Higher Spin Algebras and Universal Enveloping Algebras

Bolin Han
Supervised by: Peter Bouwknegt

May 2019

A thesis submitted for the degree of Bachelor of Science (Advanced) (Honours) of the Australian National University

Declaration

The work in this thesis is my own except where otherwise stated.

Bolin Han

Acknowledgements

First of all, I must acknowledge my supervisor Peter Bouwknegt for his support and encouragement throughout this year. He is always passionate about our project and patient to answer every single one of my questions. I have been impressed by his talented ideas many times throughout the work. Without his guidance, I would never have survived this tough year.

I would also like to thank all of my lectures and professors who lead me to the beautiful multiverse of mathematics, especially Joan Licata, who initialled my interests in algebra; Jim Borger and Vigleik Angeltveit, who broadened my eyes on the algebra universe; Griff Ware who instructed me on academic writing; Mark Bugden, who infected me with his enthusiasm on this subject and inspired me in the discussions; Amnon Neeman, who not only taught me to be a responsible writer, but also spent his time listening to my troubles with tears and generously gave me warmth.

To all my fellows and friends, you have all contributed to my year in some way or another. Especially, I want to thank for Christopher Hone and Yuzheng Yan for their generous help on some parts of this thesis. And I also need to thank for Kenny Wiratama who always stimulates me when I am pressured or depressed. I would also like to thank for Shiqiu Qiu for his accompany and inspiring discussions with me. I must also thank for Yiming Xu, Marcus Cai, Xilin Lu and Fredrick Yuan who created so many unforgettable memories in my undergraduate life.

Last but not least, I owe a huge amount of gratitude to my parents for loving me and supporting me even remotely.

Abstract

Higher spin algebras, arising from the study of the underlying global symmetries of massless higher-spin particles in physics, have become an interesting area in mathematics since people realised these algebras are deeply related to the theory of minimal representations. A well-studied special one-parameter family hs$[\lambda]$ is shown to be equivalent to a quotient of the universal enveloping algebra (UEA) of $\mathfrak{sl}_2$. In this thesis, we review the results on hs$[\lambda]$ with some modifications and then construct new higher spin algebras from the UEA of the semi-direct product $\mathfrak{sl}_2 \ltimes V_2$. In addition, we also study the centralisers in the UEA of $\mathfrak{sl}_2 \ltimes V_m$ for other values of m in preparation to construct more higher spin algebras.

Contents

Introduction

The study of massless higher spin particles in theoretical physics gives rise to higher spin algebras which describe the underlying global symmetries. The higher spin algebra that is known to be fully consistent in four-dimensional space-time was first considered in [10] from the construction of higher spin cubic interaction. Then the generalization of this algebra to any dimension has been studied based on oscillators.[28]

The unusual feature of interacting higher spin gauge theories is that they require non-flat vacuum solutions. The most symmetric vacuum they admit is known as Anti-de Sitter (AdS) space, frequently used in quantum gravity theories.[24] This brings higher spin algebras to the context of the famous AdS/CFT correspondence which proposes a duality between the AdS space formulated in terms of string theory, or M-theory, and conformal field theories (CFT) which are quantum field theories.[21]

There is a special one-parameter family of higher spin algebras known as hs[λ] which corresponds to a family of three-dimensional interacting equations.[23] It was realised that hs[λ] can also be constructed as a quotient of the universal enveloping algebra of the isometry algebra $\mathfrak{sl}_2$. And in fact, the coset construction of hs[λ] is deeply related to the minimal representations of $\mathfrak{sl}_2$ whose kernels are known as Joseph ideals.[17]

Having found the connection between higher spin algebras and the minimal representation problem, mathematicians started to generalise the notion of higher spin algebras to Lie algebras beyond the isometry algebras, while the term "spin" is often translated to the term "weight" in representation theory. The construction of higher spin algebras from the universal enveloping algebra of any semi-simple Lie algebra has been mostly studied.[18] The higher spin superalgebras are also of interests and in fact they play important roles in the area of AdS/CFT.[29] Part of this thesis will be devoted to a fairly less studied family of higher spin algebras that comes from the universal enveloping algebra of $\mathfrak{sl}_2 \ltimes V_2$ which is defined to be the semi-direct product between $\mathfrak{sl}_2$ and the two-dimensional irreducible $\mathfrak{sl}_2$-module V_2.

This thesis will be organised as follows. In Chapter 1, we investigate some properties of the one-parameter family of higher spin algebras $\mathrm{hs}[\lambda]$ based on the explicit structure constants and bilinear form. In Chapter 2, we introduce universal enveloping algebras and concentrate on the algebraic structure of $U(\mathfrak{sl}_2)$ including the commutation relations, Lie ideals and representations. In Chapter 3, we relate the higher spin algebras $\mathrm{hs}[\lambda]$ to the universal enveloping algebra $U(\mathfrak{sl}_2)$ and reveal the representations of $\mathfrak{sl}_2$ that are contained in $\mathrm{hs}[\lambda]$. In Chapter 4, we explore the structure of the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ and then construct a new one-parameter family of higher spin algebras $\widetilde{\mathrm{hs}[\lambda]}$ that contains more representations of $\mathfrak{sl}_2$, from the algebraic point of view. In Chapter 5, we determine the centralisers of universal enveloping algebras $U(\mathfrak{sl}_2 \ltimes V_m)$ for various m using tools from the representation theory, in preparation to the study of $U(\mathfrak{sl}_2 \ltimes V_m)$ in general.

Chapter 1

One-parameter Family: hs[λ]

In this chapter, we introduce the one-parameter family higher spin algebras hs[λ] and make some observations on the analytic formula of structure constants. The underlying field of algebras mentioned in this thesis will always be $\mathbb{C}$ unless elsewhere stated. To proceed the discussion, let us set up some notations first.

Notation 1.1 (Pochhammer symbols)**.** In the following, $[a]_n$ denotes the descending Pochhammer symbol and $(a)_n$ denotes the ascending Pochhammer symbol. That is,

$$[a]_n = \frac{\Gamma(a+1)}{\Gamma(a+1-n)} = a(a-1)\dots(a-n+1),$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1).$$

Notation 1.2 (Generalised hypergeometric function)**.** In the following, the generalised hypergeometric function is denoted as

$${}_pF_q\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \; ; \mathrm{z}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

Definition 1.3. The one-parameter family of higher spin Lie algebras, denoted by hs[λ], has generators $\{V^r_m : r \geq 2, |m| < r\}$, with commutation relations

$$[V^r_m, V^s_n] = \sum_{t=2,\text{ t even}}^{r+s-1} g^{rs}_t(m,n,\lambda) V^{r+s-t}_{m+n} \tag{1.1}$$

where the structure constants are given by

$$g^{rs}_t(m,n,\lambda) = \frac{2q^{t-1}}{(t-1)!} \phi^{rs}_t(\lambda) N^{rs}_t(m,n) \tag{1.2}$$

where q is a scaling constant (which is chosen to be $\frac{1}{4}$ from now on), $\lambda \in \mathbb{C}$ is the parameter, and $\phi_t^{rs}(\lambda), N_t^{rs}(m,n)$ are given by

$$\phi_t^{rs}(\lambda) = {}_4F_3\left[\begin{matrix} \frac{1}{2}+\lambda, \frac{1}{2}-\lambda, \frac{2-t}{2}, \frac{1-t}{2} \\ \frac{3}{2}-r, \frac{3}{2}-s, \frac{1}{2}+r+s-t \end{matrix} ;1\right], \tag{1.3}$$

$$N_t^{rs}(m,n) = \sum_{k=0}^{t-1}(-1)^k \binom{t-1}{k}[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k. \tag{1.4}$$

The commutation relations of hs[λ] were stated in [22]. Here we adapt the notation given in [13].

One can show that hs[λ] is indeed a Lie algebra by directly verifying the skew-symmetry and Jacobi identity on the structure constant $g_t^{rs}(m,n,\lambda)$. The skew-symmetry is fairly easy to see with noticing $\phi_t^{rs}(\lambda) = \phi_t^{sr}(\lambda)$ and $N_t^{rs}(m,n) = (-1)^{t-1}N_t^{sr}(n,m)$. The latter equality is obtained by changing the dummy index in the expression of $N_t^{rs}(m,n)$ from k to $t-1-k$. However, showing the Jacobi identity is more technical. It is subject to relating $g_t^{rs}(m,n,\lambda)$ to the so-called 6j-symbol that is well-studied in the area of quantum physics and then take advantages of some known properties of the 6j-symbol.[22]

Although not necessary, one can expect realising the abstract Lie bracket of hs[λ] as the usual commutator. In fact there exists an associative product, denoted by $\star : \text{hs}[\lambda] \times \text{hs}[\lambda] \to \text{hs}[\lambda]$, such that

$$[V_m^r, V_n^s] = V_m^r \star V_n^s - V_n^s \star V_m^r.$$

The product map $\star$ has been found explicitly in [22] and is given by

$$V_m^r \star V_n^s = \frac{1}{2}\sum_{t=1}^{r+s-1} g_t^{rs}(m,n,\lambda)V_{m+n}^{r+s-t}. \tag{1.5}$$

According to [22], the algebra hs[λ] has an invariant symmetric bilinear form given by

$$\begin{aligned}\langle V_m^r, V_n^s\rangle &= \frac{3\cdot 4^{r-3}\sqrt{\pi}q^{2r-4}\Gamma(r)(-1)^{r-m-1}}{(\lambda^2-1)\Gamma(r+\frac{1}{2})(2r-2)!}(1-\lambda)_{r-1}(1+\lambda)_{r-1}\Gamma(r+m)\Gamma(r-m)\delta^{rs}\delta_{mn} \\ &= \frac{3}{4q(\lambda^2-1)}g_{r+s-1}^{rs}(m,n,\lambda).\end{aligned} \tag{1.6}$$

More details on the bilinear form are included in Appendix B. The following statement and its proof illustrate the power of this bilinear form.

Proposition 1.4. hs[λ] *is not simple if* $\lambda \in \mathbb{Z}\backslash\{0, \pm 1\}$.

Proof. Suppose $\lambda = N \in \mathbb{Z}\backslash\{0, \pm 1\}$. From Equation (1.6) we note that when $r \geq |N| + 1$, $(1 - N)_{r-1}$ has a factor $(1 - N + |N| - 1)$ and $(1 + N)_{r-1}$ has a factor $(1 + N + |N| - 1)$, so we have either $(1 - N)_{r-1} = 0$ or $(1 + N)_{r-1} = 0$. This implies

$$\mathrm{Span}\{V_m^r : r \geq |N| + 1\} \subset \mathrm{rad}_N,$$

where rad_N denotes the radical of the bilinear form on hs[N].

When $r \leq N$, it is easy to see $\langle V_m^r, V_m^r \rangle \neq 0$ for any $|m| < r$. Hence we have

$$\mathrm{Span}\{V_m^r : r \geq |N| + 1\} = \mathrm{rad}_N.$$

Now because the bilinear form is invariant, that is, $\langle [V_m^r, V_l^t], V_n^s \rangle = \langle V_m^r, [V_l^t, V_n^s] \rangle$, we know the radical of the bilinear form is a Lie ideal of hs[N]. And since $|N| \geq 2$, $\mathrm{Span}\{V_m^r : r \geq |N| + 1\}$ is a proper ideal, then the result follows. □

We should mention that an alternative proof can be obtained from a more algebraic point of view after we relate hs[λ] to $U(\mathfrak{sl}_2)$ in Chapter 3. And in fact the converse of Proposition 1.4 is also true (see Proposition 1.10).[9]

Next, let us get a closer look at the complicated expression of the structure constants by computing some commutators explicitly.

Example 1.5. By straightforward calculation we have

$$\begin{aligned}
[V_1^2, V_0^2] &= g_2^{22}(1, 0, \lambda) V_1^2 = V_1^2, \\
[V_0^2, V_{-1}^2] &= g_2^{22}(0, -1, \lambda) V_{-1}^2 = V_{-1}^2, \\
[V_1^2, V_{-1}^2] &= g_2^{22}(1, -1, \lambda) V_0^2 = 2V_0^2.
\end{aligned}$$

which indicates that $\{V_{-1}^2, V_0^2, V_1^2\}$ forms a basis of the special linear Lie algebra $\mathfrak{sl}_2$ and hence we know hs[λ] contains $\mathfrak{sl}_2$ as a subalgebra.

Lemma 1.6. $N_t^{rs}(m, n) = 0$ *for all* $t \geq 2\min\{r, s\}$.

Proof. Without loss of generality, assume $r \leq s$ and $m \geq 0$. Recall

$$N_t^{rs}(m, n) = \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} [r - 1 + m]_{t-1-k} [r - 1 - m]_k [s - 1 - n]_{t-1-k} [s - 1 + n]_k,$$

$$[r - 1 + m]_{t-1-k} = (r - 1 + m)(r - 1 + m - 1) \cdots (r - 1 + m - (t - 1 - k) + 1),$$

$$[r - 1 - m]_k = (r - 1 - m)(r - 1 - (m + 1)) \cdots (r - 1 - (m + k - 1)).$$

Suppose $t \geq 2r$, then for each $0 \leq k \leq t-1$, we have either $k \geq r-m$ or $t-1-k \geq t-r+m \geq r+m$. If $k \geq r$, then $[r-1-m]_k$ has a factor $(r-1-(m+(r-m)-1)) = 0$, so $N_t^{rs}(m,n) = 0$. If $t-1-k \geq r+m$, then $[r-1+m]_{t-1-k}$ has a factor $(r-1+m-(r+m)+1) = 0$, so we still have $N_t^{rs}(m,n) = 0$. □

Example 1.7. Using Lemma 1.6, we have

$$[V_m^2, V_n^s] = g_t^{2s}(m,n,\lambda)V_{m+n}^s = (m(s-1)-n)V_{m+n}^s$$

which, together with Example 1.5, indicates that for each fixed $s \geq 2$, the space $\mathrm{Span}\{V_n^s : |n| < s\}$ is a $\mathfrak{sl}_2$-module.

It is well-known that the universal enveloping algebra of $\mathfrak{sl}_2$, herein denoted as $U(\mathfrak{sl}_2)$, contains all irreducible $\mathfrak{sl}_2$-modules, so this observation also suggests that hs$[\lambda]$ could be closely related to $U(\mathfrak{sl}_2)$. This turns out to be true and will be discussed in Chapter 3.

Remark 1.8. It is worth to mention that although the generalised hypergeometric function is defined as an infinite series, $\phi_t^{rs}(\lambda)$ is always a finite sum because of the presence of $\frac{2-t}{2}$ and $\frac{1-t}{2}$. When t is a positive integer, one of $\frac{2-t}{2}$ and $\frac{1-t}{2}$ must be a non-positive integer, say $-k$, and then it is easy to see that only the first k terms of $\phi_t^{rs}(\lambda)$ are non-zero.

Example 1.9. Using Lemma 1.6 and Remark 1.8, we have

$$\begin{aligned}[V_m^3, V_n^s] =& g_2^{3s}(m,n,\lambda)V_{m+n}^{s+1} + g_4^{3s}(m,n,\lambda)V_{m+n}^{s-1} \\ =& \left(m(s-1)-2n\right)V_{m+n}^{s+1} + \frac{1}{192}\phi_4^{3s}(\lambda)N_4^{3s}(m,n)V_{m+n}^{s-1}\end{aligned}$$

where

$$\phi_4^{3s}(\lambda) = 1 - \frac{(\frac{1}{2}-\lambda)(\frac{1}{2}+\lambda)}{(\frac{3}{2}-s)(s-\frac{1}{2})}$$

and

$$\begin{aligned}N_4^{3s}(m,n) =& 24m - 24m^3 + 24n - 72m^2n - 72mn^2 - 24n^3 - 76ms + 52m^3s \\ & - 48ns + 84m^2ns + 36mn^2s + 72ms^2 - 36m^3s^2 + 24ns^2 - 24m^2ns^2 \\ & - 20ms^3 + 8m^3s^3.\end{aligned}$$

It is easy to solve that $\phi_4^{3s}(\lambda) = 0$ if only if $\lambda = \pm(s-1)$. (As an aside, see Appendix D.1 for a generalisation of this.) Surprisingly, this simple calculation leads to a non-trivial result, which is the converse of Proposition 1.4, as stated below.

Proposition 1.10. hs[λ] *is simple if* $\lambda \notin \mathbb{Z}\backslash\{0, \pm 1\}$.

Proof. Suppose $\lambda \notin \mathbb{Z}\backslash\{0, \pm 1\}$ and let I be a non-zero ideal of hs[λ]. Consider the set $S = \{r : V_n^r \in I \text{ for some } n\}$. Then $S = \emptyset$ and S has a least element s.

We shall do a proof by contradiction. Suppose I is proper, then $s \geq 3$.

Note that $V_n^s \in I$ for some n actually implies $V_n^s \in I$ for all $|n| < s$ as observed in Example 1.7. In particular, for $s \geq 3$ we know $V_{-1}^s, V_{-2}^s \in I$. Then we have

$$\begin{aligned} I \ni [V_1^3, V_{-1}^s] &= g_2^{3s}(1, -1, \lambda)V_0^{s+1} + g_4^{3s}(1, -1, \lambda)V_0^{s-1} \\ &= (1+s)V_0^{s+1} - \frac{1}{16}\phi_4^{3s}(\lambda)s(s-2)(s-1)V_0^{s-1}, \\ I \ni [V_2^3, V_{-2}^s] &= g_2^{3s}(2, -2, \lambda)V_0^{s+1} + g_4^{3s}(2, -2, \lambda)V_0^{s-1} \\ &= 2(1+s)V_0^{s+1} + \frac{1}{8}\phi_4^{3s}(\lambda)s(s-1)(s+1)V_0^{s-1}. \end{aligned}$$

Now since the roots of $\phi_4^{3s}(\lambda) = 0$ are $\lambda = \pm(s-1)$ and s is an integer that is no less than 3, we see $\phi_4^{3s}(\lambda) \neq 0$ unless $\lambda \in \mathbb{Z}\backslash\{0, \pm 1\}$. And then we have

$$I \ni [V_2^3, V_{-2}^s] - 2[V_1^3, V_{-1}^s] = \frac{1}{8}\phi_4^{3s}(\lambda)s(s-1)(2s-1)V_0^{s-1} \neq 0$$

which means V_0^{s-1} is also in I, provided $\lambda \notin \mathbb{Z}\backslash\{0, \pm 1\}$. But this contradicts to the assumption that s is the least element of S. Hence I cannot be proper and the result follows. □

Chapter 2

Algebraic Structure of $U(\mathfrak{sl}_2)$

In this chapter, we review some essential definitions and results on the algebraic structure of the universal enveloping algebra $U(\mathfrak{sl}_2)$. Hereafter $\mathfrak{g}$ will always denote a Lie algebra and any additional hypothesis will be stated explicitly.

Definition 2.1 (Universal Enveloping Algebra)**.** Given a Lie algebra $\mathfrak{g}$, consider the n-fold tensor product

$$T^n\mathfrak{g} = \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n\text{times}}$$

and denote $T^0\mathfrak{g} = \mathbb{C}$. Then consider the vector space $T\mathfrak{g} = \bigoplus_{n=0}^{\infty} T^n\mathfrak{g}$ and the two-sided ideal $I\mathfrak{g}$ of $T\mathfrak{g}$ generated by $\{x \otimes y - y \otimes x - [x, y] : x, y \in \mathfrak{g}\}$. We define the universal enveloping algebra to be the quotient algebra

$$U(\mathfrak{g}) := \frac{T\mathfrak{g}}{I\mathfrak{g}}.$$

Remark 2.2. With respect to the operation of tensor product, $U(\mathfrak{g})$ is an associative algebra, and later on we shall omit $\otimes$ in the discussion. On the other hand, $U(\mathfrak{g})$ is also a Lie algebra with the Lie bracket defined to be the usual (matrix) commutator, that is,

$$[A, B]_{U(\mathfrak{g})} = AB - BA, \quad \text{for } A, B \in U(\mathfrak{g}).$$

Since $U(\mathfrak{g})$ has two multiplicative operations, when we use the word "ideal" for $U(\mathfrak{g})$, we should be careful on the terminology and notation. To be clear, we will use $(S)_\mathfrak{g}$ to denote the Lie ideal of $U(\mathfrak{g})$ generated by $S \subset U(\mathfrak{g})$ and use $\langle S \rangle$ to denote the two-sided ideal of $U(\mathfrak{g})$ as an associative algebra.

