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Invariant Distances and Metrics in Complex Analysis

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Chapter 1

Introduction and Notation

1.1 Introduction

The study of invariant metrics has played an important role in complex analysis throughout the twentieth century. In order to study domains in n-dimensional complex space $\mathbb{C}^n$, it is important to associate with these domains distance functions which are invariant under biholomorphic mappings. Such functions can then be used as a tool to determine whether domains in $\mathbb{C}^n$ are biholomorphically equivalent. It is these functions we will examine and describe in this thesis. The study of invariant metrics, while having some importance in single variable complex analysis, is of great significance in several variable complex analysis, and is a major area of research in the field. The area has many meaningful connections with the related areas of functional analysis, differential geometry, and in particular, hyperbolic geometry.

The field began to develop in the late 1800s, when Poincaré realised that the open unit disc in the complex plane could be given a metric that was distance-preserving under biholomorphic mappings from the unit disc to itself. The properties of this distance, now called the Poincaré distance, are described with the aid of a well-known lemma in one variable complex analysis called the Schwarz-Pick lemma. We begin with the study of this lemma in the following chapter.

The first distance function of this kind for arbitrary domains in $\mathbb{C}^n$ was introduced by Carathéodory in the 1930s. The Carathéodory pseudodistance, as it is now known, is distance-preserving under biholomorphic mappings from an arbitrary domain into the unit disc. One of the most significant advances in this area in the postwar period has been the introduction of a pseudodistance introduced by Shoshichi Kobayashi in the late 1960s, which utilises chains of points to describe a pseudodistance that is similarly distance-preserving under biholomorphic mappings from an arbitrary domain into the unit disc.

The main goal of this thesis is to provide a comprehensive summary of the important invariant distances and pseudodistances, and their infinitesimal counterparts. The Carathéodory and Kobayashi pseudodistances are the most widely studied, and these in particular will be the focus of our attention. Further, we summarise the methods of defining and classifying pseudodistances and pseudometrics. We assume knowledge of one variable complex analysis at the level of a standard undergraduate course. Ideas and techniques from undergraduate-level analysis are used throughout. Knowledge of some basic notions from differential geometry and the associated notation is useful. In the final chapter, we assume some knowledge of certain ideas from algebraic topology, although we review these concepts before use.

In this thesis we will focus almost exclusively on domains in n-dimensional complex space, $\mathbb{C}^n$. This is due to the fact that the pseudodistances we study are largely trivial in $\mathbb{C}$. Since the reader may be somewhat unfamiliar with certain concepts and notation in several complex variables, we now provide a summary of the basic notions of holomorphic functions in $\mathbb{C}^n$, and some other basic theorems and facts from the theory.

1.1.1 Facts from Several Complex Variables

In the following discussion, we represent a point in n-dimensional complex space as $z = (z_1, ..., z_n) \in \mathbb{C}^n$. For $z = (z_1, ..., z_m), w = (w_1, ..., w_n) \in \mathbb{C}^n$, we have the usual *Euclidean metric*, where the distance between the points z and w is given by

$$|z - w| = \sqrt{|z_1 - w_1|^2 + ... + |z_n - w_n|^2}.$$

We also consider the metric in which the distance between the points z and w is given by

$$\|z - w\| = \max_{\nu} |z_\nu - w_\nu|,$$

for $\nu = 1, ..., n$. This metric is sometimes called the *polydisc metric*. The double inequality relating the two metrics

$$\|z - w\| \leq |z - w| \leq \sqrt{n}\|z - w\|$$

shows that both metrics induce the same topology on $\mathbb{C}^n$.

We begin examination of domains in $\mathbb{C}^n$ by presenting the polydisc, one of the most common domains in several complex variables, which is analogous to the disc in one variable. For $r = (r_1, ..., r_n) \in \mathbb{R}^n$ such that $r > 0$, and $a = (a_1, ..., a_n) \in \mathbb{C}^n$, we define the *polydisc* $P(a, r)$ as

$$P(a, r) = \{z \in \mathbb{C}^n : |z_\nu - a_\nu| < r_\nu, \nu = 1, ..., n\}.$$

We call a the *centre* and r the *polyradius*, or simply the *radius* of the polydisc. The *distinguished boundary* of the polydisc is the n-dimensional set given by

$$\Gamma = \{z \in \mathbb{C}^n : |z_\nu - a_\nu| = r_\nu, \nu = 1, ..., n\},$$

and is analogous to the boundary of the disc in one variable.

We now give a fundamental definition, that of a *holomorphic* function in several variables for an arbitrary domain in $\mathbb{C}^n$. To effectively denote power series representations in several variables, we make use of the following *multi-index notation* in the definition. Let $(k_1, ..., k_n)$ be an n-tuple of non-negative integers. Then we call k a multi-index, so that

$$z^k = \prod_{j=1}^{n} z_j^{k_j}.$$

The above monomial has degree $\sum_{j=1}^n k_j$, and so we use the notation $|k| := \sum_{j=1}^n k_j$ and call this the *order* of k.

Theorem 1.1.1. *Let D be a domain in $\mathbb{C}^n$. Then the function f is holomorphic on D if and only if, for each $z \in D$, there is a power series given by*

$$f(z) = \sum_{|k|=0}^{\infty} c_k (z-a)^k$$

that converges to f for all $z \in P(a, r)$, for some polydisc $P(a, r)$.

As in single variable complex analysis, every holomorphic function can be represented locally by a power series. The study of holomorphic functions in several variables is simplified considerably by the following fundamental theorem.

Theorem 1.1.2. *(Hartogs). If the function f is holomorphic at any point of the domain $D \subset \mathbb{C}^n$ with respect to each of the variables z_ν for $\nu = 1, ..., n$, then it is holomorphic in D.*

That is, a function in several variables is holomorphic if it is holomorphic in each variable separately. A proof of this theorem can be found in [8, p.28]. Note that there is no equivalent statement in the case of real multivariable functions with respect to differentiability.

If a holomorphic function f has an inverse f^{-1} which is also holomorphic, then we say that f is *biholomorphic*. As with the one variable complex case, a holomorphic function in $\mathbb{C}^n$ is biholomorphic if and only if it is bijective. We omit the proof of this fact, which can be found in [2, p.13].

Many well-known facts and theorems from one variable complex analysis carry over easily into several variable analysis. We mention two of these here, both of which are ubiquitous in complex analysis. The first of these is the maximum modulus principle, and the second is Liouville's theorem.

Theorem 1.1.3. *(maximum modulus principle). For a function f that is holomorphic on a domain $D \in \mathbb{C}^n$, if $|f|$ attains a maximum at some point $z_0 \in D$, then f is a constant function in D.*

In common with the one variable case, the theorem states that if the maximum is attained in the interior of the domain, then the function must be constant. We now present Liouville's theorem for several variables. We note that a bounded function in $\mathbb{C}^n$ is a function such that $|f(z)| \leq M$ for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$, where $M \geq 0$.

Theorem 1.1.4. *(Liouville). If a function f is holomorphic in $\mathbb{C}^n$ and bounded, then it is constant.*

The proof of Liouville's theorem, which is not dissimilar to the proof of the theorem in one variable, can be found in [8, p.23]. We end this introductory chapter with a summary of the notation we will use throughout the body of this thesis.

1.2 Notation

We utilise notation that is standard in complex analysis. We denote the open unit disc by $\mathbb{D}$, and the closed unit disc by $\overline{\mathbb{D}}$. A *domain* $D \subset \mathbb{C}^n$ is a connected open subset of $\mathbb{C}^n$, and a *bounded domain* is a connected open set U such that there exists an $R > 0$ with $|z| < R$ for all $z \in U$. If D_1, D_2 are domains, we denote the set of all holomorphic mappings from D_1 into D_2 by $H(D_1, D_2)$. For convenience, we use the notation $H(D_1)$ to describe $H(D_1, \mathbb{C})$, that is, when D_2 is all of $\mathbb{C}$. We use the standard notation for topological concepts. In particular, the open unit ball of radius ϵ centred at z will be denoted $B_\epsilon(z)$, and the closed ball denoted $\overline{B_\epsilon(z)}$. As mentioned in the previous section, the polydisc with centre a and radius r is denoted $P(a, r)$. Beginning at Chapter 4, we use some ideas commonly seen in differential and Riemannian geometry, and standard notation will be used in these cases. The tangent space for domain D at the point p is denoted by $T_p(D)$. $T(D)$ will denote the tangent bundle of D, and an element of $T(D)$ will be denoted (p, v). Since we work with domains in finite-dimensional complex space throughout this thesis, $T(D)$ will always be given by $D \times \mathbb{C}^n$ for any domain D.

Chapter 2

One variable preliminaries

The Schwarz lemma is often seen in classical one-variable complex analysis, and is critical to the understanding of the Poincaré distance. The lemma concerns complex-valued functions holomorphic on the unit disc.

Lemma 2.0.1. (Schwarz Lemma). Let f be a complex-valued function which is holomorphic on the unit disc, such that

(i) $|f(z)| \leq 1$ for all z; and

(ii) $f(0) = 0$,

then $|f(z)| \leq |z|$, and as a corollary $|f'(0)| \leq 1$.

Moreover, if either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation: $f(z) \equiv e^{i\theta} z$ for some $\theta \in \mathbb{R}$ and all $z \in \mathbb{D}$.

Proof. We apply the maximum modulus principle to prove the result. Consider the function $g(z) = f(z)/z$, which is holomorphic on $\mathbb{D} \setminus \{0\}$. We define $g(0) = f'(0)$, since, using the power series of f at $z = 0$, we have

$$\begin{aligned}
\lim_{z \to 0} g(z) &= \lim_{z \to 0} \frac{f(z)}{z} \\
&= \lim_{z \to 0} \left(\frac{1}{z} \right) (f(0) + f'(0)z + \ldots) \\
&= f'(0).
\end{aligned}$$

Since g is continuous on $\mathbb{D}$, this implies that g has a removable singularity at $z = 0$, and therefore g is holomorphic on all of $\mathbb{D}$.

Now considering the closed disc $\overline{D}(0, 1-\epsilon)$ for $\epsilon > 0$, we have on the boundary of this disc

$$\begin{aligned}
|g(z)| &= \frac{|f(z)|}{|z|} \\
&\leq \frac{1}{1-\epsilon},
\end{aligned}$$

since $|f(z)| \leq 1$ by hypothesis. So by the maximum modulus principle, we have $|g(z)| \leq 1/(1-\epsilon)$ on the whole open disc $D(0, 1-\epsilon)$. Letting $\epsilon \to 0$ we see that $|g(z)| \leq 1$ on $\mathbb{D}$, and therefore $|f(z)| \leq |z|$ on $\mathbb{D}$.

Further, $|g(z)| \leq 1$ on $\mathbb{D}$ also implies that $|f'(0)| = |g(0)| \leq 1$.

Now, to show the moreover statement, we have $|f(z_0)| = |z_0|$ for $z_0 \neq 0$, which implies $|g(z_0)| = 1$. Therefore, by the maximum modulus principle, g is constant with modulus 1. So $g(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$, which implies $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. If $|f'(0)| = 1$ then $|g(0)| = 1$, which again implies that the function g has modulus 1, so $g(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and we have the same result. □

Since any holomorphic function f into the unit disc has $|f(z)| \leq 1$, we can capture condition (i) of the Schwarz lemma by specifying that f be a holomorphic mapping from $\mathbb{D}$ into $\mathbb{D}$. That is, to specify in the definition simply that $f \in H(\mathbb{D}, \mathbb{D})$ with $f(0) = 0$. We now drop the second assumption, and describe a generalisation of the Schwarz lemma, called the Schwarz-Pick lemma, which requires only the condition that $f \in H(\mathbb{D}, \mathbb{D})$.

2.1 The Schwarz-Pick Lemma

Lemma 2.1.1. (Schwarz-Pick). Let $f \in H(\mathbb{D}, \mathbb{D})$. Then we have

(i)
$$\left|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}\right| \leq \left|\frac{z-w}{1-z\overline{w}}\right| \text{ for all } z, w \in \mathbb{D}, \text{ and}$$

(ii)
$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2} \text{ for all } z \in \mathbb{D}.$$

Before beginning the proof of the Schwarz-Pick lemma, we define the mapping ϕ_a. A *Möbius transformation* is a rational function of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and satisfy $ad - bc \neq 0$. The following type of Möbius transformation will be used extensively. For $a \in \mathbb{D}$, we have

$$\phi_a(z) = \frac{z-a}{1-\overline{a}z}.$$

The Möbius transformation $\phi_a(z)$ is a holomorphic mapping from $\mathbb{D}$ into $\mathbb{D}$. To see this, consider that for $z \in \mathbb{D}$ we have

$$\begin{aligned} |z| < 1 &\implies |a|^2(1-|z|^2) < 1 - |z|^2, \text{ since a} \in \mathbb{D} \\ &\implies |z|^2 + |a|^2 < 1 + |a|^2|z|^2 \\ &\implies |z|^2 - 2 \text{ Re } \overline{a}z + |a|^2 < 1 - 2 \text{ Re } \overline{a}z + |a|^2|z|^2 \\ &\implies |z-a|^2 < |1 - \overline{a}z|^2 \\ &\implies \left| \frac{z-a}{1-\overline{a}z} \right| < 1. \end{aligned}$$

Proof. Now we prove the Schwarz-Pick lemma. To show each of these results, we consider the function $g = \phi_{f(w)} \circ f \circ \phi_{-w}$, given by

$$g(z) = \phi_{f(w)} \left(f \left(\frac{z+w}{1+\overline{w}z} \right) \right).$$

Since f, ϕ are holomorphic mappings from $\mathbb{D}$ into $\mathbb{D}$, then so is g. Further, $g(0) = \phi_{f(w)}(f(w)) = 0$. So the conditions of the Schwarz lemma are satisfied, and so we have $|g(\zeta)| \leq |\zeta|$ for all $\zeta \in \mathbb{D}$, and therefore it follows that $|g(\phi_w(z))| \leq |\phi_w(z)|$ for all $\zeta \in \mathbb{D}$. It is easily checked that the inverse of ϕ_w is ϕ_{-w}. Therefore, we have

$$\begin{aligned} g \circ \phi_w &= \phi_{f(w)} \circ f \circ \phi_{-w} \circ \phi_w \\ &= \phi_{f(w)} \circ f \end{aligned}$$

and so $|\phi_{f(w)}(f(z))| \leq |\phi_w(z)|$, which gives us

$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \leq \left| \frac{z-w}{1-z\overline{w}} \right| \text{ for all } z, w \in \mathbb{D}.$$

This shows (i).

We show (ii) using the second conclusion of the Schwarz lemma. We have $|g'(0)| \leq 1$, and using the chain rule, we see that

$$\begin{aligned} g'(z) &= \phi'_{f(w)}(f(\phi_{-w}(z))) \cdot f'(\phi_{-w}(z)) \cdot \phi'_{-w}(z) \\ \implies g'(0) &= \phi'_{f(w)}(f(w)) \cdot f'(w) \cdot \phi'_{-w}(0) \\ &= \left(\frac{1}{1-|f(w)|^2} \right) \cdot f'(w) \cdot (1-|w|^2). \end{aligned}$$

So we have

$$\left| \frac{f'(w)(1-|w|^2)}{1-|f(w)|^2} \right| \leq 1,$$

and finally

$$|f'(w)| \leq \frac{1-|f(w)|^2}{1-|w|^2} \text{ for all } w \in \mathbb{D}.$$

□

If we have equality in either relation, that is, if

$$\left|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}\right| = \left|\frac{z-w}{1-z\overline{w}}\right| \text{ or } |f'(z)| = \frac{1-|f(z)|^2}{1-|z|^2} \text{ for all } z, w \in \mathbb{D},$$

then f is an automorphism on $\mathbb{D}$. We explain this concept in the following section.

2.2 The Automorphism Group

We now discuss the automorphism group of the unit disc. The set of all biholomorphic automorphisms of a domain D is denoted $\mathrm{Aut}(D)$. So we have $f \in \mathrm{Aut}(D)$ if

(i) $f : D \to D$ is bijective and holomorphic; and

(ii) f^{-1} is holomorphic.

For the domains we consider in this thesis, (i) $\implies$ (ii), so it will suffice to show that f is bijective and holomorphic if we wish to show $f \in \mathrm{Aut}(D)$. We will use the above criteria to show that for the Möbius transformation $\phi_a : \mathbb{D} \to \mathbb{D}$, we have $\phi_a \in \mathrm{Aut}(\mathbb{D})$. Clearly ϕ_a is holomorphic for all $z \in \mathbb{C} \setminus \left\{\frac{1}{\overline{a}}\right\}$, and furthermore ϕ_a is bijective. To see this, note that any Möbius transformation is injective, as is easily checked. To see that ϕ_a is surjective, consider an arbitrary $\zeta \in \mathbb{D}$, and we have

$$\begin{aligned}\frac{z-a}{1-\overline{a}z} = \zeta &\implies z + \zeta\overline{a}z = a + \zeta \\ &\implies z = \frac{\zeta + a}{1 + \overline{a}\zeta} \\ &\implies z = \phi_{-a}(\zeta).\end{aligned}$$

and $\phi_{-a}(\zeta) \in \mathbb{D}$ whenever $a \in \mathbb{D}$. So $\phi_a \in \mathrm{Aut}(\mathbb{D})$.

Note that for an arbitrary domain D, $\mathrm{Aut}(D)$ equipped with the operation of function composition forms a group. This is due to the fact that the inverse of a holomorphic, one-to-one mapping is always holomorphic.

We now present the Poincaré distance, which utilises the Schwarz-Pick lemma to define a distance between points under holomorphic mappings. We firstly review the concepts of distance and pseudodistance.

2.3 Distance and Pseudodistance

Definition 2.3.1. *The function $\rho : D \times D \to [0, \infty)$ is a* pseudodistance *on D if the following three axioms are satisfied:*

(i) $\rho(z, w) \geq 0$;

(ii) $\rho(z, w) = \rho(w, z)$; *and*

(iii) $\rho(z, w) \leq \rho(z, v) + \rho(v, w)$.

Further, ρ is a distance *if it satisfies a fourth axiom, namely*

(iv) $\rho(z, w) = 0 \iff z = w$.

Note that these four axioms are the standard axioms for a metric space. In the remainder of this thesis, we will use the term *distance* in the sense described above. The term *metric* has a specific meaning, which we discuss below.

We now consider the Poincaré distance $\rho_{\mathbb{D}}$ on the unit disc, which is a central object of interest in the study of invariant distances and metrics in complex analysis.

