Monopoles and the Nahm transform

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Declaration

The work in this thesis is my own, except for where otherwise stated.

-Jaklyn Crilly.

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Contents

Introduction

Magnetic monopoles are a phenomenon arising in gauge theory as the result of a type of topological invariant existing in the theory, either due to the presence of a singularity on the base space or the effects of spontaneous symmetry breaking due to the introduction of a *Higgs field*. In particular, they arise as solutions to non-linear partial differential equations (PDE's) satisfying certain boundary conditions. This results in solutions to these equations often being hard to solve, with the majority of magnetic monopoles having no known analytic solution. The Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) [8] construction provides an alternative approach to determining $SU(2)$ BPS monopole solutions by introducing an equivalence between solutions to these monopole equations, and solutions to a set of ordinary differential equations (ODE's), given by a set of antihermitian $k \times k$ matrices $(T_1(z), T_2(z), T_3(z))$ dependent on the real variable $z \in [-1, 1]$ satisfying the *Nahm equations*

$$\frac{dT_i}{dz} = \frac{1}{2}\epsilon_{ijk}[T_j, T_k],$$

such that the T_i have simple poles at $z = -1$ and $z = 1$, whose residues define irreducible representations of the Lie algebra $\mathfrak{su}(2)$.

This equivalence will be the key focus of this thesis.

The motivation behind this thesis however comes from the idea of dualities in certain gauge theories. And in particular a type of S-duality, which on the simplest classical level corresponds to electric-magnetic duality between the Maxwell's equations in a $U(1)$-gauge theory. This duality for non-abelian gauge theories was first introduced by Goddard, Nuyts and Olive in [13], where they demonstrated that for a G-gauge theory with magnetic and electric charges, there existed a dual gauge theory whose structure group corresponds to the Langlands dual of G, denoted G^L, whose electric and magnetic charges are given respectively by the magnetic and electric charges of the original gauge theory, hence the name 'electric-magnetic' duality. It was then conjectured by Montonen and Olive in [14] that in certain gauge theories, this electric-magnetic duality corresponded to an S-duality. The simplest gauge theory in which this conjecture was found to hold true was in a twisted $\mathcal{N} = 4$ supersymmetric gauge theory, containing both bosonic and fermionic fields.

More specifically, the motivation for this thesis stemmed from the paper 'Geometric Langlands duality and the equations of Bogomolny and Nahm' by Witten [1]. In this paper, Witten is considering an $\mathcal{N} = 4$ supersymmetric G-gauge theory (i.e. a theory with this S-duality discussed above), defined on a 4-manifold with boundary, given by $M = W \times \mathbb{R}$ where $W = S^2 \times I$, and I is the closed interval $[0, L]$. On the boundary of this closed interval he considers Neumann boundary conditions at $z = 0$, and Dirichlet boundary conditions at $z = L$. This paper then considers the dual gauge theory, corresponding to the 'electric-magnetic' dual of the original theory, given by an $\mathcal{N} = 4$ supersymmetric G^L-gauge theory, and considers what the features in the original G-gauge theory correspond to in the dual G^L-gauge theory, and vice versa.

In particular, the interest for us occurs when considering what the dual of a Neumann boundary condition is. Witten shows that the dual of a Neumann boundary condition is exactly a Dirichlet boundary condition, such that all bar three components of the bosonic fields in this theory are trivial. The three non-trivial components given by $(X_1(z), X_2(z), X_3(z))$ where $z \in [0, L]$, were then shown to satisfy the Nahm equations

$$\frac{dX_i}{dz} = \frac{1}{2}\epsilon_{ijk}[X_j, X_k], \tag{0.0.1}$$

such that the fields X_i have a simple pole at $z = 0$ whose residue defines an irreducible representations of the Lie algebra $\mathfrak{su}(2)$.

So going back to the ADHMN construction, we see that there exists a correspondence between certain $SU(2)$ monopole solutions and particular sets of solutions to the Nahm equations (which we will coin the term *Nahm data*). On the other hand, considering the dual of an $\mathcal{N} = 4$ supersymmetric G-gauge theory with a Neumann boundary condition, the non-trivial bosonic fields arising in this G^L-gauge theory must satisfy a set of data which holds great similarities to the Nahm data given in the ADHMN construction. Thus, the aim of this thesis was to investigate if there existed some type of monopole correspondence for these non-trivial bosonic field solutions arising in the G^L-gauge theory. In particular, our focus was restricted to the case where we take L in the limit $L \to \infty$.

So as a summary of this thesis, Chapter 1 introduces the concept of principal bundles and connection forms. Such objects define the key features of a gauge theory and thus provide the framework on which this thesis is based. In particular, we will give a well-known non-trivial example to motivate these objects; the Hopf principal bundle.

In Chapter 2, we study electric-magnetic duality on a $U(1)$-gauge theory, which outlines the simplest example of classical S-duality. This will then naturally lead into the introduction of monopoles. For this thesis, there are two types of monopoles in which we will be interested in; those arising due to a point singularity or non-trivial base space, and those arising due to the spontaneous symmetry breaking in a non-singular theory due to the introduction of a Higgs field. Both types of monopoles will play a significant part of this thesis, and thus will be defined in detail.

Chapter 3 introduces and focuses on the ADHMN construction which, as stated above, defines an equivalence between certain types of non-singular monopole solutions and a set of matrices defined over a single variable, which satisfy the Nahm equations and have specific boundary conditions. The proof of this equivalence will be outlined and key steps will be given in detail.

Finally, in Chapter 4 we consider the solution to the Nahm equation introduced in the Witten paper, and apply a similar transform to that of the inverse Nahm transform in the ADHMN construction, to this specific case. In doing so, we show that this Nahm data corresponds to a $U(1)$ Dirac monopole under this inverse Nahm-like transform.

Chapter 1

Preliminaries

1.1 Principal bundles

All maps will be assumed to be C^∞ and all topological spaces will be Hausdorff and paracompact.

Definition 1.1.1. *A fiber bundle with fiber F, consists of a surjective projection $\pi : P \to M$ between the 'total space' P and the 'base space' M, such that*

- $\pi^{-1}(x) \cong F,\ \forall x \in M.$
- *There exists an open cover of M, given by $\{U_i\}_{i\in\mathcal{I}}$ such that on each open set U_i there exists a fiber preserving isomorphism formally known as a local trivialization given by*

$$\Psi_i : \pi^{-1}(U_i) \to U_i \times F.$$

We use the term fiber preserving here to mean that given the natural projection $\bar{\pi} : U_i \times F \to U_i$, the isomorphism Ψ_i must satisfy the following equation

$$\pi(p) = \bar{\pi} \circ \Psi_i(p),\, \forall\, p \in \pi^{-1}(U_i).$$

A useful map arising from the trivializations of the principal bundle is the transition map. Thus, consider

$$\begin{aligned} \Psi_j \circ \Psi_i^{-1} : (U_i \cap U_j) \times F &\to (U_i \cap U_j) \times F \\ (x, p) \quad &\mapsto \ (x, g_{ij}(x) f). \end{aligned} \tag{1.1.1}$$

The $g_{ij} : U_i \cap U_j \to G$ defines the *transition function* of these local trivializations.

Definition 1.1.2. *A principal G-bundle is a fiber bundle $\pi : P \to M$ consisting of a right G action on the bundle P*

$$R : P \times G \to P,$$

such that the $R(p, g) \equiv R_g(p)$ defines a free and transitive action on P, which is fiber preserving. Thus each fiber of this fiber bundle will be isomorphic to G.

In order to make sense of what these definitions mean, we will now give a non-trivial example of a principal bundle.

Example 1.1.1 (Hopf principal bundle)**.**
We are going to introduce the Hopf bundle, which is an example of a non-trivial $U(1)$ *principal bundle over the two sphere,* $X = S^2$*, such that* $P = S^3$*. Thus, the principal bundle which we will construct is given diagrammatically by*

$$\begin{array}{ccc} U(1) & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

To construct such a bundle, we first use the fact that the two sphere is isomorphic to the complex projective space given by the set of all lines in $\mathbb{C}^2$*. That is,* $S^2 \cong \mathbb{C}P^1$*. Therefore, we define*

$$\begin{aligned} S^3 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \\ &\equiv \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}, \end{aligned}$$

where we take $z_1 = x_1 + ix_2$ *and* $z_2 = x_3 + ix_4$*. Then the base space will be given by*

$$S^2 \cong \mathbb{C}P^1 = \{[z_1, z_2],\, z_1,\, z_2 \in \mathbb{C}\},$$

where $[-, -]$ *denotes the equivalence class such that* $(z_1, z_2) \sim (a, b)$ *iff* $(a, b) = (\lambda z_1, \lambda z_2)$*, for some* $\lambda \in \mathbb{C}$*.*

Now we define the projection map as follows

$$\begin{aligned} \pi : \quad S^3 &\to \mathbb{C}P^1 \\ (z_1, z_2) &\mapsto [z_1, z_2]. \end{aligned}$$

It is clear that this is a surjective map, and $\pi^{-1}([z_1, z_2]) = \{(\lambda z_1, \lambda z_2) \,|\, |\lambda|^2 = 1,\, \lambda \in \mathbb{C}\}$*, (where we have required* $|\lambda|^2 = 1$ *to ensure that* $\pi^{-1}[z_1, z_2] \in S^3$*).*

Therefore we can define a natural $U(1)$ *action on* S^3*, such that given some* $g \in U(1)$*,*

$$R_g(z_1, z_2) = (z_1\, g, z_2\, g).$$

The final step is to now show that there exists a set of local trivializations over S^2*. Now there does not exist a global trivialization, however there does exist two local trivialization. To see this, let* $U_i = \{[z_1, z_2] \,|\, z_i \neq 0\} \subset \mathbb{C}P^1$ *and consider the homomorphisms*

$$\begin{aligned} \Psi_1 : \pi^{-1}(U_1) \to &\quad U_1 \times U(1) \\ (z_1, z_2) \;\mapsto &\left([z_1, z_2], \frac{z_1}{|z_1|} \right) \\ \Psi_2 : \pi^{-1}(U_2) \to &\quad U_2 \times U(1) \\ (z_1, z_2) \;\mapsto &\left([z_1, z_2], \frac{z_2}{|z_2|} \right). \end{aligned}$$

Given the $U(1)$ action is free and transitive, we know that these homomorphisms define isomorphisms, whose transition functions are given by

$$\begin{aligned} g_{ij} : U_i \cap U_j \to & \; U(1) \\ (z_i, z_j) \mapsto & \; \frac{z_j|z_i|}{z_i|z_j|}. \end{aligned}$$

Therefore we have defined a $U(1)$ fiber bundle with a $U(1)$ action satisfying the properties of a principal bundle. This is an example of a non-trivial principal bundle

Definition 1.1.3 (Sections)**.** *Let $\pi : P \to M$ denote a principal G-bundle.*

- *A section of this bundle is a map $s : M \to P$ such that $\pi \circ s = id$, where id denotes the identity operator on M.*
- *Let U_i denote an open subset of M.*
  A local section of this bundle is a map $s_i : U_i \to \pi^{-1}(U_i)$ such that $\pi \circ s = id$, where id is the identity operator on U_i.

Proposition 1.1.4. *Let $\pi : P \to M$ denote a principal G-bundle.*

The local sections of this principal bundle, $s_i : U_i \to \pi^{-1}(U_i)$ are in bijective correspondence with the local trivializations over U_i.

Corollary 1.1.1. *Let $\pi : P \to M$ denote a principal G-bundle. A global section of this bundle exists if and only if $P \cong M \times G$.*

Theorem 1.1.5. *Any principal G-bundle over a contractible, paracompact space is isomorphic to the trivial principal G-bundle.*

In particular, any principal G-bundle over $\mathbb{R}^n$ where $n \in \mathbb{N}$ is trivial, as $\mathbb{R}^n$ is contractible.

1.1.1 Associated bundle

A particular type of bundles which we will be interested in are the vector bundles associated to a given principal bundle, called the *associated bundles*.

Definition 1.1.6. *Let $\bar{\pi} : P \to M$ denote a principal G-bundle, and $\rho : G \to GL(V)$ define a representation of G. Then the associated bundle is given by the map*

$$\begin{aligned} \pi : P \times_\rho V \to & \; M \\ [p, v] \mapsto & \; \bar{\pi}(p), \end{aligned}$$

where $P \times_\rho V = \{[p, v], p \in P, v \in \times V\}$ such that $(p, v) \sim (a, b)$ iff $(a, b) = (p \cdot \rho(g), \rho(g^{-1})v)$ for some $g \in G$. We note that the surjective map π is well-defined, as the action by g defines a fiber preserving action, that is $\bar{\pi}(p) = \bar{\pi}(\rho(g) \cdot p)$, $\forall g \in G$.

The associated bundle that will be of most use in this thesis is the *adjoint bundle*. Let $\pi : P \to M$, denote a principal G-bundle and $\mathfrak{g}$ the Lie algebra of the Lie group G. Then the adjoint bundle is the associated bundle $\bar{\pi} : P \times_{Ad} \mathfrak{g} \to M$, where

$$\begin{aligned} Ad : G &\to GL(\mathfrak{g}) \\ g &\mapsto Ad_g \end{aligned}$$

and Ad_g denotes the action on $\mathfrak{g}$ via conjugation by $g \in G$.

1.1.2 Pullback and pushforward of a bundle

Definition 1.1.7 (Pullback bundle)**.**
Let $f : N \to M$ denote a homomorphism of spaces, and $\pi : P \to M$ a principal G-bundle. Consider the space

$$f^*(P) = \{(n, p) \in N \times P \mid f(n) = \pi(p)\}.$$

Then the pullback bundle is the principal G-bundle given by the surjective homomorphism

$$\begin{aligned} \bar{\pi} : f^*(P) &\to N \\ (n, p) &\mapsto n. \end{aligned}$$

The pullback bundle will be of particular interest for when considering a principal G-bundle over $\mathbb{R}^3$, and a base space map

$$\begin{aligned} f : \mathbb{R}^4 &\to \mathbb{R}^3 \\ (x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3). \end{aligned} \tag{1.1.2}$$

Then we get that the fibers of the pullback bundle will be isomorphic to that of the original. That is, given $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, then

$$\begin{aligned} \bar{\pi}^{-1}(a) &= \{(a, p) \in \{a\} \times P | \pi(p) = (a_1, a_2, a_3)\} \\ &\cong \{p \in P | \pi(p) = (a_1, a_2, a_3)\} \\ &= \pi^{-1}((a_1, a_2, a_3)). \end{aligned}$$

In this case, the pullback bundle will define a bundle over $\mathbb{R}^4$ which is an extension of the original bundle by an extra dimension in a natural way.

Now we will introduce the pushforward of a bundle. Let TP denote the tangent space of P.

Definition 1.1.8 (Pushforward bundle)**.**
Consider a principal G-bundle, $\pi : P \to M$. Then the pushforward defines a map between the tangent spaces of P and M,

$$\pi_* : TP \to TM,$$

where TX denotes the tangent space of X.

1.2 Connection forms

Let $\pi : P \to M$ denote a principal G-bundle and consider the tangent space of the total space P, which we denote TP. Then using the projective map π, we can define a vertical tangent space $V \subset TP$ by naturally defining it as the kernel of the pushforward of π. Therefore, defining the vertical tangent subspace over $x \in M$ to be V_x, elements of V_x will be tangent to the fiber in P over the point $x \in M$, where

$$\begin{aligned} V_x &= \ker(\pi_* \mid_{(\pi^{-1}(x))}), \\ V &= \coprod_{x \in M} V_x. \end{aligned}$$

The idea of introducing a horizontal tangent subspace H_x, such that $TP_x = H_x \oplus V_x$, is not as simple in comparison with the vertical tangent space as there does not exist a naturally defined mapping onto G. If for instance P corresponded to the trivial G-bundle, $P = M \times G$, we may introduce the standard projection map $\pi_2 : X \times G \to G$. For this particular instance, we can then naturally define

$$\begin{aligned} H_x &= \ker(\pi_{2*} \mid_{(\pi^{-1}(x))}), \\ H &= \coprod_{x \in M} H_x. \end{aligned}$$

Thus, from their definition, we get $TP = H \oplus V$, and space H defines what we call a *connection* over the trivial bundle. We will now generalize this idea to non-trivial principal bundles.

Definition 1.2.1. *Let $A \in \mathfrak{g}$. The fundamental vector field at a point $p \in P$ is given by the map*

$$\begin{aligned} \sigma : P \times \mathfrak{g} &\to \chi(P) \\ (p, A) &\mapsto A^{\#} = \frac{d}{dt}\bigg|_{t=0} (p \cdot exp(tA)), \end{aligned}$$

where $\chi(P)$ denotes the space of smooth vector fields on P.

Definition 1.2.2 (Connection and connection form)**.**
A connection on a principal G-bundle $\pi : P \to M$ is the disjoint union over all $x \in M$ of the horizontal subspaces $H_x \subset T_xP$ such that $T_xP = H_x \oplus V_x$, and $(R_g)_{*p}(H_p) = H_{R_g(p)}$.

A connection form $\omega \in \Omega^1(P, \mathfrak{g})$ is then given by

$$\omega(\nu) = \begin{cases} A & \text{if } \nu = A^{\#} \\ 0 & \text{if } \nu \in H \end{cases}. \tag{1.2.1}$$

Remark 1.2.3. *In our definition we began with a connection and then used this connection to define a connection form. If instead one begins with a connection form, we can then define the connection on* P *by the space* $H = ker(\omega)$*. This is clear from the definition of the connection form.*

As with the principal bundle, we may also define a pullback on connection forms and k-forms in general.

Definition 1.2.4. *Let* $\pi : P \to M$ *denote a principal* G*-bundle with a* k*-form* $\nu \in \Omega^k(P, \mathfrak{g})$ *and let* $f : N \to M$ *denote a homomorphism. Then the pullback* k*-form* $f^*\nu \in \Omega^k(f^*(P), \mathfrak{g})$ *is defined by*

$$(f^*\nu)(v_1, ..., v_k) = \nu(f_*(v_1), ..., f_*(v_k)),$$

where $v_1, ..., v_k \in TP$*.*

Thus, for the case of the projection map from $\mathbb{R}^4$ to $\mathbb{R}^3$ introduced above, the connection form $A = a_1dx_1 + a_2dx_2 + a_3dx_3 \in \Omega^1(\mathbb{R}^3, ad(P))$ is pulled back to a connection 1-form $A = a_1dx_1 + a_2dx_2 + a_3dx_3 \in \Omega^1(\mathbb{R}^4, ad(f^*(P)))$. This is due to the projection acting via the identity on x_1, x_2, x_3 and trivially on x_4.

We are now going to introduce two equivalent ways to define such a connection one-form.

Theorem 1.2.5. *Let* $\pi : P \to X$ *denote a principal* G*-bundle. The following are equivalent.*

1. *A connection form is a 1-form* $\omega(P, \mathfrak{g})$ *as given in Definition 1.2.2.*

2. *A set of local 1-forms* $A_i \in \Omega^1(U_i, \mathfrak{g})$ *where* $\{U_i\}_{i\in\mathcal{I}}$ *defines an open cover of* M*, which are called local gauge potentials, satisfying*

$$A_{s_j} = ad_{g_{ij}^{-1}} \circ A_{s_i} + g_{ij}^{-1}dg_{ij}, \quad \text{on } U_i \cap U_j, \tag{1.2.2}$$

where $g_{ij} : U_i \cap U_j \to G$ *is the transition function. (This is equivalent to saying that the* A_i *are related by a local gauge transformation).*

3. *A gauge potential given by a 1-form* $A \in \Omega^1(M, ad(P))$*.*

In particular, given a global connection form ω, the local gauge potentials can be determined for a given local gauge. That is, given the local section $s_i : U_i \to P$,

$$A_{s_i} = s_i^*\omega \in \Omega^1(U_i, \mathfrak{g}). \tag{1.2.3}$$

The sections by which we pullback the connection onto M define what we call our choice of *gauge*. The gauge potential is uniquely determined by the connection form, however there exists many equivalent representations on which this potential can be viewed, as a result of the choice of gauge used to pull back the connection form.

In particular, given a trivial bundle which has global sections, $s : M \to P$, a gauge potential obtained via this gauge will take values globally in $\mathfrak{g}$. That is $A_s = s^*\omega \in \Omega^1(M, \mathfrak{g})$ and satisfy (2) in Theorem 1.2.5. This is convenient as we can then expand the components of the gauge potential in terms of a generating set of the Lie algebras.

Remark 1.2.6 (Notation)**.** *Throughout this thesis we will apply certain subscripts and superscripts to the connection forms to denote certain features. So given a connection form $A \in \Omega^1(U_i, \mathfrak{g})$:*

- *Let A_i denote the dx_i component of the 1-form.*
- *If A can be expressed as a matrix form, then $(A_{i,ab})$ denotes explicitly the A_i component in matrix form, and $(A_i)_{ab} \overset{def}{=} A_{i,ab}$ denotes the entry in the a-th row and b-th column of $(A_{i,ab})$=*

Definition 1.2.7. *Let $A \in \Omega^1(M, ad(\mathfrak{g}))$ denote a gauge potential on the principal G-bundle $\pi : P \to M$. The gauge field of A is given by*

$$F_A = dA + \frac{1}{2}[A, A],$$

where $[-, -]$ denotes the symmetric bilinear product consisting of the Lie bracket on g and the wedge product of one-forms.

That is, given the Lie algebra valued 1-forms $A = a_i dx_i$ and $B = b_i dx_i$ where a_i, $b_i \in \mathfrak{g}$, we get

$$[A, B] = \sum_{i,j} [a_i, b_j] dx_i \wedge dx_j.$$

Equivalently, we may express the gauge field locally in component form as

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \tag{1.2.4}$$

$$= [D_i, D_j], \tag{1.2.5}$$

where D_i denotes the dx_i component of the exterior covariant derivative $d_A = d + A$.

Now we are going to introduce some important properties of the gauge potential which will be of particular interest for later chapters.

Proposition 1.2.8. *Let $\pi : P \to M$ define a principal G-bundle with local trivialization over the open cover $\{U_i\}_{i \in \mathcal{I}}$ of M. Then on each U_i, there exists a local gauge potential $A_i \in \Omega^1(U_i, \mathfrak{g})$.*

This proposition is clear by using the fact that on each open set associated to a trivialization, there exists a natural connection form which can be determined in an analogous way to that of the horizontal tangent subspace for the trivial bundle. We can then use the fact that M is paracompact and Hausdorff, and thus has a partition of unity to identify a gauge potential. Thus we have that every bundle has a connection form.

Corollary 1.2.1. *Let $\pi : P \to M$ denote a principal G-bundle. Such a bundle admits a globally defined trivialization such that $P \cong M \times G$ if and only if there exists a globally defined gauge potential over M, taking values in $\mathfrak{g}$.*

Example 1.2.1 (Hopf principal bundle)**.** *We are now going to demonstrate the idea of a gauge potential form by introducing one to the Hopf bundle. This can be achieved by determining a connection form and then pulling it back via the local sections associated to the trivialization of our bundle, or by constructing the gauge potentials themselves. Identifying $\mathbb{C}P^1$ with S^2, there exists two local gauge potentials given by*

$$A_+ = \frac{1}{2}i(1 - cos(\varphi))d\theta \in \Omega^1(U_+, i\mathbb{R}),$$
$$A_- = -\frac{1}{2}i(1 + cos(\varphi))d\theta \in \Omega^1(U_-, i\mathbb{R}),$$

defined on the upper hemisphere S^2_+ and the lower hemisphere S^2_- respectively. This situation will be analysed more in Chapter 2.

Theorem 1.2.9. *The isomorphism classes of $U(1)$ principal bundles over S^2 are in bijective correspondence with the element of $\pi_1(U(1)) \cong \mathbb{Z}$.*

1.3 Gauge transformation

Definition 1.3.1. *Let $\pi : P \to M$ denote a principal G bundle. A gauge transformation f is given by a fiber preserving automorphism of P,*

$$f : P \to P.$$

That is, a map satisfying $\pi(f(p)) = \pi(p)$, $\forall\, p \in P$.

The set of gauge transformations of P will be denoted by the symbol $\mathcal{G}$.

This generalizes the idea of the local gauge transformation which can be viewed as an automorphism of the bundle $\pi^{-1}(U_i)$.

Now let $f \in \mathcal{G}$. The gauge transformations then act on the connection forms via a pullback. That is,

$$\omega \mapsto f^*\omega.$$

Given $f : P \to P$ defines a gauge transformation of P, let $s_i : U_i \to P$ define a local section of this principal bundle, and ω a connection form on P. Then the local potentials of

ω and $f^*(\omega)$ over the section s_i are given by $A_i = s_i^*(\omega)$ and $A_i^f = (f \circ s_i)^*(\omega)$ respectively, where A_i^f denotes the gauge transformation of A_i by $f \in \mathcal{G}$. Such local potentials are then related in the following way:

$$A_i^f = \bar{f}_i^{-1} A_i \bar{f}_i + \bar{f}_i^{-1} d\bar{f}_i \stackrel{\text{def}}{=} f \cdot A_{s_i},$$

where $\bar{f}_i : U_i \to G$ denotes the transition function such that $(f \circ s_i)(x) = s_i(x) \cdot \bar{f}_i(x)$.

Remark 1.3.2. *We will often abuse notation and simply write* $\bar{f}$ *as* f.

Definition 1.3.3. *A gauge potential* $A \in \Omega^1(M, ad(P))$ *is said to be pure gauge if there exists some gauge transformation* $f \in \mathcal{G}$ *such that* $f \cdot A = 0$.

Objects arising in a gauge theory which are related by a gauge transformation are often said to physically indistinguishable. Thus, it is often the class of gauge equivalent objects which are of interest, not the individual objects.

Chapter 2

Electric-magnetic duality & magnetic monopoles

Electric-magnetic duality is a type of duality arising in gauge theories which essentially 'swaps' the electric and magnetic charges of a theory (hence the name). This is particularly obvious for the case of a $U(1)$-gauge theory which we will prove in the following chapter gives an explicit description of electrodynamics. This will then naturally lead to the introduction of magnetic monopoles and how they arise in various gauge theories.

2.1 Gauge theories and instantons

A gauge theory is a field theory defined on a principal G-bundle in such a way that its associated action functional is invariant under a continuous group of local transformations. The objects which we are going to be interested in correspond to certain fields and k-forms arising on the principal bundle.

To be more explicit, a gauge theory consists of:

- A principal bundle $\pi : P \to M$, with structure group G (where we will always take G to be a Lie group).

- A gauge potential $A \in \Omega^1(M, ad(P))$, and its associated gauge field $F_A = dA + \frac{1}{2}[A, A] \in \Omega^2(M, ad(P))$.

- Any collection of other fields $\psi_i \in \Omega^0(M, X)$ where X denotes an associated bundle of the principal G-bundle, or more generally some bundle related to P (i.e. the spinor bundle).

- An action functional $\mathcal{S}$ depending on these fields, which is invariant under the action via the group of gauge transformations $\mathcal{G}$.

Remark 2.1.1. *For the sake of this thesis, we will always take M to be either $\mathbb{R}^3$, $\mathbb{R}^4$ or $\mathbb{R}^3 - \{0\}$.*

The gauge potentials are introduced into the theory to ensure invariance of the Lagrangian by the action of the gauge transformations. Such gauge potentials locally take values in the Lie algebra associated to the structure group, which we will denote $\mathfrak{g}$, and thus can be split locally into a generating set of $\mathfrak{g}$.

To illustrate what a gauge theory is, for the remainder of this section we are going to introduce the Yang-Mills G-gauge theory as a first example.

