Theoretical and Econometric Models of Behavioural Decision-making: Limited Attention, Framing Effects and Online Search

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A thesis submitted for the degree of

Doctor of Philosophy of

The Australian National University

September 2019

Declaration

This thesis is based on research undertaken between January 2016 and September 2019 at the Research School of Economics, Australian National University, Canberra, Australia. The material presented in this thesis is, to the best of my knowledge, original, and was not jointly authored.

Peter J Gibbard.

Peter Gibbard, 12 September, 2019.

Acknowledgements

I am particularly grateful for the guidance and insightful input of my primary supervisor Simon Grant, as well as helpful feedback from other members of my dissertation panel, Ronald Stauber and Idione Meneghel. My research also benefitted from the comments of various members of the academic staff at the RSE at the ANU, including, but not limited to, Tue Gorgens, Fedor Iskhakov, Tim Kam, Ruitian Lang and John Stachurski. I received helpful feedback on material in Chapter 2 from participants at the Australian Economic Theory Workshop in February, 2019, especially James Taylor and Matthew Ryan. In addition, my thesis benefitted in a number of respects from the instructive comments of my three anonymous examiners. Finally, I'd like to thank Rohan Alexander not only for discussions of, and comments on, my research, but also for his friendship during the PhD program.

Abstract

This thesis comprises three papers. The first paper (Chapter 2) presents a theoretical model of choice under limited attention. In this model, the decision-maker (DM) forms a consideration set, from which she chooses her most preferred alternative. Both preferences and consideration sets are stochastic. We present axiomatisations for this model. Our focus is on the following identification question: to what extent can an observer retrieve probabilities of preferences and consideration sets from observed choices? Our first conclusion is a negative one: if the observed data is choice probabilities, the model is not identified – probabilities of preferences and consideration sets cannot be retrieved from choice probabilities. To solve the identification problem, we assume that an "enriched" dataset is observed. This dataset includes choice probabilities under two frames. Given this enriched dataset, the identification problem can be solved. The intuition is that the observed data reveals how a change in attention causes a change in choice, holding preferences constant. This allows us to isolate the influence of attention on choice.

The second paper (Chapter 3) develops a theoretical model for a broad class of framing effects, which we call "resolution framing effects". In this model, a frame influences a DM's choice when she is indecisive. A DM is said to be indecisive between two alternatives when she does not have a strict preference for one over the other. A frame helps to "resolve" her indecision. Our model explains frame-dependent choices associated with a range of framing effects, including the salience effect, the endowment effect, the compromise effect and the priming effect. Axiomatisations and revealed preference relations are obtained for this broad class of framing effects. Our paper provides a response to commentators who have argued that framing effects are (i) evidence of irrationality (ii) inconsistent with the existence of a stable preference relation that is revealed in choice. In our model, frame-dependent choices are generated by a rational decision-process that is governed by a stable preference relation, which is revealed in observed choices.

The third paper (Chapter 4) presents a structural econometric model of ordered online search. Search is said to be "ordered" if alternatives are presented in a particular order. The data is a set of searches for hotels on Expedia.com, which includes searches where orderings were random rather than generated by Expedia's ranking algorithm. Even for searches with random orderings, there is a "falling purchase rate" – that is, the propensity to purchase is low at low positions in the ordering. To explain the falling purchase rate, we assume that consumers have "declining expectations":

consumers expect that, for the alternatives not yet browsed, alternatives higher in the ordering are better. Declining expectations explain the falling purchase rate, because they ensure that consumers may abandon a search part-way through. The declining expectation model generates a fall in the purchase rate similar to that in the data. Our model is an adaption of McCall's job search model. At least for modelling ordered consumer search, McCall's model is more appropriate than the commonly-used Weitzman search model.

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Chapter 1

Introduction

This dissertation comprises three papers on behavioural decision-making. The first is a theoretical model of limited attention, the second a theoretical model of framing effects and the third a structural econometric model of ordered online choice. While each of the papers is self-standing, there are several themes that cut across the three papers. In the introduction, we will first present a summary of each paper, and then highlight the themes that emerge from the dissertation.

Chapter 2 presents a theoretical model of choice under limited attention. In this model, the decision process has two stages: in the first stage, the decision-maker (DM) forms a consideration set; and in the second stage, she selects the most preferred alternative from her consideration set. Both preferences and consideration sets are assumed to be stochastic. While we present axiomatisations for our model, the focus of the paper is on the following identification question: to what extent can an observer retrieve the probabilities of preferences and consideration sets from the choice data? Our first conclusion is a negative one: if the observed data are choice probabilities, our model is not identified, in the sense that probabilities of preferences and consideration sets cannot be retrieved from choice probabilities. In order to solve the identification problem, we assume that the observer has access to an "enriched" dataset, which includes observations of choice probabilities under two distinct frames. Our main result is that, given this enriched dataset, the identification problem can be solved. The intuition is that the observed data reveal how a change in attention causes a change in choice, holding preferences constant. That is, it describes the influence of attention on choice, controlling for preferences. This allows us to isolate the influence of attention on choice.

Chapter 3 develops a theoretical model for a broad class of framing effects, which we call "resolution framing effects". In this model, a frame influences a DM's choice when she is indecisive between alternatives. A DM is said to be indecisive between two alternatives when she does not have a strict preference for one over the other. A frame helps to "resolve" her indecision. Our model explains frame-dependent choices associated with a range of well-known framing effects, including the salience effect, the endowment effect, the compromise effect and the priming effect. Axiomatisations and revealed preference relations are obtained for this broad class of framing effects. Our paper provides a response to a number of commentators who have argued that framing effects are (i) evidence of irrationality (ii) inconsistent with the existence of a stable preference relation that is revealed in choice. In our model, frame-dependent choices are generated by a rational decision-process that is governed by a stable preference relation, which is revealed in observed choices.

Chapter 4 presents a structural econometric model of ordered online search. Search is said to be "ordered" if alternatives are presented in a particular order. The data are a set of online searches for hotels on the Expedia website, which includes searches where the orderings were random rather than generated by Expedia's ranking algorithm. Even for searches with random orderings, there is a "falling purchase rate" – that is, the propensity to purchase is low at low positions in the ordering. To explain the falling purchase rate, our model assumes that consumers have "declining expectations": given their search experience, consumers expect that, for the alternatives that they have not yet browsed, alternatives higher in the ordering are better. Declining expectations explain the falling purchase rate, because they ensure that consumers may abandon a search part-way through. Our model with declining expectations generates a fall in the purchase rate of a magnitude similar to that in the data. If, instead, expectations are assumed constant, the magnitude of the fall in the purchase rate cannot be explained. Our model is an adaption of McCall's job search model. The distinguishing features of our model are that it (i) is applied to consumer search (ii) is fully structural and (iii) incorporates declining expectations. For consumer search, the Weitzman search model is far more commonly used than the McCall model, but we argue that, at least for ordered consumer search, McCall's model is more appropriate.

There are several themes that emerge from the papers in this dissertation. The first relates to the notion of rationality or optimisation in behavioural economics. Behavioural economics is sometimes described as the study of choice dispositions in which DMs fall short of rationality, or depart from optimising behaviour. This view is encapsulated by the title of the popular book on behavioural economics by Ariely (2009), *Predictably Irrational*. In contrast, our view is that, of the class of choice dispositions typically labelled as "behavioural", a considerable sub-class can, in fact, be modelled as optimising decision-making – although the optimising may be constrained by limited attention, information costs and so forth. This sub-class of behavioural choice dispositions can therefore be described as "rational", or, at least, "boundedly rational". For all of the models presented in this thesis, the decision process is rational and optimising. Chapter 2 examines departures from neoclassical economics that arise when DMs maximise utility subject to the constraint of limited attention. Chapter 3 shows that "resolution framing effects" are consistent with rational decision-making. Finally, Chapter 4 shows that, whereas "satisficing" behaviour is often presented as a departure from optimising decision-making, it may, in fact, arise when DMs maximise utility in the presence of search costs.

A second theme in this dissertation relates to the role of revealed preferences in behavioural economics. In neoclassical economics, under certain well-known assumptions about choice behaviour, a revealed preference relation can be retrieved from observed choices. Revealed preferences have a central role in neoclassical economics, underpinning choice-based welfare analysis. As noted in Chapter 3, a number of commentators have suggested that, for at least some of the choice dispositions highlighted by behavioural economics, we are unable to retrieve a DM's preferences from her observed choices. In other words, the findings of behavioural economics are claimed to undermine the notion of revealed preferences. By way of response, Chapter 3 presents a model in which, even though the DM's choices are influenced by frames, nevertheless, at least for a broad range of framing effects, a revealed preference relation can be retrieved from observed choices. Further, Chapter 2 develops a model in which, even if decision-making is constrained by limited attention, information about the DM's preferences can be retrieved from her observed choices. In both Chapters 2 and 3, these revealed preference results are obtained by using "frame-dependent choice functions". That is, the models assume that choice observations are available under various different frames. While a number of other studies have employed frame-dependent choice functions in axiomatisations, the use of such choice functions is still somewhat infrequent. It would be potentially fruitful to consider further applications of frame-dependent choice functions in theoretical models of behavioural economics.

To summarise: while acknowledging the challenges to neoclassical economic theory coming from behavioural economics, this dissertation also emphasises the potential for continuity between behavioural economics and more traditional economic theory. The axiomatisations for the models of limited attention and framing effects presented in Chapters 2 and 3 are, after all, counterparts of neoclassical axiomatisations. Further, in this dissertation, DMs are modelled as making optimising decisions, albeit in the presence of limited attention and search costs. Finally, this dissertation illustrates that revealed preferences can be retrieved from choices not only in neoclassical models but also in at least some models of behavioural decision-making.

Chapter 2

Disentangling preferences and limited attention: Random-utility models with consideration sets

2.1 Introduction

When an observer relaxes the assumption that a decision-maker (DM) has full attention, she may be unable to infer the DM's preferences from observed choices. To take a simple example, suppose that a DM faces a choice of whether or not to switch from her existing pension plan to a new pension plan. Given an assumption of full attention, the DM's choice can be interpreted as revealing her preference: the observer attributes to the DM a preference for her existing plan over the new plan if she sticks to her existing plan. Once the observer allows for the possibility of limited attention, however, such an inference may be unwarranted. There are at least two hypotheses consistent with an observation that the DM chose her existing plan: first, it may be that the DM prefers the existing plan; but second, it may be that the DM simply failed to *consider* fully the new plan. That is, the new plan was not in her *consideration set*. This simple example illustrates the identification problem that may arise in the presence of limited attention: an observer may be unable to disentangle the influence, on a DM's choices, of preferences and limited attention.

In a consideration-set model, a DM's decision has two stages. In the first stage, she forms a consideration set, which is the subset of the choice set that she has considered – that she has paid attention to. In the second stage, she selects the most preferred alternative from those alternatives in her consideration set (together with, potentially, a default option). Since the 1970s, consideration-set models have been used frequently in applied research – especially in the marketing literature – but, more recently, economic theorists have introduced consideration sets into axiomatisations of choice. In their seminal article, Masatlioglu et al. (2012) presents an axiomatisation of a consideration-set model that highlights the identification problem which, at least potentially, arises in the presence of limited attention. They find that, in the absence of restrictions on the DM's dispositions to form consideration sets, an observer is unable to retrieve from the DM's choices any information about her preferences. Accordingly, they suggest a plausible restriction on the DM's dispositions to form consideration sets. Assuming that this restriction holds, an observer may be able to draw some inferences about the DM's preferences from her choices. Nevertheless, even if such a restriction is imposed, in general, the observer will be unable to retrieve the DM's entire preference relation, and, indeed, may still be unable to infer *any* information about preferences.

In contrast to Masatlioglu et al. (2012), we model the DM's preferences and her dispositions to form consideration sets as stochastic. The DM's preferences are assumed to be generated by a probability distribution in a fashion akin to that of random-utility models. Further, for each choice set, the DM's consideration set from that choice set is generated by a probability distribution over the subsets of that choice set. The distributions of probabilities over preferences and probabilities over consideration sets determine the DM's observed choice probabilities. We focus, in particular, on the identification question that arises in this stochastic version of the consideration-set model. For such a model, we will say that it is "fully identified" if, from the observed choice probabilities, we can retrieve (i) the preference probabilities (to the same extent as in a standard full-attention random-utility model) and (ii) the consideration-set probabilities. Our first conclusion is a negative one: in Section 2.3, we show that, even if we impose strong assumptions on the consideration-set probabilities, typically, consideration-set models are not fully identified. That is, for our stochastic consideration-set model, an identification problem arises akin to that highlighted by Masatlioglu et al. (2012) for the deterministic case.

In order to solve the identification problem, we suggest that one strategy is to use an enriched dataset. Accordingly, in Section 2.4, we develop a model in which the choice dataset includes observations on the DM's choices under several different *frames* . We refer to this enriched dataset as a *frame-dependent choice function*. The recent theoretical literature on behavioural decision theory includes numerous examples of frame-dependent choice functions. For instance, in Masatlioglu and Ok (2005) and Masatlioglu and Ok (2014), the frame is the DM's initial endowment; in their models, the frame-dependent choice function records the DM's choices under various different specifications of the endowment. A *framing effect* arises if a change of frame alters the DM's choices. Our paper is concerned with a particular variety of framing effect, which we term an *attention effect*. An attention effect arises if a change in frame alters the DM's choices by influencing her attention – by changing her consideration sets. For instance, Caplin and Dean (2011) present a model of a frame-dependent choice function in which the frame is the time-period of contemplation prior to the decision. In their model, a change in the period of contemplation potentially alters the DM's choices by changing her consideration set. Attention effects may also include changes in attention arising from (a) moving some grocery item to the lowest shelf in a grocery store (b) changing the rankings in a search engine for an online bookstore and (c) creating a "display" in a retail store for a particular retail item. In contrast to Caplin and Dean (2011), however, the models that we develop do not assume that the attention effect is of a specific kind – our models are applicable to *any* attention effect. The assumption is simply that, while the DM's dispositions to form consideration sets differ under different frames, her preferences are unchanged.

Our central contribution is a finding that, given an assumption that the observer has access to such an enriched dataset, then, at least potentially, the identification problem can be solved. Our principal model, which is presented in Section 2.4, is a model without a default option. We show that the model is fully identified. This identification result is of particular interest, because, as we will argue, Section 2.4 presents a "natural" way of modelling consideration sets in the absence of a default option. Moreover, Section 2.5 and Appendix 2 present two variants of this model, which are also fully identified. Furthermore, whereas for these three models, we obtain closed-form solutions to the identification problem, there are at least some cases in which our identification strategy can be employed when no closed-form solution is available. For example, Appendix 4 presents a model in which numerical methods are used to solve the identification problem.

What is the intuition for these identification results? After all, in the simple example of the choice between pension funds, the identification problem is presented in stark terms. How does the introduction of a frame-dependent choice function allow an observer to disentangle the influence on choice of preferences and consideration sets? A frame-dependent choice function describes how a change in attention causes a change in choice, *holding preferences constant*. In other words, it describes the influence of attention on choice, *controlling for preferences*. This allows us to isolate the influence of attention on choice. The intuition can also be couched in mathematical terms. In our framework, identification is the task of solving a system of equations (the rationalisation conditions) to obtain the unknowns (preference probabilities and consideration-set probabilities). When we do not make use of frame-dependent choice functions, an identification problem arises because the number of unknowns typically exceeds the number of equations. When we introduce frame-dependent choice functions, the number of equations increases, potentially solving the identification problem.

While this strategy for solving the identification problem is successful for some models, it is not universally successful. In Appendix 3, we present a model that is not fully identified, despite the use of a frame-dependent choice function. Roughly speaking, identification fails in this model because, whereas the frame-dependent choice function supplies additional equations, many of the new equations do not impose any further *effective* constraints on the unknowns. An analogy can be made to the case of linear systems of equations: if a linearly dependent equation is added to a system of linear equations, then it does not impose any further *effective* constraints on the unknowns. In contrast, for the models presented in Sections 4.4 and 4.5, and Appendices 2 and 4, the frame-dependent choice function supplies sufficient additional effective constraints to solve for the unknowns.

As noted above, if we do not employ the richer dataset described by a frame-dependent choice function, typically an identification problem will arise. But how does this finding square with the results in the important paper by Manzini and Mariotti (2014a)? After all, they present a consideration-set model that does not make use of frame-dependent choice functions; however, in their model, consideration-set probabilities and preferences can be retrieved from observed choice probabilities. Their model makes two restrictive assumptions that we do not make. In Section 2.6, we present, in some detail, our rationale for departing from the two assumptions in Manzini and Mariotti (2014a), which can be summarised as follows. The first difference is that, whereas in our models, the default option may be chosen even if other alternatives are considered, in their model, the default option is assumed to be chosen *only if* none of the non-default alternatives are considered. In Section 2.6, we argue that, in certain specific examples, there may be evidence that warrants such an assumption, but it would not be expected to hold in general. For instance, in modelling the choice between an existing pension plan and new pension plans, it would be restrictive to impose on the model an assumption that the DM chooses the existing plan *only if* she does not consider any new plans. The restriction is that the model rules out the possibility that the DM considers at least some new plans but nevertheless chooses the existing plan. This is a cause for concern because, as we show in Section 2.6, the ability to retrieve consideration set probabilities follows immediately from the restrictive assumption. Further, the assumption is out of kilter with the empirical literature on consideration sets, which typically does not assume that the default option is selected only when the DM fails to consider other alternatives.

The second principal difference is that, whereas in our models both preferences and consideration sets are random, Manzini and Mariotti (2014a) specify consideration sets to be random but preferences to be deterministic – in other words, they treat preferences and consideration sets asymmetrically. Whereas, in certain specific examples, there may be evidence that consideration sets are "more random" than preferences, this asymmetric treatment of preferences and consideration sets is, in general, unmotivated. In Section 2.6, we present an example that highlights the restrictive character of this assumption. Whereas, as is described in Section 2.3, our paper has benefitted considerably from the insights in Manzini and Mariotti (2014a), we depart from their model by relaxing the two restrictive assumptions described above.

While our paper focusses on attention effects, the question it addresses is an instance of a more general question arising for *any* framing effect. The literature on behavioural economics has posed the following challenge for neo-classical economics: do framing effects undermine the ability to retrieve revealed preferences from choices and, therefore, undermine standard choice-based welfare economics? For instance, this challenge is explicitly issued in Ariely et al. (2003). Responding to this challenge, Salant and Rubinstein (2008) and Bernheim and Rangel (2009), as well as Chapter 3 of this thesis, present models with frame-dependent choice functions, in which preferences can be retrieved from the choice data. The current chapter can be interpreted as responding to the same challenge, but focussing on the case in which the framing effect is an attention affect. It is notable that the strategy of our response is, broadly speaking, the same as that in Salant and Rubinstein (2008), Bernheim and Rangel (2009) and Chapter 3 of this thesis. The idea is that, despite the presence of framing effects, preferences may be retrievable from choices, so long as we use an enriched dataset comprising a frame-dependent choice function.

2.2 Random-utility models: Full-attention benchmarks

2.2.1 Model without a default option

Our paper introduces consideration sets into the standard random-utility model, axiomatisations for which are provided by Falmagne (1978) and McFadden and Richter (1990). Accordingly, the standard random-utility model can be interpreted as a "full attention benchmark" for our consideration-set models: we will say that a consideration-set model faces an identification problem only if *less* information about preference probabilities is revealed by choice probabilities than in the random-utility model. Before turning to consideration-set models, therefore, we need to examine identification results for the random-utility model. We focus on the Falmagne (1978) axiomatisation because, as we will see, his axiomatisation is relatively intuitive.

The set of alternatives is assumed to be a finite non-empty set $X = \{x_1, \ldots, x_m\}$. The "set of all choice sets" is denoted $\mathcal{B} := 2^X \backslash \{\emptyset\}$. For any choice set $B \in \mathcal{B}$ and any alternative $x \in B$, a standard choice function specifies the probability of x being selected from B.

Definition 1 (*Standard choice function*)
A *standard choice function* is a mapping $c : \{(x, B) : B \in \mathcal{B}, x \in B\} \to [0, 1]$ such that for any $B \in \mathcal{B}$:

$$\sum_{x \in B} c(x, B) = 1$$

For any set of alternatives Y, a *ranking on* Y is a transitive, asymmetric and complete binary relation on Y.[1] Let $\mathcal{R}_Y$ denote the set of all rankings on Y. The DM is assumed to possess not a deterministic ranking, but rather a probability measure over the various subsets of rankings, which we will call a "ranking assignment".

[1]We say that binary relation $\succ$ is (i) *transitive* if $x \succ y$ and $y \succ z$ entails that $x \succ z$; (ii) *asymmetric* if $x \succ y$ entails that not $y \succ x$ and (iii) *complete* if $x \neq y$ entails that either $x \succ y$ or $y \succ x$.

Definition 2 (*Ranking assignment*)

A *ranking assignment on X*, denoted π, is a probability measure on the event space $(\mathcal{R}_X, 2^{\mathcal{R}_X})$.

It will be useful to introduce notation for various sets of rankings that are of particular interest. For any alternative x and any set of alternatives B that contains x, one set of rankings that is of particular interest is the set of rankings such that x is the most preferred member of B. We use the following notation to refer to such a set.

Definition 3 (*The event that alternative x is the maximum alternative in set B*)

For any non-empty $B \subset X, x \in B$, let $\mathcal{M}_{x,B} \subset \mathcal{R}_X$ denote the set of rankings given by $\mathcal{M}_{x,B} = \{\succ \in \mathcal{R}_X : \forall y \in B \backslash \{x\}, x \succ y\}$.

Using this notation, we can define a notion of rationalisation.

Definition 4 (*Stochastic rationalisation*)

Let c be a standard choice function. A ranking assignment on X, π, is said to *stochastically rationalise* c if, for every $B \in \mathcal{B}, x \in B$, we have $c(x, B) = \pi(\mathcal{M}_{x,B})$.

In his axiomatisation of the random-utility model, Falmagne (1978) defines the notion of the "Block-Marschak" polynomials. In our adaption of the Falmagne axiomatisation, we use a very similar object, which (for reasons explained below) we term the "random-utility contour function".

Definition 5 (*Random-utility contour function*)

Consider any standard choice function c. The *random-utility contour function* for c is a mapping $\kappa : \{(x, A) : A \subsetneq X, x \in X \backslash A\} \to \mathbb{R}$ such that:

$$\kappa(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} c(x, X \backslash B)$$

The Falmagne axiomatisation can then be written as follows:

Theorem 1 (*Characterisation: Random-utility model*)

Consider any standard choice function c. There exists a ranking assignment on X that stochastically rationalises c if and only if the random-utility contour function for c, κ, is non-negative.

In contrast to the axiomatisation of McFadden and Richter (1990), Falmagne's axiomatisation is relatively intuitive. As noted by Barbera and Pattanaik (1986, p.713), the random-utility contour function κ can be interpreted as follows: for an alternative x and a set A that does not contain x, $\kappa(x, A)$ can be interpreted as the revealed probability that A is the upper contour set of x. In order to formalise this observation of Barbera and Pattanaik (1986), it will be helpful to introduce the following notation that refers to "the set of rankings on which A is the upper contour set of x".

Definition 6 (*The event that A is the upper contour set of alternative x*)
For any $A \subsetneq X$ and $x \in X \backslash A$, let $\mathcal{U}_{x,A} \subset \mathcal{R}_X$ denote the set of rankings given by $\mathcal{U}_{x,A} := \{\succ \in \mathcal{R}_X : \forall y \in X, y \in A \iff y \succ x\}$.

The observation of Barbera and Pattanaik (1986) can then be formalised as follows:[2]

Theorem 2 (*Identification: Random-utility model*)
Consider any standard choice function c and any ranking assignment on X, π. If π stochastically rationalises c, then π is such that, for any $A \subsetneq X, x \in X \backslash A$, $\pi(\mathcal{U}_{x,A}) = \kappa(x, A)$. In this sense, π *is unique up to upper contours.*

Theorem 2 is the key identification result for the random-utility model. It says that, for any alternative x and any set A that does not contain x, we can retrieve from the choice probabilities the probability that A is the upper contour set of x. Note that, as was observed by Barbera and Pattanaik (1986, p.711), we cannot, in general, retrieve a unique ranking assignment from the choice probabilities. While this claim is correct, there was, however, an error in the justification for this claim presented in Barbera and Pattanaik (1986, p.711).[3] It will be helpful, therefore, to present an example that illustrates how two distinct ranking assignments can generate the same choice function.

[2] While the justification for Theorem 2 appears in Barbera and Pattanaik (1986), we present a proof of this result in Appendix 1 because our proof is relatively short and we use a similar proof for Theorem 3.

[3] Barbera and Pattanaik (1986, p.711) present an example, which they label "Example 2.2", to justify the claim that distinct ranking assignments may rationalise the same choice function. In fact, the pair of ranking assignments in Example 2.2 rationalise different choice functions. For instance, note that the probability of choosing y from the choice set $\{x, y, z\}$ is $\frac{1}{4}$ for the first ranking assignment in Example 2.2 and is zero for the second ranking assignment.

Suppose that the set of alternatives is given by $X = \{w, x, y, z\}$. Define the four rankings $\succ_1$, $\succ_2, \succ_3, \succ_4$ as follows:

$x \succ_1 w \succ_1 y \succ_1 z$

$x \succ_2 w \succ_2 z \succ_2 y$

$w \succ_3 x \succ_3 y \succ_3 z$

$w \succ_4 x \succ_4 z \succ_4 y$

Consider the ranking assignment π_a such that $\pi_a(\succ_1) = \pi_a(\succ_2) = \pi_a(\succ_3) = \pi_a(\succ_4) = \frac{1}{4}$ and the ranking assignment π_b such that $\pi_b(\succ_1) = \pi_b(\succ_4) = \frac{1}{2}$. It can easily be verified that π_a and π_b rationalise the same standard choice function.

While this example illustrates that there may be multiple ranking assignments that rationalise a choice function, nevertheless, in the sense defined in Theorem 2, ranking assignments are unique up to upper contours. Thus, in the consideration-set models developed below, we address the question: once consideration sets are introduced into the random-utility model, are ranking assignments still unique up to upper contours?

2.2.2 Model with a default option

The literature on consideration-set models often assumes that there is a designated alternative referred to as a "default option". In Sections 2.3, 2.4 and 2.5, we examine consideration-set models with and without default options. The standard random-utility model described above serves as a benchmark for consideration-set models without a default option. For models with a default option, however, we need to develop a variant of the standard utility model that can serve as a full-attention benchmark. When we introduce limited attention in the next section, the default option is interpreted as an alternative that is always attended to; but, for the purpose of the full-attention model in this subsection, the default option is simply some designated alternative that is present in every choice set. We refer to this model as the "default-option random-utility model", which is specified as follows. The set of alternatives now comprises a default option, which is denoted $\bar{x}$, and m non-default alternatives $x_1, \ldots, x_m$; the set of non-default alternatives is denoted $X = \{x_1, \ldots, x_m\}$, and the set of all alternatives is denoted $\bar{X} := X \cup \{\bar{x}\}$. For any $B \subset X$, let $\bar{B} := B \cup \{\bar{x}\}$. The "set of choice sets" is denoted $\bar{\mathcal{B}} := \{\bar{B} : B \subset X\}$. In order to specify the default-option random-utility model, we need to present a series of definitions analogous to those provided above for the standard random-utility model:

Definitions 7 (*Default-option random-utility model*)

(i) An *default-option choice function* is a mapping $\bar{c} : \{(x, B) : B \in \bar{\mathcal{B}}, x \in B\} \to [0, 1]$ such that for any $B \in \bar{\mathcal{B}}$:

$$\sum_{x \in B} \bar{c}(x, B) = 1$$

(ii) A *ranking assignment on* $\bar{X}$, denoted $\bar{\pi}$, is a probability measure on the event space $(\mathcal{R}_{\bar{X}}, 2^{\mathcal{R}_{\bar{X}}})$.

(iii) For any non-empty $B \subset \bar{X}$ and $x \in B$, let $\bar{\mathcal{M}}_{x,B} \subset \mathcal{R}_{\bar{X}}$ denote the set of rankings given by $\bar{\mathcal{M}}_{x,B} = \{\succ \in \mathcal{R}_{\bar{X}} : \forall y \in B \backslash \{x\}, x \succ y\}$.

(iv) Let $\bar{c}$ be a default-option choice function. A ranking assignment on $\bar{X}$, $\bar{\pi}$, is said to *stochastically rationalise* $\bar{c}$ if, for every $B \in \bar{\mathcal{B}}$, $x \in B$, $\bar{c}(x, B) = \bar{\pi}(\bar{\mathcal{M}}_{x,B})$.

(v) Consider any default-option choice function $\bar{c}$. The *default-option random-utility contour function* for $\bar{c}$ is a mapping $\bar{\kappa} : \{(x, A) : A \subset X, x \in \bar{X} \backslash A\} \to \mathbb{R}$ such that, for any $A \subset X, x \in \bar{X} \backslash A$:

$$\bar{\kappa}(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{c}(x, \bar{X} \backslash B)$$

(vi) For any $A \subsetneq \bar{X}$ and $x \in \bar{X} \backslash A$, let $\bar{\mathcal{U}}_{x,A} \subset \mathcal{R}_{\bar{X}}$ denote the set of rankings given by $\bar{\mathcal{U}}_{x,A} := \{\succ \in \mathcal{R}_{\bar{X}} : \forall y \in \bar{X}, y \in A \iff y \succ x\}$.

In the literature on random utility models, we are not aware of any identification and characterisation results for the random utility model with a default option. Our identification and characterisation results below are analogous to those in Theorems 1 and 2.

Theorem 3 (*Identification: Default-option random-utility model*)

Consider any default-option choice function $\bar{c}$ and any ranking assignment on $\bar{X}$, $\bar{\pi}$. If $\bar{\pi}$ stochastically rationalises $\bar{c}$, then $\bar{\pi}$ is such that, for any $A \subset X, x \in \bar{X} \backslash A$, $\bar{\pi}(\bar{\mathcal{U}}_{x,A}) = \bar{\kappa}(x, A)$. In this sense, *$\bar{\pi}$ is unique up to upper contours containing only non-default options.*

Theorem 4 (*Characterisation: Default-option random-utility model*)

Consider any default-option choice function $\bar{c}$. There exists a ranking assignment on $\bar{X}$ that stochastically rationalises $\bar{c}$ if and only if the default-option random-utility contour function for $\bar{c}$, $\bar{\kappa}$, is non-negative.

Note, however, that the identification result in the default-option random-utility model is, in a sense, weaker than the result for the standard random-utility model. In the standard model, for any alternative x and any set A not containing x, we are able to back out the probability that A is the upper contour set of x. In the default-option model, however, we are only able to retrieve such probabilities for upper contour sets *that do not contain the default alternative.* Intuitively, in the default-option model, we can retrieve less information because less choice data are available: the default option is assumed to be in every choice set; so the choice data do not include choice probabilities for those choice sets that do not contain the default option.

2.3 The identification problem arising from consideration sets

2.3.1 Rationalisation in consideration-set models

In a consideration-set model without a default option, the DM's decision has two stages: in the first stage, the DM forms a consideration set that is a subset of her choice set; and, in the second stage, she chooses the most preferred alternative in her consideration set. When faced with a choice set B, the probability that the DM's consideration set is $A \subset B$ is specified by her *attention assignment.*

Definition 8 (*Attention assignment*)
An attention assignment is a function $\alpha : \{(A, B) : A \subset B \subset X\} \to [0, 1]$ such that, for any $B \subset X$

$$\sum_{A \subset B} \alpha(A, B) = 1$$

Then $\alpha(A, B)$ is interpreted as the probability that the consideration set is A when the choice set is B. The rationalisation conditions describe how the DM's ranking assignment and attention assignment determine her choice probabilities. Whereas, in the random-utility model, we used the term "stochastic rationalisation", for consideration-set models we will simply use the term "rationalisation", which can be interpreted as an abbreviation for "rationalisation in a consideration-set model".

Definition 9 (*Rationalisation: Consideration-set model without a default option*)
Consider any standard choice function c, any ranking assignment on X, π, and attention assignment α. c is said to be *rationalised* by π and α, if, for any non-empty $B \subset X, x \in B$:

$$c(x, B) = \sum_{A \subset B : x \in A} \alpha(A, B)\pi(\mathcal{M}_{x,A}) \tag{2.1}$$

For a consideration-set model with a default option, we use the term "consideration set" to refer to those *non-default* alternatives that the DM attends to. Thus, having formed a consideration set A in the first stage of the decision process, in the second stage the DM chooses the most preferred alternative from $A \cup \{\bar{x}\}$. The assumption is that, in making her decision, the DM always attends to the default option. So, if her consideration set is empty, she necessarily chooses the default option. Accordingly, for any set of non-default alternatives B, the DM's attention assignment describes the probability, when faced with a choice set of $\bar{B}$, that the DM's consideration set is $A \subset B$. For choice functions with a default option, the following definition of rationalisation describes how the DM's choice probabilities are determined by her ranking assignment and attention assignment.

Definition 10 (*Rationalisation: Consideration-set model with a default option*)
Consider any default-option choice function $\bar{c}$, any ranking assignment on $\bar{X}$, $\bar{\pi}$, and any attention assignment α. $\bar{c}$ is said to be *rationalised* by $\bar{\pi}$ and α, if, for any $B \subset X, x \in \bar{B}$:

$$\bar{c}(x, \bar{B}) = \sum_{A \subset B : x \in \bar{A}} \alpha(A, B) \bar{\pi}(\bar{\mathcal{M}}_{x, \bar{A}}) \tag{2.2}$$

2.3.2 Restrictions on attention assignments

Attention assignments appear in both the theoretical and econometric literature on consideration sets. In two influential theoretical studies, Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017) introduce consideration sets into a revealed preference framework, retrieving, from choice probabilities, preferences and attention assignments. Examples of econometric models with consideration sets include Goeree (2008), Ho et al. (2017), Heiss et al. (2016) and Abaluck and Adams (2017), which use discrete choice models to make inferences about attention assignments from choice data.[4] In both the theoretical and the empirical literature, the authors impose restrictions on the attention assignment. For instance, Manzini and Mariotti (2014a), Goeree (2008) and Abaluck and Adams (2017) present default-option models in which attention assignments are assumed to have a certain property that we will refer to as *independence*. An independent attention assignment is constructed as follows. Each non-default alternative x_i is assigned an "attention probability" γ_i, which measures the likelihood that the DM includes x_i in her consideration set. More formally, an *attention probability vector* γ is defined to be a member of $\mathbb{R}^m$ such that, for

[4] Barseghyan et al. (2019) also present an econometric model of unobserved choice sets, which can be interpreted as a consideration set model. While preferences are only partially identified, the model imposes very weak restrictions on the process of consideration set formation.

$i = 1, \ldots, m$, $\gamma_i \in (0, 1)$. The probabilities of non-default alternatives appearing in the consideration set are assumed to be independent. Accordingly, an independent attention assignment is generated by an attention probability vector according to equation (2.3) below.

Definition 11 (*Independent attention assignment*)
Consider any attention probability vector γ. The *independent attention assignment generated by* γ is the attention assignment α such that, for any $A \subset B \subset X$:[5]

$$\alpha(A, B) = \prod_{i:x_i \in A} \gamma_i \prod_{i:x_i \in B \setminus A} (1 - \gamma_i) \tag{2.3}$$

While an assumption that an attention assignment is independent is substantive, Manzini and Mariotti (2014a) plausibly describe it as a "natural starting point" for a consideration-set model with a default option. For models without a default option, however, independent attention assignments are inapplicable. After all, for any independent attention assignment, there is always a positive probability that the consideration set is empty. For a model with a default option, the default option is assumed to be chosen when the consideration set is empty. But for models without a default option, the possibility of an empty consideration set must be ruled out. While Manzini and Mariotti (2014a) do not develop a model without a default option, they reasonably suggest that, in the absence of a default option, a "natural" restriction on attention assignments is obtained by *conditioning* equation (2.3) on the probability that at least one alternative is considered.[6] This imposes the restriction on attention assignments in the definition below:

Definition 12 (*Conditional attention assignment*)
Consider any attention probability vector γ. The *conditional attention assignment generated by* γ is the attention assignment α such that, for any non-empty $B \subset X$:
(i) $\alpha(\emptyset, B) = 0$; and
(ii) for $A \subset B$ such that $A \neq \emptyset$

$$\alpha(A, B) = \prod_{i:x_i \in A} \gamma_i \prod_{i:x_i \in B \setminus A} (1 - \gamma_i) \Bigg/ \sum_{D \subset B : D \neq \emptyset} \prod_{i:x_i \in D} \gamma_i \prod_{i:x_i \in B \setminus D} (1 - \gamma_i) \tag{2.4}$$

[5]The product over an empty set is understood to be 1.

[6]A similar suggestion is made by van Nierop et al. (2010, p.65). Horan (2018) also uses a conditional attention assignment for a model of limited attention with a default option – in this model, the default option is interpreted as the "no choice" option and thus is unobserved.

Manzini and Mariotti (2014a, p.1168) suggest the interpretation that "whenever the agent misses all alternatives, he is given the option to 'reconsider', repeating the process until he notices some alternative." Following Manzini and Mariotti (2014a), our view is that, for models without a default option, a "natural starting point" is the assumption that the DM has a conditional attention assignment. Accordingly, the principal model in our paper (presented in Section 2.4) is a model without a default option in which the DM is assumed to have a conditional attention assignment.

2.3.3 The identification problem

For their model with an independent attention assignment, Manzini and Mariotti (2014a) show that both the DM's attention assignment and her preferences can be retrieved from her observable choice probabilities. But, as noted in the introduction, their model makes two restrictive assumptions, which we relax: first, they assume that the default option is chosen only when no other alternatives are considered; and, second, they assume that preferences are deterministic. When these two assumptions are relaxed, an identification problem arises. We will illustrate this problem with a simple default-option choice function where the set of alternatives is $\bar{X} = \{x_1, x_2, \bar{x}\}$. Suppose that the choice function is rationalised by a ranking assignment on $\bar{X}$, $\bar{\pi}$, and an independent attention assignment generated by the attention probability vector (γ_1, γ_2). The rationalisation conditions are as follows:[7]

$$\bar{c}(x_1, x_1\bar{x}) = \gamma_1 \bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1\bar{x}}) \tag{2.5}$$

$$\bar{c}(x_2, x_2\bar{x}) = \gamma_2 \bar{\pi}(\bar{\mathcal{M}}_{x_2, x_2\bar{x}}) \tag{2.6}$$

$$\bar{c}(x_1, x_1x_2\bar{x}) = \gamma_1(1-\gamma_2)\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1\bar{x}}) + \gamma_1\gamma_2\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2\bar{x}}) \tag{2.7}$$

$$\bar{c}(x_2, x_1x_2\bar{x}) = \gamma_2(1-\gamma_1)\bar{\pi}(\bar{\mathcal{M}}_{x_2, x_2\bar{x}}) + \gamma_1\gamma_2\bar{\pi}(\bar{\mathcal{M}}_{x_2, x_1x_2\bar{x}}) \tag{2.8}$$

$$\bar{c}(\bar{x}, x_1\bar{x}) = (1-\gamma_1) + \gamma_1\bar{\pi}(\bar{\mathcal{M}}_{\bar{x}, x_1\bar{x}}) \tag{2.9}$$

$$\bar{c}(\bar{x}, x_2\bar{x}) = (1-\gamma_2) + \gamma_2\bar{\pi}(\bar{\mathcal{M}}_{\bar{x}, x_2\bar{x}}) \tag{2.10}$$

$$\begin{aligned}\bar{c}(\bar{x}, x_1x_2\bar{x}) = {} & (1-\gamma_1)(1-\gamma_2) + \gamma_2(1-\gamma_1)\bar{\pi}(\bar{\mathcal{M}}_{\bar{x}, x_2\bar{x}}) + \gamma_1(1-\gamma_2)\bar{\pi}(\bar{\mathcal{M}}_{\bar{x}, x_1\bar{x}}) \\ & + \gamma_1\gamma_2\bar{\pi}(\bar{\mathcal{M}}_{\bar{x}, x_1x_2\bar{x}})\end{aligned} \tag{2.11}$$

Note that equations (2.9), (2.10), and (2.11) do not impose any additional effective constraints on the unknowns over and above those in equations (2.5),(2.6), (2.7), and (2.8) because (i) choice

[7] In equation (2.5), $\bar{c}(x_1, x_1\bar{x})$ and $\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1\bar{x}})$ are to be interpreted as abbreviated ways of writing $\bar{c}(x_1, \{x_1, \bar{x}\})$ and $\bar{\pi}(\bar{\mathcal{M}}_{x_1, \{x_1, \bar{x}\}})$. This kind of abbreviation will be used throughout the paper.

probabilities must sum to one and (ii) for any $B \subset X$, $\sum_{x\in\bar{B}} \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{B}}) = 1$. Let us refer to equations (2.5),(2.6), (2.7), and (2.8) as the "non-default rationalisation conditions" and equations (2.9), (2.10), and (2.11) as the "default rationalisation conditions". More generally, for any default-option model, we use the term "non-default rationalisation condition" to refer to a rationalisation condition that imposes a constraint on the probability of choosing a non-default option $x \in X$ from a choice set $\bar{B}$, $\bar{c}(x, \bar{B})$; and we use the term "default rationalisation condition" to refer to a rationalisation condition that imposes a condition on the probability of choosing the default option $\bar{x}$ from a choice set $\bar{B}$, $\bar{c}(\bar{x}, \bar{B})$. In general, the default rationalisation conditions impose no additional effective constraints on attention assignments and ranking assignments over and above those imposed by the non-default rationalisation conditions. Accordingly, when we are attempting to identify the unknown parameters in the model, we can simply focus on the non-default rationalisation conditions.

In our example, the non-default rationalisation conditions are a system of four equations, (2.5),(2.6), (2.7), and (2.8), in six unknowns: γ_1, γ_2, $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1\bar{x}})$, $\bar{\pi}(\bar{\mathcal{M}}_{x_2,x_2\bar{x}})$, $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2\bar{x}})$ and $\bar{\pi}(\bar{\mathcal{M}}_{x_2,x_1x_2\bar{x}})$. As the number of unknowns exceeds the number of equality constraints, we are, in general, unable to identify the unknowns. This is a simple illustration of the identification problem.

To make the point more concretely, suppose that, in this example, the choice function is such that $\bar{c}(x_1, x_1\bar{x}) = \bar{c}(x_2, x_2\bar{x}) = \frac{1}{6}$ and $\bar{c}(x_1, x_1x_2\bar{x}) = \bar{c}(x_2, x_1x_2\bar{x}) = \frac{4}{27}$. Suppose that all we observe is the DM's choice function. On the basis of these observations, together with an assumption that the choice function is rationalised by some independent attention assignment and some ranking assignment, we want to make inferences about the attention assignment and ranking assignment. The identification problem is that these choice data are rationalisable by a number of different independent attention assignments and ranking assignments. For instance, the choice function is rationalised by an independent attention assignment α and a ranking assignment $\bar{\pi}$ such that (1) α is the independent attention assignment generated by $\gamma = (\frac{1}{3}, \frac{1}{3})$ and (2) $\bar{\pi}$ is the ranking assignment where, for any ranking $\succ \in \mathcal{R}_{\bar{X}}$, $\pi(\succ) = \frac{1}{6}$. But the choice function is also rationalised by an independent attention assignment $\dot{\alpha}$ and a ranking assignment $\dot{\bar{\pi}}$ such that (1) $\dot{\alpha}$ is the independent attention assignment generated by $\dot{\gamma} = (\frac{2}{3}, \frac{2}{3})$ and (2) $\dot{\bar{\pi}}$ is such that:

(i) $\dot{\bar{\pi}}$ assigns a probability of $\frac{1}{24}$ to each of the rankings $x_1 \succ x_2 \succ \bar{x}$ and $x_2 \succ x_1 \succ \bar{x}$;

(i) $\dot{\bar{\pi}}$ assigns a probability of $\frac{1}{6}$ to each of the rankings $x_1 \succ \bar{x} \succ x_2$ and $x_2 \succ \bar{x} \succ x_1$;

(i) $\dot{\bar{\pi}}$ assigns a probability of $\frac{7}{24}$ to each of the rankings $\bar{x} \succ x_1 \succ x_2$ and $\bar{x} \succ x_2 \succ x_1$.

Accordingly, we cannot retrieve from the choice function a unique independent attention assignment. Moreover, in contrast to the default-option random-utility model, ranking assignments are not unique up to upper contours of non-default alternatives.

What is the diagnosis of the source of the identification problem? As noted above, in the four non-default rationalisation conditions (2.5),(2.6), (2.7), and (2.8), there are six unknowns, which are of two types: first, there are the unknown attention probabilities γ_1 and γ_2; and, second, there are four unknowns $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1\bar{x}})$, $\bar{\pi}(\bar{\mathcal{M}}_{x_2,x_2\bar{x}})$, $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1\bar{x}})$ and $\bar{\pi}(\bar{\mathcal{M}}_{x_2,x_2\bar{x}})$, which we will refer to as *maximum probabilities*. The number of unknown maximum probabilities in the non-default rationalisation conditions is equal to the number of non-default rationalisation conditions. Accordingly, the total number of unknowns, which includes both the attention parameters and the maximum probabilities, exceeds the number of equations. Thus an identification problem arises. Moreover, we can generalise from this particular example. In general, for a default-option choice function, the number of non-default rationalisation conditions is equal to the number of unknown maximum probabilities in the equations. Thus, in general, the total number of unknowns, which includes both the attention parameters and maximum probabilities, will exceed the number of equations.

Using analogous reasoning, it is apparent that an identification problem also arises for choice functions without a default option. Consider the rationalisation conditions specified in (2.1). For any choice set B, one of the rationalisation conditions associated with B is "redundant", because the probabilities of choosing the various alternatives from B must sum to one. So, for every choice set B, we can ignore one rationalisation condition; let us refer to the remaining rationalisation conditions as the "effective rationalisation conditions". It is easy to verify that, in the effective rationalisation conditions, the number of unknown maximum probabilities is equal to the number of equations. So the total number of unknowns, which includes both the attention parameters and the maximum probabilities, exceeds the number of equations. Accordingly, an identification problem arises, which analogous to the problem for models with default options.

In summary: the identification problem arises because the choice data place insufficient equality constraints on the unknowns. But this suggests the following strategy for solving the identification problem: the strategy is to enrich the choice data so that (i) the number of equality constraints increases, but (ii) there is not a commensurate increase in the number of unknowns. In the next section, we pursue this strategy, enriching the choice data by assuming that it takes the form of a *frame-dependent choice function*.

2.4 Models without a default option

2.4.1 Frame-dependent choice functions and attention effects

Consider an example in which the set of alternatives comprises five varieties of canned tuna $X = \{x_1, x_2, x_3, x_4, x_5\}$ sold in a grocery store. Suppose that we have observations of the DM's choices under two conditions. The first condition, which we will call the "first frame", is where x_1 and x_2 are on the bottom shelf and x_3, x_4, x_5 are on the upper shelf. Suppose that, under the first frame, the DM's choice probabilities are given by the standard choice function c. The second condition, which is called the "second frame", is where x_3, x_4 and x_5 are on the bottom shelf and x_1 and x_2 are on the upper shelf. Suppose that, under the second frame, the DM's choice probabilities are given by the standard choice function c'. The change in choice probabilities from c to c' caused by a change in frames is an example of a type of a framing effect that we call an "attention effect." An attention effect is a framing effect where the change in frames influences choice probabilities by changing the DM's attention. The pair of standard choice functions (c, c') is referred to as a *frame-dependent choice function*. In our paper, a frame-dependent choice function is a pair of choice functions, where the first is interpreted as the DM's choice probabilities before an attention effect, and the second is the DM's choice probabilities after an attention effect. The notion of *rationalisation* for a frame-dependent choice function can then be defined as follows:

Definition 13 (*Rationalisation of a frame-dependent choice function*)
A frame-dependent choice function (c, c') is a pair of standard choice functions. Consider any ranking assignment π on X and pair of attention assignments (α, α'). A frame-dependent choice function (c, c') is said to be *rationalised* by π and (α, α') if c is rationalised by π and α and c' is rationalised by π and α'.

This definition captures the idea that, when an attention effect changes the DM's choices from c to c', her attention changes, but her preferences are unchanged. More formally, her attention assignment changes, but her ranking assignment is unchanged. The example above of the change is shelving conditions is a plausible candidate for an attention effect. Other examples of attention effects may include the following: (i) choice function c records choice probabilities when the DM has 1 minute to make a decision while c' records choice probabilities when the DM has 2 minutes; (ii) choice function c records choice probabilities when a special display is created for tuna variety x_1, whereas c' records choice probabilities when x_2 is on display. And so forth. As shown below, if the choice data take the form of a frame-dependent choice function, an observer may thereby be able to disentangle preferences and consideration sets. The idea is that a frame-dependent choice function may allow an observer to isolate the effect of a change in attention on choices, holding preferences constant. While our paper only explores the identification results obtained when choices are dependent on two frames, it may be fruitful to examine models in which there are more than two frames.

In this section, we present a consideration-set model without a default option, which we call the no-default-option model, or NDO model. It will be assumed that attention assignments are conditional; we argued above that this is a "natural" restriction to impose in the absence of a default option. Accordingly, this section addresses the following question: on the assumption that a frame-dependent choice function is rationalised by a pair of conditional attention assignments and a ranking assignment, to what extent can we retrieve, from observed choice probabilities, information about attention assignments and ranking assignments? In the next subsection, we show that attention assignments can be retrieved. In Section 2.4.3, we then show that ranking assignments can be retrieved (up to upper contours). Building on these two results, Section 2.4.4 presents a characterisation theorem for the NDO model.

2.4.2 Identification of attention assignments

Consider any frame-dependent choice function (c, c') that is rationalised by a ranking assignment π and a pair of conditional attention assignments (α, α') that are generated, respectively, by the probability attention vectors γ and γ'. The rationalisation conditions for this frame-dependent choice function include the following $2\binom{m}{2}$ equations:

$$c(x_i, x_ix_j) = \frac{\gamma_i(1-\gamma_j) + \gamma_i\gamma_j\pi(\mathcal{M}_{x_i,x_ix_j})}{\gamma_i(1-\gamma_j) + \gamma_j(1-\gamma_i) + \gamma_i\gamma_j} \quad i,j \in \{1,\dots,m\}, i<j \tag{2.12}$$

$$c'(x_i, x_ix_j) = \frac{\gamma'_i(1-\gamma'_j) + \gamma'_i\gamma'_j\pi(\mathcal{M}_{x_i,x_ix_j})}{\gamma'_i(1-\gamma'_j) + \gamma'_j(1-\gamma'_i) + \gamma'_i\gamma'_j} \quad i,j \in \{1,\dots,m\}, i<j \tag{2.13}$$

These equations are nonlinear in the unknown attention probabilities and maximum probabilities. It turns out, however, that we can transform the attention probabilities to obtain a linear system of equations. For any alternative x_i, the *odds* that the DM does not attend to x_i is given by $\frac{1-\gamma_i}{\gamma_i}$. Let us refer to this ratio as the *inattention odds* for x_i. We define the vector of such ratios as follows:

Definition 14 (*Inattention odds vector*)
Consider any attention probability vector γ. Its *inattention odds vector* $\iota \in \mathbb{R}^m$ is the vector such that, for $i = 1, \ldots, m$, $\iota_i = \frac{1-\gamma_i}{\gamma_i}$.

Suppose that we divide the numerator and denominator in (2.12) by $\gamma_i\gamma_j$, divide the numerator and denominator in (2.13) by $\gamma'_i\gamma'_j$, and define ι and ι' to be the inattention odds vectors for γ and γ'. Then we can rewrite (2.12) and (2.13) as the following system of equations that is linear in the unknown inattention odds and maximum probabilities:

$$(\iota_i + \iota_j + 1)c(x_i, x_ix_j) = \iota_j + \pi(\mathcal{M}_{x_i,x_ix_j}) \quad i, j \in \{1, \ldots, m\}, i < j \tag{2.14}$$

$$(\iota'_i + \iota'_j + 1)c'(x_i, x_ix_j) = \iota'_j + \pi(\mathcal{M}_{x_i,x_ix_j}) \quad i, j \in \{1, \ldots, m\}, i < j \tag{2.15}$$

The next step is the key move that allows us to obtain our identification result. Using the enriched dataset described by the frame-dependent choice function, we can isolate the effect of the change in the attention assignment on choices, controlling for preferences. In mathematical terms, we take differences between equations (2.14) and (2.15) to eliminate the unknown maximum probability $\pi(\mathcal{M}_{x_i,x_ix_j})$. The result is a system of equations in which the unknowns are simply inattention odds. If we define $\Delta_i = \iota'_i - \iota_i$, $i = 1, \ldots, m$, we can write the system of equations as follows:

$$\begin{aligned}[c(x_i, x_ix_j) - c'(x_i, x_ix_j)]\iota_j + [1 - c'(x_i, x_ix_j)]\Delta_j + [c(x_i, x_ix_j) - c'(x_i, x_ix_j)]\iota_i \\ -c'(x_i, x_ix_j)\Delta_i = c'(x_i, x_ix_j) - c(x_i, x_ix_j), \quad i, j \in \{1, \ldots, m\},\ i < j\end{aligned} \tag{2.16}$$

The equation system comprises $\binom{m}{2}$ equations in $2m$ unknowns. Accordingly, if $m \geq 5$, there are at least as many equations as unknowns. We can represent these $\binom{m}{2}$ equations in matrix notation. Denote the vector of unknowns to be $\mathbf{v} = (\iota_m, \Delta_m, \iota_{m-1}, \Delta_{m-1}, \ldots, \iota_1, \Delta_1)'$. If we fix a particular ordering of the equations, then the system of equations can be written $\mathbf{Qv} = \mathbf{b}$ where $\mathbf{Q}$ has size $\binom{m}{2} \times 2m$ and $\mathbf{b}$ has length $\binom{m}{2}$. It turns out that this system of equations does not yield a unique solution for $\mathbf{v}$. But, as is implied by Lemma 1 below, there is only one free variable in the system of equations, which allows us to solve for all variables in terms of one variable. Lemma 1 assumes that the frame-dependent choice function satisfies a certain non-zero condition, which we call the "NDO-nonzero condition". The NDO-nonzero condition is necessary to perform the row operations in the proof of Lemma 1. The case in which the NDO-nonzero condition does not hold is a special case; typically, the NDO-nonzero condition will be satisfied. Roughly speaking, if we think of attention probabilities and the ranking assignment being generated randomly from continuous distributions, then the resulting frame-dependent choice function will satisfy the NDO-nonzero condition almost surely. Appendix 5 presents an example in which the NDO-nonzero condition is satisfied.

Definition 15 (*NDO-nonzero condition*)
(i) for $i, j \in \{1, \ldots, m\}, i < j$, $c'(x_i, x_i x_j) - c(x_i, x_i x_j)$ are nonzero; (ii) the following terms, which are functions of observed choice probabilities and are defined in Appendix 1, are nonzero: $r^3_{35}, r^2_{34} - \frac{r^3_{34}}{r^3_{35}} r^2_{35}$, f and, for $i = 4, 7 \ldots, m$, d^i_{2i}.

Lemma 1 (*Reduced row echelon form for the NDO model*)
For $|X| \geq 5$, consider any frame-dependent choice function (c, c') that (i) satisfies the NDO-nonzero condition and (ii) is rationalised by a ranking assignment and a pair of conditional attention assignments. Then the matrix of coefficients for this choice function $\mathbf{Q}$ has rank $2m-1$ and, in particular, the reduced row echelon form of the augmented matrix $[\mathbf{Q}\ \mathbf{b}]$ is given by

$$\begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & s_m & -\frac{1}{2} \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & t_m & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & s_{m-1} & -\frac{1}{2} \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & t_{m-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & s_2 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & t_2 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & s_1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

where $s_i, i = 1, \ldots, m$ and $t_i, i = 2, \ldots, m$ are defined in the proof of Lemma 1 in Appendix 1.

Lemma 1 says that, using the rationalisation conditions in (2.12) and (2.13), we can solve for all the unknown attention parameters in terms of one free variable Δ_1. In particular:

$$\iota_i = -\frac{1}{2} - s_i \Delta_1, \quad i = 1, \ldots, m \tag{2.17}$$

$$\Delta_i = -t_i \Delta_1, \quad i = 2, \ldots, m \tag{2.18}$$

It remains to solve for Δ_1. Our method for doing so is specified in the proof of Theorem 5 in Appendix 1. A sketch of the method is as follows. In order to solve for Δ_1, we use the following pair of rationalisation conditions:

$$c(x_1, x_1x_2x_3) = \alpha(x_1, x_1x_2x_3) + \alpha(x_1x_2, x_1x_2x_3)\pi(\mathcal{M}_{x_1,x_1x_2})$$

$$+\alpha(x_1x_3, x_1x_2x_3)\pi(\mathcal{M}_{x_1,x_1x_3}) + \alpha(x_1x_2x_3, x_1x_2x_3)\pi(\mathcal{M}_{x_1,x_1x_2x_3})$$

$$c'(x_1, x_1x_2x_3) = \alpha'(x_1, x_1x_2x_3) + \alpha'(x_1x_2, x_1x_2x_3)\pi(\mathcal{M}_{x_1,x_1x_2})$$

$$+\alpha'(x_1x_3, x_1x_2x_3)\pi(\mathcal{M}_{x_1,x_1x_3}) + \alpha'(x_1x_2x_3, x_1x_2x_3)\pi(\mathcal{M}_{x_1,x_1x_2x_3})$$

We can rewrite these rationalisation conditions in terms of inattention odds:

$$(1 + \iota_1 + \iota_2 + \iota_3 + \iota_1\iota_2 + \iota_1\iota_3 + \iota_2\iota_3)c(x_1, x_1x_2x_3) = \iota_2\iota_3 +$$

$$\iota_3\pi(\mathcal{M}_{x_1,x_1x_2}) + \iota_2\pi(\mathcal{M}_{x_1,x_1x_3}) + \pi(\mathcal{M}_{x_1,x_1x_2x_3}) \tag{2.19}$$

$$(1 + \iota_1' + \iota_2' + \iota_3' + \iota_1'\iota_2' + \iota_1'\iota_3' + \iota_2'\iota_3')c'(x_1, x_1x_2x_3) = \iota_2'\iota_3' +$$

$$\iota_3'\pi(\mathcal{M}_{x_1,x_1x_2}) + \iota_2'\pi(\mathcal{M}_{x_1,x_1x_3}) + \pi(\mathcal{M}_{x_1,x_1x_2x_3}) \tag{2.20}$$

We then eliminate the maximum probability terms from these expressions as follows. First, we eliminate $\pi(\mathcal{M}_{x_1,x_1x_2x_3})$ by taking differences between (2.19) and (2.20). Then, from the resulting equation, we eliminate $\pi(\mathcal{M}_{x_1,x_1x_2})$ and $\pi(\mathcal{M}_{x_1,x_1x_3})$ using equations (2.14) and (2.15). This produces an equation in six unknowns: $\iota_1, \iota_2, \iota_3, \Delta_1, \Delta_2, \Delta_3$. But we can eliminate five of the unknowns, substituting them out using equations (2.17) and (2.18). The result is an equation in a single unknown, Δ_1, which is a quadratic equation of the form

$$f\Delta_1^2 + g\Delta_1 + h = 0 \tag{2.21}$$

where expressions for f, g and h are defined in Appendix 1. It turns out that g must equal 0. So the solution to this quadratic is given by $\Delta_1 = \pm\sqrt{-h/f}$. (Note that the NDO-nonzero condition assumes $f \neq 0$). How do we ascertain whether Δ_1 is the negative or positive square root? Given that (c, c') is rationalisable, we know that ι_1 must be positive; and we also know that $\iota_1 = -\frac{1}{2} - s_1\Delta_1$.

So Δ_1 must have the opposite sign to s_1. This allows us to obtain a unique solution for Δ_1, which is given as follows:

$$\hat{\Delta}_1 = \begin{cases} -\sqrt{-h/f} & \text{if } s_1 > 0 \\ \sqrt{-h/f} & \text{otherwise} \end{cases} \tag{2.22}$$

Having solved for Δ_1, we use equations (2.17) and (2.18) to specify the revealed inattention odds vector.

Definition 16 (*The NDO revealed inattention odds vectors*)
For $|X| \geq 5$, suppose that a choice function (c, c') is such that the NDO-nonzero condition is satisfied. The *NDO revealed inattention odds vectors* for c and c' are the vectors $\hat{\imath}, \hat{\imath}' \in \mathbb{C}^m$ such that, for $i = 1, 2, \ldots, m$, $\hat{\imath}_i = -\frac{1}{2} - s_i\hat{\Delta}_1$ and

$$\hat{\imath}'_i = \begin{cases} \hat{\imath}_i + \hat{\Delta}_1 & \text{for } i = 1 \\ \hat{\imath}_i - t_i\hat{\Delta}_1 & \text{for } i = 2, \ldots, m \end{cases}$$

Note that, from inspection of the definition of $\hat{\Delta}_1$ in (2.22), the revealed inattention odds may potentially be complex numbers. But if (c, c') is rationalisable, then the inattention odds must be positive real numbers. In this case, we can obtain from the revealed inattention odds the revealed attention probabilities and revealed attention assignments:

Definition 17 (*The NDO revealed attention probabilities and attention assignments*)
For $|X| \geq 5$, suppose that a choice function (c, c') is such that the NDO-nonzero condition is satisfied and that the revealed inattention odds vectors $\hat{\imath}$ and $\hat{\imath}'$ are real and positive.
(i) The *NDO revealed attention probability vectors* for c and c' are $\hat{\gamma}, \hat{\gamma}' \in \mathbb{R}^m$ such that, for $i = 1, \ldots, m$, $\hat{\gamma}_i = \frac{1}{1+\hat{\imath}_i}$ and $\hat{\gamma}'_i = \frac{1}{1+\hat{\imath}'_i}$.
(ii) The *NDO revealed attention assignments* for c and c' are the conditional attention assignments $\hat{\alpha}$ and $\hat{\alpha}'$ that are respectively generated by $\hat{\gamma}$ and $\hat{\gamma}'$.

The formal statement of our first identification result for the NDO model is then as follows.

Theorem 5 (*Identification of attention assignments in the NDO model*)
For $|X| \geq 5$, consider any frame-dependent choice function (c, c') that satisfies the NDO-nonzero condition, any ranking assignment on X, π, and any pair of conditional attention assignments (α, α'). If π and (α, α') rationalise (c, c'), then $\alpha = \hat{\alpha}$ and $\alpha' = \hat{\alpha}'$.

Appendix 5 presents a simple numerical example that illustrates how Lemma 1 and Theorem 5 can be used to retrieve attention probabilities from frame-dependent choice functions. This example also serves to verify the identification method presented above.

2.4.3 Identification of ranking assignments

Having obtained the revealed attention assignments, the next step is to retrieve, from the choice data, information relating to ranking assignments. As noted in Section 2.2, we should not expect to be able to back out from choices a unique ranking assignment, because, even in the random-utility model with full attention, ranking assignments are only unique up to upper contours. We will now show that, similarly, in the NDO model, ranking assignments are also unique up to upper contours. That is, despite the presence of consideration sets, we are able to retrieve information about ranking assignments to the same extent as we can in the model with full attention. We obtain our identification result in two steps: first, we obtain the revealed maximum probabilities; and then we obtain the revealed probabilities regarding contour sets.

We can obtain the revealed maximum probabilities simply by rearranging the rationalisation conditions in equation (2.1). Consider any alternative $x \in X$ and any $B \subset X$ that contains x. If we inspect equation (2.1), it is apparent that we can ascertain the maximum probability $\pi(\mathcal{M}_{x,B})$ if we know (i) the choice probability $c(x,B)$ (ii) the attention assignment α and (iii) the maximum probability $\pi(\mathcal{M}_{x,A})$ for every proper subset of B, A, that contains x. Accordingly, we can rearrange (2.1) to obtain a recursive algorithm for obtaining the revealed maximum probabilities. This algorithm is encapsulated in the following definition.

Definition 18 (*NDO maximum functions: Revealed maximum probabilities*)
For $|X| \geq 5$, suppose that a choice function (c, c') is such that the NDO-nonzero condition is satisfied and the revealed inattention odds vectors $\hat{\iota}$ and $\hat{\iota}'$ are real and positive. The *NDO maximum functions* for c and c' are the functions $\hat{\mu} : \{(x,B) : B \in \mathcal{B}, x \in B\} \to \mathbb{R}$ and $\hat{\mu}' : \{(x,B) : B \in \mathcal{B}, x \in B\} \to \mathbb{R}$ such that, for any $B \in \mathcal{B}, x \in B$:
(i) if $|B| = 1$, then $\hat{\mu}(x,B) = \hat{\mu}'(x,B) = 1$
(ii) if $|B| > 1$, then:

$$\hat{\mu}(x,B) = \frac{c(x,B) - \sum_{A \subsetneq B : x \in A} \hat{\alpha}(A,B)\hat{\mu}(x,A)}{\hat{\alpha}(B,B)}$$

$$\hat{\mu}'(x,B) = \frac{c'(x,B) - \sum_{A \subsetneq B : x \in A} \hat{\alpha}'(A,B)\hat{\mu}'(x,A)}{\hat{\alpha}'(B,B)}$$

$\hat{\mu}(x,B)$ and $\hat{\mu}'(x,B)$ are each to be interpreted as the revealed probability that x is the most preferred alternative in B. Accordingly, when the frame-dependent choice function is rationalisable, the two expressions must be equal. Having retrieved the revealed maximum probabilities from choice data, the next task is to obtain, for every $A \subsetneq X$, $x \in X \backslash A$, the revealed probability that A is the upper contour set of x. It turns out that this task is precisely analogous to the task performed in the proof of Theorem 2. Given that the frame-dependent choice function is rationalised, NDO maximum functions have the same formal properties as standard choice functions: namely, they are non-negative and, for a given non-empty $B \subset X$, the probabilities must sum to one. Following the method used in the full-attention case, first we define a notion of a contour function analogous to the random-utility contour function κ defined in Section 2.2.

Definition 19 (*NDO contour functions: Revealed upper contour probabilities*)
For $|X| \geq 5$, suppose that a choice function (c, c') is such that the NDO-nonzero condition is satisfied and that the revealed inattention odds vectors $\hat{\iota}$ and $\hat{\iota}'$ are real and positive. Then the *NDO contour functions* for c and c' are $\hat{\upsilon} : \{(x,A) : A \subsetneq X, x \in X \backslash A\} \to \mathbb{R}$ and $\hat{\upsilon}' : \{(x,A) : A \subsetneq X, x \in X \backslash A\} \to \mathbb{R}$ such that, for any $A \subsetneq X, x \in X \backslash A$:

$$\hat{\upsilon}(x,A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \hat{\mu}(x, X \backslash B), \quad \hat{\upsilon}'(x,A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \hat{\mu}'(x, X \backslash B)$$

Clearly these contour functions are analogous to κ; the difference is that κ is defined in terms of a choice function, whereas these contour functions are defined in terms of maximum functions. $\hat{\upsilon}(x,A)$ and $\hat{\upsilon}'(x,A)$ are to be interpreted as the revealed probability that A is the upper contour set of x. This interpretation is formalised by the following theorem. If we compare this theorem to Theorem 2, it is apparent that, in the NDO model, ranking assignments are identified to the same extent as in the full attention random-utility model.

Theorem 6 (*Identification of ranking assignments in the NDO model*)
For $|X| \geq 5$, consider any frame-dependent choice function (c, c') that satisfies the NDO-nonzero condition, any ranking assignment π and any pair of conditional attention assignments (α, α'). If π and (α, α') rationalise (c, c'), then for all $A \subsetneq X$, $x \in X \backslash A$, $\pi(\mathcal{U}_{x,A}) = \hat{\upsilon}(x,A) = \hat{\upsilon}'(x,A)$. That is, π is unique up to upper contours.

2.4.4 Characterisation

The next task is to develop a characterisation result akin to Falmagne's axiomatisation in Theorem 1. As noted above, Theorem 2 allows us to provide an intuitive interpretation of Theorem 1. Roughly speaking, Theorem 1 says that a choice function is rationalisable just in case its revealed contour probabilities are non-negative. I will refer to this kind of axiomatisation as an "identification-based characterisation". It is "identification-based" in the following sense. First, the ($\Rightarrow$) part of the characterisation theorem follows immediately from the identification result in Theorem 2. Second, the ($\Leftarrow$) part of the characterisation theorem says that, if the revealed contour probabilities are "well-formed", in a certain sense, then the choice function is rationalisable.

Similarly, we present, in Theorem 7, an identification-based characterisation result for the NDO model. It is identification-based in the following sense. First, the ($\Rightarrow$) part of the characterisation theorem follows immediately from the identification results presented in Theorems 5 and 6. Second, the ($\Leftarrow$) part of the characterisation theorem says that the choice function is rationalisable if, in a certain sense, both the revealed inattention odds vectors and the revealed contour probabilities are "well-formed".

Theorem 7 (*Characterisation for the NDO model*)
For $|X| \geq 5$, consider any choice function (c, c') that satisfies the NDO-nonzero condition. There exists a ranking assignment on X and a pair of conditional attention assignments that rationalise (c, c') if and only if:
(i) the revealed inattention odds vectors $\hat{\imath}$ and $\hat{\imath}'$ are real and positive;
(ii) the contour functions $\hat{\upsilon}$ and $\hat{\upsilon}'$ are equal and non-negative.

2.4.5 Models with minimum attention levels

As noted in Section 2.3, in the absence of a default option, a consideration-set model must rule out the possibility of an empty consideration set. A "natural" way to do this, we argued, is to assume that the DM has a conditional attention assignment. In effect, we are assuming that the DM has a *minimum attention level* – in particular, that the DM's consideration sets are always at least of size one. But then the following question arises: in at least some cases, might it be reasonable to assume that the DM has a higher minimum attention level? More precisely, let us define a minimum attention level to be some positive integer n less than m such that, if the size of the choice set is n

or less, the DM always considers all the alternatives; and if the size of the choice set is more than n, the DM only forms consideration sets whose size is at least n. If the cost of consideration is low, it might be reasonable to assume that the DM has a minimum attention level of 2 or 3 or perhaps more. In Appendix 2, we present a model in which the DM has a minimum attention level of 2. So the DM is no longer assumed to have a conditional attention assignment; we impose a somewhat different restriction on her attention assignment, which reflects the assumption that her minimum attention level is 2. For this model, we develop identification results and a characterisation result that are analogous to Theorems 5, 6 and 7. In particular, like the NDO model developed in this section, the model in Appendix 2 is fully identified. So Appendix 2 illustrates that the identification strategy used in this section can also potentially be applied when different restrictions are imposed on the DM's attention assignment.

2.5 Models with a default option

In the previous section, we used the following strategy to solve the identification problem. We enriched the choice dataset by using a frame-dependent choice function, thereby increasing the number of rationalisation conditions. This increases the number of equality constraints on the unknowns and, as a consequence, the number of equations is no longer less less than the number of unknowns. It should be apparent, however, that, for some models, this solution strategy may fail. Suppose that we have a system of equations in which the number of unknowns exceeds the number of equations, and, as a consequence, there are multiple solutions. Even if we add further equality constraints – so that the number of constraints is no longer less than the number of unknowns – this does not, of course, guarantee that we can obtain a unique solution. For example, the new equality constraints may *fail to provide any additional information about the unknowns.* An analogy can be made to the case of linear equations. If a system of linear equations has multiple solutions, the addition of another equality constraint that is linearly dependent on the original equations does not add any additional information about the unknowns. In some cases, therefore, the addition of the new equality constraint may not allow us to solve the identification problem.

In Appendix 3, we show that, for a model with a default option and an *independent* attention assignment, the use of a frame-dependent choice function does not solve the identification problem. Moreover, we present a diagnosis of the source of the identification failure. When we enrich the choice dataset by using a frame-dependent choice function, most of the new equality constraints do not add any additional information about the unknowns. The situation is analogous, therefore, to the case of adding to a system of linear equations an equality constraint that is linearly dependent on the original equations.

We suggest, however, that the identification failure that arises with an independent attention assignment is something of a special case. In Appendix 4, we present a model with a default option and a "size-effect attention assignment". A size-effect attention assignment is a simple generalization of an independent attention assignment, which accommodates the possibility that the probability of consideration sets of certain sizes may deviate systematically from the probabilities implied by an independent attention assignment. The model is fully identified for a range of frame-dependent choice functions. This model illustrates two points. First, we can interpret the identification failure for the model with an independent attention assignment as, in some sense, a special case. Second, in the size-effect model, there is not a closed-form solution for the attention parameters; they are obtained from the rationalisation conditions using numerical methods. This model illustrates that, our strategy for solving the identification problem is potentially applicable in the absence of a closed-form solution.

In the remainder of this section, we present another model with a default option that is fully identified – it is a model with a minimum attention level. For this model, we will obtain a series of identification and characterisation results akin to those in Theorems 5, 6 and 7.

