Maslov index and Spectral Flow

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Acknowledgement and Declaration

I owed my gratitude to supervisor Dr.Bai-Ling Wang (Bryan) for his persistence in guiding and providing assistance throughout this period in completing this thesis. Albeit some obstacles occurred in the process, such as changing of reference text after errors were discovered, Bryan immediately instructed me to study Salamon and Robbin's paper and other related text thoroughly. In this thesis, Chapter 2 and Chapter 3 will be mainly based on Salamon-Robbin papers. This thesis does not contain new results in the Index Theory but to introduce and connect the notion of Maslov index and spectral flow in a more comprehensible and elementary fashion. Detailed and technical proofs of certain results will be omitted for the sake of smooth reading experience and the grasp and understanding of the idea. Some detailed and long proofs would require the reader to refer to its original paper, especially the construction of map $\rho : Sp(2n) \to S^1$. Whilst if a long proof is given, it is necessarily to be written in full detail to illustrate the argument and computation.

Abstract

In [1], Arnol'd discussed the Maslov index as an intersection number of Lagrangian loop with the Maslov cycle, whereas in [17], Robbin and Salamon gave a definition of the Maslov index in term of the signature of crossing form. Furthermore, the Maslov index is characterized by axioms. The index is a homotopy invariant with fixed endpoints, and is additive for the concatenation of Lagrangian paths.

Application wise, in [8], Floer studied the case where the Hessian (a second order differential operator) of the symplectic action functional $\mathscr{A}$ is taken along a gradient flow line $x(t)$, i.e. a solution satisfying the gradient flow equation $\dot{x} = \nabla\mathscr{A}(x)$, connecting two critical endpoints $x^{\pm} = \lim_{t\to\pm\infty} x(t)$. Both the gradient $\nabla\mathscr{A}$ and Hessian $A = \nabla^2\mathscr{A}$ are taken with respect to suitable metric on the underlying manifold. In particular, Floer defined a relative index at $x^{\pm}$ and showed that the spectral flow of the Hessian of $\mathscr{A}$ is equal to the relative Morse index between $x^{\pm}$, which is also equal to the dimension of the space of trajectories of gradient flow between $x^{\pm}$.

In Morse theory (as finite dimensional case of Floer theory), such $\mathscr{A}$ is bounded below and the Hessian H has only finitely many negative eigenvalues, when treated as matrix. Then the spectral flow of $A(t)$ is the number of negative eigenvalues (counted with multiplicity) of A at x^+ minus the number of negative eigenvalues of A at x^-, which is equal to the Fredholm index of linearization operator D_A. Moreover, it happens to be the case that the unstable manifold $W^u(x^-)$ intersects the stable manifold $W^s(x^+)$ transversally if and only if D_A is surjective. Then, the moduli space $\mathcal{M} = W^u(x^-) \cap W^s(x^+)$ is a finite dimensional manifold of dimension equal to the relative Morse index.

In this thesis, we will mainly focus on both the notion of Maslov index and spectral flow and their coincidence. Roughly speaking, Maslov index can be seen as the number of times the Lagrangian paths crosses the Maslov cycle and the spectral flow can be seen as the number of eigenvalues of $A(t)$ crossing zero from negative to positive from $t = -\infty$ to $t = \infty$. The organization of this thesis will

be distributed as follows: In Chapter 1, we briefly review some backgrounds in Symplectic Geometry, that include the notion of symplectic vector space, symplectic manifold and Darboux Theorem. To prepare for later chapter, we study the relation between $Sp(2n)$ and $U(n)$ and their actions on the Lagrangian Grassmannian $\Lambda(n)$. In Chapter 2, we study the notion and computational tool for the Maslov index in the sense of crossing operator(adopting the method introduced in [17]). Other related indices such as Hörmander index and the well-known Conley-Zehnder index will also be introduced. In Chapter 3, we study the notion of Spectral flow of operator satisfying certain conditions and its coincidence with Maslov index and Fredholm index, together with several examples in which this notion applies to will be given.

Contents

Chapter 1

Linear Symplectic Geometry

1.1 Symplectic vector space

Let M^m be a $m-$dimensional vector space over the field $\mathbb{R}$. We say a map $\omega : M \times M \to \mathbb{R}$ is bilinear if ω is additive in both entries and is linear under scalar multiplication. ω is skew-symmetric if $\omega(x, y) = -\omega(y, x)$. If $\omega(x, y) = 0$ for all $y \in M$ implies $x = 0$, then ω is non-degenerate. We call a non-degenerate skew-symmetric bilinear form *symplectic form.*

Definition 1.1.1. *Let M be a vector space and ω be a symplectic form. Such a pair (M, ω) is called symplectic vector space.*

An endomorphism $J : M \to M$ is an almost complex structure satisfying the condition $J^T = J^{-1} = -Id$ and $J^2 = -1$ We say J is $\omega-$tame if $\omega(Jx, x) > 0$ for all $x \neq 0 \in M$, and J is $\omega-$compatible if J is $\omega-$tame and satisfies $\omega(Jv, Jw) = \omega(v, w)$. If J is $\omega-$compatible, then

$$\begin{aligned} g : M \times M &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto g(v, w) = \omega(Jv, w) \end{aligned}$$

defines an inner product on M. With this, we can state the following remark.

Remark 1.1.2. *Let (M, ω) be a vector space equipped with symplectic form ω. Then, the dimension of M is always even.*

Proof. Choose an inner product g on M, then there exists an almost complex structure $J : M \to M$ such that $g(\cdot, \cdot) = \omega(J\cdot, \cdot)$. Recall that J has the property $J^T = -J$, so $\dim J = (-1)^{\dim M} \dim J^T = (-1)^{\dim M} \dim J$. $\dim J$ is non-vanishing because ω is non-degenerate. The equality implies $(-1)^{\dim M} = 1$. So the dimension of M must be even. $\square$

Remark 1.1.3. *Any two symplectic vector spaces M, M' of same dimension are symplectically isomorphic. This means that there exists a linear isomorphism (in which will call symplectomorphism later) between M and M' such that it preserves the symplectic form.*

Definition 1.1.4. *A symplectic basis of a symplectic vector space (M, ω) is a set $\mathcal{B}$ of vectors*

$$\mathcal{B} = \{e_1, \ldots, e_n\} \cup \{d_1, \ldots, d_n\}$$

such that

$$\omega(e_i, e_j) = \omega(d_i, d_j) = 0$$

and

$$\omega(e_i, d_j) = \delta_{ij}$$

for $1 \leq i, j, \leq n$, where δ_{ij} is the Kronecker delta.

Note that the symplectic basis defined above is a basis in the usual sense in a vector space. In fact, one can see that if two vectors that repeating element from $\{e_i\}_{i=1}^n \cup \{d_1\}_{i=1}^n$, then ω vanishes according to the condition imposed. Hence, the linear independence of vectors spanned by $\{e_i\}_{i=1}^n \cup \{d_1\}_{i=1}^n$ holds.

We will state the theorem of the existence of symplectic basis, for detail proof, one may see [9]. The theorem is result of the Gram-Schmidt othornormalization process of Euclidean geometry in symplectic version.

Theorem 1.1.5. *Let A and B be two subsets of index set $\{1, \ldots, n\}$. Let $\mathcal{E} = \{e_i \mid i \in A\}$ and $\mathcal{D} = \{d_j \mid j \in B\}$ of (M^{2n}, ω), such that e_i and d_j satisfy*

$$\omega(e_i, e_j) = \omega(d_i, d_j) = 0$$

and

$$\omega(e_i, d_j) = \delta_{ij}$$

for $(i, j) \in A \times B$. Then, there exists a symplectic basis $\mathcal{B}$ of (M, ω) such that $\mathcal{E} \cup \mathcal{D} \subset \mathcal{B}$.

Example 1.1.1. $(\mathbb{R}^{2n}, \omega_{Ecld})$ *is a symplectic vector space with Euclidean symplectic form, with symplectic basis $e_i = (\alpha_i, 0)$ and $d_j = (0, \beta_j)$, where α_i and β_j are canonical basis of $\mathbb{R}^n$. We call this the canonical symplectic basis. Explicitly, ω_{Ecld} has the form*

$$\omega_{Ecld}(x, y) = \sum_{i=1}^{n} a_i y_i - b_i x_i$$

where $x = (x_1, \ldots, x_n, a_1, \ldots, a_n)$ *and* $y = (y_1, \ldots, y_n, b_1, \ldots, b_n)$. *In particular, if* $(e_1, \ldots, e_n)$ *is an orthonormal basis of* $\mathbb{R}^n$, *then*

$$\omega((e_i, 0), (0, e_j)) = \delta_{ij}.$$

Hence,

$$((e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_1))$$

is a symplectic basis of $(\mathbb{R}^{2n}, \omega_{Ecld})$.

Example 1.1.2. *For the case of Hermitian inner product, denoted as* $\langle v, w\rangle_{\mathbb{C}}$ *or simply* $\langle v, w\rangle$ *for* $v, w \in \mathbb{C}^n$, *by direct computation, we see that*

$$\begin{aligned}
\langle v, w\rangle_{\mathbb{C}} &= \Big\langle \sum v_i e_i, \sum w_j e_j \Big\rangle_{\mathbb{C}} \\
&= \sum v_i \overline{w_j} \langle e_i, e_j\rangle_{\mathbb{C}} \\
&= \sum v_j \overline{w_j} \\
&= \sum (a_j + ib_j)(x_j - iy_j) \\
&= \sum (a_j x_j + b_j y_j) - i \sum (a_j y_j - b_j x_j) \\
&= \langle v, w\rangle_{\mathbb{R}} - i\omega(v, w).
\end{aligned}$$

The second equality follows from the identity $\langle e_i, e_j\rangle = \delta_{ij}$. *Also,*

$$\omega(v, w) = -Im\langle v, w\rangle$$

because if we write $v = a + ib = (a, b)^T, w = x + iy = (x, y)^T$, *and*

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

then, over $\mathbb{R}$, $Jw = (y, -x)$ *and hence*

$$\omega(v, w) = \langle v, Jw\rangle_{\mathbb{R}} = v \cdot Jw = v^T(Jw) = (a, b)\begin{pmatrix} y \\ -x \end{pmatrix} = ay - bx \qquad (1.1.1)$$

Now, we show that the real bilinear form $\omega(v, w) = -Im\langle v, w\rangle_{\mathbb{C}}$ *is nondegenerate and skew-symmetric. Note that*

$$\omega(v, w) = -Im\langle v, w\rangle_{\mathbb{C}} = Im\overline{\langle v, w\rangle_{\mathbb{C}}} = -Im\langle w, v\rangle_{\mathbb{C}} = -\omega(w, v).$$

So, ω *is skew-symmetric. For all* $w \in C^n$, *such that*

$$\omega(v, w) = -Im\langle v, w\rangle_{\mathbb{C}} = 0$$

implies $Im\langle v, w\rangle_{\mathbb{C}} = 0$. *One observes that*

$$\langle v, iw\rangle_{\mathbb{C}} = -i\langle v, w\rangle_{\mathbb{C}}$$

which implies

$$Im\langle v, iw\rangle_{\mathbb{C}} = -\langle v, w\rangle_{\mathbb{R}}.$$

But,

$$Im\langle v, iw\rangle_{\mathbb{C}} = \omega(v, iw) = i\omega(v, w) = 0,$$

so $\langle v, w\rangle_{\mathbb{R}} = 0$. *Eventually, we obtain*

$$\langle v, w\rangle_{\mathbb{C}} = \langle v, w\rangle_{\mathbb{R}} - i\omega(v, w) = 0 - 0 = 0$$

which implies $v = 0$, *i.e.* ω *is nondegenerate.*

Example 1.1.3. *Suppose* V *is any vector space with its dual* V^*. *Then, the direct sum* $V \oplus V^*$ *is a symplectic vector space in which the symplectic form* ω *is given by*

$$\begin{aligned} \omega : V \oplus V^* \times V \oplus V^* &\longrightarrow \mathbb{R} \\ ((x_1, f_1), (x_1, f_1)) &\longmapsto f_1(x_2) - f_1(x_2) \end{aligned} \tag{1.1.2}$$

Proof. We only need to show that ω is non-degenerate and skew-symmetric since the bilinearity of ω follows from the linearity of f_1 and f_2. First, for skew-symmetry of ω,

$$\begin{aligned} \omega((x_1, f_1), (x_2, f_2)) &= f_2(x_1) - f_1(x_2) \\ &= -(f_1(x_2) - f_2(x_1)) \\ &= -\omega((x_2, f_2), (x_1, f_1)) \end{aligned}$$

For non-degeneracy, recall that by definition if $v \neq 0$, then there exists w such that $\omega(v, w) \neq 0$. We choose any nonzero vector $(x, f) \in V \oplus V^*$. If $x \neq 0$, then there exists some linear functional $h \in V^*$ such that $h(x) \neq 0$, take $(0, h) \in V \oplus V^*$, then

$$\omega((x, f), (0, h)) = h(x) - f(0) = h(x) \neq 0.$$

Also, take $(v, 0) \in V \oplus V^*$,

$$\omega((x, f), (v, 0)) = 0(x) - f(v) = -f(v) \neq 0.$$

Hence, ω is nondegenerate on $V \oplus V^*$. □

Definition 1.1.6. *The symplectic complement $V^\perp$ of $V \subset M$ (wrt. symplectic form ω) is defined as*

$$V^\perp = \{x \in M \mid \omega(x, y) = 0 \, \forall y \in V\}.$$

Remark 1.1.7. *$V^\perp$ is not necessarily the complement of V in general and $V^\perp$ need not to be symplectic as well. One simple counterexample would be $V = Span(e)$ for some nonzero $e \in M$. Then, $\omega(x, y) = \omega(ae, be) = ab\omega(e, e) = 0$ for $x, y \in V$. This shows that $V \subseteq V^\perp$, i.e. $V \neq (V^\perp)^c$. In this case, we say V is isotropic. Also, $\omega|_V$ degenerates. So, ω cannot be nondegenerate on whole $V^\perp$. We say V is not symplectic. This 'classification' follows from the following definition.*

Definition 1.1.8. *Let $V \subset M$ be a linear subspace, then we say V is*

- *isotropic if $V \subset V^\perp$*
- *coisotropic if $V^\perp \subset V$*
- *symplectic if $V \cap V^\perp = \{0\}$*
- *Lagrangian if $V = V^\perp$.*

One can see that if V is symplectic, then V must not be contained in $V^\perp$ but the only common element is $\{0\}$. This is equivalent to say that V is symplectic if and only if $\omega|_V \neq 0$.

Corollary 1.1.1. *Let $(V \oplus V^*, \omega)$ be the symplectic space defined above, then $V \oplus 0^*$ and $0 \oplus V^*$ are Lagrangian subspaces of V and*

$$V \oplus V^* = (V \oplus 0^*) \oplus (0 \oplus V^*).$$

Proof. We show that $V \oplus 0^*$ is Lagrangian. For all $x = (v, 0) \in V \oplus 0^*$, $\omega((v, 0), (v, 0)) = 0$ implies $(v, 0) \in V \oplus 0^*$, i.e. $V \oplus 0^* \subset (V \oplus 0^*)^\perp$. On the other hand, let $(w, f) \in (V \oplus 0^*)^\perp$, then for any $(v, 0) \in V \oplus 0^*$,

$$\omega((w, f), (v, 0)) = 0(w) - f(v) = 0$$

so $f(v) = 0$ for all $v \in V$. Then, f must be 0. This shows that $(V \oplus 0^*)^\perp \subset V \oplus 0^*$. So we have

$$V \oplus 0^* = (V \oplus 0^*)^\perp,$$

i.e. $V \oplus 0^*$ is Lagrangian. By similar argument, we show that $0 \oplus V^*$ is Lagrangian. The last claim follows directly. $\square$

Remark 1.1.9. *Suppose V is a symplectic vector space of dimension $2n$, then its Lagrangian subspaces are always of dimension n.*

Suppose (M, ω) is a symplectic vector space with symplectic form ω as defined above. Recall that we can choose a symplectic basis and express the coordinates of a vector

$$x = \sum_{i=1}^{n} x_i e_i + y_i f_i$$

in the chosen basis such that the symplectic form is

$$\omega(x, x') = \sum_{i=1}^{n} x_i y_i' - x_i' y_i.$$

In fact, we can also write ω as a differential form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Definition 1.1.10. *A symplectic form ω is called closed if $d\omega = 0$ and exact if ω is the differential of differential form with one less dimension, i.e. if ω is a differential $k-$form, then ω is closed if $\omega = d\beta$ for some $(k-1)-$form β.*

Example 1.1.4. *Let*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Then, ω is exact:

$$\omega = d\left(\sum_{i=1}^{n} x_i dy_i\right) = -d\alpha$$

where

$$\alpha = \sum_{i=1}^{n} y_i dx_i.$$

Such α is called Liouville 1-form.

1.2 Symplectic manifold

In this section, we will given a definition of symplectic manifold and explain the Darboux Theorem.

Definition 1.2.1. *Let M be a smooth manifold. Suppose a closed nondegenerate $2-$form ω is defined on M, i.e. an $\omega \in \Omega^2(M)$ such that*

1. $d\omega = 0$

2. *For all* $x \in M$, *on each tangent space* T_xM, *if*

$$\omega_x(X,Y) = 0$$

for all $Y \in T_xM$, *then* $X = 0$.

Such (M,ω) *is called symplectic manifold.*

Remark 1.2.2. *The conditions on* ω *at each* $x \in M$ *make each* T_xM *into a symplectic vector space. So, the dimension of* M *is always even. That is,*

$$\bigwedge^n \omega_x \neq 0 \in \bigwedge^{2n} T_x^*M$$

for all x, *where the term on left side is the non vanishing volume form (*$n-$*fold wedge product)* ω^n *on* M. *Then,* M *must be orientable.*

One may wonder why should we impose ω to be closed instead of exact. Suppose (M,ω) is a symplectic manifold with exact nondegenerate $2-$form ω, then M cannot be compact, which is not a desirable situation in most cases. The claim follows from the following:

Proposition 1.2.3. *Let* M *be a compact manifold. Then, there exists no* $2-$*form on* M *such that* ω *is exact and nondegenerate.*

Proof. As claimed above the $2-$form ω is nondegenerate if and only if ω^n is a volume form on M. Suppose that $\omega = d\alpha$ is exact, then so is

$$\omega^n = d(\alpha \wedge \omega^{n-1}).$$

But, by Stokes theorem, the integral of ω^n over M is positive, and we obtain

$$0 < \int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-1}) = \int_{\partial M} \alpha \wedge \omega^{n-1} = 0$$

which is a contradiction. The last equality follows from the fact that M is a closed manifold. □

Example 1.2.1. *Consider the space of cotangent bundle* T^*M *of manifold* M, *with projection* $\pi : T^*M \to M$. *On* T^*M *there exists the Liouville* $1-$*form* α *given by*

$$\alpha_{x,\varphi}(X) = \varphi(T_x\pi(X))$$

where $x \in M$, $X \in T_{(x,\varphi)}(T^*M)$ and $\varphi \in T^*_x M$ is a linear form on T_xM. Then, $\omega = -d\alpha$ is exact and hence closed because $d\omega = -d^2\alpha = 0$. In terms of coordinate: We choose $(x_1, \ldots, x_n)$ be local coordinates on M. and $(y_1, \ldots, y_n)$ be the dual coordinates of M, then we can write the local coordinates on T^*M as $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. So, in this case, we have

$$\omega = \sum dx_i \wedge dy_i, \quad \alpha = \sum y_i dx_i.$$

Example 1.2.2. *Let $M = S^2$, the unit sphere*

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \sum_i x_i^2 = 1 \right\},$$

is equipped with symplectic form (area form) ω, which is given by

$$\omega_x(\xi, \eta) = \langle x, \xi \times \eta \rangle = x \cdot (\xi \times \eta)$$

where $\xi, \eta \in T_xS^2$ and $\xi \times \eta$ is the cross product. The tangent space of S^2 at a point x is the plane orthogonal to the unit vector $\overrightarrow{x}$.

Example 1.2.3. *Recall that if ω is a closed nondegenerate symplectic form, then ω^n is a nonvanishing volume form. We say $\omega^n/n!$ is the symplectic volume form. This follows from*

$$\left(\sum dx_i \wedge dy_i \right)^n = n! dx_1 \wedge dy_i \wedge \cdots \wedge dx_n \wedge dy_n$$

This gives volume 1 to the torus $\mathbb{T}^{2n}$.

Remark 1.2.4. *When ω is nondegenerate, then there is a canonical isomorphism between the tangent and cotangent bundles of M, i.e.*

$$\begin{aligned} \omega : TM &\longrightarrow T^*M \\ X &\longmapsto \iota(X)\omega = \omega(X, \cdot). \end{aligned}$$

Definition 1.2.5. *Let (M, ω) be a symplectic manifold. Then, a symplectomorphisms of (M, ω) is a diffeomorphism $\psi \in$Diff(M) such that*

$$\psi^* \omega = \omega$$

i.e. preserves the symplectic form. We denote Symp(M) the group of symplectomorphisms.

Note that since ω is nondegenerate, then by Remark 1.2.4, at each $x \in M$,

$$\omega : T_xM \to T_x^*M, \quad v \mapsto \iota(v)\omega_x$$

is an isomorphism. There is an one-to-one correspondence between vector fields $X \in$Vect(M) and $1-$forms via Vect$(M) \to \Omega^1(M)$ defined by $X \mapsto \iota(X)\omega$. We call a vector field X symplectic if $\iota(X)\omega$ is closed, i.e. $d\iota(X)\omega = 0$. The set of all symplectic vector fields is denoted by Vect(M, ω).

Definition 1.2.6. *Let M be a smooth manifold. Let $\varphi : M \times \mathbb{R} \to M$ be a map of two parameters. Then, φ is an isotopy if for all $t \in \mathbb{R}$, $\varphi_t : M \to M$ is a diffeomorphism with initial $\varphi_0 = Id$.*

Given an isotopy φ, we can associate a family of time dependent vector fields $\{X_t\}$ such that at each $x \in M$,

$$\frac{d}{dt}\varphi_t(x) = X_t(\varphi_t(x)).$$

In fact, if M is compact, then there is an one-to-one correspondence between the set of isotopies of M and the set of time dependent vector fields on M.