2.4 The Poincaré distance

The Poincaré disc model in hyperbolic geometry refers to the unit disc in $\mathbb{C}$ equipped with the Poincaré distance function, $(\mathbb{D}, \rho_{\mathbb{D}})$. This model is a realisation of a *hyperbolic plane*. The Poincaré disc model $(\mathbb{D}, \rho_{\mathbb{D}})$ is a fundamental model in *hyperbolic geometry* (or *Bolyai-Lobachevsky* geometry), a well known non-Euclidean geometry. We study the Poincaré distance on the unit disc in this thesis because it provides an appropriate starting point for examining other distances and pseudodistances in complex analysis. In fact, this distance is used in the definition of the first pseudodistance we look at, the Carathéodory pseudodistance. Jumping straight into the definition, we now present the Poincaré distance.

Definition 2.4.1. *The Poincaré distance function is given by*

$$\rho_{\mathbb{D}}(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - z\overline{w}} \right|$$

for all $z, w \in \mathbb{D}$.

Since $\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$, the Poincaré distance $\rho_{\mathbb{D}}$ is very often seen expressed in the form

$$\rho_{\mathbb{D}}(z, w) = \frac{1}{2} \ln \frac{1 + \left| \frac{z-w}{1-z\overline{w}} \right|}{1 - \left| \frac{z-w}{1-z\overline{w}} \right|}.$$

The unit disc $\mathbb{D}$ equipped with the Poincaré distance $\rho_{\mathbb{D}}$ forms a complete metric space, and the topology generated by $\rho_{\mathbb{D}}$ coincides with the standard topology on $\mathbb{D}$.

We now show that $\rho_{\mathbb{D}}$ is a distance, by showing that it satisfies the axioms listed above.

Proposition 2.4.1. *$\rho_{\mathbb{D}}$ is a distance on $\mathbb{D}$.*

Proof. (i) For $z, w \in \mathbb{D}$, we have $|\cdot| : (\mathbb{D}, \mathbb{D}) \to [0, 1)$, and $\tanh^{-1} : [0, 1) \to [0, \infty)$, and so $\rho_{\mathbb{D}}(z, w) \geq 0$.

(ii) By observation of the definition of $\rho_{\mathbb{D}}$, clearly we have $\rho_{\mathbb{D}}(z, w) = \rho_{\mathbb{D}}(w, z)$.

(iii) Now we prove the triangle inequality $\rho_{\mathbb{D}}(z, w) \leq \rho_{\mathbb{D}}(z, v) + \rho_{\mathbb{D}}(v, w)$ is satisfied. The following proof comes from [1]. Since $\mathbb{D}$ is *homogeneous* (that is, any point in $\mathbb{D}$ can be moved biholomorphically to any other point), and $\rho_{\mathbb{D}}$ is invariant under biholomorphic mappings from $\mathbb{D}$ into $\mathbb{D}$ (a property we will show in the following chapter), it suffices to show that $\rho_{\mathbb{D}}(z, w) \leq \rho_{\mathbb{D}}(z, 0) + \rho_{\mathbb{D}}(0, w)$ for all $z, w \in \mathbb{D}$.

Let $z = re^{i\theta}$ and $w = se^{i\phi}$. Then we have

$$\begin{aligned} \rho_{\mathbb{D}}(z, w) &= \rho(re^{i\theta}, se^{i\phi}) \\ &= \rho(r, se^{i\alpha}) \text{ , where } \alpha = \phi - \theta \\ &= \tanh^{-1}\left(\left|\frac{r - se^{i\alpha}}{1 - rse^{i\alpha}}\right|\right), \end{aligned}$$

and

$$\begin{aligned} \rho_{\mathbb{D}}(z, 0) + \rho_{\mathbb{D}}(0, w) &= \tanh^{-1}(|z|) + \tanh^{-1}(|w|) \\ &= \tanh^{-1}(r) + \tanh^{-1}(s) \\ &= \tanh^{-1}\left(\frac{r + s}{1 + rs}\right), \end{aligned}$$

using the identity $\tanh^{-1}(r) + \tanh^{-1}(s) = \tanh^{-1}\left(\frac{r+s}{1+rs}\right)$ for $0 \leq r, s < 1$.

So we have to show that

$$\tanh^{-1}\left(\left|\frac{r - se^{i\alpha}}{1 - rse^{i\alpha}}\right|\right) \leq \tanh^{-1}\left(\frac{r + s}{1 + rs}\right),$$

that is,

$$\left|\frac{r - se^{i\beta}}{1 - rse^{i\beta}}\right| \leq \frac{r + s}{1 + rs} \text{ for all } \beta \in [0, 2\pi).$$

The left-hand side gives

$$\begin{aligned} \left|\frac{r - se^{i\alpha}}{1 - rse^{i\alpha}}\right|^2 &= \frac{r^2 - 2\ \mathrm{Re}(rse^{i\beta}) + s^2}{1^2 - 2\ \mathrm{Re}(rse^{i\beta}) + r^2s^2} \\ &= \frac{r^2 - 2rs\ \cos\ \beta + s^2}{1 - 2rs\ \cos\ \beta + r^2s^2} \end{aligned}$$

Now, letting the right-hand side equal $g(\beta)$, we have

$$g'(\beta) = \frac{2rs(1 - r^2)(1 - s^2)\ \sin\ \beta}{(1 - 2rs\ \cos\ \beta + r^2s^2)^2}.$$

The critical points of g occur when $\beta = 0$ and $\beta = \pi$. And $g'(\beta) > 0$ on $[0, \pi]$, implying that g is strictly increasing on this interval, with $g(0) < g(\pi)$. Further, note that

$$g(0) = \frac{r^2 - 2rs + s^2}{1 - 2rs + r^2s^2} = \left(\frac{r - s}{1 - rs}\right)^2, \text{ and}$$

$$g(\pi) = \frac{r^2 + 2rs + s^2}{1 + 2rs + r^2 s^2} = \left(\frac{r+s}{1+rs}\right)^2,$$

and so finally we have

$$g(0) = \left(\frac{r-s}{1-rs}\right)^2 \leq \left(\frac{r+s}{1+rs}\right)^2 = g(\pi)$$

for $0 \leq r, s < 1$.

Since we have shown that

$$\left|\frac{r - se^{i\beta}}{1 - rse^{i\beta}}\right| \leq \frac{r+s}{1+rs} \text{ for all } \beta \in [0, 2\pi),$$

then the triangle inequality is satisfied.

(iv) Lastly we show that $z = w \iff \rho_{\mathbb{D}}(z, w) = 0$. If $z = w$, then

$$\tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right| = \tanh^{-1}(0) = 0,$$

and if $\tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right| = 0$ then

$$\begin{aligned} \left|\frac{z-w}{1-z\overline{w}}\right| = 0 &\implies |z-w| = 0 \\ &\implies z = w, \end{aligned}$$

and so all the axioms are satisfied. $\square$

Chapter 3

Classification of pseudodistances

3.1 Finsler pseudometrics

In the previous chapter, we have considered ideas and concepts from one variable complex analysis, and the Poincaré distance, which is taken between two points in the unit disc in the complex plane. At this point in the thesis, we begin to consider several complex variable domains $D \subset \mathbb{C}^n$. As was mentioned briefly in the first chapter, in n-dimensional complex space, the tangent bundle of D is always given by $D \times \mathbb{C}^n$. We now begin to outline some of the important ways to define and classify pseudodistances, and will begin by briefly outlining the concept of a Finsler pseudometric.

Definition 3.1.1. *An Finsler pseudometric on a domain D is a non-negative function $\alpha : D \times \mathbb{C}^n \to \mathbb{R}$ such that*

(i) $\alpha(p, tv) = |t|\alpha(p, v)$ *for all* $(p, v) \in D \times \mathbb{C}^n$ *and* $t \in \mathbb{C}$,

(ii) α *is upper semicontinuous. That is, for* $\lambda \in \mathbb{R}^+$, *the set* $\{(p, v) \in D \times \mathbb{C}^n : \alpha(p, v) < \lambda\}$ *is open.*

Further, α is an Finsler metric *if a third condition is satisfied,*

(iii) $\alpha(p, v) > 0$ *for* $v \neq 0$.

Taking two arbitrary Finsler pseudometrics α_1 on D_1 and α_2 on D_2, $f : D_1 \to D_2$ is distance-decreasing if

$$\alpha_2(f(p), f'(p)(v)) \leq \alpha_1(p, v). \tag{3.1.1}$$

If we have strict equality in the above relation, that is, if

$$\alpha_2(f(p), f'(p)(v)) = \alpha_1(p, v),$$

then f is an *isometry*. We elaborate on the concepts of distance-decreasing mappings and isometries in the following section.

As it happens, in the above condition f does not have to be holomorphic, but only $\mathcal{C}^1$. However, since we deal exclusively with holomorphic functions in this thesis, this point will not be explored further.

3.2 The infinitesimal Poincaré metric

We describe the Poincaré distance in an alternative form. Consider the unit disc $\mathbb{D}$. For a vector in the tangent space, $v \in T_z(\mathbb{D})$, the *infinitesimal Poincaré metric* is given by

$$\mathfrak{p}_{\mathbb{D}}(z,v) = \frac{|v|}{1-|z|^2},$$

where $|\cdot|$ is the usual Euclidean norm. Note that in the literature, occasionally the expression for $\mathfrak{p}$ is multiplied by a scalar factor in the numerator. The infinitesimal Poincaré metric is a Finsler metric, as we show here.

Proposition 3.2.1. *The infinitesimal Poincaré metric is a Finsler metric.*

Proof. To see this, simply note that

$$\begin{aligned}\mathfrak{p}_{\mathbb{D}}(z,tv) &= \frac{|tv|}{1-|z|^2}\\ &= |t|\left(\frac{|v|}{1-|z|^2}\right)\\ &= |t|\mathfrak{p}_{\mathbb{D}}(z,v),\end{aligned}$$

and that $\mathfrak{p}_{\mathbb{D}}(z,v)$ is continuous for all $(z,v) \in \mathbb{D} \times \mathbb{C}^n$, since for all $z \in \mathbb{D}$, we have $|z| < 1$. As for the metric condition, clearly $\mathfrak{p}$ is strictly positive for non-zero v, and z in the open unit disc. □

We will show how to recover the original Poincaré distance $\rho_{\mathbb{D}}$ from the infinitesimal form given above. In order to do so, we first define the notion of length between two points in a domain D. Let $z, w \in D$ be two distinct points, and let $\gamma(t)$ be a parametrised, piecewise-smooth curve in D, with $\gamma(a) = z, \gamma(b) = w$. It follows that $t \in [a,b] \mapsto \mathfrak{p}(\gamma(t), \gamma'(t))$ is a bounded measurable function. Then the *length* of γ is given by

$$L_{\mathfrak{p}}(\gamma) = \int_a^b \mathfrak{p}(\gamma(t), \gamma'(t))\ dt.$$

If we then take the infimum of $L_{\mathfrak{p}}(\gamma)$ for all such curves γ, we have the *integrated form* of $\mathfrak{p}$, which we denote $\mathfrak{p}^i$. We will refer to such a curve γ as an *admissible curve* joining z and w, and hence we can define $\mathfrak{p}^i$ as

$$\mathfrak{p}^i(z,w) = \inf\left\{L_{\mathfrak{p}}(\gamma) : \gamma \text{ an admissible curve joining } z \text{ and } w\right\}.$$

Proposition 3.2.2. *The integrated form $\mathfrak{p}^i$ is a pseudodistance on D.*

Proof. We need to check the three axioms for a pseudodistance. Let $z, w \in D$.

(i) It is clear that $\mathfrak{p}^i(z,w) \geq 0$, since $\mathfrak{p}^i$ is a Finsler metric, and so is a non-negative function.

(ii) Clearly, we have

$$\begin{aligned}\mathfrak{p}^i(z,w) &= \inf\left\{L_{\mathfrak{p}}(\gamma) : \gamma \text{ an admissible curve joining } z \text{ and } w\right\}\\ &= \inf\left\{L_{\mathfrak{p}}(\gamma) : \gamma \text{ an admissible curve joining } w \text{ and } z\right\}\\ &= \mathfrak{p}^i(w,z)\end{aligned}$$

(iii) As for the triangle inequality, fix $v \in D$. Let Γ be an admissible curve joining z and w, and passing through the point v. Then we have

$$\begin{aligned}
\mathfrak{p}^i(z,w) &= \inf\left\{L_{\mathfrak{p}}(\gamma) : \gamma \text{ an admissible curve joining } z \text{ and } w\right\} \\
&\leq \inf\left\{L_{\mathfrak{p}}(\Gamma)\right\} \\
&= \inf\left\{L_{\mathfrak{p}}(\kappa) : \kappa \text{ an admissible curve joining } z \text{ and } v\right\} \\
&\qquad + \inf\left\{L_{\mathfrak{p}}(\tau) : \tau \text{ an admissible curve joining } v \text{ and } w\right\} \\
&= \mathfrak{p}^i(z,v) + \mathfrak{p}^i(v,w)
\end{aligned}$$

□

Consider the infinitesimal Poincaré metric on the unit disc. The integrated form of $\mathfrak{p}_{\mathbb{D}}$ is the original Poincaré distance $\rho_{\mathbb{D}}$, as shown below.

Proposition 3.2.3. *The integrated form of the infinitesimal Poincaré metric on $\mathbb{D}$ is the Poincaré distance, that is, $\mathfrak{p}^i_{\mathbb{D}} = \rho_{\mathbb{D}}$.*

Proof. Take a point r on the real line such that $0 < r < 1$, and let γ be an admissible curve in $\mathbb{D}$ joining 0 and r. Let γ_1 denote the real part of γ. Then γ_1 is also an admissible curve and we have

$$\begin{aligned}
L_{\mathfrak{p}_{\mathbb{D}}}(\gamma) &= \int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt \\
&\geq \int_0^1 \frac{\gamma_1'(t)}{1-\gamma_1(t)^2}dt \\
&\geq \int_0^r \frac{du}{1-u^2}, \text{ after substituting } u = \gamma_1(t) \\
&= \tanh^{-1}(r) \\
&= \rho_{\mathbb{D}}(0,r).
\end{aligned}$$

Now, on the other hand, consider the curve $\gamma(t) = tr$ for $t \in [0,1]$, which joins 0 and r. Then we have

$$\begin{aligned}
L_{\mathfrak{p}_{\mathbb{D}}}(\gamma) &= \int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt \\
&= \int_0^1 \frac{|r|}{1-|tr|^2}dt \\
&= \int_0^1 \frac{r}{1-t^2r^2}dt \\
&= \int_0^r \frac{du}{1-u^2}, \text{ after substituting } u = tr \\
&= \tanh^{-1}(r) \\
&= \rho_{\mathbb{D}}(0,r),
\end{aligned}$$

It follows that if we take the infimum of $L_{\mathfrak{p}_{\mathbb{D}}}$ over all curves γ joining 0 and r, then this infimum is attained when we choose the curve $\gamma(t) = tr$.

And so we have

$$\begin{aligned}\mathfrak{p}^i_{\mathbb{D}}(z,w) &= \mathfrak{p}^i_{\mathbb{D}}(0,\phi_z(w))\\ &= \mathfrak{p}^i_{\mathbb{D}}(0,|\phi_z(w)|)\\ &= \tanh^{-1}(|\phi_z(w)|)\\ &= \tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right|\\ &= \rho_{\mathbb{D}}(z,w),\end{aligned}$$

where the second equality follows from the fact that we can always find a rotation of the disc taking $\phi_z(w)$ to $|\phi_z(w)|$, and the third equality follows from the fact that the infimum is attained on the curve described above. □

This process shows how we can recover the original Poincaré distance from the infinitesimal Poincaré metric.

3.2.1 Distance-decreasing Maps

For given distances or pseudodistances ρ_1 on D_1 and ρ_2 on D_2, and a holomorphic map $f \in H(D_1, D_2)$, we say that f is *distance-decreasing* if

$$\rho_2(f(z), f(w)) \leq \rho_1(z,w) \tag{3.2.1}$$

for $z, w \in D_1$, and where ρ_1 is a pseudodistance on D_1, and ρ_2 is a pseudodistance on D_2. We also often say that f is a *contraction*. Strictly speaking, we should say that f is *distance-nonincreasing* since we do not have a strict inequality, however it has become standard practice to refer to such maps as distance-decreasing (for example, in [1], [4]), and so this is the terminology we will use here.

If we have strict equality in the above relation, that is, if

$$\rho_2(f(z), f(w)) = \rho_1(z,w),$$

then f is *distance-preserving*, or an *isometry*. Note that in the first section of this chapter, we defined an isometry with respect to a Finsler pseudometric, and it is important to keep in mind the distinction.

The above inequality (3.2.1) is sometimes called the Schwarz-Pick inequality, and plays a defining role in *Schwarz-Pick* systems of pseudodistances, which we discuss later.

A simple application of the Schwarz-Pick lemma allows us to conclude two important facts about the Poincaré distance $\rho_{\mathbb{D}}$. Firstly, $\rho_{\mathbb{D}}$ is distance-decreasing for holomorphic maps from $\mathbb{D}$ into $\mathbb{D}$. Secondly, $\rho_{\mathbb{D}}$ is an isometry for functions in the automorphism group. Showing each in turn, we have for $f \in H(\mathbb{D},\mathbb{D})$ that

$$\begin{aligned}\rho_{\mathbb{D}}(f(z), f(w)) &= \tanh^{-1}\left(\left|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}\right|\right)\\ &\leq \tanh^{-1}\left(\left|\frac{z-w}{1-z\overline{w}}\right|\right)\\ &= \rho_{\mathbb{D}}(z,w),\end{aligned}$$

using (i) of the Schwarz-Pick lemma.

This gives us the useful relation

$$\rho_{\mathbb{D}}(f(z), f(w)) \le \rho_{\mathbb{D}}(z, w) \tag{3.2.2}$$

for $f \in H(\mathbb{D}, \mathbb{D})$, which we will use often.

If f is an automorphism, i.e. $f \in \mathrm{Aut}(\mathbb{D})$, then we have equality in (i) of the Schwarz-Pick lemma, and consequently

$$\rho_{\mathbb{D}}(f(z), f(w)) = \rho_{\mathbb{D}}(z, w).$$

That is, $\rho_{\mathbb{D}}$ is *invariant* under $\mathrm{Aut}(\mathbb{D})$.

This distance-decreasing property for holomorphic maps is an important feature of the distances and pseudodistances that we study throughout this thesis. We will show later that this concept is critical in the definition of a *Schwarz-Pick* system of pseudodistances.