The definition of rationalisation in Section 2.4 can be adapted to the case of a frame-dependent choice function with a default option.

Definition 20 (*Rationalisation of a default-option frame-dependent choice function*)
A default-option frame-dependent choice function $(\bar{c}, \bar{c}')$ is a pair of default-option choice functions. Consider any ranking assignment on $\bar{X}$, $\bar{\pi}$, and any pair of attention assignments (α, α'). A default-option frame-dependent choice function $(\bar{c}, \bar{c}')$ is said to be *rationalised* by $\bar{\pi}$ and (α, α') if $\bar{c}$ is rationalised by $\bar{\pi}$ and α and $\bar{c}'$ is rationalised by $\bar{\pi}$ and α'.

We present below a model with a default option which assumes that the minimum size of a consideration set is 1. As discussed in Section 2.3, for such a case, a natural assumption is that the DM's attention assignment is *conditional*. We will refer to this model as the "default-option" model, or DO model. In order to develop models with higher minimum attention levels, we could use, instead, the restriction on attention assignments suggested in Appendix 2.

The rationalisation conditions for the DO model imply that $\bar{c}(x_i, x_i\bar{x}) = \bar{\pi}(\bar{\mathcal{M}}_{x_i,x_i\bar{x}})$, $i = 1, \dots, m$. Furthermore, the following equality constraints are also implied.

$$\bar{c}(x_i, x_ix_j\bar{x}) = \frac{\gamma_i(1-\gamma_j)\bar{c}(x_i, x_i\bar{x}) + \gamma_i\gamma_j\bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_j\bar{x}})}{\gamma_i(1-\gamma_j) + \gamma_j(1-\gamma_i) + \gamma_i\gamma_j}, i, j \in \{1, \dots, m\}, i < j \tag{2.23}$$

$$\bar{c}(x_j, x_ix_j\bar{x}) = \frac{\gamma_j(1-\gamma_i)\bar{c}(x_j, x_j\bar{x}) + \gamma_i\gamma_j\bar{\pi}(\bar{\mathcal{M}}_{x_j,x_ix_j\bar{x}})}{\gamma_i(1-\gamma_j) + \gamma_j(1-\gamma_i) + \gamma_i\gamma_j}, i, j \in \{1, \dots, m\}, i < j \tag{2.24}$$

$$\bar{c}'(x_i, x_ix_j\bar{x}) = \frac{\gamma_i'(1-\gamma_j')\bar{c}(x_i, x_i\bar{x}) + \gamma_i'\gamma_j'\bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_j\bar{x}})}{\gamma_i'(1-\gamma_j') + \gamma_j'(1-\gamma_i') + \gamma_i'\gamma_j}, i, j \in \{1, \dots, m\}, i < j \tag{2.25}$$

$$\bar{c}'(x_j, x_ix_j\bar{x}) = \frac{\gamma_j'(1-\gamma_i')\bar{c}(x_j, x_j\bar{x}) + \gamma_i'\gamma_j'\bar{\pi}(\bar{\mathcal{M}}_{x_j,x_ix_j\bar{x}})}{\gamma_i'(1-\gamma_j') + \gamma_j'(1-\gamma_i') + \gamma_i'\gamma_j'}, i, j \in \{1, \dots, m\}, i < j \tag{2.26}$$

Using the definition of inattention odds, we rewrite this equation system so that it is linear in the unknowns:

$$(1 + \iota_i + \iota_j)\bar{c}(x_i, x_ix_j\bar{x}) = \iota_j\bar{c}(x_i, x_i\bar{x}) + \bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_j\bar{x}}), i, j \in \{1, \dots, m\}, i < j \tag{2.27}$$

$$(1 + \iota_i + \iota_j)\bar{c}(x_j, x_ix_j\bar{x}) = \iota_i\bar{c}(x_j, x_j\bar{x}) + \bar{\pi}(\bar{\mathcal{M}}_{x_j,x_ix_j\bar{x}}), i, j \in \{1, \dots, m\}, i < j \tag{2.28}$$

$$(1 + \iota_i' + \iota_j')\bar{c}'(x_i, x_ix_j\bar{x}) = \iota_j'\bar{c}(x_i, x_i\bar{x}) + \bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_j\bar{x}}), i, j \in \{1, \dots, m\}, i < j \tag{2.29}$$

$$(1 + \iota_i' + \iota_j')\bar{c}'(x_j, x_ix_j\bar{x}) = \iota_i'\bar{c}(x_j, x_j\bar{x}) + \bar{\pi}(\bar{\mathcal{M}}_{x_j,x_ix_j\bar{x}}), i, j \in \{1, \dots, m\}, i < j \tag{2.30}$$

The next step is to eliminate the maximum probability terms. From (2.27) and (2.29), we can eliminate $\bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_j\bar{x}})$, to obtain (2.31) below; and from (2.28) and (2.30), we can eliminate $\bar{\pi}(\bar{\mathcal{M}}_{x_j,x_ix_j\bar{x}})$, to obtain (2.32). The result is the following system of $2\binom{m}{2}$ equations, for $i, j = 1, \dots, m$ such that $i < j$.

$$\begin{gathered}[\bar{c}(x_i, x_ix_j\bar{x}) - \bar{c}'(x_i, x_ix_j\bar{x})]\iota_j + [\bar{c}(x_i, x_i\bar{x}) - \bar{c}'(x_i, x_ix_j\bar{x})]\Delta_j + \\ [\bar{c}(x_i, x_ix_j\bar{x}) - \bar{c}'(x_i, x_ix_j\bar{x})]\iota_i + [-\bar{c}'(x_i, x_ix_j\bar{x})]\Delta_i = \bar{c}'(x_i, x_ix_j\bar{x}) - \bar{c}(x_i, x_ix_j\bar{x})\end{gathered} \tag{2.31}$$

$$\begin{gathered}[\bar{c}(x_j, x_ix_j\bar{x}) - \bar{c}'(x_j, x_ix_j\bar{x})]\iota_j + [-\bar{c}'(x_j, x_ix_j\bar{x})]\Delta_j + [\bar{c}(x_j, x_ix_j\bar{x}) - \bar{c}'(x_j, x_ix_j\bar{x})]\iota_i \\ + [\bar{c}(x_j, x_j\bar{x}) - \bar{c}'(x_j, x_ix_j\bar{x})]\Delta_i = \bar{c}'(x_j, x_ix_j\bar{x}) - \bar{c}(x_j, x_ix_j\bar{x})\end{gathered} \tag{2.32}$$

These equations contain $2m$ unknowns, so, in order to ensure that the number of equations is not less than the number of unknowns, we need to assume that the number of non-default alternatives m is such that $2\binom{m}{2} \geq 2m$. Thus we will assume that there are at least three non-default alternatives. In contrast, for the NDO model, we had to assume that there are at least five alternatives. Nevertheless, the solution strategy is parallel to that in the NDO model. The solution is presented fully in the proof of Theorem 8, but it can be outlined as follows. First, using this linear system of equations, we solve for the unknowns as affine functions of Δ_1. In particular, we find that $\iota_i = -\frac{1}{2} - \bar{s}_i\Delta_1, i = 1, \ldots, m$ and $\Delta_i = -\bar{t}_i\Delta_1, i = 2, \ldots, m$, where $\bar{s}_i, i = 1, \ldots, m$ and $\bar{t}_i, i = 2, \ldots, m$ are defined in the proof of Theorem 8 in Appendix 1. It remains then to solve for Δ_1. The proof of Theorem 8 shows that we can do so by recourse to further rationalisation conditions, which yield the following quadratic equation, analogous to to equation (2.21) in Section 2.4:

$$\bar{f}\Delta_1^2 + \bar{g}\Delta_1 + \bar{h} = 0 \tag{2.33}$$

where $\bar{f}, \bar{g}$ and $\bar{h}$ are defined in the proof of Theorem 8 in Appendix 1. This proof shows that $\bar{g} = 0$. So, from the quadratic in equation (2.33), we obtain a solution for Δ_1 analogous to the solution in equation (2.22) in Section 2.4.

$$\bar{\Delta}_1 = \begin{cases} -\sqrt{-\bar{h}/\bar{f}} & \text{if } \bar{s}_1 > 0 \\ \sqrt{-\bar{h}/\bar{f}} & \text{otherwise} \end{cases} \tag{2.34}$$

As in the NDO model, our results assume that a nonzero condition obtains:

Definition 21 (*DO-nonzero condition*)
(i) For $i, j \in \{1, \ldots, m\}, i < j$, $\bar{c}'(x_i, x_ix_j\bar{x}) - \bar{c}(x_i, x_ix_j\bar{x})$ are nonzero.
(ii) The following terms, which are functions of observed choice probabilities and defined in Appendix 1, are nonzero:$\bar{f}$ and, for $j = 2, \ldots, m$, $\bar{d}^j_{1j}$.

The following then provides definitions, for the DO model, of the revealed inattention odds, the revealed attention probabilities and the revealed attention assignments:

Definition 22 (*DO revealed inattention odds, attention probabilities, attention assignments*)
For $|X| \geq 3$, suppose that choice function $(\bar{c}, \bar{c}')$ satisfies the DO-nonzero condition.
(i) The *DO revealed inattention odds vectors* for $\bar{c}$ and $\bar{c}'$ are the vectors $\bar{\iota}, \bar{\iota}' \in \mathbb{C}^m$ such that, for $i = 1, 2, \ldots, m$, $\bar{\iota}_i = -\frac{1}{2} - \bar{s}_i\bar{\Delta}_1$ and

$$\bar{\iota}'_i = \begin{cases} \bar{\iota}_i + \bar{\Delta}_1 & \text{for } i = 1 \\ \bar{\iota}_i - \bar{t}_i\bar{\Delta}_1 & \text{for } i = 2, \ldots, m \end{cases}$$

(ii) Suppose that the revealed inattention odds vectors $\bar{\iota}$ and $\bar{\iota}'$ are real and positive. Then the *DO revealed attention probability vectors* for $\bar{c}$ and $\bar{c}'$ are $\bar{\gamma}, \bar{\gamma}' \in \mathbb{R}^m$ such that, for $i = 1, \ldots, m$, $\bar{\gamma}_i = \frac{1}{1+\bar{\iota}_i}$ and $\bar{\gamma}'_i = \frac{1}{1+\bar{\iota}'_i}$.
(iii) Suppose that the revealed inattention odds vectors $\bar{\iota}$ and $\bar{\iota}'$ are real and positive. Then the *DO revealed attention assignments* for $\bar{c}$ and $\bar{c}'$ are the conditional attention assignments $\bar{\alpha}$ and $\bar{\alpha}'$ that are respectively generated by $\bar{\gamma}$ and $\bar{\gamma}'$.

Theorem 8 justifies the interpretation of $\bar{\alpha}$ and $\bar{\alpha}'$ as the revealed attention assignments for the model.

Theorem 8 (*Identification of attention assignments in the DO model*)
For $|X| \geq 3$, consider any frame-dependent choice function $(\bar{c}, \bar{c}')$ that satisfies the DO-nonzero condition, any ranking assignment $\bar{\pi}$ on $\bar{X}$, and any pair of conditional attention assignments (α, α'). If $\bar{\pi}$ and (α, α') rationalise $(\bar{c}, \bar{c}')$, then $\alpha = \bar{\alpha}$ and $\alpha' = \bar{\alpha}'$.

Appendix 5 provides a simple numerical example that illustrates how to apply the identification method presented in the proof of Theorem 8. This example also serves to verify the identification method presented in this section.

Having obtained revealed attention assignments, we adopt the strategy from the previous section to retrieve information about the ranking assignment. First, we define maximum functions and contour functions for the DO model. These can be interpreted as specifying the revealed maximum probabilities and the revealed probabilities of upper contour sets.

Definition 23 (*DO maximum functions and contour functions*)
For $|X| \geq 3$, suppose that a choice function $(\bar{c}, \bar{c}')$ is such that the DO-nonzero condition is satisfied and that the revealed inattention odds vectors $\bar{\iota}$ and $\bar{\iota}'$ are real and positive.
(a) The *DO maximum functions* for $\bar{c}$ and $\bar{c}'$ are the functions $\bar{\mu} : \{(x, B) : B \in \bar{\mathcal{B}}, x \in B\} \to \mathbb{R}$ and $\bar{\mu}' : \{(x, B) : B \in \bar{\mathcal{B}}, x \in B\} \to \mathbb{R}$ such that:
(i) $\bar{\mu}(\bar{x}, \{\bar{x}\}) = 1$ and $\bar{\mu}'(\bar{x}, \{\bar{x}\}) = 1$; and

(ii) for any non-empty $B \subset X$, $x \in \bar{B}$:

$$\bar{\mu}(x, \bar{B}) = \frac{\bar{c}(x, \bar{B}) - \sum_{A \subsetneq B : x \in \bar{A}} \bar{\alpha}(A, B)\bar{\mu}(x, \bar{A})}{\bar{\alpha}(B, B)}$$

$$\bar{\mu}'(x, \bar{B}) = \frac{\bar{c}'(x, \bar{B}) - \sum_{A \subsetneq B : x \in \bar{A}} \bar{\alpha}'(A, B)\bar{\mu}'(x, \bar{A})}{\bar{\alpha}'(B, B)}$$

(b) The *DO contour functions* for $\bar{c}$ and $\bar{c}'$ are the functions $\bar{\upsilon} : \{(x, A) : A \subset X, x \in \bar{X} \backslash A\} \to \mathbb{R}$ and $\bar{\upsilon}' : \{(x, A) : A \subset X, x \in \bar{X} \backslash A\} \to \mathbb{R}$ such that, for any $A \subset X, x \in \bar{X} \backslash A$:

$$\bar{\upsilon}(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{\mu}(x, \bar{X} \backslash B), \quad \bar{\upsilon}'(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{\mu}'(x, \bar{X} \backslash B)$$

The following theorem says that the $\bar{\upsilon}$ and $\bar{\upsilon}'$ can be interpreted as specifying revealed contour probabilities. If we compare this theorem to Theorem 3, it says that, in the DO model, ranking assignments are revealed to the same extent that they are revealed in the full attention version of the model. In this sense, the DO model is "fully identified".

Theorem 9 (*Identification of ranking assignments in the DO model*)
For $|X| \geq 3$, consider any choice function $(\bar{c}, \bar{c}')$ that satisfies the DO-nonzero condition, any ranking assignmenton $\bar{X}$, $\bar{\pi}$, and any pair of conditional attention assignments (α, α'). If $\bar{\pi}$ and (α, α') rationalise $(\bar{c}, \bar{c}')$, then for all $A \subset X$, $x \in \bar{X} \backslash A$, $\bar{\pi}(\bar{\mathcal{U}}_{x,A}) = \bar{\upsilon}(x, A) = \bar{\upsilon}'(x, A)$. That is, $\bar{\pi}$ is unique up to upper contours of non-default alternatives.

Finally, we present a identification-based characterisation theorem for the DO model that is analogous the the characterisation result Theorem 7 in the NDO model.

Theorem 10 (*Characterisation in the DO model*)
For $|X| \geq 3$, consider any choice function $(\bar{c}, \bar{c}')$ that satisfies the DO-nonzero condition. There exists a ranking assignment on $\bar{X}$ and a pair of conditional attention assignments that rationalise $(\bar{c}, \bar{c}')$ if and only if:

(i) the revealed inattention odds vectors $\bar{\iota}$ and $\bar{\iota}'$ are real and positive; and

(ii) the contour functions $\bar{\upsilon}$ and $\bar{\upsilon}'$ are equal and non-negative.

2.6 Related literature

Since the 1970s, consideration-set models have been common in the applied literature on consumer choice, especially in research on marketing.[8] This literature divides into two strands: models which assume that observations on consideration sets are available (from, say, consumer surveys or, perhaps, through eye-tracking devices); and models in which consideration sets are unobserved. The focus of our paper is on the second strand of this literature. The pioneering paper of Masatlioglu et al. (2012) introduced consideration sets into revealed preference theory. By imposing certain assumptions on the DM's dispositions to form consideration sets, they show that, potentially, at least some information about the DM's preferences and consideration-set dispositions can be retrieved from the DM's choices. Lleras et al. (2017) present a similar model, but impose a different set of assumptions on the DM's consideration-set dispositions. As is highlighted in Masatlioglu and Nakajima (2015), however, an identification problem arises in such models. Typically, an observer can infer from the DM's choices only very limited information about her preferences and consideration-set dispositions. Note that the final section of Masatlioglu and Nakajima (2015) briefly explores the strategy of using a frame-dependent choice function to solve the identification problem in a model with deterministic preferences and deterministic consideration-set dispositions.

Whereas Masatlioglu et al. (2012) and Lleras et al. (2017) model both preferences and consideration-set dispositions as deterministic, we follow the seminal paper of Manzini and Mariotti (2014a) in modelling consideration-set dispositions as stochastic. Using our terminology, Manzini and Mariotti (2014a) assume that the DM has an independent attention assignment. Brady and Rehbeck (2016) and Aguair (2017) present generalizations of the model of Manzini and Mariotti (2014a), in which the assumption about the DM's attention assignment is less restrictive. In contrast to Masatlioglu et al. (2012) and Lleras et al. (2017), there is no identification problem in the models of Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017) – that is, the DM's attention assignment and preference relation can be retrieved from her observed choice probabilities. How are their results consistent with our finding in Section 2.3 that, in the absence of frame-dependent choice function, an identification problem typically arises in consideration-set models? The identification results in Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017) depend crucially on two restrictive assumptions described in the introduction, which we dispense

[8] For surveys of this literature, see Roberts and Lattin (1997) and Hauser (2014).

with in our paper.

The first assumption is that preferences are defined over non-default alternatives and the DM is stipulated to choose the default option only if she does not consider any other alternatives. But how are we to interpret this feature of their models? Manzini and Mariotti (2014a, p.1155) present the following interpretation: the default option is to be understood as the option of "not choosing – that is, the option of "walking away from the shop, abstaining from voting, exceeding the time limit for a move in a game of chess". We suggest that this assumption is restrictive for two reasons.

First, even if we interpret the default option as "not choosing", it is difficult to specify real-world examples that justify the assumption. Consider the example of "walking away from the shop", which is cited by Manzini and Mariotti (2014a) as an example of "not choosing". This may occur because the DM has failed to consider products sold in the shop, but, in reality, it may also occur when the DM has considered some of them, but prefers to purchase nothing. So, at least for this example, the assumption by Manzini and Mariotti (2014a) that the DM never chooses the default option when non-default options are considered, in effect, embodies an implicit assumption that the default option is the worst alternative. This is a restrictive assumption, which we would not expect to hold in general. A similar concern arises with the example cited by Manzini and Mariotti (2014a) of "abstaining from voting".

Second, more generally, the assumption is not plausible for a range of real-world examples of default options that are of interest to economists. Consider the example of the choice between an existing pension plan and new pension plans that was discussed in the introduction. The DM may choose the existing plan because she has not considered the new plans, but, in reality, she may also choose it because she considered some of the new plans, and preferred the existing plan to the new plans she considered. If an economic model specified that the existing plan is the default option and is only chosen when new plans are not considered, then the model does, in effect, incorporate an implicit assumption that the default option is the worst alternative. In general, this is a restrictive assumption. Moreover, in making this assumption, Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017) depart from an important recent strand of empirical research on consideration sets. The econometric models of consideration sets in Goeree (2008), Ho et al. (2017), Heiss et al. (2016) and Abaluck and Adams (2017) do not assume that the default option is chosen only if no other alternatives are considered.

Moreover, this restrictive assumption is of particular concern, because in Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017), the ability to disentangle the influence of consideration sets and preferences arises directly from this assumption. For instance, an immediate consequence of this assumption is that, in Manzini and Mariotti (2014a), for any non-empty $B \subset X$, the probability that the consideration set is the empty set (in our notation, $\alpha(\emptyset, B)$) is simply given by $\bar{c}(\bar{x}, \bar{B})$. Having retrieved, in this way, the probability of empty consideration sets, we can then identify the attention parameters. In the model of Manzini and Mariotti (2014a), for any non-default alternative x_i, $\bar{c}(\bar{x}, x_i\bar{x}) = \alpha(\emptyset, \{x_i\}) = 1 - \gamma_i$. So we can solve for $\gamma_i = 1 - \bar{c}(\bar{x}, x_i\bar{x})$. But the reasoning here rests crucially on the assumption that the default option is chosen only when no other alternatives are considered. Once we allow that the DM might choose $\bar{x}$ when x_i is considered, the equality $\bar{c}(\bar{x}, x_i\bar{x}) = \alpha(\emptyset, \{x_i\}) = 1 - \gamma_i$ no longer holds.[9]

The second assumption in Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017) is that, whereas consideration-set dispositions are stochastic, preferences are deterministic. We avoid this asymmetric treatment of consideration sets and preferences for the following reasons. In general, the asymmetry is unmotivated. If the choice data are deterministic, as in Masatlioglu et al. (2012), it is natural to model both the DM's consideration-set dispositions and her preferences as deterministic. But if the choice data are stochastic, surely it is natural to model both the DM's consideration-set dispositions and her preferences as stochastic. What is the motivation for modelling one as stochastic and the other as deterministic? In specific examples, it is conceivable that there is motivation for an asymmetric treatment – for instance, if, in a particular example, there is evidence that consideration sets are "more random" than preferences.[10] But we would not

[9] Manzini and Mariotti (2014a, p.1157) note that, in their model, the attention parameters can be identified even if data on choice probabilities from choice sets of size 2 are excluded from the choice function. But nevertheless, even in this case, the attention parameters can be identified in a similar fashion simply from the rationalisation conditions of the form $\alpha(\emptyset, B)) = \bar{c}(\bar{x}, \bar{B})$. This reasoning rests crucially on the assumption that the default option is chosen only if no other alternatives are considered.

[10] Brady and Rehbeck (2016, p.1203) present an example in which the researcher's data takes the form of scanner data on purchases from grocery stores. While she has access to a list of offered alternatives at grocery stores, she has reason to think that "there is random variation of alternatives that are *available* to customers that is unknown to the researcher", as she does not know if "a delivery is delayed, food is spoiled, or some alternatives are out of stock". In this example, there may a particular reason for the researcher to model the consideration sets as random, but no

expect that such an asymmetry generally arises whenever attention is limited. To highlight the restrictive character of this second assumption, let us consider an example in which (i) for the sake of argument, we allow that the first assumption is satisfied (that is, the default option is chosen only if no other alternatives are considered) and (ii) the attention assignment is assumed to be independent. Suppose that the set of alternatives is $\bar{X} = \{x_1, x_2, \bar{x}\}$ and the observer only has data on choice probabilities from the choice set $\{x_1, x_2, \bar{x}\}$: in particular, the probability of choosing x_1 from this choice set is $\frac{9}{20}$ and the probability of choosing x_2 from this choice set is also $\frac{9}{20}$. Intuitively, it might be thought that one plausible hypothesis consistent with this data are that the DM is relatively indifferent between alternatives x_1 and x_2, and the attention probabilities for each of x_1 and x_2 are close to 1. Given the assumption in Manzini and Mariotti (2014a) that preferences are deterministic, together with assumptions (i) and (ii), this plausible hypothesis is ruled out *ab initio*. In particular, given these assumptions, the choice data are consistent with only two hypotheses: first, x_1 is strictly preferred to x_2, $\gamma_1 = \frac{9}{20}$ and $\gamma_2 = \frac{9}{11}$; and second, x_2 is strictly preferred to x_1, $\gamma_2 = \frac{9}{20}$ and $\gamma_1 = \frac{9}{11}$. By way of contrast, in the DO model, these observed choice probabilities are consistent with the plausible hypothesis described above – that is, the choice probabilities can be generated by (i) a ranking assignment such that, probabilistically, the DM is equally prone to prefer x_1 over x_2 as she is to prefer x_2 over x_1 and (ii) attention probabilities that are relatively high.[11]

Manzini and Mariotti (2014a) has had a considerable influence on the literature on limited attention, and our paper, in particular, has benefitted from its insights. As discussed in Section 2.3, our treatment of attention assignments relied heavily on the analysis in Manzini and Mariotti (2014a). Nevertheless, for the reasons just discussed, we suggest that it is fruitful to explore consideration-set models which dispense with the two restrictive assumptions specified above. This gives rise to the identification problem described in Section 2.3, which motivates the use of frame-dependent choice functions.

particular reason to model preferences as random.

[11]For example, in the DO model, the ranking assignment in the table below, together with attention probabilities $\gamma_1 = \gamma_2 = \frac{4}{5}$, generate choice probabilities $\bar{c}(x_1, x_1x_2\bar{x}) = \bar{c}(x_2, x_1x_2\bar{x}) = \frac{9}{20}$. Note that the rankings on which x_1 is preferred to x_2 are equally likely as those on which x_2 is preferred to x_1.

$\frac{13}{40}$	$\frac{13}{40}$	$\frac{6}{40}$	$\frac{6}{40}$	$\frac{1}{40}$	$\frac{1}{40}$
x_1	x_2	x_1	x_2	$\bar{x}$	$\bar{x}$
x_2	x_1	$\bar{x}$	$\bar{x}$	x_1	x_2
$\bar{x}$	$\bar{x}$	x_2	x_1	x_2	x_1

Aguair et al. (2019) present a generalisation of Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017) in which there is a stochastic preference relation. But, as in each of those three papers, (i) the preference relation is only defined over non-default alternatives and (ii) they assume that "the outside option is only chosen when nothing else in the choice set is considered" (Aguair et al., 2019, p.15). In other words, they effectively assume that the outside option is the worst alternative. In addition, as in Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Aguair (2017), their identification results follow directly from this assumption.

Caplin and Dean (2011) also employ frame-dependent choice functions, modelling the attention effect resulting from changes in the time available to the DM for contemplating her decision. Our model deviates from theirs in two principal respects. First, whereas they model a specific kind of attention effect, our paper is applicable to an arbitrary attention effect. Second, they have two versions of the model: one in which both preferences and consideration-set dispositions are deterministic; and one in which preferences are deterministic and consideration-set dispositions are stochastic. In contrast, for the models presented in our paper, both preferences and consideration-set dispositions are stochastic.[12]

Cattaneo et al. (2019) present a relatively general consideration-set model, of which Manzini and Mariotti (2014a), Brady and Rehbeck (2016) and Masatlioglu et al. (2012) are special cases. They present versions with and without a default-option. As in our models, consideration sets are stochastic, but in contrast to our model, preferences are deterministic. Moreover, in contrast to our NDO and DO models, as well as the models of Manzini and Mariotti (2014a), Brady and Rehbeck (2016), Aguair (2017) and Aguair et al. (2019), in general, only partial information about preferences can be retrieved from the choice data– and, potentially no information at all.[13]

[12]Note that, whereas in our models, as well as that of Caplin and Dean (2011), frames are assumed to be observed, there are a number of stochastic models in which DM's choices are modelled as influenced by framing effects, but frames are assumed to be unobserved – thus the observed choice functions are not frame-dependent. See, for example, Tserenjigmid (2019).

[13]Another consideration-set model related to Manzini and Mariotti (2014a) is Dardanoni et al. (2018). In contrast to Manzini and Mariotti (2014a), however (i) the choice data relates to choices from a single choice set rather than from a rich collection of choice sets (ii) the data is aggregate choice shares in a population and (iii) the population is assumed to have heterogenous "cognitive types" relating to their dispositions to form consideration sets. The paper shows how the distribution of cognitive characteristics can be retrieved from aggregate choice shares.

While our focus is on consideration-set models of limited attention, there is a rich variety of other approaches to modelling limited attention. Examples include models in which the DM makes her choice on the basis of optimal sampling and models of rational inattention that specify attention as a reduction in entropy.[14] Consideration-set models are one important strand in a diverse and growing body of literature on limited attention.

2.7 Conclusions and future research

In this paper, we introduce stochastic consideration sets into a random utility model. While such a model has been used in the discrete-choice econometric literature, the theoretical literature on revealed preferences has not explored such a model. The particular focus of our paper is the following identification question: can we retrieve from observed choice probabilities (i) preference probabilities (at least to the same extent as in a full-attention random-utility model) and (ii) consideration-set probabilities? Our first conclusion is a negative one. If the choice data are not enriched by using a frame-dependent choice function, then an identification problem typically arises. That is, even if strong restrictions are imposed on the DM's attention assignments, such a model places insufficient constraints on the unknowns to retrieve from the choice data the DM's consideration-set probabilities and preference probabilities.

Our paper explores one promising strategy for solving the identification problem. If the choice data are enriched by using a frame-dependent choice function, then, at least potentially, additional constraints are placed on the unknowns, allowing us to develop fully identified models. Intuitively, we are able to disentangle the influence of consideration sets and preferences because, using a frame-dependent choice function, we can isolate the influence on choice of changes in attention, holding preferences constant.

We adopt this strategy to solve the identification problem in our principal model, the NDO model, which is presented in Section 2.4. In the NDO model, there is no default option and the DM's attention assignment is assumed to be *conditional*. Following Manzini and Mariotti (2014a), we argue that, in the absence of a default option, this assumption that the attention assignment is conditional

[14]Fudenberg et al. (2018) exemplifies the optimal sampling approach to limited attention. Sims (2003) is a seminal paper in the rational inattention literature.

is a "natural" starting point. We show that this model is fully identified. Moreover, in Section 2.5 and Appendix 2, we present two variants of this model, the DO model and the MA model, which are also fully identified. In all three of these models, we use our identification strategy to obtain a closed-form solution for the unknown consideration-set probabilities and preference probabilities. Our identification strategy can also potentially be used, however, when no closed-form solution is available. Appendix 4 provides an illustration of an application of our identification strategy in which we solve for the unknown consideration-set probabilities using numerical methods. Unsurprisingly, our strategy is not universally applicable. For instance, in Appendix 3, we show that a model with a default option and an independent attention assignment (the IDO model) is not fully identified. The following table presents a summary description of the NDO, MA, DO and IDO models, as well as the identification result for each model. The final row of the table specifies the minimum number of non-default alternatives required for the identification result.

Table 1: Summary of Models and Identification Results

	NDO	MA	DO	IDO
Default option	no	no	yes	yes
Attention assignment	conditional	size-2 conditional	conditional	independent
Fully identified	yes	yes	yes	no
$\lvert X\rvert$ required for identification	5	4	3	n/a

Given the failure of identification in the IDO model, a potential topic for future research is to ascertain whether, for such models, other identification strategies are available. There are at least two alternative strategies. The first is to impose stronger assumptions about the attention effect; in contrast to Caplin and Dean (2011), our paper is applicable to an *arbitrary* attention effect. The second strategy is to impose restrictions on the preference probabilities. Our paper makes no assumptions about the preference probabilities over and above those in the standard full-attention random-utility models. Potentially, however, if additional assumptions are imposed on the preference probabilities – such as Luce's Independence of Irrelevant Alternatives assumption – then an observer may be able infer further information about preference probabilities and consideration-set probabilities from the choice data.

Appendix 1: Proofs

Theorem 1 (*Characterisation: Random-utility model*)

For proofs of this result, see Falmagne (1978), Barbera and Pattanaik (1986) and Fiorini (2004).

Theorem 2 (*Identification in the random-utility model*)

Proof

It follows from the definitions of $\mathcal{M}_{x,X\setminus A}$ and $\mathcal{U}_{x,B}$ that, for any $x \in X$:

$$\forall A \subset X\setminus\{x\}, \quad \pi(\mathcal{M}_{x,X\setminus A}) = \sum_{B\subset A} \pi(\mathcal{U}_{x,B}) \tag{2.35}$$

But we can apply the Mobius inversion formula to this result. (See Stanley (2012, p. 267)).

(*Mobius Inversion Formula for the Power Set*)

For any finite set V and any pair of functions $f, g : 2^V \to \mathbb{R}$:

$$\forall S \subset V, \;\; g(S) = \sum_{T\subset S} f(T) \iff \forall S \subset V, \;\; f(S) = \sum_{T\subset S} (-1)^{|S\setminus T|} g(T)$$

If we fix an arbitrary $x \in X$, we can think of $\pi(\mathcal{M}_{x,X\setminus A})$ and $\pi(\mathcal{U}_{x,B})$ as defining real-valued functions on the domain $2^{X\setminus\{x\}}$. So the formula licenses us to infer the following from equation (2.35):

$$\forall A \subset X\setminus\{x\}, \quad \pi(\mathcal{U}_{x,A}) = \sum_{B\subset A} (-1)^{|A\setminus B|} \pi(\mathcal{M}_{x,X\setminus B}) \tag{2.36}$$

But given that π stochastically rationalises c, it follows that

$$\forall A \subset X\setminus\{x\}, \quad \pi(\mathcal{U}_{x,A}) = \sum_{B\subset A} (-1)^{|A\setminus B|} c(x, X\setminus B) = \kappa(x, A) \tag{2.37}$$

Theorem 3 (*Identification in the default-option random-utility model*)

Proof

It follows from the definitions of $\bar{\mathcal{M}}_{x,\bar{X}\setminus A}$ and $\bar{\mathcal{U}}_{x,B}$ that, for any $x \in \bar{X}$:

$$\forall A \subset X\setminus\{x\}, \quad \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{X}\setminus A}) = \sum_{B\subset A} \bar{\pi}(\bar{\mathcal{U}}_{x,B}) \tag{2.38}$$

But we can apply the Mobius inversion formula to this result. If we fix an arbitrary $x \in \bar{X}$, we can think of $\bar{\pi}(\bar{\mathcal{M}}_{x,\bar{X}\setminus A})$ and $\bar{\pi}(\bar{\mathcal{U}}_{x,B})$ as defining real-valued functions on the domain $2^{X\setminus\{x\}}$. So the formula licenses us to infer the following from equation (2.38):

$$\forall A \subset X\setminus\{x\}, \quad \bar{\pi}(\bar{\mathcal{U}}_{x,A}) = \sum_{B\subset A} (-1)^{|A\setminus B|} \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{X}\setminus B}) \tag{2.39}$$

But given that $\bar{\pi}$ stochastically rationalises $\bar{c}$, it follows that

$$\forall A \subset X \backslash \{x\}, \quad \bar{\pi}(\bar{\mathcal{U}}_{x,A}) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{c}(x, \bar{X} \backslash B) = \bar{\kappa}(x, A) \tag{2.40}$$

Theorem 4 (*Characterisation in the default-option random-utility model*)

Proof

The ($\Rightarrow$) part of the theorem follows immediately from Theorem 3. It remains to establish the ($\Leftarrow$) part. The first step is to extend the contour function $\bar{\kappa}$ to a function whose domain encompasses not only upper contour sets of non-default options, but also upper contour sets that include the default option. In particular, we will say that the *extended contour function* is the mapping $\bar{\kappa}^E : \{(x, A) : A \subsetneq \bar{X}, x \in \bar{X} \backslash A\} \to \mathbb{R}$ such that:

(i) for any $A \subset X, x \in \bar{X} \backslash A$, $\bar{\kappa}^E(x, A) = \bar{\kappa}(x, A)$;

(ii) for $A = \{\bar{x}\}, x \in \bar{X} \backslash A$, $\bar{\kappa}^E(x, A) = \frac{1}{m} \bar{\kappa}(\bar{x}, \emptyset)$.

(iii) for any $A = \{\bar{x}, y_1, \ldots, y_n\}$ (where $\{y_1, \ldots, y_n\} \subsetneq X$) and any $x \in \bar{X} \backslash A$, $\bar{\kappa}^E(x, A) = \frac{1}{m-n} \sum_{y \in A} \bar{\kappa}^E(y, A \backslash \{y\})$.

We establish the desired result by first defining the notion of a *contour assignment* and then establishing a proposition about contour assignments. Let $\mathscr{U}$ denote the set $\{\bar{\mathcal{U}}_{x,A} \subset \mathcal{R}_{\bar{X}} : x \in \bar{X}, A \subset \bar{X} \backslash \{x\}\}$. A function $\rho : \mathscr{U} \to \mathbb{R}_+$ is said to be a *contour assignment* if

$$\sum_{x \in \bar{X}} \rho(\bar{\mathcal{U}}_{x,\emptyset}) = 1 \tag{2.41}$$

and for any $A \subsetneq \bar{X}$, $A \neq \emptyset$

$$\sum_{x \in \bar{X} \backslash A} \rho(\bar{\mathcal{U}}_{x,A}) = \sum_{y \in A} \rho(\bar{\mathcal{U}}_{y, A \backslash \{y\}}) \tag{2.42}$$

Intuitively, suppose that $\rho(\bar{\mathcal{U}}_{x,A})$ is interpreted as the probability that A is the upper contour set of x: then (2.41) says that the sum, across each alternative x, that x is the highest ranked alternative is unity; moreover, the left-hand side and the right-hand side of equation (2.42) are two different ways of specifying the probability that the alternatives in A comprise the $|A|$ highest ranked alternatives. Define the function $\rho_{\bar{\kappa}} : \mathscr{U} \to \mathbb{R}$ such that, for any $x \in \bar{X}, A \subset \bar{X} \backslash \{x\}$, $\rho_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,A}) = \bar{\kappa}^E(x, A)$. The remainder of the proof divides into three parts.

Part (i): $\rho_{\bar{\kappa}}$ is a contour assignment.

Part (ii):For any contour assignment ρ, there exists an extension of ρ to $2^{\mathcal{R}_{\bar{X}}}$ that is a ranking assignment on $\bar{X}$.

Part (iii): Let $\bar{\pi}_{\bar{\kappa}}$ be a ranking assignment on $\bar{X}$ that is an extension of contour assignment $\rho_{\bar{\kappa}}$.

Then for any $A \in \bar{\mathcal{B}}$ and any $x \in A$, $\bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{M}}_{x,A}) = \bar{c}(x, A)$.

Proof of Part (i)

Claim 1: For any $A \subset X, B \subsetneq A$

$$\sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus B) = 1 - \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B)$$

Proof

By assumption:

$$\sum_{x \in \bar{X} \setminus B} \bar{c}(x, \bar{X} \setminus B) = 1$$

So it follows that

$$\sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus B) = \sum_{x \in \bar{X} \setminus B} \bar{c}(x, \bar{X} \setminus B) - \sum_{x \in (\bar{X} \setminus B) \setminus (\bar{X} \setminus A)} \bar{c}(x, \bar{X} \setminus B) =$$

$$1 - \sum_{x \in (\bar{X} \setminus B) \setminus (\bar{X} \setminus A)} \bar{c}(x, \bar{X} \setminus B) = 1 - \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B)$$

Claim 2:

$$\text{For } n > 0, \ \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

See Hall (1998, p. 3).

Claim 3: For any $A \subset X$, $A \neq \emptyset$

$$\sum_{B \subsetneq A} (-1)^{|A \setminus B| - 1} \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B) = \sum_{x \in A} \sum_{B \subset A \setminus \{x\}} (-1)^{|A \setminus B| - 1} \bar{c}(x, \bar{X} \setminus B)$$

This holds because $\{(x, \bar{X} \setminus B) : B \subsetneq A, x \in A \setminus B\} = \{(x, \bar{X} \setminus B) : x \in A, B \subset A \setminus \{x\}\}$

Claim 4: For any $A \subset X$, $A \neq \emptyset$

$$\sum_{x \in \bar{X} \setminus A} \bar{\kappa}(x, A) = \sum_{x \in A} \bar{\kappa}(x, A \setminus \{x\})$$

Proof

$$\sum_{x \in \bar{X} \setminus A} \bar{\kappa}(x, A) = \sum_{x \in \bar{X} \setminus A} \sum_{B \subset A} (-1)^{|A \setminus B|} \bar{c}(x, \bar{X} \setminus B)$$

$$= \sum_{B \subset A} (-1)^{|A \setminus B|} \sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus B)$$

$$= \sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus A) + \sum_{B \subsetneq A} (-1)^{|A \setminus B|} \sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus B)$$

$$= 1 + \sum_{B \subsetneq A} (-1)^{|A \setminus B|} \sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus B)$$

$$= 1 + \sum_{B \subsetneq A} (-1)^{|A \setminus B|} [1 - \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B)]$$

$$= \sum_{k=0}^{|A|} (-1)^k \binom{|A|}{k} + \sum_{B \subsetneq A} (-1)^{|A \setminus B|} [- \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B)]$$

$$= \sum_{B \subsetneq A} (-1)^{|A \setminus B|} [- \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B)]$$

$$= \sum_{B \subsetneq A} (-1)^{|A \setminus B| - 1} \sum_{x \in A \setminus B} \bar{c}(x, \bar{X} \setminus B)$$

$$= \sum_{x \in A} \sum_{B \subset A \setminus \{x\}} (-1)^{|A \setminus B| - 1} \bar{c}(x, \bar{X} \setminus B)$$

$$= \sum_{x \in A} \bar{\kappa}(x, A \setminus \{x\})$$

The first equality holds by definition of $\bar{\kappa}$.

The second and third equalities result from rearranging the sums.

The fourth equality holds as, by definition of a choice function, $\sum_{x \in \bar{X} \setminus A} \bar{c}(x, \bar{X} \setminus A) = 1$.

The fifth equality holds follows from Claim 1 .

The sixth equality results from rearranging the sums.

The seventh equality follows from Claim 2.

The eighth equality holds because, in effect, we divide twice by -1, which cancels out.

The ninth equality follows from Claim 3.

Claim 5: $\rho_{\bar{\kappa}}$ satisfies the defining properties of a contour assignment.

Proof

$\rho_{\bar{\kappa}}$ is nonnegative because $\bar{\kappa}$ is assumed to be nonnegative.

Claim 4, together with clause (i) of the definition of $\bar{\kappa}_c^E$, entail that $\rho_{\bar{\kappa}}$ satisfies equation (2.42) for $A \subset \bar{X}$ such that $\bar{x} \notin A$.

Clauses (ii) and (iii) of the definition of $\bar{\kappa}_c^E$ entail that $\rho_{\bar{\kappa}}$ satisfies equation (2.42) for $A \subset \bar{X}$ such that $\bar{x} \in A$.

The definition of $\rho_{\bar{\kappa}}$ entails that $\rho_{\bar{\kappa}}$ satisfies equation (2.41) because, by definition of $\bar{\kappa}$ and $\bar{\kappa}^E$, for any $x \in \bar{X}$, $\bar{\kappa}^E(x, \emptyset) = \bar{\kappa}(x, \emptyset) = \bar{c}(x, \bar{X})$.

Proof of Part (ii)

For non-empty subset of $\bar{X}$, A, let $\sigma(A)$ denote the set of all sequences of distinct alternatives in A. Let $\mathcal{S} := \bigcup_{A \in 2^{\bar{X}} \setminus \{\emptyset\}} \sigma(A)$.

Define the function $\phi : \mathcal{S} \to \mathbb{R}$ such that:

(i) for any $y \in \bar{X}$, $\phi(\langle y \rangle) = \rho(\bar{\mathcal{U}}_{y,\emptyset})$.

(ii) for any distinct $y_1, \dots, y_n \in \bar{X}$

$$\phi(\langle y_1, \dots, y_n \rangle) = \frac{\rho(\bar{\mathcal{U}}_{y_n, \{y_1, \dots, y_{n-1}\}})\phi(\langle y_1, \dots, y_{n-1} \rangle)}{\sum_{s \in \sigma(\{y_1, \dots, y_{n-1}\})} \phi(s)} \tag{2.43}$$

Let $\ell := m + 1$.

The desired ranking assignment, $\bar{\pi}_\rho : 2^{\mathcal{R}_{\bar{X}}} \to [0, 1]$, is defined to be such that, for any set of rankings $R \in 2^{\mathcal{R}_{\bar{X}}}$:

(i) if $R = \emptyset$, then $\bar{\pi}_\rho(R) = 0$;

(ii) if R is a singleton $\{\succ\}$ such that $y_1 \succ y_2 \succ \ldots \succ y_\ell$, then $\bar{\pi}_\rho(\{\succ\}) = \phi(\langle y_1, y_2, \dots, y_\ell \rangle)$;

(iii) otherwise

$$\bar{\pi}_\rho(R) = \sum_{\succ \in R} \bar{\pi}_\rho(\{\succ\})$$

In Claim 2, we show that $\sum_{\succ \in \mathcal{R}_{\bar{X}}} \bar{\pi}_\rho(\{\succ\}) = 1$, so $\bar{\pi}_\rho$ is, indeed, a probability measure on the event space $(\mathcal{R}_{\bar{X}}, 2^{\mathcal{R}_{\bar{X}}})$.

In Claim 6, we show that $\bar{\pi}_\rho$ is an extension of ρ.

Claim 1: Consider any $i \in \{0, 1, \dots, \ell - 1\}$.

$$\sum_{(y,A):|A|=i, y \in \bar{X} \setminus A} \rho(\bar{\mathcal{U}}_{y,A}) = 1 \tag{2.44}$$

Proof

Basis step

For $i = 0$, the desired result follows from the fact that, by definition of a contour assignment

$$\sum_{y \in \bar{X} \setminus \emptyset} \rho(\bar{\mathcal{U}}_{y,\emptyset}) = 1 \tag{2.45}$$

Induction step

The induction hypothesis is that Claim 1 holds for $i = n < \ell - 1$. We want to show that, given this hypothesis, it also holds for $i = n + 1$. For an arbitrary $A \subset X$ such that $|A| = n + 1$, the definition of a contour assignment entails that

$$\sum_{x \in \bar{X} \setminus A} \rho(\bar{\mathcal{U}}_{x,A}) = \sum_{y \in A} \rho(\bar{\mathcal{U}}_{y,A \setminus \{y\}}) \tag{2.46}$$

Summing over both sides we obtain

$$\sum_{A \subset X : |A| = n+1} \sum_{x \in \bar{X} \setminus A} \rho(\bar{\mathcal{U}}_{x,A}) = \sum_{A \subset X : |A| = n+1} \sum_{y \in A} \rho(\bar{\mathcal{U}}_{y,A \setminus \{y\}}) = \sum_{A \subset X : |A| = n} \sum_{y \in \bar{X} \setminus A} \rho(\bar{\mathcal{U}}_{y,A}) = 1 \tag{2.47}$$

The first equality is obtained by summing over both sides.

The second equality holds because $\{(y, B) :$ for some $A \subset \bar{X}$ such that $|A| = n + 1, y \in A, B = A \setminus \{y\}\ \} = \{(y, A) : |A| = n, y \in \bar{X} \setminus A\}$.

The third equality holds by the induction hypothesis.

Claim 2

$$\sum_{s \in \sigma(\bar{X})} \phi(s) = 1 \tag{2.48}$$

Therefore

$$\sum_{\succ \in \mathcal{R}_{\bar{X}}} \bar{\pi}_\rho(\{\succ\}) = 1 \tag{2.49}$$

so $\bar{\pi}_\rho$ is a ranking assignment on $\mathcal{R}_{\bar{X}}$.

Proof

By definition of ϕ:

$$\phi(\langle y_1, \ldots, y_\ell \rangle) = \frac{\rho(\bar{\mathcal{U}}_{y_\ell, \bar{X} \setminus \{y_\ell\}}) \phi(\langle y_1, \ldots, y_{\ell-1} \rangle)}{\sum_{s \in \sigma(\{y_1, \ldots, y_{\ell-1}\})} \phi(s)} \tag{2.50}$$

Summing over both sides:

$$\begin{aligned}
\sum_{s \in \sigma(\bar{X})} \phi(s) &= \sum_{y_\ell \in \bar{X}} \sum_{t \in \sigma(\bar{X} \setminus \{y_\ell\})} \frac{\rho(\bar{\mathcal{U}}_{y_\ell, \bar{X} \setminus \{y_\ell\}}) \phi(t)}{\sum_{s \in \sigma(\{y_1, \ldots, y_{\ell-1}\})} \phi(s)} \\
&= \sum_{y_\ell \in \bar{X}} \rho(\bar{\mathcal{U}}_{y_\ell, \bar{X} \setminus \{y_\ell\}}) \sum_{t \in \sigma(y \in \bar{X} \setminus \{y_\ell\})} \frac{\phi(t)}{\sum_{s \in \sigma(\bar{X} \setminus \{y_\ell\})} \phi(s)} \\
&= \sum_{y_\ell \in \bar{X}} \rho(\bar{\mathcal{U}}_{y_\ell, \bar{X} \setminus \{y_\ell\}}) = 1
\end{aligned} \tag{2.51}$$

The final equality is implied by the instance of Claim 1 where $i = \ell - 1$.

In order to specify the next claim, we first define the symbol $\oplus$ as follows: for any $y_1, \ldots, y_n \in \bar{X}$ and $k < n$, if $s = \langle y_1, \ldots, y_k \rangle$, and $s' = \langle y_{k+1}, \ldots, y_n \rangle$ let $s \oplus s'$ denote $\langle y_1, \ldots, y_n \rangle$.

Claim 3

For any $A \subsetneq \bar{X}$, $y \in \bar{X} \backslash A$:

$$\rho(\bar{\mathcal{U}}_{y,A}) = \sum_{s \in \sigma(A)} \phi(s \oplus \langle y \rangle) \tag{2.52}$$

Proof

Consider any $s \in \sigma(A)$. By definition of ϕ:

$$\phi(s \oplus \langle y \rangle) = \frac{\rho(\bar{\mathcal{U}}_{y,A})\phi(s)}{\sum_{t \in \sigma(A)} \phi(t)} \tag{2.53}$$

Summing over both sides:

$$\sum_{s \in \sigma(A)} \phi(s \oplus \langle y \rangle) = \sum_{s \in \sigma(A)} \frac{\rho(\bar{\mathcal{U}}_{y,A})\phi(s)}{\sum_{t \in \sigma(A)} \phi(t)} = \rho(\bar{\mathcal{U}}_{y,A}) \sum_{s \in \sigma(A)} \frac{\phi(s)}{\sum_{t \in \sigma(A)} \phi(t)} = \rho(\bar{\mathcal{U}}_{y,A}) \tag{2.54}$$

Claim 4

$$\sum_{x \in \bar{X} \backslash A} \rho(\bar{\mathcal{U}}_{x,A}) = \sum_{s \in \sigma(A)} \phi(s) \tag{2.55}$$

Proof

$$\sum_{x \in \bar{X} \backslash A} \rho(\bar{\mathcal{U}}_{x,A}) = \sum_{y \in A} \rho(\bar{\mathcal{U}}_{y,A \backslash \{y\}}) = \sum_{y \in A} \sum_{s \in \sigma(A \backslash \{y\})} \phi(s \oplus \langle y \rangle) = \sum_{s \in \sigma(A)} \phi(s) \tag{2.56}$$

The first equality follows from the definition of a contour function.

The second equality follows from Claim 3.

The third equality holds because $\{s \oplus \langle y \rangle : y \in A, s \in \sigma(A \backslash \{y\})\} = \sigma(A)$.

Claim 5

For any $A \subsetneq \bar{X}$ and any $s \in \sigma(A)$:

$$\sum_{s' \in \sigma(\bar{X} \backslash A)} \phi(s \oplus s') = \phi(s) \tag{2.57}$$

Proof

Basis step

$$\sum_{y \in \bar{X} \backslash A} \phi(s \oplus \langle y \rangle) = \sum_{y \in \bar{X} \backslash A} \rho(\bar{\mathcal{U}}_{y,A}) \frac{\phi(s)}{\sum_{s' \in \sigma(A)} \phi(s')} = \phi(s) \tag{2.58}$$

The first equality holds by definition of ϕ.

The second equality follows from Claim 4.

Induction step

The induction hypothesis is:

$$\sum_{y_1\in\bar{X}\setminus A}\sum_{y_2\in\bar{X}\setminus(A\cup\{y_1\})}\cdots\sum_{y_{k-1}\in\bar{X}\setminus(A\cup\{y_1,\dots,y_{k-2}\})}\sum_{y_k\in\bar{X}\setminus A(\cup\{y_1,\dots,y_{k-1}\})}\phi(s\otimes\langle y_1,\dots,y_k\rangle)=\phi(s)$$

We want to show

$$\sum_{y_1\in\bar{X}\setminus A}\sum_{y_2\in\bar{X}\setminus(A\cup\{y_1\})}\cdots\sum_{y_k\in\bar{X}\setminus(A\cup\{y_1,\dots,y_{k-1}\})}\sum_{y_{k+1}\in\bar{X}\setminus(A\cup\{y_1,\dots,y_k\})}\phi(s\otimes\langle y_1,\dots,y_k,y_{k+1}\rangle)=\phi(s)$$

Proof

$$\begin{aligned}
&\sum_{y_1\in\bar{X}\setminus A}\sum_{y_2\in\bar{X}\setminus(A\cup\{y_1\})}\cdots\sum_{y_k\in\bar{X}\setminus(A\cup\{y_1,\dots,y_{k-1}\})}\sum_{y_{k+1}\in\bar{X}\setminus(A\cup\{y_1,\dots,y_k\})}\phi(s\otimes\langle y_1,\dots,y_k,y_{k+1}\rangle)\\
&=\sum_{y_1\in\bar{X}\setminus A}\sum_{y_2\in\bar{X}\setminus(A\cup\{y_1\})}\cdots\\
&\sum_{y_k\in\bar{X}\setminus(A\cup\{y_1,\dots,y_{k-1}\})}\sum_{y_{k+1}\in\bar{X}\setminus(A\cup\{y_1,\dots,y_k\})}\rho(\bar{\mathcal{U}}_{y_{k+1},A\cup\{y_1,\dots,y_k\}})\frac{\phi(s\oplus\langle y_1,\dots,y_k\rangle)}{\sum_{s'\in\sigma(A\cup\{y_1,\dots,y_k\})}\phi(s')}\\
&=\sum_{y_1\in\bar{X}\setminus A}\sum_{y_2\in\bar{X}\setminus(A\cup\{y_1\})}\cdots\sum_{y_k\in\bar{X}\setminus(A\cup\{y_1,\dots,y_{k-1}\})}\phi(s\oplus\langle y_1,\dots,y_k\rangle)=\phi(s)
\end{aligned}$$

The first equality follows from the definition of ϕ.

The second equality follows from Claim 4.

The third equality is obtained from the induction hypothesis.

Claim 6

$\bar{\pi}_\rho$ is an extension of ρ. That is, for any $A\subsetneq\bar{X}, y\in\bar{X}\setminus A$

$$\bar{\pi}_\rho(\bar{\mathcal{U}}_{y,A})=\rho(\bar{\mathcal{U}}_{y,A})$$

Proof

$$\bar{\pi}_\rho(\bar{\mathcal{U}}_{y,A})=\sum_{s\in\sigma(A)}\sum_{s'\in\sigma(\bar{X}\setminus(A\cup\{y\}))}\phi((s\oplus\langle y\rangle)\oplus s')=\sum_{s\in\sigma(A)}\phi(s\oplus\langle y\rangle)=\rho(\bar{\mathcal{U}}_{y,A})$$

The first equality follows from the definition of $\bar{\pi}_\rho$ and the definition of the event $\bar{\mathcal{U}}_{x,A}$.

The second equality follows from Claim 5.

The third equality follows from Claim 3.

Proof of Part (iii)

From part (ii), there exists an extension of $\rho_{\bar{\kappa}}$ to $\mathcal{R}_{\bar{X}}$. Let $\bar{\pi}_{\bar{\kappa}}$ denote one such ranking on $\bar{X}$.

As $\bar{\pi}_{\bar{\kappa}}$ is an extension of $\rho_{\bar{\kappa}}$, thus for any $x \in \bar{X}, A \subset \bar{X}\backslash\{x\}$, $\bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,A}) = \rho_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,A}) = \bar{\kappa}^E(x, A)$. In particular, for any $x \in \bar{X}, A \subset X\backslash\{x\}$, $\bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,A}) = \bar{\kappa}(x, A)$. Plugging in the definition of $\bar{\kappa}$ yields:

$$\forall x \in \bar{X}, A \subset X\backslash\{x\}, \quad \bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,A}) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{c}(x, \bar{X}\backslash B)$$

Applying the Mobius inversion formula gives:

$$\forall x \in \bar{X}, A \subset X\backslash\{x\}, \quad \bar{c}(x, \bar{X}\backslash A) = \sum_{B \subset A} \bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,B})$$

But it follows from the definition of the event of x being the maximum in $\bar{X}\backslash A$ and the event of B being the upper contour set of x that

$$\forall x \in \bar{X}, A \subset X\backslash\{x\}, \quad \bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{M}}_{x,\bar{X}\backslash A}) = \sum_{B \subset A} \bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{U}}_{x,B})$$

So

$$\forall x \in \bar{X}, A \subset X\backslash\{x\}, \quad \bar{\pi}_{\bar{\kappa}}(\bar{\mathcal{M}}_{x,\bar{X}\backslash A}) = \bar{c}(x, \bar{X}\backslash A)$$

Lemma 1 (*Reduced row echelon form for the NDO model*)

Proof

We will use the following notation for the coefficients in equation (2.16). Let

$$q^i_{ij} = -c'(x_i, x_i x_j), \; q^j_{ij} = 1 - c'(x_i, x_i x_j), \; b_{ij} = c'(x_i, x_i x_j) - c(x_i, x_i x_j)$$

We can then rewrite (2.16) as follows:

$$-b_{ij}\iota_j + q^j_{ij}\Delta_j - b_{ij}\iota_i + q^i_{ij}\Delta_i = b_{ij}, \; i, j \in \{1, \ldots, m\}, i < j$$

The proof divides into two stages. First, we prove the result for the case in which there are five alternatives. Second, we then extend the proof to the case in which there are $m > 5$ alternatives. If there are five non-default options, then the equation system (2.16) comprises 10 equations. It will be helpful to order the rows of the augmented matrix for the equation system as follows:

$$[\mathbf{Q}, \mathbf{b}] = \begin{bmatrix} -b_{15} & q_{15}^5 & 0 & 0 & 0 & 0 & 0 & 0 & -b_{15} & q_{15}^1 & b_{15} \\ 0 & 0 & -b_{14} & q_{14}^4 & 0 & 0 & 0 & 0 & -b_{14} & q_{14}^1 & b_{14} \\ 0 & 0 & 0 & 0 & -b_{13} & q_{13}^3 & 0 & 0 & -b_{13} & q_{13}^1 & b_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_{12} & q_{12}^2 & -b_{12} & q_{12}^1 & b_{12} \\ 0 & 0 & 0 & 0 & -b_{23} & q_{23}^3 & -b_{23} & q_{23}^2 & 0 & 0 & b_{23} \\ -b_{25} & q_{25}^5 & 0 & 0 & 0 & 0 & -b_{25} & q_{25}^2 & 0 & 0 & b_{25} \\ 0 & 0 & -b_{24} & q_{24}^4 & 0 & 0 & -b_{24} & q_{24}^2 & 0 & 0 & b_{24} \\ -b_{35} & q_{35}^5 & 0 & 0 & -b_{35} & q_{35}^3 & 0 & 0 & 0 & 0 & b_{35} \\ 0 & 0 & -b_{34} & q_{34}^4 & -b_{34} & q_{34}^3 & 0 & 0 & 0 & 0 & b_{34} \\ -b_{45} & q_{45}^5 & -b_{45} & q_{45}^4 & 0 & 0 & 0 & 0 & 0 & 0 & b_{45} \end{bmatrix} \tag{2.59}$$

First, for each row in this matrix, we perform the row operation of dividing it by the final element. (By the NDO-nonzero condition, the final element of each row is non-zero). The result is

$$\begin{bmatrix} -1 & p_{15}^5 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & p_{15}^1 & 1 \\ 0 & 0 & -1 & p_{14}^4 & 0 & 0 & 0 & 0 & -1 & p_{14}^1 & 1 \\ 0 & 0 & 0 & 0 & -1 & p_{13}^3 & 0 & 0 & -1 & p_{13}^1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & p_{12}^2 & -1 & p_{12}^1 & 1 \\ 0 & 0 & 0 & 0 & -1 & p_{23}^3 & -1 & p_{23}^2 & 0 & 0 & 1 \\ -1 & p_{25}^5 & 0 & 0 & 0 & 0 & -1 & p_{25}^2 & 0 & 0 & 1 \\ 0 & 0 & -1 & p_{24}^4 & 0 & 0 & -1 & p_{24}^2 & 0 & 0 & 1 \\ -1 & p_{35}^5 & 0 & 0 & -1 & p_{35}^3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & p_{34}^4 & -1 & p_{34}^3 & 0 & 0 & 0 & 0 & 1 \\ -1 & p_{45}^5 & -1 & p_{45}^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $p_{ij}^j = \frac{q_{ij}^j}{b_{ij}}, p_{ij}^i = \frac{q_{ij}^i}{b_{ij}}$. Then, from row 5, subtract rows 3 and 4. From row 6, subtract rows 1 and 4. From row 7, subtract rows 2 and 4. From row 8, subtract rows 1 and 3. From row 9, subtract rows 2 and 3. From row 10, subtract rows 1 and 2. The final six rows of the matrix then become

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & d_{23}^3 & 0 & d_{23}^2 & 2 & d_{23}^1 & -1 \\ 0 & d_{25}^5 & 0 & 0 & 0 & 0 & 0 & d_{25}^2 & 2 & d_{25}^1 & -1 \\ 0 & 0 & 0 & d_{24}^4 & 0 & 0 & 0 & d_{24}^2 & 2 & d_{24}^1 & -1 \\ 0 & d_{35}^5 & 0 & 0 & 0 & d_{35}^3 & 0 & 0 & 2 & d_{35}^1 & -1 \\ 0 & 0 & 0 & d_{34}^4 & 0 & d_{34}^3 & 0 & 0 & 2 & d_{34}^1 & -1 \\ 0 & d_{45}^5 & 0 & d_{45}^4 & 0 & 0 & 0 & 0 & 2 & d_{45}^1 & -1 \end{bmatrix}$$

where $d_{ij}^1 = -p_{1i}^1 - p_{1j}^1$; for $j \neq 1, d_{ij}^j = p_{ij}^j - p_{1j}^j$, for $i \neq 1, d_{ij}^i = p_{ij}^i - p_{1i}^i$. From the final five rows, subtract row 5. The final five rows then become:

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & d_{25}^5 & 0 & 0 & 0 & -d_{23}^3 & 0 & e_{25}^2 & 0 & e_{25}^1 & 0 \\ 0 & 0 & 0 & d_{24}^4 & 0 & -d_{23}^3 & 0 & e_{24}^2 & 0 & e_{24}^1 & 0 \\ 0 & d_{35}^5 & 0 & 0 & 0 & e_{35}^3 & 0 & -d_{23}^2 & 0 & e_{35}^1 & 0 \\ 0 & 0 & 0 & d_{34}^4 & 0 & e_{34}^3 & 0 & -d_{23}^2 & 0 & e_{34}^1 & 0 \\ 0 & d_{45}^5 & 0 & d_{45}^4 & 0 & -d_{23}^3 & 0 & -d_{23}^2 & 0 & e_{45}^1 & 0 \end{bmatrix}$$

where $e_{ij}^k = d_{ij}^k - d_{23}^k$. From line 8, subtract line 6 times $\frac{d_{35}^5}{d_{25}^5}$. From line 9, subtract line 7 times $\frac{d_{34}^4}{d_{24}^4}$. The final three lines then become

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & r_{35}^3 & 0 & r_{35}^2 & 0 & r_{35}^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{34}^3 & 0 & r_{34}^2 & 0 & r_{34}^1 & 0 \\ 0 & d_{45}^5 & 0 & d_{45}^4 & 0 & -d_{23}^3 & 0 & -d_{23}^2 & 0 & e_{45}^1 & 0 \end{bmatrix}$$

where $r_{35}^1 = e_{35}^1 - e_{25}^1 \times \frac{d_{35}^5}{d_{25}^5}$, $r_{34}^1 = e_{34}^1 - e_{24}^1 \times \frac{d_{34}^4}{d_{24}^4}$, $r_{35}^2 = -d_{23}^2 - e_{25}^2 \times \frac{d_{35}^5}{d_{25}^5}$, $r_{34}^2 = -d_{23}^2 - e_{24}^2 \times \frac{d_{34}^4}{d_{24}^4}$, $r_{35}^3 = e_{35}^3 + d_{23}^3 \times \frac{d_{35}^5}{d_{25}^5}$, $r_{34}^3 = e_{34}^3 + d_{23}^3 \times \frac{d_{34}^4}{d_{24}^4}$. From line 9, subtract line 8 time $\frac{r_{34}^3}{r_{35}^3}$. Then divide line 9 by its eighth element. The final two lines become:

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t_2 & 0 \\ 0 & d_{45}^5 & 0 & d_{45}^4 & 0 & -d_{23}^3 & 0 & -d_{23}^2 & 0 & e_{45}^1 & 0 \end{bmatrix}$$

where $t_2 = [r_{34}^1 - \frac{r_{34}^3}{r_{35}^3} r_{35}^1]/[r_{34}^2 - \frac{r_{34}^3}{r_{35}^3} r_{35}^2]$. From line 8, subtract line 9 times r_{35}^2. Then divide line 8 by its sixth element. Line 8 then becomes:

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & t_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $t_3 = \frac{r^1_{35} - t_2 r^2_{35}}{r^3_{35}}$. From line 7, subtract line 9 times e^2_{24} and line 8 times $-d^3_{23}$. Then divide line 7 by its fourth element. From line 6, subtract line 9 times e^2_{25} and line 8 times $-d^3_{23}$. Then divide line 6 by its second element. From line 5, subtract line 9 times d^2_{23} and line 8 times d^3_{23}. Then divide line 5 by 2. Lines 5, 6 and 7 become:

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $t_4 = \frac{e^1_{24} - e^2_{24} t_2 + d^3_{23} t_3}{d^4_{24}}$, $t_5 = \frac{e^1_{25} - e^2_{25} t_2 + d^3_{23} t_3}{d^5_{25}}$ and $s_1 = \frac{d^1_{23} - d^2_{23} t_2 - d^3_{23} t_3}{2}$. From line 1, subtract line 6 times p^5_{15}; from line 2, subtract line 7 times p^4_{14}; from line 3, subtract line 8 times p^3_{13}; from line 4, subtract line 9 times p^2_{12}. To each of these four rows, add row 5, then multiply the result by -1. The first four rows then become:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_5 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & s_4 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & s_3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s_2 & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $s_i = t_i p^i_{1i} - p^1_{1i} - s_1$ for $i = 2, \ldots, 5$. From row 10, subtract d^5_{45} by row 6, d^4_{45} by row 7, $-d^3_{23}$ by row 8, and $-d^2_{23}$ by row 9. The result is a row of zeroes except for, potentially, the tenth element. So either (a) this element must also be zero or (b) $\Delta_1 = 0$.[15] But the second possibility is ruled out by the following claim:

<u>Claim 1</u>: $\Delta_1 \neq 0$.

[15] The matrix corresponds to a system of equations. In particular, as all of the elements in row 10 are zeros except for, potentially, the tenth element, it follows that row 10 corresponds to the equation $k\Delta_1 = 0$, where k is the tenth element in row 10. Thus either k or Δ_1 must be zero.

Proof: Suppose, for contradiction, that $\Delta_1 = 0$. From the matrix above, for $i = 2, 3, 4, 5$, $\Delta_i = -t_i \Delta_1$. So also for $i = 2, 3, 4, 5$, $\Delta_i = 0$, thus for $i = 1, 2, 3, 4, 5$, $\iota_i = \iota_i'$ and so $c = c'$. But this contradicts the NDO non-zero condition.

Reordering the rows yields the reduced row echelon form matrix.

We now extend this proof to a proof of the general case where $|X| \geq 5$. Suppose we specify the rationalisation conditions in a matrix in the following order.

(i) The first 10 equations are the 10 conditions in proof for the case where $m = 10$, and are listed in the same order as in that matrix.

(ii) The next $2(m-5)$ equations, are, respectively, the equations for the sets $\{x_1, x_6\}$, $\{x_2, x_6\}$, $\{x_1, x_7\}$, $\{x_2, x_7\}$, $\ldots \{x_1, x_m\}$, $\{x_2, x_m\}$.

(iii) The final $\binom{m}{2}$ - 2m conditions are enumerated next in any order.

Suppose that first we perform the same row operations as were performed in the example with $m = 10$. The result is the following matrix.

$$\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_5 & -\frac{1}{2} \\
0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_5 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & s_4 & -\frac{1}{2} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & s_3 & -\frac{1}{2} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & t_3 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s_2 & -\frac{1}{2} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t_2 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_1 & -\frac{1}{2} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ldots & -1 & p^6_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & p^1_{16} & 1 \\
0 & 0 & \ldots & -1 & p^6_{26} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & p^2_{26} & 0 & 0 & 1 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & p^m_{1m} & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & p^1_{1m} & 1 \\
-1 & p^m_{2m} & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & p^2_{2m} & 0 & 0 & 1 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

Perform the following row operations on rows 11 and 12.

<u>Step 1</u>: From row 12, subtract row 11, row 9, and row 8 times p^2_{26}, add row 7, then divide the resulting row by $d^6_{26} = p^6_{26} - p^6_{16}$.

Step 2: From row 11, subtract row 12 times p^6_{16}, add row 9, and multiply the result by -1.

Step 3: Move rows 11 and 12 up to respectively, rows 1 and 2, moving all the other rows down by two places.

The first and second rows then become:

$$\begin{bmatrix} 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_6 & -\frac{1}{2} \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_6 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $t_6 = \frac{s_2 - p^1_{16} - s1 - p^2_{26} t_2}{d^6_{26}}$ and $s_6 = t_6 p^6_{16} - p^1_{16} - s_1$.

Repeat similar three steps for the pair of rows 13 and 14, the pair 15 and 16, and so forth. More, precisely, for each $i \in \{7, 8, \dots, m\}$, perform the following row operations on rows $2i-1$ and $2i$.

Step 1: From row $2i$, subtract row $2i-1$, row 9, and row 8 times p^2_{2i}, add row 7, then divide the resulting row by $d^i_{2i} = p^i_{2i} - p^i_{1i}$.

Step 2: From row $2i-1$, subtract row $2i$ times p^i_{1i}, add row 9, and multiply the result by -1.

Step 3: Move rows $2i-1$ and $2i$ up to respectively, rows 1 and 2, moving all the other rows down by two places.

The result is the following matrix:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_m & -\frac{1}{2} \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_m & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_5 & -\frac{1}{2} \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_5 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & s_4 & -\frac{1}{2} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & s_3 & -\frac{1}{2} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & t_3 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s_2 & -\frac{1}{2} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t_2 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_1 & -\frac{1}{2} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where, for $i = 6, 7, \ldots, m$, $t_i = \frac{s_2 - p^1_{1i} - s1 - p^2_{2i} t_2}{d^i_{2i}}$ and $s_i = t_i p^i_{1i} - p^1_{1i} - s_1$.

It remains to show that, in the reduced row echelon form, the rows from row $2m+1$ to $\binom{m}{2}$ are all zero rows. Any such row has the form:

$$\begin{bmatrix} & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \ldots & -1 & p^j_{ij} & \ldots & -1 & p^i_{ij} & \ldots & 1 \\ & \vdots & \vdots & & \vdots & \vdots & & \vdots \end{bmatrix}$$

Suppose that, to this row, we add rows $2(m-j)+1$ and $2(m-i)+1$, and then subtract from it p^j_{ij} times row $2(m-j)+2$ and p^i_{ij} times row $2(m-i)+2$. Then all the elements of the resulting row will be zeroes except, potentially, the penultimate element. However, this penultimate element must also be zero, applying similar reasoning to that in claim 1 above.