Definition 1.2.7. *Suppose the vector field X is independent of time t, then we call the associated isotopy the exponential map or the flow of X, denoted as e^{tX} : $M \to M$ for $t \in \mathbb{R}$. Then, $\{e^{tX}\}$ is the unique smooth family of diffeomorphism such that*

$$\frac{d}{dt}(e^{tX})(x) = X \circ e^{tX(x)}$$
$$e^{tX}\Big|_{t=0} = Id.$$

Definition 1.2.8. *The Lie derivative is an operator defined on the space of differential $k-$forms on M, which is given by*

$$\begin{aligned} \mathcal{L}_X : \Omega^k(M) &\longrightarrow \Omega^k(M) \\ \omega &\longmapsto \frac{d}{dt}\Big|_{t=0} (e^{tX})^*\omega. \end{aligned}$$

Remark 1.2.9. *By Picard's theoerem, in a small neighbourhood U_x of x and for small time $t \in \mathbb{R}$, there exists an one parameter family of local diffeomorphism φ_t such that*

$$\frac{d\varphi_t}{dt} = X_t \circ \varphi_t$$

with $\varphi_0 = Id$. Then, we have the Lie derivative by X_t given by

$$\mathcal{L}_{X_t}\omega = \frac{d}{dt}\bigg|_{t=0} (\varphi_t)^*\omega.$$

One useful tool in the calculation is the so-called Cartan's formula:

$$\mathcal{L}_X\omega = \iota(X)d\omega + d\iota(X)\omega.$$

Proposition 1.2.10. *Let $\{\omega_t\}$ be a smooth family of differential $k-$forms, then*

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\bigg(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\bigg).$$

Proof.

$$\begin{aligned}\frac{d}{dt}\varphi_t^*\omega_t &= \frac{d}{dr}\bigg|_{r=t}\varphi_r^*\omega_t + \frac{d}{ds}\bigg|_{r=t}\varphi_t^*\omega_s \\ &= \varphi_r^*\big(\mathcal{L}_{X_r}\omega_t\big|_{r=t}\big) + \varphi_t^*\frac{d}{ds}\bigg|_{s=t}\omega_s \\ &= \varphi_t^*\bigg(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\bigg).\end{aligned}$$

□

Proposition 1.2.11. *Let M be a closed manifold. Suppose $\{\psi_t\} \in$ Diff(M) is a smooth family of diffeomorphisms generated by a family of vector fields $\{X_t\} \in$ Vect(M) given by*

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t$$

with $\psi_0 = Id$. Then, for all t, $\psi_t \in$ Symp(M, ω) if and only if $X_t \in$ Vect(M, ω).

Proof. By direct computation, and using Cartan's formula for Lie derivative,

$$\mathcal{L}_X\omega = \iota(X_t)d\omega + d(\iota(X_t)\omega),$$

then

$$\begin{aligned}\frac{d}{dt}\psi_t^*\omega &= \psi_t^*\bigg(\mathcal{L}_X\omega + \frac{d}{dt}\omega\bigg) \\ &= \psi_t^*(\iota(X_t)d\omega + d\iota(X_t)\omega) \\ &= \psi_t^*(d\iota(X_t)\omega).\end{aligned}$$

The last equality follows from the fact that ω is closed. So, $X_t \in$Vect(M, ω) if and only if $d(\iota(X_t)\omega) = 0$ if and only if $\mathcal{L}_X\omega = 0$. □

1.3 Darboux theorem

Roughly speaking, Darboux theorem says that every symplectic form ω on M is locally diffeomorphic to the standard symplectic form on $\mathbb{R}^{2n}$, which implies an important consequence: the symplectic structure has no local invariants. The proof of Darboux Theorem utilizes the Moser's theorem. There are two versions of the Moser's theorem:

Theorem 1.3.1 (Version 1)**.** *Let M be a compact symplectic manifold with symplectic forms ω_0 and ω_1 such that $[\omega_0] = [\omega_1]$ and a smooth family of symplectic forms $\{\omega_t\}$ given by*

$$\omega_t = (1-t)\omega_0$$

for $t \in [0,1]$. Then, there exists an isotopy $\varphi : M \times \mathbb{R} \to M$ such that $\varphi_t^* \omega_t = \omega_0$ for $t \in [0,1]$.

Theorem 1.3.2 (Version 2)**.** *Let M be a compact manifold equipped with symplectic form ω_0 and ω_1. Let $\{\omega_t\}$ be a smooth family of closed symplectic $2-$forms connecting ω_0 and ω_1 such that*

1. *(cohomology assumption) $[\omega_t]$ is independent of t, i.e. $\frac{d}{dt}[\omega_t] = \left[\frac{d}{dt}\omega_t\right] = 0$.*
2. *(nondegeneracy assumption) ω_t is nondegenerate for all t in $[0,1]$.*

We will not prove the above two versions of Moser's theorem, for detail proof, one sees [4]. However, we will first show the following Relative Moser's theorem to give a simple proof to Darboux theorem.

Theorem 1.3.3 (Relative Moser's Theorem)**.** *Let M be a manifold. Suppose there is a closed submanifold X of M such that the symplectic forms ω_0 and ω_1 agree on X. i.e.*

$$\omega_0(x) = \omega_1(x)$$

for all $x \in X$. Then, there exists a neighbourhood U_0 and U_1 of X in M and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi|_X = Id_X$ and $\varphi^*\omega_1 = \omega_0$.

Proof. Consider the path of symplectic forms $\omega_t = (1-t)\omega_0 + \omega_1$. Since ω_0 and ω_1 agree on all points on X, ω_t is constant. So we can find a small enough tubular neighbourhood U_0 of X such that ω_t is nondegenerate on U_0. Since ω_0 and ω_1 are closed, so there exists a $1-$form α on U_0 such that $\omega_1 - \omega_0 = d\alpha$. Recall that the notion of closedness of a differential form is the local notion corresponding to exactness, which is exactly why we can find such α. Another way to suggest the existence of such α is to see that U_0 is diffeomorphic to an open ball of the zero section of a vector bundle on X that retracts on X. Then, one applies the Poincare lemma to $\omega_1 - \omega_0$ which gives the form α. Back to the proof. Our goal is to find a time dependent vector field X_t such that its flow ψ_t satisfying $\psi_t^*\omega_1 = \omega_0$. Observe that

$$\begin{aligned}
\frac{d}{dt}\psi_t^*\omega_t &= \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) \\
&= \psi_t^*\left(d\alpha + \iota(X_t)d\omega_t + d\iota(X_t)\omega_t\right) \\
&= \psi_t^*\left(d(\alpha + \iota(X_t)\omega_t)\right) = 0 = d\omega_0.
\end{aligned}$$

So, we can choose the time dependent vector field X_t to be such that

$$\alpha = -\iota(X_t)\omega_t.$$

By integrating X_t we get its flow (an isotopy) $\psi_t : U_0 \times [0,1] \to M$ with $\psi_t^*\omega_t = \omega_0$. Suppose we consider such X_t vanishes on X for all t, then we get $\psi_t|_X = Id_X$. Note that $\psi_t^*\omega_t$ does not depends on time t, so we get the result by taking $\varphi = \psi_1$. In particular, at time 1, $\varphi(U_0) = U_1$. □

Theorem 1.3.4 (Darboux). *Let (M,ω) be a symplectic manifold. Then, for each $x \in M$, there exists a system of local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ with center x such that the symplectic form is given by $\omega = \sum dx_i \wedge dy_i$.*

Proof. We wish to apply the lemma above to a point x and compare ω with the constant on tangent space T_xM. But, the two forms are not defined on the same space. Hence, we create a local diffeomorphism $T_xM \to M$. Let φ be the exponential map $e_x : V_0 \to U_0$, where V_0 and U_0 are open neighbourhood of T_xM and M respectively. We can choose such neighbourhood small enough so that φ is a local diffeomorphism. Then, we define

$$\omega_0 = (\varphi^{-1})^*\omega_x.$$

The two symplectic forms ω_0 and ω_x are on neighbourhood U_0 that coincide at x. We are ready to apply the Relative Moser's theorem now. On U_0 containing x, there exists a diffeomorphism $\psi : U_0 \to M$ such that $\psi^*\omega = \omega_0$. If we choose a suitable coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in T_xM, then, locally, under the chart $\psi \circ \varphi$, we obtain a system of local coordinates in M around U_0 such that $\omega = \sum dx_i \wedge dy_i$. □

1.4 The Symplectic group $Sp(2n)$, Unitary group $U(n)$ and their relation

Definition 1.4.1. *The symplectic group $Sp(2n) = Sp(2n, \mathbb{R})$, is the group of all linear isomorphisms $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, such that it preserves the symplectic structure, i.e.*

$$\omega(\varphi x, \varphi y) = \omega(x, y)$$

for $x, y \in \mathbb{R}^{2n}$.

Notation wise, we write the symplectic structure ω above instead of ω_{Ecld}, and we would just omit the subscript from now on whenever no confuse occurs upon working in $\mathbb{R}^{2n}$. Generally, we write $Sp(M^{2n}, \omega)$ to mean the symplectic group of (M^{2n}, ω) defined in the same way. One notes that the symplectic group of any $2n-$dimensional symplectic space is isomorphic to $Sp(2n)$. In fact, we have

Proposition 1.4.2. *Let (M, ω) and (M', ω') be two symplectic vector spaces of same dimension, say $2n$. Then, their symplectic group $Sp(M, \omega)$ and $Sp(M', \omega')$ are isomorphic.*

Proof. Let $\Psi : (M, \omega) \to (M', \omega')$ be a symplectic isomorphism. Define a linear induced map $f_\Psi : Sp(M, \omega) \to (M', \omega')$ by $f_\Psi(s) = \Psi \circ s \circ \Psi^{-1}$. Then, f_Ψ is a group homomorphism since $f_\Psi(ss') = \Psi(ss')\Psi^{-1} = (\Psi s \Psi^{-1})(\Psi s' \Psi^{-1}) = f_\Psi(s) f_\Psi(s')$. Note that $f_\Psi(S) = Id_{Sp(M', \omega')}$ means $\Psi \circ S \circ \Psi^{-1} = \Psi \circ \Psi^{-1}$, so S must be equal to $Id_{Sp(M,\omega)}$, which implies f_Ψ is injective. For a $\Psi \circ s' \circ \Psi^{-1} \in Sp(M', \omega')$, there always exists an $s \in Sp(M, \omega)$ in the form $s = \Psi^{-1} \circ s' \circ \Psi$ such that $f_\Psi(s) = \Psi \circ (\Psi^{-1} \circ s' \circ \Psi) \circ \Psi^{-1} = s'$. $\square$

Let $S \in Sp(2n)$, from a direct computation from the definition,

$$\langle Sx, JSy \rangle = \omega(Sx, Sy) = \omega(x, y) = \langle x, Jy \rangle,$$

we have

$$\langle x, (S^T JS)y \rangle = \langle x, Jy \rangle,$$

i.e the relation

$$S^T JS = J.$$

In the case of $\mathbb{R}^{2n}$, we observe that the symplectic form can be rewritten in matrix form,

$$\omega(x, y) = x^T Jy$$

where x, y are vectors in $\mathbb{R}^{2n}$ with $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$, and x^T represents the transpose of x. This means that $S^T JS$ and hence S is essentially also a $2n \times 2n$ real matrix, i.e. $S \in GL(2n, \mathbb{R})$. We call matrix S a symplectic matrix if S satisfies $S^T JS = J$. The set of all such matrices, denoted by $Sp(2n, \mathbb{R})$ or by slight abuse of notation we just write $Sp(2n)$ when no confusion occurs.

It is useful to first identify the complex and real general linear group $GL(n, \mathbb{C}) \cong GL(2n, \mathbb{R})$. We shall illustrate an isomorphism between them. Take a $u \in GL(n, \mathbb{C})$, u can be written as $u = A + iB$, where A, B are real $n \times n$ invertible matrices. Let $z_1, z_2 \in \mathbb{C}^n$, so

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2.$$

Then, there exists a linear transformation, say u, such that $z_2 = uz_1$. We check that

$$x_2 + iy_2 = (A + iB)(x_1 + iy_1) = Ax_1 - By_1 + i(Ay_1 + Bx_1)$$

or in matrix form,

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Let $\Psi = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, Ψ is invertible because $\det \Psi = A^2 + B^2 > 0$. Furthermore, a direct computation by taking $x_1 = y_1 = 0$, we have the kernel of Ψ is trivial, so Ψ is indeed an isomorphism. Hence, we have defined an identification $\Psi : GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$. Now pick a $u = A + iB \in GL(n, \mathbb{C})$ and identified in $GL(2n, \mathbb{R})$, $U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. Then,

$$\begin{aligned} U^T JU &= \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \\ &= \begin{pmatrix} A^T B - B^T A & A^T A - B^T B \\ -A^T A - B^T B & A^T B - B^T A \end{pmatrix}. \end{aligned} \tag{1.4.1}$$

Suppose U satisfies $U^* U = UU^* = Id$, i.e. $U^* = U^{-1}$, where U^* is the conjugate transpose of U. Since all entries of U are real matrices,

$$U^* = \bar{U}^T = U^T = U^{-1}$$

then

$$UU^T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}.$$

implies

$$A^T B = B^T A, \quad A^T A - B^T B = Id. \tag{1.4.2}$$

substituting (1.4.2) back into (1.4.1), eventually we obtain

$$U^T J U = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} = J,$$

which shows $U \in GL(n, \mathbb{C}) \cap Sp(2n)$. Also, from $U^T = U^{-1}$, $U \in O(2n, \mathbb{R})$. In fact,

$$\det(U^T J U) = \det(U^T) \det J \det U = \det U^2 = det J = 1,$$

$\det U = \pm 1$, and the fact that $\det U > 0, \det U = 1$. So, U is in the special orthogonal group $SO(2n, \mathbb{R})$. Note that the condition U satisfies above is exactly how a unitary matrix defined. In other words, for all such U, we have the set of all unitary matrices $U(n) \subset GL(n, \mathbb{C}) \cap Sp(2n) \cap SO(2n, \mathbb{R})$. The opposite direction is very clear by checking U^* is indeed U^{-1} for U is both symplectic and orthogonal.

1.5 Lagrangian Grassmannian

In this section, we study the notion of Lagrangian Grassmannian in stratification. An excellent exposure of this notion is given in [6] and [9].

Definition 1.5.1. *Let M be a topological manifold of dimension n. We say M is stratified if there exists a stratification of M, i.e. there is a partition of M^n in a family $\{M_i^k\}_{i \in \mathcal{I}}$ of connected submanifolds (strata) for $k \leq n$, such that*

1. *$\{M_i^k\}_{i \in \mathcal{I}}$ is a locally finite partition of M^n,*
2. *For distinct i and i' in $\mathcal{I}$, $M_i^k \cap \overline{M_{i'}^{k'}} \neq \emptyset$ implies $M_i^k \subset M_{i'}^{k'}$ and $i \leq i'$.*
3. *$\overline{M_i^k} \backslash M_i^k$ is a disjoint union of strata of dimension less than k.*

Example 1.5.1. *Take $M = \Lambda(n)$. For all $L \in \Lambda(n)$, we can define a stratification of dimension $k \leq n$ at L*

$$\Lambda_L(n, k) = \{L' \in \Lambda(n) \mid dim(L \cap L') = k\}.$$

For each k, we call $\Lambda_L(n, k)$ a $k-$stratum of $\Lambda(n)$ relative to L. It is a connected submanifold of codimension $k(k+1)/2$. So, at some given L, for distinct $0 \leq k, k' \leq n$ we have

$$\Lambda_L(n, k) \cap \Lambda_L(n, k') = \emptyset.$$

Hence,

$$\Lambda(n) = \bigcup_{k=0}^{n} \Lambda_L(n,k).$$

One readily sees that if $k = 0$, then $\Lambda_L(n,0)$ is the set of all Lagrangian planes that intersect transversally to L. In this case, we define the Maslov cycle relative to L to be the closed set

$$\Lambda(n)_L = \Lambda(n)\backslash\Lambda_L(n,0) = \overline{\Lambda_L(n,1)}$$

that is the set of all Lagrangian planes that are not transverse to L.

In general, let $\Lambda(M)$ be the set of all Lagrangian subspaces of (M,ω) where M is a real symplectic vector space of dimension $2n$. Next, we show that the $0-$stratum $\Lambda_E(M,0)$ of $\Lambda(M)$ relative to $E \in \Lambda(M)$, i.e. the space of all Lagrangian subspaces in M that intersect transversally to a given reference Lagrangian E is open and dense in $\Lambda(M)$.

Theorem 1.5.2. *Let $E \in \Lambda(M)$, we define*

$$\Lambda_E(M,0) = \{L \in \Lambda(M) \mid L \cap E = 0\}.$$

$\Lambda_E(M,0)$ is diffeomorphic to $Symm(L)$ the space of all symmetric bilinear forms on L for all $L \in \Lambda_E(M,0)$. Moreover, $\Lambda_E(M,0)$ is open and dense in $\Lambda(M)$.

Proof. Fix a $E \in \Lambda(M)$, then all $L \in \Lambda(M)$ such that $L \cap E = 0$ has the form $\{x + Ax \mid x \in L'\}$ for some linear map $A : L' \to M$. The bilinear form on L is given by

$$\begin{aligned} Q_L : L' \times L' &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto Q_L(x,y) = \omega(Ax,y) \end{aligned}$$

which is symmetric if and only if L is a Lagrangian subspace (which is true for $L \in \Lambda(M)$). So, Q defines a bijection between $\Lambda_E(M,0)$ and $Symm(L')$, whilst the inverse is an embedding $Symm(L') \hookrightarrow \Lambda_E(M,0)$. In short, Q gives a set of local coordinates for $\Lambda(M)$ and 2^n many $\Lambda_E(M,0)$ cover $\Lambda(M)$. This shows $\Lambda_E(M,0)$ is diffeomorphic to $Symm(L')$. Next, $\Lambda_E(M,0)$ is open because its complement

$$\overline{\Lambda_E(M,1)} = \Lambda(M)\backslash\Lambda_E(M,0)$$

is closed. To show it is dense, we choose an $E' \in \Lambda_E(M,0) \cap \Lambda_L(M,0)$ for any $L \in \Lambda(M)$. Choose $p \in (E' \cup L)^c$, $E' + [p]$ is isotropic and $(E' + k \cdot p) \cap L = 0$. We then have a Lagrangian E' such that $E' \cap L = 0$ by induction. So, L is contained in an open neighbourhood $\Lambda_{E'}(M, 0)$, in which corresponds to the dense nonsingular elements of $Symm(E)$. $\square$

Theorem 1.5.3. *Let $E \in \Lambda(M)$. For $k \geq 1$, the set*

$$\Lambda_E(M, k) = \{L \in \Lambda(M) \mid dim(L \cap E) = k\}$$

is called the regular part of its closure

$$\overline{\Lambda_E(M, k)} = \bigcup_{l \geq k} \Lambda_E(M, l)$$

which is a connected submanifold of $\Lambda(M)$ of codimension $\frac{1}{2}k(k+1)$.

Proof. By Theorem 1.5.2, one can deduce that $\Lambda_E(M, 0) \cap \Lambda_L(M, 0)$ is also dense in $\Lambda(M)$. So, say $\Lambda_E(M, 0) \cap \Lambda_L(M, 0) \subset N \in \Lambda(M)$. Choose a coordinate given by Q. Then, for $L \in \Lambda_E(M, 0)$, must have $L' \in \Lambda_N(M, k) \cap \Lambda_E(M, 0)$ if and only if the kernel of $Q(E) - Q(L')$ is of dimension k. For instance, by definition, we have $Q(L) = 0$. Choose a basis for L, we can represent the form Q as

$$Q(E) = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad Q(L') = \begin{pmatrix} A & C \\ C^T & D' \end{pmatrix}$$

where $D \in GL(n-k, \mathbb{R})$. Then, one observes that $D - D'$ will still be invertible if D is small enough, so $rank(Q(E) - Q(L')) = n - k$ if and only if

$$A = -CS, \quad C^T = (D - D')S$$

for some S. Then, $L' \in \Lambda_E(M, k)$ is close to L is equivalent to say $A = -C(D - D')^{-1}C^T$. This shows that $\Lambda_E(M, k)$ is a smooth submanifold of codimension $\frac{1}{2}k(k+1)$. Note that if we choose a subspace $F \subset E$ with $\dim F = k$, then $F \subset E = E^{\perp} \subset F^{\perp}$. $(F^{\perp}/F, \omega_{/F})$ will be a symplectic vector space equipped with symplectic form $\omega_{/F}$ given by

$$\omega_{/F}(c_1 + F, c_2 + F) = \omega(c_1, c_2)$$

for c_1, c_2 in $F^{\perp}$. Suppose $L \in \Lambda(M)$ with $L \cap E = F$, so $F \subset L = L^{\perp} \subset F^{\perp}$. Then, one sees that the fiber over F is isomorphic to $\Lambda(F^{\perp}/F)$ via isomorphism $L \mapsto L/F$. But since $\Lambda(F^{\perp}/F)$ is connected, thus so is $\Lambda_E(M, k)$. $\square$

With all this setup, we are ready to show an important ingredient in claiming the Maslov index is an integer-valued map.

Theorem 1.5.4. $\overline{\Lambda_E(M,1)}$ *defines an oriented codimension 1 cycle* Θ *in* $\Lambda(M)$. *Furthermore, its fundamental group* $\pi_1(\Lambda(M))$ *is isomorphic to* $\mathbb{Z}$. *A generator in the cohomology* $H^1(\Lambda(M),\mathbb{Z})$ *is defined via a map*

$$\alpha : [\gamma] \to [\gamma] \cdot \Theta$$

which is the intersection number of $[\gamma] \in \pi_1(\Lambda(M))$ *with* Θ.

Proof. It is clear by definition a subvariety in $\Lambda(M)$ lie in the regular part of $\overline{\Lambda_E(M,2)}$ for k at least ≥ 2, so $\Lambda(M)\backslash\Lambda_E(M,0)$ defines a chain of codimension 1 in $\Lambda(M)$ without boundary. The orientation is determined by: it is a positive cycle at $L \in \Lambda_E(M,1)$ if the set of all $\xi \in T_L(\Lambda(M))$ with corresponding Q on L is positive definite on $L \cap E$. Next, we claim that $\pi(\Lambda(M)) \cong \mathbb{Z}$. To see this, we first show that α is an injection and there exists an $[\gamma'] \in \pi(\Lambda(M))$ such that $\alpha([\gamma']) = \gamma' \cdot \Theta = 1$, where 1 is the generator in $\mathbb{Z}$.