Yang-Mills G-gauge theory

Let $\pi : P \to M$ denote a principal G-bundle over $M = \mathbb{R}^4$. A Yang-Mills G-gauge theory over this principal bundle is a G-gauge theory whose Lagrangian is given by the Yang-Mills action functional:

$$\mathcal{S}(A) = -\int_M Tr(F^{\mu\nu}F_{\mu\nu})d^4x.$$

Remark 2.1.2.

- *Here Tr defines the trace operator, and this operator can be written in multiple, equivalent ways. The ones which will be of interest to us are the following*

$$Tr(F^{\mu\nu}F_{\mu\nu})d^4x = Tr(F_A \wedge \star F_A) := \langle F_A, F_A \rangle.$$

- *In order to avoid overcrowded notation, whenever writing the components of the curvature form F_A, we will omit the A index. It should remain clear from the context what the gauge potential associated to the gauge field is however.*

To ensure that this is in fact a gauge theory, we need to show that the Lagrangian is invariant under the action of a gauge transformation on the fields of the theory. Thus we must prove the following claim:

Claim 2.1.3. *The Yang-Mills action functional is gauge invariant under all gauge transformations $g(x) \in \mathcal{G}$.*

Proof. The action of the gauge transformation $g \in \mathcal{G}$ on the curvature from $F^{\mu\nu}$ is given by

$$g \cdot F^{\mu\nu} = gF^{\mu\nu}g^{-1}.$$

Thus, under the gauge transformation g

$$\begin{aligned}\mathcal{S} &\mapsto \mathcal{S}' \\ &= -\int_M Tr((gF^{\mu\nu}g^{-1})(gF_{\mu\nu}g^{-1}))d^4x \\ &= -\int_M Tr(gF^{\mu\nu}F_{\mu\nu}g^{-1})d^4x \\ &= -\int_M Tr(F^{\mu\nu}F_{\mu\nu})d^4x \\ &= \mathcal{S},\end{aligned}$$

where we have used the invariance of the Trace operator under permutations.

Therefore $\mathcal{S} = \mathcal{S}'$, and so the action functional is invariant under gauge transformations. □

Thus, we have defined the Yang-Mills gauge theory.

Now we want to determine the equations of motion produced from the action functional. This is achieved by considering the stationary solutions of the action, which comes down to solving the equation $\frac{\partial}{\partial t}\big|_{t=0} S(A_t) = 0$, where $A_t = A + t\zeta$, $\forall t \in (-\epsilon, \epsilon)$, for ϵ small. Thus given the 1-form A_t, the 2-form F_{A_t} is

$$\begin{aligned}F_{A_t} &= dA_t + \frac{1}{2}[A_t, A_t] \\ &= d(A + t\zeta) + \frac{1}{2}[A + t\zeta, A + t\zeta] \\ &= dA + \frac{1}{2}[A, A] + td\zeta + t[A, \zeta] + \frac{t^2}{2}[\zeta, \zeta] \\ &= F_A + td_A\zeta + \frac{t^2}{2}[\zeta, \zeta],\end{aligned}$$

were d_A denotes the *exterior covariant derivative* $d_A = d + [A, -]$.

Now solving for the equations of motion we get

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\bigg|_{t=0} \int_M \langle F_{A_t}, F_{A_t}\rangle \\ &= \int_M \frac{\partial}{\partial t}\bigg|_{t=0} \bigg(\langle F_{A_t}, F_A\rangle + t\langle F_{A_t}, d_A\zeta\rangle + \frac{t^2}{2}\langle F_{A_t}, [\zeta, \zeta]\rangle \\ &\qquad + \frac{t^2}{2}\langle d_A\zeta, d_A\zeta\rangle + \frac{t^2}{2}\langle d_A\zeta, [\zeta, \zeta]\rangle + \frac{t^4}{4}\langle [\zeta, \zeta], [\zeta, \zeta]\rangle \bigg) \\ &= \int_M \langle F_A, d_A\zeta\rangle \\ &= \int_M \langle d_A^* F_A, \zeta\rangle,\end{aligned}$$

where d_A^* denotes the adjoint of d_A.

Given ζ is an arbitrary 1-form, we get the equation of motion

$$\begin{aligned} d_A^* F_A &= 0, \\ d_A F_A &= 0 \text{ (the Bianchi identity).} \end{aligned} \tag{2.1.1}$$

Remark 2.1.4. *The Bianchi identity is included as a trivially satisfied equation for a gauge theory over $\mathbb{R}^4$. This is due to the base space being topologically trivial, and thus there existing a gauge in which F_A globally takes values in $\mathfrak{g}$. The Bianchi identity then follows from the definition of F_A.*

Now we need to determine what the adjoint of d_A is.

Claim 2.1.5.

$$d_A^* = \star d_A \star$$

Proof. We know that $d_A = d + [A, -]$. Thus, to prove the claim we will first show that $d^* = \star d \star$.

Let $v \in \Omega^2(M, ad(P))$, and $w \in \Omega^1(M, ad(P))$. Then by applying integration by parts and using the fact that $v, w \to 0$ as $\mathbf{x} \to \infty$,

$$\begin{aligned} \int_M \langle dv, w \rangle &= \int_M Tr(dv \wedge \star w) \\ &= \int_{\delta M} d\,Tr(v \wedge \star w) d^3x - \int_M Tr(v \wedge d \star w) \\ &= \int_M Tr(v \wedge \star(\star d \star w)) \\ &= \int_M \langle v, \star d \star w \rangle, \end{aligned}$$

where we have used the fact that $(\star)^2 = -1$ for our case.

For the case of $[A, -]^* = \star[A, \star -]$, we refer to [7] where an explicit derivation is given. The logic is much the same, but the key is to express the gauge potentials in their a Lie algebra basis and utilise the structure constants. □

Therefore the equations of motion given by (2.1.1) become

$$d_A \star F_A = 0 \text{ , and} \tag{2.1.2}$$

$$d_A F_A = 0 \text{ (the Bianchi identity).} \tag{2.1.3}$$

In a Yang-Mills gauge theory, the gauge potentials satisfying equations (2.1.2) and (2.1.3) with the minimum action are the self-dual / anti self-dual potential 1-forms. That is, the gauge potentials satisfying

$$F_A = \pm \star F_A. \tag{2.1.4}$$

Definition 2.1.6. *The gauge potential $A \in \Omega^1(M, ad(P))$ satisfying $F_A = \star F_A$ are formally called the instanton solutions.*

2.2 Electric-Magnetic Duality

A Yang-Mills gauge theory provides the simplest example of a classical G-gauge theory. We will now show when $G = U(1)$, the Yang-Mills gauge theory completely describes classical electromagnetics. Such a theory also has a special kind of duality, known as *electric-magnetic duality.* This provides a first example of an S-duality.

2.2.1 Sourceless Maxwell's equations

Let $M = \mathbb{R}^{1,3}$, which is just equal to $\mathbb{R}^4$ with the Minkowski metric $\eta = (-1, 1, 1, 1)$, and on this space consider an electric field E, and a magnetic field B which are real-valued functions on $\mathbb{R}^{1,3}$.

Proposition 2.2.1. *The equations of motions in a $U(1)$ Yang-Mills gauge theory, given by equations (2.1.3) and (2.1.4), are equivalent to the sourceless Maxwell's equation:*

$$\begin{aligned} \nabla \cdot E &= 0, \\ \nabla \cdot B &= 0, \\ \nabla \times E &= -\frac{\partial B}{\partial x_0}, \\ \nabla \times B &= \frac{\partial E}{\partial x_0}, \end{aligned} \tag{2.2.1}$$

where E denotes the electric field, B the magnetic field and x_0 corresponds to the time variable.

Proof. Consider the gauge potential on our $U(1)$-gauge theory given by $A = i(\phi dx_0 + \vec{A} \cdot d\mathbf{x}) = i(\phi dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3) \in \Omega^1(M, i\mathbb{R})$, where $\vec{A}$ and ϕ define the vector potential and scalar potential respectively, given by

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A}, \\ \vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial x_0}, \end{aligned}$$

where $\vec{A} = (A_1, A_2, A_3)$.

Then, from the definition of the gauge field and the fact that $U(1)$ is abelian, we get

$$\begin{aligned}
F_A &= dA \\
&= i\, d(\phi dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3) \\
&= i\left(\left(-\frac{\partial\phi}{\partial x_1} - \frac{\partial A_1}{\partial x_0}\right) dx_0 \wedge dx_1 + \left(-\frac{\partial\phi}{\partial x_2} - \frac{\partial A_2}{\partial x_0}\right) dx_0 \wedge dx_2 \right. \\
&\quad + \left(-\frac{\partial\phi}{\partial x_3} - \frac{\partial A_3}{\partial x_0}\right) dx_0 \wedge dx_3 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) dx_1 \wedge dx_2 \\
&\quad \left. + \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right) dx_1 \wedge dx_3\right) \\
&= i\vec{E} \cdot dx_0 \wedge d\mathbf{x} + i\vec{B} \cdot d\vec{S},
\end{aligned} \tag{2.2.2}$$

where $d\vec{S} = (dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2)$.

Similarly, using the definition of the Hodge dual on Minkowski space, we get

$$\begin{aligned}
\star F_A &= \star dA \\
&= \star(i\vec{E} \cdot dx_0 \wedge d\mathbf{x} + i\vec{B} \cdot d\vec{S}) \\
&= i\vec{E} \cdot d\vec{S} - i\vec{B} \cdot dx_0 \wedge d\mathbf{x}.
\end{aligned} \tag{2.2.3}$$

Now acting the exterior derivative on equations (2.2.2) and (2.2.3), we get the equations of motions for our theory are given by

$$\begin{aligned}
0 &= dF_A \\
&= -dx_0 \wedge \left(\frac{\vec{B}}{\partial x_0} + \vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} + \nabla\vec{B} \cdot d\vec{V}, \text{ and}
\end{aligned} \tag{2.2.4}$$

$$\begin{aligned}
0 &= d \star F_A \\
&= \nabla\vec{E} \cdot d\vec{V} - dx_0 \wedge \left(\frac{\vec{E}}{\partial x_0} - \nabla \times \vec{B}\right) \cdot d\vec{S}.
\end{aligned} \tag{2.2.5}$$

Therefore it is clear that the equations of motion given by equations (2.2.4) and (2.2.5) are satisfied if and only if the Maxwell's equations as given by (2.2.1) are satisfied.

□

Remark 2.2.2. *The inclusion of the factors of i for the gauge potential are required to ensure that the A_i take values in the Lie algebra of $U(1)$, given by $\mathfrak{u}(1) \cong i\mathbb{R}$. We could have instead included these factors implicitly in the magnetic charge such that the magnetic charge takes values in $i\mathbb{R}$.*

Sourceless electromagnetic duality

Consider a Yang-Mills $U(1)$-gauge theory. As proven in the previous section, the equations of motion associated to this theory are equivalent to the Maxwell's equations given in (2.2.1).

Now consider another Yang-Mills $U(1)$-gauge theory, but now whose gauge potential A' has gauge field F'_A given by the image of the following transformation of the original gauge field

$$F_A \mapsto F'_A = \star F_A.$$

By the correspondence between such solutions and the Maxwell's equation, this is equivalent to making the following transformations of the magnetic and electric fields:

$$\vec{E} \mapsto \vec{B},$$
$$\vec{B} \mapsto -\vec{E}.$$

Substituting the gauge field of this new gauge theory into the equations of motion, then equations (2.1.2) and (2.1.3) become

$$dF_A = 0,$$
$$d \star F_A = 0$$

respectively. Thus, the new gauge theory whose gauge potential is given by a transformation of the gauge potential of the original theory, produces the same equations of motion as the original theory. Given the equations of motion are the only physical laws able to be observed for the gauge theory as it is currently defined (as this gauge theory has no additional physical operators present), this results in the new $U(1)$-gauge theory (which is clearly different to the original), being physically indistinguishable to the original theory.

Given this duality corresponds to interchanging the electric and magnetic fields, this defines an electric-magnetic duality for a sourceless $U(1)$-gauge theory.

2.2.2 Maxwell's equations with sources

Now we are going to consider a $U(1)$-gauge theory with sources. Such a theory introduces a source term $J \in \Omega^1(M, ad(P))$, into the original Yang-Mills action functional, producing the Yang-Mills action functional with source:

$$\mathcal{S} = \int_M (Tr(F^{\mu\nu} F_{\mu\nu}) + Tr(A^\mu J_\mu)) d^4x,$$

which produces the following equations of motion

$$d \star F_A = j, \text{ and} \tag{2.2.6}$$
$$dF_A = k, \tag{2.2.7}$$

where $j := \star J$ and k denote 3-forms taking values in the adjoint bundle $ad(P)$.

Given the Maxwell's equations correspondence to the equations of motion in a $U(1)$ Yang-Mills gauge theory as defined in the previous section, the introduction of charges in this theory, corresponds to the introduction of electric and magnetic charges in the equivalent Maxwell's equations, with k corresponding to a magnetic source, and j corresponding to an electric source. The effect that introducing the source term j into the action functional has on the theory, is the altercation of equation (2.2.5). This source term however does not effect the remaining equation of motion given by (2.2.4). This is due to equation (2.2.7) arising as a result of the topology of the base space on which the gauge theory is defined being non-trivial, and thus this magnetic source arises due to some topological invariant. As an example consider the following lemma.

Lemma 2.2.3. *Consider a $U(1)$ Yang-Mills gauge theory over some base space M. Then given a gauge potential $A \in \Omega^1(M, ad(P))$, the following equation is always satisfied:*

$$dF_A = 0.$$

Proof. On a $U(1)$-gauge theory, we know that gauge field is invariant under acts of conjugation as $U(1)$ is abelian. Therefore we get $F_A = g^{-1}F_A g = g \cdot F_A$ for all $g \in \mathcal{G}$. Thus $F_A = dA \in \Omega^2(M, \mathfrak{u}(1))$, and so globally takes values in $\mathfrak{u}(1) \cong i\mathbb{R}$. Therefore,on M we get

$$\begin{aligned} dF_A &= d^2 A \\ &= 0, \end{aligned}$$

where we have used the fact that the $U(1)$ exterior derivative forms a chain complex on k-forms, and hence $d^2 = 0$ always. □

Now applying this lemma to the case of a magnetic charge, in order for a magnetic charge to exist it is required that the theory be defined on some non-trivial base space with some singularity. Unlike the electric charge which arose due to the introduction of a source term in the Lagrangian, thus producing a charge from the invariance of the action under some global symmetry, that is, producing a *Noether charge*, the magnetic charge arises due to non-trivial topological invariants over the base space. This will be discussed in detail in the following section, with the introduction of magnetic monopoles.

Electromagnetic duality with sources

Similar to the sourceless case, we see that there is this nice duality between the equations of motion in a Yang-Mills $U(1)$-gauge theory with electric and magnetic sources. This can be observed by considering a new $U(1)$-gauge theory with fields and sources given by the following transformed fields

$$\begin{aligned} F_A &\mapsto \star F_A, \\ \star F_A &\mapsto -F_A, \\ k &\mapsto -j, \\ j &\mapsto k, \end{aligned}$$

where we have used the fact that $(\star)^2 = -1$ on 4-dimensional Minkowski space.

By substituting these transformations into the equations of motions, we see that equations (2.2.6) and (2.2.7) transform into one another, thus reproducing a $U(1)$-gauge theory with the same equations of motion. This defines exactly an electric-magnetic duality between two distinct $U(1)$-gauge theories with sources.

2.3 The Dirac Monopole

Having now introduced the form a $U(1)$-gauge theory with electric and magnetic sources would take, we will restrict the theory to the case in which there are no electric sources, and only static magnetic monopoles. Thus we want to construct a monopole in a $U(1)$-gauge theory with magnetic field given by

$$\vec{B} = \frac{g\hat{r}}{r^2},$$

where g denotes the magnetic charge. Here we observe that the magnetic field for a monopole is of the same form as the electric field for an electron, with the electric charge q has being replaced with the magnetic charge g. This is required to ensure the electric-magnetic duality discussed earlier holds.

Definition 2.3.1 ($U(1)$ Dirac monopole)**.** *Let M denote a 4-manifold.*
A $U(1)$ monopole arising in a $U(1)$-gauge theory is given by a gauge potential $A \in \Omega^1(M, ad(P))$ whose gauge field F_A has the form

$$\begin{aligned} F_{0j} &= 0, \\ F_{ij} &= \epsilon_{ijk}\frac{g' x_k}{r^3} \end{aligned} \tag{2.3.1}$$

and satisfies the equation

$$dF_A = k, \tag{2.3.2}$$

where $k \in \Omega^3(M, ad(P))$ denotes a magnetic source.

As proven in Section 2.2, in order for a magnetic monopole to exist, it is required that the base space M be non-trivial. Given we will only be considering static monopoles with no electric field components (that is all k-forms will be independent of the variable t, and any 'dt' term will be trivial), we can pullback the principal G-bundle on which this gauge theory is defined via the embedding

$$\begin{aligned} \pi : \mathbb{R}^3 &\to \mathbb{R}^{1,3} \\ (x,y,z) &\mapsto (0,x,y,z). \end{aligned} \tag{2.3.3}$$

As a result, we will be considering a principal G-bundle over the 3-manifold consisting of only the three spatial components which we will take to have a Euclidean metric.

Remark 2.3.2. *Given this reduction of the base space to 3-dimensions, we can integrate both sides of equation (2.3.2) and equivalently define the Dirac monopole as all gauge potentials $A \in \Omega^1(M, ad(P))$ satisfying the equation*

$$\frac{1}{4\pi}\int_M F_A = g', \tag{2.3.4}$$

where $g' \in i\mathbb{R}$ denotes the magnetic charge.

This will be the more frequently used definition.

Given we are interested in topologically non-trivial base spaces, we will consider $M = \mathbb{R}^3 - \{0\} \simeq S^2$, where here $\simeq$ denotes homotopy equivalence and S^2 the unit sphere. Given our interest with monopoles lies in the topological invariants of this base space, we can thus take $M = S^2$.

Now we want to introduce a magnetic monopole. As such, applying Definition 2.3.1, we need to determine a gauge potential $A \in \Omega^1(M, ad(P))$ such that

$$\begin{aligned} F_A &= \epsilon_{ijk}\frac{1}{2}B_k dx_i \wedge dx_j \\ &= \frac{i\,g}{r^3}(x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1 + x_1 dx_2 \wedge dx_3) \\ &= \frac{i\,g}{r^3}\vec{x}\cdot d\vec{S}, \end{aligned} \tag{2.3.5}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $g \neq 0$ denotes the magnetic charge of the monopole. Here we have written $g' = ig$ where $g \in \mathbb{R}$.

Given the Lie algebra of $U(1)$ is $\mathfrak{u}(1) \cong i\mathbb{R}$, in order for the gauge potential to locally take values in $\mathfrak{g}$, we require A be pure imaginary. Thus, we require that $ig \in i\mathbb{R}$. Now given we are constructing a magnetic monopole in a $U(1)$-gauge theory, it is not the gauge field that we are interested in, but its gauge potential. Thus in order for this to be a well defined gauge theory, we need to determine some $A \in \Omega^1(S^2, ad(P))$ such that $F_A = dA$.

Proposition 2.3.3. *Let F_A be as defined in equation (2.3.5). Then there does not exist a gauge potential $A \in \Omega^1(S^2, ad(P))$ taking values globally in $\mathfrak{u}(1)$ over S^2, such that $F_A = dA$.*

Proof. Assume such a global gauge potential does exist. Let S^2_+ and S^2_- denote the upper and lower hemispheres of $M = S^2$ respectively, such that both spaces have boundary given by the equator C of S^2. Orienting C such that the positive direction is clockwise, whilst for the surfaces S^2, S^2_+ and S^2_- the positive direction is radially outwards from the origin, from Stokes' theorem we know

$$\begin{aligned} \frac{1}{4\pi}\int_S dA &= \frac{1}{4\pi}\left(\int_{S^2_+} dA + \int_{S^2_-} dA\right) \\ &= \frac{1}{4\pi}\left(\oint_C A + \oint_{-C} A\right) \\ &= \frac{1}{4\pi}\left(\oint_C A - \oint_C A\right) \\ &= 0. \end{aligned} \tag{2.3.6}$$

Now using the definition of F_A, we know that

$$\begin{aligned} \frac{1}{4\pi}\int_S dA &= \frac{1}{4\pi}\int_S F_A \\ &= \frac{1}{4\pi}\int_S \frac{i\,g}{r^2}\hat{x}\cdot d\vec{S} \\ &= \frac{i\,g}{4\pi R^2}\int_S dS \\ &= \frac{i\,g}{4\pi R^2}(4\pi R^2) \\ &= i\,g, \end{aligned} \tag{2.3.7}$$

where we have used the fact that the radius of the sphere, $R = 1$, is constant.

Thus given (2.3.6) and (2.3.7) are equal we get $0 = i\,g$. But $g \neq 0$, thus we have a contradiction. □

Remark 2.3.4. *The requirement for Stokes' theorem is generally that the local covering of M be open. By extending both hemispheres to be open spaces containing the equator, we can get open coverings and the proof is the exact same.*

Having just proven that there does not exist a gauge potential $A \in \Omega^1(S^2, ad(P))$ efined globally in $\mathfrak{u}(1)$ and satisfying $F = dA$, we now claim that there does however exist two local gauge potentials, A_+ and A_-, defined on some open neighbourhoods $U_+ = S^2\backslash\{(0,0,-1)\} \subset S^2$ and $U_- = S^2\backslash\{(0,0,1)\} \subset S^2$. This can be seen by defining

$$A_+ = \frac{i\,g}{r(z+r)}(xdy - ydx) = i\,g(1-\cos(\varphi))d\theta, \tag{2.3.8}$$

$$A_- = \frac{i\,g}{r(z-r)}(xdy - ydx) = -i\,g(1+\cos(\varphi))d\theta. \tag{2.3.9}$$

Then by explicit calculation, it is clear $dA_+ = F\,|_{U_+}$ and $dA_- = F\,|_{U_-}$, as required. Furthermore we observe that on the intersection of their domains, $U_- \cap U_+$, the two local 1-forms are related by the gauge transformation $g(\mathbf{x}) = e^{2gi\theta}$. That is,

$$\begin{aligned} A_+ &= A_- + e^{-2i\,g\theta} de^{2i\,g\theta} \\ &= A_- + 2i\,g\,d\theta. \end{aligned} \tag{2.3.10}$$

Thus there exists two local 1-forms $A_+ \in \Omega^1(U_+, \mathfrak{u}(1))$ and $A_- \in \Omega^1(U_-, \mathfrak{u}(1))$ which defines a gauge potential $A \in \Omega^1(S^2, ad(P))$.

Now in order for this connection to be well-defined, the gauge transformation must be continuous for all values of θ. Thus, we require that

$$\begin{aligned} e^{2i\,g(0)} &= e^{2i\,g(2\pi)} \\ \implies 1 &= e^{4i\,g\pi} \\ \implies 4g\pi &= 2\pi k, \quad \text{where } k \in \mathbb{Z} \end{aligned} \tag{2.3.11}$$

Thus, we get the quantization condition of the magnetic charge, such that for any $k \in \mathbb{Z}$

$$g = \frac{k}{2}. \tag{2.3.12}$$

Furthermore, we note that the integer k arises due to the mapping

$$\begin{aligned} f : S^1 &\to U(1) \cong S^1 \\ \theta &\mapsto e^{2i\,g\theta}, \end{aligned}$$

where we have taken the domain to be parametrized by $\theta \in [0, 2\pi)$. Thus, the integer k corresponds to the equivalence class of f in $\pi_1(S^1)$, and so we get $k \in \pi_1(S^1) \cong \mathbb{Z}$.

Remark 2.3.5. *From Theorem 1.2.8, we know that the isomorphism classes of $U(1)$ principal bundles are in bijective correspondence with the elements of $\pi_1(U(1)) \cong \mathbb{Z}$. Given the magnetic monopoles are defined completely by the elements of $\pi_1(U(1))$, we see that the magnetic monopoles are completely defined by the $U(1)$ principal bundles of the gauge theory. Therefore we see that the Hopf bundle introduced in the preliminaries defines (up to isomorphism) the $k = 1$ Dirac monopole.*

2.3.1 Dirac quantization condition

From the previous section we know that magnetic charge is quantized. This quantization condition is generally introduced in a slightly different form, which is known as the Dirac quantization condition which relates the electric and magnetic charges to each other.

In order to obtain this quantization condition, we assume that there exists an additional field in our theory that is coupled to the monopole with charge e. As a result of this coupling, the original uncoupled gauge potential transforms to the coupled gauge potential as

$$A \mapsto e\,A$$

and so the gauge transformation under consideration is now

$$g(\mathbf{x}) = e^{2ieg\theta}.$$

Thus, in order for this gauge transformation to be well-defined, it is required that

$$eg = \frac{k}{2}, \tag{2.3.13}$$

where $k \in \mathbb{Z}$.

This defines exactly the 'Dirac quantization condition'.

Remark 2.3.6. *Observe that from the Dirac quantization condition, we have*

$$g = \frac{k}{2e}.$$

That is, the magnetic charge is inversely related to the electric charge. By then considering the electric-magnetic duality for a $U(1)$-gauge theory with sources as introduced in Section 2.2.2, we see that the dual of a $U(1)$-gauge theory with sources given by the coupling constants e and g, is equivalent to a $U(1)$-gauge theory with inversely related coupling constants, that is $\frac{2\pi k}{e}$ and $\frac{2\pi k}{g}$ respectively. This provides an example of an S-duality for a classical $U(1)$-gauge theory, that is, a duality in which the coupling constants (e and g for this instance) are inverted when considering the dual picture.

2.4 Non-abelian monopoles

One major problem associated with the Dirac monopole is that it was imposed on the system by introducing a topologically non-trivial base, and didn't arise naturally. This means that the only property that can be calculated of this monopole is the magnetic charge (as this was the condition imposed on the monopole when introduced) and nothing else. Therefore such a monopole does not predict any exciting features. We now want to consider monopoles arising in a gauge theory with a non-abelian structure group, and whose base space need not be topologically non-trivial. In particular, we will be interested in the case when $M = \mathbb{R}^{1,3}$. Given such a space is contractible, we know from Theorem 1.1.5 that the principal bundle over $M = \mathbb{R}^{1,3}$ is isomorphic to the trivial bundle. Thus, our gauge potential can be seen to globally take values in the Lie algebra of G, by making the choice of a global section for our gauge.

So we want to generalize the idea of a monopole to a G-gauge theory for some non-abelian Lie group G with principal bundle $\pi : P \to M$, and thus we have to create some type of non-trivial topological invariant. To do this, we will introduce a new field into our theory, formally known as the *Higgs field*, denoted $\Phi \in \Omega^0(M, ad(P))$. The existence of this Higgs field results in the gauge theory having a topological charge, and hence a monopole, which arises through the process of spontaneous symmetry breaking of our structure group G to a subgroup $H \subset G$ via the Higgs mechanism. It can also be shown that a $U(1)$ monopole can arise from a non-abelian G-gauge theory, due to symmetry breaking of the symmetry group $SU(2) \to U(1)$, thus allowing for a non-singular Dirac monopole.

So consider a Yang-Mills-Higgs (YMH) G-gauge theory. This consists of a G-gauge theory, as defined in Section 2.1, with the following Yang-Mills-Higgs Lagrangian

$$L = \int_{\mathbb{R}^3} \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - V(\Phi) \right) d^3x, \tag{2.4.1}$$

where $V(\Phi) = \frac{\lambda}{4}(\Phi^a\Phi^a - v^2)^2$ denotes the *potential energy* of Φ, and D_i denotes the dx_i component of the exterior covariant derivative d_A.