Theorem 5 (*Identification of attention assignments in the NDO model*)

PROOF

The rationalisation conditions for c imply that

$$(1 + \iota_1 + \iota_2)c(x_1, x_1x_2) = \iota_2 + \pi(\mathcal{M}_{x_1,x_1x_2}) \tag{2.60}$$

$$(1 + \iota_1 + \iota_3)c(x_1, x_1x_3) = \iota_3 + \pi(\mathcal{M}_{x_1,x_1x_3}) \tag{2.61}$$

$$\begin{aligned}(1 + \iota_1 + \iota_2 + \iota_3 + \iota_1\iota_2 + \iota_1\iota_3 + \iota_2\iota_3)c(x_1, x_1x_2x_3) = \iota_2\iota_3 + \\ \iota_3\pi(\mathcal{M}_{x_1,x_1x_2}) + \iota_2\pi(\mathcal{M}_{x_1,x_1x_3}) + \pi(\mathcal{M}_{x_1,x_1x_2x_3})\end{aligned} \tag{2.62}$$

Equations (2.60) and (2.61) allow us to substitute out the terms $\pi(\mathcal{M}_{x_1,x_1x_2})$ and $\pi(\mathcal{M}_{x_1,x_1x_3})$ from (2.62), yielding

$$\begin{aligned}(1 + \iota_1 + \iota_2 + \iota_3 + \iota_1\iota_2 + \iota_1\iota_3 + \iota_2\iota_3)c(x_1, x_1x_2x_3) = \pi(\mathcal{M}_{x_1,x_1x_2x_3}) + \\ \iota_3(1 + \iota_1 + \iota_2)c(x_1, x_1x_2) + \iota_2(1 + \iota_1 + \iota_3)c(x_1, x_1x_3) - \iota_2\iota_3\end{aligned} \tag{2.63}$$

Given that c' is rationalised, we could similarly show that

$$\begin{aligned}(1 + \iota'_1 + \iota'_2 + \iota'_3 + \iota'_1\iota'_2 + \iota'_1\iota'_3 + \iota'_2\iota'_3)c'(x_1, x_1x_2x_3) = \pi(\mathcal{M}_{x_1,x_1x_2x_3}) + \\ \iota'_3(1 + \iota'_1 + \iota'_2)c'(x_1, x_1x_2) + \iota'_2(1 + \iota'_1 + \iota'_3)c'(x_1, x_1x_3) - \iota'_2\iota'_3\end{aligned} \tag{2.64}$$

If we take the difference between (2.63) and (2.64), we can eliminate $\pi(\mathcal{M}_{x_1,x_1x_2x_3})$, obtaining

$$(1 + \iota_1 + \iota_2 + \iota_3 + \iota_1\iota_2 + \iota_1\iota_3 + \iota_2\iota_3)c(x_1, x_1x_2x_3) + \iota_2\iota_3 -$$

$$(\iota_3[1+\iota_1]+\iota_2\iota_3)c(x_1,x_1x_2)-(\iota_2[1+\iota_1]+\iota_2\iota_3)c(x_1,x_1x_3)=$$

$$(1+\iota_1'+\iota_2'+\iota_3'+\iota_1'\iota_2'+\iota_1'\iota_3'+\iota_2'\iota_3')c'(x_1,x_1x_2x_3)+\iota_2'\iota_3'-$$

$$(\iota_3'[1+\iota_1']+\iota_2'\iota_3')c'(x_1,x_1x_2)-(\iota_2'[1+\iota_1']+\iota_2'\iota_3')c'(x_1,x_1x_3) \tag{2.65}$$

Rearranging

$$[1+\iota_1]c(x_1,x_1x_2x_3)+$$

$$[1+\iota_1]\iota_2(c(x_1,x_1x_2x_3)-c(x_1,x_1x_3))+$$

$$[1+\iota_1]\iota_3(c(x_1,x_1x_2x_3)-c(x_1,x_1x_2))+$$

$$\iota_2\iota_3(c(x_1,x_1x_2x_3)-c(x_1,x_1x_2)-c(x_1,x_1x_3)+1)=$$

$$[1+\iota_1']c'(x_1,x_1x_2x_3)+$$

$$[1+\iota_1']\iota_2'(c'(x_1,x_1x_2x_3)-c'(x_1,x_1x_3))+$$

$$[1+\iota_1']\iota_3'(c'(x_1,x_1x_2x_3)-c'(x_1,x_1x_2))+$$

$$\iota_2'\iota_3'(c'(x_1,x_1x_2x_3)-c'(x_1,x_1x_2)-c'(x_1,x_1x_3)+1) \tag{2.66}$$

Subtracting the right-hand side from the left-hand side

$$0=[1+\iota_1](c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3))-$$

$$\Delta_1 c'(x_1,x_1x_2x_3)+$$

$$[\iota_1+1]\iota_2(c(x_1,x_1x_2x_3)-c(x_1,x_1x_3)-c'(x_1,x_1x_2x_3)+c'(x_1,x_1x_3))+$$

$$([\iota_1+1]\iota_2-[\iota_1'+1]\iota_2')(c'(x_1,x_1x_2x_3)-c'(x_1,x_1x_3))+$$

$$[\iota_1+1]\iota_3(c(x_1,x_1x_2x_3)-c(x_1,x_1x_2)-c'(x_1,x_1x_2x_3)+c'(x_1,x_1x_2))+$$

$$([\iota_1+1]\iota_3-[\iota_1'+1]\iota_3')(c'(x_1,x_1x_2x_3)-c'(x_1,x_1x_2))+$$

$$\iota_2\iota_3(c(x_1,x_1x_2x_3)-c(x_1,x_1x_2)-c(x_1,x_1x_3)-c'(x_1,x_1x_2x_3)+c'(x_1,x_1x_2)+c'(x_1,x_1x_3))+$$

$$(\iota_2\iota_3-\iota_2'\iota_3')(c'(x_1,x_1x_2x_3)-c'(x_1,x_1x_2)-c'(x_1,x_1x_3)+1) \tag{2.67}$$

First, note that $\iota_2\iota_3-\iota_2'\iota_3'=-(\iota_2\Delta_3+\iota_3\Delta_2+\Delta_2\Delta_3)$ and for $i=2,3$, $[\iota_1+1]\iota_i-[\iota_1'+1]\iota_i'=-([\iota_1+1]\Delta_i+\iota_i\Delta_1+\Delta_1\Delta_i)$. Second, from Lemma 1, for $i=2,3,\Delta_i=-t_i\Delta_1$ and for $i=1,2,3,\iota_i=-\frac{1}{2}-s_i\Delta_1$. Substituting these results into the above equation yields

$$0=[\frac{1}{2}-s_1\Delta_1](c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3))-$$

$$\Delta_1 c'(x_1, x_1x_2x_3)+$$
$$(\frac{1}{2} - s_1\Delta_1)(-\frac{1}{2} - s_2\Delta_1)(c(x_1, x_1x_2x_3) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_3))+$$
$$(t_2(\frac{1}{2} - s_1\Delta_1)\Delta_1 + (\frac{1}{2} + s_2\Delta_1)\Delta_1 + t_2\Delta_1^2)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_3))+$$
$$(\frac{1}{2} - s_1\Delta_1)(-\frac{1}{2} - s_3\Delta_1)(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2))+$$
$$(t_3(\frac{1}{2} - s_1\Delta_1)\Delta_1 + (\frac{1}{2} + s_3\Delta_1)\Delta_1 + t_3\Delta_1^2)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_2))+$$
$$(\frac{1}{2}+s_2\Delta_1)(\frac{1}{2}+s_3\Delta_1)(c(x_1, x_1x_2x_3)-c(x_1, x_1x_2)-c(x_1, x_1x_3)-c'(x_1, x_1x_2x_3)+c'(x_1, x_1x_2)+c'(x_1, x_1x_3))+$$
$$(t_2(-\frac{1}{2}-s_3\Delta_1)\Delta_1+t_3(-\frac{1}{2}-s_2\Delta_1)\Delta_1-t_2t_3\Delta_1^2)(c'(x_1, x_1x_2x_3)-c'(x_1, x_1x_2)-c'(x_1, x_1x_3)+1) \quad (2.68)$$

The result is a quadratic equation of the form $f\Delta_1^2 + g\Delta_1 + h = 0$, where expressions for f, g and h are given as follows:

$$f = s_1s_2(c(x_1, x_1x_2x_3) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_3))+$$
$$(t_2(1 - s_1) + s_2)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_3))+$$
$$s_1s_3(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2))+$$
$$(t_3(1 - s_1) + s_3)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_2))+$$
$$s_2s_3(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2) + c'(x_1, x_1x_3))$$
$$-(s_2t_3 + s_3t_2 + t_2t_3)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_2) - c'(x_1, x_1x_3) + 1) \quad (2.69)$$
$$g = -s_1(c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3))-$$
$$c'(x_1, x_1x_2x_3)+$$
$$\frac{1}{2}(s_1 - s_2)(c(x_1, x_1x_2x_3) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_3))+$$
$$\frac{1}{2}(t_2 + 1)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_3))+$$
$$\frac{1}{2}(s_1 - s_3)(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2))+$$
$$\frac{1}{2}(t_3 + 1)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_2))+$$
$$\frac{1}{2}(s_2 + s_3)(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2) + c'(x_1, x_1x_3))-$$
$$\frac{1}{2}(t_2 + t_3)(c'(x_1, x_1x_2x_3) - c'(x_1, x_1x_2) - c'(x_1, x_1x_3) + 1) \quad (2.70)$$

$$h = \frac{1}{2}(c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3))-$$
$$\frac{1}{4}(c(x_1, x_1x_2x_3) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_3))-$$
$$\frac{1}{4}(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2))+$$
$$\frac{1}{4}(c(x_1, x_1x_2x_3) - c(x_1, x_1x_2) - c(x_1, x_1x_3) - c'(x_1, x_1x_2x_3) + c'(x_1, x_1x_2) + c'(x_1, x_1x_3)) \quad (2.71)$$

We can simplify the constant term h as follows:

$$h = [\frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}][c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3]$$
$$+[\frac{1}{4} - \frac{1}{4}][-c(x_1, x_1x_3) + c'(x_1, x_1x_3)]$$
$$+[\frac{1}{4} - \frac{1}{4}][-c(x_1, x_1x_2) + c'(x_1, x_1x_2)]$$
$$= \frac{1}{4}[c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3)]$$

Moreover, we will now show that $g = 0$.

We can rewrite the expression for g as follows:

$$g = [-s_1 + \frac{1}{2}(s_1 - s_2) + \frac{1}{2}(s_1 - s_3) + \frac{1}{2}(s_2 + s_3)](c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3))+$$
$$[-1 + \frac{1}{2}t_2 + \frac{1}{2} + \frac{1}{2}t_3 + \frac{1}{2} - \frac{1}{2}(t_2 + t_3)]c'(x_1, x_1x_2x_3)+$$
$$+\frac{1}{2}(s_1 - s_2)(-c(x_1, x_1x_3) + c'(x_1, x_1x_3))+$$
$$+(\frac{1}{2}t_2 + \frac{1}{2})(-c'(x_1, x_1x_3))+$$
$$+\frac{1}{2}(s_1 - s_3)(-c(x_1, x_1x_2) + c'(x_1, x_1x_2))+$$
$$+(\frac{1}{2}t_3 + \frac{1}{2})(-c'(x_1, x_1x_2))+$$
$$\frac{1}{2}(s_2 + s_3)(-c(x_1, x_1x_2) - c(x_1, x_1x_3) + c'(x_1, x_1x_2) + c'(x_1, x_1x_3))+$$
$$-\frac{1}{2}(t_2 + t_3)(-c'(x_1, x_1x_2) - c'(x_1, x_1x_3) + 1) \quad (2.72)$$

But the first two terms are equal to zero because $-s_1 + \frac{1}{2}(s_1 - s_2) + \frac{1}{2}(s_1 - s_3) + \frac{1}{2}(s_2 + s_3) = 0$ and $-1 + \frac{1}{2}t_2 + \frac{1}{2} + \frac{1}{2}t_3 + \frac{1}{2} - \frac{1}{2}(t_2 + t_3) = 0$. If we drop these two terms out, and rearrange, we obtain

$$g = -\frac{1}{2}((s_1 + s_3)(c(x_1, x_1x_3) - c'(x_1, x_1x_3)) - (t_3 - 1)c'(x_1, x_1x_3) + t_3+$$
$$(s_1 + s_2)(c(x_1, x_1x_2) - c'(x_1, x_1x_2)) - (t_2 - 1)c'(x_1, x_1x_2) + t_2) \quad (2.73)$$

But consider rationalisation condition (2.16) for the case where $i = 1$ and $j = 2, 3$, and suppose that we substitute in $\Delta_j = -t_j\Delta_1$, $\iota_1 = -\frac{1}{2} - s_1\Delta_1$. and $\iota_j = -\frac{1}{2} - s_j\Delta_1$. Then we obtain

$$(1 - \frac{1}{2} - s_1\Delta_1 - \frac{1}{2} - s_j\Delta_1)(c'(x_1, x_1, x_j) - c(x_1, x_1, x_j)) =$$

$$-t_j\Delta_1 - (-t_j\Delta_1 + \Delta_1)c'(x_1, x_1x_j) \tag{2.74}$$

If we divide throughout by Δ_1, and rearrange, we obtain, for $j = 2, 3$:

$$(s_1 + s_j)(c(x_1, x_1, x_j) - c'(x_1, x_1, x_j)) = (t_j - 1)c'(x_1, x_1x_j) - t_j$$

So equation (2.73) implies that $g = 0$.

Accordingly, as the NDO-nonzero condition assumes that $f \neq 0$, then Δ_1 must satisfy: $\Delta_1 = \pm\sqrt{-h/f}$.

How do we ascertain whether Δ_1 is the negative or positive square root? Given that (c, c') is rationalisable, we know that ι_1 must be positive; and we also know that $\iota_1 = -\frac{1}{2} - s_1\Delta_1$. So Δ_1 must have the opposite sign to s_1. This allows us to obtain a unique solution for Δ_1.

$$\hat{\Delta}_1 = \begin{cases} -\sqrt{-h/f} & \text{if } s_1 > 0 \\ \sqrt{-h/f} & \text{otherwise} \end{cases}$$

Theorem 6 (*Identification of ranking assignments in the NDO model*)

PROOF

In order prove this result, we first establish the following claim:

Claim 1: Given the assumptions in Theorem 6, for any $B \in \mathcal{B}, x \in B$, we have $\hat{\mu}(x, B) = \pi(\mathcal{M}_{x,B})$ and $\hat{\mu}'(x, B) = \pi(\mathcal{M}_{x,B})$.

Proof

We prove the result for $\hat{\mu}$; the proof for $\hat{\mu}'$ is analogous. The proof is by induction on the size of set B.

Basis step: Claim 1 holds for $B \in \mathcal{B}$ such that $|B| = 1$.

It follows immediately from the definition of $\hat{\mu}$ that $\hat{\mu}(x, B) = 1$. Moreover clearly $\pi(\mathcal{M}_{x,B}) = 1$. Thus $\hat{\mu}(x, B) = \pi(\mathcal{M}_{x,B})$.

Induction step

Induction hypothesis: For any integer $n > 1$, suppose that Claim 1 holds for any $B \in \mathcal{B}$ such that $|B| < n$.

We want to show that Claim 1 also holds for any $B \in \mathcal{B}$ such that $|B| = n$.

Consider any $B \in \mathcal{B}$ such that $|B| = n$. From the definition of rationalisation, for any $x \in B$, we have:

$$\pi(\mathcal{M}_{x,B}) = \frac{c(x,B) - \sum_{A \subsetneq B, x \in A} \hat{\alpha}(A,B)\pi(\mathcal{M}_{x,A})}{\hat{\alpha}(B,B)} \tag{2.75}$$

But the definition of $\hat{\mu}$ specifies that

$$\hat{\mu}(x,B) = \frac{c(x,B) - \sum_{A \subsetneq B, x \in A} \hat{\alpha}(A,B)\hat{\mu}(x,A)}{\hat{\alpha}(B,B)} \tag{2.76}$$

(2.75) and (2.76), together with the induction hypothesis, entail that $\pi(\mathcal{M}_{x,B}) = \hat{\mu}(x,B)$.

Having proved claim 1, we now turn to the proof of Theorem 6. We prove the result for $\hat{\upsilon}$; the proof for $\hat{\upsilon}'$ is analogous.

Clearly, from the definitions of the sets $\mathcal{M}_{x,B}$ and $\mathcal{U}_{x,B}$, for any $x \in X$

$$\forall A \subset X \backslash \{x\}, \quad \pi(\mathcal{M}_{x,X \backslash A}) = \sum_{B \subset A} \pi(\mathcal{U}_{x,B}) \tag{2.77}$$

Suppose we apply to equation (2.77) the Mobius inversion formula for the power set. This implies

$$\forall A \subset X \backslash \{x\}, \quad \pi(\mathcal{U}_{x,A}) = \sum_{B \subset A} (-1)^{|A \backslash B|} \pi(\mathcal{M}_{x,X \backslash A}) \tag{2.78}$$

If we substitute into equation (2.78) the result in Claim 1, we obtain the desired result:

$$\forall A \subset X \backslash \{x\}, \quad \pi(\mathcal{U}_{x,A}) = \sum_{B \subset A} (-1)^{|A \backslash B|} \hat{\mu}(x, X \backslash A) = \hat{\upsilon}(x,A) \tag{2.79}$$

Theorem 7 (*Characterisation in the NDO model*)

Proof

The ($\Rightarrow$) part follows from Theorems 5 and 6. The ($\Leftarrow$) part is established as follows. Given that $\hat{\iota}$ and $\hat{\iota}'$ are positive and real, then $\hat{\alpha}$ and $\hat{\alpha}'$ are well-defined conditional attention assignments and the NDO contour functions $\hat{\upsilon}$ and $\hat{\upsilon}'$ are also well-defined. The proof of the ($\Leftarrow$) part of Theorem 7 has five steps.

(a) for any $B \in \mathcal{B}$, $\sum_{x \in B} \hat{\mu}(x,B) = 1$.

(b) for any $B \in \mathcal{B}$, for any $x \in B$, $\hat{\mu}(x,B) \geq 0$.

(c) there exists a ranking assignment on X, π, such that, for any $B \in \mathcal{B}$ and any $x \in B$, $\hat{\mu}(x,B) = \pi(\mathcal{M}_{x,B})$.

(d) π and $\hat{\alpha}$ rationalise c.

(e) π and $\hat{\alpha}'$ rationalise c'.

Proof of step (a)

The proof is by induction over the size of the choice sets.

Basis step: Part (a) holds for any $B \in \mathcal{B}$ such that $|B| = 1$.

In this case, from the definition of $\hat{\mu}$, $1 = \hat{\mu}(x, B) = \sum_{x\in B} \hat{\mu}(x, B)$.

Induction step

Induction hypothesis: Consider any $B \in \mathcal{B}$ such that $|B| > 1$. Suppose that, for any non-empty $A \subsetneq B$, we have $\sum_{x\in A} \hat{\mu}(x, A) = 1$.

We want to show that, given that the induction hypothesis holds, $\sum_{x\in B} \hat{\mu}(x, B) = 1$.

From the definition of $\hat{\mu}$, for any $x \in B$

$$c(x, B) = \sum_{A\subset B: x\in A} \hat{\alpha}(A, B)\hat{\mu}(x, A)$$

If we sum over the alternatives in B, the definition of a choice function implies that the left-hand side is 1, yielding

$$1 = \sum_{x\in B} \sum_{A\subset B: x\in A} \hat{\alpha}(A, B)\hat{\mu}(x, A)$$

But the right-hand side of this equation is, in effect, a sum of expressions $\hat{\alpha}(A, B)\hat{\mu}(x, A)$ over the set $\{(x, A) : A \subset B, x \in A\}$. Thus

$$\begin{aligned}
1 &= \sum_{x\in B} \sum_{A\subset B: x\in A} \hat{\alpha}(A, B)\hat{\mu}(x, A) = \sum_{(x,A)\in\{(x,A): A\subset B: x\in A\}} \hat{\alpha}(A, B)\hat{\mu}(x, A) \\
&= \sum_{A\subset B: A\neq\emptyset} \sum_{x\in A} \hat{\alpha}(A, B)\hat{\mu}(x, A) = \sum_{A\subset B: A\neq\emptyset} \hat{\alpha}(A, B) \sum_{x\in A} \hat{\mu}(x, A) \\
&= \sum_{A\subsetneq B: A\neq\emptyset} \hat{\alpha}(A, B) \sum_{x\in A} \hat{\mu}(x, A) + \hat{\alpha}(B, B) \sum_{x\in B} \hat{\mu}(x, B) = \sum_{A\subsetneq B: A\neq\emptyset} \hat{\alpha}(A, B) + \hat{\alpha}(B, B) \sum_{x\in B} \hat{\mu}(x, B)
\end{aligned} \tag{2.80}$$

Note that the induction hypothesis justifies the final equality in (2.80). Moreover, from the definition of a conditional attention assignment, given that $\hat{\alpha}(\emptyset, B)$ is assumed to be zero, we have

$$1 = \sum_{A\subsetneq B: A\neq\emptyset} \hat{\alpha}(A, B) + \hat{\alpha}(B, B) \tag{2.81}$$

Subtracting (2.80) from (2.81) yields

$$0 = \hat{\alpha}(B, B)(1 - \sum_{x\in B} \hat{\mu}(x, B)) \tag{2.82}$$

But as $\hat{\alpha}(B, B) > 0$, thus $\sum_{x\in B} \hat{\mu}(x, B) = 1$

Proof of step (b)

By definition of $\hat{\upsilon}$, for any $x \in X$:

$$\forall A \subset X \backslash \{x\}, \quad \hat{\upsilon}(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \hat{\mu}(x, X \backslash B) \tag{2.83}$$

Applying the Mobius inversion formula, we obtain that, for any $x \in X$:

$$\forall A \subset X \backslash \{x\}, \quad \hat{\mu}(x, X \backslash A) = \sum_{B \subset A} \hat{\upsilon}(x, B) \tag{2.84}$$

As the NDO contour function is assumed to be non-negative, it follows that $\hat{\mu}$ is non-negative.

Proof of step (c)

The form of the justification for step (c) is precisely parallel to the justification for the ($\Leftarrow$) part of Theorem 1. This is because (i) $\hat{\mu}$ has the same domain as a standard choice function and (ii) as shown in steps (a) and (b), $\hat{\mu}$ has the same formal properties as a standard choice function – it is non-negative and probabilities over a choice set sum to one.

Proof of step (d)

It follows from the definition of a maximum function that, for all $B \in \mathcal{B}, x \in B$:

$$c(x, B) = \sum_{A \subset B : x \in A} \hat{\alpha}(A, B) \hat{\mu}(x, A) \tag{2.85}$$

If we substitute into this equation the result from step (c), then the definition of rationalisation implies that c is rationalised by π and $\hat{\alpha}$.

Proof of step (e)

In the reasoning in step (b), we can replace $\hat{\upsilon}$ and $\hat{\mu}$ by $\hat{\upsilon}'$ and $\hat{\mu}'$. Given that $\hat{\upsilon} = \hat{\upsilon}'$, (2.84) implies that, for all $x \in X$:

$$\forall A \subset X \backslash \{x\}, \quad \hat{\mu}(x, X \backslash A) = \sum_{B \subset A} \hat{\upsilon}(x, B) = \sum_{B \subset A} \hat{\upsilon}'(x, B) = \hat{\mu}'(x, X \backslash A) \tag{2.86}$$

As $\hat{\mu} = \hat{\mu}'$, in the result in step (c), we can replace $\hat{\mu}$ by $\hat{\mu}'$. Using parallel reasoning to that in step (d), we can establish that π and $\hat{\alpha}'$ rationalise c'.

Theorem 8 (*Identification of attention assignments in DO model*)

Proof

The system of linear rationalisation conditions in equations (2.31) and (2.32) can be represented in matrix form as follows. Denote the vector of unknowns to be $\bar{\mathbf{v}} = (\iota_m, \Delta_m, \iota_{m-1}, \Delta_{m-1}, \ldots, \iota_1, \Delta_1)$. If we fix a particular order of the equations, we can write the equation system as $\bar{\mathbf{Q}}\bar{\mathbf{v}} = \bar{\mathbf{b}}$ where $\bar{\mathbf{Q}}$

is a matrix of coefficients of size $2\binom{m}{2} \times 2m$, and $\mathbf{b}$ is a vector of constant terms of length length $2\binom{m}{2}$. It turns out that this linear system of equations does not allow us to solve for the inattention odds vectors ι and ι'. But, as is implied by Lemma 2 below, there is only one free variable in the system of equations, which allows us to solve for all attention parameters in terms of one attention parameter.

Lemma 2 (*Reduced row echelon form for the DO model*)
Consider any choice function $(\bar{c}, \bar{c}')$ that (i) satisfies the DO-nonzero condition and (ii) is rationalised by some ranking assignment and a pair of conditional attention assignments. Then the matrix of coefficients for this choice function $\bar{\mathbf{Q}}$ has rank $2m-1$ and, in particular, the reduced row echelon form of the augmented matrix $[\bar{\mathbf{Q}}\ \bar{\mathbf{b}}]$ is given by

$$\begin{bmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \bar{s}_m & -\frac{1}{2} \\
0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \bar{t}_m & 0 \\
0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \bar{s}_{m-1} & -\frac{1}{2} \\
0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \bar{t}_{m-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \bar{s}_2 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \bar{t}_2 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \bar{s}_1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

where the terms $\bar{t}_i$, $i = 2, \dots, m$ and $\bar{s}_i$, $i = 1, \dots, m$ are defined in the proof of the lemma below.

Proof of Lemma 2
Let us use the following notation for the coefficients in equation (2.31), which will be termed *the difference equation for* $\bar{c}(x_j, x_i x_j \bar{x})$.

$$\bar{q}^i_{i,ij} = -\bar{c}'(x_i, x_i x_j \bar{x})$$

$$\bar{q}^j_{i,ij} = \bar{c}(x_i, x_i \bar{x}) - \bar{c}'(x_i, x_i x_j \bar{x})$$

$$\bar{b}_{i,ij} = \bar{c}'(x_i, x_i x_j \bar{x}) - \bar{c}(x_i, x_i x_j \bar{x})$$

We can then write equation (2.31) as follows:

$$-\bar{b}_{i,ij}\iota_j + \bar{q}^j_{i,ij}\Delta_j - \bar{b}_{i,ij}\iota_i + \bar{q}^i_{i,ij}\Delta_i = \bar{b}_{i,ij}$$

Similarly, let us define the coefficients in equation (2.32), the difference equation for $\bar{c}(x_j, x_ix_j\bar{x})$, as follows

$$\bar{q}^i_{j,ij} = \bar{c}(x_j, x_j\bar{x}) - \bar{c}'(x_j, x_ix_j\bar{x})$$

$$\bar{q}^j_{j,ij} = -\bar{c}'(x_j, x_ix_j\bar{x})$$

$$\bar{b}_{j,ij} = \bar{c}'(x_j, x_ix_j\bar{x}) - \bar{c}(x_j, x_ix_j\bar{x})$$

We can then write equation (2.32) as follows

$$-\bar{b}_{j,ij}\iota_j + \bar{q}^j_{j,ij}\Delta_j - \bar{b}_{j,ij}\iota_i + \bar{q}^i_{j,ij}\Delta_i = \bar{b}_{j,ij}$$

The proof divides into two stages. First, we prove the result for the case in which there are only three non-default options. That is, $\bar{X} = \{x_1, x_2, x_3, \bar{x}\}$. Second, we then extend the proof to the case in which there are $m > 3$ non-default options. If there are three non-default options, then we can write the augmented matrix as follows:

$$[\bar{\mathbf{Q}}, \bar{\mathbf{b}}] = \begin{bmatrix} 0 & 0 & -\bar{b}_{1,12} & \bar{q}^2_{1,12} & -\bar{b}_{1,12} & \bar{q}^1_{1,12} & \bar{b}_{1,12} \\ 0 & 0 & -\bar{b}_{2,12} & \bar{q}^2_{2,12} & -\bar{b}_{2,12} & \bar{q}^1_{2,12} & \bar{b}_{2,12} \\ -\bar{b}_{1,13} & \bar{q}^3_{1,13} & 0 & 0 & -\bar{b}_{1,13} & \bar{q}^1_{1,13} & \bar{b}_{1,13} \\ -\bar{b}_{3,13} & \bar{q}^3_{3,13} & 0 & 0 & -\bar{b}_{3,13} & \bar{q}^1_{3,13} & \bar{b}_{3,13} \\ -\bar{b}_{2,23} & \bar{q}^3_{2,23} & -\bar{b}_{2,23} & \bar{q}^2_{2,23} & 0 & 0 & \bar{b}_{2,23} \\ -\bar{b}_{3,23} & \bar{q}^3_{3,23} & -\bar{b}_{3,23} & \bar{q}^2_{3,23} & 0 & 0 & \bar{b}_{3,23} \end{bmatrix} \tag{2.87}$$

First, we apply to this matrix the row operation of dividing each row by the final element. (By clause (i) of the DO nonzero condition, the final element of every row is nonzero.) Let us define $\bar{p}^i_{i,ij} = \bar{q}^i_{i,ij}/\bar{b}_{i,ij}$, $\bar{p}^j_{i,ij} = \bar{q}^j_{i,ij}/\bar{b}_{i,ij}$, $\bar{p}^i_{j,ij} = \bar{q}^i_{j,ij}/\bar{b}_{j,ij}$, $\bar{p}^j_{j,ij} = \bar{q}^j_{j,ij}/\bar{b}_{j,ij}$. The result is the following matrix:

$$\begin{bmatrix} 0 & 0 & -1 & \bar{p}^2_{1,12} & -1 & \bar{p}^1_{1,12} & 1 \\ 0 & 0 & -1 & \bar{p}^2_{2,12} & -1 & \bar{p}^1_{2,12} & 1 \\ -1 & \bar{p}^3_{1,13} & 0 & 0 & -1 & \bar{p}^1_{1,13} & 1 \\ -1 & \bar{p}^3_{3,13} & 0 & 0 & -1 & \bar{p}^1_{3,13} & 1 \\ -1 & \bar{p}^3_{2,23} & -1 & \bar{p}^2_{2,23} & 0 & 0 & 1 \\ -1 & \bar{p}^3_{3,23} & -1 & \bar{p}^2_{3,23} & 0 & 0 & 1 \end{bmatrix}$$

Next, for every even row, subtract from that row the odd row preceding it. If we define $\bar{d}^i_{ij} = \bar{p}^i_{j,ij} - \bar{p}^i_{i,ij}, \bar{d}^j_{ij} = \bar{p}^j_{j,ij} - \bar{p}^j_{i,ij}$, then we obtain:

$$\begin{bmatrix} 0 & 0 & -1 & \bar{p}^2_{1,12} & -1 & \bar{p}^1_{1,12} & 1 \\ 0 & 0 & 0 & \bar{d}^2_{12} & 0 & \bar{d}^1_{12} & 0 \\ -1 & \bar{p}^3_{1,13} & 0 & 0 & -1 & \bar{p}^1_{1,13} & 1 \\ 0 & \bar{d}^3_{13} & 0 & 0 & 0 & \bar{d}^1_{13} & 0 \\ -1 & \bar{p}^3_{2,23} & -1 & \bar{p}^2_{2,23} & 0 & 0 & 1 \\ 0 & \bar{d}^3_{23} & 0 & \bar{d}^2_{23} & 0 & 0 & 0 \end{bmatrix}$$

For all $j > 1$, define $\bar{t}_j = \frac{\bar{d}^1_{1j}}{\bar{d}^j_{1j}}$. Suppose that we then divide row two by $\bar{d}^2_{12}$ and row four by $\bar{d}^3_{13}$. (Note that, by the DO-nonzero condition, for $j = 2, \ldots, m$, $\bar{d}^j_{1j} \neq 0$). Then we obtain

$$\begin{bmatrix} 0 & 0 & -1 & \bar{p}^2_{1,12} & -1 & \bar{p}^1_{1,12} & 1 \\ 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\ -1 & \bar{p}^3_{1,13} & 0 & 0 & -1 & \bar{p}^1_{1,13} & 1 \\ 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\ -1 & \bar{p}^3_{2,23} & -1 & \bar{p}^2_{2,23} & 0 & 0 & 1 \\ 0 & \bar{d}^3_{23} & 0 & \bar{d}^2_{23} & 0 & 0 & 0 \end{bmatrix}$$

Next, the matrix below results if we perform the following row operations:

(i) from the first row, subtract the second row times $\bar{p}^2_{1,12}$ and multiply the result by -1;

(ii) from the third row, subtract the fourth row times $\bar{p}^3_{1,13}$ and multiply the result by -1;

(iii) from the fifth row, subtract the second row times $\bar{p}^3_{2,23}$ and the fourth row times $\bar{p}^3_{2,23}$, and multiply the result by -1.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & \bar{t}_2\bar{p}^2_{1,12} - \bar{p}^1_{1,12} & -1 \\ 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\ 1 & 0 & 0 & 0 & 1 & \bar{t}_3\bar{p}^3_{1,13} - \bar{p}^1_{1,13} & -1 \\ 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\ 1 & 0 & 1 & 0 & 0 & \bar{t}_2\bar{p}^2_{2,23} + \bar{t}_3\bar{p}^3_{2,23} & -1 \\ 0 & \bar{d}^3_{23} & 0 & \bar{d}^2_{23} & 0 & 0 & 0 \end{bmatrix}$$

We obtain the following matrix if, from row one, we subtract row five, add row three, and divide the result by 2.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \bar{s}_1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\ 1 & 0 & 0 & 0 & 1 & \bar{t}_3\bar{p}^3_{1,13} - \bar{p}^1_{1,13} & -1 \\ 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\ 1 & 0 & 1 & 0 & 0 & \bar{t}_2\bar{p}^2_{2,23} + \bar{t}_3\bar{p}^3_{2,23} & -1 \\ 0 & \bar{d}^3_{23} & 0 & \bar{d}^2_{23} & 0 & 0 & 0 \end{bmatrix}$$

where $\bar{s}_1 := \frac{1}{2}[\bar{t}_2\bar{p}^2_{1,12} - \bar{p}^1_{1,12} - \bar{t}_2\bar{p}^2_{2,23} - \bar{t}_3\bar{p}^3_{2,23} + \bar{t}_3\bar{p}^3_{1,13} - \bar{p}^1_{1,13}]$.

The matrix below results if we perform the following operations:

(i) From row three subtract row one.

(ii) From row five, subtract the row that results from the previous row operation.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \bar{s}_1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & \bar{s}_3 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & \bar{s}_2 & -\frac{1}{2} \\ 0 & \bar{d}^3_{23} & 0 & \bar{d}^2_{23} & 0 & 0 & 0 \end{bmatrix}$$

where $\bar{s}_3 = \bar{t}_3\bar{p}^3_{1,13} - \bar{p}^1_{1,13} - \bar{s}_1$ and $\bar{s}_2 = \bar{t}_2\bar{p}^2_{2,23} + \bar{t}_3\bar{p}^3_{2,23} - \bar{s}_3$.

From row six, subtract row four times $\bar{d}^3_{23}$ and row two times $\bar{d}^2_{23}$. The result is a row of zeros except for, potentially, the penultimate entry, which is $-\bar{d}^3_{23}\bar{t}_3 - \bar{d}^2_{23}\bar{t}_2$. So either (a) this penultimate element must also be zero or (b) $\Delta_1 = 0$. But the second possibility is ruled out by the following claim:

<u>Claim 1</u>: $\Delta_1 \neq 0$.

Proof Suppose, for contradiction, that $\Delta_1 = 0$. From the matrix above, for $i = 2, 3$, $\Delta_i = -\bar{t}_i\Delta_1$. So $\Delta_i = 0$. Thus, for $i = 1, \ldots, 3$, $\iota_i = \iota'_i$. So $\bar{c} = \bar{c}'$. But this violates clause (i) of the DO-nonzero condition.

Claim 1 entails that all the elements of the sixth row must be equal to zero.

If we rearrange the order of the rows, we obtain the matrix specified in Lemma 2.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \bar{s}_3 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & \bar{s}_2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & \bar{s}_1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This completes the proof for the case where $m = 3$.

We now extend this proof to a proof of the general case where $|X| = m \geq 3$. Consider the system of equations comprising the following enumeration of the rationalisation equations:
(i) the first six equations are the six difference equations represented in the augmented matrix in equation (2.87), enumerated in the order in that matrix;
(ii) the next $2(m-3)$ equations are, respectively, the difference equations for $\bar{c}(x_1, x_1x_4\bar{x})$, $\bar{c}(x_4, x_1x_4\bar{x})$, $\bar{c}(x_1, x_5\bar{x})$, $\bar{c}(x_5, x_1x_5\bar{x})$, $\ldots$, $\bar{c}(x_1, x_1x_m\bar{x})$, $\bar{c}(x_m, x_1x_m\bar{x})$;
(iii) the remaining $2\binom{m}{2} - 2m$ difference equations are enumerated beneath them in any order such that, if the equation is the difference equation for $\bar{c}(x_i, x_ix_j\bar{x})$ (for some $i, j, i > j$), then it appears in an odd row, and if it is the difference equation for $\bar{c}(x_j, x_ix_j\bar{x})$, it appears the next row.

Suppose that we undertake the same row operations specified in the proof for the case where

$m = 3$. The result is an $2\binom{m}{2}$ by $2m+1$ matrix of the following form:

$$\begin{bmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \bar{s}_3 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \bar{s}_2 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{s}_1 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & -1 & \bar{p}^4_{1,14} & 0 & 0 & 0 & 0 & -1 & \bar{p}^1_{1,14} & 1 \\
0 & 0 & \dots & 0 & 0 & 0 & \bar{d}^4_{14} & 0 & 0 & 0 & 0 & 0 & \bar{d}^1_{14} & 0 \\
0 & 0 & \dots & -1 & \bar{p}^5_{1,15} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \bar{p}^1_{1,15} & 1 \\
0 & 0 & \dots & 0 & \bar{d}^5_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d}^1_{15} & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & \bar{p}^m_{1,1m} & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \bar{p}^1_{1,1m} & 1 \\
0 & \bar{d}^m_{1m} & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d}^1_{1m} & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

Perform the following row operations on rows seven and eight.

<u>Step 1</u>: Divide the eighth row by $\bar{d}^4_{14}$.

<u>Step 2</u>: Subtract from the seventh row the eighth row times $\bar{p}^4_{1,14}$, then add to the result the fifth row, and then multiply the result by -1.

<u>Step 3</u>: Move the seventh row to the first row, the eighth row to the second row, and shift all other rows down two places.

The first and second rows then become:

$$\begin{bmatrix}
0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{s}_4 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \bar{t}_4 & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

where $\bar{t}_4 = \frac{\bar{d}^1_{14}}{\bar{d}^4_{14}}$ and $\bar{s}_4 = \bar{t}_4\bar{p}^4_{1,14} - \bar{s}_1 - \bar{p}^1_{1,14}$.

Repeat a similar three steps for the ninth and and tenth rows, then for the eleventh and twelfth rows, and so forth, up to rows $2m-1$ and $2m$. More precisely, for each $i \in \{5, 6, \dots, m\}$, perform the following row operations on rows $2i-1$ and $2i$.

<u>Step 1</u>: Divide row $2i$ by $\bar{d}^i_{1i}$.

<u>Step 2</u>: Subtract from row $2i-1$ row $2i$ times $\bar{p}^i_{1,1i}$ and add to the result the fifth row, and then

multiply the result by -1.

Step 3: Move row $2i-1$ to the first row and row $2i$ to the second row, and shift all other rows down two places.

The result is the following matrix:

$$\begin{bmatrix}
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{s}_m & -\frac{1}{2} \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}_m & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{s}_5 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}_5 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{s}_4 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \bar{t}_4 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \bar{s}_3 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{t}_3 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \bar{s}_2 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \bar{t}_2 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{s}_1 & -\frac{1}{2} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

where, for $j = 4, 5, \dots, m$, $\bar{t}_j = \frac{\bar{d}^1_{1j}}{\bar{d}^j_{1j}}$ and $\bar{s}_j = \bar{t}_j \bar{p}^j_{1,1j} - \bar{s}_1 - \bar{p}^1_{1,1j}$.

The final task is to prove that, in the reduced row echelon form, the rows from row $2m+1$ to row $2\binom{m}{2}$ are all zero rows. These rows come in pairs: that is, any odd row corresponds to a difference equation for $\bar{c}(x_i, x_i x_j \bar{x})$; and then the next row corresponds to a difference equation for $\bar{c}(x_j, x_i, x_j, \bar{x})$. Note, moreover, that $i > 1$, because, in the enumeration of the rows of the matrix, the difference equations for $\bar{c}(x_1, x_1 x_j \bar{x})$ and $\bar{c}(x_j, x_1 x_j \bar{x})$ all appeared in the first $2m$ rows. Any such pair of rows has the following form:

$$\begin{bmatrix}
 & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\dots & -1 & \bar{p}^j_{i,ij} & \dots & -1 & \bar{p}^i_{i,ij} & \dots & 1 \\
\dots & 0 & \bar{d}^j_{ij} & \dots & 0 & \bar{d}^i_{ij} & \dots & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & & \vdots
\end{bmatrix}$$

In each of the two rows depicted above, the five specified elements are in columns $2(m-j)+1, 2(m-$

$j)+2, 2(m-i)+1, 2(m-i)+2$ and $2m+1$. In the two rows, all elements not specified are zero. It remains to find the row operations that transform any such pair of rows into a pair of zero rows. Consider the second of the two rows. Subtract from it row $2(m-j+1)$ times $\bar{d}^j_{ij}$ and row $2(m-i+1)$ times $\bar{d}^i_{ij}$. These row operations create a row in which all the elements of the row are zero, except for possibly the element in column $2m$. But given that $\Delta_1 \neq 0$ (see the reasoning in Claim 1), it follows that the element in column $2m$ must also be zero.

Consider now the first of the two rows. Subtract from it row $2(m-j+1)$ times $\bar{p}^j_{i,ij}$ and row $2(m-i+1)$ times $\bar{p}^i_{i,ij}$. Then add to it rows $2(m-j)+1$ and $2(m-i)+1$. These row operations create a row in which all the elements of the row are zero, except for possibly the element in column $2m$. But given that $\Delta_1 \neq 0$, the element in column $2m$ also must be zero.

This completes the proof of Lemma 2, which provides solutions for the unknowns as a function of Δ_1: in particular, $\iota_i = -\frac{1}{2} - \bar{s}_i\Delta_1, i = 1, \ldots m$ and $\Delta_i = -\bar{t}_i\Delta_1, i = 2, \ldots, m$. It remains to solve for Δ_1. In order to do so, note that the rationalisation conditions for $\bar{c}$ imply that

$$(1+\iota_1+\iota_2)\bar{c}(x_1, x_1x_2\bar{x}) = \iota_2\bar{c}(x_1, x_1\bar{x}) + \bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2\bar{x}}) \tag{2.88}$$

$$(1+\iota_1+\iota_3)\bar{c}(x_1, x_1x_3\bar{x}) = \iota_3\bar{c}(x_1, x_1\bar{x}) + \bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_3\bar{x}}) \tag{2.89}$$

$$(1+\iota_1+\iota_2+\iota_3+\iota_1\iota_2+\iota_1\iota_3+\iota_2\iota_3)\bar{c}(x_1, x_1x_2x_3\bar{x}) = \iota_2\iota_3\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1\bar{x}})+$$

$$\iota_3\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2\bar{x}}) + \iota_2\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_3\bar{x}}) + \bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2x_3\bar{x}}) \tag{2.90}$$

Equations (2.88) and (2.89) allow us to substitute out the terms $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2\bar{x}})$ and $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_3\bar{x}})$ from (2.90), yielding

$$(1+\iota_1+\iota_2+\iota_3+\iota_1\iota_2+\iota_1\iota_3+\iota_2\iota_3)\bar{c}(x_1, x_1x_2x_3\bar{x}) = \bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2x_3\bar{x}})+$$

$$\iota_3(1+\iota_1+\iota_2)\bar{c}(x_1, x_1x_2\bar{x}) + \iota_2(1+\iota_1+\iota_3)\bar{c}(x_1, x_1x_3\bar{x}) - \iota_2\iota_3\bar{c}(x_1, x_1\bar{x}) \tag{2.91}$$

Given that $\bar{c}'$ is rationalised, we could similarly show that

$$(1+\iota'_1+\iota'_2+\iota'_3+\iota'_1\iota'_2+\iota'_1\iota'_3+\iota'_2\iota'_3)\bar{c}'(x_1, x_1x_2x_3\bar{x}) = \bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2x_3\bar{x}})+$$

$$\iota'_3(1+\iota'_1+\iota'_2)\bar{c}'(x_1, x_1x_2\bar{x}) + \iota'_2(1+\iota'_1+\iota'_3)\bar{c}'(x_1, x_1x_3\bar{x}) - \iota'_2\iota'_3\bar{c}(x_1, x_1\bar{x}) \tag{2.92}$$

If we take the difference between (2.91) and (2.92), we can eliminate $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2x_3\bar{x}})$, obtaining

$$(1+\iota_1+\iota_2+\iota_3+\iota_1\iota_2+\iota_1\iota_3+\iota_2\iota_3)\bar{c}(x_1, x_1x_2x_3\bar{x}) + \iota_2\iota_3\bar{c}(x_1, x_1\bar{x})-$$

$$(\iota_3[1+\iota_1]+\iota_2\iota_3)\bar{c}(x_1,x_1x_2\bar{x})-(\iota_2[1+\iota_1]+\iota_2\iota_3)\bar{c}(x_1,x_1x_3\bar{x})=$$

$$(1+\iota_1'+\iota_2'+\iota_3'+\iota_1'\iota_2'+\iota_1'\iota_3'+\iota_2'\iota_3')\bar{c}'(x_1,x_1x_2x_3\bar{x})+\iota_2'\iota_3'\bar{c}(x_1,x_1\bar{x})-$$

$$(\iota_3'[1+\iota_1']+\iota_2'\iota_3')\bar{c}'(x_1,x_1x_2\bar{x})-(\iota_2'[1+\iota_1']+\iota_2'\iota_3')\bar{c}'(x_1,x_1x_3\bar{x}) \tag{2.93}$$

Rearranging

$$[1+\iota_1]\bar{c}(x_1,x_1x_2x_3\bar{x})+$$

$$[1+\iota_1]\iota_2(\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}(x_1,x_1x_3\bar{x}))+$$

$$[1+\iota_1]\iota_3(\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}(x_1,x_1x_2\bar{x}))+$$

$$\iota_2\iota_3(\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}(x_1,x_1x_2\bar{x})-\bar{c}(x_1,x_1x_3\bar{x})+\bar{c}(x_1,x_1\bar{x}))=$$

$$[1+\iota_1']\bar{c}'(x_1,x_1x_2x_3\bar{x})+$$

$$[1+\iota_1']\iota_2'(\bar{c}'(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_3\bar{x}))+$$

$$[1+\iota_1']\iota_3'(\bar{c}'(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_2\bar{x}))+$$

$$\iota_2'\iota_3'(\bar{c}'(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_2\bar{x})-\bar{c}'(x_1,x_1x_3\bar{x})+\bar{c}(x_1,x_1\bar{x})) \tag{2.94}$$

Subtracting the right-hand side from the left-hand side

$$0=[1+\iota_1](\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_2x_3\bar{x}))+$$

$$-\Delta_1\bar{c}'(x_1,x_1x_2x_3\bar{x})+$$

$$[\iota_1+1]\iota_2(\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}(x_1,x_1x_3\bar{x})-\bar{c}'(x_1,x_1x_2x_3\bar{x})+\bar{c}'(x_1,x_1x_3\bar{x}))+$$

$$([\iota_1+1]\iota_2-[\iota_1'+1]\iota_2')(\bar{c}'(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_3\bar{x}))+$$

$$[\iota_1+1]\iota_3(\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}(x_1,x_1x_2\bar{x})-\bar{c}'(x_1,x_1x_2x_3\bar{x})+\bar{c}'(x_1,x_1x_2\bar{x}))+$$

$$([\iota_1+1]\iota_3-[\iota_1'+1]\iota_3')(\bar{c}'(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_2\bar{x}))+$$

$$\iota_2\iota_3(\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}(x_1,x_1x_2\bar{x})-\bar{c}(x_1,x_1x_3\bar{x})-\bar{c}'(x_1,x_1x_2x_3\bar{x})+\bar{c}'(x_1,x_1x_2\bar{x})+\bar{c}'(x_1,x_1x_3\bar{x}))$$

$$+(\iota_2\iota_3-\iota_2'\iota_3')(\bar{c}'(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_2\bar{x})-\bar{c}'(x_1,x_1x_3\bar{x})+\bar{c}(x_1,x_1\bar{x})) \tag{2.95}$$

First, note that $\iota_2\iota_3-\iota_2'\iota_3'=-(\iota_2\Delta_3+\iota_3\Delta_2+\Delta_2\Delta_3)$ and for $i=2,3$, $[\iota_1+1]\iota_i-[\iota_1'+1]\iota_i'=-([\iota_1+1]\Delta_i+\iota_i\Delta_1+\Delta_1\Delta_i)$. Second, from Lemma 2, for $i=2,3,\Delta_i=-\bar{t}_i\Delta_i$ and for $i=1,2,3,\iota_i=-\frac{1}{2}-\bar{s}_i\Delta_1$. Substituting these results into the above equation yields

$$0=[\frac{1}{2}-\bar{s}_1\Delta_1](\bar{c}(x_1,x_1x_2x_3\bar{x})-\bar{c}'(x_1,x_1x_2x_3\bar{x}))-$$

$$
\begin{gathered}
\Delta_1 \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \\
(\frac{1}{2} - \bar{s}_1\Delta_1)(-\frac{1}{2} - \bar{s}_2\Delta_1)(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) + \\
(\bar{t}_2(\frac{1}{2} - \bar{s}_1\Delta_1)\Delta_1 + (\frac{1}{2} + \bar{s}_2\Delta_1)\Delta_1 + \bar{t}_2\Delta_1^2)(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x})) + \\
(\frac{1}{2} - \bar{s}_1\Delta_1)(-\frac{1}{2} - \bar{s}_3\Delta_1)(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x})) + \\
(\bar{t}_3(\frac{1}{2} - \bar{s}_1\Delta_1)\Delta_1 + (\frac{1}{2} + \bar{s}_3\Delta_1)\Delta_1 + \bar{t}_3\Delta_1^2)(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x})) + \\
(\frac{1}{2} + \bar{s}_2\Delta_1)(\frac{1}{2} + \bar{s}_3\Delta_1)\times \\
(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) + \\
(\bar{t}_2(-\frac{1}{2} - \bar{s}_3\Delta_1)\Delta_1 + \bar{t}_3(-\frac{1}{2} - \bar{s}_2\Delta_1)\Delta_1 - \bar{t}_2\bar{t}_3\Delta_1^2)\times \\
(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x}) + \bar{c}(x_1, x_1\bar{x}))
\end{gathered} \tag{2.96}
$$

The result is a quadratic equation of the form $\bar{f}\Delta_1^2 + \bar{g}\Delta_1 + \bar{h} = 0$, where expressions for $\bar{f}, \bar{g}$ and $\bar{h}$ are given as follows:

$$
\begin{gathered}
\bar{f} = \bar{s}_1\bar{s}_2(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) + \\
(\bar{t}_2(1 - \bar{s}_1) + \bar{s}_2)(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x})) + \\
\bar{s}_1\bar{s}_3(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x})) + \\
(\bar{t}_3(1 - \bar{s}_1) + \bar{s}_3)(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x})) + \\
\bar{s}_2\bar{s}_3(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) \\
-(\bar{s}_2\bar{t}_3 + \bar{s}_3\bar{t}_2 + \bar{t}_2\bar{t}_3)(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x}) + \bar{c}(x_1, x_1\bar{x}))
\end{gathered} \tag{2.97}
$$

$$
\begin{gathered}
\bar{g} = -\bar{s}_1(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x})) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) \\
+\frac{1}{2}(\bar{s}_1 - \bar{s}_2)(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) \\
+(\frac{1}{2}\bar{t}_2 + \frac{1}{2})(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x})) \\
+\frac{1}{2}(\bar{s}_1 - \bar{s}_3)(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x})) \\
+(\frac{1}{2}\bar{t}_3 + \frac{1}{2})(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x})) + \frac{1}{2}(\bar{s}_2 + \bar{s}_3)\times \\
(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) \\
-\frac{1}{2}(\bar{t}_2 + \bar{t}_3)(\bar{c}'(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x}) + \bar{c}(x_1, x_1\bar{x}))
\end{gathered} \tag{2.98}
$$

$$
\begin{aligned}
\bar{h} &= \frac{1}{2}(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x})) \\
&-\frac{1}{4}(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) \\
&-\frac{1}{4}(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x})) \\
&\frac{1}{4}(\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x}))
\end{aligned}
\tag{2.99}
$$

We can simplify the constant term $\bar{h}$ as follows:

$$
\begin{aligned}
\bar{h} &= [\frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}][\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}] \\
&+[\frac{1}{4} - \frac{1}{4}][-\bar{c}(x_1, x_1x_3\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})] \\
&+[\frac{1}{4} - \frac{1}{4}][-\bar{c}(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x})] \\
&= \frac{1}{4}[\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x})]
\end{aligned}
$$

Moreover, we will now show that $\bar{g} = 0$.

We can rewrite the expression for $\bar{g}$ as follows:

$$
\begin{aligned}
\bar{g} = [-\bar{s}_1 + \frac{1}{2}(\bar{s}_1 - \bar{s}_2) + \frac{1}{2}(\bar{s}_1 - \bar{s}_3) + \frac{1}{2}(\bar{s}_2 + \bar{s}_3)](\bar{c}(x_1, x_1x_2x_3\bar{x}) - \bar{c}'(x_1, x_1x_2x_3\bar{x}))+ \\
[-1 + \frac{1}{2}\bar{t}_2 + \frac{1}{2} + \frac{1}{2}\bar{t}_3 + \frac{1}{2} - \frac{1}{2}(\bar{t}_2 + \bar{t}_3)]\bar{c}'(x_1, x_1x_2x_3\bar{x}) \\
+\frac{1}{2}(\bar{s}_1 - \bar{s}_2)(-\bar{c}(x_1, x_1x_3\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) \\
+(\frac{1}{2}\bar{t}_2 + \frac{1}{2})(-\bar{c}'(x_1, x_1x_3\bar{x})) \\
+\frac{1}{2}(\bar{s}_1 - \bar{s}_3)(-\bar{c}(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x})) \\
+(\frac{1}{2}\bar{t}_3 + \frac{1}{2})(-\bar{c}'(x_1, x_1x_2\bar{x})) \\
+\frac{1}{2}(\bar{s}_2 + \bar{s}_3)(-\bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}(x_1, x_1x_3\bar{x}) + \bar{c}'(x_1, x_1x_2\bar{x}) + \bar{c}'(x_1, x_1x_3\bar{x})) \\
-\frac{1}{2}(\bar{t}_2 + \bar{t}_3)(-\bar{c}'(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x}) + \bar{c}(x_1, x_1\bar{x}))
\end{aligned}
\tag{2.100}
$$

But the first two terms are equal to zero because $-\bar{s}_1 + \frac{1}{2}(\bar{s}_1 - \bar{s}_2) + \frac{1}{2}(\bar{s}_1 - \bar{s}_3) + \frac{1}{2}(\bar{s}_2 + \bar{s}_3) = 0$ and $-1 + \frac{1}{2}\bar{t}_2 + \frac{1}{2} + \frac{1}{2}\bar{t}_3 + \frac{1}{2} - \frac{1}{2}(\bar{t}_2 + \bar{t}_3) = 0$. If we drop these two terms out, and rearrange, we obtain

$$
\bar{g} = -\frac{1}{2}((\bar{s}_1 + \bar{s}_3)(\bar{c}(x_1, x_1x_3\bar{x}) - \bar{c}'(x_1, x_1x_3\bar{x})) - ((\bar{t}_3 - 1)\bar{c}'(x_1, x_1x_3\bar{x}) - \bar{t}_3\bar{c}(x_1, x_1\bar{x}))+
$$

$$(\bar{s}_1 + \bar{s}_2)(\bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}'(x_1, x_1x_2\bar{x})) - ((\bar{t}_2 - 1)\bar{c}'(x_1, x_1x_2\bar{x}) - \bar{t}_2\bar{c}(x_1, x_1\bar{x}))) \tag{2.101}$$

But consider rationalisation condition (2.31) for the case where $i = 1$ and $j = 2, 3$ suppose that we substitute in $\Delta_j = -\bar{t}_j\Delta_1$, $\iota_1 = -\frac{1}{2} - \bar{s}_1\Delta_1$. and $\iota_j = -\frac{1}{2} - \bar{s}_j\Delta_1$. Then we obtain

$$(1 - \frac{1}{2} - \bar{s}_1\Delta_1 - \frac{1}{2} - \bar{s}_j\Delta_1)(\bar{c}'(x_1, x_1x_j\bar{x}) - \bar{c}(x_1, x_1x_j\bar{x})) =$$

$$-\bar{t}_j\Delta_1\bar{c}'(x_1, x_1\bar{x}) - (-\bar{t}_j\Delta_1 + \Delta_1)\bar{c}'(x_1, x_1x_j\bar{x}) \tag{2.102}$$

Next we divide throughout by Δ_1 and rearrange. ($\Delta_1 \neq 0$ because, if it were equal to zero, then by Lemma 2, $\Delta_i = 0, i = 1, \ldots, m$. This implies $\bar{c} = \bar{c}'$, which contradicts the DO-nonzero condition.) This yields, for $j = 2, 3$:

$$(\bar{s}_1 + \bar{s}_j)(\bar{c}(x_1, x_1x_j\bar{x}) - \bar{c}'(x_1, x_1x_j\bar{x})) =$$

$$(\bar{t}_j - 1)\bar{c}'(x_1, x_1x_j\bar{x}) - \bar{t}_j\bar{c}'(x_1, x_1\bar{x}) \tag{2.103}$$

But equation (2.101) then implies that $\bar{g} = 0$.

Accordingly, given that the DO-nonzero condition stipulates that $\bar{f} \neq 0$, then Δ_1 must satisfy: $\Delta_1 = \pm\sqrt{-\bar{h}/\bar{f}}$. How to we ascertain whether Δ_1 is the negative or positive square root? Given that $(\bar{c}, \bar{c}')$ is rationalisable, we know that ι_1 must be positive; and we also know that $\iota_1 = -\frac{1}{2} - \bar{s}_1\Delta_1$. So Δ_1 must have the opposite sign to $\bar{s}_1$. This allows us to obtain a unique solution for Δ_1.

$$\bar{\Delta}_1 = \begin{cases} -\sqrt{-\bar{h}/\bar{f}} & \text{if } \bar{s}_1 > 0 \\ \sqrt{-\bar{h}/\bar{f}} & \text{otherwise} \end{cases}$$

Theorem 9 (*Identification of ranking assignments in the DO model*)

Proof

In order to establish Theorem 9, we first prove the following claims

Claim 1 For any $B \subset X$ and any $x \in \bar{B}$, we have $\bar{\mu}(x, \bar{B}) = \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{B}})$.

Proof: The proof is by induction on the size of choice sets.

Basis step: Claim 1 holds for $B \subset X$ such that $|\bar{B}| \leq 2$.

The definition of $\bar{\mu}$ implies that $\bar{\mu}(x, \bar{B}) = \bar{c}(x, \bar{B})$ and the definition of rationalisation implies that $\bar{c}(x, \bar{B}) = \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{B}})$. Thus $\bar{\mu}(x, B) = \bar{\pi}(\bar{\mathcal{M}}_{x,B})$.

Induction step

Induction hypothesis: For any integer $n > 2$, suppose that Claim 1 holds for any $B \subset X$ such that $|\bar{B}| < n$.

We want to show that Claim 1 also holds for any $B \subset X$ such that $|\bar{B}| = n$.

Consider any $B \subset X$ such that $|\bar{B}| = n$. From the definition of rationalisation, for any $x \in \bar{B}$, we have:

$$\bar{\pi}(\bar{\mathcal{M}}_{x,\bar{B}}) = \frac{\bar{c}(x,\bar{B}) - \sum_{A \subsetneq B, x \in \bar{A}} \bar{\alpha}(A,B)\bar{\pi}(\bar{\mathcal{M}}_{x,\bar{A}})}{\bar{\alpha}(B,B)} \tag{2.104}$$

But the definition of $\bar{\mu}$ specifies that

$$\bar{\mu}(x,\bar{B}) = \frac{\bar{c}(x,\bar{B}) - \sum_{A \subsetneq B, x \in \bar{A}} \bar{\alpha}(A,B)\bar{\mu}(x,\bar{A})}{\bar{\alpha}(B,B)} \tag{2.105}$$

So equations (2.104) and (2.105), together with the induction hypothesis, entail that $\bar{\pi}(\bar{\mathcal{M}}_{x,\bar{B}}) = \bar{\mu}(x,\bar{B})$.

Having proved Claim 1, we now turn to the proof of Theorem 9. We prove the result for $\bar{v}$; the proof for $\bar{v}'$ is analogous. Clearly, from the definitions of the sets $\bar{\mathcal{M}}_{x,B}$ and $\bar{\mathcal{U}}_{x,B}$, for any $x \in \bar{X}$

$$\forall A \subset X \backslash \{x\}, \quad \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{X}\backslash A}) = \sum_{B \subset A} \bar{\pi}(\bar{\mathcal{U}}_{x,B}) \tag{2.106}$$

But if we apply to equation (2.106) the Mobius inversion formula for the power set, we obtain

$$\forall A \subset X \backslash \{x\}, \quad \bar{\pi}(\bar{\mathcal{U}}_{x,A}) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{\pi}(\bar{\mathcal{M}}_{x,\bar{X}\backslash A}) \tag{2.107}$$

If we substitute into equation (2.107) the result in claim 1, we obtain the desired result:

$$\forall A \subset X \backslash \{x\}, \quad \bar{\pi}(\bar{\mathcal{U}}_{x,A}) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{\mu}(x, \bar{X} \backslash A) = \bar{v}(x,A) \tag{2.108}$$

Theorem 10 (*Characterisation in the DO model*)

Proof

The ($\Rightarrow$) part follows from Theorems 8 and 9. The ($\Leftarrow$) part is established as follows. Given that $\bar{\iota}$ and $\bar{\iota}'$ are positive and real, then $\bar{\alpha}$ and $\bar{\alpha}'$ are well-defined conditional attention assignments and the DO contour functions $\bar{v}$ and $\bar{v}'$ are also well-defined. The proof of the ($\Leftarrow$) part of Theorem 10 has five steps.

(a) for any $B \in \bar{\mathcal{B}}$, $\sum_{x \in B} \bar{\mu}(x,B) = 1$.

(b) for any $B \in \bar{\mathcal{B}}$, for any $x \in B$, $\bar{\mu}(x,B) \geq 0$.

(c) there exists a ranking assignment on $\bar{X}$, $\bar{\pi}$, such that, for any $B \in \bar{\mathcal{B}}$ and any $x \in B$, $\bar{\mu}(x,B) = \bar{\pi}(\bar{\mathcal{M}}_{x,B})$.

(d) $\bar{\pi}$ and $\bar{\alpha}$ rationalise $\bar{c}$.

(e) $\bar{\pi}$ and $\bar{\alpha}'$ rationalise $\bar{c}'$.

Proof of step (a)

The proof is by induction over the size of the choice sets.

Basis step: For any $B \subset X$ such that $|\bar{B}| \leq 2$, $\sum_{x \in \bar{B}} \bar{\mu}(x, \bar{B}) = 1$.

This follows immediately from the definition of $\bar{\mu}$ together with the definition of a choice function.

Induction step

Induction hypothesis: Consider any $B \subset X$ such that $|\bar{B}| > 2$. Suppose that, for any $A \subset B$ such that $|A| < |B|$, we have $\sum_{x \in \bar{A}} \bar{\mu}(x, \bar{A}) = 1$.

We want to show that, given that the induction hypothesis holds, $\sum_{x \in \bar{B}} \bar{\mu}(x, \bar{B}) = 1$.

From the definition of $\bar{\mu}$, for any $B \subset X, x \in \bar{B}$

$$\bar{c}(x, \bar{B}) = \sum_{A \subset B : x \in \bar{A}} \bar{\alpha}(A, B) \bar{\mu}(x, \bar{A})$$

If we sum over the alternatives in $\bar{B}$, the definition of a choice function implies that the left-hand side is 1, yielding

$$1 = \sum_{x \in \bar{B}} \sum_{A \subset B : x \in \bar{A}} \bar{\alpha}(A, B) \bar{\mu}(x, \bar{A})$$

But the right-hand side of this equation is, in effect a sum of expressions $\bar{\alpha}(A, B)\bar{\mu}(x, \bar{A})$ over the set $\{(x, A) : A \subset B, x \in \bar{A}\}$. Thus

$$1 = \sum_{x \in \bar{B}} \sum_{A \subset B : x \in \bar{A}} \bar{\alpha}(A, B) \bar{\mu}(x, \bar{A}) = \sum_{(x,A) \in \{(x,A) : A \subset B : x \in \bar{A}\}} \bar{\alpha}(A, B) \bar{\mu}(x, A)$$

$$= \sum_{A \subset B} \sum_{x \in \bar{A}} \bar{\alpha}(A, B) \bar{\mu}(x, \bar{A}) = \sum_{A \subset B} \bar{\alpha}(A, B) \sum_{x \in \bar{A}} \bar{\mu}(x, \bar{A})$$

$$= \sum_{A \subsetneq B} \bar{\alpha}(A, B) \sum_{x \in \bar{A}} \bar{\mu}(x, \bar{A}) + \bar{\alpha}(B, B) \sum_{x \in \bar{B}} \bar{\mu}(x, \bar{B}) = \sum_{A \subsetneq B} \bar{\alpha}(A, B) + \bar{\alpha}(B, B) \sum_{x \in \bar{B}} \bar{\mu}(x, \bar{B}) \quad (2.109)$$

Note that the induction hypothesis justifies the final equality in this equation. Moreover, as $\bar{\alpha}$ is an attention assignment

$$1 = \sum_{A \subsetneq B} \bar{\alpha}(A, B) + \bar{\alpha}(B, B) \quad (2.110)$$

Subtracting (2.109) from (2.110) yields

$$0 = \bar{\alpha}(B, B)(1 - \sum_{x \in \bar{B}} \bar{\mu}(x, \bar{B})) \quad (2.111)$$

But as $\bar{\alpha}(B, B) > 0$, thus $\sum_{x \in \bar{B}} \bar{\mu}(x, \bar{B}) = 1$

Proof of step (b)

The contour function for $\bar{c}$ can be specified as follows. For any $x \in \bar{X}$:

$$\forall A \subset X \backslash \{x\}, \quad \bar{v}(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \bar{\mu}(x, \bar{X} \backslash B) \tag{2.112}$$

Applying the Mobius inversion formula, we obtain that, for any $x \in \bar{X}$:

$$\forall A \subset X \backslash \{x\}, \quad \bar{\mu}(x, \bar{X} \backslash A) = \sum_{B \subset A} \bar{v}(x, B) \tag{2.113}$$

As the contour function is assumed to be non-negative, it follows that $\bar{\mu}$ is a non-negative function.

Proof of step (c)

The form of the justification for step (c) is exactly analogous to that for the ($\Leftarrow$) part of Theorem 4. This is because $\bar{\mu}$ has the same domain as a default-option choice function and (ii) as shown in parts (a) and (b), it has the same formal properties as a default-option choice function – in other words, it is non-negative and probabilities over a choice set sum to 1.

Proof of step (d)

It follows from the definition of a DO maximum function that, for all $B \subset X, x \in \bar{B}$

$$\bar{c}(x, \bar{B}) = \sum_{A \subset B : x \in \bar{A}} \bar{\alpha}(A, B) \bar{\mu}(x, \bar{B}) \tag{2.114}$$

If we substitute into this equation the result from part (c), then it follows that $\bar{c}$ is rationalised by $\bar{\pi}$ and $\bar{\alpha}$.

Proof of step (e)

In the reasoning in step (b), we can replace $\bar{v}$ and $\bar{\mu}$ by $\bar{v}'$ and $\bar{\mu}'$. Given that $\bar{v} = \bar{v}'$, (2.113) implies that, for all $x \in \bar{X}$:

$$\forall A \subset X \backslash \{x\}, \quad \bar{\mu}(x, X \backslash A) = \sum_{B \subset A} \bar{v}(x, B) = \sum_{B \subset A} \bar{v}'(x, B) = \bar{\mu}'(x, X \backslash A) \tag{2.115}$$

As $\bar{\mu} = \bar{\mu}'$, in the result in step (c), we can replace $\bar{\mu}$ by $\bar{\mu}'$. Using parallel reasoning to that in step (d), we can establish that $\bar{\pi}$ and $\bar{\alpha}'$ rationalise c'.

Appendix 2: Minimum-attention models without a default option

As discussed in Section 2.3, for a model without a default option, the attention assignment must be such that a zero probability is assigned to an event that consideration sets are of size 0. That is, it is assumed that the minimum size of a consideration set is 1. But the question then naturally arises: in some cases, might there be warrant for assuming that the minimum size is more than 1? This minimum-attention assumption might be formulated as follows: there is an integer n, which is more than 1 and less than m, and is such that: when the size of the choice set is n or less, the DM considers all alternatives; and when the size of the choice set is more than n, the DM only forms consideration sets of size at least n. Plausibly, there may be justification for such an assumption if the cost of consideration is relatively low. The following definition, which is a variant of the definition of a conditional attention assignment, incorporates a minimum-attention assumption:

Definition 24 (*Size-n conditional attention assignment*)
Consider any attention probability vector γ and any $n \in \{2, 3 \dots, m-1\}$. The *size-n conditional attention assignment generated by* γ is the attention assignment α such that, for any $B \subset X$:
(i) for $A \subsetneq B$ such that $|A| < n$, $\alpha(A, B) = 0$; and
(ii) for $A \subset B$ such that $|A| \geq n$, if $|B| > n$, then:

$$\alpha(A, B) = \prod_{i: x_i \in A} \gamma_i \prod_{j: x_j \in B \setminus A} (1 - \gamma_j) \Big/ \sum_{D \subset B: |D| \geq n} \prod_{i: x_i \in D} \gamma_i \prod_{i: x_j \in B \setminus D} (1 - \gamma_j)$$

In this section, we develop a consideration-set model without a default option in which the DM is assumed to have a size-2 conditional attention assignment. We will refer to this model as the minimum-attention model, or MA model. The first task is to retrieve the DM's attention assignment from her choice probabilities. Suppose that some frame-dependent choice function (c, c') is rationalised by a ranking assignment on X, π, and a pair of size-2 conditional attention assignments that are generated by γ and γ'. For any $i, j, k \in 1, \dots, m$ such that $i < j < k$, the rationalisation conditions for c imply the following:

$$c(x_i, x_i x_j x_k) = \frac{\gamma_i \gamma_j (1 - \gamma_k) \pi(\mathcal{M}_{x_i, x_i x_j}) + \gamma_i \gamma_k (1 - \gamma_j) \pi(\mathcal{M}_{x_i, x_i x_k}) + \gamma_i \gamma_j \gamma_k \pi(\mathcal{M}_{x_i, x_i x_j x_k})}{\gamma_i \gamma_j (1 - \gamma_k) + \gamma_i \gamma_k (1 - \gamma_j) + \gamma_j \gamma_k (1 - \gamma_i) + \gamma_i \gamma_j \gamma_k}$$
$$c(x_j, x_i x_j x_k) = \frac{\gamma_i \gamma_j (1 - \gamma_k) \pi(\mathcal{M}_{x_j, x_i x_j}) + \gamma_j \gamma_k (1 - \gamma_i) \pi(\mathcal{M}_{x_j, x_j x_k}) + \gamma_i \gamma_j \gamma_k \pi(\mathcal{M}_{x_i, x_i x_j x_k})}{\gamma_i \gamma_j (1 - \gamma_k) + \gamma_i \gamma_k (1 - \gamma_j) + \gamma_j \gamma_k (1 - \gamma_i) + \gamma_i \gamma_j \gamma_k}$$

From the definition of a size-2 conditional attention assignment, for any choice set B of size 2, and $x \in B$, $c(x, B) = \pi(\mathcal{M}_{x,B})$. So given the definition of the inattention odds vector, we can rewrite these equations

$$(1 + \iota_i + \iota_j + \iota_k) c(x_i, x_i x_j x_k) = c(x_i, x_i x_j) \iota_k + c(x_i, x_i x_k) \iota_j + \pi(\mathcal{M}_{x_i, x_i x_j x_k}) \tag{2.116}$$

$$(1 + \iota_i + \iota_j + \iota_k) c(x_j, x_i x_j x_k) = c(x_j, x_i x_j) \iota_k + c(x_j, x_j x_k) \iota_i + \pi(\mathcal{M}_{x_j, x_i x_j x_k}) \tag{2.117}$$

We can obtain an analogous pair of rationalisation equations for c':

$$(1+\iota_i'+\iota_j'+\iota_k')c'(x_i,x_ix_jx_k) = c(x_i,x_ix_j)\iota_k' + c(x_i,x_ix_k)\iota_j' + \pi(\mathcal{M}_{x_i,x_ix_jx_k}) \tag{2.118}$$

$$(1+\iota_i'+\iota_j'+\iota_k')c'(x_j,x_ix_jx_k) = c(x_j,x_ix_j)\iota_k' + c(x_j,x_jx_k)\iota_i' + \pi(\mathcal{M}_{x_j,x_ix_jx_k}) \tag{2.119}$$

To eliminate the maximum probability terms, we take differences between (2.116) and (2.118) and between (2.117) and (2.119), yielding:

$$\begin{gathered}
[c(x_i,x_ix_jx_k)-c'(x_i,x_ix_jx_k)]\iota_k + [c(x_i,x_ix_j)-c'(x_i,x_ix_jx_k)]\Delta_k + \\
[c(x_i,x_ix_jx_k)-c'(x_i,x_ix_jx_k)]\iota_j + [c(x_i,x_ix_k)-c'(x_i,x_ix_jx_k)]\Delta_j + \\
[c(x_i,x_ix_jx_k)-c'(x_i,x_ix_jx_k)]\iota_i + [-c'(x_i,x_ix_jx_k)]\Delta_i = c'(x_i,x_ix_jx_k)-c(x_i,x_ix_jx_k)
\end{gathered} \tag{2.120}$$

$$\begin{gathered}
[c(x_j,x_ix_jx_k)-c'(x_j,x_ix_jx_k)]\iota_k + [c(x_j,x_ix_j)-c'(x_j,x_ix_jx_k)]\Delta_k + \\
[c(x_j,x_ix_jx_k)-c'(x_j,x_ix_jx_k)]\iota_j + [-c'(x_j,x_ix_jx_k)]\Delta_j + \\
[c(x_j,x_ix_jx_k)-c'(x_j,x_ix_jx_k)]\iota_i + [c(x_j,x_jx_k)-c'(x_j,x_ix_jx_k)]\Delta_i = c'(x_j,x_ix_jx_k)-c(x_j,x_ix_jx_k)
\end{gathered} \tag{2.121}$$

The solution strategy is parallel to that in the NDO model. The number of equations is $2\binom{m}{3}$ and the number of unknowns is $2m$. So there are at least as many equations as unknowns so long as $|X| \geq 4$. Using this linear system of equations, we solve for each unknown as an affine function of Δ_1. In particular, we find that $\iota_i = -\frac{1}{3} - \tilde{s}_i\Delta_1, i = 1,\ldots,m$ and $\Delta_i = -\tilde{t}_i\Delta_1, i = 2,\ldots,m$, where $\tilde{s}_i, i = 1,\ldots,m$ and $\tilde{t}_i\Delta_1, i = 2,\ldots,m$ are defined in the proof of Theorem 11 at the end of this appendix. It remains then to solve for Δ_1. We do so by recourse to further rationalisation conditions, which allow us to obtain the following quadratic equation, analogous to to equation (2.21) in Section 2.4

$$\tilde{f}\Delta_1^2 + \tilde{g}\Delta_1 + \tilde{h} = 0 \tag{2.122}$$

where $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are defined in the proof of Theorem 11. As we prove that $\tilde{g} = 0$, this yields a solution for Δ_1 analogous to the solution in equation (2.22) in Section 2.4.

$$\tilde{\Delta}_1 = \begin{cases} -\sqrt{-\tilde{h}/\tilde{f}} & \text{if } \tilde{s}_1 > 0 \\ \sqrt{-\tilde{h}/\tilde{f}} & \text{otherwise} \end{cases} \tag{2.123}$$

The results in this appendix assume that the following nonzero condition obtains, which is the counterpart of the NDO-nonzero condition in the MA model.