Let $\gamma(t)$ be a differentiable closed path that represents the class $[\gamma]$, such that $\gamma(t)$ intersects transversally the stratum $\Lambda_E(M,1) \subset \Lambda(M)\backslash\Lambda_E(M,0)$ finitely many times at regular crossings t (formal definition will be given in the Chapter 2). So, the product $[\gamma]\cdot\Theta$ is given by

$$[\gamma]\cdot\Theta = \sum_{\gamma(t)\in\Lambda_E(M,1)} \pm 1$$

where $+1$ when $\gamma(t)$ crosses $\Lambda_E(M,1)$ from $-$ to $+$ and -1 the other way around. Assume that in between two opposite crossings at t_1 and t_2, $\gamma(t)$ lies in stratum $\Lambda_E(M,0)$ for $t \in (t_1,t_2)$. Choose a differentiable path $\delta \in \Lambda_E(M,1)$ that connects $\gamma(t_1)$ and $\gamma(t_2)$. For arbitrarily small $\epsilon > 0$, such that in each small neighbourhood of crossings t_1, t_2, approximate $\gamma(t)$ by two paths δ_+, δ_- such that δ_+ connects $\gamma(t_1+\epsilon)$ with $\gamma(t_2-\epsilon)$ and δ_- connects $\gamma(t_1-\epsilon)$ with $\gamma(t_2+\epsilon)$. Since $\Lambda_E(M,0)$ is connected, on $t_1+\epsilon \leq t \leq t_2-\epsilon$, the path $\gamma(t)\#(\delta_+)^{-1}$ is a contractible loop (if necessary, since $\Lambda_E(M,0)$ is dense, we may "smooth" the two endpoints by concatenate smooth paths in $\Lambda_E(M,0)$). On the other hand, there is another contractible loop: consider

$$\gamma(t)|_{t\in[t_1-\epsilon,t_1+\epsilon]} \longrightarrow \delta_+ \longrightarrow \gamma(t)|_{t\in[t_2-\epsilon,t_2+\epsilon]} \longrightarrow (\delta_-)^{-1}.$$

It is not hard to see there are 4 intersections (crossings) with $\Lambda_E(M,1)$. Then, find a path γ' that homotopic to $\gamma(t)$ for $t \leq t_1-\epsilon$ and $t \geq t_2+\epsilon$. In this case there are only 2 crossings. Repeat again to eliminate all intersections, so γ' is homotopic to a point, which leads to $[\gamma']\cdot\Theta = 0$. We now find $[\gamma_0]$ such that

$[\gamma_0]\cdot\Theta = 1$: For case $k = 2$, let $P \subset M$ be a nonisotropic subspace of dimension 2. Then, (P, ω) is a symplectic vector space such that $M = P \oplus P^{\perp}$. Using usual coordinates in P, we may represent a closed path γ_P in $\Lambda(P)$ as

$$\gamma_P(\theta) = \{(x\cos\theta, x\sin\theta) \mid x \in \mathbb{R}\}$$

for $\theta \in [0, \pi)$. It is clear that γ_P intersects the $y-$axis only at $\pi/2$. Then, we compute[1] its derivative at $\pi/2$ in terms of quadratic form:

$$y \mapsto \frac{d}{d\theta}\omega\left(\left(y\frac{\cos\theta}{\sin\theta}, 0\right), (0, y)\right)\bigg|_{\theta=\pi/2} = y^2.$$

Note that we mod $\sin\theta$ out so that ω is nondegenerate. This implies that the intersection is positive. Now, to lift to M, let L_0 and E_0 be $\in \Lambda(P)$ such that $L_0 \cap E_0 = 0$. We define

$$\gamma_0(\theta) = \gamma_P(\theta) + L_0$$

and

$$E = \gamma_P(\pi/2) + E_0.$$

Then, $\gamma_0(\theta)$ must only intersects regular $\Lambda_E(M,1) \subset \overline{\Lambda_E(M,1)}$ at $\theta = \pi/2$, in the positive direction and hence $[\gamma_0]\cdot\Theta = 1$. $\square$

1.6 Action of $Sp(2n)$ and $U(n)$ on $\Lambda(n)$

We learn two important Lie groups: the Symplectic group $Sp(2n)$ and the Unitary group $U(n)$ in the previous section. Now, we study their actions acting on the Lagrangian Grassmannian $\Lambda(n)$.

Lemma 1.6.1. *$U(n)$ acts transitively on $\Lambda(n)$.*

Proof. To show transitivity, we need to show that for all Lagrangian pair (l, l'), where $l, l' \in \Lambda(n)$, there exists $U \in U(n)$ such that $l' = Ul$. Choose a basis of $\mathbb{C}^n$ such that $l' = \mathrm{Span}\{e_1, \ldots, e_n\}$ and $l = \mathrm{Span}\{d_1, \ldots, d_n\}$. Since any two vectors in $\mathbb{C}^n$ are related by a complex linear invertible transformation, say $U \in GL(n, \mathbb{C})$. Then, under Hermitian structure, the matrix U associated to the Hermitian metrics is Hermitian, i.e. for $a \in l,\ b \in l'$

$$\langle a, b\rangle = a^T u\bar{b}.$$

[1]If we use the method in [17], i.e. let $Z(t) = (x\cos(t), x\sin(t))$ and $v = (Z(0)u)$, $Q(v) = \langle xu\cos 0, xu\cos 0\rangle - \langle xu\sin 0, xu\sin 0\rangle = x^2u^2$, choose $u = 1$ and take $x = y$ we arrive at the same answer.

Then, by equality of $\langle a,b\rangle = \overline{\langle b,a\rangle}$, we get

$$\overline{\langle b,a\rangle} = \overline{b^T U \bar{a}} = \bar{b}^T \overline{U} a = a^T \overline{U}^T \bar{b} = a^T U \bar{b} = \langle a,b\rangle.$$

Hence, U is Hermitian, i.e. $U \in U(n)$. □

Lemma 1.6.2. *The stabilizer group of the transitive $U(n)-$action on $\Lambda(n)$ is $O(n,\mathbb{R})$.*

Proof. The stabilizer group of such action consists of all $U \in U(n)$ such that $Ul_0 = l_0$. Let $l_0 = \text{Span}\{e_1,\dots,e_n\} \in \Lambda(n)$, then $Ul_0 = \text{Span}\{Ue_1,\dots,Ue_n\}$. This means that $Ue_i \in \text{Span}\{e_1,\dots e_n\}$, i.e. all entries of U must be real. We get $U^* = \bar{U}^T = U^T = U^{-1}$, so $U \in O(n,\mathbb{R})$. □

Theorem 1.6.3. $\Lambda(n) \cong U(n)/O(n)$.

Proof. This follows directly from the result of Lemma 1.6.1 and Lemma 1.6.2. □

In fact, we also have general result of transitivity of $Sp(2n)-$action on transversal Lagrangian pair on $\Lambda(n)$:

Theorem 1.6.4. *Let (l_1,l_1') and (l_2,l_2') be two Lagrangian pairs such that $l_1 \cap l_1' = l_2 \cap l_2' = 0$. Then, there exists $S \in Sp(2n)$ such that $(l,l') = (Sl,Sl')$.*

Proof. First, choose a basis $\{e_{11},\dots,e_{1n}\}$ of l_1 and a basis $\{d_{11},\dots,d_{1n}\}$ of l_1' such that $\{e_{1i}\}_{i=1}^n \cup \{d_{1j}\}_{j=1}^n$ is a symplectic basis of $(\mathbb{R}^{2n},\omega)$. We do the same for l_2 and l_2', denote the basis as $\{e_{2i}\}_{i=1}^n \cup \{d_{2j}\}_{j=1}^n$. Then, by defining a linear map $S:\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $S(e_{1i}) = e_{2i}$ and $S(d_{1j}) = d_{2j}$. we obtain

$$\begin{aligned}\omega(S(e_{1i}),S(e_{1j})) &= \omega(e_{2i},e_{2j}) = 0\\ \omega(S(d_{1i}),S(d_{1j})) &= \omega(d_{2i},d_{2j}) = 0\\ \omega(S(e_{1i}),S(d_{1j})) &= \omega(e_{2i},d_{2j}) = \delta_{ij}\end{aligned} \tag{1.6.1}$$

which is exactly saying that S is symplectic. □

1.7 Some facts about $Sp(2n)$

Recall that an invertible matrix S is in $Sp(2n)$ if and only if $S^T JS = J$.

Proposition 1.7.1. *Every symplectic matrix $S \in Sp(2n)$ has determinant 1.*

Proof. By taking determinant on both sides,

$$det(S^T JS) = det(S^T)det(J)det(S) = det^2(S)det(J) = det(J) = 1$$

which implies

$$det(S) = \pm 1.$$

But, by taking the Pfafian of $S^T JS$, we see that

$$Pf(S^T JS) = det(S)Pf(J) = Pf(J)$$

and by recalling that $Pf(J) = 1$, we have $det(S) = 1$. □

By simple manipulation, we see that

$$S^T = JS^{-1}J^{-1}$$

which means that S^T and S^{-1} are similar to each other. Also, one checks that in fact S, S^T and S^{-1} are similar to each other. Let λ be an eigenvalue of symplectic matrix S, written as $\lambda \in \sigma(S)$, then so is λ^{-1} due to similarity between S and S^{-1}, which means the eigenvalue of S always comes in pair. If $\lambda \in \sigma(S) \cap \mathbb{C}$, then so is $\bar{\lambda}$ and $\bar{\lambda}^{-1}$. It follows that we have the following result:

Proposition 1.7.2. *Let $S \in Sp(2n)$, $(\lambda, 1/\lambda)$ be a pair of eigenvalue of S. Then,*

$$det(S - \lambda Id) = \lambda^{2n} det(S - \frac{1}{\lambda} Id).$$

Proof. We have $S = -J(S^T)^{-1}J$, then

$$\begin{aligned} S - \lambda Id &= -J(S^T)^{-1}J - \lambda Id \\ &= J\big[(-S^T)^{-1} + \lambda Id\big]J \\ &= J\big[(S^T)^{-1}\big(- Id + \lambda S^T\big)\big]J \end{aligned} \tag{1.7.1}$$

So,

$$\begin{aligned} det(S - \lambda Id) &= det\big(J\big[(S^T)^{-1}\big(- Id + \lambda S^T\big)\big]J\big) \\ &= det(J)^2 det(S^T)^{-1} det\big(- Id + \lambda S^T\big) \\ &= det\big(- Id + \lambda S^T\big) \\ &= \lambda^{2n} det\big(S^T - \frac{1}{\lambda} Id\big) \\ &= \lambda^{2n} det\big(S - \frac{1}{\lambda} Id\big). \end{aligned} \tag{1.7.2}$$

The last equality follows from the fact that the characteristic polynomial of two similar matrices is the same. □

Proposition 1.7.3. *Let $Sz_1 = \lambda_1 z_1$ and $Sz_2 = \lambda_2 z_2$ where $S \in Sp(2n)$ and $\lambda_1, \lambda_2 \in \sigma(S)$. If $\lambda_1\lambda_2 \neq 1$, then $\omega(z_1, z_2) = 0$.*

Proof. This follows directly from the property of ω and S :

$$\begin{aligned}\langle z_1, Jz_2\rangle &= \langle z_1, S^T JSz_2\rangle \\ &= \langle Sz_1, JSz_2\rangle \\ &= \langle \lambda_1 z_1, J\lambda_2 z_2\rangle \\ &= \lambda_1\lambda_2\langle z_1, Jz_2\rangle = \lambda_1\lambda_2\omega(z_1, z_2)\end{aligned}$$

Since $\lambda_1\lambda_2 \neq 1$, $\omega(z_1, z_2)$ must vanish. □

In [20], Salamon and Zehnder constructed an explicit map $\rho : Sp(2n) \to S^1$, [2] which will be used in later chapter.

Theorem 1.7.4. *For all $n \in \{1, 2, \ldots\}$, there exists a continuous map*

$$\rho : Sp(2n, \mathbb{R}) \longrightarrow S^1 \tag{1.7.3}$$

such that

1. *(Naturality) Let $S, T \in Sp(2n, \mathbb{R})$. Then*

$$\rho(TST^{-1}) = \rho(S).$$

2. *(Direct sum) Let $S \in Sp(2m, \mathbb{R})$ and $S' \in Sp(2n, \mathbb{R})$. Then,*

$$\rho(S \oplus S') = \rho\begin{pmatrix} S & 0 \\ 0 & S'\end{pmatrix} = \rho(S)\rho(S')$$

3. *(Determinant) Under the identification, $U = X + iY \in U(n)$ can be written as $U = \left(\begin{smallmatrix} X & -Y \\ Y & X\end{smallmatrix}\right) \in Sp(2n, \mathbb{R})$. Then,*

$$\rho(U) = det_{\mathbb{C}}(X + iY).$$

Moreover, ρ induces an isomorphism

$$\rho_* : \pi_1(Sp(2n, \mathbb{R})) \longrightarrow \pi_1(S^1) \cong \mathbb{Z}.$$

4. *(Normalization) Let $S \in Sp(2n)$ and suppose $\sigma(S) \subset \mathbb{R}$, then*

$$\rho(S) = (-1)^{\frac{m_0}{2}}$$

where m_0 is the total multiplicity of all $\lambda \in \sigma(S) \cap \mathbb{R}_-$.

[2] Another detail proof is given in the Appendix 7.3 of [3], here we only state the particular properties that ρ satisfied without going through technical proofs.

1.8 The Lie algebra of Sp(2n)

Since $Sp(2n)$ is a Lie group, we will call the Lie algebra of $Sp(2n)$ the symplectic algebra, written as $\mathfrak{sp}(2n)$. Roughly speaking, there is an one-to-one correspondence between the one-parameter subgroups in $Sp(2n)$ and the elements of $\mathfrak{sp}(2n)$.

Definition 1.8.1. *$\mathfrak{sp}(2n)$ is the set of all real matrices X such that for all $t \in \mathbb{R}$ the exponential map e^{tX} lies in $Sp(2n)$.*

In particular, $X \in \mathfrak{sp}(2n)$ if and only if there exists a family $\{S_t = e^{tX}\}$ in $Sp(2n)$ for all $t \in \mathbb{R}$, where $\{S_t\}$ forms a group:

$$S_t \circ S_{t'} = S_{t+t'}, \quad S_t^{-1} = S_{-t}.$$

Explicitly, we can give a characterization of such matrices.

Proposition 1.8.2. *Let X be a real $2n \times 2n$ matrix. Then,*

$$X \in \mathfrak{sp}(2n) \Longleftrightarrow XJ + JX^T = 0$$

Proof. By definition, $X \in \mathfrak{sp}(2n)$ if and only if $S_t = e^{tx} \in Sp(2n)$, so we have

$$\left(e^{tX^T}\right) J \left(e^{tX}\right) = J.$$

By differentiating with respect to t,

$$\begin{aligned} 0 = \frac{d}{dt}\bigg|_{t=0} J = \frac{d}{dt}(e^{tX^T} J e^{X})\bigg|_{t=0} &= \left(X^T e^{tX^T} J e^{tX}\right)\bigg|_{t=0} + \left(e^{tX^T} JX e^{tX}\right)\bigg|_{t=0} \\ &= X^T J + JX \end{aligned}$$

So, from $X^T J + JX = 0$ one easily gets $XJ + JX^T = 0$. $\square$

On the other hand, suppose X satisfies $XJ + JX^T = 0$, it is suffice to show that $S_t = e^{tX} \in Sp(2n)$. Since $X^T = JXJ$, then,

$$\begin{aligned} S_t^T = e^{t(JXJ)} &= \sum_{k=0}^{\infty} \frac{(tJXJ)^k}{k!} \\ &= -\sum_{k=0}^{\infty} \frac{(-t)^k}{k!}(tJXJ)^k \end{aligned}$$

since

$$(JXJ)^k = (JXJ)(JXJ)\cdots(JXJ) = (-1)^{k+1} JX^k J$$

with $J^2 = -1$. So, we obtain

$$S_t^T = -J\left(\sum_{k=0}^{\infty} \frac{(-t)^k X^k}{k!}\right) J = -Je^{-tX} J.$$

Finally,

$$S_t^T J S_t = (-Je^{-tX} J) J e^{tX} = Je^{-tX} e^{tX} = J$$

which shows that $S_t \in Sp(2n)$.

Remark 1.8.3. *Equivalently, one can express $\mathfrak{sp}(2n)$ to be the set that consists of all block matrices X such that*

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

with $B = B^T$ and $C = C^T$.

Chapter 2

Maslov Index

2.1 Lagrangian frame and Crossing form

Recall that a Lagrangian plane $L \in \Lambda(n)$ is of dimension n and the symplectic form vanishes when restricted to L.

Definition 2.1.1. *Let $L \in \Lambda(n)$. A Lagrangian frame is an injective linear map $Z : \mathbb{R}^n \to \mathbb{R}^{2n}$ such that $Range(Z) = L$, i.e. a frame has the form*

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

where X, Y are $n \times n$ matrices satisfying the property

$$Y^T X = X^T Y.$$

One could view the first n or last n columns of a symplectic matrix

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n)$$

form a Lagrangian frame.

In particular, we say a Lagrangian frame is *unitary* if those columns are orthonormal in $\mathbb{R}^{2n}$. Then, the space that consists of all such unitary Lagrangian frames Z is diffeomorphic to $U(n)$ given by $Z = (X, Y) \mapsto X + iY$. So as mentioned in previous chapter, one gets a principal bundle over $\Lambda(n)$:

$$O(n) \to U(n) \to \Lambda(n) \cong U(n)/O(n)$$

More generally, we have the principal bundle

$$St(2n) \to Sp(2n) \to \Lambda(n) \cong U(n)/O(n)$$

where $St(2n)$ is the stabilizer group of all $\varphi \in Sp(2n)$, such that it fixes the vertical plane: $\varphi(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$. In this case, the stabilizer group is characterized by the block matrix $B = 0$ of φ.

Recall that if $\mathbb{R}^{2n}$ can be written as a splitting $V \oplus W$, then via symplectic form ω, W can be identified with the dual V^* of V. Next, we give a notion of quadratic form on the tangent space of $\Lambda(n)$ at fixed L.

Theorem 2.1.2. *Let $L(t)$ be a path in $\Lambda(n)$ with initial $L(0) = L$ and its associated tangent vector $\dot{L}(0) = \hat{L}$, then*

1. *Fix a Lagrangian complement W of L. We can write $v + w(t) \in L(t)$ for $v \in L$ and $w(t) \in W$ for small t, then the form is given by*

$$Q(v) = \frac{d}{dt}\bigg|_{t=0} \omega(v, w(t)).$$

Moreover, Q is independent of the choice W.

2. *Let $Z(t) = (X(t), Y(t))$ be a Lagrangian frame of $L(t)$. Then, the form is explicitly given by*

$$Q(v) = \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)u, \dot{X}(u)\rangle$$

where $v = Z(0)u$.

3. *The form Q has the naturality property: for $\varphi \in Sp(2n)$,*

$$Q(\varphi L, \varphi \hat{L}) \circ \varphi = Q(L, \hat{L}).$$

Proof. For (1), WLOG by change of coordinate so that $L(0) = \mathbb{R}^n \times 0$. Then, the Lagrangian complement of L must have the form

$$W = \{(By, y) | y \in \mathbb{R}^n\}$$

where B is a symmetric matrix. We can also write elements of $L(t)$ as

$$L(t) = \{(x, A(t)x) | x \in \mathbb{R}^n\}$$

for symmetric matrix A. So,

$$\begin{aligned} L(t) \ni v + w(t) &= (x, 0) + (By(t), y(t)) \\ &= (x + By(t), y(t)) \\ &= (x + By(t), A(t)(x + By(t))) \end{aligned}$$

By differentiating y w.r.t t,

$$\dot{y}(t) = \dot{A}(t)x + B(\dot{A}(t)y(t) + \dot{y}(t)A(t))$$

when $t = 0$, $\dot{y}(0) = \dot{A}(0)x$. Note that

$$\omega(v, w(t)) = \langle x, y(t)\rangle - \langle 0, By(t)\rangle = \langle x, y(t)\rangle.$$

We evaluate the quadratic form

$$Q(v) = \left.\frac{d}{dt}\right|_{t=0} \omega(v, w(t)) = \left\langle x, \left.\frac{d}{dt}\right|_{t=0} y(t) \right\rangle = \langle x, \dot{y}(0)\rangle = \langle x, \dot{A}(0)x\rangle.$$

For (2), assume $W = 0 \times \mathbb{R}^{2n}$, and choose a frame $Z(t) = (X(t), Y(t))$ for $L(t)$. Given $v = Z(0)u = (X(0)u, Y(0)u)$, and for any $w(t) = (0, y(t)) \in W(t)$, we have

$$\begin{aligned} v + w(t) &= (X(0)u, Y(0)u + y(t)) \\ &= (1, (Y(0)u + y(t))(X(0)u)^{-1}) \end{aligned}$$

By comparing to $Z(t) = (1, Y(t)X(t)^{-1})$, we have

$$(Y(0)u + y(t))(X(0)u)^{-1} = Y(t)X(t)^{-1}$$

rearranging them,

$$y(t)X(t) + Y(0)uX(t) = Y(t)X(0)u$$

differentiating both side w.r.t t,

$$\dot{y}(t)X(t) + y(t)\dot{X}(t) + Y(0)u\dot{X}(t) = \dot{Y}(t)X(0)u$$

So, by taking $t = 0$, we obtain

$$\begin{aligned} \dot{y}(0) &= \dot{Y}(0)X(0)uX^{-1}(0) - y(0)X^{-1}(0)\dot{X}(0) - Y(0)uX^{-1}(00\dot{(}X)(0) \\ &= \dot{Y}(0)u - Y(0)X^{-1}(0)\dot{X}(0)u \end{aligned}$$

On the other hand, we have

$$\omega(v, w(t)) = \langle X(0)u, y(t)\rangle$$

and the form is computed as

$$\begin{aligned} Q(v) &= \langle X(0)u, \dot{y}(0)\rangle \\ &= \langle X(0)u, \dot{Y}(0)u - Y(0)X^{-1}(0)\dot{X}(0)u\rangle \\ &= \langle X(0)u, \dot{Y}(0)u\rangle - \langle X(0)u, Y(0)X^{-1}(0)\dot{X}(0)u\rangle \\ &= \langle X(0)u, \dot{Y}(0)u\rangle - \langle X(0)^{-1}Y(0)^{-1}(X(0)u), \dot{X}(0)u\rangle \\ &= \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)^{-1}X(0)^{-1}(X(0)u), \dot{X}(0)u\rangle \\ &= \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)^{-1}, \dot{X}(0)u\rangle \end{aligned}$$

in which the last third equality is due to the identity X^TY is symmetric. For (3), we see that

$$
\begin{tikzcd}
(L,\hat{L}) \arrow[r, "\varphi"] \arrow[d, "Q"'] & (\varphi L, \varphi\hat{L}) \arrow[d, "Q"] \\
Q(L,\hat{L}) \arrow[r, "\cong"'] & Q(\varphi L, \varphi\hat{L})
\end{tikzcd}
$$

where $Q(L,\hat{L}) \cong Q(\varphi L, \varphi\hat{L})$ follows from the definition and the fact that ω is unchanged with the property of symplectic matrix $\varphi^T J\varphi = J$ when bringing into the metric. So, we have

$$Q(\varphi L, \varphi\hat{L}) \circ \varphi = Q(L,\hat{L}).$$

$\square$

At 'point' $\tilde{L} \in \Lambda_L(n,k)$, the tangent space to $\Lambda_L(n,k)$ is given by

$$T_{\tilde{L}}\Lambda_L(n,k) = \{\hat{\tilde{L}} \in T_L\Lambda(n) \mid Q(\tilde{L}, \hat{\tilde{L}}|_{\tilde{L}\cap L}) = 0\}. \tag{2.1.1}$$

We next give a notion of 'passing' through the Maslov cycle, called the crossing: let $l(t)$ be a path in $\Lambda(n)$, we define

Definition 2.1.3. *Let $t_0 \in [a,b]$, then t_0 is called a crossing for $l(t)$ if $l(t_0) \cap L \neq \emptyset$, that is, $l(t_0) \in \Lambda(n)_L$.*

One thing to note is that the set of crossings is compact.