Remark 2.3. $U(\mathfrak{g})$ can be considered as a $\mathfrak{g}$-module via either the left (or right) multiplication or the Lie bracket operation. In this thesis, we are more concerned with the latter case.

Theorem 2.4 (Poincaré-Birkhoff-Witt)**.** *Given a finite-dimensional Lie algebra* $\mathfrak{g}$*, let* $\{x_1, x_2, \ldots, x_n\}$ *be an ordered basis of* $\mathfrak{g}$*, then* $\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} : i_1, \ldots, i_n \in \mathbb{N}\}$ *is a basis of* $U(\mathfrak{g})$*, called Poincaré-Birkhoff-Witt (PBW) basis.*

This is a well-known result with a non-trivial proof that can be found in any standard textbook of Lie algebra and representation theory such as [7].

Let us explore some useful properties and results of the commutators in the universal enveloping algebra, using $U(\mathfrak{sl}_2)$ as an example.

Lemma 2.5. *For any* $A, B \in U(\mathfrak{g})$*,*

$$[A^n, B] = \sum_{i=1}^{n} A^{i-1}[A, B]A^{n-i}.$$

This identity could be easily proved by induction. Using this identity we could compute a family of commutators in $U(\mathfrak{sl}_2)$.

Throughout this thesis, we choose an ordered basis of $\mathfrak{sl}_2$ to be $\{F, H, E\}$ with commutation relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H. \tag{2.1}$$

By straightforward calculation we have

$$[H, F^n] = -\, 2nF^n, \tag{2.2}$$

$$[E, F^n] = nF^{n-1}H + n(n-1)F^{n-1}. \tag{2.3}$$

With some effort, we can also get

$$[F, H^j] = \sum_{i=1}^{j} (-2)^{i-1} \binom{j}{i} FH^{j-i}. \tag{2.4}$$

In order to derive more useful identities for $U(\mathfrak{sl}_2)$, we can bring in the adjoint representation to clean up the notations.

Notation 2.6. Denote the adjoint representation by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, that is,

$$(\mathrm{ad}A)\,(B) = [A, B] \text{ for all } A, B \in \mathfrak{g}.$$

And we shall denote $(\mathrm{ad}A)^n := \underbrace{(\mathrm{ad}A) \circ \cdots \circ (\mathrm{ad}A)}_{n \text{ times}}$

Bearing in mind that the representation map preserves Lie structure and taking the advantage of the adjoint representation, we can neatly derive a recursive relation that turns out to be very useful later. Note

$$\begin{aligned}
&[E, (\mathrm{ad}F)^{r-1-m}(E^{r-1})] \\
&= (\mathrm{ad}E) \circ (\mathrm{ad}F)^{r-1-m}(E^{r-1}) \\
&= (\mathrm{ad}F) \circ (\mathrm{ad}E) \circ (\mathrm{ad}F)^{r-1-(m+1)}(E^{r}) + (\mathrm{ad}H) \circ (\mathrm{ad}F)^{r-1-(m+1)}(E^{r})
\end{aligned}$$

where

$$\begin{aligned}
&(\mathrm{ad}H) \circ (\mathrm{ad}F)^{r-1-(m+1)}(E^{r}) \\
=&(r-1-(m+1))(\mathrm{ad}[H,F])(E^{r}) + (\mathrm{ad}F)^{r-1-(m+1)} \circ (\mathrm{ad}H)(E^{r-1}) \\
=&2(m+1)(\mathrm{ad}F)^{r-1-(m+1)}(E^{r-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
[E, (\mathrm{ad}F)^{r-1-m}(E^{r-1})] =&2\sum_{k=1}^{r-1-m}(m+k)(\mathrm{ad}F)^{r-1-(m+1)}(E^{r-1}) \\
=&(r+m)(r-1-m)(\mathrm{ad}F)^{r-1-(m+1)}(E^{r-1}).
\end{aligned} \tag{2.5}$$

Lemma 2.7. *For all $A, B, C \in U(\mathfrak{g})$, we have the identity*

$$(\mathrm{ad}\, C)^n(AB) = \sum_{k=0}^{n}\binom{n}{k}(\mathrm{ad}\, C)^{n-k}(A)(\mathrm{ad}\, C)^k(B) \tag{2.6}$$

The identity is again easy to prove by induction. For further use we shall do an example for $U(\mathfrak{sl}_2)$.

Example 2.8. Applying the identity above we have

$$\begin{aligned}
(\mathrm{ad}F)^n(E^r) =&(\mathrm{ad}F)^n(E^{r-1}E) \\
=&(\mathrm{ad}F)^n(E^{r-1})E - n(\mathrm{ad}F)^{n-1}(E^{r-1})H - n(n-1)(\mathrm{ad}F)^{n-2}(E^{r-1})F
\end{aligned}$$

The next thing one would care about $U(\mathfrak{sl}_2)$ is usually the ideals it contains. One of the most important Lie ideals of a universal enveloping algebra is the centre, so we shall describe the centre of $U(\mathfrak{sl}_2)$ explicitly.

Notation 2.9. Denote the centre of $U(\mathfrak{g})$ by $Z(U(\mathfrak{g}))$, that is,

$$Z(U(\mathfrak{g})) := \{A \in U(\mathfrak{g}) : [A, B] = 0, \text{ for all } B \in U(\mathfrak{g}).\}.$$

Notation 2.10. Denote

$$C = 4FE + H^2 + 2H$$

which is known as the quadratic Casimir of $U(\mathfrak{sl}_2)$.

Lemma 2.11 (Centre of $U(\mathfrak{sl}_2)$)**.** *The centre of $U(\mathfrak{sl}_2)$ is the polynomial ring generated by the quadratic Casimir C, that is, $Z(U(\mathfrak{sl}_2)) = \mathbb{C}[C]$.*

In fact this statement can be proven as a special case of Harish-Chandra's theorem which describes the centre $Z(U(\mathfrak{g}))$ for any semi-simple Lie algebra $\mathfrak{g}$.[26] However the general statement and proof of Harish-Chandra's theorem are involved, so we shall approach to an alternative proof following the idea in [4].

Proof. It is easy to check that $C \in Z(U(\mathfrak{sl}_2))$, so $\mathbb{C}[C] \subset Z(U(\mathfrak{sl}_2))$. For the converse inclusion, consider the $U(\mathfrak{sl}_2)$-algebra

$$\widetilde{U}(\mathfrak{sl}_2) := \frac{U(\mathfrak{sl}_2)[F^{-1}]}{\langle FF^{-1} - 1, F^{-1}F - 1\rangle}$$

which is also known as the localisation of $U(\mathfrak{sl}_2)$ at F. We shall formally denote $F^{-n} := (F^{-1})^n$. And one can check $\widetilde{U}(\mathfrak{sl}_2)$ is a Lie algebra with extra commutation relations extended from

$$[H, F^{-1}] = 2F^{-1}, \qquad [E, F^{-1}] = -F^{-2}H + 2F^{-2}.$$

which agree with Equations (2.2) and (2.3). It is also known that $\widetilde{U}(\mathfrak{sl}_2)$ has a basis $\{F^iH^jE^k : i \in \mathbb{Z}, j, k \in \mathbb{N}\}$. And in $\widetilde{U}(\mathfrak{sl}_2)$ we have

$$E = \frac{1}{4}F^{-1}\left(C - H^2 + 2H\right).$$

It follows that we can choose a more convenient basis of $\widetilde{U}(\mathfrak{sl}_2)$, which is

$$\{F^iH^jC^k : i \in \mathbb{Z}, j, k \in \mathbb{N}\}.$$

We claim $Z(U(\mathfrak{sl}_2)) = U(\mathfrak{sl}_2) \cap Z(\widetilde{U}(\mathfrak{sl}_2))$.

It is clear that $U(\mathfrak{sl}_2) \cap Z(\widetilde{U}(\mathfrak{sl}_2)) \subset Z(U(\mathfrak{sl}_2))$. To show the converse inclusion, take any element $z \in Z(U(\mathfrak{sl}_2))$, then it suffices to check $[z, F^{-1}] = 0$. To see this, note $[z, E] = 0$ and

$$\begin{aligned}
[z, E] =& [z, \frac{1}{4}F^{-1}\left(C - H^2 + 2H\right)] \\
=& \frac{1}{4}F^{-1}[z, C - H^2 + 2H] + [z, F^{-1}]\left(C - H^2 + 2H\right) \\
=& [z, F^{-1}]\left(C - H^2 + 2H\right)
\end{aligned}$$

Now since $C - H^2 + 2H \neq 0$, we must have $[z, F^{-1}] = 0$.

Therefore to show $Z(U(\mathfrak{sl}_2)) = \mathbb{C}[C]$, it suffices to determine $Z(\widetilde{U}(\mathfrak{sl}_2))$. Consider a generic element

$$A = \sum_{j\in\mathbb{N}} H^j a_j(F, C) \in Z(\widetilde{U}(\mathfrak{sl}_2))$$

where $a_j(F, C) = \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{N}} a_{ijk} F^i C^k$ and $a_{ijk} \in \mathbb{C}$ are some coefficients. Note only finitely many a_{ijk} are non-zero, so there exists a largest value of j such that a_{ijk} is non-zero for some i, k. Say the value is J. Then since $A, C \in Z(\widetilde{U}(\mathfrak{sl}_2))$, we have

$$\begin{aligned}
0 = [F, A] &= \sum_{j=0}^{J} [F, H^j a_j(F, C)] \\
&= \sum_{j=0}^{J} [F, H^j] a_j(F, C) \\
&= \sum_{j=0}^{J} \sum_{n=1}^{j} (-2)^{n-1} \binom{j}{n} F H^{j-n} a_j(F, C) \\
&= \sum_{n=1}^{J} \sum_{j=n}^{J} (-2)^{j-n} \binom{j}{j-n+1} F H^{n-1} a_j(F, C)
\end{aligned}$$

where we used Equation (2.4). This implies $a_j(F, C) = 0$ whenever $j \neq 0$. Therefore A can be written as

$$A = \sum_{i\in\mathbb{Z}} F^i b_i(C)$$

where $b_i(C) = \sum_{k\in\mathbb{N}} b_{ik} C^k$, with only finitely many of b_{ik} non-zero. And similar to the above we have

$$0 = [H, A] = \sum_{i\in\mathbb{Z}} [H, F^i] b_i(C) = \sum_{i\in\mathbb{Z}} (-2i) F^i b_i(C)$$

where we used Equation (2.2). This implies $b_i(C) = 0$ whenever $i \neq 0$. Hence we have

$$A = b_0(C) = \sum_{k\in\mathbb{N}} c_k C^k$$

where only finitely many c_k are non-zero. Now the result follows. □

Later on we will consider $U(\mathfrak{g})$ with $\mathfrak{g}$ not semi-simple, where the Harish-Chandra's theorem fails but this proof could be carried over as we will see in Chapter 4 and 5.

Now we introduce a quotient algebra of $U(\mathfrak{sl}_2)$ which will be our main interest in the rest of this chapter and also the next chapter.

Notation 2.12. Consider the two-sided ideal of $U(\mathfrak{sl}_2)$ generated by $C-(\lambda^2-1)$ as an associative algebra. Denote

$$U(\mathfrak{sl}_2)[\lambda] := \frac{U(\mathfrak{sl}_2)}{\langle C-(\lambda^2-1)\rangle}$$

where $\lambda \in \mathbb{C}$. The choice of this parametrisation will be explained in Chapter 3.

Remark 2.13. The ideals $\langle C-\mu\rangle$ with $\mu \in \mathbb{C}$ turn out to be the maximal ideals of $U(\mathfrak{sl}_2)$.[5]

It is claimed in [6] that there is a well-known result saying

$$U(\mathfrak{sl}_2) = [U(\mathfrak{sl}_2), U(\mathfrak{sl}_2)] \oplus Z(U(\mathfrak{sl}_2)).$$

However, contrary to one's expectation, we found this result not easy to prove. For the purpose of this thesis, it suffices to prove the weaker statement stated below.

Lemma 2.14. $U(\mathfrak{sl}_2)[\lambda] = [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]] \oplus \mathbb{C}$

To prove this, we need to introduce the Harish-Chandra character of $U(\mathfrak{g})$.

Definition 2.15 (Harish-Chandra character)**.** The Harish-Chandra character of $U(\mathfrak{g})$ is a linear function $\chi : U(\mathfrak{g}) \to \mathbb{C}$ satisfying the following conditions:
(1) $\chi(AB) = \chi(BA)$ for all $A, B \in U(\mathfrak{g})$;
(2) $\chi(1) = 1$;
(3) $\chi(z_1 z_2) = \chi(z_1)\chi(z_2)$ for all $z_1, z_2 \in Z(U(\mathfrak{g}))$.

It is proven in [14] that a Harish-Chandra character is uniquely determined by its value on the centre. Precisely speaking, we have

Lemma 2.16. *Given a semi-simple Lie algebra $\mathfrak{g}$, let χ_1, χ_2 be Harish-Chandra characters of $U(\mathfrak{g})$. If $\chi_1(z) = \chi_2(z)$ for all $z \in Z(U(\mathfrak{g}))$, then $\chi_1 = \chi_2$.*

Proof of Lemma 2.14 using Lemma 2.16. It follows from Lemma 2.11 that

$$Z(U(\mathfrak{sl}_2)[\lambda]) = \mathbb{C}.$$

And then by Lemma 2.16 we see $U(\mathfrak{sl}_2)[\lambda]$ has a unique Harish-Chandra character, say $\chi_\lambda : U(\mathfrak{sl}_2)[\lambda] \to \mathbb{C}$.

By Definition 2.15 we observe that $\chi_\lambda([A,B]) = 0$ for all $A, B \in U(\mathfrak{sl}_2)[\lambda]$ and $\chi_\lambda(c) = c$ for all $c \in \mathbb{C}$. This implies $[U(\mathfrak{sl}_2), U(\mathfrak{sl}_2)] \cap \mathbb{C} = \{0\}$.

It remains to show $U(\mathfrak{sl}_2)[\lambda] = [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]] + \mathbb{C}$ which we shall approach by contradiction.

Denote $V := [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]] + \mathbb{C}$ and suppose $V \neq U(\mathfrak{sl}_2)[\lambda]$. Then there exists a subspace $S \in U(\mathfrak{sl}_2)[\lambda]$ such that $U(\mathfrak{sl}_2)[\lambda] = V \oplus S$. Choose a basis of S, say $\{e_i : 1 \leq i < N\}$, where N is either a positive integer or infinity. Define a function $\chi'_\lambda : U(\mathfrak{sl}_2)[\lambda] \to \mathbb{C}$ by declaring

$$\chi'_\lambda(v) = \chi_\lambda(v) \text{ for all } v \in V, \qquad \chi'_\lambda(e_i) = \chi_\lambda(e_i) + 1 \text{ for all } 1 \leq i < N$$

and extending the map linearly. Then χ'_λ satisfies the conditions in Definition 2.15 because χ_λ does and $\chi'_\lambda|_V = \chi_\lambda|_V$. Therefore χ'_λ is a Harish-Chandra character of $U(\mathfrak{sl}_2)[\lambda]$. But by construction $\chi'_\lambda \neq \chi_\lambda$, which contradicts to Lemma 2.16. Hence we must have $U(\mathfrak{sl}_2)[\lambda] = V = [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]] + \mathbb{C}$. □

We may devote the last part of this chapter to a simple but yet interesting result that reveals the structure of $U(\mathfrak{sl}_2)$ (and consequently $U(\mathfrak{sl}_2)[\lambda]$) via its representations.

Proposition 2.17. *Let V_n be an n-dimensional irreducible module of $U(\mathfrak{sl}_2)$ with the representation homomorphism $\pi : U(\mathfrak{sl}_2) \to \mathfrak{gl}_n$ satisfying $\pi(C) = (n^2 - 1)\, \mathbb{1}$, where $\mathbb{1}$ denotes the identity operator in $\mathfrak{gl}_n$. Then*

$$U(\mathfrak{sl}_2)/\ker(\pi) \cong \mathfrak{gl}_n.$$

Bearing the first isomorphism theorem in mind, this follows immediately from the famous classical theorem stated below.

Theorem 2.18 (Burnside)**.** *Let V be a finite-dimensional vector space over an algebraically closed field. Let $\mathfrak{gl}(V)$ be the operator algebra consisting of all linear operators on V. If a non-trivial subalgebra $\mathcal{A} \subset \mathfrak{gl}(V)$ is irreducible, then $\mathcal{A} = \mathfrak{gl}(V)$.*

Burnside's theorem of matrix algebras was first proved in 1905. Interestingly, people have made many efforts on simplifying its proof. Here we adapt the proof given in [25], which is claimed to be the simplest proof.

Lemma 2.19. *Any irreducible algebra $\mathcal{A}$ contains a non-zero non-invertible operator S.*

Proof. We assumed $\mathcal{A} \neq \{0\}$. Note the algebra $\mathcal{U} = \{kI : k \in \mathbb{C}\}$, where I denotes the identity operator, is not irreducible unless $\dim(V) = 1$. If $\dim(V) = 1$, then the theorem is trivially true, so we may assume $\dim(V) > 1$. Now by irreducibility of $\mathcal{A}$, we have $\mathcal{A} \neq \mathcal{U}$. Thus there exists a transformation $T \in \mathcal{A}$ such that $T \notin \mathcal{U}$.

If T is not invertible, then we are done with $S = T$. Suppose T is invertible, then T has an eigenvalue λ such that $T - \lambda I$ is not invertible and not the zero-operator. If $I \in \mathcal{A}$, then we are done with $S = T - \lambda I$. Suppose $I \notin \mathcal{A}$, let $S = T(T - \lambda I) = T^2 - \lambda T$, which belongs to $\mathcal{A}$, and is not invertible (because if it is, then $(S^{-1}T)(T - \lambda I) = I$, which contradicts with $T - \lambda I$ not invertible), and is not the zero-operator. This proves the lemma. □

Lemma 2.20. *Any irreducible algebra $\mathcal{A}$ contains a rank-one operator.*

Proof. We shall prove it by induction on the dimension of V. For the base case $\dim V = 1$, the lemma is trivial. Now suppose the lemma holds when $\dim V \in \{1, 2, \cdots, n-1\}$, then for $\dim V = n > 1$, consider the subspace $V_0 = S(V)$ and the family $\mathcal{A}_0 = \{(SA)|_{V_0} : A \in \mathcal{A}\}$, where S is a non-zero non-invertible operator in $\mathcal{A}$. Since S is non-invertible, $\dim V_0 < n$.

Note $\mathcal{A}_0$ is a subalgebra of $\mathfrak{gl}(V_0)$. We claim $\mathcal{A}_0$ is irreducible. It's well-known that transitivity is equivalent to irreducibility, so it suffices to show $\mathcal{A}_0$ is transitive on V_0. Take $0 \neq v \in V_0$, then $\mathcal{A}v = V$ because $\mathcal{A}$ is irreducible hence transitive. Then $\mathcal{A}_0 v = S(\mathcal{A}v) = S(V) = V_0$, so $\mathcal{A}_0$ is transitive on V_0.

Now by the induction hypothesis, $\mathcal{A}_0$ contains a rank-one operator, which means there exists a operator $T \in \mathcal{A}$ such that $(ST)|_{V_0}$ has rank one. Thus $\dim STS(V) = 1$, which means $STS \in \mathcal{A}$ has rank one. □

Lemma 2.21. *If a subalgebra $\mathcal{A}$ is irreducible, then its adjoint-algebra $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ is also irreducible.*

Proof. We shall prove the contrapositive statement. Suppose $\mathcal{A}^*$ has a non-trivial $\mathcal{A}^*$-invariant subspace $M \subset V$, that is, $\forall v \in M, \forall A^* \in \mathcal{A}^*$, $A^*v \in V$. We claim the orthogonal complement $M^\perp$ is a non-trivial $\mathcal{A}$-invariant subspace. To see this, take any $u \in M^\perp$ and any $A \in \mathcal{A}$, then $\langle Au, v\rangle = \langle u, A^*v\rangle = 0$ for all $v \in M$ since $A^*v \in M$ for all $v \in M$. This means $Au \in M^\perp$. And $M^\perp$ is non-trivial because M is proper. □

Proof of Burnside's Theorem. Let $S \in \mathcal{A}$ be a rank-one operator, then there exist non-zero vectors $v, w \in V$ such that S takes $x \in V$ to $\langle x, w\rangle\, v$. We shall denote this as $S = v \oplus w$. It can be checked that for any $T \in \mathcal{A}$, we have $T(v \oplus w)T = (Tv) \oplus (T^*w)$, which implies $(Tv) \oplus (T^*w) \in \mathcal{A}$.