Remark 2.4.1. *Here we have used the fact that F takes values globally (due to our choice of gauge) in the Lie algebra, and have expanded the curvature form as $F^{\mu\nu} = F^{a\mu\nu}T^a$ where $\{T^a\}$ defines an orthonormal basis for the adjoint representation of $\mathfrak{g}$. We could have equivalently written the terms inside the Lagrangian as*

$$Tr(X^{\mu\nu}Y_{\mu\nu}) = \frac{1}{2} X^{a\mu\nu} Y^a_{\mu\nu} := \langle X, Y \rangle.$$

This Lagrangian then produces the following equations of motion:

$$\begin{aligned} D_\mu D^\mu \Phi &= \lambda(1 - |\Phi|^2)\Phi, \\ D_\mu F^{\mu\nu} &= [D^\nu \Phi, \Phi]. \end{aligned} \tag{2.4.2}$$

In addition, given on M, F_A globally takes values in $\mathfrak{g}$, then straight from the definition of the exterior derivative and the gauge field, we get that the Bianchi identity $d_A F_A = 0$ is satisfied on M. In component form, the Bianchi identity can be written as $\epsilon_{\sigma\eta\nu\mu} D^\eta F_{\nu\mu} = 0$.

Now since we are only looking at classical gauge theories, the Lagrangian can be split into a natural kinetic term K which contains all of the time dependence of our Lagrangian, and a potential term V, such that $L = T - V$. From this, we can then determine the total energy of the system, which is equivalent to $E = T + V$. Thus, for the YMH Lagrangian, we get

$$T = \int \left(-\frac{1}{2} F^a_{i0} F^a_{i0} - \frac{1}{2} D_0 \Phi^a D^0 \Phi^a \right) d^3x, \tag{2.4.3}$$

$$\begin{aligned} V &= \int \left(-\frac{1}{4} F^a_{ij} F^a_{ij} - \frac{1}{2} D_i \Phi^a D^i \Phi^a + V(\Phi) \right) d^3x \\ &= \frac{1}{2} \int \left(B^a_i B^a_i + D_i \Phi^a D_i \Phi^a + 2V(\Phi) \right) d^3x, \end{aligned} \tag{2.4.4}$$

where we have used the results from Proposition 2.2.1 to write $F_{ij} = \epsilon_{ijk}B_k$ where B_k denotes the generalized magnetic field components.

Now applying the formula for the energy, we get

$$E = \frac{1}{2}\int \left(E_i^a E_i^a + B_i^a B_i^a + D^0\Phi^a D_0\Phi^a + D_i\Phi^a D_i\Phi^a + 2V(\Phi)\right) d^3x.$$

From this point on, we are now going to restrict to looking at only static solutions to the equations of motion. That is, solutions which don't depend on $t = x^0$. Given this independence on the variable x_0 and the fact that we have taken A to have no dx_0 components, we may pullback the bundle over $\mathbb{R}^4$ (via an arbitrary choice of $a \in x_0$ given the x_0 invariance) to a bundle over $\mathbb{R}^3$, producing a G gauge theory over $\mathbb{R}^3$, whose equations of motion are given by

$$\begin{aligned} D_iD_i\Phi &= -\lambda(1 - |\Phi|^2)\Phi, \\ D_iF_{ij} &= -[D_j\Phi, \Phi], \\ 0 &= D^i\epsilon_{ijk}F_{jk}, \end{aligned} \tag{2.4.5}$$

where i, j, k denote the three spatial components.

Furthermore, the kinetic terms of the energy will now be zero due to there being no time dependence. As a result, the energy simplifies down to

$$E = \frac{1}{2}\int \left(B_i^a B_i^a + D_i\Phi^a D_i\Phi^a + 2V(\Phi)\right) d^3x. \tag{2.4.6}$$

Spontaneous symmetry breaking.

We are now going to introduce a type of spontaneous symmetry breaking to our gauge theory. To do this, we impose the condition that the total energy of our gauge theory must be finite, and as a result of this, we impose the conditions on our fields that as $r \to \infty$

$$\begin{aligned} B_i &\to 0, \\ D_i\Phi^a &\to 0, \\ V(\Phi) &\to 0 \end{aligned} \tag{2.4.7}$$

sufficiently fast (fastness to be defined in the coming chapter), to ensure $E < \infty$.

In particular we see that at spatial infinity, the conditions imposed on the Higgs field are $D_i\Phi^a = 0$ and $|\Phi|^2 = 1$. Such solutions of the Higgs field take values in what we will call the *Higgs vacuum*, M_H, at spatial infinity. In order to define explicitly what the Higgs vacuum is, we must make a choice of some fixed $\Phi_0 \in M_H$ which will denote our standard reference vacuum point. The Higgs vacuum is then given by

$$M_H = \{\Phi \mid \Phi = \Omega \cdot \Phi_0, \Omega \in G\}.$$

It is clear that by making a choice of a standard reference vacuum point we have imposed a new condition on our theory. By making a choice of Φ_0, our gauge theory now requires not only that the Lagrangian be invariant under local G-gauge transformations, but that the standard reference Φ_0 is also invariant under the action of G. As a result, we get that as $r \to \infty$, the symmetry group G is broken down to

$$H = \{h \in G | \Phi_0 = h \cdot \Phi_0\}.$$

Thus the requirement that at spatial infinity the Higgs field takes values in M_H, results in spontaneous symmetry breaking of the G-gauge theory to a H-gauge theory.

Lemma 2.4.2. *Let $H \subset G$ and let G/H denote the quotient group. The vacuum manifold M_H is isomorphic to G/H. That is,*

$$G/H \cong M_H.$$

This follows directly from the first isomorphism theorem

Definition 2.4.3. *Consider a YMH G-gauge theory which experiences spontaneous symmetry breaking of the symmetry group G to $H \subset G$.*

A monopole is given by a gauge potential and Higgs field pair (A, Φ) which define solutions to the YMH equations of motion, given by (2.4.5), and which also satisfy the finite energy conditions given in (2.4.7).

2.4.1 Magnetic charge of a monopole

Having now defined a magnetic monopole in a non-abelian gauge theory, we are now going to define the magnetic charge of such a monopole. The dynamical approach in which we will be taking to determining this charge is similar to the method used for the abelian monopole. Furthermore, we will assume that asymptotically

$$F_{ij} = \epsilon_{ijk} \frac{g\, x_k}{r^3},$$

where $g \in \mathfrak{h}$ denotes the magnetic charge.

In doing so, by making appropriate gauge transformations and fixing of degrees of freedom as detailed in [4], we find from solving our equations of motion, the local gauge potentials at spatial infinity (that is, on the space $S_\infty = \lim_{R \to \infty} S^2_R$, where S^2_R denotes the 2-sphere of radius R) are given by

$$\begin{aligned} A_+ &= g(1 - \cos(\varphi))d\theta, \text{ on } U_+ \\ A_- &= -g(1 + \cos(\varphi))d\theta, \text{ on } U_-, \end{aligned}$$

where U_+ is the space S_∞ minus the south pole at spatial infinity. This is analogous to the abelian case, however now the magnetic charge g takes values in the Lie algebra of the unbroken group H.

As a result of this, we see that the two local potential forms are related on their intersection by the gauge transformation

$$k(x) = e^{2g\theta},$$

where we have the exponential of a matrix valued g. This is true as

$$A_+ = k^{-1} A\, k + k^{-1} dk,$$

where we have used the fact that $k^{-1} A\, k = A$ due to A and k commuting with each other, which is clear once we expand $k(x)$ as a power series.

Analogous to the abelian case, we then get that this gauge potential is only well defined if the gauge transformation $k(x)$ at $\theta = 0$ and $\theta = 2\pi$ is the same. Thus, applying the same logic as we did in (2.3.11) we get the generalized quantization condition for non-abelian monopoles

$$e^{4\pi g} = 1 \in H, \tag{2.4.8}$$

where we have taken the domain to be parametrized by $\theta \in [0, 2\pi)$.

Furthermore, we note that the quantization condition arises due to the mapping

$$\begin{aligned} f : S^1 &\to H \\ \theta &\mapsto e^{2g\theta}. \end{aligned}$$

Therefore, we get that the magnetic charge is characterized by the elements of $\pi_1(H)$.

Remark 2.4.4. *As in the case for the Dirac monopole, the quantization condition can equivalently be written as*

$$exp(4\pi e g) = 1 \in H$$

with the introduction of some electric charge e taking value in the root space of H. This is the generalization of the Dirac quantization condition to the case of non-abelian gauge theories, and is given in detail in [13].

2.4.2 Topological charge of a monopole

From the data given in Definition 2.4.3 which is required for a non-singular magnetic monopole to exist, the map

$$\Phi|_\infty : S^2_\infty \to M_H$$

provides the defining map which ensures the existence of a non-abelian monopole. This map ensures that at spatial infinity the Higgs field takes values in the Higgs vacuum, and that the G-gauge theory therefore has finite energy and experiences spontaneous symmetry breaking down to a H gauge theory.

In particular, we observe that such a map is characterised by the topological invariant given by the second homotopy class of $M_H \cong G/H$, $\pi_2(G/H)$. This defines what we call the topological charge of the monopole.

2.4.3 Equivalence of the magnetic and topological charge

Thus far, both the magnetic and topological charges of a non-abelian monopole have been defined. Both of these charges arise by taking completely different approaches, with the magnetic charge characterization coming from the group $\pi_1(H)$ by analysing the gauge potential, and the topological classification arising as an element of the group $\pi_2(G/H)$ by considering features of the Higgs field. We are now going to prove that these two types of charges are in fact equivalent.

Proposition 2.4.5. *Let G be a compact, simply-connected Lie group, with subgroup $H \subset G$.*
Then

$$\pi_2(G/H) \cong \pi_1(H).$$

Proof. Consider the following fiber bundle with structure group G, and $H \subset G$

$$\begin{array}{ccc} H & \hookrightarrow & G \\ & \searrow & \downarrow \\ & & G/H \end{array}$$

Associated to it is the following long exact sequence of a fibration

$$\cdots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \cdots$$

Given G is compact and semisimple, we know that $\pi_2(G) = 0$, whilst the fact that G is simply-connected, by its definition, means that $\pi_1(G) = 0$. Thus, a section of the long exact sequence reduces down to

$$0 \to \pi_2(G/H) \xrightarrow{i} \pi_1(H) \xrightarrow{j} 0 \to \pi_1(G/H) \to \cdots$$

From exactness, it is known that the

$$\ker(i) = 0,$$
$$\operatorname{im}(i) = \pi_1(H).$$

Thus, the map i is both injective and surjective, and thus defines an isomorphism. Therefore

$$\pi_2(G/H) \cong \pi_1(H).$$

□

Thus, it is clear that when G is both compact and simply connected, magnetic and topological charges are equivalent. In particular, the case which we will be focusing on is when $G = SU(2)$, which is indeed simply-connected and compact. A generalization of this proposition can be proven for the case in which G is not simply connected.

2.4.4 The ’t Hooft-Polyakov monopole

We are now going to look exclusively at the $SU(2)$ ’t Hooft-Polyakov monopole. Such a monopole arises in a gauge theory with structure group $SU(2)$ being spontaneously broken down to $U(1)$ where for the $SU(2)$ case, we will always take our standard reference Higgs field to be

$$\Phi_0 = \lim_{x_3 \to \infty} \Phi(0,0,x_3) = i\sigma_3. \tag{2.4.9}$$

where σ_3 denotes the 3rd Pauli matrix.

By an appropriate gauge choice, we want the long range magnetic field produced by this nonsingular monopole to have the standard, singular monopole form

$$B_k = \epsilon_{ijk} F_{ij} = \frac{g x_k}{r^3}, \tag{2.4.10}$$

where g denotes the magnetic charge and takes values in the Lie algebra of the unbroken structure group, $U(1) \subset SU(2)$. Furthermore $\pi_1(U(1)) \cong \mathbb{Z} \cong \pi_2(SU(2)/U(1))$ and so the magnetic charge g will take values in $\mathbb{Z}$. To analyse this charge in greater detail, we are now going to construct a formula of the magnetic charge in terms of the Higgs field. Due to the boundary conditions of the Higgs field, we know that $D_i\Phi = 0$ and $|\Phi|^2 = 1$ at spatial infinity. By solving these equations in terms of the Higgs field, we can determine an expression for the gauge potential in terms of the Higgs field at spatial infinity. Doing so, we get the Lie algebra components (i.e. A^a corresponds to the $T^a \in \mathfrak{su}(2)$ coefficient) of the gauge potential are

$$A_i^a = -\epsilon_{abc} \Phi^b \partial_i \Phi^c + \Phi^a A_i. \tag{2.4.11}$$

From this we can then determine the gauge field F_{ij}, which is given by

$$\begin{aligned} F_{ij}^a &= F_{ij}\Phi^a = \epsilon_{ijk} B_k \Phi^a \\ &= (\partial_i A_j - \partial_j A_i + \epsilon_{bcd} \Phi^b D_i \Phi^c D_j \Phi^d)\Phi^a. \end{aligned}$$

And in particular we observe that $F_{ij} = \epsilon_{ijk} B_k^a \Phi^a$. Thus, we get that the charge is

$$\begin{aligned} g &= \frac{1}{4\pi} \int_{S^2_\infty} F \\ &= \frac{1}{4\pi} \int_{S^2_\infty} B_k^a \Phi^a \, dS_k \\ &= \frac{ik'}{2}, \end{aligned} \tag{2.4.12}$$

where $k' \in \mathbb{Z}$ and dS_k denotes the k-th component of $d\vec{S} = (dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2)$

This defines the type of monopole and magnetic charge arising in an $SU(2)$-gauge theory experiencing spontaneous symmetry breaking from $SU(2)$ to $U(1)$.

The Prasad-Sommerfield monopole

As a first example of the explicit form of a monopole solution, we will determine the famous Prasad-Sommerfield monopole [6] which is a charge $k = 1$, 't Hooft-Polyakov monopole taken in the limit $\lambda \to 0$. These monopoles define static solutions to the equations of motion given by equations (2.4.5), where we take $\lambda = 0$ whilst still requiring the Higgs field satisfies the boundary conditions for finite energy, (i.e. still allowing for the existence of a monopole).

Thus, under these conditions the equations of motion reduce down to

$$D_i D_i \Phi = 0, \tag{2.4.13}$$
$$D_i F_{ij} = -[D_j \Phi, \Phi]. \tag{2.4.14}$$

Given we are looking for a $k = 1$ monopole solution, we will thus apply the ansatz that the monopole solution will be both spherically symmetric and reflectively symmetric about the origin. Therefore, let

$$\begin{aligned} \Phi &= h(r)\frac{x_a}{r} t_a, \\ A_i &= -(1 - k(r))\epsilon_{ija}\frac{x_j}{r^2} t_a, \end{aligned}$$

where $h(r)$ and $k(r)$ are functions depending only on $r = |\mathbf{x}|$, and $t_a = i\sigma_a$.

Then, substituting these formulas into equations (2.4.13) and (2.4.14), we get that the equations of motion in terms of $h(r)$ and $k(r)$ are given by

$$\begin{aligned} r^2\frac{d^2h}{dr^2} + 2r\frac{dh}{dr} &= 2k^2 h, \\ \frac{d^2k}{dr^2} &= kh^2 + \frac{k}{r^2}(k^2 - 1). \end{aligned}$$

The solution to this set of differential equations is then given by

$$\begin{aligned} h(r) &= \coth(r) - \frac{1}{r}, \\ k(r) &= \frac{r}{\sinh(r)}. \end{aligned}$$

Substituting these now into the definition of Φ and A_i, we get the monopole solution

$$\begin{aligned} \Phi &= \left(\coth(r) - \frac{1}{r}\right)\frac{x_a}{r} t_a, \\ A_i &= -\left(1 - \frac{r}{\sinh(r)}\right)\epsilon_{ija}\frac{x_j}{r^2} t_a. \end{aligned} \tag{2.4.15}$$

Furthermore given $\lim_{x_3\to\infty} \coth(x_3) \to 1$, we see that the standard reference Higgs field is exactly that as required by equation (2.4.9), as

$$\begin{aligned} \Phi_0 &= \lim_{x_3\to\infty}\left(\coth(x_3) - \frac{1}{x_3}\right) t_3 \\ &= t_3. \end{aligned}$$

This defines exactly the Prasad-Sommerfield monopole.

Now consider the gauge transformation given by $g(x) \in \mathcal{G}$ such that

$$g(x) = \begin{bmatrix} \cos(\frac{\varphi}{2}) & -e^{-i\theta}\sin(\frac{\varphi}{2}) \\ e^{i\theta}\sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{bmatrix},$$

where we are considering spherical coordinates (r, θ, φ), and we observe that $g(x) \in SU(2)$ as required. We note that when this gauge transformation is converted to Cartesian coordinates, this gauge transformation is singular on the line $\mathbb{R}^3 \backslash \{(0,0,x_3)|x_3 \geq 0\}$.

Then taken in the limit $r \to \infty$ under this gauge transformation, the connection form A of the Prasad-Sommerfield monopole as given in equation (2.4.15) becomes

$$\begin{aligned} g^{-1}Ag - g^{-1}dg &= \begin{bmatrix} \frac{i}{2}(1-\cos(\varphi))d\theta & 0 \\ 0 & -\frac{i}{2}(1-\cos(\varphi))d\theta \end{bmatrix} \\ &= \frac{i}{2}(1-\cos(\varphi))d\theta\, \sigma_3. \end{aligned}$$

Thus the gauge potential at spatial infinity where spontaneous symmetry breaking of the structure group $SU(2) \to U(1) \subset SU(2)$ is undertaken, can be seen to be equivalent to the pure imaginary gauge potential of the $k = 1$ Dirac monopole, given by

$$A = \frac{i}{2}(1-\cos(\varphi))d\theta,$$

which is defined on $\mathbb{R}^3 \backslash \{(0,0,x_3)|x_3 \geq 0\}$.

2.5 GNO duality

As with the case of abelian gauge theories, there also exists a type of duality between certain non-abelian gauge theories. This duality was first introduced by Goddard-Nuyts-Olive (GNO) in [13], and is a type of electric-magnetic duality. We are now going to very briefly outline the ideas and results behind this paper. Restricting to the case in which H is simply connected (for this explanation), the idea is that in a G-gauge theory being spontaneously broken down to a H-gauge theory, the electric charges which arise as a Noether charge take values in the root lattice of the structure group H, whilst the magnetic charges take values in the weight lattice of the dual group $H^\vee$.

As we saw in the previous section, non-abelian monopoles must satisfy the generalized Dirac quantization condition given by

$$\exp(4\pi eg) = 1. \tag{2.5.1}$$

Now we know that the electric charges arise as a Noether charge, and thus arises due to global invariants of the Lagrangian. Therefore they take values in the generators of the Lie group, given by the basis elements of the Cartan subalgebra. Furthermore, we know that g takes values in the Lie algebra of H, and as a result of this, we may perform a gauge transformations to the charge terms such that they can be expressed as a linear combination of the generating elements $\{T_i\}_{i\in\mathbb{N}_{\leq r}}$ of the Cartan subalgebra of $\mathfrak{h}$, which we denote $\mathfrak{h}_0$, given by

$$eg \to \sum_{i=1}^{r}\beta_i T_i \stackrel{\text{def}}{=} \beta \cdot T,$$

where r denotes the rank of $\mathfrak{h}$, and β_i denotes the 'coefficients' of the generating element T_i, that will define the magnetic charge g, and which we shall term the *magnetic weights*. Substituting this into equation (2.5.1), the quantization condition becomes

$$\exp(4\pi\beta \cdot T) = 1. \tag{2.5.2}$$

Now, associated to each $E_{\alpha_j} \in \mathfrak{h}_{\alpha_j}$ as defined in Appendix B, we get the root $\alpha_j(H)$ such that for all $H \in \mathfrak{h}_0$, $[H, E_{\alpha_j}] = \alpha_j(H)E_{\alpha_j}$. Thus, commuting both sides of equation (2.5.2) with the elements of $\mathfrak{h}_{\alpha_j}$, we get

$$2\beta \cdot \alpha \stackrel{\text{def}}{=} 2\beta_i\,\alpha_i \in \mathbb{Z}. \tag{2.5.3}$$

From Appendix 2, we know that this equation is equivalent to the defining equation for the weight space of the dual Lie algebra $\mathfrak{h}^\vee$. Thus the elements β are given exactly by the elements of the dual weight space. That is

$$2\beta \in \Lambda(\mathfrak{h}^\vee) = \left\{w \,\middle|\, w \cdot \alpha \in \mathbb{Z},\, \alpha \in \Phi(\mathfrak{h})\right\}. \tag{2.5.4}$$

Therefore we find that the electric charges are characterized by elements of the root space of the Lie algebra $\mathfrak{h}$, whilst the magnetic charges are characterized by elements in the weight space of the dual system with Lie algebra $\mathfrak{h}^\vee$, or equivalently (via Proposition B.2.2), elements of the coroot system of $\mathfrak{h}$.

The idea is that we can then construct a *dual* group to H, with structure group given by the *Langlands dual* of H which we denote H^L, such that the root space of H^L is equal to the coroot space of H, and the coroot space of H^L is equal to the root space of H. Given the electric charges are given by the roots and magnetic charges by the coroots, by considering the dual group, we are essentially swapping the magnetic and electric charge. This defines exactly the GNO duality, which by its definition, defines a type of *electric-magnetic duality.*

Montonen and Olive conjecture

From the GNO paper, Montonen and Olive [14] then expanded on this duality by conjecturing that in some gauge theories, there exists not only this electric-magnetic duality, but a type of symmetry such that the dual theory will also have its coupling constants inversely related to those of the original theory, producing an S-duality. The simplest gauge theory on which this conjecture was found to exist was a 4-dimensional, $\mathcal{N} = 4$ supersymmetric gauge theory.

Chapter 3

The ADHMN construction

In the following chapter, we introduce the equivalence arising from the ADHMN construction between certain $SU(2)$ monopole solutions and a set of solutions to the Nahm equations known as the Nahm data. We will outline in detail some of the keys steps of this construction and give a simple example of how the Prasad-Sommerfield monopole introduced in Chapter 2 can be constructed from the $k = 1$ Nahm data.

We emphasize the point that throughout this chapter, all double subscripts and superscripts are implicitly summed over, unless stated otherwise. That is,

$$X_a Y_a \stackrel{\text{def}}{=} \sum_a X_a Y_a.$$

3.1 BPS monopoles

The Bogomolny-Prasad-Sommerfield (BPS) monopole is a particular example of a non-singular monopole taken in the Prasad-Sommerfield limit. Such a limit occurs by considering $\lambda \to 0$ whilst still requiring the total energy be finite (and hence ensuring the existence of a monopole). By then restricting the solutions to consider only those which occur at the total energy minimum, we can show that this is equivalent to considering solutions of the *Bogomolny equation*. Thus, these such solutions define exactly the BPS monopoles.

For the remainder of this thesis we will restrict to the case $G = SU(2)$ unless stated otherwise.

So consider a YMH $SU(2)$-gauge theory over the base space $M = \mathbb{R}^3$, with spontaneous symmetry breaking to the subgroup $U(1) \subset SU(2)$. Given the principal bundles of this gauge theory is defined over a contractible spaces, we know that the bundle must be isomorphic to the trivial bundle (by Theorem 1.1.5), and thus there exists a global section on the principal bundle. Under this choice of gauge, our Higgs field and gauge potential over $\mathbb{R}^3$ will globally take values in $\mathfrak{g}$.

We now want to explore the Prasad-Sommerfield limit of this theory. We know the total energy can be written classically in terms of a kinetic and a potential energy term of the Lagrangian, as $E = T + V$. Given we will only be interested in static solutions, the kinetic energy term will be trivial, and thus only the potential energy term V will be of interest. Thus to determine when the energy is at a minimum is equivalent to determining when the potential energy is at a minimum, and now from (2.4.6) we know the energy is given by

$$E = \frac{1}{2}\int_M (B_i^a B_i^a + D_i\Phi^a D_i\Phi^a)\, d^3x. \tag{3.1.1}$$

Proposition 3.1.1. *The total energy given by equation (3.1.1) has a global minimum, given by*

$$E \geq 2\pi k,$$

where $k \in \pi_1(U(1)) \cong \mathbb{Z}$ corresponds to the charge of the monopole produced by the theory.

Proof. It is clear that we can express the total energy term as follows:

$$\begin{aligned} E &= \frac{1}{2}\int_M (B_i^a B_i^a + D_i\Phi^a D_i\Phi^a)\, d^3x \\ &= \frac{1}{2}\int_M (B_i^a - D_i\Phi^a)(B_i^a - D_i\Phi^a) d^3x + \int_M B_i^a D_i\Phi^a d^3x. \end{aligned} \tag{3.1.2}$$

The first term of the total energy is always positive. This is clear if we rewrite this term in terms of inner products:

$$\frac{1}{2}\int_M (B_i^a - D_i\Phi^a)(B_i^a - D_i\Phi^a) d^3x = \frac{1}{2}\int_M \langle B_i - D_i\Phi, B_i - D_i\Phi\rangle d^3x$$

and by the properties of an inner product, this must always be greater than or equal to zero.

For the second term in (3.1.1), we observe that

$$\begin{aligned} \int_M B_i^a D_i\Phi^a d^3x &= \int_M B_i^a D_i\Phi^a d^3x + \int_M D_i B_i^a \Phi^a d^3x && (3.1.3) \\ &= \int_M \partial_i(B_i^a\Phi^a) d^3x \\ &= \int_{S^2_\infty} B_i^a \Phi^a d^2x \\ &= 2\pi k, && (3.1.4) \end{aligned}$$

where $k \in \mathbb{Z}$. Here we have applied the Bianchi identity $D_i B_i^a = 0$ to obtain line (3.1.3), and the definition of the magnetic charge given in (2.4.12) for line (3.1.4).

Now substituting these into the total energy given by equation (3.1.2), we get

$$\begin{aligned} E &= \frac{1}{2}\int_M \langle B_i - D_i\Phi, B_i - D_i\Phi\rangle d^3x + 2\pi k \\ &\geq 2\pi k. \end{aligned}$$

□

We therefore see that the energy is bounded below, and has a minima at $E = 2\pi k$ which occurs when $B - D\Phi = 0$. That is, the energy bound is saturated when the monopole data (A, Φ) satisfies the *Bogomolny equation*

$$\star F_A = D\Phi,$$

where we have used the fact that $B = \star F_A$. Note that given $(\star)^2 = 1$ on our Euclidean base space $M = \mathbb{R}^3$, we may have equivalently written the Bogomolny equation as $F_A = \star D\Phi$. Solutions to this equation also satisfy the equations of motion of the YMH-gauge theory introduced in Section 2.4, and thus under the required boundary conditions,these solutions define a type of monopole.

Remark 3.1.2. *By applying the Bogomolny equation to the $SU(2)$ Prasad-Sommerfield monopole determined in Section 2.4.4, we see that this monopole is in fact a solution to the Bogomolny equation.*

We are now going to formally define a static, BPS monopole with topological charge k. This will be the definition of a monopole used in the ADHMN equivalence to be defined in Section 3.4.