Definition 2.1.4. *Let $t_0 \in [a,b]$ be a crossing for $l(t)$. The crossing form is defined by*

$$\Gamma(l, L, t_0) = Q(l(t_0), \dot{l}(t_0))|_{l(t_0)\cap L}. \tag{2.1.2}$$

Remark 2.1.5. *$\Gamma(l,L,t_0)$ is an example of the quadratic form Q defined above. So, it satisfies the naturality property: for any symplectic matrix $\varphi \in Sp(2n)$,*

$$\Gamma(\varphi l, \varphi L, t_0) \circ \varphi = \Gamma(l, L, t_0). \tag{2.1.3}$$

Theorem 2.1.6. *There exists a unique function μ that assigns each continuous Lagrangian path $L \in \Lambda(n)$ an integer such that it satisfies the following properties:*

1. *(**Naturality**) Let $\varphi \in Sp(2n)$ and $L, L' \in \Lambda(n)$. Then,*

$$\mu(\varphi L, \varphi L') = \mu(L, L') \tag{2.1.4}$$

2. ***(Concatenation)*** *Suppose $c \in (a,b) \subset \mathbb{R}$ and L concatenate with L' at c, then*

$$\mu(L\#_c L') = \mu(L|_{[a,c]}, L'|_{[a,c]}) + \mu(L|_{[c,b]}, L'|_{[c,b]}) \tag{2.1.5}$$

3. ***(Homotopy)*** *Let $L, L' : [a,b] \to \Lambda(n)$ with $L(a) = L'(a)$ and $L(b) = L'(b)$ are homotopic with fixed endpoints if and only if their Maslov index are the same.*

4. ***(Zero)*** *Let $L, L' : [a,b] \to \Lambda_L(n,k)$, then*

$$\mu(L, L') = 0 \tag{2.1.6}$$

5. ***(Direct sum)*** *Let $M = M_1 \oplus M_2$, then*

$$\mu(L_1 \oplus L_2, L'_1 \oplus L'_2) = \mu(L_1, L'_1) + \mu(L_2, L'_2) \tag{2.1.7}$$

6. ***(Normalization)*** *Consider $(\mathbb{R}^{2n}, \omega)$, $L(t) = Gr(A(t))$ for $A(t) \in \mathbb{R}^{n\times n}$ a path of symmetric matrices and $L'(t) = \mathbb{R}^n \times 0$. Then,*

$$\mu(L, L') = \frac{1}{2} sgn\ A(b) - \frac{1}{2} sgn\ A(a). \tag{2.1.8}$$

Computational wise, we may use the notion of Lagrangian frame and crossing operator introduced before.

Remark 2.1.7. *Let denote $V = \mathbb{R}^n \times 0$. and let $Z(t) = (X(t), Y(t))$ be a Lagrangian frame for $L(t)$ and $v \in Z(t)u$, that is $v = (X(t)u, 0)$, then the crossing operator at crossing t is given by*

$$\Gamma(L, V, t)(v) = \langle X(t)u, \dot{Y}(t)u\rangle - \langle Y(t)u, \dot{X}(t)u\rangle = \langle X(t)u, \dot{Y}(t)u\rangle \tag{2.1.9}$$

because $Y(t)u = 0$. Conversely, when $V = 0 \times \mathbb{R}^n$, then $X(t)u = 0$ and for $v = (0, Y(t)u)$, we have

$$\Gamma(L, V, t)(v) = \langle X(t)u, \dot{Y}(t)u\rangle - \langle Y(t)u, \dot{X}(t)u\rangle = -\langle Y(t)u, \dot{X}(t)u\rangle. \tag{2.1.10}$$

Remark 2.1.8. *Remark 2.1.7 actually make sense for the Lagrangian path crosses a plane in certain orientation and gives different Maslov indices. For instance, take $n = 1$, $V = \mathbb{R}$. For the case $V = \mathbb{R}^n \times 0$, one can think of $L(t)$ passes the $x-$axis and move in anti-clockwise direction towards he $y-$axis, so the Maslov index is positive. Conversely, for $V = 0 \times \mathbb{R}^n$, $L(t)$ crosses the $y-$axis and in clockwise to the $x-$axis, so the Maslov index is negative.*

Remark 2.1.9. *Certainly, a loop is a special case of a continuous path. Let $L(t) = L(t+1)$ be a Lagrangian loop and choose $Z(t) = (X(t), Y(t))$ as a lift of unitary frame. Then, for any Lagrangian subspace V, the Maslov index for Lagrangian loop is given by*

$$\mu(L, V) = \frac{\alpha(1) - \alpha(0)}{\pi}$$

where $\alpha(t)$ is a continuous map satisfying

$$det(X(t) + iY(t)) = e^{i\alpha(t)}.$$

(Compare this with Definition 2.3.3).

Remark 2.1.10. *Theorem 2.1.6 can be translated to the case for paths of symplectic matrices $\varphi(t) \in Sp(2n)$, in which we consider $Sp(2n)$ as the union of strata of codimension $k(k+1)/2$*

$$Sp_k(2n) = \{\varphi \in Sp(2n) \mid dim(\varphi V \cap V) = k\}$$

for $V = 0 \times \mathbb{R}^n$. The Maslov index is given by

$$\mu(\varphi) = \mu(\varphi V, V).$$

It can be seen as the intersection number of $\varphi(t)$ with the Maslov cycle

$$\overline{Sp}_1(2n) = Sp(2n) \backslash Sp_0(2n) = \bigcup_{i=1}^{n} Sp_k(2n).$$

In particular, $Sp_k(2n)$ is related to $\Lambda_L(n, k)$ by the fibration

$$\begin{aligned} Sp(2n) &\longrightarrow \Lambda(n) \\ \varphi &\longmapsto \varphi V \end{aligned}$$

To be more explicit, $\varphi \in Sp_k(2n)$ if and only if rank $B = n - k$ where B is the block matrix in

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In the case $k = 0$, $\varphi \in Sp_0(2n)$ if and only if $det(B) \neq 0$. Suppose $(B, D)^T$ be a Lagrangian frame for φV, then the corresponding crossing form is

$$\begin{aligned} \Gamma(\varphi, t) : ker\ B(t) &\longrightarrow \mathbb{R} \\ y &\longmapsto -\langle D(t)y, \dot{B}(t)y \rangle. \end{aligned}$$

The Maslov index $\mu(\varphi)$ satisfies the axioms

1. ***(Homotopy)*** *Let* $\varphi_1, \varphi_2 : [a,b] \to Sp(2n)$. *If* φ_1 *is homotopic with* φ_2 *with fixed endpoints, i.e.* $\varphi_1 \simeq \varphi_2$ *with* $\varphi_1(a) = \varphi_2(a)$ *and* $\varphi_1(b) = \varphi_2(b)$, *then*
$$\mu(\varphi_1) = \mu(\varphi_2).$$
2. ***(Concatenation)*** *Let* $\varphi : [a,b] \to Sp(2n)$ *and let* $c \in (a,b)$, *such that* $\varphi(t) = \varphi_1(t) \#_c \varphi_2(t)$, *then*
$$\mu(\varphi) = \mu\left(\varphi\big|_{[a,c]}\right) + \mu\left(\varphi\big|_{[c,b]}\right).$$
3. ***(Zero)*** *For all* k, *let* $\varphi \in Sp_k(2n)$, *then* $\mu(\varphi) = 0$.
4. ***(Direct sum)*** *Let* $\varphi = \varphi_1 \oplus \varphi_2 \in Sp(2n) \times Sp(2m)$, *then*
$$\mu(\varphi) = \mu(\varphi_1) + \mu(\varphi_2).$$
5. ***(Normalization)*** *Let* $\varphi = \begin{pmatrix} Id & B(t) \\ 0 & Id \end{pmatrix}$, *for* $t \in [a,b]$, *then*
$$\mu(\varphi) = \frac{1}{2} sgn\ B(b) - \frac{1}{2} sgn\ B(a).$$

Definition 2.1.11. *Given a Lagrangian pair* $(L, L') : [a,b] \in \Lambda(n)$, *and they intersect each other* $L(t) \cap L'(t)$ *for some* t, *we define the relative crossing form on the intersection as*

$$\Gamma(L, L', t) = \Gamma(L, L'(t), t) - \Gamma(L', L(t), t) \tag{2.1.11}$$

Definition 2.1.12. *If* $\Gamma(L, L', t) \neq 0$ *for some* t, *then such crossing* t *is said to be regular.*

Definition 2.1.13. *Let* (L, L') *be a Lagrangian pair with only regular crossing* $t \in \mathbb{R}$. *Then, the relative Maslov index is defined by*

$$\mu(L, L') = \frac{1}{2} sgn\ \Gamma(L, L', a) + \sum_{t \in (a,b)} sgn\ \Gamma(L, L', t) + \frac{1}{2} sgn\ \Gamma(L, L', b) \tag{2.1.12}$$

Remark 2.1.14. *Let* $\varphi(t) \in Sp(2n)$ *be a path of symplectic matrices acting on Lagrangians* L *and* L'. *Recall that the Maslov index is invariant under symplectic action, i.e.*
$$\mu(\varphi L, \varphi L') = \mu(L, L').$$

To see this, it all boils down to study the crossing form by (2.1.12)*. Since* Γ *is natural in the sense of* (2.1.3)*, and by* (2.1.2)*, the pair* $(\varphi L, \varphi L')$ *has only regular crossings if* (L, L') *has only regular crossings. So,*

$$\begin{aligned}
\mu(\varphi L, \varphi L') &= \frac{1}{2} sgn\, \Gamma(\varphi L, \varphi L', a) + \sum_{t \in (a,b)} sgn\, \Gamma(\varphi L, \varphi L', t) + \frac{1}{2} sgn\, \Gamma(\varphi L, \varphi L', b) \\
&= \frac{1}{2} sgn\, \Gamma(L, L', a) + \sum_{t \in (a,b)} sgn\, \Gamma(L, L', t) + \frac{1}{2} sgn\, \Gamma(L, L', b) \\
&= \mu(L, L').
\end{aligned}$$

Example 2.1.1. *Consider the symplectic space* $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ *with symplectic form* $\bar{\omega} = -\omega \times \omega$*. Let* $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ *be the diagonal, then the relative Maslov index is given by*

$$\mu(L, L') = \mu(\Delta, L \times L'). \tag{2.1.13}$$

Since we know the the relative Maslov index μ *is the sum of crossing form* $\Gamma(L, L', t)$ *over all regular crossings* t *with half integer valued at the endpoint of an interval, we only need to show that*

$$\Gamma(\Delta, L \times L', t)(\bar{v}) = \Gamma(L, L', t)(v).$$

We first denote $\bar{L}(t) = L(t) \times L'(t)$*. One readily sees that*

$$\Delta \cap \bar{L}(t) = \{\bar{v} = (v, v) \mid v \in L(t) \cap L'(t)\}.$$

Now, choose a Lagrangian subspace $\bar{W} = W \times W'$ *such that* $W \cap L(t) = 0$ *and* $W' \cap L'(t) = 0$*. For some* $s \in [t - \epsilon, t + \epsilon]$ *we choose* $w(s) \in W$ *and* $w'(s) \in W'$ *such that* $v + w(s) \in L(t)$ *and* $v + w'(s) \in L'$ *and so* $\bar{v} + \bar{w}(s) \in \bar{L}(s)$ *where* $\bar{v} \in \Delta, \bar{w}(s) = (w(s), w'(s))$*. Note that*

$$\bar{\omega}(\bar{v}, \bar{w}(s)) = -\omega(v, w(s)) + \omega(v, w'(s))$$

in which we differentiate w.r.t s *at* t

$$\left.\frac{d}{ds}\right|_{s=t} \bar{\omega}(\bar{v}, \bar{w}(s)) = -\left.\frac{d}{ds}\right|_{s=t} \omega(v, w(s)) + \left.\frac{d}{ds}\right|_{s=t} \omega(v, w'(s))$$

and taking $t = 0$ *we obtain*

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \bar{\omega}(\bar{v}, \bar{w}(t)) &= -\left.\frac{d}{dt}\right|_{t=0} \omega(v, w(t)) + \left.\frac{d}{dt}\right|_{t=0} \omega(v, w'(t)) \\
\Longrightarrow \Gamma(\Delta, L \times L', t)(\bar{v}) &= -\Gamma(L', L(t), t)(v) + \Gamma(L, L'(t), t)(v) \\
&= \Gamma(L, L', t)(v)
\end{aligned}$$

Example 2.1.2. *Now consider instead the symplectic space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with symplectic form $\bar{\omega} = -\omega \times \omega$, together with symplectomorphism $\varphi(t)$ of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Then, the relative Maslov index is given by*

$$\mu(\varphi L, L') = \mu(Gr(\varphi), L \times L').$$

To show this, we define a symplectomorphism

$$\begin{aligned} \bar{\varphi}(t) : \mathbb{R}^{2n} \times \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n} \\ (z, z') &\longmapsto \varphi(t)(z, z') = (z, \varphi(t)z') \end{aligned}$$

then obviously

$$\bar{\varphi}(t)\Delta = \{(z, \varphi(t)z) \mid (z, z) \in \Delta\} = Gr(\varphi).$$

On the other hand, we have

$$\bar{\varphi}(L \times \varphi^{-1}L') = L \times \varphi\varphi^{-1}L' = L \times L'.$$

Hence, we have

$$\begin{aligned} \mu(Gr(\varphi), L \times L') &= \mu(\bar{\varphi}\Delta, \bar{\varphi}(L \times \varphi^{-1}L')) \\ &= \mu(\Delta, L \times \varphi^{-1}L') \\ &= \mu(L, \varphi^{-1}L') \\ &= \mu(\varphi L, L') \end{aligned}$$

where the second and last equality is due to the naturality of μ, third equality follows from previous example.

A Lagrangian homotopy $L(s,t) : [0,1] \times [a,b] \to \Lambda(n)$ is called a stratum homotopy w.r.t $L' \in \Lambda(n)$ if both $L(s,a)$ and $L(s,b)$ lie in the same stratum. That is, there exists $k_a, k_b \in \mathbb{Z}$ correspond to $L(s,a)$ and $L(s,b)$ respectively, such that

$$L(s,a) \in \Lambda_V(n,a), \quad L(s,b) \in \Lambda_V(n,b).$$

In particular, we have the following theorem.

Theorem 2.1.15. *Two Lagrangian paths $L(t)$ and $L'(t)$ are stratum homotopic w.r.t V if and only if they have the same μ, k_a and k_b. They are related by*

$$\mu + \frac{k_a - k_b}{2} \in \mathbb{Z}.$$

For the case of Lagrangian loop, $k_a = k_b$, so the Maslov index is an integer.

It is natural to ask what if we have two Lagrangian pair such that they are homotopic to each other? Will they have the same relative Maslov index?

Corollary 2.1.1. *Let $(L(t), L'(t))$ be a Lagrangian pairs for on an interval $[a, b]$, such that they are transversal at the endpoints:*

$$L(a) \cap L'(a) = 0 \quad L(b) \cap L'(b) = 0.$$

Let $(L(s,t), L'(s,t))$ be a homotopy with $s \in [0,1]$ satisfying the endpoint condition, then

$$\begin{gathered} (L(0,t), L'(0,t)) \simeq (L(1,t), L'(1,t)) \\ \Updownarrow \\ \mu(L(0,t), L'(0,t)) = \mu(L(1,t), L'(1,t)) \end{gathered}$$

Proof. " $\Leftarrow$ " By assumption, the endpoints are transversal

$$L(i,a) \cap L'(i,a) = 0 = L(i,b) \cap L'(i,b) \ \text{ for } i = 0,1$$

and $\mu(L(0,t), L'(0,t)) = \mu(L(1,t), L'(1,t))$. Our aim is to find such homotopy. We first define a smooth path in $Sp(2n)$:

$$\begin{aligned} H_\varphi : [0,1] \times [a,b] &\longrightarrow Sp(2n) \\ (s,t) &\longmapsto \varphi(s,t) \end{aligned}$$

such that

$$\varphi(0,t)L'(0,t) = V, \quad \varphi(1,t)L'(1,t) = V,$$

then we have

$$\begin{aligned} \varphi(i,a)L(i,a) \cap \underbrace{\varphi(i,a)L'(i,a)}_{=V} = 0 \\ \varphi(i,b)L(i,b) \cap \underbrace{\varphi(i,b)L'(i,b)}_{=V} = 0 \end{aligned}$$

for $i = 0, 1$. Then, we have

$$\begin{aligned} &\mu(\varphi(0,t)L(0,t), \varphi(0,t)L'(0,t)) \\ &\quad = \mu(L(0,t), L'(0,t)) \\ &\quad = \mu(L(1,t), L'(1,t)) \\ &\quad = \mu(\varphi(1,t)L(1,t), \varphi(1,t)L'(1,t)) \end{aligned}$$

i.e.

$$\mu(\varphi(0,t)L(0,t),V) = \mu(\varphi(1,t)L(1,t),V)$$

Then, by Theorem 2.1.15, $\varphi(0,t)L(0,t)$ is homotopic to $\varphi(1,t)L(1,t)$ w.r.t V, i.e. we define a homotopy (stratum) from $L(0,t)$ to $L(1,t)$

$$\begin{aligned} H_L : [0,1] \times [a,b] &\longrightarrow \Lambda(n) \\ (s,t) &\longmapsto L(s,t) \end{aligned}$$

such that

$$\varphi(s,a)L(s,a) \cap V = 0, \quad \varphi(s,b)L(s,b) \cap V = 0$$

Lastly, to express the homotopy in terms of Lagrangian pair $(L(s,t), L'(s,t))$, one simply take $L'(s,t) = \varphi^{-1}(s,t)V$. The other direction follows directly from the last example. $\square$

2.2 Hörmander index

In this section, we learn the so-called Hörmander index which is closely related to the Maslov index.

Corollary 2.2.1 (**Hörmander index**[6][11])**.** *Let V_0, V_1, L_0, L_1 be any four Lagrangian subspaces, and suppose there is a path $L(t)$ such that*

$$L(0) = L_0, \quad L(1) = L_1.$$

Then, a half-integer-valued map $\mu_H : \Lambda(n)^4 \to \mathbb{Z}$ given by

$$\mu_H(V_0, V_1; L_0, L_1) = \mu(L, V_1) - \mu(L, V_0).$$

In particular, μ_H is independent of the choice of such $L(t)$ so μ_H is well-defined.

Remark 2.2.1. *μ_H satisfies the following two properties:*

1. *(antisymmetry)*

$$\mu_H(V_0, V_1; L_0, L_1) = -\mu_H(V_0, V_1; L_1, L_0) = \mu_H(V_1, V_0; L_1, L_0)$$

2. *(cocycle)*

$$\mu_H(V_0, V_1; L_0, L_1) + \mu_H(V_0, V_1; L_1, L_2) - \mu_H(V_0, V_1; L_0, L_2) = 0.$$

Definition 2.2.2. *Let $V_0, V_1 \in \Lambda(n)$ be transversal Lagrangian subspaces. Assume that L is transversal to V_1, if $L = \{x + \varphi x \mid x \in V_0\}$ for $\varphi : V_0 \to V_1$, then we define the signature $sgn(V_0, V_1; L)$ to be the signature of the symmetric bilinear form*

$$\begin{aligned} Q_L : V_0 \times V_0 &\longrightarrow \mathbb{Z} \\ (x, y) &\longmapsto \omega(\varphi x, y) \end{aligned}$$

More generally, we have

Lemma 2.2.3. *Let V_0, V_1, L_0, L_1 be 4 Lagrangian subspaces of (M, ω). Suppose V_0 and V_1 are transversal and L_i are transversal to V_j, for all $i, j = 0, 1$. Then the index becomes*

$$\mu_H(V_0, V_1; L_0, L_1) = \frac{1}{2} sgn(V_0, V_1; L_0) - \frac{1}{2} sgn(V_0, V_1; L_1).$$

Proof. Geometrically $\mu_H(V_0, V_1; L_0, L_1)$ can be viewed as the intersection number of smooth path $\gamma(t) : L_1 \to L_0$ in stratum $\Lambda_{V_1}(M, 0)$ with the Maslov cycle $\overline{\Lambda_{V_0}(M, 1)} = \Lambda(M) \backslash \Lambda_{V_0}(M, 0)$ such that $\gamma(t)$ intersect $\overline{\Lambda_{V_0}(M, 1)}$ transversally at the regular part $\Lambda_{V_0}(M, 1)$. So, $sgn(V_0, V_1; \gamma(t))$ changes when $\gamma(t)$ crosses $\Lambda_{V_0}(M, 1)$, in which we show that it changes by ± 2 depending on the crossings. In the case of positive crossings t_0, choose suitable coordinates in V_0 so that $\gamma(t_0)$ intersects V_0 at $x-$axis and the bilinear form is given by

$$Q(\gamma(t_0)) = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$$

where $D \in GL(n-1, \mathbb{R})$ and is symmetric. Note that we can diagonalize the following matrix

$$\begin{aligned} &\begin{pmatrix} I & -C(D')^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & C \\ C' & D' \end{pmatrix} \begin{pmatrix} I & 0 \\ -(D')^{-1} & I \end{pmatrix} \\ = &\begin{pmatrix} A - C(D')^{-1}C' & 0 \\ C' & D' \end{pmatrix} \begin{pmatrix} I & 0 \\ -(D')^{-1} & I \end{pmatrix} \\ = &\begin{pmatrix} A - C(D')^{-1}C' & 0 \\ 0 & D' \end{pmatrix} \end{aligned}$$

for invertible D'. So, we have

$$sgn \begin{pmatrix} A & C \\ C' & D' \end{pmatrix} = sgn\ D' + sgn\ (A - C(D')^{-1}C').$$

What we want is to see the change of signature when $\gamma(t_0)$ has crossing at t_0, so $A(t_0) = C(t_0) = 0$ and $D'(t_0) = D$. Also, $C(t)$ vanishes at t_0 implies the term $C(t_0)(D'(t_0))^{-1}C'(t_0)$ vanishes of second order. Hence, the signature changes by $+2$ if $\dot{A}(t_0)$ is positive and -2 if $\dot{A}(t_0)$ is negative, whilst $sgn\ D'(t)$ is constant. Furthermore, we can establish exactly when $\dot{A}(t_0)$ is positive. We can include the situation at crossing t_0:

$$\gamma(t) = \{x + \rho(t)x \mid x \in V_0\} = \{y + \tilde{\rho}(t)y \mid y \in \gamma(t_0)\}$$

where $\rho(t) : V_0 \to V_1$ and $\tilde{\rho}(t) : \gamma(t_0) \to V_1$. Then,

$$x + \rho(t_0)x + \tilde{\rho}(t)(x + \rho(t_0)x) = x + [\rho(t_0) + \tilde{\rho}(t)(id + \rho(t_0))]x$$

i.e.

$$\rho(t) = \rho(t_0) + \tilde{\rho}(t)(id + \rho(t_0))$$

implies

$$\begin{aligned}\omega(\rho(t)x, y) &= \omega(\rho(t_0)x + \tilde{\rho}(t)x + \tilde{\rho}(t)\rho(t_0)x, y)\\ &= \underbrace{\omega(\rho(t_0)x, y)}_{=0} + \omega(\tilde{\rho}(t)x, y) + \underbrace{\omega(\tilde{\rho}(t)\rho(t_0)x, y)}_{=0}\end{aligned}$$

So,

$$\left.\frac{d}{dt}\right|_{t=t_0} \omega(\rho(t)x, y) = \left.\frac{d}{dt}\right|_{t=t_0} \omega(\tilde{\rho}(t)x, y)$$

for $x, y \in \gamma(t_0) \cap V_0$. This means that $\dot{A}(t_0)$ is positive if and only if the path $\gamma(t)$ intersects $\Lambda_{V_0}(M, 1)$ in the positive direction. $\square$

2.3 Conley-Zehnder index

In this section, we will discuss about paths in the symplectic group $Sp(2n)$ and associate an integer to it, in which we call the Conley-Zehnder index, introduced in [18]. Roughly speaking, the symplectic path that the Conley-Zehnder index associated to is a path of symplectic matrices which starts from the identity and ends at a symplectic matrix that does not have 1 as an eigenvalue.