Note $\mathcal{A}$ is transitive, so $\mathcal{A}v = V$, which means $x \oplus (T^*w) \in \mathcal{A}$ for any $x \in V$. Now note $\mathcal{A}^*$ is also transitive, so $\mathcal{A}^*w = V$ and hence $x \oplus y \in \mathcal{A}$ for any $x, y \in V$, that is, $\mathcal{A}$ contains all rank-one operators which span $\mathfrak{gl}(V)$, so $\mathcal{A} = \mathfrak{gl}(V)$. This proves the theorem. □

Proof of Proposition 2.17 using Burnside's theorem. By the first isomorphism theorem, we have $U(\mathfrak{sl}_2)/\ker(\pi) \cong \operatorname{im}(\pi) \subset \mathfrak{gl}_n$. Note that V_n being an irreducible module is equivalent to $\operatorname{im}(\pi)$ being an irreducible subalgebra. And the constraint that $\pi(C) = (n^2 - 1)I$ ensures $\operatorname{im}(\pi)$ is non-trivial. Hence $\operatorname{im}(\pi) = \mathfrak{gl}_n$ by Burnside's theorem of matrix algebra. □

Besides of taking the advantage of the theorem, here we also provide a more constructive proof of Proposition 2.17 in the case of $\mathfrak{sl}_2$. And in Chapter 4 we will be able to use the same approach to prove a new result.

Constructive proof of 2.17. Let V_n be the set of polynomials of single variable, say x, with degree less than n, denoted by $\mathbb{C}[x]_n$. Choose an ordered basis of $\mathbb{C}[x]_n$ to be $\{1, x, \ldots, x^{n-1}\}$. Then we have a realization of $U(\mathfrak{sl}_2)$ as differential operator given by

$$\pi(E) = \frac{\partial}{\partial x}, \qquad \pi(H) = -2x\frac{\partial}{\partial x} + (n-1), \qquad \pi(F) = -x^2\frac{\partial}{\partial x} + (n-1)x.$$

We shall denote $A \cdot v := \pi(A)v$ for $A \in U(\mathfrak{sl}_2), v \in \mathbb{C}[x]_n$.

One can routinely check this is indeed a well-defined $U(\mathfrak{sl}_2)$-representation. Here we only comment that, by construction we have

$$F \cdot x^{n-1} = -x^2\frac{\partial\left(x^{n-1}\right)}{\partial x} + (n-1)x^n = -(n-1)x^n + (n-1)x^n = 0$$

which shows that F acting on x^{n-1} does not cause any trouble. We may check

$$C = 4FE + HH + 2H = (n-1)^2 + 2(n-1) = n^2 - 1$$

as desired. Also note that for $k \in \{0, 1, \ldots, n-1\}$

$$E \cdot x^k = kx^{k-1} \tag{2.7}$$

$$H \cdot x^k = (-2k + n - 1)x^k \tag{2.8}$$

$$F \cdot x^k = (-k + n - 1)x^{k+1}. \tag{2.9}$$

Let M_{ij} denote the basis operator of $\mathfrak{gl}_n$ such that

$$M_{ij}x^i = x^j, \qquad M_{ij}x^k = 0, \text{ for all } k \neq i.$$

Now we can construct an element $A_{ij} \in U(\mathfrak{sl}_2)$ such that $\pi(A_{ij}) = M_{ij}$.

For the special case $i = j$, note that

$$A_{ii} := \prod_{k \neq i, k < n} \frac{1}{2(k-i)} (H + 2k + 1 - n)\,.$$

satisfies

$$A_{ii}x^i = x^i, \qquad A_{ii}x^k = 0, \text{ for all } k \neq i.$$

according to Equation (2.8). Now to construct A_{ij} for $j \neq i$, we only need to care about how to send x^i to x^j. An obvious choice of A_{ij} for $i < j$ is

$$A_{ij} := \frac{(n-1-j)!}{(n-1-i)!} F^{j-i} A_{ii}$$

and if $i > j$ we may take

$$A_{ij} := \frac{j!}{i!} E^{i-j} A_{ii}.$$

Let B_{ij} denote the equivalence class of A_{ij} in the quotient algebra $U(\mathfrak{sl}_2)/\ker(\pi)$. Then $\{B_{ij} : i, j \in \{0, 1, \ldots, n-1\}\}$ forms a basis of $\mathfrak{gl}_n$. □

Corollary 2.22. *With the notation used in Proposition 2.17, for* $n \geq 2$*, consider* V_n *as a* $[U(\mathfrak{sl}_2)[n], U(\mathfrak{sl}_2)[n]]$*-module with induced representation homomorphism*

$$\tilde{\pi} : [U(\mathfrak{sl}_2)[n], U(\mathfrak{sl}_2)[n]] \to \mathfrak{gl}_n.$$

Then

$$[U(\mathfrak{sl}_2)[n], U(\mathfrak{sl}_2)[n]] \,/\, \ker(\tilde{\pi}) \cong \mathfrak{sl}_n$$

Proof. The unique Harish-Chandra character χ_n of $U(\mathfrak{sl}_2)[n]$ allows us to define an invariant symmetric bilinear form on $U(\mathfrak{sl}_2)[n]$, which we shall call the trace, given by

$$Tr(AB) = \chi_n(AB)$$

for all $A, B \in U(\mathfrak{sl}_2)[n]$. Then the corollary is an immediate consequence of Proposition 2.17 by taking the traceless part of both sides. □

Chapter 3

Construction of hs[λ] from $U(\mathfrak{sl}_2)$

As mentioned in Example 1.7, it is known that hs[λ] is indeed isomorphic to the derived algebra of the quotient algebra $U(\mathfrak{sl}_2)[\lambda]$ constructed from $U(\mathfrak{sl}_2)$. Although this result is stated in many articles on higher spin algebras, to the best of our knowledge, there is no paper that gives a complete proof. The main content of this chapter is to organize a comprehensive proof. For clarity, we will not present all details when they can be found in the literature or can be done by straightforward computation.

Notation 3.1. Let

$$V_m^r = \frac{(r+m-1)!}{(2r-2)!} (\mathrm{ad}F)^{r-1-m} (E^{r-1}) \in U(\mathfrak{sl}_2)$$

for $r \geq 2$, $|m| < r$, and denote the identity element of $U(\mathfrak{sl}_2)$ by V_0^1, that is, $V_0^1 := 1 \in U(\mathfrak{sl}_2)$.

Proposition 3.2. *The set*

$$\{C^k V_m^r : r \geq 1, |m| < r, k \geq 0\}$$

forms a basis of $U(\mathfrak{sl}_2)$*.*

Proof. Recall a PBW basis of $U(\mathfrak{sl}_2)$ is $\{F^i H^j E^k : i,j,k \geq 0\}$. Define the degree of a monomial $F^i H^j E^k$ to be $\deg(F^i H^j E^k) := i+j+k$, then as a vector space, $U(\mathfrak{sl}_2)$ can be decomposed into the direct sums of graded pieces via the degree, that is,

$$U(\mathfrak{sl}_2) = \bigoplus_{t=0}^{\infty} \mathrm{Gr}_t(\mathfrak{sl}_2)$$

where $\mathrm{Gr}_t(\mathfrak{sl}_2) := \mathrm{Span}\{F^i H^j E^k : \deg(F^i H^j E^k) = t\}$. (This decomposition can be formalised by using the machinery tool called filtration [7] which takes the Lie structure of $U(\mathfrak{sl}_2)$ into account, but for the purpose of this proof it is not harm to restrict ourselves to the decomposition of $U(\mathfrak{sl}_2)$ as a vector space.) We may also denote

$$U(\mathfrak{sl}_2)_n = \bigoplus_{t=0}^{n} \mathrm{Gr}_t(\mathfrak{sl}_2).$$

And we shall then perform the proof by inducting on n, that is, we are going to prove that for each n, $\{C^k V_m^r : k \geq 0, r \geq 1, 2k + r - 1 = n, |m| < r\}$ is a basis of $U(\mathfrak{sl}_2)_n$.

For $n = 0$, this holds trivially. For $n = 1$, we see k could only be 0 and it is easy to compute $V_1^2 = E, V_0^2 = -\frac{1}{2}H$ and $V_{-1}^2 = -F$, so it is clear that $\{V_{-1}^2, V_0^2, V_{-1}^2\}$ is a basis of $U(\mathfrak{sl}_2)_1 \cong \mathfrak{sl}_2$.

For $n \geq 2$, suppose $\{C^k V_m^r : k \geq 0, r \geq 1, 2k + r - 1 = s, |m| < r\}$ is a basis of $U(\mathfrak{sl}_2)_s$ for all $s < n$. Then to show $\{C^k V_m^r : k \geq 0, r \geq 1, 2k+r-1 = n, |m| < r\}$ is a basis of $U(\mathfrak{sl}_2)_n$, it suffices to check the vectors $\{C^k V_m^r : 2k + r - 1 = n\}$ are independent, because then the set is automatically a basis of $U(\mathfrak{sl}_2)_n$ by counting the dimension. To check the independence we need to look at the explicit expression of V_m^{n+1}.

By recursively using Example 2.8, we see

$$V_m^{n+1} = \frac{1}{(2n)!} \sum_{i+j+k=n} (-1)^{i+j} [n + m]_{2k+j} [n - m]_{2i+j} P\left(F^i H^j E^k\right) \tag{3.1}$$

where $[n + m]_{2k+j}$ is the Pochhammer symbol and $P\left(F^i H^j E^k\right)$ denotes the sum of all possible configurations of F, H, E's such that F appears exactly i times, H appears exactly j times and E appears exactly k times. For example, $P\,(FHE)$ is $FHE + FEH + EFH + EHF + HEF + HFE$.

For fixed i, j, k with $i + j + k = n$, the total number of all such possible configurations is $\binom{n}{i}\binom{n-i}{j}$. Now write V_m^{n+1} in terms of the PBW basis, we have

$$V_m^{n+1} = \sum_{i+j=0}^{n} \frac{(-1)^{i+j}}{(2n)!} \binom{n}{i} \binom{n-i}{j} [n + m]_{2n-2i-j} [n - m]_{j+2i} F^i H^j E^{n-i-j} + U(\mathfrak{sl}_2)_{n-1}$$

where "$+\ U(\mathfrak{sl}_2)_{n-1}$" means "+ some element in $U(\mathfrak{sl}_2)_{n-1}$". Since we are only concerned with the highest degree part of V_m^{n+1}, hereafter we secretly use V_m^{n+1} to denote V_m^{n+1} modulo $U(\mathfrak{sl}_2)_{n-1}$.

Now note that for any fixed m, the term $F^i H^j E^{n-i-j}$ is non-vanished only when $2i + j = n - m$, so we have

$$V_m^{n+1} = \sum_{i=\max\{-m,0\}}^{\lfloor (n-m)/2 \rfloor} \frac{(n + m)!(n - m)!(-1)^{n-m-i}}{(2n)!} \binom{n}{i} \binom{n-i}{m+i} F^i H^{n-m-2i} E^{m+i}$$

For each $\max\{-m, 0\} < i \leq \lfloor (n-m)/2 \rfloor$, we see

$$F^i H^{n-m-2i} E^{m+i} = FE(F^{i-1} H^{n-m-2i} E^{m+i-1}) + U(\mathfrak{sl}_2)_{n-1}$$
$$F^{i-1} H^{n-m-2i+2} E^{m+i-1} = H^2 (F^{i-1} H^{n-m-2i} E^{m+i-1}) + U(\mathfrak{sl}_2)_{n-1}.$$

Therefore with the cost of adding extra elements in $U(\mathfrak{sl}_2)_{n-1}$, we can factor out the quadratic Casimir $C = 4FE + H^2 + 2H$, that is,

$$\begin{aligned} V_m^{n+1} = & \left(\sum_{k=d_m}^{f_{nm}} \frac{(-1)^{n-m-d_m}}{4^{k-d_m}} c_{knm} \right) F^{d_m} H^{n-m-2d_m} E^{m+d_m} \\ & + C \sum_{i=d_m+1}^{f_{nm}} \left(\sum_{k=i}^{f_{nm}} \frac{(-1)^{n-m-i}}{4^{k-i}} c_{knm} \right) F^{i-1} H^{n-m-2i} E^{m+i-1} \end{aligned} \tag{3.2}$$

where $d_m = \max\{-m, 0\}, f_{nm} = \lfloor (n-m)/2 \rfloor, c_{knm} = \dfrac{(n+m)!(n-m)!}{(2n)!} \binom{n}{k} \binom{n-k}{m+k}$. And it can be computed with the help of Mathematica™ that the coefficient of the first term in Equation (3.2) is either $(-1)^n 2^{-m-n}$ or $(-2)^{n-m}$, which are non-zero for all $|m| < n+1$.

By the inductive hypothesis, the second term in Equation (3.2) can be written as a linear combination of the vectors in $\{C^k V_m^r : k, r \geq 1, 2k+r-1 = n, |m| < r\}$. Then because of the presence of the term in (3.2), we see V_m^{n+1} is independent on the vectors in $\{C^k V_m^r : k, r \geq 1, 2k+r-1 = n, |m| < r\}$.

Also by the inductive hypothesis we see the vectors in $\{V_m^r : r-1 < n, |m| < r\}$ are independent and hence the vectors $\{C^k V_m^r : k, r \geq 1, 2k+r-1 = n, |m| < r\}$ are also independent. Now by the PBW Theorem 2.4 we can see the vectors in $\{V_m^{n+1} : |m| < n+1\}$ are independent. Therefore we can conclude that all the vectors in $\{C^k V_m^r : k \geq 0, r \geq 1, 2k+r-1 = n, |m| < r\}$ are independent, as desired. □

Notation 3.3. In $U(\mathfrak{sl}_2)[\lambda]$, V_m^r denotes the corresponding equivalence class.

Corollary 3.4.

$$U(\mathfrak{sl}_2)[\lambda] = \operatorname{Span}\{V_m^r : r \geq 2, |m| < r\} \oplus \mathbb{C}$$

Proof. This is an immediate consequence of Proposition 3.2. □

Corollary 3.5.

$$\operatorname{Span}\{V_m^r : r \geq 2, |m| < r\} = [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]]$$

Proof. This follows from Corollary 3.4 and Lemma 2.14. □

Proposition 3.6. *The set of generators* $\{V_m^r : r \geq 2, |m| < r\} \subset U(\mathfrak{sl}_2)[\lambda]$ *satisfies the commutation relation* (1.1) *given in the definition of* hs[λ], *which we shall recall here.*

$$[V_m^r, V_n^s] = \sum_{t=2,\text{ t even}}^{r+s-1} g_t^{rs}(m,n,\lambda) V_{m+n}^{r+s-t}$$

The key point to prove proposition 3.6 is to figure out the composition law of $U(\mathfrak{sl}_2)[\lambda]$ under the basis $\{V_m^r : r \geq 1, |m| < r\}$.

From Proposition 2.17 we see when λ is an integer, $U(\mathfrak{sl}_2)[\lambda]$ is closely related to the matrix algebra, so it is not surprising that we could get some power from the theory of matrix algebras. Indeed, under a new basis introduced by Racah, a composition law of the matrix algebra obtained as an application of the Wigner-Eckart theorem [3] is helpful. It is proved in [11] that one can derive a manifest expression of the composition law between $\{V_m^r\}$ from the Racah composition law for integer N, and one can then obtain the law for any $\lambda \in \mathbb{C}$ by the analytic continuation. To state the manifest expression, we first introduce a few notations.

Notation 3.7 (Clebsch-Gordan coefficient)**.** There are many different notations for Clebsch-Gordan coefficients, here we adapt the convention in [3] and set

$$\begin{aligned} C_{m_1 m_2 m}^{j_1 j_2 j} &:= \delta(m_1 + m_2, m) \\ &\times \left(\frac{(2j+1)(j_1+j_2-j)!(j_1-m_1)!(j_2-m_2)!(j-m)!(j+m)!}{(j_1+j_2+j+1)!(j+j_1-j_2)!(j+j_2-j_1)!(j_1+m_1)!(j_2+m_2)!} \right)^{\frac{1}{2}} \\ &\times \sum_t \frac{(-1)^{j_1-m_1+t}(j_1+m_1+t)!(j+j_2-m_1-t)!}{t!(j-m-t)!(j_1-m_1-t)!(j_2-j+m_1+t)!} \end{aligned}$$

where t goes from $|\min\{j_2 - j + m_1, 0\}|$ to $\min\{j - m, j_1 - m_1, j + j_2 - m_1\}$.

Notation 3.8 (Triangle coefficient)**.** Let (a, b, c) be a triple obeying the triangle conditions. Define the triangle coefficient

$$\Delta(a,b,c) := \sqrt{\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}}.$$

Notation 3.9 (Modified $6j$-symbol)**.** Let $\begin{bmatrix} s & s' & s'' \\ j & j & j \end{bmatrix}$ denote the modified $6j$-

symbol defined in [11] which is explicitly given by

$$\begin{bmatrix} s & s' & s'' \\ j & j & j \end{bmatrix} := s!s'!s''!\Delta(s,s',s'')\sum_t \frac{(-1)^t(2j+1+s''+t)!(2j+1-s''-1)!}{(2j-s-s'+t)!(2j+1+s'')!} \times \frac{1}{t!(t+s''-s)!(t+s''-s')!(s+s'-s''-t)!(s-t)!(s'-t)!}$$

where $\Delta(s,s',s'')$ is the triangle coefficient.

Now according to [11], the composition law of $U(\mathfrak{sl}_2)[\lambda]$ under our chosen basis is given by

$$V^r_m V^s_n = \sum_{s',m'} \frac{(-1)^{r+s+s'}(2s')!(s-1-n)!(r-1-m)!}{(2s-2)!(2r-2)!(s'+m')!} \times f(r-1,s-1,s'|\lambda) C^{(r-1)(s-1)s'}_{(-m)(-n)m'} V^{s'+1}_{-m'} \tag{3.3}$$

where $C^{(r-1)(s-1)s'}_{(-m)(-n)m'}$ are the Clebsh-Gordan coefficients and

$$f(r-1,s-1,s'|\lambda) = \sqrt{(2r-1)(2s-1)} \begin{bmatrix} r & s & s' \\ \frac{\lambda-1}{2} & \frac{\lambda-1}{2} & \frac{\lambda-1}{2} \end{bmatrix}. \tag{3.4}$$

It is possible to directly relate Equation (3.3) to Equation (1.5) to prove Proposition 3.6, since the hypergeometric functions have occurred in the study of Clebsh-Gordan coefficients [20] and the relation between hypergeometric functions and the $6j$-symbol has also been partly revealed in [22]. However, approaching the proof in this way requires a large amount of knowledge in special functions, so we decided to outline a more practical proof here.

Proof Outline of Proposition 3.6. We shall prove it by a double induction, where the primary induction is on r and the secondary induction in on m.

The Equation (3.3) with $r=2, m=-1$ reads

$$V^2_{-1}V^s_n = \sum_{s',m'} \frac{(-1)^{s+s'}(2s')!(s-1-n)!}{(2s-2)!(s'+m')!} f(1,s-1,s'|\lambda) C^{1(s-1)s'}_{1(-n)m'} V^{s'+1}_{-m'} \tag{3.5}$$

Note the delta-function $\delta(1-n,m')$ in the Clebsh-Gordan coefficient forces $m' = 1-n$. And the triangle condition that is secretly required in $f(1,s-1,s'|\lambda)$ implies s' could only be $s, s-1$ or $s-2$. Then it remains to calculate the coefficients of V^{s+1}_{n-1}, V^s_{n-1} and V^{s-1}_{n-1} on the right hand side of Equation (3.5). The calculation is omitted. Up to a normalisation factor, the result should be

$$\begin{aligned} V_{-1}^2 V_n^s =& V_{n-1}^{s+1} - \frac{(s-1+n)}{2} V_{n-1}^s \\ &+ \frac{(s-2+n)(s-1+n)(s-1-\lambda)(s-1+\lambda)}{4(2s-3)(2s-1)} V_{n-1}^{s-1} \\ =& g_1^{2s}(-1,n,\lambda) V_{n-1}^{s+1} + g_2^{2s}(-1,n,\lambda) V_{n-1}^s + g_3^{2s}(-1,n,\lambda) V_{n-1}^{s-1} \end{aligned} \tag{3.6}$$

which agrees with Equation (1.5).