We now present an important lemma which shows that if f is distance-decreasing with respect to Finsler pseudometrics, then it is distance-decreasing with respect to the integrated forms of these pseudometrics.

Lemma 3.2.4. *If the holomorphic function $f : (D_1, \alpha_1) \to (D_2, \alpha_2)$ is distance decreasing, then $f : (D_1, \alpha_1^i) \to (D_2, \alpha_2^i)$ is also distance-decreasing. That is, if*

$$\alpha_2(f(z), f(w)) \le \alpha_1(z, w),$$

then

$$\alpha_2^i(f(z), f(w)) \le \alpha_1^i(z, w),$$

for points $z, w \in D_1$.

Proof. If γ is a curve joining z and w in D_1, then $f \circ \gamma$ is a curve joining $f(z)$ and $f(w)$ in D_2. That is, if $\gamma \in C_{D_1}(z, w)$ then $f \circ \gamma \in C_{D_2}(f(z), f(w))$. So we have

$$\begin{aligned}
\alpha_2^i(f(z), f(w)) &= \inf \{L_{\alpha_2}(f \circ \gamma)\} \\
&\le L_{\alpha_2}(f \circ \gamma) \\
&= \int_0^1 \alpha_2\left((f \circ \gamma)(t), (f \circ \gamma)'(t)\right) dt \\
&= \int_0^1 \alpha_2\left(f(\gamma(t)), f'(\gamma(t))(\gamma'(t))\right) dt \\
&\le \int_0^1 \alpha_1(\gamma(t), \gamma'(t)) dt, \text{ by (5.1)} \\
&= L_{\alpha_1}(\gamma).
\end{aligned}$$

Since the above implies that $\alpha_2^i(f(z), f(w)) \le L_{\alpha_1}(\gamma)$ for an arbitrary curve γ, by taking the infimum of the right-hand side over all curves joining z and w, we see that

$$\begin{aligned}
\alpha_2^i(f(z), f(w)) &\le \inf \{L_{\alpha_1}(\gamma) : \gamma \in \mathcal{C}_{D_1}(z, w)\} \\
&= \alpha_1^i(z, w).
\end{aligned}$$

And so we see that $\alpha_2^i(f(z), f(w)) \le \alpha_1^i(z, w)$. □

We have examined the Poincaré distance and its infinitesimal form, the infinitesimal Poincaré metric, which are applied to points in the unit disc. We now present the Carathéodory pseudodistance, a pseudodistance between points in an arbitrary complex domain. We will also consider its infinitesimal form, and consider further important ways of classifying distances and pseudodistances.

Chapter 4

The Carathéodory pseudodistance

We now consider an important pseudodistance, the Carathédory pseudodistance, which was introduced by Carathéodory in 1926. This pseudodistance, in common with the Kobayashi pseudodistance, generalises the concept of the Poincaré distance in arbitrary domains. That is, it is defined for points in $D \subset \mathbb{C}^n$, rather than simply for points in the unit disc as in the case of the Poincaré distance. In fact, the Carathéodory and Kobayashi pseudodistances can be defined for arbitrary complex *manifolds*, which we will examine briefly in the final chapter. Much of the notation and some definitions in section 4.2 are from [5].

We now define the Carathéodory pseudodistance on a domain D as follows:

Definition 4.0.1. *The Carathéodory pseudodistance function is given by*

$$C_D(z, w) = \sup \{\rho_{\mathbb{D}}(f(z), f(w)) : f \in H(D, \mathbb{D})\}$$

for $z, w \in D$.

Note that the Poincaré distance $\rho_{\mathbb{D}}$ is used in its definition, and the supremum is taken over all holomorphic mappings from the domain D into the unit disc $\mathbb{D}$.

Proposition 4.0.1. C_D *is a pseudodistance on* D.

Proof. To prove the proposition, we utilise the fact that $\rho_{\mathbb{D}}$ satisfies the pseudodistance axioms, and use the corresponding relations. Considering each axiom, we have for $f \in H(D, \mathbb{D})$:

(i) Since $\rho_{\mathbb{D}}(p, q) \geq 0$ for any $p, q \in \mathbb{D}$, then $\rho_{\mathbb{D}}(f(z), f(w)) \geq 0$, which implies that $\sup_f \rho_{\mathbb{D}}(f(z), f(w)) \geq 0$.

(ii) Since $\rho_{\mathbb{D}}(q, p) = \rho_{\mathbb{D}}(p, q)$ for $p, q \in \mathbb{D}$, then clearly $\sup_f \rho_{\mathbb{D}}(f(z), f(w)) = \sup_f \rho_{\mathbb{D}}(f(w), f(z))$.

(iii) Since the triangle inequality holds for $\rho_{\mathbb{D}}$, we have

$$\begin{aligned} C_D(f(z), f(w)) &= \sup_f \rho_{\mathbb{D}}(f(z), f(w)) \\ &\leq \sup_f [\rho_{\mathbb{D}}(f(z), f(v)) + \rho_{\mathbb{D}}(f(v), f(w))] \\ &\leq \sup_f \rho_{\mathbb{D}}(f(z), f(v)) + \sup_f \rho_{\mathbb{D}}(f(v), f(w)) \\ &= C_D(f(z), f(v)) + C_D(f(v), f(w)). \end{aligned}$$

showing that the triangle inequality holds for C_D. □

We now show that C_D is not necessarily a distance. That is, axiom (iv) may not be satisfied. We do so with the aid of Liouville's theorem, a result that was presented in the introductory chapter. By this theorem, any function $f \in H(\mathbb{C}^n, \mathbb{D})$ must be constant. Therefore, such an f maps two distinct points $z, w \in \mathbb{C}^n$ to the same point in $\mathbb{D}$, and we have $\rho_{\mathbb{D}}(f(z), f(w)) = 0$ for any $z, w \in \mathbb{C}^n$. So $z \neq w \implies C_{\mathbb{C}^n}(z, w) = 0$, and axiom (iv) is not satisfied. Since this holds true for all $z, w \in \mathbb{C}^n$, we have $C_{\mathbb{C}^n} \equiv 0$.

4.1 Schwarz-Pick systems

The Carathédory pseudodistance is an example of a *Schwarz-Pick system* which we define in this section.

Definition 4.1.1. *A Schwarz-Pick system is an assignment of a pseudodistance to each domain in $\mathbb{C}^n$ for all n, satisfying the following two conditions:*

(i) the pseudodistance assigned to $\mathbb{D}$ is the Poincaré distance; and

(ii) if ρ_1 and ρ_2 are the pseudodistances assigned to $D_1 \subseteq \mathbb{C}^n$ and $D_2 \subseteq \mathbb{C}^k$ respectively, and $f \in H(D_1, D_2)$, then

$$\rho_2(f(z), f(w)) \leq \rho_1(z, w).$$

The above inequality is referred to as the *Schwarz-Pick inequality.* We show that the Carathéodory pseudodistances form a Schwarz-Pick system.

Proposition 4.1.1. *The Carathéodory pseudodistances C_D form a Schwarz-Pick system.*

Proof. (i) By the distance-decreasing property (3.2.2), for $z, w \in \mathbb{D}$ we have

$$\rho_{\mathbb{D}}(f(z), f(w)) \leq \rho_{\mathbb{D}}(z, w),$$

for $f \in H(\mathbb{D}, \mathbb{D})$, so we see that

$$\begin{aligned} C_{\mathbb{D}}(z, w) &= \sup \{\rho_{\mathbb{D}}(f(z), f(w)) : f \in H(\mathbb{D}, \mathbb{D})\} \\ &= \rho_{\mathbb{D}}(z, w), \end{aligned}$$

if we take f to be the identity mapping $f(z) = z$, showing condition (i) is satisfied.

(ii) As for the second condition, let $f \in H(D_1, D_2)$. If $h \in H(D_2, \mathbb{D})$ then $h \circ f \in H(D_1, \mathbb{D})$, and so for any $z, w \in D_1$ we have

$$\rho_{\mathbb{D}}(h \circ f(z), h \circ f(w)) \leq C_{D_1}(z, w),$$

and therefore

$$\begin{aligned} C_{D_2}(f(z), f(w)) &= \sup \{\rho_{\mathbb{D}}(h(f(z)), h(f(w))) : h \in H(D_2, \mathbb{D})\} \\ &\leq C_{D_1}(z, w). \end{aligned}$$

which shows that condition (ii) is satisfied.

Thus we have shown that the Carathéodory pseudodistances form a Schwarz-Pick system. □

The conclusion that $C_{D_2}(f(z), f(w)) \leq C_{D_1}(z, w)$, above, shows that C_D is distance-decreasing with respect to holomorphic mappings. Moreover, $C_{D_2}(f(z), f(w)) = C_{D_1}(z, w)$ for biholomorphic mappings. We show this using a simple application of the distance-decreasing property. If $f \in H(D_1, D_2)$ is biholomorphic, we of course have

$$C_{D_2}(f(z), f(w)) \leq C_{D_1}(z, w).$$

But we can also apply this relation to $f^{-1} \in H(D_2, D_1)$, to see that

$$C_{D_1}(f^{-1}(\zeta), f^{-1}(\xi)) \leq C_{D_2}(\zeta, \xi),$$

for $\zeta, \xi \in D_2$. Now simply letting $\zeta = f(z)$ and $\xi = f(w)$, we see that

$$C_{D_1}(z, w) \leq C_{D_2}(f(z), f(w)),$$

from which we conclude that

$$C_{D_2}(f(z), f(w)) = C_{D_1}(z, w).$$

So we see that C_D is invariant for biholomorpic mappings.

The fact that C_D is distance-decreasing for holomorphic mappings, and invariant for biholomorphic mappings is a property shared with the Poincaré distance.

Progressing in a similar manner to the previous section on the Poincaré distance, we now examine the infinitesimal Carathéodory pseudometric, which will be denoted with a lowercase c. As with the infinitesimal Poincaré metric, it is defined at a point in the domain, and a vector in the tangent space.

4.1.1 The infinitesimal Carathéodory pseudometric

Definition 4.1.2. *For a domain D and $(p, v) \in D \times \mathbb{C}^n$, the infinitesimal Carathéodory pseudometric is given by*

$$c_D(p, v) = \sup \{|df(p)(v)| : f \in H(D, \mathbb{D})\}.$$

Proposition 4.1.2. *c_D is a continuous Finsler pseudometric.*

Proof. We will show the two conditions for a Finsler pseudometric are satisfied.

(i) Considering the first condition, since

$$\begin{aligned}|df(p)(\lambda v)| &= |f'(p)| \cdot |\lambda v| \\ &= |\lambda| \cdot |df(p)(v)|,\end{aligned}$$

we see that $c_D(p, \lambda v) = |\lambda| c_D(p, v)$ for all $\lambda \in \mathbb{C}$, and all $(p, v) \in D \times \mathbb{C}^n$.

(ii) As for the second condition, we will not merely show upper semicontinuity, but that c_D is locally Lipschitz. This of course implies that c_D is a continuous Finsler pseudometric. Firstly note that for $(p, v) \in T(D)$ and a fixed z in $\mathbb{D}$, we have

$$c_D(p, v) = \sup \left\{ \frac{|df(p)(v)|}{1 - |f(p)|^2} : f \in H(D, \mathbb{D}), f(p) = z \right\}.$$

We omit the proof, which can be found in [1, p.55]. Now, let $x \in D$ and suppose $B_r(x) \subset D$. Let $y, z \in B_{r/4}(x), v \in \mathbb{C}^n$, and suppose that $c_D(y, v) \geq c_D(z, v)$. Then we have

$$\begin{aligned}c_D(y, v) - c_D(z, v) &= \sup \{|df(y)(v)| : f \in H(D, \mathbb{D}), f(y) = 0\} \\ &\quad - \sup \left\{ \frac{|df(z)(v)|}{1 - |f(z)|^2} : f \in H(D, \mathbb{D}), f(y) = 0 \right\} \\ &\leq \sup \left\{ |df(y)(v)| - \frac{|df(z)(v)|}{1 - |f(z)|^2} : f \in H(D, \mathbb{D}), f(y) = 0 \right\} \\ &\leq \sup \{|df(y)(v)| - |df(z)(v)| : f \in H(D, \mathbb{D}), f(y) = 0\} \\ &\leq \sup \{|df(y)(v) - df(z)(v)| : f \in H(D, \mathbb{D}), f(y) = 0\} \\ &\leq \left(\frac{2}{r}\right)^2 \|y - z\| \cdot \|v\|,\end{aligned}$$

so we can conclude that

$$|c_D(y, v) - c_D(z, v)| \leq \left(\frac{2}{r}\right)^2 \|y - z\| \cdot \|v\|$$

for all $y, z \in B_{r/4}(x)$.

Further, if $z \in B_{r/4}(x)$ and $v, w \in T_z(D)$ then

$$\begin{aligned}|c_D(z, v) - c_D(z, w)| &\leq c_D(z, v - w) \\ &= \sup \{|df(z)(v - w)| : f \in H(D, \mathbb{D}), f(z) = 0\} \\ &\leq \frac{2}{r} \|v - w\|\end{aligned}$$

So using the triangle inequality, we see that

$$\begin{aligned}|c_D(y, v) - c_D(z, w)| &\leq |c_D(y, v) - c_D(z, v)| + |c_D(z, v) - c_D(z, w)| \\ &\leq \left(\frac{2}{r}\right)^2 \|y - z\| \cdot \|v\| + \frac{2}{r} \|v - w\|\end{aligned}$$

showing c_D is locally Lipschitz. $\square$

We can also define the infinitesimal Carathéodory pseudometric as

$$c_D(p,v) = \sup\left\{\mathfrak{p}_{\mathbb{D}}(f(p), df(p)(v)) : f \in H(D,\mathbb{D})\right\},$$

since we have

$$\begin{aligned} c_D(p,v) &= \sup_f \mathfrak{p}_{\mathbb{D}}(f(p), df(p)(v)) \\ &= \sup_f \frac{|df(p)(v)|}{1-|p(v)|^2} \\ &= \sup_f |df(p)(v)|, \end{aligned}$$

choosing f such that $f(p) = 0$.

In a similar vein to condition (i) in the definition of a Schwarz-Pick system, we show that $c_{\mathbb{D}}$ is the infinitesimal Poincaré metric. That is, the infinitesimal Carathéodory pseudometric applied on the unit disc gives the infinitesimal Poincaré metric.

Lemma 4.1.3. *$c_{\mathbb{D}}$ is the infinitesimal Poincaré metric.*

Proof. We prove the result by showing $c_{\mathbb{D}}(p,v) \leq \mathfrak{p}_{\mathbb{D}}(p,v)$, and then showing $c_{\mathbb{D}}(p,v) \geq \mathfrak{p}_{\mathbb{D}}(p,v)$. Considering the former inequality first, we have

$$\begin{aligned} |df(p)(v)| &= |f'(p)| \cdot |v| \\ &\leq \frac{1-|f(p)|^2}{1-|p|^2} \cdot |v| \\ &\leq \frac{1}{1-|p|^2} \cdot |v| \\ &= \mathfrak{p}_{\mathbb{D}}(p,v), \end{aligned}$$

where the first inequality in the above computation follows from part (ii) of the Schwarz-Pick lemma.

As for the latter inequality, we show it by taking the mapping f to be the Möbius transformation for the disc. So taking this transformation ϕ_p, and using the fact that its derivative is given by

$$\phi_p'(z) = \frac{1-p\overline{p}}{(1-z\overline{p})^2},$$

we have

$$\begin{aligned} c_{\mathbb{D}}(p,v) &= \sup\left\{|df(p)(v)| : f \in H(\mathbb{D},\mathbb{D})\right\} \\ &\geq |d\phi_p(p)(v)| \\ &= |\phi_p'(p)||v| \\ &= \frac{1-p\overline{p}}{(1-p\overline{p})^2}|v| \\ &= \frac{|v|}{1-|p|^2} \\ &= \mathfrak{p}_{\mathbb{D}}(p,v), \end{aligned}$$

and this shows the inequality. □

We show a second fact about the infinitesimal Carathéodory pseudometric, namely that it has the distance-decreasing property.

Lemma 4.1.4. *For $f \in H(D_1, D_2)$, we have*

$$c_{D_2}(f(p), df(p)(v)) \leq c_{D_1}(p, v)$$

for $(p, v) \in T(D_1)$.

Proof. We have $f \in H(D_1, D_2)$, as above. Let $h \in H(D_2, \mathbb{D})$, and so $h \circ f \in H(D_1, \mathbb{D})$. So for $(p, v) \in T(D_1)$ we have

$$\begin{aligned} |d(h \circ f)(p)(v)| &= |(dh(f(p))) \circ (df(p)(v))| \\ &\leq c_{D_1}(p, v). \end{aligned}$$

So if we set $q = f(p)$ and $w = df(p)(v)$, then we have
$c_{D_1}(p, v) \geq \sup\{|dh(q)(w)| : h \in H(D_2, \mathbb{D})\}$, and so we see the result, that
$c_{D_1}(p, v) \geq c_{D_2}(f(p), df(p)(v))$. □

We can use the above two lemmas to show that the integrated form of the infinitesimal Carathéodory pseudometric forms a Schwarz-Pick system.

Lemma 4.1.5. *$\{c_D^i\}$ form a Schwarz-Pick system of pseudodistances.*

Proof. Checking each condition of Definition 4.1.1, we have:

(i) Considering $\{c_{\mathbb{D}}^i\}$, we know from Lemma 4.1.3 that $c_{\mathbb{D}}$ is the infinitesimal Poincaré metric, and we know from Proposition 3.2.3 that the integrated form of the infinitesimal Poincaré metric is the Poincaré distance.

(ii) We know from the previous Lemma that c_D is distance-decreasing for $f \in H(D_1, D_2)$. Therefore, simply applying Lemma 3.2.4 tells us that its integrated form c_D^i is also distance-decreasing. □

The last point we wish to make in this section concerns the fact that the Carathéodory pseudodistances are smaller than any other Schwarz-Pick system. That is, for all domains D and all $z, w \in D$, we have $C_D(z, w) \leq \rho_D(z, w)$ for an arbitrary Schwarz-Pick system ρ_D.