Definition 25 (*MA-nonzero condition*)

A choice function (c, c') is said to satisfy the *MA-nonzero condition* if:

(i) for any positive integers i, j, k such that $i < j < k \leq m$, $c'(x_i, x_ix_jx_k) - c(x_i, x_ix_jx_k)$ are nonzero and $c'(x_j, x_ix_jx_k) - c(x_j, x_ix_jx_k)$ are nonzero;

(ii) the following terms, which are functions of observed choice probabilities and defined in the proof of Theorem 11, are nonzero: $\tilde{d}^2_{123}\tilde{d}^3_{134}\tilde{d}^4_{124} + \tilde{d}^2_{124}\tilde{d}^3_{123}\tilde{d}^4_{134}$, $\tilde{f}$, for $i = 3, \ldots, m$, $\tilde{d}^i_{12i}$.

Then the following definitions specify, for the MA model, the revealed inattention odds, the revealed attention probabilities and the revealed attention assignments:

Definition 26 (*MA revealed inattention odds, attention probabilities, attention assignments*)

For $|X| \geq 4$, suppose that choice function (c, c') satisfies the MA-nonzero condition.

(i) The *MA revealed inattention odds vectors* for c and c' are the vectors $\tilde{\iota}, \tilde{\iota}' \in \mathbb{C}^m$ such that, for $i = 1, 2, \ldots, m$, $\tilde{\iota}_i = -\frac{1}{3} - \tilde{s}_i\tilde{\Delta}_1$ and

$$\tilde{\iota}'_i = \begin{cases} \tilde{\iota}_i + \tilde{\Delta}_1 & \text{for } i = 1 \\ \tilde{\iota}_i - \tilde{t}_i\tilde{\Delta}_1 & \text{for } i = 2, \ldots, m \end{cases}$$

(ii) Suppose that the revealed inattention odds vectors $\tilde{\iota}$ and $\tilde{\iota}'$ are real and positive. Then the *MA revealed attention probability vectors* for c and c' are $\tilde{\gamma}, \tilde{\gamma}' \in \mathbb{R}^m$ such that, for $i = 1, \ldots, m$, $\tilde{\gamma}_i = \frac{1}{1+\tilde{\iota}_i}$ and $\tilde{\gamma}'_i = \frac{1}{1+\tilde{\iota}'_i}$.

(iii) Suppose that the revealed inattention odds vectors $\tilde{\iota}$ and $\tilde{\iota}'$ are real and positive. Then the *MA revealed attention assignments* for c and c' are the size-2 conditional attention assignments $\tilde{\alpha}$ and $\tilde{\alpha}'$ that are respectively generated by $\tilde{\gamma}$ and $\tilde{\gamma}'$.

Theorem 11 then states the identification result for attention assignments in the MA model.

Theorem 11 (*Identification of attention assignments in the MA model*)

For $|X| \geq 4$, consider any choice function (c, c') that satisfies the MA-nonzero condition, any ranking assignment on X, π and any pair of size-2 conditional attention assignments (α, α'). If π and (α, α') rationalise (c, c'), then $\alpha = \tilde{\alpha}$ and $\alpha = \tilde{\alpha}'$.

Having obtained revealed attention assignments, we can adopt the strategy applied in the NDO model to retrieve the ranking assignment up to upper contours. First, we define the maximum functions and the contour functions for the MA model.

Definition 27 (*MA maximum functions and contour functions*)
For $|X| \geq 4$, suppose that a choice function (c, c') is such that the MA-nonzero condition is satisfied and that the revealed inattention odds vectors $\tilde{\iota}$ and $\tilde{\iota}'$ are real and positive.
(a) The *MA maximum functions* for c and c' are the functions $\tilde{\mu} : \{(x, B) : B \in \bar{\mathcal{B}}, x \in B\} \to \mathbb{R}$ and $\tilde{\mu}' : \{(x, B) : B \in \mathcal{B}, x \in B\} \to \mathbb{R}$ such that, for any $B \in \mathcal{B}, x \in B\}$:
(i) if $|B| = 1$, then $\tilde{\mu}(x, B) = 1$ and $\tilde{\mu}'(x, B) = 1$; and
(ii) if $|B| > 1$, then:

$$\tilde{\mu}(x, B) = \frac{c(x, B) - \sum_{A \subsetneq B : x \in \bar{A}} \tilde{\alpha}(A, B)\tilde{\mu}(x, \bar{A})}{\tilde{\alpha}(B, B)}$$

$$\tilde{\mu}'(x, B) = \frac{c'(x, B) - \sum_{A \subsetneq B : x \in \bar{A}} \tilde{\alpha}'(A, B)\tilde{\mu}'(x, \bar{A})}{\tilde{\alpha}'(B, B)}$$

(b) The *MA contour functions* for c and c' are the functions $\tilde{\upsilon} : \{(x, A) : A \subsetneq X, x \in X \backslash A\} \to \mathbb{R}$ and $\tilde{\upsilon}' : \{(x, A) : A \subsetneq X, x \in X \backslash A\} \to \mathbb{R}$ such that, for any $A \subsetneq X, x \in X \backslash A$:

$$\tilde{\upsilon}(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \tilde{\mu}(x, X \backslash B), \quad \tilde{\upsilon}'(x, A) = \sum_{B \subset A} (-1)^{|A \backslash B|} \tilde{\mu}'(x, X \backslash B)$$

The following identification theorem allows us to interpret the MA contour functions as specifying revealed upper contour probabilities.

Theorem 12 (*Identification of ranking assignments in the MA model*)
For $|X| \geq 4$, consider any frame-dependent choice function (c, c') that satisfies the MA-nonzero condition, any ranking assignment π, any pair of size-2 conditional attention assignments (α, α'). If π and (α, α') rationalise (c, c'), then for all $A \subsetneq X$, $x \in X \backslash A$, $\pi(\mathcal{U}_{x,A}) = \tilde{\upsilon}(x, A) = \tilde{\upsilon}'(x, A)$. That is, π is unique up to upper contours.

We can then obtain an identification-based characterisation result that is analogous to the result in Theorem 7.

Theorem 13 (*Characterisation for the MA model*)
For $|X| \geq 4$, consider any choice function (c, c') that satisfies the MA-nonzero condition. There exists a ranking assignment on X and a pair of size-2 conditional attention assignments that rationalise (c, c') if and only if:

(i) the revealed inattention odds vectors $\tilde{\iota}$ and $\tilde{\iota}'$ are real and positive;

(ii) the contour functions $\tilde{\upsilon}$ and $\tilde{\upsilon}'$ are equal and non-negative.

We do not provide a proof of Theorem 12 nor a proof of Theorem 13, because the former is exactly parallel to the proof of Theorem 6 and the latter is exactly parallel to the proof of Theorem 7. The proof of Theorem 11 is given below.

Proof of Theorem 11

Let us refer to equation (2.120) as the *difference equation for* $c(x_i, x_i x_j x_k)$ and (2.121) as the *difference equation for* $c(x_j, x_i x_j x_k)$ and, collectively, (2.120) and (2.121) as the difference equations for the choice set $\{x_i, x_j, x_k\}$. Consider the system of equations comprising the difference equations across the various choice sets of size 3. When there are m alternatives, the number of subsets of size three is $\binom{m}{3}$. So the number of difference equations is $2\binom{m}{3}$. We can represent the system of difference equations in matrix notation. The vector of unknowns is given by $\tilde{\mathbf{v}} = (\iota_m, \Delta_m, \iota_{m-1}, \Delta_{m-1}, \ldots, \iota_1, \Delta_1)'$. If we fix a particular enumeration of the equations, then they can be written $\tilde{\mathbf{Q}}\tilde{\mathbf{v}} = \tilde{\mathbf{b}}$, where matrix $\tilde{\mathbf{Q}}$ has size $2\binom{m}{3} \times 2m$ and vector $\tilde{\mathbf{b}}$ has length $2\binom{m}{3}$. First, we establish the following lemma about this equation system.

Lemma 3 (*Reduced row echelon form*)

Consider any choice function (c, c') that (i) satisfies the MA-nonzero condition and (ii) is rationalisable. Then the matrix of coefficients for this choice function $\tilde{\mathbf{Q}}$ has rank $2m-1$ and, in particular, the reduced row echelon form of the augmented matrix $[\tilde{\mathbf{Q}}\ \tilde{\mathbf{b}}]$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \tilde{s}_m & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \tilde{t}_m & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \tilde{s}_{m-1} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \tilde{t}_{m-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \tilde{s}_2 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \tilde{t}_2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \tilde{s}_1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the terms $\tilde{t}_i$, $i = 2, \dots, m$ and $\tilde{s}_i$, $i = 1, \dots, m$ are defined in the proof below.

Proof of Lemma 3

Let us define the coefficients in the difference equation for $c(x_i, x_i x_j x_k)$ as follows

$$\tilde{q}^i_{i,ijk} = -c'(x_i, x_i x_j x_k)$$

$$\tilde{q}^j_{i,ijk} = c(x_i, x_i x_k) - c'(x_i, x_i x_j x_k)$$

$$\tilde{q}^k_{i,ijk} = c(x_i, x_i x_j) - c'(x_i, x_i x_j x_k)$$

$$\tilde{b}_{i,ijk} = c'(x_i, x_i x_j x_k) - c(x_i, x_i x_j x_k)$$

We can then write the difference equation for $c(x_i, x_i x_j x_k)$ as follows

$$-\tilde{b}_{i,ijk}\iota_k + \tilde{q}^k_{i,ijk}\Delta_k - \tilde{b}_{i,ijk}\iota_j + \tilde{q}^j_{i,ijk}\Delta_j - \tilde{b}_{i,ijk}\iota_i + \tilde{q}^i_{i,ijk}\Delta_i = \tilde{b}_{i,ijk}$$

Similarly, let us define the coefficients in the difference equation for $c(x_j, x_i x_j x_k)$ as follows

$$\tilde{q}^i_{j,ijk} = c(x_j, x_j x_k) - c'(x_j, x_i x_j x_k)$$

$$\tilde{q}^j_{j,ijk} = -c'(x_j, x_i x_j x_k)$$

$$\tilde{q}^k_{j,ijk} = c(x_j, x_i x_j) - c'(x_j, x_i x_j x_k)$$

$$\tilde{b}_{j,ijk} = c'(x_j, x_i x_j x_k) - c(x_j, x_i x_j x_k)$$

We can then write the difference equation for $c(x_j, x_i x_j x_k)$ as follows

$$-\tilde{b}_{j,ijk}\iota_k + \tilde{q}^k_{j,ijk}\Delta_k - \tilde{b}_{j,ijk}\iota_j + \tilde{q}^j_{j,ijk}\Delta_j - \tilde{b}_{j,ijk}\iota_i + \tilde{q}^i_{j,ijk}\Delta_i = \tilde{b}_{j,ijk}$$

The proof divides into two stages. First, we prove the result for the case in which there are only four alternatives. That is, $X = \{x_1, x_2, x_3, x_4\}$. Second, we then extend the proof to the case in which there are $m > 4$ alternatives. If there are four alternatives, then we can write the augmented matrix for the equation system as follows:

$$[\tilde{\mathbf{Q}}, \tilde{\mathbf{b}}] = \begin{bmatrix} 0 & 0 & -\tilde{b}_{1,123} & \tilde{q}^3_{1,123} & -\tilde{b}_{1,123} & \tilde{q}^2_{1,123} & -\tilde{b}_{1,123} & \tilde{q}^1_{1,123} & \tilde{b}_{1,123} \\ 0 & 0 & -\tilde{b}_{2,123} & \tilde{q}^3_{2,123} & -\tilde{b}_{2,123} & \tilde{q}^2_{2,123} & -\tilde{b}_{2,123} & \tilde{q}^1_{2,123} & \tilde{b}_{2,123} \\ -\tilde{b}_{1,124} & \tilde{q}^4_{1,124} & 0 & 0 & -\tilde{b}_{1,124} & \tilde{q}^2_{1,124} & -\tilde{b}_{1,124} & \tilde{q}^1_{1,124} & \tilde{b}_{1,124} \\ -\tilde{b}_{2,124} & \tilde{q}^4_{2,124} & 0 & 0 & -\tilde{b}_{2,124} & \tilde{q}^2_{2,124} & -\tilde{b}_{2,124} & \tilde{q}^1_{2,124} & \tilde{b}_{2,124} \\ -\tilde{b}_{1,134} & \tilde{q}^4_{1,134} & -\tilde{b}_{1,134} & \tilde{q}^3_{1,134} & 0 & 0 & -\tilde{b}_{1,134} & \tilde{q}^1_{1,134} & \tilde{b}_{1,134} \\ -\tilde{b}_{3,134} & \tilde{q}^4_{3,134} & -\tilde{b}_{3,134} & \tilde{q}^3_{3,134} & 0 & 0 & -\tilde{b}_{3,134} & \tilde{q}^1_{3,134} & \tilde{b}_{3,134} \\ -\tilde{b}_{2,234} & \tilde{q}^4_{2,234} & -\tilde{b}_{2,234} & \tilde{q}^3_{2,234} & -\tilde{b}_{2,234} & \tilde{q}^2_{2,234} & 0 & 0 & \tilde{b}_{2,234} \\ -\tilde{b}_{3,234} & \tilde{q}^4_{3,234} & -\tilde{b}_{3,234} & \tilde{q}^3_{3,234} & -\tilde{b}_{3,234} & \tilde{q}^2_{3,234} & 0 & 0 & \tilde{b}_{3,234} \end{bmatrix}$$

First, we apply to this matrix the row operation of dividing each row by the final element. (By clause (i) of the MA-nonzero condition, the final element of every row is non-zero.) Let us define $\tilde{p}^i_{i,ijk} = \tilde{q}^i_{i,ijk}/\tilde{b}_{i,ijk}$, $\tilde{p}^j_{i,ijk} = \tilde{q}^j_{i,ijk}/\tilde{b}_{i,ijk}$, $\tilde{p}^k_{i,ijk} = \tilde{q}^k_{i,ijk}/\tilde{b}_{i,ijk}$, $\tilde{p}^i_{j,ijk} = \tilde{q}^i_{j,ijk}/\tilde{b}_{j,ijk}$, $\tilde{p}^j_{j,ijk} = \tilde{q}^j_{j,ijk}/\tilde{b}_{j,ijk}$, $\tilde{p}^k_{j,ijk} = \tilde{q}^k_{j,ijk}/\tilde{b}_{j,ijk}$. The result is the following matrix:

$$\begin{bmatrix} 0 & 0 & -1 & \tilde{p}^3_{1,123} & -1 & \tilde{p}^2_{1,123} & -1 & \tilde{p}^1_{1,123} & 1 \\ 0 & 0 & -1 & \tilde{p}^3_{2,123} & -1 & \tilde{p}^2_{2,123} & -1 & \tilde{p}^1_{2,123} & 1 \\ -1 & \tilde{p}^4_{1,124} & 0 & 0 & -1 & \tilde{p}^2_{1,124} & -1 & \tilde{p}^1_{1,124} & 1 \\ -1 & \tilde{p}^4_{2,124} & 0 & 0 & -1 & \tilde{p}^2_{2,124} & -1 & \tilde{p}^1_{2,124} & 1 \\ -1 & \tilde{p}^4_{1,134} & -1 & \tilde{p}^3_{1,134} & 0 & 0 & -1 & \tilde{p}^1_{1,134} & 1 \\ -1 & \tilde{p}^4_{3,134} & -1 & \tilde{p}^3_{3,134} & 0 & 0 & -1 & \tilde{p}^1_{3,134} & 1 \\ -1 & \tilde{p}^4_{2,234} & -1 & \tilde{p}^3_{2,234} & -1 & \tilde{p}^2_{2,234} & 0 & 0 & 1 \\ -1 & \tilde{p}^4_{3,234} & -1 & \tilde{p}^3_{3,234} & -1 & \tilde{p}^2_{3,234} & 0 & 0 & 1 \end{bmatrix}$$

Next, for every even row, subtract from that row the odd row preceding it. The following equation results, where we define $\tilde{d}^i_{ijk} = \tilde{p}^i_{j,ijk} - \tilde{p}^i_{i,ijk}, \tilde{d}^j_{ijk} = \tilde{p}^j_{j,ijk} - \tilde{p}^j_{i,ijk}, \tilde{d}^k_{ijk} = \tilde{p}^k_{j,ijk} - \tilde{p}^k_{i,ijk}$.

$$\begin{bmatrix} 0 & 0 & -1 & \tilde{p}^3_{1,123} & -1 & \tilde{p}^2_{1,123} & -1 & \tilde{p}^1_{1,123} & 1 \\ 0 & 0 & 0 & \tilde{d}^3_{123} & 0 & \tilde{d}^2_{123} & 0 & \tilde{d}^1_{123} & 0 \\ -1 & \tilde{p}^4_{1,124} & 0 & 0 & -1 & \tilde{p}^2_{1,124} & -1 & \tilde{p}^1_{1,124} & 1 \\ 0 & \tilde{d}^4_{124} & 0 & 0 & 0 & \tilde{d}^2_{124} & 0 & \tilde{d}^1_{124} & 0 \\ -1 & \tilde{p}^4_{1,134} & -1 & \tilde{p}^3_{1,134} & 0 & 0 & -1 & \tilde{p}^1_{1,134} & 1 \\ 0 & \tilde{d}^4_{134} & 0 & \tilde{d}^3_{134} & 0 & 0 & 0 & \tilde{d}^1_{134} & 0 \\ -1 & \tilde{p}^4_{2,234} & -1 & \tilde{p}^3_{2,234} & -1 & \tilde{p}^2_{2,234} & 0 & 0 & 1 \\ 0 & \tilde{d}^4_{234} & 0 & \tilde{d}^3_{234} & 0 & \tilde{d}^2_{234} & 0 & 0 & 0 \end{bmatrix}$$

From the sixth row, subtract the second row times $\frac{\tilde{d}^3_{134}}{\tilde{d}^3_{123}}$. (The MA-nonzero condition condition ensures that d^3_{123} is non-zero). Row six then becomes:

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \tilde{d}^4_{134} & 0 & 0 & 0 & -\frac{\tilde{d}^3_{134}}{\tilde{d}^3_{123}}\tilde{d}^2_{123} & 0 & \tilde{d}^1_{134} - \frac{\tilde{d}^3_{134}}{\tilde{d}^3_{123}}\tilde{d}^1_{123} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Then, from the sixth row, subtract the fourth row times $\frac{\tilde{d}^4_{134}}{\tilde{d}^4_{124}}$, yielding the following row. (The MA-nonzero condition condition ensures that $\tilde{d}^4_{124}$ is non-zero).

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -\frac{\tilde{d}^3_{134}}{\tilde{d}^3_{123}}\tilde{d}^2_{123} - \frac{\tilde{d}^4_{134}}{\tilde{d}^4_{124}}\tilde{d}^2_{124} & 0 & \tilde{d}^1_{134} - \frac{\tilde{d}^3_{134}}{\tilde{d}^3_{123}}\tilde{d}^1_{123} - \frac{\tilde{d}^4_{134}}{\tilde{d}^4_{124}}\tilde{d}^1_{124} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

If we divide the sixth row by the sixth element in the row, we obtain the following row. (The MA-nonzero condition condition ensures that the sixth element in the row is non-zero).

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $\tilde{t}_2 := \frac{\tilde{d}^1_{123}\tilde{d}^3_{134}\tilde{d}^4_{124} + \tilde{d}^1_{124}\tilde{d}^3_{123}\tilde{d}^4_{134} - \tilde{d}^1_{134}\tilde{d}^3_{123}\tilde{d}^4_{124}}{\tilde{d}^2_{123}\tilde{d}^3_{134}\tilde{d}^4_{124} + \tilde{d}^2_{124}\tilde{d}^3_{123}\tilde{d}^4_{134}}$.

The matrix below results if we next perform the following row operations:

(i) from the second row subtract the sixth row times $\tilde{d}^2_{123}$, and divide the result by $\tilde{d}^3_{123}$ (the MA-nonzero condition condition ensures that $\tilde{d}^3_{123}$ is non-zero);

(ii) from the fourth row subtract the sixth row times $\tilde{d}^2_{124}$, and divide the result by $\tilde{d}^4_{124}$ (the MA-nonzero condition condition ensures that $\tilde{d}^4_{124}$ is non-zero).

$$\begin{bmatrix} 0 & 0 & -1 & \tilde{p}^3_{1,123} & -1 & \tilde{p}^2_{1,123} & -1 & \tilde{p}^1_{1,123} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & t_3 & 0 \\ -1 & \tilde{p}^4_{1,124} & 0 & 0 & -1 & \tilde{p}^2_{1,124} & -1 & \tilde{p}^1_{1,124} & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 \\ -1 & \tilde{p}^4_{1,134} & -1 & \tilde{p}^3_{1,134} & 0 & 0 & -1 & \tilde{p}^1_{1,134} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & t_2 & 0 \\ -1 & \tilde{p}^4_{2,234} & -1 & \tilde{p}^3_{2,234} & -1 & \tilde{p}^2_{2,234} & 0 & 0 & 1 \\ 0 & \tilde{d}^4_{234} & 0 & \tilde{d}^3_{234} & 0 & \tilde{d}^2_{234} & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{t}_3 = \frac{\tilde{d}^1_{123} - \tilde{d}^2_{123} t_2}{\tilde{d}^3_{123}}$ and $t_4 = \frac{\tilde{d}^1_{124} - \tilde{d}^2_{124} t_2}{\tilde{d}^4_{124}}$.

Next, the matrix below results if we perform the following row operations:

(i) from the first row, subtract the second row times $\tilde{p}^3_{1,123}$ and the sixth row times $\tilde{p}^2_{1,123}$, and multiply the result by -1;

(ii) from the third row, subtract the fourth row times $\tilde{p}^4_{1,124}$ and the sixth row times $\tilde{p}^2_{1,124}$, and multiply the result by -1;

(iii) from the fifth row, subtract the fourth row times $\tilde{p}^4_{1,134}$ and the second row times $\tilde{p}^3_{1,134}$, and multiply the result by -1;

(i) from the seventh row, subtract the second row times $\tilde{p}^3_{2,234}$, the fourth row times $\tilde{p}^4_{2,234}$ and the sixth row times $\tilde{p}^2_{2,234}$, and multiply the result by -1.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & \tilde{e}_1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \tilde{t}_3 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & \tilde{e}_2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_4 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & \tilde{e}_3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & \tilde{e}_4 & -1 \\ 0 & \tilde{d}^4_{134} & 0 & \tilde{d}^3_{234} & 0 & \tilde{d}^2_{234} & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{e}_1 = \tilde{p}^2_{1,123}\tilde{t}_2 + \tilde{p}^3_{1,123}\tilde{t}_3 - \tilde{p}^1_{1,123}$, $\tilde{e}_2 = \tilde{p}^2_{1,124}\tilde{t}_2 + \tilde{p}^4_{1,124}\tilde{t}_4 - \tilde{p}^1_{1,124}$, $\tilde{e}_3 = \tilde{p}^3_{1,134}\tilde{t}_3 + \tilde{p}^4_{1,134}\tilde{t}_4 - \tilde{p}^1_{1,134}$ and $\tilde{e}_4 = \tilde{p}^2_{2,234}\tilde{t}_2 + \tilde{p}^3_{2,234}\tilde{t}_3 + \tilde{p}^4_{2,234}\tilde{t}_4$.

Then the matrix below results if we perform the following row operations:

(i) from the third row, subtract row 5 and multiply the result by -1;

(ii) from the first row, subtract the third row and divide by two;

(iii) from the seventh row, subtract the fifth row.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{\tilde{e}_1+\tilde{e}_2-\tilde{e}_3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \tilde{t}_3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & \tilde{e}_3-\tilde{e}_2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_4 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & \tilde{e}_3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & \tilde{e}_4-\tilde{e}_3 & 0 \\ 0 & \tilde{d}^4_{134} & 0 & \tilde{d}^3_{234} & 0 & \tilde{d}^2_{234} & 0 & 0 & 0 \end{bmatrix}$$

Next, we obtain the matrix below when we perform the following row operations:

(i) from the seventh row, subtract the first row and multiply the result by $-\frac{2}{3}$;

(ii) from the first row, subtract the seventh row times $\frac{1}{2}$;

(iii) to the third row, add the first row;

(iv) from the fifth row, subtract the third and seventh rows.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tilde{s}_2 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \tilde{t}_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \tilde{s}_3 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_4 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \tilde{s}_1 & -\frac{1}{3} \\ 0 & \tilde{d}^4_{134} & 0 & \tilde{d}^3_{234} & 0 & \tilde{d}^2_{234} & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{s}_1 = (\tilde{e}_1+\tilde{e}_2+\tilde{e}_3-2\tilde{e}_4)/3$, $\tilde{s}_2 = (\tilde{e}_1+\tilde{e}_2+\tilde{e}_4-2\tilde{e}_3)/3$,

$\tilde{s}_3 = (\tilde{e}_1+\tilde{e}_3+\tilde{e}_4-2\tilde{e}_2)/3$ and $\tilde{s}_4 = (\tilde{e}_2+\tilde{e}_3+\tilde{e}_4-2\tilde{e}_1)/3$.

Finally, we obtain the matrix below if we perform the following row operations:

(i) exchange the first and fifth rows and also exchange the second and fourth rows;

(ii) from the eighth row, subtract the second row times $\tilde{d}^4_{234}$, the fourth row times $\tilde{d}^3_{234}$ and the sixth row times $\tilde{d}^2_{234}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_4 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \tilde{s}_3 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \tilde{t}_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tilde{s}_2 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \tilde{s}_1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Why are all the elements of the eighth row equal to zero? The above row operations imply that all the elements of the eighth row are equal to zero except possibly for the eighth element, which is equal to $-\tilde{d}^4_{234}\tilde{t}_4 - \tilde{d}^3_{234}\tilde{t}_3 - \tilde{d}^2_{234}\tilde{t}_2$. It follows that either (a) the eighth element must also be zero or (b) $\Delta_1 = 0$. But the second possibility is ruled out by the following claim:

Claim 1: $\Delta_1 \neq 0$.

Proof Suppose, for contradiction, that $\Delta_1 = 0$. From the matrix above, for $i = 2, 3, 4$, $\Delta_i = -\tilde{t}_i\Delta_1$. So $\Delta_i = 0$. Thus, for $i = 1, \ldots, 4$, $\iota_i = \iota'_i$. So $c = c'$. But this violates clause (i) of the MA-nonzero condition.

Claim 1 entails that all the elements of the eighth row must be equal to zero. This completes the proof for the case where $|X| = 4$.

We now extend this proof to a proof of the general case where $|X| = m \geq 4$. Consider the system of equations comprising the following enumeration of the rationalisation equations:

(i) the first eight equations are the eight difference equations represented in the augmented matrix above, enumerated in the order in that matrix;

(ii) the next $2(m-4)$ equations are, respectively, the difference equations for $c(x_1, x_1x_2x_5)$, $c(x_2, x_1x_2x_5)$, $c(x_1, x_1x_2x_6)$, $c(x_2, x_1x_2x_6)$, $\ldots$, $c(x_1, x_1x_2x_m)$, $c(x_2, x_1x_2x_m)$; and

(iii) the remaining $2\binom{m}{3} - 2m$ difference equations are enumerated beneath them in any order such that, if, for some i, j, k, the equation is a difference equation for $c(x_i, x_ix_jx_k)$, then it appears in an odd row, and if it is a difference equation for $c(x_j, x_ix_jx_k)$, it appears the next row. Suppose that we undertake the same row operations specified above. The result is a $2\binom{m}{3}$ by $2m + 1$ matrix of the following form:

$$
\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_4 & -\frac{1}{3} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_4 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \tilde{s}_3 & -\frac{1}{3} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \tilde{t}_3 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tilde{s}_2 & -\frac{1}{3} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \tilde{s}_1 & -\frac{1}{3} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & -1 & \tilde{p}^5_{1,125} & 0 & 0 & 0 & 0 & -1 & \tilde{p}^2_{1,125} & -1 & \tilde{p}^1_{1,125} & 1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \tilde{d}^5_{125} & 0 & 0 & 0 & 0 & 0 & \tilde{d}^2_{125} & 0 & \tilde{d}^1_{125} & 0 \\
0 & 0 & \ldots & -1 & \tilde{p}^6_{1,126} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \tilde{p}^2_{1,126} & -1 & \tilde{p}^1_{1,126} & 1 \\
0 & 0 & \ldots & 0 & \tilde{d}^6_{126} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{d}^2_{126} & 0 & \tilde{d}^1_{126} & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & \tilde{p}^m_{1,12m} & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \tilde{p}^2_{1,12m} & -1 & \tilde{p}^1_{1,12m} & 1 \\
0 & \tilde{d}^m_{12m} & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{d}^2_{12m} & 0 & \tilde{d}^1_{12m} & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

Perform the following row operations on rows nine and ten.

Step 1: Subtract from the tenth row the sixth row times $\tilde{d}^2_{125}$, and divide the result by $\tilde{d}^5_{125}$.

Step 2: Subtract from the ninth row the tenth row times $\tilde{p}^5_{1,125}$ and the sixth row times $\tilde{p}^2_{1,125}$, then add to the result the fifth row and the seventh row, and, finally, multiply the result by -1.

Step 3: Move the ninth row to the first row, the tenth row to the second row, and shift all other rows down two places.

The first and second rows then become:

$$
\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_5 & -\frac{1}{3} \\
0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_5 & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

where $\tilde{t}_5 = (\tilde{d}^1_{125} - \tilde{d}^2_{125}\tilde{t}_2)/\tilde{d}^5_{125})$ and $\tilde{s}_5 = -(\tilde{p}^1_{1,125} - \tilde{p}^5_{1,125}\tilde{t}_5 - \tilde{p}^2_{1,125}\tilde{t}_2 + \tilde{s}_1 + \tilde{s}_2)$

Repeat a similar three steps for the eleventh and twelfth rows, then for the thirteenth and fourteenth rows, and so forth, up to rows $2m-1$ and $2m$. More precisely, for each $i \in \{6, 7, \ldots, m\}$, perform the following row operations on rows $2i-1$ and $2i$.

Step 1: Subtract from row $2i$ the sixth row times d^2_{12i}, and divide the result by d^i_{12i}.

Step 2: Subtract from row $2i-1$ row $2i$ times $p^i_{1,12i}$ and the sixth row times $p^2_{1,12i}$, then add to the result the fifth row and the seventh row, and, finally, multiply the result by -1.

Step 3: Move row $2i-1$ to the first row and row $2i$ to the second row, and shift all other rows down two places.

The result is the following matrix:

$$\begin{bmatrix}
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_m & -\frac{1}{3} \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_m & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_6 & -\frac{1}{3} \\
0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_6 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_5 & -\frac{1}{3} \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_5 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{s}_4 & -\frac{1}{3} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \tilde{t}_4 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \tilde{s}_3 & -\frac{1}{3} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \tilde{t}_3 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tilde{s}_2 & -\frac{1}{3} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \tilde{t}_2 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \tilde{s}_1 & -\frac{1}{3} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

where, for $i = 5, 6, \dots, m$, $\tilde{t}_i = (\tilde{d}^1_{12i} - \tilde{d}^2_{12i}\tilde{t}_2)/\tilde{d}^i_{12i}$ and $\tilde{s}_i = -(\tilde{p}^1_{1,12i} - \tilde{p}^i_{1,12i}\tilde{t}_i - \tilde{p}^2_{1,12i}\tilde{t}_2 + \tilde{s}_1 + \tilde{s}_2)$.

The final task is to prove that, in the reduced row echelon form, the rows from row $2m+1$ to row $2\binom{m}{3}$ are all zero rows. These rows come in pairs: that is, any odd row corresponds to a difference equation for $c(x_i, x_i x_j x_k)$; and then the next row corresponds to a difference equation for $c(x_j, x_i, x_j, x_k)$. Any such pair of rows has the following form:

$$\begin{bmatrix}
 & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\dots & -1 & \tilde{p}^k_{i,ijk} & \dots & -1 & \tilde{p}^j_{i,ijk} & \dots & -1 & \tilde{p}^i_{i,ijk} & \dots & 1 \\
\dots & 0 & \tilde{d}^k_{ijk} & \dots & 0 & \tilde{d}^j_{ijk} & \dots & 0 & \tilde{d}^i_{ijk} & \dots & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots
\end{bmatrix}$$

where the elements specified above appear in, respectively, columns $2(m-k)+1$, $2(m-k)+2$, $2(m-j)+1$, $2(m-j)+2$, $2(m-i)+1$, $2(m-i)+2$ and $2m+1$. All other elements in the two rows are zero. It remains to find the row operations that transform any such pair of rows into a pair of zero rows. There are two cases to consider: if $i=1$, the required row operations are slightly different from the case in which i does not equal 1.

<u>Case 1</u>: $i>1$

Consider the second of the two rows. Subtract from it row $2(m-k+1)$ times $\tilde{d}^k_{ijk}$, row $2(m-j+1)$ times d^j_{ijk} and row $2(m-i+1)$ times $\tilde{d}^i_{ijk}$. These row operations create a row in which all the elements of the row are zero, except for possibly the element in column $2m$. But given that Δ_1 cannot be equal to zero (see the reasoning in Claim 1), it follows that this element must also be zero.

Consider now the first of the two rows. Subtract from it row $2(m-k+1)$ times $\tilde{p}^k_{i,ijk}$, row $2(m-j+1)$ times $\tilde{p}^j_{i,ijk}$ and row $2(m-i+1)$ times $\tilde{p}^i_{i,ijk}$. Then add to it rows $2(m-k)+1$, $2(m-j)+1$ and $2(m-i)+1$. These row operations create a row in which all the elements of the row are zero, except for possibly the element in column $2m$. But given that $\Delta_1 \neq 0$, it follows that this element must also be zero.

<u>Case 2</u>: $i=1$

Consider the second of the two rows. Subtract from it row $2(m-k+1)$ times $\tilde{d}^k_{ijk}$ and row $2(m-j+1)$ times $\tilde{d}^j_{ijk}$. These row operations create a row in which all the elements of the row are zero, except for possibly the element in column $2m$. But given that $\Delta_1 \neq 0$, it follows that this element must also be zero.

Consider now the first of the two rows. Subtract from it row $2(m-k+1)$ times $\tilde{p}^k_{i,ijk}$ and row $2(m-j+1)$ times $\tilde{p}^j_{i,ijk}$. Then add to it rows $2(m-k)+1$, $2(m-j)+1$ and $2(m-i)+1$. These row operations create a row in which all the elements of the row are zero, except for possibly the element in column $2m$. But given that $\Delta_1 \neq 0$, it follows that this element must also be zero. This completes the proof of Lemma 3.

Lemma 3 allows us to solve for all of the unknowns as the following affine functions of Δ_1: $\iota_i = -\frac{1}{3} - \tilde{s}_i\Delta_1, i = 1,\ldots,m$ and $\Delta_i = -\tilde{t}_i\Delta_1, i = 2,\ldots,m$. It remains to solve for Δ_1. In order to do so, we will use the following rationalisation condition:

$$c(x_1, x_1x_2x_3x_4) = [\gamma_1\gamma_2(1-\gamma_3)(1-\gamma_4)c(x_1, x_1x_2) + \gamma_1\gamma_3(1-\gamma_2)(1-\gamma_4)c(x_1, x_1x_3)$$

$$+\gamma_1\gamma_4(1-\gamma_2)(1-\gamma_3)c(x_1,x_1x_4)+\gamma_1\gamma_2\gamma_3(1-\gamma_4)\pi(\mathcal{M}_{x_1,x_1x_2x_3})$$

$$+\gamma_1\gamma_2\gamma_4(1-\gamma_3)\pi(\mathcal{M}_{x_1,x_1x_2x_4})+\gamma_1\gamma_3\gamma_4(1-\gamma_2)\pi(\mathcal{M}_{x_1,x_1x_3x_4})+\gamma_1\gamma_2\gamma_3\gamma_4\pi(\mathcal{M}_{x_1,x_1x_2x_3x_4})]/$$

$$[\gamma_1\gamma_2(1-\gamma_3)(1-\gamma_4)+\gamma_1\gamma_3(1-\gamma_2)(1-\gamma_4)+\gamma_1\gamma_4(1-\gamma_2)(1-\gamma_3)$$

$$[\gamma_2\gamma_3(1-\gamma_1)(1-\gamma_4)+\gamma_2\gamma_4(1-\gamma_1)(1-\gamma_3)+\gamma_3\gamma_4(1-\gamma_1)(1-\gamma_2)$$

$$+\gamma_1\gamma_2\gamma_3(1-\gamma_4)+\gamma_1\gamma_2\gamma_4(1-\gamma_3)+\gamma_1\gamma_3\gamma_4(1-\gamma_2)+\gamma_2\gamma_3\gamma_4(1-\gamma_1)+\gamma_1\gamma_2\gamma_3\gamma_4]$$

If we rearrange and apply the definition of inattention odds, we obtain the following, which will be referred to as the *rationalisation equation for* $c(x_1, x_1x_2x_3x_4)$.

$$(1+\iota_1+\iota_2+\iota_3+\iota_4+\iota_1\iota_2+\iota_1\iota_3+\iota_1\iota_4+\iota_2\iota_3+\iota_2\iota_4+\iota_3\iota_4)c(x_1,x_1x_2x_3x_4)=$$

$$\iota_3\iota_4c(x_1,x_1x_2)+\iota_2\iota_4c(x_1,x_1x_3)+\iota_2\iota_3c(x_1,x_1x_4)+$$

$$\iota_4\pi(\mathcal{M}_{x_1,x_1x_2x_3})+\iota_3\pi(\mathcal{M}_{x_1,x_1x_2x_4})+\iota_2\pi(\mathcal{M}_{x_1,x_1x_3x_4})+\pi(\mathcal{M}_{x_1,x_1x_2x_3x_4}) \tag{2.124}$$

Suppose we obtain the instances of (2.116) for $i=1, j=2, k=3$, $i=1, j=2, k=4$, $i=1, j=3, k=4$. We can rearrange these three equations as follows:

$$\pi(\mathcal{M}_{x_1,x_1x_2x_3})=(1+\iota_1+\iota_2+\iota_3)c(x_1,x_1x_2x_3)-\iota_3c(x_1,x_1x_2)-\iota_2c(x_1,x_1x_3)$$

$$\pi(\mathcal{M}_{x_1,x_1x_2x_4})=(1+\iota_1+\iota_2+\iota_4)c(x_1,x_1x_2x_4)-\iota_4c(x_1,x_1x_2)-\iota_2c(x_1,x_1x_4)$$

$$\pi(\mathcal{M}_{x_1,x_1x_3x_4})=(1+\iota_1+\iota_3+\iota_4)c(x_1,x_1x_3x_4)-\iota_4c(x_1,x_1x_3)-\iota_3c(x_1,x_1x_4)$$

If we substitute these three equations into the rationalisation equation for $c(x_1, x_1x_2x_3x_4)$ we obtain:

$$(1+\iota_1+\iota_2+\iota_3+\iota_4+\iota_1\iota_2+\iota_1\iota_3+\iota_1\iota_4+\iota_2\iota_3+\iota_2\iota_4+\iota_3\iota_4)c(x_1,x_1x_2x_3x_4)=$$

$$\iota_4(1+\iota_1+\iota_2+\iota_3)c(x_1,x_1x_2x_3)+\iota_3(1+\iota_1+\iota_2+\iota_4)c(x_1,x_1x_2x_4)+\iota_2(1+\iota_1+\iota_3+\iota_4)c(x_1,x_1x_3x_4)$$

$$-\iota_3\iota_4c(x_1,x_1x_2)-\iota_2\iota_4c(x_1,x_1x_3)-\iota_2\iota_3c(x_1,x_1x_4)+\pi(\mathcal{M}_{x_1,x_1x_2x_3x_4})$$

The rationalisation equation for $c'(x_1, x_1x_2x_3x_4)$ can be written in an analogous fashion. If we take the difference between the two rationalisation equations, we eliminate $\pi(\mathcal{M}_{x_1,x_1x_2x_3x_4})$ and obtain the following, which is termed *the difference equation for* $c(x_1, x_1x_2x_3x_4)$:

$$(\iota_1+1)[c(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3x_4)]-\Delta_1c'(x_1,x_1x_2x_3x_4)+$$

$$(\iota_1+1)\iota_2[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)]+$$

$$[(\iota_1+1)\iota_2-(\iota_1'+1)\iota_2'][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)]+$$

$$(\iota_1+1)\iota_3[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_4)]+$$

$$[(\iota_1+1)\iota_3-(\iota_1'+1)\iota_3'][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_4)]+$$

$$(\iota_1+1)\iota_4[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_3)]+$$

$$[(\iota_1+1)\iota_4-(\iota_1'+1)\iota_4'][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3)]+\iota_2\iota_3\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_4)]$$

$$+[\iota_2\iota_3-\iota_2'\iota_3'][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_4)+c(x_1,x_1x_4)]+\iota_2\iota_4\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_3)]$$

$$+[\iota_2\iota_4-\iota_2'\iota_4'][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_3)]+\iota_3\iota_4\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_4)+c'(x_1,x_1x_2x_3)]$$

$$+[\iota_3\iota_4-\iota_3'\iota_4'][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_2)]=0$$

In the difference equation for $c(x_1,x_1x_2x_3x_4)$, there are eight unknowns: $\iota_1,\iota_2,\iota_3,\iota_4,\Delta_1,\Delta_2,\Delta_3,\Delta_4$. But using Lemma 1, we can eliminate seven of these eight unknowns, substituting in the equations $\iota_i=-\frac{1}{3}-\tilde{s}_i\Delta_1, i=1,2,3,4$ and $\Delta_i=-\tilde{t}_i\Delta_1, i=2,3,4$. When substituting in these equations, we are able to simplify using the following equality: $\iota_i\iota_j-\iota_i'\iota_j'=\iota_i\iota_j-\iota_i\iota_j'+\iota_i\iota_j'-\iota_i'\iota_j'=-\iota_i\Delta_j-\iota_j'\Delta_i=-\iota_i\Delta_j-\iota_j'\Delta_i+\iota_j\Delta_i-\iota_j\Delta_i=-(\iota_i\Delta_j+\iota_j\Delta_i+\Delta_i\Delta_j)$.

The result of substituting these equations into the difference equation is as follows:

$$(\frac{2}{3}-\tilde{s}_1\Delta_1)[c(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3x_4)]-\Delta_1c'(x_1,x_1x_2x_3x_4)+$$

$$(\frac{2}{3}-\tilde{s}_1\Delta_1)(-\frac{1}{3}-\tilde{s}_2\Delta_1)[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)]+$$

$$[\tilde{t}_2\Delta_1(\frac{2}{3}-\tilde{s}_1\Delta_1)+\Delta_1(\frac{1}{3}+\tilde{s}_2\Delta_1)+\tilde{t}_2\Delta_1^2][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)]+$$

$$(\frac{2}{3}-\tilde{s}_1\Delta_1)(-\frac{1}{3}-\tilde{s}_3\Delta_1)[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_4)]+$$

$$[\tilde{t}_3\Delta_1(\frac{2}{3}-\tilde{s}_1\Delta_1)+\Delta_1(\frac{1}{3}+\tilde{s}_3\Delta_1)+\tilde{t}_3\Delta_1^2][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_4)]+$$

$$(\frac{2}{3}-\tilde{s}_1\Delta_1)(-\frac{1}{3}-\tilde{s}_4\Delta_1)[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_3)]+$$

$$[\tilde{t}_4\Delta_1(\frac{2}{3}-\tilde{s}_1\Delta_1)+\Delta_1(\frac{1}{3}+\tilde{s}_4\Delta_1)+\tilde{t}_4\Delta_1^2][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3)]+$$

$$\{(\frac{1}{3}+\tilde{s}_2\Delta_1)(\frac{1}{3}+\tilde{s}_3\Delta_1)\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_4)]\}$$

$$-\{[(\tilde{t}_3\Delta_1(\frac{1}{3}+\tilde{s}_2\Delta_1)+(\tilde{t}_2\Delta_1(\frac{1}{3}+\tilde{s}_3\Delta_1)+\tilde{t}_2\tilde{t}_3\Delta_1^2]\times$$

$$[c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_4)+c(x_1,x_1x_4)]\}+$$

$$\{(\frac{1}{3}+\tilde{s}_2\Delta_1)(\frac{1}{3}+\tilde{s}_4\Delta_1)\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_3)]\}$$

$$-\{[(\tilde{t}_4\Delta_1(\frac{1}{3}+\tilde{s}_2\Delta_1)+(\tilde{t}_2\Delta_1(\frac{1}{3}+\tilde{s}_4\Delta_1)+\tilde{t}_2\tilde{t}_4\Delta_1^2]\times$$

$$[c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_3)]\}+$$

$$\{(\frac{1}{3}+\tilde{s}_3\Delta_1)(\frac{1}{3}+\tilde{s}_4\Delta_1)\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_4)+c'(x_1,x_1x_2x_3)]\}$$

$$-\{[(\tilde{t}_4\Delta_1(\frac{1}{3}+\tilde{s}_3\Delta_1)+(\tilde{t}_3\Delta_1(\frac{1}{3}+\tilde{s}_4\Delta_1)+\tilde{t}_3\tilde{t}_4\Delta_1^2]\times$$

$$[c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_2)]\}=0$$

The result is a quadratic equation of the form $\tilde{f}\Delta_1^2+\tilde{g}\Delta_1+\tilde{h}=0$, where expressions for $\tilde{f},\tilde{g}$ and $\tilde{h}$ are given as follows:

$$\tilde{f}=\tilde{s}_1\tilde{s}_2[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)]+$$

$$[\tilde{s}_2+\tilde{t}_2(1-\tilde{s}_1)][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)]+$$

$$\tilde{s}_1\tilde{s}_3[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_4)]+$$

$$[\tilde{s}_3+\tilde{t}_3(1-\tilde{s}_1)][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_4)]+$$

$$\tilde{s}_1\tilde{s}_4[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_3)]+$$

$$[\tilde{s}_4+\tilde{t}_4(1-\tilde{s}_1)][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3)]+\tilde{s}_2\tilde{s}_3\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_4)]$$

$$-[\tilde{s}_2\tilde{t}_3+\tilde{s}_3\tilde{t}_2+\tilde{t}_2\tilde{t}_3][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_4)+c(x_1,x_1x_4)]+\tilde{s}_2\tilde{s}_4\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_3)]$$

$$-[\tilde{s}_2\tilde{t}_4+\tilde{s}_4\tilde{t}_2+\tilde{t}_2\tilde{t}_4][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_3)]+\tilde{s}_3\tilde{s}_4\times$$

$$[c(x_1,x_1x_2x_3x_4)-c(x_1,x_1x_2x_4)-c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3x_4)+c'(x_1,x_1x_2x_4)+c'(x_1,x_1x_2x_3)]$$

$$-[\tilde{s}_3\tilde{t}_4+\tilde{s}_4\tilde{t}_3+\tilde{t}_3\tilde{t}_4][c'(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_2)] \tag{2.125}$$

$$\tilde{g}=-\tilde{s}_1[c(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3x_4)]-c'(x_1,x_1x_2x_3x_4)+$$

$$[\frac{1}{3}\tilde{s}_1 - \frac{2}{3}\tilde{s}_2][c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_3x_4) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_3x_4)]+$$

$$[\frac{2}{3}\tilde{t}_2 + \frac{1}{3}][c'(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_3x_4)]+$$

$$[\frac{1}{3}\tilde{s}_1 - \frac{2}{3}\tilde{s}_3][c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_2x_4) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_2x_4)]+$$

$$[\frac{2}{3}\tilde{t}_3 + \frac{1}{3}][c'(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_2x_4)]+$$

$$[\frac{1}{3}\tilde{s}_1 - \frac{2}{3}\tilde{s}_4][c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_2x_3)]+$$

$$[\frac{2}{3}\tilde{t}_4 + \frac{1}{3}][c'(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_2x_3)] + \frac{1}{3}[\tilde{s}_2 + \tilde{s}_3]\times$$

$$[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_3x_4) - c(x_1, x_1x_2x_4) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_3x_4) + c'(x_1, x_1x_2x_4)]$$

$$-\frac{1}{3}[\tilde{t}_2 + \tilde{t}_3][c'(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_3x_4) - c'(x_1, x_1x_2x_4) + c(x_1, x_1x_4)] + \frac{1}{3}[\tilde{s}_2 + \tilde{s}_4]\times$$

$$[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_3x_4) - c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_3x_4) + c'(x_1, x_1x_2x_3)]$$

$$-\frac{1}{3}[\tilde{t}_2 + \tilde{t}_4][c'(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_3x_4) - c'(x_1, x_1x_2x_3) + c(x_1, x_1x_3)] + \frac{1}{3}[\tilde{s}_3 + \tilde{s}_4]\times$$

$$[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_2x_4) - c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_2x_4) + c'(x_1, x_1x_2x_3)]$$

$$-\frac{1}{3}[\tilde{t}_3 + \tilde{t}_4][c'(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_2x_4) - c'(x_1, x_1x_2x_3) + c(x_1, x_1x_2)]$$

$$\tilde{h} = \frac{2}{3}[c(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_2x_3x_4)]$$

$$-\frac{2}{9}[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_3x_4) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_3x_4)]$$

$$-\frac{2}{9}[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_2x_4) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_2x_4)]$$

$$-\frac{2}{9}[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_2x_3)]$$

$$+\frac{1}{9}[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_3x_4) - c(x_1, x_1x_2x_4) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_3x_4) + c'(x_1, x_1x_2x_4)]$$

$$+\frac{1}{9}[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_3x_4) - c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_3x_4) + c'(x_1, x_1x_2x_3)]$$

$$+\frac{1}{9}[c(x_1, x_1x_2x_3x_4) - c(x_1, x_1x_2x_4) - c(x_1, x_1x_2x_3) - c'(x_1, x_1x_2x_3x_4) + c'(x_1, x_1x_2x_4) + c'(x_1, x_1x_2x_3)]$$

We can simplify the constant term $\tilde{h}$ as follows:

$$\tilde{h} = [\frac{2}{3} - 3 \times \frac{2}{9} + 3 \times \frac{1}{9}][c(x_1, x_1x_2x_3x_4) - c'(x_1, x_1x_2x_3x_4)]$$

$$+[2 \times \frac{1}{9} - \frac{2}{9}][-c(x_1, x_1x_3x_4) + c'(x_1, x_1x_3x_4)]$$

$$+[2 \times \frac{1}{9} - \frac{2}{9}][-c(x_1, x_1x_2x_4) + c'(x_1, x_1x_2x_4)]$$

$$+[2\times\frac{1}{9}-\frac{2}{9}][-c(x_1,x_1x_2x_3)+c'(x_1,x_1x_2x_3)]$$
$$=\frac{1}{3}[c(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3x_4)]$$

Moreover, we will now show that $\tilde{g}=0$.

We can rewrite the expression for $\tilde{g}$ as follows:

$$\tilde{g}=[-\tilde{s}_1+\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_2+\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_3+\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_4+\frac{1}{3}[\tilde{s}_2+\tilde{s}_3]+\frac{1}{3}[\tilde{s}_2+\tilde{s}_4]+\frac{1}{3}[\tilde{s}_3+\tilde{s}_4]]$$
$$\times[c(x_1,x_1x_2x_3x_4)-c'(x_1,x_1x_2x_3x_4)]+$$
$$[-1+\frac{2}{3}\tilde{t}_2+\frac{1}{3}+\frac{2}{3}\tilde{t}_3+\frac{1}{3}+\frac{2}{3}\tilde{t}_4+\frac{1}{3}-\frac{1}{3}[\tilde{t}_2+\tilde{t}_3]-\frac{1}{3}[\tilde{t}_2+\tilde{t}_4]-\frac{1}{3}[\tilde{t}_3+\tilde{t}_4]]c'(x_1,x_1x_2x_3x_4)+$$
$$[\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_2][-c(x_1,x_1x_3x_4)+c'(x_1,x_1x_3x_4)]+$$
$$[\frac{2}{3}\tilde{t}_2+\frac{1}{3}][-c'(x_1,x_1x_3x_4)]+$$
$$[\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_3][-c(x_1,x_1x_2x_4)+c'(x_1,x_1x_2x_4)]+$$
$$[\frac{2}{3}\tilde{t}_3+\frac{1}{3}][-c'(x_1,x_1x_2x_4)]+$$
$$[\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_4][-c(x_1,x_1x_2x_3)+c'(x_1,x_1x_2x_3)]+$$
$$[\frac{2}{3}\tilde{t}_4+\frac{1}{3}][-c'(x_1,x_1x_2x_3)]+$$
$$\frac{1}{3}[\tilde{s}_2+\tilde{s}_3][-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_4)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_4)]$$
$$-\frac{1}{3}[\tilde{t}_2+\tilde{t}_3][-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_4)+c(x_1,x_1x_4)]+$$
$$\frac{1}{3}[\tilde{s}_2+\tilde{s}_4][-c(x_1,x_1x_3x_4)-c(x_1,x_1x_2x_3)+c'(x_1,x_1x_3x_4)+c'(x_1,x_1x_2x_3)]$$
$$-\frac{1}{3}[\tilde{t}_2+\tilde{t}_4][-c'(x_1,x_1x_3x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_3)]+$$
$$\frac{1}{3}[\tilde{s}_3+\tilde{s}_4][-c(x_1,x_1x_2x_4)-c(x_1,x_1x_2x_3)+c'(x_1,x_1x_2x_4)+c'(x_1,x_1x_2x_3)]$$
$$-\frac{1}{3}[\tilde{t}_3+\tilde{t}_4][-c'(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_3)+c(x_1,x_1x_2)]$$

The first two terms drop out, because $-\tilde{s}_1+\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_2+\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_3+\frac{1}{3}\tilde{s}_1-\frac{2}{3}\tilde{s}_4+\frac{1}{3}[\tilde{s}_2+\tilde{s}_3]+\frac{1}{3}[\tilde{s}_2+\tilde{s}_4]+\frac{1}{3}[\tilde{s}_3+\tilde{s}_4]=0$ and $-1+\frac{2}{3}\tilde{t}_2+\frac{1}{3}+\frac{2}{3}\tilde{t}_3+\frac{1}{3}+\frac{2}{3}\tilde{t}_4+\frac{1}{3}-\frac{1}{3}[\tilde{t}_2+\tilde{t}_3]-\frac{1}{3}[\tilde{t}_2+\tilde{t}_4]-\frac{1}{3}[\tilde{t}_3+\tilde{t}_4]=0$. If we remove the first two terms and rearrange, we obtain

$$\tilde{g}=-\frac{1}{3}[\{(\tilde{s}_1+\tilde{s}_3+\tilde{s}_4)[c(x_1,x_1x_3x_4)-c'(x_1,x_1x_3x_4)]-$$

$$[(-1+\tilde{t}_3+\tilde{t}_4)c'(x_1,x_1x_3x_4)-\tilde{t}_4c(x_1,x_1x_3)-\tilde{t}_3c(x_1,x_1x_4)]\}$$

$$+\{(\tilde{s}_1+\tilde{s}_2+\tilde{s}_4)[c(x_1,x_1x_2x_4)-c'(x_1,x_1x_2x_4)]-$$

$$[(-1+\tilde{t}_2+\tilde{t}_4)c'(x_1,x_1x_2x_4)-\tilde{t}_4c(x_1,x_1x_2)-\tilde{t}_2c(x_1,x_1x_4)]\}$$

$$+\{(\tilde{s}_1+\tilde{s}_2+\tilde{s}_3)[c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3)]-$$

$$[(-1+\tilde{t}_2+\tilde{t}_3)c'(x_1,x_1x_2x_3)-\tilde{t}_3c(x_1,x_1x_2)-\tilde{t}_2c(x_1,x_1x_3)]\}] \tag{2.126}$$

But if we substitute into the difference equation for $c(x_1, x_1x_2x_3)$ the expressions $\iota_i = -\frac{1}{3} - \tilde{s}_i\Delta_1$ for $i = 1, 2, 3$ and $\Delta_i = -\tilde{t}_i\Delta_1$ for $i = 2, 3$, then we obtain

$$(1-\frac{1}{3}-\tilde{s}_1\Delta_1-\frac{1}{3}-\tilde{s}_2\Delta_1-\frac{1}{3}-\tilde{s}_3\Delta_1)[c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3)] =$$

$$-(-1+\tilde{t}_2+\tilde{t}_3)\Delta_1c'(x_1,x_1x_2x_3)+\tilde{t}_3\Delta_1c(x_1,x_1x_2)+\tilde{t}_2\Delta_1c(x_1,x_1x_3)$$

So

$$(\tilde{s}_1+\tilde{s}_2+\tilde{s}_3)[c(x_1,x_1x_2x_3)-c'(x_1,x_1x_2x_3)] =$$

$$(-1+\tilde{t}_2+\tilde{t}_3)c'(x_1,x_1x_2x_3)-\tilde{t}_3c(x_1,x_1x_2)-\tilde{t}_2c(x_1,x_1x_3)$$

So in equation (2.126) , the third term in the curly brackets is zero. Similar reasoning can be used to establish that each of the first two terms in the curly brackets are zero. So $\tilde{g} = 0$. Therefore Δ_1 is the square root of $-\tilde{h}/\tilde{f}$.

How do we ascertain whether Δ_1 is the negative or positive square root? Given that (c, c') is rationalisable, we know that ι_1 must be positive; and we also know that $\iota_1 = -\frac{1}{3} - \tilde{s}_1\Delta_1$. So Δ_1 must have the opposite sign to $\tilde{s}_1$. This allows us to obtain a unique solution for Δ_1.

$$\tilde{\Delta}_1 = \begin{cases} -\sqrt{-\tilde{h}/\tilde{f}} & \text{if } \tilde{s}_1 > 0 \\ \sqrt{-\tilde{h}/\tilde{f}} & \text{otherwise} \end{cases}$$

Appendix 3: Model with an independent attention assignment

This appendix examines a consideration-set model in which there is a default option and the attention assignment is independent. We will refer to this model as the IDO model. Suppose that the set of alternatives is $\bar{X} = \{x_1, x_2, x_3, \bar{x}\}$. In Section 2.5, we showed that, under an assumption that the attention assignment is conditional, a model with three non-default alternatives is fully identified. We now show, however, that, under the assumption that the attention assignment is independent, the model with three non-default alternatives is not fully identified. To see this, consider the four attention probability vectors $\gamma, \gamma', \dot{\gamma}$ and $\dot{\gamma}'$ which are defined as follows: $\gamma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\gamma' = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, $\dot{\gamma} = (\frac{5}{18}, \frac{5}{18}, \frac{5}{18})$ and $\dot{\gamma}' = (\frac{5}{9}, \frac{5}{9}, \frac{5}{9})$. Moreover, consider two ranking assignments $\bar{\pi}$ and $\dot{\bar{\pi}}$, which are defined as follows. First, $\bar{\pi}$ is such that, for any ranking on $\bar{X}$, $\succ$, $\bar{\pi}(\succ) = \frac{1}{24}$. Second, $\dot{\bar{\pi}}$ is such that:

(i) for any ranking $\succ$ such that $\bar{x}$ is the first ranked, then $\dot{\bar{\pi}}(\succ) = \frac{26}{750}$;

(ii) for any ranking $\succ$ such that $\bar{x}$ is the second ranked, then $\dot{\bar{\pi}}(\succ) = \frac{9}{250}$;

(iii) for any ranking $\succ$ such that $\bar{x}$ is the third ranked, then $\dot{\bar{\pi}}(\succ) = \frac{6}{250}$;

(iv) for any ranking $\succ$ such that $\bar{x}$ is the fourth ranked, then $\dot{\bar{\pi}}(\succ) = \frac{18}{250}$.

The following rationalisation conditions can be used to calculate (a) the choice function rationalised by $\bar{\pi}$ and the independent attention assignments generated by γ and γ' and (b) the choice function rationalised by the ranking assignment $\dot{\bar{\pi}}$ and the independent attention assignments generated by $\dot{\gamma}$ and $\dot{\gamma}'$.

$$\bar{c}(x_i, x_i\bar{x}) = \gamma_i \bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}), \quad i \in \{1,2,3\} \tag{2.127}$$

$$\bar{c}(x_i, x_ix_j\bar{x}) = \gamma_i(1-\gamma_j)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) + \gamma_i\gamma_j\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_j\bar{x}}) \quad \text{for distinct } i,j \in \{1,2,3\} \tag{2.128}$$

$$\bar{c}(x_i, x_ix_jx_k\bar{x}) = \gamma_i(1-\gamma_j)(1-\gamma_k)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) + \gamma_i\gamma_j(1-\gamma_k)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_j\bar{x}}) +$$
$$\gamma_i\gamma_k(1-\gamma_j)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_k\bar{x}}) + \gamma_i\gamma_j\gamma_k\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_jx_k\bar{x}}) \ \text{ for distinct } i,j,k \in \{1,2,3\} \tag{2.129}$$

$$\bar{c}'(x_i, x_i\bar{x}) = \gamma'_i \bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}), \quad i \in \{1,2,3\} \tag{2.130}$$

$$\bar{c}'(x_i, x_ix_j\bar{x}) = \gamma'_i(1-\gamma'_j)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) + \gamma'_i\gamma'_j\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_j\bar{x}}) \quad \text{for distinct } i,j \in \{1,2,3\} \tag{2.131}$$

$$\bar{c}'(x_i, x_ix_jx_k\bar{x}) = \gamma'_i(1-\gamma'_j)(1-\gamma'_k)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) + \gamma'_i\gamma'_j(1-\gamma'_k)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_j\bar{x}}) +$$
$$\gamma'_i\gamma'_k(1-\gamma'_j)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_k\bar{x}}) + \gamma'_i\gamma'_j\gamma'_k\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_ix_jx_k\bar{x}}) \quad \text{for distinct } i,j,k \in \{1,2,3\} \tag{2.132}$$

Using these equations, it is easy to verify that the frame-dependent choice function rationalised by $\bar{\pi}$ and the independent attention assignments generated by γ and γ' is the same as that rationalised by the ranking assignment $\dot{\bar{\pi}}$ and the independent attention assignments generated by $\dot{\gamma}$ and $\dot{\gamma}'$. Accordingly, the model is not fully identified.

The question then arises: intuitively, what is the *diagnosis* of the source of the identification failure? Why does identification succeed in the models in Section 2.4, in Section 2.5, in Appendix 2 and Appendix 4, but fail for the model in this appendix? To answer this question, we must first consider the reason why identification succeeds when it succeeds. The reason is that, in general, by using a frame-dependent choice function, we increase the number of rationalisation conditions and thus increase the number of equality constraints on the unknowns. As a consequence, the number of equality constraints is no longer less less than the number of unknowns. Accordingly, the use of a frame-dependent choice function potentially solves the identification problem.

But this solution strategy will not always work. Suppose that we have a system of equations in which the number of unknowns exceeds the number of equation, and, as a consequence, there are multiple solutions. Even if we add further equality constraints – so that the number of constraints is no longer less than the number of unknowns – this does not, of course, guarantee that we can obtain a unique solution. For example, the new equality constraints may fail to provide any additional information about the unknowns. Thus, in a system of linear equations that has multiple solutions, the addition of another equality constraint that is linearly dependent on the original equations does not add any additional information about the unknowns. Therefore, the addition of the new equality constraint does not allow us to solve the identification problem. We suggest that a similar diagnosis is applicable for the failure of identification in the model with an independent attention assignment. Even though the use of a frame-dependent choice function imposes additional equality constraints, most of the new equality constraints do not provide any additional information about the unknowns.

To see this, note that the rationalisation conditions imply that for distinct alternatives i and j:[16]

$$\bar{c}'(x_i, x_i x_j \bar{x}) = \bar{c}'(x_i, x_i \bar{x}) + \frac{\bar{c}'(x_i, x_i \bar{x})\bar{c}'(x_j, x_j \bar{x})}{\bar{c}(x_i, x_i \bar{x})\bar{c}(x_j, x_j \bar{x})}[\bar{c}(x_i, x_i x_j \bar{x}) - \bar{c}(x_i, x_i \bar{x})] \tag{2.133}$$

According to this equation, the observation $\bar{c}'(x_i, x_i x_j \bar{x})$ is determined by the observations in $\bar{c}$ together with the observations $\bar{c}'(x_i, x_i \bar{x})$ and $\bar{c}'(x_j, x_j \bar{x})$. Therefore, the rationalisation condition associated with $\bar{c}'(x_i, x_i x_j \bar{x})$ provides no additional constraint on the unknowns over and above that provided by the rationalisation conditions for $\bar{c}$ and for $\bar{c}'(x_i, x_i \bar{x}), \bar{c}'(x_j, x_j \bar{x})$.

Note also that the rationalisation conditions entail that for distinct i, j and k:

$$\begin{gathered}\bar{c}'(x_i, x_i x_j x_k \bar{x}) = \bar{c}'(x_i, x_i x_j \bar{x}) + \bar{c}'(x_i, x_i x_k \bar{x}) - \bar{c}'(x_i, x_i \bar{x}) + \\ \frac{\bar{c}'(x_i, x_i \bar{x})\bar{c}'(x_j, x_j \bar{x})\bar{c}'(x_k, x_k \bar{x})}{\bar{c}(x_i, x_i \bar{x})\bar{c}(x_j, x_j \bar{x})\bar{c}(x_k, x_k \bar{x})}[\bar{c}(x_i, x_i x_k x_k \bar{x}) - \bar{c}(x_i, x_i x_j \bar{x}) - \bar{c}(x_i, x_i x_k \bar{x}) + \bar{c}(x_i, x_i \bar{x})]\end{gathered} \tag{2.134}$$

[16] Equations (2.127) and (2.128) imply $\bar{c}(x_i, x_i x_j \bar{x}) - \bar{c}(x_i, x_i \bar{x}) = \gamma_i \gamma_j [\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i x_j \bar{x}}) - \bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i \bar{x}})]$ and equations (2.130) and (2.131) imply $\bar{c}'(x_i, x_i x_j \bar{x}) - \bar{c}'(x_i, x_i \bar{x}) = \gamma_i' \gamma_j' [\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i x_j \bar{x}}) - \bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i \bar{x}})]$. Dividing the second of these equations by the first yields: $[\bar{c}'(x_i, x_i x_j \bar{x}) - \bar{c}'(x_i, x_i \bar{x})]/[\bar{c}(x_i, x_i x_j \bar{x}) - \bar{c}(x_i, x_i \bar{x})] = [\gamma_i'/\gamma_i][\gamma_j'/\gamma_j]$. We can then obtain equation (2.133) by using the fact that, from equations (2.127) and (2.130), $\gamma_i'/\gamma_i = \bar{c}'(x_i, x_i \bar{x})/\bar{c}(x_i, x_i \bar{x})$.

From equation (2.133), we know that the terms $\bar{c}'(x_i, x_i x_j \bar{x})$ and $\bar{c}'(x_i, x_i x_k \bar{x})$ in this equation are determined by the observations in $\bar{c}$ together with the observations $\bar{c}'(x_i, x_i \bar{x})$, $\bar{c}'(x_j, x_j \bar{x})$ and $\bar{c}'(x_k, x_k \bar{x})$. Therefore equation (2.134) says that, similarly, the observation $\bar{c}'(x_i, x_i x_j x_k \bar{x})$ is determined by the observations in $\bar{c}$ together with the observations $\bar{c}'(x_i, x_i \bar{x}), \bar{c}'(x_j, x_j \bar{x}), \bar{c}'(x_k, x_k \bar{x})$. So the rationalisation condition associated with $\bar{c}'(x_i, x_i x_j x_k \bar{x})$ provides no additional constraint on the unknowns over and above that provided by the rationalisation conditions for $\bar{c}$ together with the observations $\bar{c}'(x_i, x_i \bar{x})$, $\bar{c}'(x_j, x_j \bar{x})$ and $\bar{c}'(x_k, x_k \bar{x})$.

In summary: the model with a default option and an independent assignment exemplifies a special case in which the restriction imposed on the attention assignment ensures that, when the choice data in $\bar{c}$ are supplemented by the additional choice data in $\bar{c}'$, the new rationalisation equations provide insufficient additional information about the unknowns to allow identification. By way of contrast, for the models presented in Section 2.4, Section 2.5, Appendix 2 and Appendix 4, the additional rationalisation equations provide sufficient additional information about the unknowns to enable identification.

Appendix 4: Identification in the absence of a closed-form solution

This appendix presents a simple model with a default option that is designed to illustrate two points. First, the identification failure in the model in Appendix 3 (the model with a default option and an independent attention assignment) is something of a special case. We show that the attention parameters may be retrieved if the DM is assumed to have an attention assignment that is a simple generalization of the independent attention assignment. Second, this appendix illustrates that the identification strategy outlined in our paper may be applicable even in the absence of a closed-form solution.

In order to illustrate these two points, we define a "size-effect attention assignment", which is a straightforward generalization of an independent attention assignment. Whereas an independent attention assignment is generated by an attention probabilities vector γ, a size-effect attention assignment is generated by an attention probabilities vector γ together with a size parameter $\delta \in [0, 1]$. If $\delta = 1$, the size-effect attention assignment is an independent attention assignment. If $\delta < 1$, then the size-effect attention assignment is such that (i) it places more weight on consideration sets of

size 2 than an independent attention assignment generated by γ and (ii) it places less weight on empty consideration sets. Intuitively, the size-effect attention assignment accommodates the possibility that the DM has a greater tendency to form consideration sets of size 2 than is implied by an independent attention assignment. More formally, let α_I denote an independent attention assignment generated by γ and let α denote a size-effect attention assignment generated by γ and size parameter δ. The size-effect attention assignment is understood to be an attention assignment that satisfies the following properties. For any $A \subset B \subset X$:
(i) if $|A| \neq 2$, $|A| \neq 0$, then $\alpha(A, B) = \alpha_I(A, B)$;
(ii) if $|A| = 2$, then $\alpha(A, B) = \alpha_I(A, B)^{\delta}$;
(iii) if $|A| = 0$, then $\alpha(A, B) = 1 - \sum_{D \subset B: D \neq \emptyset} \alpha(D, B)$.