Recall that the symplectic group $Sp(2n)$ consists of all invertible $2n \times 2n-$real matrices S with characterisation $S^T JS = J$, where $J = \left(\begin{smallmatrix} 0 & Id \\ -Id & 0 \end{smallmatrix}\right)$. We set

$$Sp^*(2n) = \{S \in Sp(2n) \mid det(S - Id) \neq 0\}$$

which is the complement $Sp(2n)\backslash\Sigma$ where

$$\Sigma = \{S \in Sp(2n) \mid det(S - Id) = 0)\}.$$

We also need the following set of paths

$$\mathcal{S} = \{\psi : [0,1] \to Sp(2n) \mid \psi(0) = Id,\ det(\psi(1) - Id) \neq 0\}.$$

Remark 2.3.1. *$Sp^*(2n)$ is not connected. It is the union of two disjoint open sets $Sp^+(2n) \cup Sp^-(2n)$ in which $Sp^+(2n)$ and $Sp^-(2n)$ correspond to $det(S - Id) > 0$ and $det(S - Id) < 0$ respectively. In application, the sign of $det(S - Id)$ coincides with the sign of $\prod(\lambda_i - 1)$ for $\lambda \in \sigma(S) \cap \mathbb{R}_+$.*

We explain the above remark in a more detail fashion.

Lemma 2.3.2. *Fix an $S \in Sp^\pm(2n)$. Then, there exists a path $\psi(t)$ in $Sp^\pm(2n)$ such that connects S to a matrix R, in which all eigenvalues of R are distinct. Moreover, if $S \in Sp^+(2n)$, R has no real positive eigenvalue; otherwise, R has exactly two real positive eigenvalues.*

Proof. For a given $S \in Sp^\pm(2n)$, we claim that we can find a path ψ such that $\psi(0) = S$,

$$\psi(1) = \begin{cases} -Id \in Sp^+(2n) & \text{if } S \in Sp^+(2n), \\ \begin{pmatrix} \begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix} & 0 \\ 0 & -Id \end{pmatrix} & \text{if } S \in Sp^-(2n) \end{cases}$$

The idea in doing this is to first connect S to R followed by either of the two matrices given above. We choose a suitable basis of eigenvectors of R such that under complex conjugation, it is symplectic and invariant. Suppose $\lambda \notin \sigma(R) \cap \mathbb{R}_+$, choose a path $\lambda(t)$ of eigenvalues such that $\lambda(0) = \lambda$ and $\lambda(1) = -1$ without crosses 1. Recall that if λ is a complex eigenvalue of R, then so are $\bar{\lambda}, 1/\lambda$ and $1/\bar{\lambda}$. So, the path for $\bar{\lambda}$ is $\bar{\lambda}(t)$ and so on. Then, one can define the path $R(t)$ as

$$R(t)\xi_\lambda = \begin{cases} \lambda(t)\xi_\lambda & \text{if } \lambda \notin \mathbb{R}_+, \\ \lambda\xi_\lambda & \text{if } \lambda \in \mathbb{R}_+ \end{cases}$$

with $R(0) = R$, and $R(1)$ equals to $-Id$ if $\lambda \notin \sigma(R) \cap \mathbb{R}_+$ and if $\lambda \in \sigma(R) \cap \mathbb{R}_+$,

$$R(1) = \begin{pmatrix} \begin{smallmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{smallmatrix} & 0 \\ 0 & -Id \end{pmatrix}$$

Note that such $R(t)$ exists in $Sp^{\pm}(2n)$ because $R(t)$ does not touches 1. Now for the latter case, that is $R(1)$ has two (distinct) real positive eigenvalues λ and $1/\lambda$, since $Sp(2n,\mathbb{R})$ is path connected, we can find a path $R'(t)$ that connects $R(1) = R'(0)$ to the matrix of same form in canonical basis, then to

$$R'(1) = \begin{pmatrix} \begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix} & 0 \\ 0 & -Id \end{pmatrix}.$$

So, the desired claimed path is $\psi = R' \circ R$. This shows that $Sp^{\pm}(2n)$ are connected components of $Sp^*(2n)$. □

Example 2.3.1. *We look at the case $n = 1$, that is the group $Sp(2)$, which is equivalent to $SL(2,\mathbb{R})$. The elements are real matrices $S = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that $AD - BC = 1$. By polar decomposition, it retracts onto the $U(1)$ that consists of matrices $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. The hypersurface Σ that corresponds to the term $det(S - Id) = 0$ is*

$$\Sigma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, AD - BC = 1,\ Tr(S) = A + D = 2 \right\}.$$

It is smooth outside of Id and the complement of Σ is the union of two open sets characterized by $Tr(S) < 2$ and $Tr(S) > 2$. Then, one can observe that the set with $Tr(S) > 2$ consists of matrices that are similar to the matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} (\lambda > 0).$$

On the other hand, the matrices in the set with $Tr(S) < 2$ are similar to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ or \ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} (\lambda < 0) \ or \ \begin{pmatrix} -1 & F \\ 0 & -1 \end{pmatrix}$$

where all these matrices can be connected to $-Id$.

Corollary 2.3.1. *The inclusion $i : Sp^*(2n,\mathbb{R}) \hookrightarrow Sp(2n,\mathbb{R})$ induces the trivial morphism on the fundamental group.*

Proof. Note that there exists two continuous lift maps $\tilde{\rho}^{\pm} : Sp^{\pm}(2n) \to \mathbb{R}$ such that

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\exp"] & S^1 \\ Sp^{\pm}(2n,\mathbb{R}) \arrow[u, "\tilde{\rho}^{\pm}"] \arrow[r, "i"'] & Sp(2n,\mathbb{R}) \arrow[u, "\rho"'] \end{tikzcd}$$

commutes, so the induced morphisms are

$$\exp_* \circ \tilde{\rho}^{\pm}_* = \rho_* \circ i_* = 0$$

which implies $i_* : \pi_1(Sp^{\pm}(2n, \mathbb{R})) \to \pi_1(Sp(2n, \mathbb{R}))$ must be zero. $\square$

Now, we give the definition of Maslov index associated to a symplectic path:

Definition 2.3.3. *Let $\gamma : [0,1] \to Sp(2n)$ and*

$$\begin{aligned} \alpha : [0,1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \rho(\gamma(t)) = e^{i\alpha(t)} \end{aligned}$$

be a lift of composed map $\rho \circ \gamma$, i.e.

$$\begin{tikzcd} & & \mathbb{R} \arrow[d] \\ I \arrow[r, "\gamma"] \arrow[urr, "\alpha"] & Sp(2n) \arrow[r, "\rho"] & S^1 \end{tikzcd}$$

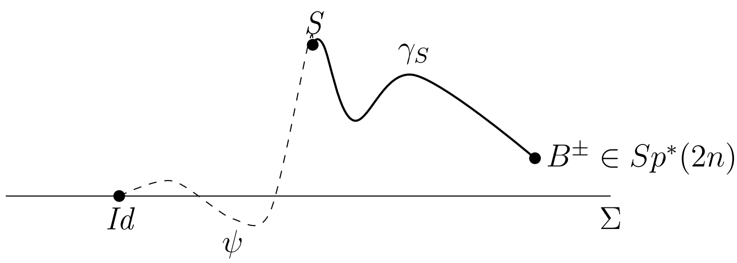
Then, we define

$$\Delta_s(\gamma) = \frac{\alpha(s) - \alpha(1)}{\pi}$$

Remark 2.3.4. *Suppose $S \in Sp^*(2n)$, then we can choose a path γ_S that connects S to either $B^+ = -Id$ or $B^- = diag(2, \frac{1}{2}, -1, \dots)$ as discussed above so that the whole path lie within the connected component of $Sp^*(2n)$. The homotopy class of γ_S is hence well defined, so $\Delta_1(\gamma_S)$ is independent of the choice of γ_S. Clearly, γ_S depends only on the choice of S. Now, suppose there is a path $\psi(t) \in Sp(2n)$ such that $\psi(0) = Id \in \Sigma$ and $\psi(1) = S = \gamma(0)$. Then, in this case the Maslov index is given by*

$$\mu_s(\psi) = \Delta_s(\psi) + \Delta_1(\psi(s))$$

This can be seen as the number of clockwise "half turns" traced on S^1 via the composition $\rho \circ \gamma_s \circ \psi$.



Proposition 2.3.5. *The Maslov index μ is an integer. In particular, for two paths ψ, ψ', $\mu(\psi) = \mu(\psi')$ if and only if $\psi \simeq \psi'$ with endpoint in $Sp^*(2n)$. Furthermore, we have*

1. *sign $det(\psi(1) - Id) = (-1)^{\mu(\psi)-n}$*
2. *Let $S \in GL(2n, \mathbb{R})$ with $S = S^T$ and $\|S\| < 2\pi$. Also, let path $\psi(t) = e^{tJS}$, then*
$$\mu(\psi) = \nu(S) - n$$
where $J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ and $\nu(S)$ is the total number of negative eigenvalues of S.

Proof. Let $\psi(s) \in \mathcal{S}$, for $s \in [0, 1]$. Extend $\psi(s)$ to a smooth path $\gamma : [0, 2] \to Sp(2n)$, i.e. $\psi\big|_{[0,1]} = \gamma\big|_{[0,1]}$. Then, since $\rho(B^{\pm}) = \pm 1$, which can be determined by condition (3) of Theorem 1.7.4, we have $\mu_1(\psi) = \Delta_2(\gamma) \in \mathbb{Z}$. For homotopic property, follow by Lemma 2.3.2, as long as the endpoints in $Sp^*(2n)$ are fixed, then paths ψ and ψ' are homotopic if and only if the extended path γ and γ' are homotopic. Moreover, since ρ induces an isomorphism on their fundamental groups, we have $\Delta_2(\gamma) = \Delta_2(\gamma')$ and $\mu_1(\psi) = \mu_1(\psi')$.

Next, consider a path γ as the extension of ψ as above. We have $\psi(1) = S$ and $\gamma(2) = B^+ = -Id \in Sp^+(2n)$, then, by (2) of Theorem 1.7.4,
$$\rho(B^+) = \underbrace{\rho(-Id_{2\times 2})\rho(-Id_{2\times 2}) \cdots \rho(-Id_{2\times 2})}_{n}$$

by (3), take $X = -1$ and $Y = 0$, so $\rho(-Id_{2\times 2}) = -1$, implies $\rho(B^+) = (-1)^n$, which corresponds to $det(\psi(1) - Id) > 0$, so $\mu(\psi) - n$ must be even, or in other word they have the same parity. Similarly, for the case $\Psi(1) = B^- = diag(2, \frac{1}{2}, -1, \ldots)$, one gets $\rho(B^-) = (-1)^{n-1}$, which corresponds to the case $\mu(\psi) - n$ have different parity.

For the second point, we use the fact that if S is symmetric, then e^{tJS} is symplectic (a non trivial result) and if S is symmetric, then S is diagonalizable in an orthonormal basis. We can define a path $s \mapsto R(s)$ in $O^+(2n)$ for $s \in [0, 1]$ that connects $R(0) = Id$ to $S(1) = R^T(1)SR(1)$ which is a diagonal matrix. We may also define a path
$$S(s) = R^T(s)SR(s)$$

and

$$\psi(s,t) = e^{tJS(s)}.$$

Note that $e^{tJS(s)}$ does not have 1 as eigenvalue is ensured by the assumption $\|S\| < 2\pi$. Moreover, $\mu(\psi(s,t))$ does not depend on the parameter s and is well defined for any paths that lie in the connected component of $Sp^*(2n)$. In other words, we may directly calculate $\mu(e^{tJS})$ without worrying about s. So, we may only consider the case where a symmetric matrix is diagonalized and have eigenvalues π or $-\pi$. To simplify the calculation further, we treat $\mathbb{R}^{2n}$ as the direct sum of $n-$many $\mathbb{R}^2$ planes, i.e. to consider only three types of matrices

$$\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}, \begin{pmatrix} -\pi & 0 \\ 0 & -\pi \end{pmatrix}, \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}$$

where each is in $Sp(2)$ and the eigenvalues are placed on the diagonal when in $\mathbb{R}^{2n}$, in which we can then apply property (3) of Theorem 1.7.4. Now, we examine the first matrix: when $S = diag(\pi, \pi)$, and J is treated as multiplication by i in the exponential , so e^{tJS} corresponds to a rotation matrix by πt

$$e^{tJS} = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix},$$

i.e. the unitary matrix $e^{\pi i t} \in U(1) \cong S^1$. Its image is $e^{\pi i t} \in S^1$ under the map ρ. Its associated Maslov index is computed as

$$\mu(e^{tJS}) = \nu(S) - n = 0 - 1 = -1$$

since S has no negative eigenvalues. For the second case, $S = diag(-\pi, -\pi)$, in a similar fashion, by considering the matrix

$$e^{tJS} = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix},$$

we obtain

$$\mu(e^{tJS}) = \nu(S) - n = 2 - 1 = 1.$$

Lastly, for the case $S = diag(\pi, -\pi)$, the matrix exponential is given by

$$e^{tJS} = \begin{pmatrix} \cosh \pi t & \sinh \pi t \\ \sinh \pi t & \cosh \pi t \end{pmatrix}$$

whose eigenvalues are $e^{\pi t}, e^{-\pi t} \in \mathbb{R}_+$. By property (4) of Theorem 1.7.4, we get ± 1 under the map ρ. So,

$$\mu(e^{tJS}) = \nu(S) - n = 1 - 1 = 0.$$

Then, one just take ρ of the direct sums of all such S and apply μ to the product of ρ. $\square$

Definition 2.3.6. *Let $\Psi \in \mathcal{S}$ on interval $[0,2]$ be an extension of ψ as above, i.e. Ψ agrees with ψ on $[0,1]$ and $\Psi(s) \in Sp^*(2n), s \geq 1$ and $\Psi(2) = B^{\pm}$. Let $\rho : Sp(2n) \to S^1$ as before, then the Conley-Zehnder index is defined by*

$$\mu_{CZ}(\Psi) = deg(\rho^2 \circ \Psi)$$

Remark 2.3.7. *We know that $\rho(B^{\pm}) = \pm 1$, then $\rho^2 \circ \Psi : [0,2] \to S^1$ is a loop. So, $\mu_{CZ}(\Psi)$ can be seen as the winding number of $\rho^2 \circ \Psi$ going round in S^1. Moreover, μ_{CZ} is an integer-valued functor satisfying the following properties:*

1. ***(Naturality)*** *Let $\Psi, \psi : [0,1] \to Sp(2n)$, then*

$$\mu_{CZ}(\psi \circ \Psi \circ \psi^{-1}) = \mu_{CZ}(\Psi).$$

2. ***(Homotopy)*** *If Ψ is homotopic to Ψ', i.e. $\Psi \simeq \Psi'$, then*

$$\mu_{CZ}(\Psi) = \mu_{CZ}(\Psi').$$

3. ***(Direct sum)*** *Let $\Psi \in Sp(2m), \Psi' \in Sp(2n)$, then,*

$$\mu_{CZ}(\Psi \oplus \Psi') = \mu_{CZ}(\Psi) + \mu_{CZ}(\Psi').$$

4. ***(Zero)*** *For $s > 0$, if $\Psi(s)$ has no eigenvalue on S^1, then $\mu_{CZ}(\Psi) = 0$.*

5. ***(Loop)*** *Let $\Psi, \psi : [0,1] \to Sp(2n)$, such that ψ is a loop, i.e. $\psi(0) = \psi(1) = Id$, then*

$$\mu_{CZ}(\psi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\psi).$$

6. ***(Determinant)*** *sign $det(\Psi(1) - id) = (-1)^{\mu_{CZ}(\Psi) - n}$.*

7. ***(Signature)*** *Let $S \in GL(2n, \mathbb{R})$ such that $S = S^T$ and $\|S\| < 2\pi$. Let $\Psi(t) = e^{tJS}$, then*

$$\mu_{CZ}(\Psi) = \frac{1}{2} sgn(S)$$

   where $sgn(S)$ is the signature of matrix S, i.e. the net number of negative eigenvalues.

8. ***(Inverse)*** *Let $\Psi : [0,1] \to Sp(2n)$, and let Ψ^{-1} and Ψ^T be the inverse and transpose of Ψ respectively, then*

$$\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}(\Psi^T) = -\mu_{CZ}(\Psi).$$

Definition 2.3.8. *Consider the symplectic space* $(\mathbb{R}^{2n}\times\mathbb{R}^{2n}, (-\omega)\times\omega)$. *Let* $\Psi \in \mathcal{S}$ *on interval* $[a,b]$, *i.e.* $\Psi(a) = Id$ *and* $det(\Psi(b)-Id) \neq 0$, *then the Conley-Zehnder index is given by*

$$\mu_{CZ}(\Psi) = \mu(Gr(\Psi), \Delta)$$

where $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ *is the diagonal.*

Example 2.3.2. *Consider the rotation matrix as an example of case* $n = 1$,

$$\varphi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $t \in [0, \epsilon]$. *Then, we claim that* $\mu_{CZ}(\varphi) = 1$. *We first choose a splitting*

$$\mathbb{R}^2 \times \mathbb{R}^2 = \bar{V} \oplus \bar{W}$$

equipped with symplectic form $\bar{\omega} = (-\omega)\times\omega$, *where* $\bar{V} = \Delta$ *and* $\bar{W} = 0\times\mathbb{R}\times\mathbb{R}\times 0$. *Explicitly, express*

$$\begin{aligned} \bar{v} &= \left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) = (v, v) \in \bar{V}, \\ \bar{w}(t) &= \left(\begin{pmatrix} 0 \\ \eta(t) \end{pmatrix}, \begin{pmatrix} \xi(t) \\ 0 \end{pmatrix} \right) = (w(t), w'(t)) \in \bar{W} \end{aligned}$$

such that

$$\bar{v} + \bar{w}(t) = \left(\begin{pmatrix} x_0 \\ y_0 + \eta(t) \end{pmatrix}, \begin{pmatrix} x_0 + \xi(t) \\ y_0 \end{pmatrix} \right) \in Gr(\varphi(t)),$$

i.e.

$$\begin{pmatrix} x_0 + \xi(t) \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 + \eta(t) \end{pmatrix}$$

so we have

$$x_0 + \xi(t) = x_0 \cos t - (y_0 + \eta(t)) \sin t, \quad y_0 = x_0 \sin t + (y_0 + \eta(t)) \cos t$$

and by differentiating both sides w.r.t t *at* $t = 0$, *with* $\xi(0) = \eta(0) = 0$, *we have*

$$\begin{aligned} \dot{\xi}(0) &= -x_0 \sin 0 - y_0 \cos 0 + \dot{\eta}(0) \sin 0 + \eta(0) \cos 0 = -y_0 \\ 0 &= x_0 \cos 0 - y_0 \sin 0 + \dot{\eta}(0) \cos 0 - \eta(0) \sin 0 = x_0 + \dot{\eta}(0) \end{aligned}$$

that is $\dot{\xi}(0) = -y_0$ *and* $\dot{\eta}(0) = -x_0$. *By using the identity* $\omega(x, y) = x^T J y$, *one computes*

$$\begin{aligned}\omega(v, w(t)) &= (x_0, y_0) \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta(t) \end{pmatrix} \\ &= (-y_0, x_0) \begin{pmatrix} 0 \\ \eta(t) \end{pmatrix} = x_0 \eta(t)\end{aligned}$$

Similarly, one gets $\omega(v, w'(t)) = -y_0 \xi(t)$. *Applying the crossing form operator at* $t = 0$, *we have*

$$\begin{aligned}Q(\bar{v}) = \Gamma(Gr(\varphi), \Delta, t)|_{t=0}(\bar{v}) &= \frac{d}{dt}|_{t=0} \bar{\omega}(v, w(t)) \\ &= -\frac{d}{dt}|_{t=0} \omega(v, w(t)) + \frac{d}{dt}|_{t=0} \omega(v, w'(t)) \\ &= -\frac{d}{dt}|_{t=0} x_0 \eta(t) - \frac{d}{dt}|_{t=0} y_0 \xi(t) \\ &= -x_0 \dot{\eta}(0) - y_0 \dot{\xi}(0) \\ &= x_0^2 + y_0^2\end{aligned}$$

In conclusion, the signature is 2 and the Conley-Zehnder index is

$$\mu_{CZ}(\varphi) = \mu(Gr(\varphi), \Delta) = \frac{1}{2} sgn\ Q = \frac{1}{2} \cdot 2 = 1.$$

Chapter 3

Spectral flow

3.1 Fredholm operator

We have studied the notion and properties of Maslov index, Hӧrmander index and Conley-Zehnder index in the previous chapter. In this chapter, we will study a certain kind of Fredholm operator, acting on Sobolev space and the notion of spectral flow (in the sense of crossings) of that operator satisfying three conditions. The latter will then be shown to be equal to the index of a Fredholm operator.

