The Selberg Trace Formula & Prime Orbit Theorem

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May 2019

A thesis submitted for the degree of Honours
of the Australian National University

Declaration

The work in this thesis is my own except where otherwise stated.

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Acknowledgements

I would like to express my gratitude to my supervisor, Prof.Andrew Hassell, for his dedication to my honours project. Without his inspiring and patient guidance, non of this could have be accomplished.

I would like also to thank my colleague Kelly Maggs, for his help and encouragement, so I can work with confidence and delight, even in hard times.

Furthermore, I would like to thank my colleague Dean Muir and Diclehan Erdal, for their precious advice about writing styles for this thesis.

Finally, I would like to thank my families for their warming support to my study, despite their holding a suspicious attitude about whether I can find a job after graduating as a mathematician.

Abstract

The purpose of this thesis is to study the asymptotic property of the primitive length spectrum on compact hyperbolic surface S of genus at least 2, defined as a set with multiplicities:

$$\mathcal{L}_S = \{l(\gamma) : \gamma \text{ is a primitive oriented closed geodesic on } S\}.$$

where $l(\gamma)$ denotes the length of γ. In particular, we will prove the Prime Orbit Theorem. That is, for the counting function of exponential of primitive lengths, defined as

$$\pi_0(x) = \#\{l : l \in \mathcal{L}_S \text{ and } e^l \leq x\},$$

we have the asymptotic behavior of $\pi_0(x)$:

$$\pi_0(x) \sim \frac{x}{\ln(x)}.$$

The major tool we will use is the Selberg trace formula, which states the trace of a certain compact self-adjoint operator on $L_2(S)$ can be expressed as a sum over conjugacy classes in hyperbolic Fuchsian groups. We will generalize the ideas of Poisson Summation formula and Laplacian eigenvalue counting problem on the torus to prove the Selberg trace formula.

Furthermore, we will prove the Prime Number Theorem, which states for the prime counting function

$$\pi(x) = \#\{p : p \leq x \text{ and } p \text{ is a prime}\}.$$

we have the asymptotic behavior

$$\pi(x) \sim \frac{x}{\ln(x)}.$$

We will elucidate the similarity between the Prime Number Theorem and the Prime Orbit Theorem, and use the analogy to produce a proof of the Prime Orbit Theorem, based on the Wiener-Ikehara Tauberian Theorem.

Contents

Notation and terminology

The notation used in the thesis is fairly standard. Some other notations are rarely used and will be introduced upon using.

Notation

$\mathbb{C}$	The complex plane
$\mathbb{H}$	The upper-half plane
$\mathbb{U}$	The unit disk
Γ	A Fuchsian group
$\mathcal{F}$	A fundamental region of a Fuchsian group (There are more than one fundamental regions of a Fuchsian group, we usually work with a compact one.)
$\mathcal{O}(f(x))$	The standard big-O notation. $g(x) = \mathcal{O}(f(x))$ means $\lvert g(x)\rvert \leq Mf(x)$ for some constant M as $x \to \infty$.
$f(x) \sim g(x)$	An equivalent relation for functions $\mathbb{R} \to \mathbb{R}$, $f(x) \sim g(x)$ as $x \to a$ means $\lim_{x\to a} f(x)/g(x) = 1$, where $a \in \mathbb{R} \cup \{\infty\}$.
$\mathbf{PSL}(2, \mathbb{R})$	$\mathbf{SL}(2, \mathbb{R})/(\pm\, I)$, where I is the identity element.
Δ	The Laplacian operator on $\mathbb{R}^2$, $\Delta f = f_{xx} + f_{yy}$
$C_c^k(E)$	The set of $\mathbb{C}$-valued C^k functions on E that have compact support
$\mathbb{H}/\Gamma$	The Riemann surface constructed by quotient action of Γ on $\mathbb{H}$

$\mathcal{Z}_\Gamma(T)$	The centralizer of T, $\{S \in \Gamma \mid ST = TS\}$. That is, the group members in Γ that commutes with T.
$L_2(E)$	The Hilbert space of all square integrable functions on (E)
$\langle f, g\rangle$	The Hilbert space inner product of f and g.

Terminology

Fuchsian Group	A discrete subgroup of $\mathbf{PSL}(2, \mathbb{R})$
Discrete set	In this thesis, a set is discrete means there is no accumulation point. It is slightly stronger than the definition of each point being an isolated point.

Chapter 1

Introduction

The main purpose of this thesis is to prove the Prime Orbit Theorem (sometimes called the Prime Geodesic Theorem) on compact Riemann surfaces of genus at least two, with hyperbolic metric. We will achieve this goal in five chapters. The main tool we will use is the Selberg trace formula.

In chapter 2 we will introduce some background to hyperbolic geometry. The main objects we will deal with in chapters 5 and 6 are all closely related to hyperbolic geometry, including the Prime Orbit Theorem itself. Throughout the thesis, we will mainly use the upper-half plane $\mathbb{H}$ with the Poincaré metric as the model of hyperbolic plane, but the unit disk model will be occasionally used when convenient.

The crucial tool we use in hyperbolic geometry is the isometry group of upper-half plane, which isomorphic to the group $\mathbf{PSL}(2,\mathbb{R})$. In particular, we will focus on the discrete subgroups of isometry group $\mathbf{PSL}(2,\mathbb{R})$, namely Fuchsian groups Γ. We will discuss in details about the the functions on quotient space $\mathbb{H}/\Gamma$. Group elements of $\mathbf{PSL}(2,\mathbb{R})$ are classified by their fixed points as automorphism of Riemann sphere $\mathbb{C}_\infty$. There are three kinds of elements: elliptic, parabolic and hyperbolic. We mainly focus on the hyperbolic elements, which by definition have two fixed point on the boundary of $\mathbb{H}$. The reason for this is that the quotient space $\mathbb{H}/\Gamma$ is compact if and only if all non-identity elements in Γ are hyperbolic. We will discuss this in detail at the beginning of chapter 5. Throughout the entire thesis, the readers should keep in mind that the Fuchsian group we use is a strictly hyperbolic one. The reason for this is also given at the beginning of chapter 5.

There are some specific properties for a hyperbolic element in $\mathbf{PSL}(2,\mathbb{R})$. We will discuss the axis and the displacement length of a hyperbolic element. Since a hyperbolic element T fixes two point on the boundary of $\mathbb{H}$, it preserves the unique geodesic connecting those two fixed points. We call the geodesic $\alpha(T)$, the *axis* of T. By conjugation, we can convert T into the standard dilation form $z \mapsto e^{lz}$ for some $l > 0$. We call this number l the *displacement length*. We will see that the elements in the same conjugacy class in Γ have the same displacement length, and it is the distance of the points on $\alpha(T)$ traversed under the action of T. We shall see in chapter 6 that $\alpha(T)$ represents a closed geodesic in $\mathbb{H}/\Gamma$ of length l.

There are two useful kinds of objects connected to Fuchsian groups, namely fundamental domains and primitive elements. A fundamental domain $\mathcal{F}$ of a Fuchsian group Γ is a region in $\mathbb{H}$ such that its images under Γ tessellate the whole upper-half plane and have disjoint interior. Any such fundamental domain $\mathcal{F}$ is isometrically related to the quotient space $\mathbb{H}/\Gamma$, thus we can transfer the problem of functions on $\mathbb{H}/\Gamma$ to the ones on $\mathcal{F}$. In particular, when $\mathbb{H}/\Gamma$ is compact, we can choose $\mathcal{F}$ to be a compact polygon with its boundary being geodesic segments, so the geometric properties are nice. This is analogous to a flat torus, viewed as a rectangle with opposite sides identified. We will utilize this property in chapter 5. The other useful objects, primitive elements in Γ, are defined to be elements that are not a higher power of another element in Γ. At the end of chapter 2, we will prove that every element in Γ can be expressed uniquely as a positive power of a primitive element. We will see at the beginning of chapter 6 that primitive elements in Γ correspond to primitive geodesics on $\mathbb{H}/\Gamma$, which are the geodesics that are not an iterate of a shorter geodesic. We treat geodesics as paths with an orientation. It should not be surprising that a primitive element and its inverse, which is another primitive element, correspond to the same geodesic curve with opposite orientation.

At the beginning of chapter 5 (Theorem 5.1), we will use the big result from algebraic topology, namely the Uniformization Theorem, to prove the connection between the the quotient space $\mathbb{H}/\Gamma$ and compact Riemann surfaces. It is revealed that any compact Riemann surface S of genus ≥ 2 can be expressed as the quotient space $\mathbb{H}/\Gamma$ with Γ being a strictly hyperbolic Fuchsian group. This result allows us to utilize the properties of Fuchsian groups and fundamental regions we proved in chapter 3. This give us one of two major tools we need in 5 to prove the goal of that chapter, the Selberg trace formula (Theorem 5.4).

The other major tool we will use in chapter 5 is given in chapter 4, namely determining the distribution of eigenvalues of the Laplacian. We will show in chapter 4 that the Laplace-Beltrami operator on compact Riemann surface has the eigenvalues forming an increasing non-negative sequence $\lambda_0, \lambda_1, \ldots$ tends to infinity. This induces the study of asymptotic behavior of the eigenvalue counting function, which counts the number of eigenvalues below a certain value, it is defined as following:

$$N(x) = \#\{n : \lambda_n \leq x\} \qquad \text{for } x > 0.$$

The eigenvalue counting problem is closely related to the study of heat kernel on Riemann surfaces. We will use some conclusions and techniques obtained by study on the heat kernels in [17] chapter 8. In chapter 4 we will discuss in details about the eigenvalue counting problem on a torus $\mathcal{T}$, which is a compact Riemann surface of genus 1. We will construct $\mathcal{T}$ as the quotient space of $\mathbb{C}$ over a integer lattice. The methods and ideas we used there are very inspiring to our study of eigenvalues of Laplacian on the genus at least 2 cases in chapter 5.

The essential idea in chapter 4 is to convert the eigenvalue counting problem to the classic Gaussian circle problem, which is the study of asymptotic behavior of the lattice counting function:

$$\begin{aligned} r(n) &= \#\{(a,b) \in \mathbb{Z} \times \mathbb{Z} : n = a^2 + b^2\}, \\ A(x) &= \sum_{0 \leq n \leq x} r(n) = \pi x + \mathcal{R}(x). \end{aligned}$$

where $A(x)$ is the counting function of the number of integer lattice points within a circle of radius $\sqrt{x}$. We will use the Poisson summation formula to prove Theorem 4.5, which allows us to obtain an accurate estimation to the remainder function $\mathcal{R}(x)$. The derivation of Poisson summation formula comes from the construction of function in $L_2(\mathcal{T})$ in two ways, one is via Fourier series, and the other is to start with a compactly supported function and translate it using the groups of integer translations.

With the both tools mentioned above, we are able to prove the Selberg trace formula (Theorem 5.4) in chapter 5. Following from the analogous idea as we did in chapter 4, we will construct the function in $L_2(\mathcal{F})$ in two different ways. One is through the eigenbasis of Laplacian in $L_2(\mathcal{F})$ and the other is using a compactly supported real-valued function on $\mathbb{R}$ and the hyperbolic distance formula (2.5). Using the rotational invariance of the Laplace-Beltrami operator, we are able to write the function constructed above as the trace of a compact self-adjoint operator on $L_2(\mathcal{F})$ (for backgrounds about the trace of an operator, see Appendix

A). The result is shown in equation (5.9). Then we expand this trace in two different ways: We can derive a explicit formula for one side of (5.9), and for the other side we use the conjugacy classes in Γ to expand the sum. Combining the result obtained in two ways together, we get the expression for the Selberg trace formula.

With the Selberg trace formula, we acquire a very powerful mathematical tool on the surface S, but it is not yet clear why we would expect a conclusion like the Prime Orbit Theorem. The reason is given in two aspects. Firstly, at the beginning of chapter 6, we will prove that there is a one to one correspondence between conjugacy classes in Γ and closed geodesic in S. With the discussion above about axis and displacement length of hyperbolic elements, the readers should not be surprised by the fact that the displacement length of a conjugacy class is precisely the length of its corresponding geodesics, and the primitive elements correspond to primitive geodesics. So the sum we expanded in (5.9) using conjugacy classes in Γ can now be utilized to reflect the information about length spectrum of geodesic on S. We construct $\mathcal{L}_S$, the primitive length spectrum on S, being the collection of length of primitive closed geodesics on S. We will discuss the details in chapter 6. The other aspect mentioned above is the inspiration of the Prime Orbit Theorem: we will prove the Prime Number Theorem (Theorem 3.3) in chapter 3.

The Prime Number Theorem (Theorem 3.3), is not only the origin of the name "Prime Orbit Theorem", but also supplies us the major idea of how we would accomplish the proof. In chapter 3, we use the analytic properties of Riemann zeta function to prove the Prime Number Theorem. We prove that the Riemann zeta function $\zeta(s)$ is meromorphic on the half-plane $\mathrm{Re}(s) > 0$ with only one simple pole at $s = 1$. We also prove that $\zeta(s)$ has no zeros in the half-plane $\mathrm{Re}(s) \geq 1$. We convert the problem to the asymptotic behavior of the logarithm sum function:

$$\theta(x) := \sum_{p \leq x} \ln(p),$$

where p denotes the prime numbers. By the Wiener-Ikehara Tauberian theorem (Theorem 3.16), ultimately we convert the problem to the analyticity of the function

$$\Phi(s) := \sum_{p} \frac{\ln(p)}{p^s},$$

along the line $\mathrm{Re}(s) = 1$. In Lemma 3.18, we related the analyticity of function

$\Phi(s)$ to the fraction $\zeta'(s)/\zeta(s)$. This shows $\Phi(s)$ is meromorphic on $\mathrm{Re}(s) \geq 1$ with only a simple pole at $s = 1$. Then the Wiener-Ikehara Tauberian theorem implies the result $\theta(x) \sim x$ and eventually proves the Prime Number Theorem.

The proof of the Prime Orbit Theorem is chapter 6 is analogous. By applying the Selberg trace formula to the heat trace as implied at the beginning of chapter 4, we see the asymptotic behavior of the result related to the analyticity of the function Φ_0, defined as the following:

$$\Phi_0(s) := \sum_{j=1}^{\infty} \frac{\ln(\xi_j)}{\xi_j^s},$$

where $\xi_j = e^{l_j}$, with l_j being the j^{th} length in the primitive length spectrum, ordered in an ascending manner. The readers should have noticed the function Φ_0 is defined in a very similar way to the function Φ of primes, with e^{l_j} playing the role of j^{th} prime, ordered from lower to higher as well. The result of asymptotic analysis above implies Φ_0 has a simple pole at $s = 1$. We are then encouraged to proceed the similar steps as in chapter 3 and apply the Wiener-Ikehara Tauberian theorem. To do this we need to check the analyticity of Φ_0 at all the other points excluding $s = 1$ on the half-plane $\mathrm{Re}(s) \geq 1$, and this is achieved by applying the Selberg trace formula again to the Resolvent of Laplacian. Then the Prime Orbit Theorem (Theorem 6.4) is proved.

The main idea of approaching the Prime Orbit Theorem in above manner is inspired by D.Hejhal's paper "The Selberg Trace Formula and the Riemann Zeta Function" [9] and D.Borthwick's book "Spectral Theory of Infinite-area Hyperbolic Surfaces"[5]. The succinct proof of the Prime Number Theorem in chapter 3 is partly from the notes written by M.Baker and D.Clark [3], which is based on the previous work of D.Newman and D.Zagier. The Selberg trace formula was first proved by A.Selberg himself around 1950 [15].

Chapter 2

Hyperbolic Geometry & Fuchsian Group

2.1 Hyperbolic geometry

2.1.1 The upper-half plane model

It is known that there is a unique simply connected hyperbolic surface up to isometry, namely the hyperbolic plane. We first introduce the upper half-plane model, with the standard Poincaré metric.

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} \,:\, y > 0\}, \qquad \mathrm{d}s = \frac{\sqrt{dx^2 + dy^2}}{y}, \qquad \mathrm{d}\mu(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

The geodesics on $\mathbb{H}$ are the arcs of circles that intersect $\partial\mathbb{H}$ orthogonally, including straight lines orthogonal to the real axis.

We can also define the *geodesic polar* coordinates $(r, \theta) \in \mathbb{R}_+ \times S^1$, which is asymptotic to Euclidean polar coordinates as $r \to 0$:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2(r)\,\mathrm{d}\theta^2, \qquad\qquad \mathrm{d}\mu(z) = \sinh(r)\,\mathrm{d}r\,\mathrm{d}\theta.$$

The geodesic polar coordinates allows us to conveniently calculate the area of a ball in $\mathbb{H}$, defined as:

$$B(w, R) := \{z : d(z, w) < R\}.$$

Then the area of $B(w, R)$ is:

$$\mathcal{A}(B(w, R)) = \int_0^{2\pi}\int_0^R \sinh(r)\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi(\cosh(R) - 1).$$

It follows that:

$$\mathcal{A}(B(w, R)) = \mathcal{O}(e^R). \tag{2.1}$$

2.1.2 The unit disk model

Another well-known model of hyperbolic surface is the unit disk model.

$$\mathbb{U} := \{z \in \mathbb{C} \,:\, |z| < 1\} \qquad\qquad \mathrm{d}s = \frac{2|\mathrm{d}z|}{1-|z|^2}.$$

From complex analysis, we know $\mathbb{H}$ and $\mathbb{U}$ are conformally equivalent. In particular, there are conformal map between $\mathbb{H}$ and $\mathbb{U}$ that are Möbius transformations. For example, we can choose:

$$\begin{aligned} f &: \mathbb{U} \to \mathbb{H} \qquad z \mapsto \frac{z+i}{iz+1}, \\ g &: \mathbb{H} \to \mathbb{U} \qquad z \mapsto \frac{z-i}{-iz+1}. \end{aligned}$$

Note that f and g are inverse of each other.

2.1.3 Isometry on upper-half plane

We care about a particular kind of map on $\mathbb{H}$, namely the members of $\mathbf{PSL}(2,\mathbb{R})$, with the correspondence as following:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad z \mapsto Tz := \frac{az+b}{cz+d}. \tag{2.2}$$

And the following theorem shows the reason.

Theorem 2.1. *The group of orientation-preserving isometries of* $\mathbb{H}$ *is precisely the group* $\mathbf{PSL}(2,\mathbb{R})$.

In the proof of the theorem, we adapt a very common definition:

Definition 2.2. A *Möbius transformation* is a map such that:

$$f(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

The definition implies that a Möbius transformation can be represented by a matrix in $\mathbf{GL}(2,\mathbb{C})$. Since rescaling by a non-zero constant on the matrix does not change the map, we can assume determinant of T is 1, which narrows down the case to $\mathbf{PSL}(2,\mathbb{C})$.

Proof of Theorem 2.1 (Sketch). Since the Euclidean metric and hyperbolic metric are conformal related, an isometry of $\mathbb{H}$ must preserve Euclidean angles. It follows from definition that these maps must be a conformal automorphism of $\mathbb{H}$. Using the Schwarz lemma, we can show all conformal automorphisms of $\mathbb{H}$ must be a Möbius transformation. A simple calculation shows that a Möbius transformation preserves the upper half-plane if and only if all of its entries are real. Thus the maps satisfying all the requirements are precisely those represented by $\mathbf{PSL}(2,\mathbb{R})$. On the other hand, suppose $T \in \mathbf{PSL}(2,\mathbb{R})$. A short calculation shows

$$T'(z) = \frac{1}{(cz+d)^2} \qquad \operatorname{Im}(Tz) = \frac{\operatorname{Im}(z)}{|cz+d|^2}. \tag{2.3}$$

Hence the pullback of the metric is

$$T^*(\mathrm{d}s^2) = \frac{|T'(z)\mathrm{d}z|^2}{(\operatorname{Im}(Tz))^2} = \frac{|dz|^2}{\operatorname{Im}(z)^2} = \mathrm{d}s^2,$$

which shows T is an isometry. □

We classify non-identity members of $\mathbf{PSL}(2,\mathbb{R})$ by their fixed points, which are closely related to their matrix trace. Solving the equation $z = Tz$ will lead to the polynomial equation:

$$cz^2 + (d-a)z - b$$

, with discriminant:

$$(d-a)^2 + 4bc = (d+a)^2 + 4bc - 4ad = (d+a)^2 - 4 = (\operatorname{tr} T)^2 - 4.$$

Thus the trace of T determines the type of fixed point in $\mathbb{H}$ and its boundary.

Definition 2.3. A transformation T represented by a matrix in $\mathbf{PSL}(2,\mathbb{R})$ is:

1. *elliptic* if $\operatorname{tr} T < 2$, implying one fixed point within $\mathbb{H}$ (with a matching one in the lower half plane.)

2. *parabolic* if $\operatorname{tr} T = 2$, implying a single double-root fixed point in $\partial\mathbb{H}$.

3. *hyperbolic* if $\operatorname{tr} T > 2$, implying two distinct fixed points in $\partial\mathbb{H}$.

Remark 2.4. We call a subgroup H of $\mathbf{PSL}(2,\mathbb{R})$ *elliptic* if all non-identity elements in H are elliptic, similarly for parabolic and hyperbolic cases.

With the information of isometry on $\mathbb{H}$, we can prove the following lemma, which gives us a very useful expression for hyperbolic distance.