Definition 3.1.3. *Consider an $SU(2)$-gauge theory with a YMH-Lagrangian consisting of a gauge potential $A \in \Omega^1(\mathbb{R}^3, ad(P))$, and a Higgs field $\Phi \in \Omega^0(\mathbb{R}^3, ad(P))$, and let $SU(2)$ be spontaneously broken to $U(1) \subset SU(2)$, with standard reference Higgs field given by $\Phi_0 = i\sigma_3$.*

An $SU(2)$ BPS monopole with topological charge $k \in \pi_1(U(1)) \cong \mathbb{Z}$ is a solution to the Bogomolny equation

$$F_A = \star D\Phi, \tag{3.1.5}$$

which in addition satisfies the following boundary conditions in the limit $r \to \infty$:

- $|\Phi| = 1 - \frac{k}{r} + O(r^{-2})$,
- $|D\Phi| = O(r^{-2})$
- $\frac{\partial|\Phi|}{\partial\Omega} \overset{def}{=} \left(\left(\frac{\partial|\Phi|}{\partial\theta}\right)^2 + sin^2\theta\left(\frac{\partial|\Phi|}{\partial\varphi}\right)^2\right)^{1/2} = O(r^{-2})$,

where we have used spherical coordinates (r, θ, φ)*.*

The first two boundary conditions are simply the requirements that the total energy of the system is finite, and the monopole has charge k. The third requirement ensures that the angular derivative of the Higgs field is convergent.

3.2 The Bogomolny equation and instantons

We are now going to show that all solutions to the Bogomolny equation on $\mathbb{R}^3$ are in fact equivalent to time-invariant instantons on $\mathbb{R}^4$. This plays a significant role in the *ADHMN construction* of monopoles as it allows the ADHM construction of instanton to be adapted to the case of BPS monopoles.

Proposition 3.2.1. *There exists a bijective correspondence between the solutions of the Bogomolny equation in a* G*-gauge theory over* $\mathbb{R}^3$*, and the time-invariant instanton solutions in a* G*-gauge theory over* $\mathbb{R}^4$*.*

Before we prove this proposition, we will first construct explicitly the solutions of the Bogomolny equation and the equations of motion for time-invariant instanton solutions.

Let (A, Φ) be a solution to the Bogomolny equation over $\mathbb{R}^3$. Then by definition of the gauge field form and covariant exterior derivative, in component form we get

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j],$$
$$(\star D\Phi)_{ij} = \epsilon_{ijk} D_k \Phi = \epsilon_{ijk} \left(\frac{\partial \Phi}{\partial x^k} + [A_k, \Phi] \right).$$

Therefore the Bogomolny equation $F_A = \star D\Phi$, can be expressed in its component form as

$$\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] = \epsilon_{ijk} \left(\frac{\partial \Phi}{\partial x^k} + [A_k, \Phi] \right). \tag{3.2.1}$$

This can then be expanded into the following 3 independent equations:

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_1} + [A_1, \Phi] &= \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} + [A_3, A_2], \quad \text{if } (i,j) \in \{(2,3),(3,2)\}, \\
\frac{\partial \Phi}{\partial x_2} + [A_2, \Phi] &= \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} + [A_1, A_3], \quad \text{if } (i,j) \in \{(3,1),(1,3)\}, \\
\frac{\partial \Phi}{\partial x_3} + [A_3, \Phi] &= \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_2, A_1], \quad \text{if } (i,j) \in \{(1,2),(2,1)\}.
\end{aligned} \tag{3.2.2}$$

Now let $\bar{A}$ denote an instanton solution which is time-invariant. That is $\bar{A} \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$

defines a connection form satisfying the self-duality equation $F_{\bar{A}} = \star F_{\bar{A}}$, and time invariance equation $\frac{\partial \bar{A}_i}{\partial x^0} = 0,\ \forall i \in \{0, ..., 3\}$. In component form, the self duality equation then becomes

$$\frac{\partial \bar{A}_j}{\partial x_i} - \frac{\partial \bar{A}_i}{\partial x_j} + [\bar{A}_i, \bar{A}_j] = \frac{1}{2}\epsilon_{ijkl}\left(\frac{\partial \bar{A}_l}{\partial x_k} - \frac{\partial \bar{A}_k}{\partial x_l} + [\bar{A}_k, \bar{A}_l]\right). \tag{3.2.3}$$

This can also be expanded into 3 independent equations, given by:

$$\begin{aligned} \frac{\partial \bar{A}_0}{\partial x_1} + [A_1, \bar{A}_0] &= \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} + [A_3, A_2], \quad \text{if } (i,j) \in \{(0,1),(1,0),(2,3),(3,2)\}, \\ \frac{\partial \bar{A}_0}{\partial x_2} + [A_2, \bar{A}_0] &= \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} + [A_1, A_3], \quad \text{if } (i,j) \in \{(0,2),(2,0),(3,1),(1,3)\}, \\ \frac{\partial \bar{A}_0}{\partial x_3} + [A_3, \bar{A}_0] &= \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_2, A_1], \quad \text{if } (i,j) \in \{(0,3),(3,0),(1,2),(2,1)\}. \end{aligned} \tag{3.2.4}$$

Now we will prove the previous proposition.

Proof of Proposition 3.2.1. First consider a principal G-bundle $\pi : P \to \mathbb{R}^4$ with an instanton solution $\bar{A} = \bar{A}_0 dx_0 + ... + \bar{A}_3 dx_3 \in \Omega^1(M, ad(P))$, which is time-invariant (that is, $\partial_{x_0} A_i = 0$, $\forall i$). Then given $\bar{A}$ defines a gauge potential, a gauge transformation $q \in \mathcal{G}$ acts on $\bar{A}$ in the following way

$$A \mapsto q \cdot A = q^{-1}Aq + q^{-1}dq.$$

Now restrict the bundle to $\mathbb{R}^3$ by considering the pullback of P by the following embedding

$$\begin{aligned} g : \mathbb{R}^3 &\to \mathbb{R}^4 \\ (x_1, x_2, x_3) &\mapsto (0, x_1, x_2, x_3), \end{aligned}$$

where we note that the choice of the fixed $x_0 = 0$ is irrelevant and any other element in $\mathbb{R}$ could have been chosen. This is a result of the bundle over $\mathbb{R}^4$ being x_0-invariant.

Then, the pullback G-bundle over $\mathbb{R}^3$ induced by this map, $\pi' : g^*(P) \to \mathbb{R}^3$ is isomorphic to the restriction of P to those fibers with $x_0 = 0$. On the pullback bundle, we now define the following $\mathfrak{g}$-valued k-forms

$$\begin{aligned} A &= g^*(A) = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 \in \Omega^1(\mathbb{R}^3, \mathfrak{g}), \\ \Phi &= \bar{A}_0\big|_{\{0\}\times\mathbb{R}^3} \in \Omega^1(\mathbb{R}^3, \mathfrak{g}), \end{aligned}$$

where $A_i = g^*(\bar{A}_i)$.

Then it is clear that A defines a gauge potential on $\mathbb{R}^3$, as it is the pullback of a gauge potential. For the Higgs field, we first acknowledge that as a restriction, this is well-defined given on $\mathbb{R}^4$ it had no x_0 dependence. Furthermore, the gauge transformation must act on Higgs field via conjugation, d and, but $(g^{-1}dg)_{x_0} = 0$ for all $g \in \mathcal{G}$.

Furthermore, substituting $\bar{A}_0 = \Phi$ and $\bar{A}_i = A_i$ into equations (3.2.4), we see that the Bogomolny equations given by (3.2.2) are satisfied, given that $\bar{A}$ satisfies the x_0-invariant self-duality equations.

Now consider a principal G-bundle, $\pi : P \to \mathbb{R}^3$, with a gauge potential $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 \in \Omega^1(\mathbb{R}^3, ad(P))$ and a Higgs field $\Phi \in \Omega^0(\mathbb{R}^3, ad(P))$, satisfying the Bogomolny equation.

Now consider the projection map introduced in Chapter 1

$$f : \mathbb{R}^4 \to \mathbb{R}^3$$
$$(x_0, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3).$$

Then on the pullback bundle $\pi' : f^*(P) \to \mathbb{R}^4$, we claim the following defines a gauge potential which is invariant under translation in the x_0 direction:

$$\bar{A} = \Phi dx_0 + f^*(A) = \bar{\Phi} dx_0 + \bar{A}_1 dx_1 + \bar{A}_2 dx_2 + \bar{A}_3 dx_3 \in \Omega^1(\mathbb{R}^4, \mathfrak{g}),$$

where $\bar{X} = f^*(X)$.

It is clear that the gauge potential is invariant under translation in the x_0 direction as $\partial_i \bar{A}_i = 0$. Furthermore under a gauge transformation, the gauge potential $\bar{A}$ transforms as required.

Thus this gauge potential $\bar{A}$ is $\mathbb{R}$ invariant in the x_0 direction, and substituting it's components into equations (3.2.2), we see that it satisfies the self-duality equations, given (A, Φ) satisfy the Bogomolny equations.

□

3.3 The ADHM construction

Before we get to the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction, which defines an equivalence between certain BPS monopoles and a set of Nahm data, we will first introduce the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction, from which the ADHMN construction was derived. The ADHM construction defines an equivalence between all instanton solutions on $\mathbb{R}^4$ and a set linear algebraic data.

Definition 3.3.1 (ADHM data)**.** *Let M be a 4-dimensional space with a complex structure, and define the complex coordinates on M to be $(z_1, z_2) \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. A set of ADHM data consists of:*

1. *Complex vector spaces V and W of dimension k and n respectively.*
2. *Complex $k \times k$ matrices which we will denote B_1 and B_2, a complex $k \times n$ matrix I and a complex $n \times k$ matrix J,*

such that the following are satisfied

- *B_1, B_2, I, J satisfy the ADHM equations given by*

$$[B_1, B_2] + IJ = 0,$$
$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger = 0.$$

- *For all $(x, y) \neq (0, 0) \in M^2$, where $x = (z_1, z_2)$, $y = (w_1, w_2)$, the map $\alpha_{(x,y)} : V \to W \oplus V \otimes M$*

$$\alpha_{(x,y)} = \begin{bmatrix} w_2 J - w_1 I^\dagger \\ -w_2 B_1 - w_1 B_2^\dagger - z_1 \\ w_2 B_2 - w_1 B_1^\dagger + z_2 \end{bmatrix}$$

is injective, while $\beta_{(x,y)} : W \oplus V \otimes M \to V$ given by

$$\beta_{(x,y)} = \begin{bmatrix} w_2 I + w_1 J^\dagger & w_2 B_2 - w_1 B_1^\dagger + z_2 & w_2 B_1 + w_1 B_2^\dagger + z_1 \end{bmatrix}$$

is surjective.

Lemma 3.3.2. *Let $g \in U(k)$, $h \in SU(n)$ act on (B_1, B_2, I, J) by*

$$(B_1, B_2, I, J) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gIh^{-1}, hJg^{-1}).$$

If (B_1, B_2, I, J) satisfy the ADHM equations, then so do their transformed matrices defined above.

Theorem 3.3.3 (ADHM construction)**.** *There exists a one-to-one correspondence between the equivalence classes of ADHM data under the action of the groups $U(k)$ and $SU(n)$, and gauge equivalent classes of anti-self-dual $SU(n)$ gauge potentials A over $\mathbb{R}^4$, with instanton number $-k$*

We will now briefly outline the process from which the instanton data can be constructed from the ADHM data. The first step involves **constructing an operator** $\Delta^\dagger$ which is used to construct the instanton data.

For the case at hand $\Delta_{(x,y)}$ and in particular, its adjoint operator $\Delta^\dagger_{(x,y)}$, are given by

$$\Delta_{(x,y)} = \begin{bmatrix} \beta_{(x,y)} \\ \alpha^\dagger_{(x,y)} \end{bmatrix} : W \oplus V \otimes U \to V \times V,$$
$$\Delta^\dagger_{(x,y)} = \begin{bmatrix} \beta^\dagger_{(x,y)} & \alpha_{(x,y)} \end{bmatrix} : V \times V \to W \oplus V \otimes U.$$

The next step is **determining the kernel** of the operator $\Delta^\dagger$, and its dimension. This will require determining the kernel of Δ and the index of $\Delta^\dagger$ and then applying the index formula. We then consider a set of orthonormal solutions $\{\psi_i\}_{\forall i \in \mathcal{I}}$ of the equation

$$\Delta^\dagger_x \psi_i = 0.$$

In addition to the fact that $\ker\Delta = 0$, it can be shown that there are exactly n solutions to this equation. That is, $i \in \mathbb{N}_{\leq n}$.

Now we want to **construct the instanton data** and then **prove that the data satisfies the required equations**. The instanton data of charge $-k$ is given, in component form, by the gauge potential

$$A_{i,ab} = \psi_a \frac{\partial}{\partial x_i} \psi_b,$$

which can be shown to satisfy the self duality equation.

Lastly we need to show that **this solution is unique up to gauge transformations**.

We will see that this process is replicated in the ADHMN construction for both the Nahm transform and its inverse. This is due to the ADHMN construction being derived from the ADHM construction under the equivalence proven in Section 3.2, and thus the construction for monopoles is obtained by using similar methods to that for the instanton case.

The ADHM transform as a Fourier transform

Before we go on to the ADHMN construction, we will briefly emphasize an important perspective on the ADHM transforms. In particular, that they behave as Fourier transforms.

Consider the standard Fourier transform between functions of a single variable. So consider $f : \mathbb{R} \to \mathbb{C}$ some integrable function of variable x. Then the Fourier transform of f which we denote $\hat{f}(\eta)$, over the variable $\eta \in \mathbb{R}$ is given by

$$\hat{f}(\eta) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} f(x) e^{-ix\eta} dx,$$

and what we see is that under this transformation,

$$x \mapsto \frac{d}{d\eta},$$
$$\frac{d}{dx} \mapsto \eta.$$

Therefore, the Fourier transform maps algebraic operators to differential operators and vice-versa. This allows for a differential problem to be Fourier transformed to an algebraic problem which will generally be much simpler to solve.

The transforms arising in the ADHM construction can equivalently be viewed as a type of Fourier transform between two sets of data with four variables. The transform from the vector space data to the instanton data focuses around the operator $\Delta^\dagger$ in its construction. In particular, this operator is an algebraic operator, with four algebraic variables (x_1, x_2, x_3, x_4). When considering the ADHM construction in the reverse direction, that is the transform from the instanton data to the vector space data, the focus is on a differential operator, which we denote $\not{D}^\dagger$. This operator is a differential operator in four variables (x_1, x_2, x_3, x_4), with no algebraic operators.

Thus what we see is that the transforms involved in this theory are mapping the algebraic operators to differential ones, which is a generalization of the above Fourier transform to four variables. This allows for a simpler method of solving for instanton solutions rather than just solving directly the self-duality equations, which correspond to non-linear PDE's, to instead solving algebraic equations and then applying this Fourier-like transform to obtain the instanton solutions.

What we will now see is that this Fourier transform generalizes to a Fourier-like transform between a set of data consisting of one differential variable and three algebraic variables (the Nahm data), and a set of data consisting of one three differential variables and one algebraic variable (the monopole data). This is the case of the ADHMN construction.

3.4 The ADHMN construction

Given the correspondence proven in Section 3.2, we know that there exists an equivalence between the solutions of the Bogomolny equation and the time-invariant instantons. As a result of this, the ADHM construction can be adapted to certain types of monopole solutions. The ADHMN construction (see [8], [9]) is exactly Nahm's adaption of the ADHM construction to the case of the BPS monopoles given by Definition 3.1.1. This construction allows for one to find BPS monopole solutions, and thus solutions to the non-linear PDE's given by the Bogomolny equations, by instead solving a set of non-linear ODE's given by the Nahm equations, and vice versa.

Theorem 3.4.1 (ADHMN construction [9]). *The following sets of data are equivalent:*

1. *The gauge equivalent $SU(2)$ BPS monopoles as defined in Section 3.1.*
2. *The $O(k)$ conjugacy classes of solutions to the ordinary differential equation*

$$\frac{dT_i}{dz} = \frac{1}{2}\sum_{j,k} \epsilon_{ijk}[T_j, T_k], \quad z \in (-1, 1) \tag{3.4.1}$$

for $k \times k$ matrices $T_i(z)$ where $i \in \{1, 2, 3\}$ and z is a real variable, satisfying the conditions:

   (a) $T_i^\dagger = -T_i$,

   (b) $T_i(-z) = T_i(z)^T$ (were the superscript T denotes the transpose),

   (c) Each T_i has simple poles at $z = -1$ and $z = 1$, and is analytic on $z \in (-1, 1)$,

(d) at each pole the residues of (T_1, T_2, T_3) *define an irreducible representation of* $\mathfrak{su}(2)$.

Remark 3.4.2.

- *This theorem as given in Hitchin states that the residues correspond to an* $SU(2)$ *representation, not an* $\mathfrak{su}(2)$ *representation. Given however* $SU(2)$ *is simply connected, we know that the representations of* $SU(2)$ *and* $\mathfrak{su}(2)$ *are in bijective correspondence. Thus the two statements are equivalent.*

- *We will often refer to the set of data satisfying* (1) *as the monopole data, and the set of data satisfying* (2) *as the Nahm data.*

The ADHMN construction as a Fourier transform

Like with the ADHM construction, the ADHMN construction can be viewed as a type of generalized Fourier transform. To see this, we will consider the Nahm transform which corresponds to the transformation of the monopole data to the Nahm data, and its inverse.

As we will soon see, the Nahm transform focuses on the defining differential operator $\not{D}^\dagger$ which consists of one algebraic variable z, and three variables (x_1, x_2, x_3) which arise as differential operators in the covariant exterior derivative of the connection form A, in the operator $\not{D}^\dagger$. From this operator the Nahm data is constructed by the elements of its kernel. On the other hand, the inverse Nahm transform focuses on the operator $\Delta^\dagger$ which consists of three algebraic variables (x_1, x_2, x_3) and one differential operator in the variable z. Similarly, the monopole data is constructed from this operator by considering elements of its kernel.

Thus, what we see is that the number of algebraic and differential variables for the defining operators of the Nahm and inverse Nahm transform are swapped. This means that applying one of these transforms essentially applies a Fourier transform to the data, emphasizing the point that the Nahm transforms and its inverse can be viewed as a type of Fourier transform

The significance of such a transform comes into play when trying to solve for solutions to the monopole data. Given there are three differential objects, solving a solution to the Bogomolny equations means solving a non-linear PDE. If instead we were to look for solutions of the Nahm equations, there exists only one differential variable, and thus this equates to solving a non-linear ODE; generally a much simpler feat than solving a PDE. Thus the ADHMN construction allows one to instead solve for the Nahm data and then apply the transform to obtain the monopole solutions.

3.4.1 The inverse Nahm transform

We are now going to prove the statement that (2) $\Longrightarrow$ (1), and thus construct the monopole data from the Nahm data. This transform between the two sets of data is formally known as the inverse Nahm transform.

So assume we have a set of Nahm data satisfying (2) of Theorem 3.4.1.

Constructing the operator $\Delta^\dagger$.

As in [9], consider the following two spaces

$$V = \mathcal{L}^2[-1,1] \otimes \mathbb{C}^k \otimes \mathbb{C}^2, \text{ and}$$
$$W = \{f \in H^1 \otimes \mathbb{C}^k | f(-1) = f(1) = 0\},$$

where H^1 denotes the Sobolev space of functions of $[-1,1]$ whose derivatives are in $\mathcal{L}^2$.

Then the standard $\mathcal{L}^2$ hermitian inner product on V, such that given ψ_1, $\psi_2 \in V$,

$$\langle \psi_1, \psi_2 \rangle = \int \psi_1^\dagger \psi_2 \, ds$$

defines a compatible hermitian structure on the space V.

Now similar to the ADHM construction we are first going to consider the following *Weyl Operator* Δ_z given by

$$\Delta_z = i1_{2k} \frac{d}{dz} + i\, 1_k \otimes x_i \sigma_i + T_i \otimes \sigma_i : W \to V, \tag{3.4.2}$$

where (x_1, x_2, x_3) denote three spatial coordinates in $\mathbb{R}^3$, and σ_i denotes the i-th Pauli matrix.

The focus of the ADHMN construction lies however not on the operator Δ_z, but on its adjoint operator $\Delta_z^\dagger$. In order to define such an operator, we must first define what the adjoint of an operator is.

Definition 3.4.3. *Let V denote a Hilbert space, W a Banach space and $\Delta : W \to V$ a Weyl operator. Then the adjoint operator of Δ, given by $\Delta^\dagger : V \to W^*$, where W^* denotes the dual space of W, is the operator satisfying*

$$\langle \Delta^\dagger f, v \rangle = f(\Delta s).$$

Applying this definition to our Weyl operator Δ, we get

$$\Delta_z^\dagger = i1_{2k} \frac{d}{dz} - i1_k \otimes x_i \sigma_i - T_i \otimes \sigma_i : V \to W^*, \tag{3.4.3}$$

where we have used the property $T_i^\dagger = -T_i$.

Determining the kernel of $\Delta^\dagger$.

Now the first thing we are going to do is show that Δ is a positive operator, and thus has trivial kernel. To do this we will show that $\Delta^\dagger\Delta$ defines a positive operator. Given all solutions of the equation $\Delta v = 0$ are also solutions of the equation $\Delta^\dagger\Delta w = 0$, if the kernel of the operator $\Delta^\dagger\Delta$ is trivial, then it must be that the kernel of the operator Δ is also trivial.

Proposition 3.4.4. *The operator $\Delta^\dagger\Delta : W \to W^*$ is a positive operator.*

Proof. First we will expand $\Delta^\dagger\Delta$ into its individual elements. Thus

$$\begin{aligned}\Delta_z^\dagger\Delta_z &= \left(i1_{2k}\frac{d}{dz} - i1_k \otimes x_i\sigma_i - T_i \otimes \sigma_i\right)\left(i1_{2k}\frac{d}{dz} + i\,1_k \otimes x_j\sigma_j + T_j \otimes \sigma_j\right)\\ &= -1_{2k}\frac{d^2}{dz^2} + i\frac{dT_i}{dz}\otimes\sigma_i + (1_k\otimes x_i\sigma_i)(1_k\otimes x_j\sigma_j) - i(1\otimes x_i\sigma_i)(T_j\otimes\sigma_j)\\ &\quad - i(T_i\otimes\sigma_i)(1\otimes x_j\sigma_j) - (T_i\otimes\sigma_i)(T_j\otimes\sigma_j). \end{aligned} \tag{3.4.4}$$

Now we are going to focus on the individual terms of the final line (3.4.4). Using the property of the Pauli matrices that for some fixed $a \neq b$, $\sigma_a\sigma_b = -\sigma_b\sigma_a$ and $\sigma_a^2 = 1_2$, we know that

$$\begin{aligned}(1_k\otimes x_i\sigma_i)(1_k\otimes x_j\sigma_j) &= x_i^2\,1_{2k}, \quad \text{and}\\ -i(1_k\otimes x_i\sigma_i)(T_j\otimes\sigma_j) - i(T_i\otimes\sigma_i)(1_k\otimes x_j\sigma_j) &= -i\,T_i\otimes x_i 1_2 - i\,T_j\otimes x_j 1_2\\ &= -2i\,T_i\otimes x_i 1_2.\end{aligned}$$

Furthermore, the T_i satisfy the Nahm equations, $\frac{dT_i}{dz} = \frac{1}{2}\epsilon_{ijk}[T_j, T_k]$. Then given the Pauli matrices satisfy the commutation relations $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$, we get

$$i\frac{d}{dz}T_i\otimes\sigma_i - (T_i\otimes\sigma_i)(T_j\otimes\sigma_j) = -(T_i\otimes 1_2)(T_i\otimes 1_2).$$

Substituting these back into equation (3.4.4), we find

$$\Delta_z^\dagger\Delta_z = -1_{2k}\frac{d^2}{dz^2} + ((iT_i - 1_k x_i)\otimes 1_2)((iT_i - 1_k x_i)\otimes 1_2). \tag{3.4.5}$$

Now let $f \in W$ such that $\Delta_z^\dagger\Delta_z f = 0$. Then

$$\begin{aligned}0 &= \langle\Delta_z^\dagger\Delta_z f, f\rangle\\ &= \left\langle -1_{2k}\frac{d^2}{dz^2}f, f\right\rangle + \langle((iT_i - 1_k x_i)\otimes 1_2)((iT_i - 1_k x_i)\otimes 1_2)\,f, f\rangle\\ &= \left\langle 1_{2k}\frac{d}{dz}f, \left(-1_{2k}\frac{d}{dz}\right)^\dagger f\right\rangle + \langle(iT_i - 1_k x_i)\otimes 1_2\,f, ((iT_i - 1_k x_i)\otimes 1_2)^\dagger f\rangle\\ &= \left\langle 1_{2k}\frac{d}{dz}f, 1_{2k}\frac{d}{dz}f\right\rangle + \langle(iT_i - 1_k x_i)\otimes 1_2\,f, (iT_i - 1_k x_i)\otimes 1_2\,f\rangle\\ &\geq 0,\end{aligned} \tag{3.4.6}$$

where we have used the positive-definite property of an inner product, and that $T_i^\dagger = -T_i$.

Given $f \in W$ and is thus zero on the boundary of z by its definition, we get that equation (3.4.6) holds iff $f = 0$. Therefore $\Delta_z^\dagger \Delta_z$ must define a positive operator. □

Given $\Delta_z^\dagger \Delta_z$ defines a positive operator, we know that $\ker(\Delta_z^\dagger \Delta_z) = 0$. Thus, by the argument used at the beginning of this subsection, we can conclude $\ker(\Delta_z) = 0$.

We are now going to prove that the kernel of $\Delta^\dagger$ is a 2-dimensional space over the complex numbers. This fact will then be used to construct the bundle on which the monopole data is to be defined.

Lemma 3.4.5. *There exist exactly two independent, normalizable solutions to the following equation*

$$\Delta_z^\dagger v_i(z) = 0. \tag{3.4.7}$$

That is, two solutions $v_1(z)$ and $v_2(z)$ to equation (3.4.7), satisfying

$$\int v_i^\dagger(z) v_j(z)\, ds = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta.

The following proof will detail that given in [9].

Proof. The Nahm matrices have simple poles at $z = -1$ and $z = 1$ such that the residues define an irreducible $\mathfrak{su}(2)$ representation, as stated in Theorem 3.4.1. We will now focus on the pole at $z = -1$, however the exact same arguments can be applied to the pole at $z = 1$. Therefore, on some neighbourhood about the point $z = -1$, we may write the Nahm matrices as

$$\begin{aligned} T_i(z) &= -\frac{a_i}{z+1} + b_i(z) \qquad (3.4.8)\\ \implies \frac{dT_i}{dz} &= \frac{a_i}{(z+1)^2} + \frac{d\, b_i(z)}{dz}, \end{aligned}$$

where the $b_i(z)$ are analytic on this neighbourhood and the T_i define $k \times k$ matrices.

By now applying the Nahm equation to (3.4.8), we find that the a_i must satisfy

$$a_i = \frac{1}{2}\epsilon_{ijk}[a_j, a_k].$$

Furthermore, by the properties of the Nahm data, we know that these a_i define an irreducible, k-dimensional representation, given by

$$\begin{aligned} \rho_{k-1} : \mathfrak{su}(2) &\to \mathfrak{gl}(S^{k-1})\\ h_1 t_1 + h_2 t_2 + h_3 t_3 &\mapsto h_1 a_1 + h_2 a_2 + h_3 a_3, \end{aligned}$$

where $h_i \in \mathbb{C}$, $S^{k-1} = \mathbb{C}^k$, and the t_i define the generators of $\mathfrak{su}(2)$ as given by equation (A.1.2) in Appendix A. Furthermore, $\rho_{k-1}(t_i)$ will be isomorphic to the $\rho_{k-1}(t_i)$ as defined by equations (A.1.3)-(A.1.5) in Appendix A. Without lose of generality, we will assume that these representations are in fact equal.