Let W and H be real separable Hilbert spaces such that

$$W \subset H = H^* \subset W^*$$

where we identify H with its dual H^*. In particular, the identification is given by an inner product on H, that is, $H \to H^*, h \mapsto \varphi_h(v) = \langle v, h\rangle$ for $\varphi \in H^*$ and all $v \in H$. But we do not identify W with its dual. So, here we write $\langle \xi, \eta\rangle$ to mean the pairing of $\xi \in W$ and $\eta \in W^*$. Let $\mathcal{L}(W, H)$ be the set of bounded linear operators from W to H. We denote $A : \mathbb{R} \to \mathcal{L}(W, H)$ to be a family of bounded linear operators with parameter $t \in \mathbb{R}$.

Definition 3.1.1. *Let $A(t) \in \mathcal{L}(W, H)$ for $t \in \mathbb{R}$. For a differentiable curve $\xi : \mathbb{R} \to W$, we define $D_A\xi : \mathbb{R} \to H$ by*

$$(D_A\xi)(t) = \dot{\xi}(t) - A(t)\xi(t) \tag{3.1.1}$$

Hereby we list three conditions that we need A to possess: (in which in future we will write **C1**,**C2** and **C3**)

C1 $A(t)$ is bounded continuously differentiable, i.e. $A(t)$ is continuously differentiable in the weak operator topology and there is an $k > 0$ such that

$$\|A(t)\xi\|_H + \|\dot{A}(t)\xi\|_H \leq k\|\xi\|_W \tag{3.1.2}$$

C2 $A(t)$ is uniformly self-adjoint, i.e. $A(t)$ is self-adjoint when $A(t)$ is considered as an unbounded operator on H with domain W. Moreover, there is an $k > 0$ such that

$$\|\xi\|_W^2 \leq k(\|A(t)\xi\|_H^2 + \|\xi\|_H^2) \tag{3.1.3}$$

C3 There exists an invertible limit operator $A^{\pm} \in \mathcal{L}(W, H)$ such that $A(t)$ converges to $A^{\pm}$ in the operator norm, i.e.

$$\lim_{t \to \pm\infty} \|A(t) - A^{\pm}\|_{\mathcal{L}(W,L)} = 0. \tag{3.1.4}$$

We assume W to be Sobolev space $W^{1,2}$ (by convention 1 as for regularity and 2 as for integrability) and H to be the space of square integrable L^2 which is a Hilbert space, both defined on, for instance a closed odd-dimensional manifold, then $A(t)$ is a first order linear elliptic differential operator on a closed odd-dimensional manifold with coefficients depend on variable $t \in \mathbb{R}$ smoothly.

Furthermore, we define Hilbert spaces

$$\mathcal{H} = L^2(\mathbb{R}, H)$$

equipped with the norm

$$\|\xi\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} \|\xi(t)\|_H^2 dt$$

and

$$\mathcal{W} = L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H)$$

with

$$\|\xi\|_{\mathcal{W}}^2 = \int_{-\infty}^{\infty} \|\xi(t)\|_W^2 + \|\dot{\xi}(t)\|_H^2 dt.$$

Note that $\mathcal{W} \hookrightarrow \mathcal{H}$ is a bounded linear injection such that the range is dense. By inclusion here, albeit the topologies are different in both mentioned space, it is an injection with respect to respective own structure.

Remark 3.1.2. *By **C1**, the bound on $A(t)$ allows us to consider the bounded linear operator $D_A : \mathcal{W} \to \mathcal{H}$. So, we will show that such D_A is a Fredholm operator (and hence the Fredholm index makes sense).*

If we consider only a compact interval $[-T,T]$ instead of $\mathbb{R}$ for $\mathcal{W}$ and $\mathcal{H}$, in which we will write as $\mathcal{W}(T)$ and $\mathcal{H}(T)$, then we have the following two results[1]:

Lemma 3.1.3. *The inclusion $\mathcal{W}(T) \hookrightarrow \mathcal{H}(T)$ is compact for every $T > 0$. Moreover, the estimate is given by*

$$\|\xi\|_{\mathcal{W}} \le c\left(\|\xi\|_{\mathcal{H}(T)} + \|D_A\xi\|_{\mathcal{H}}\right)$$

for $T > 0$ and constant $c > 0$.

We also need the following useful theorem.

Theorem 3.1.4. *(**Abstract Closed Range**) Let X, Y and Z be Banach spaces such that $D : X \to Y$ be a bounded linear operator and $K : X \to Z$ be a compact linear operator. Further assume that $x \in X$ is bounded in the sense*

$$\|x\|_X \le c(\|Dx\|_Y + \|Kx\|_Z).$$

Then, D has a closed range and its kernel is of finite dimensional.

Proof. See Appendix. □

Corollary 3.1.1. *By Lemma 3.1.3 and Theorem 3.1.4, it follows that D_A has kernel of finite dimensional and a closed range, with the inclusion being the compact operator and D_A being the bounded linear operator.*

But still, we are lacking of information on the cokernel of operator D_A. However, we have a powerful theorem giving the so called elliptic regularity. Roughly speaking, since D_A is not self-adjoint (although we assume A is) we want to use the formal adjoint operator of D_A to give a notion of weak solution. It tells us that if ξ and ζ in $\mathcal{W}$ satisfy certain equation, then $-D_{-A}$ is the formal adjoint operator of D_A.

Theorem 3.1.5. *[18] Let $\xi, \eta \in \mathcal{H}$ such that $\langle \dot{\zeta} + A\zeta, \xi\rangle_{\mathcal{H}} + \langle \zeta, \eta\rangle_{\mathcal{H}} = 0$ for any $\zeta \in C^\infty(\mathbb{R}, W)$. Then, $\xi \in \mathcal{W}$ and $D_A\xi = \eta$.*

Remark 3.1.6. *The formal adjoint operator of D_A in the above theorem can be written as*

$$\begin{aligned} D_{-A} : \mathcal{W} &\longrightarrow \mathcal{H} \\ \zeta &\longmapsto -\dot{\zeta} - A\zeta \end{aligned}$$

for $\zeta \in \mathcal{W}$ and self-adjoint $A \in \mathcal{L}(W, H)$. This implies that the kernel of D_{-A} is the cokernel of D_A and is of finite dimensional.

[1] See [18] for the (technical) proofs so we do not shift away too much the main story.

Corollary 3.1.2. *D_A is a Fredholm operator.*

Proof. It follows from Corollary 3.1.1 that it has finite dimensional kernel and closed range. Its cokernel is also of finite dimensional from Remark 3.1.6. The Fredholm index is well-defined. □

Corollary 3.1.3. *Let $A(t)$ be a bijection for all $t \in \mathbb{R}$, then D_A has Fredholm index* 0.

3.2 The spectral flow of $A(t)$

Definition 3.2.1. *Let $A(t) : \mathbb{R} \to \mathcal{L}(W, H)$, then the crossing operator for $A(t)$ is defined as*

$$\Gamma(A, t) = P\dot{A}(t)P|_{ker\ A(t)}$$

for fixed $t \in \mathbb{R}$ and $P : H \to H$ is the orthogonal projection onto ker $A(t)$.

Definition 3.2.2. *A $t \in \mathbb{R}$ is called a crossing for A if $A(t)$ is not injective.*

Definition 3.2.3. *A crossing $t \in \mathbb{R}$ is called regular if $\Gamma(A, t)$ is nonsingular, i.e. $\Gamma(A, t)$ is non-vanishing at t.*

Definition 3.2.4. *A crossing $t \in \mathbb{R}$ is called simple if t is regular and the dimension of kernel of A is 1.*

Definition 3.2.5. *Let $t_0 \in \mathbb{R}$ be a simple crossing. Then, a continuously differentiable function $\lambda(t)$ is called crossing eigenvalue function such that $\lambda(t_0) = 0$ and $\lambda(t)$ is an eigenvalue of A for each $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.*

Remark 3.2.6. *If $t_0 \in \mathbb{R}$ is a simple crossing, then such $\lambda(t)$ always exists and is unique.*

Definition 3.2.7. *We denote the space of bounded operators from W to H by $\mathcal{L}(W, H)$. Then, the space of bounded symmetric operators from W to H is given by*

$$\mathcal{L}_{sym}(W, H) = \{A(t) \in \mathcal{L}(W, H) \mid A^*|_W = A\}. \tag{3.2.1}$$

In particular, an open subset $Res(W, H)$ of $\mathcal{L}_{sym}(W, H)$ that consists of operators with non empty resolvent set i.e. for $\lambda \notin \sigma(A)$ the map $\lambda Id - A$ from W to H is bijective.

Definition 3.2.8. *Let $A(t) \in \mathcal{L}_{sym}(W,H)$ such that $A(t)$ is continuous in norm topology and*

$$A^{\pm} = \lim_{t\to\infty} A(t).$$

We denote the space of all such $\{A(t)\}_{t\in\mathbb{R}}$ by $\mathcal{B}(W,H)$. Let $\mathcal{B}'(W,H)$ be a subset of $\mathcal{B}(W,H)$ consisting of A that are continuously differentiable in the norm topology and satisfying

$$\|A\|_{\mathcal{B}'} = \sup_{t\in\mathbb{R}}(\|A(t)\| + \|\dot{A}(t)\|) < \infty.$$

Definition 3.2.9. *Let $\mathcal{A}(W,H) \subset \mathcal{B}(W,H)$ be a subset consisting of operator A such that $A(t) \in Res(W,H)$ for all $t \in \mathbb{R}$ and the limit operators $A^{\pm} : W \to H$ are bijective.*

Remark 3.2.10. *From Definition 3.2.9, one readily sees that $\mathcal{A}(W,H)$ contains all continuous $A(t) \in \mathcal{L}(W,H)$ and satisfying **C2** and **C3**.*

Now, our intention is to introduce $A(t) \in \mathcal{L}(W,H)$ such that $A(t)$ satisfies **C1**,**C2** and **C3**. We can do that by defining the following

Definition 3.2.11. *Let $A(t) \in \mathcal{A}(W,H)$. We denote*

$$\mathcal{A}'(W,H) = \mathcal{A}(W,H) \cap \mathcal{B}'(W,H)$$

to be the set of operators $A(t)$ that inherited continuous differentiability and sup norm bound from $\mathcal{B}(W,H)$ with bijective limit operator and invertible resolvent set.

Remark 3.2.12. *$\mathcal{A}'(W,H)$ consists of all operators $A(t)$ satisfying **C1**,**C2**, **C3** and are continuously differentiable (C^1) in the norm topology.*

Corollary 3.2.1. *For all $A(t) \in \mathcal{A}'(W,H)$ and $\xi(t) \in W$, then the linearization $D_A\xi(t) = \dot{\xi}(t) - A(t)\xi(t)$ is a Fredholm operator by Theorem 3.1.2.*

Theorem 3.2.13. *There exists a unique map $\mu : \mathcal{A}(W,H,\mathbb{R}) \to \mathbb{Z}$ called the spectral flow of A for every compact dense injection $W \hookrightarrow H$ such that*

1. *(**Homotopy**) If $A_1(t)$ is homotopic to $A_2(t)$, then $\mu(A_1(t)) = \mu(A_2(t))$.*
2. *(**Constant**) If A is constant, then $\mu(A) = 0$.*
3. *(**Direct sum**) If $A = A_1 \oplus A_2$, then $\mu(A) = \mu(A_1) + \mu(A_2)$.*
4. *(**Concatenation**) If $A = A_1 \# A_2$, then $\mu(A) = \mu(A_1) + \mu(A_2)$.*

5. *(**Normalization**) If $W = H = \mathbb{R}$ and $A(t) = tan^{-1}(x)$, then $\mu(A) = 1$.*

Remark 3.2.14. *For any $A(t) \in \mathcal{A}(W, H)$, we can always find $A_1, A_2 \in \mathcal{A}'(W, H)$ such that*

$$\sup_{t\in\mathbb{R}} \|A(t) - A_i(t)\|_{\mathcal{L}(W,H)} < \epsilon$$

for $i = 1, 2$ and some $\epsilon > 0$. Then, we define a homotopy

$$\begin{aligned} A : [0, 1] \times \mathbb{R} &\longrightarrow \mathcal{A}'(W, H) \\ (s, t) &\longmapsto A(s, t) = (1 - s)A_1(t) + sA_2(t) \end{aligned}$$

One interesting thing worth mentioning is that if there exists another continuous homotopy between A_1 and A_2 in $\mathcal{A}(W, H)$, then they are homotopic by a continuously differentiable homotopy in $\mathcal{A}'(W, H)$. Consequently, the homotopy invariant in $\mathcal{A}'(W, H)$ can be canonically extend to the corresponding homotopy invariant in $\mathcal{A}(W, H)$.

Remark 3.2.15. *To be more explicit, for $A(t)$ to be the concatenation of operators $A_1(t)$ and $A_2(t)$, we mean*

$$A(t) = \begin{cases} A_1(t) & \text{for } t \leq 0 \\ A_2(t) & \text{for } t \geq 0 \end{cases}$$

and $A(0) = A_1(t) = A_2(-t)$ for $t > 0$. This means that we 'join' the right end of $A_1(t)$ and left end of $A_2(t)$ at $t = 0$ and assign values $A(0)$ to the respective paths outside its defined region.

Remark 3.2.16. *The concatenation axiom follows from the homotopy, direct sum and constant axioms. To see this, suppose $A_1, A_2 \in \mathcal{A}(\mathbb{R}, W, H)$ with $A_1(t) = A_2(t)$ for $t \geq -1$. Also, let L_c be a constant operator. Then, the homotopy $A(s) \in (\mathbb{R}, \mathcal{W} \oplus \mathcal{W}, \mathcal{H} \oplus \mathcal{H})$ between*

$$A_0 = A_1 \oplus A_2, \quad A_1 = A_1 \# A_2 \oplus L$$

is given by

$$A(s,t) = \begin{cases} A_1(t) \oplus A_2(t) & \text{for } t \leq 0 \\ g\left(-\frac{s\pi}{2}\right) \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix} g\left(\frac{s\pi}{2}\right) & \text{for } t \geq 0 \end{cases}$$

where $g(\theta)$ is an isomorphism on $H \oplus H$ defined by the rotation matrix

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Whilst on constant $L \oplus L$, $g(\theta)$ commutes. So, $A_1 \simeq A_2$. Applying μ, we get

$$\mu(A_1 \oplus A_2) = \mu(A_1 \# A_2 \oplus L) = \mu(A_1 \# A_2) + \mu(L) = \mu(A_1 \# A_2)$$

but $\mu(A_1 \oplus A_2) = \mu(A_1) + \mu(A_2)$. So, $\mu(A_1 \# A_2) = \mu(A_1) + \mu(A_2)$.

In application, we intend to make the spectral flow of $A(t) \in \mathcal{A}(W, H)$ computable by applying similar notion as defined in previous chapter.

Definition 3.2.17. *For all operator $L \in Res(W, H)$ with kernel of dimension k, we write $L \in Res_k(W, H)$. In particular, $Res_k(W, H)$ is a submanifold of codimension $k(k+1)/2$. We may express $Res(W, H)$ by*

$$Res(W, H) = \bigcup_{k=0}^{\infty} Res_k(W, H).$$

Then, we write the tangent space of $Res_k(W, H)$ at L as

$$T_L Res_k(W, H) = \{\tilde{L} \in \mathcal{L}_{sym}(W, H) \mid P\tilde{L}P = 0\}$$

[2] *where $P : H \to H$ is the orthogonal projection onto the kernel of L. Compare with* (2.1.1)*, the corresponding form Q is the crossing operator given in Definition 3.2.1. In particular, we call*

$$Res_{\mathfrak{M}}(W, H) = Res(W, H) \backslash Res_0(W, H) = \overline{Res_1(W, H)}$$

the Maslov cycle of resolvent set where $Res_0(W, H)$ consists of all injective $A(t)$ with nonempty resolvent set.

Remark 3.2.18. *$A(t) \in \mathcal{A}(W, H)$ is tangent to $Res_k(W, H)$ for some k at $t = 0$ if and only if $A(0) \in Res_k(W, H)$ and the crossing operator $\Gamma(A, 0)$ vanishes.*

Remark 3.2.19. *Let $A(t) \in \mathcal{A}(W, H)$ satisfying **C1**, **C2** and **C3**, then a crossing $t \in \mathbb{R}$ has only regular crossing $t \in \mathbb{R}$ if and only if it is transversal to $\overline{Res_1(W, H)}$, i.e. the crossing operator $\Gamma(A, t)$ is non vanishing and its derivative $\dot{A}(t)$ at crossing t does not lie in the tangent cone at $L = A(t)$*

$$T_L \overline{Res_1(W, H)} = \{\tilde{L} \in \mathcal{L}_{sym}(W, H) \mid 0 \in \sigma(P\tilde{L}P)|_{ker\ L}\}.$$

[2]Recall from equation (3.2.1) $\tilde{L}$ means $L \in \mathcal{L}(W, H)$ such that $L^*|_W = L$, where L^* is the adjoint of L.

Lemma 3.2.20. *Let $A(t)$ be an operator in $Res(W,H)$. Then, $A(t)$ has a simple crossing $t \in \mathbb{R}$ if and only if $A(t)$ is transverse to $Res_k(W,H)$ for all $k > 1$.*

Proof. Clearly $Res_1(W,H)$ is a submanifold of codimension 1, and $A(t)$ has simple crossing if $A(t) \in Res_1(W,H)$ and t is a regular crossing. So, for t a simple crossing, $A(t)$ is transverse to $Res_k(W,H)$ means that $A(t)$ must not intersect with stratum $Res_k(W,H)$, i.e. $A(t)$ cannot have kernel of dimension k other than 1. $\square$

Theorem 3.2.21. *Let $A(t) \in \mathcal{L}(W,H)$ satisfying **C1**,**C2** and **C3** with only regular crossing. Then, the spectral flow of A is given by*

$$\mu(A) = \sum_{t\in\mathbb{R}} sgn\ \Gamma(A,t)$$

where the sum is over all crossings $t \in \mathbb{R}$ and sgn denotes the signature, that is, the number of positive eigenvalues minus the number of negative eigenvalues. The summation is well-defined since the total number of crossings $t \in \mathbb{R}$ is finite.

Proof. Let $A(t)$ be as described in Remark 3.2.19. Pick a crossing $t_0 \in \mathbb{R}$. The assumption of regular t_0 implies that 0 is an eigenvalue of $A(t_0)$ with multiplicity k. Let $\lambda \in \sigma(A(t))$, we choose a $c > 0$ so small that it ensures in the vicinity of λ bounded by c has no other eigenvalue other than λ. We can always find an $\epsilon > 0$ such that the interval $t_0 - \epsilon \le t_0 \le t_0 + \epsilon$ corresponds to that small neighbourhood of λ. By 3.2.3, there exists k many C^1-paths of eigenvalues

$$\lambda_1, \dots \lambda_k : [t_0 - \epsilon, t_0 + \epsilon] \longrightarrow (-c, c)$$

such that at regular t_0, $\dot{\lambda}_i(t_0)$ is non vanishing. In other words, the path $\lambda_i(t)$ passing through the interval $[t_0 - \epsilon, t_0 + \epsilon]$ at only one point t_0. So, all crossings t are isolated and $\{t\}$ is finite. Hence the summation is well-defined $\sum_{t\in\mathbb{R}} \dot{\lambda}(t)$ is well-defined. Now, WLOG we may shrink ϵ to make sure $\lambda_i(t) \neq 0$ for $0 < |t - t_0| < \epsilon$ so that

$$sgn\ \dot{\lambda}_i(t_0) = sgn\ \lambda_i(t_0 + \epsilon) = -sgn\ \lambda_i(t_0 - \epsilon).$$

Then, the spectral flow of A is just the total finite sum of signature [3] over all

[3] By slight abuse of notation, we write $sgn\ \dot{\lambda}_i(t_0)$ to mean: $sgn\ \dot{\lambda}_i(t_0) = +1$ if $sgn\ \lambda_i(t_0+\epsilon) > 0$ and $sgn\ \lambda_i(t_0 - \epsilon < 0)$, $sgn\ \dot{\lambda}_i(t_0) = -1$ if $sgn\ \lambda_i(t_0 + \epsilon) < 0$ and $sgn\ \lambda_i(t_0 - \epsilon) > 0$. So the summation of $sgn\ \dot{\lambda}_i(t_0)$ over all regular crossings is by counting the total number of $+1$ minus that of -1. On the other hand, $sgn\ \Gamma(A,t)$ is the number of positive eigenvalues minus the number of negative eigenvalues of crossing operator $\Gamma(A,t)$, in which two signatures eventually boil down to the same number.

crossing t

$$\begin{aligned}\mu(A(t)) &= \#\{i \mid 0 < \lambda_i(t_0+\epsilon) < \delta\} - \#\{i \mid 0 < \lambda_i(t_0-\epsilon) < \delta\} \\ &= \sum_{t_0\in\mathbb{R}} sgn\ \dot{\lambda}_i(t_0) = \sum_{crossing\ t\in\mathbb{R}} sgn\ \Gamma(A,t)\end{aligned}$$

□

Corollary 3.2.2. *Since $A(t) \in \mathcal{A}'(W,H)$ satisfies $C1, C2, C3$, if furthermore $A(t)$ has only regular crossings, then Theorem 3.2.21 applies.*

Now, we know that we can define spectral flow for operator $A(t) \in \mathcal{A}'(W,H)$ with regular crossing t. Naturally one may ask is the operator with regular crossings abundance in existence? Else, it might not be useful in application. Fortunately, we have the following result.

Theorem 3.2.22. *Let $A(t) \in \mathcal{A}'$ with regular crossing $t \in \mathbb{R}$. Then, for almost every $\delta \in \mathbb{R}$, the path $A - \delta id$ has only regular crossing.*

Proof. Consider the $\{(t, \lambda(t)|\lambda(t) \in \sigma(A(t))\}$ and cover it by countably many graphs $\{(t, \lambda_i(t)\}$ on each interval $[a_i, b_i]$. On the other hand, $A - \delta id$ has only regular crossings if and only if δ is a common regular value of $\lambda_i(t)$, i.e. $\delta = \lambda_i(t)$ for some t and i. Then, by Sard's theorem, $A - \delta id$ has only regular crossings for almost every $\delta \in \mathbb{R}$, i.e. its complement has measure zero. □

Lemma 3.2.23 (Kato Selection[12]). *Let $W = H = \mathbb{R}^{2n}$ and $A(t) \in \mathcal{A}'(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. Then, there exists a C^1 path of diagonal matrices*

$$\begin{aligned}\Delta : \mathbb{R} &\longrightarrow \mathbb{R}^{2n} \\ t &\longmapsto diag(\lambda_1(t), \ldots, \lambda_n(t))\end{aligned}$$

such that $\Delta(t) \sim A(t)$ where $\sim$ denotes the similarity of real matrices. Consequently, the crossing operators are also in similarity

$$\Gamma(\Delta, t) \sim \Gamma(A, t)$$

for all t and eigenvalues λ.

