An Introductory View of the Weak Solution of the p-Laplacian Equation

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June 2017

A thesis submitted for the degree of Honours
of the Australian National University

Australian National University

Declaration

The work in this thesis is my own except where otherwise stated.

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Acknowledgements

The author wishes to thank Prof.John Urbas for supervising this thesis and providing numerous helps and guides throughout the honours year; the Mathematical Science Institute at the Australian National University for providing an office and all related amenities.

Abstract

In this paper we will explore the various property of the weak solutions of the p-Laplacian equation:

$$\operatorname{div}(|Du|^{p-2}Du) = 0$$

for $1 < p < 2$, including existence, uniqueness theory, differentiability and regularity results.

Contents

Notation and terminology

In this paper I will mainly follow the notation convention used in [2]. Following are some main comment worth pointing out.

- I will stick to the symbol Du and avoid using ∇u for the gradient of the function u, and the letter U for an open subset of $\mathbb{R}^n$.
- The letter C denotes various constants, it may not be the same constant when appearing in different lines of computation. In cases when I need to keep track of the constant, I will use $C_1, C_2, \dots$ to represent different constants.
- For function spaces that specifies functions with compact support, I will stick to the subscript 0 notation instead of subscript c as preferred by some author. For example, $C_0^k(U)$ denotes functions in $C^k(U)$ with compact support.
- For functions that are not pointwisely defined, we write "sup" and "inf" notation to mean the essential supreme and infimum, which are defined as:

$$\begin{aligned} \sup f &\equiv \operatorname{ess\,sup} f \equiv \inf\{\mu \in \mathbb{R} |\ \operatorname{meas}\{f > \mu\} = 0\} \\ \inf f &\equiv \operatorname{ess\,inf} f \equiv -\sup(-f). \end{aligned}$$

  Since this definition coincides with the normal supremum and infimum in the classical sense, using the same notation causes no trouble.

Other common notations are listed below.

Notation

$\triangle_p u$ the p-Laplacian operator, defined as

$$\triangle_p u \equiv \operatorname{div}(|Du|^{p-2} Du) = 0.$$

$[u]_{C^{0,\alpha}(U)}$ the α^{th}-Holder seminorm of u on U, defined as

$[u]_{C^{0,\alpha}(U)} \equiv \sup_{x,y\in U} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

Usually I will write $[u]_{C^\alpha(U)}$ for $[u]_{C^{0,\alpha}(U)}$.

$\|u\|_{C^{0,\alpha}(U)}$ α^{th}-Holder norm, defined as

$\|u\|_{C^{0,\alpha}(U)} \equiv \sup_{x\in U} |u(x)| + [u]_{C^{0,\alpha}(U)} = \|u\|_{C(U)} + [u]_{C^{0,\alpha}(U)}$.

$(u)_U$ Average of u on U, defined as

$(u)_U = \fint_U u(x)dx = \frac{1}{|U|}\int_U u(x)dx$.

$\|u\|_{W^{k,p}(U)}$ $W^{k,p}$ norm of $u : U \to \mathbb{R}$, defined as

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha\leq k|}\int_U |D^\alpha u|^p dx\right)^{1/p}, & (1\leq p<\infty) \\ \sum_{|\alpha|\leq k} \text{ess sup}_U |D^\alpha u|, & (p=\infty) \end{cases}.$$

$\omega(n)$ Volume of n-dimensional unit ball. (this term will normally be absorbed into the constant term so we do not need the formula)

Terminology

compact embedding Let A and B be Banach spaces, we say A is compactly embedded in B, written

$$A \subset\subset B$$

provided

1. $\|a\|_B \leq C\|a\|_A$ $(a\in A)$ for some constant C and
2. each bounded sequence in A is precompact in B.

Chapter 1

Weak Solutions and Preliminary Results

The p-Laplacian equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0 \tag{1.1}$$

reduces to the well-known Laplacian equation when $p = 2$, suggesting which it is generalised from, as the Laplacian equation is the Euler-Lagrange equation* for the Diricihlet integral

$$D(u) = \frac{1}{2}\int_U |Du|^2 dx. \tag{1.2}$$

Changing the square to a p^{th} power we have the integral

$$I[u] = \int_U L(Du(x), u(x), x)dx = \frac{1}{p}\int_U |Du|^p dx, \quad 1 < p < \infty. \tag{1.3}$$

We now demonstrate that (1.1) is a the Euler-Lagrange equation for (1.3). Let $\eta \in C_0^\infty(U)$ and suppose u is a minimizer of (1.3). Let

$$i(\tau) := I[u + \tau\eta] \quad (\tau \in \mathbb{R}). \tag{1.4}$$

*See [2, Section 8.1 and 8.2] for a systematic introduction on Euler-Lagrange equation.

Computing the first variation (i.e. the derivative) explicitly, we get

$$\begin{aligned} i(\tau) &= \frac{1}{p}\int_U |Du+\tau D\eta|^p dx \\ &= \frac{1}{p}\int_U \left(\sum_i (D_i u+\tau D_i\eta)^2\right)^{p/2} dx \\ \Rightarrow i'(\tau) &= \int_\Omega \frac{1}{2}\left(\sum_i (D_i u+\tau D_i\eta)^2\right)^{\frac{p-2}{2}} \left(\sum_i 2(D_i u+\tau D_i\eta)D_i\eta\right) dx \\ &= \int_U |Du+\tau D\eta|^{p-2}\left(Du\cdot D\eta+\tau\sum_i |D_i\eta|^2\right) dx. \end{aligned}$$

Since $i(\cdot)$ has a minimum at $\tau=0$, we have

$$i'(0)=0=\int_U |Du|^{p-2}\, Du\cdot D\eta dx. \tag{1.5}$$

If we assume, for now, that u is a smooth solution, then using integration by parts, we have

$$\begin{aligned} \int_U |Du|^{p-2}\, Du\cdot D\eta dx &= \int_U \eta \operatorname{div}(|Du|^{p-2}Du)dx \\ &= 0, \end{aligned} \tag{1.6}$$

there is no boundary term as $\eta\in C_0^\infty$.

Since (1.6) has to hold for all test functions η, we must have

$$\triangle_p u \equiv \operatorname{div}(|Du|^{p-2}Du)=0 \tag{1.7}$$

showing that the minimizer of (1.3) is a solution of (1.1).

Note that the computation above gives us a motivation for defining a weak solution as requiring only $u\in C^\infty$ or even $u\in C^1$ is too narrow for the treatment of such problem and clearly the less smoothness we assume of u to start with, more theory can be developed, hence a notion of weak derivative is suitable in this case:

Definition 1.1. Suppose $u,v\in L^1_{loc}(U)$. and α ia a multiindex. We say that v is the α^{th}-*weak partial derivative* of u, written

$$D^\alpha u = v,$$

provided

$$\int_U uD^\alpha\phi\ dx = (-1)^{|\alpha|}\int_U v\phi\ dx$$

for all test functions $\phi\in C_0^\infty(U)$.

To study function with this property systematically, we define the function space called the Sobolev space:

Definition 1.2. The *Sobolev space*, denoted by $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α wth $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$, equipped with the norm

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx\right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

Remark 1.3.
- From the definition, it is clear that $W^{k,p}$ is a subspace of L^p and hence functions in Sobolev space are defined up to sets of measure zero. Moreover, the Sobolev space is also Banach and, in particular, is Hilbert when $p = 2$.
- The weak derivative bears many properties the normal derivative has such as linearity (with respect to addition and constant multiplication); $u \in W^{k,p} \Rightarrow D^\alpha u \in W^{k-|\alpha|,p}$ for $|\alpha| \leq k$; product rule and chain rule. For a detailed exploration of the properties of Sobolev space, see [2, chapter 5].

It is clear that for the right side of (1.5) to make sense, the integrand needs to be in L^1, in other words, we need at least $Du \in L^{p-1}_{\mathrm{loc}}(U)$, which means the natural space to seek a weak solution is $W^{1,p-1}_{\mathrm{loc}}(U)$. Unfortunately, little is known about the weak solution in this space, thus for the interest of this paper, we will instead study weak solutions in the space $W^{1,p}_{\mathrm{loc}}(U)$. We now have a precise meaning of the weak solution:

Definition 1.4. Let U be a domain in $\mathbb{R}^n$. We say that $u \in W^{1,p}$ is a *weak solution* of (1.1) in U, if

$$\int_U |Du|^{p-2} Du \cdot D\eta \, dx = 0 \tag{1.8}$$

for each $\eta \in C_0^\infty(U)$. If, in addition, u is continuous, then we say that u is a *p-harmonic function.*

We conclude the above computation in the next theorem:

Theorem 1.5. *The following conditions are equivalent for $u \in W^{1,p}(U)$:*

(i). *u is minimizing :*

$$\int |Du|^p dx \leq \int |Dv|^p dx, \text{ when } v - u \in W_0^{1,p}(U)$$

(ii). *the first variation vanishes:*

$$\int |Du|^{p-2} Du \cdot D\eta \, dx = 0, \text{ when } \eta \in W_0^{1,p}(U)$$

If, in addition, $\triangle_p u$ *is continuous, the the conditions are equivalent to* $\triangle_p u$ *in* U*.*

Proof. "(i) $\Rightarrow$ (ii)" is already shown in the above computation.

"(ii) $\Rightarrow$ (i)" Recall that for a convex function $f : \mathbb{R} \to \mathbb{R}$, f is convex if and only if for any $a, b \in \mathbb{R}$

$$f(b) \geq f(a) + f'(a)(b - a).$$

In the case where $f : \mathbb{R}^n \to \mathbb{R}$, the inequality becomes:

$$f(b) \geq f(a) + Df(a) \cdot (b - a).$$

Since $|\cdot|^p$ is convex for $p \geq 1$, then for $f : x \mapsto |x|^p$, we have

$$|b|^p \geq |a|^p + p|a|^{p-2} a \cdot (b - a) \tag{1.9}$$

It follows that

$$\int_U |Dv|^p dx \geq \int_U |Du|^p dx + p \int_U |Du|^{p-2} Du \cdot D(v - u).$$

By letting $\eta = v - u$, (ii) implies

$$\int_U |Dv|^p dx \geq \int_U |Du|^p dx,$$

which is (i) as needed.

Finally, the equivalence of (ii) and $\triangle_p u$ follow from (1.6).

$\square$

Remark 1.6. Note that since a typical function in $W^{1,p}(U)$ may not be pointwise defined in U, so we need some notion of assigning boundary values. Different resources I refer to in this paper uses different notions to deal with this problem, some author uses the so-called *trace operator* (see [2, Section 5.5]) and others avoids this notation and instead defines that $u \leq 0$ on ∂U if $u^+ \equiv \max\{u, 0\} \in$

$W_0^{1,p}(U)$ (other inequalities follows accordingly, see [6, Section 8.1]). According to [2, Section 5.5, Theorem 2], we know that these two notion are in fact equivalent.

In this paper I will stick to the latter notion, as I did in the preceding Theorem where $v - u \in W_0^{1,p}(U)$ means $u - v = 0$ on ∂U, in other words, u agrees with v on ∂U.

Next result we introduce now involves the concept of weak supersolutions and weak subsolutions, which i very useful in studying the viscosity solutions. However, within the scope of this paper we will mostly apply these lemma on weak solutions which, by definition, is both weak supersolution and weak subsolution. Nevertheless, we still include the definitions here for completeness purpose and reader's interest.

Definition 1.7. $v \in W_{loc}^{1,p}(U)$ is said to be a *weak supersolution* (*weak subsolution*) in U, if

$$\int_U |Dv|^{p-2} Dv \cdot D\eta \; dx \geq (\leq) 0 \tag{1.10}$$

for all nonnegative $\eta \in C_0^\infty(U)$.

Lemma 1.8. *If $v > 0$ is a weak supersolution in U, then*

$$\int_U \zeta^p |D(\log v)|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_U |D\zeta|^p dx$$

whenever $\zeta \in C_0^\infty(U)$, $\zeta \geq 0$

Proof. (Sketch)

The result follows from (1.10) by choosing $\eta = \zeta^p v^{1-p}$.

For a detailed proof, see [8, p 10, Lemma 2.14].

□

We now examine the *existence* and *uniqueness* of a p-harmonic function with given boundary values, which is given in the following theorem:

Theorem 1.9. *Suppose that $g \in W^{1,p}(U)$, where U is a bounded domain in $\mathbb{R}^n$, is given. There exists a unique $u \in W^{1,p}(U)$ with boundary values $u - g \in W_0^{1,p}(U)$ such that for all v satisfying (1.8) and $v - g \in W_0^{1,p}(U)$,*

$$\int_U |Du|^p dx \leq \int_U |Dv|^p dx.$$

This u is a weak solution.