Lemma 2.5. *For $z, z' \in \mathbb{H}$,*

$$d(z, z') = \cosh^{-1}\left(1 + \frac{|z - z'|^2}{2yy'}\right). \tag{2.4}$$

Proof. First we show that the right hand side of (2.4) is invariant under isometry. Let T be an isometry as in (2.3), from the first equation in (2.3):

$$|Tz - Tz'|^2 = \frac{(ad - bc)|z - z'|^2}{|cz + d|^2|cz' + d|^2} = |T'(z)T'(z')(z - z')^2|,$$

where we used $ad - bc = 1$ for T. Then use this and the second equation in (2.3), we have:

$$\begin{aligned} \frac{|Tz - Tz'|^2}{2\mathrm{Im}(Tz)\mathrm{Im}(Tz')} &= \frac{|T'(z)T'(z')(z - z')^2|}{\mathrm{Im}(z)\mathrm{Im}(z')}|cz + d|^2|cz' + d|^2 \\ &= \frac{|z - z'^2|}{\mathrm{Im}(z)\mathrm{Im}(z')}. \end{aligned}$$

For any two points in $\mathbb{H}$, it is easy for us to construct an isometry in $\mathbf{PSL}(2, \mathbb{R})$ to map those two points onto y-axis. Then since by definition $d(z, z')$ is invariant under isometry, it suffices to show the formula holds for two points ia and ib for $b > a > 0$ on y-axis. A direct calculation from metric shows that $d(ia, ib) = \ln(b/a)$, which is consistent with (2.4) by another simple calculation. □

2.2 Fuchsian group

2.2.1 General properties

It is natural to consider the hyperbolic surface obtained by quotient $\mathbb{H}/\Gamma$, where Γ is a subgroup of $\mathbf{PSL}(2, \mathbb{R})$. To obtain a well-defined metric space, we need some condition on Γ. This is given by the following definition.

Definition 2.6. Let a group G acting on metric space X, we say the action is *properly discontinuous* if any compact subset of X only intersects with finitely many orbit points of the action.

From some basic analysis, we can conclude that $\mathbb{H}/\Gamma$ is a well-defined metric space if and only if the action of Γ on $\mathbb{H}$ is properly discontinuous.

Now we are prepared to define one of the major tools towards Selberg trace formula in chapter 4: the *Fuchsian group*.

Definition 2.7. A *Fuchsian group* is a discrete subgroup of $\mathbf{PSL}(2, R)$

The reason for making such a definition is given by the following proposition:

Proposition 2.8. *A subgroup* $\Gamma \subset \mathbf{PSL}(2, \mathbb{R})$ *acts properly discontinuously on* $\mathbb{H}$ *if and only if* Γ *is a Fuchsian group.*

Proof. We follow the idea in [5], chapter 2. First suppose Γ is a Fuchsian group, then it follows from basic algebra that an orbit Γz is discrete. Let S be a compact subset of $\mathbb{H}$, then $S \cap \Gamma z$ is a discrete compact set and therefore finite. Thus Γ acts properly discontinuously.

For the converse, suppose Γ act properly discontinuously. We first show that there are points which are not fixed by any non-identity element of Γ. Let $T \neq I \in \Gamma$. If $Tw = w$ for some $w \in \mathbb{H}$, then for any $z \in \mathbb{H}$, from triangle inequality and T being an isometry, we have

$$d(Tz, z) \leq d(Tz, Tw) + d(Tw, z) = 2d(z, w).$$

Then since every neighborhood of z can only contains finitely many point s of the form Tz, every neighborhood of z only contains finitely many such w fixed by some T as well. We pick $z_0 \in \mathbb{H}$ not fixed by any non-identity element. Suppose Γ is not discrete for contradiction, then there exist a sequence $\{T_k\}$ of distinct elements such that $T_k \to I$. The sequence $\{T_k z_0\}$ contains only distinct points by construction and $\{T_k z_0\} \to z_0$. This gives an accumulation point of orbit and hence contradicts the proper discontinuity of the action. $\square$

An important object that relates to the Fuchsian group is the *fundamental domain*, which will be discussed more in chapter 4.

Definition 2.9. A *fundamental domain* $\mathcal{F} \subset \mathbb{H}$ for a Fuchsian group Γ is a closed set such that

$$\Gamma\mathcal{F} := \bigcup_{T \in \Gamma} = \mathbb{H},$$

and for each $T \neq I$, the interior of $\mathcal{F}$ and $T\mathcal{F}$ does not intersect.

It is clear from the definition that if $\mathcal{F}$ is a fundamental domain, so is $T\mathcal{F}$ for any $T \in \Gamma$. Since each $T \in \Gamma$ is an isometry, all the fundamental domains have the same area (if they are finite). There is a particular kind of fundamental domain, called the *Dirichlet domain*, defined as:

$$\mathcal{D}_w := \{z \in \mathbb{H} : d(z, w) \leq d(z, Tw) \text{ for all } T \in \Gamma\}.$$

In the case that w is not an elliptic fixed point of Γ, $\mathcal{D}_w$ is a fundamental domain for Γ. Furthermore, $\mathcal{D}_w$ is convex and its boundary is a union of geodesics (See [5], Chapter 2).

2.2.2 Hyperbolic element in Fuchsian group

Now we will discuss some special properties of hyperbolic elements in $\mathbf{PSL}(2, R)$, since those will be the object of discussion in chapter 4. The specific reason why we care about hyperbolic case particularly is given by Theorem 5.1.

In the previous section, we know that a hyperbolic element T in $\mathbf{PSL}(2, R)$ has two fixed points on $\partial\mathbb{H}$. Then there is a unique geodesic connecting those two fixed points. We call this geodesic $\alpha(T)$ the *axis* of T. Note that T preserves $\alpha(T)$.

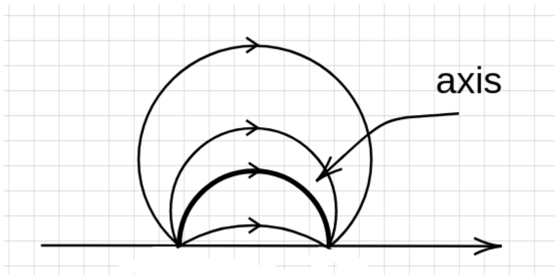


Figure 2.1: Hyperbolic fixed point and axis.

For such a hyperbolic element T, one of the fixed point is attracting and the other is repelling, in the sense that T transfers the point on the axis away from the repelling fixed point towards the attracting one (See figure 2.1). By conjugation, we can transfer the repelling fixed point to 0 and the attracting one to ∞. Since conjugation preserves trace, we will have another hyperbolic element $S = PTP^{-1}$ with $\alpha(S)$ being the y-axis. For the resulting map $S = \frac{az+b}{cz+d}$, by considering the fixed points we have $b = c = 0$, $S(z) = \frac{az}{d}$. We have $a/d > 1$ since S fixes $\mathbb{H}$ and 0 is a repelling fixed point. Then $a/d = e^l$ for some $l > 0$, we have S is a dilation. By construction, this l is unique. We make the following definition:

Definition 2.10. Let $T \in \mathbf{PSL}(2, R)$ be a hyperbolic element. Then there exists unique $S \in \mathbf{PSL}(2, R)$ conjugate to T, with $S(z) = e^l z$. We call this number $l = l(T)$ the *displacement length* of T.

Since $l(T)$ is invariant under conjugation, we can use it to classify the conjugation class of T. We also has the following Proposition that is crucial in chapter 5.

Proposition 2.11. *For a hyperbolic element $T \in \boldsymbol{PSL}(2, R)$, we have:*

$$l(T) = \min_{z \in \mathbb{H}} d(z, Tz), \tag{2.5}$$

with the minimum is achieved if and only if $z \in \alpha(T)$.

Proof. Let $S(z) = e^l z$ be the map after conjugation as above, $S = PTP^{-1}$. Then note that $d(z, Sz) = d(P^{-1}z, TP^{-1}z)$, thus $\min_{z\in\mathbb{H}} d(z, Tz) = \min_{z\in\mathbb{H}} d(z, Sz)$.

Then apply (2.4) to $d(z, e^l z)$, we have:

$$\cosh(d(z, e^l z)) = 1 + \frac{|z - e^l z|^2}{2y^2 e^l} = 1 + \frac{1 - e^l}{2e^l}\frac{x^2 + y^2}{y^2}.$$

Note that the function $\cosh(u)$ is injective and increasing for $u \geq 0$. Then it's clear that the right hand side of above equation is minimized when $x = 0$.

$$\min_{z\in\mathbb{H}} d(z, e^l z) = \cosh^{-1}\big(1 + \frac{1 - e^l}{2e^l}\big).$$

But the right hand side of above equation is precisely l by definition of cosh. Note that a points z minimize the distance if and only $z \in \alpha(S)$. Since isometry preserves geodesics and P, P^{-1} are invertible maps, we have

$$\alpha(PTP^{-1}) = P(\alpha(T)), \tag{2.6}$$

so P must maps $\alpha(T)$ to $\alpha(S)$. Thus $P^{-1}z \in \alpha(T)$. □

Remark 2.12. This proposition reveals that $l(T)$ is the length of the a geodesic connecting z and Tz for some $z \in \mathbb{H}$ on $\alpha(T)$. We will discuss how this can be used for geometric properties in chapter 5. Note that $l(T)$ is precisely the distance of points on $\alpha(T)$ being transfered under T, away from the repelling fixed point of T towards the attracting one.

Another useful concept in chapter 5 is *primitive* elements in the Fuchsian group, which means the element is not a positive power of some other element. To define it formally:

Definition 2.13. Let Γ be a Fuchsian group, we say an element T is *primitive* if it cannot be written as S^k with $|k| > 1$ for any $S \in \Gamma$.

The following proposition holds in general for any Fuchsian group, but we only prove it for strictly hyperbolic case since it is sufficient for our purpose.

Proposition 2.14. *Let Γ be a strictly hyperbolic Fuchsian group. Then each element $S \in \Gamma$ can be written uniquely as a power T^k for $T \in \Gamma$ primitive and $k \geq 1$. Furthermore, $\mathcal{Z}_\Gamma(S) = \langle T \rangle$, the cyclic group generated by T.*

Proof. It suffices to prove the proposition for the conjugated element $S(z) = e^l z$, since if $S = T^k$ then $R = PSP^{-1} = (PTP^{-1})^k$. Suppose $R \in \mathcal{Z}_\Gamma(S)$, then $RS = SR$ implies the matrix for R satisfies:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix} = \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.7}$$

which means $b = c = 0$. Then $R = \frac{a}{d} z = e^t z$ for some $t \in \mathbb{R}$, note $t = t(R)$ might be negative. It's clear $t(I) = 0$ and $t(RR') = t(R) + t(R')$ for any $R, R' \in \mathcal{Z}_\Gamma(S)$, so the map $R \mapsto t(R)$ is a group homomorphism $\mathcal{Z}_\Gamma(S) \to \mathbb{R}$. Since Γ is discrete, the image of $\mathcal{Z}_\Gamma(S)$ is a discrete subset of $\mathbb{R}$ hence must be a lattice $t_0\mathbb{Z}$ for some minimum $t_0 > 0$. Then the choice $T(z) = e^{t_0} z$ uniquely satisfies the condition in the proposition and it follows $k = l/t_0$. It is clear that T generates $\mathcal{Z}_\Gamma(S)$. $\square$

Chapter 3

Riemann Zeta Function & the Prime Number Theorem

In this chapter, we will prove the Prime Number Theorem. The major tools we will use are some analytic properties of the Riemann zeta function. The proof of the Prime Number Theorem is instructive as to how should we approach the Prime Orbit Theorem in chapter 5.

To state the Prime Number theorem, we first make a few definitions.

Definition 3.1. We define the function $\pi(x)$ on $\mathbb{R}$ to be the counting function of number of primes less than or equal to x.

$$\pi(x) = \#\{p : p \leq x \text{ and } p \text{ is a prime}\}.$$

Definition 3.2. Let $f, g : \mathbb{R} \to \mathbb{R}$, we define:

1. $f = \mathcal{O}(g)$ if $\exists M \in \mathbb{R}$ such that $|f| \leq Mg$.
2. $f \sim g$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. We will usually simplify this to write $f \sim g$ to mean $f \sim g$ as $x \to \infty$, unless otherwise stated.

The Prime Number theorem then stated as the following:

Theorem 3.3 (the Prime Number Theorem)**.**

$$\pi(x) \sim \frac{\ln(x)}{x}. \tag{3.1}$$

We will denote the Prime Number Theorem as PNT from here.

3.1 The Riemann zeta function

To prove PNT, we will need some analytic properties of the Riemann zeta function. The well-known Riemann zeta function is defined as follows.

Definition 3.4 (the Riemann zeta function)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for} \quad \mathrm{Re}(s) > 1.$$

By the Weierstrass M-test, we see $\zeta(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. It is clear that for any $\delta > 0$, the partial sums converge absolutely and uniformly on any compact subset of the half-plane $\mathrm{Re}(s) > 1 + \delta$, so we conclude that $\zeta(s)$ is analytic on $\mathrm{Re}(s) > 1$. ([2] Theorem 7.6)

At first glance it is not obvious why the Riemann zeta function is deeply connected with the distribution of primes. The next Lemma, the Euler product formula, would illuminate the link between these two.

Lemma 3.5 (The Euler product formula for ζ)**.**

$$\zeta = \prod_p \frac{1}{1 - p^{-s}} \qquad \text{for} \quad \mathrm{Re}(s) > 1, \tag{3.2}$$

where p denotes primes.

Before we prove Lemma 3.5, let us introduce some basic concepts about infinite products. The proof can be found in [2] chapter 17.

Proposition 3.6. *If $\sum_{k=1}^{\infty} |z_k|$ is convergent, $\prod_{k=1}^{\infty}(1 + z_k)$ converges to a non-zero limit.*

Definition 3.7. We say $\prod_{k=1}^{\infty}(1 + z_k)$ is *absolutely convergent* if $\prod_{k=1}^{\infty}(1 + |z_k|)$ is convergent.

Proposition 3.8. *An absolutely convergent product is convergent.*

Similar to the infinite sum case, if a product is absolutely convergent, we can rearrange its term without affecting its convergence [10]. With this information about infinite products, we are now prepared to prove Lemma 3.5.

Proof of Lemma 3.5. We have

$$\zeta = \prod_p \frac{1}{1 - p^{-s}} = \prod_p (1 + \frac{p^{-s}}{1 - p^{-s}}) \leq 2 \prod_p (1 + p^{-s}).$$

Then since for $\mathrm{Re}(s) > 1$, we have:

$$\sum_p |p^{-s}| = \sum_p p^{-\mathrm{Re}(s)} \leq \sum_{n=1}^{\infty} n^{-\mathrm{Re}(s)} < \infty.$$

we know that the infinite product in (3.2) converge absolutely by Proposition 3.6 and Proposition 3.8. Then we can use the relation $\frac{1}{1-x} = 1 + x + x^2 \ldots$ in the product, which gives:

$$\begin{aligned} \prod_p \frac{1}{1-p^{-s}} &= \frac{1}{1-2^{-s}} \frac{1}{1-3^{-s}} \frac{1}{1-5^{-s}} \cdots \\ &= (1 + 2^{-s} + 2^{-2s} \ldots)(1 + 3^{-s} + 3^{-2s} \ldots)(1 + 5^{-s} + 5^{-2s} \ldots). \end{aligned}$$

Now we want to expand the last expression into an infinite sum. Each summand is a product of one term from each parenthesis. If infinitely many of these terms are not 1, their product will be zero. So all but finitely many terms contributing to each term will be 1. Then, since every natural number admits a unique prime factorization, we conclude

$$\prod_p \frac{1}{1-p^{-s}} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \ldots,$$

as desired. □

From Proposition 3.6 and the proof of Lemma 3.5, we have an immediate corollary:

Corollary 3.9. $\zeta(s) \neq 0$ *for* $\mathrm{Re}(s) > 1$.

Lemma 3.10. *The function* $\zeta(s) - \frac{1}{s-1}$*, initially defined for* $\mathrm{Re}(s) > 1$*, extends to an analytic function on the half-plane* $\mathrm{Re}(s) > 0$.

Proof. From complex analysis, it is easy to derive the following formula:

$$\int_1^{\infty} x^{-s}\,\mathrm{d}x = \frac{1}{s-1} \qquad \text{for} \quad \mathrm{Re}(s) > 1, \tag{3.3}$$

$$\int_n^{x} \frac{s}{u^{s+1}}\,\mathrm{d}u = \frac{1}{n^s} - \frac{1}{x^s} \qquad \text{for} \quad \mathrm{Re}(s) > 0 \text{ and } n \in \mathbb{N}. \tag{3.4}$$

Use (3.3), we have for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} \zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) \mathrm{d}x. \end{aligned}$$

Now we want to show the series in the above equation converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > 0$, hence converges to an analytic function. Using (3.4) and the M-L inequality, we have:

$$\begin{aligned}\left|\int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)\mathrm{d}x\right| &= \left|\int_n^{n+1}\int_n^x \frac{s\,\mathrm{d}u}{u^{s+1}}\,\mathrm{d}x\right| \\ &\le \max_{n\le x\le n+1}\left|\int_n^x \frac{s\,\mathrm{d}u}{u^{s+1}}\right| \\ &\le \max_{n\le u\le n+1}\left|\frac{s}{u^{s+1}}\right| \\ &= \frac{|s|}{n^{\mathrm{Re}(s)+1}}.\end{aligned}$$

It is clear that the sum $\sum_{n=1}^{\infty}\frac{|s|}{n^{\mathrm{Re}(s)+1}}$ converges absolutely and uniformly on compact subset of the half-plane $\mathrm{Re}(s) > 0$, then the lemma follows. □

The next, and the last property of $\zeta(s)$ we want to prove is that $\zeta(s)$ does not have zeros for $\mathrm{Re}(s) \ge 1$. This is a well-known and important fact in analytic number theory, with many different methods of proof. Here we adapt a particularly succinct one from [3].

Lemma 3.11. *For all $x, y \in \mathbb{R}$ with $x > 1$, we have:*

$$|\zeta^3(x)\,\zeta^4(x+iy)\,\zeta^2(x+2iy)| \ge 1.$$

Proof. Using the Euler Product formula (3.2), it suffices to prove that for each prime number p, we have

$$\left|\left(1-\frac{1}{p^x}\right)^3\left(1-\frac{1}{p^{x+iy}}\right)^4\left(1-\frac{1}{p^x}\right)^2\right| \le 1.$$

We make the substitution $r = 1/p^x$ and $e^{i\theta} = 1/p^{iy}$, then $0 < r < 1$ and the above inequality becomes:

$$\left|(1-r)^3(1-re^{i\theta})^4(1-re^{2i\theta})^2\right| \le 1$$

Then it suffices to prove that for fixed $0 < r < 1$, and $\theta \in \mathbb{R}$, we have

$$f(\theta) := \left|(1-re^{i\theta})^4(1-re^{2i\theta})^2\right| \le \frac{1}{(1-r)^3}. \tag{3.5}$$

From a simple calculation, we have:

$$f(\theta) = \big(1+r^2-2r\cos(\theta)\big)^2\big(1+r^2-2r\cos(2\theta)\big).$$

Letting $u = \cos(\theta)$ and using the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we can rewrite the above equation as:

$$f(\theta) = g(u) := (1 + r^2 - 2ru)^2(1 + r^2 + 2r - 4ru^2).$$

Now we use the inequality of arithmetic and geometric means, which states:

$$\frac{x_1 + x_2 \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Let $x_1 = x_2 = (1 + r^2 - 2ru)$, $x_3 = 1 + r^2 + 2r - 4ru^2$, we have:

$$g(u) \leq h(u) := \frac{\big(3 + 3r^2 - 2r(2u^2 + 2u - 1)\big)^3}{27}.$$

Some basic calculus shows that $h(u)$ is minimized at $u = -1/2$. Then by a simple Taylor expansion, we have:

$$\max_\theta f(\theta) \leq h(-1/2) = (1 + r + r^2)^3 < (1 + r + r^2 + \ldots)^3 = \frac{1}{(1-r)^3},$$

which proves (3.5) and hence the lemma. □

Corollary 3.12. $\zeta(s) \neq 0$ *for* $\mathrm{Re}(s) \geq 1$.

Proof. From Corollary 3.9, we know that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$. So it suffices to consider the zeros on the line $\mathrm{Re}(s) = 1$.

Suppose that $\zeta(s)$ has a zero at $s_0 = 1 + iy_0$. From Lemma 3.10, we know that $\zeta(s)$ is analytic at $s = 1 + 2iy_0$ and has a simple pole at $s = 1$. Thus we obtain:

$$\lim_{x \to 1^+} \zeta^3(x)\zeta^4(x + iy_0)\zeta^2(x + 2iy_0) = 0,$$

which contradicts with Lemma 3.11. The corollary hence follows. □

3.2 The Prime Number Theorem

In addition to $\zeta(s)$, we have one another function which will be useful:

$$\theta(x) = \sum_{p \leq x} \ln(p).$$

The reason is given by the following lemma:

Lemma 3.13. *The Prime Number Theorem holds if* $\theta(x) \sim x$.

Instead of proving this lemma directly, we will prove a more general statement in the next proposition, since we will encounter the same situation in chapter 5.