Corollary 3.2.3. *[12] Let $A(t) \in \mathcal{A}(W,H)$ satisfying **C1** and **C2**. Let $t_0 \in \mathbb{R}$ be a crossing of $A(t)$ and $c > 0$ such that $\pm c$ are not eigenvalues of $A(t_0)$. Then, there exists an $\epsilon > 0$ and C^1-paths of diagonal matrices $\Delta(t)$ (defined in Lemma 3.2.23) such that for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ and $\lambda \in (-c, c)$, we have*

$$\Gamma(\Delta - \lambda id, t) \sim \Gamma(A - \lambda id, t)$$

The main result of this section is given below:

Theorem 3.2.24. *Suppose operator A satisfies condition **C1**,**C2** and **C3** with only regular crossings. Let D_A be as defined in* (3.1.1)*. Then, the set of crossings $\{t\}$ is finite and the Fredholm index of operator D_A is given by*

$$index(D_A) = -\sum_{t\in\mathbb{R}} sgn\,\Gamma(A,t) \tag{3.2.2}$$

Proof. First, consider in particular the case $A \in \mathcal{A}'(\mathbb{R}, W, H)$. Such A satisfies the assumption. So, we need to show that $-index(D_A)$ satisfies Theorem 3.2.13. In the Appendix, we have stated some properties that Fredholm index of D_A possesses, such as direct sum (the index of a direct sum is the sum of the indexes) and homotopy (stable under compact perturbations), which correspond to direct sum and homotopy axioms. The constant axiom is due to if A is constant and hence bijective, then $index(D_A) = 0$. The Fredholm index also satisfies the concatenation axiom by Remark 3.2.16. The normalization axiom would require to refer to next subsection regarding an example in finite dimensional case, in which the index is given explicitly. So, we have

$$index(D_A) = -\mu(A).$$

In general, we can approximate A by a curve in $\mathcal{A}'(\mathbb{R}, W, H)$, then we link to crossing form by Theorem 3.2.21

$$index(D_A) = -\mu(A) = -\sum_{t\in\mathbb{R}} sgn\,\Gamma(A,t).$$

□

Corollary 3.2.4. *If a path of operator $A(t)$ has only simple crossing, then the index is reduced to*

$$index(D_A) = -\sum_{t\in\mathbb{R}} sgn\dot{\lambda}(t) \tag{3.2.3}$$

where $\lambda(t)$ is the crossing eigenvalue function at t.

3.3 Example: Morse index theorem

Consider the second order Jacobi equation in differential geometry

$$Av = -\frac{d^2}{dt^2}v - \kappa(t)v = 0 \tag{3.3.1}$$

where $t \in [0,1], \kappa(t)$ is symmetric matrix representing the curvature, and input $u \in \mathbb{R}^n$. The fundamental solution $\xi(t) \in \mathbb{R}^{n^2}$ to equation (3.3.1) satisfies

$$\frac{d^2\xi}{dt^2} + \kappa(t)\xi = 0, \quad \xi(0) = 0, \quad \dot{\xi}(0) = Id.$$

Then, the Lagrangian plane in this case is given by

$$L(t) = Range \begin{pmatrix} \xi(t) \\ \dot{\xi}(t) \end{pmatrix}.$$

Let $t_0 \in (0,1]$. It is called a conjugate point of A if there exists a non trivial solution v satisfying (3.3.1) and

$$v(t_0) = v(0) = 0. \tag{3.3.2}$$

We then define the multiplicity of the conjugate point t_0 to be the dimension of vector space consists of all v satisfying (3.3.1). In particular, we denote the total number of conjugate points (counted with multiplicity) of A by $\nu(A)$. Then, we have the following result:

Proposition 3.3.1. *Let $\xi(t) \in \mathbb{R}^{n^2}$ be defined as above such that $det(\xi(1)) \neq 0$. Also, let $V = 0 \times \mathbb{R}^n$. Then,*

$$\mu(L,V) = -\nu(A) - \frac{n}{2}.$$

Proof. By direct application of the second case of Remark 2.1.7, let

$$v = (0, \dot{\xi}(t_0)u_0), u_0 \in ker\ \xi(t_0),$$

the crossing form is given by

$$\Gamma(L,V,t_0)(v) = -\langle \dot{\xi}(t_0)u_0, \dot{\xi}(t_0)u_0 \rangle$$

where t_0 is a crossing of multiplicity m_0. Note that $\dot{\xi}(t_0)$ is injective on $ker\ \xi(t_0)$ so Γ is nonsingular (in fact it is negative definite) and so all crossings are regular. In particular, for regular crossing $t_0 = 0$, the crossing index is the rank of Γ, which is $m_0 = n$. So, we obtain

$$\mu(L,V) = -\frac{1}{2}dim(ker\ \xi(0)) - \sum_{\substack{t_0 \in (0,1) \\ det\ \xi(t_0)=0}} dim(ker\ \xi(t_0)) = -\frac{n}{2} - \nu(A).$$

□

Corollary 3.3.1. *Let* $V = 0 \times \mathbb{R}^n$ *and let* $u(s) = \xi(s,t)u_0, v = (0, \partial_s\xi(1,t)u_0)$ *with* $\xi(1,t)u_0 = 0$, *then*

$$\Gamma(A,t)(u) = \Gamma(L(1,\cdot), V, t)(v).$$

Consider the map

$$\begin{aligned} A : W^{2,2}([0,1],\mathbb{R}^n) \cap W_0^{1,2}([0,1],\mathbb{R}^n) &\longrightarrow L^2([0,1],\mathbb{R}^n) \\ \zeta &\longmapsto A(t)\zeta = -\frac{d^2}{ds^2}\zeta - Q(s,t)\zeta \end{aligned}$$

To translate **C1**,**C2** and **C3** to this case, we further assume $Q(s,t)$ is continuously differentiable on closed strip $[0,1]\times\mathbb{R}$ and is independent of t for $|t| \geq T$. To ensure limit $A^{\pm}$ are invertible, we assume 1 is not a conjugate point for $A^{\pm}$.

Proposition 3.3.2. *The spectral flow of operator* $A(t)$ *is*

$$\mu(A) = \nu(A^-) - \nu(A^+)$$

Proof. Let $\xi \in \mathbb{R}^{n^2}$ satisfying the equation

$$\frac{\partial^2\xi}{\partial s^2} + Q\xi = 0$$

with $Q(0,t) = 0$ and associated vector $\frac{\partial\xi}{\partial s}(0,t) = Id$. As a routine we notice that $A(t)$ is injective if and only if $det\ \xi(1,t) \neq 0$, so the kernel of $A(t)$ consists of $v(s)$ such that $u(s) = \xi(s,t)u_0$ and $\xi(1,t)u_0 = 0$. The Lagrangian frame is given by

$$L(t) = Range \begin{pmatrix} \xi(s,t) \\ \partial_s\xi(s,t) \end{pmatrix}.$$

The crossing form on $ker\ A(t)$ is then given by

$$\begin{aligned} \Gamma(A,t)(v) &= -\langle \partial_s\xi(1,t)u_0, \partial_s\xi(1,t)u_0 \rangle \\ &= -\int_0^1 \langle u(s), \partial_t Q(s,t)u(s) \rangle. \end{aligned}$$

So,

$$\begin{aligned} \mu(A) = \mu(L(1,t),V) &= \mu(L^+,V) - \mu(L^-,V) \\ &= -\frac{n}{2} - \nu(A^+) - (-\frac{n}{2} - \nu(A^-)) \\ &= \nu(A^-) - \nu(A^+). \end{aligned}$$

□

Corollary 3.3.2 (Morse index)**.** *Let operator* $A(t)$ *be defined above. Then its associated Morse index is equal to the number of conjugate points,* $\nu(A)$.

3.4 Example: D_A as the Linearization of Cauchy-Riemann operator

In this section, we consider the following perturbed Cauchy-Riemann operator

$$\bar{\partial}_{S,L}\zeta = \frac{\partial \zeta}{\partial t} - J\frac{\partial \zeta}{\partial s} + S\zeta \tag{3.4.1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}^{2n}$ such that ζ satisfies nonlocal boundary condition

$$(\zeta(0,t), \zeta(1,t)) \in L(t) \tag{3.4.2}$$

where $L(t)$ is a Lagrangian path in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. $S(s,t)$ is a family of real matrices in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

Remark 3.4.1. *$\bar{\partial}_{S,L}$ has the form $D_A = \frac{d}{dt} - A(t)$ in which in this case $A(t) : W(t) \to H$ given by*

$$A(t) = J\frac{\partial}{\partial s} - S(s,t) \tag{3.4.3}$$

But, we cannot directly deduce all the result from above on $\bar{\partial}_{S,L}$ because the domain of operator $A(t)$ depends on t.

Translating **C1**,**C2** and **C3** to this case, we refer them as **CR1**,**CR2** and **CR3** respectively:

CR1 $L : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n\times 2n}, (-\omega) \times \omega)$ is continuously differentiable. There exists a constant $T > 0$ and constant Lagrangian subspaces $L^{\pm}$ such that

$$L(t) = \begin{cases} L^- & \text{for } t \leq -T \\ L^+ & \text{for } t \geq T \end{cases}$$

CR2 $S : [0,1] \times \mathbb{R} \to \mathbb{R}^{2n\times 2n}$ is continuous and there are symmetric matrices $S^{\pm} : [0,1] \to \mathbb{R}^{2n\times 2n}$ that depends only on s such that

$$\lim_{t\to\pm\infty} \sup_{s\in[0,1]} \|S(s,t) - S^{\pm}\| = 0$$

CR3 Given a symplectic path $\varphi^{\pm} : [0,1] \to Sp(2n)$ such that

$$\frac{\partial \varphi^{\pm}}{\partial s} + J_0 S^{\pm}\varphi^{\pm} = 0$$

with initial $\varphi^{\pm}(0) = id$, then the graph[4] of $\varphi^{\pm}(1)$ is transverse to the Lagrangian planes $L^{\pm}$.

[4] The graph Gr of $\varphi^{\pm}$ is defined as $Gr(\varphi^{\pm}(t)) = \{(x, \varphi^{\pm}(t)x) \mid x \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}\}$.

Remark 3.4.2. *Clearly, in **CR1** we see that $L(t)$ is independent of t for $|t| \geq T$. Since $S(s,t)$ can be (lim sup) estimated by symmetric matrices $S^{\pm}$ when $t \to \pm\infty$, this makes the corresponding limit operator $A^{\pm}$ self-adjoint.*

Also, in this case, the spaces $\mathcal{W}$ and $\mathcal{H}$ are respectively

$$\begin{aligned}
&\mathcal{H} = L^2([0,1] \times \mathbb{R}, \mathbb{R}^{2n}) \\
&\mathcal{W} = W_L^{1,2} = \left\{ \xi \in W^{1,2}([0,1] \times \mathbb{R}, \mathbb{R}^{2n}) | \ (\xi(0,t), \xi(1,t)) \in L(t) \right\}
\end{aligned}$$

We then claim that $\bar{\partial}_{S,L} : \mathcal{W} \to \mathcal{H}$ is Fredholm. In particular, the Fredholm index of $\bar{\partial}_{S,L}$ is given by

$$\text{index } \bar{\partial}_{S,L} = \mu(Gr(\varphi^-), L^-) - \mu(\Delta, L) - \mu(Gr(\varphi^+), L^+) \tag{3.4.4}$$

This is a highly non trivial result. We discuss this in several cases. Let us first consider (3.4.3), where $A(t)$ is

$$\begin{aligned}
A(t) : W_0^{1,2}([0,1], \mathbb{R}^n) \times W^{1,2}([0,1], \mathbb{R}^n) &\to L^2([0,1], \mathbb{R}^{2n}) \\
\xi &\longmapsto J_0 \frac{d}{ds}\xi - S\xi
\end{aligned}$$

Assume $S = S^T$ is continuously differentiable and suppose $\varphi(s,t) \in Sp(2n)$ satisfying

$$\frac{\partial \varphi}{\partial s} + J_0 S \varphi = 0 \tag{3.4.5}$$

with initial $\varphi(0,t) = Id$. We first note that **CR2** for φ implies that the limit $\varphi^{\pm}$ exists, whereas **CR3** implies that for sufficiently large T, the Maslov index for path $\varphi(s, \pm T)$ is equal to the Maslov index for $\varphi^{\pm}$. So, to show (3.4.4), we only need to show

$$\textit{index } \bar{\partial}_{S,L} = -\mu(Gr(\varphi(1,\cdot)), L)$$

where $L(s,t) = L(t)$ and $L'(s,t) = Gr(\varphi(s,t))$ are Lagrangian loops over $[0,1] \times [-T,T]$.

At crossings t, $A(t)$ is injective if and only if $\varphi(1,t)V$ intersect V transversally. If we consider φ in the form of block matrix

$$\varphi(s,t) = \begin{pmatrix} A(s,t) & B(s,t) \\ C(s,t) & D(s,t) \end{pmatrix},$$

then, $\xi(s)$ lie in the kernel of $A(t)$ if it can be expressed as

$$\xi(s) = \varphi(s,t)v, \quad \text{for some } v = (0,y) \in V, \text{ and } B(1,t)y = 0 \tag{3.4.6}$$

and the crossing operator is given by

$$\Gamma(A,t)(\xi) = -\int_0^1 \langle \xi(s), \frac{\partial}{\partial t}S(s,t)\xi(s)\rangle.$$

On the kernel of $A(t)$, by applying φ, we have

$$S\varphi = J_0\frac{\partial}{\partial s}\varphi \tag{3.4.7}$$

differentiating both sides w.r.t t,

$$\Big(\frac{\partial}{\partial t}S\Big)\varphi + S\Big(\frac{\partial}{\partial t}\varphi\Big) = J_0\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi\Big)$$

multiplying both sides by φ^T,

$$\varphi^T\Big(\frac{\partial}{\partial t}S\Big)\varphi + \varphi^T S\Big(\frac{\partial}{\partial t}\varphi\Big) = \varphi^T J_0\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi\Big) \tag{3.4.8}$$

by transposing (3.4.7),

$$\varphi^T S^T = -\frac{\partial}{\partial s}\varphi^T J_0 \tag{3.4.9}$$

substituting (3.4.9) into (3.4.8) and using $S = S^T$, we have

$$\begin{aligned}\varphi^T\Big(\frac{\partial}{\partial t}S\Big)\varphi &= \varphi^T J_0\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi\Big) - \varphi^T S\Big(\frac{\partial}{\partial t}\varphi\Big)\\ &= \varphi^T J_0\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi\Big) + \Big(\frac{\partial}{\partial s}\varphi^T J_0\Big)\Big(\frac{\partial}{\partial t}\varphi\Big).\end{aligned}$$

Now, integrate both side w.r.t s,

$$\begin{aligned}\int_0^1 \varphi^T\Big(\frac{\partial}{\partial t}S\Big)\varphi ds &= \int_0^1 \varphi^T J_0\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi\Big)ds + \int_0^1 \Big(\frac{\partial}{\partial s}\varphi^T\Big)\Big(J_0\frac{\partial}{\partial s}\varphi\Big)ds\\ &= \int_0^1 \frac{\partial}{\partial s}\Big(\varphi^T(s,t)J_0\frac{\partial}{\partial t}\varphi(s,t)\Big)ds\\ &= \varphi^T(1,t)J_0\frac{\partial}{\partial t}\varphi(1,t).\end{aligned}$$

Now, multiply both sides by $v = (0,y)$ on the left and right, with $B(1,t)y = 0$ and rewritten in the form of pairing, we obtain

$$\begin{aligned}\Gamma(A,t)(\xi) &= -\int_0^1 \langle \xi(s), \frac{\partial}{\partial t}S(s,t)\xi(s)\rangle ds\\ &= -\int_0^1 \langle \varphi(s,t)v, \frac{\partial}{\partial t}S(s,t)\varphi(s,t)v\rangle ds\\ &= -\langle \varphi(1,t)v, J_0\frac{\partial}{\partial t}\varphi(1,t)v\rangle\\ &= -\langle D(1,t)y, \dot{B}(1,t)y\rangle\\ &= \Gamma(\varphi(1,t),t)(y)\end{aligned}$$

Hence, we have shown that

$$\begin{aligned}
\text{index } \bar{\partial}_{S,L} &= -\mu(A) = -\sum_t sgn\ \Gamma(A,t) \\
&= -\sum_t sgn\ \Gamma(\varphi(1,t),t) \\
&= -\mu_{CZ}(\varphi(1,t)) \\
&= -\mu(Gr(\varphi(1,t)),L) \\
&= -\mu(Gr(\varphi^+),L^+) - \mu(\Delta,L) + \mu(Gr(\varphi^-),L^-)
\end{aligned}$$

where $L = V \oplus V$ and $t \in \mathbb{R}$ are regular crossings of $A(t)$.

Furthermore, we can make adjustment for S by introducing cutoff function: Choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(t) = 0$ for $t \leq -T$ and $\beta(t) = 1$ for $t \geq T$. Then, create a new $\tilde{S}$ as

$$\tilde{S}(s,t) = \beta(s)S^+(s) + (1-\beta(t))S^-(s)$$

in which $S - \tilde{S}$ induces a compact operator, i.e. the new operator $\tilde{A}$ would become

$$\tilde{A}(t) = A(t) + \tilde{S}(s,t).$$

Then, $\bar{\partial}_{\tilde{S},L}$ will still be a Fredholm operator. Thanks to the its unique feature, the new Fredholm index is invariant under small perturbation of compact operator

$$index\ \bar{\partial}_{\tilde{S},L} = index\ \bar{\partial}_{S,L}.$$

But, $\tilde{S}$ might not be symmetric and the derivation above does not apply. To overcome this, we choose a perturbation to get $\tilde{\tilde{S}}$ such that $\tilde{\tilde{S}} = \tilde{\tilde{S}}^T$ and is continuously differentiable and its corresponding symplectic matrix $\tilde{\tilde{\varphi}}(1,t)$ solving (3.4.5) will has only regular crossings (this is crucial in the definition of spectral flow $\mu(\tilde{\tilde{A}})$). Then, every steps as derived before works and the indices would be equal

$$index\ \bar{\partial}_{\tilde{\tilde{S}},L} = index\ \bar{\partial}_{S,L}.$$

So far we have discussed the case for $L(t) = V \oplus V$. We will also show that (3.4.4) holds for the case $L(t) = L_1(t) \oplus L_2(t)$, where $L_i \in \Lambda(n)$. Identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$, and choose a continuously differentiable unitary matrix $\Psi(s,t) \in U(n)$ such that for sufficiently large T, $\Psi(s,t)$ is independent of t for $|t| \geq T$. Then, let

$$S' = \Psi^{-1}\Big(\frac{\partial\Psi}{\partial t} - J_0\frac{\partial\Psi}{\partial s} + S\Psi\Big),$$

we have

$$\bar{\partial}_{S,L} \circ \Psi = \Psi \circ \bar{\partial}_{S',L'},$$

where the corresponding Lagrangian paths are

$$L'(t) = L_1'(t) \oplus L_2'(t)$$

with

$$L_1'(t) = \Psi^{-1}(0,t)L_1(t), \quad L_2'(t) = \Psi^{-1}(1,t)L_2(t).$$

For $|t| \geq T$, since $\Psi(s,t)$ is independent of t and the fact that $\Psi^{-1} = \Psi^T$, the matrix S' is symmetric. Then, one can obtain the corresponding symplectic matrices $\varphi'(s,t) \in Sp(2n)$ by conjugation

$$\varphi'(s,t) = \Psi^{-1}(s,t)\varphi(s,t)\Psi(s,t).$$

Now consider the symplectic group on $\mathbb{R}^{2n}_{\Delta} = (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega) \times \omega)$, and let

$$\tilde{\Psi}(t) = \Psi(0,t) \oplus \Psi(1,t) \in Sp(\mathbb{R}^{2n}_{\Delta}).$$

To set up for the Maslov index, we have the corresponding graph $Gr(\varphi'(1,t)) = \tilde{\Psi}^{-1}(t)Gr(\varphi(1,t))$ relative to the Lagrangian plane $L'(t) = \tilde{\Psi}^{-1}(t)L(t)$. Hence,

$$\begin{aligned}\mu(Gr(\varphi(1,t)), L) &= \mu(\tilde{\Psi}Gr(\varphi'(1,t)), \tilde{\Psi}L') \\ &= \mu(Gr(\varphi'(1,t)), L')\end{aligned}$$

where the second equality follows from the naturality axiom of the Maslov index. So, to relate back to the Fredholm index as derived previously, it boils down to the choice of $\Psi \in U(n)$: we choose Ψ such that the corresponding Lagrangian $L' = V \oplus V$, then every step of derivation follows. In short, (3.4.4) works not only for $L(t) = V \oplus V$, but also in the case of local boundary condition [5] $L'(t) = L_1'(t) \oplus L_2'(t)$.