Proof. We first show the uniqueness. Suppose there were two minimizers , u_1 and u_2. Let $v = (u_1 + u_2)/2$. If $Du_1 \neq Du_2$ in a set of positive measure, then we have

$$\left|\frac{Du_1 + Du_2}{2}\right|^p < \frac{|Du_1|^p + |Du_2|^p}{2}$$

in that set. It follows that

$$\begin{aligned}
\int_U |Du_2|^p\,dx &\leq \int_U \left|\frac{Du_1 + Du_2}{2}\right|^p dx \\
&< \frac{1}{2}\int_U |Du_1|^p + \frac{1}{2}\int_U |Du_2|^p\,dx \\
&= \int_U |Du_2|^p\,dx
\end{aligned}$$

which is a clear contradiction. Thus $Du_1 = Du_2$ a.e. in U and hence $u_1 = u_2$+Constant. Since $u_2 - u_1 \in W_0^{1,p}(U)$, the constant part of the integration is zero. Hence the minimizer is unique.

The existence of a minimizer is obtained through the so-called direct method. Let

$$I_0 = \inf \int_U |Dv|^p dx \leq \int_U |Dg|^p dx < \infty.$$

Choose admissible functions v_j such that

$$\int_U |Dv_j|^p < I_0 + \frac{1}{j}, \quad j = 1, 2, 3, \ldots \tag{1.11}$$

The goal is to bound the sequence $\|v_j\|_{W^{1,p}}$. Let $w = v_j - g$, then by Poincare's inequality (Cf. [2], Theorem 3, p. 265) we have

$$\begin{aligned}
\|v_j - g\|_{L^p(U)} &\leq C_U \|D(v_j - g)\|_{L^p(U)} \\
&\leq C_U \left(\|Dv_j\|_{L^p(U)} + \|Dg\|_{L^p(U)}\right) \\
&\leq C_U \left((I_0 + 1)^{\frac{1}{p}} + \|Dg\|_{L^p(U)}\right) \qquad (1.12)
\end{aligned}$$

Since $\|v_j - g\|_{L^p(U)} \geq \left|\|v_j\|_{L^p(U)} - \|g\|_{L^p(U)}\right|$ by triangle inequality, then combining with (1.12) we get

$$\|v_j\|_{L^p(U)} \leq M \ (j = 1, 2, 3, \ldots) \tag{1.13}$$

where the constant M is independent of j. (1.11) and (1.13) are the bounds we need.

Now, by the Rellich-Kondrachov Compactness Theorem (Cf [2],Theorem 1, p. 272), we know $\{v_j\}_j$ and $\{Dv_j\}_j$ are precompact in $L^p(U)$, which means there exists a function $u \in W^{1,p}(U)$ and a subsequence such that

$$v_{j_\nu} \to u,\ Du_{j_\nu} \to Du \text{ weakly in } L^p(U).$$

As L^p is Banach (i.e. L^p is a complete and normed space) and $W_0^{1,p}(U)$ is , by definition, closed under weak convergence (it is defined as the closure of $C_0^\infty(U)$), we have $u - g \in W_0^{1,2}(U)$, thus u is an admissible function. To see that u is the minimizer of (1.3), we use inequality (1.9) to obtain

$$\int_U |Dv_{j_\nu}|^p \geq \int_U |Du|^p\, dx + p \int_U |Du|^{p-2}\, Du \cdot (Dv_{j_\nu} - Du)dx$$

and by the weak convergence we have

$$\lim_{\nu\to\infty} \int_U |Du|^{p-2}\, Du \cdot (Dv_{j_\nu} - Du)dx = 0$$

which proves the claim.

□

Note that the theorem gives the existence and uniqueness of a minimizer for (1.3), it is by Theorem 1.5 that we know this is equivalent to that of a weak solution.

Chapter 2

Regularity Theory

Now that we have shown the existence and uniqueness of the weak solution. It is natural to examine whether a weak solution u of (1.1) is in fact smooth, or how much smoothness can we expect, this is called the *regularity problem* for weak solutions.

We need first a quantitative formulation of the continuity,

Definition 2.1. Let x_0 be a point in $\mathbb{R}^n$ and f a function defined on a bounded set D containing x_0. Then f is *Hölder continuous with exponent α at x_0* if

$$[f]_{\alpha;x_0} = \sup_D \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < \infty, \ 0 < \alpha < 1$$

$[f]_{\alpha;x_0}$ is called the *α-Hölder coefficient of f at x_0*.

We say f is *uniformly Hölder continuous with exponent α in D* if

$$[f]_{\alpha;D} = \sup_{\substack{x,y\in D\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty, \ 0 < \alpha \leq 1.$$

The main result of this chapter is :

Theorem 2.2. *Suppose that $u \in W^{1,p}_{loc}(U)$ is a weak solution to the p-harmonic equation. Then u is Hölder continuous, which means*

$$[u]_{\alpha,U} = \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq L$$

for a.e. $x, y \in B_r(x_0)$ provided that $B_{2r}(x_0) \subset\subset U$. The exponent $\alpha > 0$ depends only on n and p, while L also depend on $\|u\|_{L^p(B_{2r})}$.

We show first the fact that it suffices to prove the following theorem (proved later), which is the so-called Harnack's inequality.

Theorem 2.3. *(Harnack's inequality) Suppose that* $u \in W^{1,p}_{loc}(U)$ *is a weak solution and that* $u \geq 0$ *in* $B_{2r} \subset U$*. Then the quantities*

$$m(r) = \inf_{B_r} u, \quad M(r) = \sup_{B_r} u$$

satisfy

$$M(r) \leq Cm(r)$$

where $C = C(n, p)$*.*

Applying the Harnack inequality to the two non-negative weak solutions $u(x) - m(2r)$ and $M(2r) - u(x)$ for small enough r, we have

$$\begin{aligned} M(r) - m(2r) &\leq C(m(r) - m(2r)), \\ M(2r) - m(r) &\leq C(M(2r) - M(r)). \end{aligned}$$

Adding two inequalities, we get

$$\begin{aligned} (M(r) - m(r)) + (M(2r) - m(2r)) &\leq C((M(2r) - m(2r)) - (M(r) - m(r))) \\ \Rightarrow \quad (1 + C)(M(r) - m(r)) &\leq (C - 1)(M(2r) - m(2r)) \\ \Rightarrow \quad M(r) - m(r) &\leq \frac{C-1}{C+1}(M(2r) - m(2r)) \\ \Rightarrow \quad \omega(r) &\leq \frac{C-1}{C+1}\omega(2r) \qquad (2.1) \end{aligned}$$

where $\omega(r) = M(r) - m(r)$ is the oscillation of u over $B_r(x_0)$. Since

$$\lambda = \frac{C-1}{C+1} < 1,$$

we can iterate (2.1) to get

$$\omega(2^{-k}r) \leq \lambda^k \omega(r).$$

In order to get the estimate for any radius, we require the following lemma:

Lemma 2.4. *Let* ω *be a non-decreasing function on an interval* $(0, R_0]$ *satisfying, for all* $R \leq R_0$*, the inequality*

$$\omega(\tau R) \leq \gamma\omega(R) + \sigma(R)$$

where σ *is also non-decreasing and* $0 < \gamma$*,* $\tau < 1$*. Then, for any* $\mu \in (0, 1)$ *and* $R \leq R_0$*, we have*

$$\omega(R) \leq C\left(\left(\frac{R}{R_0}\right)^\alpha \omega(R_0) + \sigma(R^\mu R_0^{1-\mu})\right)$$

where $C = C(\gamma, \tau)$ *and* $\alpha = (1 - \mu)\frac{\log\gamma}{\log\tau}$ *are positive constants.*

Proof. See [6, Chapter 8, Lemma 8.23]. ☐

By choosing $\tau = 2^{-k}$, $\gamma = \lambda^k$ and $\sigma \equiv 0$, lemma 2.4 implies that

$$\omega(\rho) \leq A \left(\frac{\rho}{r}\right)^{\alpha} \omega(r), \; 0 < \rho < r \tag{2.2}$$

for some $\alpha = \alpha(n,p) > 0$ and $A = A(n,p)$. Assuming for now that solutions are locally bounded (we will prove this later and eliminate the possibility of $\omega(r) = \infty$), then we have Hölder continuity.

☐

There is a very important property that follows from this theorem, the Strong Maximum Principle.

Corollary 2.5. (Strong Maximum Principle) If a p-harmonic function attains its maximum at an interior point, then it reduces to a constant.

Proof. If $u(x_0) = \max_{x\in U} u(x)$ for some $x_0 \in U$, then applying Harnack inequality on the function $v(x) = u(x_0) - u(x)$ (which is surely non-negative) gives

$$\begin{aligned} u(x_0) - m(r) &\leq C(u(x_0) - M(r)) \\ &= C(u(x_0) - u(x_0)) = 0 \end{aligned}$$

which implies $\max_{x\in U} = u(x_0) = m(r)$ for $2|x - x_0| < \operatorname{dist}(x_0, \partial U)$. For arbitrary points in U, simply applying this argument on a chain of intersecting balls from a point x_0 satisfying $2|x - x_0| < \operatorname{dist}(x_0, \partial U)$ to the point of interest completes the proof.

☐

We will show the Hölder continuity of the weak solution in three cases, for $p < n$, $p = n$ and $p > n$ where n is the dimension.

2.1 The case $p > n$

We start with the following lemma:

Lemma 2.6. *Let $u \in W_0^{1,p}(U)$, $p > n$. Then for any ball $B = B_R$,*

$$osc_{U\cap B_R} \leq CR^{1-n/p}\|Du\|_p.$$

Proof. See [6, p 163, Theorem 7.17].

☐

Proof of Theorem 2.2. Let $v \in W^{1,p}(B)$ where B is a ball in $\mathbb{R}^n$. Applying Lemma 2.6 on the set $B \cap U \cap B_{|x-y|}(y)$ where $x, y \in B$, we get

$$\begin{aligned} |v(y) - v(x)| &\leq \operatorname{osc}_{B\cap U\cap B_{|x-y|}(y)} v \\ &\leq C_1 |x-y|^{1-\frac{n}{p}} \|Dv\|_{L^p(B)} \end{aligned} \tag{2.3}$$

As $\|Dv\|_{L^p(B)}$ is finite, v Hölder continuous with exponent $\alpha = 1 - \frac{n}{p}$.

If u is a positive weak solution or supersolution, it then follows from Lemma 1.7, by choosing ζsuch that $D\zeta = r^{-1}$, that

$$\begin{aligned} \|D(\log u)\|_{L^p(B_r)} &\leq \frac{p}{p-1} \left(\int_{B_r} r^{-p} dx \right)^{1/p} \\ &= C_2 r^{\frac{n-p}{p}} \end{aligned} \tag{2.4}$$

assuming $u > 0$ in B_{2r}. For $v = \log u$ we have

$$\begin{aligned} \left| \log \frac{u(y)}{u(x)} \right| &= |\log u(y) - \log u(x)| \\ &= |v(y) - v(x)| \\ &\leq C_1 |x-y|^{1-n/p} \|Dv\|_{L^p(B_r)} \text{ by (2.3)} \\ &\leq C_1 C_2 |x-y|^{1-n/p} r^{\frac{n-p}{p}} \text{ by (2.4).} \end{aligned} \tag{2.5}$$

By choosing x and y such that $u(x) = \sup_{B_r} u$ and $u(y) = \inf_{B_r} u$, the inequality above implies the Harnack's inequality with the constant $C(n,p) = e^{C_1 C_2}$, which in turn implies the Hölder continuity of u in this case.

2.2 The case $p = n$

The proof provided in this section is based on the so-called the hole filling technique. Lemma 2.6 is not good enough for this case and we need a generalised version, which is given by the following lemma:

Lemma 2.7. (Morrey) Assume that $u \in W^{1,p}(U)$, $1 \leq p < \infty$. Suppose that

$$\int_{B_r} |Du|^p dx \leq K r^{n-p+p\alpha}$$

whenever $B_{2r} \subset U$. Here $0 < \alpha < 1$ and K are independent of the ball B. Then $u \in C^{\alpha}_{loc}(U)$. In fact,

$$osc_{B_r}(u) \leq \frac{4}{\alpha} \left(\frac{K}{\omega_n} \right)^{1/p} r^{\alpha}, \;\; B_{2r} \subset U.$$

Proof. See [6,Theorem 7.19]. $\square$

Proof of Theorem 2.2. Let $B_{2r} = B_{2r}(x_0) \subset U$. Select a test function ζ such that $0 \le \zeta \le 1$, $\zeta = 1$ in B_r, $\zeta = 0$ outside B_{2r} and $|D\zeta| \le r^{-1}$. Choose

$$\eta(x) = \zeta(x)^n (u(x) - a)$$

in the n-harmonic equation. Then we have

$$\begin{aligned}
& \int_U \zeta^n |Du|^n dx \\
= \;& -n \int_U \zeta^{n-1}(u-a)|Du|^{n-2} Du \cdot D\zeta dx \text{ (integration by parts)} \\
\le \;& n \int_U |\zeta Du|^{n-1} |(u-\alpha) D\zeta| dx \text{ (Cauchy-Schwarz inequality)} \\
\le \;& n \left(\int_U \zeta^n |Du|^n dx \right)^{1-\frac{1}{n}} \left(\int_U |u-a|^n |D\zeta|^n dx \right)^{\frac{1}{n}}. \text{ (Hölder inequality)}
\end{aligned}$$

Rearranging the inequality, we have

$$\begin{aligned}
\left(\int_U \zeta^n |Du|^n dx \right)^{\frac{1}{n}} &\le n \left(\int_U |u-a|^n |D\zeta|^n dx \right)^{\frac{1}{n}} \\
\Rightarrow \left(\int_U \zeta^n |Du|^n dx \right) &\le n^n \left(\int_U |u-a|^n |D\zeta|^n dx \right) \\
&\le n^n r^{-n} \left(\int_U |u-a|^n |dx \right).
\end{aligned}$$

Let a denote the average

$$a = \frac{1}{H(r)} \int_{H_r} u(x) dx$$

of u taken over the annulus $H(r) = B_{2r} \backslash B_r$. The Poincare inequality

$$\int_{H(r)} |u(x) - a|^n dx \le C r^n \int_{H_r} |Du|^n dx$$

implies

$$\int_{B_r} |Du|^n dx \le C n^n \int_{H(r)} |Du|^n \tag{2.6}$$

Adding $Cn^n \int_{B_r} |Du|^n dx$ to both sides of (2.6), we get

$$(1 + Cn^n) \int_{B_r} |Du|^n dx \le C n^n \int_{B_{2r}} |Du|^n dx$$

which means

$$D(r) \leq \lambda D(2r), \ \lambda < 1 \tag{2.7}$$

holds for the Dirichlet integral

$$D(r) = \int_{B_r} |Du|^n dx$$

with the constant

$$\lambda = \frac{Cn^n}{1 + Cn^n}. \tag{2.8}$$

Now we can iterate (2.7) to get

$$D(2^{-k}) \leq \lambda^k D(r), \ k = 1, 2, 3, ...$$

Then lemma 2.4 implies that

$$D(\rho) \leq 2^\delta \left(\frac{\rho}{r}\right)^\delta D(r), \quad 0 < \rho < r$$

with $\delta = \frac{\log(1/\lambda)}{\log 2}$, when $B_{2r} \subset U$, which give Hölder continuity.