Now suppose for all $2 \leq k \leq r$, it is true that

$$V_{-k+1}^k V_n^s = \sum_{t=1}^{k+s-1} g_t^{ks}(-k+1,n,\lambda) V_{n-k+1}^{k+s-t}.$$

Then to show the formula also holds for $k = r+1$, we note that $V_{-r}^{r+1} = V_{-1}^2 V_{-r+1}^r$ by Equation (3.6), so it amounts to show

$$\begin{aligned} 2g_t^{(r+1)s}(-r,n,\lambda) = & 2g_t^{rs}(1-r,n,\lambda) - (s+n-t+1) g_{t-1}^{rs}(1-r,n,\lambda) \\ &+ \frac{(s+n-t+1)(s+n-t+2)(r+s-t-\lambda+1)(r+s-t+\lambda+1)}{2(2r+2s-2t+1)(2r+2s-2t+3)} \\ &\times g_{t-2}^{rs}(1-r,n,\lambda) \end{aligned}$$

for all t. Recall $g_t^{rs}(m,n,\lambda) = \frac{2}{4^{t-1}(t-1)!} \phi_t^{rs}(\lambda) N_t^{rs}(m,n)$ and note that

$$\begin{aligned} N_t^{(r+1)s}(-r,n) =& -\frac{1}{(2r-t)(s+n-t+1)} N_t^{rs}(1-r,n) \\ N_{t-2}^{rs}(1-r,n) =& \frac{1}{(2r-t)(2r-t+1)(s+n-t+1)(s+n-t+2)} N_t^{rs}(1-r,n), \end{aligned}$$

we see it suffices to show

$$\begin{aligned} 0 =& \left(r - \frac{t}{2} + \frac{1}{2}\right)\left(r - \frac{t}{2}\right)\left(r+s-t+\frac{1}{2}\right)\left(r+s-t+\frac{3}{2}\right)\phi_t^{rs}(\lambda) \\ &+2\left(\frac{t}{2}-\frac{1}{2}\right)\left(r-\frac{t}{2}+\frac{1}{2}\right)\left(r+s-t+\frac{1}{2}\right)\left(r+s-t+\frac{3}{2}\right)\phi_{t-1}^{rs}(\lambda) \\ &-r\left(r-\frac{1}{2}\right)\left(r+s-t+\frac{1}{2}\right)\left(r+s-t+\frac{3}{2}\right)\phi_t^{(r+1)s}(\lambda) \\ &+\left(\frac{t}{2}-\frac{1}{2}\right)\left(\frac{t}{2}-1\right)(r+s-t-\lambda+1)(r+s-t+\lambda+1)\phi_{t-2}^{rs}(\lambda). \end{aligned} \tag{3.7}$$

which can be done by using the following contiguous relations of the hypergeometric functions.[1]

$$0 = b_4F_3\left[\begin{matrix} a, b+1, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ;1\right] - a_4F_3\left[\begin{matrix} a+1, b, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ;1\right] + (a-b)_4F_3\left[\begin{matrix} a, b, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ;1\right]$$

$$0 = c_4F_3\left[\begin{matrix} a, a_2, a_3, a_4 \\ c, b_2, b_3 \end{matrix} ;1\right] - a_4F_3\left[\begin{matrix} a+1, a_2, a_3, a_4 \\ c+1, b_2, b_3 \end{matrix} ;1\right] + (a-c)_4F_3\left[\begin{matrix} a, a_2, a_3, a_4 \\ c+1, b_2, b_3 \end{matrix} ;1\right]$$

$$0 = d_4F_3\left[\begin{matrix} a_1, a_2, a_3, a_4 \\ c+1, d, b_3 \end{matrix} ;1\right] - c_4F_3\left[\begin{matrix} a_1, a_2, a_3, a_4 \\ c, d+1, b_3 \end{matrix} ;1\right] + (a-c)_4F_3\left[\begin{matrix} a_1, a_2, a_3, a_4 \\ c+1, d+1, b_3 \end{matrix} ;1\right]$$

Hence $V^r_{-r+1}V^s_n = \sum_{t=1}^{r+s-1} g^{rs}_t(-r+1, n, \lambda)V^{r+s-t}_{n-k+1}$ is true for all $r \geq 2$. Now as observed in Chapter 1 that $\phi^{rs}_t = \phi^{sr}_t$ and $N^{rs}_t(m,n) = (-1)^{t-1}N^{sr}_t(n,m)$ for all t, we know $g^{rs}_t(-r+1,n,\lambda) = (-1)^{t-1}g^{sr}_t(n,-r+1,\lambda)$ and therefore

$$[V^r_{-r+1}, V^s_n] = \sum_{t=2,\text{ t even}}^{r+s-1} g^{rs}_t(-r+1, n, \lambda)V^{r+s-t}_{-r+1+n}.$$

Suppose for each fixed r, $[V^r_k, V^s_n] = \sum_{t=2,\text{ t even}}^{r+s-1} g^{rs}_t(k,n,\lambda)V^{r+s-t}_{k+n}$ holds for all $-r+1 \leq k < m$. Then to show it also holds for $k = m$, we first note that using the recursive relation $[E, V^r_{m-1}] = (r-m)V^r_m$ read from Equation (2.5) together with the Jacobi identity, we have

$$(r-m)[V^r_m, V^s_n] = [E, [V^r_{m-1}, V^s_n]] - [V^r_{m-1}, [E, V^s_n]],$$

so it suffices to show

$$0 = \frac{r-m+s-t-n}{r-m}N^{rs}_t(m-1,n) - \frac{s-1-n}{r-m}N^{rs}_t(m-1,n+1) - N^{rs}_t(m,n) \tag{3.8}$$

for all t even, with noting that other factors in $g^{rs}(m,n,\lambda)$ are all independent on m and n. This may be done by rewriting the function $N^{rs}_t(m,n)$ in terms of hypergeometric functions in the type of ${}_3F_2$ and then again using some contiguous relations. Alternatively, we are also able to prove it by fairly straightforward calculation (see Appendix C.1). Therefore the commutation relation

$$[V^r_m, V^s_n] = \sum_{t=2,\text{ t even}}^{r+s-1} g^{rs}_t(m,n,\lambda)V^{r+s-t}_{m+n}$$

holds for all required values of r, m, s and n. □

Theorem 3.10. *For all* $\lambda \in \mathbb{C}$, $hs[\lambda] \cong [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]]$.

Proof. This follows from Corollary 3.5 and Proposition 3.6. □

With Theorem 3.10 in hands, now as promised in Chapter 1, we are able to prove that hs$[\lambda]$ is not simple when $\lambda \in \mathbb{Z}\backslash\{0, \pm 1\}$ (Proposition 1.4) using the algebraic property of $U(\mathfrak{sl}_2)[\lambda]$. To some extent, this proof suggests that we can actually get benefits from the algebraic aspect of hs$[\lambda]$.

Alternative proof of Proposition 1.4. If $\lambda = N \in \mathbb{Z}\backslash\{0, \pm 1\}$, then Corollary 2.22 can be applied and we have $\text{hs}[N]/\ker(\tilde{\pi}) \cong \mathfrak{sl}_N$. Note since hs$[N]$ is infinite dimensional while $\mathfrak{sl}_N$ is finite dimensional, it is clear that $\text{hs}[N] \ncong \mathfrak{sl}_N$ and therefore $\ker(\tilde{\pi})$ is ensured to be non-trivial. Also $\ker(\tilde{\pi})$ cannot be the entire hs$[N]$ because $\mathfrak{sl}_N$ is non-trivial. Hence $\ker(\tilde{\pi})$ is a non-trivial proper Lie ideal of hs$[N]$. □

Chapter 4

An Analogue of hs$[\lambda]$ Constructed from $U(\mathfrak{sl}_2 \ltimes V_2)$

So far we have seen that $U(\mathfrak{sl}_2)[\lambda] = \mathrm{hs}[\lambda] \oplus \mathbb{C}$ as an $\mathfrak{sl}_2$-module can be decomposed into the direct sum of all irreducible $\mathfrak{sl}_2$-modules of odd dimension (see Remark 1.7, Remark 2.3 and Proposition 3.2), that is,

$$\mathrm{hs}[\lambda] \oplus \mathbb{C} \cong \bigoplus_{n=1}^{\infty} V_{2n-1}$$

where V_{2n-1} is the $(2n-1)$-dimensional irreducible $\mathfrak{sl}_2$-module. Then a natural question would be: Can we construct an analogue of hs$[\lambda]$ that can be decomposed into the direct sum of all irreducible $\mathfrak{sl}_2$-modules, including the ones of even dimension? That is, we attempt to construct an algebra, hereafter denoted as $\widetilde{\mathrm{hs}[\lambda]}$, such that

$$\widetilde{\mathrm{hs}[\lambda]} \oplus \mathbb{C} \cong \bigoplus_{n=1}^{\infty} V_n.$$

To this goal, we shall first introduce a family of Lie algebras affiliated to $\mathfrak{sl}_2$.

Notation 4.1 ($\mathfrak{sl}_2 \ltimes V_m$)**.** Let V_m be the m-dimensional irreducible $\mathfrak{sl}_2$-module, viewed as an abelian Lie algebra. Then denote the semi-direct product of $\mathfrak{sl}_2$ and V_m by $\mathfrak{sl}_2 \ltimes V_m$. Choose a basis of V_m, say $\{X_0, X_1, \ldots, X_{m-1}\}$, then we have an ordered basis $\{F, H, E, X_0, X_1, \ldots, X_{m-1}\}$ for $\mathfrak{sl}_2 \ltimes V_m$. Explicitly, the commutation relations of $\mathfrak{sl}_2 \ltimes V_m$ are given by

$$\begin{aligned} &[E, F] = H, && [H, E] = 2E, && [H, F] = -2F, \\ &[H, X_n] = (m - 2n - 1)X_n, && [E, X_n] = n(m - n)X_{n-1}, && [F, X_n] = X_{n+1}. \end{aligned}$$

Example 4.2. In this chapter, we are particularly interested in $\mathfrak{sl}_2 \ltimes V_2$, which has an ordered basis $\{F, H, E, X_0, X_1\}$ for $\mathfrak{sl}_2 \ltimes V_2$ and commutation relations

$$\begin{aligned} &[E,F]=H, && [H,E]=2E, && [H,F]=-2F,\\ &[E,X_0]=0, && [H,X_0]=X_0, && [F,X_0]=X_1,\\ &[E,X_1]=X_0, && [H,X_1]=-X_1, && [F,X_1]=0. \end{aligned}$$

Let us first investigate the structure of $U(\mathfrak{sl}_2 \ltimes V_2)$ as we did for $U(\mathfrak{sl}_2)$ in Chapter 2. As promised before, some methods developed in Chapter 2 and 3 can be inherited here.

Example 4.3. A set of commutation relations can be obtained with the help of Lemma 2.5. We shall list some useful ones here for later reference.

$$\begin{aligned} &[X_1,H^j]=\sum_{n=1}^{j}\binom{j}{n}H^{j-n}X_1, && [X_0,H^j]=\sum_{n=1}^{j}(-1)^n\binom{j}{n}H^{j-n}X_0\\ &[X_1,E^k]=-kE^{k-1}X_0, && [X_0,F^i]=-iF^{i-1}X_1,\\ &[F,X_0^i]=iX_0^{i-1}X_1, && [E,X_1^j]=jX_0X_1^{j-1},\\ &[H,X_0^i]=iX_0^i, && [H,X_1^j]=-jX_1^j. \end{aligned}$$

Lemma 4.4 (Centre of $U(\mathfrak{sl}_2 \ltimes V_2)$)**.** *The centre of $U(\mathfrak{sl}_2 \ltimes V_2)$ is the polynomial ring generated by $Z := FX_0^2 - HX_0X_1 - EX_1^2$, that is,*

$$Z(U(\mathfrak{sl}_2 \ltimes V_2)) = \mathbb{C}[Z].$$

Proof. It is easy to check $\mathbb{C}[Z] \subset Z(U(\mathfrak{sl}_2 \ltimes V_2))$. To show the converse inclusion, similar to the proof of Lemma 2.11, consider the localisation of $U(\mathfrak{sl}_2 \ltimes V_2)$ at X_1 defined as

$$\widetilde{U}(\mathfrak{sl}_2 \ltimes V_2) := \frac{U(\mathfrak{sl}_2 \ltimes V_2)[X_1^{-1}]}{\langle X_1X_1^{-1}-1, X_1^{-1}X_1-1\rangle}$$

and denote $X_1^{-n} := \left(X_1^{-1}\right)^n$. Note in $\widetilde{U}(\mathfrak{sl}_2 \ltimes V_2)$ we have

$$E = \left(FX_0^2 - HX_0X_1 - Z\right)X_1^{-2}.$$

With the similar reasoning as in the proof of Lemma 2.11, again it suffices to determine $Z(\widetilde{U}(\mathfrak{sl}_2 \ltimes V_2))$.

Expand a generic element $z \in Z(\widetilde{U}(\mathfrak{sl}_2 \ltimes V_2))$ as

$$z = \sum_{j\in\mathbb{N}} H^j a_j(F,Z,X_0,X_1)$$

where $a_j(F,Z,X_0,X_1) = \sum_{i,k,m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} a_{ijkmn}F^iZ^kX_0^mX_1^n$ with finitely many non-zero $a_{ijkmn} \in \mathbb{C}$. Then

$$0 = [X_1,z] = \sum_{j\in\mathbb{N}}\sum_{n=1}^{j}\binom{j}{n}H^{j-n}X_1a_j(F,Z,X_0,X_1)$$

implies $a_j(F, Z, X_0, X_1) = 0$ unless $j = 0$, so z can be written as

$$z = \sum_{m\in\mathbb{N}} X_0^m b_m(F, Z, X_1)$$

where $b_m(F, Z, X_1) = \sum_{i,k\in\mathbb{N}} \sum_{n\in\mathbb{Z}} b_{ikmn} F^i Z^k X_1^n$ with finitely many non-zero $b_{ikmn} \in \mathbb{C}$. Now

$$0 = [F, z] = \sum_{m\in\mathbb{N}} m X_0^{m-1} X_1 b_m(F, Z, X_1)$$

implies $b_m(F, Z, X_1) = 0$ unless $m = 0$. Therefore z is in the form of

$$z = \sum_{i\in\mathbb{N}} F^i c_i(Z, X_1)$$

where $c_i(Z, X_1) = \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{Z}} c_{ikn} Z^k X_1^n$ with finitely many non-zero $c_{ikn} \in \mathbb{C}$.

$$0 = [X_0, z] = -\sum_{i\in\mathbb{N}} i F^{i-1} X_1 c_i(Z, X_1)$$

then implies $c_i(Z, X_1) = 0$ unless $i = 0$, so

$$z = \sum_{n\in\mathbb{Z}} X_1^n d_n(Z)$$

where $d_n(Z) = \sum_{k\in\mathbb{N}} d_{nk} Z^k$. Finally

$$0 = [E, z] = \sum_{n\in\mathbb{Z}} n X_0 X_1^{n-1} d_n(Z)$$

implies $d_n(Z) = 0$ unless $n = 0$. Hence

$$z = \sum_{k\in\mathbb{N}} p_k Z^k$$

with finitely many non-zero $p_k \in \mathbb{C}$. □

One would also like to have a result that is analogous to Proposition 2.17. Unfortunately, the classification of $U(\mathfrak{sl}_2 \ltimes V_2)$-modules is not as simple as that of $U(\mathfrak{sl}_2)$, so it is almost impossible to rely on a general theorem such as the Burnside's theorem.[2] This is the place that our constructive proof of 2.17 could be motivating.

Consider the space of polynomials of two variables, say x_0 and x_1, that have degree less than $m \in \{1, 2, \ldots, \}$, denoted as $W_m := \mathbb{C}[x_0, x_1]_m$. Then one can check that W_m is a $(\mathfrak{sl}_2 \ltimes V_2)$-module with the representation homomorphism $\tau : \mathfrak{sl}_2 \ltimes V_2 \to W_m$ determined by

$$\tau(H) = x_0\frac{\partial}{\partial x_0} - x_1\frac{\partial}{\partial x_1}, \qquad \tau(E) = x_0\frac{\partial}{\partial x_1}, \qquad \tau(F) = x_1\frac{\partial}{\partial x_0},$$
$$\tau(X_0) = \frac{\partial}{\partial x_1}, \qquad \tau(X_1) = -\frac{\partial}{\partial x_0}.$$

And clearly (W_m, τ) can be extended to a $U(\mathfrak{sl}_2 \ltimes V_2)$-module. Fortunately, even though these modules are not irreducible, it turns out that we indeed have a result parallel to Proposition 2.17.

Proposition 4.5. *Consider the $U(\mathfrak{sl}_2 \ltimes V_2)$-module W_m defined above. Then*

$$U(\mathfrak{sl}_2 \ltimes V_2)/\ker(\tau) \cong \mathfrak{p}_{1,2,\ldots,m}$$

where $\mathfrak{p}_{d_1,\ldots,d_n}$ denotes the parabolic subalgebra of $\mathfrak{gl}_{d_1+\cdots+d_n}$ with blocks of dimension $d_1, \ldots, d_n$.

Before proving the proposition, we first need to fix an ordering of the monomials in W_m. For convenience, we choose the graded reverse lexicographic ordering here.[19]

Definition 4.6 (Graded reverse lexicographic ordering)**.** Given a polynomial ring $R[x_0, x_1, \ldots, x_n]$, where R is a ring. Let $t = x_0^{e_0} \cdots x_n^{e_n}$ and $t' = x_0^{e_0'} \cdots x_n^{e_n'}$ be monomials in $R[x_0, x_1, \ldots, x_n]$. And let $deg(t) := \sum_{i=0}^{n} e_i$. The graded reverse lexicographic ordering, also known as grevlex, is given by saying that $t \leq t'$ if $t = t'$ or $deg(t) < deg(t')$, or $deg(t) = deg(t')$ and $e_i > e_i'$ for the largest index i with $e_i \neq e_i'$.

Example 4.7. For W_3, the grevlex ordering gives $1 \leq x_1 \leq x_0 \leq x_1^2 \leq x_0x_1 \leq x_0^2$, which suggests we may choose $\{1, x_1, x_0, x_1^2, x_0x_1, x_0^2\}$ to be an ordered basis of W_3.

Proof of Proposition 4.5. Choose an ordered basis of W_m to be the set of monomials with graded reverse lexicographic ordering. By the first isomorphism theorem, $U(\mathfrak{sl}_2 \ltimes V_2)/\ker(\tau) \cong \operatorname{im}(\tau)$, so it suffices to show $\operatorname{im}(\tau) = \mathfrak{p}_{d_1,\ldots,d_n}$ (with respect to the chosen basis of W_m).

From the explicit homomorphism given above, we see none of the generators of $U(\mathfrak{sl}_2 \ltimes V_2)$ could increase the degree of polynomials in W_m, so $\operatorname{im}(\tau) \subset \mathfrak{p}_{1,2,\ldots,m}$. To show the converse inclusion, we may construct all basis operators of $\mathfrak{p}_{1,2,\ldots,m}$ as we did in the constructive proof of Proposition 2.17.