Thus, on the neighbourhood about $z = -1$ we get

$$\begin{aligned} T_i \otimes \sigma_i &= -\frac{1}{z+1} a_i \otimes \sigma_i + b_i(z) \otimes \sigma_i \\ &= -\frac{1}{z+1} \rho_{k-1}(t_i) \otimes \sigma_i + b_i(z) \otimes \sigma_i \\ &= \frac{T}{z+1} + b_i(z) \otimes \sigma_i, \end{aligned}$$

where $T = -\rho_{k-1}(t_i) \otimes \sigma_i = -2i\rho_{k-1}(t_i) \otimes t_i$ denotes the residue T_i at $z = -1$.

Now from Proposition A.1.10 from Appendix A, we know that

$$C(S^{k-1} \otimes S^1) = C(S^{k-1}) \otimes 1_2 + 2\rho_{k-1}(t_i) \otimes t_i + 1_k \otimes C(S^1).$$

Therefore we can express $\rho_{k-1}(t_i) \otimes t_i$, and hence T, in terms of Casimir operators, C. Doing so, we get

$$T = i(C(S^{k-1}) \otimes 1_2 + 1_k \otimes C(S^1) - C(S^{k-1} \otimes S^1)).$$

Also proven in Appendix A, we know that $S^{k-1} \otimes S^1 \cong S^k \oplus S^{k-2}$, and $C(S^k) = -\frac{1}{4}k(k+2)$. Thus, considering T on $S^k \oplus S^{k-2}$, we can simplify T such that:

$$\begin{aligned} T &= \frac{i}{4}(-(k-1)(k+1) - 3 + k(k+2)) \\ &= \frac{i}{2}(k-1), \text{ on } S^k, \\ T &= \frac{i}{4}(-(k-1)(k+1) - 3 + (k-2)k) \\ &= -\frac{i}{2}(k+1), \text{ on } S^{k-2}. \end{aligned}$$

Having now determined T in its simplest form, we are going to consider the operator

$$\tilde{\Delta}_z = i\frac{d}{dz} + \left(\frac{1}{z+1} + \frac{1}{z-1}\right) T : W \to V$$

and in particular we want to consider the elements of its kernel, and thus solutions f of the equation

$$\tilde{\Delta} f = 0.$$

By the definition of T, it is clear that this equation has a solution space with dimension $\dim(S^k) = (k+1)$ of the form $f = a(z^2-1)^{-(k-1)/2}$, and a solution space with dimension $\dim(S^{k-2}) = (k-1)$ of the form $f = a(z^2-1)^{(k+1)/2}$.

Given all these solutions are $\mathcal{L}^2$ integrable, they are all normalizable, and so we get $\dim_{\mathbb{C}}(\ker\tilde{\Delta}) = (k-1)$ and $\dim_{\mathbb{C}}(\ker\tilde{\Delta}^\dagger) = (k+1)$. Given $\tilde{\Delta}$ is bounded operator, by finiteness of the kernel and cokernel, we know that $\tilde{\Delta}$ is a Fredholm operator, and thus we can determine its index by the standard index formula. Therefore we get

$$\begin{aligned}\text{index}\tilde{\Delta} &= \dim_{\mathbb{C}}(\ker\tilde{\Delta}) - \dim_{\mathbb{C}}(\ker\tilde{\Delta}^\dagger) \\ &= (k-1) - (k+1) \\ &= -2,\end{aligned}$$

where we have used the fact that $\dim_{\mathbb{C}}(\ker\tilde{\Delta}^\dagger) = \dim_{\mathbb{C}}(\text{coker}\tilde{\Delta})$

Now focusing on the boundary points at $z=-1$ and $z=1$, we know that the residue at these points define an irreducible $\mathfrak{su}(2)$ representation. Thus, by Schur's lemma we know there exists an isomorphism between these two representations. Furthermore, by the properties satisfied by the Nahm matrices, we know that $T_i(z) = T_i^T(-z)$. Thus we get

$$\begin{aligned}\operatorname*{Res}_{z=-1} T_i &= \operatorname*{Res}_{z=-1} T_i^T(-z) \\ &= \operatorname*{Res}_{z=1} T_i^T(z) \\ &= \operatorname*{Res}_{z=1} P^{-1}T_iP \\ \implies T &= \operatorname*{Res}_{z=1} P^{-1}T_iP,\end{aligned}$$

where $P \in U(k)$ by the properties of the transpose of an antihermitian matrix.

Now let Q be an anti-hermitian matrix, that is $Q \in \mathfrak{u}(k)$, such that $P = e^{2Q}$. Then

$$\Delta = e^{-(z+1)Q}\tilde{\Delta}e^{(z+1)Q} + K,$$

where K is some matrix valued function which is analytic on $[-1,1]$.

Now quoting the argument used in [9], the matrix K is compact and thus from the properties of a Fredholm operator the index of $\tilde{\Delta}$ must be invariant under the deformation $\tilde{\Delta} \mapsto \Delta$. Therefore by the invariance of the index, we find

$$\text{index}\Delta = \text{index}\tilde{\Delta} = -2.$$

Now we know from the previous proposition that $\dim_{\mathbb{C}}(\ker\Delta) = 0$ and so by applying the index formula we get

$$\dim_{\mathbb{C}}(\ker\Delta^\dagger) = 2.$$

□

Constructing the monopole data.

We have just shown that $\ker_{\mathbb{C}}(\Delta^\dagger) = 2$, and thus equivalently $\ker_{\mathbb{H}}(\Delta^\dagger) = 1$, where $\mathbb{H}$ denotes the quaternions which have basis given by the set $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ where we note that $i\sigma_j$ is antihermitian.

Now we want to construct the monopole data. In order to do this we need to first introduce the bundle on which the monopole data takes its values. So let $E_x = \ker(\Delta^\dagger(x))$ and $E = \ker(\Delta^\dagger) \subset V$. Therefore, $E \to \mathbb{R}^3$ defines a 1-dimensional, quaternionic fiber bundle over $\mathbb{R}^3$, where E_x denotes the fiber over $x \in \mathbb{R}^3$.

At last, we are able to define the monopole data.

Proposition 3.4.6. *Let $\{v_1, v_2\}$ denote an ordered basis of normalizable solutions of equation 3.4.7, and define*

$$(A_i)_{ab} = \int_{-1}^{1} v_a^\dagger(z)\partial_i v_b(z)dz, \tag{3.4.9}$$

$$(\Phi)_{ab} = i\int_{-1}^{1} z v_a^\dagger(z)v_b(z)dz. \tag{3.4.10}$$

Then the monopole data (A, Φ) (whose matrix components are given by (3.4.9) and (3.4.10) respectively) taking values on the bundle $E \to \mathbb{R}^3$ satisfies the Bogomolny equation.

Before we prove the above proposition, we need to define what the Green's function, and the *backwards operator* of a differential operator are. These objects will play a significant role in the proof of Proposition 3.4.6.

Definition 3.4.7 (Green's function)**.**

Let $D(z) : V \to W$ denote a linear differential operator with trivial kernel. Then the Green's function of $D(z)$, denoted $G(z,t)$, where t is a variable over the same domain as z, is an operator satisfying

$$D(z)G(z,t) = \delta(z - t).$$

In the context for which we will be considering, our operator will be $\Delta^\dagger\Delta$ and thus its Green's function $G(z,t)$, will satisfy

$$\Delta_z^\dagger \Delta_z G(z,t) = \delta(z - t). \tag{3.4.11}$$

Furthermore, given the space on which the Green's function maps to takes values in W for a fixed value of t, the Green's function must be zero at $z = -1, 1$. That is, $G(-1,t) = 0 = G(1,t)$.

Now we are going to define what the *backwards operator*, $\overleftarrow{D}$, of an operator D is. Given our interest is in matrix valued differential operators, we will restrict the definition to precisely these operators.

Let D_z denote an $n \times k$ matrix valued differential operator in the variable z. Then we may write $D = (d_{x,y})$, where $x \in \mathbb{N}_{\leq n}$ and $y \in \mathbb{N}_{\leq k}$. Now D acts on $k \times m$ matrices, $v(z) = (v_{p,q})$ where $p \in \mathbb{N}_{\leq k}$ and $q \in \mathbb{N}_{\leq m}$, producing an $n \times m$ matrix $D_z v(z)$.

Definition 3.4.8 (Backwards operator)**.**

Let $\overleftarrow{D}_z$ denote the backwards operator of D_z defined above. Then $\overleftarrow{D}_z = (\overleftarrow{D}_{x,y})$ acts backwards on an $m \times n$ matrix $v(z) = (v_{p,q})$ in the following way:

$$v(z)\overleftarrow{D}_z = (v_{p,x})(\overleftarrow{D}_{x,y}) = (D_{x,y})(v_{p,x}) = D_z^T v(z).$$

Thus $\overleftarrow{D}$ is just the forward operator D, acting on a matrix by the outer product.

Observe that when D is just a matrix (and not a differential operator), we get $\overleftarrow{D} = D$.

Now we can continue to prove the original proposition.

Author note: Up until this point we have used k to denote the magnetic charge. We will now let $k = N$, and let N denote the magnetic charge for the remainder of this section. This is due to k being used for a different purpose which is common to the literature, and we do not want to create a confusion.

Proof of Proposition 3.4.6. In order to prove the following proposition, we will show via explicit calculation that the monopole data given in Proposition 3.4.6 satisfies the Bogomolny equation

$$F = \star d_A \Phi. \tag{3.4.12}$$

In order to do this, we will need to determine $\partial_i v_a(z)$ and $\partial_i v_a^\dagger(z)$ in terms of the elements $v_i(z)$. From the definition of the orthonormal basis $\{v_1(z), v_2(z)\}$, we know that $\forall a \in \{1, 2\}$,

$$0 = \Delta_z^\dagger v_a(z) \tag{3.4.13}$$

$$\implies 0 = v_a^\dagger(z)\overleftarrow{\Delta}_z, \tag{3.4.14}$$

where equation (3.4.14) was obtained by taking the adjoint of both sides of equation (3.4.13).

Now acting the partial derivative with respect to x_i on both sides of equations (3.4.13), and using the fact that $\partial_i \Delta_z^\dagger = -i1_N \otimes \sigma_i$, we get

$$\begin{aligned} 0 &= \partial_i(\Delta_z^\dagger v_a(z)) \\ &= -i1_N \otimes \sigma_i v_a(z) + \Delta_z^\dagger \partial_i v_a(z) \\ \implies \Delta_z^\dagger \partial_i v_a(z) &= i1_N \otimes \sigma_i v_a(z). \end{aligned} \tag{3.4.15}$$

And similarly for equation (3.4.14), we get

$$\begin{aligned} 0 &= \partial_i(v_a^\dagger(z)\overleftarrow{\Delta}_z) \\ &= \partial_i v_a^\dagger(z)\overleftarrow{\Delta}_z + iv_a^\dagger(z)1_N \otimes \sigma_i \\ \implies \partial_i v_a^\dagger(z)\overleftarrow{\Delta}_z &= -iv_a^\dagger(z)1_N \otimes \sigma_i. \end{aligned} \tag{3.4.16}$$

Therefore, the partial derivatives of $v_i(z)$ and $v_i^\dagger(z)$ are

$$\begin{aligned} \partial_i v_a(z) &= i\int_{-1}^1 \Delta_z G(z,t)1_N \otimes \sigma_i v_a(t)dt + v_c(z)\int_{-1}^1 v_c^\dagger(t)\partial_i v_a(t)dt, \\ \partial_i v_a^\dagger(z) &= -i\int_{-1}^1 v_a^\dagger(t)1_N \otimes \sigma_i G(t,z)\overleftarrow{\Delta}_z^\dagger dt + \int_{-1}^1 \partial_i v_a^\dagger(t)v_c(t)dt\, v_c^\dagger(z), \end{aligned}$$

where the first term in each expression is derived from equations (3.4.15) and (3.4.16) respectively, and the second term are elements of the kernel of $\Delta^\dagger$ and $\overleftarrow{\Delta}$ respectively, arising due to the kernel of both of these operators being non-trivial, and thus not producing unique solutions to (3.4.13) and (3.4.14). The choice of coefficients for these kernel elements is to ensure the partial derivatives take value on the correct space.

Now to prove the proposition. Given A and Φ are matrix valued, we will explicitly calculate the a,b-th component of the matrix for both sides of the Bogomolny equation and show that they are equal for all $a,b \in \{1,2\}$, thus proving the matrices themselves are equal. We will first focus on the RHS of the Bogomolny equation as given in equation (3.4.12).

$$\begin{aligned} (d_{A_k}\Phi)_{ab} &= \partial_k \Phi_{ab} + [A_k,\Phi]_{ab} \\ &= \partial_k\Phi_{ab} + A_{k,ac}\Phi_{cb} - \Phi_{ac}A_{k,cb} \\ &= i\int_{-1}^1 s\partial_k v_a^\dagger(s)v_b(s)ds + i\int_{-1}^1 sv_a^\dagger(s)\partial_k v_b(s)ds + A_{k,ac}\Phi_{cb} - \Phi_{ac}A_{k,cb}. \end{aligned} \tag{3.4.17}$$

Now we will focus on the first two terms in line (3.4.17). For the first term we will expand the partial derivative, and then apply integration by parts. The fact that the Green's function vanishes on $s=-1,1$ and that $\partial_i(\int_{-1}^1 v_a^\dagger(s)v_c(s)ds) = \partial_i(\delta_{a,c}) = 0$ will be used.

Thus

$$
\begin{aligned}
&i\int_{-1}^{1} s\,\partial_k v_a^\dagger(s) v_b(s) ds \\
&\quad = \iint_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) \overleftarrow{\Delta}_s^\dagger\, s\, v_b(s) ds dt + i\iint_{-1}^{1} \partial_k v_a^\dagger(t) v_c(t)\, s\, v_c(s) v_b(s) ds dt \\
&\quad = i\int_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) s\, v_b(s) dt\Big|_{s=-1}^{s=1} - \iint_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) \Delta_s^\dagger (s\, v_b(s)) ds dt \\
&\qquad + i\int_{-1}^{1} \partial_k \left(\int_{-1}^{1} v_a^\dagger(t) v_c(t) dt\right) s\, v_c(s) v_b(s) ds - i\int_{-1}^{1} v_a^\dagger(t) \partial_k v_c(t)\, s\, v_c(s) v_b(s) ds dt \\
&\quad = -i\iint_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) v_b(s) ds dt - A_{k,ac}\Phi_{cb},
\end{aligned}
$$

where we have used the fact that $\Delta_s^\dagger(s v_b(s)) = i v_b(s) + \Delta_s^\dagger v_b(s)$.

Applying the same techniques for the second term we get

$$
i\int_{-1}^{1} s v_a^\dagger(s) \partial_k v_b(s) ds = -i\iint_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) v_b(s) ds dt + \Phi_{ac} A_{k,cb}.
$$

Thus, substituting these back into equation (3.4.17), we find

$$
\begin{aligned}
(d_{A_k}\Phi)_{ab} &= -i\iint_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) v_b(s) dt ds - i\iint_{-1}^{1} v_a^\dagger(s) G(s,t) 1_N \otimes \sigma_k v_b(t) dt ds \\
&= -2i\iint_{-1}^{1} v_a^\dagger(t) 1_N \otimes \sigma_k G(t,s) v_b(s) dt ds, \qquad (3.4.18)
\end{aligned}
$$

where we have used the fact that the Green's function commutes with $1_N \otimes \sigma_k$.

Now we will explicitly calculate the components of the gauge field.

$$
\begin{aligned}
(F_{ij})_{ab} &= (\partial_i A_j - \partial_j A_i + [A_i, A_j])_{ab} \\
&= \partial_i A_{j,ab} - \partial_j A_{i,ab} + A_{i,ac} A_{j,cb} - A_{j,ac} A_{i,cb} \\
&= \int_{-1}^{1} \partial_i v_a^\dagger(s) \partial_j v_b(s) ds + \int_{-1}^{1} v_a^\dagger(s) \partial_i \partial_j v_b(s) ds - \int_{-1}^{1} \partial_j v_a^\dagger(s) \partial_i v_b(s) ds \\
&\quad - \int_{-1}^{1} v_a^\dagger(s) \partial_j \partial_i v_b(s) ds + A_{i,ac} A_{j,cb} - A_{j,ac} A_{i,cb} \\
&= \int_{-1}^{1} \partial_i v_a^\dagger(s) \partial_j v_b(s) ds - \int_{-1}^{1} \partial_j v_a^\dagger(s) \partial_i v_b(s) ds + A_{i,ac} A_{j,cb} - A_{j,ac} A_{i,cb}, \quad (3.4.19)
\end{aligned}
$$

where we have used the commutativity of the partial derivatives. That is, that $\partial_i\partial_j = \partial_j\partial_i$. Now we are going to focus on the first term in line (3.4.19). Given the second term is identical up to a change of subscripts, the exact same logic can be applied to the second term.

$$\begin{aligned}\int_{-1}^{1} \partial_i v_a^\dagger(s)\partial_j v_b(s)ds &= i\iint_{-1}^{1} \partial_i v_a^\dagger(s)\Delta_s G(s,t)1_N \otimes \sigma_j\, v_b(t)dsdt \\ &\quad + \iint_{-1}^{1} \partial_i v_a^\dagger(s)v_c(s)\, v_c^\dagger(t)\partial_j v_b(t)dsdt \\ &= \iiint_{-1}^{1} v_a^\dagger(z)1_N \otimes \sigma_i G(z,s)\overleftarrow{\Delta}_s^\dagger \Delta_s G(s,t)1_N \otimes \sigma_j\, v_b(t)dsdtdz \\ &\quad + i\iiint_{-1}^{1} \partial_i v_c^\dagger(z)v_c(z)v_a^\dagger(s)\Delta_s G(s,t)1_N \otimes \sigma_j\, v_b(t)dsdtdz \\ &\quad + \iint_{-1}^{1} \partial_i v_a^\dagger(s)v_c(s)\, v_c^\dagger(t)\partial_j v_b(t)dsdt.\end{aligned}$$

Now look at the individual terms of the last line. For the first term we will use integration by parts on the s variable of the derivative term in $\Delta_s^\dagger$, and then apply the definition of the Green's function to simplify the expression. Therefore, for the first term we get

$$\begin{aligned}&\iiint_{-1}^{1} v_a^\dagger(z)1_N \otimes \sigma_i G(z,s)\overleftarrow{\Delta}_s^\dagger \Delta_s G(s,t)1_N \otimes \sigma_j\, v_b(t)dsdtdz \\ &\quad = i\iint_{-1}^{1} v_a^\dagger(z)1_N \otimes \sigma_i G(z,s)\Delta_s G(s,t)1_N \otimes \sigma_j\, v_b(t)dtdz\Big|_{s=-1}^{s=1} \qquad (3.4.20)\\ &\quad\quad - \iiint_{-1}^{1} v_a^\dagger(z)1_N \otimes \sigma_i G(z,s)\Delta_s^\dagger \Delta_s G(s,t)1_N \otimes \sigma_j\, v_b(t)dsdtdz \\ &\quad = -\iiint_{-1}^{1} v_a^\dagger(z)1_N \otimes \sigma_i G(z,s)\delta(s,t)1_N \otimes \sigma_j\, v_b(t)dsdtdz \qquad (3.4.21)\\ &\quad = -\iint_{-1}^{1} v_a^\dagger(z)1_N \otimes \sigma_i G(z,s)1_N \otimes \sigma_j\, v_b(s)dsdz,\end{aligned}$$

where line (3.4.20) was obtained by applying integration by parts. We have then used the fact that the Green's function is trivial on the boundary of s to obtain line (3.4.21).

For the second term, by applying integration by parts to the s variable, and using the fact that $\Delta^\dagger v_a(s) = 0$, we get that this term is equal to zero.

For the third and final term, we are going to use the fact that the $v_i(z)$ form an orthonormal basis, and thus $\partial_i(\int_{-1}^{1} v_a^\dagger(s)v_c(s)ds) = \partial_i(\delta_{a,c}) = 0$. Therefore we get

$$\begin{aligned}\iint_{-1}^{1} \partial_i v_a^\dagger(s)v_c(s)\, v_c^\dagger(t)\partial_j v_b(t)dsdt &= \int_{-1}^{1} \partial_i \left(\int_{-1}^{1} v_a^\dagger(s)v_c(s)ds\right) v_c^\dagger(t)\partial_j v_b(t)dt \\ &\quad - \iint_{-1}^{1} v_a^\dagger(s)\partial_i v_c(s)\, v_c^\dagger(t)\partial_j v_b(t)dsdt \\ &= -A_{i,ac}A_{j,cb}.\end{aligned}$$

Thus, substituting these results back into equation (3.4.19) and using the Green's functions commutativity with the term $1_N \otimes \sigma_k$, we get

$$\begin{aligned}(F_{ij})_{ab} &= -\iint_{-1}^{1} v_a^\dagger(z) 1_N \otimes \sigma_i G(z,s) 1_N \otimes \sigma_j \, v_b(s) ds dz \\ &\quad + \iint_{-1}^{1} v_a^\dagger(z) 1_N \otimes \sigma_j G(z,s) 1_N \otimes \sigma_i \, v_b(s) ds dz \\ &= -\iint_{-1}^{1} v_a^\dagger(z) 1_N \otimes [\sigma_i, \sigma_j] G(z,s) \, v_b(s) ds dz \\ &= -2i\epsilon_{ijk} \iint_{-1}^{1} v_a^\dagger(z) 1_N \otimes \sigma_k G(z,s) v_b(s) ds dz \\ &= ((\star D)_{ij} \Phi)_{ab}\end{aligned}$$

where we have used our result from equation (3.4.18).

Therefore, given this holds for all a and b, it is true that the Bogomolny equation is satisfied by the monopole data defined in equations (3.4.9) and (3.4.10). □

Showing A and Φ define the required a gauge potential and Higgs field (resp.) on an $SU(2)$ bundle.

Having now determined the explicit form we want our Higgs field and gauge potential to take, we will refer to [9] for the proof of the monopole data satisfying the boundary conditions given in Definition 3.1.3. What we will now show is that the monopole data does indeed take values in $\mathfrak{su}(2)$, and thus defines a set of monopole data on a principal $SU(2)$-bundle.

From the definition of the gauge potential and Higgs field, we know that these objects were constructed from an orthonormal basis of $\ker(\Delta^\dagger)$, and thus must take values in the 1-dimensional quaternionic space. Now we want to show that A and Φ take values in the subalgebra of the quaternions isomorphic to $\mathfrak{su}(2)$. The proof of the following lemma will be based on a similar proof for the instanton case given in [7].

Lemma 3.4.9. *The gauge potential A and the Higgs field Φ take values in $\mathfrak{su}(2)$.*

Proof. First we will show that both fields are anti-hermitian, and thus take values in the Lie algebra $\mathfrak{u}(n) \subset \mathbb{H}$, whose basis is given by $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ (i.e. consisting of the antihermitian elements of $\mathbb{H}$). We will then show that the trace of A and Φ are zero, proving the lemma.

Now using the fact that $0 = \partial_i(\delta_{a,b}) = \partial_i(\int_{-1}^{1} v_b^\dagger(z) v_a(z) dz)$, then from the definition of the 1-form A we get

$$\begin{aligned}
(A_i)^\dagger_{ab} &= \left(\int_{-1}^{1} v_a^\dagger(z)\partial_i v_b(z)dz\right)^\dagger \\
&= \int_{-1}^{1} \partial_i v_b^\dagger(z) v_a(z)dz \\
&= -\int_{-1}^{1} v_b^\dagger(z)\partial_i v_a(z)dz \\
&= -(A_i)_{ba} \qquad (3.4.22)\\
\implies A^\dagger &= \begin{bmatrix} (A)_{11} & (A)_{12} \\ (A)_{21} & (A)_{22} \end{bmatrix}^\dagger \\
&= \begin{bmatrix} (A)^\dagger_{11} & (A)^\dagger_{21} \\ (A)^\dagger_{12} & (A)^\dagger_{22} \end{bmatrix} \\
&= -A.
\end{aligned}$$

Applying the same logic to Φ, we find

$$\begin{aligned}
(\Phi)^\dagger_{ab} &= \left(i\int_{-1}^{1} v_a^\dagger(z)z\, v_b(z)dz\right)^\dagger \\
&= -i\int_{-1}^{1} z v_b^\dagger(z) v_a(z)dz \\
&= -\Phi_{ba} \\
\implies \Phi^\dagger &= \begin{bmatrix} (\Phi)_{11} & (\Phi)_{12} \\ (\Phi)_{21} & (\Phi)_{22} \end{bmatrix}^\dagger \\
&= \begin{bmatrix} (\Phi)^\dagger_{11} & (\Phi)^\dagger_{21} \\ (\Phi)^\dagger_{12} & (\Phi)^\dagger_{22} \end{bmatrix} \\
&= -\Phi.
\end{aligned}$$

Thus, A and Φ are both anti-hermitian, and so $A, \Phi \in \mathfrak{u}(2)$.

Now we need to show that the trace of A and Φ is zero. First observe that the trace of these two fields is pure imaginary in the quaternions $\mathbb{H}$. That is, $\mathrm{Re}(A) = 0 = \mathrm{Re}(\Phi)$, where Re(−), denotes the real component. This is because

$$\begin{aligned}
\mathrm{Re}(Tr(A)) &= \frac{1}{2}(Tr(A) + \overline{Tr}(A)) \\
&= \frac{1}{2}(Tr(A) + Tr(A^\dagger)) \\
&= 0,
\end{aligned}$$

where we have used that A is anti-hermitian, and the linearity of the trace.

The exact same process can also be applied to Φ, and thus, both A and Φ have no real component, and must be pure imaginary. Now we are going to split the A and Φ into two components, given as follows

$$A = A' + A'' \equiv A' + \frac{1}{2}Tr(A)1_2, \tag{3.4.23}$$

$$\Phi = \Phi' + \Phi'' \equiv \Phi' + \frac{1}{2}Tr(\Phi)1_2, \tag{3.4.24}$$

where 1_2 denotes the 2×2 identity matrix. Then the second terms in both (3.4.23) and (3.4.24) are diagonal, and by taking the trace of both sides of the two equations, it is clear that A' and Φ' are both traceless, and thus take values in $\mathfrak{su}(2)$. The second terms A'' and Φ'' on the other hand are diagonal, and thus commutative.

Now, we know that (A, Φ) satisfy the Bogomolny equation. Thus

$$\begin{aligned} F_A &= \star d_A \Phi \\ \implies dA + [A, A] &= \star d\Phi + \star[A, \Phi] \\ \implies dA' + dA'' + [A', A'] &= \star d\Phi' + \star d\Phi'' + \star[A', \Phi'] \\ \implies F_{A'} + F_{A''} &= \star d_{A'}\Phi' + \star d_{A''}\Phi''. \end{aligned} \tag{3.4.25}$$

Given that the first terms on both sides of equation (3.4.25) are traceless, and the second terms diagonal, by linear independence, equation (3.4.25) splits into the following two equations

$$\begin{aligned} F_{A'} &= \star d_{A'}\Phi', \\ F_{A''} &= \star d_{A''}\Phi''. \end{aligned} \tag{3.4.26}$$

Now focusing on $F_{A''}$ and using the fact that A'' is abelian, we get

$$\begin{aligned} \langle F_{A''}, F_{A''} \rangle &= \langle \star d_{A''}\Phi'', F_{A''} \rangle \\ &= \langle \star d\Phi'', dA'' \rangle \\ &= \langle d^* \star \Phi'', dA'' \rangle \\ &= \langle \star\Phi'', d^2 A'' \rangle \\ &= 0 \end{aligned}$$

where we have used $(\star)^2 = 1$ and $d^2 = 0$.