Let us begin with some setup:

Definition 3.14. Let $(y_n)_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers. Define the counting function:

$$\tilde{\pi}(x) := \#\{y_n : y_n \leq x\},$$

and the summation function:

$$\tilde{\theta}(x) := \sum_{y_n \leq x} \ln(y_n).$$

For the use of Lemma 3.13, we make the prime numbers into a non-decreasing sequence $(p_n)_{n=1}^{\infty}$, with $p_1 = 2$ and p_{n+1} being the subsequent prime of p_n. Note that in particular $\pi(x) \leq x$ for all x. Then Lemma 3.13 clearly follows from the following proposition:

Proposition 3.15. *With the setup in Definition 3.14, if we have:*

$$\tilde{\theta}(x) \sim x \qquad \text{and} \qquad \tilde{\pi}(x) \leq Cx,$$

For some constant $C > 0$, Then

$$\tilde{\pi}(x) \sim \frac{x}{\ln(x)}.$$

Proof. Since $\tilde{\theta}(x)$ has at most $\tilde{\pi}(x)$ summands and $\ln(x)$ is increasing, for $x \geq 1$, we have:

$$0 \leq \tilde{\theta}(x) \leq \tilde{\pi}(x) \ln(x).$$

Dividing by x, we have

$$\frac{\tilde{\theta}(x)}{x} \leq \tilde{\pi}(x) \frac{\ln(x)}{x}. \tag{3.6}$$

On the other hand, for arbitrary $\epsilon > 0$, we have

$$\tilde{\theta}(x) \geq \sum_{x^{1-\epsilon} < y_n \leq x} \ln(y_n),$$

since the right hand side of above inequality has less summand. Then since $\ln(x)$ is increasing, we have the lower bound:

$$\tilde{\theta}(x) \geq (1 - \epsilon) \ln(x) \big(\tilde{\pi}(x) - \tilde{\pi}(x^{1-\epsilon})\big).$$

Using the condition $\tilde{\pi}(x) \leq Cx$, we have:

$$\tilde{\theta}(x) \geq (1-\epsilon)\ln(x)\big(\tilde{\pi}(x) - Cx^{1-\epsilon}\big).$$

which can be rearranged as:

$$\tilde{\pi}(x) \leq \frac{1}{1-\epsilon}\frac{\tilde{\theta}(x)}{\ln(x)} + Cx^{1-\epsilon}. \tag{3.7}$$

Combining (3.6) and (3.7), we have:

$$\frac{\tilde{\theta}(x)}{x} \leq \tilde{\pi}(x)\frac{\ln(x)}{x} \leq \frac{1}{1-\epsilon}\frac{\tilde{\theta}(x)}{x} + C\frac{\ln(x)}{x^{\epsilon}}.$$

For each $\epsilon > 0$, we have $C\frac{\ln(x)}{x^{\epsilon}} \to 0$ as $x \to \infty$. Then we easily see as $x \to \infty$, $\frac{\tilde{\theta}(x)}{x} \to 1$ implies $\tilde{\pi}(x)\frac{\ln(x)}{x} \to 1$ □

With Lemma 3.13, our goal now is to show:

$$\theta(x) \sim x. \tag{3.8}$$

To approach (3.8), let us introduce the following Tauberian theorem from [20] chapter 3.

Theorem 3.16 (Wiener-Ikehara Tauberian Theorem)**.** *Let $\mu(x)$ be a monotone increasing function on $\mathbb{R}$, and let*

$$f(u) := \int_1^{\infty} t^{-u}\, \mathrm{d}\mu(t),$$

converge for $\mathrm{Re}(u) > 1$*. Let*

$$g(u) := f(u) - \frac{A}{u-1},$$

for some constant A. Suppose $g(u)$ converge uniformly over any finite interval of the line $\mathrm{Re}(u) = 1$ *to a finite limit as* $\mathrm{Re}(u) \searrow 1$*. Then*

$$\mu(x) \sim Ax \qquad \text{as} \quad x \to \infty.$$

With Wiener-Ikehara Tauberaian Theorem, we are encouraged to define the function:

Definition 3.17.

$$\Phi(s) := \sum_{p} \frac{\ln(p)}{p^s} \qquad \text{for} \quad \mathrm{Re}(s) > 1$$

The reason of defining such a function Φ is that if we let $\mu(x) = \theta(x)$ and $A = 1$ in Theorem 3.16, the theorem gives precisely the result $\theta(x) \sim x$ as we want, with function f in the theorem being Φ. Thus to prove PNT, what remains is proving the following Lemma.

Lemma 3.18. *The function* $\Phi(x) - \frac{1}{s-1}$*, initially defined as an analytic function for* $\mathrm{Re}(s) > 1$*, extends to an analytic function on the half-plane* $\mathrm{Re}(s) \geq 1$*.*

Proof. First we want to show $\Phi(x) - \frac{1}{s-1}$ is analytic for $\mathrm{Re}(s) > 1$. It suffices to show $\Phi(x)$ is analytic for $\mathrm{Re}(s) > 1$. For each $\delta > 0$ and $\mathrm{Re}(s) > 1 + \delta$, we have:

$$\left|\Phi(s)\right| = \left|\sum_p \frac{\ln(p)}{p^s}\right| \leq \sum_{n=1}^{\infty} \left|\frac{\ln(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^{1+\delta/2}} \frac{\ln(n)}{n^{\delta/2}}\right|.$$

Since $\ln(n)/n^{\delta/2}$ is bounded for $n \geq 1$, we easily see $\Phi(s)$ converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > 1 + \delta$. This holds for all $\delta > 0$, $\Phi(s)$ is an analytic function on for $\mathrm{Re}(s) > 1$.

Now we show Φ extends to an analytic function on the half-plane $\mathrm{Re}(s) \geq 1$. For $\mathrm{Re}(s) > 1$, using the Euler product formula (3.2), taking the logarithm and differentiating, then multiplying by a factor -1, we obtain:

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}s} \ln(\zeta(s)) = -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\ln\left(\prod_p \frac{1}{1-p^{-s}}\right)\right) \\ &= \sum_p \frac{\mathrm{d}}{\mathrm{d}s}\left(\ln\left(1-p^{-s}\right)\right) \\ &= \sum_p \frac{p^{-s}\ln(p)}{1-p^{-s}} \\ &= \sum_p \frac{\ln(p)}{p^s - 1}. \end{aligned}$$

Here we can take the term-by-term differentiation since the sum is absolutely convergent. Then we use the identity:

$$\frac{1}{p^s - 1} = \frac{1}{p^s} - \frac{1}{p^s(p^s-1)}.$$

Combining above, we have;

$$-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) - \sum_p \frac{\ln(p)}{p^s(p^s-1)}. \tag{3.9}$$

Consider the infinite sum on the right hand side of equation (3.9). By comparing it with the series $\sum_{n=0} \frac{ln(n)}{n^{2s}}$, we see the sum converges to an analytic function for $\text{Re}(s) > 1/2$. Now consider the left hand side of (3.9). By taking derivative in Lemma 3.10, we know $\zeta'(s)$ is meromorphic on $\text{Re}(s) > 0$ with possibly one pole of order at most 2 at $s = 1$. Then since $\zeta(s)$ only has a simple pole at $s = 1$, $\frac{\zeta'(s)}{\zeta(s)}$ is meromorphic on $\text{Re}(s) > 0$, with $s = 1$ a pole of order at most 1. Since by Corollary 3.12, $\zeta(s)$ has no zero for $\text{Re}(s) \geq 1$, we conclude that $\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is analytic for $\text{Re}(s) \geq 1$. Then the lemma follows. □

3.3 The li function

There is a useful function, called the logarithm integral function, which relates to the Prime Number Theorem. This function, denoted by $\mathbf{li}(x)$, as indicated by its name, is defined as following:

Definition 3.19. For $x \in [2, \infty)$,

$$\mathbf{li}(x) = \int_2^x \frac{\mathrm{d}t}{\ln(t)},$$

and we define $\mathbf{li}(x) = 0$ for $x \in (-\infty, 2)$, so it is defined on the whole real line.

The **li** function provides us an equivalent formulation of the Prime Number Theorem:

Theorem 3.20.

$$\pi(x) \sim \mathbf{li}(x). \tag{3.10}$$

The reason why we want this new estimation is that numerical result shows that $\mathbf{li}(x)$ is a closer estimation to $\pi(x)$ than the original function $x/\ln(x)$. To prove (3.10), by PNT, it suffices to prove the following Lemma:

Lemma 3.21.

$$\frac{x}{\ln(x)} \sim \mathbf{li}(x).$$

Proof. by applying integration by parts, we obtain:

$$\mathbf{li}(x) = \int_2^x \frac{\mathrm{d}t}{\ln(t)} = \frac{x}{\ln(x)} + \frac{2}{\ln(2)} + \int_2^x \frac{\mathrm{d}t}{(\ln(t))^2}.$$

We can write:

$$\int_2^x \frac{\mathrm{d}t}{(\ln(t))^2} = \int_2^e \frac{\mathrm{d}t}{(\ln(t))^2} + \int_e^x \frac{\mathrm{d}t}{(\ln(t))^2},$$

where the first term at right hand side is just a constant. Then it suffices to show

$$\frac{\ln(x)}{x}\int_e^x \frac{\mathrm{d}t}{(\ln(t))^2} \to 0 \qquad \text{as} \quad x \to \infty.$$

But we have:

$$\begin{aligned}\int_e^x \frac{\mathrm{d}t}{(\ln(t))^2} &= \int_e^{x^{1/2}} \frac{\mathrm{d}t}{(\ln(t))^2} + \int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\ln(t))^2} \\ &\leq x^{1/2} + x\Big(\frac{2}{\ln(x)}\Big)^2,\end{aligned}$$

where we overestimate the length of two intervals by $x^{1/2}$ and x, and use the fact that $1/\ln(t)^2$ is a decreasing function. This implies:

$$\frac{\ln(x)}{x}\int_e^x \frac{\mathrm{d}t}{(\ln(t))^2} \leq \frac{\ln(x)}{x^{1/2}} + \frac{4}{\ln(x)},$$

which tend to 0 as $x \to \infty$ as desired. $\square$

Chapter 4

Eigenvalue Counting Problem on the Torus

In chapters 4 and 5, we will consider some eigenvalue problems of the Laplacian operator on compact Riemannian manifolds. In chapter 4, we will mainly discuss the case of the torus, and the next chapter will deal with more general cases.

4.1 The Laplace operator on compact Riemann surfaces

First let us introduce the general Laplacian eigenvalue problem on compact Riemann surfaces. Let F be a compact Riemann surface with a C^∞ metric:

$$\mathrm{d}s^2 = \sum g_{ij}\mathrm{d}x_i\mathrm{d}x_j.$$

Then the Laplace-Beltrami operator on F is:

$$\mathbf{D}u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ik}\frac{\partial u}{\partial x_k}\right), \tag{4.1}$$

as indicated in [6]. The eigenvalue problem is to find λ which satisfies:

$$\mathbf{D}u + \lambda u = 0, \tag{4.2}$$

where u is not zero.

Since F is a compact surface, there is no boundary condition. It can be shown that $\mathbf{D}$ is self-adjoint and has compact resolvent ([18], Chapter 5, Proposition 1.2).

The inverse of Laplace-Beltrami operator is self-adjoint and compact, thus from the spectral theorem, the eigenvalues of $\mathbf{D}^{-1}$ form a real sequence converging to 0. Thus the eigenvalues of $\mathbf{D}$ are all real positive and form an increasing sequence to ∞ [14]. We order the eigenvalues from lowest to highest towards infinity:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{4.3}$$

Note that $\lambda_0 = 0$ corresponds to constant eigenfunctions. The eigenfunctions of $\mathbf{D}$ are smooth and form an orthonormal basis of $L_2(F)$ [8][14].

We are interested in the eigenvalue counting problem: that is, the behavior of the function

$$N(x) = \#\{n : \lambda_n \leq x\} \qquad \text{for } x > 0. \tag{4.4}$$

A good method to approach the problem is an asymptotic expansion of the heat kernel on F. We quote the following theorem from [17], section 8.3:

Theorem 4.1. *With the above setup, we have*

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} = \frac{\mathcal{A}}{4\pi t} + \sum_{n=0}^{N} b_n t^n + \mathcal{O}(t^{N+1}), \tag{4.5}$$

where $\mathcal{A}$ denotes the area of F and b_n are constants.

By the following Tauberian theorem, we can extract a very useful result about $N(x)$ from (4.5).

Theorem 4.2 (Karamata's Tauberian Theorem)**.** *Suppose μ is a positive measure on $[0,\infty)$, $\alpha \in (0,\infty)$. Suppose further that*

$$\int_0^\infty e^{-t\lambda}\, \mathrm{d}\mu(\lambda) \sim at^{-\alpha} \qquad \text{as} \quad t \searrow 0.$$

Then we have

$$\int_0^x \mathrm{d}\mu(\lambda) \sim \frac{ax^\alpha}{\Gamma(\alpha+1)} \qquad \text{as} \quad x \to \infty.$$

Applying the Karamata's Tauberian theorem to the measure $\mathrm{d}N(x)$ on (4.5), we have:

$$N(x) \sim \frac{\mathcal{A}}{4\pi} x \qquad \text{as} \quad x \to \infty. \tag{4.6}$$

4.2 Eigenvalue counting on the Torus

4.2.1 The lattice counting function

We are now ready to apply (4.6) to a torus. It is well-known that a torus $\mathcal{T}$ can be regarded as the quotient space $\mathbb{C}/\Omega$ with Ω a lattice in $\mathbb{C}$. We choose the classic one: $\Omega = \{a + ib : a, b \in \mathbb{Z}\}$. With the metric being the same as on $\mathbb{C}$, the Laplace-Beltrami operator on $\mathcal{T}$ is just Δ, we see that the solution of (4.2) is;

$$u = e^{2\pi imx}e^{2\pi iny} \qquad \lambda = 4\pi^2(m^2+n^2) \quad (m,n) = \mathbb{Z}^2.$$

Then for the purpose of (4.6), it is natural to define the integer lattice counting function as following:

Definition 4.3.

$$r(n) = \#\{(a,b) \in \mathbb{Z}\times\mathbb{Z} : n = a^2+b^2\}, \tag{4.7}$$

$$A(x) = \sum_{0\le n\le x} r(n) = \pi x + \mathcal{R}(x). \tag{4.8}$$

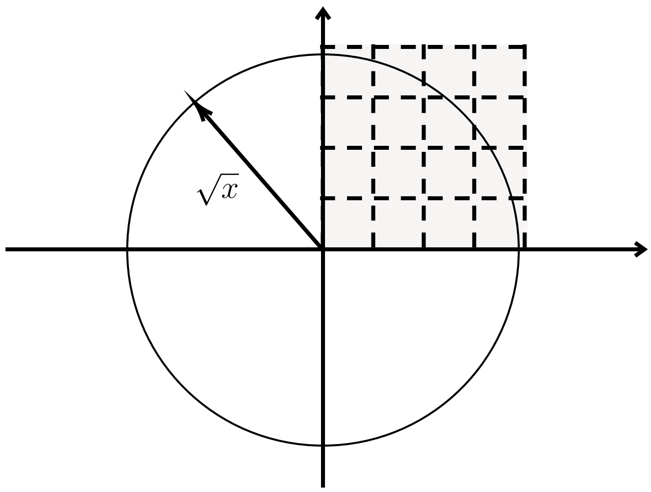


Figure 4.1: The lattice counting function $A(x)$

Note the function A is the counting function of integer lattice points within a circle of radius $\sqrt{x}$ (visualized in Figure 3.1 below). Thus we expect the function $A(x)$ to behave as πx, with some remainder function $\mathcal{R}$. A very rough estimation can be given as

$$\pi(\sqrt{x}-\sqrt{2})^2 \le A(x) \le \pi(\sqrt{x}+\sqrt{2})^2. \tag{4.9}$$

Since a circle of radius $\sqrt{x}+\sqrt{2}$ is big enough to cover the whole square touched by circle of radius $\sqrt{x}$. This immediately gives us the estimation of the remainder:

$$\mathcal{R}(x) = \mathcal{O}(\sqrt{x}).$$

in the last part of this chapter, we will prove a better estimation to $\mathcal{R}(x)$ than this.

Using definition 4.3, since for $\mathcal{T}$ we have $\mathcal{A} = 1$, it follows that:

$$N(x) = A\Big(\frac{x}{4\pi^2}\Big). \tag{4.10}$$

4.2.2 The Poisson summation formula

From equation 4.10, we see that the unknown asymptotic property of $N(x)$ is now completely included in that of $\mathcal{R}(x)$. To estimate the behavior of $\mathcal{R}(x)$, we introduce the Poisson summation formula on the torus. Note that $\mathcal{T}$ is compact, then it is fairly obvious that $L_2(\mathcal{T})$ consists of doubly-periodic functions on $\mathbb{C}$.

The Poisson summation formula will appear when we construct doubly-periodic function in two different ways.

One way here is to take the Fourier expansion, we have:

$$L_2(\mathcal{T}) = \Big\{ f:\mathbb{C}\to\mathbb{C} : f(x+iy) = \sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty} c_{ab}e^{2\pi iax}e^{2\pi iby}\Big\}.$$

where $c_{ab}\in\mathbb{C}$ coefficients form a sequence in $l_2(\mathbb{C}^2)$.

Another way is to periodise a compactly supported function. Let $g\in C_c^2(\mathbb{R}^2)$, then we create a doubly-periodic function by defining:

$$G(z) = \sum_{w\in\Omega} g(z+w) = \sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty} g(z+a+ib). \tag{4.11}$$

Since g is compactly supported, for each z we are taking a finite sum, thus the convergence of G is ensured.

To acquire the Poisson summation formula, we take the Fourier series of G, with:

$$G(z) = \sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty} c_{ab}e^{2\pi iax}e^{2\pi iby}. \tag{4.12}$$

For a C^2 function, this converges absolutely, and the coefficients satisfy:

$$\begin{aligned} c_{ab} &= \iint_S G(z)e^{-2\pi iax}e^{-2\pi iby}\,\mathrm{d}x\,\mathrm{d}y = \sum_{w\in\Omega}\iint_S g(z+w)e^{-2\pi iax}e^{-2\pi iby}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{w\in\Omega}\iint_{S+w} g(z')e^{-2\pi iax'}e^{-2\pi iby'}\,\mathrm{d}x'\,\mathrm{d}y' = \iint_{\mathbb{C}} g(z)e^{-2\pi iax}e^{-2\pi iby}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

where S is the unit square. Then we observe that $\widehat{g}(a,b) = c_{ab}$, where $\widehat{g}$ is the Fourier transform of g. Then calculate $G(0)$ in (4.11) and (4.12) will give the desired result.

Theorem 4.4. *Let $g \in C_c^2(\mathbb{R}^2)$ and $\widehat{g}$ denotes the Fourier transform of g. We have the **Poisson summation formula**:*

$$\sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} g(a+bi) = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \widehat{g}(a,b). \tag{4.13}$$

It turns out the Poisson summation formula has a deep connection with our problem above, if we consider the radially symmetric smooth g:

$$g(x+iy) = f(x^2+y^2) \qquad \text{with} \qquad f \in C_c^{\infty}(\mathbb{R}),$$

we then have the relation:

$$\sum_{n=0}^{\infty} r(n) f(n) = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} g(a+bi) = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \widehat{g}(a,b), \tag{4.14}$$

which reflects a lot of information about $A(x)$ if we apply with different functions. In particular, if we take f to be the characteristic function of interval $[0,x]$, the left hand side of above equation is just the function $A(x)$ itself. That will be discussed further later. For now we firstly want to derive an explicit formula for $\tilde{g}$ in terms of f. We use the formula:

$$a\cos(\theta) + b\sin(\theta) = \sqrt{a^2+b^2}\cos(\theta+\theta_0),$$

for some θ_0. Then:

$$\begin{aligned}
\widehat{g}(a,b) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x^2+y^2) \exp(-2\pi i(ax+by))\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\infty} \int_0^{2\pi} f(r^2) \exp(-2\pi i r[a\cos(\theta) + b\sin(\theta)])\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^{\infty} \int_0^{2\pi} f(r^2) \exp(-2\pi i r\sqrt{a^2+b^2}\cos(\theta+\theta_0))\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^{\infty} \int_0^{2\pi} f(r^2) \exp(-2\pi i r\sqrt{a^2+b^2}\cos(\phi))\,\mathrm{d}\phi\,\mathrm{d}r,
\end{aligned}$$

where we used polar coordinates and the linear combination formula for sin and cos.

To proceed further, we define

$$\mathcal{U}(z) = e^{(2\pi i\sqrt{a^2+b^2}\,\mathrm{Re}(z))}.$$

Then using the rotational invariance of ϕ integration, we have

$$\int_0^{2\pi} \mathcal{U}(re^{i\phi})\,\mathrm{d}\phi = \int_0^{2\pi} \mathcal{U}(ze^{i\phi})\,\mathrm{d}\phi \qquad \text{for} \qquad |z| = r. \tag{4.15}$$

Now we use the rotational invariance of the Laplace operator. The function $\mathcal{U}$ is an eigenfunction of Δ, and it will still be a eigenfunction of Δ after we rotated the argument, with the same eigenvalue. Then we can use the rotated result on the right hand side of the equation to replace the original one. After the ϕ integration, we still have a rotationally symmetric eigenfunction with the same eigenvalue. Follow from simple calculation, we note that the right hand side of the above equation is a radially symmetric solution of differential equation $\Delta u + 4\pi^2(a^2+b^2)u = 0$. Write in polar form, we have the ODE:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + 4\pi^2(a^2+b^2)u = 0.$$

which is a Bessel's equation. With some knowledge of ODEs (see [1] chapter 10), we have:

$$\int_0^{2\pi} \mathcal{U}(re^{i\phi})\,\mathrm{d}\phi = 2\pi J_0(2\pi r\sqrt{a^2+b^2}),$$

where J_0 is the Bessel function of order 0. Substitute in the formula above for $\tilde{g}$, we have:

$$\widehat{g}(a,b) = 2\pi \int_0^{\infty} r f(r^2) J_0(2\pi r\sqrt{a^2+b^2})\,\mathrm{d}r,$$

with change of variable $x = r^2$. Then using the relation (4.14) above, we have the following theorem:

Theorem 4.5. *Let $f \in C_c^{\infty}(\mathbb{R})$, then:*

$$\sum_{n=0}^{\infty} r(n)f(n) = \pi \sum_{n=0}^{\infty} r(n) \int_0^{\infty} f(x) J_0(2\pi\sqrt{nx})\,\mathrm{d}x. \tag{4.16}$$

4.2.3 A better estimation to $\mathcal{R}(x)$

In this section, we will apply Theorem 4.5 to find a better estimation of $\mathcal{R}(x)$. We will need a few properties of Bessel functions, which are included in Appendix B.