By straightforward calculation we have

$$C \cdot x_0^i x_1^j = (i+j)(i+j+2) x_0^i x_1^j,$$
$$H \cdot x_0^i x_1^j = (i-j) x_0^i x_1^j.$$

Let $A_{(p,q)}^{(i,j)} \in U(\mathfrak{sl}_2 \ltimes V_2)$ with $p+q \leq i+j$ be an element such that $\tau\left(A_{(p,q)}^{(i,j)}\right)$ satisfies

$$\tau\left(A_{(p,q)}^{(i,j)}\right) x_0^i x_1^j = x_0^p x_1^q,$$
$$\tau\left(A_{(p,q)}^{(i,j)}\right) x_0^k x_1^l = 0, \text{ for all } (k,l) \in \{(k,l) : k \neq i \text{ or } l \neq j\}.$$

For the special case $p = i, q = j$, up to a normalisation factor, we can take

$$A_{(i,j)}^{(i,j)} := \prod_{\substack{k+l=i+j \\ k-l \neq i-j}} (H - k + l) \prod_{\substack{t<m \\ t \neq i+j}} (C - t(t+2)).$$

When $p \leq i, q \leq j$, up to a normalisation factor, we can take

$$A_{(p,q)}^{(i,j)} := X_0^{j-q} X_1^{i-p} A_{(i,j)}^{(i,j)}.$$

When $p < i, q > j$, up to a normalisation factor, we can take

$$A_{(p,q)}^{(i,j)} := F^{q-j} X_1^{i+j-p-q} A_{(i,j)}^{(i,j)}.$$

When $p > i, q < j$, up to a normalisation factor, we can take

$$A_{(p,q)}^{(i,j)} := E^{p-i} X_0^{i+j-p-q} A_{(i,j)}^{(i,j)}.$$

Now the corresponding equivalence classes of $A_{(p,q)}^{(i,j)}$ in $U(\mathfrak{sl}_2 \ltimes V_2)/\ker(\tau)$ forms a basis of $\mathfrak{p}_{1,2\ldots,m}$. □

In Chapter 2 and 3, one of the reason we were interested in the ideals $\langle C - \mu \rangle$ with $\mu \in \mathbb{C}$ (where $C = 4FE + HH + 2H$ is the quadratic Casimir) is that they are the maximal ideals of $U(\mathfrak{sl}_2)$. In fact, it is shown in [2] that they are also maximal in $U(\mathfrak{sl}_2 \ltimes V_2)$. This suggests that we might be able to construct $\widetilde{\mathrm{hs}[\lambda]}$ by considering the quotient algebra

$$U(\mathfrak{sl}_2 \ltimes V_2)[\lambda] := \frac{U(\mathfrak{sl}_2 \ltimes V_2)}{\langle C - (\lambda^2 - 1) \rangle}.$$

We should mention that the two-sided ideal $\langle C - (\lambda^2 - 1) \rangle$ contains the Lie ideal of $U(\mathfrak{sl}_2 \ltimes V_2)$ generated by C. Unlike the situation for $U(\mathfrak{sl}_2)$ where C is a central element, now the Lie ideal $(C)_{\mathfrak{sl}_2 \ltimes V_2}$ considerably enriches the structure of $\langle C - (\lambda^2 - 1)\rangle$. It is not completely obvious now what kind of elements are contained in $\langle C - (\lambda^2 - 1)\rangle$. In the following discussion, we will find a fairly illustrating generating set of $\langle C - (\lambda^2 - 1)\rangle$ to help us understand the structure of $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$.

Lemma 4.8. *The set*

$$\{C^k V_m^r X_0^i X_1^j : i, j, k \geq 0, r \geq 1, |m| < r\}$$

is a basis of $U(\mathfrak{sl}_2 \ltimes V_2)$.

Proof. This follows from Proposition 3.2 and PBW Theorem 2.4. □

Proposition 4.9. *With the notation above, the set consists of all elements in the following forms:*

$$C^{k+1} V_m^r X_0^i X_1^j,\ V_m^r X_0^{i+2} X_1^j,\ V_m^r X_0 X_1^{j+1},\ V_m^r X_1^{j+2},\ V_m^r [C, X_0],\ V_m^r [C, X_1],$$

where $k \geq 0, r \geq 1, m < |r|, i, j \geq 0$, *is a generating set of* $\langle C\rangle$.

Proof. Denote the set given above by S_C from now on. We first show all the elements in S_C are contained in $\langle C\rangle$.

It is clear that $C^{k+1} V_m^r X_0^i X_1^j \in \langle C\rangle$ for all required k, r, m, i, j. Also note $V_m^r [C, X_0] = [C, V_m^r X_0]$ and $V_m^r [C, X_1] = [C, V_m^r X_1]$ are in $(C)_{\mathfrak{sl}_2 \ltimes V_2} \subset \langle C\rangle$ for all required r, m. Now note that

$$(C)_{\mathfrak{sl}_2 \ltimes V_2} \ni [[C, X_0], X_0] = X_0^2,$$

which implies $X_0 X_1, X_1^2$ are also in $(C)_{\mathfrak{sl}_2 \ltimes V_2}$ by commuting with F and therefore $V_m^r X_0^{i+2} X_1^j$, $V_m^r X_0 X_1^{j+1}$, $V_m^r X_1^{j+2}$ are contained in $\langle C\rangle$ for all required r, m, i, j.

To show $\langle C\rangle \subset \mathrm{Span}(S_C)$, note that by Lemma 4.8 we know

$$\{C^{k'} V_{m'}^{r'} X_0^{i'} X_1^{j'} C^{k+1} V_m^r X_0^i X_1^j : k, k', i, i', j, j' \geq 0, r, r' \geq 1, |m| < r, |m'| < r'\}.$$

is a generating set of $\langle C\rangle$. We now try to rewrite each generator above in terms of the elements in S_C.

If $i' = j' = 0$, then by Proposition 3.2 we know

$$C^{k'} V_{m'}^{r'} C^{k+1} V_m^r X_0^i X_1^j = C^{k'+k+1} V_{m'}^{r'} V_m^r X_0^i X_1^j \in \mathrm{Span}\{C^{k+1} V_m^r X_0^i X_1^j\} \subset \mathrm{Span}(S_C)$$

If $i'+j'\geq 2$, then since $\mathrm{Span}\{C^kV_m^rX_0^{i+2}X_1^j,\ C^kV_m^rX_0X_1^{j+1},\ C^kV_m^rX_1^{j+2}\}$ is a Lie ideal of $U(\mathfrak{sl}_2\ltimes V_2)$ as one can check, we see

$$\begin{aligned}&C^{k'}V_{m'}^{r'}X_0^{i'}X_1^{j'}C^{k+1}V_m^rX_0^iX_1^j\\ =&C^{k'+k+1}V_{m'}^{r'}V_m^rX_0^{i'+i}X_1^{j'+j}+C^{k'}V_{m'}^{r'}[X_0^{i'}X_1^{j'},C^{k+1}V_m^r]X_0^iX_1^j\\ \in&\mathrm{Span}\{C^{k+1}V_m^rX_0^iX_1^j,V_m^rX_0^{i+2}X_1^j,V_m^rX_0X_1^{j+1},V_m^rX_1^{j+2}\}\subset\mathrm{Span}(S_C)\end{aligned}\tag{4.1}$$

If $i'+j'=1, k'\geq 1$, then from Equation (4.1) and Lemma 4.8 we can conclude

$$C^{k'}V_{m'}^{r'}X_0^{i'}X_1^{j'}C^{k+1}V_m^rX_0^iX_1^j\in\mathrm{Span}\{C^{k+1}V_m^rX_0^iX_1^j\}\subset\mathrm{Span}(S_C)$$

If $i'+j'=1$ and $k'=0$, we need to look more closer at $[X_n,C^{k+1}V_m^r]$ where $n\in\{0,1\}$. Without loss of generality we may assume $k=0$, then

$$[X_n,CV_m^r]=-V_m^r[C,X_n]-[[V_m^r,X_n],C]+C[X_n,V_m^r]$$

where we used the Jacobi identity.

One might be able to easily observe that $[V_m^r,X_n]$ could be written in the form of $aV_{m'}^{r-1}X_0+bV_{m'+1}^{r-1}X_1$ for some $a,b\in\mathbb{C},|m'|<r-1$. But here we decide to present some careful calculation for later reference.

By recursively using the $[F,V_m^r]=(m+r-1)V_{m-1}^r$, we have

$$\begin{aligned}[V_m^r,X_1]&=\frac{1}{(m+r)_{r-1-m}}(\mathrm{ad}F)^{r-1-m}([V_{r-1}^r,X_1])\\ &=\frac{r-1}{(m+r)_{r-1-m}}(\mathrm{ad}F)^{r-2-(m-1)}(E^{r-2}X_0)\end{aligned}$$

where $(m+r)_{r-1-m}$ is the Pochhammer symbol. And by Lemma 2.8 we have

$$\begin{aligned}&(\mathrm{ad}F)^{r-2-m}(E^{r-2}X_0)\\ =&(\mathrm{ad}F)^{r-2-m}(E^{r-2})X_0+(r-2-m)(\mathrm{ad}F)^{r-3-m}(E^{r-2})[F,X_0]\\ =&\frac{(2r-4)!}{(m+r-1)!}\left((m+r-1)V_m^{r-1}X_0+(r-2-m)V_{m+1}^{r-1}X_1\right),\end{aligned}\tag{4.2}$$

and therefore we see

$$[V_m^r,X_1]=\frac{(m+r-1)}{2(2r-3)}\left((m+r-2)V_{m-1}^{r-1}X_0+(r-1-m)V_m^{r-1}X_1\right)\tag{4.3}$$

$$\begin{aligned}[V_m^r,X_0]=&\frac{1}{(m+r)_{r-1-m}}(\mathrm{ad}F)^{r-1-m}([V_{r-1}^r,X_0])-\frac{(r-1-m)}{2(2r-3)}[V_{m+1}^r,X_1]\\ =&-\frac{(r-1-m)}{2(2r-3)}\left((m+r-1)V_m^{r-1}X_0+(r-2-m)V_{m+1}^{r-1}X_1\right)\end{aligned}\tag{4.4}$$

And using the formula given in Equation (1.5) one could easily compute

$$\begin{aligned} [C, X_0] &= -4V_0^2 X_0 + 4V_1^2 X_1 - 3X_0 \\ [C, X_1] &= -4V_{-1}^2 X_0 + 4V_0^2 X_1 - 3X_1. \end{aligned} \tag{4.5}$$

Now combining all the results given in Equation (4.4), (4.3), (4.5) and using Proposition 3.2, we see if $i + j \geq 1$, then

$$\begin{aligned} &V_{m'}^{r'} X_0^{i'} X_1^{j'} C^{k+1} V_m^r X_0^i X_1^j \\ \in &\mathrm{Span}\{C^{k+1} V_m^r X_0^i X_1^j, V_m^r X_0^{i+2} X_1^j, V_m^r X_0 X_1^{j+1}, V_m^r X_1^{j+2}\} \subset \mathrm{Span}(S_C), \end{aligned}$$

and if $i = j = 0$, then

$$V_{m'}^{r'} X_0^{i'} X_1^{j'} C^{k+1} V_m^r \in \mathrm{Span}\{C^{k+1} V_m^r X_0^i X_1^j, V_m^r [C, X_0], V_m^r [C, X_1]\} \subset \mathrm{Span}(S_C).$$

□

From Proposition 4.9 analysed above, it should be recognised that

$$U(\mathfrak{sl}_2 \ltimes V_2) - \langle C - (\lambda^2 - 1) \rangle \subset \mathrm{Span}\{V_m^r, V_m^r X_0, V_m^r X_1 : r \geq 1, |m| < r\}.$$

As remarked at the start of this chapter, we know for each r, $\mathrm{Span}\{V_m^r : |m| < r\}$ is a $(2r-1)$-dimensional irreducible $\mathfrak{sl}_2$-module, which we denote by V_{2r-1}. Now we also notice that

$$\mathrm{Span}\{V_m^r, V_m^r X_0, V_m^r X_1 : r \geq 1, |m| < r\} = \bigoplus_{r=1}^{\infty} \left(V_{2r-1} \oplus (V_{2r-1} \otimes V_2)\right).$$

It is known that $V_{2r-1} \otimes V_2$ can be decomposed as $V_{2r-2} \oplus V_{2r}$. This can be shown via some machinery tools of the representation theory of $\mathfrak{sl}_2$ such as the Young tableau.[12] We shall include a constructive proof here with giving the highest weight vectors for later use.

Lemma 4.10. *Let $V_{2r-1} := Span\{V_m^r : |m| < r\}$ and $V_2 := Span\{X_0, X_1\}$. Then*

$$V_{2r-1} \otimes V_2 = V_{2r-2} \oplus V_{2r}.$$

Moreover, up to a normalisation factor, the highest weight vector of V_{2r} is $V_{r-1}^r X_0$ and for $r \geq 2$ the highest weight vector of V_{2r-2} is $V_{r-2}^r X_0 - V_{r-1}^r X_1$.

Proof. Let $aV_m^r X_0 + bV_{m+1}^r X_1$ be an element in $V_{2r-1} \otimes V_2$, where $a, b \in \mathbb{C}$. For this element to be a highest weight vector, we must have

$$\begin{aligned} 0 = [E, aV_m^r X_0 + bV_{m+1}^r X_1] =& a[E, V_m^r] X_0 + b[E, V_{m+1}^r] X_1 + bV_{m+1}^r [E, X_1] \\ =& \left(a(r-1-m) + b\right) V_{m+1}^r X_0 + b(r-2-m) V_{m+2}^r X_1 \end{aligned}$$

which requires $a(r-1-m)+b=0$ and either $b=0$ or $m=r-2$. If $b=0$, then $m=r-1$, so $V^r_{r-1}X_0$ is the highest weight vector of V_{2r}. If $b\neq 0$ then $m=r-2$ and $a+b=0$, so $V^r_{r-2}X_0 - V^r_{r-1}X_1$ is the highest weight vector of V_{2r-2}.

To show the independence, let us first explore all basis vectors of V_{2r} and V_{2r-2}. Recall Equation (4.2), we see the basis vectors of V_{2r} are

$$(\mathrm{ad}F)^{r-1-m}(V^r_{r-1}X_0) = \frac{(2r-2)!}{(m+r)!}\left((m+r)V^r_m X_0 + (r-1-m)V^r_{m+1}X_1\right) \quad (4.6)$$

where $-r\le m\le r-1$. And using $[F, V^r_m] = (m+r-1)V^r_{m-1}$, it is not hard to find the basis vectors of V_{2r-2} which are given by

$$(\mathrm{ad}F)^{r-1-m}\left(V^r_{r-2}X_0 - V^r_{r-1}X_1\right) = [2r-3]_{r-1-m}\left(V^r_{m-1}X_0 - V^r_m X_1\right) \quad (4.7)$$

where $-r+2\le m\le r-1$ and $[2r-3]_{r-1-m}$ is the Pochhammer symbol. Now by matching the weights, it suffices to check $(\mathrm{ad}F)^{r-1-m}\left(V^r_{r-2}X_0 - V^r_{r-1}X_1\right)$ and $(\mathrm{ad}F)^{r-m}(V^r_{r-1}X_0)$ are independent for $-r+2\le m\le r-1$. Suppose

$$0 = c\left((m+r-1)V^r_{m-1}X_0 + (r-m)V^r_m X_1\right) + d\left(V^r_{m-1}X_0 - V^r_m X_1\right),$$

then $c(m+r-1)+d = c(r-m)-d = 0$, which implies $c(2r-1)=0$. This requires $c=0$ since r is an integer. But then $d=c(r-m)=0$. This shows the sum of $V_{2r-2}+V_{2r}$ is direct. And then the result follows by counting the dimension of both sides. □

Now we have found that $\mathrm{Span}\{V^r_m, V^r_m X_0, V^r_m X_1 : r\ge 1, |m|<r\}$ contains each odd-dimensional irreducible $\mathfrak{sl}_2$-module once and each non-trivial even dimensional irreducible $\mathfrak{sl}_2$-module twice. From the explicit analysis of the generating set S_C of $\langle C-(\lambda^2-1)\rangle$, we know that for any $r\ge 1$ neither $V^{r-1}_{r-2}X_0$ nor $V^r_{r-2}X_0 - V^r_{r-1}X_1$ is contained in $\langle C-(\lambda^2-1)\rangle$, so the $2r$-dimensional $\mathfrak{sl}_2$-modules are not trivial in $U(\mathfrak{sl}_2\ltimes V_2)[\lambda]$.

Remember our intention is to construct $\widetilde{\mathrm{hs}[\lambda]}$ that contains each non-trivial irreducible $\mathfrak{sl}_2$-module once, so it would be good if we can get rid of one copy of V_{2r} for each $r\ge 1$. Fortunately, $U(\mathfrak{sl}_2\ltimes V_2)[\lambda]$ indeed accomplishes this goal in an elegant manner.

Notice that the element $V^r_{r-1}[C, X_0]\in\langle C-(\lambda^2-1)\rangle$ expands to

$$V^{r-1}_{r-2}[C, X_0] = -4V^r_{r-2}X_0 + 4V^r_{r-1}X_1 - (2r-1)V^{r-1}_{r-2}X_0$$

according to Equation (4.5) and (1.5). Together with Lemma 4.10, we see for each $r\ge 1$ the highest weight vector $V^{r-1}_{r-2}X_0$ of $V_{2r}\subset V_{2r-1}\otimes V_2$ is identified with the highest weight vector $V^r_{r-2}X_0 - V^r_{r-1}X_1$ of $V_{2r} \subset V_{2r+1} \otimes V_2$ up to a normalisation factor. By the nature of the $\mathfrak{sl}_2$-modules, we can then conclude that the two $2r$-dimensional $\mathfrak{sl}_2$-modules are identified as the same in $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$ (see Appendix C.2 for calculations). Hereafter we shall denote the highest weight vector of this $V_{2r} \subset U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$ by U^r_{r-1} and define $U^r_m := \frac{(r+m-2)!}{(2r-3)!}(\mathrm{ad}F)^{r-1-m}(U^r_{r-1})$ for $-r \leq m \leq r-1$.

It remains to check that in $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$ the sums between even-dimensional $\mathfrak{sl}_2$-modules are still direct, that is, we want to show

$$\sum_{i=0}^{\min\{m+r,\, r-1-m\}} c_i U^{r-i}_m = 0 \text{ only if } c_i = 0 \text{ for all } i.$$

Suppose $\sum_{i=0}^{\min\{m+r,\, r-1-m\}} c_i U^{r-i}_m = 0$ and there exists $c_i \neq 0$. Consider the set $I = \{i : c_i \neq 0\}$ and let j be the largest element in I, then

$$0 = (\mathrm{ad}E)^{r-j-1-m}\left(\sum_{i=0}^{\min\{m+r,\, r-1-m\}} c_i U^{r-i}_m\right) = c\, U^{r-j}_{r-j-1}$$

where the coefficient $c \in \mathbb{C}$ is non-zero. But this implies $U^{r-j}_{r-j-1} = 0$, which is impossible.

Although we have not yet found a manifest expression of the structure constants for $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$ analogous to the one given in Equation (1.1) (see Appendix C.3 for some progress), from Equation (4.3) and (4.4) we can at least deduce that $[U^r_m, U^s_n] = 0$ for any r, s, m, n and

$$[V^r_m, U^s_n] \in \mathrm{Span}\{U^r_m : r \geq 1, -r \leq m \leq r-1\}.$$

This implies

$$[U(\mathfrak{sl}_2 \ltimes V_2)[\lambda], U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]] \cap \mathbb{C} = \{0\}.$$

Hence we may let $\widetilde{\mathrm{hs}[\lambda]} := [U(\mathfrak{sl}_2 \ltimes V_2)[\lambda], U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]]$. And finally we are ready to state the new result.

Theorem 4.11. *The Lie algebra $\widetilde{hs[\lambda]}$ with a basis*

$$\{V^r_m : r \geq 2, |m| < r\} \cup \{U^s_n :, s \geq 1, -s \leq n \leq s-1\}$$

where

$$V^r_m := \frac{(r+m-1)!}{(2r-2)!}(adF)^{r-1-m}(E^{r-1})$$

and

$$U_n^s := \frac{(s+n-2)!}{(2s-3)!}(adF)^{s-1-n}(V_{s-2}^s X_0 - V_{s-1}^s X_1)$$

decomposes, as a $\mathfrak{sl}_2$*-module, into the direct sum of irreducible* $\mathfrak{sl}_2$*-modules of dimension larger than* 1*. That is,*

$$\widetilde{hs[\lambda]} = \bigoplus_{k=2}^{\infty} V_k$$

where for k *odd,* V_k *is the* k*-dimensional* $\mathfrak{sl}_2$*-module generated by* $V_{(k-1)/2}^{(k+1)/2}$ *and for* k *even,* V_k *is the* k*-dimensional* $\mathfrak{sl}_2$*-module generated by* $U_{k/2-1}^{k/2}$*.*

Chapter 5

Centralisers in $U(\mathfrak{sl}_2 \ltimes V_m)$

The success on constructing an interesting algebra from $U(\mathfrak{sl}_2 \ltimes V_2)$ motivates us to explore more on the structure of $U(\mathfrak{sl}_2 \ltimes V_m)$ for generic m. We decided to start with investigating the centre of $U(\mathfrak{sl}_2 \ltimes V_m)$, which is one of the most natural Lie ideals.

