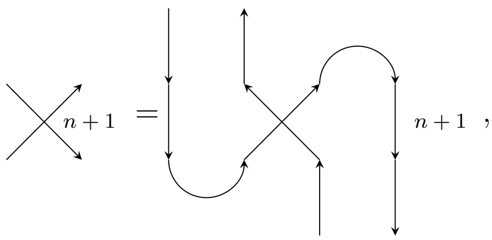
To do this, we send $f \otimes_{n+1} g \otimes_n v$ to



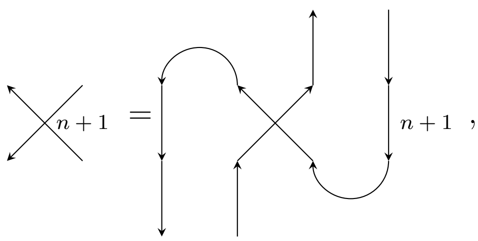
$\otimes_{n+1} 1 \otimes_n v$

and extend linearly. It is routine to check that this gives a well defined R_{n+2}-module map and that these component maps assemble into a natural transformation. We omit these details.

Once we have defined natural transformations for one oriented crossing we get natural transformations for all other oriented crossings. This is achieved by rotating the crossing using the various caps and cups. For example

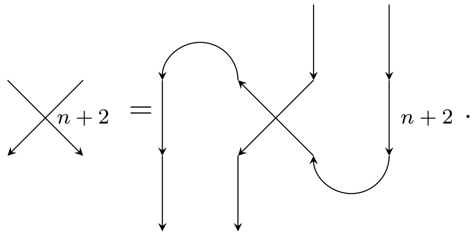


(5.1)



(5.2)

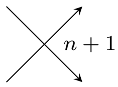
and



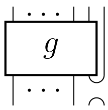
(5.3)

It will be helpful to have explicit formulae for some of these rotated crossings.

Lemma 5.1.1. *For any R_{n+1}-module V, the V-component of the natural transformation represented by*



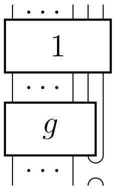
is the R_{n+1}-module map $R_{n+1} \otimes_n V \to R_{n+2} \otimes_{n+1} V$ sending $g \otimes_n v$ to



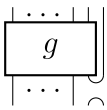
$\otimes_{n+1} v.$

Proof. We can essentially read this off from (5.1). The effect of the V-component of the right hand side on $g \otimes_n v \in R_{n+1} \otimes_n V$ is

$g \otimes_n v \mapsto 1 \otimes_{n+1} g \otimes_n v \mapsto$



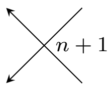
$\otimes_{n+1} 1 \otimes_n v \mapsto$



$\otimes_{n+1} v.$

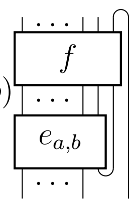
□

Lemma 5.1.2. *For any R_{n+1}-module V, the V-component of the natural transformation represented by*



is the R_{n+1}-module map $R_{n+2} \otimes_{n+1} V \to R_{n+1} \otimes_n V$ sending $f \otimes_{n+1} v$ to

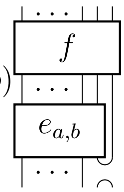
$\sum_{a,b \in P_{n+1}} r(b)$

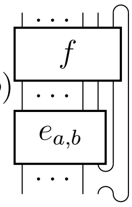
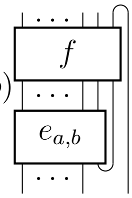


$\otimes_n e_{b,a} v.$

Proof. Refer to (5.2). The effect of the V-component of the right hand natural transformations on $f \otimes_{n+1} v \in R_{n+2} \otimes_{n+1} V$ is

$$f \otimes_{n+1} v \mapsto f \otimes_{n+1} \left(\sum_{a,b \in P_{n+1}} r(b) e_{a,b} \otimes_n e_{b,a} v \right)$$

$\mapsto \sum_{a,b \in P_{n+1}} r(b)$  $\otimes_{n+1} 1 \otimes_n e_{b,a} v$

$\mapsto \sum_{a,b \in P_{n+1}} r(b)$  $\cdot (1 \otimes_n e_{b,a} v) = \sum_{a,b \in P_{n+1}} r(b)$  $\otimes_n e_{b,a} v.$

This proves the claim. □

5.2 A family of functors out of $\mathcal{G}_X$

The set-up is complete and we can now state and prove the main theorem of this chapter.