Consider an example in which the set of alternatives is $\bar{X} = \{x_1, x_2, x_3, \bar{x}\}$. Assume that a frame-dependent choice function $(\bar{c}, \bar{c}')$ is rationalised by a ranking assignment $\bar{\pi}$ on $\bar{X}$ and a pair of size-effect attention assignments (α, α') generated respectively by attention probability vectors (γ, γ') together with a size parameter δ. This appendix will address the following identification equation: how do we retrieve the attention parameters γ, γ' and δ from the observed choice probabilities. Note that, once we have retrieved the attention assignments, we can also retrieve upper contour probabilities using the method outlined in the proof of Theorem 9. The rationalisation equations imply the following equality constraints:

$$\bar{c}(x_i, x_i\bar{x}) = \gamma_i \bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) \ , \quad i \in \{1,2,3\} \tag{2.135}$$

$$\bar{c}(x_i, x_i x_j \bar{x}) = \gamma_i(1-\gamma_j)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) + [\gamma_i\gamma_j]^{\delta}\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i x_j \bar{x}}) \ , \quad i, j \in \{1,2,3\}, i \neq j \tag{2.136}$$

$$\begin{aligned} \bar{c}(x_1, x_1x_2x_3\bar{x}) = {} & \gamma_1(1-\gamma_2)(1-\gamma_3)\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1\bar{x}}) + [\gamma_1\gamma_2(1-\gamma_3)]^{\delta}\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2\bar{x}}) + \\ & [\gamma_1\gamma_3(1-\gamma_2)]^{\delta}\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_3\bar{x}}) + \gamma_1\gamma_2\gamma_3\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2x_3\bar{x}}) \end{aligned} \tag{2.137}$$

$$\bar{c}'(x_i, x_i\bar{x}) = \gamma'_i \bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) \ , \quad i \in \{1,2,3\} \tag{2.138}$$

$$\bar{c}'(x_i, x_i x_j \bar{x}) = \gamma'_i(1-\gamma'_j)\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i\bar{x}}) + [\gamma'_i\gamma'_j]^{\delta}\bar{\pi}(\bar{\mathcal{M}}_{x_i, x_i x_j \bar{x}}) \ , \quad i, j \in \{1,2,3\}, i \neq j \tag{2.139}$$

$$\begin{aligned} \bar{c}'(x_1, x_1x_2x_3\bar{x}) = {} & \gamma'_1(1-\gamma'_2)(1-\gamma'_3)\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1\bar{x}}) + [\gamma'_1\gamma'_2(1-\gamma'_3)]^{\delta}\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2\bar{x}}) + \\ & [\gamma'_1\gamma'_3(1-\gamma'_2)]^{\delta}\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_3\bar{x}}) + \gamma'_1\gamma'_2\gamma'_3\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2x_3\bar{x}}) \end{aligned} \tag{2.140}$$

Consider the ranking assignment on $\bar{X}$, $\bar{\pi}$, such that for any $\succ \in \mathcal{R}_X$:

$$\bar{\pi}(\succ) = \begin{cases} \frac{5}{48} & \text{if } \forall y \in \bar{X} \backslash \{x_1\},\ x_1 \succ y \\ \frac{1}{48} & \text{otherwise} \end{cases}$$

Consider the attention parameters $\gamma_1 = 0.3, \gamma_2 = 0.2, \gamma_3 = 0.8, \gamma_1' = 0.8, \gamma_2' = 0.6, \gamma_3' = 0.4, \delta = 0.95$. The frame-dependent choice function that is rationalised by this ranking assignment and the size-effect attention assignments generated by these attention parameters is such that:
$\bar{c}(x_1, x_1\bar{x}) = 0.225, \bar{c}(x_2, x_2\bar{x}) = 0.1, \bar{c}(x_3, x_3\bar{x}) = 0.4, \bar{c}'(x_1, x_1\bar{x}) = 0.6, \bar{c}'(x_2, x_2\bar{x}) = 0.3, \bar{c}'(x_3, x_3\bar{x}) = 0.2, \bar{c}(x_1, x_1x_2\bar{x}) \approx 0.2260, \bar{c}(x_2, x_1x_2\bar{x}) \approx 0.0815, \bar{c}(x_1, x_1x_3\bar{x}) \approx 0.2168,\ \bar{c}(x_3, x_1x_3\bar{x}) \approx 0.3230, \bar{c}(x_1, x_1x_2, x_3\bar{x}) \approx 0.2150, \bar{c}'(x_1, x_1x_2\bar{x}) \approx 0.5720,\ \bar{c}'(x_2, x_1x_2\bar{x}) \approx 0.1430, \bar{c}'(x_1, x_1x_3\bar{x}) \approx 0.5858, \bar{c}'(x_3, x_1x_3\bar{x}) \approx 0.0965, \bar{c}'(x_1, x_1x_2, x_3\bar{x}) \approx 0.5629$.

Suppose an observer has this choice data, and wants to retrieve the attention parameters. The following method can be used. First, define $r_i = \frac{\bar{c}'(x_i, x_i\bar{x})}{\bar{c}(x_i, x_i\bar{x})}$ for $i = 1, 2, 3$. So $r_1 = 2\frac{2}{3}, r_2 = 3, r_3 = \frac{1}{2}$. The observer can infer that:

$$\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2\bar{x}}) \neq 0 \tag{2.141}$$

Why? Suppose, for contradiction, $\bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_2\bar{x}}) = 0$. Then rationalisation conditions (2.135), (2.136),(2.138),(2.139) imply $r_1 r_2(\bar{c}(x_1, x_1x_2\bar{x}) - \bar{c}(x_1, x_1\bar{x})) = \bar{c}'(x_1, x_1x_2, \bar{x}) - \bar{c}'(x_1, x_1, \bar{x})$. But this is not the case. Similar reasoning allows the observer to conclude that

$$\bar{\pi}(\bar{\mathcal{M}}_{x_2, x_1x_2\bar{x}}) \neq 0, \bar{\pi}(\bar{\mathcal{M}}_{x_1, x_1x_3\bar{x}}) \neq 0, \bar{\pi}(\bar{\mathcal{M}}_{x_3, x_1x_3\bar{x}}) \neq 0 \tag{2.142}$$

Given (2.141) and (2.142), rationalisation conditions (2.135), (2.136),(2.138),(2.139) imply that for $j = 2, 3$

$$\frac{\bar{c}'(x_1, x_1x_j\bar{x}) - (1 - r_j\gamma_j)\bar{c}'(x_1, x_1\bar{x})}{\bar{c}(x_1, x_1x_j\bar{x}) - (1 - \gamma_j)\bar{c}(x_1, x_1\bar{x})} = \frac{\bar{c}'(x_j, x_1x_j\bar{x}) - (1 - r_1\gamma_1)\bar{c}'(x_j, x_j\bar{x})}{\bar{c}(x_j, x_1x_j\bar{x}) - (1 - \gamma_1)\bar{c}(x_j, x_j\bar{x})} = (r_1 r_j)^{\delta} \tag{2.143}$$

From (2.143), we can obtain expressions for γ_2 and γ_3 as a function of γ_1. It turns out that this function is affine. In particular, for $j = 2, 3$, we have

$$\gamma_j = a_j + b_j\gamma_1 \tag{2.144}$$

where

$$a_j = [\bar{c}'(x_j, x_1x_j\bar{x})\bar{c}(x_1, x_1x_j\bar{x}) - \bar{c}'(x_j, x_1x_j\bar{x})\bar{c}(x_1, x_1\bar{x}) - \bar{c}(x_1, x_1x_j\bar{x})\bar{c}'(x_j, x_j\bar{x})$$

$$-\bar{c}(x_j, x_1x_j\bar{x})\bar{c}'(x_1, x_1x_j\bar{x}) + \bar{c}'(x_1, x_1x_j\bar{x})\bar{c}(x_j, x_j\bar{x}) +$$

$$\bar{c}'(x_j, x_j\bar{x})\bar{c}(x_1, x_1\bar{x}) - \bar{c}'(x_1, x_1\bar{x})\bar{c}(x_j, x_j\bar{x}) + c'(x_1, x_1\bar{x})\bar{c}(x_j, x_1x_j\bar{x})]/$$

$$[\bar{c}(x_1, x_1\bar{x})(\bar{c}'(x_j, x_j\bar{x}) - \bar{c}'(x_j, x_1x_j\bar{x})) - \bar{c}'(x_1, x_1\bar{x})(\bar{c}'(x_j, x_j\bar{x}) - r_j\bar{c}(x_j, x_1x_j\bar{x}))] \tag{2.145}$$

$$b_j = [r_1\bar{c}'(x_j, x_j\bar{x})\bar{c}(x_1, x_1x_j\bar{x}) - \bar{c}'(x_1, x_1\bar{x})\bar{c}'(x_j, x_j\bar{x})$$

$$-\bar{c}(x_j, x_j\bar{x})\bar{c}'(x_1, x_1x_j\bar{x}) + \bar{c}'(x_1, x_1\bar{x})\bar{c}'(x_j, x_j\bar{x})]/$$

$$[\bar{c}(x_1, x_1\bar{x})(\bar{c}'(x_j, x_j\bar{x}) - \bar{c}'(x_j, x_1x_j\bar{x})) - \bar{c}'(x_1, x_1\bar{x})(\bar{c}'(x_j, x_j\bar{x}) - r_j\bar{c}(x_j, x_1x_j\bar{x}))] \tag{2.146}$$

These functions are well-defined because the denominators in (2.145) and (2.146) are nonzero.[17] Given that $b_2 > 0$ and $\gamma_2 > 0$, (2.144) implies that $\gamma_1 > -\frac{a_2}{b_2} = 0.1875$. Also, given the assumption that $\delta \geq 0$ and given the fact that $r_1, r_2 > 1$, equation (2.143) implies that:

$$\frac{\bar{c}'(x_2, x_1x_2\bar{x}) - (1 - r_1\gamma_1)\bar{c}'(x_2, x_2\bar{x})}{\bar{c}(x_2, x_1x_2\bar{x}) - (1 - \gamma_1)\bar{c}(x_2, x_2\bar{x})} \geq 1 \tag{2.147}$$

Let γ_1^* denote the value of γ_1 such that (2.147) holds with equality, which can be calculated to be $\gamma_1^* \approx 0.1979$. The fact that $\gamma_1 > -\frac{a_2}{b_2} = 0.1875$ implies that the denominator of (2.147) is positive, so (2.147) entails that $\gamma_1 \geq \gamma_1^*$. Also, as $\gamma_3 \leq 1$ and $b_3 > 0$ thus $\gamma_1 \leq \frac{1-a_3}{b_3} \approx 0.3281$. In summary, then, $\gamma_1 \in [\gamma_1^*, \frac{1-a_3}{b_3}]$, where $\gamma_1^* \approx 0.1979$ and $\frac{1-a_3}{b_3} \approx 0.3281$. Let us denote this interval by $\mathcal{D}$. We can then rearrange the second equality in equation (2.143) to express δ as a function of γ_1 on the domain $\mathcal{D}$. In particular, we specify the function $\xi : \mathcal{D} \to \mathbb{R}$ to be the function such that, for any $\gamma_1 \in \mathcal{D}$:

$$\xi(\gamma_1) = \log(\frac{\bar{c}'(x_2, x_1x_2\bar{x}) - (1 - r_1\gamma_1)\bar{c}'(x_2, x_2\bar{x})}{\bar{c}(x_2, x_1x_2\bar{x}) - (1 - \gamma_1)\bar{c}(x_2, x_2\bar{x})})/\log(r_1r_2) \tag{2.148}$$

Note that this function is well-defined and continuous for all γ_1 in the domain $\mathcal{D}$, and is such that $\delta = \xi(\gamma_1)$. If we rearrange equations (2.137) and (2.140) to eliminate the term $\bar{\pi}(\bar{\mathcal{M}}_{x_1,x_1x_2x_3\bar{x}})$, then use (2.135), (2.136) to eliminate the terms $\bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_j\bar{x}})$, $\bar{\pi}(\bar{\mathcal{M}}_{x_i,x_ix_k\bar{x}})$ and $\bar{\pi}(\bar{\mathcal{M}}_{x_i,x_i\bar{x}})$, we obtain the equation:

$$\zeta(\gamma_1) = 0$$

where the function $\zeta : \mathcal{D} \to \mathbb{R}$ is such that, for any $\gamma_1 \in \mathcal{D}$:

$$\zeta(\gamma_1) = \bar{c}(x_1, x_1x_2x_3\bar{x}) - (1 - (a_2 + b_2\gamma_1)(1 - (a_3 + b_3\gamma_1))\bar{c}(x_1, x_1\bar{x}) -$$

[17] If δ were unity, however, as is the case when the attention assignment is independent, then the denominators of (2.145) and (2.146) would be zero, and the identification method presented in this appendix could not be employed. To see why, in this case, the denominators in (2.145) and (2.146) are zero, note that if we set $\delta = 1$ in the second equality in (2.143) and rearrange, we obtain: $\bar{c}'(x_j, x_1x_j\bar{x}) - (1 - r_1\gamma_1)\bar{c}'(x_j, x_j\bar{x}) = r_1r_j[\bar{c}(x_j, x_1x_j\bar{x}) - (1 - \gamma_1)\bar{c}(x_j, x_j\bar{x})]$. Plugging in the definitions of r_j and r_1 and rearranging yields: $\bar{c}(x_1, x_1\bar{x})[\bar{c}'(x_j, x_1x_j\bar{x}) - \bar{c}'(x_j, x_j\bar{x})] = \bar{c}'(x_1, x_1\bar{x})[r_j\bar{c}(x_j, x_1x_j\bar{x}) - \bar{c}'(x_j, x_j\bar{x})]$. This implies that the denominators in (2.145) and (2.146) are zero.

$$(1-(a_3+b_3\gamma_1))^{\xi(\gamma_1)}[\bar{c}(x_1,x_1x_2\bar{x})-(1-(a_2+b_2\gamma_1))\bar{c}(x_1,x_1\bar{x})]-$$
$$(1-(a_2+b_2\gamma_1))^{\xi(\gamma_1)}[\bar{c}(x_1,x_1x_3\bar{x})-(1-(a_3+b_3\gamma_1))\bar{c}(x_1,x_1\bar{x})]-$$
$$-\frac{1}{r_1r_2r_3}[\bar{c}'(x_1,x_1x_2x_3\bar{x})-(1-r_2(a_2+b_2\gamma_1)(1-r_3(a_3+b_3\gamma_1))\bar{c}'(x_1,x_1\bar{x})-$$
$$(1-r_3(a_3+b_3\gamma_1))^{\xi(\gamma_1)}[\bar{c}'(x_1,x_1x_2\bar{x})-(1-r_2(a_2+b_2\gamma_1))\bar{c}'(x_1,x_1\bar{x})]-$$
$$(1-r_2(a_2+b_2\gamma_1))^{\xi(\gamma_1)}[\bar{c}'(x_1,x_1x_3\bar{x})-(1-r_3(a_3+b_3\gamma_1))\bar{c}'(x_1,x_1\bar{x})]] \tag{2.149}$$

Note that this function is well-defined and continuous on the domain $\mathcal{D}$. In order to obtain the value of the attention parameter γ_1, therefore, we need to find the root of the function ζ on domain $\mathcal{D}$. As there is no obvious closed-form solution to this problem, we turn to numerical methods for root finding. We employ the simple method of "incremental search" (see Chapra (2012, p.131)), and find that ζ has a unique root at $\gamma_1 \approx 0.3000$. Thus the method successfully retrieves the value of $\gamma_1 = 0.3$ that was used to generate the choice data. We can then obtain parameter δ from (2.148) and parameters γ_2 and γ_3 from (2.144). In order to retrieve the parameters γ_1', γ_2' and γ_3', we use that fact that, for $i = 1, 2, 3, \gamma_i' = r_i\gamma_i$. Note that this method cannot be used if the attention assignment is independent, because in the special case where $\delta = 1$ the denominators in equations (2.145) and (2.146) are zero.

Appendix 5: Numerical illustrations of the identification method

This appendix presents two examples – one for the NDO model and one for the DO model – of the method presented in Sections 2.4 and 2.5 for backing out inattention odds vectors from observed choice probabilities. Each example begins by specifying a ranking assignment and two vectors of attention probabilities γ and γ'. These generate a frame-dependent choice function. We then apply our identification method to retrieve, from the frame-dependent choice function, two inattention odds vectors. Our identification method can then be verified by checking whether the revealed inattention odds vectors obtained using this method are the same as the inattention odds vectors associated with the attention probability vectors used to generate the frame-dependent choice functions.

A5.1. Example for NDO model

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and π denote the ranking assignment on $\mathcal{R}_X$ such that, for any $\succ \in \mathcal{R}_X$

$$\pi(\succ) = \begin{cases} \frac{1}{240} & \text{if } x_2 \succ x_1 \\ \frac{3}{240} & \text{otherwise} \end{cases}$$

Suppose that the attention probabilities are $\gamma = (\frac{2}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2})$ and $\gamma' = (\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2})$. Then the inattention odds vectors are $\iota = (\frac{1}{2}, 1, \frac{1}{2}, 1, 1)$ and $\iota' = (1, \frac{1}{2}, 1, \frac{1}{2}, 1)$. This ranking assignment and these attention probabilities generate a frame-dependent choice function (c, c') such that: $c(x_1, x_1x_2) = \frac{7}{10}$, $c(x_1, x_1x_3) = \frac{13}{24}$, $c(x_1, x_1x_4) = c(x_1, x_1x_5) = \frac{19}{30}$, $c(x_2, x_2x_3) = \frac{11}{30}$, $c(x_2, x_2x_4) = c(x_2, x_2x_5) = \frac{17}{36}$, $c(x_3, x_3x_4) = c(x_3, x_3x_5) = \frac{3}{5}$, $c(x_4, x_4x_5) = \frac{1}{2}$, $c(x_1, x_1x_2x_3) = \frac{47}{102}$, $c'(x_1, x_1x_2) = \frac{1}{2}$, $c'(x_1, x_1x_3) = c'(x_1, x_1x_5) = \frac{19}{36}$, $c'(x_1, x_1x_4) = \frac{13}{30}$, $c'(x_2, x_2x_3) = c'(x_2, x_2x_5) = \frac{17}{30}$, $c'(x_2, x_2x_4) = \frac{11}{24}$, $c'(x_3, x_3x_4) = \frac{2}{5}$, $c'(x_3, x_3x_5) = \frac{1}{2}$, $c'(x_4, x_4x_5) = \frac{3}{5}$, $c'(x_1, x_1x_2x_3) = \frac{49}{132}$.

In order to retrieve the inattention odds vectors from these choice probabilities, the first step is to obtain the reduced row echelon form of the augmented matrix $[\mathbf{Q}, \mathbf{b}]$. As specified in the proof of Lemma 1, this augmented matrix is determined by the choice function (c, c'). It can be verified that the NDO-nonzero condition is satisfied. We use a Matlab function to obtain the following reduced row echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that all of the unknowns can be specified as linear functions of the unknown Δ_1; in particular, for $i = 1, \ldots, 5$, $\iota_i = -\frac{1}{2} - s_i\Delta_1$, for $i = 2, \ldots, 5$, $\Delta_i = -t_i\Delta_1$, where the values of s_i and t_i can be obtained from column 10 of the reduced row echelon form. In particular, $s_1 = -2$,

$t_2 = 1, s_2 = -3$, $t_3 = -1, s_3 = -2$, $t_4 = 1, s_4 = -3$, $t_5 = 0, s_5 = -3$.[18]

The second step is to obtain a value for Δ_1. In order to do so, we calculate f and h, using the expressions for these two terms specified in the proof of Theorem 5. It turns out that $f \approx -0.0896, h \approx 0.0224$. As $s_1 < 0$, equation (2.22) implies that $\hat{\Delta}_1 = \sqrt{-h/f} \approx 0.5000$. (Note that this is consistent with the fact that $\iota_1 = \frac{1}{2}$ and $\iota'_1 = 1$). In order to retrieve the revealed inattention odds vector $\hat{\iota}$, we use the fact that for $i = 1, \ldots, 5$, $\hat{\iota}_i = -\frac{1}{2} - s_i\hat{\Delta}_1$. To retrieve the revealed inattention odds vector $\hat{\iota}'$, we use the fact that $\hat{\iota}'_1 = \hat{\Delta}_1 + \hat{\iota}_1$ and for $i = 2, \ldots, 5$, $\hat{\iota}'_i = \hat{\iota}_i - t_i\hat{\Delta}_1$. It is easy to verify that the revealed inattention odds vectors $\hat{\iota}$ and $\hat{\iota}'$ obtained in this fashion are equal to the inattention odds vectors ι and ι' that were used to generate c and c'.

A5.2. Example for DO model

Suppose that the set of alternatives is $\bar{X} = \{x_1, x_2, x_3, \bar{x}\}$ and let $\bar{\pi}$ denote the ranking assignment on $\mathcal{R}_{\bar{X}}$ such that for all $\succ \in \mathcal{R}_{\bar{X}}$, $\bar{\pi}(\succ) = \frac{1}{24}$. Suppose that the attention probabilities are $\gamma = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$ and $\gamma' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{4})$, so the inattention odds vectors are $\iota = (1, 1, 2)$ and $\iota' = (2, 2, 3)$. This ranking assignment and these attention probabilities generate a frame-dependent choice function $(\bar{c}, \bar{c}')$ that satisfies the following:
$\bar{c}(x_1, x_1\bar{x}) = \bar{c}(x_2, x_2\bar{x}) = \bar{c}(x_3, x_3\bar{x}) = \bar{c}'(x_1, x_1\bar{x}) = \bar{c}'(x_2, x_2\bar{x}) = \bar{c}'(x_3, x_3\bar{x}) = \frac{1}{2}$; $\bar{c}(x_1, x_1x_2\bar{x}) = \bar{c}(x_2, x_1x_2\bar{x}) = \frac{5}{18}$; $\bar{c}'(x_1, x_1x_2\bar{x}) = \bar{c}'(x_2, x_1x_2\bar{x}) = \frac{4}{15}$; $\bar{c}(x_1, x_1x_3\bar{x}) = \bar{c}(x_2, x_2, x_3\bar{x}) = \frac{1}{3}$; and $\bar{c}'(x_1, x_1x_3\bar{x}) = \bar{c}'(x_2, x_2, x_3\bar{x}) = \frac{11}{36}$; $\bar{c}(x_3, x_1x_3\bar{x}) = \bar{c}(x_3, x_2, x_3\bar{x}) = \frac{5}{24}$; $\bar{c}'(x_3, x_1x_3\bar{x}) = \bar{c}'(x_3, x_2, x_3\bar{x}) = \frac{2}{9}$; $\bar{c}(x_1, x_1x_2x_3\bar{x}) = \frac{9}{40}$, $\bar{c}'(x_1, x_1x_2x_3\bar{x}) = \frac{59}{288}$. The following uses the two-step method for retrieving inattention odds vectors that is outlined in the proof of Theorem 8.

The first step is to calculate the reduced row echelon form of the augmented matrix $[\bar{\mathbf{Q}}, \bar{\mathbf{b}}]$ that is specified in the proof of Theorem 8. Note that that the DO-nonzero condition is satisfied. Using Matlab, we calculate the reduced row echelon form of $[\bar{\mathbf{Q}}, \bar{\mathbf{b}}]$ to be the following:

[18] It is easy to check that these values for s_i and t_i, which were obtained from the reduced row echelon form, are the same as the expressions for s_i and t_i defined in the proof of Lemma 1. This verifies Lemma 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2.5 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1.5 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1.5 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus for $i = 1, 2, 3$, $\iota_i = -\frac{1}{2} - \bar{s}_i \Delta_1$, for $i = 2, 3$, $\Delta_i = -\bar{t}_i \Delta_1$, where $\bar{s}_1 = -1.5$, $\bar{t}_2 = -1, \bar{s}_2 = -1.5$, $\bar{t}_3 = -1, \bar{s}_3 = -2.5$.[19]

The next step is to find Δ_1 using the definitions of $\bar{f}$ and $\bar{h}$ specified in the proof of Theorem 8. It is easy to check that $\bar{f} \approx -0.0050$ and $\bar{g} \approx 0.0050$. Given $\bar{s}_1 < 0$, equation (2.34) entails that $\bar{\Delta}_1 = \sqrt{-\bar{h}/\bar{f}} \approx 1.0000$. (Note that this is consistent with the fact that $\iota_1 = 1$ and $\iota'_1 = 2$). The revealed inattention odds vector $\bar{\iota}$, by definition, is calculated as follows: $i = 1, 2, 3$, $\bar{\iota}_i = -\frac{1}{2} - \bar{s}_i \bar{\Delta}_1$. The revealed inattention odds vector $\bar{\iota}'$ is then obtained as follows: $\bar{\iota}'_1 = \bar{\Delta}_1 + \bar{\iota}_1$ and for $i = 2, 3$, $\bar{\iota}'_i = \bar{\iota}_i - t_i \bar{\Delta}_1$. It is easily checked that the revealed inattention odds vectors $\bar{\iota}$ and $\bar{\iota}'$ obtained in this fashion are equal to the inattention odds vectors ι and ι' that were used to generate $\bar{c}$ and $\bar{c}'$.

[19]It can be verified that these values for $\bar{s}_i$ and $\bar{t}_i$, obtained from the reduced row echelon form, are equal to the values calculated using the definitions of $\bar{s}_i$ and $\bar{t}_i$ in Lemma 2. This serves to verify Lemma 2.

Chapter 3

Framing effects with indecisive decision-makers

3.1 Introduction

In their study of framing effects, Salant and Rubinstein (2008, p.1287) describe a *frame* as a condition "which is irrelevant in the rational assessment of the alternatives but nonetheless affects behaviour". For instance, framing effects may include the influence on choice of the location of goods in a supermarket; the effect of specifying an alternative as a "default option"; or the influence on online shopping of web-wallpaper. Drawing on evidence of the prevalence of framing effects, a number of commentators have challenged some of the central tenets of neoclassical economics. First, whereas neoclassical economics assumes that decision-makers (DMs) have stable underlying preferences, framing effects are seen as evidence that preferences are "radically unstable", "labile" and "constructed". Second, framing effects have been presented as evidence of irrationality in decision-making. Third, it has been argued that framing effects undermine neoclassical choice-based welfare economics, on the grounds that, if choices are influenced by framing effects, revealed preferences cannot be retrieved from choice data. As these three conclusions challenge central tenets of traditional neoclassical theory, we refer to them as *skeptical* conclusions.[1]

Framing effects, of course, come in different shapes and sizes. Our paper develops a model of

[1] For an overview of the literature on "preference construction", see Lichtenstein and Slovic (2006). For popular articulations of the view that framing effects are evidence of irrationality, see Thaler and Sunstein (2008) and Ariely (2009). See Ariely et al. (2003, p. 102) for the claim that framing effects undermine choice-based welfare economics.

one important class of framing effects, which are labelled *resolution framing effects*. According to our model, frames influence a DM's choice when she is indecisive between alternatives. A DM is said to be *indecisive* between two alternatives if she does not have a strict preference for one alternative over the other.[2] As we use the term, a *resolution framing effect* is a framing effect that influences a DM by "tipping the balance" when she is indecisive between alternatives. We refer to this type of framing effect as a <u>resolution</u> framing effect because, at least potentially, the framing effect helps to <u>resolve</u> the decision of indecisive DMs. Section 3.3 of the paper presents a formal definition of a resolution framing effect – it is defined to be a certain type of mapping from frames to relations. We will argue that, at least for resolution framing effects, there is no warrant for drawing the three skeptical conclusions articulated above.

In our model, the observed data are the frame-dependent choices of a DM. We refer to the observed data as a *frame-dependent choice function*. Whereas a neoclassical choice function is a mapping from budget sets to choices, a frame-dependent choice function is a mapping from budget sets <u>and frames</u> to choices. Frame-dependent choice functions have been modelled in a number of papers in the behavioural economics literature: they appear in the models of the endowment effect of Masatlioglu and Ok (2005) and Masatlioglu and Ok (2014), in the model of choices from lists of Rubinstein and Salant (2006), and in the more general models of Salant and Rubinstein (2008) and Bernheim and Rangel (2009).

According to our model, frame-dependent choice functions are generated by a two-stage process.[3] In the first stage, an alternative is eliminated if another available alternative is strictly preferred to it. The DM is, therefore, indecisive between alternatives that survive the first stage. The second stage of the decision process depends upon the frame. A resolution framing effect determines how, in the second stage, the frame influences the decision. The resolution framing effect, which is a mapping from frames to relations, determines, for each frame, what "tips the balance" in that frame when the DM is indecisive between alternatives. In this two-stage model, the DM's preferences, together

[2] As discussed in Section 3.2, we use the term "indecisive" rather than "indifferent", because we do not assume that the weak preference relation is complete.

[3] In using frame-dependent choice functions, our paper is distinguished from a number of other models of two-stage decisions. Whereas Manzini and Mariotti (2007), Manzini and Mariotti (2012) and Horan (2016) model two-stage decision processes that are, in some respects, similar to ours, in their papers the choice data are a neoclassical choice function, while in our paper it is a frame-dependent choice function.

with the resolution framing effect, generate a frame-dependent choice function. Section 3.5 of our paper develops an extension to the two-stage model, adding a third stage to the decision process. The three-stage model can accommodate a broader class of frame-dependent choice functions.

We provide axiomatisations and revealed preference results for the broad class of resolution framing effects. An attractive feature of these results is their generality. Rather than developing a bespoke model for each specific kind of framing effect, we present more general results which imply, as a corollaries, axiomatisations for specific framing effects. For instance, we show that our model yields axiomatisations for the salience effect, the endowment effect, the priming effect and the compromise effect. Moreover, it would be fruitful to investigate whether other types of framing effects can be modelled as resolution framing effects, including the anchoring effect and asymmetric dominance. While generality is a virtue in any scientific enterprise, it is particularly important in behavioural economics, which has been criticised as fragmented in comparison to the unified framework of neo-classical economics.[4]

Our model provides a response to the skeptical conclusion that frame-dependent choices imply that preferences are "radically unstable", "labile" and "constructed." According to our model, frame-dependent choices are generated by a stable underlying preference relation together with a resolution framing effect. Moreover, our revealed preference results constitute a response to the skeptical conclusion that frame-dependent choices preclude the possibility of retrieving preferences from choice data. We demonstrate that, if resolution framing effects satisfy certain "richness" conditions, a unique revealed preference relation can be backed out from the DM's frame-dependent choices. Section 3.6 compares this revealed preference relation to two other influential proposals for backing out ranking relations from frame-dependent choice data – the proposals in Salant and Rubinstein (2008) and Bernheim and Rangel (2009). As discussed in Section 3.6, both papers present different models to ours, and are asking somewhat different questions to us. In contrast to our model, Bernheim and Rangel (2009) do not postulate a general decision process that generates frame-dependent choice functions. Moreover, whereas Salant and Rubinstein (2008) can be interpreted as presenting a "multi-self" model, our model accommodates frame-dependent choices that are inconsistent with a multi-self model.[5] Despite these differences, there are parallels between

[4]See, for example, Caplin (2008).

[5]Roughly speaking, for a "multi-self" model, the DM can be attributed, for each frame, a ranking of the alternatives specific to that frame, which generates the choices in that frame.

our revealed preference results and the findings of Bernheim and Rangel (2009) and Salant and Rubinstein (2008), which are highlighted in Section 3.6.

Section 3.7 presents a response to the skeptical conclusion that framing effects are evidence of irrationality. At least for resolution framing effects, there is no reason for classifying frame-dependent choices as irrational. On the one hand, for a resolution framing effect, the DM's choice is influenced by a non-rational factor – namely, a frame. On the other hand, this does not render her choice irrational, because *the non-rational factor only affects her choice among alternatives between which she is indecisive*. We acknowledge that, if, in fact, a non-rational factor caused a DM to choose an inferior alternative over a superior alternative, then, at least potentially, there may be an argument for concluding that she is irrational. But, for a resolution framing effect, the frame only affects the DM's choice when she is indecisive between alternatives, so there is no basis for deeming such choices irrational.

In summary: for resolution framing effects, there are no grounds for accepting the three skeptical conclusions articulated above. First, although choices are influenced by frames, this does not imply that preferences are "radically unstable": we show that a frame-dependent choice function can be generated by a preference relation. Second, even if choices are influenced by non-rational factors, this does not render them irrational. Third, framing effects do not preclude the possibility of choice-based welfare analysis: for resolution framing effects, our revealed preference relation can serve as an individual welfare relation. In the literature on behavioural economics, there are a number of other responses to the skeptical challenges. One response is that, if DMs have limited attention, apparently irrational behaviour may nevertheless be boundedly rational.[6] A second response is to argue that frames may be informative: framing effects can be construed as rational if features of the context convey information relevant to the rational assessment of the alternatives.[7] The model in our paper supplements such responses to the skeptical challenge.

The structure of our paper is as follows. Section 3.2 presents minor variants of two well-known axiomatisations from standard choice theory. All the results in our models of framing effects are counterparts of these two key results. Section 3.3 presents our two-stage model of resolution framing

[6] Moreover, Masatlioglu et al. (2012) and Masatlioglu and Nakajima (2015) present models of limited attention which show that, at least to some extent, revealed preferences can be retrieved from choice data.

[7] For example, see the response in Gigerenzer (2015) to Thaler and Sunstein (2008).

effects, illustrating the model with a specific type of resolution framing effect – the salience effect. Section 3.4 argues that two other framing effects – the priming effect and the compromise effect – can also be modelled as resolution framing effects. Section 3.5 presents an extension to the two-stage model of resolution framing effects, adding a third stage. This three-stage decision process is used to model the endowment effect. Section 3.6 compares the revealed preference results in our model and the findings in Salant and Rubinstein (2008) and Bernheim and Rangel (2009). Section 3.7 then presents our argument that resolution framing effects are consistent with rational decision making.

3.2 Neoclassical choice with incomplete preferences

Let X denote the set of alternatives, which throughout the paper is assumed to be finite. We take as primitive a strict preference relation $\succ$ on X, and present axiomatisations for strict preferences that are (i) asymmetric and transitive and (ii) asymmetric and acyclic.[8] By way of contrast, a common approach in textbook treatments of choice is to take the weak preference relation $\succeq$ as primitive, and to assume that the relation is transitive and complete; this is equivalent to assuming that the strict preference relation is asymmetric and negatively transitive. Given that transitivity and acyclicity are weaker assumptions than negative transitivity, we are imposing weaker restrictions on the preference relation than this textbook treatment. Our motivation for deviating from this textbook treatment is that, as a number of commentators have argued, completeness of the weak preference relation is not a requirement of rationality.[9] For any relation $\succ$ on X, define the relation $I_\succ$ as follows: for any $x, y \in X, xI_\succ y \iff x \not\succ y$ and $y \not\succ x$. In the textbook treatment of choice described above, where the strict preference relation $\succ$ satisfies asymmetry and negative transitivity, the relation $I_\succ$ satisfies transitivity and is referred to as the indifference relation. In contrast, if the relation $\succ$ merely satisfies (i) asymmetry and transitivity, or (ii) asymmetry and acyclicity, then the relation $I_\succ$ need not satisfy transitivity. To indicate this difference from the

[8] Throughout this paper, the term “relation” will refer to a binary relation. Consider any relation R on a set X. R is said to be (i) *transitive* if, for all $x, y, z \in X$, xRy and yRz entails xRz (ii) *asymmetric* if, for all $x, y \in X$, xRy entails not yRx (iii) *acyclic* if there are no alternatives $x_1, x_2, \dots, x_n \in X$ such that $x_1Rx_2R\dots Rx_nRx_1$ (iv) *complete* if, for all $x, y \in X$, either xRy or yRx (v) *negatively transitive* if not xRy and not yRz entails not xRz (vi) *a linear ordering* if R is transitive and, for any $x, y \in X$, exactly one of the following holds: $xRy, yRx, x = y$. For $S \subset X$, the expression $\max(S, R)$ refers to the set of maximal elements in S with respect to R. So if R is asymmetric, $\max(S, R) = \{x \in S : \text{for no } y \in S, yRx\}$.

[9] For a discussion of why completeness is not required for rationality, see Eliaz and Ok (2006).

textbook treatment, if a DM's preference relation $\succ$ is such that $xI_{\succ}y$, we will say that she is "indecisive" between x and y rather than "indifferent".[10]

Before turning to consider frame-dependent choices in Section 3.3, it will be helpful to review some results from neoclassical choice theory. A standard choice function, which maps budget sets to choices, will be referred to as a "neoclassical choice function". The "set of all non-empty budget sets" is denoted $\mathcal{B} := 2^X \backslash \{\emptyset\}$. In this section, it is assumed that $|X| \geq 3$. The domain of a neoclassical choice function $\mathscr{D}$ can then be defined as any subset of $\mathcal{B}$ that contains all members of $\mathcal{B}$ of size two or three. A *neoclassical choice function* is a mapping $C : \mathscr{D} \to \mathcal{B}$ such that, for any $B \in \mathscr{D}$, $C(B) \subset B$. C is said to be *rationalised* by a relation $\succ$ if, for all $B \in \mathcal{D}, C(B) = \max(B, \succ)$. The axioms Contraction, Expansion and Weak IIA (independence of irrelevant alternatives) are defined as follows:

Contraction

Consider any $B \in \mathscr{D}$ and $x, y \in B$. If $x \in C(B)$, then $x \in C(\{x, y\})$.

Expansion

Consider any $B \in \mathscr{D}$ and $x \in B$. If, for all $y \in B \backslash \{x\}$, $x \in C(\{x, y\})$, then $x \in C(B)$.

Weak IIA

Consider any distinct $x, y, z \in X$. If $C(\{x, y, z\}) = \{x\}$, then $C(\{x, y\}) = \{x\}$.

Note that the IIA property is often stated in a stronger form: for any $A, B \in \mathscr{D}$, if $C(A) \subset B \subset A$, then $C(B) \subset C(A)$. It is easy to verify that this stronger form of the IIA property entails Weak IIA. We use a weaker form of the axiom, because, as shown below, this weaker form has a counterpart in axiomatisations of framing effects. In the axiom Weak IIA, intuitively, the alternative z is the "irrelevant alternative"; the idea is that the removal of the irrelevant alternative z from the budget set does not change the DM's disposition to choose x over y.

Next, we define the *base relation* for a neoclassical choice function C as follows: the *base rela-*

[10] Eliaz and Ok (2006) also introduce a concept of "indecisiveness" in their analysis of incomplete preferences, although their concept is somewhat different to the definition of indecisiveness in our paper.

tion P is the relation on X such that, for any $x, y \in X$, xPy if and only if $\{x\} = C(\{x, y\})$. Then the following two theorems are minor variants of standard results in the literature.[11]

Theorem 1A (*Neoclassical model: Acyclic preferences*)
Consider any neoclassical choice function C with domain $\mathcal{B}$.
(a) There exists an asymmetric, acyclic relation $\succ$ that rationalises C if and only if C satisfies Contraction and Expansion.
(b) If C satisfies Contraction and Expansion, then P is asymmetric, acyclic and the unique relation that rationalises C.

Theorem 1B (*Neoclassical model: Transitive preferences*)
Consider any neoclassical choice function C.
(a) There exists an asymmetric, transitive relation $\succ$ that rationalises C if and only if C satisfies Contraction, Expansion and Weak IIA.
(b) If C satisfies Contraction, Expansion and Weak IIA, then P is asymmetric, transitive and the unique relation that rationalises C.

In the next section, we will establish a pair of theorems that are counterparts to Theorems 1A and 1B, but where the theorems apply to frame-dependent choices.

3.3 A model of resolution framing effects

To illustrate our concept of a "resolution framing effect", it will be helpful to consider a simple example. A shopkeeper sells two types of chocolates, x and y, which are placed next to the cash-register. The set of alternatives then is $X = \{x, y\}$. For a customer standing at the cash-register, the closer alternative is more salient. The customer's choices are observed under two frames: the state in which x is closer than y and hence more salient, which is denoted by $f_{\triangleright_{xy}}$; and the state in which y is closer than x and thus more salient, which is denoted by $f_{\triangleright_{yx}}$. Suppose that the customer has a two-stage decision process. In the first stage, she eliminates an alternative if there is another available alternative strictly preferred to it. In the second stage, her choice is dependent on the frame in which she makes her decision: of the alternatives that survive the first stage, she is

[11] Versions of these results can be found, for example, in Moulin (1985).

disposed to choose the alternative that is most salient. Her choice dispositions in the second stage of this process can be described by a function from the set of frames $\mathcal{F}_1 := \{f_{\rhd xy}, f_{\rhd yx}\}$ to the set of relations on X, $\mathcal{R}$. In particular, the second stage can be described by the function $r_1 : \mathcal{F}_1 \to \mathcal{R}$ such that:

(i) the relation $r_1(f_{\rhd xy})$ is such that $xr_1(f_{\rhd xy})y$ and not $yr_1(f_{\rhd xy})x$;

(ii) the relation $r_1(f_{\rhd yx})$ is such that $yr_1(f_{\rhd yx})x$ and not $xr_1(f_{\rhd xy})y$.

Function r_1 is to be interpreted as saying that, if x and y both survive the first stage, the DM's choice depends on the frame in the following way: if her decision is made in frame $f_{\rhd xy}$, where x is closer, and thus more salient, the DM is disposed to select x over y; if her decision is made in frame $f_{\rhd yx}$, where y is more salient than x, the DM is disposed to select y over x. If the DM's preference relation is given by $\succ$ and a decision is made in frame $f \in \mathcal{F}_1$, then the two-stage decision process generates a choice that is given by $\max(\max(\{x, y\}, \succ), r_1(f))$. Assume that $\succ$ is such that the DM is indecisive between x and y; that is, $xI_{\succ}y$. Then her choice in frame $f_{\rhd xy}$ is given by $\{x\} = \max(\max(\{x, y\}, \succ),), r_1(\rhd_{xy}))$ and her choice in frame $f_{\rhd yx}$ is given by $\{y\} = \max(\max(\{x, y\}, \succ), r_1(\rhd_{yx}))$.

The DM's choices from each budget set and in each frame can be described by a "frame-dependent choice function". Whereas a neoclassical choice function is a mapping from budget sets to choices, a frame-dependent choice function is a mapping from budget sets <u>and frames</u> to choices:

Definition 1 (*Frame-dependent choice function*)

Let X denote a set of alternatives and $\mathcal{F}$ a set of frames. The domain, $\mathcal{D}$, is a non-empty subset of $\mathcal{B} \times \mathcal{F}$. A *frame-dependent choice function* is a mapping $c : \mathcal{D} \to \mathcal{B}$ such that for any $(B, f) \in \mathcal{D}$, $c(B, f) \subset B$.

In the chocolate example, the DM's choices can be described by a frame-dependent choice function c with domain $\mathcal{B} \times \mathcal{F}_1$ such that $c(\{x, y\}, f_{\rhd xy}) = \{x\}$ and $c(\{x, y\}, f_{\rhd yx}) = \{y\}$. In general, a frame is understood to be an observable state which, as Salant and Rubinstein (2008) put it, "is irrelevant in the rational assessment of the alternatives but nonetheless affects behaviour". The frame-dependent choice function is assumed to be the observable choice data: that is, an observer has data not only on the choices made from the various budget sets, but also on the frames in which the choice was made. For ease of exposition, the term "choice function" henceforth will be understood as an abbreviation for "frame-dependent choice function".

In the chocolate example, the DM's choice function is generated by her preference relation together with the function r_1, which describes her second-stage choice dispositions. The function r_1 is a simple example of what we will call a "resolution framing effect". Intuitively, a resolution framing effect determines how, in each frame, the DM resolves her choice from alternatives between which she is indecisive. That is, a resolution framing effect determines what "tips the balance", in each frame, when the DM is indecisive. To describe a particular resolution framing effect, we must specify, for each frame, the relation that determines how, in that frame, the DM makes a selection from alternatives between which she is indecisive. In other words, *a resolution framing effect is a mapping from frames to relations.* More formally:

Definition 2 (*Resolution framing effect*)
Consider a set of alternatives X, and a set of frames $\mathcal{F}$. Let $\mathcal{R}$ denote the set of relations on X. A *resolution framing effect* is an injective function $r : \mathcal{F} \to \mathcal{R}$ such that, for all $f \in \mathcal{F}$, the relation $r(f)$ is asymmetric and transitive. For any $f \in \mathcal{F}$ and $x, y \in X$:
(i) if $xr(f)y$, frame f is said to *favour x over y* and
(ii) if neither $xr(f)y$ nor $yr(f)x$, frame f is said to be *neutral between x and y.*

The function r_1 exemplifies a specific type of resolution framing effect – we will call this type of resolution framing effect a "salience effect". The resolution framing effect r_1 is the salience effect for the case where the set of alternatives has two members. It will be helpful to define a salience effect for an arbitrary set of alternatives X. First, we need to define the set of frames for a salience effect: a frame is an observable state in which the set of alternatives are presented in a linear order $\rhd$, which is interpreted to mean "is more salient than". In the example presented above, the salience ordering $\rhd$ was the relation "is closer to the customer than", but it could also mean, for example, "has brighter packaging than". If the relation $\rhd$ is an arbitrary linear order on X, then the frame $f_{\rhd}$ denotes the state in which the salience ordering of the alternatives in X is $\rhd$. We can then define a salience framing effect as follows.

Definition 3 (*Salience effect*)
Consider a set of alternatives X. The *salience set of frames* for X is given by $\mathcal{F}^s = \{f_{\rhd} :\rhd$

is a linear ordering on $X\}$. The *salience effect* for X is the resolution framing effect $s_X : \mathcal{F}^s \to \mathcal{R}$ such that, for all $f_{\triangleright} \in \mathcal{F}^s$, $s_X(f_{\triangleright}) = \triangleright$.

By way of example, for the case where $X = \{x, y\}$, the salience effect for X is $s_X = r_1$. Intuitively, a salience effect is a resolution framing effect such that a frame favours alternative x over y when, in that frame, x is more salient than y; moreover, there are no frames in which the frame is neutral between the alternatives.

The next task is to provide an account of how a resolution framing effect generates a choice function. We want an account that applies not only to salience framing effects but also, more generally, to an arbitrary resolution framing effect r. Our proposal is the following definition:

Definition 4 (*r-rationalisation*)
Consider a set of alternatives X, a resolution framing effect $r : \mathcal{F} \to \mathcal{R}$, a choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$ and a relation $\succ$ on X. c is said to be *r-rationalised* by $\succ$ if, for all $(B, f) \in \mathcal{D}, c(B, f) = \max(\max(B, \succ), r(f))$.

According to this definition, the choice made from a given budget set B in a given frame f is generated by a two-stage decision process. The first stage involves the application of the DM's preference relation: an alternative x in B is eliminated if another alternative in B is strictly preferred to x. In the second stage, an alternative that survives the first stage, y, will be eliminated if there is another alternative that survives the first stage, x, such that the frame f favours x over y – that is, $xr(f)y$. For example, it is easy to verify that in the chocolate example, the choice function is s_X-rationalised by the preference relation $\succ_1$ on X such that $xI_{\succ_1}y$.

To illustrate the concepts introduced above, it will be helpful to consider a variant of the chocolate example, for the case where there are three alternatives. Suppose now that a shopkeeper sells three varieties of chocolates: $X = \{x, y, z\}$. Suppose that we have observations of the DM's choice dispositions under six frames; the salience set of frames is given by $\mathcal{F}^s = \{f_{\triangleright_{xyz}}, f_{\triangleright_{xzy}}, f_{\triangleright_{yxz}}, f_{\triangleright_{yzx}}, f_{\triangleright_{zxy}}, f_{\triangleright_{zyx}}\}$. For any distinct $u, v, w \in X$, the frame $f_{\triangleright_{uvw}}$ is understood to be the state in which u is closest the customer, w is furthest away and v is at an intermediate distance. So the salience ordering $\triangleright_{uvw}$ for that state is such that $u \triangleright_{uvw} v \triangleright_{uvw} w$. Suppose that the DM's observed choice function is described by Table 1:

Table 1: Salience effect

B	$f_{\triangleright_{xyz}}$	$f_{\triangleright_{xzy}}$	$f_{\triangleright_{yxz}}$	$f_{\triangleright_{yzx}}$	$f_{\triangleright_{zxy}}$	$f_{\triangleright_{zyx}}$
x, y, z	x	x	x	z	z	z
x, y	x	x	y	y	x	y
x, z	x	x	x	z	z	z
y, z	z	z	z	z	z	z

It can be verified that the choice function in Table 1 is s_X-rationalised by the preference relation $\succ_2$, where $\succ_2$ is such that $z \succ_2 y, xI_{\succ_2}y, xI_{\succ_2}z$. Consider, for instance, the top-left cell of Table 1, which says that $c(\{x, y, z\}, f_{\triangleright_{xyz}}) = \{x\}$. As c is s_X-rationalised by $\succ_2$, it follows that $\max(\max(\{x, y, z\}, \succ_2), s_X(f_{\triangleright_{xyz}})) = \{x\}$. Intuitively, we can think of this expression as saying that, if (1) the budget set is $\{x, y, z\}$ and (2) x is positioned closest and z is furthest way, then the DM's choice of x is generated by the following two-stage process. In the first stage, the DM eliminates y because z is preferred to y. Two alternatives survive the first stage, x and z, between which the DM is indecisive. In the second stage, the frame "tips the balance" in favour of x over z, causing the DM to choose x, because, in frame $f_{\triangleright_{xyz}}$, x is more salient than z.

As is illustrated by the example above, the notion of s_X-rationalisation provides an account of what it means for a choice function to be generated by a salience effect. But the salience effect is only one of a number of resolution framing effects explored in this paper. We will also show that the priming effect, the compromise effect and the endowment effect can also be interpreted as types of resolution framing effects. Accordingly, we will show that, for each of these types of framing effects, the definition of r-rationalisation provides an account of what is means for a choice function to be generated by that framing effect. In this sense, our model is a relatively general model of resolution framing effects.

While our model is relatively general, it will not be applied to all conceivable resolution framing effects. In particular, we will restrict our attention to a certain interesting class of framing effects that possess a "richness' property". Formally, this richness property is defined as follows:

Definition 5 (*Rich framing effect*)

Consider any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$, and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$. The resolution framing effect r is said to be *rich in* c if, for any pair of distinct alternatives $(x, y) \in X^2$, there exists an $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}$ and not $xr(f)y$.

For example, if $X = \{x, y, z\}$, the resolution framing effect s_X is rich in the choice function of Table 1. To see why, consider any pair of alternatives $(u, v) \in X^2$ and let $\rhd'$ be a linear ordering of X such that $v \rhd' u$; then $vs_X(f_{\rhd'})u$, so not $us_X(f_{\rhd'})v$. Intuitively, a framing effect is rich in a choice function if, for any pair of alternatives (u, v), it is possible to observe the DM's choice between u and v in a frame that does not favour u over v.

To understand the significance of rich framing effects, consider the challenge of backing out preference relations from a frame-dependent choice function c. Suppose that the researcher has reason to think that a particular framing effect $r : \mathcal{F} \to \mathcal{R}$, such as the salience effect, is operating, and, given this assumption, is attempting to back out the DM's preferences from observed choices. Consider a particular pair of alternatives $x, y \in X$, and suppose that the researcher observes the DM's choices from the budget set $\{x, y\}$ in every frame (that is, $\forall f \in \mathcal{F}$, $(\{x, y\}, f)$ is in the domain of c). Suppose that the researcher observes that, in all frames, x is the DM's unique choice from the budget set $\{x, y\}$ (that is, $\forall f \in \mathcal{F}, c(\{x, y\}, f) = \{x\}$). Consider two cases.

The first case is where r is not rich in c, and, in particular, in every frame, the frame favours x over y (that is, $\forall f \in \mathcal{F}, xr(f)y$). There are potentially two hypotheses consistent with the observation that, in all frames, x is chosen over y when the budget set is $\{x, y\}$: (1) it may be that x is strictly preferred to y; or (2), it may be that that the DM is indecisive between x and y, but, in every frame, the DM chooses x over y because every frame favours x over y.

In the second case, r is rich in c. In this case, by definition of richness, there must be a frame $f' \in \mathcal{F}$ such that f' does not favour x over y (that is, not $xr(f')y$), even though $c(\{x, y\}, f') = \{x\}$. In this case, the fact that the DM chooses x over y even in frame f' implies that any preference relation $\succ$ that r-rationalises the choice function must satisfy $x \succ y$. It cannot be the case that the DM is indecisive between x and y. So, given that r is rich in c, the choices reveal that x is preferred to y. For this reason, our revealed preference results rely crucially on the restriction to rich framing effects. Moreover, despite the restriction of our model to rich framing effects, the scope of the model is still relatively broad; for all of the examples considered in this paper, framing effects are rich in the choice functions.

The discussion above suggests a plausible candidate for a revealed preference relation. So long as a framing effect is rich in a choice function, then, if the DM chooses x uniquely from the budget set $\{x, y\}$ regardless of the frame, there is reason to think that the DM prefers x to y. This motivates the following definition of the counterpart to the neoclassical base relation, which we will term the "framing base relation":

Definition 6 (*Framing base relation* $\hat{P}$).
Consider a set of alternatives X and a choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$. The *framing base relation*, which is denoted $\hat{P}$, is the asymmetric relation on X such that, for any distinct alternatives $x, y \in X$, $x\hat{P}y \iff c(\{x, y\}, f) = \{x\}$ for all $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}$.

That is, $x\hat{P}y$ if, regardless of the frame, x is always chosen over y from a budget set $\{x, y\}$. In Theorems 2A and 2B below, we will formally establish that $\hat{P}$ is the revealed preference relation in our model.

In order to specify our axiomatisation, we will develop counterparts to the three neoclassical axioms presented in Section 3.2. The counterpart to the neoclassical contraction axiom is as follows.

Framing contraction
Consider a choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, a budget set $B \in \mathcal{B}$ and alternatives $x, y \in B$. If there is a frame $f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D}$ and $x \in c(B, f)$, then there is a frame $f' \in \mathcal{F}$ such that $(\{x, y\}, f') \in \mathcal{D}$ and $x \in c(\{x, y\}, f')$.

The plausibility of the Framing contraction axiom can be motivated by a comparison to its neoclassical counterpart. Like the neoclassical contraction axiom, Framing contraction says that, if an alternative x is chosen from a budget set containing alternative y, then x will also be chosen from $\{x, y\}$. To obtain a more precise motivation for Framing contraction, however, it is helpful to consider the contrapositive of Framing contraction. The contrapositive says that, for $x, y \in B$, if $x \notin c(\{x,y\},f)$ for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}$, then $x \notin c(B,f)$ for all $f \in \mathcal{F}$ such that $(B,f) \in \mathcal{D}$. This contrapositive is plausible when r is rich in c. Given that r is rich in c, the 'if clause' of the contrapositive – the proposition that $x \notin c(\{x,y\},f)$ for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}$ – entails that y is revealed preferred to x. The fact that y is preferred to x, in turn, entails the 'then clause' in the contrapositive – the proposition that $x \notin c(B,f)$ for all $f \in \mathcal{F}$ such that $(B,f) \in \mathcal{D}$.

Note that, even if a choice function satisfies Framing contraction, it is possible that, for a given frame, the choices made within that frame violate the neoclassical contraction property. For example, in Table 1, consider the frame $f_{\triangleright_{yxz}}$. In this frame, whereas x is the choice from budget set $\{x,y,z\}$, y is the choice from $\{x,y\}$. So *within this frame*, the neoclassical contraction axiom is violated. In this respect, our model of framing effects differs from so-called "multi-self" models of framing effects. Roughly speaking, a multi-self model of framing effects attributes to the DM, for each frame, a preference relation specific to that frame, which generates frame-dependent choices.[12] Accordingly, if a multi-self model assumes frame-dependent preferences are transitive and asymmetric, the neoclassical axioms from the previous section must hold *within each frame*. Given that, in our model, the neoclassical contraction property may be violated within a frame, then, in this respect, our model differs from multi-self models of framing effects.

In our model, the counterpart to the neoclassical axiom "Weak IIA" is the following:

Framing weak IIA

Consider any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$ and any $x,y,z \in X$. If, for all $f \in \mathcal{F}$ such that $c(\{x,y,z\},f) \in \mathcal{D}$, $c(\{x,y,z\},f) = \{x\}$, then, for all $f \in \mathcal{F}$ such that $c(\{x,y\},f) \in \mathcal{D}$, $c(\{x,y\},f) = \{x\}$.

Like Weak IIA, Framing weak IIA licenses the inference that x is the unique alternative chosen from $\{x,y\}$ from the fact that x is the unique alternative chosen from $\{x,y,z\}$. But Framing weak IIA licenses this inference only if, in every frame, x is the unique alternative chosen from $\{x,y,z\}$.

[12]For an example of a multi-self model, see Sagi (2006). Furthermore, any frame-dependent choice function that satisfies the axiomatisation in Salant and Rubinstein (2008) is a multi-self choice function – this is apparent from the specification of their "salient consideration" property.

In order to specify a counterpart of the neoclassical expansion axiom, we first define the notion of the 'revealed non-inferior set'.

Definition 7 (*The revealed non-inferior set N_B*)
Consider any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$ and any budget set $B \in \mathcal{B}$. The set $N_B = \{x \in B : \forall y \in B \backslash \{x\}, \exists f \in \mathcal{F}, x \in c(\{x, y\}, f)\}$ is said to be *the revealed non-inferior set for* B.

The intuition for revealed non-inferiority is as follows. Suppose that, for a given alternative x in set B, it holds that, for every other alternative y in B, x is chosen from $\{x, y\}$ in at least one frame. Then x has been revealed not to be inferior to any other member of this set. So we describe x as a member of the 'revealed non-inferior set' for B. We then specify the counterpart to the neoclassical expansion axiom as follows.[13]

***r*-Expansion**
Consider any set of alternatives X, any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$, any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, any $B \in \mathcal{B}$ and any $x \in B$.
If, for all $y \in B \backslash \{x\}$, there exists $f' \in \mathcal{F}$ such that $(\{x, y\}, f') \in \mathcal{D}$ and $x \in c(\{x, y\}, f')$, then $(\forall f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D})[x \in c(B, f) \iff x \in \max(N_B, r(f))]$.

Note that, unlike the Framing contraction and Framing weak IIA axioms, there is a different expansion axiom for each type of resolution framing effect. For example, we obtain the expansion property for a model of the salience effect by replacing r with s_X in the r-Expansion expansion axiom, yielding:

s_X-Expansion
Consider any set of alternatives X, any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}^s$, any $B \in \mathcal{B}$ and any $x \in B$.

[13] The axiom is articulated in a way that highlights the similarity to its neoclassical counterpart. A more economical way of expressing the second sentence in the axiom is as follows: If $x \in N_B$, then $(\forall f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D})[x \in c(B, f) \iff x \in \max(N_B, r(f))]$.

If, for all $y \in B\backslash\{x\}$, there exists $f_{\rhd'} \in \mathcal{F}^s$ such that $(\{x, y\}, f_{\rhd'}) \in \mathcal{D}$ and $x \in c(\{x, y\}, f_{\rhd'})$, then $(\forall f_{\rhd} \in \mathcal{F}^s$ such that $(B, f_{\rhd}) \in \mathcal{D})[x \in c(B, f_{\rhd}) \iff x \in \max(N_B, \rhd)]$.

Like the neoclassical expansion property, s_X-Expansion licenses the following kind of inference: if x is chosen from $\{x, y\}$ for every $y \in B\backslash\{x\}$, then we can infer that x is chosen from B. But, unlike the neoclassical expansion property, the s_X-Expansion axiom places a caveat on when this inference can be made. In particular, s_X-expansion says that this inference can be made only when there no other alternative that (i) is revealed non-inferior in B and (ii) is more salient than x.

We now present a pair of characterisation and revealed preference results in Theorems 2A and 2B, which are, respectively, the counterparts of the neoclassical results in Theorems 1A and 1B.

Theorem 2A (*Framing effect model: Acyclic preferences*)
Consider any set of alternatives X such that $|X| \geq 3$, any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, where r is rich in c. Suppose that, for any $B \in \mathcal{B}$, there is a frame $f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D}$.
(a) There exists an asymmetric and acyclic relation $\succ$ that r-rationalises c if and only if c satisfies Framing contraction and r-Expansion.
(b) If c satisfies Framing contraction and r-Expansion, then $\hat{P}$ is asymmetric, acyclic and is the unique relation that r-rationalises c.

Theorem 2B (*Framing effect model: Transitive preferences*)
Consider any set of alternatives X such that $|X| \geq 3$, any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, where r is rich in c. Suppose that, for any frame $f \in \mathcal{F}$ and any $B \in \mathcal{B}$ with $|B| \leq 3$, $(B, f) \in \mathcal{D}$.
(a) There exists an asymmetric, transitive relation $\succ$ that r-rationalises c if and only if c satisfies Framing contraction, r-Expansion and Framing weak IIA.
(b) If c satisfies Framing contraction, r-Expansion and Framing weak IIA, then $\hat{P}$ is asymmetric, transitive and is the unique relation that r-rationalises c.

Theorems 2A and 2B can be viewed as providing axiomatisations and revealed preference results for each of the types of resolution framing effects considered in our paper – in particular, for the salience effect, the priming effect, the compromise effect and the endowment effect. For example, if, in the two theorems, we replace $r : \mathcal{F} \to \mathcal{R}$ by $s_X : \mathcal{F}^s \to \mathcal{R}$, we obtain axiomatisations and revealed preference results for the salience effect. In this case, clause (a) of Theorem 2B provides a test, in terms of observable properties, for whether a choice function – such as that in Table 1 – is generated by the salience effect. More precisely, clause (a) provides a test for whether a choice function is s_X-rationalised by a transitive, asymmetric relation. The test is that Framing Contraction, s_X-Expansion and Framing weak IIA must be satisfied. It is easy to verify that the choice function in Table 1 satisfies these three properties. Clause (b) of Theorem 2B says that, if this test is satisfied, $\hat{P}$ is the unique preference relation that s_X-rationalises the choice function. For the choice function in Table 1, $\hat{P}$ is the preference relation such that the DM prefers z over y, but is indecisive between x and y and x and z. Again, it can be verified that the choice function in Table 1 is s_X-rationalised by $\hat{P}$; moreover, clause (b) of Theorem 2B guarantees that $\hat{P}$ is the unique preference relation to do so.

In the next section of the paper, we will define the priming effect and the compromise effect. Theorems 2A and 2B can then be used to yield axiomatisations and revealed preference results for these two types of framing effects.

3.4 Further examples of resolution framing effects

3.4.1 The priming effect

In their study of the psychological concept of "priming", Mandel and Johnson (2002) investigate the influence on online choice of the background wallpaper of a web page. In one experiment, subjects faced a choice between two sofas: a low-cost, less comfortable sofa; and a high-cost, more comfortable sofa. When making their choice, one group of subjects was exposed to a web page with wallpaper designed to prompt selection of the low-cost sofa – green wallpaper decorated with images of pennies. This stimulus is referred to as the "money prime". The other group was presented with wallpaper intended to prompt selection of the more comfortable sofa – blue wallpaper covered with images of clouds. This stimulus is referred to as the "comfort prime". Mandel and Johnson (2002) found that the prime had a significant influence on choice: the low-cost sofa was chosen by 56 per cent of subjects exposed to the money prime, but only by 39 per cent of subjects exposed to the comfort prime. This behaviour exemplifies "feature priming", which refers to the following propensity: if a DM is exposed to a prime associated with a particular feature of the alternatives, then that feature is given more weight when the DM evaluates the alternatives.

The results of this experiment suggest that, for some subjects, choice depends upon the prime. Consider the choice dispositions of such a subject. Let y denote the low-cost sofa and z the more comfortable sofa. Thus, when choosing from the budget set $\{y, z\}$, this subject is disposed to choose y when exposed to the money prime and z when exposed to the comfort prime. Moreover, suppose that, in addition, there is a third alternative x that has the same cost and comfort of the low-cost sofa y but has a fold-out bed capability. Suppose that the DM values the fold-out bed capability sufficiently that, regardless of the prime, she is disposed to choose x over y and z whenever x is available.

We will say that low-cost sofas are "primed' by the frame in which subjects are exposed to the money prime, and that comfortable sofas are "primed" by the frame in which subjects are exposed to the comfort prime. The set of alternatives $X = \{x, y, z\}$ can be partitioned into the set $A = \{x, y\}$ of alternatives that are primed by the frame in which subjects are exposed to the money prime and the set $A^c = \{z\}$ of alternatives that are primed by the frame in which subjects are exposed to the comfort prime. Accordingly, we will denote the frame in which subjects are exposed to the money prime as f_A; the notation f_A can then be interpreted as denoting the frame that primes the alternatives in A. Similarly, the frame in which subjects are exposed to the comfort prime is denoted f_{A^c}; it is the frame that primes the alternatives in A^c. Then the DM's choice dispositions are described in Table 2.

Table 2: Priming effect

B	f_A	f_{A^c}
x, y, z	x	x
x, y	x	x
x, z	x	x
y, z	y	z

We will now formulate a simple definition of the priming effect for a set of alternatives X of arbitrary finite size. It is assumed the set of alternatives X is partitioned into two sets A and A^c and there are two frames f_A and f_{A^c}. As in the example above, f_A is interpreted as the frame that primes the alternatives in A and f_{A^c} is the frame that primes the alternatives in A^c. Accordingly, our simple definition of the priming effect is applicable only to sets of alternatives where every alternative is primed by exactly one of the two frames.[14] The following defines the priming effect for a given set of alternatives X and a given partition $\{A, A^c\}$.

Definition 8 (*Priming effect*)
Consider a set of alternatives X, a partition of X, $\{A, A^c\}$, and a set of frames $\mathcal{F}^p = \{f_A, f_{A^c}\}$. The *priming effect* for the set of alternatives X and the partition $\{A, A^c\}$ is the resolution framing effect $p_X^A : \mathcal{F}^p \to \mathcal{R}$ such that for all $x, y \in X$:
(i) $xp_X^A(f_A)y \iff x \in A$ and $y \in A^c$;
(ii) $xp_X^A(f_{A^c})y \iff x \in A^c$ and $y \in A$.

Intuitively, this definition says that, if the priming effect is operating, then, in the second stage of the decision process, the DM's choice depends upon the frame in the following way: (i) if the decision is made in frame f_A, which is the frame that primes the alternatives in A, then the frame favours an alternative in A over an alternative in A^c; (ii) if the decision is made in frame f_{A^c}, which is the frame that primes the alternatives in A^c, then the frame favours an alternative in A^c over an alternative in A. In the example of Table 2, the set of alternatives X is $\{x, y, z\}$ and the partition $\{A, A^c\}$ is such that $A = \{x, y\}$ and $A^c = \{z\}$. So the priming effect for the set of alternatives X and the partition $\{A, A^c\}$ is given by the function $p_X^A : \mathcal{F}^p \to \mathcal{R}$ which is such that (i) $xp_X^A(f_A)z$ (ii) $yp_X^A(f_A)z$ (iii) $zp_X^A(f_{A^c})x$ (iv) $zp_X^A(f_{A^c})y$ and (v) neither $xp_X^A(f_A)y$ nor $yp_X^A(f_A)x$ nor $xp_X^A(f_{A^c})y$ nor $yp_X^A(f_{A^c})x$.

Theorems 2A and 2B provide axiomatisations and revealed preference results for the priming effect. For instance, suppose that, in Theorem 2B, we replace the resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ by the priming effect $p_X^A : \mathcal{F}_X^p \to \mathcal{R}$.[15] Then clause (a) of the theorem tells us that the test for the choice function being p_X^A-rationalised is that Framing Contraction, p_X^A-Expansion and Framing

[14] More general and complex definitions of the priming effect could, of course, be specified. We present a simple definition because our principal goal is to illustrate how Theorems 2A and 2B can be applied to the framing effect described by Mandel and Johnson (2002).

[15] The theorem can be applied to the choice function in Table 2 because the priming effect p_X^A is rich in the choice function.

weak IIA hold. It can be verified that the choice function in Table 2 satisfies these three properties, entailing that the choice function is p_X^A-rationalised by a transitive and asymmetric relation. Clause (b) then tells us that, given the test is satisfied, the unique preference relation that p_X^A-rationalises the choice function is $\hat{P}$, which is the relation $\succ_3$ such that $x \succ_3 y$, $x \succ_3 z$ and $yI_{\succ_3}z$. It is easy to check that this preference relation p_X^A-rationalises the choice function. Intuitively, this suggests the following explanation for the choice reversal described in the final row of Table 2. The explanation is that the DM is indecisive between y and z, but exposure to the money prime tips the balance in favour of y, whereas exposure to the comfort prime tips the balance in favour of z.

We now have two illustrations of how Theorems 2A and 2B allow us to obtain axiomatisations and revealed preference results for specific resolution framing effects – for the salience effect and for the priming effect. The next section will illustrate how the theorems provide an axiomatisation and revealed preference results for another type of framing effect – the compromise effect.

3.4.2 The compromise effect

In his seminal study of the compromise effect, Simonson (1989) presents two kinds of evidence – we will label these *description evidence* and *budget-set evidence*. An example of an experiment providing *description evidence* involved three goods: a battery with an expected life of 12 hours and a probability of corrosion of 2 per cent (b_2^{12}); one with an expected life of 14 hours and a probability of corrosion of 4 per cent (b_4^{14}); and one with an expected life of 16 hours and a probability of corrosion of 6 per cent (b_6^{16}). Only the first two of these alternatives were available for choice. That is, the budget set was $\{b_2^{12}, b_4^{14}\}$. But the choice from that budget set was presented under several frames: in one frame, only the two items in the budget set were described, and the subject was asked to choose between them; in another frame, all three items were described, but the subject was then asked to choose between the two in the budget set. In the first frame, 34 per cent of subjects chose b_4^{14}, but in the second frame, 60 percent chose b_4^{14}. The explanation for greater propensity to choose b_4^{14} in the second frame is that, in this frame, b_4^{14} is a *compromise item.*

How might we describe the results of this experiment in a choice function? Suppose that the set of alternatives is $X = \{i, x, y\}$, where i is the *compromise item.* Let f_2 denote the frame in which only two items in the budget set are described and let f_3 denote the frame in which all three items are described. The following two kinds of choice behaviour are consistent with the compromise effect:

(a) $c(\{i, x\}, f_3) = \{i\}$ and $c(\{i, x\}, f_2) = \{i, x\}$;

(b) $c(\{i, x\}, f_3) = \{i\}$ and $c(\{i, x\}, f_2) = \{x\}$.

The decision maker with the choice dispositions described by (a) is equally disposed to choose x and i when i is not presented as the compromise item, but chooses i when i is so presented. The decision maker with choice dispositions described by (b) switches from choosing x to choosing i when i is presented as the compromise item.

Description evidence is based upon a comparison between choices in two different frames: one in which two alternatives are described; one in which three items are described. In contrast, *budget-set evidence* is based on a comparison of choices from two different budget sets. The kind of experiment giving rise to *budget-set evidence* is as follows. There are three goods, i, x and y, where i is the compromise item. The subjects' choice from budget set $\{i, x, y\}$ is compared to their choice from budget set $\{i, x\}$ (where, for the second choice, i and x are the only two items described). Simonson (1989) finds that subjects had a greater propensity to select i from budget set $\{i, x, y\}$ than from $\{i, x\}$. If at least some subjects display either of the following two kinds of choice dispositions, this is consistent with budget-set evidence for the compromise effect:

(a) $c(\{i, x, y\}, f_3) = \{i\}$ and $c(\{i, x\}, f_2) = \{i, x\}$;

(b) $c(\{i, x, y\}, f_3) = \{i\}$ and $c(\{i, x\}, f_2) = \{x\}$.

Accordingly, Tables 3 and 4 present two different choice functions consistent with the compromise effect. In each table, the first two rows, in particular, display choices consistent with description evidence and budget-set evidence for the compromise effect. Note that the top right cell of the tables is empty, because it does not make sense to say that the DM had a choice between all three alternatives i, x and y, but only two alternatives were described.

Table 3: Compromise effect

B	f_3	f_2
i, x, y	i	
i, x	i	i, x
i, y	i	i
x, y	x	x

Table 4: Compromise effect with leaning

B	f_3	f_2
i, x, y	i	
i, x	i	x
i, y	i	i
x, y	x	x

The next task is to define the compromise effect as a type of resolution framing effect. First, we will impose a condition that the set of alternatives must have three members.[16] In defining the compromise effect, the primitives are (i) a set of alternatives X of size three (ii) a designated compromise item $i \in X$ and (iii) the compromise set of frames $\mathcal{F}^q = \{f_2, f_3\}$.[17] The following defines the compromise effect for a given set of alternatives X and a compromise item $i \in X$:

Definition 9 (*Compromise effect*)
Consider a set of alternatives X with $|X| = 3$, a compromise item $i \in X$ and set of frames $\mathcal{F}^q = \{f_2, f_3\}$. The *compromise effect* for the set of alternatives X and the compromise item $i \in X$ is the resolution framing effect $q^i_X : \mathcal{F}^q \to \mathcal{R}$ such that:
(i) $q^i_X(f_2) = \emptyset$ and
(ii) for any $x, y \in X$, $xq^i_X(f_3)y \iff x = i$ and $y \neq i$.

Intuitively speaking, this definition states that, when the compromise effect operates, the DM's choice in the second stage of the decision depends upon the frame in the following way: (i) if the decision is made in frame f_2, which is the frame where i is not perceived as the compromise item, then the frame is neutral between all of the alternatives; (ii) if the decision is made in frame f_3, which is the frame where i is perceived as the compromise item, then the frame favours i over the other alternatives.

We will first discuss how the choice function in Table 3 can be interpreted as generated by the compromise effect, and then consider Table 4. It is easy to verify that the choice function in Table 3 is q^i_X-rationalised by the preference relation $\succ_4$ such that $i \succ_4 y$, $x \succ_4 y$ and $iI_{\succ_4}x$. As both i and x are preferred to y, y is never chosen. In frame f_2, when choosing from budget set $\{i, x\}$, the compromise effect does not favour one alternative over another; so, given that the DM is indecisive

[16] Much of the empirical research on the compromise effect has been restricted to budget sets with three or fewer members – see Kivetz et al. (2004).