Thus, by the positive definiteness of the inner product, $F_{A''} = 0$. Therefore there exists a gauge transformation (which we will discuss in the next subsection) $g \in \mathcal{G}$, such that $g \cdot A'' = 0$. Under this gauge $A = A'$ and so A must be anti-hermitian and traceless, and thus take values in $\mathfrak{su}(2)$.

Furthermore, using the fact that $F_{A''} = 0$ and $d_{A''} = d$, from (3.4.26) we get

$$d\Phi'' = 0.$$

Given Φ'' is a 0-form however, this simply means that Φ has no dependence on any variable and thus, must be a constant. Therefore, $Tr(\Phi)$ is constant. Now from the boundary conditions on Φ, we know that the standard reference Higgs field $\Phi_0 = i\sigma_3$ is traceless. Given $\lim_{x_3\to\infty} \Phi(0,0,x_3) = \Phi_0$, we know that Φ is traceless at some point in the limit in which $r \to \infty$. But we know $Tr(\Phi)$ is constant everywhere and thus we get that $Tr(\Phi) = 0$ everywhere.

Therefore Φ is both antihermitian and traceless, and so $\Phi \in \Omega^0(M, \mathfrak{su}(2))$. □

Uniqueness of data up to gauge equivalence.

Finally, we observe that in making our choice of orthonormal basis $\{v_1, v_2\}$, we have some degree of freedom. Let $V(z)$ denote the 2×2 matrix whose i-th column is v_i. Then it is clear that

$$\Delta_z^\dagger V(z) = 0 \tag{3.4.27}$$

and by the orthonormality of the v_i,

$$\int_{-1}^{1} V^\dagger(z)V(z)dz = 1_2. \tag{3.4.28}$$

Now consider the transformation

$$V(z) \mapsto V(z)g(x), \tag{3.4.29}$$

where $g(x)$ is a unit quaternion in 2×2 matrix form satisfying $g^\dagger = g^{-1}$.

Then given $V(z)$ defines a normalized solution to equation (3.4.27), it is clear that $V(z)g(x)$ will also define such a normalized solution.

Now in terms of $V(z)$, it is clear from the definition of A and Φ given in equations (3.4.9) and (3.4.10) that we can define them in non-component form as

$$A_i = \int_{-1}^{1} V^\dagger(z)\partial_i V(z)dz,$$
$$\Phi = i\int_{-1}^{1} zV^\dagger(z)V(z)dz.$$

Thus applying the transformation given in equation (3.4.29) we get

$$\begin{aligned}(A_i) \mapsto &\int_{-1}^{1} (V(z)g(x))^\dagger \partial_i(V(z)g(x))dz \\ &= \int_{-1}^{1} (g^\dagger(x)V^\dagger(z)\partial_i V(z)\, g(x) + g^\dagger(x)V^\dagger(z)V(z)\partial_i g(x))dz \\ &= g^{-1}(x)A_i g(x) + g^{-1}(x)\partial_i g(x) \\ &= g \cdot A_i,\end{aligned}$$

where we have used $g^\dagger = g^{-1}$ and equation (3.4.28).

Similarly for Φ we get

$$\begin{aligned}\Phi \mapsto i\int_{-1}^{1} z(V(z)g(x))^\dagger(V(z)g(x))dz \\ = i\int_{-1}^{1} zg^{-1}(x)V(z)^\dagger V(z)g(x)dz \\ = g^{-1}(x)\Phi g(x).\end{aligned}$$

Thus, we find that the transformation on V by $g(x)$ is exactly equivalent to a gauge transformation on A and Φ by $g(x) \in \mathcal{G}$. This means that the monopole data determined by the inverse Nahm transform is unique up to gauge transformation, with the monopole data transforming under gauge transformations in the required way to ensure that the gauge potential and Higgs field are well-defined.

This finishes the proof that (2) $\implies$ (1) for the equivalence given in Theorem 3.4.1.

3.4.2 The Nahm transform

Now we want to prove that for Theorem 3.4.1, (1) $\implies$ (2) and hence that the monopole data implies the Nahm data. This is known as the Nahm transform, and we will see that the construction will be very similar to that of the inverse Nahm transform.

Constructing the operator $\not{D}$.

For the Nahm transform we want to consider a Dirac operator which acts on the space of Weyl spinor fields $S = S^+ \oplus S^-$. Consider the standard inner product on this space given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathbb{R}^3} \psi_1^\dagger \psi_2 d\mathbf{x}.$$

Now let (A, Φ) denote a set of monopole data which has charge k, and consider the *Dirac operator* $\not{D}$ and its adjoint, $\not{D}^\dagger$, given by

$$\not{D} = i(\sigma_i D_i + z - i\Phi) : S^+ \to S^-, \tag{3.4.30}$$

$$\not{D}^\dagger = i(\sigma_i D_i - z + i\Phi) : S^- \to S^+, \tag{3.4.31}$$

where D_i denotes the i-th component of the covariant exterior derivative of A, and we have used the fact that A and Φ are antihermitian.

The operator $\not{D}^\dagger$ will be the focus for the construction of the Nahm data, but first we need to determine its kernel.

Determining the kernel of $\not{D}^\dagger$.

We will first prove that the dimension of ker$\not{D}$ is trivial, and from this deduce the dimension of ker$\not{D}^\dagger$. To do this we will consider the operator $\not{D}^\dagger\not{D} : S^+ \to S^+$.

Proposition 3.4.10. *The operator $\not{D}$ has trivial kernel.*

Proof. In order to prove this result, we will first show that $\not{D}^\dagger\not{D}$ is a positive operator and therefore has trivial kernel. The result then follows by using the fact that $\ker(\not{D}) \subset \ker(\not{D}^\dagger\not{D})$.

From the properties of the Pauli matrices, we know that for some fixed $a \neq b$, $\sigma_a\sigma_b = -\sigma_b\sigma_a$ and $\sigma_a^2 = 1$. Then for some fixed a and b, we know that

$$(\sigma_a D_a)(\sigma_b D_b) = \sigma_a\sigma_b D_a D_b = \begin{cases} D_a^2 & \text{if } a = b \\ -\sigma_b\sigma_a D_a D_b & \text{if } a \neq b \end{cases},$$

where we emphasize that we **are not** summing over the a and b as they are fixed.

Now using the fact that $[D_i, D_j] = F_{ij}$, and $\sigma_i\sigma_j = \epsilon_{ijk} i\sigma_k$ when $i \neq j$, we find

$$\begin{aligned} \not{D}^\dagger\not{D} &= -(\sigma_i D_i - z + i\Phi)(\sigma_j D_j + z - i\Phi) \\ &= -D_i^2 - \Phi\Phi - 2iz\Phi + z^2 + i\sigma_k(-(\star F)_k + (D\Phi)_k) \\ &= -D_i^2 + (i\Phi - z)^2, \end{aligned}$$

where we have used the fact that (A, Φ) satisfy the Bogomolny equations and $(\star)^2$=1.

Now, we know that $D^\dagger = -D$, and Φ is antihermitian, so by the same analysis as used in Proposition 3.4.4, we get that $\not{D}^\dagger\not{D}$ is a positive operator, and thus $\ker(\not{D}^\dagger\not{D}) = 0$. □

So we know that $\dim_{\mathbb{C}}(\ker\not{D}) = 0$. Now applying the results given in [12], the index of the operator $\not{D}^\dagger$ can be determined by the Atiyah-Singer index theorem to be k (which corresponds to the charge of the monopole data). So by the index formula we get

$$\begin{aligned} \dim_{\mathbb{C}}(\ker\not{D}^\dagger) &= \text{index}(\not{D}^\dagger) - \dim_{\mathbb{C}}(\ker\not{D}) \\ &= k. \end{aligned}$$

Therefore the kernel of $\not{D}^\dagger$ has a k-dimensional, normalizable solution space. We will thus take $\{\psi_1, ..., \psi_k\}$ to denote an orthonormal basis of $\ker(\not{D}^\dagger)$, so that for all $i \in \mathbb{N}_{\leq k}$

$$\not{D}^\dagger\psi_i = 0 \tag{3.4.32}$$

$$\implies \psi_i^\dagger\overleftarrow{\not{D}} = 0. \tag{3.4.33}$$

Constructing the Nahm data.

Now we can split the space of odd spinors into the direct sum of the k-dimensional subspace of S^- taking values in the kernel of $\not{D}^\dagger$, which we will denote S_0^-, and its orthogonal complement $\tilde{S}^-$. Then restricting $\not{D}^\dagger$ to the space on which its kernel is trivial, we get

$$\not{D}^\dagger = i(\sigma_i D_i - z + i\Phi) : \tilde{S}^- \to S^+, \tag{3.4.34}$$

$$\not{D} = i(\sigma_i D_i + z - i\Phi) : S^+ \to \tilde{S}^-. \tag{3.4.35}$$

Therefore $\not{D}^\dagger$ is an invertible operator, and given $\not{D}^\dagger \not{D}$ is also invertible (as proven above), it must be that $\not{D}$ is also invertible.

Given all these operators are invertible, they must all have a Green's function associated to them. Let $G(\mathbf{x}, \mathbf{y})$ denote the Green's function of $\not{D}^\dagger \not{D}$. That is,

$$\not{D}^\dagger \not{D} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}).$$

Then, the Green's function of $\not{D}$ is $G(\mathbf{x}, \mathbf{y}) \overleftarrow{\not{D}}^\dagger$. By the definition of the Green's function, $\not{D}$ acting on its Green's function is given by the identity operator on $\tilde{S}^-$, which is just the identity operator on S^- minus the identity components on S_0^-. Using the fact that $\{\psi_a\}_{a \in \mathbb{N}_{\leq k}}$ defines an orthonormal basis of S_0^-, we get

$$\not{D} G(\mathbf{x}, \mathbf{y}) \overleftarrow{\not{D}}^\dagger = \delta(\mathbf{x} - \mathbf{y}) - \psi_a(\mathbf{x}) \psi_a^\dagger(\mathbf{x}).$$

We are now going to use the orthonormal basis of $\ker(\not{D}^\dagger)$, given by $\{\Psi_i\}_{i \in \mathbb{N}_{\leq k}}$, to construct the Nahm matrices. Thus we claim that the Nahm matrices, in component form, are given by

$$(T_i)_{ab} = -i \int_{\mathbb{R}^3} x_i \psi_a^\dagger(\mathbf{x}) \psi_b(\mathbf{x}) d^3\mathbf{x}. \tag{3.4.36}$$

Therefore we need to prove that these matrices do in fact satisfy the Nahm equations.

Proposition 3.4.11. *The matrices (T_1, T_2, T_3) as defined by equation (3.4.36) satisfy the Nahm equations.*

The following proof will detail that given in [3] and [11].

Proof. We will prove this by explicitly calculating the matrix components of both sides of the Nahm equation, and show that they are equal. So for the RHS

$$\begin{aligned}
(T_i T_j)_{ab} &= (T_i)_{ac} (T_j)_{cb} \\
&= -\int_{\mathbb{R}^6} x_i y_j \psi_a^\dagger(\mathbf{x}) \psi_c(\mathbf{x}) \psi_c^\dagger(\mathbf{y}) \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} \\
&= -\int_{\mathbb{R}^6} x_i y_j \psi_a^\dagger(\mathbf{x}) \left(\delta(\mathbf{x} - \mathbf{y}) - \not{D}_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \overleftarrow{\not{D}}^\dagger \right) \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} \\
&= -\int_{\mathbb{R}^3} x_i x_j \psi_a^\dagger(\mathbf{x}) \psi_b(\mathbf{x}) d^3\mathbf{x} + \int_{\mathbb{R}^6} x_i y_j \psi_a^\dagger(\mathbf{x}) \not{D}_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \overleftarrow{\not{D}}^\dagger \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x}.
\end{aligned} \tag{3.4.37}$$

Now we want to focus on the second term in line (3.4.37). Using integration by parts on the x and y variables, we know that the total derivative terms will be zero as $\psi(\mathbf{z}) \to 0$ sufficiently fast in the limit $\mathbf{z} \to \infty$ (to ensure that the ψ_i are normalizable). Thus, we get

$$\begin{aligned}
\int_{\mathbb{R}^6} x_i y_j \psi_a^\dagger(\mathbf{x}) \not{D}_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) \overleftarrow{\not{D}}^\dagger \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} &= -\int_{\mathbb{R}^6} y_j (\not{D}_{\mathbf{x}}^\dagger (x_i \psi_a(\mathbf{x})))^\dagger G(\mathbf{x},\mathbf{y}) \overleftarrow{\not{D}}^\dagger \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} \\
&= -\int_{\mathbb{R}^6} y_j (i\sigma_i \psi_a(\mathbf{x}) + x_i \not{D}_{\mathbf{x}}^\dagger \psi_a(\mathbf{x}))^\dagger G(\mathbf{x},\mathbf{y}) \overleftarrow{\not{D}}^\dagger \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} \\
&= i\int_{\mathbb{R}^6} y_j \psi_a^\dagger \sigma_i(\mathbf{x}) G(\mathbf{x},\mathbf{y}) \overleftarrow{\not{D}}^\dagger \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} \\
&= -i\int_{\mathbb{R}^6} \psi_a^\dagger(\mathbf{x}) \sigma_i G(\mathbf{x},\mathbf{y}) \not{D}_{\mathbf{y}}^\dagger (y_j \psi_b(\mathbf{y})) d^3\mathbf{y} d^3\mathbf{x} \\
&= \int_{\mathbb{R}^6} \psi_a^\dagger(\mathbf{x}) \sigma_i G(\mathbf{x},\mathbf{y}) \sigma_j \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x}.
\end{aligned}$$

Substituting this back into equation (3.4.37) we find

$$(T_i T_j)_{ab} = -\int_{\mathbb{R}^3} x_i x_j \psi_a^\dagger(\mathbf{x}) \psi_b(\mathbf{x}) d^3\mathbf{x} + \int_{\mathbb{R}^6} \psi_a^\dagger(\mathbf{x}) \sigma_i G(\mathbf{x},\mathbf{y}) \sigma_j \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x}.$$

Therefore the RHS of the Nahm equation is given by

$$\begin{aligned}
[T_i, T_j]_{ab} &= (T_i)_{ac}(T_j)_{cb} - (T_j)_{ac}(T_i)_{cb} && (3.4.38) \\
&= \int_{\mathbb{R}^6} \psi_a^\dagger(\mathbf{x}) [\sigma_i, \sigma_j] G(\mathbf{x},\mathbf{y}) \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x} && (3.4.39) \\
&= 2i\epsilon_{ijk} \int_{\mathbb{R}^6} \psi_a^\dagger(\mathbf{x}) \sigma_k G(\mathbf{x},\mathbf{y}) \psi_b(\mathbf{y}) d^3\mathbf{y} d^3\mathbf{x}. && (3.4.40)
\end{aligned}$$

Now we want to explicitly calculate the LHS of the Nahm equation, given by the term $(\partial_z T_k)_{ab}$. Before we do this however we need to determine $\partial_z \psi_a$ and $\partial_z \psi_a^\dagger$ in terms of the basis elements ψ_b. We know that

$$\begin{aligned}
0 &= \not{D}_{\mathbf{x}}^\dagger \psi_a(\mathbf{x}) \\
\implies 0 &= \partial_z \not{D}_{\mathbf{x}}^\dagger \psi_a(\mathbf{x}) + \not{D}_{\mathbf{x}}^\dagger \partial_z \psi_a(\mathbf{x}) \\
&= -i\psi_a(\mathbf{x}) + \not{D}_{\mathbf{x}}^\dagger \partial_z \psi_a(\mathbf{x}).
\end{aligned}$$

Then by the definition of the Green's function, we get

$$\frac{d\psi_a(\mathbf{x})}{dz} = i\int_{\mathbb{R}^3} \not{D}_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) \psi_a(\mathbf{y}) d^3y. \tag{3.4.41}$$

Similarly, if we apply the same process to (3.4.33), we find that

$$\frac{d\psi_a^\dagger(\mathbf{x})}{dz} = -i\int_{\mathbb{R}^3} \psi_a^\dagger(\mathbf{y}) G(\mathbf{y},\mathbf{x}) \overleftarrow{\not{D}}^\dagger{}_{\mathbf{x}} d^3y. \tag{3.4.42}$$

Therefore the LHS of the Nahm equations are given by

$$\begin{aligned}\left(\frac{dT_k}{dz}\right)_{ab} &= -i\int_{\mathbb{R}^3} x_k \frac{d\psi_a^\dagger(\mathbf{x})}{dz}\psi_b(\mathbf{x})d^3\mathbf{x} - i\int_{\mathbb{R}^3} x_k\psi_a^\dagger(\mathbf{x})\frac{d\psi_b(\mathbf{x})}{dz}d^3\mathbf{x}\\
&= -\int_{\mathbb{R}^6}\psi_a^\dagger(\mathbf{y})G(\mathbf{y},\mathbf{x})\overleftarrow{\not{D}^\dagger}_{\mathbf{x}} x_k\psi_b(\mathbf{x})d^3\mathbf{x}d^3\mathbf{y}\\
&\qquad + \int_{\mathbb{R}^6} x_k\psi_a^\dagger(\mathbf{x})\not{D}_{\mathbf{x}}G(\mathbf{x},\mathbf{y})\psi_b(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y}\\
&= \int_{\mathbb{R}^6}\psi_a^\dagger(\mathbf{y})G(\mathbf{y},\mathbf{x})\not{D}_{\mathbf{x}}^\dagger(x_k\psi_b(\mathbf{x}))d^3\mathbf{x}d^3\mathbf{y}\\
&\qquad - \int_{\mathbb{R}^6}(\not{D}_{\mathbf{x}}^\dagger(x_k\psi_a(\mathbf{x})))^\dagger G(\mathbf{x},\mathbf{y})\psi_b(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y}\\
&= i\int_{\mathbb{R}^6}\psi_a^\dagger(\mathbf{y})G(\mathbf{y},\mathbf{x})\sigma_k\psi_b(\mathbf{x})d^3\mathbf{x}d^3\mathbf{y}\\
&\qquad + i\int_{\mathbb{R}^6}\psi_a(\mathbf{x})^\dagger\sigma_k G(\mathbf{x},\mathbf{y})\psi_b(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y}\\
&= 2i\int_{\mathbb{R}^6}\psi_a(\mathbf{x})^\dagger\sigma_k G(\mathbf{x},\mathbf{y})\psi_b(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y}.\end{aligned}$$

Thus, given the $\mathbf{x}$ and $\mathbf{y}$ are dummy variables, we see that for all $a, b \in \mathbb{N}_{\leq k}$

$$\begin{aligned}\left(\frac{dT_k}{dz}\right)_{ab} &= \left(\frac{1}{2}\epsilon_{ijk}[T_i,T_j]\right)_{ab}\\
\implies \frac{dT_k}{dz} &= \frac{1}{2}\epsilon_{ijk}[T_i,T_j].\end{aligned}$$

□

Showing the boundary conditions are satisfied by the Nahm matrices.

It is clear by their definition that the Nahm matrices are anti-hermitian. This is because:

$$\begin{aligned}(T_i)_{ab}^\dagger &= \left(-i\int_{\mathbb{R}^3} x_i\psi_a^\dagger(\mathbf{x})\psi_b(\mathbf{x})d^3\mathbf{x}\right)^\dagger\\
&= i\int_{\mathbb{R}^3} x_i\psi_b^\dagger(\mathbf{x})\psi_b(\mathbf{x})d^3\mathbf{x}\\
&= -(T_i)_{ba}\\
\implies T_i^\dagger &= -T_i.\end{aligned}$$

Now we need to prove that $T_i(z) = \overline{T}_i(-z)$. To do this we will use the fact that the T_i are antihermitian and prove instead that the Nahm matrices satisfy the equivalent equation $\overline{T}_i = -T_i(-z)$, where $\overline{T_i}$ denotes the complex conjugate of T_i.

Proposition 3.4.12. *Let T_i denote the Nahm matrices which we have constructed thus far. Then*

$$\overline{T}_i = -T_i(-z).$$

Proof. Let $g = i\sigma_2$, which is an antihermitian matrix. Then denoting the complex conjugate of the Pauli matrices σ_i, by $\bar{\sigma}_i$, it is clear that

$$-g^{-1}\,\sigma_i\,g = \bar{\sigma}_i.$$

Now given Φ is antihermitian, we know $i\Phi$ will be hermitian. Thus we can express $i\Phi$ in terms of the Pauli matrices (which we know form a basis for the set of 2×2 antihermitian matrices) with real coefficients a_i. Doing so, we get

$$\begin{aligned} i\Phi &= a_i\sigma_i \\ \implies \overline{i\Phi} &= a_i\bar{\sigma}_i \\ &= -g^{-1}(i\Phi)g. \end{aligned}$$

Now consider the element ψ_a in the kernel of $\not{D}^\dagger$, such that $\{\psi_b\}_{b\in\mathbb{N}_{\leq k}}$ forms an orthonormal basis of this space. Then we know that

$$0 = \not{D}_z^\dagger\psi_{z,a} = i(\sigma_i D_i - z + i\Phi)\psi_{z,a}.$$

where the z in $\psi_{z,a}$ is included to show explicitly the z dependence of ψ_a.

Now taking the complex conjugate of this equation, and using the properties of the Pauli matrices given by $g\sigma_i = -\sigma_i g$ when $i \neq 2$, and $g\sigma_2 = \sigma_2 g$, we get

$$\begin{aligned} 0 &= \overline{\not{D}_z^\dagger}\overline{\psi}_{z,a} \\ &= -i(\bar{\sigma}_i D_i - z + (\overline{i\Phi}))\overline{\psi}_{z,a} \\ &= -i((g^{-1}g)\sigma_1 D_1 - (g^{-1}g)\sigma_2 D_2 + (g^{-1}g)\sigma_3 D_3 - (g^{-1}g)z - g^{-1}i\Phi g)\overline{\psi}_{z,a} \\ &= -ig^{-1}(-\sigma_1 g D_1 - \sigma_2 g D_2 - \sigma_3 g D_3 - zg - i\Phi g)\overline{\psi}_{z,a} \\ &= ig^{-1}(\sigma_i D_i + z + i\Phi)g\overline{\psi}_{z,a} \\ &= g^{-1}\not{D}_{-z}^\dagger(g\overline{\psi}_{-z,a}). \end{aligned}$$

Now consider the 'conjugation operator' C given by $C(\psi_{z,a}) = g\overline{\psi}_{-z,a}$, where we note that $C^2 = -1$. Now given $C(\psi_{z,a})$ is an element of $\ker(\Delta_{-z}^\dagger)$, we know that we can expand $C(\psi_{z,a})$ in terms of the orthonormal basis $\{\psi_{-z,a}\}_{a\in\mathbb{N}_{\leq k}}$ of $\ker(\Delta_{-z}^\dagger)$ such that for some $k \times k$ matrix A, we get

$$C(\psi_{z,a}) = A_{ab}\psi_{-z,b}$$

where we know that $A^2 = -1$ from the fact that $C^2 = -1$. Therefore, eigenbasis of eigenvectors with eigenvalue $\pm i$ of A, we get

$$\begin{aligned} C(\psi_{z,a}) &= \pm i\psi_{z,a} \\ \implies \overline{\psi}_{z,a} &= \pm ig^{-1}\psi_{-z,a} \end{aligned}$$

Now using this result and the fact that $g^{-1} = -g = g^\dagger$, from the expression of the Nahm matrices in terms of ψ_a given in (3.4.36), we get

$$\begin{aligned}(\overline{T}_i(z))_{ab} &= -i \int_{\mathbb{R}^3} x_i \overline{\psi}^\dagger_{z,a}(\mathbf{x}) \overline{\psi}_{z,b}(\mathbf{x}) d^3x \\ &= i \int_{\mathbb{R}^3} x_i (\pm i g^{-1} \psi_{-z,a}(\mathbf{x}))^\dagger (\pm i g^{-1} \psi_{-z,b}(\mathbf{x})) d^3x \\ &= i \int_{\mathbb{R}^3} x_i \psi^\dagger_{-z,a}(\mathbf{x}) (g^{-1})^\dagger g^{-1} \psi_{-z,b}(\mathbf{x}) d^3x \\ &= i \int_{\mathbb{R}^3} x_i \psi^\dagger_{-z,a}(\mathbf{x}) \psi_{-z,b}(\mathbf{x}) d^3x \\ &= -T_i(-z).\end{aligned}$$

□

There remains one last boundary condition to prove which is to do with the residue of the Nahm matrices at the boundaries. We refer to [10] for the details of this proof, and will omit the proof from this thesis.

Uniqueness of data up to conjugation by elements of $O(k)$.

Similar to the case for the inverse Nahm transform, our choice of basis for the kernel of $\not{D}$ is unique up to a transformation. Let Ψ denote the $k \times k$ matrix whose i-th column is given by the vector ψ_a. Then given Ψ is a normalized element in the kernel of $\not{D}^\dagger$, under the transformation

$$\Psi \mapsto \Psi\, X, \tag{3.4.43}$$

where $X \in U(k)$ and thus satisfies $X^\dagger X = 1$, $\Psi_a\, X$ will also define a normalizable solution in the kernel of $\not{D}^\dagger$.

Using the construction of the Nahm matrices given by equation (3.4.36), under the transformation given in (3.4.43), the Nahm matrices are transformed in the following way

$$T_i \mapsto X^{-1} T_i\, X. \tag{3.4.44}$$

Furthermore, if we let $T_i' = X^\dagger T_i X$, we know that these Nahm matrices must satisfy the condition $T_i'(-z) = T_i'^T(z)$, and therefore

$$\begin{aligned}X^{-1} T_i^T(z) X &= X^{-1} T_i^T(-z) X \\ &= T_i'(-z) \\ &= (T'(z))^T \\ &= X^T (T_i(z))^T (X^{-1})^T.\end{aligned}$$

But this is is true if and only if $X^T = X^{-1}$, which in turn is true if and only if X is orthogonal. Therefore, the Nahm data is unique up to conjugacy classes of the orthogonal group $O(k)$.

This completes the proof of (1) $\implies$ (2), and thus completes the proof of Theorem 3.4.1

3.4.3 $k = 1$ BPS monopole

As a first example of the ADHMN construction in action, we are now going to explicitly determine the $SU(2)$ BPS monopole of charge $k = 1$ by first determining the $k = 1$ Nahm data and then applying the inverse Nahm transform.

For $k = 1$, the Nahm matrices $T_i(z)$ are 1×1 complex matrices, and therefore must be commutative. So from the Nahm equations we know that

$$\frac{dT_i}{dz} = 0.$$

This means that each T_i is independent of z, and thus the T_i must be constant elements of $\mathbb{C}$. So we let $\vec{T} = \vec{a}$ where each $a_i \in \mathbb{C}$. Furthermore, given $T_i^\dagger = -T_i$ we know that each a_i must be pure imaginary, and thus all the conditions on $\vec{T}$ are satisfied. In particular, we observe that the only 1-dimensional, irreducible representation of $\mathfrak{su}(2)$ is the unique trivial representation. Given the residue of the simple poles must be defined by this representation, we get that the coefficient of the simple poles are zero, and thus there are no simple poles.

Therefore, given the Nahm matrices are constant, the equation $\Delta^\dagger v_i(z) = 0$ reduces to

$$\left(i1_2\frac{d}{dz} - iy\cdot\sigma\right)v_i(z) = 0, \tag{3.4.45}$$

where $y_i = x_i + a_i$ and $y\cdot\sigma = y_i\sigma_i$.