Definition 5.2.1. Define a category $\mathcal{S}_X$ as in Chapter 0. The objects of $\mathcal{S}_X$ are compositions of the functors $\mathrm{Ind}_1^2, \mathrm{Res}_1^2, \mathrm{Ind}_2^3, \mathrm{Res}_2^3, \ldots$. The morphisms between two such composites are those natural transformations that can be built (by the two types of composition) using the fixed biadjointness natural transformations, the natural transformations representing the crossings, and the identity natural transformations for the functors.

Theorem 5.2.2. *For each $n \geq 1$, there is a functor $\mathcal{F}_n : \mathcal{G}_X \to \mathcal{S}_X$. These functors are collectively surjective – everything in $\mathcal{S}_X$ is in the image of on of these $\mathcal{F}_n$.*

The proof of this theorem will take the entirety of this section. To start, we need to define the functors.

As in Chapter 0, for each $n \geq 1$, there is a functor $\mathcal{F}_n : \mathcal{G}_X \to \mathcal{S}_X$. If w is a word in $\{+, -\}$ with at least $n+1$ more minuses than pluses, we define $\mathcal{F}_n(w) = 0$. Otherwise, w gets sent to some composition of induction and restriction functors. We start at the level n and work from right to left – a plus contributes an induction functor and a minus contributes a restriction functor. For instance,

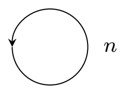
$$\mathcal{F}_n(++-+) = \mathrm{Ind}_{n+1}^{n+2} \circ \mathrm{Ind}_n^{n+1} \circ \mathrm{Res}_n^{n+1} \circ \mathrm{Ind}_n^{n+1}$$

On morphisms, $\mathcal{F}_n$ takes a diagram in $\mathcal{C}$, labels the rightmost region by n and interprets the resulting diagram as a morphism between compositions of induction and restriction functors. If a region within this diagram ends up labelled with a nonpositive integer, then we interpret the resulting diagram as the zero morphism.

First, we need to check that these functors are well defined. Once we have established this, the collective surjectivity is an immediate consequence of the definition of $\mathcal{S}_X$. There is no issue on the level of objects, but we need to be more careful on the level of morphisms. For each n, we want to show that applying one of the defining local relations of $\mathcal{G}_X$ to a morphism in $\mathcal{G}_X$ does not change the image of said morphism under $\mathcal{F}_n$.

To establish well-definedness, it suffices to check that the local relations allowed in $\mathcal{G}_X$ also hold in $\mathcal{S}_X$ when we label the rightmost region by any positive integer. We have already seen that the caps and cups constitute the data of a biadjunction, establishing that local isotopies of arcs are allowed in $\mathcal{S}_X$ for any labelling. For the remainder of this section we check that the same is true for the other defining relations of $\mathcal{G}_X$.

Proposition 5.2.3. *For any $n \geq 1$, we have*



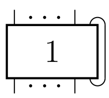
$= \dim X,$

where the right hand side is interpreted as $\dim X$ *times the identity functor on* R_n-**mod**.

Proof. Again we work in components. If V is any R_n-module, then the V-component of the natural transformation represented by the left hand side is the R_n-module map

$$V \longrightarrow \mathrm{Res}_n^{n+1} \mathrm{Ind}_n^{n+1}(V) = R_{n+1} \otimes_n V \longrightarrow V$$

sending $v \in V$ to $1 \otimes_n v$, which is then sent to

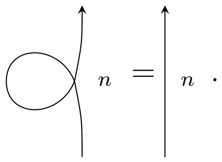


$\cdot\, v.$

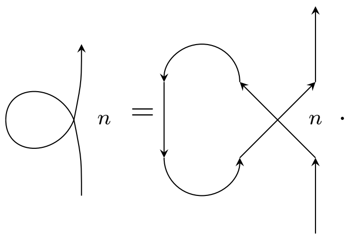
Here 1 denotes the identity of R_{n+1}, which is depicted graphically as $n+1$ vertical strands. Thus, taking the partial trace as above leaves us with n

vertical strands and a circle. The n vertical strands are the identity of R_n, so the above becomes $\dim X \cdot v$. This means that the map is $\dim X$ times the identity map on V, and the result follows. $\square$