Proposition 4.1.6. *The Carathéodory pseudodistances form the smallest Schwarz-Pick system.*

Proof. Let $\{\rho_D\}$ be a Schwarz-Pick system. Then for $z, w \in D$, and $f \in H(D, \mathbb{D})$ we have

$$\rho_{\mathbb{D}}(f(z), f(w)) \leq \rho_D(z, w),$$

by condition (ii) of Schwarz-Pick systems. Therefore,

$$\begin{aligned} C_D(z, w) &= \sup\{\rho_{\mathbb{D}}(f(z), f(w)) : f \in H(D, \mathbb{D})\} \\ &\leq \rho_D(z, w). \end{aligned}$$

□

This concept will be discussed further on, when we discuss the Kobayashi pseudodistance and pseudometric. We will show that the Kobayashi pseudodistances form the largest Schwarz-Pick system.

4.2 Inner pseudodistances

At this point, we will introduce a further concept that is relevant to the classification of pseudodistances, and of particular importance to the Carathéodory and Kobayashi pseudodistances. We define innerness for a pseudodistance on a domain $D \subset \mathbb{C}^n$ that is *locally bounded by the norm*, that is, for all $u \in D$, there exists an $M > 0$ and $\delta > 0$ such that

$$\rho(z, w) \leq M\|z - w\| \text{ for all } z, w \in B_\delta(u) \cap D.$$

This property implies that the pseudodistance ρ is continuous. Every pseudodistance we will refer to from now on will be assumed to have this property. We now define an inner pseudodistance.

Definition 4.2.1. *For a pseudodistance ρ on a domain D, with $z, w \in D$, the inner pseudodistance $\tilde{\rho}$ associated with ρ is defined as*

$$\tilde{\rho}(z, w) = \inf \{L_\rho(\gamma) : \gamma \text{ an admissible curve joining } z \text{ and } w\}.$$

In the above definition, $\gamma(t)$ is a parametrised, piecewise-smooth curve in D, with $t \in [0, 1]$ such that $\gamma(0) = z, \gamma(1) = w$. Further, $L_\rho(\gamma)$ in the above definition is given by the following:

For a piecewise-differentiable curve in D given by $\gamma(t)$, with $a \leq t \leq b$ in D, the length of $\gamma(t)$ is defined to be

$$L_\rho(\gamma) = \sup \sum_{i=1}^{k} \rho(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < ... < t_k = b$ of the interval $[a, b]$. The quantity $L_\rho(\gamma)$ is sometimes referred to as the ρ-length of γ. The definition of an inner pseudodistance is similar to the definition of an integrated form of a Finsler pseudometric. The difference is that a pseudodistance is used in the definition of the length L of the curve, as opposed to a Finsler pseudometric. A pseudodistance itself is said to be *inner* if $\rho = \tilde{\rho}$. A question arises as to whether $\tilde{\rho}$ is finite for any pseudodistance ρ. We show that this is the case, and $\tilde{\rho}$ is always finite. For any δ, we take a partition small enough so that both endpoints of every line segment in the partition are in $B_\delta(u)$ for some u. So we have

$$\begin{aligned}
\sum_{i=1}^{k} \rho(\gamma(t_{i-1}), \gamma(t_i)) &\leq M \sum_{i=1}^{k} \|\gamma(t_{i-1}) - \gamma(t_i)\| \\
&= M \sum_{i=1}^{k} \left(\sum_{i=1}^{k} |\gamma(t_{i-1}) - \gamma(t_i)|^2 \right)^{\frac{1}{2}} \\
&= M \sum_{i=1}^{k} \left(\sum_{i=1}^{k} |\gamma(\xi_j^k)||t_{i-1} - t_i|^2 \right)^{\frac{1}{2}} \\
&\leq MM' \left(\sum_{i=1}^{k} |t_{i-1} - t_i|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

for M' such that $\left(\sum_{i=1}^{k} |\gamma_k(\xi_j^k)|\right)^{\frac{1}{2}} \leq M'$ for some intermediate value ξ^k. Since the partition is finite, this shows that $\tilde{\rho}$ is finite.

Since holomorphic mappings map admissible curves onto admissible curves, it follows that for an arbitrary Schwarz-Pick system ρ_D, $\{\tilde{\rho}_D\}$ is also a Schwarz-Pick system. Further, $\rho_D \leq \tilde{\rho}_D$ for any domain D. To see this, note that

$$\rho_D(z, w) \leq L_\rho(\gamma)$$

for any admissible curve γ joining z and w. By taking the infimum of the right-hand side over all curves of this kind, it follows that $\rho_D \leq \tilde{\rho}_D$. We will later show with an example that the Caratheéodory pseudodistance is not necessarily inner. That is, we do not necessarily have $C_D = \tilde{C}_D$ for all domains.

We know from Lemma 4.1.5 that $\{c_D^i\}$ form a Schwarz-Pick system of pseudodistances, and from Proposition 4.1.6 that $\{C_D\}$ form the smallest Schwarz-Pick system. So we can immediately conclude that $C_D \leq c_D^i$ for any domain D. Further, from the remarks in the above paragraph, we can also conclude that $C_D \leq \tilde{C}_D$. In the argument that follows, it is not entirely clear that $\tilde{C}$ is even finite, as we have not yet shown that it is bounded above by the norm. We address this midway through the following chapter, and take it as given for now. Before showing that the Carathéodory pseudodistance is not necessarily inner, we firstly present two propositions that will help us do this.

Proposition 4.2.1. *If ρ is a pseudodistance on a domain D then $\tilde{\rho}$ is the largest of all pseudodistances d on D such that $L_\rho(\gamma) = L_d(\gamma)$ for all admissible curves.*

Proof. Consider a pseudodistance d on D such that $L_d(\gamma) = L_\rho(\gamma)$ for every admissible curve γ. Taking infimum of both sides implies $\tilde{d} = \tilde{\rho}$. Since $\rho \leq \tilde{\rho}$ for any pseudodistance ρ, we have $d \leq \tilde{d} = \tilde{\rho}$. Let γ denote the piecewise-differentiable parametrisation of the admissible curve γ, and let $\{t_0, t_1, ..., t_n\}$ be a partition of $[0, 1]$ such that $0 = t_0, ..., t_n = 1$. Further, let $\gamma_k(t) = \gamma((1-t)t_{k-1} + tt_k)$ for $k = 1, ..., n$. Note that $\gamma_k(0) = t_{k-1}$ and $\gamma_k(1) = t_k$, so γ_k partitions the curve γ into k parts. Therefore, we see that

$$\begin{aligned} \sup \sum_{k=0}^{n-1} \tilde{\rho}(\gamma(t_k), \gamma(t_{k+1})) &\leq \sum_{k=1}^{n} L_\rho(\gamma_k([0, 1])) \\ &= L_\rho(\gamma). \end{aligned}$$

where the inequality follows from the definition of $\tilde{\rho}$. Considering the left-hand side of the above inequality, letting n tend to infinity allows us to take finer and finer partitions, and so the left-hand side tends to $L_{\tilde{\rho}}(\gamma)$. So we have $L_{\tilde{\rho}}(\gamma) \leq L_\rho(\gamma)$. Since $\rho \leq \tilde{\rho}$ (as mentioned above), we also have $L_\rho(\gamma) \leq L_{\tilde{\rho}}(\gamma)$. It follows that $L_\rho(\gamma) = L_{\tilde{\rho}}(\gamma)$ for all admissible curves γ, which proves the proposition. □

Proposition 4.2.2. *The integrated form of an Finsler pseudometric is inner.*

Proof. Take α to be the Finsler pseudometric, with integrated form ρ. With a similar calculation to that in the proof of the previous proposition, we have

$$\begin{aligned}\sup\sum_{k=0}^{n-1}\rho(\gamma(t_k),\gamma(t_{k+1})) &\leq \sum_{k=1}^{n} L_\alpha(\gamma_k([0,1])) \\ &= L_\alpha(\gamma).\end{aligned}$$

Again, letting n tend to infinity on the left-hand side to get finer and finer partitions, we have $L_\rho(\gamma) \leq L_\alpha(\gamma)$ for admissible curves γ between z and w. We use this relation, and the definition of $\tilde{\rho}$ to see easily that

$$\begin{aligned}\tilde{\rho}(z,w) &= \inf\{L_\rho(\gamma) : \gamma \text{ an admissible curve}\} \\ &\leq \inf\{L_\alpha(\gamma) : \gamma \text{ an admissible curve}\} \\ &= \rho(z,w),\end{aligned}$$

where the last line is simply the definition of the integrated form of α. Since we always have $\rho \leq \tilde{\rho}$, we conclude that $\rho = \tilde{\rho}$, showing that the integrated form ρ is inner. □

So at this point we have that $C_D \leq \tilde{C}_D \leq c_D^i$ for any domain D. We now provide an example of a domain which illustrates that, in general, the Carathéodory pseudodistance is not an inner pseudodistance. That is, in general we do not have $C_D = \tilde{C}_D$. In this example we use a domain in several complex variables, specifically in $\mathbb{C}^2$. We begin by considering the Carathéodory pseudodistance on the bidisc $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$.

Proposition 4.2.3. *The Carathéodory pseudodistance on the bidisc $\mathbb{D}^2$ is given by*

$$C_{\mathbb{D}^2}((z_1,w_1),(z_2,w_2)) = \sup\{\rho_{\mathbb{D}}(z_1,z_2),\rho_{\mathbb{D}}(w_1,w_2)\}$$

for $(z_1,w_1),(z_2,w_2) \in \mathbb{D}^2$.

Proof. We firstly will show that $C_{\mathbb{D}^2}((0,0),(z,w)) = \sup\{\rho_{\mathbb{D}}(0,z),\rho_{\mathbb{D}}(0,w)\}$, by showing $C_{\mathbb{D}^2}((0,0),(z,w)) \leq \sup\{\rho_{\mathbb{D}}(0,z),\rho_{\mathbb{D}}(0,w)\}$, and then $C_{\mathbb{D}^2}((0,0),(z,w)) \geq \sup\{\rho_{\mathbb{D}}(0,z),\rho_{\mathbb{D}}(0,w)\}$. We do so with the aid of the following two mappings. Let $\|(\cdot,\cdot)\| : \mathbb{D}^2 \to \mathbb{D}$ be given by

$$\|(z,w)\| = \sup(|z|,|w|).$$

Further, for fixed $(z,w) \neq (0,0)$, let $\psi : \mathbb{D} \to \mathbb{D}^2 \setminus \{(0,0)\}$ be given by

$$\psi(\zeta) = \frac{\zeta\cdot(z,w)}{\|(z,w)\|},$$

which is holomorphic in $\mathbb{D}$. So we have

$$\begin{aligned}C_{\mathbb{D}^2}((0,0),(z,w)) &= C_{\mathbb{D}^2}(\psi(0),\psi(\|(z,w)\|)) \\ &\leq \rho_{\mathbb{D}}(0,\|(z,w)\|) \\ &= \rho_{\mathbb{D}}(0,\sup(|z|,|w|)) \\ &= \tanh^{-1}|\sup(|z|,|w|)| \\ &= \sup(\tanh^{-1}|z|,\tanh^{-1}|w|) \\ &= \sup(\rho_{\mathbb{D}}(0,z),\rho_{\mathbb{D}}(0,w)),\end{aligned}$$

where the inequality in the above computation follows from the fact that $C_{\mathbb{D}^2}$ is distance-decreasing for holomorphic mappings from $\mathbb{D}$ into $\mathbb{D}^2$.

Now, we show the second inequality. The projection mappings $(z, w) \to z$ and $(z, w) \to w$ are clearly holomorphic, and so both are in $H(\mathbb{D}^2, \mathbb{D})$. Considering the first, we see that

$$\begin{aligned} C_{\mathbb{D}^2}((0,0),(z,w)) &= \sup\left\{\rho_{\mathbb{D}}(f(0,0), f(z,w)) : f \in H(\mathbb{D}^2, \mathbb{D})\right\} \\ &\geq \rho_{\mathbb{D}}(0, z). \end{aligned}$$

Similarly, we see that $C_{\mathbb{D}^2}((0,0),(z,w)) \geq \rho_{\mathbb{D}}(0,w)$ using the second projection mapping. So clearly we have $C_{\mathbb{D}^2}((0,0),(z,w)) \geq \sup(\rho_{\mathbb{D}}(0,z), \rho_{\mathbb{D}}(0,w))$. This shows that

$$C_{\mathbb{D}^2}((0,0),(z,w)) = \sup(\rho_{\mathbb{D}}(0,z), \rho_{\mathbb{D}}(0,w)) \tag{4.2.1}$$

as required. Now, using this fact to show the conclusion, consider $(z_1, w_1), (z_2, w_2) \in \mathbb{D}^2$ and $(\phi_{z_1}, \phi_{w_1}) \in \mathrm{Aut}(\mathbb{D}^2)$, where ϕ_{z_1} and ϕ_{w_1} are the usual Möbius transformations. Since $C_{\mathbb{D}^2}$ is invariant under automorphisms, we have

$$\begin{aligned} C_{\mathbb{D}^2}((z_1,w_1),(z_2,w_2)) &= C_{\mathbb{D}^2}((\phi_{z_1},\phi_{w_1})(z_1,w_1),(\phi_{z_1},\phi_{w_1})(z_2,w_2)) \\ &= C_{\mathbb{D}^2}((0,0),(\phi_{z_1}(z_2),\phi_{w_1}(w_2))) \\ &= \sup\left\{\rho_{\mathbb{D}}(0,\phi_{z_1}(z_2)), \rho_{\mathbb{D}}(0,\phi_{w_1}(w_2))\right\} \text{ using equation (4.2.1)} \\ &= \sup\left\{\rho_{\mathbb{D}}(z_1,z_2), \rho_{\mathbb{D}}(w_1,w_2)\right\}, \end{aligned}$$

and finally we have the result. □

We now provide the example that illustrates the above claim, that in general $C_D \neq \tilde{C}_D$. Specifically, in this example $C_D < \tilde{C}_D$.

We will require the following useful relation:

$$\rho_{\mathbb{D}}(z,w) \geq \rho_{\mathbb{D}}(|z|,|w|) \tag{4.2.2}$$

for $z, w \in \mathbb{D}$.

Showing this relation, we have

$$\begin{aligned} \rho_{\mathbb{D}}(z,w) &= \rho_{\mathbb{D}}(|z|, e^{i\theta}|w|) \text{ for some } \theta \in \mathbb{R} \\ &= \tanh^{-1}\left|\frac{|z| - e^{i\theta}|w|}{1 - |z||w|e^{i\theta}}\right| \\ &\geq \tanh^{-1}\left|\frac{|z| - |w|}{1 - |z||w|}\right| \\ &= \rho_{\mathbb{D}}(|z|,|w|). \end{aligned}$$

The inequality in the above computation is arrived at as follows: recall that in the proof of the triangle inequality in proposition 2.4.1 we showed that

$$g(0) = \left(\frac{r-s}{1-rs}\right)^2 \leq \left|\frac{r - se^{i\beta}}{1 - rse^{i\beta}}\right|^2 = g(\beta)$$

for all $\beta \in [0, 2\pi)$ and real positive $r, s < 1$. So we simply let $r = |z|$ and $s = |w|$ to see that

$$\left|\frac{|z| - |w|}{1 - |z||w|}\right| \leq \left|\frac{|z| - |w|e^{i\theta}}{1 - |z||w|e^{i\theta}}\right|,$$

and the inequality follows.

Example 4.2.1. Consider the domain in two complex variables given by $\mathcal{D} = \mathbb{D}^2 \setminus \{(z, w) : |z| < r \text{ and } |w| > s\}$ for fixed r, s such that $0 < r < 1$ and $0 < s < 1$.

Using the Cauchy integral formula, a function holomorphic on $\mathcal{D}$ can be extended to the bidisc $\mathbb{D}^2$. Since the distinguished boundary of the bidisc is $\{(z, w) : |z| = 1, |w| = 1\}$, we obtain the extended function f using the Cauchy integral formula in two variables, to give

$$g(\zeta) = \frac{1}{(2\pi i)^2} \int_{|z|=r'} \int_{|w|=s'} \frac{f(z, w)}{(z - \zeta_1)(w - \zeta_2)} dw dz,$$

for $r < r' < 1$ and $0 < s' < 1$, since we integrate over the boundary of a slightly smaller bidisc, since f is not necessarily defined on the boundary. We will show that g coincides with f on the domain $\{(z, w) : |z| > r, |w| < s\}$. Considering the inside integral, we have a one-dimensional Cauchy integral. So for fixed z with $|z| = r' > r$, and all ζ_2 with $|\zeta_2| < s'$, we have

$$f(z, \zeta_2) = \frac{1}{2\pi i} \int_{|w|=s'} \frac{f(z, w)}{(w - \zeta_2)} dw.$$

So therefore

$$g(\zeta) = \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z, \zeta_2)}{(z - \zeta_1)} dz.$$

Now f is holomorphic for all (z, ζ_2) when $|\zeta_2| < s$. So we have

$$g(\zeta) = \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z, \zeta_2)}{(w - \zeta_1)} dz = f(\zeta),$$

so g is holomorphic. By the identity theorem, g coincides with f for all ζ. Letting r', s' approach 1, we see g is holomorphic for the entire bidisc. By the maximum modulus principle, we see that $|g(z, w)| < 1$, implying that g is onto the unit disc. So each $f \in H(D, \mathbb{D})$ can be extended to an element of $H(\mathbb{D}^2, \mathbb{D})$.

Consider $\rho_{\mathbb{D}}(s_0, s)$ as $s_0 \to 1$. We have

$$\begin{aligned} \rho_{\mathbb{D}}(s_0, s) &= \tanh^{-1}\left|\frac{s - s_0}{1 - s\overline{s_0}}\right| \\ &\to \tanh^{-1}\left|\frac{s - 1}{1 - s}\right| \\ &\to \infty \end{aligned} \tag{4.2.3}$$

as $s_0 \to 1$, since $\tanh^{-1} x \to \infty$ as $x \to 1$. So for a fixed r_0 such that $r < r_0 < 1$ we can choose an $s_0 > s$ such that $\rho_{\mathbb{D}}(s_0, s) > \rho_{\mathbb{D}}(r_0, 0)$. We let $p = (r_0, s_0)$ and $q = (-r_0, s_0)$, which by construction lie in $\mathcal{D}$. We will show that $\tilde{C}_D(p, q) > C_D(p, q)$.