We are now going to prove (3.4.4) in the general case. Define operator

$$\begin{aligned} T : L^2([0,1] \times \mathbb{R}, \mathbb{R}^{2n}) &\longrightarrow L^2([0,1] \times \mathbb{R}, \mathbb{R}^{2n} \times \mathbb{R}^{2n}) \\ \xi(s,t) &\longmapsto T\xi(s,t) = \eta(s,t) \end{aligned}$$

[5]This condition means if we assume $S = 0$ and Lagrangian split $L'(t) = L_1'(t) \oplus L_2'(t)$, where $L_i'(t) \in \Lambda(n)$, and $L_1'(\pm T) \cap L_2'(\pm T) = 0$. The index is given by *index* $\bar{\partial}_{S',L'} = -\mu(Gr(\varphi'(1,t)), L') = -\mu(L_1', L_2')$.

where

$$\eta(s,t) = \Big(\underbrace{\xi\Big(\frac{1-s}{2},\frac{t}{2}\Big)}_{\eta_1},\underbrace{\xi\Big(\frac{1+s}{2},\frac{t}{2}\Big)}_{\eta_2}\Big).$$

Suppose $\xi \in W^{1,2}_L = \{\xi \in W^{1,2}([0,1]\times\mathbb{R},\mathbb{R}^{2n}) \mid (\xi(0,t),\xi(1,t)) \in L(t)\}$, then under the map T, the image η satisfies the local boundary condition, i.e.

$$\eta(0,t) \in \Delta = \{(x,x) \mid x \in \mathbb{R}^{2n}\}, \quad \eta(1,t) \in L(t).$$

Consider the operator $\tilde{\bar{\partial}}_{S',L'} = T \circ \bar{\partial}_{S,L} \circ T^{-1}$ given by

$$\tilde{\bar{\partial}}_{S',L'}(\xi) = (\zeta_1,\zeta_2)$$

where

$$\begin{aligned}
\zeta_1 &= \frac{\partial}{\partial t}\eta_1 + J_0\frac{\partial}{\partial t} + \frac{1}{2}S\left(\frac{1-s}{2},\frac{t}{2}\right)\eta_1,\\
\zeta_2 &= \frac{\partial}{\partial t}\eta_2 - J_0\frac{\partial}{\partial t} + \frac{1}{2}S\left(\frac{1+s}{2},\frac{t}{2}\right)\eta_2.
\end{aligned}$$

So, $\tilde{\bar{\partial}}_{S',L'}$ is a Cauchy Riemann operator, in which the associated complex structure is $\bar{J} = (-J_0)\times J_0$ compatible with symplectic form $\bar{\omega} = (-\omega)\times\omega$. On the other hand, the symplectic matrix solving (3.4.5) is $\bar{\Psi}(s,t) = \Psi_1(s,t)\oplus\Psi_2(s,t)$ given by

$$\begin{aligned}
\Psi_1(s,t) &= \Psi\left(\frac{1-s}{2},\frac{t}{2}\right)\Psi^{-1}\left(\frac{1}{2},\frac{t}{2}\right),\\
\Psi_2(s,t) &= \Psi\left(\frac{1+s}{2},\frac{t}{2}\right)\Psi^{-1}\left(\frac{1}{2},\frac{t}{2}\right).
\end{aligned}$$

Now, for $s=1$, $\Psi(0,\frac{t}{2}) = 0$, and $\Delta = \{(z,z) \in \mathbb{R}^{2n}\times\mathbb{R}^{2n}\}$, we have

$$\bar{\Psi}(1,t)\Delta = Gr\left(\Psi\left(1,\frac{t}{2}\right)\right),$$

thus

$$\mu(Gr(\Psi(1,\cdot)),L) = \mu(\bar{\Psi}(1,\cdot)\Delta,L).$$

Follow the result in last case (i.e. $\tilde{\bar{\partial}}_{S',L'}\circ T = T\circ\bar{\partial}_{S,L}$), $\tilde{\bar{\partial}}_{S',L'}$ is a Fredholm operator. It follows that its Fredholm index is given by

$$index\ \tilde{\bar{\partial}}_{S',L'} = -\mu(\bar{\Psi}(1,\cdot)\Delta,L) = -\mu(Gr(\Psi(1,\cdot)),L) = index\ \bar{\partial}_{S,L}.$$

3.5 Example: Finite dimensional $\mathbb{R}^n$

In finite dimensional, we consider the linearization operator

$$D_A : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \longrightarrow L^2(\mathbb{R}, \mathbb{R}^n)$$

given by

$$D_A\xi(t) = \dot{\xi}(t) - A(t)\xi(t) \tag{3.5.1}$$

along a connecting orbit generated by a gradient flow of Morse function $f : \mathbb{R}^n \to \mathbb{R}$. In our case, we assume that the orbit concerned connects hyperbolic [6] critical points at two ends. So the limit matrix

$$A^{\pm} = \lim_{t\to\infty} A(t)$$

exists and are hyperbolic.

We claim and show that the linearization given by (3.5.1) is a Fredholm operator, by claiming D_A satisfies the estimation

$$\|\xi\|_{W^{1,2}(\mathbb{R})} \leq c(\|\xi\|_{L^2(I)} + \|D_A\xi\|_{L^2(\mathbb{R})}) \tag{3.5.2}$$

where $I = [-T, T] \subset \mathbb{R}$ for sufficiently large T. The estimate is true for $T = \infty$, that is $I = \mathbb{R}$. In particular, we investigate the case in which $A(t) \equiv A_0$ is constant. $\mathbb{R}^n$ can be decomposed into the direct sum such that each summand has all eigenvalues lie in either one of the two half planes[7]. WLOG, we may consider the case where they lie in the "negative" half planes, i.e. A_0 has eigenvalues that have negative real part. Now, pick an element $\zeta \in L^2(\mathbb{R}, \mathbb{R}^n)$ such that for $\xi \in L^2(\mathbb{R}, \mathbb{R}^n)$, $\zeta = D_{A_0}\xi$. This is possible because we consider $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ as an inclusion into $L^2(\mathbb{R}, \mathbb{R}^n)$. The unique solution $\xi \in L^2(\mathbb{R}, \mathbb{R}^n)$ is given by

$$\xi(t) = \int_{-\infty}^{t} e^{A_0(t-s)}\zeta(s)ds = \Phi(t) * \zeta(t)$$

where

$$\Phi(t) = \begin{cases} e_0^A t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

[6]Here the term hyperbolic means nondegenerate or nonsingular. For instance, if $A^{\pm}(t) = \lim_{t\to\infty} A(t)$ are $n \times n$ matrices, then hyperbolic critical points $A^{\pm}$ mean $A^{\pm}$ are nonsingular, i.e. invertible matrices.

[7]This means that for a decomposition of $\mathbb{R}^n$, say $\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^s$ such that $r + s = n$, the half planes of $\mathbb{R}^r$ are $\mathbb{R}^r_+ = \{(x_1, \ldots, x_r) \mid x_i \in \mathbb{R}_+ \text{ for } i = 1, \ldots, r\}$ and $\mathbb{R}^r_- = \{(x_1, \ldots, x_r) \mid x_i \in \mathbb{R}_- \text{ for } i = 1, \ldots, r\}$. Same applies to other summands of $\mathbb{R}^n$ under decomposition.

and by Young's inequality[8],

$$\|\xi\|_{L^2} \leq \|e^{A_0 t}\|_{L^1}\|\zeta\|_{L^2}.$$

Then,

$$\begin{aligned}\|\dot{\xi}\|_{L^2} &= \|A_0\xi + \zeta\|_{L^2}\\ &\leq \|A_0\|_{L^2}\|\xi\|_{L^2} + \|\zeta\|_{L^2}\\ &\leq \|\zeta\|_{L^2}(\|A_0\|_{L^2}\|e^{A_0 t}\|_{L^1} + 1).\end{aligned}$$

One notes that since A_0 is constant (independent of time t), the kernel of operator D_{A_0} consists of all ξ which is an exponential (so that $D_{A_0}e^{A_0 t} = A_0 e^{A_0 t} - A_0 e^{A_0 t} = 0$). So ξ can only lie in L^2 if $\xi = 0$, else ξ would escape to infinity as $t \to \infty$. That is, D_{A_0} is bijective and $\zeta = 0$, so we must have $\|\dot{\xi}\|_{L^2} = 0$ and $\|\xi\|_{L^2} = 0$ The estimate (3.5.2) reduces to

$$\|\xi\|_{W^{1,2}(\mathbb{R})} \leq c(\|D_{A_0}\xi\|_{L^2(\mathbb{R})}). \tag{3.5.3}$$

To show the inequality in general case, we first note that from estimate (3.5.3), there exists constants $T > 0$ and $c > 0$ such that for all $\xi \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$, if $\xi(t)$ is vanishing for $|t| \leq T - 1$, then the estimate is

$$\|\xi\|_{W^{1,2}(\mathbb{R})} \leq c(\|D_A\xi\|_{L^2(\mathbb{R})}). \tag{3.5.4}$$

Next, we define a cutoff function $\beta : \mathbb{R} \to [0, 1]$ satisfying: fix a $T > 0, \beta(t) = 0$ for $|t| \geq T$, smoothing by a smooth curve connecting 0 and 1 for $|t| \in (T - 1, T)$ and $\beta(t) = 1$ for $|t| \leq T - 1$. Then, we obtain

$$\begin{aligned}\|\xi\|_{W^{1,2}} &\leq \|\beta\xi\|_{W^{1,2}} + \|(1-\beta)\xi\|_{W^{1,2}}\\ &\leq k(\|\beta\xi\|_{L^2} + \|D_A\beta\xi\|_{L^2} + \|D_A(1-\beta)\xi\|_{L^2})\\ &\leq k_1(\|\beta\xi\|_{L^2} + \|D_A\xi\|_{L^2})\\ &= k_1(\|\xi\|_{L^2[-T,T]} + \|D_A\xi\|_{L^2})\end{aligned}$$

where the second inequality follows by applying estimate (3.5.2) (for $I = \mathbb{R}$) to $\beta\xi$ and (3.5.4) to $(1-\beta)\xi$. Third inequality follows by expanding and cancelling the term of $D_A\beta\xi$. The last equality follows from the fact that under cutoff function

[8]Recall the definition of the Young's inequality for convolution: Suppose $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ satisfying the equation $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, with $1 \leq p, q, r \leq \infty$. Then $\|f * g\|_r \leq \|f\|_p\|g\|_q$. In our case, $p = 1$ and $q = 2$, deduce that $r = 2$, so the inequality $\|\xi(t)\|_{L^2} = \|\Phi(t) * \zeta(t)\|_{L^2} \leq \|\Phi(t)\|_{L^1}\|\zeta(t)\|_{L^2}$ follows directly.

β, the norm of ξ only survive on the interval $|t| \leq T$. So, we have shown (3.5.2). The restriction operator $D_A : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2([-T,T], \mathbb{R}^n)$ is compact, and thanks to (Abstract Closed Range) Theorem 3.1.4, the kernel and cokernel of D_A are of finite dimensional, i.e. D_A is a Fredholm operator.

The index of D_A also need some work. Unlike the case above in which $A(t)$ is constant, in general, the kernel of D_A consists of ξ such that $\dot{\xi} - A\xi = 0$ and $\xi(t) \to 0$ as $t \to \pm\infty$. Hence, we consider $\zeta(s,t) \in \mathbb{R}^{n^2}$ such that

$$\frac{\partial \zeta(s,t)}{\partial s} = A(s)\zeta(s,t)$$

with $\zeta(s,s) = Id$ and $\zeta(s,s') \circ \zeta(s',s'') = \zeta(s,s'')$. Then, we define the stable and unstable subspaces at time t_0 as

$$\begin{aligned} W^s(t_0) &= \left\{ \xi_0 \in \mathbb{R}^n \middle| \lim_{t \to +\infty} \zeta(s,t_0)\xi_0 = 0 \right\} \\ W^u(t_0) &= \left\{ \xi_0 \in \mathbb{R}^n \middle| \lim_{t \to -\infty} \zeta(s,t_0)\xi_0 = 0 \right\}. \end{aligned}$$

such that $W^s(t) \to W^s(A^+)$ when $t \to \infty$ and $W^u(t) \to W^u(A^-)$ when $t \to -\infty$. This is possible under the map $W^s(s) = \zeta(s,t)W^s(t)$, similarly for W^u. So, they have dimensions

$$\begin{aligned} dim\ W^u(t) &= dim\ W^u(A^-) \\ dim\ W^s(t) &= n - dim\ W^u(A^+) \end{aligned}$$

Now, suppose $\xi(t)$ is a solution satisfying $\dot{\xi} = A\xi$ where $\xi(s) = \zeta(s,t)\xi(t)$ for all $s,t \in \mathbb{R}$. Note that $\xi \in ker\ D_A$ if and only if $\xi(s) = \zeta(s,t)\xi(t)$ and $\xi(s)$ lie in $W^s(s) \cap W^u(s)$. The latter is because if $\xi(s) \in W^s(s)$ then for $s \to \infty$, $|\xi(s)| \to 0$ by definition. Similarly for the case $\xi(s) \in W^u(s)$ and $s \to -\infty$. We must have the kernel of D_A to be finite dimensional.

We are left to show the cokernel of D_A is of finite codimension. First, let $\eta \in L^2(\mathbb{R}, \mathbb{R}^n)$ be orthogonal to $Range(D_A)$. Then, for $\xi \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$, we have the pairing $\langle \eta, \dot{\xi} - A\xi \rangle = 0$ and can deduce that $\dot{\eta} + A^T\eta = 0$. One obtains the result $\eta \perp Range(D_A)$ if and only if

$$\eta(t) = \zeta^T(s,t)\eta(s) \quad \eta(t) \perp W^s(t) + W^u(t)$$

so $coker\ D_A$ is of finite codimension. Lastly, the index of D_A can be computed as

$$\begin{aligned}
index\ D_A &= dim(ker\ D_A) - dim(coker\ D_A) \\
&= dim(W^s \cap W^u) - dim(W^s + W^u)^{\perp} \\
&= dim(W^s \cap W^u) - (n - dim(W^s + W^u)) \\
&= dim\ W^s + dim\ W^u - n \\
&= n - dim\ W^u(A^+) + dim\ W^u(A^-) - n \\
&= dim\ W^u(A^-) - dim\ W^u(A^+)
\end{aligned}$$

Corollary 3.5.1. *If $A^{\pm}$ is symmetric, the the index of D_A computed above can be expressed by*

$$index\ D_A = \frac{1}{2}(sgn\ A^- - sgn\ A^+)$$

where sgn A is the usual signature of matrix A, that is the total net number of negative eigenvalues.

Appendix A

Appendix

A.1 A bit of Analysis

[1]

We say V is a Banach space if V is a complete normed space, where norm on V is positive function $\|\cdot\|$ such that for all $v \in V, c \in \mathbb{C}$ or $\mathbb{R}$,

$$\|v\| \geq 0, \quad \|cv\| = |c| \cdot |v|, \quad \|v + w\| \leq \|v\| + \|w\|$$

Let V and W be Banach spaces. $\mathcal{L}(V, W)$ is defined to be the space of continuous linear operators from $V \to W$. A linear operator $T : V \to W$ is continuous if and only if T is also a bounded operator: there exists a constant $c < \infty$ such that $\|Tv\| \leq c\|v\|$ for all $v \in V$. In particular, we define the norm of T to be the the infimum over all such c :

$$\|T\| = sup\{\|Tv\| \mid \|v\| \leq 1\}.$$

Let $U \subset \mathbb{R}^n$ be an open set. Denote $C^\infty(\tilde{U})$ to be the space of all smooth functions on $\mathbb{R}^n$ restricted to U. In particular, $C^\infty_{cpt}(\tilde{U}) \subset C^\infty(\tilde{U})$ consists of all such functions with compact support on U. The Sobolev space $W^{s,p}$ is a completion of $C^\infty(\tilde{U})$ in the sense that $W^{s,p}$ consists of all smooth functions on U whose derivative up to order s are in the space $L^p(U)$, where the norm $\|\cdot\|_{s,p}$ defined for smooth functions on U is given by

$$\|\phi\|_{s,p} = \left(\int_U \sum_{|\alpha| \leq s} |\partial^\alpha \phi(x)|^p dx \right)^{\frac{1}{p}}.$$

In particular, $W^{s,p}_0(U) = \overline{C^\infty_{cpt}(U)}$.

[1]See [3].

Proposition A.1.1. *Let $U \subset \mathbb{R}^n$ be an open subset whose boundary is a continuously differentiable submanifold. Let $\psi \in L^p(U)$. Then, $\psi \in W^{s,p}(U)$ if and only if for all α with $|\alpha| \leq s$, there exists $\phi_\alpha \in L^p(U)$ such that for all $\psi \in C^\infty_{cpt}(U)$,*

$$\int_U \psi(x)\partial^\alpha \phi(x) ds = (-1)^{|\alpha|} \int_U \psi_\alpha(x)\phi(x) dx.$$

Theorem A.1.2. *Let U be bounded open subset of $\mathbb{R}^n$. Suppose $p > n$, then $W^{1,p}(U;\mathbb{R}^m)$ is a subspace of $C^0(U;\mathbb{R}^m)$ and the inclusion*

$$W^{1,p}(U;\mathbb{R}^m) \hookrightarrow C^0(U;\mathbb{R}^m)$$

is a compact operator.

Remark A.1.3. *When $p = 2$ and over $\mathbb{R}$, all elements of $W^{1,2}$ are continuous functions.*

Theorem A.1.4 (Rellich). *Let U be bounded open subset of $\mathbb{R}^n$ whose boundary is of class C^1. Let $k \in \mathbb{Z}^+$ and $p \in (1, \infty)$, then*

1. *If $p > \frac{n}{k}$, then $W^{k,p}(U) \hookrightarrow L^q(U)$ with*

$$q = \frac{np}{n - kp}.$$

2. *If $p = \frac{n}{k}$, then $W^{k,p}(U) \hookrightarrow L^q(U)$ for all $q \geq p$.*

3. *If $p < \frac{n}{k}$, then $W^{k,p}(U) \hookrightarrow L^q(U)$.*

All the inclusions are continuous. For $\psi \in W^{k,p}(U)$, if $|\alpha| \leq k$, then,

$$\left\| \frac{\partial^\alpha \psi}{\partial x^\alpha} \right\|_{L^\infty(U)} \leq C\|\psi\|_{W^{k,p}(U)}.$$

In particular, for $m = [k - n/p]$ (the integer part), then the inclusion

$$W^{k,p}(U) \hookrightarrow C^m(U)$$

is continuous.

A.2 A bit of Fredholm Theory

Let $T : V \to W$ be an operator from V to W which are both Banach spaces. We say T is Fredholm if T has an index, in the sense that if $ker(T)$ is a finite dimensional subspace of V and $Im(T)$ is a finite codimensional subspace of W, then the index is given by

$$Ind(T) = dim\ ker(T) - dim\ coker(T)$$

where $coker(T) = W/Im(T)$.

Remark A.2.1. *Let $T : V \to W$ is a Fredholm operataor, then $index(T)$ coincides with the Euler characteristic of the chain complex*

$$0 \longrightarrow V \xrightarrow{T} W \longrightarrow 0$$

that is $\chi = dim(ker(T)/0) - dim(W/Im(T))$.

Proposition A.2.2. *Let $T_1 : V_1 \to W_1, T_2 : V_2 \to W_2, T : V \to W, T' : W \to X$ be Fredholm operators, then*

1. *$T_0 \oplus T_1$ is a Fredholm operator and its index is*

$$index(T_0 \oplus T_1) = index(T_0) + index(T_1).$$

2. *Let H^m be a $m-$dimensional vector space, then $T \otimes Id_H$ is a Fredholm operator and*

$$index(T \otimes Id_H) = m \cdot index(T).$$

3. *The composition $T' \circ T$ is a Fredholm operator and its index is*

$$index(T' \circ T) = index(T') + index(L).$$

Proposition A.2.3. *Let $T : V \to W$ be a Fredholm operator, i.e. $dim\ ker(T) < \infty$ and $dim\ coker(T) < \infty$, then, $Im(T)$ is closed.*

The following theorem is an important feature that a Fredholm operator possesses: the index is invariant under small perturbation by a compact operator.

Theorem A.2.4. *Let $T : V \to W$ be a Fredholm operator. Then, there exists a constant $\epsilon > 0$ such that for all operator $u : V \to W$ with $\|u\| < \epsilon$, the operator $T + u$ is Fredholm and the index is*

$$index(T + u) = index(T)$$

In particular, for all compact operator $K : V \to W$, the operator $T+K$ is Fredholm and the index is given by

$$index(T + K) = index(T)$$

Remark A.2.5. *Let $\mathcal{F}$ be the set of all Fredholm operators from $V \to W$. $\mathcal{F}$ is an open set of the set of all operators from $V \to W$. By Theorem A.2.4, index is a continuous map on $\mathcal{F}$.*

Let V, W be Banach spaces. A map $F : V \to W$ is called a Fredholm map if for all $v \in V$, there exists a continuous differential map $T_vF : V \to W$, which is also a Fredholm operator. By Theorem A.2.4 and Remark A.2.5, $index(T_vF)$ is independent of v. We state a relevant result below.

Theorem A.2.6. *Let $F : V \to W$ be a Fredholm map. For $w \in W$ such that $v = F^{-1}(w) \in V$, and the differential map $T_vF : V \to W$ is a surjection, then $F^{-1}(w)$ is a manifold of dimension $index(F)$. Its tangent space at v is $ker\ T_vF$.*

A.3 Abstract Closed Range

We restate and give a proof of Theorem 3.1.4

Theorem. 3.1.4 *Let X, Y and Z be Banach spaces such that $D : X \to Y$ be a bounded linear operator and $K : X \to Z$ be a compact linear operator. Further assume that $x \in X$ is bounded in the sense*

$$\|x\|_X \leq c(\|Dx\|_Y + \|Kx\|_Z).$$

Then, D has a closed range and its kernel is of finite dimensional.

Proof. We follow the flow of proof given in [3]. First, we show that $ker(D)$ is a finite-dimensional subspace of X. To see this, we show its unit ball is compact. Take a sequence $\{x_n\}$ in X, such that

$$\|x_n\|_X \leq 1, \quad D(x_k) = 0.$$

Then, by assumption, we have

$$\|x_n\|_X \leq C\|Kx_n\|.$$

So the image of the unit ball under the operator K is relatively compact. We may extract a subsequence if necessary to ensure $\{Kx_n\}$ converges in Z, so $\{x_n\}$

is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges in X and hence in the unit ball, i.e. the unit ball is compact.
Next, we show that $Im(D)$ is a closed subspace of Y. Take a sequence $\{x_n\}$ in X such that $\{Lx_k\}$ converges to a limit $y \in Y$. We are left to show that $y \in Im(L)$. There are two possibilities: either $\{x_k\}$ is bounded or unbounded. Suppose $\{x_k\}$ is bounded, then by assumption

$$\|x_k\|_X \leq C(\|Lx_k\|_Y + \|Kx_k\|_Z)$$

where $Lx_k \to y \in Y$ whilst Kx_k lies in a compact subset of Z. Then, there is a convergent subsequence $\{Kx_{k_i}\}$ of $\{Kx_k\}$. The corresponding subsequence $\{x_{k_i}\}$ is necessarily Cauchy and converges, say to x. Such limit x must satisfies $Lx = y$. On the other hand, suppose that $\{x_k\}$ is an unbounded sequence in X, there must exists a subsequence $\{x_{k_i}\}$ such that

$$\lim_{k \to +\infty} \|x_{k_i}\| = +\infty.$$

Proven above that $ker(D)$ is finite dimensional, we can find a complement X^c of X such that $x_k \in X^c$ for all k. Let

$$v_k = \frac{x_k}{\|x_k\|_X} \in X^c.$$

Then, by applying the inequality, we have

$$\|v_k\|_X \leq C(\|Lv_k\|_Y + \|Kv_k\|_Z).$$

$\{Kv_k\}$ admits a convergent subsequence, whereas L sends v_k to zero. So, we may assume $\{v_k\}$ is Cauchy and converges in X^c with limit

$$v \in ker(L) \cap X^c = 0,$$

that is saying $v = 0$ but having norm 1. Contradiction. So, $Im(L)$ contains all the limit points of any sequences in X, that is $Im(L)$ is closed in Y.

□

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