As indicated as above, we want to apply Theorem 4.5 to the characteristic function:

$$f_c(y) = \begin{cases} 1 & \text{if } 0 \le y \le x, \\ 0 & \text{otherwise,} \end{cases}$$

since this will restore $A(x)$ on the left hand side of the equation. But there is a obvious problem, namely Theorem 4.5 only applying to smooth functions, but the characteristic function above is not smooth. This will indeed give us a problem. Let us expand the right hand side of (4.16) with the above characteristic function. For $n > 0$ terms in the sum, we have:

$$\begin{aligned}
\pi r(n) \int_0^x J_0(2\pi\sqrt{ny})\,\mathrm{d}y &= \pi \frac{r(n)}{2\pi^2 n} \int_0^{2\pi\sqrt{nx}} uJ_0(u)\,\mathrm{d}u \\
&= \frac{r(n)}{2\pi n} \Big[uJ_1(u)\Big]_0^{2\pi\sqrt{nx}} && \text{(apply (B.2))} \\
&= \sqrt{x}\frac{r(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}). && \text{(since } J_1(0) = 0\text{)}
\end{aligned}$$

For the $n = 0$ term, we use the following limit, which directly follows from the definition. (See Appendix B).

$$\lim_{n\to 0} \frac{J_1(2\pi\sqrt{(nx)})}{\sqrt{n}} = \lim_{n\to 0} \frac{\pi\sqrt{nx}}{\sqrt{n}} + \mathcal{O}(n) = \pi\sqrt{x}.$$

Combining all these terms together, and writing it in a sum, we obtain:

$$A(x) = \pi x + \sqrt{x} \sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{(n)}} J_1(2\pi\sqrt{nx}). \tag{4.17}$$

Comparing to definition 4.3, we have

$$\mathcal{R}(x) = \sqrt{x} \sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{(n)}} J_1(2\pi\sqrt{nx}).$$

But since the characteristic function f fails the smooth condition, the sum in the above equation may not converge. Indeed, we have a problem with convergence here. To see this, we quote some analytic properties from [19], namely $J_n(z)$ is an entire function with

$$J_n(z) \sim \sqrt{\frac{2}{\pi}} \frac{\cos(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}{\sqrt{z}} \qquad \text{for} \quad z \to \infty, \quad |\arg(z)| < \pi. \tag{4.18}$$

Thus $\mathcal{R}(x)$ behaves like

$$\frac{x^1/4}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{3}{4}\pi\right).$$

where the convergence of the sum is not guaranteed. To solve this problem, we use the following fact:

Lemma 4.6.
$$\int_0^x A(t)\,\mathrm{d}t = \sum_{n\leq x}(x-n)r(n). \tag{4.19}$$

Proof. From definition of $A(x)$, we have

$$\int_0^x A(t)\mathrm{d}t = \int_0^x \sum_{0\leq n\leq t} r(n)\,\mathrm{d}t.$$

Let m be the largest integer such that $m \leq x$. Note that $\sum_{0\leq n\leq t} r(n)$ is constant on interval the $[k-1,k)$ for each positive integer k as well as on the interval $[m,x]$. Then we have:

$$\begin{aligned}
\int_0^x \sum_{0\leq n\leq t} r(n)\,\mathrm{d}t &= \sum_{k=1}^{m}\int_{k-1}^{k}\sum_{0\leq n\leq t} r(n)\,\mathrm{d}t + \int_m^x \sum_{0\leq n\leq t} r(n)\,\mathrm{d}t \\
&= \sum_{k=1}^{m}\left[t\sum_{0\leq n\leq t} r(n)\right]_{k-1}^{k} + \left[t\sum_{0\leq n\leq t} r(n)\right]_m^x \\
&= \sum_{k=1}^{m}\sum_{0\leq n\leq k-1} r(n) + (x-m)\sum_{0\leq n\leq m} r(n).
\end{aligned}$$

Then for each particular $n \leq x$, observe that the coefficient of $r(n)$ is $m-n$ from the first summand and $x-m$ from the second summand. Then from the last line above we have

$$\int_0^x \sum_{0\leq n\leq t} r(n)\,\mathrm{d}t = \sum_{n\leq x}(x-n)r(n).$$

which establishes (4.19). $\square$

This encourages us to apply Theorem 4.5 to the function:

$$f(y) = \begin{cases} x-y & \text{if } 0\leq y\leq x, \\ 0 & \text{otherwise.} \end{cases}$$

We define the function $\tilde{f}$:

$$\tilde{f}(y) = \begin{cases} 1-|y| & \text{if } 0\leq |y|\leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The advantage of $\tilde{f}$ is that unlike the original f, it has no jump discontinuity. We also scale by a factor of $1/x$ to simplify the calculation, which is valid since $x > 0$. We want to show that the Poisson summation holds for $\tilde{f}$ and the sum in

4.13 converges. If we proved this, then clearly the same holds for f, and following the same step as above we can derive Theorem 4.5 for f.

Let $\tilde{g}(z) = \tilde{f}(x^2 + y^2)$, it suffices to show the Poisson summation formula converges for $\tilde{g}$. If we directly apply the Poisson summation formula for $\tilde{g}$, since it is not C^2, we might have some convergence problem. So instead we will take the convolution of $\tilde{g}$ with a standard bump function, which makes the resulting function smooth. Let $\psi(x, y)$ be the standard bump function on $\mathbb{R}^2$, that is, a smooth compactly supported function with integration over $\mathbb{R}^2$ equal to 1. Let $\psi_\epsilon(z) = \psi(x/\epsilon, y/\epsilon)/\epsilon^2$ for $\epsilon > 0$. Then we want to take the convolution of $\tilde{g}$ and ψ_ϵ, since the Poisson summation formula is valid for $\tilde{g} * \psi_\epsilon$ for each $\epsilon > 0$. We have:

$$
\begin{aligned}
\lim_{\epsilon \to 0} \tilde{g}(x', y') * \psi_\epsilon(x', y') &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \tilde{g}(x' - x, y' - y)\psi_\epsilon(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \tilde{g}(x' - x, y' - y)\frac{\psi(x/\epsilon, y/\epsilon)}{\epsilon^2}\, \mathrm{d}x\, \mathrm{d}y \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \tilde{g}(x' - \epsilon\bar{x}, y' - \epsilon\bar{y})\psi(\bar{x}, \bar{y})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} \\
&= \int_{\mathbb{R}^2} \tilde{g}(x', y')\psi(\bar{x}, \bar{y})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} \\
&= \tilde{g}(x', y').
\end{aligned}
$$

where $\bar{x} = x/\epsilon$, $\bar{y} = y/\epsilon$ and we use the dominated convergence theorem since $g(x' - \epsilon\bar{x}, y' - \epsilon\bar{y})$ is uniformly bounded for all ϵ. Each ψ_ϵ is continuous and compactly supported, so it is uniformly continuous. Then since $\tilde{g}$ is obviously uniformly continuous, follows from a similar calculation to above, the left hand side of (4.13) satisfies:

$$
\sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \tilde{g}(a, b) * \psi_\epsilon(a, b) \to \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \tilde{g}(a, b) \quad \text{as} \quad \epsilon \to 0.
$$

Let $\widehat{h}$ denote the Fourier transform of the function h and $r = \sqrt{a^2 + b^2}$, then by convolution theorem, we have:

$$
\widehat{\tilde{g} * \psi_\epsilon}(r) = \widehat{\tilde{g}}(\xi)\widehat{\psi_\epsilon}(\xi).
$$

We know that $\widehat{\psi_\epsilon}(x, y) \to 1$ pointwisely. Now to estimate $\widehat{\tilde{g}}(\xi)$, we consider the non-smooth singular point of $\tilde{g}(r)$. We use the argument similar to [16] (chapter IV, section 2.2). For the singularity at $r = 0$, if $\tilde{g}(r) \sim r^\alpha$ near $r = 0$ for $\alpha > 0$, then $\widehat{\tilde{g}}(\xi) \sim |\xi|^{-2-\alpha}$ as $|\xi| \to \infty$. In our case $\alpha = 1$, implies $\widehat{\tilde{g}}(\xi) \sim |\xi|^{-3}$. For the singularity at $r = 1$, if $\tilde{g}(r) \sim (r-1)^{\beta}$ near $r = 1$ for $\beta > 0$, then $\widehat{\tilde{g}}(\xi) \sim |\xi|^{-3/2-\beta}$ as $|\xi| \to \infty$. So in our case $\widehat{\tilde{g}}(\xi) \sim |\xi|^{-5/2}$. In conclusion, as $|\xi| \to \infty$, $\widehat{\tilde{g}}(\xi)$ decays as $\mathcal{O}(|\xi|^{-5/2})$. Thus the product $\widehat{\tilde{g}}(a,b)\widehat{\psi_{\epsilon}}(a,b)$ is uniformly bounded for any ϵ. By the dominated convergence theorem for sums, we have:

$$\lim_{\epsilon \to 0} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \widehat{\tilde{g}}(a,b)\widehat{\psi_{\epsilon}}(a,b) = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \widehat{\tilde{g}}(a,b).$$

where the decaying condition implies the sum on the right hand side of the above equation is absolutely convergent. So by the convolution theorem, we have:

$$\sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \tilde{g}(a,b) = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \widehat{\tilde{g}}(a,b).$$

So we have established the Poisson summation formula for $\tilde{g}$. This establishes the convergence argument we need to apply Theorem 4.5 on f. So we apply Theorem 4.5 top obtain:

$$\begin{aligned}
\sum_{n \le x} (x-n) r(n) &= \sum_{n=0}^{\infty} r(n) f(n) \\
&= \pi \sum_{n=0}^{\infty} r(n) \int_0^{\infty} f(y) J_0(2\pi\sqrt{ny})\,\mathrm{d}y \\
&= \pi \sum_{n=0}^{\infty} r(n) \int_0^{x} x J_0(2\pi\sqrt{ny}) - y J_0(2\pi\sqrt{ny})\,\mathrm{d}y.
\end{aligned}$$

Splitting the integrand, for the first part, similar to above, we have:

$$\int_0^x x J_0(2\pi\sqrt{ny})\,\mathrm{d}y = \frac{x^{3/2}}{\pi\sqrt{n}} J_1(2\pi\sqrt{nx}). \tag{4.20}$$

For the second part, we have:

$$\begin{aligned}
-\int_0^x y J_0(2\pi\sqrt{ny})\,\mathrm{d}y &= \frac{-1}{8\pi^4 n^2} \int_0^{2\pi\sqrt{nx}} u^3 J_0(u)\,\mathrm{d}u \\
&= \frac{-1}{8\pi^4 n^2} \int_0^{2\pi\sqrt{nx}} 2u^2 J_1(u) - u^3 J_2(u)\,\mathrm{d}u \\
&\qquad\qquad \text{(apply (B.3),(B.4))} \\
&= \frac{-1}{8\pi^4 n^2} \Big[-2u^2 J_2(u) - u^3 J_1(u) \Big]_0^{2\pi\sqrt{nx}} \\
&\qquad\qquad \text{(apply (B.4) again)} \\
&= -\frac{x^{3/2}}{\pi\sqrt{n}} J_1(2\pi\sqrt{nx}) + \frac{x}{\pi^2 n} J_2(2\pi\sqrt{nx}).
\end{aligned}$$

The first summand canceled with (4.20). For the $n = 0$ term, we have the limit:

$$\lim_{n\to 0} \frac{J_2(2\pi\sqrt{nx})}{n} = \lim_{n\to 0} \frac{\pi^2 nx}{2n} + \mathcal{O}(n^2) = \frac{\pi^2 x}{2}.$$

Combining above, and apply the previous Lemma. Then we obtain:

$$\int_0^x A(t)\,\mathrm{d}t = \frac{\pi x^2}{2} + \frac{x}{\pi}\sum_{n=1}^{\infty} \frac{r(n)}{n} J_2(2\pi\sqrt{nx}). \tag{4.21}$$

With (4.21), we can now move further to explore the asymptotic behavior of $\mathcal{R}(x)$ as $x \to \infty$. But to utilize this equation, we first need an estimation for the infinite series involving $r(n)$. It was given by the following lemma:

Lemma 4.7. *For $s > 1$, the sum $\sum_{n=1}^{\infty} \frac{r(n)}{n^s}$ converges absolutely.*

Proof. From Definition (4.3), we can always overestimate the sum $\sum_{n=0}^{x} r(n)$ by $\pi(x+\sqrt{2})$, the area of circle that covers all the lattice points involved in the sum. Then we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\frac{r(n)}{n^s}\right| &= \sum_{j=1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \left|\frac{r(n)}{n^s}\right| \\
&\leq \sum_{j=1}^{\infty} 2^{-js} \sum_{n=2^j}^{2^{j+1}-1} |r(n)| \\
&\leq \sum_{j=1}^{\infty} 2^{-js} C(2^{j+1} + \sqrt{2}) \\
&= 2C \sum_{j=1}^{\infty} 2^{j(1-s)} + \sqrt{2}C \sum_{n=0}^{\infty} 2^{-js}.
\end{aligned}$$

which converges to a finite constant since $s > 1$. □

With the above Lemma, we are now prepared to prove the following theorem.

Theorem 4.8.

$$\mathcal{R}(x) = \mathcal{O}(x^{1/3}). \tag{4.22}$$

Proof. First we try to find the upper and lower bounds of $\mathcal{R}(x)$. We use Definition

4.3 and the fact that $A(x)$ is a increasing function. For $h > 0$ we have:

$$\begin{aligned}
\mathcal{R}(x) &= \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(x)\,\mathrm{d}t \\
&= \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(t)\,\mathrm{d}t + \frac{1}{h}\int_{x-h}^{x} \big[\mathcal{R}(x) - \mathcal{R}(t)\big]\,\mathrm{d}t \\
&= \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(t)\,\mathrm{d}t + \frac{1}{h}\int_{x-h}^{x} \big[A(x) - A(t) + \pi(t-x)\big]\,\mathrm{d}t \\
&\geq \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(t)\,\mathrm{d}t + \frac{1}{h}\int_{x-h}^{x} \pi(t-x)\,\mathrm{d}t \\
&= \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(t)\,\mathrm{d}t - \frac{\pi h}{2} \\
&= \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(t)\,\mathrm{d}t + \mathcal{O}(h).
\end{aligned}$$

So we have a lower bound for h. Similarly, take the integral on interval $[x, x+h]$, we will obtain an upper bound for h. Combining these two, we have the following inequality:

$$\mathcal{O}(h) + \frac{1}{h}\int_{x-h}^{x} \mathcal{R}(t)\,\mathrm{d}t \leq \mathcal{R}(x) \leq \mathcal{O}(h) + \frac{1}{h}\int_{x}^{x+h} \mathcal{R}(t)\,\mathrm{d}t. \tag{4.23}$$

Compare (4.17) and (4.21), it is clear that

$$\int_{0}^{x} \mathcal{R}(t)\,\mathrm{d}t = \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n} J_2(2\pi\sqrt{nx}).$$

To prove the theorem, we choose $h = x^{1/3}$ in (4.23). We have:

$$\int_{x}^{x+x^{1/3}} \mathcal{R}(t)\,\mathrm{d}t = \frac{x + x^{1/3}}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n} J_2\Big(2\pi\sqrt{n(x + x^{1/3})}\Big) - \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n} J_2(2\pi\sqrt{n(x)}).$$

To estimate the right hand side of the above equation, we add and subtract the term

$$\frac{x + x^{1/3}}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n} J_2(2\pi\sqrt{n(x)}).$$

and then pair the terms. Then the question is to estimate the following two terms:

$$\frac{x^{1/3}}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n} J_2(2\pi\sqrt{n(x)}), \tag{a}$$

$$\frac{x + x^{1/3}}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n}\Big[J_2\Big(2\pi\sqrt{n(x + x^{1/3})}\Big) - J_2(2\pi\sqrt{n(x)})\Big]. \tag{b}$$

From (4.18), use the fact that $|\cos(x)| \leq 1$ for $x \in \mathbb{R}$, we have $|J_k(x) \leq Cx^{-1/2}|$ for a constant C as $x \to \infty$. Using above lemma, for (a) we have:

$$\frac{x^{1/3}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} J_2(2\pi\sqrt{n(x)}) \leq \frac{x^{1/3}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} C(nx)^{-1/4} = \mathcal{O}(x^{1/12}).$$

Now consider the sum in (b), by Taylor expansion we have:

$$\sqrt{1 + x^{-2/3}} = 1 + D(x^{-2/3}) + \text{lower order terms.}$$

where D is a constant. Then we can drop the lower order terms to get:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{r(n)}{n} \left[J_2\left(2\pi\sqrt{n(x + x^{1/3})}\right) - J_2(2\pi\sqrt{nx}) \right] \\
&= \sum_{n=1}^{\infty} \frac{r(n)}{n} \left[J_2\left(2\pi\sqrt{nx}\sqrt{1 + x^{-2/3}}\right) - J_2(2\pi\sqrt{nx}) \right] \\
&= \sum_{n=1}^{\infty} \frac{r(n)}{n} \left[J_2\left(2\pi\sqrt{nx}(1 + D(x^{-2/3}))\right) - J_2(2\pi\sqrt{nx}) \right].
\end{aligned}$$

Define

$$\mathcal{J}(x) = \left[J_2\left(2\pi\sqrt{nx}(1 + D(x^{-2/3}))\right) - J_2(2\pi\sqrt{nx}) \right].$$

Next we split up the sum as:

$$\sum_{n=1}^{\infty} \frac{r(n)}{n} \mathcal{J}(x) = \sum_{n=1}^{x^{1/3}/M} \frac{r(n)}{n} \mathcal{J}(x) + \sum_{n=x^{1/3}/M}^{\infty} \frac{r(n)}{n} \mathcal{J}(x). \tag{4.24}$$

for a large constant M. Then for the first summand in (4.24) we have $x \leq x^{1/3}/M$

$$2\pi\sqrt{nx}(1 + D(x^{-2/3})) - 2\pi\sqrt{nx} \approx 2\pi\sqrt{\frac{x^{4/3}}{M}} D(x^{-2/3}) = \frac{2\pi D}{\sqrt{M}}.$$

We choose M large enough so that $2\pi D/\sqrt{M} \ll 1$. Then we can approximate $\mathcal{J}$ by the derivative of J_2. Note that from (4.18) for $J_2'(x)$ we also have the bound $|J_2'(x)| \leq Cx^{1/2}$ for large x. Then:

$$\mathcal{J}(x) \leq |J_2'(2\pi\sqrt{nx})| D(x^{-2/3}) \leq D' n^{1/4} x^{-5/12},$$

where D' is a constant. Then up to some constant the first summand in (4.24) becomes

$$x^{-5/12} \sum_{n=1}^{x^{1/3}/M} \frac{1}{n^{3/4}},$$

where the $r(n)$ term is dropped by the above lemma.

For the second summand in (4.24), we can directly use the bound $|J_2(x)| \leq C(nx)^{-1/4}$. Then we have $|\mathcal{J}(x)| \leq 2C(nx)^{-1/4}$ and up to a constant the second summand becomes:

$$x^{-1/4} \sum_{n=x^{1/3}/M}^{\infty} \frac{1}{n^{5/4}}.$$

Now we use the integral to estimate the sum and omit the constants, for (4.24) we have:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{r(n)}{n} \mathcal{J}(x) &= x^{-5/12} \int_{1}^{x^{1/3}/M} \frac{1}{n^{3/4}} \,\mathrm{d}n + x^{-1/4} \int_{x^{1/3}/M}^{\infty} \frac{1}{n^{5/4}} \,\mathrm{d}n \\ &= x^{-5/12} \big[n^{1/4}\big]_{1}^{x^{1/3}/M} + x^{-1/4} \big[n^{-1/4}\big]_{x^{1/3}/M}^{\infty} \\ &= \mathcal{O}(x^{-1/3}).\end{aligned}$$

Then for part (b) we have:

$$\frac{x + x^{1/3}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} \mathcal{J}(x) = \mathcal{O}(x^{-2/3}).$$

Eventually, for (4.23), we have

$$\mathcal{R}(x) \leq \mathcal{O}(x^{1/3}) + \frac{1}{x^{1/3}} \big(\mathcal{O}(x^{1/12}) + \mathcal{O}(x^{2/3})\big) = \mathcal{O}(x^{1/3}).$$

For the other part of the inequality in (4.23), we can get a same result from very similar steps. Thus we can conclude that:

$$\mathcal{R}(x) = \mathcal{O}(x^{1/3}).$$

$\square$

Remark 4.9. Note that in equation (4.24), if we increased the limit of the first sum as well as the lower bound of the second, the first sum will be bigger and the second one will be smaller. So we can balance the two sums to obtain a better result. In this case, for both sums we have the estimation $\mathcal{O}(x^{1/3})$, so the two sums are balanced, indicating that the the choice of limit of sums we made is optimal.

Chapter 5

The Selberg Trace Formula

In the previous chapter, we discussed the eigenvalue counting problem on a torus, which is a compact Riemann surface of genus 1. In this chapter, we want to generalize to the compact Riemann surface with genus ≥ 2 and derive the Selberg trace formula on such surfaces.

5.1 Motivation: The classification of compact Riemann surface

First we state a theorem from algebraic topology, which classifies the compact Riemann surfaces. Using this theorem and combining with some knowledge about Fuchsian group in chapter 2, we can discuss the results that hold for arbitrary compact Riemann surface of genus at least 2 without going through the specific geometric properties of each surface.