[17] In specifying that the compromise item is a primitive in the model, we are introducing a simplification. Models of the compromise effect typically construe the alternatives as bundles of attributes. Whether an alternative is the compromise item is then determined by the attributes of the alternative compared to the attributes of the other alternatives – as is illustrated by Simonson's experiment on battery choice described above. We adopt this simplification because our main aim is to illustrate how the compromise effect can be interpreted as coming within the scope of Theorem 2A.

between i and x, she is equally disposed to choose i and x. By way of contrast, in frame f_3, when choosing from $\{i, x\}$ or $\{i, x, y\}$, the compromise effect "tips the balance" in favour of i, causing the DM to select i over x.

Suppose that, in Theorems 2A and 2B, we replace the resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ by the compromise effect $q_X^i : \mathcal{F}_X^q \to \mathcal{R}$. Note that the compromise effect is rich in the choice function in Table 3, which ensures that Theorem 2A is applicable. Theorem 2B is not applicable, however, as the requirement on the domain is not satisfied – the requirement is violated because (X, f_2) is absent from the domain. Given that, as noted above, the choice function in Table 3 is q_X^i-rationalised by $\succ_4$, clause (a) of Theorem 2A tell us that the choice function in Table 3 must satisfy Framing contraction and q_X^i-Expansion. It is easy to verify that the choice function does satisfy these two properties. Moreover, clause (b) of Theorem 2A says that $\hat{P}$ is the unique preference relation that q_X^i-rationalises the choice function. Clearly, $\hat{P}$ is the relation $\succ_4$ defined above. On the other hand, as Theorem 2B is not applicable, our two theorems allow for the possibility that the choice function fails to satisfy Framing weak IIA. Indeed, it turns out that the choice function in Table 3 does violate Framing weak IIA.

Whereas the choice function in Table 3 is q_X^i-rationalisable by an acyclic and asymmetric relation, the choice function in Table 4 is not. To see this, note that, given the DM's choice from budget set $B = \{i, x\}$ is $\{i\}$ in frame f_3 and $\{x\}$ in frame f_2, it follows that any preference relation $\succ$ which q_X^i-rationalises the choice function must be such that $iI_{\succ}x$. But given that $q_X^i(f_2) = \emptyset$, thus $\max(\max(\{i, x\}, \succ), q_X^i(f_2)) = \{i, x\}$. As this is distinct from $c(\{i, x\}, f_2) = \{x\}$, the choice function cannot be q_X^i-rationalised. The failure of the model in Section 3.3 to accommodate the choice function in Table 4 motivates an extension to the model, which allows DMs not only to have preferences for alternatives but also "leanings".

3.5 Framing effects with "leanings"

Before presenting the extension of the model, it will be helpful to introduce the endowment effect, in order to motivate the concept of a "leaning". One of the simplest illustrations of the endowment effect is an experiment reported in Knetsch (1989). In this experiment, three groups of subjects were offered a choice between a mug and a chocolate bar: the first group was initially endowed with a mug and given the opportunity to exchange; the second group was initially endowed with a chocolate bar and offered the opportunity to exchange; the third group was not provided with an initial endowment but was simply offered a choice between the two alternatives. For the two groups with an initial endowment, about 90 per cent opted to retain the endowment. In the third group, close to half chose the mug and half the chocolate bar. Let y denote the mug, z the chocolate bar, f_y the frame in which the mug is the endowment, f_z the frame in which the chocolate was the endowment and $f_\diamond$ the frame in which there is no endowment. We can then record the choice dispositions of the "typical subject" in the Knetsch experiment as follows: $c(\{y,z\}, f_y) = \{y\}, c(\{y,z\}, f_z) = \{z\}, c(\{y,z\}, f_\diamond) = \{y,z\}$. The "choice reversal" induced by a change from frame f_y to f_z is characteristic of the endowment effect.

Consider a hypothetical variant of the Knetsch experiment in which there is a third alternative – a pen x. Let f_x denote the frame in which the pen is the endowment. So the set of alternatives now is $X = \{x, y, z\}$. Suppose that the pen is of sufficiently high quality that, whenever x is available, the DM is disposed to choose only x. The choice dispositions of the DM are given in Table 5. The first three lines of the table reflect the fact that the DM chooses x whenever it is available. In the fourth line, the second, third and fourth cells record the choice dispositions from the Knetsch experiment. The first cell in the fourth line specifies the choice of the DM from budget set $\{y,z\}$ when the endowment is x. At least in this example, it is unclear what this means. We simply stipulate that, if alternative u is not in budget set B, then "the DM's choice from budget set B under endowment u" is to be interpreted as meaning "the DM's choice in the absence of an endowment", so that $c(B, f_u) := c(B, f_\diamond)$.[18]

Table 5: Endowment effect

B	f_x	f_y	f_z	$f_\diamond$
x,y,z	x	x	x	x
x,y	x	x	x	x
x,z	x	x	x	x
y,z	y,z	y	z	y,z

Table 6: Endowment effect with leaning

B	f_x	f_y	f_z	$f_\diamond$
x,y,z	x	x	x	x
x,y	x	x	x	x
x,z	x	x	x	x
y,z	y	y	z	y

[18] This stipulation ensures that the domain is rectangular. As a consequence, the domain satisfies the conditions in Theorems 2B above and 3B below, which would not be satisfied without this stipulation.

In order to define the endowment effect for an arbitrary finite set of alternatives X, let us first define the endowment set of frames. The endowment set of frames for X is given by $\mathcal{F}^e = \{f_x : x \in X\} \cup \{f_\diamond\}$ where, for any $x \in X$, f_x is interpreted as the state in which x is the endowment, and $f_\diamond$ is the state in which there is no endowment. The endowment effect can then be defined as follows:

Definition 10 (*Endowment effect*)
Consider an arbitrary set of alternatives X and the endowment set of frames $\mathcal{F}^e = \{f_x : x \in X\} \cup \{f_\diamond\}$. The *endowment effect* for X is the resolution framing effect $e_X : \mathcal{F}^e \to \mathcal{R}$ such that (i) $e_X(f_\diamond) = \emptyset$ (ii) for any distinct $x, y \in X$, $xe_X(f_x)y$ and (iii) for any distinct $x, y, z \in X$, neither $ze_X(f_x)y$ nor $ye_X(f_x)z$.

Intuitively speaking, the endowment effect is a resolution framing effect such that, for any alternative x, a frame in which x is the endowment favours x over other alternatives. The frame $f_\diamond$, in which there is no endowment, is neutral between all the alternatives. It can be verified that the choice function in Table 5 satisfies Framing contraction, e_X-Expansion and Framing weak IIA. Accordingly, Theorem 2B entails that the framing base relation $\hat{P}$ is the unique preference relation that e_X-rationalises the choice function. $\hat{P}$ is the relation $\succ_5$ such that $x \succ_5 y$, $x \succ_5 z$ and $yI_{\succ_5}z$. The intuitive explanation for the choice reversal that occurs when the DM is choosing from the budget set $\{y, z\}$ is as follows: given that the DM is indecisive between y and z, her choice is influenced by framing; the frame where y is the endowment "tips the balance" in favour of y; and the frame where z is the endowment "tips the balance" in favour of z.

The choice dispositions in Table 6 differ from those in Table 5 in respect of the bottom right and bottom left cells. (Recall that, by stipulation, the bottom left cell is identical to the bottom right cell.) Whereas, in Table 5, the DM is equally inclined to choose y and z in the absence of an endowment, in Table 6, the DM uniquely chooses y when there is no endowment. The choices in Table 6 cannot be accommodated by the model in Section 3.3. That is, whereas the choice function in Table 5 is e_X-rationalisable, there is no preference relation that e_X-rationalises the choice function in Table 6. After all, for the choice function in Table 6, given that the DM's choice from budget set $B = \{y, z\}$ is $\{y\}$ in frame f_y and $\{z\}$ in frame f_z, it follows that any preference relation $\succ$ which e_X-rationalises the choice function must be such that $yI_\succ z$. But given that $e_X(f_\diamond) = \emptyset$, thus $\max(\max(\{y, z\}, \succ), e_X(f_\diamond)) = \{y, z\}$. But this is distinct from $c(\{y, z\}, f_\diamond) = \{y\}$, so the choice function in Table 6 cannot be e_X-rationalised.

Note that the model of status quo bias in Masatlioglu and Ok (2005) accommodates choice dispositions akin to those in Table 6.[19] Moreover, our view is that the model in Section 3.3 is excessively restrictive in failing to accommodate the choice dispositions in Table 6. Intuitively, a DM might be revealed to be indecisive between x and y – by choosing x from $\{x, y\}$ in at least one frame, and y in at least one frame – and yet have a disposition to choose y uniquely in a frame that is neutral between x and y. In other words, it is possible that a DM is revealed both (1) to be indecisive between two alternatives but (2) to have a *leaning* towards one of them. The notion of a "leaning", as distinct to a preference, can be illustrated by the account of "preferences for flexibility" in Kreps (1979). In Kreps's model, it is possible that the DM has a leaning towards one alternative over another, but is, nevertheless, uncertain about her preferences between them. Kreps (1979, p. 566) gives an example in which a DM "prefers a menu containing only steak to one containing only chicken. But he strictly prefers a menu with both steak and chicken to either of the first two, because it gives him greater flexibility." In Kreps's model, this choice pattern reveals the DM's preference for flexibility. Intuitively, we might also say that it reveals that, whereas the DM is uncertain about his preferences between steak and chicken, currently he has a leaning towards steak. Similarly, we might say that the choice pattern in Table 6 reveals that, whereas the DM is indecisive between y and z, nevertheless she has a leaning towards y.

To develop a model that accommodates leaning, the first task is to modify the definition of r-rationalisation. Intuitively, this notion of rationalisation describes a two-stage decision process. In the first stage, an alternative y is eliminated if there is another available alternative that is strictly preferred to y. In the second stage, an alternative y that survives the first stage is eliminated if there is another alternative that survives the first stage, x, such that the frame favours x over y. Our model of "leaning" adds a third stage to the decision process. The DM also has a leaning relation, ℓ. Intuitively, $x \ell y$ means that, while the DM does not have a settled preference between x and y, she has a leaning towards x over y. In the third stage, if alternatives x and y survive the first two stages, but the agent has a leaning towards x over y, then y is eliminated. The following

[19] More precisely, the axioms in the SQB model of Masatlioglu and Ok (2005) do not rule out the possibility that, for a DM choosing from a budget set $\{y, z\}$, y is the unique choice when y is the status quo, z is the unique choice when z is the status quo and y is the unique choice when there is no status quo.

definition of rationalisation formalises this three-stage decision process:

Definition 11 (*r-rationalisation**)
Consider a set of alternatives X, a resolution framing effect $r : \mathcal{F} \to \mathcal{R}$, a choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$ and relations $\succ$ and ℓ on X. c is said to be *r-rationalised** by $(\succ, \ell)$ if:
(1) for all $(B, f) \in \mathcal{D}, c(B, f) = \max(\max(\max(B, \succ), r(f)), \ell)$; and
(2) for any $x, y \in X$, if $x\ell y$, then $xI_{\succ}y$.

Clause (1) of the definition is simply a formal specification of the three-stage decision process described above. Clause (2) can be interpreted as a formalisation of a key feature of the concept of *leaning* in our paper. It encapsulates the idea that, according to the concept of *leaning* in our paper, it only makes sense to say that the DM has a leaning towards x over y if the DM is indecisive between x and y; the DM is not to be attributed a leaning towards x over y if she strictly prefers x over y or y over x.

The definition of r-rationalisation* can be illustrated by the choice functions in Tables 5 and 6. The choice function in Table 6 can be shown to be e_X-rationalised* by the pair $(\succ_5, \ell_1)$ where $\succ_5$ was defined above and ℓ_1 is given by $xI_{\ell_1}y, xI_{\ell_1}z, y\ell_1 z$. Intuitively, the explanation for why, when choosing from the budget set $\{y, z\}$, the DM reverses her choice between frame f_z and the frame $f_\diamond$ is that (1) the DM is indecisive between y and z (2) in frame f_z, as z is the endowment, this "tips the balance" in favour of z, causing the DM to choose z in that frame and (3) the frame $f_\diamond$ is neutral between y and z, but, as the DM has a leaning towards y over z, she chooses y in that frame. In contrast, the choice function in Table 5 is e_X-rationalised* by the pair $(\succ_5, \ell_2)$, where $\succ_5$ was defined above and $\ell_2 = \emptyset$. This illustrates a general feature of the relationship between the concepts of r-rationalisation from Section 3.3 and r-rationalisation* developed in this section: if a choice function is r-rationalised by $\succ$, then it is r-rationalised* by $(\succ, \ell)$ where $\ell = \emptyset$. In other words, if a choice function is r-rationalised, the DM's choices are entirely explained by the framing effect and the DM's preferences; we do not need to attribute to the DM any "leanings" to explain her choices.

If we return now to the compromise effect, the concept of r-rationalisation* can be used to explain how the choice function in Table 4 is generated. It can be verified that choice function is

q_X^i-rationalised* by $(\succ_4, \ell_3)$, where $\succ_4$ was defined in Section 3.4.2 and the leaning relation ℓ_3 is such that $x\ell_3 i$, $xI_{\ell_3}y$ and $iI_{\ell_3}y$. When the DM chooses from budget set $\{i, x\}$, the intuition for why the DM reverses her choice when deciding in frame f_3 rather than in f_2 is as follows: (1) she is indecisive between i and x (2) in frame f_3 she chooses i over x because in this frame i is perceived as the compromise item and (3) in frame f_2 she chooses x over i because, although f_2 is neutral between x and i, the DM has a leaning towards x over i.

What is the key feature of the choice dispositions in Tables 4 and 6 that motivate the concept of a leaning? Consider Table 6. The motivation is that the DM reverses her choice from the budget set $\{y, z\}$ when choosing from the following two frames:
(i) the frame $f_\diamond$ that is neutral between y and z (that is, neither $ye_X(f_\diamond)z$ nor $ze_X(f_\diamond)y$);
(ii) the frame f_z that favours z over y (that is, $ze_X(f_z)y$).
Accordingly, for the version of our model with leanings, we introduce a stronger richness condition, which requires that, for any pair of alternatives (x, y), the observed choice data include choices from a budget set $\{x, y\}$ both in a frame that is neutral between x and y and also in a frame that favours x over y.

Definition 12 (*Rich* framing effect*)
Consider any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$, and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$. The resolution framing effect r is said to be *rich* in c* if, for any pair of distinct alternatives $(x, y) \in X^2$, there exists
(i) a frame $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}$ and f is neutral between x and y; and
(ii) a frame $f' \in \mathcal{F}$ such that $(\{x, y\}, f') \in \mathcal{D}$ and f' favours x over y.

For example, for the choice function in Table 6, the endowment effect e_X is rich* in the choice function, whereas, for the choice function in Table 4, the compromise effect c_X^i is not rich* in the choice function. As will be established in Theorems 3A and 3B below, this means that a unique preference relation and leaning relation can be backed out from the choices in Table 6. In contrast, several distinct $(\succ, \ell)$ pairs c_X^i-rationalise the choice function in Table 4. Clearly, richness* is a stronger property than richness: that is, if a resolution framing effect r is rich* in a choice function c, then r is also rich in c.

Theorems 3A and 3B will establish that, under the assumption of richness*, the revealed preference relation is the framing base relation. Moreover, it turns out that the revealed leaning relation is given as follows.

Definition 13 (*Leaning base relation* $\hat{L}$)
Consider any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$. The *leaning base relation* $\hat{L}$ is the relation on X such that, for any distinct alternatives $x, y \in X$, $x\hat{L}y \iff$ (i) $xI_{\hat{P}}y$ and (ii) for all $f \in \mathcal{F}$ such that $(\{x,y\}, f) \in \mathcal{D}$ and f is neutral between x and y, we have $c(\{x,y\}, f) = \{x\}$.

What is the appropriate set of axioms for the model with leaning? Whereas the Framing contraction and Framing weak IIA axioms do not need to be modified, the expansion property must be amended. Consider any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$ and any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ that is rich* in c. First, we define a subset of the revealed non-inferior set N_B.

Definition 14 (*The revealed non-dominated set* $N_{B,f}$)
Consider any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, and any $B \in \mathcal{B}, f \in \mathcal{F}$. The set $N_{B,f} = \{x \in N_B : \forall y \in N_B \backslash \{x\},\ \text{not } yr(f)x\}$ is said to be *the revealed non-dominated set for* B *in frame* f.

Intuitively, an alternative x from N_B is also in $N_{B,f}$ so long as frame f does not favour some member of N_B over x. Then the modified expansion property is defined as follows.[20]

r-Expansion*
Consider any set of alternatives X, any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$, any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, any $B \in \mathcal{B}$ and any $x \in B$. Suppose that r is rich* in c.
If, for all $y \in B \backslash \{x\}$, there exists an $f' \in \mathcal{F}$ such that $(\{x,y\}, f') \in \mathcal{D}$ and $x \in c(\{x,y\}, f')$, then $(\forall f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D})[x \in c(B, f) \iff$ (i) $x \in \max(N_B, r(f))$ and (ii) for all $y \in N_{B,f} \backslash \{x\}$, for some $f'' \in \mathcal{F}$ neutral between x and y, $(\{x,y\}, f'') \in \mathcal{D}$ and $x \in c(\{x,y\}, f'')]$.

[20] The axiom has been written to highlight the similarity to the r-Expansion axiom. A more economical articulation of the second sentence in the axiom is as follows: If $x \in N_B$, then $(\forall f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D})[x \in c(B, f) \iff$ (i) $x \in \max(N_B, r(f))$ and (ii) for all $y \in N_{B,f} \backslash \{x\}$, for some $f'' \in \mathcal{F}$ neutral between x and y, $(\{x,y\}, f'') \in \mathcal{D}$ and $x \in c(\{x,y\}, f'')]$.

The intuition for r-Expansion* is similar to that for r-Expansion. The motivation for the latter is follows: if x is chosen from $\{x, y\}$ for every $y \in B\backslash\{x\}$, then x may be chosen from B in a given frame f so long as *there is no other non-inferior alternative* y *where* f *favours* y *over* x. The r-Expansion* axiom, in effect, changes the condition italicised in the previous sentence to require, in addition, that, for any alternative y revealed to be non-dominated for B in frame f, there is a frame f'' that is neutral between x and y such that $x \in c(\{x, y\}, f'')$. The motivation for this additional requirement is to ensure that there is not a leaning towards y over x. The following two theorems then provide characterisation and revealed preference results for the version of the model with leaning.

Theorem 3A (*Model of framing effects with leaning: Acyclic preferences*)
Consider any set of alternatives X such that $|X| \geq 3$, any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, where r is rich* in c. Suppose that, for every $B \in \mathcal{B}$, there exists a frame $f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D}$.
(a) There exists an asymmetric, acyclic relation $\succ$ and asymmetric relation ℓ such that $(\succ, \ell)$ r-rationalises* c if and only if c satisfies Framing contraction and r-Expansion*.
(b) If c satisfies Framing contraction and r-Expansion*, then $\hat{P}$ is asymmetric and acyclical, $\hat{L}$ is asymmetric, and $(\hat{P}, \hat{L})$ is the unique pair that r-rationalises* c.

Theorem 3B (*Model of framing effects with leaning: Transitive preferences*)
Consider any set of alternatives X such that $|X| \geq 3$, any resolution framing effect $r : \mathcal{F} \to \mathcal{R}$ and any choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$, where r is rich* in c. Suppose that, for any $f \in \mathcal{F}$ and any $B \in \mathcal{B}$ with $|B| \leq 3$, $(B, f) \in \mathcal{D}$.
(a) There exists an asymmetric, transitive relation $\succ$ and an asymmetric relation ℓ such that $(\succ, \ell)$ r-rationalises* c if and only if c satisfies Framing contraction, r-Expansion* and Framing weak IIA.
(b) If c satisfies Framing contraction, r-Expansion* and Framing weak IIA, then $\hat{P}$ is asymmetric and transitive, $\hat{L}$ is asymmetric, and $(\hat{P}, \hat{L})$ is the unique pair that r-rationalises* c.

Theorems 3A and 3B can be used to obtain axiomatisations for models of specific framing effects with leaning. For example, if we replace $r : \mathcal{F} \to \mathcal{R}$ in the two theorems by the endowment effect $e_X : \mathcal{F}^e \to \mathcal{R}$, we obtain axiomatisations for a model of the endowment effect with leaning.

Consider the choice function in Table 6. The endowment effect e_X is rich* in that choice function, so the theorems are applicable. We saw above that the choice function is e_X-rationalised* by the pair $(\succ_5, \ell_1)$. Clause (a) Theorem 3B then entails that the choice function satisfies Framing contraction, e_X-Expansion* and Framing weak IIA. It is easy to check that, indeed, the choice function satisfies these three properties. Moreover, clause (b) of Theorem 3B then says that $(\hat{P}, \hat{L})$ is the unique pair that e_X-rationalises the choice function; it can be verified that $\hat{P} = \succ_5$ and $\hat{L} = \ell_1$.

How does our model of the endowment effect compare with other models in the literature? One influential theory is the "loss aversion" account in Tversky and Kahneman (1991): if a DM is loss averse, then her endowment may influence her choice by affecting what she perceives as gains and losses. Another prominent theory is presented in Masatlioglu and Ok (2005) and Masatlioglu and Ok (2014): they argue that a DM's endowment may influence her choice if (1) she is subject to a psychological constraint regarding the set of alternatives that she considers; and (2) this constraint depends upon the endowment. Rival models have distinct implications for choice behaviour; accordingly, they can be evaluated on the basis of their predictive and explanatory power.[21] Our model of the endowment effect is, perhaps, most similar to the model of status quo bias in Masatlioglu and Ok (2005). But in their model, endowments influence choice by affecting a psychological constraint, whereas, in our model, endowments influence choice by "tipping the balance" when the DM is indecisive.

3.6 Framing effects and choice-based welfare economics

Responding to skeptical arguments that framing effects undermine choice-based welfare analysis, Bernheim and Rangel (2009) present an approach for backing out a revealed "individual welfare relation" from frame-dependent choices. One of their proposed welfare relations is the relation P^*, which they interpret as specifying strict improvements in individual welfare. In the framework of our paper, their proposed relation can be defined as follows:

Definition 15 (*Bernheim-Rangel relation P^**)

Consider any set of alternatives X and a choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$. Suppose

[21] See Masatlioglu and Uler (2013) for a comparison of the implications of (1) a constant loss aversion model (2) a more general loss aversion model (3) the status quo bias model in Masatlioglu and Ok (2005) and (4) the more general model in Masatlioglu and Ok (2014).

that, for any $B \in \mathcal{B}$, there is a frame $f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D}$. Then $P^* = \{(y, x) \in X^2 :$ for all $(B, f) \in \mathcal{D}$ such that $x, y \in B, x \notin c(B, f)\}$.

Intuitively, yP^*x obtains if x is never chosen when y is available. The authors suggest that P^* can be interpreted as an individual welfare relation in the sense that "it is possible to strictly improve upon a choice $x \in X$ if there exists $y \in X$ such that yP^*x."

The Bernheim and Rangel (2009) approach is ambitious in its generality. It is intended to apply regardless of the decision process that generates the frame-dependent choice function. For this reason, the literature describes their proposal as a "model-free" approach. A number of commentators have criticised their model-free approach, arguing that it has implausible implications, especially for cases of limited attention.[22] Such critics have concluded that a method for retrieving welfare relations from choice data should be "model-based" rather than model-free (Manzini and Mariotti (2014b)).

On the one hand, we agree that a method for obtaining welfare relations should be "model-based". On the other hand, we regard the goal of Bernheim and Rangel (2009) as laudable: as discussed in the introduction, generality is desirable in the fragmented discipline of behavioural decision theory. Our proposal is that, at least for *resolution* framing effects, the revealed preference relation $\hat{P}$ can serve as an individual welfare relation. While our approach is model-based, it is nevertheless relatively general: it applies to any resolution framing effect that satisfies the restrictions specified in Theorems 2A, 2B, 3A or 3B. Furthermore, our approach provides a partial vindication of the Bernheim-Rangel proposal – in particular, it vindicates the application of their approach to the class of resolution framing effects. In clause (b) of Theorems 2A and 3A, we could replace $\hat{P}$ by P^*. This is entailed by the following lemma:

[22] Manzini and Mariotti (2014b) and Rubinstein and Salant (2012) observe that the relation P^* is excessively coarse-grained. Moreover Masatlioglu et al. (2012) present a plausible example in which the Bernheim-Rangel relation has precisely the opposite implication for welfare to that which it should have. In their example, a DM with limited attention prefers an alternative x over y, but yP^*x. Bernheim and Rangel (2009) appear to have anticipated the challenges to their approach that arise in cases of limited attention. In Section 7 of their paper, they suggest that, for data points involving "imperfect information processing", the datapoints should be "deleted" from the choice data. But this response is unsatisfactory (see Manzini and Mariotti (2014b)); it is an *ad hoc* response to a broad class of choice anomalies.

Lemma 1 (*$\hat{P}$ and the Bernheim-Rangel relation P^**)
Consider any set of alternatives X and a choice function c with domain $\mathcal{D} \subset \mathcal{B} \times \mathcal{F}$. Suppose that, for any $B \in \mathcal{B}$, there is a frame $f \in \mathcal{F}$ such that $(B, f) \in \mathcal{D}$. If c satisfies Framing contraction, then $\hat{P} = P^*$.

Our revealed preference results also have parallels in the findings of Salant and Rubinstein (2008). In their paper, a preference relation is constructed from frame-dependent choices. Moreover, Salant and Rubinstein (2008, p. 1292) note that this constructed preference relation is "similar" to the Bernheim-Rangel relation P^*. It is also similar to our proposed revealed preference relation $\hat{P}$. They are, however, asking a somewhat different question to us, and so this constructed preference relation has a different interpretation to our revealed preference relation.[23] Moreover, as noted in Section 3.3, while our two-stage model accommodates frame-dependent choice functions that are not "multi-self" (for example, the choice function in Table 1), such choice functions violate the axiomatisation in Salant and Rubinstein (2008).

3.7 Framing effects and rationality

A number of commentators have claimed that framing effects are evidence of irrationality.[24] This claim is not merely abstract philosophising; it is presented as part of the case for policy intervention. In this section, we criticise this broad conclusion. At least for resolution framing effects, frame-dependent choices are consistent with rationality. What, precisely, are the grounds for deeming such framing effects irrational? After all, as we have shown, if a frame-dependent choice function is r-rationalisable, it can be interpreted as generated by a process involving a transitive or acyclic preference relation. When commentators judge framing effects to be irrational, they seem to have in mind the following kind of argument. Frame-dependent choices are irrational on the grounds that DMs are influenced by factors that are "arbitrary", factors that are irrelevant to the rational

[23] Salant and Rubinstein (2008) introduce the notion of "the neoclassical choice correspondence that is induced by a frame-dependent choice function", and then ask the question: what axioms must be satisfied by a frame-dependent choice function to ensure that the induced neoclassical choice correspondence is rationalisable by a preference relation? They construct a preference relation that rationalises the induced neoclassical choice correspondence when the axioms are satisfied. So this constructed preference relation has a different interpretation to the revealed preference relation $\hat{P}$ in our model.

[24] See Thaler and Sunstein (2003), Thaler and Sunstein (2008) and Ariely (2009).

assessment of the alternatives. Let us label such factors *non-rational*. For example, in the priming experiment, the primes are non-rational factors that influence choice. In summary: a principle underlying the view that framing-effects are irrational is that *a DM is irrational if non-rational factors influence her choice*. Let us call this the *irrationality principle*.

To evaluate the irrationality principle, we need to distinguish between two cases. In the first case, the DM strictly prefers x to y, but a non-rational factor causes her to choose y over x. In this case, at least potentially, the irrationality principle has some plausibility, because the fact that she prefers x to y gives her a reason to choose x over y. In the second case, the DM is indecisive between x and y, and a non-rational factor causes her to choose y over x. In this case, there are simply no grounds for deeming the DM to be irrational, because she did not have a reason to choose x rather than y. So the irrationality principle is not plausible when the DM is indecisive between alternatives. But, for the case of resolution framing effects, the non-rational factors – the frames – only influence the DM's choice among alternatives between which she is indecisive. After all, frames affect choice only in the second stage of the decision process and, necessarily, the DM is indecisive between the alternatives that survive the first stage. Accordingly, the irrationality principle is not plausible for resolution framing effects.

To highlight the implausibility of the irrationality principle, consider the following example. Suppose that the DM faces a choice between two slices of cheesecake – New York cheesecake and Chicago cheesecake – but she does not have a preference for one over the other. Nevertheless, she should pick one of them and picks the Chicago cheesecake. An observer may posit an explanation of what caused her to pick the Chicago cheesecake: perhaps it was on her left and she is left-handed; perhaps it was slightly closer; and so forth. But the fact that a non-rational factor influenced her choice surely does not render the DM irrational. After all, she should pick one of the two slices, and there were no rational grounds for selecting one over the other. So it was inevitable that her selection would resolved by a non-rational factor.

3.8 Conclusions and future research

Bernheim and Rangel (2009, p. 59) decide that "in the interests of greater precision" they will use the term "choice reversal" to refer to the phenomenon that is typically denoted in the literature as a “preference reversal”. Throughout our paper, we have also used the term “choice reversal” rather than “preference reversal”, because it is our contention that, contrary to the skeptical view, choice reversals need not be construed as evidence of changing preferences. An archetypal choice reversal is where, under one frame, x is the unique choice from a budget set comprising x and y, but under another frame, y is the unique choice from that budget set. The prevalence of this form of choice reversal has been a key source of evidence for skeptical conclusions. But, whereas the skeptic construes a choice reversal as evidence that a change of frame “reverses” the DM’s preference, we instead conclude that such a choice reversal may simply reveal that the DM is indecisive between the two alternatives. Whereas the skeptic concludes that preferences are “radically unstable”, “labile” and “constructed”, we regard such choice reversals as potentially consistent with a stable underlying preference relation, albeit characterised by indecisiveness. Moreover, if the DM’s choices satisfy the conditions specified in Theorem 2A, 2B, 3A or 3B, we are able to retrieve the DM’s preferences from her choices.

An attractive feature of our approach, we suggest, is its relative generality. It is applicable to a broad range of resolution framing effects, which include the salience effect, the endowment effect, the priming effect and the compromise effect. For each of these framing effects, axiomatisations and revealed preference relations can be obtained. Moreover, our study highlights the parallels between these diverse framing effects. The choice functions generated by these framing effects satisfy similar properties, and, in each case, the revealed preference relation is the framing base relation $\hat{P}$.

There are a number of potentially fruitful lines of future research. One question is whether other types of framing effects can be analysed using our models. For example, can the type of “anchoring effect” described in Ariely et al. (2003) be modelled as a resolution framing effect? Can asymmetric dominance be modelled as a resolution framing effect, using an approach similar to our analysis of the compromise effect? A second line of future research is an extension to a stochastic model. After all, in reality, choice data typically take the form of *frequencies* of choices. Can stochastic frame-dependent choice functions be modelled as generated by stochastic resolution framing effects?

Appendix: Proofs

Even though Theorems 1A and 1B are minor variants of results in the literature (see Moulin (1985)), we prove them here because they are specified in terms of Weak IIA rather than the standard IIA axiom. Moreover, the proofs of Theorems 2A and 3A are parallel to the proof of 1A and the proofs of 2B and 3B are parallel to the proof of 1B.

Theorem 1A

Proof of (a) ($\Leftarrow$): Suppose that C satisfies Contraction and Expansion. Clearly P is asymmetrical. It remains to show that it is acyclic and rationalises C.

Claim 1: P rationalises C.

$x \notin C(B) \iff \exists y \in B \backslash \{x\}$ such that $\{y\} = C(\{x, y\}) \iff \exists y \in B \backslash \{x\}$ such that yPx.

The first biconditional follows from Contraction and Expansion, the second from the definition of P.

Claim 2: P is acyclic.

Suppose, for contradiction, that there exists a cycle in P. So for some $x_1, x_2, \ldots, x_m \in X$, $x_1 P x_2 P \ldots P x_m P x_1$. As P rationalises C, $C(\{x_1, \ldots, x_m\}) = \max(\{x_1, \ldots, x_m\}, P)$ is empty.

($\Rightarrow$): Suppose that there exists an asymmetric, acyclic relation $\succ$ that rationalises C.

Claim 3: $\succ = P$.

For any $x, y \in X$, $xPy \iff \{x\} = C(\{x, y\}) \iff \{x\} = \max(\{x, y\}, \succ) \iff x \succ y$.

Claim 4: C satisfies Contraction and Expansion.

As $\succ = P$ and $\succ$ rationalises C, it follows that: $x \in C(B) \iff$ for no $y \in B \backslash \{x\}, yPx \iff$ for no $y \in B \backslash \{x\}, \{y\} = C(\{x, y\})$. That is, $x \in C(B) \iff$ for all $y \in B \backslash \{x\}, x \in C(\{x, y\})$. But this biconditional is equivalent to the conjunction of Contraction and Expansion.

Proof (b): This is proved in the course of proving (a).

Theorem 1B

Proof of (a) ($\Leftarrow$): Suppose that C satisfies Contraction, Expansion and Weak IIA.

Applying the reasoning in the proof of Claim 1 in Theorem 1A, it follows that P rationalises C. It remains to show that P is transitive. Consider any $x, y, z \in X$ such that xPy, yPz . In other words, suppose that $\{x\} = C(\{x, y\}), \{y\} = C(\{y, z\})$. By Contraction (i) as $z \notin C(\{y, z\})$, thus $z \notin C(\{x, y, z\})$ and (ii) as $y \notin C(\{x, y\})$, thus $y \notin C(\{x, y, z\})$. Therefore $C(\{x, y, z\}) = \{x\}$. So Weak IIA implies that $C(\{x, z\}) = \{x\}$. Thus xPz.

($\Rightarrow$): Suppose that there exists an asymmetric, transitive relation $\succ$ that rationalises C. Applying the reasoning in the proof in Theorem 1A, it follows that Contraction and Expansion are satisfied. It remains to show that Weak IIA is satisfied. Consider any $x, y, z \in X$, and suppose that $C(\{x, y, z\}) = \{x\}$. We want to show that $C(\{x, y\}) = \{x\}$. Suppose, for contradiction, $y \in C(\{x, y\})$. As $\succ$ rationalises C, $x \not\succ y$. So, as $y \notin C(\{x, y, z\})$, $z \succ y$. Given $z \succ y$, as $z \notin C(\{x, y, z\})$, $x \succ z$. In summary $x \succ z, z \succ y$ and $x \not\succ y$, which violates transitivity.

Proof of (b): This is proved in the course of proving (a).

Theorem 2A

Proof of (a) ($\Leftarrow$): Suppose that c satisfies Framing contraction and r-Expansion. Clearly $\hat{P}$ is asymmetric. It remains to show that it r-rationalises c and is acyclic.

Claim 1: $N_B = \max(B, \hat{P})$

This follows from the definitions of $\hat{P}$ and N_B.

Claim 2: $\hat{P}$ r-rationalises c. That is, for all $(B, f) \in \mathcal{D}, c(B, f) = \max(\max(B, \hat{P}), r(f))$.

Step (i): show $x \in c(B, f) \Rightarrow x \in \max(\max(B, \hat{P}), r(f))$.

Suppose that $x \in c(B, f)$. Then, by Framing contraction, for all $y \in B\backslash\{x\}$, there exists an $f' \in \mathcal{F}$ such that $x \in c(\{x, y\}, f')$. That is, for all $y \in B\backslash\{x\}$, not $y\hat{P}x$. So $x \in \max(B, \hat{P})$. As $x \in c(B, f)$, r-Expansion implies that, for all $y \in N_B\backslash\{x\}$, not $yr(f)x$. So by Claim 1, for all $y \in \max(B, \hat{P})$, not $yr(f)x$. So $x \in \max(\max(B, \hat{P}), r(f))$.

Step (ii): show $x \in \max(\max(B, \hat{P}), r(f)) \Rightarrow x \in c(B, f)$.

Suppose that $x \in \max(\max(B, \hat{P}), r(f))$. Therefore $x \in \max(B, \hat{P})$. So, for all $y \in B\backslash\{x\}$, not $y\hat{P}x$. Therefore, for all $y \in B\backslash\{x\}$, there exists an $f' \in \mathcal{F}$ such that $x \in c(\{x, y\}, f')$. Also, for all $y \in \max(B, \hat{P})$, not $yr(f)x$. So by Claim 1, for all $y \in N_B\backslash\{x\}$, not $yr(f)x$. So, by r-Expansion, $x \in c(B, f)$.

Claim 3: $\hat{P}$ is acyclic.

Suppose, for contradiction, that there exists a cycle in $\hat{P}$. That is, for some $x_1, x_2, \ldots, x_m \in X$, $x_1\hat{P}x_2\hat{P}\ldots\hat{P}x_m\hat{P}x_1$. Consider any $f \in \mathcal{F}$ such that $(\{\{x_1, \ldots, x_m\}, f) \in \mathcal{D}$. (Note that, given the assumption about the domain of c, such a frame exists.) Clearly $\max(\{x_1, \ldots, x_m\}, \hat{P})$ is empty. Therefore $\max(\max(\{x_1, \ldots, x_m\}, \hat{P}), r(f))$ is empty. So it follows from Claim 2 that $c(\{x_1, \ldots, x_m\}, f)$ is empty.

($\Rightarrow$): Suppose that there exists an asymmetric and acyclic relation $\succ$ that r-rationalises c. That is, for all $(B, f) \in \mathcal{D}, c(B, f) = \max(\max(B, \succ), r(f))$.

Claim 4: $\succ = \hat{P}$.

That is, for any $x, y \in X$ (a) $x \succ y \Rightarrow x\hat{P}y$; and (b) $x\hat{P}y \Rightarrow x \succ y$.

Proof of (a): Suppose that $x \succ y$. So, given that $\succ$ r-rationalises c, it follows that, for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}$, $c(\{x,y\},f) = \max(\max(\{x,y\},\succ),r(f)) = \max(\{x\},r(f)) = \{x\}$. So, by definition of $\hat{P}$, $x\hat{P}y$.

Proof of (b): Suppose that $x\hat{P}y$. Thus, for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}, c(\{x,y\},f) = \{x\}$. Given that $\succ$ r-rationalises c, thus, for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}, \max(\max(\{x,y\},\succ),r(f)) = \{x\}$. So $x \in \max(\{x,y\},\succ)$. Thus either $x \succ y$ or $xI_{\succ}y$. Suppose, for contradiction, that $xI_{\succ}y$. Then $\{x,y\} = \max(\{x,y\},\succ)$. So, given that for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}, \max(\max(\{x,y\},\succ),r(f)) = \{x\}$, thus, for for all $f \in \mathcal{F}$ such that $(\{x,y\},f) \in \mathcal{D}$, $xr(f)y$. This contradicts the assumption that r is rich in c. So $x \succ y$.

Claim 5: c satisfies Framing contraction.

Suppose that $(B,f) \in \mathcal{D}$, $x,y \in B$ and $x \in c(B,f)$. As $\succ$ r-rationalises c and $\succ = \hat{P}$ (from Claim 4), it follows that $x \in c(B,f) = \max(\max(B,\hat{P}),r(f))$. So, in particular, $x \in \max(B,\hat{P})$. Therefore not $y\hat{P}x$. Therefore, by definition of $\hat{P}$, there exists a $f' \in \mathcal{F}$ such that $x \in c(\{x,y\},f')$.

Claim 6: c satisfies r-Expansion.

Suppose that, for all $y \in B\backslash\{x\}$, there exists an $f \in \mathcal{F}$ such that $x \in c(\{x,y\},f)$. We want to show that $x \in c(B,f') \iff$ not $yr(f')x$ for $y \in N_B\backslash\{x\}$.

First, note that, for all $y \in B\backslash\{x\}$, not $y\hat{P}x$. That is, $x \in \max(B,\hat{P})$. Thus:

$x \in c(B,f') \iff x \in \max(\max(B,\succ),r(f')) \iff x \in \max(\max(B,\hat{P}),r(f'))$

$\iff$ not $yr(f')x$ for $y \in (\max(B,\hat{P})) \iff$ not $yr(f')x$ for $y \in N_B\backslash\{x\}$.

The first equivalence holds because $\succ$ r-rationalises c. The second equivalence holds because, as shown in Claim 4, $\succ = \hat{P}$. The third equivalence holds because, as noted above, $x \in \max(B,\hat{P})$. The fourth equivalence follows from Claim 1.

But then $x \in c(B,f') \iff$ not $yr(f')x$ for $y \in N_B\backslash\{x\}$, which is what we wanted to show.

Proof (b): This is proved in the course of proving (a).

Theorem 2B

Proof of (a) ($\Leftarrow$): Suppose that c satisfies Framing contraction, r-Expansion and Framing weak IIA. Applying the reasoning in the proof of Claim 2 in Theorem 2A, it follows that $\hat{P}$ r-rationalises c. It remains to show that $\hat{P}$ is transitive. Suppose that, for any $x,y,z \in X$, $x\hat{P}y$ and $y\hat{P}z$. That is, suppose that $\forall f \in \mathcal{F}, \{x\} = c(\{x,y\},f)$ and $\forall f \in \mathcal{F}, \{y\} = c(\{y,z\},f)$. We want to show that $\forall f \in \mathcal{F}, \{x\} = c(\{x,z\},f)$. But given that $\forall f \in \mathcal{F}, \{x\} = c(\{x,y\},f)$, the contrapositive of Framing contraction entails that $\forall f \in \mathcal{F}, y \notin c(\{x,y,z\},f)$. Moreover, given that $\forall f \in \mathcal{F}, \{y\} = c(\{y,z\},f)$, the contrapositive of Framing contraction also entails that

$\forall f \in \mathcal{F}, z \notin c(\{x,y,z\},f)$. So $\forall f \in \mathcal{F}, c(\{x,y,z\},f) = \{x\}$. So Framing weak IIA implies that $\forall f \in \mathcal{F}, c(\{x,z\},f) = \{x\}$.

($\Rightarrow$): Suppose that there exists an asymmetric, transitive relation $\succ$ that r-rationalises c. Applying the reasoning in the proof of Theorem 2A, it follows that c satisfies Framing contraction and r-Expansion. It remains to show that c satisfies Framing weak IIA. Consider any $x, y, z \in X$ such that, for all $f \in \mathcal{F}$, $c(\{x,y,z\},f) = \{x\}$. We want to show that for all $f \in \mathcal{F}$, $c(\{x,y\},f) = \{x\}$. Suppose, for contradiction, that there exists a frame $f' \in \mathcal{F}$ such that $y \in c(\{x,y\},f')$. But given that $\succ$ r-rationalises c, we have:

$$\forall f \in \mathcal{F}, \{x\} = c(\{x,y,z\},f) = \max(\max(\{x,y,z\},\succ),r(f)) \tag{3.1}$$

$$y \in c(\{x,y\},f') = \max(\max(\{x,y\},\succ),r(f')) \tag{3.2}$$

From (3.1), it follows that

$$y \not\succ x \tag{3.3}$$

$$z \not\succ x \tag{3.4}$$

From (3.2), it follows that

$$x \not\succ y \tag{3.5}$$

So from (3.3) and (3.5), we have

$$xI_{\succ}y \tag{3.6}$$

Moreover, given (3.2) and (3.6)

$$\neg xr(f')y \tag{3.7}$$

Given (3.4) and (3.6), there are six remaining cases: (1) $x \succ z$ and $z \succ y$ (2) $x \succ z$ and $y \succ z$ (3) $x \succ z$ and $yI_{\succ}z$ (4) $xI_{\succ}z$ and $y \succ z$ (5) $xI_{\succ}z$ and $z \succ y$ and (6) $xI_{\succ}z$ and $yI_{\succ}z$.

<u>Case 1</u>: By transitivity of $\succ$, $x \succ y$, which contradicts (3.6).

<u>Cases 2, 3, 4</u>: In these cases, (3.6) implies that $\max(\{x,y,z\},\succ) = \{x,y\}$. But given (3.7), $y \in \max(\max(\{x,y,z\},\succ),r(f'))$. But this contradicts (3.1).

<u>Case 5</u>: $xI_{\succ}z$ and $z \succ y$ together with (3.6) imply that $\max(\{x,y,z\},\succ) = \{x,z\}$. But (3.1) implies that $\forall f \in \mathcal{F}, z \notin \max(\max(\{x,y,z\},\succ),r(f))$. Given $\max(\{x,y,z\},\succ) = \{x,z\}$, it follows that $\forall f \in \mathcal{F}, z \notin \max(\{x,z\},r(f))$, which implies that, for all $f \in \mathcal{F}$, $xr(f)z$. This violates the assumption that r is rich in c.

<u>Case 6</u>: $xI_{\succ}z$ and $yI_{\succ}z$ together with (3.6) imply that $\max(\{x,y,z\},\succ) = \{x,y,z\}$. But (3.1) entails

that $y \notin \max(\max(\{x,y,z\},\succ),r(f'))$. For y to fail to be in $\max(\max(\{x,y,z\},\succ),r(f'))$, it must be that either $xr(f')y$ or $zr(f')y$. But from (3.7), not $xr(f')y$, so it must be that $zr(f')y$. But (3.1) also entails that $z \notin \max(\max(\{x,y,z\},\succ),r(f'))$. For z to fail to be in $\max(\max(\{x,y,z\},\succ),r(f'))$, it must be that either $xr(f')z$ or $yr(f')z$. But we have found that $zr(f')y$, thus not $yr(f')z$. So it must be the case that $xr(f')z$. But then by transitivity of $r(f')$, $zr(f')y$ and $xr(f')z$ entail that $xr(f')y$. But this contradicts (3.7).

Proof of (b): This is proved in the course of proving (a).

Theorem 3A

Proof (a) ($\Leftarrow$): Suppose that c satisfies Framing contraction and r-Expansion*. Clearly $\hat{P}$ and $\hat{L}$ are asymmetric. It remains to show that $(\hat{P},\hat{L})$ r-rationalises* c and $\hat{P}$ is acyclic.

Claim 1: The set $N_B = \max(B,\hat{P})$

This follows from the definitions of $\hat{P}$ and N_B.

Claim 2: For all $(B,f) \in \mathcal{D}, c(B,f) = \max(\max(\max(B,\hat{P}),r(f)),\hat{L})$.

Step (i): $x \in c(B,f) \Rightarrow x \in \max(\max(\max(B,\hat{P}),r(f)),\hat{L})$.

Suppose that $x \in c(B,f)$. Then, by Framing contraction, for all $y \in B\backslash\{x\}$, there exists an $f' \in \mathcal{F}$ such that $x \in c(\{x,y\},f')$. That is, for all $y \in B\backslash\{x\}$, not $y\hat{P}x$. So $x \in \max(B,\hat{P})$. As $x \in c(B,f)$, r-Expansion* implies that, for all $y \in N_B\backslash\{x\}$, not $yr(f)x$ and for all $y \in N_{B,f}\backslash\{x\}$ there is a frame $f'' \in \mathcal{F}$ that is neutral between x and y such that $x \in c(\{x,y\},f'')$. So by Claim 1, $\forall y \in \max(B,\hat{P})$, not $yr(f)x$. So $x \in \max(\max(B,\hat{P}),r(f))$. Also, given that for all $y \in N_{B,f}\backslash\{x\}$, there is a frame f'' neutral between x and y such that $x \in c(\{x,y\},f'')$, the definition of $N_{B,f}$ implies that $\forall y \in \max(\max(B,\hat{P}),r(f))\backslash\{x\}$ there is a frame f'' neutral between x and y such that $x \in c(\{x,y\},f'')$. So, by definition of $\hat{L}$, $\forall y \in \max(\max(B,\hat{P}),r(f))$, not $y\hat{L}x$. So $x \in \max(\max(\max(B,\hat{P}),r(f)),\hat{L})$.

Step (ii): $x \in \max(\max(\max(B,\hat{P}),r(f)),\hat{L}) \Rightarrow x \in c(B,f)$.

Suppose that $x \in \max(\max(\max(B,\hat{P}),r(f)),\hat{L})$. Thus $x \in \max(B,\hat{P})$. So $\forall y \in B\backslash\{x\}$, not $y\hat{P}x$. Thus, $\forall y \in B\backslash\{x\}$, there exists an $f' \in \mathcal{F}$ such that $x \in c(\{x,y\},f')$. Also, $\forall y \in \max(B,\hat{P})\backslash\{x\}$, not $yr(f)x$ and $\forall y \in \max(\max(B,\hat{P}),r(f))\backslash\{x\}$, not $y\hat{L}x$. Given the former conjunct and claim 1, $\forall y \in N_B\backslash\{x\}$, not $yr(f)x$ Given the latter conjunct, the definitions of $\hat{L}$ and $N_{B,f}$ imply that $\forall y \in N_{B,f}\backslash\{x\}$, there is a frame $f'' \in \mathcal{F}$ that is neutral between x and y such that $x \in c(\{x,y\},f'')$. So by r-Expansion*, $x \in c(B,f)$.

Claim 3: $(\hat{P},\hat{L})$ r-rationalises* c.

Given Claim 2, it remains to show that the second clause in the definition of r-rationalisation* is satisfied. The second clause specifies that $(\succ, \ell)$ r-rationalises* c only if: if $x\ell y$, then $xI_{\succ}y$. So we need to show that if $x\hat{L}y$, then $xI_{\hat{P}}y$. But this follows immediately from the definition of $\hat{L}$.

Claim 4: $\hat{P}$ is acyclic.

Suppose, for contradiction, that there exists a cycle in $\hat{P}$. So for some $x_1, x_2, \dots, x_m \in X$, $x_1\hat{P}x_2\hat{P}\dots\hat{P}x_m\hat{P}x_1$. Thus $\max(\{x_1, \dots, x_m\}, \hat{P})$ is empty. So for all $f \in \mathcal{F}$, $\max(\max(\max(\{x_1, \dots, x_m\}, \hat{P}), r(f)), \hat{L})$ is empty. But it is assumed that there exists an $f' \in \mathcal{F}$ such that $(\{x_1, \dots, x_m\}, f') \in \mathcal{D}$. So Claim 3 implies that $c(\{x_1, \dots, x_m\}, f')$ is empty. Contradiction.

($\Rightarrow$): Suppose that there exists an asymmetric, acyclic relation $\succ$ and an asymmetric relation ℓ such that $(\succ, \ell)$ r-rationalises* c. That is, for all $(B, f) \in \mathcal{D}, c(B, f) = \max(\max(\max(B, \succ), r(f)), \ell)$.

Claim 5: $\succ = \hat{P}$

That is, for any $x, y \in X$ (a) $x \succ y \Rightarrow x\hat{P}y$; and (b) $x\hat{P}y \Rightarrow x \succ y$.

Proof of (a): Suppose that $x \succ y$. So, given that $(\succ, \ell)$ r-rationalises* c, it follows that, for all $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}$, $c(\{x, y\}, f) = \max(\max(\max(\{x, y\}, \succ), r(f)), \ell) = \max(\max(\{x\}, r(f)), \ell) = \{x\}$. So, by definition of $\hat{P}$, $x\hat{P}y$.

Proof of (b): Suppose that $x\hat{P}y$. It follows that, for all $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}, c(\{x, y\}, f) = \{x\}$. Given that $(\succ, \ell)$ r-rationalises c, thus, for all $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}, \max(\max(\max(\{x, y\}, \succ), r(f)), \ell) = \{x\}$. Therefore $x \in \max(\{x, y\}, \succ)$. Thus either $x \succ y$ or $xI_{\succ}y$. Suppose, for contradiction, that $xI_{\succ}y$. Then $\{x, y\} = \max(\{x, y\}, \succ)$. But, as r is rich* in c, there exists a frame $f' \in \mathcal{F}$ such that $(\{x, y\}, f') \in \mathcal{D}$ and $yr(f')x$. So $x \notin \max(\max(\{x, y\}, \succ), r(f'))$. Contradiction.

Claim 6: $\ell = \hat{L}$

That is, for any $x, y \in X$ (a) $x\ell y \Rightarrow x\hat{L}y$ and (b) $x\hat{L}y \Rightarrow x\ell y$.

Proof of (a): Suppose $x\ell y$. As r is rich* in c, there is at least one frame f that satisfies the following property: $(\{x, y\}, f) \in \mathcal{D}$ and f is neutral between x and y. Let f' denote an arbitrary frame with this property. As $(\succ, \ell)$ r-rationalises*, we know, in particular, that $c(\{x, y\}, f') = \max(\max(\max(\{x, y\}, \succ), r(f')), \ell)$. But from clause (2) in the definition of r-rationalisation*, we have $xI_{\succ}y$. So $c(\{x, y\}, f') = \max(\max(\{x, y\}, r(f')), \ell)$. Moreover, as f' is neutral between x and y, thus $c(\{x, y\}, f') = \max(\{x, y\}, \ell)$. So as $x\ell y$, thus $c(\{x, y\}, f') = \{x\}$. Moreover, from Claim 5, $\succ = \hat{P}$, so $xI_{\hat{P}}y$. So by definition of $\hat{L}$, $x\hat{L}y$.

Proof of (b): Suppose that $x\hat{L}y$. As r is rich* in c, there is at least one frame f that satisfies the following property: $(\{x, y\}, f) \in \mathcal{D}$ and f is neutral between x and y. Let f' denote an arbitrary frame with this property. So by definition of $\hat{L}$, $xI_{\hat{P}}y$ and $c(\{x, y\}, f') =$

$\{x\}$. But as $(\succ, \ell)$ r-rationalises* c, we have $c(\{x,y\}, f') = \max(\max(\max(\{x,y\}, \succ), r(f')), \ell)$. From Claim 5, $c(\{x,y\}, f') = \max(\max(\max(\{x,y\}, \hat{P}), r(f')), \ell)$. But as $xI_{\hat{P}}y$, $c(\{x,y\}, f') = \max(\max(\{x,y\}, r(f')), \ell)$. Moreover as f' is neutral between x and y, thus $c(\{x,y\}, f') = \max(\{x,y\}, \ell)$. But given $c(\{x,y\}, f') = \{x\}$, it follows that $x\ell y$.

Claim 7: c satisfies Framing contraction.

Consider any $(B, f) \in \mathcal{D}$ and any $x, y \in B$. Suppose that $x \in c(B, f)$. As $(\succ, \ell)$ r-rationalises* c and $\succ = \hat{P}$, thus $x \in c(B, f) = \max(\max(\max(B, \hat{P}), r(f)), \ell)$. So not $y\hat{P}x$. Thus, by definition of $\hat{P}$, there is an $f' \in \mathcal{F}$ such that $(\{x,y\}, f') \in \mathcal{D}$ and $x \in c(\{x,y\}, f')$.

Claim 8: c satisfies r-Expansion*.

Consider any $(B, f') \in \mathcal{D}$. Suppose that, for all $y \in B\backslash\{x\}$, there exists an $f \in \mathcal{F}$ such that $x \in c(\{x,y\}, f)$. We want to show that $x \in c(B, f') \iff$ (i) not $yr(f')x$ for $y \in N_B\backslash\{x\}$ and (ii) for $y \in N_{B,f'}\backslash\{x\}$ there exists a frame $f'' \in \mathcal{F}$ such that f'' is neutral between x and y and $x \in c(\{x,y\}, f'')$. First, note that, for all $y \in B\backslash\{x\}$, not $y\hat{P}x$. That is, $x \in \max(B, \hat{P})$. Thus $x \in c(B, f') \iff x \in \max(\max(\max(B, \succ), r(f')), \ell) \iff x \in \max(\max(\max(B, \hat{P}), r(f')), \hat{L})$ $\iff$ (i) not $yr(f')x$ for $y \in \max(B, \hat{P})\backslash\{x\}$ and (ii) for all $y \in \max(\max(B, \hat{P}), r(f'))\backslash\{x\}$, there is a frame $f'' \in \mathcal{F}$ such that f'' is neutral between x and y and $x \in c(\{x,y\}, f'') \iff$ (i) not $yr(f')x$ for $y \in N_B\backslash\{x\}$ and (ii) for all $y \in N_{B,f'}\backslash\{x\}$, there is a frame $f'' \in \mathcal{F}$ such that f'' is neutral between x and y and $x \in c(\{x,y\}, f'')$.

The first equivalence holds because $(\succ, \ell)$ r-rationalises* c. The second equivalence holds because, from Claims 5 and 6, $\succ = \hat{P}$ and $\ell = \hat{L}$. The third equivalence holds because $x \in \max(B, \hat{P})$, together with the definition of $\hat{L}$. The fourth equivalence follows from the definitions of N_B and $N_{B,f}$.

Proof (b): Clause (b) is proven in the course of the proof of (a).

Theorem 3B

Proof (a) ($\Leftarrow$): Suppose that c satisfies Framing contraction, r-Expansion* and Framing weak IIA. Applying the reasoning in the proof of Theorem 3A, it follows that $(\hat{P}, \hat{L})$ r-rationalises* c. It remains to show that $\hat{P}$ is transitive. Suppose, for any $x, y, z \in X$, $x\hat{P}y$ and $y\hat{P}z$. That is, suppose $\forall f \in \mathcal{F}, \{x\} = c(\{x,y\}, f)$ and $\forall f \in \mathcal{F}, \{y\} = c(\{y,z\}, f)$. We want to show that $\forall f \in \mathcal{F}, \{x\} = c(\{x,z\}, f)$, which would entail that $x\hat{P}z$. But given $\forall f \in \mathcal{F}, \{x\} = c(\{x,y\}, f)$, the contrapositive of Framing contraction entails $\forall f \in \mathcal{F}, y \notin c(\{x,y,z\}, f)$. Moreover, given that $\forall f \in \mathcal{F}, \{y\} = c(\{y,z\}, f)$, the contrapositive of Framing contraction entails that $\forall f \in \mathcal{F}, z \notin c(\{x,y,z\}, f)$. So $\forall f \in \mathcal{F}, c(\{x,y,z\}, f) = \{x\}$. So Framing weak IIA implies that

$\forall f \in \mathcal{F}, c(\{x,z\},f) = \{x\}$.

($\Rightarrow$): Suppose that there exists an asymmetric, transitive relation $\succ$ and asymmetric relation ℓ such that $(\succ, \ell)$ r-rationalises* c. That is, for all $(B,f) \in \mathcal{D}$, $c(B,f) = \max(\max(\max(B,\succ), r(f)), \ell)$. Applying the reasoning in the proof of Theorem 3A, it follows that c satisfies Framing contraction, and r-Expansion*. It remains to show that c satisfies Framing weak IIA. Consider any $x,y,z \in X$ such that, for all $f \in \mathcal{F}$, $c(\{x,y,z\},f) = \{x\}$. We want to show that for all $f \in \mathcal{F}$, $c(\{x,y\},f) = \{x\}$. Suppose, for contradiction, that there exists a frame $f' \in \mathcal{F}$ such that $y \in c(\{x,y\},f')$. But given that $(\succ, \ell)$ r-rationalises* c, we have:

$$\forall f \in \mathcal{F}, \{x\} = c(\{x,y,z\},f) = \max(\max(\max(\{x,y,z\},\succ), r(f)), \ell) \tag{3.8}$$

$$y \in c(\{x,y\},f') = \max(\max(\max(\{x,y\},\succ), r(f')), \ell) \tag{3.9}$$

From (3.8) it follows that

$$y \not\succ x \tag{3.10}$$

$$z \not\succ x \tag{3.11}$$

From (3.9) it follows that

$$x \not\succ y \tag{3.12}$$

So from (3.10) and (3.12), we have

$$x I_{\succ} y \tag{3.13}$$

But (3.8) and (3.13) imply

$$z \succ y \tag{3.14}$$

To see why (3.14) must hold, suppose, for contradiction $z \not\succ y$. From (3.13), $y \in \max(\{x,y,z\},\succ)$. As r is rich* in c, there is a frame $f'' \in \mathcal{F}$ such that $yr(f'')x$. So $x \notin \max(\max(\{x,y,z\},\succ), r(f''))$. But this contradicts (3.8). But (3.14) and (3.13) then imply

$$x \not\succ z \tag{3.15}$$

To see why (3.15) holds, suppose, for contradiction $x \succ z$. Then (3.14), given the assumed transitivity of $\succ$, entails $x \succ y$, which contradicts (3.13). But (3.15) and (3.11) yield

$$x I_{\succ} z \tag{3.16}$$

(3.16) and (3.14) then entail that $z \in \max(\{x,y,z\},\succ)$. But as r is rich* in c, there exists a frame $f'' \in \mathcal{F}$ such that $zr(f'')x$. So $x \notin \max(\max(\{x,y,z\},\succ), r(f''))$. But this contradicts (3.8).

Proof (b): This is proved in the course of proving (a).

Lemma 1

Suppose that c satisfies Framing contraction. We want to show that, for any $x, y \in X$ (a) $xP^*y \Rightarrow x\hat{P}y$ and (b) $x\hat{P}y \Rightarrow xP^*y$.

Proof of (a): Suppose that xP^*y. Then, by definition of P^*, for all $(B, f) \in \mathcal{D}$ such that $x, y \in B, y \notin c(B, f)\}$. But this holds, in particular, for $B = \{x, y\}$. So, by definition of $\hat{P}$, $x\hat{P}y$.

Proof of (b): Suppose that $x\hat{P}y$. So for all $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}$, $c(\{x, y\}, f) = \{x\}$. Thus for all $f \in \mathcal{F}$ such that $(\{x, y\}, f) \in \mathcal{D}$, $y \notin c(\{x, y\}, f)$. So, by the contrapositive of Framing contraction, for any $(B, f) \in \mathcal{D}$ such that $x, y \in B$, $y \notin c(B, f)$. So xP^*y.

Chapter 4

Structural econometric models of ordered online search

4.1 Introduction

In recent years, online retail sales have been expanding rapidly, with e-commerce accounting for approximately 8 per cent of US retail trade sales in 2016, which is about double the ratio seen in 2009.[1] Accordingly, in both economics and marketing, researchers are showing an increasing interest in modelling purchases made by consumers in ordered online search. We say that search is "ordered" if search alternatives are presented to the decision maker (DM) in a particular order, which occurs, for example, on the websites amazon.com and expedia.com. We assume that DMs search in the order in which the alternatives are presented. A stylised fact about ordered online search is that, for alternatives with a lower position in the search results, DMs have a lower propensity to purchase. We will refer to this stylised fact as the "falling purchase rate". Plausibly, the falling purchase rate is partly explained by the operation of ranking algorithms – the algorithms that determine the position of alternatives in the search results. Online retailers design ranking algorithms to ensure that more "relevant" alternatives are assigned a higher position in the search results. In our paper, however, we use a dataset where the positions of alternatives in the search were generated randomly – it is a dataset of searches for hotels on Expedia.[2] In our dataset, despite the random rankings, lower ranked alternatives have substantially lower purchase rates. So there is a need to consider other candidate explanations for the falling purchase rate. Our paper estimates

[1] See United States Census Bureau (2018), Table 4.

[2] This dataset was first used in academic research by Ursu (2018).

a structural econometric model to explain the fall in the purchase rate.

A natural way of modelling ordered online search, we suggest, is to adapt the labour search model developed in McCall (1970). In McCall's model, the DM makes the following trade-off: if she continues the search then, on the one hand, there is a possibility of obtaining a superior outcome, but, on the other hand, searching is costly. The optimal strategy for an unemployed worker is to continue searching until she receives a wage in excess of a certain reservation wage. When this model is adapted to ordered online search, the optimal policy of a consumer is to continue searching until she encounters an alternative yielding a utility in excess of a threshold reservation utility.

We develop a version of the McCall model in which DMs have "declining expectations". That is, for search alternatives that the DM has not yet browsed, she expects lower ranked alternatives to be worse. Even though the DM's belief is incorrect in our dataset (because rankings are random), it is plausible that a DM would have such a belief on the basis of her previous experience of searching online. Section 4.2 presents a theoretical result entailing that, for a DM with declining expectations, her threshold utility – the minimum utility required to make a purchase – is lower at lower positions in the search. Intuitively, at lower positions in the search, the value of continuing the search is less, because the value of the remaining alternatives is believed to be relatively poor. We refer to models with this property as "declining expectations models".

Our model also includes an unobserved outside option, which is understood to be the option of either abandoning the search, or shifting to a different search on another platform. The inclusion of an outside option, together with the assumption of declining expectations, is the principal driver of the falling purchase rate in our model. At early positions in the search, when the value of continuing the search is high, relatively few DMs will depart the search for the outside option. However, as the search progresses, the value of continuing falls, ensuring that many DMs depart the search part-way through to take the outside option. So a considerable fraction of DMs never reach later positions in the search, giving rise to a relatively low purchase rate later in the search. Section 4.5 of the paper shows that our model generates a substantial fall in the purchase rate, which is of a magnitude similar to that observed in the data.

In the literature, there are broadly two kinds of structural econometric models of ordered online choice: (i) those based on McCall's theory; and (ii) those based on the model of search in Weitzman (1979).[3] Other than our paper, as far as we are aware, Chan and Park (2015) is the only paper that presents a structural econometric model of ordered online search that is based on McCall's theory of search. Our model has two principal advantages over that in Chan and Park (2015). First, the theoretical foundation for Chan and Park (2015) is a version of the McCall model in which the DM's beliefs about the probabilities of the value of unbrowsed alternatives do not change with position. We refer to this type of theory as a "constant expectations model", which is to be contrasted with a "declining expectations model". Before estimating our declining expectations model, we first estimate a constant expectations model. The fundamental problem with this type of model is revealed by a comparison of Figures 4.2 and 4.3 in Sections 4.4 and 4.5: whereas the declining expectations model yields a substantial fall in the purchase rate, which is similar to the magnitude observed in the data, the constant expectations model generates only a very slight fall in the purchase rate. The reason why the pair of models generate such different results is as follows. In our declining expectations model, there are two drivers of the falling purchase rate. Low-rank alternatives are less likely to be purchased, because:

(A) a DM may leave the search before reaching low-rank alternatives if she purchases an alternative;

(B) a DM may leave the search before reaching low-rank alternatives if she takes the outside option.

By way of contrast, in the constant expectations model, (A) is the only driver the falling purchase rate. But typically for ordered online search – and certainly for our dataset – driver (A) only generates a very slight fall in the purchase rate. So if the actual fall in the purchase rate is substantial, constant expectations models provide a poor explanation of the falling purchase rate. We regard this as a major problem with assuming constant expectations models in McCall models of ordered online choice. After all, a central task of any model of ordered online search is to explain the falling purchase rate.

The second advantage of our model over Chan and Park (2015) is that ours is a *fully* structural model. In contrast, when estimating the McCall model, Chan and Park (2015) introduce a "reduced form" specification of the utility threshold that does not explicitly depend upon an information cost. So the empirical implementation of their model does not incorporate the relationship – implied by

[3] De los Santos et al. (2012) estimates a structural econometric model of online search that fits into neither of these categories, but it is not a model of *ordered* online search. There is also a considerable literature on *reduced-form* econometric models of ordered online choice – see, for example, Ghose et al. (2014).

McCall's theory of search – between the threshold, the information cost and the believed distribution of the product attributes. In contrast, we estimate a value for the information cost, as well as all of the other deep structural parameters.

Over the past decade, a number of papers have presented structural econometric models of ordered online search that are based not on McCall's theory of search but rather on the theory presented by Weitzman (1979). In Weitzman's model, the DM is assumed, before beginning the search, to have a belief, for each alternative, regarding (i) the distribution of the value of that alternative and (ii) the cost of discovering the actual value of that alternative. Weitzman shows that the optimal strategy is to assign a reservation price to each alternative, which determines the order of the search: the DM first searches the alternative with the highest reservation price, and moves downwards. Search is terminated when the actual value of an alternative exceeds its reservation price. The Weitzman model is the basis of the structural models of ordered online choice in Kim et al. (2010), Kim et al. (2017), Chen and Yao (2017), Ghose et al. (2019) and Ursu (2018).

While, in general, Weitzman's theory is a useful model of the process of information gathering, there are a number of concerns with applying it to model the type of *ordered* online search that takes place on, say, amazon.com or expedia.com. First, in the Weitzman model, it is assumed that, in advance of the search, the consumer has knowledge, for each alternative, of a distribution of values *specific to that alternative*. So, for example, when Ursu (2018) applies the Weitzman model to online search for hotels on expedia.com, she is assuming, in effect, that the consumer knows, in advance of the search, for each hotel, the attributes (price, star rating and so forth) that appear on the search results page. We suggest, however, that it is inappropriate to attribute this knowledge to the consumer *in advance of the search*. In our model, the consumer gains such information in the course of her search; moreover, in the majority of searches, the DM departs the search part-way through and thus, for alternatives that are sufficiently low in the search results, the DM never obtains this information. A second concern arises from the fact that, in the Weitzman model, the order of the search is endogenous. It was not designed to model *ordered* online search (such as searches on amazon.com or expedia.com) where, arguably, the task of the model is to explain how the propensity to purchase is affected by the *given* search order. We suggest, therefore, that McCall models of search – such as ours and the model in Chan and Park (2015) – are, perhaps, more appropriate for modelling *ordered* online search, in comparison with the model of Weitzman.[4]

Whereas, in a number of respects, our declining expectations model introduces complexities that are not present Chan and Park (2015) and the Weitzman models discussed above, in one important respect our model is simpler. While we model the decision to purchase as optimising behaviour, we do not model "clicking" as optimising behaviour. Our model can be interpreted as making the simplifying assumption that most of the information relevant to choice is present on the search results page; little additional information is obtained by clicking on a hotel to obtain more photos, reviews and so forth. We adopt this simplification because (i) in other respects, our model is relatively complicated and (ii) as discussed in Section 4.5.1, an attempt to provide a single optimising model of both clicking and purchasing behaviour gives rise to a risk of a certain type of specification error.[5] That said, one possible extension of our declining expectations model is to incorporate an optimising model of clicking behaviour.

Our paper is structured as follows. Section 4.2 presents two theorems describing the optimal strategy in the constant expectations and declining expectations models. Section 4.3 describes the Expedia dataset. Then Sections 4.4 and 4.5 present estimates, respectively, of the constant expectations and declining expectations models.

4.2 Two theoretical models of falling purchase rates

4.2.1 Constant expectations model

In our paper, search is modelled as an optimal stopping problem. After the DM initiates a search, she is presented with search results in the form of a list. Initially, the DM browses information relating to the first item that appears on the list. Having browsed it, the DM is said to have "considered" this alternative; this alternative is now the unique item in her "consideration set". Having

[4]This is not a criticism of the application of the Weitzman model to searches which are not *ordered* search, such as the application to the auto insurance industry by Honka and Chintagunta (2017). Note also that, whereas Ursu (2018, p.543) defends her use of the Weitzman theory to model ordered online search, she acknowledges the attractions of using a McCall model, such as that of Chan and Park (2015).

[5]While the model in Chan and Park (2015) incorporates both optimising clicking and purchasing behaviour, their dataset does not include data on purchases but only on clicks. So they cannot assess the fit of the model in relation to actual purchasing decisions.

considered the alternative, the DM faces the choice of whether or not to purchase. If she purchases the first alternative, the search stops then. If she does not, she incurs an information cost of $\chi > 0$ in order to consider the second alternative. In this case, the DM's consideration set now comprises both the first and second alternatives, and she faces the choice of whether to purchase one of these two alternatives. And so forth. For simplicity, it is assumed that the search list includes an infinite number of alternatives.

If the DM purchases the alternative in position $i = 1, 2, \ldots$, she obtains a utility of U^i. In advance of incurring the information cost of χ for considering an alternative, the DM does not know the value of the utility yielded by the alternative, but she knows that the utilities of the various alternatives are generated by independent and identical random variables $\mathbf{U}^i, i = 1, 2, \ldots$, with a distribution F. As the random variables are identical, we refer to this model as the "constant expectations model" – the DM's expectation of the utility is the same for each position $i = 1, 2, \ldots$. The decision problem is precisely analogous to a model of labour search that was developed in McCall (1970). Accordingly, the value function for the optimal stopping problem has the same form as the value function in McCall's infinite horizon model without discounting. If, for $i = 1, 2, \ldots$, we define $u^i := \max\{U^1, \ldots, U^i\}$, we can write the value function as follows:

$$v_i(u^i) = \max_{\text{stop, continue}} \{u^i, -\chi + \mathbb{E}[v_{i+1}(\max\{u^i, \mathbf{U}^{i+1}\})]\}, \quad i = 1, 2, \ldots \tag{4.1}$$

That is, the DM stops after browsing alternative i if and only if the maximum utility she has encountered up to and including position i, u^i, exceeds the continuation value at position i, which is given by $c_i(u^i) = -\chi + \mathbb{E}[v_{i+1}(\max\{u^i, \mathbf{U}^{i+1}\})]$. In McCall's model of job search, the solution to the optimal stopping problem is that an unemployed worker continues to search until she receives a wage in excess of a certain reservation wage. Similarly, in the constant expectations model of consumer search, the optimal strategy is that the DM continues to search until she obtains a utility in excess of a certain threshold $\mathcal{T}$, which can be referred to as her reservation utility. The following theorem formalises this well-known result: it says that the condition that the DM's utility is in excess of $\mathcal{T}$ is equivalent to the condition that the DM's utility is in excess of the value of continuing.