2.3 The case $1 < p < n$

The last case is most difficult to prove, we will be using the Moser's proof. The idea is to reach Harnack's inequality through the limits

$$\begin{aligned} \sup_B u &= \lim_{q \to \infty} \left(\int_B u^q dx \right)^{\frac{1}{q}} \\ \inf_B u &= \lim_{q \to -\infty} \left(\int_B u^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

In order for the proof to be carried out, we need some Lemmas:

Lemma 2.8. *Let $u \in W^{1,p}_{loc}(U)$ be a weak subsolution. Then*

$$\sup_B (u_+) \leq C_\beta \left(\frac{1}{(R-r)^n} \int_{B_R} u_+^\beta dx \right)^{\frac{1}{\beta}} \tag{2.9}$$

for $\beta > p - 1$ when $B_r \subset\subset U$. Here $u_+ = \max(u(x), 0)$ and $C_\beta = C(n, p, \beta)$.

Proof. (Sketch)

The idea is to use the test function $\eta = \zeta^p u_+^{\beta-(p-1)}$ to yield

$$\left(\int_{B_r} u_+^{\kappa\beta} dx\right)^{\frac{1}{\kappa\beta}} \leq C^{\frac{1}{\beta}} \left(\frac{2\beta - p + 1}{\beta - p + 1}\right)^{\frac{p}{\beta}} \frac{1}{(R-r)^n} \left(\int_{B_R} u_+^{\beta} dx\right)^{\frac{1}{\beta}}$$

where $\kappa = n/(n-p)$ and $\beta > p-1$. Next iterating the above estimate so that $\kappa\beta, \kappa^2\beta, \kappa^3\beta, \ldots$ are reached, while the radii shrink and by choosing $\alpha = \beta-(p-1)$ and $D\eta = p\zeta^{p-1}u_+^{\alpha} D\zeta + \alpha u_+^{\alpha-1}\zeta^p Du_+$ to yield

$$\alpha \int_U \zeta^p u_+^{\alpha-1} |DU_+|^p dx \leq -p \int_U \zeta^{p-1} u_+^{\alpha} |Du_+|^{p-2} DU_+ \cdot D\zeta dx$$

Then after some calculation we have

$$\left(\int_U |D(\zeta u^{\beta/p})|^{\kappa p} dx\right)^{\frac{1}{\kappa}} \leq S^p \int_U |D(\zeta u^{\beta/p})|^p dx$$

where $S = S(n,p)$. Since $|D\zeta| \leq 1/(R-r)$ and $\zeta = 1$ in B_r. It follows that

$$\left(\int_{B_r} u^{\kappa\beta} dx\right)^{\frac{1}{\kappa\beta}} \leq \left(\left(S \frac{2\beta - p + 1}{\beta - p + 1} \frac{1}{R-r}\right)^p \int_{B_R} u^{\beta} dx\right)^{\frac{1}{b}}.$$

Fix a β, say $\beta \leq \beta_0 > p-1$Again, we can iterate the estimate and replace the radii R and r with r_j and r_{j+1} where $r_j = r + 2^{-j}(R-r)$ to obtain

$$\|u\|_{L^{\kappa^{j+1}\beta_0}(B_{r_{j+1}})} \leq \left(\frac{Sb}{R-r}\right)^{p\beta_0^{-1}\sum k\kappa^{-k}} \|u\|_{L^{\beta_0}(B_{r_0})}$$

where index k is being summed over $1, 2, \ldots, j$. The proof is then concluded by

$$\|u\|_{L^{\kappa^{j+1}\beta_0}(B_r)} \leq \|u\|_{L^{\kappa^{j+1}\beta_0}(B_{r_{j+1}})}$$

For detailed proof, see [8, chapter 3, page 20]. □

Remark 2.9. Since we need this lemma to conclude that arbitrary solutions are locally bounded, we do not assume positivity here, which is why positive part needs to appear in the result. By doing so, we have the following corollary:

Corollary 2.10. *The weak solutions to the p-harmonic equation are locally bounded.*

Proof. Let $\beta = p$ and apply the lemma to u and $-u$ (recall $\inf u = -\sup -u$). Then we have bounds for both the supremum and infimum. □

Lemma 2.11. *Let $u \in W^{1,p}_{loc}(U)$ be a non-negative weak supersolution. Then for $\kappa = n/(n-p)$,*

$$\left(\frac{1}{(R-r)^n}\int_{B_r} v^\beta dx\right)^{\frac{1}{\beta}} \leq C(\varepsilon,\beta)\left(\frac{1}{(R-r)^n}\int_{B_R} v^\varepsilon dx\right)^{\frac{1}{\varepsilon}}$$

when $0 < \varepsilon < \beta < \kappa(p-1) = n(p-1)/(n-p)$ and $B_R \subset\subset U$.

Proof. (Sketch)

The calculation is somewhat similar to that of Lemma 2.8. Use

$$\eta = \zeta^p v^{\beta-(p-1)}$$

to obtain,

$$\left(\int_{B_r} v^{\kappa\beta} dx\right)^{\frac{1}{\kappa\beta}} \leq C^{\frac{1}{\beta}}\left(\frac{p-1}{p-1-\beta}\right)^{\frac{p}{\beta}}\frac{1}{(R-r)^{p/\beta}}\left(\int_{B_R} v^\beta dx\right)$$

for $0 < \beta < p-1$

For a detailed proof, see [8, chapter 3, page 23]. □

In the next lemma $\beta < 0$.

Lemma 2.12. *Suppose that $v \in W^{1,p}_{loc}(U)$ is a non-negative supersolution. Then*

$$\left(\frac{1}{(R-r)^n}\int_{B_R} v^\beta dx\right)^{\frac{1}{\kappa\beta}} \leq C\inf_{B_r} v \tag{2.10}$$

when $\beta < 0$ and $B_R \subset\subset U$. The constant C is of the form $C(n,p)^{-1/\beta}$.

Proof. See [8, chapter 3, page 24]. □

Combing lemma 2.11 and 2.12, we have the following bounds for non-negative weak solutions:

$$\begin{aligned}\sup_{B_r} &\leq C_1(\varepsilon,n,p)\left(\frac{1}{(R-r)^n}\int_{B_R} u^\varepsilon dx\right)^{\frac{1}{\varepsilon}}\\ \inf_{B_r} &\geq C_2(\varepsilon,n,p)\left(\frac{1}{(R-r)^n}\int_{B_R} u^{-\varepsilon} dx\right)^{-\frac{1}{\varepsilon}}\end{aligned}$$

for all $\varepsilon > 0$. For simplicity we can take $R = 2r$. Upon inspection, we still need the inequality

$$\left(\frac{1}{(R-r)^n}\int_{B_R} u^\varepsilon dx\right)^{\frac{1}{\varepsilon}} \leq \left(\frac{1}{(R-r)^n}\int_{B_R} u^{-\varepsilon} dx\right)^{-\frac{1}{\varepsilon}}$$

to obtain the Harnack inequality. To prove this inequality, we need the following lemma:

Lemma 2.13. (John-Nirenberg) Let $w \in L^1_{loc}(U)$. Suppose that there is a constant K such that

$$\fint_{B_r} |w(x) - w_{B_r}|\, dx \le K \tag{2.11}$$

holds whenever $B_{2r} \subset U$. Then there exists a constant $\nu = \nu(n) > 0$ such that

$$\fint_{B_r} e^{\nu |w(x) - w_{B_r}|/K} dx \le 2 \tag{2.12}$$

whenever $B_{2r} \subset U$ (and even when $\overline{B_{2r}} \subset U$).

From (2.12) we immediately have two inequalities:

$$\fint_{B_r} e^{\pm\nu (w(x) - w_{B_r})/K} dx \le 2.$$

Multiplying them together we have

$$\begin{aligned} & \fint_{B_r} e^{\nu(w(x)-w_{B_r})/K} dx \fint_{B_r} e^{-\nu(w(x)-w_{B_r})/K} dx \\ = & \fint_{B_r} e^{\nu w(x)/K} dx \fint_{B_r} e^{-\nu w(x)/K} dx \le 4. \end{aligned} \tag{2.13}$$

Proof. See [6, Chapter 7, Lemma 7.16 and Lemma 7.20]. □

Let $w = \log u$, we aim to show that w satisfy (2.12). To prove this, we assume for now that $u > 0$ is a weak solution. Combining the Poincare inequality

$$\int_{B_r} |\log u(x) - (\log u)_{B_r}|^p\, dx \le C_1 r^p \int_{B_r} |D \log u|^p dx$$

with the estimate

$$\int_{B_r} |D \log u|^p dx \le C_2 r^{n-p}$$

which follows from lemma 1.8 (by choosing ζ such that $D\zeta \le r^{-1}$), we have for $B_{2r} \subset U$

$$\fint_{B_r} |w - w_{B_r}|^p dx \le C_1 C_2 \omega_n^{-1} = K.$$

Now that we have shown estimate needed to apply the John-Nirenberg theorem, then it follows from (2.13) that

$$\fint_{B_r} u^{\nu/K} dx \fint_{B_r} u^{-\nu/K} dx \le 4.$$

Setting $\varepsilon = \nu/K$, we get

$$\left(\fint_{B_r} u^{\nu/K} dx \right)^{\frac{1}{\varepsilon}} \leq 4^{\frac{1}{\varepsilon}} \left(\int_{B_r} u^{-\nu/K} dx \right)^{-\frac{1}{\varepsilon}}$$

for $B_{2r} \subset\subset U$. Then we have the Harnack inequality which, in turn, implies Hölder continutiy.

Chapter 3

Differentiability

We have shown that the weak solution are Hölder continuous, we want to seek more regularity of the weak solution. In fact, even the gradients are locally Hölder continuous. However, this result is very difficult to prove, hence in this chapter, we study some simpler result as stated below:

which leads to the main result we are going to prove in this chapter:

1. For $1 < p \leq 2$, we have $u \in W^{2,p}_{loc}(U)$, which means u had second Sobolev derivatives.

2. For $p \geq 2$, then $|Du|^{(p-2)/2}Du$ belongs to $W^{1,2}_{loc}(U)$. Thus the Sobolev derivatives
$$\frac{\partial}{\partial x_j}\left(|Du|^{\frac{p-2}{2}}\frac{\partial u}{\partial x_i}\right)$$
exist.