Theorem 5.1. *Let S be a compact Riemann surface with genus at least 2. Then $S \cong \mathbb{H}/\Gamma$, where Γ is a strictly hyperbolic Fuchsian group, and $\pi_1(S) = \Gamma$. Furthermore, such Γ has a compact fundamental region $\mathcal{F}$.*

Proof. (*Sketch*) By the Uniformization theorem of the Riemann surfaces, the universal covering space of S is one of these three: the Riemann sphere, the complex plane, or the unit disk. As in [21], using Gauss-Bonnet theorem, we can conclude for genus at least 2 case the covering space is the unit disk, which is conformally equivalent to $\mathbb{H}$ as showed in chapter 2. Then S can be expressed as $\mathbb{H}/\Gamma$, the quotient space of $\mathbb{H}$ by a group of isometric automorphisms of $\mathbb{H}$. (See [4] Proposition B.1.6 and Theorem B.1.7)

Now, for the resulting quotient space to be a well-defined metric space, we need the group to act properly discontinuously, hence the group Γ must be a Fuchsian group, as shown in Proposition 2.8. For the resulting Riemann surface to be compact, we need Γ to be strictly hyperbolic, since an elliptic element will make the quotient space an orbifold, and an parabolic element will induce a cusp, which is not compact. (For details see [5] section 2.2 and 2.4). Similarly such Γ has a compact fundamental region, as proved in chapter 3 of [11].

□

Remark 5.2. Note that throughout this chapter, the Fuchsian group we discuss will be strictly hyperbolic as indicated in the theorem above.

It should not be surprising that the compact surface $S \cong \mathbb{H}/\Gamma$ has a deep connection with the compact fundamental domain $\mathcal{F}$ of Γ. Let S inherit the quotient metric from $\mathbb{H}$, then $\mathcal{F}$ has the same metric as $\mathbb{H}$. From definition 2.9, we see that the image of $\mathcal{F}$ under group action covers $\mathbb{H}$, so S should be isomorphic to a subset of $\mathcal{F}$. But since $\mathcal{F}$ and $T\mathcal{F}$ have disjoint interiors, the subset must contain every point, except for some point in the boundary of $\mathcal{F}$. Thus S and $\mathcal{F}$ are "almost" isomorphic, in the sense that a function defined on S can be isometrically transfered to $\mathcal{F}$, except for the boundary, which has measure 0. This allows us to consider the integral of functions on S as integrals on $\mathcal{F}$, hence we can utilize the convenient geometric properties of $\mathcal{F}$. If we choose $\mathcal{F}$ to be a Dirichlet domain, then $\mathcal{F}$ is a convex compact polygon with finitely many edges of geodesic segments, and the opposite edges are paired with side-pairing congruences, being the generators of Γ (See [5] Chapter 2). This is very reminiscent to what we have in the Torus case, where the Torus is obtained by gluing the opposite edges of a square in $\mathbb{C}$.

It follows from above that S and $\mathcal{F}$ have the same area. We can deduce a explicit formula for the area of $\mathcal{F}$. Since the Poincaré metric has curvature -1, by Gauss-Bonnet theorem we obtain:

$$\mu(\mathcal{F}) = 4\pi(g-1),$$

where $\mu(\mathcal{F})$ is the area of $\mathcal{F}$ and g denotes the genus of S.

To state the Selberg trace formula, we will need some information about the Laplace-Beltrami operator $\mathbf{D}$ on S. Recall from chapter 4 that $\mathbf{D}$ is a self-adjoint operator with compact resolvent. Apply 4.1 to the Poincaré metric, we have:

$$\mathbf{D}f = y^2(f_{xx} + f_{yy}) = y^2 \Delta f,$$

with the usual behavior of eigenvalues:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \qquad \lim_{n\to\infty} \lambda_n = \infty.$$

To simplify the equations, we make a definition;

Definition 5.3.

$$\begin{aligned} r_n^2 &= \lambda_n - \frac{1}{4} \qquad \text{with} \quad \arg(r_n) \in \{0, \frac{-\pi}{2}\}, \\ s_n &= \frac{1}{2} + ir_n \qquad \tilde{s_n} = \frac{1}{2} - ir_n. \end{aligned}$$

Since $\lambda_n - 1/4$ is real, its square roots are always real or purely imaginary. Thus each r_n is uniquely determined.

Moreover, recall from chapter 2 that for each hyperbolic element $P \in \Gamma$, we have a unique *displacement length* $l(P)$ of P acquired by conjugation. Furthermore such hyperbolic element P can be written as a power of a unique primitive element P_0. From now we will use P_0 to denote the unique primitive element corresponding to P.

Now we are prepared to state the Selberg trace formula;

Theorem 5.4 (the Selberg trace formula)**.** *With the condition above, let $h(z)$ be an analytic function on the strip $\{z : \mathrm{Im}(z) \leq \frac{1}{2} + \delta\}$ such that;*

$$h(-z) = h(z) \qquad \text{and} \qquad |h(z)| \leq A(1 + |r|)^{-2-\delta},$$

with $A, \delta > 0$, and define $g(u)$ for $u \in \mathbb{R}$ by the Fourier transformation of h:

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(z) e^{-izu} \, \mathrm{d}z.$$

Then:

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(\mathcal{F})}{4\pi} \int_{-\infty}^{\infty} z h(z) \tanh(\pi z) \, \mathrm{d}z + \sum_{\{P\}} \frac{l(P_0)}{e^{l(P)/2} - e^{-l(P)/2}} \, g\big[l(P)\big]. \quad (5.1)$$

Where the $\{P\}$ sum is taken over distinct conjugacy classes in Γ excluding the class of identity. Furthermore, the sums and integrals are all absolutely convergent.

5.2 The Selberg trace formula

5.2.1 The construction of the trace

In this section we will prove the Selberg trace formula, as stated in Theorem 5.4. Recall from chapter 4 we derived the Poisson summation formula on the torus by considering two different ways to construct functions in $L_2(\mathcal{T})$, one is from the Fourier basis and the other is directly from compactly supported functions. Now we want to do similar things here.

Following from the above argument regarding the close relation between S and $\mathcal{F}$, we have:

$$L_2(S) = L_2(\mathbb{H}/\Gamma) = \{f \in L_2(\mathcal{F}) \,|\, f(Tz) = f(z) \text{ for all } T \in \Gamma \text{ and } z \in \mathbb{H}\}.$$

We call such an f is *automorphic*. It is easy to show $L_2(S)$ is a Hilbert space with the inner product:

$$\langle f_1, f_2 \rangle = \int_{\mathcal{F}} f_1(z)\overline{f_2(z)} \,\mathrm{d}\mu(z).$$

To construct functions in $L_2(S)$, one way is to use the eigenbasis of $\mathbf{D}$. Let $\{\psi_n\}$ denotes a eigenbasis of $\mathbf{D}$, which is chosen to be real-valued without loss of generality. Note that each ψ_n is automorphic. We have:

$$G(z) = \sum_{n=0}^{\infty} c_n \psi_n(z), \tag{5.2}$$

where each $c_n = \langle G, \psi_n \rangle$ is a constant.

Another way is to use compactly supported functions.

$$G(z) = \sum_{T\in\Gamma} f(Tz) \qquad \text{for } f \in C_c^{\infty}(\mathbb{H}). \tag{5.3}$$

Recall from Proposition 2.8 that the action of Γ is properly discontinuous. Thus any compact set can only intersect with finitely many points in the orbit Γz for any z. Thus the sum in $G(z)$ is finite for each z, $G(z)$ is absolutely convergent.

By construction, $G \in C^{\infty}(S)$. Then by compactness of S, $G \in L_2(S)$. Thus we can combine (5.2) and (5.3) to obtain:

$$\begin{aligned} c_n = \langle G, \psi_n \rangle &= \int_{\mathcal{F}} G(z)\psi_n(z)\,\mathrm{d}\mu(z) \\ &= \sum_{T\in\Gamma} \int_{\mathcal{F}} f(Tz)\psi_n(z)\,\mathrm{d}\mu(z) \\ &= \sum_{T\in\Gamma} \int_{T(\mathcal{F})} f(w)\psi_n(w)\,\mathrm{d}\mu(w) \\ &= \int_{\mathbb{H}} f(w)\psi_n(w)\,\mathrm{d}\mu(w). \end{aligned}$$

Similar to (4.14), we consider the radially symmetric f in particular. According to the distance formula in Lemma 2.5, a such function should be of the following form:

$$f(z) = \Phi\left[\frac{|z-z_0|^2}{y\,y_0}\right] \qquad \text{with } \Phi \in C_c^\infty(\mathbb{R}),$$

with $z = x + iy, z_0 = x_0 + iy_0 \in \mathbb{H}$. Note f is radially symmetric on $\mathbb{H}$ with respect to z_0. We can choose Φ to be real-valued without loss of generality, to express $G(z)$ explicitly:

$$G(z) = \sum_{T\in\Gamma} \Phi\left[\frac{|Tz-z_0|^2}{\operatorname{Im}(Tz)\operatorname{Im}(z_0)}\right]. \tag{5.4}$$

Our next goal is to calculate c_n, where

$$c_n = \int_{\mathbb{H}} \Phi\left[\frac{|z-z_0|^2}{y\,y_0}\right]\psi_n(z)\,\mathrm{d}\mu(z) \qquad \text{with } \Phi \in C_c^\infty(\mathbb{R}).$$

We first show that the required information of ψ_n can be reduced to its value at z_0. In a similar manner to in 4.15, we use the rotational invariance of the Laplace-Beltrami operator to construct the solution of the ODE. This is achieved by the following lemma:

Lemma 5.5. *Suppose that $\Phi \in C_c^\infty(\mathbb{R})$, $\psi \in C^\infty(\mathbb{H})$. Suppose further that $\mathbf{D}\psi + \lambda\psi = 0$. Then we have*

$$\int_{\mathbb{H}} \Phi\left[\frac{|z-z_0|^2}{y\,y_0}\right]\psi(z)\,\mathrm{d}.\mu(z) = H(\lambda)\psi(z_0) \tag{5.5}$$

*where $H(\lambda)$ is a function that **only** depends on Φ and λ.*

Proof. Let M be the Möbius transformation that maps $\mathbb{H} \leftrightarrow \mathbb{U}$ conformally, with $w \leftrightarrow z$ for $z \in \mathbb{H}$ and $M(z_0) = 0$. Then we have

$$\frac{|dz|}{y} = \frac{2d|w|}{1-|w|^2}.$$

After a short calculation, we have the Laplace-Beltrami on $\mathbb{U}$ is $\widetilde{\mathbf{D}}f = \frac{1}{4}(1-|w|^2)^2\Delta f$. If we define $\phi(w) = \psi(z)$, then we have $\widetilde{\mathbf{D}}\phi + \lambda\phi = 0$. Letting η denote the area element on $\mathbb{U}$ and defining $P(|w|) = \Phi\left[\frac{4|w|^2}{1-|w|^2}\right]$, we convert equation (5.5) into the following:

$$\int_{\mathbb{H}} P(|w|)\phi(w)\,\mathrm{d}\eta(w) = H(\lambda)\phi(0).$$

Converting the integral using polar coordinates, and letting $r = |w|$, we have

$$\int_0^1 P(r)\frac{4r}{(1-r^2)^2}\left[\int_0^{2\pi}\phi(re^{i\theta})\,\mathrm{d}\theta\right]\mathrm{d}r. \tag{5.6}$$

The θ integral is invariant under rotation, that is

$$\int_0^{2\pi}\phi(re^{i\theta})\,\mathrm{d}\theta = \int_0^{2\pi}\phi(we^{i\theta})\,\mathrm{d}\theta. \tag{5.7}$$

Similar to (4.15), by the rotational invariance of Laplace-Beltrami operator, we can rotate the argument and the resulting function will still be an eigenfunction of $\mathbf{D}$ with the same eigenvalue. So we can replace the original eigenfunction by the right-hand side of (5.7) which is a radially symmetric C^∞ solution of $\widetilde{\mathbf{D}}f + \lambda f = 0$. That is, a solution of the differential equation:

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \frac{4\lambda}{(1-r^2)^2}f = 0. \tag{5.8}$$

The ODE has a regular singular point at $r = 0$. We apply method of Frobenius on (5.8). If we define:

$$p(r) = \frac{1}{r} \qquad q(r) = \frac{4\lambda}{(1-r^2)^2},$$

then take the limits

$$p_0 = \lim_{r\to 0} rp(r) = 1 \qquad q_0 = \lim_{r\to 0} r^2q(r) = 0,$$

we have the indicial equation of the ODE:

$$s^2 + (p_0 - 1)s + q_0 = s^2 = 0.$$

So we have two solutions:

$$f_1(r) = \sum_{n=0}^{\infty} a_n r^n \qquad f_2(r) = f_1 \ln(r) + \sum_{n=0}^{\infty} b_n r^n,$$

where a_n, b_n are constants involving λ. The solution of ODE is a linear combination of f_1 and f_2. Since f is smooth, and f_2 is not smooth at $r = 0$, the coefficient of f_2 must be 0. Furthermore, the constants a_n are determined by a_0 recursively, and from the series in f_1 we clearly have $a_0 = f(0)$. Thus we conclude f is uniquely determined by $f(0)$. Thus we obtain:

$$\int_0^{2\pi} \phi(we^{i\theta})\,\mathrm{d}\theta = \phi(0)E_\lambda(r).$$

Note that $E_\lambda(r)$ is an eigenfunction of $\widetilde{\mathbf{D}}$, and $\phi(0)$ is a constant. With $\Phi(x)$ being a fixed function, the value of the definite integral in (5.6) is uniquely determined by λ. Taking out the constant $\phi(0)$ in (5.6) then gives the desired result. $\square$

To simplify the calculation, we now make the following definitions:

Definition 5.6. We define for $(z, w) \in \mathbb{H} \times \mathbb{H}$, where $w = x' + y'$

$$k(z, w) := \Phi\left[\frac{|z - w|^2}{y\,y'}\right],$$
$$K(z, w) := \sum_{T \in \Gamma} k(Tz, w).$$

Note by the hyperbolic distance formula in Definition 2.5, $k(z, w) = k(Tz, Tw)$ for all $T \in$**PSL**$(2, \mathbb{R})$.

Remark 5.7. We discussed above that G is a well-defined C^∞ function on S. Similarly $K(z, w)$ is a well-defined C^∞ function on S for each fixed w.

Similar to what we did in chapter 4, the next step is to get rid of the eigenfunctions ψ_n. We apply the Lemma 5.5 we just proved. With above definition and (5.3), then take $z_0 = w$ to be an arbitrary fixed point, we then have:

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} c_n \psi_n(z) = \sum_{T \in \Gamma} k(Tz, w) \\ &= \sum_{n=0}^{\infty} H(\lambda_n)\psi_n(w)\psi_n(z) = K(z, w). \end{aligned}$$

Then to eliminate ψ_n:

$$\int_{\mathcal{F}} K(z,z)\,\mathrm{d}\mu(z) = \sum_{n=0}^{\infty} H(\lambda_n)\int_{\mathcal{F}} \psi(z)^2\,\mathrm{d}\mu(z) = \sum_{n=0}^{\infty} H(\lambda_n), \tag{5.9}$$

where we used the fact that $\{\psi_n\}$ are real-valued and orthonormal on $L_2(\mathcal{F})$.

Remark 5.8. The Selberg trace formula arises when we expand (5.9) in two different ways. The first way is to find a more explicit formula for H and use it to express the rightmost sum of (5.9). The other is to split the leftmost integral of (5.9) into summand of distinct conjugacy classes in Γ.

5.2.2 Two ways of expanding the trace

In this section, we will calculate two expansions of (5.9) as indicated in Remark 5.8. This will lead us to the equation of the Selberg trace formula. We will discuss the bound and convergence of the Selberg trace formula when applied to general analytic functions in the next section.

First we show that $\sum_{n=1}^{\infty} H(\lambda_n)$ can indeed be regarded as a trace. To see more about trace of an operator on Hilbert space, see Appendix A. We construct the operator L:

$$Lf(z) = \int_{\mathbb{H}} k(z,w)f(w)\,\mathrm{d}\mu(w) \qquad \text{for } f \in L_2(S).$$

Apply Lemma 5.5, we have $L\psi_n(z) = H(\lambda_n)\psi_n(z)$. Then apply the definition of trace:

$$Tr(L) = \sum_{n=0}^{\infty}\langle \psi_n, L\psi_n\rangle = \sum_{n=0}^{\infty} H(\lambda_n).$$

Following from Definition (2.9), for automorphic functions f, we can decompose the integral over $\mathbb{H}$ using the images of $\mathcal{F}$ under action of Γ. Thus L can be write in the following form:

$$\begin{aligned} Lf(z) &= \int_{\mathbb{H}} k(z,w)f(w)\,\mathrm{d}\mu(w) = \sum_{T\in\Gamma}\int_{T(\mathcal{F})} k(z,w)f(w)\,\mathrm{d}\mu(w) \\ &= \int_{\mathcal{F}}\sum_{T\in\Gamma} k(T^{-1}z,w)f(w)\,\mathrm{d}\mu(w) = \int_{\mathcal{F}} K(z,w)f(w)\,\mathrm{d}\mu(w). \end{aligned}$$

Since we constructed K using real-valued Φ, we see L is self-adjoint. By Remark (5.7) we see L is a Hilbert-Schmidt operator, hence being compact([13] chapter

IV). Thus we conclude that L is a self-adjoint compact operator on $L_2(S)$.

The next step is to find an explicit formula for H. Note for $z = x + iy$ the function $f(z) = y^s$ for $s \in \mathbb{C}$ satisfies $\mathbf{D}f + s(1-s)f = 0$. Then take $z_0 = i$ in Lemma 3.3, we have

$$\int_{\mathbb{H}} k(z,i) y^s \,\mathrm{d}\mu(z) = H[s(1-s)] \qquad \forall s \in \mathbb{C}. \tag{5.10}$$

To simplify calculation, we make the following definition:

Definition 5.9.

$$s = \frac{1}{2} + ir, \qquad\qquad \lambda = s(1-s) = \frac{1}{4} + r^2, \tag{5.11}$$

$$Q(v) := \int_v^\infty \frac{\Phi(t)}{\sqrt{t-v}} \,\mathrm{d}t, \qquad\qquad \forall v \geq 0, \tag{5.12}$$

$$g(u) := Q(e^u + e^{-u} - 2), \qquad\qquad \forall u \in \mathbb{R}, \tag{5.13}$$

with λ is the same as in Lemma 5.5. Note that (5.11) is consistent with our Definition 5.3 above.

Since Φ is smooth and compactly supported, we see Q is absolutely convergent for all $v \geq 0$. Applying Leibniz rule to Q, we obtain that Q is smooth as well.

We can then calculate H:

Corollary 5.10.

$$H(\lambda) = h(r) := \int_{-\infty}^{\infty} g(u) e^{iru} \,\mathrm{d}u. \tag{5.14}$$

Proof. First we use change of variable

$$t = \frac{|z-i|^2}{y} = \frac{x^2 + (y-1)^2}{y}, \qquad\qquad y = e^u.$$

Then

$$dt = \frac{2x\,\mathrm{d}x}{y}, \qquad du = \frac{dy}{y}, \qquad d\mu(z) = \frac{du\,\mathrm{d}t}{2e^{u/2}\sqrt{t - e^u - e^{-u} + 2}}.$$

We then have:

$$\begin{aligned}
H(\lambda) = \int_{\mathbb{H}} k(z,i) y^s \,\mathrm{d}\mu(z) &= \int_0^\infty \int_{-\infty}^\infty \Phi\left[\frac{|z-i|^2}{y}\right] y^s \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^\infty \int_{e^u + e^{-u} - 2}^\infty \frac{\Phi(t) e^{us}}{e^{u/2}\sqrt{t - e^u - e^{-u} + 2}} \,\mathrm{d}t\,\mathrm{d}u \\
&= \int_{-\infty}^\infty Q(e^u + e^{-u} + 2)\, e^{(1/2 + ir)u - u/2} \,\mathrm{d}u \\
&= \int_{-\infty}^\infty g(u)\, e^{iru} \,\mathrm{d}u,
\end{aligned}$$

where we used the fact t is a even function of x. $\square$

Remark 5.11. Note that both g and h are even functions, and $g \in C_c^\infty(\mathbb{R})$.

For future use, we prove the following inversion formulas:

Corollary 5.12.

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\, e^{-iru}\, \mathrm{d}r, \tag{5.15}$$

$$\Phi(x) = -\frac{1}{\pi}\int_{x}^{\infty} \frac{dQ(t)}{\sqrt{t-x}}. \tag{5.16}$$

Proof. Note that (5.15) is just the inverse Fourier transform applied to h.