Theorem 1 (*Solution to constant expectations model*)
Suppose that the DM believes that her utilities at positions $i = 1, 2, \ldots$ are generated by independent and identical variables $\mathbf{U}^1, \mathbf{U}^2, \ldots$, with a distribution function F that is strictly increasing

and continuously differentiable, and where $-\infty < \mathbb{E}(\mathbf{U}^i) < \infty$ for $i = 1, 2, \ldots$. Then there exists a unique $\mathcal{T} \in \mathbb{R}$ such that for $i = 1, 2, \ldots$

$$u^i \geq \mathcal{T} \iff u^i \geq c_i(u^i)$$

where $\mathcal{T}$ satisfies

$$\chi = \int_{\mathcal{T}}^{\infty} (U - \mathcal{T}) \mathrm{d}F \tag{4.2}$$

In this model, the DM faces the following tradeoff: if she continues the search then, on the one hand, there is a possibility of obtaining a superior outcome, but, on the other hand, searching is costly. Equation (4.2) says that the threshold $\mathcal{T}$ will be such that the marginal cost of searching, χ, equals the marginal benefit.

Note that the DM's decision rule in the constant expectations model bears some similarity to Herbert Simon's account of satisficing behaviour. According to Simon, a DM is said to satisfice if her criterion for selecting an alternative is that it yields utility in excess of a certain threshold, which Simon terms her "aspiration level".[6] Whereas Simon presents satisficing behaviour in opposition to maximising behaviour, the constant expectations model implies that such a decision rule is consistent with optimisation. That is, it is consistent with optimisation that has regard not only to the utilities of the alternatives but also to search costs.

4.2.2 Declining expectations model

As is established in Section 4.4, the constant expectations model provides a poor fit for our data – in particular, it generates only a slight fall in the purchase rate, which is at odds with the substantial falls seen in the data. The central problem with the model is that the only way the DM can end the search is by purchasing. That is, the only reason why the propensity to purchase is high for alternatives that are high in the search ordering is that, at such a position, it is less likely that someone will have purchased a more highly ranked alternative. But given that, in our dataset, only about 5 percent of searches have a purchase on the first page of search results, it follows that, in the constant expectations model, the position of an alternative should only have a small effect on observed purchasing rates. But this is at odds with the data, in which the purchase rate falls substantially.

[6] See, for example Simon (1959).

While this is the principal concern with the constant expectations model, a second concern is that model imposes an implausible restriction that DMs never "backtrack" – that is, DMs do not return to purchase an earlier alternative after browsing a later alternative. Arguably, this feature of the model is intuitively unappealing, and, moreover, there is at least some evidence against it. In our dataset, of the searches with purchases that also include clicks at other positions, clicks fall after purchases about as frequently as they occur before purchases. A third concern with the constant expectations models is that it is implausible given the DMs' past experience with online search. Typically, for online searches, the search engine ranks alternatives according to some criterion of relevance. While this does not occur in our dataset, for which the rankings are random, DMs presumably would expect, on the basis of evidence from previous online searches, that the utility of alternatives falls with position.

In response to these three problems with the constant expectations model, we develop a "declining expectations model", in which the DM believes that, for alternatives that have not yet been browsed, later alternatives will tend to be worse than earlier alternatives. In contrast to the constant expectations model, the declining expectations model assumes that the search is of finite length – there are M alternatives that are searched. Furthermore, the DM has available an outside option and she knows its utility throughout the search. (Henceforth, the term "alternative" will be used to refer only to the DM's M options other than the outside option). Accordingly, we introduce a "position 0", at which the DM undertakes a decision in advance of the search. Let U^0 denote the utility from the outside option, which the DM knows. The DM commences the search if and only if the expected value of searching exceeds U^0. As the DM's expectations are declining, she does not believe that utilities from the M alternatives are generated by identical random variables, but she does believe that the random variables are independent. Suppose that the DM believes that, for positions $i = 1, \ldots, M$, her utility from the alternative at that position, U^i, is generated by a random variable $\mathbf{U}^i$ with a distribution F_i. In order to capture the idea that the DM expects utilities of earlier alternatives to be be better than utilities of later alternatives, we assume that there is a vector of $M-1$ positive real numbers $(\delta_1, \ldots, \delta_{M-1}) \in \mathbb{R}^{M-1}_{++}$ such that, for any $i = 1, \ldots, M-1$ and for any $U \in \mathbb{R}$, $F_i(U + \delta_i) = F_{i+1}(U)$.[7] The assumption ensures that, in advance of browsing

[7]Intuitively, the vector of utilities $(U^1, \ldots, U^M)$ can be thought of as generated by the following process. From the distribution F_M, obtain M independent draws $(D^1, ..., D^M)$. Set $U^M = D^M$ and for $i = 1, ..., M-1$, set $U^i = D^i + \sum_{j=i}^{M-1} \delta_j$.

position i, the expected utility of alternative i is δ_i more than the expected utility of the subsequent alternative $i+1$. That is, $\mathbb{E}(\mathbf{U}^i) - \mathbb{E}(\mathbf{U}^{i+1}) = \delta_i$.

After the DM browses position i, the utility level that governs the DM's decision of whether to stop is the maximum of the utilities that the DM has observed, which is denoted $u^i := \max\{U^0, U^1, \ldots, U^i\}$ for $i = 0, 1, \ldots, M$. Accordingly, the value function for the optimal stopping problem can be defined recursively as follows:

$$v_M(u^M) = u^M \tag{4.3}$$

$$v_i(u^i) = \max_{\text{stop,continue}} \{u^i, -\chi + \mathbb{E}[v_{i+1}(\max\{u^i, \mathbf{U}^{i+1}\})]\}, \quad i = 0, 1, , \ldots, M-1 \tag{4.4}$$

where the continuation value at position i is $c_i(u^i) = -\chi + \mathbb{E}[v_{i+1}(\max\{u^i, \mathbf{U}^{i+1}\})]$. Whereas, in the constant expectations model, there is a single threshold $\mathcal{T}$ that defines the optimal policy, in the declining expectations model, the policy is defined by a vector of thresholds $(\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^{M-1})$. The following theorem summarises the solution to the optimum stopping problem.

Theorem 2 (*Solution to declining expectations model*)[8]

Suppose that, for $i = 1, \ldots, M$, the DM believes that her utility at position i is generated by a random variable $\mathbf{U}^i$ with distribution function F_i that is strictly increasing and continuously differentiable and $-\infty < \mathbb{E}(\mathbf{U}^i) < \infty$. Suppose that $\mathbf{U}^1, \ldots, \mathbf{U}^M$ are independent and, for some $(\delta_1, \ldots, \delta_{M-1}) \in \mathbb{R}^{M-1}_{++}$, for $i = 1, \ldots, M-1$, for any $U \in \mathbb{R}$, $F_i(U + \delta_i) = F_{i+1}(U)$. Then there exists a unique vector $(\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^{M-1}) \in \mathbb{R}^M$ such that, for $i = 0, \ldots, M-1$:

$$u^i \geq \mathcal{T}^i \iff u^i \geq c_i(u^i)$$

where $\mathcal{T}^{M-1}$ satisfies

$$\chi = \int_{\mathcal{T}^{M-1}}^{\infty} (U - \mathcal{T}^{M-1}) \mathrm{d}F_M \tag{4.5}$$

and for $i = 0, \ldots, M-2$, $\mathcal{T}^i$ satisfies

$$\mathcal{T}^i = \mathcal{T}^{i+1} + \delta_{i+1} \tag{4.6}$$

In Sections 4.4 and 4.5, we estimate constant and declining expectations models. There are, however, a number of other candidate models that might explain the falling purchase rate. First, the falling

[8]While we are not aware of research that presents this specific result, there is a large literature that builds upon the McCall labour search model, and, in particular, a number of models have been developed that depart from an assumption that workers have constant expectations about wages. (See, for example, Sant (1977)).

purchase rate might be explained by a model with learning. In such a model, the DM uses Bayesian updating to alter her assessment of the utility distribution in the course of the search. In a model with learning, the DM may depart the search in advance of a purchase if she learns that the distribution is such that the likelihood of high utilities is relatively poor. A second candidate model is as follows. After a DM browses an alternative, she may assign it a higher utility than she would if it were lower ranked, on the grounds that, given her previous experience with online searching, higher ranked alternatives tend to have higher utilities. This second explanation is distinct from the declining expectations model, which assumes that the DM's previous experience with online search only affects her evaluation of alternatives *that have not yet been browsed*. In future research, both of these explanations warrant further investigation. Regarding the second explanation, however, there is at least some preliminary evidence that it is not a major cause of the fall in purchase rate in the Expedia dataset. In particular, Ursu (2018, p.535) shows that the conversion rate – the ratio of purchases to clicks – does not appear to be substantially less at lower ranked positions.

4.3 Description of the data

The data analysed in this paper was provided by the online travel agency Expedia to a data-science competition on Kaggle.com. It pertains to hotel searches on the Expedia website between November 1, 2012 and June 30, 2013. For each search, the data include the selected destination, as well as the following information for each hotel that appeared on the first page of search results: (i) the position of the hotel in the search results (ii) an indicator of whether it was clicked on (iii) an indicator of whether it was booked (iv) its price (v) the number of stars (vi) its review score (vii) a score for the desirability of its location (viii) an indicator of whether it was offering a promotion (xi) an indicator of whether it has a "brand" (whether it is a major chain). Expedia provided data on two kinds of searches. The first kind we term "random searches", where the ordering of the hotels was generated randomly, for experimental purposes. For the second kind, the ordering was generated by Expedia's relevance algorithm. The data indicate whether a search is random.

For random searches, the ranking of a hotel can be treated as exogenous in econometric analysis; accordingly, our dataset is restricted to random searches. Appendix 6 describes our data cleaning process. The resulting dataset comprises 6,044 searches, which include 176,263 observations of hotels. After the data cleaning process, the maximum search length (that is, the number of hotels on the first page of search results) is 33, and the average search length is 29.2. The number of bookings is 304, so approximately 5.03 per cent of searches lead to a booking on the first page of search results.

While the availability of random searches is a particularly attractive feature of the Expedia dataset, there are several limitations of the data. First, the dataset only includes searches for which there is at least one click. Accordingly, the results can be interpreted as applicable to the class of consumers who click, and may not be applicable to the broader class of consumers. The second limitation of the data is that searches resulting in purchases were oversampled. While the oversampling is substantial for the non-random searches, the oversampling for random searches appears to be considerably less substantial. In our dataset, which only includes random searches, a fraction of about 5 per cent of searches result in a purchase, which is of a broadly similar magnitude to the fraction in another comparable dataset for an online travel agency.[9] Given that our dataset includes only random searches, this mitigates concerns arising from this second limitation.

To estimate a random utility model, we need to specify the attributes that appear in the utility function. We use the six attributes in Table 1. The final two columns show that the mean price in the overall dataset is somewhat higher than the mean price for observations with bookings. Conversely, for the other attributes, the mean in the overall dataset was lower than the mean for observations with bookings, although for review scores and brand this difference was small.

Table 1: Hotel attributes

Attribute	Minimum	Maximum	Median	Mean (Overall)	Mean (Booking)
Price (US$)	11.54	1000	167	196.140	159.815
Stars	2	5	4	3.533	3.740
Location	1	5	5	3.723	4.125
Promotion	0	1	0	0.341	0.549
Review score	0	5	4	3.923	3.985
Brand	0	1	1	0.690	0.697

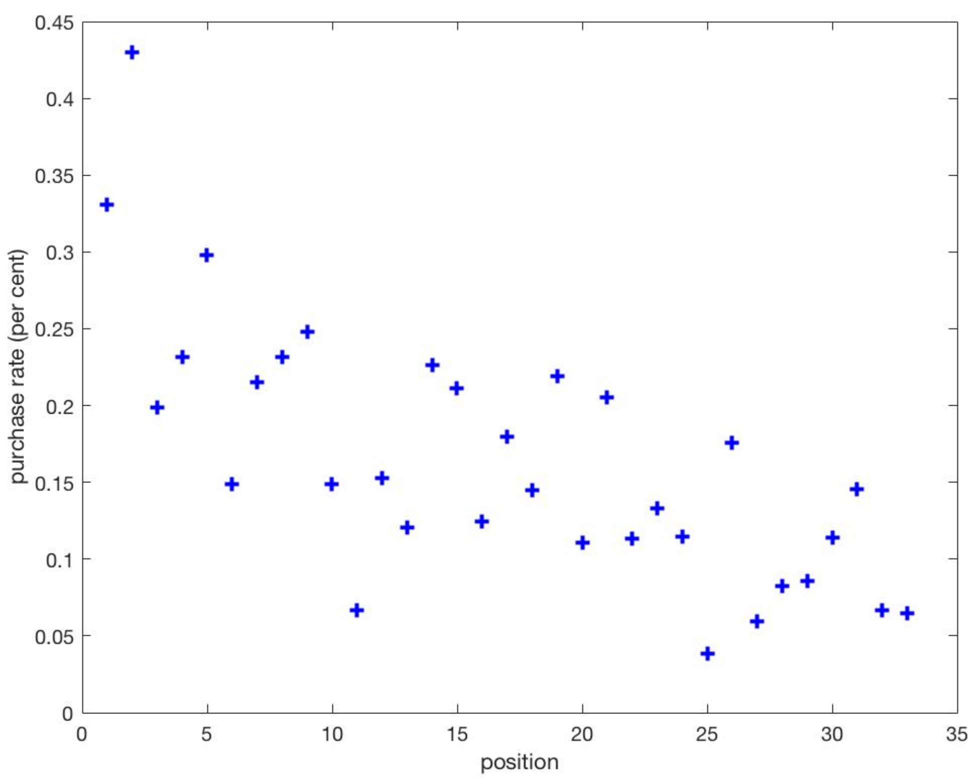
As noted in Appendix 6, we restrict our dataset to searches involving the "top ten" destinations – that is, the ten destinations with the most searches. Appendix 5 assesses the sensitivity of our results to this restriction, estimating the model on datasets for the "top five" destinations and for the

[9]The online appendix to Ursu (2018) describes a dataset of online hotel searches for a major online travel agent for a time period in 2009. Of the 3,560 searches with at least one click, approximately 3 percent result in a purchase.

most popular destination. We do not know what these destinations are, as the data are anonymised.

Figure 4.1 depicts how the decision to purchase depends upon the position of the hotel. The "purchase rate" for a given position is the percentage of the hotels appearing at that position which are booked. As noted above, in our dataset, about 5 per cent of searches lead to a purchase; moreover, as Figure 4.1 shows, at any given position, the purchase rate is less than one per cent. Clearly, there is a greater propensity to purchase earlier alternatives. For example, the average purchase rate for the first 17 positions is 0.21 per cent , whereas the average for the last 17 positions is 0.12 per cent. The central task of the next two sections to model econometrically this fall in the purchase rate with position. The next section will show that the constant expectations model cannot account for the magnitude of the fall in the purchase rate. Section 4.5 will then argue that the declining expectations model performs considerably better in modelling the fall in the purchase rate.

Figure 4.1: Average purchase rate by position



4.4 Econometric model: Constant expectations

4.4.1 Set up

Consider a dataset of searches $s = 1, \ldots, S$. Each search is assumed to be undertaken by a different DM. For any search s, the dataset includes observations on the first M_s alternatives. For the alternative at position i in search s, there is data on J attributes of the alternative, which is denoted by the attribute vector $x_s^i = (x_s^{1,i}, \ldots, x_s^{J,i})$. For example, in the Expedia dataset, there are six attributes, which are specified in Table 1. For $s = 1, \ldots, S, i = 1, \ldots, M_s$, the vectors x_s^i are generated by i.i.d. random variables with a distribution $F_x(x; \lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_K)$ is a vector of the parameters. Appendix 2 specifies a particular functional form of F_x for the Expedia data. Let X_s denote the values of the attributes for search s; more precisely, X_s is the $J \times M_s$ matrix such that column i of the matrix is the transpose of the attribute vector x_s^i. X denotes the vector of matrices $(X_1, \ldots, X_S)$.

For the DM engaging in search s, her utility from alternative $i \in \{1, \ldots, M_s\}$ is assumed to be given by a conventional random utility specification:

$$\mathbf{U}_s^i = \mathcal{V}_s^i + \varepsilon_s^i = x_s^i \beta + \varepsilon_s^i \tag{4.7}$$

where ε_s^i is an i.i.d. error term with a standard logistic distribution F_ε. $\mathcal{V}_s^i$ is the "observed" component of utility, which is given by $\mathcal{V}_s^i = x_s^i \beta$, where β is a J-dimensional vector of attribute coefficients. Suppose that the DM has considered alternative i. In this case, we adopt the assumption that is typically made in random utility models – that, while ε_s^i is unknown to the researcher, it is known to the DM. In contrast, x_s^i is known to both the researcher and the DM.

As discussed in Appendix 2, the distribution function of the random utility $\mathbf{U}_s^i$ satisfies the conditions in Theorem 1. Accordingly, Theorem 1 entails that the DM's optimal strategy is to purchase the first alternative with a utility in excess of the threshold $\mathcal{T}$, which is determined by equation (4.2). Given our random utility model, equation (4.2) in Theorem 1 becomes the following:

$$\chi = \int \int_{x\beta + \varepsilon > \mathcal{T}} (x\beta + \varepsilon - \mathcal{T}) \; \mathrm{d}F_\varepsilon(\varepsilon) \mathrm{d}F_x(x|\lambda) \tag{4.8}$$

Equation (4.8) determines an implicit function that yields the value of the threshold $\mathcal{T}$ as a function of the information cost χ, attribute parameters β and the parameters of F_x, λ, which we will write as $\psi(\beta, \lambda, \chi)$.

For any search s, the DM's choices can lead to $M_s + 1$ distinct observable outcomes: she may purchase one of the M_s observable alternatives, or she may fail to purchase any of them, continuing to search after alternative M_s. So the dependent variable y_s is specified as taking $M_s + 1$ possible values: for $i = 1, \ldots, M_s$, $y_s = i$ indicates a purchase of alternative i, whereas $y_s = 0$ indicates no observed purchase. Let y denote the vector $(y_1, \ldots, y_S)'$. Then for $s = 1, .., S$, y_s will be such that:

$$y_s = \begin{cases} 0 & \text{if } U^k \leq \mathcal{T}, \quad k = 1, \ldots, M_s \\ 1 & \text{if } U^1 > \mathcal{T} \\ i \in \{2, \ldots, M_s\} & \text{if } U^i > \mathcal{T} \ \& \ U^k \leq \mathcal{T}, \quad k = 1, \ldots, i-1 \end{cases}$$

4.4.2 Likelihood functions

Given ε_s^i has an i.i.d logistic distribution, the conditional probabilities of the various outcomes can be written as follows:

$$P(y_s|X_s, \beta, \lambda, \chi) = \begin{cases} \prod_{k=1}^{M_s} \frac{1}{1+\exp(x_s^k\beta - \psi(\beta,\lambda,\chi))} & \text{if } y_s = 0 \\ \frac{1}{1+\exp(\psi(\beta,\lambda,\chi) - x_s^1\beta)} & \text{if } y_s = 1 \\ \frac{1}{1+\exp(\psi(\beta,\lambda,\chi) - x_s^i\beta)}) \prod_{k=1}^{i-1} \frac{1}{1+\exp(x_s^k\beta - \psi(\beta,\lambda,\chi))} & \text{if } y_s = i \in \{2, \ldots, M_s\} \end{cases} \tag{4.9}$$

These conditional probabilities only depend on λ and χ through the function $\mathcal{T} = \psi(\beta, \lambda, \chi)$. As will be shown below, it will therefore prove useful to define the following specification of the conditional probabilities which are conditionalised not on λ and χ but instead on the threshold $\mathcal{T}$.

$$\breve{P}(y_s|X_s, \beta, \mathcal{T}) = \begin{cases} \prod_{k=1}^{M_s} \frac{1}{1+\exp(x_s^k\beta - \mathcal{T})} & \text{if } y_s = 0 \\ \frac{1}{1+\exp(\mathcal{T} - x_s^1\beta)} & \text{if } y_s = 1 \\ \frac{1}{1+\exp(\mathcal{T} - x_s^i\beta)}) \prod_{k=1}^{i-1} \frac{1}{1+\exp(x_s^k\beta - \mathcal{T})} & \text{if } y_s = i \in \{2, \ldots, M_s\} \end{cases} \tag{4.10}$$

The full likelihood function L for this model can be written as the product of two partial likelihood functions L^y and L^x:

$$L(\beta, \lambda, \chi|y, X) = L^y(\beta, \lambda, \chi|y, X) \ L^x(\lambda|X) \tag{4.11}$$

where L^y is given by

$$L^y(\beta, \lambda, \chi|y, X) = \prod_{s=1}^{S} P(y_s|X_s, \beta, \lambda, \chi) \tag{4.12}$$

and P was defined in equation (4.9). The likelihood function L^x is given by

$$L^x(\lambda|X) = \prod_{s=1}^{S} \prod_{i=1}^{M_s} f_x(x_s^i|\lambda) \tag{4.13}$$

where f_x is the density function associated with the distribution F_x that generates the attribute vectors. A particular functional form for f_x is specified in Appendix 2.[10]

We now demonstrate that maximum likelihood estimates can be obtained not only by maximising the full likelihood function L, but also by maximising certain partial likelihood functions.

Proposition 1

Let β^*, λ^* and χ^* denote the values that yield a unique maximum for the full likelihood function L, and let $\mathcal{T}^* := \psi(\beta^*, \lambda^*, \chi^*)$.

(a) Suppose $\hat{\lambda}$ maximises the partial likelihood function $L^x(\lambda|X)$. Then $\hat{\lambda} = \lambda^*$.

(b) Suppose that $\hat{\beta}$ and $\hat{\mathcal{T}}$ maximise the partial likelihood function $\breve{L}^y(\beta, \mathcal{T}|y, X)$ that is defined as follows:

$$\breve{L}^y(\beta, \mathcal{T}|y, X) = \prod_{s=1}^{S} \breve{P}(y_s|X_s, \beta, \mathcal{T}) \tag{4.14}$$

Then $\hat{\beta} = \beta^*$ and $\hat{\mathcal{T}} = \mathcal{T}^*$.

Part (a) of this proposition says that, even though λ is a parameter in the partial likelihood function L^y, we can obtain maximum likelihood estimates of λ for the full likelihood function L simply by maximising the partial likelihood function L^x. Part (b) says that, in order to obtain maximum likelihood estimates of β and $\mathcal{T}$ for the full likelihood function L, we can restrict our attention to maximising the partial likelihood function $\breve{L}^y$. One implication is that the maximum likelihood estimates of β and $\mathcal{T}$ do not depend on assumptions about the particular functional form of the distribution of attributes F_x. Relying on Proposition 1, we will adopt the following three step approach to estimating the parameters in the model. First, in Section 4.4.3, we will maximise $\breve{L}^y(\beta, \mathcal{T}|y, X)$ to obtain maximum likelihood estimates of β and $\mathcal{T}$. Second, we maximise L^x to obtain the maximum likelihood estimate of λ. The second step is presented in Appendix 2, because the approach used to estimate the distribution of attributes is specific to our dataset, and uses standard statistical

[10] This paper assumes that the attribute vector is exogenous, and, in particular, does not explore issues relating to the potential endogeneity of price. Ursu (2018, p.546) provides some reassurance that price endogeneity may not be a major concern in the Expedia dataset. But, in any event, this topic should be pursued in future research.

methods. Third, we use equation (4.8) to obtain a maximum likelihood estimate of χ. The third step is also presented in Appendix 2, because the simulation results below do not depend upon χ – they are generated using the maximum likelihood estimates of β, $\mathcal{T}$ and λ.

4.4.3 Estimating the random utility coefficients β and threshold $\mathcal{T}$

Substituting the conditional probabilities in (4.10) into likelihood function $\breve{L}^y$ in (4.14), and taking logarithms, we obtain we obtain the following log likelihood function:

$$\log \breve{L}^y(\beta, \mathcal{T}|y, X) = \sum_{s=1}^{S} \log \breve{P}(y_s|X_s, \beta, \mathcal{T}) \tag{4.15}$$

where for $s = 1, \ldots, S$:

$$\log \breve{P}(y_s|X_s, \beta, \mathcal{T}) = \begin{cases} -\sum_{k=1}^{M_s} \log(1 + \exp(x_s^k \beta - \mathcal{T})) & \text{if } y_s = 0 \\ -\log(1 + \exp(\mathcal{T} - x_s^1 \beta)) & \text{if } y_s = 1 \\ -\log(1 + \exp(\mathcal{T} - x_s^i \beta)) & \\ \qquad - \sum_{k=1}^{i-1} \log(1 + \exp(x_s^k \beta - \mathcal{T})) & \text{if } y_s = i \in \{2, \ldots, M_s\} \end{cases} \tag{4.16}$$

Maximizing $\log \breve{L}^y$ yields the maximum likelihood estimates of β and $\mathcal{T}$ reported in Table 2.[11]

Table 2: Estimates of random utility coefficients and threshold

Estimate of:	Estimate (6 attributes)	Standard Error	Estimate (4 attributes)	Standard Error
threshold, $\mathcal{T}$	5.1051**	0.5769	4.8227**	0.5146
coefficient of:				
log price, β_1	-0.8991**	0.1205	-0.8528**	0.1162
stars, β_2	0.4270**	0.0993	0.4845**	0.0915
location,β_3	0.2056**	0.0507	0.2014**	0.0504
promotion, β_4	0.4716**	0.1252	0.4717**	0.1232
review score, β_5	0.1827	0.1225		
brand, β_6	-0.0243	0.1301		

(* $p < 0.05$, ** $p < 0.01$)

The first column reports estimates for the model with six attributes. The coefficients on log price,

[11]Standard errors are obtained from the outer products of gradients, using the BHHH estimator. See Greene (2017, p.550). Expressions for the gradient are presented in Appendix 4.

stars, location score, promotions and review score all have the expected signs.[12] (It is not obvious what is the expected sign on brand). Moreover, the coefficients on log price, stars, location and promotion are significantly different from zero, whereas those on review score and brand are not. Note that this is consistent with our finding in Table 1 that the means for booked hotels of price, stars, location and promotion are all substantially different than the overall means, whereas the means of review score and brand for booked hotels are close to their overall means. For the purpose of estimating the distribution F_x of attributes in Appendix 2, it is helpful to reduce the number of attributes. So we drop from our regression the two insignificant attributes, review score and brand. The third column of Table 2 reports estimates for the four-attribute version of the model.

4.4.4 Simulation and identification

Using our estimates of the joint distribution of attributes in Appendix 2, together with the estimates of β and $\mathcal{T}$ from the previous section, we generate a simulated dataset of 100,000 searches, each with a length of $M_s = 33$ positions. In order to assess whether our method of estimating β and $\mathcal{T}$ is identified, we maximise the log likelihood function $\log \breve{L}^y$ for the simulated dataset to obtain another set of estimates. The new estimates for the simulated data are reported in the first column of Table 3. The third column records the parameters used to generate the data, which are the original estimates from Table 2. All of the estimates are within a 95 per cent confidence interval of the underlying parameters, indicating that the estimators are identified.

Table 3: Estimates from simulated data for constant expectations model

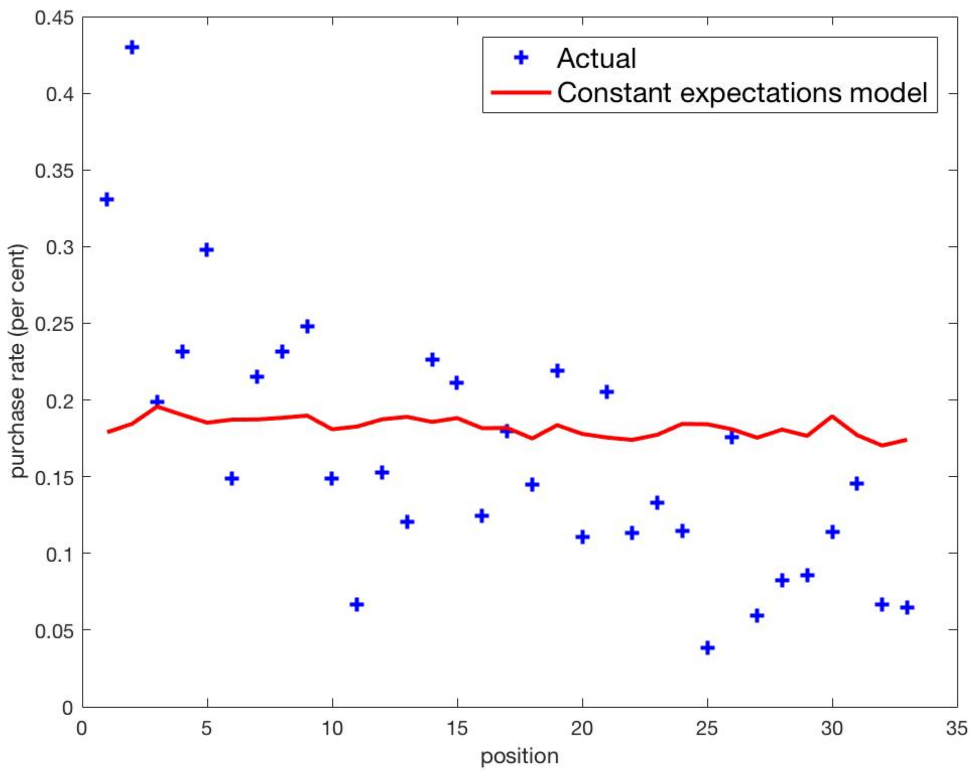
Estimate of:	Estimate	Standard Error	Parameter
threshold, $\mathcal{T}$	4.8907**	0.0766	4.8227
coefficient of:			
log price, β_1	-0.8513**	0.0143	-0.8528
stars, β_2	0.5070**	0.0170	0.4845
location, β_3	0.1997**	0.0107	0.2014
promotion, β_4	0.4533**	0.0259	0.4717

(* $p < 0.05$, ** $p < 0.01$)

[12] Our dataset encompasses a considerable range of prices, from 11 to a 1000 U.S. dollars. Intuitively, the use of log price rather than price embodies an assumption that an increase in price from, say, 20 to 30 dollars has a greater impact on the probability of purchase than an increase from 900 to 910 dollars.

We can use simulated data to assess how well the fall in the purchase rate, depicted in Figure 4.1, is modelled by the constant expectations model. In order to ensure that we obtain a relatively smooth average purchase rate (average purchases per position), we simulate the model for a larger dataset of 1,000,000 searches, each with a length of $M_s = 33$ positions. The result is depicted in Figure 4.2. As foreshadowed in Section 4.2, the constant expectations model predicts a very slight

Figure 4.2: Purchase rate: Actual versus simulation in constant expectation model



decline in the purchase rate. The average purchase rate is approximately flat, sloping downwards only slightly. (The average of the first five positions is 0.187 percent, whereas the average of the last five positions is 0.178 percent). In contrast, average purchase rates in the actual data tend to fall substantially with position. The reason why, in the simulated model, the decline in the purchase rate is so modest is that (1) the only driver of a falling purchase rate is the propensity for DMs to depart searches after making a purchase and (2) only about 5 per cent of searches result in a purchase on the first page of search results. This shortcoming of the constant expectations model is the principal motivation for developing the declining expectations model, which, as we will show, generates a far more substantial decline in the purchase rate.

4.5 Econometric model: Declining expectations

4.5.1 Set up

In the declining expectations model, we assume that all DMs undertake a search comprising M alternatives but, for any given search s, we only have observations on the first $M_s \leq M$ alternatives, which is the number of alternatives that appear on the first page of search results. As in the constant expectations model, attribute vectors x_s^i are generated by i.i.d. random variables with a distribution $F_x(x; \lambda)$ that is specified in Appendix 2. The DM's random utility is given by an equation of the same form as equation (4.7) in the constant expectations model, except that now the error terms ζ_s^i are assumed to be i.i.d. with standard Gumbel distributions rather than logistic distributions.[13]

$$\mathbf{U}_s^i = \mathcal{V}_s^i + \zeta_s^i = x_s^i \beta + \zeta_s^i \tag{4.17}$$

In the constant expectations model, we made an assumption about the DMs' beliefs in order to solve for the threshold $\mathcal{T}$. The assumption, in effect, was that DMs have rational expectations. That is, the DMs' beliefs about the random variables that determine the utilities of alternatives are correct – DMs believe these utilities are determined by the true random utility equation (4.7) and the true distribution of alternatives F_x. But in the declining expectations model, by definition, DMs do not have rational expectations – they do not have correct beliefs. Whereas, in reality, alternatives are generated randomly for experimental purposes, DMs believe that, for unbrowsed alternatives, the alternatives will become worse, on average, as the DM moves further down the search results (as is typically the case in online search). In a rational expectations model, there is only one way, broadly speaking, of specifying the DMs' beliefs – they have true beliefs. Given that we are dropping the rational expectations assumption, the question then arises: how, precisely, should we specify the incorrect beliefs of DMs?

We will, in effect, present two versions of the model: one which makes a relatively weak assumption about the DMs' beliefs, and one which makes a stronger assumption. The weak assumption will be denoted "Assumption W". Assumption W has been designed to ensure that the distribution of the DMs' beliefs satisfy the condition in Theorem 2.

[13] The assumption that utility has a Gumbel distribution allows us to obtain analytic expressions for choice probabilities.

Assumption W

For $i = 1, \ldots, M$, the DM believes that her utility at position i is generated by a random variable $\mathbf{W}^i$ with distribution function $F_i(U; \beta, \delta, \lambda)$ that is strictly increasing and continuously differentiable, and $-\infty < \mathbb{E}(\mathbf{W}^i) < \infty$, where $\mathbf{W}^1, \ldots, \mathbf{W}^M$ are independent, and, for some $\delta \in \mathbb{R}_{++}$, for $i = 1, \ldots, M-1$, and for any $U \in \mathbb{R}$, $F_i(U + \delta) = F_{i+1}(U)$.[14]

The assumption about the DM's beliefs in Theorem 2 differs from Assumption W in one key respect: whereas Theorem 2 allows that the fall in expectation δ_i may differ across positions $i = 0, \ldots, M-1$, Assumption W assumes the fall is constant – that is, for $i = 0, \ldots, M-1$, $\delta_i = \delta$. Given Assumption W, Theorem 2 allows us to conclude that there is a vector of thresholds $\mathcal{T} = (\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^{M-1})$ such that the DM continues her search when the threshold associated with her current position is above the maximum utility for any alternative observed thus far and for the outside option. Further, this vector of thresholds satisfies the property that, for $i = 0, \ldots, M-2$, $\mathcal{T}^i = \mathcal{T}^{i+1} + \delta$. It turns out that, simply on the basis of this weak assumption, we can make considerable headway in estimating the model. In particular, as we will prove below, Assumption W allows us to obtain maximum likelihood estimators for the threshold vector, as well as all the structural parameters in the model except for the information cost χ. As the parameter χ is not required to generate the simulations presented in this paper, we relegate its estimation to Appendix 2. In order to estimate χ, we need to introduce a stronger assumption than W, which is specified in Appendix 2. The stronger assumption is, in effect, that the distributions $F_i(U; \beta, \delta, \lambda), i = 1, \ldots, M$ in Assumption W have a particular functional form.

In the declining expectations model, we assume that the DM has an outside option. While the random utility equation (4.17) specifies how utilities are determined for alternatives at positions $1, \ldots, M$, we have not yet specified how the utility of the outside option is determined. DMs are assumed to have heterogenous outside options: the utilities of the DMs' outside options $U_1^0, \ldots, U_S^0$ are generated by i.i.d. random variables $\mathbf{U}_1^0, \ldots, \mathbf{U}_S^0$ with a distribution F_0. At position 0, a DM is assumed to know the utility of her outside option, while the researcher, of course, does not.

[14]Why are β, δ and λ specified as parameters in the distributions of utility F_i, $i = 1, \ldots, M$? The distribution depends upon λ because utility depends upon the attributes. The distribution depends upon β because this determines how the attributes affect utility. The distribution depends upon δ because this is a measure of the fall in expected utility with position. Further, the motivation for specifying λ, β and δ as parameters is apparent from inspecting the particular examples of F_i that are presented in Appendix 2.

Whereas DMs who assign a relatively high utility to their outside option tend to leave the search early, DMs who place a lower value on their outside option tend to stay on the search for longer. Accordingly, the assumption about the functional form of the distribution of utilities of the outside option, F_0, has a crucial effect on the manner in which the purchase rate declines with position. For simplicity, we assume that, roughly speaking, the distribution is uniform. In further research, it would be instructive to consider alternative specifications of the distribution.

For the distribution of utilities of the outside option, we assume that the lower bound is given by a parameter $\omega < \mathcal{T}^{33}$. (If the lower bound exceeded $\mathcal{T}^{33}$, then all of the DMs in our dataset would have departed the search before the end of the first page of search results when $M_s = 33$). Regarding the upper bound, a DM will only commence a search if $\mathcal{T}^0$ is greater than the utility from her outside option, so we assume that the utility of the outside option, conditional on the DM beginning the search, is distributed uniformly from ω to $\mathcal{T}^0$. More accurately, we assume that this distribution of the utility of the outside option is a specific discretization of the uniform distribution from ω to $\mathcal{T}^0$. In particular, we assume that the probability mass function of $\mathbf{U}_s^0$, f_0, is given as follows. Let ρ be defined as $\delta/(\mathcal{T}^0 - \omega)$ and H be defined as the least integer that is greater than or equal to $1/\rho$. Then $\mathbf{U}_s^0$ has the following probability mass function:

$$f_0(U_s^0) = \begin{cases} \rho & \text{if } U_s^0 = \mathcal{T}^0 - k\delta, \text{ for } k \in \{1, \dots, H-1\} \\ 1 - (H-1)\rho & \text{if } U_s^0 = \omega \\ 0 & \text{otherwise} \end{cases} \tag{4.18}$$

If $H = 1/\rho$, then this distribution is exactly a uniform discrete distribution, in which the utilities $\mathcal{T}^0 - \delta, \mathcal{T}^0 - 2\delta, \dots, \mathcal{T}^0 - (H-1)\delta, \omega$ are all assigned a probability ρ. If $H > 1/\rho$, then this distribution is approximately a uniform discrete distribution, in which the utilities $\mathcal{T}^0 - \delta, \mathcal{T}^0 - 2\delta, \dots, \mathcal{T}^0 - (H-1)\delta$ are assigned a probability ρ, but ω is assigned a somewhat lower probability. The motivation for choosing this particular discretization of the uniform distribution is that $\Pr(U_s^0 = \mathcal{T}^i) = \rho$ for $i = 1, \dots, 33$. This means that, in the absence of purchasing an alternative, for any given position, ρ is the probability that a DM departs the search by taking the outside option at that position.

The next question is: how should data on clicks be incorporated in the model? Broadly speaking, given a declining expectations model of searching, click data could potentially provide the researcher with two types of information about DMs:

(i) *Information about the utility of alternatives*

If a DM clicks on an alternative, this suggests that, on the basis of browsing the attributes, the DM expects that it will be worth incurring the cost of learning more about the alternative, by looking at additional photos, reviews and so forth. Thus clicking data provide information about the coefficients in the utility function, over and above the information provided by purchase data.

(ii) *Information about the utility of the outside option*

Suppose that a substantial fraction of searches have clicks at relatively low positions in the search. This would provide evidence that many DMs tend to persist with searching rather than departing the search for the outside option, which indicates that DMs, on average, tend to have a low valuation of the outside option. More generally, clicking data provide information relevant to assessing the distribution of DMs' valuations of the outside option.

In a complete model of online choice, the researcher would use click data to obtain both (i) and (ii). In the McCall model of Chan and Park (2015), DMs make optimising decisions not only about purchasing but also about clicking, so their model uses click data to obtain (i). Their model does not, however, use click data to obtain (ii), as there is no outside option. By way of contrast, our model uses click data to obtain (ii), but not (i). Clicking is not modelled as the outcome of an optimising decision. Rather, clicks are assumed to be determined by a very simple process: if a DM decides to purchase, she must by necessity click; and if a DM decides not to purchase and also has not departed the search for an outside option, there is nevertheless a fixed probability κ that she will click.

We defend this simple model of clicking as follows. In our model, the principal explanation of the falling purchase rate is that DMs leave the search to take the outside option. Moreover, clicking data provide crucial evidence of whether a DM has departed the search for the outside option. So it is important that our model makes use of clicking data to obtain information about DMs' valuations of the outside option. On the other hand, for two reasons, we don't follow Chan and Park (2015) in explaining the decision to click as optimising behaviour. First, it would make an already complicated model more complicated. Second, there is a risk that the estimated model would be poor at explaining purchasing behaviour. After all, in our dataset, the ratio of clicks to purchases is approximately 22. Accordingly, if not only purchases but also clicks are modelled as utility maximising, then, when the parameters in the utility function are estimated to provide the best fit to the data, there is a risk (if the model is imperfectly specified) that the estimated model would be good at explaining clicks but poor at explaining purchases. Our focus is on the explanation of purchases, so we use a model that avoids the risk of this kind of specification error.[15] Nevertheless, one promising topic for future research is to develop an extension of the declining expectations model that makes use of both of the two types of information described in (i) and (ii).

In the declining expectations model, the dependent variable incorporates data both on purchases and clicks. For any search $s \in \{1, \ldots, S\}$, the dependent variable y_s consists of a pair (i, N), where i denotes the position at which a purchase takes place; if no purchase occurs, $i = 0$. N denotes the set of positions at which clicks takes place, excluding any click on a purchased alternative. If the DM clicks on no alternatives other than a purchased alternative, then N is the empty set.

4.5.2 Likelihood functions

In contrast to the constant expectations model, the declining expectations model allows "backtracking". Backtracking occurs when a DM decides, after browsing alternative k, to go back and purchase an earlier alternative $i < k$, because, in the period between browsing i and k, the DM's purchase threshold has fallen. The notation $\mathcal{P}_s^{i,k}$ denotes the probability that the DM in search s purchases the alternative at position i when browsing position k, conditional on $U_s^0 \leq \mathcal{T}^k$. $\mathcal{P}_s^{i,k}$ is a function of X_s, β, as well as the threshold vector $\mathcal{T}$; Appendix 3 specifies an expression for that function. For example, in search s, $\mathcal{P}_s^{1,2}$ is the probability that the DM purchases the alternative at position 1 when browsing at position 2, conditional on $U_s^0 \leq \mathcal{T}^2$. This is the probability that:
(1) $U_s^1 \leq \mathcal{T}^1$ (otherwise the DM would have purchased alternative 1 at position 1); and
(2) $U_s^1 > \mathcal{T}^2$ (the DM purchases an alternative only if its utility is exceeds the threshold); and
(3) $U_s^1 > U_s^2$ (otherwise the DM would purchase alternative 2 at position 2).
Note that, as we only have data for the first page of search results, we cannot obtain the probability $\mathcal{P}_s^{i,k}$ if (1) position i is on the first page of search results ($i \leq M_s$) but (2) position k is on a later page ($k > M_s$). So we make the simplifying assumption that for $i \leq M_s < k$, $\mathcal{P}_s^{i,k} = 0$. That is, while DMs can backtrack, they tend not to backtrack across search pages. For a dataset that is not restricted to the first page of search results, however, this assumption is not required.

[15] As noted in the introduction, whereas the model of Chan and Park (2015) incorporates optimising decisions about both clicking and purchasing, the dataset to which they apply their model only includes data on clicking and not on purchasing. So they are unable to evaluate the fit of their model in relation to actual purchasing decisions.

Three more pieces of notation must be introduced in order to specify the probabilities in the likelihood functions. First, $\mathcal{N}_s^k$ denotes the probability of not purchasing an alternative in the first k positions, conditional on $U_s^0 \leq \mathcal{T}^k$. It is given by $\mathcal{N}_s^k = \Pr(\max\{U_s^1, \ldots, U_s^k\} < \mathcal{T}^k)$. Like $\mathcal{P}_s^{i,k}$, $\mathcal{N}_s^k$ is a function of X_s, β and $\mathcal{T}$. An expression for this function is provided in Appendix 3. Second, the probability of clicking on a set N of non-purchased alternatives, given that $m \geq |N|$ non-purchased alternatives have been browsed, is denoted by $\mathcal{C}_{|N|,m} = \kappa^{|N|}(1-\kappa)^{m-|N|}$. Third, let n_N denote the highest position in N; if N is empty, n_N is specified to be 1.

The conditional probability of the dependent variable y_s is then given as follows:

$$\breve{P}(y_s|X_s,\beta,\delta,\kappa,\omega,\mathcal{T}^0) = \begin{cases} \mathbb{P}_s^{0,N}(X_s,\beta,\delta,\kappa,\omega,\mathcal{T}^0) & \text{if } y_s = (0,N), N \subset \{1,\ldots,M_s\} \\ \mathbb{P}_s^{i,N}(X_s,\beta,\delta,\kappa,\omega,\mathcal{T}^0) & \text{if } y_s = (i,N), i \in \{1,\ldots,M_s\}, \\ & \qquad\qquad N \subset \{1,\ldots,M_s\}\backslash\{i\} \end{cases} \tag{4.19}$$

where

$$\mathbb{P}_s^{0,N} = \sum_{m=n_N}^{M_s-1} \rho \mathcal{N}_s^m \mathcal{C}_{|N|,m} + (1-(M_s-1)\rho)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s} \tag{4.20}$$

$$\mathbb{P}_s^{i,N} = \sum_{m=\max\{i,n_N\}}^{M_s-1} \rho \sum_{k=\max\{i,n_N\}}^{m} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} + (1-(M_s-1)\rho) \sum_{k=\max\{i,n_N\}}^{M_s} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} \tag{4.21}$$

In equation (4.20), if $n_N = M_s$, the equation is interpreted as reducing just to the second term, and in (4.21), if $\max\{i, n_N\} = M_s$, again the equation is interpreted as reducing to the second term.

To illustrate equation (4.21), consider the probability that, for a search s with observations $M_s = 33$, a purchase occurs at position 32, and the only click occurs at that position. This probability is written $\mathbb{P}_s^{32,\emptyset}$. First, note that such an outcome is inconsistent with the utility of the DM's outside option being $U_s^0 = \mathcal{T}^k$ for $k \in \{1,\ldots,31\}$, otherwise the DM would have departed the search before position 32. The outcome can be seen as the union of the following three mutually exclusive events.
(a) The utility of the outside option is $U_s^0 = \mathcal{T}^{32}$; the DM purchases alternative 32 at position 32; and the DM clicks on no non-purchased alternatives of the 31 non-purchased alternatives that she browsed. This event has a probability of $\rho\mathcal{P}_s^{32,32}\mathcal{C}_{0,31}$.
(b) The utility of the outside option is $U_s^0 \leq \mathcal{T}^{33}$; the DM purchases alternative 32 at position 32; and the DM clicks on no non-purchased alternatives of the 31 non-purchased alternatives that she browsed. This event has a probability of $(1-32\rho)\mathcal{P}_s^{32,32}\mathcal{C}_{0,31}$.

(c) The utility of the outside option is $U_s^0 \leq \mathcal{T}^{33}$; the DM purchases alternative 32 at position 33; and the DM clicks on no non-purchased alternatives of the 32 non-purchased alternatives that she browsed. This event has a probability of $(1-32\rho)\mathcal{P}_s^{32,33}\mathcal{C}_{0,32}$.
Accordingly, $\mathbb{P}_s^{32,\emptyset}$ is the sum of the probabilities of events (a),(b) and (c).

To illustrate equation (4.20), consider the probability that, for search s with observations $M_s = 33$, there are no purchases and there is only one click, at position 32. The probability of this outcome is written $\mathbb{P}_s^{0,\{32\}}$. The outcome is the union of the following two mutually exclusive events.
(a) The utility of the outside option is $U_s^0 = \mathcal{T}^{32}$; the DM does not purchase any of the 32 browsed alternatives, but takes the outside option at position 32; the DM clicks on one non-purchased alternative, alternative 32, of the 32 browsed alternatives. This event has probability $\rho\mathcal{N}_s^{32}\mathcal{C}_{1,32}$.
(b) The utility of the outside option is $U_s^0 \leq \mathcal{T}^{33}$; the DM does not purchase any of the 33 alternatives that she browsed; and the DM clicks on one non-purchased alternative, namely alternative 32, of the 33 alternatives that she browsed. This event has probability $(1-32\rho)\mathcal{N}_s^{33}\mathcal{C}_{1,33}$.
So $\mathbb{P}_s^{0,\{32\}}$ is the sum of the probabilities of events (a) and (b).

Note that the probability in equation (4.19) is conditional on $X_s, \beta, \delta, \kappa, \omega$ and $\mathcal{T}^0$ because (1) $\mathcal{C}_{m,n}$ is a function of κ (2) $\mathcal{P}_s^{i,k}$ and $\mathcal{N}_s^k$ are functions of $X_s, \beta, \mathcal{T}^0$ and δ and (3) ρ is a function of $\mathcal{T}^0$, ω and δ. Equation (4.19) is the counterpart of equation (4.10) for the constant expectations model. For the constant expectations model, however, we also specified, in equation (4.9), the conditional probability conditional on λ and χ instead of conditional on the threshold, because the threshold is a function ψ of β, λ and χ. Analogously, in the declining expectations model, the threshold $\mathcal{T}^0$ is a function of β, δ, λ and χ. Let us refer to this function as ϕ, which is the implicit function defined by the following pair of equations: first, $\mathcal{T}^0 = \mathcal{T}^{M-1} + (M-1)\delta$ and, second, from Theorem 2:

$$\chi = \int_{\mathcal{T}^{M-1}}^{\infty} (U - \mathcal{T}^{M-1}) \mathrm{d}F_M(U; \beta, \delta, \lambda) \tag{4.22}$$

The first equation specifies $\mathcal{T}^0$ as a function of $\mathcal{T}^{M-1}$ and δ, while, according to Theorem 2, the second equation yields $\mathcal{T}^{M-1}$ as a function of β, δ, λ and χ. Accordingly, $\mathcal{T}^0$ is a function ϕ of β, δ, λ and χ. Thus, in the declining expectations model, we can write the conditional probability as conditional on λ and χ instead of $\mathcal{T}^0$, where

$$P(y_s|X_s, \beta, \delta, \kappa, \omega, \lambda, \chi) = \breve{P}(y_s|X_s, \beta, \delta, \kappa, \omega, \mathcal{T}^0 = \phi(\beta, \delta, \lambda, \chi)) \tag{4.23}$$

We are now in a position to write the full likelihood function for the declining expectations model, which is the counterpart of equation (4.11) in the constant expectations model:

$$L(\beta, \delta, \kappa, \omega, \lambda, \chi|y, X) = L^y(\beta, \delta, \kappa, \omega, \lambda, \chi|y, X)\ L^x(\lambda|X) \tag{4.24}$$

where L^y is given by

$$L^y(\beta, \delta, \kappa, \omega, \lambda, \chi|y, X) = \prod_{s=1}^{S} P(y_s|X_s, \beta, \delta, \kappa, \omega, \lambda, \chi) \tag{4.25}$$

and P was defined in equation (4.23). The likelihood function L^x has the same specification as in the constant expectations model, and so is given by equation (4.13). In the constant expectations model, our method for estimating the parameters relied on Proposition 1. We will now present an analogous proposition for the declining expectations model.

Proposition 2

Let β^*, δ^*, κ^*, ω^*, λ^* and χ^* denote the values that yield a unique maximum for the full likelihood function L, and let $\mathcal{T}^{0*} := \phi(\beta^*, \delta^*, \lambda^*, \chi^*)$

(a) Suppose $\hat{\lambda}$ maximises the partial likelihood function $L^x(\lambda|X)$. Then $\hat{\lambda} = \lambda^*$.

(b) Suppose that $\hat{\beta}$, $\hat{\delta}$, $\hat{\kappa}$, $\hat{\omega}$ and $\hat{\mathcal{T}}^0$ maximise the partial likelihood function $\breve{L}^y(\beta, \delta, \kappa, \omega, \mathcal{T}^0|y, X)$ that is defined as follows:

$$\breve{L}^y(\beta, \delta, \kappa, \omega, \mathcal{T}^0|y, X) = \prod_{s=1}^{S} \breve{P}(y_s|X_s, \beta, , \delta, \kappa, \omega, \mathcal{T}^0) \tag{4.26}$$

Then $\hat{\beta} = \beta^*$, $\hat{\delta} = \delta^*$, $\hat{\kappa} = \kappa^*$, $\hat{\omega} = \omega^*$ and $\hat{\mathcal{T}}^0 = \mathcal{T}^{0*}$.

Part (a) states that, despite the fact that L^y depends on λ, maximum likelihood estimates of λ for the full likelihood function L can be obtained by maximising the partial likelihood function L^x. Part (b) states that we can obtain maximum likelihood estimates of $\beta, \delta, \kappa, \omega$ and $\mathcal{T}^0$ for the full likelihood function L by simply maximising the partial likelihood function $\breve{L}^y$. Based on this proposition we will adopt a recursive approach to obtaining estimates analogous to the three-step approach in the constant expectations model. First, as described in the Section 4.5.3, we maximise $\breve{L}^y(\beta, \delta, \kappa, \omega, \mathcal{T}^0|y, X)$ to obtain maximum likelihood estimates of $\beta, \delta, \kappa, \omega$, and $\mathcal{T}^0$. Second, L^x is maximised to obtain an estimate of λ. For this second step, we can use the results for the constant expectations model presented in Appendix 2. Third, we use equation (4.22) to obtain an estimate of χ; the third step is also presented in Appendix 2, as the simulation results below do not depend upon the estimate of χ.

While the method specified above provides a general approach for estimating a model of ordered online search, a minor adjustment must be made to accommodate a feature specific to our dataset. As noted in Section 4.3, Expedia excluded from the data all searches that lacked a click. Accordingly, the probabilities in equation (4.19) must be made conditional on the event that there is at least one click in the search. But the probability of this event is one minus the probability that there are no clicks in the search, which, in our terminology, is written $1 - \mathbb{P}_s^{0,\emptyset}$. So, in equation (4.19), which determines the conditional probability $\breve{P}$, we replace $\mathbb{P}_s^{0,N}$ by $\mathbb{P}_s^{0,N}/(1 - \mathbb{P}_s^{0,\emptyset})$ and $\mathbb{P}_s^{i,N}$ by $\mathbb{P}_s^{i,N}/(1 - \mathbb{P}_s^{0,\emptyset})$. Let us denote this new conditional probability $\breve{P}_T$, where the T indicates that this is the conditional probability when the data are truncated. So in expression (4.26), which specifies the partial likelihood function, we replace $\breve{P}$ by $\breve{P}_T$. Let us denote the resulting partial likelihood function by $\breve{L}_T^y$.

4.5.3 Estimating β, δ, κ, ω and $\mathcal{T}^0$

The maximum likelihood estimates of the parameters presented in Table 4 are obtained by maximizing the log of the partial likelihood function $\breve{L}_T^y$. Column 1 contains the estimates for the model with all six attributes. First, note that the estimates of the utility parameters β are of a broadly similar magnitude to those in the constant expectations model, so the comments made in Section 4.4.3 about the interpretation of those parameters also apply here. Moreover, as in the constant expectation model, all of the parameters are significantly different from zero except for two attributes – review score and brand. Accordingly, for the reasons given in Section 4.4.3, we drop these two attributes from the model. For the rest of the paper, our discussion of the declining expectations model will be restricted to the four-attribute version.

Table 4: Estimates of β, δ, κ, ω and $\mathcal{T}^0$

Estimate of:	Estimate (6 attributes)	Standard Error	Estimate (4 attributes)	Standard Error
initial threshold, $\mathcal{T}^0$	6.8883**	0.5482	6.5946**	0.5106
expectation fall, δ	0.02284**	0.006384	0.02299**	0.006360
click probability, κ	0.01076**	0.0001650	0.01076**	0.0001642
lower bound, ω	5.9749**	0.5483	5.6753**	0.4958
coefficient of:				
log price, β_1	-0.9204**	0.1192	-0.8700**	0.1137
stars, β_2	0.4310**	0.0999	0.4920**	0.0921
location, β_3	0.2103**	0.0510	0.2057**	0.0507
promotion, β_4	0.4648**	0.1242	0.4647**	0.1225
review score, β_5	0.1945	0.1221		
brand, β_6	-0.0244	0.1299		

(* $p < 0.05$, ** $p < 0.01$)

The estimate of κ is approximately 0.01, which means that, for any position that DM browses without purchasing, the probability that the DM clicks on the alternative is about one percent. Recall also that ρ is defined as $\delta/(\mathcal{T}^0 - \omega)$, so, from Table 4, the estimate of ρ is approximately 0.0250. This means that, for each position that the DM browses without making a purchase, there is about a 2.5 per cent probability that the DM will depart the search to take the outside option. So, for a search with 33 positions on the first page of results that does not lead to a purchase, there is around an 83 per cent probability that the DM will have left the search at some point on the first page, and thus does not continue the search on the second page.

To interpret the coefficients in the random utility function in the four-attribute model, it is helpful to calculate some partial effects, which are recorded in Table 5. Line 1 in the table says that the average, across all observations, of the elasticity of the purchase probability to price (not log price) is -0.87. Line 2 is obtained as follows: (i) for each observed hotel, replace the value for the attribute stars by 4, and obtain the purchase probability for the observation that is implied by the model; (ii) do the same as step (i), but replacing instead the attribute stars by 3; (iii) find, for each observation, the proportional change in the purchase probabilities obtained in (i) and (ii); and

(iv) take the average of this proportional change across observations. Lines 3 and 4 in the table are obtained by undertaking an analogous exercise for location scores and promotions.[16] Line 5 is obtained by first finding the average purchase probability at each position, and then calculating the average proportional change between average purchase probabilities at adjacent positions.

Table 5: Average partial effects in four-attribute model

Partial effect	Estimate
Elasticity of purchase probability to price	-0.870
Proportional increase in purchase probability from:	
Increasing stars from 3 to 4	0.620
Increasing location score from 3 to 5	0.498
Introducing a promotion	0.578
Moving one position higher in the ranking	0.037

4.5.4 Simulation and identification

Using the estimated parameters in Section 4.5.3 and the distribution of attributes estimated in Appendix 2, we generate a simulated dataset of 300,000 searches, each with 33 positions. Applying the same truncation as Expedia, we delete all searches with no clicks, yielding a truncated dataset of 58,432 searches. In order to evaluate whether our estimator is identified, we apply the estimator to the simulated dataset. The first column of Table 6 reports the new estimates. The parameters used to generate the data (the original estimates from Table 4) are presented in column 3. All of the estimates are within a 95 per cent confidence interval of the parameter, except for the estimate of the click probability κ, which is within a 99 per cent confidence interval of the parameter. This provides evidence that our estimator of the parameters is identified.

[16] As noted in Appendix 6, the location score is discretised to three values, 1, 3 and 5, which is why Table 5 reports the increase from 3 to 5.

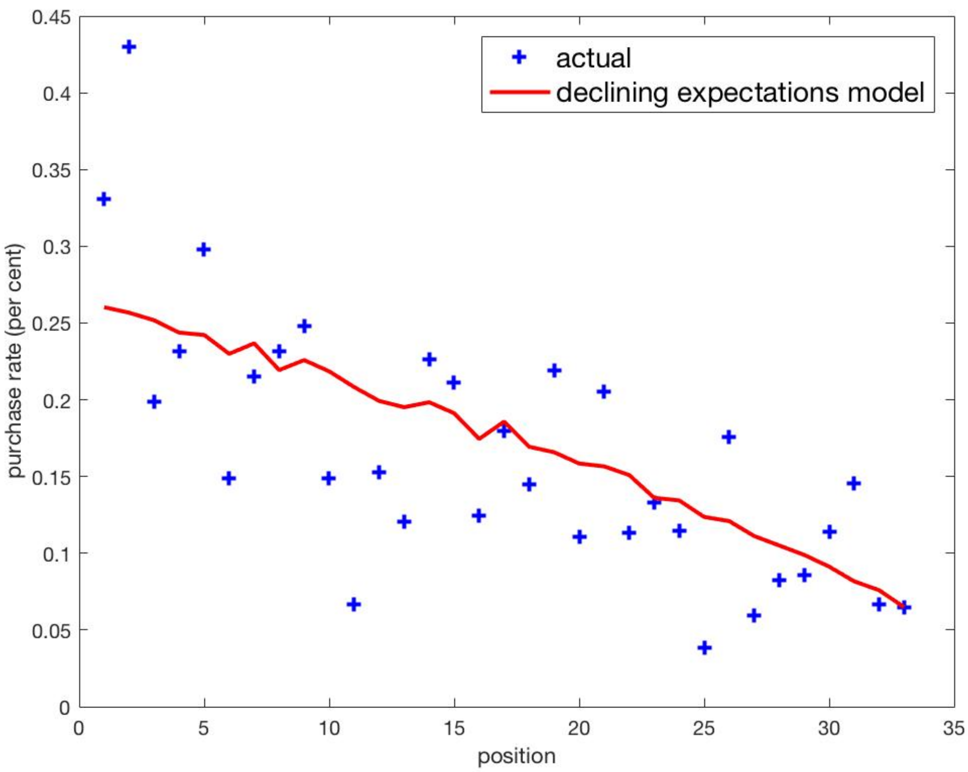
Table 6: Estimates from simulated data for declining expectations model

Estimate of:	Estimate	Standard Error	Parameter
initial threshold, $\mathcal{T}^0$	6.6797**	0.1221	6.5946
expectation fall, δ	0.02445**	0.002481	0.02299
click probability, κ	0.01061**	0.0000660	0.01076
lower bound, ω	5.7824**	0.1053	5.6753
coefficient of:			
log price, β_1	-0.8696**	0.01921	-0.8700
stars, β_2	0.4946**	0.02245	0.4920
location, β_3	0.2189**	0.01450	0.2057
promotion, β_4	0.4818**	0.03474	0.4647

(* $p < 0.05$, ** $p < 0.01$)

We also use simulated data to evaluate how well the fall in the purchase rate is modelled by the declining expectations model. To do this, we generate a simulated dataset of 3,000,000 searches. After truncating the dataset by eliminating searches without clicks, we are left with a dataset of 586,282 searches. Figure 4.3 compares, for each position, the purchase rates in the simulated data with the purchase rates in the actual data. A comparison of Figure 4.3 and Figure 4.2 reveals the key motivation for developing the declining expectations model. Whereas the constant expectations model only yields a slight fall in the purchase rate with position, the declining expectations model generates a substantial fall in the purchase rate, which is of a magnitude similar to that observed in the data. The explanation for this difference is that, while, in the constant expectations model, the only source of the falling purchase rate is departures from searching caused by purchases, in the declining expectations model, DMs also depart searches when they choose the outside option.

Figure 4.3: Purchase rate: Actual versus simulation in declining expectation model



4.6 Conclusions and future research

When estimating models of ordered online search, a central task of the model is to explain the stylised fact that the propensity to purchase is lower at lower positions in the search results. While, in general, the falling purchase rate may be explained partly by the operation of ranking algorithms, in our dataset, the purchase rate falls substantially with position despite the fact that search orderings are randomly generated. In order to explain the falling purchase rate in our dataset, we develop a McCall search model in which DMs expect that alternatives at lower positions are worse – that is, DMs have "declining expectations". In this model, there are two reasons for the falling purchase rate. In particular, low alternatives are less likely to be purchased because:

(i) DMs may leave a search before reaching lower alternatives to purchase a higher alternative;

(ii) DMs may leave a search before reaching lower alternatives to take the outside option.

If the fraction of searches with a purchase is relatively low (it is around 5 per cent in our dataset), then reason (i) can only explain a very slight fall in the purchase rate. Accordingly, in our dataset, reason (ii) is by far the more important explanation of the falling purchase rate.

By way of contrast, in the version of our model where DMs have constant expectations, the only reason for the falling purchase rate is (i). So, at least in datasets where only a small fraction of searches result in a purchase, the constant expectations model may be inadequate to explain substantial falls in the purchase rate. In our dataset, whereas Figure 4.2 shows that the constant expectations model implies only a slight fall in the purchase rate, Figure 4.3 shows that the declining expectations model generates a substantial fall in the purchase rate, which is of a magnitude similar to that observed in the data.

While declining expectations provide one explanation for why a DM may abandon a search for the outside option, there are other candidate explanations. One promising avenue for future research is to develop models in which the DM learns, during the search, about the probabilities of the utilities of unbrowsed alternatives. In the literature on job search, there are models in which the DM updates her belief about the wage distribution as she observes outcomes of the distribution in the course of her search.[17] It would be potentially fruitful to explore whether such models of job search can be applied to ordered search by consumers. A second promising avenue for future research is to adapt the declining expectations model so that it can be applied to datasets where the search positions are generated by ranking algorithms. A third promising extension would be incorporate into our declining expectations model a utility-maximizing decision to click. The result would be an approach that combines features of our declining expectations model with the model in Chan and Park (2015). While such an approach is theoretically appealing, it does come with a risk, described in Section 4.5.1, that any specification errors may lead to a model that provides a good fit for clicks and a poor fit for purchases.

Appendix 1: Proofs

Theorem 1

This is a standard result. See, for example, McCall (1970) and McCall and McCall (2008, p.73-74).

[17] For an early contribution to this literature, see Sant (1977); for a textbook treatment, see Ljungqvist and Sargent (2012, p.179).

Theorem 2

For $i = 1, \ldots, M$, as F_i is continuously differentiable and strictly increasing, then $\mathbf{U}^i$ has a continuous probability density function with support $\mathbb{R}$, which will be denoted f_i. From equation (4.4), for $i = 0, \ldots, M-1$:

$$\begin{aligned} v_i(u^i) &= \max\{u^i, -\chi + \mathbb{E}[v_{i+1}(\max\{u^i, \mathbf{U}^{i+1}\})]\} \\ &= \max\{u^i, -\chi + v_{i+1}(u^i)\int_{-\infty}^{u^i} f_{i+1}(U)\mathrm{d}U + \int_{u^i}^{\infty} v_{i+1}(U)f_{i+1}(U)\mathrm{d}U\} \end{aligned}$$

<u>Decision at position $M-1$</u>

$$\begin{aligned} v_{M-1}(u^{M-1}) &= \max\{u^{M-1}, -\chi + v_M(u^{M-1})\int_{-\infty}^{u^{M-1}} f_M(U)\mathrm{d}U + \int_{u^{M-1}}^{\infty} v_M(U)f_M(U)\mathrm{d}U\} \\ &= \max\{u^{M-1}, -\chi + u^{M-1}\int_{-\infty}^{u^{M-1}} f_M(U)\mathrm{d}U + \int_{u^{M-1}}^{\infty} Uf_M(U)\mathrm{d}U\} \end{aligned}$$

The second equality follows from equation (4.3). So:

$$c_{M-1}(x) = -\chi + x\int_{-\infty}^{x} f_M(U)\mathrm{d}U + \int_{x}^{\infty} Uf_M(U)\mathrm{d}U$$

Define the function $s_{M-1} : \mathbb{R} \to \mathbb{R}$ such that

$$s_{M-1}(x) := x - c_{M-1}(x) = \chi - \int_{x}^{\infty} (U-x)f_M(U)\mathrm{d}U$$

Intuitively, s_{M-1} is a "stopping function" for position $M-1$: a DM stops at position $M-1$ if the maximum utility obtained thus far, x, renders this function positive. We can rewrite s_{M-1} in terms of the function $g_{M-1} : \mathbb{R} \to \mathbb{R}$ such that

$$g_{M-1}(x) := \int_{x}^{\infty} (U-x)f_M(U)\mathrm{d}U$$

So

$$s_{M-1}(x) = x - c_{M-1}(x) = \chi - g_{M-1}(x)$$

Claim 1: g_{M-1} is differentiable and strictly decreasing, with $g'_{M-1}(x) = -\int_x^\infty f_M(U)\mathrm{d}U$.

Proof: As f_M is assumed to be continuous, the fundamental theorem of calculus implies that the function $\int_x^\infty f_M(U)\mathrm{d}U$ is differentiable and

$$\frac{\partial}{\partial x}\int_{x}^{\infty} f_M(U)\mathrm{d}U = -f_M(x)$$

(See Rudin (1976, p.133-4)). As $Uf_M(U)$ is continuous, $\int_x^\infty Uf_M(U)\mathrm{d}U$ is differentiable and

$$\frac{\partial}{\partial x}\int_{x}^{\infty} Uf_M(U)\mathrm{d}U = -xf_M(x)$$

Therefore g_{M-1} is differentiable and

$$\frac{\partial}{\partial x} g_{M-1}(x) = -x f_M(x) - [\int_x^\infty f_M(U)\mathrm{d}U - x f_M(x)] = -\int_x^\infty f_M(U)\mathrm{d}U$$

But this is negative because F_M is assumed to be strictly increasing.

Claim 1 immediately implies the following claim:

Claim 2: s_{M-1} is differentiable and strictly increasing.

Claim 3: $\lim_{x\to\infty} s_{M-1}(x) = \chi > 0$.

$$\lim_{x\to\infty} s_{M-1}(x) = \lim_{x\to\infty} [\chi - g_{M-1}(x)] = \chi - \lim_{x\to\infty} \int_x^\infty (U-x) f_M(U)\mathrm{d}U = \chi$$

Claim 4: $\lim_{x\to-\infty} s_{M-1}(x) = -\infty$.

$$\begin{aligned}\lim_{x\to-\infty} s_{M-1}(x) &= \lim_{x\to-\infty} [\chi - g_{M-1}(x)] = \chi - \lim_{x\to-\infty} \int_x^\infty (U-x) f_M(U)\mathrm{d}U \\ &= \chi - \mathbb{E}(\mathbf{U^M}) + \lim_{x\to-\infty} [x \int_x^\infty f_M(U)\mathrm{d}U] = -\infty\end{aligned}$$

relying on the assumption in Theorem 2 that $-\infty < \mathbb{E}(\mathbf{U^M}) < \infty$.

Claims 2, 3 and 4 then imply:

Claim 5: There is a unique $x \in \mathbb{R}$ such that $s_{M-1}(x) = 0$.

Denote this value of x by $\mathcal{T}^{M-1}$. That is, $\mathcal{T}^{M-1}$ satisfies $s_{M-1}(\mathcal{T}^{M-1}) = 0$, which immediately implies equation (4.5) in Theorem 2. Moreover, by definition of the stopping function s_{M-1} and the fact that it is strictly increasing:

$$u^{M-1} \geq \mathcal{T}^{M-1} \iff s_{M-1}(u^{M-1}) \geq 0 \iff u^{M-1} \geq c_{M-1}(u^{M-1}) \tag{4.27}$$

In other words, at position $M-1$, the DM will stop if and only if the maximum observed utility at that position u^{M-1} exceeds the threshold $\mathcal{T}^{M-1}$.

Decision at position $M-2$

From equation (4.4):

$$v_{M-2}(u^{M-2}) = \max\{u^{M-2}, -\chi + v_{M-1}(u^{M-2})\int_{-\infty}^{u^{M-2}} f_{M-1}(U)\mathrm{d}U + \int_{u^{M-2}}^{\infty} v^{M-1}(U)f_{M-1}(U)\mathrm{d}U\}$$

But by definition of v_{M-1}, for any $x \in \mathbb{R}$, $v_{M-1}(x) = \max\{x, c_{M-1}(x)\}$. So

$$v_{M-2}(u^{M-2}) = \max\{u^{M-2}, -\chi + \max\{u^{M-2}, c_{M-1}(u^{M-2})\}\int_{-\infty}^{u^{M-2}} f_{M-1}(U)\mathrm{d}U$$
$$+ \int_{u^{M-2}}^{\infty} \max\{U, c_{M-1}(U)\}f_{M-1}(U)\mathrm{d}U\} \tag{4.28}$$

So the function describing the value of continuing at $M-2$, $c_{M-2} : \mathbb{R} \to \mathbb{R}$, is such that

$$c_{M-2}(x) = -\chi + \max\{x, c_{M-1}(x)\}\int_{-\infty}^{x} f_{M-1}(U)\mathrm{d}U$$
$$+ \int_{x}^{\infty} \max\{U, c_{M-1}(U)\}f_{M-1}(U)\mathrm{d}U \tag{4.29}$$

The "stopping function" for $M-2$, $s_{M-2} : \mathbb{R} \to \mathbb{R}$, is the function such that the DM stops so long as it is positive, which satisfies:

$$s_{M-2}(x) = x - c_{M-2}(x)$$

Suppose the DM is at position $M-2$ and has observed a maximum of x. Let $c^1_{M-2}(x)$ be interpreted as the value of the strategy of continuing for one more period and then stopping. Then $c^1_{M-2} : \mathbb{R} \to \mathbb{R}$ is the function such that

$$c^1_{M-2}(x) := -\chi + x\int_{-\infty}^{x} f_{M-1}(U)\mathrm{d}U + \int_{x}^{\infty} Uf_{M-1}(U)\mathrm{d}U \tag{4.30}$$

The following claim about c^1_{M-2} is the key to the remainder of the proof:

Claim 6:

$$c^1_{M-2}(x) := -\chi + x\int_{-\infty}^{x-\delta_{M-1}} f_{M}(U)\mathrm{d}U + \int_{x-\delta_{M-1}}^{\infty} (U + \delta_{M-1})f_{M}(U)\mathrm{d}U \tag{4.31}$$

Proof

The substitution rule says that for real-valued functions g and h, if $U = g(y)$ is an invertible function, then

$$\int_a^b h(U)\mathrm{d}U = \int_{g^{-1}(a)}^{g^{-1}(b)} h(g(y))g'(y)\mathrm{d}y$$

Suppose we specify the function g to be such that $U = g(y) = y + \delta_{M-1}$, noting that $g'(y) = 1$, and we apply the substitution rule to the pair of integrals in equation (4.30). This yields:

$$c^1_{M-2}(x) := -\chi + x \int_{-\infty}^{g^{-1}(x)} f_{M-1}(g(y))g'(y)\mathrm{d}y + \int_{g^{-1}(x)}^{\infty} g(y)f_{M-1}(g(y))g'(y)\mathrm{d}y$$

$$= -\chi + x \int_{-\infty}^{x-\delta_{M-1}} f_{M-1}(y + \delta_{M-1})\mathrm{d}y + \int_{x-\delta_{M-1}}^{\infty} (y + \delta_{M-1})f_{M-1}(y + \delta_{M-1})\mathrm{d}y \tag{4.32}$$

But Theorem 2 assumes that, for any $y \in \mathbb{R}$, $F_{M-1}(y + \delta_{M-1}) = F_M(y)$.
So $f_{M-1}(y + \delta_{M-1}) = f_M(y)$. Substituting this equality into equation (4.32) yields Claim 6.