Now we can determine the solutions to this equation. It should be noted that, opposed to how it was introduced in the previous section, we are going to take $V(z)$ to be a 2×2 matrix whose i-th column corresponds to the orthonormal solution of (3.4.45) given by $v_i(z)$. Thus

$$\begin{aligned} 0 &= \left(i\frac{d}{dz} + iy\cdot\sigma\right)V(z) \\ &= \frac{d}{dz}(e^{-y\cdot\sigma z}V(z)) \\ \implies V(z) &= V_0 e^{y\cdot\sigma z}, \end{aligned}$$

where V_0 denotes the constant coefficient (with respect to z). Now applying the normalizability condition, we get

$$V(z) = \left(\frac{r}{\sinh(r)}\right)^{\frac{1}{2}} e^{y\cdot\sigma z}.$$

Having determined a solution to $\Delta^\dagger V(z) = 0$, the monopole data (A, Φ) can now be established. First we will determine the explicit form of $\partial_i V(z)$.

$$\begin{aligned}\partial_i V(z) &= \partial_i \left(\left(\frac{r}{\sinh(r)} \right)^{\frac{1}{2}} e^{y\cdot\sigma z} \right) \\ &= \frac{1}{2} \left(\frac{y_i}{r\sinh(r)} - \frac{y_i \cosh(r)}{\sinh(r)^2} \right) \left(\frac{\sinh(r)}{r} \right)^{\frac{1}{2}} e^{y\cdot\sigma} z \\ &+ \left(\frac{r}{\sinh(r)} \right)^{\frac{1}{2}} \Big(\sinh(rt) \frac{y_i z}{r} + \cosh(rz) \frac{y_i z \, y\cdot\sigma}{r^2} \\ &\qquad - \frac{y_i y\cdot\sigma}{r^5} \sinh(rz) + \frac{\sigma_i}{r} \sinh(rz) \Big).\end{aligned}$$

Now focusing on the Higgs field. Given $t_a = i\sigma_a$ we get:

$$\begin{aligned}\Phi &= i \int_{-\frac{1}{2}}^{\frac{1}{2}} z V^\dagger(z) V(z) dz \\ &= \left(\coth(r) - \frac{1}{r} \right) \frac{y_a}{r} t_a.\end{aligned} \tag{3.4.46}$$

And determining the potential A_i, we find

$$\begin{aligned}A_i &= \int_{-\frac{1}{2}}^{\frac{1}{2}} V^\dagger(z) \partial_i V(z) dz \\ &= -\left(1 - \frac{r}{\sinh(r)} \right) \epsilon_{ijk} \frac{y_j}{r^2} t_k.\end{aligned} \tag{3.4.47}$$

Thus, we see that the $k = 1$ BPS monopole which has been determined via the inverse Nahm transform corresponds exactly to the Prasad-Sommerfield monopole we determined in Section 2.4.2. This was to be expected as the monopole solution in Section 2.4.2 was determined with $\lambda = 0$ and whose energy is 2π, which corresponds exactly to the energy of the $k = 1$ solution in the BPS limit.

3.4.4 $k = 2$ BPS monopole

Now we will briefly consider the case in which $k = 2$. The $k = 2$ monopole can be solved for explicitly provided the monopole satisfies certain symmetries. The corresponding Nahm data will have, unlike the $k = 1$ case, a non-trivial irreducible $\mathfrak{su}(2)$ representations at the simple poles of the boundary.

For the case in which $k = 2$ (and for all cases $k > 1$), there does not exist any spherically symmetric $SU(2)$ *BPS* monopoles [5]. The $k = 2$ BPS monopoles are however axially symmetric along a certain axis, and will have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. We will use this to simplify the solution.

First we need to determine the Nahm data for a $k = 2$ BPS monopole. The Nahm matrices will be 2×2 matrices and may therefore be written as linear combinations of the Pauli matrices. Furthermore, given the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, the Nahm matrices may be written in terms of three functions f_1, f_2, f_3 such such that

$$T_1(z) = f_1(z)\left(\frac{-i\sigma_1}{2}\right), \quad T_2(z) = f_2(z)\left(\frac{-i\sigma_2}{2}\right), \quad T_3(z) = f_3(z)\left(\frac{-i\sigma_3}{2}\right).$$

Given the Nahm matrices must satisfy the Nahm equations, then we know that the $f_i(z)$ must satisfy

$$\frac{df_1(z)}{dz} = f_2(z)f_3(z), \quad \frac{df_2(z)}{dz} = f_3(z)f_1(z), \quad \frac{df_3(z)}{dz} = f_1(z)f_2(z). \tag{3.4.48}$$

Furthermore, we see that the T_i are antihermitian, and from the boundary conditions invoked on the Nahm data, the f_i must have simple poles at $z = \pm 1$ with residue ± 1, such that f_1 and f_3 are even functions, whilst f_2 is an odd function.

Equations (3.4.48) can then be written in terms of integration constants as

$$f_2^2 - f_1^2 = A^2\gamma^2, \quad f_2^2 - f_3^1 = A^2, \quad f_1^2 - f_3^2 = A^2(1 - \gamma^2),$$

where γ and A denote constants in z.

Then the equations given in (3.4.48) define a version of the well-known Euler equations. The solutions can thus be given in terms of the Jacobian elliptic functions, as follows

$$f_1 = \frac{-A\,dn_\gamma(u)}{sn_\gamma(u)}, \quad f_2 = \frac{-A}{sn_\gamma(u)}, \quad f_3 = \frac{-A\,cn_\gamma(u)}{sn_\gamma(u)},$$

where $u = z + t_0$ and t_0 is some fixed constant chosen to ensure the requirement that $T_i^T = T_i(-z)$.

Thus, the inverse Nahm data can now be applied to the Nahm data defined above to obtain the $k = 2$ SU(2) BPS monopole, where the separation between the two monopoles is indicated by the constant γ. Here $\gamma = 0$ indicates the two monopole charges are defined at the same spot in space, whilst $\gamma \to 1$ corresponds to the distance between the two charges approaching infinity. The details can be found in [15].

As the value of k increases, the monopole data becomes increasingly difficult to solve, and so does the inverse Nahm transform. Unlike with the ADHM construction, where all instanton solutions are able to be determined by applying the ADHM equivalence, not all $SU(2)$ monopoles have been determined.

Chapter 4

Monopoles arising in geometric Langlands duality

Now we want to reconsider the motivation behind this thesis which we discussed in the introduction, and focus on the conditions of the non-trivial bosonic fields arising in the dual of an $\mathcal{N} = 4$ supersymmetric gauge theory with a Neumann boundary condition. In particular, we are interested in the three non-zero components (X_1, X_2, X_3) of the bosonic field which must obey the Nahm equations on the closed interval $z \in [0, L]$ such that at $z = 0$, the X_i have a simple pole whose residues define an irreducible representation of $\mathfrak{su}(2)$. In particular, to remove the boundary conditions that are occurring at $z = L$, we will consider L in the limit $L \to \infty$, and thus take our non-trivial bosonic field to lie on the half-line $[0, \infty)$.

One interesting feature of the boundary condition required of the non-trivial bosonic field, is its great similarity to the Nahm data introduced in Chapter 3, which we showed to be equivalent to the $SU(2)$ BPS monopole. This leads to the natural question of whether these bosonic fields satisfying the Nahm equations on the half-line of the gauge theory have any type of monopole correspondence.

4.1 The Nahm pole

Theorem 4.1.1. *Consider the set of solutions to the ordinary differential equations*

$$\frac{dT_i}{dz} = \frac{1}{2}\sum_{j,k} \epsilon_{ijk}[T_j, T_k], \quad z \in [0, \infty) \tag{4.1.1}$$

for $k \times k$ matrices $T_i(z)$ where $i, j, k \in \{1, 2, 3\}$, satisfying the conditions that:

- *each T_i has a simple pole at $z = 0$ and is analytic for $z \in (0, \infty)$,*
- *at the pole $z = 0$, the residue of (T_1, T_2, T_3) define an irreducible representation of $\mathfrak{su}(2)$.*

- *As* $z \to \infty$, $T_i(z) \to 0$.

Then this set of Nahm data defines a $U(1)$ *Dirac monopole of charge* $-k$*, arising due to a point singularity.*

In order to prove this theorem, we will apply an inverse Nahm-like transform, which we will call the '$U(1)$ inverse Nahm transform'. This transform is a generalization of the inverse Nahm transform to the $G = U(1)$ case which applies a similar logic to the inverse Nahm transform.

4.1.1 The $U(1)$ inverse Nahm transform

The Nahm data we are considering for the current case looks extremely similar to that used in the ADHMN construction, however now we do not have the condition that the Nahm matrices need to be antihermitian. As a result of this we cannot simply apply a generalization of the inverse Nahm transform to the case at hand, due to the antihermitian condition being used for many of the arguments (i.e. showing that the complex dimension of the kernel of $\Delta^\dagger$ is 2). What we will do instead is utilise a theorem by Kronheimer given in [16] which will allow us to prove the previous theorem explicitly.

So first consider the spaces

$$\begin{aligned} V &= \{v \in \mathcal{L}^2[0,\infty] \otimes \mathbb{C}^k \otimes \mathbb{C}^2 | f(0) = 0\}, \text{ and} \\ W &= \{f \in H^1 \otimes \mathbb{C}^k | f(0) = 0\}, \end{aligned}$$

and on such spaces consider the operator, which we will denote $\Delta^\dagger$, given by

$$\Delta_z^\dagger = i 1_{2k} \frac{d}{dz} - i 1_k \otimes x_i \sigma_i - T_i \otimes \sigma_i : V \to W^*. \tag{4.1.2}$$

Now on the spaces V and W^* we will define the standard inner product, given by

$$\langle x, y \rangle = \int_0^\infty x^\dagger(z) y(z) dz,$$

where $x(z), y(z) \in V$ or W.

Now analogous to the inverse Nahm transform, we want to consider the normalizable solutions $v(z) \in V$ to the equation

$$\Delta^\dagger v(z) = 0. \tag{4.1.3}$$

Proposition 4.1.2. *On* $\mathbb{R}^3 \backslash \{0\}$*, there exists exactly one normalizable solution to equation (4.1.3), whilst at the point* $x = (0,0,0)$*, equation (4.1.3) has no normalizable solutions. That is, given* $r = \sqrt{x^2 + y^2 + z^2}$

$$dim_{\mathbb{C}}(ker \Delta^\dagger) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

To prove this proposition, we will calculate the dimension of $\Delta^\dagger$ explicitly. We can do this by applying a result given in [16], which investigates the solutions to the Nahm's equations on the half-line. We will now state the following theorem without proof.

Theorem 4.1.3. *There exists a unique (up to isomorphism) set of $k \times k$ Nahm matrices $T_i(z)$ where $z \in [0, \infty)$, satisfying the Nahm equations*

$$\frac{dT_i}{dz} = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} [T_j, T_k]$$

and the conditions given for the Nahm data in Theorem 4.1.1.

Thus, consider the set of Nahm matrices given by the *Nahm pole* solutions

$$T_i(z) = -\frac{\rho_{k-1}(t_i)}{z}, \tag{4.1.4}$$

where $(\rho_{k-1}(t_1), \rho_{k-1}(t_2), \rho_{k-1}(t_3))$ define the irreducible k-dimensional representation of $\mathfrak{su}(2)$ given in Appendix 1. Then these matrices define a solution to the Nahm data given in Theorem 4.1.1, and therefore defines the unique solution up to isomorphism. This simplifies Theorem 4.1.1 down to proving the theorem for the case when the T_i are as given in (4.1.4). We will now show that for this set of Nahm matrices, Proposition 4.1.2 is true, and therefore by the above theorem, is true in general.

Proof of Proposition 4.1.2. We are going to explicitly determine the solutions to the equation $\Delta_z^\dagger v(z) = 0$, and thus to the equation

$$\left(1_{2k}\frac{d}{dz} - 1_k \otimes x_i\sigma_i - i\,\frac{\rho_{k-1}(t_i)}{z} \otimes \sigma_i\right) v(z) = 0. \tag{4.1.5}$$

Now we will simplify this equation down by considering only the spherically symmetric solutions about the origin and using the fact that we may apply an $SU(2)$ rotation to the system such that as opposed to considering (x_1, x_2, x_3), we can consider instead the coordinates $(0, 0, r)$ where $r \geq 0$. By now substituting in explicitly the matrices $\rho_k(t_i)$ and σ_i as defined in Appendix A, and writing $v(z)$ as a $2k \times 1$ matrix with components $v(z)_{n,1} = v_n$, then we get that equation (4.1.5) reduces down to the following set of equations

$$0 = \begin{bmatrix} \partial_z v_1 \\ \partial_z v_2 \\ \partial_z v_3 \\ \partial_z v_4 \\ \vdots \\ \partial_z v_{2k-2} \\ \partial_z v_{2k-1} \\ \partial_z v_{2k} \end{bmatrix} - r \begin{bmatrix} v_1 \\ -v_2 \\ v_3 \\ -v_4 \\ v_5 \\ \vdots \\ -v_{2k-2} \\ v_{2k-1} \\ -v_{2k} \end{bmatrix} + \frac{1}{z} \begin{bmatrix} 0 \\ -av_3 \\ -v_2 \\ (-a+1)v_5 \\ -2v_4 \\ \vdots \\ -v_{2k-1} \\ -av_{2k-2} \\ 0 \end{bmatrix} - \frac{1}{2z} \begin{bmatrix} av_1 \\ -av_2 \\ (a-2)v_3 \\ -(a-2)v_4 \\ (a-4)v_5 \\ \vdots \\ (a-2)v_{2k-2} \\ -av_{2k-1} \\ av_{2k} \end{bmatrix},$$

where $a = k - 1$. These equations can then be written compactly as

$$0 = \frac{dv_1}{dz} - rv_1 - \frac{av_1}{2z}, \tag{4.1.6}$$

$$0 = \frac{dv_n}{dz} + \frac{-a + \frac{n}{2} - 1}{z} v_{n+1} + \left(\frac{a - n + 2}{2z} + r\right) v_n, \tag{4.1.7}$$

$$0 = \frac{dv_{n+1}}{dz} - \frac{n}{2z} v_n - \left(\frac{a-n}{2z} + r\right) v_{n+1}, \tag{4.1.8}$$

$$0 = \frac{dv_{2k}}{dz} + rv_{2k} - \frac{av_{2k}}{2z}, \tag{4.1.9}$$

where $n \in 2\mathbb{Z}$ such that $2 \leq n \leq 2k - 2 = 2a$.

It is clear that there are explicit solutions for v_1 and v_{2k} given by

$$v_1 = X_1 z^{\frac{a}{2}} e^{rz},$$
$$v_{2k} = X_{2k} z^{\frac{a}{2}} e^{-rz},$$

where X_1 and X_2 denote the constant coefficients.

We observe that given $r \geq 0$, $v_1(z)$ is never normalizable and therefore the constant coefficient X_1 must equal zero, meaning $v_1(z) = 0$.

On the other hand, v_{2k} is normalizable for all $r > 0$, but is not normalizable at $r = 0$. Considering only $r > 0$, by normalizing v_{2k} we get

$$v_{2k} = \frac{\sqrt{2r}}{\sqrt{(N-1)!}} (2rz)^{\frac{N-1}{2}} e^{-rz}. \tag{4.1.10}$$

Now we want to consider the remaining $2k - 2$ equations. To do this, we will first prove the following claim before continuing on with our proof.

Claim 4.1.4. *Consider equations (4.1.7) and (4.1.8). The only solution to these equations are the trivial solutions. That is, $v_i = 0$, for all $i < 2k$.*

Proof. In order to prove this claim we will re-express equations (4.1.7) and (4.1.8) as differential equations in only one variable. From this we will show that the differential operator corresponding to each variable is a negative operator, and thus solutions to its kernel must be trivial.

Expressing equation (4.1.8) in terms of v_n and then taking its derivative, we can substitute these values into equation (4.1.7) to obtain a differential in terms of the single variable v_{n+1}. In doing so we get the following ODE

$$0 = \left(\left(z\frac{d}{dz} + \frac{1}{2}\right)^2 - \left(zr + 1 - \frac{n}{2} + \frac{a}{2}\right)^2 + \frac{a}{2} + \frac{n^2}{4} - \frac{na}{2} - n + \frac{3}{4}\right) v_{n+1}$$
$$\overset{\text{def}}{=} \Psi_{n+1} v_{n+1}.$$

Now let $f \in V$ such that $\Psi_{n+1} f = 0$. Then

$$\begin{aligned} 0 &= \langle \Psi_{n+1} f, f \rangle \\ &= \left\langle \left(z\frac{d}{dz} + \frac{1}{2} \right)^2 f, f \right\rangle - \left\langle \left(zr + 1 - \frac{n}{2} + \frac{a}{2} \right)^2 f, f \right\rangle \\ &\quad + \langle \frac{1}{4} \left(2a + n^2 - 2na - 4n + 3 \right) f, f \rangle \\ &= - \left\langle \left(z\frac{d}{dz} + \frac{1}{2} \right) f, \left(z\frac{d}{dz} + \frac{1}{2} \right) f \right\rangle - \left(zr + 1 - \frac{n}{2} + \frac{a}{2} \right)^2 \langle f, f \rangle \\ &\quad + \frac{1}{4} \left(n^2 - 2na - 4n + 2a + 3 \right) \langle f, f \rangle, \end{aligned} \tag{4.1.11}$$

where we have used the fact that $\left(z\frac{d}{dz} + \frac{1}{2} \right)^\dagger = - \left(z\frac{d}{dz} + \frac{1}{2} \right)$ which is proven in Appendix C.

Given that $\frac{1}{4} \left(n^2 - 2na - 4n + 2a + 3 \right) \leq 0$ whenever $a \geq \frac{n}{2}$ as proven in Appendix C (note that these are the only values of a and n we are interested in), we know that the RHS of (4.1.11) is always less than or equal to zero as Ψ_{n+1} is a negative operator. Therefore, equation (4.1.11) can hold only if $f = 0$, and thus $\ker(\Psi_{n+1}) = 0$. This implies that $v_{k+1} = 0$ for all $n \in 2\mathbb{N}$ such that $2 \leq n \leq 2k - 2$.

By now substituting the solutions $v_{n+1} = 0$ into (4.1.7) and (4.1.8), we see that the only solutions for the v_n are the trivial solutions, and so $v_n = 0$ for all $n \in 2\mathbb{Z}$ such that $2 \leq n \leq 2k - 2$.

This proves the claim. □

Proof of Proposition 4.1.2 continued.
So we have shown that at $r = 0$ there are no normalizable solutions of (4.1.4), whilst on $r > 0$ there exists exactly one normalizable solution to equation (4.1.4), given by

$$v(z) = \frac{\sqrt{2r}}{\sqrt{(k-1)!}} (2rz)^{\frac{N-1}{2}} e^{-rz} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4.1.12}$$

Hence, we get

$$\dim_{\mathbb{C}}(\ker \Delta^\dagger) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

□

Now as in Section 3.4, we can construct a vector bundle given by

$$E \stackrel{\text{def}}{=} \ker \Delta^\dagger \to \mathbb{R}^3 - \{0\}$$

and from this, construct the monopole data where

$$A_i = \int_0^\infty z v^\dagger(z)\partial_i v(z) dz, \tag{4.1.13}$$

$$\Phi = i\int_0^\infty v^\dagger(z) v(z) dz, \tag{4.1.14}$$

such that at $r = 0$, there exists a point singularity.

Now we need to show that this set of monopole data does indeed satisfy the Bogomolny equation. If we can show that the adjoint operator of $\Delta^\dagger$, given by Δ, has trivial kernel then analogous to Section 3.4.2 we can consider the operator $\Delta^\dagger\Delta$ and its Green's function. This would resolve the problem of the T_i not being antihermitian, as antihermitiacy only comes into play when considering dimensions and the existence of the Green's function, for the proof of the monopole data satisfying the Bogomolny equations. So we have that the adjoint operator of $\Delta^\dagger$ is given by

$$\Delta = 1_{2k}\frac{d}{dz} + 1_k \otimes x_i\sigma_i - i\frac{\rho_{k-1}(t_i)^\dagger}{z}\otimes\sigma_i : W \to V. \tag{4.1.15}$$

Claim 4.1.5. *Consider the operator Δ given in (4.1.15). Then it is true that*

$$dim_{\mathbb{C}}(ker\Delta) = 0.$$

Assuming the above claim is correct, the only remaining difference between the proof given in Section 3.4.2 and that required for the given case, is the domain of the variable z. This comes into play in the proof of Proposition 3.4.6 as it ensures the total derivatives be trivial as the Green's function is trivial at the boundary points $z = -1, 1$. For the case at hand however, we get that $G(z, s)$ is trivial at $z = 0$, whilst $v(z)$ equals zero in the limit $z \to \infty$ by the requirement that $v(z)$ is $\mathcal{L}^2$ integrable. This requirement on $v(z)$ in the limit $z \to \infty$, and the triviality of the Green's function at $z = 0$ ensures that the total derivatives arising for the current calculation remain trivial. Thus, by then applying the same analysis as in Proposition 3.4.6, we get that A and Φ as defined by equations (4.1.13) and (4.1.14) do indeed satisfy the Bogomolny equations.

Proof of Claim 4.1.5. Substituting the explicit values of $\rho_{k-1}(t_i)^\dagger$ and σ_i into equation (4.1.5), the differential equation reduces down to the following set of equations

$$0 = \frac{dv_1}{dz} + rv_1 + \frac{av_1}{2z},$$

$$0 = \frac{dv_n}{dz} + \frac{n}{2z}v_{n+1} + \left(\frac{a-n+2}{2z} - r\right)v_n, \tag{4.1.16}$$

$$0 = \frac{dv_{n+1}}{dz} + \frac{a - \frac{n}{2} + 1}{z}v_n - \left(\frac{a-n}{2z} - r\right)v_{n+1}, \tag{4.1.17}$$

$$0 = \frac{dv_{2k}}{dz} - rv_{2k} + \frac{av_{2k}}{2z},$$

where $n \in 2\mathbb{Z}$ such that $2 \leq n \leq 2k-2 = 2a$.

It is clear that there are explicit solutions for v_1 and v_{2k} given by

$$v_1 = X_1 z^{-\frac{a}{2}} e^{-rz},$$
$$v_{2k} = X_{2k} z^{-\frac{a}{2}} e^{rz}.$$

However it is clear that on the interval $(0, \infty)$ both of these solutions diverge, and thus there does not exist a normalizable solution for v_1 or v_{2k}. Therefore, we get $v_1 = 0 = v_{2k}$.

Furthermore, by analogy to the proof of Claim 4.1.4, we can show that equations (4.1.16) and (4.1.17) can be rearranged such that we get

$$0 = \left(z\frac{d}{dz} + \frac{1}{2}\right) - \frac{1}{4}(2zr - a + n)^2 - \frac{1}{4}(2a + 2an - n^2 + 1)v_n.$$

This is indeed a negative operator, given $n \in 2\mathbb{Z}$ such that $2 \leq n \leq 2k-2 = 2a$, and thus its only solution is $v_n = 0$. Substituting this into equation (4.1.16) we also get that $v_{n+1} = 0$.

Thus we get that for all $1 < i < 2k$, $v_i = 0$, and so the only element in $\ker\Delta$ is the trivial element. Therefore

$$\dim_{\mathbb{C}}(\ker\Delta) = 0.$$

□

Now to determine the values on which the image of the gauge potential and Higgs field take, consider the following lemma:

Lemma 4.1.6. *The gauge potential and Higgs field as given by equations (4.1.13) and (4.1.14) are pure imaginary, and thus take values in* $\mathfrak{u}(1) \cong i\mathbb{R}$.

Proof. Given A_i and Φ are complex valued 1×1 matrices, taking their adjoint is equivalent to taking their conjugate. Thus

$$\begin{aligned} \overline{A_i} &= (A_i)^\dagger \\ &= -A_i, \qquad \text{and} \\ \overline{\Phi} &= (\Phi)^\dagger \\ &= -\Phi. \end{aligned}$$

where we have used the same process as in Lemma 3.4.9 to show that $(A_i)^\dagger = -A_i$ and $(\Phi)^\dagger = -\Phi$. □

Thus, A_i and Φ take values in $\mathfrak{u}(1) \cong i\mathbb{R}$.

Now we note that the choice of $v(z)$ is not unique. Given some complex valued 1×1 matrix $g(x)$ satisfying $g(x)^\dagger = g^{-1}(x)$ (i.e. for any fixed x, $g(x) \in U(1)$), then the image of the transformation

$$v(z) \mapsto v(z)g(x)$$

also defines a normalizable solution of equation (4.1.4).

Under such a transformation, we get that the gauge potential and Higgs field transform as follows:

$$\begin{aligned} A_i \mapsto g^{-1} A_i g + g^{-1} dg = A_i + g^{-1} dg, \\ \Phi \mapsto g^{-1} \Phi g = \Phi, \end{aligned}$$

where we have used the commutativity of a 1×1 matrix.

These transformations define exactly the $U(1)$ gauge transformations. Thus the monopole solutions are defined uniquely, up to $U(1)$ gauge transformation, and so are defined on a $U(1)$ gauge theory.

4.1.2 The Dirac monopole

We are now going to finish the proof of Theorem 4.1.1 and show that the Nahm data creates a $U(1)$ monopole. Given that such a $U(1)$ monopole arises due to some topologically non-trivial base space (and hence not usually a Higgs field), the significance of the Higgs field for this case is to ensure the existence of a gauge potential satisfying the defining equation of a $U(1)$ monopole, given by equation (2.3.4).

Final part of proof to Theorem 4.1.1.
From the previous section, we constructed a set monopole data from the $U(1)$ Nahm data. Now we want to determine explicitly the Higgs field and the gauge potential of this theory and show that they are equivalent to the Dirac monopole data. Substituting solution (4.1.12) into equation (4.1.14) we find the Higgs field is given by

$$\Phi = \frac{i\,k}{2r} \in \Omega^0(M, ad(P)), \tag{4.1.18}$$

where $k \in \mathbb{N}$ corresponds in the Nahm data picture, to the size of the Nahm matrix (that is, the $k \times k$ matrix).

Now, as opposed to explicitly determining the gauge potential A, we will instead substitute our value of Φ into the Bogomolny equation, which we know is satisfied as proven in the previous section.

Therefore we get

$$\begin{aligned} F_A &= \star d\Phi \\ \implies dF_A &= d \star d\Phi \\ &= \Delta_L \Phi \, dx \wedge dy \wedge dz \\ &= \frac{i\,k}{2} \Delta_L \left(\frac{1}{r}\right) dV \\ &= -2\, i\pi k\, \delta^3(r)\, dV, \quad k \in \mathbb{Z} \end{aligned} \tag{4.1.19}$$

where $\Delta_L = \partial_1^2 + \partial_2^2 + \partial_3^2$ denotes the Laplace operator, and we have used the fact that $d \star d = \Delta_L dV$ over the 3-dimensional base space, $\mathbb{R}^3$.

This emphasizes the presence of a singularity at the point $r = 0$, which was established when determining the kernel of the operator $\Delta^\dagger$. Now integrating both sides of (4.1.19) over the base space $M = \mathbb{R}^3$ we get

$$\int_M F = -2i\pi k \equiv g'. \tag{4.1.20}$$

This is exactly the defining equation for a $U(1)$ monopole as given in Chapter 2, whose gauge field defined on $r > 0$ is given by

$$F_{ij} = \epsilon_{ijk} \frac{g' x_k}{4\pi r^3}.$$

The presence of the singularity ensures a topologically non-trivial base space, and thus allows for such a monopole to exist. Therefore the gauge potential A must be singular at the origin on $\mathbb{R}^3$, and define exactly the gauge potential for a Dirac monopole.