To prove (5.16), we do some change of variables. Let $t = v + \xi^2$ in (5.12) and $t = x + \eta^2$ in (5.16). We take $\xi, \eta \in [0, \infty)$. Then (5.12) becomes:

$$Q(v) = 2\int_0^\infty \Phi(v+\xi^2)\, \mathrm{d}\xi.$$

Then apply Leibniz rule on the smooth function Q to obtain

$$Q'(v) = 2\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}v}\Phi(v+\xi^2)\, \mathrm{d}\xi.$$

With the change of variable and the above, the right hand side of (5.16) becomes:

$$\begin{aligned}
-\frac{1}{\pi}\int_x^\infty \frac{\mathrm{d}Q(t)}{\sqrt{t-x}} &= -\frac{1}{\pi}\int_x^\infty \frac{Q'(t)\,\mathrm{d}t}{\sqrt{t-x}} \\
&= -\frac{2}{\pi}\int_0^\infty Q'(x+\eta^2)\,\mathrm{d}\eta \\
&= -\frac{4}{\pi}\int_0^\infty\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}v}\Phi(x+\xi^2+\eta^2)\,\mathrm{d}\xi\mathrm{d}\eta \\
&= -\frac{4}{\pi}\int_0^{\pi/2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}v}\Phi(x+r^2)r\,\mathrm{d}r\mathrm{d}\theta \\
&= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}v}\Phi(x+r^2)\,\mathrm{d}(r^2) \\
&= -\big[\Phi(x+y)\big]_{y=0}^{y=\infty} \\
&= \Phi(x).
\end{aligned}$$

We used $\mathrm{d}Q(t) = Q'(t)\mathrm{d}t$ since Q is smooth. Note that by taking $\xi, \eta \in [0, \infty)$, we are integrating over a quarter of the whole plane, thus when we transform to polar coordinates the range of the integral for θ is from 0 to $\pi/2$. We use the fact that Φ goes to 0 at infinity since it is compactly supported. □

Applying Corollary 5.10, we obtain one method to expand (5.9) mentioned in Remark 5.8:

$$\int_{\mathcal{F}} K(z,z)\,\mathrm{d}\mu(z) = \sum_{n=0}^{\infty} h(r_n). \tag{5.17}$$

Now we try to expand the left hand side of above equation using some algebraic techniques. From basic algebra, we know conjugacy classes in Γ partition the group. We can express the conjugacy class of an element P as $\{P\} := \{SPS^{-1} : S \in \Gamma\}$. Now we consider the subgroup that commutes with P, $\mathcal{Z}_\Gamma(P)$, which is a Fuchsian group itself. From now we write $\mathcal{Z}(P)$ for $\mathcal{Z}_\Gamma(P)$ since we always work within Γ.

Suppose two elements $S_1PS_1^{-1}$ and $S_2PS_2^{-1}$ are the same in the conjugacy class of P. Then $S_1PS_1^{-1} = S_2PS_2^{-1}$ implies $S_1^{-1}S_2P = PS_1^{-1}S_2$, so $S_1^{-1}S_2 \in \mathcal{Z}(P)$. Then S_1 and S_2 are in the same coset of $\mathcal{Z}(P)$ in Γ. The converse of the above statement, saying that if two elements are in the same coset of $\mathcal{Z}(P)$, then the conjugation of P by those two elements are the same, is obviously true. So conjugating by two elements in Γ of P are distinct if and only if those two elements are in the different coset on $\mathcal{Z}(P)$, that is, can be represented by different elements in $\Gamma/\mathcal{Z}(P)$. More explicitly,

$$\{P\} := \{T : T = SPS^{-1}, S \in \Gamma\} = \{T = RPR^{-1} : R\mathcal{Z}(P) \in \Gamma/\mathcal{Z}(P)\}.$$

For each coset $R\mathcal{Z}(P)$ we fix a unique representative $R \in \Gamma$. From now on we will abuse the notation a little bit to use $R \in \Gamma/\mathcal{Z}(P)$ to denote those unique representations of each coset.

Now we want to discuss the fundamental domain of $\mathcal{Z}(P)$. Let $\mathcal{F}$ denotes the fundamental domain of Γ as usual, we claim:

$$\bigcup_{R\in\Gamma/\mathcal{Z}(P)} R\mathcal{F} = \mathcal{G}_P.$$

where $\mathcal{G}_P$ is a fundamental domain of $\mathcal{Z}(P)$, which is not necessarily compact.

To see this, first we show that $\bigcup_{S\in\mathcal{Z}(P)} S\mathcal{G}_P = \mathbb{H}$. Since right cosets partition the group, for each $T \in \Gamma$, $T \in \mathcal{Z}(P)R$ for some $R \in \Gamma/\mathcal{Z}(P)$. Then $T = SR$ for some $S \in \mathcal{Z}(P)$. Since $\mathcal{F}$ is a fundamental domain of Γ, for each $w \in \mathbb{H}$ we have $w = Tz$ for some $T \in \Gamma$ and $z \in \mathcal{F}$. Then $w = SRz \in \bigcup_{S\in\mathcal{Z}(P)} S\mathcal{G}_P$ as claimed.

Then we show that interior of $\mathcal{G}_P$ and $S\mathcal{G}_P$ do not intersect for $S \neq I$. Both $\mathcal{G}_P$ and $S\mathcal{G}_P$ are a union of fundamental domains. For a point z be in interior of $\mathcal{G}_P$, all fundamental domains it intersects must be in the union $\mathcal{G}_P$. Thus it suffices to show that $\mathcal{G}_P$ and $S\mathcal{G}_P$ does not share any common fundamental domain. Since for two distinct element $T, T' \in \Gamma$ the fundamental domain $T\mathcal{F}, T'\mathcal{F}$ are distinct, it suffices to show for any $S \neq I \in \mathcal{Z}(P)$ and $R_1, R_2 \in \Gamma/\mathcal{Z}(P)$, we have $SR_1 \neq R_2$. Suppose not for contradiction. Then since $S \neq I$, we have $R_1 \neq R_2$. But then the fact $SR_1 = R_2$ implies R_1 and R_2 are in the same coset of $\mathcal{Z}(P)$, so $R_1 = R_2$ since the representation we choose for each coset is unique, contradiction. Thus we get the desired result.

With those algebraic techniques above, we are now prepared to expand the left hand side of (5.17).

$$\begin{aligned}
\int_{\mathcal{F}} K(z,z)\,\mathrm{d}\mu(z) &= \sum_{T\in\Gamma}\int_{\mathcal{F}} k(Tz,z)\,\mathrm{d}\mu(z) \\
&= \sum_{\{P\}}\sum_{T\in\{P\}}\int_{\mathcal{F}} k(Tz,z)\,\mathrm{d}\mu(z) \\
&= \sum_{\{P\}}\sum_{R\in\Gamma/\mathcal{Z}(P)}\int_{\mathcal{F}} k(R^{-1}PRz,z)\,\mathrm{d}\mu(z) \\
&= \sum_{\{P\}}\sum_{R\in\Gamma/\mathcal{Z}(P)}\int_{\mathcal{F}} k(PRz,Rz)\,\mathrm{d}\mu(z) \\
&= \sum_{\{P\}}\sum_{R\in\Gamma/\mathcal{Z}(P)}\int_{R\mathcal{F}} k(Pw,w)\,\mathrm{d}\mu(w) \\
&= \sum_{\{P\}}\int_{\mathcal{G}_P} k(Pw,w)\,\mathrm{d}\mu(w).
\end{aligned}$$

where the $\{P\}$ sum is taken over distinct conjugacy classes.

To make the formula more explicit, we want to express above result using only function g and h. To do this we need to solve the integration over $\mathcal{G}_P$. First we consider the relatively simple case with $P = I$. Then $\mathcal{G}_I = \mathcal{F}$ and we obtain:

$$\int_{\mathcal{F}} k(w,w)\,\mathrm{d}\mu(w) = \mu(\mathcal{F})\Phi(0).$$

We will apply (5.16) to calculate $\Phi(0)$ and use the following integration formula: (Proved in Appendix C, Lemma C.1)

$$\int_0^\infty \frac{\sin(az)}{\sinh(z)}\,\mathrm{d}z = \frac{\pi}{2}\tanh(\frac{\pi a}{2}).$$

and for an arbitrary odd function $\eta(x)$, since $\cos(x)$ is even, we have

$$\int_{-\infty}^{\infty} \eta(x)e^{ix}\,\mathrm{d}x = i\int_{-\infty}^{\infty} \eta(x)\sin(x)\,\mathrm{d}x.$$

Then we have:

$$\begin{aligned}
\Phi(0) &= \frac{1}{\pi}\int_0^{\infty} \frac{Q'(t)\,\mathrm{d}t}{\sqrt{t}} \\
&= \frac{1}{\pi}\int_0^{\infty} \frac{g'(u)}{\sqrt{e^u+e^{-u}-2}}\,\mathrm{d}u \\
&= \frac{1}{\pi}\int_0^{\infty} \frac{1}{e^{u/2}-e^{-u/2}}\left[\frac{-i}{2\pi}\int_{-\infty}^{\infty} rh(r)e^{-iru}\,\mathrm{d}r\right]\mathrm{d}u \\
&= \frac{1}{4\pi^2}\int_0^{\infty} \frac{1}{\sinh(u/2)}\left[\int_{-\infty}^{\infty} rh(r)\sin(ru)\,\mathrm{d}r\right]\mathrm{d}u \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} rh(r)\left[\int_0^{\infty} \frac{\sin(ru)}{\sinh(u/2)}\,\mathrm{d}u\right]\mathrm{d}r \\
&= \frac{\mu(\mathcal{F})}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,\mathrm{d}r.
\end{aligned}$$

Note that since $h(r)$ is a even function, $rh(r)$ is an odd function, and that since $Q(t)$ is smooth and compactly supported, so is $Q'(t)$. Thus the integration is taken over a finite interval. Since $Q'(t)$ is bounded, and the integration of $1/\sqrt{t}$ is convergent on any finite interval $[0, M]$, the integral in finite. Thus we can use Fubini's theorem to interchange the integral.

Thus we have

$$\int_{\mathcal{F}} k(w,w)\,\mathrm{d}\mu(w) = \frac{1}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,\mathrm{d}r. \tag{5.18}$$

Now we deal with the conjugacy classes of non-identity hyperbolic elements. Recall from the proof of Proposition 2.14, for each hyperbolic element P there is a unique corresponding primitive element P_0 such that $P = P_0^k$. This P_0 takes the form $P_0(z) = e^{l(P_0)}z$ for $l(P_0) > 0$ and $\mathcal{Z}_\Gamma(P) = \langle P_0\rangle$. Since $k(Pz, z)$ is invariant under conjugation of P, we can assume P takes the form $P(z) = e^{l(P)}z$. To simplify the equation, we use ξ to denote $e^{l(P)}$ and ξ_0 to denote $e^{l(P_0)}$.

With the simple formula of P_0, we can easily construct the fundamental domain of $\langle P_0\rangle$. Since $l(P_0) > 0$, we have $\xi_0 > 1$. It is easy to verify that the strip:

$$S_0 = \{z : 1 \leq \mathrm{Im} z < \xi_0\},$$

is a fundamental domain of $\langle P_0\rangle$.

With the condition above, we can then expand the integral for hyperbolic conjugacy classes $\{P\}$:

$$\begin{aligned}\int_{\mathcal{G}_P} k(Pw,w)\,\mathrm{d}\mu(w) &= \int_{S_0} k(\xi z, z)\,\mathrm{d}\mu(z)\\ &= \int_1^{\xi_0}\int_{-\infty}^{\infty}\Phi\left[\frac{(\xi-1)^2(x^2+y^2)}{\xi y^2}\right]\frac{\mathrm{d}x\mathrm{d}y}{y^2}\\ &= \int_1^{\xi_0}\int_{-\infty}^{\infty}\Phi\left[\frac{(\xi-1)^2}{\xi}(1+\eta^2)\right]\frac{\mathrm{d}\eta\mathrm{d}y}{y.}\end{aligned}$$

where we used the change of variable $\eta = x/y$. Now we use another change of variable;

$$t = \frac{(\xi-1)^2}{\xi}(1+\eta^2) \qquad \mathrm{d}\eta = \frac{\xi}{2\eta(\xi-1)^2}\,\mathrm{d}t.$$

Note that

$$\begin{aligned}\sqrt{t-(\xi+\frac{1}{\xi}-2)} &= \left[\frac{(\xi-1)^2}{\xi}(1+\eta^2)-(\xi+\frac{1}{\xi}-2)\right]^{1/2}\\ &= \left[\frac{1}{\xi}\eta^2\Big(\xi^2-2\xi+1\Big)\right]^{1/2}\\ &= \eta\big(\xi+\frac{1}{\xi}-2\big)^{1/2}.\end{aligned}$$

And evaluate the y-integral directly:

$$\int_1^{\xi_0}\frac{1}{y}\,\mathrm{d}y = \ln(\xi_0).$$

We then have;

$$\begin{aligned}\int_{\mathcal{G}_P} k(Pw,w)\,\mathrm{d}\mu(w) &= l(P_0)\int_{\xi+1/\xi-2}^{\infty}\frac{\xi\Phi(t)}{\eta(\xi-1)^2}\,\mathrm{d}t\\ &= l(P_0)\int_{\xi+1/\xi-2}^{\infty}\frac{\Phi(t)}{\eta(\xi+1/\xi-2)}\,\mathrm{d}t\\ &= \frac{l(P_0)}{\big(\xi+1/\xi-2\big)^{1/2}}\int_{\xi+1/\xi-2}^{\infty}\frac{\Phi(t)}{\eta(\xi+1/\xi-2)^{1/2}}\,\mathrm{d}t\\ &= \frac{l(P_0)}{\xi^{1/2}-\xi^{-1/2}}\int_{\xi+1/\xi-2}^{\infty}\frac{\Phi(t)}{\sqrt{t-(\xi+1/\xi-2)}}\,\mathrm{d}t\\ &= \frac{l(P_0)}{\xi^{1/2}-\xi^{-1/2}}g\big[\ln(\xi)\big]. \qquad (5.19)\end{aligned}$$

Now composing (5.17),(5.18) and (5.19), we obtain the formula in Theorem 5.4.

5.2.3 Generalization to analytic functions

Note that till now we were working on the function h constructed from the real-valued C_{00}^{∞} function Φ. Now we want to generalize the formula obtained for analytic functions on a strip with certain decaying condition, as stated in Theorem 5.4. In general, we want a function $h(z)$ to satisfy some conditions so that if we construct Φ backward through (5.15),(5.13) and (5.16), the sum of $G(z)$ in (5.4) is convergent.

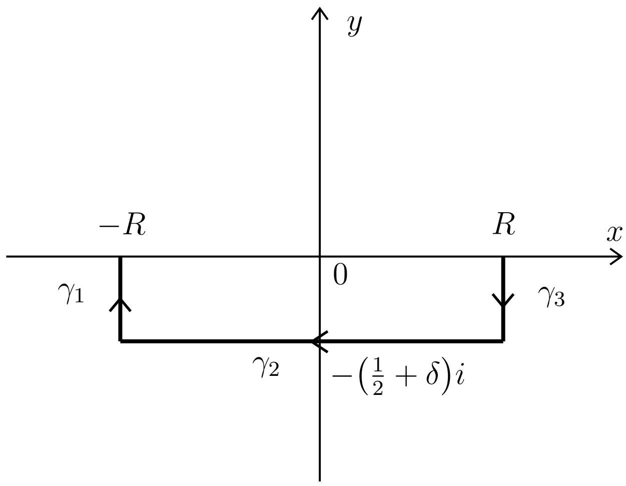
As indicated in Theorem 5.4, suppose we have an analytic function $h(z)$ on the strip $\{z : \mathrm{Im}(z) \leq \frac{1}{2} + \delta\}$, with the condition

$$h(-z) = h(z) \qquad \text{and} \qquad |h(z)| \leq A(1+|r|)^{-2-\delta}. \tag{5.20}$$

for some $A, \delta > 0$. First we want to estimate the size of g, which is the Fourier transform of h as defined above.

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\, e^{-iru}\, \mathrm{d}r.$$

For $u > 0$, shift the contour to $\mathrm{Im}(r) = -(1/2 + \delta)$ for the integral, as shown in the graph below:



Since h is analytic in the strip that contains the rectangle contour, the integral over the rectangle is 0. Then we have:

$$\int_{-R}^{R} h(r)e^{-iru} = -\int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} h(r)e^{-iru}\, \mathrm{d}r,$$

where the direction of path as indicated on the figure. As $R \to \infty$, from the bound on h in (5.20), the integrand $h(r)e^{-iru}$ goes to 0. Thus the integral over γ_1 and γ_3 goes to 0. Thus we have:

$$
\begin{aligned}
\int_{-\infty}^{\infty} h(r)e^{-iru} &= -\int_{\gamma_2} h(r)e^{-iru}\,\mathrm{d}r \\
&= \int_{-\infty}^{\infty} h(r-\Big(\frac{1}{2}+\delta\Big)i)e^{-iu(r-(1/2+\delta)i)}\,\mathrm{d}r \\
&= e^{-u(1/2+\delta)}\int_{-\infty}^{\infty} h(r-\Big(\frac{1}{2}+\delta\Big)i)e^{-iur}\,\mathrm{d}r \\
&\leq e^{-u(1/2+\delta)}\int_{-\infty}^{\infty} \frac{A}{(1+|r|)^{2+\delta}}\,\mathrm{d}r \\
&= 2Ae^{-u(1/2+\delta)}\int_{1}^{\infty} \frac{1}{r^{2+\delta}}\,\mathrm{d}r \\
&= Ce^{-u(1/2+\delta)}.
\end{aligned}
$$

From now on we use C to represent a constant, which might change from equation to equation, but will not affect our final result. From basic calculus wee know the integral in the second last line converges. Similarly for $u < 0$, by shifting the contour to $\mathrm{Im}(r) = 1/2 + \delta$, we have that $g(u) \leq Ce^{u(1/2+\delta)}$. So eventually:

$$
g(u) \leq Ce^{-(1/2+\delta)|u|}.
$$

Recall from (5.13) that, since Q is increasing, for $x > 0$ large we have:

$$
Q(x) \leq g\big(\ln(x)\big) \leq Cx^{-(1/2+\delta)}.
$$

Then we can use (5.16) to estimate function Φ, as $x \to \infty$:

$$
\begin{aligned}
\Phi(x) &= C\int_{x}^{\infty} \frac{\mathrm{d}Q(t)}{\sqrt{t-x}} \\
&\sim \int_{x}^{\infty} \frac{t^{-(3/2+\delta)}}{\sqrt{t-x}}\,\mathrm{d}t \\
&\sim \int_{1}^{\infty} \frac{(\tilde{t}x)^{-(3/2+\delta)}}{\sqrt{x}\sqrt{\tilde{t}-1}}x\,\mathrm{d}\tilde{t} \\
&\sim x^{-(1+\delta)}\left[\int_{1}^{2} \frac{\tilde{t}^{-(3/2+\delta)}}{\sqrt{\tilde{t}-1}}\,\mathrm{d}t + \int_{2}^{\infty} \frac{\tilde{t}^{-(3/2+\delta)}}{\sqrt{\tilde{t}-1}}\,\mathrm{d}t\right] \\
&\sim x^{-(1+\delta)}\left[\int_{1}^{2} \frac{1}{\sqrt{\tilde{t}-1}}\,\mathrm{d}t + \int_{2}^{\infty} \tilde{t}^{-(3/2+\delta)}\,\mathrm{d}t\right] \\
&\sim x^{-(1+\delta)}.
\end{aligned}
$$

where both integrals in the second last step converges by basic calculus.

Now we are prepared to go back to the sum in (5.4). If we order the orbit points Γz as a sequence $\{z_i\}$ by their distance to the chosen point z_0, then apply the distance formula (2.5) and above, we obtain:

$$\begin{aligned} G(z) &= \sum_{T\in\Gamma} \Phi\left[\frac{|Tz - z_0|^2}{\operatorname{Im}(Tz)\operatorname{Im}(z_0)}\right] \\ &= \sum_{z_i} \Phi\left[2\cosh(d(z_i, z_0)) - 2\right] \\ &\approx \sum_{z_i} \Phi\left[e^{d(z_i,z_0)}\right]. \end{aligned}$$

So the problem breaks down to estimating how many orbit points are there in a hyperbolic ball certain radius. Note that each fundamental domain of Γ contains 1 orbit point. Recall that each fundamental domain of Γ has the same finite area, and from (2.1) we know the hyperbolic ball $B(z_0, R)$ has area $\mathcal{O}(e^R)$, then we know the number of fundamental domains in the ball $B(z_0, R)$ is approximately $\mathcal{O}(e^R)$. More precisely, we can enlarge the radius to cover all the fundamental domains that intersect $B(z_0, R)$ are covered by the larger ball $B(z_0, R + C)$ for some constant C. So the number of orbit points within the ball $B(z_0, R)$ is no more than $\mathcal{O}(e^{R+C})$. Similarly the number of orbit points in the ball $B(z_0, R)$ is no less than $\mathcal{O}(e^{R-C})$. Then for each n, consider the orbit points lying in the annulus with inner radius $\ln(n-1)$ and outer radius $\ln(n)$, we know that there are at most $2C + 1$ of them from above, and each of them is of distance at least $\ln(n-1)$ away from z_0. So the sum becomes:

$$\begin{aligned} \sum_{z_i} \Phi\left[e^{d(z_i,z_0)}\right] &\leq \sum_{n=1}^{\infty}(2C+1)\Phi(n-1) \\ &\sim (2C+1)\sum_{n=1}^{\infty} \frac{1}{(n-1)^{-(1+\delta)}}, , \end{aligned}$$

which we know converges to a finite number. This shows Theorem 5.4 holds with the assumption (5.20).

In this chapter, we proved the Selberg trace formula for compact Riemann surface S of genus ≥ 2. To construct functions in $L_2(S)$, we transfer the situation to the fundamental region of $\mathcal{F}$ Fuchsian group Γ, and constructed the functions in $L_2(\mathcal{F})$ instead. Similar to in chapter 4 we expressed functions in $L_2(\mathcal{F})$ in two ways: one is to sum the value of a compactly supported function on $\mathcal{R}$ over all orbit points, the other is to use the eigenbasis of Laplace-Beltrami operator. Using the rotational invariance of Laplace-Beltrami operator, we write the function constructed as the trace of a self-adjoint compact operator. Then we expand the trace in two ways as explained in Remark 5.8. Combining the equation we get through these two ways, we obtain the Selberg trace formula. At the end we proved the formula holds for a large branch of analytic function besides the kind we used to construct the formula. In the next chapter, we will use the Selberg trace formula on various functions, to derive the Prime Orbit Theorem, which is, just as its name implies, very analogous to the Prime Number Theorem we proved in chapter 3.