Before we start the main results, there are some elementary inequalities we will need in this chapter, we put the list here so they can be referred to when needed:

$$\begin{aligned} \frac{4}{p^2}\left||b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a\right|^2 &\leq \left(|b|^{p-2}b - |a|^{p-2}a\right)\cdot(b-a) && (3.1)\\ \left||b^{p-2}b - |a|^{p-2}a|\right| &\leq (p-1)\left(|a|^{\frac{p-2}{2}} + |b|^{\frac{p-2}{2}}\right)\left||b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a\right| && (3.2)\end{aligned}$$

and for $1 < p < 2$

$$(|b|^{p-2}b - |a|^{p-2}a)\cdot(b-a) \geq (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} \tag{3.3}$$

Proof. See [8, Page 71]. $\square$

We start by look at the second case first, let

$$F(x) = |Du(x)|^{(p-2)/2} Du(x)$$

Theorem 3.1. (Bojarski- Iwaniec) Let $p \geq 2$. If u is p-harmonic in U, then $F \in W^{1,2}(U)$. For each subdomain $V \subset\subset U$,

$$\|DF\|_{L^2(V)} \leq \frac{C(n,p)}{dist(V, \partial U)} \|F\|_{L^2(U)}.$$

Proof. The proof is based on integrated difference quotients. Let $\zeta \in C_0^\infty(U)$ be a cut-off function so that $0 \leq \zeta \leq 1$, $\zeta = 1$ in V and $|D\zeta| \leq C_n / dist(V, \partial U)$. Let h be a constant vector such that $|h| < \text{dist}(\text{supp}\zeta, \partial U)$. Define $u_h = u(x+h)$. Clearly, u_h is p-harmonic when $x + h \in U$. We choose the test function η as

$$\eta(x) = \zeta(x)^2 (u(x+h) - u(x))$$

in the equations

$$\int_U |Du|^{p-2} Du(x) \cdot D\eta(x) dx = 0, \tag{3.4}$$

$$\int_U |Du(x+h)|^{p-2} Du(x+h) \cdot D\eta(x) dx = 0. \tag{3.5}$$

Then after subtraction we have

$$\int_U \left(|Du(x+h)|^{p-2} Du(x+h) - |Du(x)|^{p-2} Du(x) \right) \cdot D\eta(x) dx = 0. \tag{3.6}$$

It follows that

$$\begin{aligned}
&\int_U \zeta(x)^2 \left(|Du(x+h)|^{p-2} Du(x+h) - |Du(x)|^{p-2} Du(x) \right) \cdot \left(Du(x+h) - Du(x) \right) dx \\
= \; & -2 \int_U \zeta(x) \left(u(x+h) - u(x) \right) \left(\left(|Du(x+h)|^{p-2} Du(x+h) - |Du(x)|^{p-2} Du(x) \right) \cdot D\zeta(x) \right) dx \\
& \text{(product rule on } D\eta \text{ and rearranging)} \\
\leq \; & 2 \int_U \zeta(x) \left| u(x+h) - u(x) \right| \left| |Du(x+h)|^{p-2} Du(x+h) - |Du(x)|^{p-2} Du(x) \right| \left| D\zeta(x) \right| dx \\
\leq \; & 2 \int_U \zeta(x) \left| u(x+h) - u(x) \right| \left| |Du(x+h)|^{p-2} Du(x+h) - |Du(x)|^{p-2} Du(x) \right| \left| D\zeta(x) \right| dx
\end{aligned}$$

By choosing $b = Du(x+h)$ and $a = Du(x)$, inequality (3.2) implies

$$\begin{aligned}
&2 \int_U \zeta(x) \left| u(x+h) - u(x) \right| \left| |Du(x+h)|^{p-2} Du(x+h) - |Du(x)|^{p-2} Du(x) \right| \left| D\zeta(x) \right| dx \\
\leq \; & 2(p-1) \int_U \zeta(x) \left| u(x+h) - u(x) \right| \left| |Du(x+h)|^{\frac{p-2}{2}} + |Du(x)|^{\frac{p-2}{2}} Du(x) \right| \\
& \qquad \left| |Du(x+h)|^{\frac{p-2}{2}} Du(x+h) - |Du(x)|^{\frac{p-2}{2}} Du(x) \right| \left| D\zeta(x) \right| dx
\end{aligned}$$

By choosing $b = Du(x+h)$ and $a = Du(x)$, inequality (3.1) and (3.2) implies

$$
\begin{aligned}
& \frac{4}{p^2}\int_U \zeta^2(x)|F(x+h)-F(x)|^2 dx \\
\leq\ & \frac{4}{p^2}\int_U \zeta^2(x)\left(Du(x+h)^{p-2}Du(x+h)-Du(x)^{p-2}Du(x)\right)\cdot(Du(x+h)-Du(x))dx \\
& \quad \text{by (3.1)} \\
\leq\ & 2(p-1)\int_U |u(x+h)-u(x)||D\zeta(x)|\left(|Du(x+h)|^{\frac{p-2}{2}}+|Du(x)|^{\frac{p-2}{2}}\right) \\
& \qquad \zeta(x)|F(x+h)-F(x)|dx \quad \text{by (3.2)} \\
\leq\ & 2(p-1)\left(\int_U |u(x+h)-u(x)|^p|D\zeta(x)|^p dx\right)^{\frac{1}{p}}\left(\int_U \zeta^2(x)|F(x+h)-F(x)|^2 dx\right)^{\frac{1}{2}} \\
& \quad \left(\int_{\text{supp }\zeta}\left(|Du(x+h)|^{\frac{p-2}{2}}+|Du(x)|^{\frac{p-2}{2}}\right)^{\frac{2p}{p-2}} dx\right)^{\frac{p-2}{2p}} \\
& \quad \text{by general Hölder inequality with exponents } p, 2 \text{ and } \frac{2p}{p-2}.
\end{aligned}
$$

We estimate the last integral by the Minkowski's inequality:

$$
\begin{aligned}
& \left(\int_{\text{supp }\zeta}\left(|Du(x+h)|^{\frac{p-2}{2}}+|Du(x)|^{\frac{p-2}{2}}\right)^{\frac{2p}{p-2}} dx\right)^{\frac{p-2}{2p}} \\
\leq\ & \left(\int_{\text{supp }\zeta}|Du(x+h)|^p dx\right)^{\frac{p-2}{2p}}+\left(\int_{\text{supp }\zeta}|Du(x)|^p dx\right)^{\frac{p-2}{2p}} \\
\leq\ & 2\left(\int_{\text{supp }\zeta}|Du(x)|^p dx\right)^{\frac{p-2}{2p}} \quad \text{for sufficiently small } h \\
=\ & 2\left(\int_{\text{supp }\zeta}|F(x)|^2 dx\right)^{\frac{p-2}{2p}}
\end{aligned}
$$

Combining the results so far:

$$
\begin{aligned}
& \frac{4}{p^2}\int_U \zeta^2(x)|F(x+h)-F(x)|^2 dx \\
\leq\ & 4(p-1)\left(\int_U |u(x+h)-u(x)|^p|D\zeta(x)|^p dx\right)^{\frac{1}{p}}\left(\int_U \zeta^2(x)|F(x+h)-F(x)|^2 dx\right)^{\frac{1}{2}} \\
& \quad \left(\int_{\text{supp }\zeta}|F(x)|^2 dx\right)^{\frac{p-2}{2p}}
\end{aligned}
$$

Dividing both sides by $\left(\int_U \zeta^2(x)|F(x+h)-F(x)|^2dx\right)^{\frac{1}{2}}$, we have

$$\begin{aligned}&\frac{1}{p^2}\left(\int_U \zeta^2(x)|F(x+h)-F(x)|^2dx\right)^{\frac{1}{2}}\\ \leq\ &(p-1)\left(\int_U |u(x+h)-u(x)|^p|D\zeta(x)|^pdx\right)^{\frac{1}{p}}\left(\int_{\text{supp }\zeta}|F(x)|^2dx\right)^{\frac{p-2}{2p}}\end{aligned}\tag{3.7}$$

To proceed, we need the characterization of Sobolev spaces in terms of integrated difference quotients studied in [2, Section 5.8.2] and [6, Section 7.11] , in particular, we need the following lemma:

Lemma 3.2. (Difference quoteients and weak derivatives)

(i). Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. Then $\|\frac{u(x+he_i)-u(x)}{h}\|_{L^p(V)} \in L^p(V)$ for any $V \subset\subset U$ satisfying $h \leq \text{dist}(V, \partial U)$, $h \neq 0$ and we have

$$\|\frac{u(x+he_i)-u(x)}{h}\|_{L^p(V)} \leq \|D_iu\|_{L^p(U)}.$$

(ii). Assume $1 < p < \infty$, $u \in L^p(V)$, and there exists a constant C such that

$$\|\frac{u(x+he_i)-u(x)}{h}\|_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2}\text{dist }(V, \partial U)$. Then

$$u \in W^{1,p}(V), \text{with } ||Du||_{L^p(V)} \leq C.$$

Proof. (Sketch)

(i). We prove for $u \in C^1(U) \cap W^{1,p}(U)$ and the general case is obtained by an approximation argument using [6, Theorem 7.9]. We have

$$\begin{aligned}&\frac{u(x+he_i)-u(x)}{h}\\ =\ &\frac{1}{h}\int_0^h D_iu(x_1,\ldots,x_{i-1},x_i+\xi,x_{i+1},\ldots,x_n)d\xi.\end{aligned}$$

By Hölder inequality

$$\left|\frac{u(x+he_i)-u(x)}{h}\right|^p \leq \frac{1}{h}\int_0^h |u(x_1,\ldots,x_{i-1},x_i+\xi,x_{i+1},\ldots,x_n)|^p\,d\xi.$$

and the result follows.

For a detailed proof, see [6, Lemma 7.23].

(ii). Let $\{h_m\}$ be a sequence converging to 0 and $v \in L^p(U)$ with $\|v\|_p \leq C$ satisfy for all $\varphi \in C_0^1(U)$

$$\int_U \varphi \triangle^{h^m} u dx \longrightarrow \int_U \varphi v dx.$$

For $h_m < \text{dist}(\text{supp } \varphi, \partial U)$, we have

$$\int_U \varphi \triangle^{h^m} u dx = -\int_U u \triangle^{-h_m} \varphi dx \longrightarrow -\int_U u D_i \varphi.$$

Hence

$$\int_U \varphi v dx = -\int_U u D_i \varphi dx$$

which implies $v = D_i u$.

Note that for the sequence $\triangle^{h_m} u$ to exist, we require the following analysis fact:

A bounded sequence in a separable, reflexive Banach space contains a weakly convergent subsequence.

Reader can refer to [6, Chapter 5] for related knowledge.

□

From part (i) of the lemma 3.2 we have

$$\begin{aligned}
\left(\int_V \left|\frac{u(x+he_i)-u(x)}{h}\right|^p\right)^{\frac{1}{p}} &= \left(\int_U \left|\frac{u(x+he_i)-u(x)}{h}\right|^p |D\zeta(x)|^p\right)^{\frac{1}{p}} \\
&\leq \frac{C(n)}{\text{dist}(V, \partial U)} \left(\int_U |Du(x)|^p dx\right)^{\frac{1}{p}}
\end{aligned}$$

Combining with (3.8), we have

$$\begin{aligned}
\left(\int_U \zeta^2(x)|F(x+h)-F(x)|^2 dx\right)^{\frac{1}{2}} &\leq \frac{C(n,p)}{\text{dist}(V, \partial U)} \left(\int_U |Du(x)|^p dx\right)^{\frac{1}{p}} \\
&= \frac{C(n,p)}{\text{dist}(V, \partial U)} \left(\int_U |F(x)|^2 dx\right)^{\frac{1}{p}} \\
&\leq C
\end{aligned}$$

Then part two of lemma 3.2 tells us that $F \in W^{1,2}(V)$, this completes the proof.

□

Now we turn to the second case, the result is formally stated in the following theorem:

Theorem 3.3. *Let $1 < p \leq 2$. If u is p-harmonic in U, then $u \in W^{2,p}_{loc}(U)$. Moreover*

$$\int_V \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^p dx \leq C_V \int_U |Du|^p dx$$

when $V \subset\subset U$.

Proof. We start with the following lemma:

Lemma 3.4. *Let $f \in L^1_{loc}(U)$. Then*

$$\int_U \varphi(x) \frac{f(x + he_k) - f(x)}{h} dx = - \int_U \frac{\partial \varphi}{\partial x_k} \left(\int_0^1 f(x + the_k dt) \right) dx$$

holds for all $\varphi \in C_0^\infty(U)$.

Proof. (Sketch)

For a smooth function f,

$$\frac{\partial}{\partial x_k} \int_0^1 f(x + the_k) dt = \frac{f(x + te_k) - f(x)}{h}$$

holds by the infinitesimal calculus. As the set of smooth function is dense in L^1, the general case follows easily by an approximation argument. □

Notation 3.5. Regarding the x_k-axis as the chosen direction, we use the abbreviation

$$\triangle^h f = \triangle^h f(x) = \frac{f(x + he_k) - f(x)}{h}.$$

Choosing $f = |Du|^{p-2} Du$ and by the lemma above we have

$$\triangle^h (|Du|^{p-2} Du) = \frac{\partial}{\partial x_k} \int_0^1 |Du(x + the_k)|^{p-2} Du(x + the_k) dt, \tag{3.8}$$

then

$$\int \zeta^2 \triangle^h (|Du|^{p-2} Du) \cdot \triangle^h (Du) dx$$

Proof of Theorem 3.3.