Finally, we see that the affect of requiring the monopole solutions be symmetric about the origin in our above proof is that all monopole charges are 'located' at the origin, which is evident from the singularity at $r = 0$. If instead we removed this condition, this would affect the monopoles with charge $k > 1$ as the monopole charges need no longer be centered at the origin. □

Therefore the $U(1)$ inverse Nahm transform applied to the Nahm data given in Theorem 4.1.1 completely defines a singular $U(1)$ monopole. The significance of the Higgs field for this case is to ensure the existence of a gauge potential satisfying the defining equation for a Dirac monopole, and thus ensuring the existence of a monopole.

4.1.3 The $U(1)$ Nahm transform

We now make the following conjecture:

Conjecture 4.1.1. *The following two sets of data are equivalent:*

1. *Solutions to the ordinary differential equation*

$$\frac{dT_i}{dz} = \frac{1}{2} \sum_{j,k} \epsilon_{ijk}[T_j, T_k], \quad z \in [0, \infty) \tag{4.1.21}$$

for $k \times k$ matrices $T_i(z)$ where $i \in \{1, 2, 3\}$, satisfying the conditions:

- *T_i have a simple pole at $z = 0$ and is analytic for $z \in (0, \infty)$,*
- *at the pole $z = 0$, the residue of (T_1, T_2, T_3) defines an irreducible representation of $\mathfrak{su}(2)$.*
- *As $z \to \infty$, $T_i(z) \to 0$.*

2. *$U(1)$ Dirac monopoles with topological charge $k \in -\mathbb{N}$, arising due to point singularities on the base space $M = \mathbb{R}^3$.*

We have proven this equivalence in the direction (1) $\implies$ (2). We now conjecture that (2) $\implies$ (1). The proof to show that solutions of the $U(1)$ BPS monopole correspond to solutions of the Nahm equations is essentially the same as in the proof given in Section 3.4 using the Nahm transform, by applying a similar explicit calculation proof as we did for the case of (1) $\implies$ (2). The part which we have not yet proven is that at $z = 0$ there is a simple pole with residue defining an irreducible $\mathfrak{su}(2)$ representation. We expect that the proof however can be given along the same lines as for the Nahm transform case in [?].

4.2 The Nahm pole and Langlands duality

So what we have shown is that in the limit $L \to \infty$, the non-zero bosonic fields correspond to a Dirac monopole arising from a point singularity, which suggest there exists an abelian Nahm-like transform between certain Dirac monopoles and such Nahm data. Relating this back to the motivation, this introduces an interesting limiting case for the non-trivial bosonic field components. It is inconclusive however the effect that this result has in the Langlands dual picture, and further investigation of the significance of the boundary condition at $z = L$ where L is finite would need to be undertaken.

Appendix A

Representation theory of $\mathfrak{su}(2)$

The irreducible representation of $\mathfrak{su}(2)$ plays a significant role in the ADHMN construction, with the Nahm data defining such a representation. We will now outline the theory behind such representations, and detail in particular the case of $\mathfrak{su}(2)$.

A.1 Irreducible representations

Definition A.1.1. *Let V denote a vector space over $\mathbb{C}$. A complex representation of the Lie algebra $\mathfrak{g}$ is given by an algebra homomorphism*

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V),$$

where $\mathfrak{gl}(V)$ denotes the Lie algebra of endomorphism on V. That is, a map ρ satisfying

$$\rho([g_1, g_2]) = [\rho(g_1), \rho(g_2)],\ \forall g_1, g_2 \in \mathfrak{g}.$$

Given a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we call the space V a $\mathfrak{g}$-module.

Definition A.1.2. *Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a Lie algebra representation.*

- *A $\mathfrak{g}$-submodule of V is a linear subspace $W \subset V$ which is $\mathfrak{g}$-invariant, that is, $\forall w \in W, g \in \mathfrak{g},\ \rho(g)w \in W$. The homomorphism $\rho|_W : \mathfrak{g} \to \mathfrak{gl}(W)$ then defines a subrepresentation of ρ.*
- *V is an irreducible $\mathfrak{g}$-module if the only $\mathfrak{g}$-submodules of V are $\{0\} \subset V$ and V itself. If this is true, we say ρ defines an irreducible representation.*

Given our interest in representations will be solely on irreducible representations, we are able to make use of Schur's Lemma.

Definition A.1.3. *Let $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ denote representations of a Lie algebra $\mathfrak{g}$. A homomorphism between the modules V_1 and V_2 is given by a linear map*

$$\psi : V_1 \to V_2,$$

such that for all $v \in V_1$, $g \in \mathfrak{g}$

$$\rho_2(g_1)\psi(v) = \psi(\rho_1(g_1)v). \tag{A.1.1}$$

Lemma A.1.4 (Schur's Lemma)**.** *Any non-zero homomorphism between irreducible modules is an isomorphism.*

Corollary A.1.1. *If* V *is an irreducible and finite-dimensional module over* $\mathbb{C}$*, then any module endomorphism*

$$\psi : V \to V$$

is a multiple of the identity map.

Now we want to determine the finite dimensional, irreducible representations of $\mathfrak{su}(2)$.

Example A.1.1. *The Lie algebra* $\mathfrak{su}(2)$ *consists of the* 2×2 *complex matrices which are anti-hermitian with trace zero, and thus* $\mathfrak{su}(2)$ *defines a 3-dimensional Lie algebra. (We will be interested only in the complexification of this Lie algebra, and thus when we state* $\mathfrak{su}(2)$*, we will implicitly mean* $\mathfrak{su}(2, \mathbb{C})$*). Now letting* σ_i *denote the i-th Pauli matrix. A generating set of* $\mathfrak{su}(2)$ *is then given by* $\{t_1, t_2, t_3\} = \{-\frac{i}{2}\sigma_1, -\frac{i}{2}\sigma_2, -\frac{i}{2}\sigma_3\}$

$$t_1 = -\frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad t_2 = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad t_3 = -\frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{A.1.2}$$

The Lie bracket on this generating set then satisfies the following equations

$$[t_i, t_j] = \epsilon_{ijk} t_k,$$

where ϵ_{ijk} *denotes the structure constant, which for* $\mathfrak{su}(2)$ *is just the skew-symmetric Levi-Civita symbol.*

Now applying Definition 1.2.1, a representation of $\mathfrak{su}(2)$ is given by an $\mathfrak{su}(2)$-module V over $\mathbb{C}$ and a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that

$$[\rho(t_i), \rho(t_j)] = \epsilon_{ijk} \rho(t_k).$$

Theorem A.1.5. *There exists a unique (up to isomorphism) irreducible* $(k+1)$*-dimensional representation of the Lie algebra* $\mathfrak{su}(2)$*, which we will denote* $\rho_k : \mathfrak{g} \to \mathfrak{gl}(S^k)$*, where* $S^k = \mathbb{C}^{k+1}$*.*

Such an irreducible $(k+1)$-dimensional representation of $\mathfrak{su}(2)$ is given by

$$\rho_k(t_1) = -\frac{i}{2}\begin{bmatrix} 0 & k & 0 & \dots & 0 \\ 1 & 0 & k-1 & \ddots & \vdots \\ 0 & 2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & k & 0 \end{bmatrix}, \tag{A.1.3}$$

$$\rho_k(t_2) = -\frac{1}{2}\begin{bmatrix} 0 & k & 0 & \dots & 0 \\ -1 & 0 & k-1 & \ddots & \vdots \\ 0 & -2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & -k & 0 \end{bmatrix}, \tag{A.1.4}$$

$$\rho_k(t_3) = -\frac{i}{2}\begin{bmatrix} k & 0 & 0 & \dots & 0 \\ 0 & k-2 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -(k-2) & 0 \\ 0 & \dots & 0 & 0 & -k \end{bmatrix}. \tag{A.1.5}$$

The irreducibility of this representation can be seen by rewriting the representation in terms of the basis

$$\begin{aligned} E_k &= \rho_k(it_1 - t_2) = i(\rho_k(t_1) + i\,\rho_k(t_2)), \\ F_k &= \rho_k(it_1 + t_2) = i(\rho_k(t_1) - i\,\rho_k(t_2)), \\ H_k &= \rho_k(2it_3) = 2i\,\rho_k(t_3), \end{aligned} \tag{A.1.6}$$

where

$$[E_k, F_k] = H_k, \quad [H_k, E_k] = 2E_k, \quad [H_k, F_k] = -2F_k. \tag{A.1.7}$$

Then given the highest weight vector v_1 (that is, the vector satisfying $E_k v_1 = 0$), the set of elements $F_k^n v_1$ for all $n \in \mathbb{N}_{\leq k}$ have distinct eigenvectors of H_k, and thus span the module. Thus, there cannot exists a non-trivial submodule, and so this representation must be irreducible.

A.1.1 Tensor products and direct sums

We now want to introduce the concept of the tensor product and direct sum of $\mathfrak{g}$-modules.

Proposition A.1.6. *Let $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W)$ denote two representations of the Lie algebra $\mathfrak{g}$, and let $\{v_1, ..., v_p\}$ and $\{w_1, ..., w_k\}$ define bases for V and W respectively.*

1. *The tensor product of two representations is itself a representation. That is, $\rho : \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$ defines a representation of $\mathfrak{g}$, such that $\forall h \in \mathfrak{g}$*

$$\rho(t) \equiv \rho_V(t) \otimes 1_k + 1_p \otimes \rho_W(t).$$

Furthermore, the set $\{(v_m, w_n)\}_{m \in \mathbb{N}_{\leq p}, n \in \mathbb{N}_{\leq k}}$ defines a basis of $V \otimes W$.

2. *The direct sum of two representations is itself a representation. That is,* $\rho : \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ *defines a representation of* $\mathfrak{g}$*, such that* $\forall t \in \mathfrak{g}$

$$\rho(t) \equiv \rho_V(t) \oplus \rho_W(t).$$

Furthermore, the set $\{v_m, w_n\}_{m\in\mathbb{N}_{\le p}, n\in\mathbb{N}_{\le k}}$ *defines a basis of* $V \oplus W$*.*

A useful result to do with the irreducible representations of $\mathfrak{su}(2)$ which we use in Chapter 3, is the following:

Proposition A.1.7. *Let* $\rho_k : \mathfrak{su}(2) \to \mathfrak{gl}(S^k)$ *denote the irreducible,* $(k+1)$*-dimensional representation of* $\mathfrak{su}(2)$*. Then*

$$S^{k-1} \otimes S^1 \cong S^k \oplus S^{k-2}. \tag{A.1.8}$$

In order to prove this proposition, we will construct an explicit isomorphism.

Proof. Let $\{v_1, ..., v_k\}$ and $\{w_1, w_2\}$ denote bases of S^{k-1} and S^1 respectively, such that v_k and w_2 denote the highest weight vectors. That is, v_k and w_2 satisfy $E_k v_k = 0 = E_2 w_2$

From this, we can construct a basis of the representation $S^{k-1} \otimes S^1$, given by

$$\{(v_m, w_n)\}_{m\in\mathbb{N}_{\le k}, n\in\{1,2\}}.$$

Similarly, the representation $S^k \oplus S^{k-2}$ has basis given by

$$\{x_a, y_b\}_{a\in\mathbb{N}_{\le k+1}, b\in\mathbb{N}_{\le k-1}},$$

where $\{x_1, ..., x_{k+1}\}$ and $\{y_1, ..., y_{k-1}\}$ define a basis of S^k and S^{k-2} respectively.

Then we claim the following map defines an isomorphism between $S^k \oplus S^{k-2}$ and $S^{k-1} \otimes S^1$

$$\begin{aligned} f : S^k \oplus S^{k-2} &\to S^{k-1} \otimes S^1 \\ x_p \quad &\mapsto (v_p, w_1) + (v_{p-1}, w_2) \\ y_q \quad &\mapsto \frac{k-q}{q}(v_q, w_2) - (v_{q+1}, w_1), \end{aligned}$$

where $p \in \mathbb{N}_{\le k+1}$ and $q \in \mathbb{N}_{\le k-1}$

This is clearly an isomorphism as the map is bijective and satisfies equation (A.1.1). □

A.1.2 Casimir operator

Definition A.1.8. *Let* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *define a representation of* $\mathfrak{g}$*, and consider the generating set* $\{t_1, ..., t_n\}$ *of the Lie algebra* $\mathfrak{g}$*, with structure constant* f_{ijk} *given by*

$$[t_i, t_j] = f_{ijk} t_k.$$

The Casimir operator C of such a representation is given by

$$C(V) := \sum_{i,j} g_{ij}\rho(t_i)\rho(t_j), \tag{A.1.9}$$

where $g_{ij} = \frac{1}{2}\sum_{l,m} f_{ilm} f_{lmj}$.

Example A.1.2 (Casimir operator of $\mathfrak{su}(2)$)**.**

Let $\mathfrak{g} = \mathfrak{su}(2)$, and let $\rho_k : \mathfrak{su}(2) \to \mathfrak{gl}(S^k)$ denote the irreducible $(k+1)$-dimensional representation as given by (A.1.3)-(A.1.5).

Given the generating set $\{t_1, t_2, t_3\}$ of $\mathfrak{su}(2)$ as defined in (A.1.2), we know the structure constants are given by

$$f_{ijk} = \epsilon_{ijk}.$$

Then applying the definition of g_{ij}, the only non-zero values of g_{ij} occur when $i = j$ and these are given by

$$g_{11} = g_{22} = g_{33} = 1.$$

Therefore the Casimir operator is

$$C(S^k) = \rho_k(t_1)^2 + \rho_k(t_2)^2 + \rho_k(t_3)^2 = \rho_k(t_i)^2. \tag{A.1.10}$$

Now we want to determine the Casimir operator in the representation basis $\{E_k, H_k, F_k\}$. Using (A.1.6), we can determine the $\rho_k(t_i)$ in terms of this new basis. By then substituting these into the Casimir operator given by equation (A.1.10), we get

$$C(S^k) = -\frac{1}{4}(H_k^2 + 4F_kE_k + 2H_k). \tag{A.1.11}$$

Now we will prove the following claim:

Claim A.1.9.

$$C(S^k) = -\frac{k}{4}\,(k+2)\,. \tag{A.1.12}$$

Proof. First observe that $C(S^k)$ commutes with F_k, as

$$\begin{aligned} C(S^k)F_k &= -\frac{1}{4}(H_k^2F_k + 4F_kE_kF_k + 2H_kF_k) \\ &= -\frac{1}{4}(H_kF_kH_k + 4F_k^2E_k + 4F_kH_k) \\ &= -\frac{1}{4}(F_kH_k^2 - 2F_kH_k + 4F_k^2E_k + 4F_kH_k) \\ &= F_kC(S^k). \end{aligned}$$

Now we know that $\{v, F_1 v, ..., F_k v\}$ defines a basis for S^k such that $E_k v = 0$ and $H_k v = kv$. Acting the Casimir operator on these basis vectors, we get

$$\begin{aligned} C(S^k)F_k^i v &= F_k^i C(S^k)v \\ &= F_k^i \left(-\frac{1}{4}(H_k^2 + 4F_k E_k + 2H_k)v\right) \\ &= F_k^i \left(-\frac{1}{4}(k^2 + 2k)v\right) \\ &= -\frac{k}{4}(k+2)F_k^i v. \end{aligned}$$

Given any element of S^k can be written as a linear combination of the basis elements $F_k^i v$, and using the fact that Casimir operator is clearly linear, we get that

$$C(S^k) \equiv -\frac{k}{4}(k+2).$$

□

Proposition A.1.10. *Let $\rho_k : \mathfrak{su}(2) \to \mathfrak{gl}(S^k)$ denote the irreducible, $(k+1)$-dimensional $\mathfrak{su}(2)$ representation, and $\rho : \mathfrak{su}(2) \to \mathfrak{gl}(S^{k-1} \otimes S^1)$ the tensor product representation of $\mathfrak{su}(2)$. Then*

$$C(S^{k-1} \otimes S^1) = C(S^{k-1}) \otimes 1_2 + 2\rho_{k-1}(t_i) \otimes t_i + 1_k \otimes C(S^1).$$

Before we prove this proposition, we will first prove the following claim:

Claim A.1.11. *Let $\rho_{k-1} : \mathfrak{su}(2) \to \mathfrak{gl}(S^{k-1})$ denote the irreducible, k-dimensional $\mathfrak{su}(2)$ representation. Then*

$$(\rho_{k-1}(t_i) \otimes 1_2)(1_k \otimes \rho_1(t_i)) + (1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2) = 2\rho_{k-1}(t_i) \otimes t_i. \tag{A.1.13}$$

Proof. We know that $(1_k \otimes \rho_1(t_i))$ is a square matrix which is block diagonal with size 2×2 blocks, and that all of these diagonal blocks are all equal to $\rho_1(t_i)$. This means that $(1_k \otimes \rho_1(t_i))$ will commute with another block matrix, A, with size 2×2 blocks if and only if all of the individual 2×2 blocks of A commute with $\rho_1(t_i)$.

Now we know that $(\rho_{k-1}(t_i) \otimes 1_2)$ is a block matrix with size 2×2 blocks which are each some multiple of the identity matrix. Therefore all of these individual blocks will commute with $\rho_1(t_i)$. Thus these two matrices must commute, and so we get

$$(1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2) = (\rho_{k-1}(t_i) \otimes 1_2)(1_k \otimes \rho_1(t_i)).$$

Now we want to show that $(1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2) = (\rho_{k-1}(t_i) \otimes \rho_1(t_i))$. Focusing on the LHS, we know from our previous argument that multiplying $(\rho_{k-1}(t_i) \otimes 1_2)$ by $(1_k \otimes \rho_1(t_i))$ will just multiply all the individual 2×2 blocks of $(\rho_{k-1}(t_i) \otimes 1_2)$ by $\rho_{k-1}(t_i)$. Given the blocks of $(\rho_{k-1}(t_i) \otimes 1_2)$ are just a constant multiple of the identity, we know that the individual blocks of $(1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2)$ are now given by the same constant times $\rho_1(t_i)$. But this defines exactly the matrix $(\rho_{k-1}(t_i) \otimes \rho_1(t_i))$, which is what we wanted.

Applying these results, and using the fact that $\rho_2(t_i) = t_i$ we get

$$\begin{aligned}(\rho_{k-1}(t_i) \otimes 1_2)(1_k \otimes \rho_1(t_i)) + (1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2) &= 2(\rho_{k-1}(t_i) \otimes 1_2)(1_k \otimes \rho_1(t_i)) \\ &= 2\rho_{k-1}(t_i) \otimes \rho_1(t_i) \\ &= 2\rho_{k-1}(t_i) \otimes t_i.\end{aligned}$$

□

Proof of Proposition A.1.10.
The Casimir operator for the representation $\rho_{k-1} : \mathfrak{su}(2) \to \mathfrak{gl}(S^{k-1})$ of $\mathfrak{su}(2)$ is

$$C(S^{k-1}) = \rho_{k-1}(t_i)^2.$$

Now applying the result from Claim A.1.11 and the definition of the tensor product of two $\mathfrak{su}(2)$-representations, we get

$$\begin{aligned}C(S^{k-1} \otimes S^1) &= \rho(t_i)\rho(t_i) \\ &= (\rho_{k-1}(t_i) \otimes 1_2 + 1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2 + 1_k \otimes \rho_1(t_i)) \\ &= \rho_{k-1}(t_i)^2 \otimes 1_2 + (\rho_{k-1}(t_i) \otimes 1_2)(1_k \otimes \rho_1(t_i)) + (1_k \otimes \rho_1(t_i))(\rho_{k-1}(t_i) \otimes 1_2) \\ &\quad + 1_k \otimes \rho_1(t_i)^2 \\ &= C(S^{k-1}) \otimes 1_2 + 2\rho_{k-1}(t_i) \otimes t_i + 1_k \otimes C(S^1).\end{aligned}$$

□

Appendix B

Root systems of a Lie algebra

Let $\mathfrak{g}$ denote the Lie algebra of some Lie group G. We want to introduce the concept of a root system and weight space.

Now any Lie algebra $\mathfrak{g}$ may be decomposed via the *triangle decomposition* given by

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-,$$

where $\mathfrak{g}_0$ denotes the *Cartan subalgebra* of $\mathfrak{g}$, and $\mathfrak{g}_\pm$ the eigenvectors with respect to the Cartan subalgebra, such that

- $[\mathfrak{g}_0, \mathfrak{g}_0] = 0$.
- $[\mathfrak{g}_0, \mathfrak{g}_\pm] = \mathfrak{g}_\pm$.
- There exists an homomorphism $\gamma : \mathfrak{g}_+ \to \mathfrak{g}_-$ that is invertible and which reverses the Lie bracket, that is
$$\gamma([x, y]) = [\gamma(y), \gamma(x)].$$

B.1 Root system

Definition B.1.1. *A root vector of the Lie algebra $\mathfrak{g}$ is given by an element*

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, [H, x] = \alpha(H)x,\, \forall H \in \mathfrak{g}_0,\, \alpha \neq 0\}.$$

The root of the root vector $\mathfrak{g}_\alpha$ is given by $\alpha : \mathfrak{g}_0 \to \mathbb{C}$ which take values in the dual space of the Cartan subalgebra, $\mathfrak{g}_0^*$.

Definition B.1.2. *A root system on some vector space V consists of a finite set, which we will denote $\Phi(V)$, of non-zero vectors called roots such that*

- *The roots span V*
- *Given $x \in \Phi$, $ax \in \Phi$ iff $a = -1, 1$.*

- *If* $x, y \in \Phi$, $y - 2\frac{x \cdot y}{x^2} x \in \Phi$.
- *If* $x, y \in \Phi$, $2\frac{x \cdot y}{y \cdot y} \in \mathbb{Z}$.

Lemma B.1.3. *The roots of a Lie algebra define a root system.*

B.1.1 Dual root system

Definition B.1.4. *The coroot* $\alpha^\vee \in \Phi^\vee(\mathfrak{g})$ *of the root* $\alpha \in \mathfrak{g}^*$ *is given by*

$$\alpha^\vee = \frac{2\alpha}{\alpha^2},$$

where here the inner product is defined naturally by the *Killing form* of the Lie algebra.

Lemma B.1.5. *The set of coroots* $\Phi^\vee(\mathfrak{g})$ *is given by the set of elements* $x \in \mathfrak{g}$ *such that*

$$x \cdot \alpha \in \mathbb{Z}, \quad \forall \alpha \in \Phi(\mathfrak{g})$$

Lemma B.1.6. *The set of coroots* $\Phi^\vee$ *define a root system.*

We are now going to assume the existence of some *dual* Lie algebra $\mathfrak{g}^\vee$ such that the root system of $\mathfrak{g}^\vee$ is given by

$$\Phi(\mathfrak{g}^\vee) = \{\alpha^\vee | \alpha \in \Phi(\mathfrak{g})\}.$$

Such a dual Lie algebra always exist, due to the existence of the *Langlands dual*.

B.2 Weight spaces

Now we are going to introduce the abstract definition of a weight.

Definition B.2.1. *Let* $\Phi^\vee(\mathfrak{g})$ *denote the set of coroots of the Lie algebra* $\mathfrak{g}$. *The weight space of* $\mathfrak{g}$ *is given by the set of elements* $\lambda \in \mathfrak{g}^*$ *such that*

$$\lambda \cdot \alpha^\vee \in \mathbb{Z}, \quad \forall \alpha^\vee \in \Phi^\vee(\mathfrak{g})$$

Proposition B.2.2. *The weight space of the dual Lie algebra* $\mathfrak{g}^\vee$ *is equivalent to the space of coroots in* $\mathfrak{g}$.

Proof.

$$\begin{aligned} \Lambda(\mathfrak{g}^\vee) &= \{w | w \cdot (\alpha^\vee)^\vee \in \mathbb{Z}\} \\ &= \{w | w \cdot \frac{2\alpha^\vee}{(\alpha^\vee)^2} \in \mathbb{Z}\} \\ &= \{w | w \cdot \alpha \in \mathbb{Z}\} \\ &= \Phi^\vee(\mathfrak{g}). \end{aligned}$$

Thus, the coroots of the Lie algebra $\mathfrak{g}$ are equivalent to the weights of the dual Lie algebra $\mathfrak{g}^\vee$. □

Appendix C

Negative terms from Section 4.1.1

C.1 The function $\gamma(n, a)$

We want to prove the claim given in Section 4.1.1, associated to the function

$$\gamma(n, a) = n^2 - 2na - 4n + 2a + 3 \leq 0.$$

We observe that if we fix n and continue to increase a, given $\gamma(n, a)$ is a decreasing function in the variable a, there will exist some $m \in \mathbb{Z}$ such that $\forall a \geq m$, $\gamma(n, a) \leq 0$. We will now determine these bounds.

Lemma C.1.1 (Lemma C.1)**.** *Let $n = 2p$, where $p \in \mathbb{N}$. Then $\forall\, a \geq p - 1$,*

$$\gamma(n, a) = n^2 - 2na - 4n + 2a + 3 \leq 0$$

and $\gamma(n, p) = -3n + 3$.

Proof. We will prove the following lemma by induction.

For the case $n = 2$

$$\gamma(2, a) = -2a - 1 \leq 0, \;\; \forall a \geq 0.$$

Furthermore $\gamma(2, 1) = -3 = -3(2) + 3$.

Now assume the lemma is true for $n = 2q$.

Then for the case $n = 2(q + 1)$, let $a \geq q$. Under our assumption, we get

$$\begin{aligned}
\gamma(2(q+1), a) &= ((2q)^2 - 2(2q)a - 4(2q) + 2a + 3) + 8q - 4a - 4 \\
&\leq (-6q + 3) + 8q - 4a - 4 \\
&= 2q - 4a - 1 \\
&\leq 2q - 4q - 1 \\
&= -2q - 1 \\
&\leq 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}\gamma(2(q+1), q+1) &= -6q-3 \\ &= -3(2(q+1))+3.\end{aligned}$$

□

C.2 Adjoint of the operator $\left(z\frac{d}{dz}+\frac{1}{2}\right)$

We will now prove the following result which is also used in Section 4.1.1.

Lemma C.2.1.

$$\left(z\frac{d}{dz}+\frac{1}{2}\right)^{\dagger} = -\left(z\frac{d}{dz}+\frac{1}{2}\right). \tag{C.2.1}$$

Proof. Let $v, w \in V$ or W. Then

$$\begin{aligned}\left\langle v, \left(z\frac{d}{dz}+\frac{1}{2}\right) w\right\rangle &= \int_{-1}^{1} v^{\dagger}(z) z\,\partial_z w(z) dz + \frac{1}{2}\int_{-1}^{1} v^{\dagger}(z) w(z) dz \\ &= \left(\int_0^{\infty} \partial_z (v^{\dagger}(z) z\, w(z)) dz - \int_{-1}^{1} \partial_z v^{\dagger}(z) z\, w(z) dz - \int_0^{\infty} v^{\dagger}(z)\, w(z) dz\right) \\ &\quad + \frac{1}{2}\int_0^{\infty} v^{\dagger}(z) w(z) dz \\ &= -\int_0^{\infty} z\,\partial_z v^{\dagger}(z)\, w(z) dz - \frac{1}{2}\int_0^{\infty} v^{\dagger}(z)\, w(z) dz \\ &= \left\langle -\left(z\frac{d}{dz}+\frac{1}{2}\right) v, w\right\rangle\end{aligned}$$

where we have applied integration by parts to obtain the first line and then gone on to apply the requirement that on the boundary, $v(z) = 0 = w(z)$.

Thus, by the definition of the adjoint, we get exactly equation (C.2.1). □

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