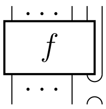
Proposition 5.2.4. *For any $n \geq 1$, we have*

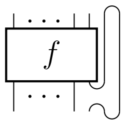


Proof. Breaking the curl up into smaller pieces, we write



Then for any R_n-module V, the effect of the V-component on $f \otimes_n v \in R_{n+1} \otimes_n V$ is

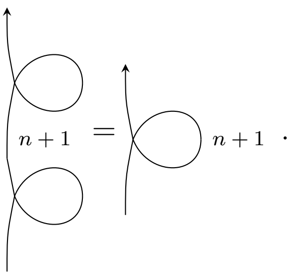
$f \otimes_n v \mapsto 1 \otimes_{n+1} f \otimes_n v \mapsto$ 

$\otimes_{n+1} 1 \otimes_n v \mapsto$ 

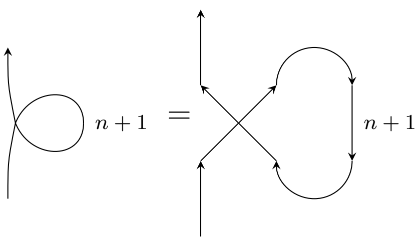
$\cdot (1 \otimes_n v) = f \otimes_n v.$

This is the identity on $R_{n+1} \otimes_n V$, which is exactly the V-component of the identity natural transformation on Ind_n^{n+1}. This proves the relation. $\square$

Proposition 5.2.5. *For any $n \geq 1$, we have*

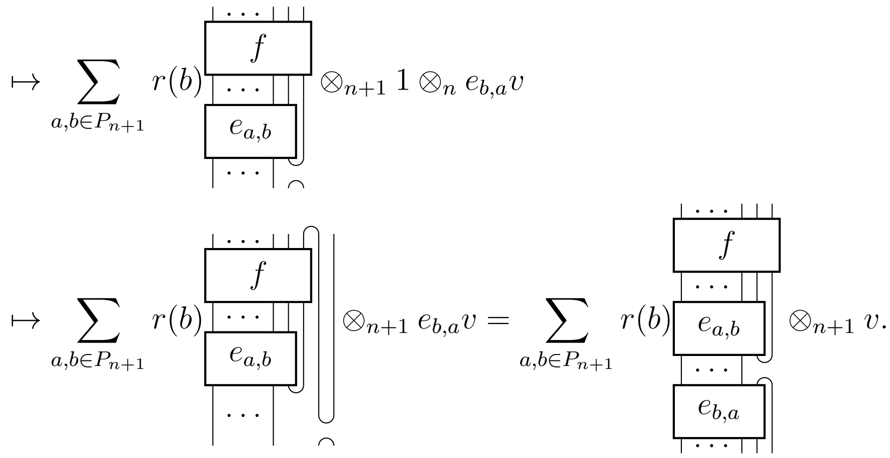


Proof. We begin by breaking the right curl up into simpler pieces:

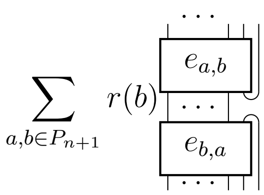


Then for any R_{n+1}-module V, the effect of the V-component on $f \otimes_{n+1} v \in R_{n+2} \otimes_{n+1} V$ is

$$f \otimes_{n+1} v \mapsto f \otimes_{n+1} \left(\sum_{a,b \in P_{n+1}} r(b) e_{a,b} \otimes_n e_{b,a} v \right)$$



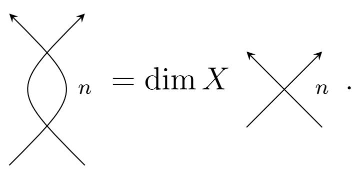
We see that the effect of this map is to insert



below f. Therefore, to establish the desired relation, it suffices to show that this element is an idempotent. This follows from a straightforward but tedious calculation using Corollary 4.2.16 and the same logic as the calculations in Section 4.3. As such, we omit the details.