Again, choosing the test function

$$\eta = \zeta(x)^2 \triangle^h u(x)$$

in the equations (3.5) gives

$$\begin{aligned}\int_U \triangle^h(|Du|^{p-2}Du)\cdot D(\zeta^2\triangle^h u) &= 0\\ \Rightarrow \int_U \triangle^h(|Du|^{p-2}Du)\cdot(\zeta^2\triangle^h(Du)+2\zeta\triangle^h uD\zeta) &= 0.\end{aligned}$$

Rearranging the equation, we get

$$\begin{aligned}&\int_U \zeta^2\triangle^h(|Du|^{p-2}Du)\cdot\triangle^h(Du)\\ =\ & -2\int_U \zeta\triangle^h u\triangle^h(|Du|^{p-2}Du)\cdot D\zeta\\ =\ & 2\int\left(\int_0^1 |Du(x+the_k)|^{p-2}Du(x+the_k)dt\right)\cdot\frac{\partial}{\partial x_k}\left(\triangle^h u\cdot\zeta D\zeta\right)dx\\ =\ & 2\int\left(\int_0^1 |Du(x+the_k)|^{p-2}Du(x+the_k)dt\right)\\ &\cdot\left(\zeta D\zeta\triangle^h u_{x_k}+\triangle^h u(\zeta_{x_k}D\zeta+\zeta D\zeta_{x_k})\right)dx \end{aligned} \tag{3.9}$$

where the second last equality is just integration by parts with respect to x_k with the aid of (3.8).

Let ζ be a cutoff function such that $0\le\zeta\le 1$, $\zeta=1$ in B_R, $\zeta=0$ outside B_{2R} and

$$|D\zeta|\le R^{-1},\quad |D^2\zeta|\le CR^{-2}.$$

Then we can bound the $\zeta_{x_k}D\zeta$ by $|D\zeta|^2$ and $\zeta D\zeta_{x_k}$ by $|D^2\zeta|$, hence it follow from (3.9) that

$$\begin{aligned}&\int_U \zeta^2\triangle^h(|Du|^{p-2}Du)\cdot\triangle^h(Du)\\ \le\ & \frac{2}{R}\int_U \zeta Y|\triangle^h u_{x_k}|dx+\frac{c}{R^2}\int_{B_{2R}}|\triangle^h u|Y dx\end{aligned} \tag{3.10}$$

where we use the abbreviation

$$Y(x)=\int_0^1 |Du(x+the_k)|^{p-1}dt.$$

Now, by choosing $b=Du(x+he_k)$ and $a=Du(x)$ in the elementary inequality (3.3), we can estimate the left side of (3.10) from below:

$$\begin{aligned}&\int_U \zeta^2\triangle^h(|Du|^{p-2}Du)\cdot\triangle^h(Du)\\ \ge\ & (p-1)\int_U |\triangle^h(Du)|^2(1+|Du(x)|^2+|Du(x+he_k)|^2)^{\frac{p-2}{2}}dx \quad \text{by (3.3)}\\ =\ & (p-1)\int_U |\triangle^h(Du)|^2 W^{p-2}dx\end{aligned}$$

where we used the abbreviation

$$W(x)^2 = 1 + |Du(x)|^2 + |Du(x + he_k)|^2.$$

Combining with (3.10) and using $|\triangle^h u_{x_k}| \le |\triangle^h (Du)|$, we have

$$\begin{aligned}
& (p-1)\int_U |\triangle^h (Du)|^2 W^{p-2} dx \\
\le\ & \frac{2}{R}\int_U \zeta Y |\triangle^h u_{x_k}| dx + \frac{c}{R^2}\int_{B_{2R}} |\triangle^h u| Y dx \\
\le\ & \frac{2}{R}\int_U \zeta Y |\triangle^h (Du)| dx + \frac{c}{R^2}\int_{B_{2R}} |\triangle^h u| Y dx \qquad (3.11)
\end{aligned}$$

The trick here is to absorb the first term on the right hand side by the following computation:

$$\begin{aligned}
& 2R^{-1}\zeta Y |\triangle^h (Du)| \\
=\ & \left(W^{(p-2)/2}\zeta |\triangle^h (Du)|\right)\left(2W^{(2-p)/2} Y R^{-1}\right) \\
\le\ & \epsilon \left(W^{(p-2)/2}\zeta |\triangle^h (Du)|\right)^2 + \frac{1}{4}\epsilon^{-1}\left(2W^{(2-p)/2} Y R^{-1}\right)^2 \\
& \quad \text{(by Young's inequality with exponents 2)} \\
=\ & \epsilon \left(W^{p-2}\zeta^2 |\triangle^h (Du)|^2\right) + \epsilon^{-1}\left(W^{2-p} Y^2 R^{-2}\right),
\end{aligned}$$

take $\epsilon = (p-1)/2$, then we get from (3.11) that

$$\begin{aligned}
& (p-1)\int_U |\triangle^h (Du)|^2 W^{p-2} dx \\
\le\ & \int_U \left(\frac{p-1}{2} W^{p-2}\zeta^2 |\triangle^h (Du)|^2 dx + \frac{2}{p-1} W^{2-p} Y^2 R^{-2}\right) dx \\
& \quad + \frac{c}{R^2}\int_{B_{2R}} |\triangle^h u| Y dx \\
\le\ & \int_{B_R} \left(\frac{p-1}{2} W^{p-2} |\triangle^h (Du)|^2\right) dx + \int_{B_{2R}} \left(\frac{2}{p-1} W^{2-p} Y^2 R^{-2}\right) dx \\
& \quad + \frac{c}{R^2}\int_{B_{2R}} |\triangle^h u| Y dx,
\end{aligned}$$

by rearranging, we have

$$\begin{aligned}
& \frac{p-1}{2}\int_U |\triangle^h (Du)|^2 W^{p-2} dx \\
\le\ & \int_{B_{2R}} \left(\frac{2}{p-1} W^{2-p} Y^2 R^{-2}\right) dx + \frac{c}{R^2}\int_{B_{2R}} |\triangle^h u| Y dx.
\end{aligned}$$

Using the elementary inequality

$$\begin{aligned} |\triangle^h(Du)|^p &\leq W^{p-2}|\triangle^h(Du)|^2 + W^p, \\ W^{2-p}Y^2 &\leq W^p + Y^{p/(p-1)}, \\ |\triangle^h u|Y &\leq |\triangle^h u|^p + Y^{p/(p-1)}, \end{aligned}$$

we have

$$\int_{B_R} |\triangle^h(Du)|^p dx \leq C_1 \int_{B_{2R}} W^p dx + C_2 \int_{B_{2R}} Y^{\frac{p}{p-1}} dx + C_3 \int_{B_{2R}} |\triangle^h u|^p dx$$

where the constants depend on R. It remains to bound the three integrals as $h \to 0$. The first one is easy:

$$\begin{aligned} \int_{B_{2R}} W^p dx &= \int_{B_{2R}} \left(1 + |Du(x)|^2 + |Du(x+he_k)|^2\right)^{p/2} dx \\ &\leq \int_{B_{2R}} 1 + |Du(x)|^p + |Du(x+he_k)|^p dx \\ &\leq CR^n + C\int_{B_{2R}} |Du|^p dx. \end{aligned}$$

The second integral is bounded as follows:

$$\begin{aligned} \int_{B_{2R}} Y^{\frac{p}{p-1}} dx &= \int_{B_{2R}} \left(\int_0^1 |Du(x+the_k)|^{p-1} dt\right)^{\frac{p}{p-1}} dx \\ &\leq \int_{B_{2R}} \left(\int_0^1 |Du(x+the_k)|^p dt\right) dx \\ &\leq \int_{B_{3R}} |Du|^p dx \end{aligned}$$

The bound for the last integral

$$\int_{B_{2R}} |\triangle^h u|^p dx \leq \int_{B_{3R}} |Du|^p dx \tag{3.12}$$

simply follows from Lemma 3.2. Collecting the bounds, we have

$$\int_{B_R} |\triangle^h(Du)|^p dx \leq C(n,p,R)\int_{B_{3R}} |Du|^p dx$$

and the theorem follows.

□

Chapter 4

Regularity of the Derivatives

We have shown that the weak solutions admits derivatives for different values of p, in this chapter, we study even further regularity and prove $C^{1,\alpha}_{loc}$ estimates for solutions $u \in W^{1,p+2}$ for $p > 0$ (for computational convenience we replace p by $p+2$ in 1.1 for this section). This problem is studied in [1] and extended in [4]. The proof introduced in this chapter is given in [1].

The main result is stated in the following theorem:

Theorem 4.1. *Suppose that $u \in W^{1,p+2}(U)$ is a weak solution of* (1.1). Then there exists a constant $\alpha = \alpha(p,n) > 0$ and, for each $V \subset\subset U$, a constant $C(V) = C(V, p, n, ||u||_{W^{1,p}})$ such that

$$\max_V |Du| \leq C(V)$$

and

$$[Du]_{C^\alpha(V)} \leq C(V)$$

where $[Du]_{C^\alpha(V)}$ is an abbreviation defined as

$$[Du]_{C^\alpha(V)} \equiv \sup_{\substack{x,y\in V \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

4.1 An apriori estimate on the oscillations of $|Du|$

In this section, we present a formal derivation of an estimate on the Hölder continuity of Du near a point where $Du = 0$, also called a point a degeneracy and then indicate how to modify the estimates to cover a related, approximate problem.

Suppose for now that u is a smooth solution of (1.1) in some ball B_{R_0}, that

$$Du(0) = 0 \tag{4.1}$$

and

$$\max_{B_{R_0}} |Du| \leq K. \tag{4.2}$$

Define

$$M(R) \equiv \max_{B_R} |Du| \quad (0 < R < R_0). \tag{4.3}$$

We aim to show that (4.1) forces $M(R)$ to grow no faster than some fractional power of R. This idea is properly stated in the following proposition:

Proposition 4.2. *There exist constants $C_1 = C_1(p, n)$ and $\beta = \beta(p, n) > 0$ such that*

$$M(R) \leq C_1 K \left(\frac{R}{R_0}\right)^{\beta} \quad (0 < R < R_0).$$

Fix some $0 < R < R_0$ and define

$$M_k^{\pm}(R) \equiv \max_{B_R} \pm u_{x_k}, \quad (k = 1, 2, ..., n)$$

Since $|Du| = (u_{x_1}^2 + \cdots + u_{x_n}^2)^{1/2}$, there exists some index i such that u_{x_i} is greater than the average of $M(R)$. In other words, there exists i such that either

$$M_i^+(R) \geq \frac{1}{\sqrt{n}} M(R) \tag{4.4}$$

or

$$M_i^-(R) \geq \frac{1}{\sqrt{n}} M(R).$$

Therefore it is safe to assume, upon relabelling the coordinate axes if necessary,

$$M_1^+(R) \geq \frac{1}{\sqrt{n}} M(R) > 0 \tag{4.5}$$

We will need several lemmas to get to the main result, the first one is the following:

Lemma 4.3. *There exists a constant $\varepsilon_0 = \varepsilon_0(p,n) > 0$ such that*

$$\fint_{B_R} (M_1^+ - u_{x_1})^{+^2} dx \le \varepsilon_0 M_1^+(R)^2$$

implies

$$\min_{B_{R/2}} u_{x_1} \ge \frac{M_1^+(R)}{2}.$$

Proof. Regarding the premise of this lemma, we temporarily drop assumption (4.1).

Let $M_1 = M_1^+(R)$, $v \equiv M_1 - u_{x_1}$. Differentiating (1.1) with respect to x_1, we get

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left(\operatorname{div}\left(|Du|^p Du\right)\right) &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_1}(|Du|^p u_{x_i}) \\
&= \frac{\partial}{\partial x_i}\left(p|Du|^{p-2}u_{x_j}u_{x_j x_1}u_{x_i} + |Du|^p u_{x_i x_1}\right) \\
&= \frac{\partial}{\partial x_i}\left(p|Du|^{p-2}u_{x_j}u_{x_j x_1}u_{x_i} + \delta_{ij}|Du|^p u_{x_j x_1}\right) \\
&= \frac{\partial}{\partial x_i}\left(|Du|^p u_{x_j x_1}\left(p|Du|^{-2}u_{x_j}u_{x_i} + \delta_{ij}\right)\right) \\
&= -\left(a_{ij}|Du|^p v_{x_j}\right)_{x_i}.
\end{aligned}$$

We see that v solves the P.D.E.

$$-\left(a_{ij}|Du|^p v_{x_j}\right)_{x_i} = 0 \quad \text{in } B_{R_0} \tag{4.6}$$

where

$$a_{ij} \begin{cases} \equiv \delta_{ij} + p\frac{u_{x_i}u_{x_j}}{|Du|^2} & \text{if } Du \neq 0 \\ \equiv \delta_{ij} & \text{if } Du = 0. \end{cases} \tag{4.7}$$

Let ζ be a smooth cutoff function such that $0 \le \zeta \le 1$ and $\zeta = 0$ outside B_R and k a constant satisfying

$$0 \le k \le \frac{M_1}{2}. \tag{4.8}$$

Multiply (4.6) by $\zeta^2(v-k)^+$ and then integrate both sides over B_R, we have

$$\begin{aligned}
-(a_{ij}|Du|^p v_{x_i})_{x_j}\zeta^2(v-k)^+ &= 0 \\
\Rightarrow \quad \int_{B_R} -(a_{ij}|Du|^p v_{x_i})_{x_j}\zeta^2(v-k)^+ &= 0 \\
\Rightarrow \int_{B_R\cap\{v>k\}} -(a_{ij}|Du|^p v_{x_i})_{x_j}\zeta^2(v-k) &= 0,
\end{aligned}$$