We let $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \mathcal{D}$ denote a piecewise-smooth parametrisation of an admissible curve joining p and q in $\mathcal{D}$. Since the second variable is s_0 for both p and q, there are two possibilities to consider:

(a) First, when γ goes 'under' the hole in the annulus $r \leq |z| < 1$, we have $|\gamma_1(t)| < r$ for some $t \in [0,1]$. Further, since $\gamma(t) \in \mathcal{D}$, we must also have $|\gamma_2(t)| < s$ for the same $t \in [0,1]$. So for $|\gamma_1(t)| < r$ we have

$$\begin{aligned} L_{C_D}(\Gamma) &\geq C_{\mathcal{D}}(p, \gamma(t)) + C_{\mathcal{D}}(\gamma(t), q) \\ &= \sup_{f \in H(\mathcal{D},\mathbb{D})} \rho_{\mathbb{D}}(f(p), f(\gamma(t))) + \sup_{f \in H(\mathcal{D},\mathbb{D})} \rho_{\mathbb{D}}(f(\gamma(t)), f(q)) \\ &\geq \rho_{\mathbb{D}}(s_0, \gamma_2(t)) + \rho_{\mathbb{D}}(\gamma_2(t), s_0) \\ &\geq 2\rho_{\mathbb{D}}(|\gamma_2(t)|, s_0) \qquad \text{using inequality (4.2.2)} \\ &\geq 2\rho_{\mathbb{D}}(s, s_0), \qquad \text{since } |\gamma_2(t)| < s \end{aligned}$$

where we have taken the projection mapping $(z, w) \to w$ in the definition of the Carathéodory pseudodistance to conclude the second inequality.

(b) The second possibility is when γ goes around the hole, and so is entirely contained in the annulus $r \leq |z| < 1$, and we have $|\gamma_1(t)| \geq r$ for all $t \in [0,1]$. For $t \in \mathbb{R}$ and $r \leq t < 1$, we see that $z = it$ lies inside the annulus $r \leq |z| < 1$, since $|z| = |it| = t$. So we have

$$\begin{aligned} L_{C_D}(\Gamma) &\geq C_{\mathcal{D}}(p, \gamma(t)) + C_{\mathcal{D}}(\gamma(t), q) \\ &= \sup_{f \in H(\mathcal{D},\mathbb{D})} \rho_{\mathbb{D}}(f(p), f(\gamma(t))) + \sup_{f \in H(\mathcal{D},\mathbb{D})} \rho_{\mathbb{D}}(f(\gamma(t)), f(q)) \\ &\geq \rho_{\mathbb{D}}(r_0, \gamma_1(t)) + \rho_{\mathbb{D}}(\gamma_1(t), -r_0) \\ &\geq \inf_{r \leq t < 1} \left\{ \rho_{\mathbb{D}}(r_0, it) + \rho_{\mathbb{D}}(it, -r_0) \right\}, \end{aligned}$$

where, in a similar manner to the previous computation, we take the projection mapping $(z, w) \to z$ to conclude the second inequality.

The infimum in the above expression is realised since $\rho_{\mathbb{D}}$ is continuous, and because, by a similar calculation to (4.2.3) above, $\rho_{\mathbb{D}}(r_0, it) \to \infty$ as $t \to 1$. Hence there exists a t_0 with $r \leq t_0 < 1$, such that

$$\begin{aligned} L_{C_D}(\Gamma) &\geq \rho_{\mathbb{D}}(r_0, it_0) + \rho_{\mathbb{D}}(it_0, -r_0) \\ &= 2\rho_{\mathbb{D}}(r_0, it_0). \end{aligned}$$

Since $\rho_{\mathbb{D}}(r_0, it) \to \infty$ as $t \to 1$, we can choose $t = t_0$ such that $\rho_{\mathbb{D}}(r_0, it_0) > \rho_{\mathbb{D}}(r_0, 0)$.

So finally, we have

$$\begin{aligned}
\tilde{C_D}(p,q) &= \inf\{L_{C_D}(\Gamma) : \Gamma \text{ an admissible curve joining } p \text{ and } q\} \\
&\geq 2\inf\{\rho_{\mathbb{D}}(s,s_0), \rho_{\mathbb{D}}(r_0, it_0)\} \\
&> 2\rho_{\mathbb{D}}(r_0, 0) \\
&= \rho_{\mathbb{D}}(r_0, -r_0) \text{ since } r_0 \in \mathbb{R} \\
&= \sup\{\rho_{\mathbb{D}}(r_0,-r_0), \rho_{\mathbb{D}}(s_0,s_0)\} \text{ by Proposition (4.2.3)} \\
&= C_{\mathbb{D}^2}((r_0,s_0),(-r_0,s_0)) \\
&= C_{\mathcal{D}}(p,q).
\end{aligned}$$

So we have shown that the Carathéodory pseudodistance C_D is not an inner pseudodistance in general. Specifically, we have shown that $C_D < \tilde{C}_D$ for some pairs of points. It follows from the above that $C_{\mathcal{D}} < c^i_{\mathcal{D}}$ for some pairs of points. We show one final result in this chapter, namely that the inner Carathéodory pseudodistance is equal to the integrated form of the infinitesimal Carathéodory pseudometric, that is $\tilde{C}_D = c^i_D$. Conceptually, this could be done by integrating c_D, however, we will instead show the result by differentiating C_D. In order to do so, we now make precise the idea of a derivative of a pseudodistance.

Definition 4.2.2. *A continuous pseudodistance ρ on a domain D is differentiable or $\mathcal{C}^1$ if there is a non-negative real-valued function α on $D \times \mathbb{C}^n$ such that*

(i) *$\alpha(p,tv) = |t|\alpha(p,v)$ for all $(p,v) \in D \times \mathbb{C}^n$ and $t \in \mathbb{C}$,*

(ii) *for every $u \in D$ and $\epsilon > 0$, there exists $\delta > 0$ such that the open ball $B_\delta(u) \subset D$ and*

$$|\rho(x+v,x) - \alpha(x,v)| \leq \epsilon\|v\| \tag{4.2.4}$$

when $\|x-u\| < \delta$ and $\|v\| < \delta$.

Then α is the *derivative* of ρ, and it follows from the above definition that

$$\alpha(x,v) = \lim_{t\to 0} \frac{\rho(x+tv,x)}{|t|}. \tag{4.2.5}$$

To see this, we fix x and v, and let $|t| < \frac{\epsilon}{\|v\|}$. Then in equation (4.2.4) we have

$$\begin{aligned}
&|\rho(x+tv,x) - \alpha(x,tv)| \leq \epsilon\|tv\| \\
\iff &|\rho(x+tv,x) - |t|\alpha(x,v)| \leq |t|\epsilon\|v\| \\
\iff &|\frac{\rho(x+tv,x)}{|t|} - \alpha(x,v)| \leq \epsilon\|v\|,
\end{aligned}$$

and we see the result.

Condition (i) in the above definition is identical to condition (i) for an Finsler pseudometric. Recall that the second condition was that α be upper semicontinuous. Since the above limit exists locally uniformly in both x and v, it follows that this condition is also satisfied, and α is a continuous Finsler pseudometric.

Before proving that the derivative of C_D is c^i_D, we need to present an important proposition relating the ρ-length and α-length of an admissible curve in D.

Proposition 4.2.4. *If ρ is a $\mathcal{C}^1$ pseudodistance with derivative α on a domain D then $L_\rho(\gamma) = L_\alpha(\gamma)$ for every admissible curve γ in D.*

Proof. We will assume without loss of generality that γ admits a $\mathcal{C}^1$ parametrisation $\gamma : [0,1] \to \gamma$. For r, s and $t \in [0,1]$ such that $r < s$ we have the following relation,

$$\|\gamma(s) - \gamma(r) - \gamma'(t)(s-r)\| \leq (s-r) \sup_{r \leq \theta \leq s} \|\gamma'(\theta) - \gamma'(t)\|, \tag{4.2.6}$$

which can be seen from the following computation:

$$\begin{aligned} \|\gamma(s) - \gamma(r) - \gamma'(t)(s-r)\| &= (s-r) \left\| \frac{\gamma(s) - \gamma(r)}{s-r} - \gamma'(t) \right\| \\ &= (s-r)\|\gamma'(\theta) - \gamma'(t)\| \end{aligned}$$

for some θ between r and s. Taking the supremum over θ, we have the relation. Using this relation, along with (4.2.5) and the fact that α is continuous on D, it follows that the function ψ on $[0,1] \times [0,1]$ defined by

$$\psi(s,t) = \begin{cases} \alpha(\gamma(s), \gamma'(s)) - \frac{\rho(\gamma(s),\gamma(t))}{|s-t|} & \text{if } s \neq t \\ 0 & \text{if } s = t \end{cases}$$

is continuous and hence uniformly continuous.

To see this, let us consider the second term in the above equation when $s \neq t$, and in particular the expression $\rho(\gamma(s), \gamma(t))$. We see this can be written $\rho(\gamma(s), \gamma(t)) = \rho(\gamma(t) + (\gamma(s) - \gamma(t)), \gamma(t))$. Therefore, by setting $x = \gamma(t)$ and $v = \gamma(s) - \gamma(t)$, and substituting into equation (4.2.4) above, we have

$$|\rho(\gamma(s), \gamma(t)) - \alpha(\gamma(t), \gamma(s) - \gamma(t))| \leq \epsilon \|\gamma(s) - \gamma(t)\|$$

for s and t sufficiently close, and $s \neq t$, so that

$$\left| \frac{\rho(\gamma(s), \gamma(t))}{|s-t|} - \alpha\left(\gamma(t), \frac{\gamma(s) - \gamma(t)}{|s-t|}\right) \right| \leq \epsilon \left\| \frac{\gamma(s) - \gamma(t)}{|s-t|} \right\|, \tag{4.2.7}$$

for $s \neq t$. Now, using equation (4.2.6) above, we have

$$\left\| \frac{\gamma(s) - \gamma(t)}{s-t} - \gamma'(a) \right\| \leq \sup_{t \leq \theta \leq s} \|\gamma'(\theta) - \gamma'(a)\|,$$

for some value a. So then, as $s, t \to a$, we have $\frac{\gamma(s)-\gamma(t)}{s-t} \to \gamma'(a)$, and so by equation (4.2.7) above, we have

$$\left| \frac{\rho(\gamma(s), \gamma(t))}{(s-t)} - \alpha\left(\gamma(t), \frac{\gamma(s) - \gamma(t)}{(s-t)}\right) \right| \to 0$$

as $s, t \to a$. But

$$\alpha\left(\gamma(t), \frac{\gamma(s) - \gamma(t)}{s-t}\right) \to \alpha(\gamma(a), \gamma'(a))$$

as $s, t \to a$. Therefore, we have

$$\left|\frac{\rho(\gamma(s), \gamma(t))}{(s-t)} - \alpha(\gamma(s), \gamma'(s))\right| \to 0$$

as $s, t \to a$. So we see that $\psi(s, t) \to 0$ as $s, t \to a$, showing the continuity of ψ when $s = t$. Clearly, continuity holds when $s \neq t$, and we have the result.

So we see that $\psi(s, t)$ is continuous and hence uniformly continuous. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $|\psi(s, t)| < \epsilon$ when $s, t \in [0, 1]$ and $|s - t| < \delta$.

Now, we choose δ_1 such that

$$\left|L_\alpha(\Gamma) - \sum_{k=1}^{n} \alpha(\gamma(t_k), \gamma'(t_k))(t_k - t_{k-1})\right| < \epsilon$$

where $t_0, ..., t_n$ is a partition of $[0, 1]$ satisfying $t_k - t_{k-1} < \delta$ for $k = 1, ..., n$. For any such partition, we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n} \alpha(\gamma(t_k), \gamma'(t_k))(t_k - t_{k-1}) - \sum_{k=1}^{n} \rho(\gamma(t_k), \gamma(t_{k-1}))\right| \\
&\quad = \left|\sum_{k=1}^{n} \psi(t_k, t_{k-1})(t_k - t_{k-1})\right| \\
&\quad < \epsilon
\end{aligned}$$

by the definition of ψ above. By refining this partition if necessary, we can also suppose

$$\left|L_\rho(\Gamma) - \sum_{k=1}^{n} \rho(\gamma(t_k), \gamma(t_{k-1}))\right| < \epsilon.$$

Now, after applying the triangle inequality and combining the above three inequalities we see that

$$\begin{aligned}
|L_\alpha(\Gamma) - L_\rho(\Gamma)| \leq &\left|L_\alpha(\Gamma) - \sum_{k=1}^{n} \alpha(\gamma(t_k), \gamma'(t_k))(t_k - t_{k-1})\right| \\
&+ \left|\sum_{k=1}^{n} \alpha(\gamma(t_k), \gamma'(t_k))(t_k - t_{k-1}) - \sum_{k=1}^{n} \rho(\gamma(t_k), \gamma(t_{k-1}))\right| \\
&+ \left|\sum_{k=1}^{n} \rho(\gamma(t_k), \gamma(t_{k-1})) - L_\rho(\Gamma)\right| \\
< 3\epsilon &
\end{aligned}$$

and this computation shows the result, since ϵ was arbitary. So we have $L_\alpha(\Gamma) = L_\rho(\Gamma)$. $\square$

Now finally, we show that the derivative of C_D is given by c_D. In the proof, we will use the following proposition.

Proposition 4.2.5. *The infinitesimal Carathéodory pseudodistance c_D is continuous and locally Lipschitz. Further, for $x, v \in D$ and $B_r(x) \subset D$, if $c_D(y,v) \geq c_D(z,v)$ then we have*

$$|c_D(y,v) - c_D(z,v)| \leq \left(\frac{2}{r}\right)^2 \|y-z\| \cdot \|v\|$$

for all $y, z \in B_{r/4}(x)$.

Proof. This fact was established in the proof of Proposition 4.1.2, where we showed that c_D was a continuous Finsler pseudometric. □

Now we show that the derivative of C_D is given by c_D.

Proposition 4.2.6. *On a domain D the Carathéodory pseudodistance C_D is a $\mathcal{C}^1$ pseudodistance with derivative c_D. Furthermore*

$$L_{C_D}(\Gamma) = L_{\tilde{C}_D}(\Gamma) = L_{c^i_D}(\Gamma)$$

for any admissible curve Γ and $\left\{\tilde{C}_D\right\} = \{c^i_D\}$ form the largest Schwarz-Pick system of inner pseudodistances.

Proof. For a fixed $u \in D$, we choose $r > 0$ such that $B_{2r}(u) \subset D$. Let $x \in B_r(u)$. By Proposition 4.2.5 above, we have

$$|c_D(y,v) - c_D(z,v)| \leq \frac{4}{r^2}\|y-x\| \cdot \|v\| \tag{4.2.8}$$

for all $y \in B_{r/2}(x)$.

Let Γ denote the curve with parametrisation $\gamma(t) = x + tv$, with $0 \leq t \leq 1$ and $\|v\| < r$. Note that the endpoints of this curve are given by $\gamma(0) = x$ and $\gamma(1) = x+v$. That is, from the point x to a point an infinitesimal length away in the direction of the tangent vector v. Considering the Carathéodory pseudodistance between these two points, we have

$$
\begin{aligned}
C_D(x+v,x) &\le L_{C_D}(\Gamma) \\
&\le L_{c_D^i}(\Gamma) \\
&= \sup \sum_{j=1}^{k} c_D^i(\gamma(t_{j-1}),\gamma(t_j)) \text{ where } t_0=0 \text{ and } t_k=1 \\
&= \sup \sum_{j=1}^{k} \left(\inf_{\gamma} \int_{t_{j-1}}^{t_j} c_D(\gamma(t),\gamma'(t))dt \right) \\
&= \sup \sum_{j=1}^{k} \left(\int_{t_{j-1}}^{t_j} c_D(x+tv,v)dt \right) \\
&\le \int_0^1 c_D(x+tv,v)dt \\
&= c_D(x,v) - c_D(x,v) + \int_0^1 c_D(x+tv,v)dt \\
&= c_D(x,v) - \int_0^1 c_D(x,v)dt + \int_0^1 c_D(x+tv,v)dt \\
&= c_D(x,v) + \int_0^1 [c_D(x+tv,v) - c_D(x,v)]dt \\
&\le c_D(x,v) + \int_0^1 |c_D(x+tv,v) - c_D(x,v)|dt \\
&\le c_D(x,v) + \int_0^1 \frac{4}{r^2}\|tv\| \cdot \|v\| dt \\
&= c_D(x,v) + \frac{4\|v\|^2}{r^2},
\end{aligned}
$$

where the first two equalities in the above computation simply follow from the definitions of $L_{c_D^i}$ and c_D^i. The final inequality follows from substituting $x+tv$ in for y in (4.2.8).

Therefore, by the computation above, we have the following relation:

$$
C_D(x+v,x) \le \tilde{C}_D(x+v,x) \le c_D^i(x+v,x) \le c_D(x,v) + \frac{4\|v\|^2}{r^2}. \tag{4.2.9}
$$

On the other hand, if $g \in H(D,\mathbb{D})$ and $g(x)=0$, and using the fact that

$$
\begin{aligned}
g(x+v) &= \sum_{|k|=0}^{\infty} \frac{d^k g(x)}{k!} v^k \\
&= g(x) + dg(x)(v) + \sum_{|k|=2}^{\infty} \frac{d^k g(x)}{k!} v^k,
\end{aligned}
$$

then we have

$$\begin{aligned}
|g(x+v)-g(x)-dg(x)(v)| &= \sum_{|k|=2}^{\infty} \left| \frac{1}{k_1!...k_m!} \frac{d^k g}{\partial z_1^{k_1}...\partial z_m^{k_m}} v_1^{k_1}...v_m^{k_m} \right| \\
&\leq \sum_{|k|=2}^{\infty} \left| \frac{1}{k_1!...k_m!} \right| \left| \frac{d^k g}{\partial z_1^{k_1}...\partial z_m^{k_m}} \right| (|v_1|+...+|v_m|)^k \\
&\leq \sum_{|k|=2}^{\infty} \frac{1}{k!} \left(\max_k \left| \frac{d^k g}{\partial z_1^{k_1}...\partial z_m^{k_m}} \right| \right) \left(m \max_{1\leq j\leq m} |v_j| \right)^k \\
&= \sum_{|k|=2}^{\infty} \frac{1}{k!} \|d^k g(x)\| \cdot c^k \|v\|^k \\
&\leq \sum_{|k|=2}^{\infty} \frac{1}{k!} \left(\frac{k!}{r^k} \right) \cdot c^k \|v\|^k \\
&= \sum_{|k|=2}^{\infty} \left(\frac{c\|v\|}{r} \right)^k
\end{aligned}$$

where the final inequality follows from the Cauchy estimates. Note also that since m is an arbitrary index, which is constant, we have relabelled it c in the above computation. For the above geometric series to converge, we must have the following hold:

$$\left(\frac{c\|v\|}{r} \right)^2 \geq \sum_{|k|=3}^{\infty} \left(\frac{c\|v\|}{r} \right)^k,$$

So we see that

$$\sum_{|k|=2}^{\infty} \left(\frac{c\|v\|}{r} \right)^k \leq 2 \left(\frac{c\|v\|}{r} \right),$$

implying

$$|g(x+v)-g(x)-dg(x)(v)| \leq 2 \left(\frac{c\|v\|}{r} \right).$$

Hence, as $g(x)=0$, we have

$$\begin{aligned}
|g(x+v)-g(x)-dg(x)(v)| \leq 2\frac{c^2\|v\|^2}{r^2} &\implies |dg(x)(v)-g(x+v)| \leq 2\frac{c^2\|v\|^2}{r^2} \\
&\implies ||dg(x)(v)|-|g(x+v)|| \leq 2\frac{c^2\|v\|^2}{r^2} \\
&\implies |dg(x)(v)| \leq |g(x+v)| + 2\frac{c^2\|v\|^2}{r^2}.
\end{aligned}$$

So we see that

$$
\begin{aligned}
|dg(x)(v)| &\le |g(x+v)| + 2\frac{c^2\|v\|^2}{r^2} \\
&\le \tanh\left(\sup\left\{\tanh^{-1}|g(x+v)| : g \in H(D,\mathbb{D}), g(x)=0\right\}\right) + 2\frac{c^2\|v\|^2}{r^2} \\
&= \tanh\left(\sup\left\{\rho_{\mathbb{D}}(g(x+v),0) : g \in H(D,\mathbb{D}), g(x)=0\right\}\right) + 2\frac{c^2\|v\|^2}{r^2} \\
&= \tanh\left(\sup\left\{\rho_{\mathbb{D}}(g(x+v),g(x)) : g \in H(D,\mathbb{D}), g(x)=0\right\}\right) + 2\frac{c^2\|v\|^2}{r^2} \\
&= \tanh C_D(x+v,x) + 2\frac{c^2\|v\|^2}{r^2} \\
&\le C_D(x+v,x) + 2\frac{c^2\|v\|^2}{r^2}
\end{aligned}
$$

since $\tanh x \le x$ for all $x \ge 0$.