5.1 Centre of $U(\mathfrak{sl}_2 \ltimes V_m)$

The centres of $U(\mathfrak{sl}_2 \ltimes V_1) \cong U(\mathfrak{sl}_2)$ and $U(\mathfrak{sl}_2 \ltimes V_2)$ have been found in Lemmas 2.11 and 4.4 respectively. Another studied case is for $m = 3$. And in fact the proof given in [4] is the prototype of our proofs for $m = 1, 2$, so here we only present the result and briefly sketch the proof.

Lemma 5.1. *Let* $Z_1 := 2X_0X_2 - X_1^2$, $Z_2 := EX_2 + HX_1 - 2FX_0$. *Then*

$$Z(U(\mathfrak{sl}_2 \ltimes V_3)) = \mathbb{C}[Z_1, Z_2].$$

Proof outline. Consider a localisation of $U(\mathfrak{sl}_2 \ltimes V_3)$ at X_2, which we hereafter denote by $\widetilde{U}(\mathfrak{sl}_2 \ltimes V_3)$, where we can make sense of X_2^{-1} so that $X_0 = \frac{1}{2}\left(Z_1 + X_1^2\right) X_2^{-1}$ and $E = (Z_2 + 2FX_0 - HX_1)X_2^{-1}$. Then a basis of $\widetilde{U}(\mathfrak{sl}_2 \ltimes V_3)$ is

$$\{F^iH^jX_1^kX_2^lZ_1^mZ_2^n : i, j, k, m, n \in \mathbb{N}, l \in \mathbb{Z}\}.$$

Now for a generic central element $z \in \widetilde{U}(\mathfrak{sl}_2 \ltimes V_3)$, using $[X_2, z] = 0$ and X_2 commutes with every basis element except for those with $j \geq 1$, we can deduce z could only be a linear combination of $\{F^iX_1^kX_2^lZ_1^mZ_2^n : i, k, m, n \in \mathbb{N}, l \in \mathbb{Z}\}$. And then $[F, z] = 0$ implies $z \in \mathrm{Span}\{F^iX_2^lZ_1^mZ_2^n : i, m, n \in \mathbb{N}, l \in \mathbb{Z}\}$. Now $[X_1, z] = 0$ gives $z \in \mathrm{Span}\{X_2^lZ_1^mZ_2^n : m, n \in \mathbb{N}, l \in \mathbb{Z}\}$ and finally $[H, z] = 0$

allows us to conclude $z \in \mathrm{Span}\{Z_1^m Z_2^n : m, n \in \mathbb{N}\} = \mathbb{C}[Z_1, Z_2]$. Then the result follows from $Z(\widetilde{U}(\mathfrak{sl}_2 \ltimes V_3)) \cap U(\mathfrak{sl}_2 \ltimes V_3) = Z(U(\mathfrak{sl}_2 \ltimes V_3))$. □

The idea behind this proof is simple and beautiful, but it is based on knowing some central elements in advance. However, for larger m, to find some central elements of $U(\mathfrak{sl}_2 \ltimes V_m)$ is already problematical. And even if we have some central elements, because there are too many non-trivial commutation relations between the generators of $\mathfrak{sl}_2 \ltimes V_m$, we cannot expect that we could always find a generator that commutes with almost everything to start the proof.

5.2 Centralisers of $\{F, H, E\}$ in $U(\mathfrak{sl}_2 \ltimes V_m)$

The failure of the existing method forces us to pursue a new route to determine the centre. Instead of looking for a one-step solution, we may slightly relax the constraints and first determine the centraliser of $\{F, H, E\}$. As an $\mathfrak{sl}_2$-module, the centraliser can be decomposed to a sum of 1-dimensional irreducible $\mathfrak{sl}_2$-modules (also known as $\mathfrak{sl}_2$-singlets). Fortunately, to find $\mathfrak{sl}_2$-singlets we have a useful tool called the algebraic character.

Definition 5.2 (Algebraic character)**.** Given a Lie algebra $\mathfrak{g}$ and a weighted $\mathfrak{g}$-module M. For a weight λ, let M_λ be the corresponding weight space and define e^λ to be the formal exponential of λ. Then the algebraic character of M is

$$\mathrm{ch}(M) := \sum_{\lambda} \dim(M_\lambda) e^\lambda$$

where the sum is over all weights of M.

If M is graded, say $M = \bigoplus_{k=0}^{\infty} M(k)$, where $M(k)$ are the homogeneous parts of M, then we can define a graded character to be

$$\mathrm{ch}(M)(q) := \sum_{k=0}^{\infty} \mathrm{ch}(M(k)) q^k.$$

Now consider $U(\mathfrak{sl}_2 \ltimes V_m)$ as an $\mathfrak{sl}_2$-module via the adjoint representation. It can be graded under the PBW basis using the filtration as we mentioned in the proof of Proposition 4.8. Set $z = e^\alpha$, where α is the fundamental weight of $\mathfrak{sl}_2$. Use q to track the degree contributed by $\mathfrak{sl}_2$ and use qu to track the degree contributed by V_m, then one can calculate the graded character of $U(\mathfrak{sl}_2 \ltimes V_m)$, which turns out to be

$$\mathrm{ch}\left(U(\mathfrak{sl}_2 \ltimes V_m)\right)(q,u,z) := \frac{1}{(1-qz^2)(1-q)(1-qz^{-2})} \times \prod_{j=0}^{m-1} \frac{1}{(1-quz^{m-1-2j})} \tag{5.1}$$

The right-hand side of Equation (5.1) can be expanded as a formal Laurent series of z, say

$$\mathrm{ch}\left(U(\mathfrak{sl}_2 \ltimes V_m)\right)(q,u,z) = \sum_{n\in\mathbb{Z}} A_n(q,u)z^n.$$

Then it is known from the representation theory of $\mathfrak{sl}_2$ (see [15]) that the character of $\mathfrak{sl}_2$-singlets in $U(\mathfrak{sl}_2 \ltimes V_m)$, which we shall denote by $\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_m))$, can be calculated by

$$\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_m))(q,u,z) = A_0(q,u) - A_{-2}(q,u). \tag{5.2}$$

For small values of m, one can get the Laurent series of $\mathrm{ch}\left(U(\mathfrak{sl}_2 \ltimes V_m)\right)$ by expanding each factor as a Laurent series and then do the product of formal Laurent series.

Examples 5.3. The graded character of $U(\mathfrak{sl}_2)$ can be expanded as

$$\begin{aligned}\mathrm{ch}(U(\mathfrak{sl}_2))(q,z) &= \frac{1}{(1-qz^2)(1-q)(1-qz^{-2})} \\ &= \frac{1}{(1-q)}\left(\sum_{i=0}^{\infty} q^i z^{2i}\right)\left(\sum_{j=0}^{\infty} q^j z^{-2j}\right) \\ &= \frac{1}{(1-q)(1-q^2)}\left(\sum_{k=0}^{\infty} q^k z^{2k} + \sum_{k=1}^{\infty} q^k z^{-2k}\right)\end{aligned} \tag{5.3}$$

Therefore by Equation (5.2) we have

$$\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_m))(q,u,z) = \frac{1}{(1-q)(1-q^2)}(1-q) = \frac{1}{(1-q^2)} \tag{5.4}$$

which agrees with the result in Lemma 2.11 saying $Z(U(\mathfrak{sl}_2)) = \mathbb{C}[C]$ since the quadratic Casimir $C := 4FE + HH + 2H$ has degree 2.

Similarly the graded character of $U(\mathfrak{sl}_2 \ltimes V_3)$ can be expanded as

$$\begin{aligned}&\mathrm{ch}(U(\mathfrak{sl}_2 \ltimes V_3))(q,u,z) \\ =&\frac{1}{(1-qz^2)(1-q)(1-qz^{-2})(1-quz^2)(1-qu)(1-quz^{-2})} \\ =&\frac{(1-q^2u^2-u+q^2u)+\sum_{k=1}^{\infty}\left(q^k-q^{k+2}u^2-q^ku^{k+1}+q^{k+2}u^{k+1}\right)(z^{2k}+z^{-2k})}{(1-q)(1-u)(1-qu)(1-q^2)(1-q^2u)(1-q^2u^2)}\end{aligned} \tag{5.5}$$

which gives

$$\begin{aligned}\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_3))(q,u) =& \frac{(1-q^2u^2-u+q^2u)-(q-q^3u^2-qu^2+q^3u^2)}{(1-q)(1-u)(1-qu)(1-q^2)(1-q^2u)(1-q^2u^2)} \\ =& \frac{1}{(1-q^2)(1-q^2u)(1-q^2u^2)}.\end{aligned} \tag{5.6}$$

This agrees with Lemma 5.1 as we can see the factor $(1-q^2u^2)$ corresponds to the generator $Z_1 := 2X_0X_2 - X_1^2$ and the factor $(1-q^2u)$ corresponds to the generator $Z_2 := EX_2 + HX_1 - 2FX_0$. Also the factor $(1-q^2)$ corresponds to the quadratic Casimir C, but C is not central in $U(\mathfrak{sl}_2 \ltimes V_3)$ as one can check.

It can be checked that the calculated result of $\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_2))$ also agrees with Lemma 4.4.

From the examples above, one can foresee that calculating the Laurent series of $\mathrm{ch}(U(\mathfrak{sl}_2 \ltimes V_m))$ in this way for larger values of m would not be efficient in practice. Usually, to extract the coefficients from a Laurent series, one would like to use Cauchy's residue theorem, so it would be good if we could pretend that z is a complex number and apply the theorem here. Let us briefly review the residue theorem and some of its consequences [16] and then have a try.

Theorem 5.4 (Cauchy's Residue Theorem). *Let $D \subset \mathbb{C}$ be a bounded Jordan domain and $a_1, \dots, a_n$ points in D. Assume that $f : D\backslash\{a_1, \dots, a_n\} \to \mathbb{C}$ is holomorphic. Then*

$$\oint_{\partial D} f dz = 2\pi i \sum_{i=1}^{n} res_{a_i} f$$

where $res_{a_i} f$ is the residue of f at a_i, that is, the coefficient of $(z-a_i)^{-1}$ in the Laurent expansion of $f(z)$ at a_i.

The points $a_1, \dots, a_n$ are the singularities of f, also known as the poles. The residue could be hard to calculate in general, but for a special kind of poles known as the simple poles, it is easy to calculate the residues. Loosely speaking, if a complex-valued function $f(z)$ is in the form of $h(z)/g(z)$, where h, g are some holomorphic complex-valued functions such that $g(a_0) = 0$, $g'(a_0) \neq 0$, then residue of f at a_0 is

$$\mathrm{res}_{a_0} f = \frac{h(a_0)}{g'(a_0)}.$$

Note the formal Laurent expansion of $\mathrm{ch}(U(\mathfrak{sl}_2 \ltimes V_m))$ converges only when $|q| < 1$, $|q|^2 < z < |q|^{-2}$, $|qu| < 1$ and $|qu|^{m-1} < z < |qu|^{-m+1}$. We may choose the values of q and u such that $|q|^2 < |qu|^{m-1}$ for all $m \geq 2$. Then by Equation (5.2) and (5.1), we see to get the character of $\mathfrak{sl}_2$-singlets in $U(\mathfrak{sl}_2 \ltimes V_m)$ we need to evaluate the integral of

$$F_m(z) := \frac{1-z^2}{z} \frac{1}{(1-qz^2)(1-q)(1-qz^{-2})} \times \prod_{j=0}^{m-1} \frac{1}{(1-quz^{m-1-2j})} \tag{5.7}$$

around a contour γ_m centred at 0 with radius R such that $|qu|^{m-1} < R < |qu|^{-m+1}$. The poles contained in the chosen contour γ_m are $\pm\sqrt{q}$ and the roots of $(qu)^{m-1-2j} = 1$ where j runs from 0 to $\lfloor (m-1)/2 \rfloor$ and they are all simple poles. Therefore by Cauchy's Residue Theorem 5.4, we have

$$\oint_{\gamma_m} F_m(z) \frac{dz}{2\pi i} = \mathrm{res}_{\sqrt{q}} F_m + \mathrm{res}_{-\sqrt{q}} F_m + \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \sum_{k=1}^{m-1-2j} \mathrm{res}_{e^{\frac{i2k\pi}{m-1-2j}} (qu)^{\frac{1}{m-1-2j}}} F_m \tag{5.8}$$

Example 5.5. Apply Equation (5.8) to $m = 2$ we have

$$\begin{aligned} \mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_2))(q,u) &= \oint_{\gamma_2} F_2(z) \frac{dz}{2\pi i} \\ =&\mathrm{res}_{\sqrt{q}} F_2(z) + \mathrm{res}_{-\sqrt{q}} F_2(z) + \mathrm{res}_{qu} F_2(z) = \frac{1}{(1-q^2)(1-q^3u^2)} \end{aligned}$$

where the factor $(1-q^3u^2)$ corresponds to the generator $FX_0^2 - HX_0X_1 - EX_1^2$ given in Lemma 4.4 and the factor $(1-q^2)$ corresponds to the quadratic Casimir C which is not central in $U(\mathfrak{sl}_2 \ltimes V_2)$.

One can also check $\oint_{\gamma_1} F_1(z) \frac{dz}{2\pi i}$ and $\oint_{\gamma_3} F_3(z) \frac{dz}{2\pi i}$ agree with Equation (5.3) and Equation (5.5) respectively. These results made us confident to calculate $\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_m))$ using Equation (5.8) for larger m.

For $m = 4$ we have

$$\begin{aligned} \mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_4))(q,u) &= \oint_{\gamma_4} F_4(z) \frac{dz}{2\pi i} \\ =&\mathrm{res}_{\sqrt{q}} F_4(z) + \mathrm{res}_{-\sqrt{q}} F_4(z) + \mathrm{res}_{qu} F_4(z) \\ &+ \mathrm{res}_{(qu)^{\frac{1}{3}}} F_4(z) + \mathrm{res}_{e^{\frac{2\pi i}{3}} (qu)^{\frac{1}{3}}} F_4(z) + \mathrm{res}_{e^{\frac{-2\pi i}{3}} (qu)^{\frac{1}{3}}} F_4(z) \\ =&\frac{1-q^{14}u^8}{(1-q^2)(1-q^3u^2)(1-q^5u^2)(1-q^4u^4)(1-q^7u^4)} \end{aligned} \tag{5.9}$$

We should mention that this equation confirmed the conjectured expression of $\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_4))$ obtained by expanding $\mathrm{ch}(U(\mathfrak{sl}_2 \ltimes V_4))$ to sufficiently high order of z and q in Mathematica™.

Now we examine the corresponding generator for each factor of the denominator in Equation (5.9). There is no doubt that $(1-q^2)$ corresponds to the quadratic Casimir C, which is not central in $U(\mathfrak{sl}_2 \ltimes V_4)$ as one can check. The generator corresponding to $(1-q^4u^4)$ consists of X_0, X_1, X_2, X_3 only and has weight 0, so it must be in the form of

$$A_{4,4} := a_1 X_0^2 X_3^2 + a_2 X_0 X_1 X_2 X_3 + a_3 X_1^2 X_2^2 + a_4 X_0 X_2^3 + a_5 X_1^3 X_3$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C}$. By requiring $[E, A_{4,4}] = [F, A_{4,4}] = 0$ we find $A_{4,4}$ is (proportional to)

$$-9X_0^2X_3^2 + 18X_0X_1X_2X_3 + 3X_1^2X_2^2 - 6X_0X_2^3 - 8X_1^3X_3$$

which is central in $U(\mathfrak{sl}_2 \ltimes V_4)$. To find the other generators, we need the lemma stated below.

Lemma 5.6. *Consider $U(\mathfrak{sl}_2 \ltimes V_m)$ as a $\mathfrak{sl}_2$-module via the adjoint representation. Let $U, V \in U(\mathfrak{sl}_2 \ltimes V_m)$ be two highest weight vectors with the same weight n. Then the element*

$$W := \sum_{i=0}^{n} (-1)^i (adF)^i (U) \cdot (adF)^{n-i} (V)$$

commutes with F, H, E.

Proof. We see W commutes with H because it has weight 0 by construction. And $[F, W] = [E, W] = 0$ can be shown by straightforward calculation. We have

$$\begin{aligned}
&(\mathrm{ad}F)\left(\sum_{i=0}^{n}(-1)^i (\mathrm{ad}F)^i (U) \cdot (\mathrm{ad}F)^{n-i} (V)\right)\\
=&\sum_{i=0}^{n}(-1)^i (\mathrm{ad}F)^{i+1} (U) \cdot (\mathrm{ad}F)^{n-i} (V) + \sum_{i=0}^{n}(-1)^i (\mathrm{ad}F)^i (U) \cdot (\mathrm{ad}F)^{n-i+1} (V)\\
=&\sum_{i=1}^{n+1}(-1)^{i-1} (\mathrm{ad}F)^i (U) \cdot (\mathrm{ad}F)^{n-i+1} (V) + \sum_{i=0}^{n}(-1)^i (\mathrm{ad}F)^i (U) \cdot (\mathrm{ad}F)^{n-i+1} (V)\\
=&(-1)^n (\mathrm{ad}F)^{n+1} (U) + (\mathrm{ad}F)^{n+1} (V)\\
=&0
\end{aligned}$$

The last equality holds because the lowest possible weight for the elements generated by a highest weight vector with weight n is $-n$. Similarly one can show $[E, W] = 0$. □

With Lemma 5.6, by looking for the $\mathfrak{sl}_2$-highest weight vectors of desired degree in $U(\mathfrak{sl}_2)$ and $U(V_4)$, we find the generators corresponding to $(1 - q^3u^2)$, $(1 - q^5u^2)$, $(1 - q^7u^4)$ are

$$A_{3,2} := \sum_{i=0}^{2} (-1)^i (\mathrm{ad}F)^i (E) \cdot (\mathrm{ad}F)^{2-i} (3X_0X_2 - 2X_1^2)$$
$$A_{5,2} := \sum_{i=0}^{6} (-1)^i (\mathrm{ad}F)^i (E^3) \cdot (\mathrm{ad}F)^{6-i} (X_0^2)$$

and

$$A_{7,4} := \sum_{i=0}^{6} (-1)^i (\mathrm{ad}F)^i (E^3) \cdot (\mathrm{ad}F)^{6-i} (9X_0^2X_1X_2 - 9X_0^3X_3 - 4X_0X_1^3)$$

respectively. One can check $A_{3,2}, A_{5,2}, A_{7,4}$ are not central in $U(\mathfrak{sl}_2 \ltimes V_4)$.

We should mention that owe to the presence of the non-trivial numerator in Equation (5.9), the centraliser of $\{F, H, E\}$ in $U(\mathfrak{sl}_2 \ltimes V_4)$ is not free, and the relation between generators is not easy to calculate. However, since $U(\mathfrak{sl}_2 \ltimes V_4)$ is an integral domain, we know there is no non-trivial relation in $\mathbb{C}[A_{4,4}]$ when included in $U(\mathfrak{sl}_2 \ltimes V_4)$. Therefore we can conclude:

Lemma 5.7. *The centre of* $U(\mathfrak{sl}_2 \ltimes V_4)$ *is*

$$Z(U(\mathfrak{sl}_2 \ltimes V_4)) = \mathbb{C}[A_{4,4}]$$

where $A_{4,4} := -9X_0^2X_3^2 + 18X_0X_1X_2X_3 + 3X_1^2X_2^2 - 6X_0X_2^3 - 8X_1^3X_3$.