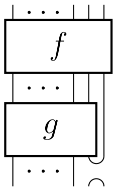
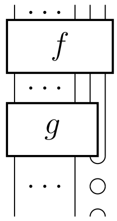
□

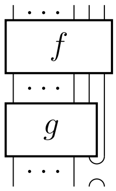
Proposition 5.2.6. *For any $n \geq 1$, we have*



Proof. The diagram on the left is built by stacking two upwards oriented crossings on top of each other. Thus, for any R_n-module V, the effect of the

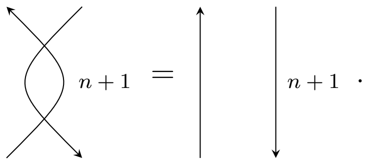
V-component of the right hand side on some $f \otimes_{n+1} g \otimes_n v \in R_{n+2} \otimes_{n+1} R_{n+1} \otimes_n V$ is

$f \otimes_{n+1} g \otimes_n v \mapsto$  $\otimes_{n+1} 1 \otimes_n v \mapsto$  $\otimes_{n+1} 1 \otimes_n v$

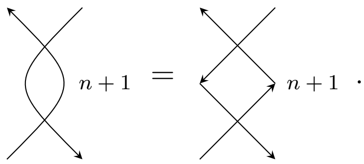
$= \dim X \cdot$  $\otimes_{n+1} 1 \otimes_n v.$

This completes the proof. $\square$

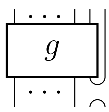
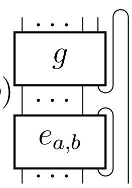
Proposition 5.2.7. *For any $n \geq 1$, we have*



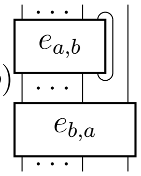
Proof. The diagram on the left is built by stacking the two oriented crossings from Lemmas 5.1.1 and 5.1.2:



From the aforementioned lemmas, we have formulae for these two pieces. Thus, on an R_{n+1}-module V, the V-component of the left hand side has the following effect on $g \otimes_n v \in R_{n+1} \otimes_n V$:

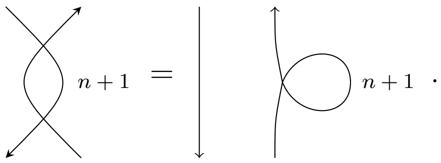
$g \otimes_n v \mapsto$  $\otimes_{n+1} v \mapsto \sum_{a,b \in P_{n+1}} r(b)$  $\otimes_n e_{b,a} v.$

Now, pulling the partial traced $e_{a,b}$ through the tensor product and using the bilinearity of $\otimes_n$, this last expression simplifies to

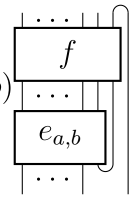
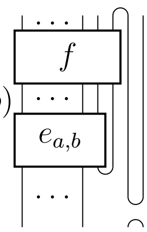
$$g \otimes_n \Bigg(\sum_{a,b} r(b)$$

$$\cdot v \Bigg).$$

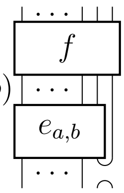
Then by equation (4.12), this simplifies to $g \otimes_n v$. Thus, the V-component for the right hand side is exactly the V-component of the left hand side. The result follows. □

Proposition 5.2.8. *For any $n \geq 1$, we have*

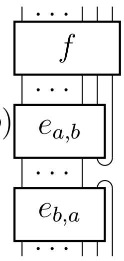


Proof. The diagram on the left is built by stacking the two oriented crossings from Lemmas 5.1.1 and 5.1.2 again, but this time the other way around. On any R_{n+1}-module V, the V-component of the left hand side has the following effect on $f \otimes_{n+1} v \in R_{n+2} \otimes_{n+1} V$:

$$f \otimes_{n+1} v \mapsto \sum_{a,b \in P_{n+1}} r(b)$$

$$\otimes_n e_{b,a} v \mapsto \sum_{a,b \in P_{n+1}} r(b)$$

$$\otimes_{n+1} e_{b,a} v$$

$$= \sum_{a,b \in P_{n+1}} r(b)$$

$$\otimes_{n+1} e_{b,a} v.$$

Since $e_{b,a} \in R_{n+1}$, we can pull it through the tensor product to find that this last expression is equal to

$$\sum_{a,b \in P_{n+1}} r(b)$$

$$\otimes_{n+1} v.$$

In the proof of Proposition 5.2.5 we computed the components of the right handed loop. Referring to this, we see that the V-component of the right hand side in the statement is the same as the map just computed. This completes the proof. $\square$

Thus, the defining relations of $\mathcal{G}_X$ still hold when labelled by arbitrary positive numbers and viewed as morphisms in $\mathcal{S}_X$. This completes the proof of Theorem 5.2.2.

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