Since we are temporarily drop assumption (4.1), it is safe to use $a_{ij} = \delta_{ij} + p\frac{u_{x_i}u_{x_j}}{|Du|^2}$ in the following calculation:

$$\begin{aligned}
0 &= -\int_{B_R\cap\{v>k\}} -a_{ij}|Du|^p v_{x_i}\left(2\zeta\zeta_{x_j}(v-k)+\zeta^2(v-k)\right) \\
&= \int_{B_R\cap\{v>k\}} \left(\left(\delta_{ij}+p\frac{u_{x_i}u_{x_j}}{|Du|^2}\right)|Du|^p v_{x_i}\right)\left(2\zeta\zeta_{x_j}(v-k)+\zeta^2(v-k)\right) \\
&\quad \text{(integration by parts)} \\
&= I_1 + I_2 \qquad (4.9)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{B_R\cap\{v>k\}} a_{ij}|Du|^p v_{x_i} 2\zeta\zeta_{x_i}(v-k)dx \\
I_2 &= a_{ij}|Du|^p v_{x_i}\zeta^2(v-k) \\
&= \int_{B_R\cap\{v>k\}} \zeta^2|Du|^p|Dv|^2 + p|Du|^{p-2}\zeta^2 u_{x_i}u_{x_j}v_{x_i}(v-k)dx.
\end{aligned}$$

Rearranging (4.9), we have

$$\begin{aligned}
&\int_{B_R\cap\{v>k\}} \zeta^2|Du|^p|Dv|^2dx \\
= &-\int_{B_R\cap\{v>k\}} a_{ij}|Du|^p 2(\zeta v_{x_i})((v-k)\zeta_{x_j})dx - I_2 \\
\le &\left|\int_{B_R\cap\{v>k\}} |Du|^p\left(2\varepsilon\zeta^2|Dv|^2 + C_\varepsilon(v-k)^2|D\zeta|^2\right)dx\right| + |I_2| \\
\le &\; C\left|\int_{B_R\cap\{v>k\}} |Du|^p\zeta^2|Dv|^2dx\right| + C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right| + |I_2| \\
\le &\; C\left|\int_{B_R\cap\{v>k\}} |Du|^p\zeta^2|Dv|^2dx\right| + C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right| \\
&\quad + C\left|\int_{B_R\cap\{v>k\}} p|Du|^{p-2}\zeta^2u_{x_i}u_{x_j}v_{x_i}(v-k)dx\right|.
\end{aligned}$$

Putting the first term on right side to left side and getting rid of the constant on left side,

$$\begin{aligned}
&\int_{B_R\cap\{v>k\}} \zeta^2|Du|^p|Dv|^2dx \\
\le &\; C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right| + C\left|\int_{B_R\cap\{v>k\}} p|Du|^{p-2}\zeta^2u_{x_i}u_{x_j}v_{x_i}(v-k)dx\right| \\
\le &\; C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right| + C\left|\int_{B_R\cap\{v>k\}} p|Du|^{p-2}\zeta^2(\varepsilon u_{x_i}^2 + C_\varepsilon u_{x_j}^2)v_{x_i}(v-k)dx\right| \\
\le &\; C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right| + C\left|\int_{B_R\cap\{v>k\}} p|Du|^p\zeta^2\left(\varepsilon|Dv|^2 + C_\varepsilon(v-k)^2\right)dx\right|.
\end{aligned}$$

Now by selecting value for ε such that C_ε is small enough, we can absorb the second integral into the left side, which leave us with

$$\begin{aligned} & \int_{B_R\cap\{v>k\}} \zeta^2|Du|^p|Dv|^2dx \\ \leq\ & C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right|. \end{aligned} \tag{4.10}$$

Recall that we defined $M(R)=\max_{B_R}|Du|$, then

$$\begin{aligned} & C\left|\int_{B_R\cap\{v>k\}} |Du|^p(v-k)^2|D\zeta|^2dx\right| \\ \leq\ & CM(R)^p\int_{B_R\cap\{v>k\}} (v-k)^2|D\zeta|^2dx. \end{aligned}$$

Combining with (4.10),we have

$$\begin{aligned} & \int_{B_R\cap\{v>k\}} \zeta^2|Du|^p|Dv|^2dx \\ \leq\ & CM(R)^p\int_{B_R\cap\{v>k\}} (v-k)^2|D\zeta|^2dx. \end{aligned} \tag{4.11}$$

Also the left side of (4.10) is greater than or equal to

$$\int_{B_R\cap\{k<v<k+M_1/4\}} |Dv|^2|Du|^p\zeta^2dx$$

simply due to the fact that the set being integrated over is smaller. Now if $v=M_1-u_{x_1}<k+M_1/4$, we have

$$\begin{aligned} u_{x_1} &\geq \frac{3}{4}M_1-k\geq\frac{1}{4}M_1\geq\frac{1}{4}M_1 \quad \text{by (4.8)} \\ &\geq \frac{1}{4\sqrt{n}}M(R) \quad \text{by (4.4)}, \end{aligned}$$

then

$$|Du|^p\geq CM(R)^p \ \ (C>0) \tag{4.12}$$

on $\{k<v<k+M_1/4\}$. Using this estimate in (4.11) gives

$$M(R)^p\int_{B_R\cap\{k<v<k+M_1/4\}} |Dv|^2\zeta^2dx\leq C\int_{B_R\cap\{v>k\}}(v-k)^2|D\zeta|^2dx,$$

and therefore, after cancellation,

$$\int_{B_R}|D\phi_k(v)|^2\zeta^2dx\leq C\int_{B_R\cap\{v>k\}}(v-k)^2|D\zeta|^2dx, \tag{4.13}$$

where

$$\phi_k(x) \equiv \begin{cases} 0, & x < k \\ x - k, & k \le x \le k + M_1/4 \\ M_1/4, & k + M_1/4 < x. \end{cases}$$

We now try to bound the left side of (4.13):

$$\begin{aligned}
& \int_{B_R} |D(\phi_k(v)\zeta)|^2 \, dx \\
= & \int_{B_R} |D\phi_k(v)Dv\zeta + \phi_k(v)D\zeta|^2 \, dx \\
= & \int_{B_R} |D\phi_k(v)|^2 |Dv|^2\zeta^2 + \phi_k(v)^2 |D\zeta|^2 + 2D\phi_k(v)Dv\zeta\phi_k(v)D\zeta dx \\
\le & \int_{B_R} |D\phi_k(v)|^2 |Dv|^2\zeta^2 + \phi_k(v)^2 |D\zeta|^2 + 2\left(\varepsilon |D\phi_k(v)|^2 |Dv|^2\zeta^2 + C_\varepsilon |\phi_k(v)|^2 |D\zeta|^2\right) dx \\
\le & \int_{B_R} C |D\phi_k(v)|^2 |Dv|^2\zeta^2 + C'\phi_k(v)^2 |D\zeta|^2 \, dx,
\end{aligned}$$

then by Sobolev's inequality (with exponent 2), we have

$$\begin{aligned}
& \left(\int_{B_R} \left|(\phi_k(v)\zeta)^{\frac{2n}{n-2}}\right| dx\right)^{\frac{n-2}{n}} \\
\le & \int_{B_R} |D(\phi_k(v)\zeta)|^2 \, dx \\
\le & \int_{B_R} C |D\phi_k(v)|^2 |Dv|^2\zeta^2 + C'\phi_k(v)^2 |D\zeta|^2 \, dx \\
\le & C \int_{B_R \cap \{v>k\}} (v-k)^2 |D\zeta|^2 dx \\
\le & C \max_{B_R} |D\zeta|^2 \int_{B_R \cap \{v>k\}} (v-k)^2 dx
\end{aligned} \tag{4.14}$$

Next, define for $m = 0, 1, 2, \ldots$

$$\begin{aligned}
k_m &\equiv \frac{M_1}{2}(1 - \frac{1}{2^m}) < \frac{M_1}{2} \quad \text{by (4.8)} \\
R_m &\equiv \frac{R}{2}(1 + \frac{1}{2^m}),
\end{aligned}$$

and choose smooth cutoff functions ζ_m such that $0 \le \zeta_m \le 1$, $\zeta_m \equiv 1$ on $B_{R_{m+1}}, \zeta_m \equiv 0$ outside B_{R_m} and $|D\zeta_m| \le \frac{C2^m}{R}$. Set $R = R_m$, $\zeta = \zeta_m$, $k = k_m$ in (4.14), we have

$$\left(\int_{B_{R_{m+1}}} \phi_{k_m}(v)^{2n/(n-2)} dx\right)^{(n-2)/n} \le \frac{C4^m}{R^2} \int_{B_{R_m}} (v - k_m)^2 dx. \tag{4.15}$$

Define

$$J_m = \int_{B_{R_m}} \phi_{k_m}(v)^2 dx \quad (m = 0, 1, \dots),$$

by definition of ϕ_k, we know that

$$J_m = \begin{cases} 0, & v < k_m \\ \int_{B_{R_m}} |v - k_m|^2 dx, & k_m \le v \le k_m + M_1/4 \\ \frac{M_1^2}{16} \text{meas}\{B_{R_m}\}, & k_m + M_1/4 < v \end{cases}$$

then we have the estimate

$$J_m \le \int_{B_{R_m}} (v - k_m)^2 dx \le C J_m, \tag{4.16}$$

and on the set $\{v > M_1/4 + k_m\}$ we have

$$(v - k_m)^{+^2} \le CM(R)^2 \le CM_1^2 \le C\phi_{k_m}^2.$$

Furthermore

$$\begin{aligned} & \text{meas}\{x \in B_{R_{m+1}} | \phi_{k_{m+1}}(v) > 0\} \\ = \;& \text{meas}\{x \in B_{R_{m+1}} | v > k_{m+1}\} \\ \le \;& \frac{1}{(k_{m+1} - k_m)^2} \int_{B_{R_m}} (v - k_m)^{+^2} dx \\ \le \;& \frac{C4^m}{M_1^2} J_m \quad \text{by (4.16)} \end{aligned}$$

Consequently

$$\begin{aligned} J_{m+1} &= \int_{B_{R_{m+1}}} \phi_{k_{m+1}}(v)^2 dx \\ &\le \left(\int_{B_{R_{m+1}}} \phi_{k_{m+1}}(v)^{2n/(n-2)} \right)^{(n-2)/n} \times \left(\text{meas}\{x \in B_{R_{m+1}} | \phi_{k_{m+1}}(v) > 0\} \right)^{2/n} \\ &\le \frac{C_2 C_3^m}{R^2 M_1^{4/n}} J_m^{1+2/n} \quad (m = 0, 1, 2, \dots) \end{aligned}$$

where the first inequality follows from Hölder inequality with exponents $\frac{n}{n-2}$ and $\frac{n}{2}$. To make use of this recursive relation, we require the following lemma:

Lemma 4.4. *Suppose that a sequence y_i, for $i = 0, 1, 2, \dots$ of non-negative numbers satisfies the recursive relation*

$$y_{i+1} \le c b^i y_i^{1+\varepsilon}, i = 0, 1, 2, \dots$$

where c, b and ε are positive constants and $b > 1$. Then

$$y_i \leq c^{\frac{(1+\varepsilon)^i-1}{\varepsilon}} b^{\frac{(1+\varepsilon)^i-1}{\varepsilon^2}-\frac{1}{\varepsilon}} y_0^{(1+\varepsilon)^i}.$$

and in particular, if

$$y_0 \leq \theta = c^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}},$$

then

$$y_i \leq \theta b^{1\frac{1}{\varepsilon}},$$

and consequently, $y_i \to 0$ as $i \to \infty$.

Proof. Since this is an auxiliary lemma, we emit the proof here and refer to [3, Lemma 4.7]. $\square$

Now, if

$$\begin{aligned} J_0 &\equiv \int_{B_R} \phi_0(v)^2 dx \\ &\leq \int_{B_R} v^{+^2} = \int_{B_R} (M_1 - u_{x_1})^{+^2} dx \\ &\leq \varepsilon_0 M_1^2 \text{meas} B_R, \end{aligned}$$

then we can apply the lemma to obtain

$$J_m \xrightarrow{m\to\infty} 0,$$

in which case we have

$$\begin{aligned} \max_{B_{R/2}} v &\leq \frac{M_1}{2} \\ \Leftrightarrow \min_{B_{R/2}} u_{x_1} &\geq \frac{M_1}{2} \end{aligned}$$

which is the desired result.

$\square$

The proof of this lemma involves some tricky computation so readers may find it difficult to follow, but the idea of the lemma is simply saying that if u_{x_1} is on average very close to its positive maximum $M_1^+(R)$ on B_R, then u_{x_1} is strictly positive on $B_{R/2}$.

The second lemma we introduce states that u_{x_1} must be strictly less than its maximum $M_1^+(R)$ on some proper subset of B_R.

Lemma 4.5. *Under assumption (4.2) and (4.5) there exist constants* $0 < \lambda, \mu < 1$ *such that*

$$meas\{x \in B_R | u_{x_1}(x) \leq \lambda M_1^+(R)\} \geq \mu \ meas B_R$$

with λ *and* μ *depend only on* p *and* n.