Now we do similar analysis for $m = 5$. Using Equation 5.8 we have

$$\begin{aligned}
&\mathrm{ch}_0(U(\mathfrak{sl}_2 \ltimes V_5))(q, u) = \oint_{\gamma_5} F_5(z) \frac{dz}{2\pi i} \\
=&\mathrm{res}_{\sqrt{q}} F_5(z) + \mathrm{res}_{-\sqrt{q}} F_5(z) + \mathrm{res}_{\sqrt{qu}} F_5(z) + \mathrm{res}_{-\sqrt{qu}} F_5(z) \\
&+ \mathrm{res}_{(qu)^{\frac{1}{4}}} F_5(z) + \mathrm{res}_{i(qu)^{\frac{1}{4}}} F_5(z) + \mathrm{res}_{-(qu)^{\frac{1}{4}}} F_5(z) + \mathrm{res}_{-i(qu)^{\frac{1}{4}}} F_5(z) \\
=&\frac{1 - q^{12}u^6}{(1 - q^2)(1 - q^2u^2)(1 - q^3u)(1 - q^3u^3)(1 - q^4u^2)(1 - q^6u^3)}
\end{aligned} \tag{5.10}$$

which is again consistent with the numerical result given by Mathematica$^{\mathrm{TM}}$.

Looking at the factors in the denominator of (5.9), we see $(1 - q^2)$, as always, corresponds to the generator C which is not central in $U(\mathfrak{sl}_2 \ltimes V_5)$. And $(1 - q^2u^2)$ corresponds to the generator $B_{2,2} := 2X_0X_4 - 2X_1X_3 + X_2^2$, which is central (In fact, it is not hard to see that when m is odd, the element $\sum_{i=0}^{m-1} (-1)^i X_i X_{m-1-i}$ is always central in $U(\mathfrak{sl}_2 \ltimes V_m)$). Also, using the undetermined coefficients method, we find the generator corresponding to $(1-q^3u^3)$ is

$$B_{3,3} := 12X_0X_2X_4 - 9X_1^2X_4 - 6X_0X_3^2 + 6X_1X_2X_3 - 2X_2^3$$

which is central.

Using Lemma 5.6, the generators corresponding to $(1-q^3u)$, $(1-q^4u^2)$, $(1-q^6u^3)$ are found to be

$$B_{3,1} := \sum_{i=0}^{4} (-1)^i \left(\mathrm{ad}F\right)^i (E^2) \cdot \left(\mathrm{ad}F\right)^{4-i} (X_0)$$

$$B_{4,2} := \sum_{i=0}^{4} (-1)^i \left(\mathrm{ad}F\right)^i (E^2) \cdot \left(\mathrm{ad}F\right)^{4-i} (4X_0X_2 - 3X_1^2)$$

and

$$B_{6,3} := \sum_{i=0}^{6} (-1)^i \left(\mathrm{ad}F\right)^i (E^3) \cdot \left(\mathrm{ad}F\right)^{6-i} (6X_0X_1X_2 - 4X_0^2X_3 - 3X_1^3)$$

respectively. One can check $B_{3,2}, B_{5,2}, B_{7,4}$ are not central in $U(\mathfrak{sl}_2 \ltimes V_5)$.

Once again, although it is hard to find the explicit relation corresponding to the $1-q^{12}u^6$ that appears in the numerator of (5.10), from its degree we can already see the relation is not between $B_{2,2}$ and $B_{3,3}$. Hence we have:

Lemma 5.8. *The centre of* $U(\mathfrak{sl}_2 \ltimes V_5)$ *is*

$$Z(U(\mathfrak{sl}_2 \ltimes V_5)) = \mathbb{C}[B_{2,2}, B_{3,3}]$$

where

$$B_{2,2} := 2X_0X_4 - 2X_1X_3 + X_2^2$$

and

$$B_{3,3} := 12X_0X_2X_4 - 9X_1^2X_4 - 6X_0X_3^2 + 6X_1X_2X_3 - 2X_2^3.$$

The calculations we have done so far illustrate that the central elements in $U(\mathfrak{sl}_2 \ltimes V_m)$ for larger m are very likely only contributed by the symmetric product of V_m. This suggests that we can only focus on the factors $\prod_{j=0}^{m-1} \frac{1}{(1-quz^{m-1-2j})}$ in the graded character (5.1) and replace qu by a single variable, say v, to make the computation easier to handle. Again using Equation (5.8) we find

$$\mathrm{ch}_0(U(V_6))(v) = \frac{1-v^{36}}{(1-v^4)(1-v^8)(1-v^{12})(1-v^{18})}$$

and

$$\mathrm{ch}_0(U(V_7))(v) = \frac{1 - v^{30}}{(1-v^2)(1-v^4)(1-v^6)(1-v^{10})(1-v^{15})}$$

Now the corresponding generators can be determined by the method of undetermined coefficient as before. Also we can see from these characters that there exist non-trivial relations between the central elements in $U(\mathfrak{sl}_2 \ltimes V_6)$ and $U(\mathfrak{sl}_2 \ltimes V_7)$. We believe one can find the character for arbitrary m by routinely doing the calculations.

These results further indicate the complexity of the structure of centres in $U(\mathfrak{sl}_2 \ltimes V_m)$ and one would not expect that the general theorems (for example, Harish-Chandra's theorem) which are used to the determine centres of semi-simple Lie algebras could be directly adapted in this case, although we will not be surprised if one could possibly achieve by only slightly modifying those general theorems.

5.3 Centralisers of the element H in $U(\mathfrak{sl}_2 \ltimes V_m)$

Since H spans the Cartan subalgebra of $\mathfrak{sl}_2$, the centralisers of H in $U(\mathfrak{sl}_2 \ltimes V_m)$ are also of interests. Let $C_H(\mathfrak{sl}_2 \ltimes V_m)$ denote the centraliser of the element H in $U(\mathfrak{sl}_2 \ltimes V_m)$. Because we consider $U(\mathfrak{sl}_2 \ltimes V_m)$ as an $\mathfrak{sl}_2$-module via the adjoint representation, it is clear that the character of $C_H(\mathfrak{sl}_2 \ltimes V_m)$ is just the zeroth coefficient $A_0(q,u)$ of the Laurent series of $\mathrm{ch}(U(\mathfrak{sl}_2 \ltimes V_m))$.

From Equation (5.3), we can read

$$\mathrm{ch}(C_H(\mathfrak{sl}_2))(q) = \frac{1}{(1-q)(1-q^2)}$$

which suggests $C_H(\mathfrak{sl}_2) = \mathbb{C}[H, FE]$ as we would expect. By similar calculation we also get

$$\mathrm{ch}(C_H(\mathfrak{sl}_2 \ltimes V_2))(q,u) = \frac{1 - q^6u^4}{(1-q)(1-q^2)(1-q^2u^2)(1-q^3u^2)^2}$$

which gives

$$C_H(\mathfrak{sl}_2 \ltimes V_3) = \frac{\mathbb{C}[H, FE, X_0X_1, FX_0^2, EX_1^2]}{\langle (FE)(X_0X_1)^2 - (FX_0^2)(EX_1^2)\rangle}$$

that is consistent with the result given in [2]. Also, from Equation (5.5), we can read that

$$\mathrm{ch}(C_H(\mathfrak{sl}_2 \ltimes V_3))(q,u) = \frac{1 - q^4u^2}{(1-q)(1-qu)(1-q^2)(1-q^2u)^2(1-q^2u^2)}$$

which indicates

$$C_H(\mathfrak{sl}_2 \ltimes V_3) = \frac{\mathbb{C}[H, FE, X_1, X_0X_2, FX_0, EX_2]}{\langle (FE)(X_0X_2) - (FX_0)(EX_2) \rangle}.$$

One might guess we can get $\mathrm{ch}\,(C_H(\mathfrak{sl}_2 \ltimes V_m))\,(q, u)$ for larger m using the complex analysis techniques introduced in the last section and then determine $C_H(\mathfrak{sl}_2 \ltimes V_m)$. However, the situation here is in fact more complicated than we expected. Let us get our hands on $C_H(\mathfrak{sl}_2 \ltimes V_4)$ as an example.

Lemma 5.9. *The centraliser of H in $U(\mathfrak{sl}_2 \ltimes V_4)$ is a quotient of the polynomial ring generated by the set of generators*

$$G := \{H, FE, X_0X_3, X_1X_2, EX_2^2, FX_1^2, EX_1X_3, FX_0X_2, E^2X_2X_3, F^2X_0X_1, X_1^3X_3, X_0X_2^3, E^3X_3^2, F^3X_0^2\}.$$

Proof sketch. It is easy to check each of the generators in G commutes with H. To show that $C_H(\mathfrak{sl}_2 \ltimes V_4) \subset \mathbb{C}[G]$, it suffices to show all monomials in $C_H(\mathfrak{sl}_2 \ltimes V_4)$ are in $\mathbb{C}[G]$. By the PBW theorem 2.4, a generic monomial $M \in C_H(\mathfrak{sl}_2 \ltimes V_4)$ can be written as $M = F^iH^jE^kX_0^mX_1^nX_2^pX_3^q$ where $i, j, k, m, n, p, q \in \mathbb{N}$ and $-2i + 2k + 3m + n - p - 3q = 0$. If $j > 0$, then since $F^iE^kX_0^mX_1^nX_2^pX_3^q$ is also in $C_H(\mathfrak{sl}_2 \ltimes V_4) \subset \mathbb{C}[G]$ and $H \in G$, we do not need M as a generator. If $k \geq i$, then since $FE \in G$, we only need to consider $E^{k-i}X_0^mX_1^nX_2^pX_3^q$. Now if $q \geq m$, then since $X_0X_3 \in G$, it remains to consider $E^{k-i}X_1^nX_2^pX_3^{q-m}$. Then if $n \geq p$, it is sufficient to consider $E^{k-i}X_1^{n-p}X_3^{q-m}$. Note that $2(k-i) + (n-p) - 3(q-m)$, so $k - i = 0$ if and only if $n - p = 3(q - m)$. Therefore, if $k - i = 0$ then we see $X_1^{n-p}X_3^{q-m} = (X_1^3X_3)^{q-m} \in \mathbb{C}[G]$, which implies $M \in \mathbb{C}[G]$. And otherwise we have $E^{k-i}X_1^{n-p}X_3^{q-m} = (EX_1X_3)^{n-p}\,(E^3X_3^2)^{\frac{q-m-n+p}{2}} \in \mathbb{C}[G]$. Similar analysis can be done if $k < i$ and/or $q < m$ and/or $n < p$. □

Having found all the generators, we know the graded character of $C_H(\mathfrak{sl}_2 \ltimes V_4)$ should be

$$\frac{P(q, u)}{(1-q)(1-q^2)(1-q^2u^2)^2(1-q^3u^2)^4(1-q^4u^2)^2(1-q^4u^4)^2(1-q^5u^2)^2}$$

where $P(q, u)$ is a polynomial (because of Hilbert's syzygy theorem [8]). Now by expanding $\mathrm{ch}(U(\mathfrak{sl}_2 \ltimes V_4))$ to a sufficiently high order of z and q in Mathematica™, we found that

$$\begin{aligned} P(q, u) =&(1-q^4u^4)(1-q^4u^2)^2(1-q^3u^2)^3(1-q^5u^2) \\ &\times(1 + 3q^3u^2 + 2q^4u^2 + q^5u^2 + q^4u^4 - q^8u^4 - q^7u^6 - 2q^8u^6 - 3q^9u^6 - q^{12}u^8) \end{aligned}$$

which has total degree 62. We can see there are common factors in the numerator and denominator, so if we attempted to use the simplified expression of the character to determine the generators and relations, some information would actually be hidden behind the formula. Nevertheless it is worth to notice that the centralisers in $U(\mathfrak{sl}_2 \ltimes V_m)$ are so much richer even though V_m is only an abelian Lie algebra.

Appendix A

Background Knowledge

In this appendix, we list some primary definitions and results on Lie algebras and representation theory to clarify any possible confusion on the terminologies used in this thesis. [15]

Definition A.1 (Lie algebra). A Lie algebra $\mathfrak{g}$ is an algebra with a bilinear map (often called the Lie bracket) $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the antisymmetry

$$[x, y] + [y, x] = 0, \forall x, y \in \mathfrak{g}$$

and the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \forall x, y, z \in \mathfrak{g}.$$

Definition A.2 (Lie subalgebra). A Lie subalgebra of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ such that

$$[x, y] \in \mathfrak{h}, \forall x, y \in \mathfrak{h}.$$

Definition A.3 (Lie ideal). A Lie ideal of $\mathfrak{g}$ is a vector subspace $\mathfrak{i} \subset \mathfrak{g}$ such that

$$[x, y] \in \mathfrak{i}, \forall x \in \mathfrak{i}, y \in \mathfrak{g}.$$

Definition A.4 (Centre). The centre $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the Lie ideal

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0, \forall y \in \mathfrak{g}\}.$$

Definition A.5 (Derived algebra). The derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{[x, y] : x, y \in \mathfrak{g}\}.$$

Definition A.6 (Lie algebra homomorphism)**.** A homomorphism between Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ that preserves the Lie bracket, that is,

$$\phi([x, y]) = [\phi(x), \phi(y)], \forall x, y \in \mathfrak{g}_1.$$

Definition A.7 (Bilinear form)**.** A symmetric invariant bilinear form of a Lie algebra $\mathfrak{g}$ is a bilinear map $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which is symmetric, that is,

$$\langle x, y \rangle = \langle y, x \rangle, \forall x, y \in \mathfrak{g}$$

and invariant, that is,

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0, \forall x, y, z \in \mathfrak{g}.$$

Definition A.8 (Radical)**.** The (right) radical of a bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the set

$$\{x \in \mathfrak{g} : \langle x, y \rangle = 0, \forall y \in \mathfrak{g}\}$$

Definition A.9 (Representation and $\mathfrak{g}$-module)**.** Let V be a vector space and $\mathfrak{gl}(V)$ be the Lie algebra of linear maps from V to itself. A representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$.

The vector space V is called a $\mathfrak{g}$-module. The action of $\mathfrak{g}$ on V is usually abbreviated as $xv := \pi(x)v$ where $x \in \mathfrak{g}, v \in V$.

Definition A.10 ($\mathfrak{g}$-submodule and invariant subspace)**.** A submodule (or an invariant subspace) W of a $\mathfrak{g}$-module V is a vector subspace that is preserved by the Lie algebra action, that is,

$$w \in W \implies xw \in W, \forall x \in \mathfrak{g}.$$

Definition A.11 (Irreducible $\mathfrak{g}$-module)**.** A $\mathfrak{g}$-module is irreducible if it has no proper non-trivial submodule.

Remark A.12 (Irreducible subalgebra)**.** In Burnside's theorem of matrix algebras 2.18, we say a subalgebra $\mathcal{A}$ of $\mathfrak{gl}(V)$ is irreducible if there is no proper non-trivial subspace of V that is invariant under $\mathcal{A}$. In other words, we think of the vector space V as a module of its operator algebra $\mathfrak{gl}(V)$.

Definition A.13 (Nilpotent Lie subalgebra)**.** Given a Lie algebra $\mathfrak{g}$, let $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_n = [\mathfrak{g}_{n-1}, \mathfrak{g}]$ for all $n \geq 2$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is nilpotent if there exists $n \in \mathbb{N}$ such that $\mathfrak{g}_n = 0$.

Definition A.14 (Cartan subalgebra)**.** A Cartan subalgebra $\mathfrak{g}_0$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra that satisfies

$$[x, y] \in \mathfrak{g}_0, \forall x \in \mathfrak{g}_\circ \implies y \in \mathfrak{g}_\circ.$$

Definition A.15 (Weight)**.** Given a Lie algebra $\mathfrak{g}$, let V be a $\mathfrak{g}$-module. If there exists a non-zero vector $v \in V$ and a linear functional $\lambda \in \mathfrak{g}_0^*$, where $\mathfrak{g}_0^*$ is the dual of the Cartan subalgebra of $\mathfrak{g}$, that satisfies

$$hv = \lambda(h)v, \forall h \in \mathfrak{g}_0,$$

then the vector v is called a weight vector and the linear functional λ is called the weight of this weight vector. The space spanned by all weight vectors for a given weight is called a weight space.

Definition A.16 (Weighted module)**.** A $\mathfrak{g}$-module V is a weighted module if it has a basis whose elements are all weight vectors.

Theorem A.17. *A finite-dimensional irreducible $\mathfrak{g}$-module is a weighted module.*

Theorem A.18 (Classification of $\mathfrak{sl}_2$-modules)**.** *Let V be a $(d+1)$-dimensional irreducible $\mathfrak{sl}_2$-module, where $d \in \mathbb{N}$, then V is spanned by weight vectors*

$$\{v_d^{(i)} : i \in \mathbb{N}, 0 \leq i \leq d\}$$

where $v_d^{(i)}$ corresponds to weight $d - 2i$.

Appendix B

Bilinear Form of hs$[\lambda]$

Suppose $\langle\cdot,\cdot\rangle : \text{hs}[\lambda]\times\text{hs}[\lambda]\to\mathbb{C}$ is an invariant bilinear form on hs$[\lambda]$, then in particular we have

$$0 = \left\langle [V_0^2, V_m^r], V_n^s\right\rangle + \left\langle V_m^r, [V_0^2, V_n^s]\right\rangle = -(m+n)\left\langle V_m^r, V_n^s\right\rangle$$

where we used Example 1.7. This implies $\langle V_m^r, V_n^s\rangle = 0$ for $m+n\neq 0$.

Now we only need to consider $\left\langle V_m^r, V_{-m}^s\right\rangle$. Note that using Example 1.7 we also have

$$\begin{aligned}0 &= \left\langle [V_{-1}^2, V_m^r], V_{-(m-1)}^s\right\rangle + \left\langle V_m^r, [V_{-1}^2, V_{-(m-1)}^s]\right\rangle \\ &= (-(r-1)-m)\left\langle V_{m-1}^r, V_{-(m-1)}^s\right\rangle + (-(s-1)+(m-1))\left\langle V_m^r, V_{-m}^s\right\rangle\end{aligned}$$

which gives

$$\left\langle V_m^r, V_{-m}^s\right\rangle = -\frac{r+m-1}{s-m}\left\langle V_{m-1}^r, V_{-(m-1)}^s\right\rangle.$$

But similarly $0 = \left\langle [V_1^2, V_{m-1}^r], V_{-m}^s\right\rangle + \left\langle V_{m-1}^r, [V_1^2, V_{-m}^s]\right\rangle$ gives

$$\left\langle V_m^r, V_{-m}^s\right\rangle = -\frac{s+m-1}{r-m}\left\langle V_{m-1}^r, V_{-(m-1)}^s\right\rangle$$

so we have $\dfrac{r+m-1}{s-m} = \dfrac{s+m-1}{r-m}$ which holds if and only if $r=s$. Hence $\left\langle V_m^r, V_{-m}^s\right\rangle = 0$ for $r\neq s$.

For $r=s$, using $\left\langle V_m^r, V_{-m}^r\right\rangle = -\dfrac{r+m-1}{r-m}\left\langle V_{m-1}^r, V_{-(m-1)}^r\right\rangle$ recursively we have

$$\left\langle V_m^r, V_{-m}^r\right\rangle = \frac{(-1)^{r-m-1}}{(2r-2)!}\Gamma(r+m)\Gamma(r-m)\left\langle V_{r-1}^r, V_{-(r-1)}^r\right\rangle.$$

Denote $N_r := \left\langle V_{r-1}^r, V_{-(r-1)}^r\right\rangle$. Then using

$$0 = \left\langle [V_1^3, V_{r-2}^{r-1}], V_{-(r-1)}^r\right\rangle + \left\langle V_{r-2}^{r-1}, [V_1^3, V_{-(r-1)}^r]\right\rangle$$

and the structure constants given in Equation (1.1), we get a recurrence relation

$$\begin{aligned} 0 =& g_2^{3(r-1)}(1, r-2; \lambda) N_r + g_r^{3r}(1, -(r-1); \lambda) N_{r-1} \\ \implies N_r =& \frac{8q^2(r-1)(r-1-\lambda)(r-1+\lambda)}{2r-1} N_{r-1}. \end{aligned}$$

which can be solved by

$$N_r = \frac{3 \cdot 4^{r-3}\sqrt{\pi} q^{2r-4}\Gamma(r)}{(\lambda^2-1)\Gamma(r+\frac{1}{2})}(1-\lambda)_{r-1}(1+\lambda)_{r-1}.$$

This gives the first expression of the bilinear form in Equation (1.6). This process can be carried over to construct a symmetric invariant bilinear form on $\widetilde{\mathrm{hs}[\lambda]}$ once the structure constants are found explicitly.