So taking the supremum of the left-had side gives us the infinitesimal Carathéodory pseudometric (by definition), and we see that

$$c_D(x,v) - 2\frac{c^2\|v\|^2}{r^2} \le C_D(x+v,x).$$

This implies also that

$$c_D(x,v) - 4\frac{c^2\|v\|^2}{r^2} \le C_D(x+v,x), \tag{4.2.10}$$

and so we see that

$$
\begin{aligned}
c_D(x,v) - 4\frac{c^2\|v\|^2}{r^2} &\le C_D(x+v,x) \le \tilde{C}_D(x+v,x) \\
&\le c_D^i(x+v,x) \le c_D(x,v) + 4\frac{c^2\|v\|^2}{r^2}.
\end{aligned}
$$

since $c \ge 1$.

By (4.2.9), and (4.2.10) above, for $x \in B_r(u)$ and $\epsilon > 0$ such that $\|v\| \le \frac{\epsilon r^2}{4c^2}$ (ie. so that $\epsilon \ge 4\frac{c^2\|v\|}{r^2}$), we have

$$
\begin{aligned}
c_D(x,v) - \epsilon\|v\| &\le C_D(x+v,x) \le \tilde{C}_D(x+v,x) \\
&\le c_D^i(x+v,x) \le c_D(x,v) + \epsilon\|v\|.
\end{aligned}
$$

Hence,

$$|C_D(x+v,v) - c_D(x,v)| \le \epsilon\|v\|$$

for $x \in B_r(u)$ and $v \in B_{\epsilon r^2/4c^2}(0)$.

So we see by (4.2.4) that this implies that C_D is a $\mathcal{C}^1$ pseudodistance with derivative c_D.

By direct application of Proposition 4.2.4, we have $L_{C_D}(\Gamma) = L_{c_D}(\Gamma)$. Taking the infimum of both sides over all admissible curves Γ, we have $\tilde{C}_D = c_D^i$. Together with Proposition 4.2.1 we have the result, that

$$L_{C_D}(\Gamma) = L_{\tilde{C}_D}(\Gamma) = L_{c_D^i}(\Gamma).$$

for any admissible curve Γ.

Since $\left\{\tilde{C}_D\right\}$ is a Schwarz-Pick system of inner pseudodistances, and $\{C_D\}$ is the smallest Schwarz-Pick system, it follows that $\left\{\tilde{C}_D\right\}$ is the smallest Schwarz-Pick system of inner pseudodistances. $\square$

Recall that the Carathéodory pseudodistances were defined with the use of holomorphic mappings from an arbitrary domain D into $\mathbb{D}$. We now consider pseudodistances that can be constructed using holomorphic mappings from $\mathbb{D}$ into an arbitrary domain D, and determine whether these pseudodistances form a Schwarz-Pick system. We now introduce the Kobayashi pseudodistance.

Chapter 5

The Kobayashi pseudodistance

For two points z, w in an arbitrary domain D, we define a *chain* from z to w as a collection of points such that $z = p_0, p_1, ..., p_{k-1}, p_k = w$. Further, we choose points $a_1, ..., a_k, b_1, ..., b_k$ in $\mathbb{D}$, and holomorphic mappings $f_1, ..., f_k$ from $\mathbb{D}$ into D such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for all $i = 1, ..., k$. Note that we can always find such a chain due to the connectedness of D.

The Kobayashi pseudodistance is then defined as follows.

Definition 5.0.1. *The Kobayashi pseudodistance function is given by*

$$K_D(z,w) = \inf \left\{ \sum_{i=1}^{k} \rho_{\mathbb{D}}(a_i, b_i) : \{z = p_0, p_1, ..., p_{k-1}, p_k = w\} \subset D \right\}$$

for $z, w \in D$.

Clarifying this definition, we have points $a_1, ..., a_k, b_1, ..., b_k$ in $\mathbb{D}$ that are mapped to D by the holomorphic mappings $f_1, ..., f_k$. The f_i map these points to points in D that make up a chain between z and w. To determine the Kobayashi pseudodistance, we take the quantity

$$\rho_{\mathbb{D}}(a_1, b_1) + ... + \rho_{\mathbb{D}}(a_k, b_k),$$

and take the infimum over all chains from z to w.

It is straightforward to show that K_D is a pseudodistance, by showing the three axioms are satisfied.

Proposition 5.0.1. *K_D is a pseudodistance on D.*

Proof. In common with the proof that C_D is a pseudodistance on D, we utilise the fact that $\rho_{\mathbb{D}}$ satisfies the pseudodistance axioms, and use the corresponding relations. Considering each axiom, we have:

(i) Since $\rho_{\mathbb{D}}(p,q) \geq 0$ for any p, q in $\mathbb{D}$, then clearly the quantity $\inf\{\rho_{\mathbb{D}}(a_1, b_1) + ... + \rho_{\mathbb{D}}(a_k, b_k)\} \geq 0$ for any a_i, b_i in $\mathbb{D}$.

(ii) Taking the infimum over all chains from z to w is clearly equivalent to taking the infimum of all chains from w to z, and so we have $K_D(z,w) = K_D(w,z)$.

(iii) As for the triangle inequality, consider $z, w, v \in D$ and the chains $\{z = p_0, p_1, ..., p_{k-1}, p_k = v\}$ and $\{v = q_0, q_1, ..., q_{k-1}, q_k = w\}$, both of which are in D. Also consider the holomorphic mappings $f_1, ..., f_k$ from $\mathbb{D}$ into D such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for all $i = 1, ..., k$, and the holomorphic mappings $g_1, ..., g_k$ from $\mathbb{D}$ into D such that $g_i(c_i) = q_{i-1}$ and $g_i(d_i) = q_i$ for all $i = 1, ..., k$. Then we have

$$K_D(z, w) \leq \sum_{i=1}^{m} \rho_{\mathbb{D}}(a_i, b_i) + \sum_{j=1}^{n} \rho_{\mathbb{D}}(c_j, d_j).$$

Taking the infimum of the first sum over the chains $\{z = p_0, p_1, ..., p_{k-1}, p_k = v\}$, and the infimum of the second sum over the chains $\{v = q_0, q_1, ..., q_{k-1}, q_k = w\}$, we have

$$K_D(z, w) \leq K_D(z, v) + K_D(v, w),$$

showing that the triangle inequality is satisfied. □

The Kobayashi pseudodistance can be described with the use of the following function:

$$\delta_D(z, w) = \inf \{\rho_{\mathbb{D}}(u, v) : f \in H(\mathbb{D}, D), f(u) = z, f(v) = w\},$$

for $z, w \in D$.

We introduce this function as an aid in understanding the Kobayashi pseudodistance conceptually, and because it helps to make some upcoming calculations easier. The Kobayashi pseudodistance is described with the use of the δ function as follows:

Proposition 5.0.2. *The Kobayashi pseudodistance function is given by*

$$K_D(z, w) = \inf \left\{ \sum_{i=1}^{n} \delta_D(w_i, w_{i+1}) : n, \{z = w_1, ..., w_{n+1} = w\} \subset D \right\}$$

for $z, w \in D$.

Note that the infimum is taken over all n, and over all chains from z to w. With this notation, we avoid the need to define multiple functions from $\mathbb{D}$ to D, and the a_i, b_i notation.

Utilising this definition, we will now show that the Kobayashi pseudodistances form a Schwarz-Pick system.

Proposition 5.0.3. *The Kobayashi pseudodistances K_D form a Schwarz-Pick system.*

Proof. We show that the two axioms for a Schwarz-Pick system of pseudodistances are satisfied.

(i) We have to show that $K_{\mathbb{D}} = \rho_{\mathbb{D}}$ for $z, w \in \mathbb{D}$. We do so by showing that $K_{\mathbb{D}} \leq \rho_{\mathbb{D}}$ and $\rho_{\mathbb{D}} \leq K_{\mathbb{D}}$.

If, in the definition of δ,

$$\delta_{\mathbb{D}}(z, w) = \inf \{\rho_{\mathbb{D}}(u, v) : f \in H(\mathbb{D}, \mathbb{D}), f(u) = z, f(v) = w\},$$

we take the mapping f to be the identity mapping $f(\zeta) = \zeta$ for any $\zeta \in \mathbb{D}$, we see that

$$\delta_{\mathbb{D}}(z, w) \leq \rho_{\mathbb{D}}(z, w).$$

Since $K_{\mathbb{D}}$ is the infimum of $\delta_{\mathbb{D}}$ taken over all chains from z to w, we have $K_{\mathbb{D}} \leq \rho_{\mathbb{D}}$.

Now, showing the reverse, consider $u, v \in \mathbb{D}$. For any f such that $f \in H(\mathbb{D}, \mathbb{D})$ and $f(u) = z$ and $f(v) = w$, we have

$$\begin{aligned} \rho_{\mathbb{D}}(z, w) &= \rho_{\mathbb{D}}(f(u), f(v)) \\ &\leq \rho_{\mathbb{D}}(u, v), \end{aligned}$$

since f is distance-decreasing with respect to the Poincaré distance. Therefore, we can conclude that

$$\begin{aligned} \rho_{\mathbb{D}}(z, w) &\leq \inf \{\rho_{\mathbb{D}}(u, v) : f \in H(\mathbb{D}, \mathbb{D}), f(u) = z, f(v) = w\} \\ &\leq \delta_{\mathbb{D}}(z, w) \end{aligned}$$

showing that $\rho_{\mathbb{D}} \leq \delta_{\mathbb{D}}$. Since the Poincaré distance satisfies the triangle inequality, then $\delta_{\mathbb{D}} = K_{\mathbb{D}}$. So we have $\rho_{\mathbb{D}} \leq K_{\mathbb{D}}$.

So we conclude that $K_{\mathbb{D}} = \rho_{\mathbb{D}}$ for $z, w \in \mathbb{D}$.

Now, it remains to show condition (ii), that K is distance-decreasing from D_1 into D_2, that is, $K_{D_2}(f(z), f(w)) \leq K_{D_1}(z, w)$. In a similar fashion to the above calculations, we firstly show that $\delta_{D_2}(f(z), f(w)) \leq \delta_{D_1}(z, w)$.

For two arbitrary domains D_1 and D_2, and holomorphic $f \in H(D_1, D_2)$, we have the following: for $z, w \in D_1$, consider a holomorphic function $g \in H(\mathbb{D}, D_1)$ such that $g(u) = z$ and $g(v) = w$. Then we have $f \circ g \in H(\mathbb{D}, D_2)$, with $(f \circ g)(u) = f(z)$ and $(f \circ g)(v) = f(w)$.

Noting that in the definition of δ_{D_2}, the infimum is taken over all holomorphic mappings from $\mathbb{D}$ into D_2, we therefore have

$$\delta_{D_2}(f(z), f(w)) \leq \rho_{\mathbb{D}}(u, v),$$

Now, taking the infimum of $\rho_{\mathbb{D}}(u, v)$ over all holomorphic mappings from $\mathbb{D}$ into D_1, we see that

$$\delta_{D_2}(f(z), f(w)) \leq \delta_{D_1}(z, w). \tag{5.0.1}$$

So it is finally left to show that the above inequality implies that $K_{D_2}(f(z), f(w)) \leq K_{D_1}(z, w)$. To see this, simply note that if $\{z = w_1, ..., w_{n+1} = w\}$ is a chain from z to w in D_1, then $\{f(z) = f(w_1), ..., f(w_{n+1}) = f(w)\}$ is a chain from $f(z)$ to $f(w)$ in D_2. Therefore, we have

$$\begin{aligned} K_{D_2}(f(z), f(w)) &\leq \sum_{i=1}^{n} \delta_{D_2}(f(w_i), f(w_{i+1})) \\ &\leq \sum_{i=1}^{n} \delta_{D_1}(w_i, w_{i+1}), \end{aligned}$$

where the second inequality follows from the above inequality (5.0.1). By taking the infimum of the right-hand side over all holomorphic mappings from $\mathbb{D}$ into D_1, finally we have $K_{D_2}(f(z), f(w)) \leq K_{D_1}(z, w)$, which shows that the Kobayashi pseudodistances K_D form a Schwarz-Pick system. □

As the above calculation shows, K_D is distance-decreasing for holomorphic mappings, and further, is invariant for biholomorphic mappings. This can be realised by employing an identical argument to that used for the Carathéodory pseudodistance. So K_D shares these properties with the Poincaré distance $\rho_{\mathbb{D}}$ and the Carathéodory pseudodistance C_D. We now show that the Kobayashi pseudodistances form the largest Schwarz-Pick system, in the sense that for any arbitrary pseudodistance ρ_D on a domain D, we have $K_D \geq \rho_D$ for all pairs of points.

Proposition 5.0.4. *The Kobayashi pseudodistances form the largest Schwarz-Pick system.*

Proof. We show this using property (ii) for Schwarz-Pick systems, which we remind the reader is given by the following:

If ρ_1 and ρ_2 are the pseudodistances assigned to D_1 and D_2 respectively, and $f \in H(D_1, D_2)$, then

$$\rho_2(f(z), f(w)) \leq \rho_1(z, w).$$

Now beginning the proof, let $\{\rho_D\}$ be an arbitrary Schwarz-Pick system of pseudodistances. Consider $z, w \in D$, where D is an arbitrary domain, and $u, v \in \mathbb{D}$. Consider also the holomorphic mapping $f \in H(\mathbb{D}, D)$ such that $f(u) = z$ and $f(v) = w$. Then we have

$$\rho_D(z, w) \leq \rho_{\mathbb{D}}(u, v).$$

By simply taking the infimum of the right-hand side over all holomorphic mappings f, we see that

$$\rho_D(z, w) \leq \delta_D(z, w).$$

By taking the infimum of the right-hand side once more, this time with respect to all chains from z to w, we have

$$\rho_D(z, w) \leq K_D(z, w),$$

showing that $\{K_D\}$ forms the largest Schwarz-Pick system of pseudodistances. □

Recall that for an arbitrary Schwarz-Pick system $\{\rho_D\}$, $\{\tilde{\rho}_D\}$ is also a Schwarz-Pick system, and $\rho_D \leq \tilde{\rho}_D$ for every domain D. Therefore, $K_D \leq \tilde{K}_D$, and since $\{K_D\}$ is the largest Schwarz-Pick system, we must have $K_D = \tilde{K}_D$ for all D.

Before we introduce the infinitesimal Kobayashi pseudometric, we make one final point about the Kobayashi pseudodistance, namely that K_D is not always a true distance. It shares this property with the Carathéodory pseudodistance. We take two distinct points $z, w \in \mathbb{C}$, and show that we can find certain holomorphic mappings such that $K_{\mathbb{C}}(z, w) = 0$. Consider the mappings $f_n \in H(\mathbb{D}, \mathbb{C})$ given by

$$f_n(\lambda) = z + n\lambda(w - z).$$

Then in the definition of $\delta_{\mathbb{C}}$, we have $f_n(u) = z$ and $f_n(v) = w$, implying that $nu(w-z) = 0$ and $(nv-1)(w-z) = 0$. So either $w = z$, or $u = 0$ and $v = \frac{1}{n}$.

We cannot have $w = z$, since z and w are distinct points. Therefore we must have $u = 0$ and $v = \frac{1}{n}$. So after taking the infimum of $\rho_{\mathbb{D}}$ over these f_n, we get

$$\begin{aligned}\delta_{\mathbb{C}}(z,w) &= \inf_n \rho_{\mathbb{D}}(0, 1/n) \\ &= 0,\end{aligned}$$

showing that $K_{\mathbb{C}} \equiv 0$. Thus we have shown that the Kobayashi pseudodistance is not a true distance.

We show one more relation in this section. Consider the mapping $\phi : \mathbb{D} \to B_r(x) \subset D$ given by

$$\phi(\lambda) \to x + r\frac{(y-x)}{\|y-x\|}\lambda.$$

Note that $\phi(0) = x$, and $\phi(\frac{\|y-x\|}{r}) = y$. Then for $x, y \in D$ we have

$$\begin{aligned}K_D(x,y) &\leq \delta_D(x,y) \\ &\leq \rho_{\mathbb{D}}\left(0, \frac{\|y-x\|}{r}\right) \\ &= \tanh^{-1}\left(\frac{\|y-x\|}{r}\right) \\ &\leq \frac{\|y-x\|}{r}\end{aligned}$$

since $\tanh^{-1} x \leq x$ for small, positive x. This shows that $K_D(x,y) \leq \frac{1}{r}\|y-x\|$, and hence that K_D is locally bounded by the norm. A similar argument (see [1, pp. 47-48]) shows that the Carathéodory pseudodistance C_D is locally bounded by the norm.