Proof. Set $M_1 = M_1^+(R)$. Suppose otherwise, in other words, for all constants $0 < \lambda, \mu < 1$ we have

$$\text{meas}\{x \in B_R | u_{x_1}(x) \leq \lambda M_1^+(R)\} < \mu \ \text{meas} B_R,$$

then in particular, for $0 < \mu$ small enough and $\lambda < 1$ close enough to 1, we have

$$\begin{aligned}
& \fint_{B_R} (M_1 - u_{x_1})^{+^2} dx \\
= & \fint_{B_R \cap \{u_{x_1} < \lambda M_1\}} (M_1 - u_{x_1})^{+^2} dx + \fint_{B_R \cap \{\lambda M_1 \leq u_{x_1} \leq M_1\}} (M_1 - u_{x_1})^{+^2} dx \\
\leq & \ C M_1^2 \frac{\text{meas}\{u_{x_1} < \lambda M_1\}}{R^n} + C(1-\lambda)^2 M_1^2 \\
\leq & \ C(\mu + (1-\lambda)^2) M_1^2 \\
\leq & \ \varepsilon_0 M_1^2.
\end{aligned}$$

Hence the premise of Lemma 4.3 is verified and we thus have

$$\min_{B_{R/2}} u_{x_1} \geq \frac{M_1^+(R)}{2} > 0,$$

which is a contradiction to (4.1).

□

Next lemma builds on the preceding one and asserts that $M_1^+(R/2)$ is then strictly less than $M_1^+(R)$.

Lemma 4.6. *There exists a positive constant* $\gamma = \gamma(p, n) < 1$ *such that*

$$M_1^+ \left(\frac{R}{2} \right) \leq \gamma M_1^+(R).$$

Proof. (Sketch)

Once more we set $M_1 = M_1^+(R)$. Define for $\delta > 0$

$$\phi(x) = \phi_\delta(x) \equiv \left(-\log \left(\frac{M_1 - x + \delta}{M_1(1-\lambda)} \right) \right)^+ \quad \text{for } x \leq M_1.$$

It is easy to check that ϕ is nondecreasing and convex and

$$\begin{cases} (\phi')^2 = \phi'', & \text{for } x \neq M_1\lambda + \delta, \\ \phi = 0, & \text{for } x < M_1\lambda + \delta. \end{cases} \tag{4.17}$$

Let

$$w \equiv \phi(u_{x_1}). \tag{4.18}$$

Then lemma 4.5 and (4.17) imply

$$\text{meas}\{x \in B_R | w = 0\} \geq \mu \ \text{meas} B_R,$$

and so

$$\text{meas}\{x \in B_{\theta R} | w = 0\} \geq \frac{\mu}{2} \text{meas} B_R \tag{4.19}$$

for some $\theta = \theta(\mu, n)$, $\frac{3}{4} < \theta < 1$.

To proceed, we required the following lemma from [5, Lemma 2]:

Lemma 4.7. *Let w be defined in $|x| < \rho$ and $w \in H^1$. Then there exists a constant C_n which depends on n and the choice of C_0 such that*

$$\left(\rho^{-n} \int_{|x|<\rho} w^{2k}\right)^{1/k} \leq C_n \left(\rho^{-n+2} \int_{|x|<\rho} |Dw|^2 dx + \rho^{-n} \int_N w^2 dx\right)$$

for every $1 \leq k \leq n/(n-1)$. Here N is any measurable set in $|x| < \rho$ of measure $m(N) \geq C_0^{-1}\rho^n$.

Proof. See [5, Lemma 2]. □

Choose $\rho = \theta R$, $k = 1$ and $N = \{x \in B_{\theta R} | w = 0\}$, it follows from the lemma and (4.19) that

$$⨍_{B_{\theta R}} w^2 dx \leq CR^2 ⨍_{B_{\theta R}} |Dw|^2 dx, \ C = C(\mu, \theta, n). \tag{4.20}$$

Furthermore, since ϕ satisfies (4.17) and $v = u_{x_1}$ solves (4.6), then w is a non-negative weak subsolution of the same equation:

$$-(a_{ij}|Du|^p w_{x_i})_{x_j} = -a_{ij}|Du|^p w_{x_i} w_{x_j} \leq 0. \tag{4.21}$$

In addition on the set where $w > 0$, we have

$$\begin{aligned} \left(-\log\left(\frac{M_1 - u_{x_1} + \delta}{M_1(1-\lambda)}\right)\right)^+ &> 0 \\ \Rightarrow \frac{M_1 - u_{x_1} + \delta}{M_1(1-\lambda)} &\leq 1 \\ \Rightarrow u_{x_1} &\geq \delta + M_1\lambda \geq M_1\lambda, \end{aligned}$$

hence using 4.5 we have

$$M(R)^p \leq CM_1^p \leq C|Du|^p \leq CM(R)^p \tag{4.22}$$

where the constant C depends only on n and p. As a consequence we can apply the Moser iteration method* to (4.21), using (4.22) to estimate the $|Du|^p$ terms in the integrals and cancelling the resulting expressions $M(R)^p$, and arrive therefore at the estimate

$$\max_{B_{R/2}} w^2 \leq C \fint_{B_{\theta R}} w^2 dx, \quad C = C(p, n, \theta) \tag{4.23}$$

Choosing the cutoff function ζ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_{\theta R}$, $\zeta = 0$ near ∂B_R and $|D\zeta| \leq \frac{C}{(1-\theta)R}$. Then by calculations similar to the proof of lemma 4.3, we obtain

$$\fint_{B_{\theta R}} |Dw|^2 dx \leq \frac{C}{R^2}, \quad C = C(p, n, \theta) \tag{4.24}$$

Combining (4.23), (4.20) and (4.24), we have

$$\max_{B_{R/2}} w \leq C_4, \quad C_4 = C(p, n, \theta, \mu). \tag{4.25}$$

Then (4.25) implies that, for $x \in B_{R/2}$,

$$\begin{aligned} u_{x_1}(x) &\leq M_1(1 - (1-\lambda)e^{-w(x)}) + \delta \\ &\leq \gamma M_1 + \delta \end{aligned}$$

for $\gamma \equiv 1 - (1-\lambda)e^{C_4} < 1$. Then as $\delta \to 0$, we have

$$M_1^+\left(\frac{R}{2}\right) \leq \gamma M_1 = \gamma M_1^+(R).$$

*Moser iteration method is a common method in proofs of P.D.E. problems where estimates involving recursive relations are iterated to yield the desired result, the trick is to control the constant term so that it remains finite for intended iteration, this method is introduced several times in [6, Section 8.5]. In particular, for readers with interest in this method, [6, Theorem 8.15] provides detailed demonstration of this method.

which concludes the proof.

For emitted details, reader can refer to [5].

□

The last assertion we need is a lemma stated in [7, lemma 12.5, p. 273], this lemma is the bridge that connects all the result we have proved so far to the main result of this section. Since we also use it auxiliary tool, the proof is emitted.

Lemma 4.8. *Let $\omega_1, \dots, \omega_M$ and $\bar{\omega}_1, \dots, \bar{\omega}_N$ be nonnegative nondecreasing functions on an interval $(0, R_0)$. Suppose there exist constants $\delta_0 > 0$, $0 < \sigma < 1$, $1 < \eta < 1$, such that for each $1 < R < R_0$,*

(i) $\delta_0 \max\limits_{1 \le i \le M} \omega_i(R) \le \bar{\omega}_i(R)$ *for some $i \in \{1, \dots, N\}$.*

(ii) $\bar{\omega}_i(\eta R) \le \sigma \bar{\omega}_i(R)$.

Then there exist constants $C = C(N, M, \delta_0, \sigma, \eta)$ and $\beta = \beta(N, M, \delta_0, \sigma, \eta) > 0$ such that for all $i = 1, 2, \dots, M$

$$\omega_i(R) \le C \left(\frac{R}{R_0} \right)^{\beta} \max_{1 \le i \le N} \bar{\omega}_i(R_0) \quad (1 < R < R_0).$$

Proof of Propsition 4.2. It is clear that the premise of lemma 4.8 is guaranteed by lemma 4.3 and 4.6, thus applying the lemma with $M = 1$, $N = 2n$, $\eta = \frac{1}{2}$ and

$$\begin{aligned} \omega_1(R) = M(R), \quad \bar{\omega}_i(R) &= M_i^+(R) \quad \text{for } i = 1, \dots, n, \\ \bar{\omega}_i(R) &= M_{i-n}^- \quad \text{for } i = n+1, \dots, 2n \\ \delta_0 = \frac{1}{\sqrt{n}}, \quad \sigma &= \gamma \quad \text{(the same } \gamma \text{ as in lemma 4.6).} \end{aligned}$$

□

Remark 4.9. Since we do not know that the weak solution of (1.1) is smooth, later when we prove the main result, we will need to regularise the problem so that a smooth solution exist, in other words, we will study a sequence of approximate problems of the form

$$\operatorname{div}(|Du|^p Du) + \varepsilon \triangle u = 0 \quad (\varepsilon > 0)$$

in some ball B_{R_0}. Therefore when we apply the results achieved in this section, we will be applying the following modified version:

Proposition 4.10. *There exist constants $C_1 = C_1(p,n)$ and $\beta = \beta(p,n) > 0$ such that*

$$M(R) \leq C_1 K \left(\frac{R}{R_0}\right)^{\beta} \quad (0 < R < R_0);$$

where C_1 and β do not depend on ε.

Proof. The proof of this version is very similar to what we have done so far, except that we need to change the term $M(R)^p$ to $M(R)^p + \varepsilon$, which also cancels in the proofs of lemma 4.3 and 4.6. □

4.2 An A Priori Hölder Estimate for Du

In the preceding section we have shown the Hölder estimate on the oscillation of Du near a point of degeneracy, where $Du = 0$. In this section, we extend this result to all interior points.

For the purpose of this section, we still assume u is a smooth solution of (2.1) in some ball B_{R_0}, as well as estimate (4.2) but dropping assumption (4.1). The main result is stated below, the modified version of this result for the approximate problems will be stated at the end.

Proposition 4.11. *There exist constants $C_5 = C_5(R_0, p, n, K)$ and $\alpha = \alpha(p,n) > 0$ such that*

$$[Du]_{C^{\alpha}(B(R_0/2))} \leq C_5.$$

Proof. We know from proposition 4.2 that Du is Hölder contunuous with exponent β at any point $x_0 \in B_{R_0/2}$ at which $Du = 0$. Suppose now instead

$$|Du(x_0)| > 0 \tag{4.26}$$

Define for $k = 1, 2, \ldots, n$, $0 < R < R_0 < 2$:

$$\begin{aligned}
M(R) &\equiv \max_{B_R(x_0)} |Du|, \\
M_k^{+}(R) &\equiv \max_{B_R(x_0)} \pm u_{x_k}, \\
\mathrm{osc}_{B_R(x_0)} u_{x_k} &\equiv \max_{B_R(x_0)} u_{x_k} - \min_{B_R(x_0)} u_{x_k} \\
&= M_k^{+}(R) + M_k^{-}(R).
\end{aligned}$$

Let $\gamma < 1$ be the constant from lemma 4.6. Define R_1 to be the supremum of the set of numbers $0 < R \leq R_0/2$ for which

$$M_k^{\varepsilon}\left(\frac{R}{2}\right) \leq \gamma M_k^{\varepsilon}(R) \tag{4.27}$$

fails for some choice of $k \in \{1, 2, \ldots, n\}$ and ε represents either $+$ or $-$, such that

$$M_k^\varepsilon(R) \geq \frac{1}{\sqrt{n}} M(R) > 0. \quad \text{(see also (4.5))} \tag{4.28}$$

Then $R_1 > 0$ for otherwise we would have, as in previous section, that

$$M(R) \leq C \left(\frac{R}{R_0}\right)^\beta \qquad \left(0 < R \leq \frac{R_0}{2}\right)$$

which is a contradiction to (4.26). Consequently, there exists $R_1/2 < R_2 \leq R_1$ such that

$$M_1^+(R_2) \geq \frac{1}{\sqrt{n}} M(R_2) > 0,$$

and (4.27) fails for $R = R_2$ and some choice of k and ε, say $k = 1$ and $\varepsilon = +$. Then the hypotheses of lemma 4.3 must hold for otherwise lemma 4.6 would imply (4.27) with $R = R_2$, $k = 1$ and $\varepsilon = +$. Therefore lemma 4.3 implies

$$\min_{B_{R_2/2}(x_0)} u_{x_1} \geq \max_{B_{R_2}(x_0)} u_{x_1} \geq \frac{1}{2\sqrt{n}} M(R_2) > 0. \tag{4.29}$$

Accordingly, calculation in the proof of lemma 4.3 implies $v = u_{x_k}$ satisfies the nondegenerate equation

$$-(a_{ij}|Du|^p v_{x_i})_{x_j} = 0 \quad \text{in } B_{R_2/2}(x_0), \tag{4.30}$$

with a_{ij} defined by (4.7). Also, it follows from 4.29 that

$$M(R_2)^p \geq |Du|^p \geq \left(\frac{1}{\sqrt{n}} M(R_2)\right)^p \quad \text{in } B_{R_2/2}(x_0). \tag{4.31}$$