By Theorem 3.10, we expect that the bilinear on $\mathrm{hs}[\lambda] \cong [U(\mathfrak{sl}_2)[\lambda], U(\mathfrak{sl}_2)[\lambda]]$ given in the proof of Corollary 2.22 is proportional to $g^{rs}_{r+s-1}(m, n, \lambda)$ according to Equation (1.5), and the normalisation factor can be determined by explicit calculation of $V^r_m \star V^s_n$ for some r, s, m, n. This explains the second equality in Equation (1.6).

Appendix C

Some Calculation Details

C.1 Proof of Equation (3.8)

Expand Equation (3.8) using (1.4), then we see it amounts to show

$$\begin{aligned}0 =&\frac{r-m+s-t-n}{r-m}\\&\times\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}[r-2+m]_{t-1-k}[r-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\\&-\frac{s-1-n}{r-m}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}[r-2+m]_{t-1-k}[r-m]_k[s-2-n]_{t-1-k}[s+n]_k\\&-\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\end{aligned}\tag{C.1}$$

for all t even. Rewrite the first term in (C.1) as

$$\begin{aligned}&\frac{r-m+s-t-n}{r-m}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}[r-2+m]_{t-1-k}[r-m]_k\\&\times[s-1-n]_{t-1-k}[s-1+n]_k\\=&\frac{(r-m+s-t-n)(r+m-t)}{(r-m)(r-1+m)}[r-1+m]_{t-1}[s-1-n]_{t-1}\\&+\sum_{k=1}^{t-1}\frac{(r-m+s-t-n)(r+m-t+k)}{(r-1+m)(r-m-k)}(-1)^k\binom{t-1}{k}\\&\times[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\end{aligned}$$

Rewrite the second term in (C.1) as

$$
\begin{aligned}
&\frac{s-1-n}{r-m}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}[r-2+m]_{t-1-k}[r-m]_k[s-2-n]_{t-1-k}[s+n]_k\\
=&\frac{(r+m-t)(s-n-t)}{(r-m)(r-1+m)}[r-1+m]_{t-1}[s-1-n]_{t-1}\\
&+\sum_{k=1}^{t-1}\frac{(r+m-t+k)(s-n-t+k)(s+n)}{(r-1+m)(r-m-k)(s+n-k)}(-1)^k\binom{t-1}{k}\\
&\times[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\\
=&\frac{(r+m-t)(s-n-t)}{(r-m)(r-1+m)}[r-1+m]_{t-1}[s-1-n]_{t-1}\\
&+\sum_{k=1}^{t-1}\frac{(r+m-t+k)(s-n-t+k)}{(r-1+m)(r-m-k)}(-1)^k\binom{t-1}{k}\\
&\times[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\\
&+\sum_{k=1}^{t-1}\frac{k(r+m-t+k)(s-n-t+k)}{(r-1+m)(r-m-k)(s+n-k)}(-1)^k\binom{t-1}{k}\\
&\times[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\\
=&\frac{(r+m-t)(s-n-t)}{(r-m)(r-1+m)}[r-1+m]_{t-1}[s-1-n]_{t-1}\\
&+\sum_{k=1}^{t-1}\frac{(r+m-t+k)(s-n-t+k)}{(r-1+m)(r-m-k)}(-1)^k\binom{t-1}{k}\\
&\times[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k\\
&-\sum_{k=0}^{t-2}\frac{k+1}{r-1+m}(-1)^k\binom{t-1}{k+1}[r-1+m]_{t-1-k}[r-1-m]_k[s-1-n]_{t-1-k}[s-1+n]_k
\end{aligned}
$$

Now we only need to handle the coefficients, note for $k=0$ we have

$$
\begin{aligned}
&\frac{(r-m+s-t-n)(r+m-t)}{(r-m)(r-1+m)}-\frac{(r+m-t)(s-n-t)}{(r-m)(r-1+m)}+\frac{1}{r-1+m}\binom{t-1}{1}-1\\
=&\frac{(r-m)(r+m-t)}{(r-m)(r-1+m)}+\frac{t-1}{r-1+m}-\frac{r-1+m}{r-1+m}\\
=&\frac{r+m-t+t-1-(r-1+m)}{r-1+m}\\
=&0
\end{aligned}
$$

And for $k = t-1$ we have

$$\begin{aligned}
&\frac{(r-m+s-t-n)(r+m-1)}{(r-1+m)(r-m-t+1)} - \frac{(r+m-1)(s-n-1)}{(r-1+m)(r-m-t+1)} - 1 \\
=&\frac{r-m+s-t-n-(s-n-1)}{r-m-t+1} - 1 \\
=&0
\end{aligned}$$

For $2 \le k \le t-2$ we have

$$\begin{aligned}
&\left(\frac{(r-m+s-t-n)(r+m-t+k)}{(r-1+m)(r-m-k)} - \frac{(r+m-t+k)(s-n-t+k)}{(r-1+m)(r-m-k)} - 1\right) \\
&\times \binom{t-1}{k} + \frac{k+1}{r-1+m}\binom{t-1}{k+1} \\
=&\frac{r+m-t+k-(r-1+m)}{r-1+m}\binom{t-1}{k} + \frac{k+1}{r-1+m}\binom{t-1}{k+1} \\
=&\frac{k+1}{r-1+m}\binom{t-1}{k} + \frac{k+1}{r-1+m}\binom{t-1}{k+1} - \frac{t}{r-1+m}\binom{t-1}{k} \\
=&\frac{k+1}{r-1+m}\binom{t}{k+1} - \frac{k+1}{r-1+m}\binom{t}{k+1} \\
=&0
\end{aligned}$$

C.2 Identification of even-dimensional $\mathfrak{sl}_2$-modules in $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$

As we have done all the explicit calculations we need, we may also re-confirm that for each $r \ge 1$, the two $2r$-dimensional $\mathfrak{sl}_2$-modules are identified as the same in $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$. Again using Equation (4.5) and (1.5) we have

$$\begin{aligned}
\big(\langle C-(\lambda^2-1)\rangle\big)_{\mathfrak{sl}_2\ltimes V_2} \ni & V^r_{m+1}[C,X_1] - V^r_m[C,X_0] \\
=&\frac{C-r(r-2)}{2r-3}\left((r+m-1)V^{r-1}_m X_0 + (r-2-m)V^{r-1}_{m+1}X_1\right) \\
&+(2r-3)\left(-V^r_m X_0 + V^r_{m+1}X_1\right)
\end{aligned}$$

for all $-r+1 \le m \le r-2$. Compared with the calculated weight vectors in Equation (4.6) and (4.7), our conclusion above is confirmed.

C.3 Another basis of $\widetilde{\mathbf{hs}[\lambda]}$

Modulo the elements in $\{V^r_m[C,X_0], V^r_m[C,X_1] : r \geq 1, |m| < r\} \subset \langle C - (\lambda^2-1)\rangle$, we can find a set of elements that can replace the basis elements

$$U^s_n := \frac{(s+n-2)!}{(2s-3)!}(\mathrm{ad}F)^{s-1-n}(V^s_{s-2}X_0 - V^s_{s-1}X_1) = V^s_{n-1}X_0 - V^s_n X_1$$

defined in Theorem 4.11. We observe from Equation (4.5) that for all required r and m, $V^r_m X_0$ can be replaced by a linear combination of the elements in $\{V^r_{r-1}X_0, V^r_m X_1 : r \geq 1, |m| < r\}$.

To present the explicit calculation results, let us define

$$\begin{aligned}
h^{rm}_k :=& \frac{1}{2} g^{(r-k)2}_2(m,1,\lambda) \\
=& -\frac{r-k-m-1}{2} \\
f^{rm}_k :=& \frac{1}{2} g^{(r-k)2}_3(m,1,\lambda) \\
=& \frac{(r-m-1-k)(r-m-2-k)(r-1-k-\lambda)(r-1-k+\lambda)}{4(2r-3-2k)(2r-1-2k)} \\
g^{rm}_k :=& -\frac{1}{2} g^{(r-k)2}_3(m,0,\lambda) \\
=& \frac{(r-m-1-k)(r+m-1-k)(r-1-k-\lambda)(r-1-k+\lambda)}{4(2r-3-2k)(2r-1-2k)}
\end{aligned}$$

and define $p^{rm}_n \in \mathbb{C}$ by a recurrence relation

$$p^{rm}_n = p^{rm}_{n-2} g^{rm}_{n-1} + p^{rm}_1 p^{rm}_{n-1} \tag{C.2}$$

with $p^{rm}_0 = 1$ and $p^{rm}_1 = -\frac{3+2m}{4}$.

Now the change of basis for $V^r_m X_0$ is listed as follows. We should mention that the equalities below means, strictly speaking, the left-hand side and the right-hand side are equivalent in $U(\mathfrak{sl}_2 \ltimes V_2)[\lambda]$.

If $m = r-1$, keep $V^r_{r-1}X_0$.

If $m = r-2$, then

$$V^r_{r-2}X_0 = V^r_{r-1}X_1 - \frac{2r-1}{4} V^{r-1}_{r-2} X_0.$$

If $m = r-3$, then

$$V^r_{r-3}X_0 = V^r_{r-2}X_1 + \left(p^{r(r-3)}_0 h^{r(r-3)}_1 + p^{r(r-3)}_1\right) V^{r-1}_{r-2} X_1 + p^{r(r-3)}_2 V^{r-2}_{r-3} X_0.$$

If $r-4 \geq m \geq 0$, then

$$\begin{aligned} V^r_m X_0 =& V^r_{m+1} X_1 + \left(p^{rm}_0 h^{rm}_1 + p^{rm}_1\right) V^{r-1}_{m+1} X_1 \\ &+ \sum_{k=2}^{r-m-2} \left(p^{rm}_{k-2} f^{rm}_{k-1} + p^{rm}_{k-1} h^{rm}_k + p^{rm}_k\right) V^{r-k}_{m+1} X_1 + p^{rm}_{r-m-1} V^{m+1}_m X_0. \end{aligned}$$

If $0 \geq m \geq -r+4$, then

$$\begin{aligned} V^r_m X_0 =& V^r_{m+1} X_1 + \left(p^{rm}_0 h^{rm}_1 + p^{rm}_1\right) V^{r-1}_{m+1} X_1 \\ &+ \sum_{k=2}^{r+m-2} \left(p^{rm}_{k-2} f^{rm}_{k-1} + p^{rm}_{k-1} h^{rm}_k + p^{rm}_k\right) V^{r-k}_{m+1} X_1 \\ &+ \left(p^{rm}_{r+m-3} f^{rm}_{r+m-2} + p^{rm}_{r+m-2} h^{rm}_{r+m-1}\right) V^{-m+1}_m X_1 \\ &+ p^{rm}_{r+m-2} f^{rm}_{r+m-1} V^{-m}_{m+1} X_1 + p^{rm}_{r+m-1} V^{-m+1}_m X_0. \end{aligned}$$

If $m = -r+3$, then

$$\begin{aligned} V^r_{-r+3} X_0 =& V^r_{-r+4} X_1 + \left(p^{r(-r+3)}_0 h^{r(-r+3)}_1 + p^{r(-r+3)}_1\right) V^{r-1}_{-r+4} X_1 \\ &+ \left(p^{r(-r+3)}_0 f^{r(-r+3)}_1 + p^{r(-r+3)}_1 h^{r(-r+3)}_2\right) V^{r-2}_{-r+3} X_1 \\ &+ p^{r(-r+3)}_1 f^{r(-r+3)}_2 V^{r-3}_{-r+4} X_1 + p^{r(-r+3)}_2 V^{r-2}_{-r+3} X_0. \end{aligned}$$

If $m = -r+2$, then

$$\begin{aligned} V^r_{-r+2} X_0 =& V^r_{-r+3} X_1 + p^{r(-r+2)}_0 h^{r(-r+3)}_1 V^{r-1}_{-r+3} X_1 + p^{r(-r+2)}_0 f^{r(-r+3)}_1 V^{r-2}_{-r+3} X_1 \\ &+ p^{r(-r+2)}_1 V^{r-1}_{-r+2} X_0. \end{aligned}$$

If $m = -r+1$,then

$$V^r_{-r+1} X_0 = V^r_{-r+2} X_1 + \frac{1-2r}{4} V^{r-1}_{-r+2} X_1.$$

With this new basis, it might be easier to figure out the manifest expression of the structure constants for $\widetilde{\text{hs}[\lambda]}$. This requires find a solution of the recurrence relation (C.2) in a closed form, and is left as part of the future work.

Appendix D

Further Conjectures

D.1 Roots of $\phi_{2(r-1)}^{rs}(\lambda)$

Motivated by the calculation of Example 1.9, we observed an interesting fact that for any fixed $r \in \mathbb{N}, r \geq 3$, the solutions of $\phi_{2(r-1)}^{rs}(\lambda) = 0$ are

$$\{\lambda = \pm(s-k) : k \in \mathbb{N}, 1 \leq k \leq r-2\}.$$

We believe this result can be confirmed for any given r, but we have not attempted a rigorous proof for the general statement, so we decided to leave it as a conjecture for this moment. If this conjecture turned out to be true, then we might be able to come up with yet another proof of Proposition 1.4. In spite of that, this is also an interesting property of generalized hypergeometric functions.

D.2 Explicit inverse map of the symmetrisation

Let $S(\mathfrak{g})$ denote the symmetric algebra of $\mathfrak{g}$, that is,

$$S(\mathfrak{g}) := \frac{T\mathfrak{g}}{\langle \{x \otimes y - y \otimes x : x, y \in \mathfrak{g}\} \rangle}.$$

It is known that there exists a natural bijection $\omega : S(\mathfrak{g}) \to U(\mathfrak{g})$ defined by

$$\omega(x_1 x_2 \cdots x_n) = \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)} \tag{D.1}$$

where $n \geq 1$, $x_1, \cdots, x_n \in \mathfrak{g}$ and S_n is the permutation group. The bijectivity of ω is proved in [7] using abstract tools.

Take $\mathfrak{g} = \mathfrak{sl}_2$, we note the right-hand side of Equation (D.1) is closely related to the explicit expression of the basis elements V_m^r of $U(\mathfrak{sl}_2)[\lambda]$ given in (3.1). We tried to construct ω^{-1} from a purely combinatoric view in order to approach an alternative proof of Proposition 3.6, but the expression turned out to be extremely tedious and not very useful in practice. On the other hand, inspecting the proof of Proposition 3.6, we would expect the explicit map of ω^{-1} could be written in terms of the Clebsh-Gordan coefficients.

D.3 More possible higher spin algebras

The higher spin algebras $\widetilde{\mathrm{hs}[\lambda]}$ we constructed in Chapter 4 accomplishes our initial goal from the mathematical point of view. However, since the commutators between the half-spin generators U^r_m are all trivial, it is not that interesting from the view of physics. A interesting generalisation of $\widetilde{\mathrm{hs}[\lambda]}$ could potentially be obtained by considering the possible Lie superalgebras related to $\mathfrak{sl}_2 \ltimes V_2$, by imposing anti-commutator relations between the half-integer spin generators, as this is the way higher spin superalgebras arise in theoretical physics.

Another straightforward generalisation is to replace $\mathfrak{sl}_2 \ltimes V_2$ by a Hecke algebra which has the same basis $\{F, H, E, X_0, X_1\}$ and same commutation relations as given in Example 4.2 with an additional relation $[X_0, X_1] = z$, where $z \in Z(U(\mathfrak{sl}_2))$.[27] We would expect the corresponding higher spin algebras, as $\mathfrak{sl}_2$-modules, still decompose into the sum of all irreducible $\mathfrak{sl}_2$-modules under the same chosen basis $\{V^r_m : r \geq 2, |m| < r\} \cup \{U^s_n :, s \geq 1, -s \leq n \leq s-1\}$ and moreover, allow non-trivial commutators and bilinear forms on the subspace spanned by $\{U^s_n :, s \geq 1, -s \leq n \leq s-1\}$, which would make the structure more interesting.

Bibliography

[1] W. N. Bailey. Contiguous hypergeometric functions of the type ${}_3F_2(1)$. *Proceedings of the Glasgow Mathematical Association*, 2(2):6265, 1954.

[2] V. Bavula and T. Lu. The universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$, its prime spectrum and a classification of its simple weight modules. *Journal of Lie Theory*, 28:525–560, 01 2018.

[3] L. Biedenharn, J. Louck, and P. Carruthers. Angular momentum in quantum physics: Theory and application. *Cambridge University Press*, 2009.

[4] Y. Cai, R. Shen, and J. Zhang. Whittaker modules and quasi-whittaker modules for the euclidean lie algebra $\mathfrak{e}(3)$. *Journal of Pure and Applied Algebra*, 220(4):1419 – 1433, 2016.

[5] S. Catoiu. Ideals of the Enveloping Algebra $U(\mathfrak{sl}_2)$. *Journal of Algebra*, 202(1):142–177, 1998.

[6] J. Dixmier. Quotients simples de l'algèbre enveloppante de $\mathfrak{sl}_2$. *Journal of Algebra*, 24(3):551–564, 1973.

[7] J. Dixmier. Enveloping algebras. *North-Holland Publishing Company*, 1977.

[8] D. Eisenbud and P. Eisenbud. Commutative algebra: With a view toward algebraic geometry. *Springer*, 1995.

[9] B. L. Feigin. The Lie algebras $\mathfrak{gl}(\lambda)$ and cohomologies of Lie algebras of differential operators. *Russian Mathematical Surveys*, 43(2):169–170, 1988.

[10] E. Fradkin and M. Vasiliev. Candidate for the role of higher-spin symmetry. *Annals of Physics*, 177(1):63 – 112, 1987.

[11] E. S. Fradkin and V. Y. Linetsky. Infinite-dimensional generalizations of finite-dimensional symmetries. *Journal of mathematical physics*, 32(5):1218–1226, 1991.

[12] W. Fulton and J. Harris. Representation theory: a first course. *Springer-Verlag*, 1991.

[13] M. R. Gaberdiel and T. Hartman. Symmetries of holographic minimal models. *Journal of High Energy Physics*, 2011(5):31–56, 2011.

[14] Harish-Chandra. On some applications of the universal enveloping algebra of a semisimple lie algebra. *Transactions of the American Mathematical Society*, 70(1):28–96, 1951.

[15] J. Humphreys. Introduction to Lie algebras and representation theory. *Springer*, 1972.

[16] A. Isaev. Twenty-one lectures on complex analysis. *Springer*, 2017.

[17] A. Joseph. Minimal realizations and spectrum generating algebras. *Communications in Mathematical Physics*, 36(4):325–338, 1974.

[18] E. Joung and K. Mkrtchyan. Notes on higher-spin algebras: minimal representations and structure constants. *Journal of High Energy Physics*, 2014(5):103, 2014.

[19] G. Kemper. A course in commutative algebra. *Springer Berlin Heidelberg*, 2010.

[20] S. D. Majumdar. The Clebsch-Gordan Coefficients. *Progress of Theoretical Physics*, 20(6):798–803, 12 1958.

[21] J. Maldacena. The large-n limit of superconformal field theories and supergravity. *International journal of theoretical physics*, 38(4):1113–1133, 1999.

[22] C. Pope, X. Shen, and L. Romans. $W(\infty)$ and the Racah-Wigner algebra. *Nuclear Physics B*, 339(1):191–221, 1990.

[23] S. Prokushkin and M. Vasiliev. 3d higher-spin gauge theories with matter. *arXiv preprint hep-th/9812242*, 1998.

[24] S. Prokushkin and M. A. Vasiliev. Higher-spin gauge interactions for massive matter fields in 3d ads space-time. *Nuclear physics B*, 545(1-3):385–433, 1999.

[25] J. Shapiro. Burnside's theorem on matrix algebras. *Lecture notes, Portland State University Analysis Seminar*, 2014.

[26] S. Sternberg. Lie algebras. *Lecture notes, University Press of Florida*, 2009.

[27] A. Tikaradze and A. Khare. Center and representations of infinitesimal Hecke algebras of $\mathfrak{sl}_2$. *Communications in Algebra*, 38(2):405–439, 2010.

[28] M. A. Vasiliev. Higher spin gauge theories in any dimension. *Comptes Rendus Physique*, 5(9-10):1101–1109, 2004.

[29] M. A. Vasiliev. Higher spin superalgebras in any dimension and their representations. *Journal of High Energy Physics*, 2004(12):046, 2005.