5.1 The Kobayashi-Royden pseudometric

The *infinitesimal Kobayashi pseudometric* was first introduced by Kobayashi in 1967, and comprehensively studied by Royden in the years after its introduction. For this reason, it is often called the *Kobayashi-Royden* pseudometric. In common with the other infinitesimal forms we have looked at, it is defined for a point in the domain and vector in the tangent space.

Definition 5.1.1. *For a domain D and $(p,v) \in D \times \mathbb{C}^n$, the infinitesimal Kobayashi pseudometric k_D is given by*

$$k_D(p,v) = \inf\{\eta > 0; \exists f \in H(\mathbb{D}, D), f(0) = p, df(0)(\eta) = v\}$$

for $(p,v) \in D \times \mathbb{C}^n$.

We prove two important propositions before showing the main conclusion in this section, namely that for any domain D the Kobayashi pseudodistance is equal to the integrated form of the Kobayashi-Royden pseudometric, that is, $K_D = k_D^i$.

Proposition 5.1.1. *The Kobayashi-Royden pseudometric is a Finsler pseudometric.*

Proof. To show that k_D is an infinitesimal Finsler pseudometric, we need to show that each of the two conditions in definition (3.1.1) are satisfied. Condition (i) says we must have $k_D(p, tv) = |t| k_D(p, v)$ for $(p, v) \in D \times \mathbb{C}^n$ and $t \in \mathbb{C}$. Considering this, we have

$$\begin{aligned} k_D(p, tv) &= \inf \{\eta > 0; \exists f \in H(\mathbb{D}, D), f(0) = p, df(0)(\eta) = tv\} \\ &= |t| k_D(p, v), \end{aligned}$$

since we must have $t \in \mathbb{R}^+$.

To show upper semicontinuity, condition (ii), let D be a domain. For $\lambda \in \mathbb{R}^+$ let $W_\lambda = \{(p, v) \in T(D) : k_D(p, v) < \lambda\}$. We show upper semicontinuity by showing that W_λ is open in the tangent bundle $T(D)$ for all λ.

Let $(p, v) \in W_\lambda$. There there exists η with $0 < \eta < \lambda$, and $f \in H(\mathbb{D}, D)$ such that $f(0) = p$ and $df(0)(\eta) = v$. For r such that $0 < r < 1$, consider the function $f_r : \overline{\mathbb{D}} \to X$ given by

$$f_r(z) = f(rz).$$

Then clearly $f_r\big|_{\mathbb{D}} \in H(\mathbb{D}, D)$, f_r is continuous on $\overline{\mathbb{D}}$, and $df_r(0)(\frac{\eta}{r}) = v$. To see this, observe that

$$\begin{aligned} df_r(0)\left(\frac{\eta}{r}\right) &= f_r'(0) \cdot \frac{\eta}{r} \\ &= f'(0) \cdot \eta \\ &= v. \end{aligned}$$

Now, we fix r such that $\eta/r < \lambda$. Since $f_r(\overline{\mathbb{D}})$ is a compact subset of D, there exists $\delta > 0$ such that $f_r(\overline{\mathbb{D}}) + B_\delta(0) \subset D$. If

$$\|(p, v) - (q, w)\| := \|p - q\| + \frac{r}{\eta}\|v - w\| < \delta$$

then

$$\tilde{f}(z) := f_r(z) + (q - p) + \frac{zr}{\eta}(w - v)$$

defines an element of $H(\mathbb{D}, D)$ with $\tilde{f}(0) = q$ and $d\tilde{f}(0)\left(\frac{\eta}{r}\right) = w$. To see each of these, we have

$$\begin{aligned} \tilde{f}(0) &= f_r(0) + (q - p) \\ &= f(0) + (q - p) \\ &= q, \end{aligned}$$

and

$$\begin{aligned} d\tilde{f}(0)\left(\frac{\eta}{r}\right) &= \tilde{f}'(0) \cdot \frac{\eta}{r} \\ &= \left(f_r'(0) + \frac{r}{\eta}(w - v)\right) \cdot \frac{\eta}{r} \\ &= \left(r \cdot f'(0) + \frac{rw}{\eta} - \frac{rv}{\eta}\right) \cdot \frac{\eta}{r} \\ &= \left(r \cdot \frac{v}{\eta} + \frac{rw}{\eta} - \frac{rv}{\eta}\right) \cdot \frac{\eta}{r} \\ &= w. \end{aligned}$$

Hence, $k_D(q, w) \leq \frac{\eta}{r} < \lambda$, showing that k_D is upper semicontinuous. So we have shown that the infinitesimal Kobayashi pseudometric is an infinitesimal Finsler pseudometric. $\square$

Proposition 5.1.2. *The system $\{k_D^i\}$ is a Schwarz-Pick system of pseudodistances.*

Proof. We firstly consider condition (i) for Schwarz-Pick systems, that is, that the integrated form of the infinitesimal Kobayashi pseudometric and the Poincaré distance coincide on the unit disc. Recall that part (ii) of the Schwarz-Pick lemma states that for $f \in H(\mathbb{D}, \mathbb{D})$, we have

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

So for $f \in H(\mathbb{D}, \mathbb{D})$ such that $f(0) = p$ and $df(0)(\eta) = v \neq 0$, this inequality implies

$$\frac{\left|\frac{v}{\eta}\right|}{1 - |p|^2} \leq 1.$$

Therefore,

$$\eta \geq \frac{|v|}{1 - |p|^2} = \mathfrak{p}_{\mathbb{D}}(p, v),$$

where $\mathfrak{p}_{\mathbb{D}}$ is the infinitesimal Poincaré metric. Therefore, by taking the infimum of the left-hand side, we see that $k_{\mathbb{D}}(p, v) \geq \mathfrak{p}_{\mathbb{D}}(p, v)$.

On the other hand, if we take $\eta = \frac{v}{1-|p|^2}$ for $v > 0$, and the function $\phi_{-p}(z)$, then we have $\phi_{-p}(0) = p$. Also, we have $d\phi_{-p}(0)(\eta) = v$, since

$$\begin{aligned} d\phi_{-p}(0)(\eta) &= \phi_{-p}'(0) \cdot \eta \\ &= \phi_{-p}'(0) \cdot \left(\frac{v}{1 - |p|^2}\right) \\ &= \frac{1 - |-p|^2}{(1 - (0)(\overline{-p}))^2} \cdot \left(\frac{v}{1 - |p|^2}\right) \\ &= (1 - |p|^2) \cdot \left(\frac{v}{1 - |p|^2}\right) \\ &= v. \end{aligned}$$

This shows that $k_{\mathbb{D}}(p,v) \leq \frac{|v|}{1-|p|^2} = \mathfrak{p}_{\mathbb{D}}(p,v)$. So we have $k_{\mathbb{D}} = \mathfrak{p}_{\mathbb{D}}$. Since the Poincaré distance is the integrated form of the infinitesimal Poincaré metric, we see that $k^i_{\mathbb{D}} = \rho_{\mathbb{D}}$.

As for condition (ii), if $f \in H(D_1, D_2), h \in H(\mathbb{D}, D_1), h(0) = p$, and $dh(0)(\eta) = v$ we have $f \circ h \in H(\mathbb{D}, D_2), (f \circ h)(0) = f(p)$, and

$$\begin{aligned} d(f \circ h)(0)(\eta) &= df(h(0))(dh(0)(\eta)) \\ &= df(p)(v). \end{aligned}$$

Hence,

$$\begin{aligned} k_{D_2}(f(p), df(p)(v)) &= \inf \{\eta > 0; \exists g \in H(\mathbb{D}, D_2), g(0) = f(p), dg(0)(\eta) = df(p)(v)\} \\ &\leq \eta, \end{aligned}$$

for all η. Now taking the infimum of the right-hand side over all h with $h(0) = p$ and $dh(0)(\eta) = v$, we have

$$k_{D_2}(f(p), df(p)(v)) \leq k_{D_1}(p, v).$$

So by lemma (3.2.4), we have

$$k^i_{D_2}(f(p), df(p)(v)) \leq k^i_{D_1}(p, v),$$

and condition (ii) is satisfied. Therefore, we have shown that $\{k^i_D\}$ is a Schwarz-Pick system of pseudodistances. $\square$

We give one further proposition before proving that $K_D = k^i_D$ for any domain D. Recall from the previous chapter that a pseudodistance ρ on a domain D is *locally bounded above by the norm* if for each $u \in D$ there exists an $M > 0$ and $\delta > 0$ such that $\rho(x, y) \leq M\|x - y\|$ for all $x, y \in B_\delta(u) \cap D$. Any pseudodistance assigned by a Schwarz-Pick system is locally bounded above by the norm. The proposition is as follows.

Proposition 5.1.3. *If α is an infinitesimal Finsler pseudometric on a domain D, ρ is a pseudodistance on D locally bounded above by the norm, and for all $(p, v) \in D \times \mathbb{C}^n$ we have*

$$\limsup_{t \to 0} \frac{\rho(p + tv, p)}{|t|} \leq \alpha(p, v),$$

then $\rho \leq \alpha^i$.

Proof. Let $x, y \in D$, and suppose $\gamma : [0, 1] \to D$ is a smooth parametrisation of an admissible curve joining x and y. Then $\phi(t) = \alpha(\gamma(t), \gamma'(t))$ is upper semicontinuous on $[0, 1]$ and hence there exists a decreasing sequence of continuous functions on $[0, 1]$ converging to ϕ. For an arbitrary $\epsilon > 0$, by the monotone convergence theorem there is a continuous function h on $[0, 1]$, with $h > \phi$ such that

$$\int_0^1 h(t)dt < \int_0^1 \phi(t)dt + \epsilon.$$

For $s, t \in [0,1]$ and $|s-t|$ sufficiently small, since ρ obeys the triangle inequality, we have

$$\begin{aligned} d(s,t) &:= \rho(\gamma(s),\gamma(t)) \\ &\leq \rho(\gamma(s),\gamma(s)+\gamma'(s)(t-s)) + \rho(\gamma(s)+\gamma'(s)(t-s),\gamma(t)) \\ &\leq \rho(\gamma(s),\gamma(s)+\gamma'(s)(t-s)) + M\|\gamma(s)-\gamma(t)+\gamma'(s)(t-s)\|, \end{aligned}$$

since ρ is locally bounded by the norm. Therefore, by hypothesis and the above computation, we see that

$$\begin{aligned} \lim_{t\to s}\frac{d(s,t)}{|s-t|} &\leq \lim_{t\to s}\frac{\rho(\gamma(s)+(t-s)\gamma'(s),\gamma(s))}{|s-t|} \\ &\leq \alpha(\gamma(s),\gamma'(s)) \\ &< h(s) \end{aligned}$$

for all $s \in [0,1]$.

We will now show that

$$d(1,0) \leq \int_0^1 h(s)ds.$$

By the continuity of h, there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} h(t_k^*)(t_k - t_{k-1}) \leq \int_0^1 h(t)dt + \epsilon$$

whenever $\{t_0, ..., t_n\}$ is a partition of $[0,1]$ satisfying $t_k - t_{k-1} < \delta$ for $k = 1, ..., n$ and $t_{k-1} \leq t_k^* \leq t_k$ for all k. For each $s \in [0,1]$ there is an open interval $I(s)$ with centre s and radius less than δ such that $d(s,t) < h(s)|s-t|$ whenever $t \in I(s)$ and $t \neq s$. Since $[0,1]$ is compact, there is a finite subset S of $[0,1]$ such that $\{I(s) : s \in S\}$ is an open cover of $[0,1]$. By removing points from S we can assume that no proper subset gives an open cover. Let $S = \{s_1, ..., s_n\}$ where $0 \leq s_1 \leq s_2 ... < s_n \leq 1$. We have $0 \in I(s_1), 1 \in I(s_n)$, and $I(s_k) \cap I(s_{k+1})$ non-empty for $k = 1, ..., n-1$. For each k such that $1 \leq k \leq n-1$, choose $r_k \in I(s_k) \cap I(s_{k+1})$ with $s_k < r_k < s_{k+1}$, and let $r_0 = 0$ and $r_n = 1$. Let $x_{2k} = r_k$ for $k = 0, ..., n$ and $x_{2k-1} = s_k$ for $k = 1, ..., n$ and take $x_k^* = x_{k-1}$ when k is even and $x_k^* = x_k$ when k is odd. By construction, $0 = x_0 \leq x_1 < x_2 ... < x_{2n-1} \leq x_{2n} = 1$. So if $k = 1, ..., 2n$ then x_k or x_{k-1} is an element s of S and $x_k, x_{k-1} \in I(S)$. Hence, $d(x_k, x_{k-1}) \leq h(x_k^*)(x_k - x_{k-1})$ and $x_k - x_{k-1} < \delta$. Therefore, since $d(s,t)$ is a pseudodistance on $[0,1]$ we have

$$\begin{aligned} d(1,0) &\leq \sum_{k=1}^{2n} d(x_k, x_{k-1}) \\ &\leq \sum_{k=1}^{2n} h(x_k^*)(x_k - x_{k-1}) \\ &\leq \int_0^1 h(x)dx + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows

$$d(1,0) \leq \int_0^1 h(s)ds,$$

as claimed.

So finally, we have

$$\begin{aligned}\rho(x,y) &= \rho(\gamma(0),\gamma(1)) \\ &= d(0,1) \\ &\leq \int_0^1 h(t)dt \\ &\leq \int_0^1 \phi(t)dt + \epsilon \\ &= L_\alpha(\gamma) + \epsilon\end{aligned}$$

which shows that $\rho(x,y) \leq L_\alpha(\gamma)$. On taking the infimum of the right-hand side over all admissible curves joining x and y, we see that $\rho \leq \alpha^i$ □

Proposition 5.1.4. *For any domain D, $K_D = k_D^i$.*

Proof. By proposition 5.1.2, proved above, $\{k_D^i\}$ is a Schwarz-Pick system of pseudodistances, and since $\{K_D\}$ is the largest Schwarz-Pick system we have $k_D^i \leq K_D$ for all D. By proposition 5.1.1, we know that k_D is an infinitesimal Finsler pseudometric, so by the proposition immediately above, we need only show that for $(x,v) \in D \times \mathbb{C}^n$

$$\limsup_{t\to 0} \frac{K_D(x+tv,x)}{|t|} \leq k_D(x,v),$$

and we have our result, that $K_D \leq k_D^i$.

Beginning the proof, let $f \in H(\mathbb{D},D)$ with $f(0) = x$ and $df(0)(\eta) = v$ with $\eta > 0$. Then

$$f(z) = x + \frac{vz}{\eta} + O(|z|^2),$$

since clearly $f(0) = x$, and $df(0)(\eta) = f'(0)\cdot\eta = \frac{v}{\eta}\cdot\eta = v$. Since K_D is locally bounded above by the norm, there exists $M > 0$ such that for t close to 0 in $\mathbb{D}$ we have

$$K_D(x+tv, f(t\eta)) \leq M\|x+tv-f(t\eta)\| = M\|\phi(t)\|.$$

for $\phi(t) := x + tv - f(t\eta)$. Now, we have

$$\phi(t) = x + tv - f(t\eta) = x + tv - x - vt + O(|t|^2),$$

so $\phi(t) = O(|t|^2)$. Therefore, we can see that

$$\begin{aligned}\limsup_{t\to 0} \frac{K_D(x+tv,f(t\eta))}{|t|} &\leq \limsup_{t\to 0} \frac{M\|\phi(t)\|}{|t|} \\ &= \lim_{t\to 0} O(|t|) \\ &= 0.\end{aligned}$$

Further, we have

$$\begin{aligned}
\limsup_{t\to 0} \frac{K_D(x+tv,x)}{|t|} &\le \limsup_{t\to 0} \frac{K_D(x+tv,f(t\eta))}{|t|} + \limsup_{t\to 0} \frac{K_D(f(t\eta),f(0))}{|t|} \\
&\le \limsup_{t\to 0} \frac{\rho_{\mathbb{D}}(t\eta,0)}{|t|} \\
&= \lim_{t\to 0} \frac{\tanh^{-1}(|t|\eta)}{|t|} \\
&= \lim_{t\to 0+} \frac{\tanh^{-1}(t\eta)}{t} \quad \text{for } t\in\mathbb{R} \\
&= \lim_{t\to 0+} \eta\cdot\left(\frac{1}{1-(t\eta)^2}\right) \\
&= \eta,
\end{aligned}$$

where the first inequality follows from the fact that $f(0) = x$ and K_D obeys the triangle inequality (since it is a pseudodistance). The second inequality follows from the fact that K_D is distance-decreasing for holomorphic mappings $f \in H(\mathbb{D}, D)$. The above computation shows that

$$\begin{aligned}
\limsup_{t\to 0} \frac{K_D(x+tv,x)}{|t|} &\le \inf\{\eta > 0; f\in H(\mathbb{D},D), f(0)=x, df(0)(\eta)=v\} \\
&= k_D(x,v),
\end{aligned}$$

and the result follows. □

Since we have shown that $K_D = k_D^i$, we can make a final remark about the various relationships between each of the pseudodistances we have demonstrated in this chapter. We have

$$C_D \le \tilde{C}_D = c_D^i \le k_D^i = \tilde{K}_D = K_D.$$

Chapter 6

Hyperbolic manifolds

Up until this chapter, we have focused exclusively on pseudodistances and pseudometrics between points in arbitrary domains. All of these can be generalised for connected complex manifolds. In this chapter, we briefly review some results on pseudodistances between points on finite-dimensional complex manifolds, with a particular emphasis on the Kobayashi pseudodistance.

We will look at situations in which a given pseudodistance ρ on a domain D generates the original topology. If this is the case, then D is said to be *ρ-hyperbolic*. Further, if (D, ρ) is a complete metric space, then we say that D is *complete ρ-hyperbolic*. We begin by looking at the conditions for ρ-hyperbolicity when D is a bounded domain.

Proposition 6.0.1. *If D is a bounded domain in $\mathbb{C}^n$ and ρ is the pseudodistance assigned to D by a Schwarz-Pick system then D is ρ-hyperbolic.*

Proof. Suppose $D \subseteq B_r(x)$ for some $r > 0$. Let $y \in D \setminus \{x\}$ be arbitrary. There exists a linear functional ϕ such that $\phi(y - x) = \|y - x\|$ and $\|\phi\| = 1$. So the mapping $\left(\psi : z \to \frac{\phi(z-x)}{r}\right) \in H(D, \mathbb{D})$ and by the distance-decreasing property for Schwarz-Pick systems (condition (ii)), we have

$$\begin{aligned}\tanh^{-1}\left(\frac{\|y-x\|}{r}\right) &= \rho_{\mathbb{D}}\left(0, \frac{\|y-x\|}{r}\right)\\ &= \rho_{\mathbb{D}}(\psi(x), \psi(y))\\ &\leq \rho(x, y).\end{aligned}$$

Further, since ρ is locally bounded above by the norm, then $\rho(x, y) \leq M\|x - y\|$ for all $x, y \in B_r(x)$, and so we see that the two topologies coincide. □

The following lemma establishes the criteria for a distance to induce a given topology.