Since the coefficients a_{ij} are bounded, it follows that

$$\lambda|\xi|^2 \leq a_{ij}|Du|^p \xi_i \xi_j \leq \mu|\xi|^2 \quad (\xi \in \mathbb{R}^n)$$

for

$$\lambda \equiv \left(\frac{1}{2\sqrt{n}} M(R_2)\right)^p$$

and

$$\mu \equiv (1 + p) M(R_2)^p.$$

Then

$$1 > \frac{\lambda}{\mu} = (2^p n^{p/2}(p+1))^{-1} > 0,$$

hence we can apply the DeGiorgi-Moser estimate (cf. [5, page 465]) to prove the existence of a constant $\delta = \delta(p, n) < 1$) such that

$$\mathrm{osc}_{B_{R/4}(x_0)} u_{x_k} \leq \delta \mathrm{osc}_{B_R(x_0)} u_{x_k} \quad \left(k = 1, 2, \ldots ; 0 < R < \frac{R_2}{2}\right). \tag{4.32}$$

Now, since the premise of lemma 4.8 is guaranteed by (4.27), (4.28) and (4.32), choosing $M = n$, $N = 3n$, $\eta = \frac{1}{4}$, $\delta_0 = \frac{1}{2\sqrt{n}}$, $\sigma = \max(\delta, \gamma)$ and

$$\begin{aligned} \omega_i(R) &= \bar{\omega}_i(R) = \mathrm{osc}_{B_R(x_0)} u_{x_i}, \quad (i = 1, 2, \ldots, n), \\ \bar{\omega}_i(R) &= M^+_{i-n}(R) \quad (i = n+1, \ldots, 2n), \\ \bar{\omega}_i(R) &= M^-_i(R) \quad (i = 2n+1, \ldots, 3n), \end{aligned}$$

the lemma implies that if $0 < R < R_0/2$,

$$\mathrm{osc}_{B_R(x_0)} u_{x_k} \leq C \left(\frac{R}{R_0}\right)^{\alpha} \quad (k = 1, 2, \ldots, n) \tag{4.33}$$

for certain positive constants C, α depending only on known quantities. Since estimate (4.33) holds for $x_0 \in B_{R_0/2}$ where $Du \neq 0$, the result follows. □

Again, when we use result of this section in the proof of our main result, we will be using the following version of the proposition:

Proposition 4.12. *If u is a smooth solution of the regularised problem in B_{R_0} and estimate (4.2) holds, then there exist constants $C_5 = C_5(R_0, p, n, K)$ and $\alpha = \alpha(p, n) > 0$, independent of $\varepsilon > 0$ such that*

$$[Du]_{C^{\alpha}(B_{R_0/2})} \leq C_5.$$

4.3 Proof of Theorem 4.1

As mention previously, the result proved so far base on the assumption that u is a smooth weak solution, while the weak solution constructed by variational principle is known a priori only to exist in $W^{1,p}(U)$. The way we resolve this problem is to regularise problem (1.1) so that smooth solution exists and then approach (1.1) by approximation arguments.

Firstly we want to work with smoother boundary conditions, which is achieved by means of mollification†. Assume the ball B_{R_0} lies in U, and define

$$g = g_{\delta} = \rho_{\delta} * u \tag{4.34}$$

†mollification is a common way to approximate the subject map with smooth functions. Details can be found in [2] and most elementary analysis texts.

where ρ_δ is the standard mollifier, $\delta > 0$.

Next, we regularise (1.1) by adding adding the $\varepsilon\triangle u^\varepsilon$ term, in other words, we now consider the probem

$$\begin{cases} \operatorname{div}(|Du^\varepsilon|^p Du^\varepsilon) + \varepsilon\triangle u^\varepsilon = 0 & \text{in } B_{R_0} \\ u^\varepsilon = g & \text{on } \partial B_{R_0} \end{cases}. \tag{4.35}$$

We have done most of the work already, it remains to make sure that the bounds we put on gradient of u is independent of δ and ε so that when we later let $\delta, \varepsilon \to 0$ to recover the general case, the bounds remains valid. The first lemma we prove

Lemma 4.13. (i) *There exists a constant $C = C(R_0, \delta)$ such that*

$$\max_{B_{R_0}} |Du^\varepsilon| \leq C,$$

if u^ε is a smooth solution of (4.35); C is independent on ε.

(ii) Problem (4.35) has a unique smooth solution.

Proof. (Sketch)

(i). Let x^* be the 'south pole' of ∂B_{R_0}, namely $x^* = (0, 0, \ldots, -R_0)$. Define

$$w(x) \equiv g(x^*) + \sum_{i=1}^{n-1} g_{x_{x_i}}(x^*)x_i + \max|D^2 g| \sum_{i=1}^{n-1} x_i^2 + \mu(x_n + R_0) - \lambda(x_n + R_0)^2$$

for $\mu, \lambda > 0$. Choose $\mu > 0$ so that $w_{x_n} \geq 1$, then we have

$$\begin{aligned} & -\operatorname{div}(|Dw|^p Dw) - \varepsilon\triangle w \\ = \;& -(|Dw|^p + \varepsilon)\triangle w - p|Dw|^p w_{x_i} w_{x_j} w_{x_i x_j} \\ = \;& -(|Dw|^p + \varepsilon)(2(n-1)\max|D^2 g| - 2\lambda) \\ & \quad - (p-2)|Dw|^{p42} \left(\sum_{i=1}^{n-1} 2w_{x_i}^2 \max|D^2 g| - 2w_{x_n}^2 \lambda \right) \\ \geq \;& 0 \end{aligned}$$

for large enough $\lambda > 0$. By the maximum principle, we have

$$w \geq u \quad \text{in } \partial B_{R_0}$$

also $w(x^*) = g(x^*) = u(x^*)$ implies $\frac{\partial u}{\partial n}(x) \geq \frac{\partial w}{\partial n}(x^*) = -\mu$. An upper bound is derived similarly. The global bound can be obtained by similar calculations as the proof of lemma 4.3 only that we need to multiply

$$-(a_{ij}|Du|^p u_{x_k x_i})_{x_j} + \varepsilon\triangle u_{x_k} = 0 \tag{4.36}$$

by $(\pm u_{x_k} - C_6)^+$ instead of $\zeta^2(v-k)^+$.

(ii). Existence and uniqueness of the solution follow from the uniform ellipticity of (4.35) and standard quasilinear elliptic theory (cf. [6, Chapter 13]).

For a detailed proof, see [1]. □

Next lemma gives estimates independent of δ.

Lemma 4.14. (i) *There exists a constant $C = C(R_0)$ such that*

$$\int_{B_{R_0}} |Du^\varepsilon|^p dx \le C\left(\int_U |Du|^{p+2}dx + 1\right). \tag{4.37}$$

(ii) *There exists $C_7 = C_7(R_0)$ such that*

$$\max_{B_{R_0/2}} |Du^\varepsilon| \le C_7. \tag{4.38}$$

These constants here are independent of ε or δ.

Proof. (Sketch)

(i). Multiply (4.35) by $u^\varepsilon - g$ and integrate by parts gives

$$\int_{B_{R_0}} (|Du^\varepsilon|^p + \varepsilon)|Du^\varepsilon|^2 \le \int_{B_{R_0}} (|Du^\varepsilon|^p + \varepsilon)|Dg|^2 dx.$$

Hence

$$\begin{aligned}\int_{B_{R_0}} |Du^\varepsilon|^{p+2} + \varepsilon|Du^\varepsilon|^2 dx \quad le \quad & \int_{B_{R_0}} |Dg|^{p+2} + \varepsilon|Dg|^2 dx \\ \le \quad & C\left(\int_{B_{R_0}} |Dg|^{p+2}dx + 1\right) \\ \le \quad & C\left(\int_U |Du|^{p+2}dx + 1\right).\end{aligned}$$

(ii). Let $W = W_\varepsilon \equiv \{x \in B_{R_0} | |Du^\varepsilon(x)| > 1\}$ and $w = w_\varepsilon = |Du^\varepsilon|^{p+2}$. Equation (4.36) implies

$$-(b_{ij}|Du^\varepsilon|^p u_{x_k x_i})_{x_j} = 0 \quad \text{on } W \tag{4.39}$$

where

$$b_{ij} \equiv a_{ij} + \frac{\varepsilon\delta_{ij}}{|Du^\varepsilon|^p} \quad (1 \le i, j \le n).$$

Since

$$|\xi|^2 \le b_{ij}\xi_i\xi_j \le (1+p+\varepsilon)|\xi|^2 \quad (\xi \in \mathbb{R}^n). \tag{4.40}$$

on W, then

$$\begin{aligned} -(b_{ij}w_{x_i})_{x_j} &= -(p+2)(b_{ij}|Du^\varepsilon|^p u^\varepsilon_{x_kx_i}u^\varepsilon_{x_k})_{x_j} \\ &= -(P+2)b_{ij}|Du^\varepsilon|^p u^\varepsilon_{x_kx_i}u^\varepsilon_{x_kx_j} \quad \text{by (4.39)} \\ &\le 0 \quad \text{on } W. \end{aligned} \tag{4.41}$$

Therefore

$$-(b_{ij}v_{x_i})_{x_j} \le 0 \ \text{ in } B_{R_0} \tag{4.42}$$

for $v \equiv (w-1)^+$. By a standard elliptic estimate, we have

$$\max_{B_{R_0/2}} v \le C(q)\left(⨍_{B_{3R_0/4}} v^q dx\right)^{1/q} \quad \text{for any } q > 1, \tag{4.43}$$

Let $q = 1^* = \frac{n}{n-1}$. Using Sobolev inequalities (cf. [2, Section 5.6, Theroem 2 and 3]), we have

$$\begin{aligned} \left(⨍_{B_{3R_0/4}} v^{1^*}dx\right)^{1/1^*} &\le C⨍_{B_{3R_0/4}} |Dv| + vdx \\ &\le C\left(⨍_{B_{3R_0/4}} |Du^\varepsilon|^{p+2}|D^2u^\varepsilon| + |Du^\varepsilon|^{p+2}dx\right) \\ &\le C\left(⨍_{B_{3R_0/4}} |Du^\varepsilon|^{p}|D^2u^\varepsilon|^2 + |Du^\varepsilon|^{p+2}dx\right) \end{aligned} \tag{4.44}$$

Multiplying (4.41) by a smooth cutoff fucntion ζ^2 and integrate by parts, where $\zeta \equiv 1$ on $B_{3R_0/4}$ and $\zeta \equiv 0$ near ∂B_{R_0}. Then we have

$$⨍_{B_{3R_0/4}}⨍_{B_{3R_0/4}} |Du^\varepsilon|^p|D^2u^\varepsilon|^2dx \le \frac{C}{R_0^2}⨍_{B_{R_0}} |DUu^\varepsilon|^{p+2}dx.$$

The result then follows from this estimate, (4.43) and (4.44).

For details, see [1].

□

Now we are in position to prove Theorem 4.1. By part (ii) of lemma 4.14 and proposition 4.12, we have

$$[Du^\varepsilon]_{C^\alpha_{B_{R_0/4}}} \le C \tag{4.45}$$

for some $\alpha > 0$, the constants independent of ε and δ. Moreover, we have

$$\max_{U''} |Du^\varepsilon| + [Du^\varepsilon]_{C^\alpha(U'')} \leq C(U'')$$

for any $U'' \subset\subset B_{R_0}$ and $u = v$ on ∂B_{R_0}. Thus there exists a subsequence (denoted as "u" for convenience) converging as $\varepsilon, \delta \to 0$ to a fucntion v,

$$\begin{aligned} u^\varepsilon \to v \qquad & \text{weakly in } W^{1,p+2}(B_{R_0}), \\ u^\varepsilon \to v \qquad & \text{uniformaly on each } U'' \subset\subset B_{R_0}, \\ Du^\varepsilon \to Dv \qquad & \text{uniformly on each } U'' \subset\subset B_{R_0} \end{aligned}$$

since we also have $u^\varepsilon \to u$ in L^p norm and $g = u_\delta \to u$ in $W^{1,p}$ norm, then

$$\|u^\varepsilon - u\|_{L^p(\partial B_{R_0})} \leq \|u^\varepsilon - g\|_{L^p(\partial B_{R_0})} + \|u_\delta - u\|_{L^p(\partial B_{R_0})}$$

as the first term on the right side converges to 0, we have

$$\begin{aligned} \|u^\varepsilon - u\|_{L^p(\partial B_{R_0})} &\leq \|u_\delta - u\|_{L^p(\partial B_{R_0})} \\ &\leq C\|u_\delta - u\|_{W^{1,p}(B_{R_0})} \xrightarrow{\delta \to 0} 0. \end{aligned}$$

Hence by uniqueness of limit $u = v$. Since v solves (1.1), so does u. It follows from part (ii) of lemma 4.14 and (4.45) that

$$\max_{B_{R_0/2}} |Du| \leq C, \quad [Du]_{C^\alpha(B_{R_0/4})} \leq C.$$

which proves the assumption in previous sections. Finally, since we can cover any compact subset of U by a chain of intersecting balls contained in U, then the previous estimates generalise to all compact subset without difficulty, which gives us the theorem.

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