Chapter 6

The Prime Orbit Theorem

The Selberg trace formula derived in the last chapter is a very powerful tool, we will apply it in this chapter to obtain an important geometric fact on compact Riemann surfaces.

The setup in this chapter is the same as in chapter 5. We have $S \cong \mathbb{H}/\Gamma$ being a compact Riemann surface of genus ≥ 2, carries the Poincar metric, with Γ being a strictly hyperbolic Fuchsian group.

6.1 Geometric interpretation of conjugacy class in Γ

Let $T \in \Gamma$ be a group element, which is of course hyperbolic. In Remark 2.12 we have already noticed $l(T)$, the displacement length of T, is the length of geodesic in $\mathbb{H}$ connecting T and Tz. But it is precisely such a geodesic on $\mathbb{H}$ projects down to closed geodesic on S. Thus we expect some correspondence between such T and the closed geodesics on S. With some sophistication, we formalize this idea in the next Proposition.

Proposition 6.1. *There is a one-to-one correspondence between the closed, oriented geodesics of the surface $S \cong \mathbb{H}/\Gamma$ and the conjugacy classes in Γ. The length of the geodesic corresponding to the conjugacy class $\{T\}$ is the displacement length $l(T)$.*

Proof. Let $\pi : \mathbb{H} \to S$ be the quotient map. Let $T \in \Gamma$. Then the axis $\alpha(T)$ is preserved by T, hence projects to a closed geodesic under π to S. Since T is an isometry, the projection will preserve the tangent vectors. Thus $\alpha(T)$ and $\alpha(T^{-1})$, being the same curve in $\mathbb{H}$ with opposite direction of traversing, will project down to closed geodesics with opposite orientation. From Proposition 2.11, we know the length of $\pi(\alpha(T))$ is equal to $l(T)$. Note that $\alpha(T)$ has a natural orientation, from the repelling fixed point towards the attracting one. The projected geodesic inherits the orientation. From (2.6) we know that for any other element conjugate to T, its axis will also project to $\pi(\alpha(T))$. So each conjugacy class corresponds to a unique geodesic of S.

For the other direction, suppose γ is a closed oriented geodesic in S, with $\gamma(t) = \gamma(t+l)$ for some l. We can construct a complete oriented geodesic $\tilde{\gamma}$ in $\mathbb{H}$ from the successive lifts of γ. Then there is a unique hyperbolic element $T \in \mathbf{PSL}(2, R)$ with $\alpha(T) = \tilde{\gamma}$, and by Proposition 2.11 the displacement length of T is l. Now we need to show $T \in \Gamma$. Re-parametrize $\tilde{\gamma}$. Note that $\tilde{\gamma}(t)$ and $\tilde{\gamma}(t+l)$ project to the same point on S, so for some fixed t_0 we must have $\tilde{\gamma}(t_0 + l) = R\gamma(t_0) = T\gamma(t_0)$ for some $R \in \Gamma$. Let $z = \tilde{\gamma}(t_0)$, then consider the sequence $(z_n) = (\tilde{\gamma}(t_0 + 1/n))$ in $\mathbb{H}$, similarly we have sequence (R_n) in Γ such that $\tilde{\gamma}(t_0 + l + 1/n) = R_n\tilde{\gamma}(t_0 + 1/n) = T\tilde{\gamma}(t_0 + 1/n)$. By the triangle inequality, for each n we have:

$$d(R_n z, Rz) \leq d(R_n z, R_n z_n) + d(R_n z_n, Rz_n) + d(Rz_n, Rz).$$

As $n \to \infty$, the first and last term in the right hand side of the above inequality goes to 0 by continuity of R_n and R. For the term in middle, use triangle inequality again, we have:

$$d(R_n z_n, Rz_n) \leq d(R_n z_n, Tz_n) + d(Tz_n, Tz) + d(Tz, Rz_n).$$

From construction, we have $Tz_n = R_n z_n$ and $Tz = Rz$. Then the first term in the right hand side of the above inequality varnishes, and the other two tends to 0 as $n \to \infty$ by continuity of T and R. Thus $d(R_n z, Rz)$ tends to 0 as $n \to \infty$. Since the action of Γ is properly discontinuous, we have $R_n = R$ for all but finitely many n. Then since these two analytic functions agree on a sequence with an accumulation point, from complex analysis, we have $T = R$. Thus $T \in \Gamma$. For any other lift of γ, again from (2.6) we see it must be the axis of a hyperbolic transformation conjugate to T in Γ. □

Remark 6.2. It can be shown that for such compact surface S with negative curvature, each free homotopy class has a unique closed geodesic, being the shortest representative in that class. ([7] chapter 1). Thus we can also interpret the precedent proposition as there is a one-to-one correspondence between free homotopy classes on S and conjugacy classes in Γ. This conclusion is consistent with the result $\pi_1(S) \cong \Gamma$ in Theorem 5.1.

Given a closed geodesic in S, we can generate a family of paths that traverse it multiple times. Like what we did in chapter 2 for an element in the Fuchsian group, we define a *primitive* closed geodesic to be the root of the family, that is, a closed geodesic which is not an iterate of a shorter closed geodesic.

With the precedent proposition, we would expect there to be some connection between the primitive elements in Γ and primitive closed geodesic in S. This is indeed the case. It is clear that for $T \in \Gamma$ with corresponding primitive element T_0, the fixed points of T and T_0 are the same, so they have the same axis. Then after they are projected down to S, the image curve of closed geodesic corresponds is the same, but the lengths are $l(T)$ and $l(T_0)$ respectively. Suppose $T = T_0^k$, we have $l(T) = l(T_0)^k$ from Proposition 2.14. Thus the geodesic corresponding to T traverses k times along the geodesic corresponding to T_0. Following from the precedent proposition, we see that conjugacy classes of primitive elements in Γ correspond to primitive closed geodesics in S.

6.2 The Prime Orbit Theorem

We are particularly interested in the properties of primitive closed geodesics. We make the following definition:

Definition 6.3. Let T_0 denote a primitive element in Γ,

$$\pi_0(x) = \#\{\text{distinct } \{T_0\} : e^{l(T_0)} \leq x\}. \tag{6.1}$$

So the function $\pi_0(x)$ counts the number of distinct conjugacy classes of primitive elements with displacement length up to x. With the above discussion we can realize that it is also the counting function of primitive closed geodesics with length up to $\ln(x)$.

Our goal, in this chapter, is to prove the Prime Orbit Theorem:

Theorem 6.4 (the Prime Orbit Theorem)**.**

$$\pi_0(x) \sim \mathbf{li}(x),$$

where the function $\mathbf{li}(x)$ is defined as in Definition 3.19.

We can have a rough estimation on $\pi_0(x)$ even without the help of trace formula. The result will also be helpful when we prove the Prime Orbit Theorem.

Lemma 6.5.

$$\pi_0(x) = \mathcal{O}(x). \tag{6.2}$$

Proof. Let γ be a primitive closed geodesic on S. Then it can be lifted to a geodesic $\tilde{\gamma}$ in $\mathbb{H}$. We have that $\tilde{\gamma}$ will pass through some point z_0 in one of the fundamental domains $\mathcal{F}$ of Γ. The geodesic $\tilde{\gamma}$ is the axis of some primitive element T_0 with displacement length $l_0 = l(T_0)$. From above, we know the length of γ is l_0. Then pick a point $w \in \mathcal{F}$ and some $R > 0$ such that the hyperbolic ball $B(w, R)$ covers $\mathcal{F}$. Such R can be chosen since each fundamental domain is compact has the same area. Recall from Proposition 2.11 that the distance z_0 traverse under T_0 is precisely l_0. Then by triangle inequality:

$$\begin{aligned} d(w, T_0 w) &\leq d(w, z_0) + d(z_0, T z_0) + d(T z_0, T w) \\ &\leq 2R + l. \end{aligned}$$

This result implies that for each primitive closed geodesic of length l_0, there is a transformation $T_0 \Gamma$ that maps w to a point of distance less than $l_0 + 2R$ away of w. Since each copy of fundamental domain can only have one image of w, we can use the number of images of $\mathcal{F}$ within the ball of radius $l_0 + 3R$ centered at w, which gives the bound:

$$\pi_0(e^x) \leq \frac{\mathcal{A}(B(w, x + 3R))}{\mathcal{A}(\mathcal{F})} = \mathcal{O}(e^x).$$

which follows from (2.1). Taking the logarithm on both sides of the above equation gives the desired result. □

Now we are prepared to apply the Selberg trace formula. Theorem 4.1 encourage us to take;

$$h(r) = e^{-(1/4 + r^2)T} \qquad \text{for} \quad T > 0,$$

where r is taken to be consistent with the definition $\lambda_n = (1/4 + r_n^2)$ for eigenvalues. The Fourier transform of h is a Gaussian integral, so we have

$$g(u) = \frac{e^{-T/4}}{\sqrt{4\pi T}} e^{u^2/4T}.$$

Recall the definition of $N(x)$ from (4.4), by applying the Selberg trace formula, we obtain;

$$\begin{aligned} \int_0^\infty e^{-Tx} \, \mathrm{d}N(x) = \sum_{n=0}^{\infty} e^{-\lambda_n T} &= \frac{\mu(\mathcal{F})}{4\pi} \int_{-\infty}^{\infty} z e^{-(z^2 + 1/4)T} \tanh(\pi z) \, \mathrm{d}z \\ &+ \sum_{\{P\}} \frac{l(P_0)}{e^{l(P)/2} - e^{-l(P)/2}} \frac{e^{-T/4}}{\sqrt{4\pi T}} e^{-l(P)^2/4T}. \end{aligned} \tag{6.3}$$

where the sums and integrals are absolutely convergent by Theorem 5.4.

First we want to discuss the asymptotic behavior of $N(x)$. Similar to (4.6), we want to apply Karamata's Tauberian theorem to measure $\mathrm{d}N(x)$ as $T \searrow 0$.

As $T \searrow 0$, since e^{-x} decays faster than any polynomial as $x \to \infty$, we see the sum over conjugacy classes in above equation goes to 0. Thus the result is solely determined by the tanh integral. Note that tanh has the limit:

$$\lim_{x\to\infty} \tanh(x) = 1 \qquad \lim_{x\to-\infty} \tanh(x) = -1.$$

Using a change of variable, we see as $T \searrow 0$:

$$\begin{aligned}
\int_{-\infty}^{\infty} z e^{-(z^2+1/4)T} \tanh(\pi z)\,\mathrm{d}z &= \int_{-\infty}^{\infty} \frac{\tilde{z}}{\sqrt{T}} e^{-(\tilde{z}^2+T/4)} \tanh(\frac{\pi \tilde{z}}{\sqrt{T}}) \frac{\mathrm{d}\tilde{z}}{\sqrt{T}} \\
&= \frac{1}{T} \int_{-\infty}^{\infty} e^{-\tilde{z}^2} |\tilde{z}|\,\mathrm{d}\tilde{z} \\
&= \frac{2}{T} \int_{0}^{\infty} e^{-\tilde{z}^2} \tilde{z}\,\mathrm{d}\tilde{z} \\
&= -\frac{1}{T} \left[e^{-\tilde{z}^2} \right]_0^{\infty} \\
&= \frac{1}{T},
\end{aligned}$$

where $\tilde{z} = z\sqrt{T}$. Then we have

$$\int_0^{\infty} e^{-Tx}\,\mathrm{d}N(x) \sim \frac{\mu(\mathcal{F})}{4\pi} \frac{1}{T} \qquad \text{as} \quad T \searrow 0. \tag{6.4}$$

Applying Karamata's Tauberian theorem, we obtain:

$$N(x) \sim \frac{\mu(\mathcal{F})}{4\pi} x \qquad \text{as} \quad x \to \infty,$$

which is consistent with (4.6).

Now to discuss the asymptotic behavior of $\pi_0(x)$, we consider the limit $T \to \infty$ in (6.3). From above we have $N(x) \sim Cx$ for some constant C, so we must have $\lambda_n > cn$ for some $c > 0$ and all n. For T sufficiently large, we can assume that $e^{-cT} \leq 1/2$, so for the sum over eigenvalue in (6.3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} e^{-\lambda_n T} &\leq \sum_{n=0}^{\infty} e^{-cnT} \\
&= 1 + \frac{e^{-cT}}{1-e^{-cT}} \\
&\leq 1 + 2e^{-cT}.
\end{aligned} \tag{6.5}$$

So the sum contributes $1+\mathcal{O}(e^{-cT})$. To discuss the contribution of tanh integral, assume $T > 0$ and note that $|\tanh(x)| < 1$ for real x, then we have:

$$\begin{aligned}\int_{-\infty}^{\infty} ze^{-(z^2+1/4)T}\tanh(\pi z)\,\mathrm{d}z &= e^{-T/4}\int_{-\infty}^{\infty} ze^{-z^2T}\tanh(\pi z)\,\mathrm{d}z \\ &\le e^{-T/4}\int_{-\infty}^{\infty} |z|e^{-z^2T}\,\mathrm{d}z \\ &= e^{-T/4}2\int_{0}^{\infty} ze^{-z^2T}\,\mathrm{d}z \\ &\le Ce^{-T/4}. \end{aligned} \tag{6.6}$$

for some constant $C > 0$ since the integral clearly converges. So the tanh integral contributes $\mathcal{O}(e^{-T/4})$.

To discuss the contribution of the sum over the conjugacy classes, we make the following definition:

Definition 6.6. The *primitive length spectrum* on S is a set with multiplicities:

$$\mathcal{L}_S = \{l(\gamma) : \gamma \text{ is a primitive oriented closed geodesic on } S\}.$$

where $l(\gamma)$ denotes the length for γ and repeated lengths are counted with multiplicity.

Remark 6.7. Note that lengths always come is pairs, corresponding to two different orientations of the geodesic.

By Proposition 6.1, we can convert the sum over conjugacy classes of Γ in (6.3) into sum over length of closed oriented geodesics. Since each hyperbolic element can be expressed as a positive power of primitive elements, $P = P_0^k$, with the displacement length satisfies $l(P) = kl(P_0)$, we can actually express the sum as over $\mathcal{L}_S$ with multiplicities. Note that by definition of displacement length, two hyperbolic elements are conjugate if and only if they have the same displacement length. Thus it is clear that the conjugacy classes $\{P_0^k\}$ of P_0 primitive are mutually disjoint for different k. Thus we have:

$$\sum_{\{P\}} \frac{l(P_0)}{e^{l(P)/2} - e^{-l(P)/2}} \frac{e^{-T/4}}{\sqrt{4\pi T}} e^{-l(P)^2/4T} = \frac{e^{-T/4}}{\sqrt{4\pi T}} \sum_{l\in\mathcal{L}_S} \sum_{k=1}^{\infty} \frac{l}{2\sinh(kl/2)} e^{-l^2/4T}.$$

Our next goal is to show that:

$$\frac{e^{-T/4}}{\sqrt{4\pi T}} \sum_{l\in\mathcal{L}_S} \sum_{k=1}^{\infty} \frac{l}{2\sinh(kl/2)} e^{-l^2/4T} = \frac{e^{-T/4}}{\sqrt{4\pi T}} \sum_{l\in\mathcal{L}_S} \frac{l}{e^{l/2}} e^{-l^2/4T} + \mathcal{O}(e^{-\eta T}). \tag{6.7}$$

for some $\eta > 0$ as $T \to \infty$.

To estimate the difference of two sums, we want to be able to estimate the sum over $\mathcal{L}_S$. We number the length in $\mathcal{L}_S$ from lower to higher, this will give us a non-decreasing sequence $(l_j)_{j=1}^{\infty}$. From (6.2) we know that $\pi_0(x) \sim Cx$ for some constant C, so similar to above, we have $e^{l_j} \geq cj$ for some $c > 0$ and all j. In particular, we have $e^{l_j/2} \geq 2e^{-l_j/2}$ for large enough j.

With above information, we can estimate the difference in sums in (6.7). We will do so by considering different values of k on the left hand side. We can exchange the sum since the sum is absolutely convergent.

First we want to show that all the $k \geq 2$ terms only contribute $\mathcal{O}(c'T)$ to the error term for some $c' > 0$. We have:

$$2\sinh(x) = e^x - e^{-x} \geq \frac{1}{2}e^x \qquad \text{for} \quad x \geq \ln(\sqrt{2}).$$

Then since $l_j \geq \ln(cj)$ for all j, for large enough j we can overestimate $(2\sinh(kl_j/2))^{-1}$ by $1/e^{kl_j/2}$ in the sum. Then we have:

$$\begin{aligned}\sum_{j\geq 1}\sum_{k\geq 2}\frac{1}{2\sinh(kl_j/2)}l_j e^{-kl_j^2/4T} &\sim \sum_{j\geq 1}\sum_{k\geq 2}\frac{1}{e^{kl_j/2}}l_j e^{-kl_j^2/4T} \\ &\leq \sum_{j\geq 1}\sum_{k\geq 2}\frac{1}{e^{kl_j/2}}l_j e^{-l_j^2/4T} \\ &= \sum_{j\geq 1} l_j e^{-l_j^2/4T}\sum_{k\geq 2} e^{-kl_j/2}. \end{aligned} \tag{6.8}$$

But for the k-sum in the last part we have:

$$\sum_{k\geq 2} e^{-kl_j/2} = \frac{1}{1-e^{-kl_j/2}} - \frac{1-e^{-kl_j}}{1-e^{-kl_j/2}} = \frac{e^{-kl_j}}{1-e^{-kl_j/2}} \leq Ce^{-l_j}.$$

for some constant C and all j, by the summation formula of geometric series. So the last step of (6.8) becomes:

$$\sum_{j\geq 1} l_j e^{-l_j^2/4T}\sum_{k\geq 2} e^{-kl_j/2} \leq C\sum_{j\geq 1}\frac{l_j}{e^{l_j}}e^{-l_j^2/4T}.$$

From above, we have $e_j^l \geq cj$, for some constant $c > 0$, implies $l_j \geq \ln(cj)$. Since the function x/e^x is decreasing when $x \geq 1$, and $e^{-l_j^2/4T}$ is strictly decreasing, we can approximate the sum in the above inequality as:

$$\begin{aligned}
\sum_{j\geq 1}\frac{l_j}{e^{l_j}}e^{-l_j^2/4T} &\sim \sum_{j\geq 1}\frac{\ln(cj)}{cj}e^{-\ln(cj)^2/4T} \\
&= \frac{1}{c}\sum_{m\geq 1}^{\infty}\sum_{j\in[e^m,e^{m+1}]}\frac{\ln(cj)}{j}e^{-\ln(cj)^2/4T} \\
&\leq C\sum_{m\geq 1}^{\infty}(e^{m+1}-e^m)\frac{m+\ln(c)}{e^m}e^{-(m+\ln(c))^2/4T} \\
&= C'\sum_{m\geq 1}^{\infty}(m+\ln(c))e^{-(m+\ln(c))^2/4T},
\end{aligned}$$

where C, C' are some constants. Then we can use integral to estimate the sum. Since c is fixed, the convergence is decided by the tail of the integral. We have:

$$\begin{aligned}
\sum_{m\geq 1}^{\infty}(m+\ln(c))e^{-(m+\ln(c))^2/4T} &\sim \int_0^{\infty} xe^{-x^2/4T}\,\mathrm{d}x \\
&= -2T\left[e^{-x^2/4T}\right]_0^{\infty} \\
&= -2T.
\end{aligned}$$

Then with the term $\frac{e^{-T/4}}{\sqrt{4\pi T}}$ in front, we can conclude the $k \geq 2$ sum contribute $\mathcal{O}(e^{-c'T})$ for some constant $c' > 0$.

Then we want to show the $k = 1$ term on the left hand side of (6.7) contributes to the sum on the right hand side of that equation, with some error of size $\mathcal{O}(e^{-c''T})$ for some $c'' > 0$. For $k = 1$, note the the function $xe^{-x^2/T}$ is uniformly bounded for any $T \geq 1$. Then we have

$$\begin{aligned}
\sum_{j=1}^{\infty}\left(\frac{l_j}{2\sinh(l_j/2)}-\frac{l_j}{e^{l_j/2}}\right)e^{-l_j^2/4T} &= \sum_{j=1}^{\infty}l_je^{-l_j^2/4T}\left(\frac{1}{e^{l_j/2}-e^{-l_j/2}}-\frac{1}{e^{l_j/2}}\right) \\
&\leq C\sum_{j=1}^{\infty}\frac{e^{-l_j}}{e^{l_j/2}-e^{-l_j/2}} \leq C\sum_{j=1}^{\infty}\frac{e^{-l_j}}{\frac{1}{2}e^{l_j/2}} \\
&= 2C\sum_{j=1}^{\infty}e^{-3l_j/2} \leq 2C\sum_{j=1}^{\infty}(cj)^{-3/2}.
\end{aligned}$$

Thus the sum converges to a finite constant. With the term $\frac{e^{-T/4}}{\sqrt{4\pi T}}$ in front, we see the sum decays as $\mathcal{O}(e^{-c''T})$ for some $c'' > 0$ as $T \to \infty$. With the discussion of $k = 1$ case and $k \geq 2$ case above, we proved equation (6.7).

Now we have examined the behavior of all terms in (6.3). Combining (6.5), (6.6) and (6.7) and writing the sum in right hand side of (6.7) as a integral, then for $T > 0$, we obtain:

$$1+\mathcal{O}(e^{-\epsilon T}) = \frac{e^{-T/4}}{\sqrt{4\pi T}}\int_1^\infty \frac{\ln(x)}{\sqrt{x}}e^{-\ln(x)^2/4T}\,\mathrm{d}\pi_0(x). \tag{6.9}$$

For some $\epsilon > 0$.