Lemma 6.0.2. *Let ρ be an inner distance on a path-wise connected, complex manifold M. Then ρ induces the topology on M.*

Proof. Since the preimage of an open set under a continuous function is open, it will suffice to show that each open subset V of M is open with respect to ρ.

Let $p \in V$. Since M is manifold, it is a locally compact Hausdorff space, and so every point of M has a relatively compact neighbourhood. So there exists a relatively compact open neighbourhood of p, which we will call W, contained in V. Let r be the minimum value of the positive, continuous function $\rho(p, \cdot)$ on ∂W. Since $p \notin \partial W$ then $r > 0$. If we let $B = \{y \in M : \rho(p, y) < r\}$, then we see that $B \cap \partial W$ is empty. Since M is path-connected by hypothesis, B is path-connected and therefore connected. Since $p \in B$, we have $B \subset W \subset V$. So V is open with respect to ρ. □

Note that ρ must be a *distance*, that is, must separate points (axiom (iv) in definition 3.2.1). Using the above lemma, we immediately conclude an important fact about the Kobayashi pseudodistance. Note that we denote the Kobayashi distance between two points p and q on a finite-dimensional complex manifold M by $K_M(p, q)$.

Proposition 6.0.3. *If M is a finite dimensional complex manifold and K_M is a distance on M, then M is K-hyperbolic.*

Proof. In the previous chapter, we showed that $K_D = \tilde{K}_D$. So if K_D is a distance in the given topology, then by the above lemma K_D induces this topology on M, and so M is K-hyperbolic. □

We now briefly review some concepts seen in algebraic topology. A *covering space* of a complex manifold M, denoted $(\tilde{M}, \pi)$, is a complex manifold $\tilde{M}$ together with a surjective mapping $\pi : \tilde{M} \to M$ (the *covering projection*) such that for all $p \in M$ there exists an open neighbourhood N_p of p with the property that each connected component of $\pi^{-1}(N_p)$ is mapped biholomorphically onto N_p by π.

Every complex manifold admits a unique (up to biholomorphism) *simply connected* complex manifold as a covering space (its *universal cover*). If $(\tilde{M}, \pi)$ is a covering space of M, then for any domain D, any $f \in H(D, M), p \in M, \tilde{p} \in \tilde{M}$ with $\pi(\tilde{p}) = p$ and $a \in D$ with $f(a) = p$ there exists $\tilde{f} \in H(D, \tilde{M})$ such that $\pi \circ \tilde{f} = f$ and $\tilde{f}(a) = \tilde{p}$. The situation is illustrated in the commutative diagram, shown below.

$$\begin{tikzcd} & \tilde{M} \arrow[d, "\pi"] \\ D \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}$$

The mapping $\tilde{f}$ is called a *lifting* of f to $\tilde{M}$.

Proposition 6.0.4. *Let $\tilde{M}$ be a covering space of a complex manifold M with covering projection $\pi : \tilde{M} \to M$. Let $p, q \in M$, and $\tilde{p} \in \tilde{M}$, such that $\pi(\tilde{p}) = p$. Then*

$$K_M(p, q) = \inf \left\{ K_{\tilde{M}}(\tilde{p}, \tilde{q}) : \tilde{q} \in \pi^{-1}(q). \right\}$$

Proof. The covering projection $\pi : \tilde{M} \to M$ is holomorphic. Since K_D is distance-decreasing for holomorphic maps, we have $K_M(p,q) \leq K_{\tilde{M}}(\tilde{p},\tilde{q})$ for all $\tilde{q} \in \pi^{-1}(q)$. Now, suppose there exists an $\epsilon > 0$ such that

$$K_M(p,q) + \epsilon \leq K_{\tilde{M}}(\tilde{p},\tilde{q})$$

for all $\tilde{q} \in \pi^{-1}(q)$. Then by the definition of K_D, there exist points $\{a_0, ..., a_n\} \in \mathbb{D}$ and $f_1, ..., f_n \in H(\mathbb{D}, M)$ such that $p = f_1(a_0), f_1(a_1) = f_2(a_1), ..., f_{n-1}(a_{n-1}) = f_n(a_{n-1}), f_n(a_n) = q$, and

$$K_M(p,q) + \epsilon > \sum_{i=0}^{n-1} \rho_{\mathbb{D}}(a_i, a_{i+1}).$$

We can lift $f_1, ..., f_n$ to holomorphic mappings $\tilde{f}_1, ..., \tilde{f}_n \in H(\mathbb{D}, \tilde{M})$ such that $\tilde{p} = \tilde{f}_1(a_0), \tilde{f}_1(a_1) = \tilde{f}_2(a_1), ..., \tilde{f}_{n-1}(a_{n-1}) = \tilde{f}_n(a_{n-1}), \tilde{f}_n(a_n) = \tilde{q}$. Then $\pi(\tilde{q}) = q$, so that $\tilde{q} \in \pi^{-1}(q)$, and we have

$$K_{\tilde{M}}(\tilde{p},\tilde{q}) \leq \sum_{i=0}^{n-1} \rho_{\mathbb{D}}(a_i, a_{i+1}) < K_M(p,q) + \epsilon,$$

which contradicts the assumption at the beginning of the proof. So we have $K_M(p,q) = \inf \{K_{\tilde{M}}(\tilde{p},\tilde{q}) : \tilde{q} \in \pi^{-1}(q).\}$. □

We will use the result just proven to obtain an important fact, that a complex manifold M is Kobayashi-hyperbolic if and only if $\tilde{M}$ is Kobayashi-hyperbolic. In the proof of the following proposition, we utilise an important result. Firstly, a hyperbolic manifold M is *complete* with respect to K_M if for each point $p \in M$ and every $r > 0$, the closed ball of radius r centered at p is a compact subset of M. This definition is equivalent to the usual definition of completeness, that is, that all Cauchy sequences in M converge. We prove this fact now. Let us denote the open ball of radius r centred at a with respect to K_M as follows. Given a subset A of a complex manifold M, and $r > 0$, let $U(A,r)$ be the open set defined by

$$U(A,r) = \{p \in M : K_M(p,a) < r \text{ for some point } a \in A\}.$$

Proposition 6.0.5. *A complex manifold M is complete in the sense that every Cauchy sequence converges if and only if the closure $\bar{U}(o,r)$ of $U(o,r)$ is compact for all $o \in M$ and all $r > 0$.*

Proof. We only have to prove that if M is Cauchy complete, then $\bar{U}(o,r)$ is compact. The reverse implication is trivial. In order to show this, we need the following lemma.

Lemma 6.0.6. *The closed ball $\bar{U}(o,r)$ is compact if there exists a $b > 0$ such that $\bar{U}(p,b)$ is compact for every $p \in U(o,r)$.*

Proof. Since M is a manifold, it is locally compact, and so there exists an $s < r$ such that $\bar{U}(o,s)$ is compact. It will suffice to show that if $\bar{U}(o,r)$ is compact, then so is $\bar{U}(o, r+\frac{b}{2})$. Let $p_1, p_2, ...$ be a sequence of points in $\bar{U}(o, s+\frac{b}{2})$. Let $q_1, q_2, ...$ be a sequence of points in $\bar{U}(o, s)$ such that $d(p_i, q_i) < \frac{3}{4}b$. Since $\bar{U}(o, s)$ is compact, we can assume (by passing to a subsequence if necessary) that $q_1, q_2, ...$ converges to some point, say q, of $\bar{U}(o, s)$. Then $\bar{U}(q, b)$ contains all p_i for large i. Since $\bar{U}(q, b)$ is compact by assumption, a suitable subsequence of $p_1, p_2, ...$ converges to a point p of $\bar{U}(q, b)$. Since $\bar{U}(o, s+\frac{b}{2})$ and $\bar{U}(o, s+\frac{b}{2})$ is a closed set, the limit point p lies in $\bar{U}(o, s+\frac{b}{2})$, and this completes the proof. □

So by the above lemma, if we can show there exists a $b > 0$ such that $\bar{U}(p, b)$ is compact for all $p \in M$, then we have the result. Assume for a contradiction that there does not exist such a $b > 0$. Then there is a point $p_1 \in M$ such that $\bar{U}(p_1, \frac{1}{2})$ is noncompact. Applying the above lemma to $\bar{U}(p_1, \frac{1}{2})$, we see that there exists a point $p_2 \in \bar{U}(p_1, \frac{1}{2})$ such that $\bar{U}(p_2, \frac{1}{2^2})$ is noncompact. Continuing in this way, we obtain a Cauchy sequence $p_1, p_2, p_3, ...$ such that $p_k \in \bar{U}(p_{k-1}, \frac{1}{2^{k-1}})$ and such that $\bar{U}(p_k, \frac{1}{2^k})$ is noncompact. Let p be the limit point of the Cauchy sequence $p_1, p_2, ...$ Since M is locally compact, for a suitable $c > 0$, $\bar{U}(p, c)$ is compact. For a sufficiently large k, $\bar{U}(p_k, \frac{1}{2^k})$ is a closed set contained in $\bar{U}(p, c)$, and hence is compact. We have our contradiction. □

We also remark on the concept of geodesics in the unit disc. A curve joining the points a and b in the unit disc is called a *geodesic* if its length is equal to the distance between its endpoints. That is, a geodesic is a length-minimising path between a and b. Geodesics in the unit circle that minimise length under the Poincaré distance are either straight lines through the origin or circular arcs that intersect the boundary of the unit disc orthogonally. For a greater exposition of these concepts, see for instance [1, pp. 39-40].

We now prove that a complex manifold M is Kobayashi-hyperbolic if and only if $\tilde{M}$ is Kobayashi-hyperbolic.

Proposition 6.0.7. *Suppose $(\tilde{M}, \pi)$ is a covering space of a complex manifold M, with covering projection π. Then M is K-hyperbolic (respectively complete K-hyperbolic) if and only if $\tilde{M}$ is K-hyperbolic (respectively complete K-hyperbolic).*

Proof. We first will show that if $\tilde{M}$ is K-hyperbolic, then M is K-hyperbolic. Assuming that $\tilde{M}$ is K-hyperbolic, we let $p, q \in M$ and $K_M(p, q) = 0$. We aim to show that K_M is a distance, that is, if $K_M(p, q) = 0$ then $p = q$. Let $\tilde{p}$ be a point of $\tilde{M}$ such that $\pi(\tilde{p})$. By Proposition 6.0.4, there exists a sequence of points $\tilde{q}_1, ..., \tilde{q}_i, ...$ of $\tilde{M}$ such that $\pi(\tilde{q}_i) = q$ and $\lim_{i\to\infty} K_{\tilde{M}}(\tilde{p}, \tilde{q}_i) = 0$. Then the sequence $\{\tilde{q}_i\}$ converges to $\tilde{p}$, and hence $\pi(\tilde{q}_i)$ converges to p. Since $\pi(\tilde{q}_i) = q$, we have $p = q$.

Now we show completeness. Assuming that $\tilde{M}$ is complete K-hyperbolic, let $\tilde{B}_r$ be the closed ball of radius r around $\tilde{p} \in \tilde{M}$ with respect to $K_{\tilde{M}}$, that is, $\tilde{B}_r = \left\{\tilde{q} \in \tilde{M} : K_{\tilde{M}}(p, q) \leq r\right\}$. Similarly, let B_r be the closed ball of radius r around $p = \pi(\tilde{p}) \in M$ with respect to K_M. As a consequence of Proposition 6.0.4, we have $B_r \subset \pi(\tilde{B}_{r+\delta})$ for $\delta > 0$. Since $\tilde{B}_{r+\delta}$ is compact by assumption and B_r is closed, B_r must be compact. Hence M is complete.

Now we show the reverse implication. Assume that M is K-hyperbolic. Let $\tilde{p}, \tilde{q} \in \tilde{M}$ and $K_{\tilde{M}}(\tilde{p}, \tilde{q}) = 0$. Since the projection π is distance-decreasing, we have $K_M(\pi(\tilde{p}), \pi(\tilde{q})) = 0$, implying $\pi(\tilde{p}) = \pi(\tilde{q})$. Let $\tilde{U}$ be a neighbourhood of $\tilde{p}$ in $\tilde{M}$ such that $\pi : \tilde{U} \to \pi(\tilde{U})$ is a biholomorphism and $\pi(\tilde{U})$ is an ϵ-neighbourhood of $\pi(\tilde{p})$ with respect to K_M. In particular, $\tilde{U}$ does not contain $\tilde{q}$ unless $\tilde{p} = \tilde{q}$. Since $K_{\tilde{M}}(\tilde{p}, \tilde{q}) = 0$ by assumption, there exist points $a_1, ..., a_k, b_1, ..., b_k$ in $\mathbb{D}$ and holomorphic mappings $f_1, ..., f_k \in H(\mathbb{D}, \tilde{M})$ such that $\tilde{p} = f_1(a_1), f_i(b_i) = f_{i+1}(a_{i+1})$ for $i = 1, ..., k-1$ and $f_k(b_k) = \tilde{q}$ and that $\sum_{i=1}^{k} \rho_{\mathbb{D}}(a_i, b_i) < \epsilon$. Let $\widehat{a_i b_i}$ denote the geodesic from a_i to b_i in $\mathbb{D}$. Joining the curves $f_1(\widehat{a_1 b_1}), ..., f_k(\widehat{a_k b_k})$ in $\tilde{M}$, we obtain a curve from $\tilde{p}$ to $\tilde{q}$ in $\tilde{M}$, which we will denote by $\tilde{C}$. Since $\pi \circ f_1, ..., \pi \circ f_k$ are distance-decreasing from $\mathbb{D}$ into M, and $\widehat{a_1 b_1}, ..., \widehat{a_k b_k}$ are geodesics in $\mathbb{D}$, every point of the curve $\pi(\tilde{C})$ remains in the ϵ-neighbourhood $\pi(\tilde{U})$ of $\pi(\tilde{p})$. Hence, the endpoint $\tilde{q}$ coincides with $\tilde{p}$.

Now, to show completeness of $\tilde{M}$. Assume that M is complete hyperbolic. We will prove that every Cauchy sequence with respect to $K_{\tilde{M}}$ converges. Due to the remarks made before this proposition, this is equivalent to completeness in the sense we are trying to show. Let $\{\tilde{p}_i\}$ be a Cauchy sequence in $\tilde{M}$. Since π is distance-decreasing, $\{\pi(\tilde{p}_i)\}$ is a Cauchy sequence in M and hence converges to a point $p \in M$. Let $\epsilon > 0$ and U the 2ϵ-neighbourhood of p. For small ϵ, we assume that π induces a homeomorphism of each connected component of $\pi^{-1}(U)$ onto U. Let N be an integer large enough so that $\pi(p_i)$ is within the ϵ-neighbourhood of p for $i > N$. Then every point outside U is at least ϵ away from $\pi(\tilde{p}_i)$. Let $\tilde{U}_i$ be the connected component of $\pi^{-1}(U)$ containing $\tilde{p}_i$. We now show that the ϵ-neighbourhood of $\tilde{p}_i$ lies in $\tilde{U}_i$ for $i > N$. Let $\tilde{q}$ be a point of $\tilde{M}$ with $K_{\tilde{M}}(\tilde{p}_i, \tilde{q}) < \epsilon$. Then we can find points $a_1, ..., a_k, b_1, ..., b_k$ in $\mathbb{D}$ and holomorphic mappings $f_1, ..., f_k \in H(\mathbb{D}, \tilde{M})$ such that $\sum_{j=1}^{k} \rho_{\mathbb{D}}(a_j, b_j) < \epsilon$. We let $\tilde{C}$ be the curve from $\tilde{p}_i$ to $\tilde{q}$ obtained by joining $f_1(\widehat{a_1 b_1}), ..., f_k(\widehat{a_k b_k})$ in M, and let $C = \pi(\tilde{C})$. Then C is contained in the ϵ-neighbourhood of $\pi(p_i)$ and therefore is contained in U. It follows that $\tilde{C}$ lies in $\tilde{U}_i$. Lastly, we let $\tilde{p}$ be the point of $\tilde{U}_i$ defined by $p = \pi(\tilde{q})$. Then $\{\tilde{p}_j\}$ converges to $\tilde{p}$. □

Lastly, we present one application of the Kobayashi pseudodistance, and show that for a K-hyperbolic complex manifold M, the group $\mathrm{Aut}(M)$ can be given the structure of a real Lie group whose topology agrees with the compact-open topology. In a similar manner to [3, p.550], we only give a sketch of the proof. This is due to the fact that some results from the theory of Lie groups are utilised, which are outside the scope of this thesis. We need the following lemma, the proof of which can be found in [7, pp.46-50], in order to show the result.

Lemma 6.0.8. *The group $I(M)$ of isometries of a connected, locally compact metric space M is locally compact with respect to the compact-open topology, and its stabilizer subgroup $I_p(M)$ is compact for each $p \in M$. If M is moreover compact, then $I(M)$ is compact.*

We now give an outline of the result.

Proposition 6.0.9. *If M is a K-hyperbolic complex manifold,* the group $\mathrm{Aut}(M)$ can be given the structure of a real Lie group whose topology agrees with the compact-open topology, and for every $p \in M$, the stabilizer $I_p(M) := \{f \in \mathrm{Aut}(M) : f(p) = p\}$ of p in $\mathrm{Aut}(M)$ is compact.

Proof. Since K_M is inner, it induces the topology of M and thus turns M into a locally compact metric space. By the above lemma, the group $G(M)$ of isometries of M is locally compact, and the stabilizer $St_p(M) = \{f \in G(M) : f(p) = p\}$ of p in $G(M)$ is compact for every $p \in M$ with respect to the compact-open topology. Since $\mathrm{Aut}(M)$ and $I_p(M)$ are closed in $G(M)$ and $St_p(M)$ respectively, it follows that $\mathrm{Aut}(M)$ is locally compact and $I_p(M)$ is compact. A theorem of Bochner and Montgomery states that a locally compact group of differentiable transformations of a manifold is a Lie transformation group. This proves the result. $\square$

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