To proceed, we multiply $e^{-(\beta+\beta^2)T}$ on both side of (6.9) and integrating over T over the interval $(0,\infty)$. The reason is that the right hand side of equation (6.9) has a factor of $e^{-T/4}$, so multiplying by $e^{-(\beta+\beta^2)T}$ will complete the square and give us $e^{-(\beta+1/2)^2}$. This is convenient since we have the following integration formula (Proved in Appendix C, Lemma C.2), which we will use when integrating the right hand side of (6.9). For $a, d > 0$, we have:

$$\int_0^\infty \frac{e^{-d^2/4T}}{\sqrt{4\pi T}}e^{-a^2T}\,\mathrm{d}T = \frac{e^{-ad}}{2a}.$$

First let us consider the easier part. Integrating the left hand side of (6.9), we obtain:

$$\int_0^\infty e^{-(\beta+\beta^2)T} + \mathcal{O}(e^{(\epsilon+\beta+\beta^2)T})\,\mathrm{d}T = \frac{1}{\beta+\beta^2} + \mathcal{O}(\frac{1}{\epsilon+\beta+\beta^2}). \tag{6.10}$$

For the right hand side of (6.9), from (6.10), we see that the integration is absolutely convergent. So we can use Fubini's theorem to exchange the integral. We have:

$$\int_1^\infty \frac{\ln(x)}{\sqrt{x}}\int_0^T \frac{e^{-T/4}}{\sqrt{4\pi T}}e^{-\ln(x)^2/4T}e^{-(\beta+\beta^2)T}\,\mathrm{d}T\,\mathrm{d}\pi_0(x). \tag{6.11}$$

The T integral gives:

$$\begin{aligned}\int_0^T \frac{e^{-T/4}}{\sqrt{4\pi T}}e^{-\ln(x)^2/4T}e^{-(\beta+\beta^2)T}\,\mathrm{d}T &= \int_0^T \frac{e^{-\ln(x)^2/4T}}{\sqrt{4\pi T}}e^{-(\beta+1/2)^2T}\,\mathrm{d}T \\ &= \frac{x^{-(\beta+1/2)}}{2\beta+1}.\end{aligned}$$

Substitute back into (6.11) and make a change of variable, we have:

$$\int_1^\infty \frac{\ln(x)}{\sqrt{x}}\frac{x^{-(\beta+1/2)}}{2\beta+1}\,\mathrm{d}\pi_0(x) = \int_0^\infty u\frac{e^{-(1+\beta)u}}{1+2\beta}\,\mathrm{d}\pi_0(e^u), \tag{6.12}$$

where $u = \ln(x)$.

Then combine (6.10) and (6.12), as $\beta \searrow 0$, we obtain:

$$\frac{1}{\beta} + \mathcal{O}(1) = \int_0^\infty ue^{-(1+\beta)u}\,\mathrm{d}\pi_0(e^u),$$

which is equivalent to

$$\sum_{\{P_0\}} \frac{l(P_0)}{(e^{l(P_0)})^{1+\beta}} = \frac{1}{\beta} + \mathcal{O}(1).$$

Use the sequence (l_j) from $\mathcal{L}_S$, we can rewrite the above sum as:

$$\sum_{j=1}^{\infty} \frac{l_j}{e^{l_j(1+\beta)}} = \frac{1}{\beta} + \mathcal{O}(1). \tag{6.13}$$

Let $\xi_j = e^{l_j}$, and make the following definition;

Definition 6.8.

$$\Phi_0(s) := \sum_{j=1}^{\infty} \frac{\ln(\xi_j)}{\xi_j^s} \qquad \text{for} \quad \mathrm{Re}(s) > 1.$$

Note that this definition is very analogous to Definition 3.17 in chapter 3, with ξ_j playing the role of prime numbers.

Then equation (6.13) can be interpreted as $\Phi_0(s)$ having a simple pole as $s \searrow 1$. Now we are in a situation that is very similar to chapter 3, section 3.2. Similar to Lemma (3.18), we first show that $\Phi_0(s)$ is analytic for $\mathrm{Re}(s) \geq 1$. For any $\delta > 0$ and $\mathrm{Re}(s) > 1 + \delta$, recall from (6.2) we have $\xi_j \geq cj$, and $\ln(x)/x$ is a decreasing function for $x > e$, so we have:

$$\left|\Phi_0(s)\right| = \left|\sum_{j=1}^{\infty} \frac{\ln(\xi_j)}{\xi_j^s}\right| \sim \left|\sum_{j=1}^{\infty} \frac{\ln(cj)}{(cj)^s}\right| \leq \sum_{j=1}^{\infty} \left|\frac{\ln(cj)}{(cj)^{(\delta/2)}} \frac{1}{(cj)^{(1+\delta/2)}}\right|.$$

Since $\frac{\ln(cj)}{(cj)^{(\delta/2)}}$ is bounded, we see that $\Phi_0(s)$ converges absolutely and uniformly on any compact set of $\mathrm{Re}(s) > 1 + \delta$. Thus $\Phi_0(s)$ is analytic on $\mathrm{Re}(s) > 1$.

Now in the spirit of Lemma (3.18), if we can show that $\Phi_0(x)$ can be extend analytically on $\{s : \mathrm{Re}(s) = 1, \mathrm{Im}(s) \neq 0\}$, then we can apply the Wiener-Ikehara Tauberian Theorem 3.16 on Φ_0 with $A = 1$, as what we did in chapter 3 for Φ. This will give us:

$$\theta_0(x) := \sum_{\xi_j \leq x} \ln(\xi_j) \sim x.$$

Then with (6.2) and Definition 3.14, we can apply Proposition 3.15 on θ_0, which finishes the proof of the Prime Orbit Theorem 6.4.

Now our last goal is to prove the following proposition:

Proposition 6.9. *The function $\Phi_0(s)$, initially defined as an analytic function for $\mathrm{Re}(s) > 1$, extends to an meromorphic function on $\mathrm{Re}(s) \geq 1$, with only a simple pole at $s = 1$.*

With the discussion above, for the proposition it suffices to prove that Φ_0 is analytic on the set $\{s : \mathrm{Re}(s) = 1, \mathrm{Im}(s) \neq 0\}$. Let $a = s - 1/2$, $\mathrm{Re}(a) > 1/2$. For the Laplace-Beltrami operator $\mathbf{D}$, we want to discuss the resolvent. In the Selberg trace formula 5.4, by Definition 5.3, we see r_n are the eigenvalues of $\sqrt{\mathbf{D}-1/4}$. So we have:

$$\sum_{n=0}^{\infty}\frac{1}{r_n^2+a^2}=\operatorname{Tr}h(\sqrt{\mathbf{D}-1/4})=\operatorname{Tr}(\mathbf{D}-1/4+a^2)^{-1},$$

where $h(r)=\frac{1}{r^2+a^2}$. But the operator $\operatorname{Tr}(\mathbf{D}-1/4+a^2)^{-1}$ is not trace class. Recall from (4.6) that the eigenvalue counting function $N(x)\sim Cx$. Thus $\lambda_n\sim cn$ for some $c>0$, $r_n^2\sim c'n$ for some $c'>0$, so the sum in the above equation does not converge. It can be observed that the $h(r)$ we used in the above equation also fails the decaying condition in Theorem 5.4.

To fix the problem, we fix $b\geq 1$ and take

$$h(r)=\frac{1}{r^2+a^2}-\frac{1}{r^2+b^2}. \tag{6.14}$$

One can easily calculate the Fourier transform: (See Appendix C, Lemma C.3)

$$g(u)=\frac{1}{2a}e^{-a|u|}-\frac{1}{2b}e^{-b|u|}.$$

We see then $h(r)$ decays as $\mathcal{O}(r^{-4})$, which satisfies the decaying condition in Theorem 5.4.

Proof of Proposition 6.9. Applying the Selberg trace formula to $h(r)$ in (6.14), we have:

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\frac{1}{r_n^2+(s-\frac{1}{2})^2}-\frac{1}{r_n^2+b^2}\right]&=\frac{\mu(\mathcal{F})}{4\pi}\int_{-\infty}^{\infty}z\left[\frac{1}{z^2+(s-\frac{1}{2})^2}-\frac{1}{z^2+b^2}\right]\tanh(\pi z)\,\mathrm{d}z\\&+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\ln(\xi_j)}{\xi_j^{k/2}-\xi_j^{-k/2}}\left[\frac{1}{2s-1}\frac{1}{\xi_j^{(s-1/2)k}}-\frac{1}{2b}\frac{1}{\xi_j^{kb}}\right],\end{aligned} \tag{6.15}$$

where we substitute back $a=s-1/2$. Recall from (4.3) that (λ_n) forms a increasing real sequence. From (4.6) we know that $\lambda_n\geq cn$ for some $c>0$. Thus only finitely many λ_n can be less than $1/4$. Let M be the largest index such that $\lambda_M<1/4$. Then there are finitely many r_n, namely $r_1,\dots,r_M$, are purely imaginary, and the rest are purely real (See Definition 5.3). Since $\lambda_n\geq 0$ for all n, we see $r_n^2\geq -1/4$ for all n. Now we want to discuss the analyticity of each term in (6.15) on the half-plane $\operatorname{Re}(s)>1/2$.

First consider the sum on the left hand side of (6.15). Since $b\geq 1$, $r_n^2+b^2$ is never 0. Now suppose $r_n^2+(s-1/2)^2=0$. We have $s=1/2\pm ir_n$. By the choice of r_n in Definition 5.3, we see that the values of s that satisfy the condition on $\mathrm{Re}(s) > 1/2$ are the points $s_1, \ldots, s_M$. Each of these points are real and satisfy $s_i < 1$.

Now consider the tanh integral on the right hand side of (6.15). If $\mathrm{Im}(s) \neq 0$, the term $z^2 + (s-1/2)^2 \neq 0$ since it will have some imaginary part. If $\mathrm{Im}(s) = 0$, then $(s - 1/2) > 0$ on $\mathrm{Re}(s) > 1/2$, so $z^2 + (s - 1/2)^2 \neq 0$ also holds. From Theorem 5.4, we know the integral converges. Then the tanh integral is an analytic function on $\mathrm{Re}(s) > 1/2$.

Finally consider the sum term on the right hand side of (6.15). We see that all the $k \geq 2$ term are analytic on $\mathrm{Re}(s) > 1/2$. Now we consider the term $k = 1$. The fraction involving b is clearly analytic and irrelevant to s, so we consider solely the first term with s in the square parentheses. We have:

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\ln(\xi_j)}{\xi_j^{1/2} - \xi_j^{-1/2}} \frac{1}{2s-1} \frac{1}{\xi_j^{s-1/2}} &= \frac{1}{2s-1} \sum_{j=1}^{\infty} \frac{\ln(\xi_j)\xi_j^{1/2-s}\xi_j^{-1/2}}{1-\xi_j^{-1}} \\
&= \frac{1}{2s-1} \sum_{j=1}^{\infty} \big(\ln(\xi_j)\xi_j^{-s}\big)(1 + \xi_j^{-1} f(\xi_j^{-1})) \\
&= \frac{1}{2s-1} \Phi_0(s) + \tilde{f}(s),
\end{aligned}$$

where we took the Taylor expansion of $\frac{1}{1-\xi_j^{-1}}$ in terms of ξ_j^{-1}. The result function f is an analytic function in ξ_j^{-1} for $\xi_j > 0$. It follows that $\tilde{f}s$ is an analytic function on $\mathrm{Re}(s) > 1/2$.

Now we have discussed the analyticity of all terms in (6.15). Combining all of the above, we conclude that $\Phi_0(s)$ is a meromorphic function in s on $\mathrm{Re}(s) > 1/2$, with simple poles at the points $s_0, \ldots, s_M$. The Proposition then follows. □

To summarize, in this chapter, to prove the Prime Orbit theorem, we used the Selberg trace formula twice. The first use is to apply on the heat kernel. We obtained 6.3. Take both limits as $T \to 0$ and $T \to \infty$, we derived the behavior of function Φ_0 on the half-plane $\mathrm{Re}(s) > 1$, and the fact that $\Phi_0(s)$ has a simple pole, at $s = 1$, as the function Φ in chapter 3. The second use of the Selberg trace formula is to apply on the resolvent of Laplacian, which allow us to prove the analyticity of Φ_0 on $\mathrm{Re}(s) = 1$ excluding $s = 1$. This allows us to apply Wiener-Ikehara Tauberian theorem as in chapter 3 to prove the Prime Orbit theorem.

Appendix A

Trace Class Operator

In this section we define the trace class operators on a Hilbert space.

Definition A.1. Let $\mathcal{H}$ be a Hilbert space. An operator $B \in \mathcal{L}(\mathcal{H})$ is called *positive* if $\langle Bx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. We denote this by $B \geq 0$.

We quote the following theorem from [13] chapter VI, which comes naturally from vector space structure of $\mathcal{H}$.

Theorem A.2 (square root lemma)**.** *Let $A \in \mathcal{L}(\mathcal{H})$ and $A \geq 0$. Then there exists a unique $B \in \mathcal{L}(\mathcal{H})$ with $B \geq 0$ such that $B^2 = A$. We denote this fact by $B = \sqrt{A}$.*

Note that A^*A, where A^* denotes the adjoint of A, is always a positive operator. Then we can make the following definition:

Definition A.3. Let $A \in \mathcal{L}(\mathcal{H})$. Then we define $|A| := \sqrt{A^*A}$

Again we quote a theorem from [13] chapter VI, the proof of the theorem is just a simple calculation.

Theorem A.4. *Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{\phi_n\}_{n=0}^{\infty}$. Let A be a positive operator. Then the number (possibly be ∞) $\sum_{n=0}^{\infty} \langle \phi_n, A\,\phi_n \rangle$ is independent of the orthonormal basis chosen. We hence define this number to be the* ***trace*** *of A, denoted by $Tr(A)$.*

It is then reasonable to make the following definition:

Definition A.5. An operator $A \in \mathcal{L}(\mathcal{H})$ is called *trace class* if and only if $\mathrm{Tr}\,(A) < \infty$.

Appendix B

Bessel function

For the purpose of this thesis, we only discuss the Bessel functions with integer order. Consider the Bessel's equation with integer order:

$$x^2u'' + xu' + (x^2 - n^2)u = 0 \qquad \text{with} \qquad n \in \mathbb{Z}$$

A solution will be of the form

$$J_n = (\frac{x}{2})^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}(\frac{x}{2})^{2k}. \tag{B.1}$$

The derivative of such functions are not hard to derive. With some direct calculation, we have the following relations:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n J_n = x^n J_{n-1} \tag{B.2}$$

$$J_{-n}(x) = (-1)^n J_n(x) \tag{B.3}$$

$$xJ_{n+1} = 2nJ_n - xJ_{n-1} \tag{B.4}$$

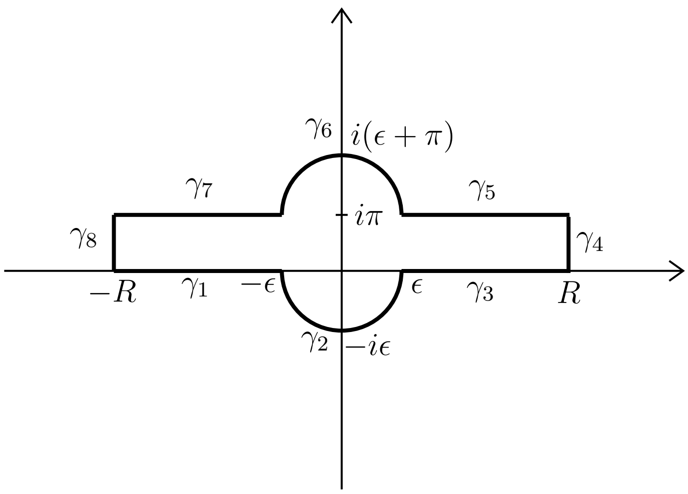
$$\frac{\mathrm{d}}{\mathrm{d}x}x^n J_v = x^n J_{n-1} \tag{B.5}$$

Appendix C

Integration formulas

Lemma C.1.
$$\int_0^\infty \frac{\sin(az)}{\sinh(z)}\,\mathrm{d}z = \frac{\pi}{2}\tanh(\frac{\pi a}{2}).$$

Proof. The convergence of the integral clearly follows if we write down the integrand in the form $\frac{2\sin(x)}{e^x - e^{-x}}$. We prove this integration formula by contour integration. Since $\frac{\sin(az)}{\sinh(z)}$ is even, we can integrate over the whole $\mathbb{R}$ axis. Consider the contour shown in the figure: We integrate $\frac{\sin(az)}{\sinh(z)}$ along the contour, and let



$R \to \infty$ and $\epsilon \to 0$. We will use the following equations in trigonometry:

$$\sin(a(z+i\pi)) = \sin(az)\cos(ai\pi) + \cos(az)\sin(ai\pi), \tag{C.1}$$
$$\sinh(z+i\pi) = -\sinh(z). \tag{C.2}$$

Since the integrand is even, we know the integration along γ_1 and γ_3 are equal.

For γ_5 and γ_7, we have:

$$\begin{aligned}
\int_{\gamma_5} + \int_{\gamma_7} &= -\int_{-\infty}^{\infty} \frac{\sin(a(z+i\pi))}{\sinh(z+i\pi)}\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \frac{\sin(az)\cos(ai\pi) + \cos(az)\sin(ai\pi)}{\sinh(z)}\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \frac{\sin(az)\cos(ai\pi)}{\sinh(z)}\,\mathrm{d}z \\
&= \cos(ai\pi)\left[\int_{\gamma_1} + \int_{\gamma_3}\right].
\end{aligned}$$

where we used the fact that $\frac{\cos(az)}{\sinh(z)}$ is odd. The convergence holds due to similar reasons. Note that as $R \to \infty$, $\int_{\gamma_4}$ and $\int_{\gamma_8}$ goes to 0. The value of $\int_{\gamma_2}$ and $\int_{\gamma_6}$ is just $i\pi$ times the residue of integrand at point 0 and $i\pi$ respectively ([12] P196). Then we have $\int_{\gamma_2} = 0$ since 0 is a removable singularity and $\int_{\gamma_6} = -i\pi\sin(ai\pi)$ by a simple Laurent expansion.

Now we have examined the integration along the entire contour. Since $z = i\pi$ is the only non-removable singularity within the contour and the residue there is $-\sin(ai\pi)$, by the Residue theorem, we have:

$$-2i\pi\sin(ai\pi) = (2 + 2\cos(ai\pi))\int_{\gamma_1} -i\pi\sin(ai\pi).$$

Thus we finally obtain:

$$\int_0^{\infty} \frac{\sin(az)}{\sinh(z)}\,\mathrm{d}z = \frac{-i\pi\sin(ai\pi)}{2+2\cos(ai\pi)} = \frac{\pi}{2}\tanh(\frac{\pi a}{2}).$$

□

Lemma C.2. *For $a, d > 0$,*

$$\int_0^{\infty} \frac{e^{-d^2/4T}}{\sqrt{4\pi T}} e^{-a^2 T}\,\mathrm{d}T = \frac{e^{-ad}}{2a}.$$

Proof. Take the inverse Fourier transform in d. The convergence of the integral is clear, so we can apply Fubini's theorem to exchange the integral. Using the integral formula for Gaussian functions, we have

$$\begin{aligned}
\text{LHS} &= \int_0^{\infty}\int_{-\infty}^{\infty} \frac{e^{-y^2/4T}}{\sqrt{4\pi T}} e^{-a^2 T} e^{iy\xi}\,\mathrm{d}y\,\mathrm{d}T \\
&= \int_0^{\infty} e^{-T\xi^2} e^{-Ta^2}\,\mathrm{d}T \\
&= \frac{1}{\xi^2 + a^2}.
\end{aligned}$$

Since we have the restriction $d > 0$, we extend the function on the right hand side to $\frac{e^{-a|d|}}{2a}$ on the whole real line. Then we have:

$$\begin{aligned}
\text{RHS} &= \int_{-\infty}^{\infty} e^{iy\xi} \frac{e^{-a|y|}}{2a} \,\mathrm{d}y \\
&= \int_{0}^{\infty} \frac{e^{-(a-i\xi)y}}{2a} \,\mathrm{d}y - \int_{0}^{\infty} \frac{e^{(a+i\xi)y}}{2a} \,\mathrm{d}y \\
&= \frac{1}{2a}\left(\frac{1}{a-i\xi} + \frac{1}{a+i\xi}\right) \\
&= \frac{1}{\xi^2 + a^2}.
\end{aligned}$$

Hence the lemma is proved. □

Lemma C.3. *For* $\mathrm{Re}(a) > 0$*,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{r^2 + a^2} e^{-iru} \,\mathrm{d}r = \frac{1}{2a} e^{-a|u|}.$$

Proof. Assume that $u > 0$, consider the semicircular contour of radius R in the upper-half plane, centered at origin. We integrate along this contour counter-clockwise. Integrating the circular arc part of contour contribute 0 as $R \to \infty$. The integrand has only one pole at $z = ai$, and the residue there is $\frac{e^{-ak}}{2ai}$ by a simple Laurent expansion. Then by the Residue theorem, we have:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{r^2 + a^2} e^{-iru} \,\mathrm{d}r = \frac{1}{2\pi} 2\pi i \frac{e^{-au}}{2ai} = \frac{1}{2a} e^{-au}.$$

Similarly, for $u < 0$, we take the contour in the lower-half plane, with a simple pole at $z = -ai$. The integral will give the result $\frac{1}{2a}e^{au}$. Thus for arbitrary u the result will be $\frac{1}{2a}e^{-a|u|}$. The lemma then follows. □

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