Claim 6 is then used to establish the following claim, which says, intuitively, that if there is a threshold for the decision at $M - 2$, it cannot be less than the threshold for the decision at $M - 1$.

Claim 7: For all $x \leq \mathcal{T}^{M-1}$, $s_{M-2}(x) < 0$.

Proof

Suppose, for contradiction, there exists $x^* \leq \mathcal{T}^{M-1}$, such that $s_{M-2}(x^*) \geq 0$.
As $s_{M-2}(x^*) \geq 0$, thus $x^* \geq c_{M-2}(x^*)$. But $c_{M-2}(x^*) \geq c^1_{M-2}(x^*)$. (It is intuitively clear that the continuation value cannot be less than the value of continuing for one period; it is also clear from inspection of equations (4.29) and (4.30)). Thus:

$$x^* \geq c_{M-2}(x^*) \geq c^1_{M-2}(x^*)$$

$$= -\chi + x^* \int_{-\infty}^{x^*-\delta_{M-1}} f_M(U)\mathrm{d}U + \int_{x^*-\delta_{M-1}}^{\infty} (U + \delta_{M-1})f_M(U)\mathrm{d}U$$

$$= -\chi + x^* \int_{-\infty}^{x^*} f_M(U)\mathrm{d}U + \int_{x^*}^{\infty} Uf_M(U)\mathrm{d}U$$

$$+[\int_{x^*-\delta_{M-1}}^{x^*} (U - (x^* - \delta_{M-1}))f_M(U)\mathrm{d}U + \delta_{M-1} \int_{x^*}^{\infty} f_M(U)\mathrm{d}U]$$

$$> -\chi + x^* \int_{-\infty}^{x^*} f_M(U)\mathrm{d}U + \int_{x^*}^{\infty} (U)f_M(U)\mathrm{d}U$$

$$= c_{M-1}(x^*)$$

The first equality follows from Claim 6. In the second equality, the right hand side is simply a rearrangement of the left hand side. Given that $x^* > c_{M-1}(x^*)$ and given that $s_{M-1}(x) = x - c_{M-1}(x)$, it follows that $s_{M-1}(x^*) > 0$. But as s_{M-1} is strictly increasing, and $s_{M-1}(\mathcal{T}^{M-1}) = 0$, thus $x^* > \mathcal{T}^{M-1}$. Contradiction.

The following claim says that if, at position $M-2$, the maximum observed utility is not less than the threshold at $M-1$, then the stopping problem at position $M-2$ reduces to the one-period stopping problem. Intuitively, this is because the DM knows that if she continues at $M-2$ but does not purchase at position $M-2$, she will stop at $M-1$.

Claim 8: For all $x \geq \mathcal{T}^{M-1}$, $s_{M-2}(x) = x - c^1_{M-2}(x)$.

Proof

Suppose $x \geq \mathcal{T}^{M-1}$. It follows that $s_{M-1}(x) \geq 0$ (by definition of $\mathcal{T}^{M-1}$ and the fact that s_{M-1} is increasing). So $x \geq c_{M-1}(x)$, For similar reasons, if $U \geq x \geq \mathcal{T}^{M-1}$, we have $U \geq c_{M-1}(U)$. Then, from inspection of equations (4.29) and (4.30)), it is apparent that $c_{M-2}(x) = c^1_{M-2}(x)$. So $s_{M-2}(x) = x - c_{M-2}(x) = x - c^1_{M-2}(x)$.

Claim 9: For all $x \geq \mathcal{T}^{M-1}$, $s_{M-2}(x)$ is differentiable and strictly increasing.

Proof

Given Claim 8, when the domain of $s_{M-2}(x)$ is restricted to $x \geq \mathcal{T}^{M-1}$, the function has a similar form to $s_{M-1}(x)$, so the reasoning used to establish, in Claim 2, that s_{M-1} is differentiable and strictly increasing also applies to s_{M-2}.

Claim 10: $s_{M-2}(\mathcal{T}^{M-1} + \delta_{M-1}) = 0$.

Proof

From Claims 6 and 8:

$$s_{M-2}(\mathcal{T}^{M-1} + \delta_{M-1}) = \mathcal{T}^{M-1} + \delta_{M-1} - c^1_{M-2}(\mathcal{T}^{M-1} + \delta_{M-1})$$

$$= \mathcal{T}^{M-1} + \delta_{M-1} + \chi - (\mathcal{T}^{M-1} + \delta_{M-1}) \int_{-\infty}^{\mathcal{T}^{M-1}} f_M(U)\mathrm{d}U - \int_{\mathcal{T}^{M-1}}^{\infty} (U + \delta_{M-1}) f_M(U)\mathrm{d}U$$

$$= \mathcal{T}^{M-1} + \chi - \mathcal{T}^{M-1} \int_{-\infty}^{\mathcal{T}^{M-1}} f_M(U)\mathrm{d}U - \int_{\mathcal{T}^{M-1}}^{\infty} U f_M(U)\mathrm{d}U$$

$$= \mathcal{T}^{M-1} - c_{M-1}(\mathcal{T}^{M-1}) = s_{M-1}(\mathcal{T}^{M-1}) = 0$$

Claims 7 and 9 imply that if there exists a value $\mathcal{T}^{M-2} \in \mathbb{R}$ such that $s_{M-2}(\mathcal{T}^{M-2}) = 0$, then that value is unique. Claim 10 then establishes that that value is $\mathcal{T}^{M-2} = \mathcal{T}^{M-1} + \delta_{M-1}$. $\mathcal{T}^{M-2}$ serves as the threshold for the optimal policy at position $M-2$ for the following reason. Given (i)

s_{M-2} is strictly increasing for $x \geq \mathcal{T}^{M-1}$ (ii) $s_{M-2}(x) < 0$ for $x \leq \mathcal{T}^{M-1}$ and (iii) $\mathcal{T}^{M-2}$ renders s_{M-2} zero, it follows that $u^{M-2} \geq \mathcal{T}^{M-2} \iff s_{M-2}(u^{M-2}) \geq 0 \iff u^{M-2} \geq c_{M-2}(u^{M-2})$. Continuing in this fashion, we can show, in addition, that for any $i \in \{0, 1, \ldots, M-2\}$, $u^i \geq \mathcal{T}^i \iff u^i \geq c_i(u^i)$, where $\mathcal{T}^i = \mathcal{T}^{i+1} + \delta_{i+1}$.

Proof of Proposition 1(a)

Equation (4.8) can be used to define χ as a function g of β, $\mathcal{T}$ and λ. That is:

$$\chi = g(\beta, \mathcal{T}, \lambda) = \int \int_{x\beta+\varepsilon>\mathcal{T}} (x\beta + \varepsilon - \mathcal{T}) \, \mathrm{d}F_\varepsilon(\varepsilon)\mathrm{d}F_x(x|\lambda) \tag{4.33}$$

Suppose, for contradiction, that $\hat{\lambda} \neq \lambda^*$. Then $L^x(\hat{\lambda}|X) > L^x(\lambda^*|X)$. Define $\hat{\chi} = g(\beta^*, \mathcal{T}^*, \hat{\lambda})$. Then $\mathcal{T}^* = \psi(\beta^*, \hat{\lambda}, \hat{\chi}) = \psi(\beta^*, \lambda^*, \chi^*)$. But it is clear from inspection of equations (4.9) and (4.12) that for any $\beta, \lambda, \lambda', \chi$ and χ', if $\psi(\beta, \lambda, \chi) = \psi(\beta, \lambda', \chi')$, then $L^y(\beta, \lambda, \chi|y, X) = L^y(\beta, \lambda', \chi'|y, X)$. Thus $L^y(\beta^*, \hat{\lambda}, \hat{\chi}|y, X) = L^y(\beta^*, \lambda^*, \chi^*|y, X)$. Thus

$$L(\beta^*, \hat{\lambda}, \hat{\chi}|y, X) = L^y(\beta^*, \hat{\lambda}, \hat{\chi}|y, X)L^x(\hat{\lambda}|X) > L^y(\beta^*, \lambda^*, \chi^*|y, X)L^x(\lambda^*|X) = L(\beta^*, \lambda^*, \chi^*|y, X)$$

which contradicts the assumption that β^*, λ^* and χ^* maximise the likelihood function L.

Proof of Proposition 1(b)

Suppose, for contradiction, that it's not the case that both $\hat{\beta} = \beta^*$ and $\hat{\mathcal{T}} = \mathcal{T}^*$. Then

$$\breve{L}^y(\hat{\beta}, \hat{\mathcal{T}}|y, X) > \breve{L}^y(\beta^*, \mathcal{T}^*|y, X) = L^y(\beta^*, \lambda^*, \chi^*|y, X)$$

Define $\hat{\chi} = g(\hat{\beta}, \hat{\mathcal{T}}, \lambda^*)$. Then $\hat{\mathcal{T}} = \psi(\hat{\beta}, \lambda^*, \hat{\chi})$ and so

$$L^y(\hat{\beta}, \lambda^*, \hat{\chi}|y, X) = \breve{L}^y(\hat{\beta}, \hat{\mathcal{T}}|y, X) > \breve{L}^y(\beta^*, \mathcal{T}^*|y, X) = L^y(\beta^*, \lambda^*, \chi^*|y, X)$$

So

$$L(\hat{\beta}, \lambda^*, \hat{\chi}|y, X) = L^y(\hat{\beta}, \lambda^*, \hat{\chi}|y, X)L^x(\lambda^*|X) > L^y(\beta^*, \lambda^*, \chi^*|y, X)L^x(\lambda^*|X) = L(\beta^*, \lambda^*, \chi^*|y, X)$$

which contradicts the assumption that β^*, λ^* and χ^* maximise the likelihood function L.

Proof of Proposition 2(a)

Note that the proof of Proposition 2 is parallel to the proof of Proposition 1. Suppose, for contradiction, that $\hat{\lambda} \neq \lambda^*$. Then $L^x(\hat{\lambda}|X) > L^x(\lambda^*|X)$. Rearranging $\mathcal{T}^0 = \mathcal{T}^{M-1} + (M -$

$1)\delta$, we obtain $\mathcal{T}^{M-1} = \mathcal{T}^0 - (M-1)\delta$. If we substitute this into equation (4.22), we can define χ as a function of $\beta, \delta, \mathcal{T}^0$ and λ. Let us refer to this function as g. Then define $\hat{\chi} = g(\beta^*, \delta^*, \mathcal{T}^{0*}, \hat{\lambda})$. Then $\mathcal{T}^{0*} = \phi(\beta^*, \delta^*, \hat{\lambda}, \hat{\chi}) = \phi(\beta^*, \delta^*, \lambda^*, \chi^*)$. But clearly, for any $\beta, \delta, \lambda, \lambda', \chi$ and χ', if $\phi(\beta, \delta, \lambda, \chi) = \phi(\beta, \delta, \lambda', \chi')$, then $L^y(\beta, \delta, \kappa, \omega, \lambda, \chi|y, X) = L^y(\beta, \delta, \kappa, \omega, \lambda', \chi'|y, X)$. Thus $L^y(\beta^*, \delta^*, \kappa^*, \omega^*, \hat{\lambda}, \hat{\chi}|y, X) = L^y(\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*, \chi^*|y, X)$. Thus

$$L(\beta^*, \delta^*, \kappa^*, \omega^*, \hat{\lambda}, \hat{\chi}|y, X) = L^y(\beta^*, \delta^*, \kappa^*, \omega^*, \hat{\lambda}, \hat{\chi}|y, X)L^x(\hat{\lambda}|X) >$$

$$L^y(\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*, \chi^*|y, X)L^x(\lambda^*|X) = L(\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*, \chi^*|y, X)$$

which contradicts the assumption that $\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*$ and χ^* maximise the likelihood function L.

Proof of Proposition 2(b)

Suppose, for contradiction, that one or more of the following are false: $\hat{\beta} = \beta^*$; $\hat{\delta} = \delta^*$; $\hat{\kappa} = \kappa^*$; $\hat{\omega} = \omega^*$; $\hat{\mathcal{T}}^0 = \mathcal{T}^{0*}$. Then

$$\breve{L}^y(\hat{\beta}, \hat{\delta}, \hat{\kappa}, \hat{\omega}, \hat{\mathcal{T}}^0|y, X) > \breve{L}^y(\beta^*, \delta^*, \kappa^*, \omega^*, \mathcal{T}^{0*}|y, X) = L^y(\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*, \chi^*|y, X)$$

Define $\hat{\chi} = g(\hat{\beta}, \hat{\delta}, \hat{\mathcal{T}}^0, \lambda^*)$. Then $\hat{\mathcal{T}}^0 = \phi(\hat{\beta}, \hat{\delta}, \lambda^*, \hat{\chi})$ and so

$$L^y(\hat{\beta}, \hat{\delta}, \hat{\kappa}, \hat{\omega}, \lambda^*, \hat{\chi}|y, X) = \breve{L}^y(\hat{\beta}, \hat{\delta}, \hat{\kappa}, \hat{\omega}, \hat{\mathcal{T}}^0|y, X) >$$

$$\breve{L}^y(\beta^*, \delta^*, \kappa^*, \omega^*, \mathcal{T}^{0*}|y, X) = L^y(\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*, \chi^*|y, X)$$

But this implies $L(\hat{\beta}, \hat{\delta}, \hat{\kappa}, \hat{\omega}, \lambda^*, \hat{\chi}|y, X) > L(\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*, \chi^*|y, X)$, which contradicts the assumption that $\beta^*, \delta^*, \kappa^*, \omega^*, \lambda^*$ and χ^* maximise the likelihood function L.

Proof of Proposition 3

Proposition 3 is presented in Appendix 2. The inner integral of equation (4.53) is given by

$$\int_{\mathcal{T}-x\beta}^{\infty} (x\beta + \varepsilon - \mathcal{T}) \ \mathrm{d}F_\varepsilon \tag{4.34}$$

Let v denote $\mathcal{T} - x\beta$. To establish Proposition 3, it suffices to show

$$\int_v^\infty (\varepsilon - v) \ \mathrm{d}F_\varepsilon = -v\int_v^\infty \mathrm{d}F_\varepsilon + \int_v^\infty \varepsilon \ \mathrm{d}F_\varepsilon = \log(1 + \exp(-v)) \tag{4.35}$$

To show this, we need to obtain the two integrals in (4.35), $\int_v^\infty \mathrm{d}F_\varepsilon$ and $\int_v^\infty \varepsilon \ \mathrm{d}F_\varepsilon$. Given that ε has a standard logistic distribution, the first integral is clearly as follows:

$$\int_v^\infty \mathrm{d}F_\varepsilon = \frac{1}{1 + \exp(v)} \tag{4.36}$$

We will now show that the second integral is given as follows.

$$\int_v^\infty \varepsilon \ \mathrm{d}F_\varepsilon = \int_v^\infty \varepsilon \frac{\exp(\varepsilon)}{(1+\exp(\varepsilon))^2} \ \mathrm{d}\varepsilon = \log(1+\exp(v)) - \frac{v\exp(v)}{1+\exp(v)} \tag{4.37}$$

The first equality holds because the probability density function of the logistic distribution is $f(\varepsilon) = \exp(\varepsilon)/(1+\exp(\varepsilon))^2$. It remains to show that the second equality in (4.37) holds. Define $z = \exp(\varepsilon)$, so $\mathrm{d}z = \exp(\varepsilon)\mathrm{d}\varepsilon$. Then by substitution

$$\int_v^\infty \varepsilon \frac{\exp(\varepsilon)}{(1+\exp(\varepsilon))^2} \ \mathrm{d}\varepsilon = \int_{\exp(v)}^\infty \frac{\log z}{(1+z)^2} \ \mathrm{d}z \tag{4.38}$$

The integration by parts formula can be written $\int g'(z)f(z)\mathrm{d}z = f(z)g(z) - \int f'(z)g(z)\mathrm{d}z$. Suppose we set $f(z) = \log(z)$ so $f'(z) = 1/z$, and $g(z) = -1/(1+z)$ so $g'(z) = 1/(1+z)^2$. Then

$$\int_{\exp(v)}^\infty \frac{\log z}{(1+z)^2} \ \mathrm{d}z = \Big[\frac{-\log z}{(1+z)}\Big]_{\exp(v)}^\infty - \int_{\exp(v)}^\infty -\frac{1}{z(1+z)} \ \mathrm{d}z$$

But $\lim_{z\to\infty} \log z/(1+z) = 0$ by l'Hopital's rule, because $\lim_{z\to\infty}(1+z) = \infty$ and taking derivatives of the numerator and denominator, $\lim_{z\to\infty}(1/z)/1 = \lim_{z\to\infty} 1/z = 0$. So

$$\int_{\exp(v)}^\infty \frac{\log z}{(1+z)^2} \ \mathrm{d}z = \frac{v}{(1+\exp(v))} + \int_{\exp(v)}^\infty \frac{1}{z(1+z)} \ \mathrm{d}z$$

But $\frac{1}{z(z+1)} = \frac{z+1}{z(z+1)} - \frac{z}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$. So

$$\int_{\exp(v)}^\infty \frac{\log z}{(1+z)^2} \ \mathrm{d}z = \frac{v}{(1+\exp(v))} + \int_{\exp(v)}^\infty \frac{1}{z} - \frac{1}{z+1} \ \mathrm{d}z = \frac{v}{(1+\exp(v))} + \Big[\log|z| - \log|z+1|\Big]_{\exp(v)}^\infty$$

But as $\lim_{z\to\infty} \log|z| - \log|z+1| = 0$, and $\exp(v) > 0$, it follows:

$$\int_{\exp(y)}^\infty \frac{\log z}{(1+z)^2} \ \mathrm{d}z = \frac{v}{1+\exp(v)} - v + \log(1+\exp(v)) = \log(1+\exp(v)) - \frac{v\exp(v)}{1+\exp(v)} \tag{4.39}$$

Equations (4.38) and (4.39) establish (4.37). Then (4.36) and (4.37) entail:

$$-v\int_v^\infty \mathrm{d}F_\varepsilon + \int_v^\infty \varepsilon \ \mathrm{d}F_\varepsilon = \frac{-v}{1+\exp(v)} + \log(1+\exp(v)) - \frac{v\exp(v)}{1+\exp(v)} \tag{4.40}$$

$$= \log(1+\exp(v)) - v = \log(1+\exp(-v)) \tag{4.41}$$

This establishes equation (4.35), which is what we were required to show.

Proof of Proposition 4

Proposition 4 is presented in Appendix 2. As specified in Appendix 2, Assumption S is that an arbitrary DM believes that her utility at position M is determined as follows

$$\mathbf{W}^M = x\beta + \zeta - \delta M + 17\delta - \eta \tag{4.42}$$

where subscripts and superscripts have been dropped for ease of exposition. Let ν denote a random variable with a standard Gumbel distribution. As η is assumed to have a zero expectation and a scale parameter of 1, we can replace η by $\nu - \gamma$, where γ is the Euler-Mascheroni constant.

$$\mathbf{W}^M = x\beta + \zeta - \delta M + 17\delta + \gamma - \nu$$

But $\zeta - \nu$ then has a standard logistic distribution. If we let ε denote a random variable with a standard logistic distribution, then we can write

$$\mathbf{W}^M = x\beta - \delta M + 17\delta + \gamma + \varepsilon \tag{4.43}$$

Assumption S implies that the distribution F_M in equation (4.22) is the distribution of the random variable $\mathbf{W}_M$ specified in (4.43). Equation (4.22) can then be rewritten

$$\chi = \int_{-\infty}^{\infty} \int_{\mathcal{T}^{M-1}-x\beta-\gamma+(M-17)\delta}^{\infty} (x\beta + \gamma - (M-17)\delta + \varepsilon - \mathcal{T}^{M-1})\ \mathrm{d}F_\varepsilon(\varepsilon)\mathrm{d}F_x(x|\lambda) \tag{4.44}$$

where F_ε is the standard logistic distribution function. But from Theorem 2, $\mathcal{T}^0 = \mathcal{T}^{M-1} + (M-1)\delta$. If we substitute this into (4.44), we obtain

$$\chi = \int_{-\infty}^{\infty} \int_{\mathcal{T}^{0}-x\beta-16\delta-\gamma}^{\infty} (x\beta + \gamma + 16\delta + \varepsilon - \mathcal{T}^{0})\ \mathrm{d}F_\varepsilon(\varepsilon)\mathrm{d}F_x(x|\lambda) \tag{4.45}$$

Using equation (4.45), equation (4.58) in Proposition 4 can be obtained using a proof exactly analogous to the proof of Proposition 3.

Proof of Proposition 5

Proposition 5 is presented in Appendix 2. We give the proof for F_i; the proof for G_i is similar. Dropping the superscripts and subscripts, the random utility equation for the constant expectations model can be written $\mathbf{U} = \mathcal{V} + \varepsilon$, where $\mathcal{V} = x\beta = \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4$. As the error term ε has a logistic distribution, its distribution function is strictly increasing and continuously differentiable, and the expectation of ε is finite. So it remains to show that the distribution function F_V of $\mathcal{V}$ is strictly increasing and continuously differentiable, and the expectation of $\mathcal{V}$ is finite. As specified in equation (4.50), $h(z; a, b)$ is the probability density function of the negative Gumbel. Let $H(z; a, b)$ denote its distribution function. Given that $\beta_1 \neq 0$, there are two cases to consider: β_1 is positive and β_1 is negative. Consider first the case where β_1 is positive. Then

$$F_V(v) = \Pr(\mathcal{V} < v) = \Pr(\beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 < v) \tag{4.46}$$

$$= \sum_{k\in\{0,1\}} \sum_{(m,n)\in D} \Pr(\beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 < v | x^2 = m, x^3 = n, x^4 = k) q_k p_{m,n} \tag{4.47}$$

$$= \sum_{k\in\{0,1\}} \sum_{(m,n)\in D} \Pr(x^1 < (v - \beta_2 m - \beta_3 n - \beta_4 k)/\beta_1) q_k p_{m,n} \tag{4.48}$$

$$= \sum_{k\in\{0,1\}} \sum_{(m,n)\in D} H((v - \beta_2 m - \beta_3 n - \beta_4 k)/\beta_1; a_{m,n}, b) q_k p_{m,n} \tag{4.49}$$

As the distribution function for the negative Gumbel, H, is strictly increasing and continuously differentiable, it follows that F_V is. Also as a negative Gumbel random variable has a finite expectation, so does $\mathcal{V}$. The argument is analogous for the case where β_1 is negative.

Appendix 2: Estimating the information cost and the distribution of attributes

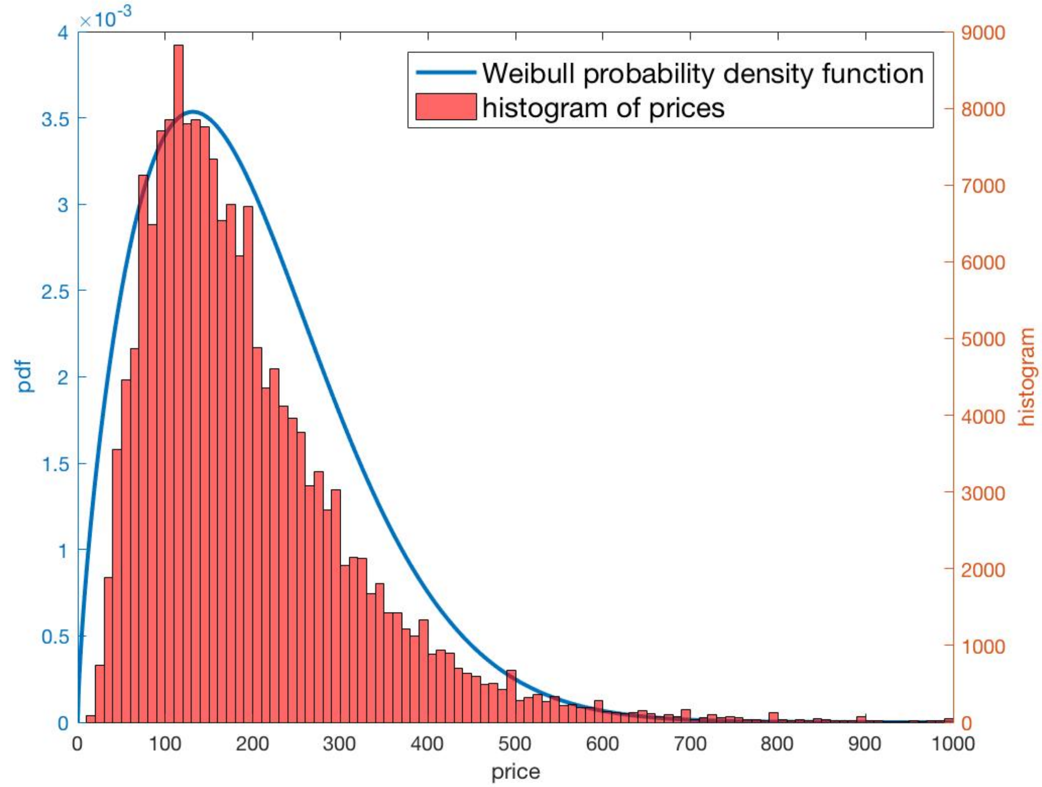
A2.1 Estimating the parameters λ in the distribution of attributes

This section presents estimates of the joint distribution of the four attributes log price, stars, location score and promotion. We might expect price, stars and location to be positively correlated with one another. This expectation is borne out by the estimates of correlation coefficients: the coefficient for log price and stars is 0.551, for log price and location is 0.310, and for stars and location is 0.335. In contrast, it is not obvious what would be the correlation, if any, between the appearance of a promotion, and the variables of price, stars and location. It turns out that promotion has relatively low correlation coefficients with log price, stars and location – respectively -0.148, 0.117 and 0.171. According, in order to reduce the number of parameters in the specification of the distribution of attribute vectors, we will assume that promotion is independent of the other three variables. On the other hand, we will allow for correlation between log price, stars and location.

Promotion, stars and location are discrete variables: promotion is a dummy variable; stars is categorical, taking on values 2, 3, 4 and 5; and location is categorical, taking on values 1, 3 and 5. In contrast, price is continuous. We assume that price has a Weibull distribution, for which the support is the positive reals. Figure 4.4 displays the histogram of prices together with the probability density function of the Weibull distribution estimated for the price data. The assumption that price is Weibull entails that the log of price has a negative Gumbell distribution.[18] The probability

[18] See Forbes et al. (2011). We will say that a random variable Y has a "negative Gumbell distribution" if it has the same distribution as $-Z$, where Z is a random variable with a Gumbell distribution.

Figure 4.4: Histogram of prices and a fitted Weibull pdf



density function for a negative Gumbell distribution is given as follows, where a is the location parameter and b is the scale parameter.

$$h(z; a, b) = \frac{1}{b} \exp((z-a)/b) \exp(-\exp((z-a)/b)) \tag{4.50}$$

Denote the variables log price, stars, location and promotion by, respectively, LOGP, STAR, LOC and PRO. The attribute vectors are then assumed to be generated by the following process. Given the assumption that PRO is independent of the other variables, we assume it is generated by Bernoulli trials, where q_1 represents the probability that PRO $= 1$, and q_0 the probability that PRO $= 0$. STAR and LOC have a joint probability distribution such that, for $m = 2, 3, 4, 5,\ n = 1, 3, 5$, $\Pr(\text{STAR} = m, \text{LOC} = n) = p_{m,n}$. Finally, for $m = 2, 3, 4, 5,\ n = 1, 3, 5$, we specify that, conditional on STAR $= m$ and LOC $= n$, LOGP has a negative Gumbell distribution with probability density function $h(z; a_{m,n}, b)$. That is, the location parameter $a_{m,n}$ in the conditional distribution depends on STAR and LOC, allowing for correlation between price and these two variables. For parsimony, however, we assume that the location variable b is constant across these conditional density functions. Consider an arbitrary attribute vector given by $x = (\text{LOGP} = z, \text{STAR} = m, \text{LOC} = n, \text{PRO} = k)$. Then the density function for the distribution F_x that generates the attribute vectors is given by

$$f_x(x;\lambda) = f_x((z,m,n,k);\lambda) = q_k \; p_{m,n} \; h(z; a_{m,n}, b) \tag{4.51}$$

The vector of parameters λ will be specified as follows. First q_0 is not in λ, because it will be given by $q_0 = 1 - q_1$. Second, $p_{5,5}$ is not in λ because, if we define $D := \{(m,n) | m \in \{2,3,4,5\}, n \in \{1,3,5\}\}$, $p_{5,5}$ can be given by:

$$p_{5,5} = 1 - \sum_{(m,n) \in D \setminus \{(5,5)\}} p_{m,n} \tag{4.52}$$

Accordingly, we define the vector λ of parameters for the density function as follows:
$\lambda = (b, a_{2,1}, a_{2,3}, a_{2,5}, a_{3,1} \ldots, a_{5,1}, a_{5,3}, a_{5,5}, p_{2,1}, p_{2,3}, p_{2,5}, p_{3,1} \ldots, p_{5,1}, p_{5,3}, q_1)'$. The vector λ has dimension $K = 25$. If we substitute the expression for the density function (4.51) into (4.13), we obtain the partial likelihood function L^x. We obtain an estimate of λ by maximising the log of L^x, which, as Proposition 1 states, is also the maximum likelihood estimator of the full likelihood function L. The resulting estimates are reported in Table A1.

Table A1: Estimates of the parameters λ in the distribution of attributes

Parameter	Estimate	Standard Error ($\times 10^{-3}$)	Parameter	Estimate ($\times 10^{-2}$)	Standard Error ($\times 10^{-4}$)
$a_{2,1}$	4.6856**	3.705	$p_{2,1}$	3.8051**	4.559
$a_{3,1}$	5.0129**	4.024	$p_{3,1}$	7.2494**	6.190
$a_{4,1}$	5.4368**	3.716	$p_{4,1}$	3.6899**	4.493
$a_{5,1}$	6.0111**	14.47	$p_{5,1}$	0.86689**	2.208
$a_{2,3}$	4.6893**	3.248	$p_{2,3}$	5.1996**	5.292
$a_{3,3}$	4.9972**	2.229	$p_{3,3}$	15.330**	8.592
$a_{4,3}$	5.3916**	2.792	$p_{4,3}$	9.8353**	7.094
$a_{5,3}$	5.7303**	6.791	$p_{5,3}$	2.2495**	3.533
$a_{2,5}$	4.9903 **	3.980	$p_{2,5}$	3.4772**	4.365
$a_{3,5}$	5.3089**	2.730	$p_{3,5}$	12.583**	7.903
$a_{4,5}$	5.5955**	2.116	$p_{4,5}$	25.431**	10.38
$a_{5,5}$	5.9308**	3.107	q_1	34.128**	11.36
b	0.47486**	0.6315			

(* $p < 0.05$, ** $p < 0.01$)

All estimates are significantly different from zero at a one percent level. Note that the estimates of the parameters $a_{m,n}$ in the conditional distributions of log price have the following property: for any given value of LOC $= n$, $a_{m,n}$ is increasing in m, the number of stars. This reflects the strong positive correlation between price and stars. The maximum likelihood estimates of q_1 and the $p_{m,n}$ parameters, correspond to the empirical frequencies in the data. That is, the estimate of q_1 is the fraction of observations with a promotion, and the estimate of $p_{m,n}$ is the fraction of observations with m stars and a location of n. We now use the estimates of λ to obtain the information cost in the constant expectations model.

A2.2 Estimating the information cost for the constant expectations model

Equation (4.8), which determines the threshold, can be rewritten as follows:

$$\chi = \int_{-\infty}^{\infty} \int_{\mathcal{T}-x\beta}^{\infty} (x\beta + \varepsilon - \mathcal{T}) \ \mathrm{d}F_{\varepsilon}(\varepsilon)\mathrm{d}F_x(x|\lambda) \tag{4.53}$$

Appendix 1 establishes the following proposition, which says that, if we obtain the inner integral in equation (4.53), the equation simplifies to equation (4.54).

Proposition 3

The relationship between χ, $\mathcal{T}$, β and λ can be characterised as follows:

$$\chi = \int_{-\infty}^{\infty} \log(1 + \exp(x\beta - \mathcal{T}))\mathrm{d}F_x(x|\lambda) \tag{4.54}$$

Suppose, again, that we write an arbitrary attribute vector as $x = (\mathrm{LOGP} = z, \mathrm{STAR} = m, \mathrm{LOC} = n, \mathrm{PRO} = k)$. For $(m, n) \in D$, let $d(m, n)$ denote an indicator variable that has value 1 if STAR $= m$ and LOC $= n$, and is zero otherwise. For $k \in \{0, 1\}$, let $c(k)$ be an indicator variable with value 1 if PRO $= k$ and zero otherwise. Then we can rewrite equation (4.54) as follows:

$$\chi = \sum_{k\in\{0,1\}} c(k) \sum_{(m,n)\in D} d(m, n) \int_{-\infty}^{\infty} \log(1 + \exp((z, m, n, k)\beta - \mathcal{T}))f_x((z, m, n, k)|\lambda) \ \mathrm{d}z \tag{4.55}$$

If we substitute into this equation the expression for the density f_x in equation (4.51), we obtain

$$\chi = \sum_{k\in\{0,1\}} c(k)q_k \sum_{(m,n)\in D} d(m, n)p_{m,n} \int_{-\infty}^{\infty} \log(1 + \exp((z, m, n, k)\beta - \mathcal{T}))h(z; a_{m,n}, b) \ \mathrm{d}z \tag{4.56}$$

where h is the density of the negative Gumbell distribution, given by equation (4.50). There does not appear to be a closed form for the integral in equation (4.56), but it is straightforward to calculate it numerically. Accordingly, we obtain the maximal likelihood estimate of the information cost χ by substituting into equation (4.56) the maximum likelihood estimates, obtained above, of β, $\mathcal{T}$ and λ. In order to obtain its standard error, define the vector θ as follows $\theta = (\mathcal{T}, \beta_1, \ldots, \beta_J, \lambda_1, \ldots, \lambda_K)'$. Equation (4.56) determines χ as a function of θ; we will denote this function by g. So equation (4.56) can be rewritten $\chi = g(\theta)$, and we will denote the derivative of this function with respect to θ by g_θ. If the delta method is employed, an estimate of the variance of χ is given by $(g_\theta(\hat{\theta}))'\hat{\text{Cov}}(\theta)g_\theta(\hat{\theta})$, where $\hat{\text{Cov}}(\theta)$ is obtained using the BHHH method. Appendix 4 presents expressions for the elements of $g_\theta(\theta)$. In this fashion, we obtain an estimate for χ of 1.880×10^{-3}, and its standard error of 0.121×10^{-3}. Accordingly, χ is significantly different from zero at a one percent significance level.

Whereas equation (4.56) depends on the specific parametric assumptions made about the attribute distribution, equation (4.54) holds for any arbitrary distribution of attributes. So the technique of obtaining χ that is specified above can be seen as an instance of a more general methodology.

A2.3 Estimating the information cost for the declining expectations model

As discussed in Section 4.5.1, in order to estimate parameters $\mathcal{T}^0$, δ, κ, ω and β, it is sufficient to make weak assumption W about the DM's beliefs. In order to estimate the information cost, however, we need to introduce a stronger assumption about DMs beliefs which, in effect, specifies a particular functional form for the distributions F_i $(i = 1, \ldots, M)$ defined in assumption W.

For the constant expectations model, we assumed that the DM believes that the distribution of the utilities for unbrowsed alternatives is given by the equation generating the random utilities (4.7). For the declining expectations model, we cannot assume similarly that the DM believes the distribution is given by the equation generating the random utilities (4.17), because this is inconsistent with assumption W, which entails that the DM has declining expectations. Instead, we assume that the DM believes that the distribution is given by equation (4.57) below, which is a variant of (4.17), but adjusted to ensure that the expected value of the utility falls as position i increases.

Assumption S

For any search s, the DM believes that, for any unbrowsed alternatives, her utilities at positions $1, \ldots, M$ are generated by random variables $W_s^1, \ldots, W_s^M$ such that

$$\mathbf{W}_s^i = x_s^i \beta + \zeta_s^i - [\delta(i - 17) + \eta_s^i] \tag{4.57}$$

where attributes x are generated by $F_x(x;\lambda)$ and η_s^i is an error term with a Gumbel distribution whose scale parameter is 1 and expected value is zero.

The error term η_s^i can be interpreted as reflecting the DM's uncertainty about the decline in utility.[19] The assumption is labelled "Assumption S" to indicate that it is stronger than Assumption W. Note, in particular, that the distributions of random variables $\mathbf{W}^1, \ldots, \mathbf{W}^M$ satisfy the conditions on distributions $F_1, \ldots, F_M$ in Assumption W: Proposition 5 below establishes that F_i is strictly increasing, continuously differentiable and $\mathbf{W}^i$ has a finite expectation; moreover, clearly, for any $U \in \mathbb{R}$, $F_i(U+\delta) = F_{i+1}(U)$; and $\mathbf{W}^1, \ldots, \mathbf{W}^M$ are independent. Accordingly, the specification of the random variables in Assumption S provides a particular functional form for the class of distributions specified in Assumption W.

Given that the DM's utilities are, in fact, generated by equation (4.17), thus the beliefs specified in equation (4.57) represent a departure from rational expectations. Note, however, that at position $i = 17$, the expected value of the random variable $\mathbf{W}_s^{17}$ (given by equation (4.57)), which the DM *believes* is generating the utility of alternative 17, is equal to the expected value of the random variable $\mathbf{U}_s^{17}$ (given by equation (4.17)), which is *actually* generating the utility of alternative 17. In this sense, the DM has the correct expectation about the alternative at position 17. This position was chosen because it was mid-way down the first-page of results for the maximum first-page length of 33 positions. Intuitively, we have made a specific assumption about the character of the DM's departure from rational expectations: roughly speaking, we are assuming that, for alternatives in the first half of the first-page search results, the DM tends to overestimate the utility of alternatives; and for the alternatives in the second half, the DM tends to underestimate. Accordingly, we will refer to position 17 as the "switching position" – the position where the DM switches from overestimating to underestimating. Clearly, it is somewhat arbitrary to select position 17 as the switching position. So it is reassuring that Proposition 2 proves that, for all of the parameters in the model other than χ, the estimators simply depend upon the weaker assumption W. Moreover, we will evaluate the sensitivity of our estimate of χ to this assumption, calculating how our estimate would change if instead of selecting position 17 as the switching position, we had

[19] As η_s^i has a scale parameter of 1 and expected value of zero, it follows that its location parameter is $-\gamma$, where γ is the Euler-Mascheroni constant, with a value of approximately 0.5772. The motivation for introducing the disturbance term η_s^i with a Gumbel distribution is that $\zeta_s^i - \eta_s^i$ has a logistic distribution with a scale parameter 1 and location parameter γ, which will allow us to obtain Proposition 4.

selected (i) position 7 or (ii) position 27.

As noted above, the estimator for the parameters λ are identical to that for the constant expectations model. So, to obtain the estimate of the information cost χ, we can use the estimates for λ presented in Table A1. As is proven in Appendix 1, Assumption S implies the following proposition, which is the counterpart to Proposition 3 in the constant expectations model.

Proposition 4

Given Assumption S, the relationship between χ, $\mathcal{T}^0$, δ, β and λ can be characterised as follows:

$$\chi = \int_{-\infty}^{\infty} \log(1 + \exp(x\beta + \gamma + 16\delta - \mathcal{T}^0))\mathrm{d}F_x(x|\lambda) \tag{4.58}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Using equation (4.58), the maximum likelihood estimate of χ and its standard error are obtained in a parallel fashion to the estimates for the constant expectations model. The estimate of χ is 7.927×10^{-4} and its standard error its 6.961×10^{-5}. So the estimate of χ is significantly different from zero at a 5 per cent significance level. Note that if, instead of assuming the switching position is 17, we had assumed it were position 7, then the estimate of χ would be slightly lower, equal to 6.301×10^{-4}. If we had assumed the switching position were 27, the estimate of χ would be higher, equal to 9.974×10^{-4}.[20]

Note that, throughout this Appendix, it has been assumed that the distribution functions associated with random variable $\mathbf{U}^i$ defined in equation (4.7) satisfy the conditions in Theorem 1, and the distribution functions associated with random variable $\mathbf{W}^i$ defined in equation (4.57) satisfy the conditions in Theorem 2. But we have not yet established that these distribution functions are strictly increasing and continuously differentiable and the random variables have finite expectations. In order to discharge that debt, Appendix 1 presents a proof of the following proposition.

Proposition 5

Let F^i denote the distribution function of the random variable $\mathbf{U}^i$ defined in equation (4.7). Let G^i denote the distribution function for the random variable $\mathbf{W}^i$ defined in equation (4.57). Assume

[20] If the switching position were 7 or 27, then in equation (4.57), 17 is, respectively, replaced by 7 or 27, and in equation (4.58), 16 is replaced by 6 or 26.

that $\beta_1 \neq 0$. Then F^i and G^i are strictly increasing and continuously differentiable, and $\mathbf{U}^i$ and $\mathbf{W}^i$ have finite expectations.

Appendix 3: Choice probabilities in the declining expectations model

A3.1 Probability of purchasing

For $s \in \{1, \ldots, S\}, i, k \in \{1, \ldots, M_s\}$, $\mathcal{P}_s^{i,k}$ denotes the probability that in search s the DM purchases the alternative at position i when browsing position k, conditional on $U_s^0 \leq \mathcal{T}^k$. In obtaining expression for these probabilities, the following notation will be used: $\mathcal{V}_s^i = x_s^i \beta$; $\mathcal{V}_s^{k,i} = \log \sum_{m \in \{1,\ldots,k\}\backslash\{i\}} \exp(\mathcal{V}_s^m)$; $U_s^{k,i} = \max(\{U_s^1, \ldots, U_s^k\}\backslash\{U_s^i\})$; and $\mathcal{S}_s^k = \sum_{m=1}^k \exp(\mathcal{V}_s^m)$. There are four cases to consider. We will show that the probabilities in these four cases are given as follows:

<u>Case 1</u>: $i > k$

$\mathcal{P}_s^{i,k} = 0$.

<u>Case 2</u>: $i < k$

$$\mathcal{P}_s^{i,k} = \frac{\exp(\mathcal{V}_s^i)}{\mathcal{S}_s^k} \Big[\exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})} \Big) - \exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)} \Big) \Big] \tag{4.59}$$

<u>Case 3</u>: $1 < i = k$

$$\begin{aligned} \mathcal{P}_s^{k,k} = &\exp\Big(- \frac{\mathcal{S}_s^{k-1}}{\exp(\mathcal{T}^{k-1})} \Big) - \exp\Big(- \frac{\mathcal{S}_s^{k-1}}{\exp(\mathcal{T}^k)} \Big) \\ &- \frac{\mathcal{S}_s^{k-1}}{\mathcal{S}_s^k} \Big[\exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})} \Big) - \exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)} \Big) \Big] \\ &+ (1 - \exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k))) \exp\Big(- \frac{\mathcal{S}_s^{k-1}}{\exp(\mathcal{T}^k)} \Big) \end{aligned} \tag{4.60}$$

<u>Case 4</u>: $i = k = 1$

$$\mathcal{P}_s^{1,1} = \Pr(U_s^1 > \mathcal{T}^1) = 1 - \exp(-\exp(\mathcal{V}_s^1 - \mathcal{T}^1))$$

The result for case 1 follows from the fact that the DM cannot purchase alternative i if she has only browsed alternatives up to the earlier position k. The result for case 4 follows immediately from the fact that U_s^1 has a Gumbel distribution with location parameter $\mathcal{V}_s^1$ and scale parameter 1. The expressions in Cases 2 and 3 will be obtained using the following three properties of the Gumbel distribution.

1. The maximum property

Suppose that $\varepsilon_1, \ldots, \varepsilon_J$ are independent random variables with Gumbel distributions with parameters $(a_1, 1), .., (a_J, 1)$ respectively. Then $\epsilon := \max(\varepsilon_1, \ldots, \varepsilon_J)$ has a Gumbel distribution with parameters $(\log \sum_{j=1}^{J} \exp(a_j), 1)$.

2. The logistic property

Suppose that ε_1 and ε_2 are independent random variables with Gumbel distributions with parameters $(a_1, 1), (a_2, 1)$ respectively. Define the difference $\varepsilon^* = \varepsilon_2 - \varepsilon_1$. Then ε^* is logistically distributed; it has the distribution function

$$F(x) = \frac{1}{1 + \exp(-[x - (a_2 - a_1)])}$$

Moreover, as $\Pr(\varepsilon_1 = \max(\varepsilon_1, \varepsilon_2)) = \Pr(\varepsilon_1 > \varepsilon_2) = \Pr(\varepsilon_2 - \varepsilon_1 < 0) = F(0)$, it follows that:

$$\Pr(\varepsilon_1 = \max(\varepsilon_1, \varepsilon_2)) = \frac{1}{1 + \exp(a_2 - a_1)}$$

3. The generalised logistic property

Suppose that ε_1 and ε_2 are independent random variables with Gumbel distributions with parameters $(a_1, 1), (a_2, 1)$ respectively. Suppose $s, t \in \mathbb{R}$ are such that $s > t$. Then

$$\Pr(t < \varepsilon_1 < s \;\&\; \varepsilon_1 = \max(\varepsilon_1, \varepsilon_2)) = \frac{1}{1 + \exp(a_2 - a_1)} \times$$

$$\Big[\exp\Big(-\exp(a_1 - s)(1 + \exp(a_2 - a_1))\Big) - \exp\Big(-\exp(a_1 - t)(1 + \exp(a_2 - a_1))\Big)\Big]$$

The first two properties are standard properties of the Gumbel distribution (see, for example Garrow (2010, pp.28-32)). The third property is proven at the end of this appendix. We call it the "generalised" logistic property, because, as s approaches ∞ and t approaches $-\infty$, the expression for $\Pr(t < \varepsilon_1 < s \;\&\; \varepsilon_1 = \max(\varepsilon_1, \varepsilon_2))$ approaches the expression, in the logistic property, for $\Pr(\varepsilon_1 = \max(\varepsilon_1, \varepsilon_2))$.

We now prove the expressions for $\mathcal{P}_s^{k,i}$ in equations (4.59) and (4.60) for cases 2 and 3.

<u>Case 2</u>: $i < k$

$$\begin{aligned}
\mathcal{P}_s^{i,k} &= \Pr(\mathcal{T}^k < U_s^i < \mathcal{T}^{k-1}, U_s^i = \max\{U_s^1, \ldots, U_s^k\}) \\
&= \Pr(\mathcal{T}^k < U_s^i < \mathcal{T}^{k-1}, U_s^i = \max\{U_s^i, U_s^{k,i}\}) \\
&= \frac{1}{1 + \exp(\mathcal{V}_s^{k,i} - \mathcal{V}_s^i)} \Big[\exp\Big(-\exp(\mathcal{V}_s^i - \mathcal{T}^{k-1})\Big[1 + \exp(\mathcal{V}_s^{k,i} - \mathcal{V}_s^i)\Big]\Big) \\
&\quad - \exp\Big(-\exp(\mathcal{V}_s^i - \mathcal{T}^k)\Big[1 + \exp(\mathcal{V}_s^{k,i} - \mathcal{V}_s^i)\Big]\Big)\Big]
\end{aligned}$$

$$= \frac{\exp(\mathcal{V}_s^i)}{\mathcal{S}_s^k}\Big[\exp\Big(-\frac{\exp(\mathcal{V}_s^i)}{\exp(\mathcal{T}^{k-1})}\frac{\mathcal{S}_s^k}{\exp(\mathcal{V}_s^i)}\Big) - \exp\Big(-\frac{\exp(\mathcal{V}_s^i)}{\exp(\mathcal{T}^k)}\frac{\mathcal{S}_s^k}{\exp(\mathcal{V}_s^i)}\Big)\Big]$$

$$= \frac{\exp(\mathcal{V}_s^i)}{\mathcal{S}_s^k}\Big[\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big]$$

The third equality follows from the generalised logistic property, together with the fact that the maximum property implies that $U_s^{k,i}$ has a Gumbel distribution with location parameter $\mathcal{V}_s^{k,i}$. The fourth equality holds because $\exp(\mathcal{V}_s^{k,i}) = \mathcal{S}_s^k - \exp(\mathcal{V}_s^i)$, so $\exp(\mathcal{V}_s^{k,i} - \mathcal{V}_s^i) = [\mathcal{S}_s^k - \exp(\mathcal{V}_s^i)]/\exp(\mathcal{V}_s^i) = [\mathcal{S}_s^k/\exp(\mathcal{V}_s^i)] - 1$.

Case 3: $1 < i = k$

$$\mathcal{P}_s^{k,k} = \Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1}, U_s^k = \max\{U_s^1, \ldots, U_s^k\}) + \Pr(U_s^k > \mathcal{T}^k)\Pr(U_s^{k,k} < \mathcal{T}^k)$$

The first term is obtained as follows.

$$\Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1}, U_s^k = \max\{U_s^1, \ldots, U_s^k\}) = \Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1}, U_s^k = \max\{U_s^{k,k}, U_s^k\})$$

$$= \Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1}) - \Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1},\ U_s^{k,k} = \max\{U_s^{k,k}, U_s^k\})$$

where

$$\Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1}) = \exp(-\exp(\mathcal{V}_s^{k,k} - \mathcal{T}^{k-1})) - \exp(-\exp(\mathcal{V}_s^{k,k} - \mathcal{T}^k))$$

$$= \exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)$$

$$= \exp\Big(-\frac{\mathcal{S}_s^{k-1}}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^{k-1}}{\exp(\mathcal{T}^k)}\Big)$$

and

$$\Pr(\mathcal{T}^k < U_s^{k,k} < \mathcal{T}^{k-1},\ U_s^{k,k} = \max\{U_s^{k,k}, U_s^k\})$$

$$= \frac{1}{1 + \exp(\mathcal{V}_s^k - \mathcal{V}_s^{k,k})}\Big[\exp\Big(-\exp(\mathcal{V}_s^{k,k} - \mathcal{T}^{k-1})\Big[1 + \exp(\mathcal{V}_s^k - \mathcal{V}_s^{k,k})\Big]\Big)$$
$$- \exp\Big(-\exp(\mathcal{V}_s^{k,k} - \mathcal{T}^k)\Big[1 + \exp(\mathcal{V}_s^k - \mathcal{V}_s^{k,k})\Big]\Big)\Big]$$

$$= \frac{1}{\frac{\mathcal{S}_s^k}{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}}\Big[\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big[\frac{\mathcal{S}_s^k}{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}\Big]\Big)$$
$$- \exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_k)}{\exp(\mathcal{T}^k)}\Big[\frac{\mathcal{S}_s^k}{\mathcal{S}_s^k - \exp(\mathcal{V}_k)}\Big]\Big)\Big]$$

$$= \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\mathcal{S}_s^k}\Big[\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big]$$

$$= \frac{\mathcal{S}_s^{k-1}}{\mathcal{S}_s^k}\Big[\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big]$$

The first equality relies on the generalised logistic property. The second equality holds because

$$1+\exp(\mathcal{V}_s^k - \mathcal{V}_s^{k,k}) = 1 + \frac{\exp(\mathcal{V}_s^k)}{\mathcal{V}_s^{k,k}} = 1 + \frac{\exp(\mathcal{V}_s^k)}{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)} = \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k) + \exp(\mathcal{V}_s^k)}{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)} = \frac{\mathcal{S}_s^k}{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}$$

The second term is obtained as follows.

$$\Pr(U_s^k > \mathcal{T}^k)\Pr(U_s^{k,k} < \mathcal{T}^k) = (1 - \exp(-\exp(-(\mathcal{T}^k - \mathcal{V}_s^k)))\exp\Big(-\exp(-(\mathcal{T}^k - \mathcal{V}_s^{k,k}))\Big)$$

$$= \Big[1-\exp\Big(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)\Big)\Big]\exp\Big(-\exp(\mathcal{V}_s^{k,k} - \mathcal{T}^k)\Big) = \Big[1-\exp\Big(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)\Big)\Big]\exp\Big(-\frac{\mathcal{S}_s^{k-1}}{\exp(\mathcal{T}^k)}\Big)$$

A3.2 Probability of no purchase

$\mathcal{N}_s^k$ denotes the probability not purchasing an alternative conditional on $U_s^0 \leq \mathcal{T}^k$, and is given by

$$\mathcal{N}_s^m = \Pr(\max\{U_s^1, \ldots, U_s^m\} < \mathcal{T}^m) = \exp(-\exp(-(\mathcal{T}^m - \log\mathcal{S}_s^m)))$$

$$= \exp(-\exp(\log\mathcal{S}_s^m - \mathcal{T}^m)) = \exp(-\mathcal{S}_s^m\exp(-\mathcal{T}^m))$$

The second equality holds because, by the maximum property, $\max\{U_s^1, \ldots, U_s^m\}$ has a Gumbel distribution with location parameter $\log\mathcal{S}_s^m$ and scale parameter 1.

A3.3 Proof of the generalised logistic property

We can write $\varepsilon_1 = a_1 + \eta_1, \varepsilon_2 = a_2 + \eta_2$, where η_1 and η_2 are two standard Gumbel distributions. It will be helpful to introduce the following notation: let $\check{t} := \exp(-t)$; $\check{s} := \exp(-s)$; for $i = 1, 2$, $\check{\eta}_i := \exp(-\eta_i)$; $\check{a}_i := \exp(-a_i)$. For $i = 1, 2$, as η_i have standard Gumbell distributions, it follows that $\check{\eta}_i$ each have the exponential pdf $f(x) = \exp(-x)$ (see Forbes et al. (2011)). So:

$$\Pr(t < \varepsilon_1 < s \;\;\&\;\; \varepsilon_1 = \max\{\varepsilon_1, \varepsilon_2\}) = \Pr(t < \varepsilon_1 < s \;\;\&\;\; \varepsilon_2 < \varepsilon_1)$$

$$= \Pr(t - a_1 < \eta_1 < s - a_1 \;\;\&\;\; a_2 + \eta_2 < a_1 + \eta_1)$$

$$= \Pr(a_1 - s < -\eta_1 < a_1 - t \;\;\&\;\; -\eta_2 > a_2 - a_1 - \eta_1)$$

$$= \Pr\Big(\frac{\check{s}}{\check{a}_1} < \check{\eta}_1 < \frac{\check{t}}{\check{a}_1} \;\;\&\;\; \check{\eta}_2 > \frac{\check{a}_1}{\check{a}_2}\check{\eta}_1\Big)$$

$$= \int_{\frac{\check{s}}{\check{a}_1}}^{\frac{\check{t}}{\check{a}_1}} \int_{\frac{\check{a}_1}{\check{a}_2}\check{\eta}_1}^{\infty} \exp(-\check{\eta}_1)\exp(-\check{\eta}_2)\; d\check{\eta}_2 d\check{\eta}_1$$

$$= \int_{\frac{\check{s}}{\check{a}_1}}^{\frac{\check{t}}{\check{a}_1}} \exp(-\check{\eta}_1)\Big[\int_{\frac{\check{a}_1}{\check{a}_2}\check{\eta}_1}^{\infty} \exp(-\check{\eta}_2)\ d\check{\eta}_2\Big] d\check{\eta}_1$$

$$= \int_{\frac{\check{s}}{\check{a}_1}}^{\frac{\check{t}}{\check{a}_1}} \exp(-\check{\eta}_1)\Big[-\exp(-\check{\eta}_2)\Big]_{\frac{\check{a}_1}{\check{a}_2}\check{\eta}_1}^{\infty} d\check{\eta}_1$$

$$= \int_{\frac{\check{s}}{\check{a}_1}}^{\frac{\check{t}}{\check{a}_1}} \exp(-\check{\eta}_1)\Big[\exp(-\frac{\check{a}_1}{\check{a}_2}\check{\eta}_1)\Big] d\check{\eta}_1$$

$$= \int_{\frac{\check{s}}{\check{a}_1}}^{\frac{\check{t}}{\check{a}_1}} \exp\Big(-\check{\eta}_1\Big[1+\frac{\check{a}_1}{\check{a}_2}\Big]\Big) d\check{\eta}_1$$

$$= \Big[-\frac{1}{1+\frac{\check{a}_1}{\check{a}_2}}\exp\Big(-\check{\eta}_1\Big[1+\frac{\check{a}_1}{\check{a}_2}\Big]\Big)\Big]_{\frac{\check{s}}{\check{a}_1}}^{\frac{\check{t}}{\check{a}_1}}$$

$$= \frac{1}{1+\frac{\check{a}_1}{\check{a}_2}}\Big[\exp\Big(-\frac{\check{s}}{\check{a}_1}\Big[1+\frac{\check{a}_1}{\check{a}_2}\Big]\Big) - \exp\Big(-\frac{\check{t}}{\check{a}_1}\Big[1+\frac{\check{a}_1}{\check{a}_2}\Big]\Big)\Big]$$

$$= \frac{1}{1+\exp(a_2-a_1)}\Big[\exp\Big(-\exp(a_1-s)\Big[1+\exp(a_2-a_1)\Big]\Big)$$
$$-\exp\Big(-\exp(a_1-t)\Big[1+\exp(a_2-a_1)\Big]\Big)\Big]$$

Appendix 4: Gradients

A4.1 Gradient vectors for the likelihood functions

For $s = 1,\ldots,S$, if we define $\breve{\ell}^y_s(\beta,\mathcal{T}|y_s,X_s) := \log \check{P}(y_s|X_s,\beta,\mathcal{T})$, then we can rewrite equation (4.15) as $\log \check{L}^y(\beta,\mathcal{T}|y,X) = \sum_{s=1}^{S} \breve{\ell}^y_s(\beta,\mathcal{T}|y_s,X_s)$. Let us call $\breve{\ell}^y_s$ the "log likelihood for observation s". To apply the BHHH method for estimating standard errors of estimates, it is necessary to obtain, for each observation, a vector of partial derivatives of the log likelihood for that observation with respect to the parameters (see Greene (2017, p.550)). The first section below specifies, for an arbitrary observation in the constant expectations model, the partial derivatives of the log likelihood function with respect to the parameters. The second section does the same, but for the declining expectations model. The third section then provides such expressions for the likelihood function of the attribute distribution.

1.1 Partial likelihood function $\breve{L}^y$ in the constant expectations model

For each observation $s = 1, \ldots, S$, the partial derivatives of the log likelihood for that observation with respect to the parameters are given as follows:

$$\frac{\partial}{\partial \beta_j} \log \breve{P}(y_s|X_s, \beta, \mathcal{T}) = \begin{cases} -\sum_{k=1}^{M_s} x_s^{k,j} \frac{\exp(x_s^k\beta - \mathcal{T})}{1+\exp(x_s^k\beta-\mathcal{T})} & \text{if } y_s = 0 \\ x_s^{1,j} \frac{\exp(\mathcal{T} - x_s^1\beta)}{1+\exp(\mathcal{T}-x_s^1\beta)} & \text{if } y_s = 1 \\ x_s^{i,j} \frac{\exp(\mathcal{T} - x_s^i\beta)}{1+\exp(\mathcal{T}-x_s^i\beta)} & \\ \quad - \sum_{k=1}^{i-1} x_s^{k,j} \frac{\exp(x_s^k\beta - \mathcal{T})}{1+\exp(x_s^k\beta-\mathcal{T})} & \text{if } y_s = i \in \{2, \ldots, M_s\} \end{cases} \tag{4.61}$$

and

$$\frac{\partial}{\partial \mathcal{T}} \log \breve{P}(y_s|X_s, \beta, \mathcal{T}) = \begin{cases} \sum_{k=1}^{M_s} \frac{\exp(x_s^k\beta - \mathcal{T})}{1+\exp(x_s^k\beta-\mathcal{T})} & \text{if } y_s = 0 \\ -\frac{\exp(\mathcal{T} - x_s^1\beta)}{1+\exp(\mathcal{T}-x_s^1\beta)} & \text{if } y_s = 1 \\ \frac{-\exp(\mathcal{T} - x_s^i\beta)}{1+\exp(\mathcal{T}-x_s^i\beta)} & \\ \quad + \sum_{k=1}^{i-1} \frac{\exp(x_s^k\beta - \mathcal{T})}{1+\exp(x_s^k\beta-\mathcal{T}))} & \text{if } y_s = i \in \{2, \ldots, M_s\} \end{cases} \tag{4.62}$$

1.2 Partial likelihood function $\breve{L}_T^y$ in the declining expectations model

The log of the partial likelihood function $\breve{L}_T^y$, which is defined in Section 4.5.2, can be written as follows

$$\log \breve{L}_T^y = \sum_{s=1}^{S} \breve{\ell}_s^y \tag{4.63}$$

where the log likelihood function for observations s, $\breve{\ell}_s^y$, is defined as follows. For a search s with no purchase and with clicks at positions in the set N, it is given by

$$\breve{\ell}_s^y := \log \frac{\mathbb{P}_s^{0,N}}{1 - \mathbb{P}_s^{0,\emptyset}} = \log \mathbb{P}_s^{0,N} - \log(1 - \mathbb{P}_s^{0,\emptyset})$$

For a search s with a purchase at position $i > 0$ and with clicks at positions in the set N, $\breve{\ell}_s^y$ is given by:

$$\breve{\ell}_s^y := \log \frac{\mathbb{P}_s^{i,N}}{1 - \mathbb{P}_s^{0,\emptyset}} = \log \mathbb{P}_s^{i,N} - \log(1 - \mathbb{P}_s^{0,\emptyset})$$

where if $n_N < M_s$

$$\mathbb{P}_s^{0,N} = \sum_{m=n_N}^{M_s-1} \rho \mathcal{N}_s^m \mathcal{C}_{|N|,m} + (1 - (M_s - 1)\rho)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s}$$

if $n_N = M_s$

$$\mathbb{P}_s^{0,N} = (1 - (M_s - 1)\rho)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s}$$

if $i > 0$ and $\max\{i, n_N\} < M_s$

$$\mathbb{P}_s^{i,N} = \sum_{m=\max\{i,n_N\}}^{M_s-1} \rho \sum_{k=\max\{i,n_N\}}^{m} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} + (1-(M_s-1)\rho) \sum_{k=\max\{i,n_N\}}^{M_s} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1}$$

and if $i > 0$ and $\max\{i, n_N\} = M_s$

$$\mathbb{P}_s^{i,N} = (1-(M_s-1)\rho)\mathcal{P}_s^{i,M_s}\mathcal{C}_{|N|,M_s-1}$$

To implement the BHHH method, we need to obtain, for observations $s = 1, \ldots, S$, the partial derivatives with respect to the parameters of the log likelihood function $\breve{\ell}_s^y$ for that observation. If we let θ denote the vector of parameters, therefore, the task is to obtain the following. For a search s with no purchase, the required vector of partial derivatives is:

$$\frac{\partial \breve{\ell}_s^y}{\partial \theta} = \frac{1}{\mathbb{P}_s^{0,N}}\frac{\partial \mathbb{P}_s^{0,N}}{\partial \theta} + \frac{1}{1-\mathbb{P}_s^{0,\emptyset}}\frac{\partial \mathbb{P}_s^{0,\emptyset}}{\partial \theta}$$

and a search s with a purchase at position $i > 0$, the required vector of partial derivatives is:

$$\frac{\partial \breve{\ell}_s^y}{\partial \theta} = \frac{1}{\mathbb{P}_s^{i,N}}\frac{\partial \mathbb{P}_s^{i,N}}{\partial \theta} + \frac{1}{1-\mathbb{P}_s^{0,\emptyset}}\frac{\partial \mathbb{P}_s^{0,\emptyset}}{\partial \theta}$$

The remaining task, then, is to obtain the partial derivatives, with respect to the parameters, of $\mathbb{P}_s^{0,N}$ and $\mathbb{P}_s^{i,N}$.

1. Derivatives with respect to ω

$$\rho = \frac{\delta}{\mathcal{T}^0 - \omega}$$
$$\frac{\partial \rho}{\partial \omega} = \frac{\delta}{(\mathcal{T}^0 - \omega)^2}$$

Derivatives for searches with no purchases:

If $n_M < M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \omega} = \frac{\partial \rho}{\partial \omega}\Big[\sum_{m=n_N}^{M_s-1} \mathcal{N}_s^m \mathcal{C}_{|N|,m} - (M_s-1)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s}\Big]$$

If $n_M = M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \omega} = -\frac{\partial \rho}{\partial \omega}(M_s-1)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s}$$

Derivatives for searches with purchases:

If $\max\{i, n_N\} < M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \omega} = \frac{\partial \rho}{\partial \omega}\Big[\sum_{m=\max\{i,n_N\}}^{M_s-1} \sum_{k=\max\{i,n_N\}}^{m} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} - (M_s-1) \sum_{k=\max\{i,n_N\}}^{M_s} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1}\Big]$$

If $\max\{i, n_N\} = M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \omega} = -\frac{\partial \rho}{\partial \omega}(M_s - 1)\mathcal{P}_s^{i,M_s}\mathcal{C}_{|N|,M_s-1}$$

<u>2. Derivatives with respect to β</u>

Derivatives for searches with no purchases:

If $n_M < M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \beta_j} = \sum_{m=n_N}^{M_s-1} \rho \frac{\partial \mathcal{N}_s^m}{\partial \beta_j}\mathcal{C}_{|N|,m} + (1 - (M_s - 1)\rho)\frac{\partial \mathcal{N}_s^{M_s}}{\partial \beta_j}\mathcal{C}_{|N|,M_s}$$

where

$$\mathcal{N}_s^i = \exp(-\mathcal{S}_s^i \exp(-\mathcal{T}^i)), \quad \mathcal{S}_s^i = \sum_{m=1}^{i} \exp(\mathcal{V}_s^m)$$

$$\frac{\partial \mathcal{S}_s^i}{\partial \beta_j} = \sum_{m=1}^{i} x_s^{j,m} \exp(\mathcal{V}_s^m)$$

$$\frac{\partial \mathcal{N}_s^i}{\partial \beta_j} = -\exp(-\mathcal{T}^i)\mathcal{N}_s^i \frac{\partial \mathcal{S}_s^i}{\partial \beta_j}$$

If $n_M = M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \beta_j} = (1 - (M_s - 1)\rho)\frac{\partial \mathcal{N}_s^{M_s}}{\partial \beta_j}\mathcal{C}_{|N|,M_s}$$

Derivatives for searches with purchases:

If $\max\{i, n_N\} < M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \beta_j} = \sum_{m=\max\{i,n_N\}}^{M_s-1} \rho \sum_{k=\max\{i,n_N\}}^{m} \frac{\partial \mathcal{P}_s^{i,k}}{\partial \beta_j}\mathcal{C}_{|N|,k-1} + (1 - (M_s - 1)\rho) \sum_{k=\max\{i,n_N\}}^{M_s} \frac{\partial \mathcal{P}_s^{i,k}}{\partial \beta_j}\mathcal{C}_{|N|,k-1}$$

If $\max\{i, n_N\} = M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \beta_j} = (1 - (M_s - 1)\rho)\frac{\partial \mathcal{P}_s^{i,M_s}}{\partial \beta_j}\mathcal{C}_{|N|,M_s-1}$$

Expressions for the $\mathcal{P}_s^{i,k}$ probabilities are given in Appendix 3. Taking derivatives:

<u>Case 2</u>: $i < k$

$$\frac{\partial \mathcal{P}_s^{i,k}}{\partial \beta_j} = \frac{\exp(\mathcal{V}_s^i)}{\mathcal{S}_s^k}\Big(\Big[x_s^{j,i} - \frac{1}{\mathcal{S}_s^k}\frac{\partial \mathcal{S}_s^k}{\partial \beta_j}\Big]\Big[\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big]$$
$$+\frac{\partial \mathcal{S}_s^k}{\partial \beta_j}\Big[\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big) - \exp(-\mathcal{T}^{k-1})\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big)\Big]\Big)$$

where

$$\frac{\partial \mathcal{S}_s^k}{\partial \beta_j} = \sum_{m=1}^{k} x_s^{j,m} \exp(\mathcal{V}_s^m)$$

Case 3: $1 < i = k$

If we define:

$$A = \exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)$$

$$B = \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\mathcal{S}_s^k}\Big[\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big]$$

$$C = (1 - \exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)))\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)$$

Then we can write

$$\mathcal{P}_s^{k,k} = A - B + C$$

So

$$\frac{\partial A}{\partial \beta_j} = \Big[\frac{\partial \mathcal{S}_s^k}{\partial \beta_j} - x_s^{j,k}\exp(\mathcal{V}_s^k)\Big]\Big[\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big) - \exp(-\mathcal{T}_m^{k-1})\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big)\Big]$$

$$\begin{aligned}\frac{\partial B}{\partial \beta_j} &= \frac{\partial \mathcal{S}_s^k}{\partial \beta_j}\Big[\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big) - \exp(-\mathcal{T}^{k-1})\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big)\Big] \\ &- \frac{\exp(\mathcal{V}_s^k)}{\mathcal{S}_s^k}\Big(\Big[x_s^{j,k} - \frac{1}{\mathcal{S}_s^k}\frac{\partial \mathcal{S}_s^k}{\partial \beta_j}\Big]\Big[\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big) - \exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big] \\ &- \frac{\partial \mathcal{S}_s^k}{\partial \beta_j}\Big[\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big) - \exp(-\mathcal{T}^{k-1})\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big)\Big]\Big)\end{aligned}$$

$$\frac{\partial C}{\partial \beta_j} = x_s^{j,k}\exp(\mathcal{V}_s^k - \mathcal{T}^k)\exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k))\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big) +$$

$$(-(\frac{\partial \mathcal{S}_s^k}{\partial \beta_j} - x_s^{j,k}\exp(\mathcal{V}_s^k))\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)(1 - \exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)))\Big)$$

$$= \exp(\mathcal{V}_s^k - \mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)\times$$

$$\Big(x_s^{j,k}\exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)) - (\frac{\partial \mathcal{S}_s^k}{\partial \beta_j}\exp(-\mathcal{V}_s^k) - x_s^{j,k})(1 - \exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)))\Big)$$

$$\frac{\partial \mathcal{P}_s^{k,k}}{\partial \beta_j} = \frac{\partial A}{\partial \beta_j} - \frac{\partial B}{\partial \beta_j} + \frac{\partial C}{\partial \beta_j}$$

Case 4: $i = k = 1$

$$\frac{\partial \mathcal{P}_s^{1,1}}{\partial \beta_j} = x_s^{j,1}\exp(\mathcal{V}_s^1 - \mathcal{T}^1)\exp(-\exp(\mathcal{V}_s^1 - \mathcal{T}^1))$$

3. Derivatives with respect to κ

$$\mathcal{C}_{n,m} = \kappa^n (1-\kappa)^{m-n}$$

so

$$\frac{\partial \mathcal{C}_{n,m}}{\partial \kappa} = n\kappa^{n-1}(1-\kappa)^{m-n} - (m-n)(1-\kappa)^{m-n-1}\kappa^n$$

Derivatives for searches with no purchases:

If $n_M < M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \kappa} = \sum_{m=n_N}^{M_s-1} \rho \mathcal{N}_s^m \frac{\partial \mathcal{C}_{|N|,m}}{\partial \kappa} + (1-(M_s-1)\rho)\mathcal{N}_s^{M_s} \frac{\partial \mathcal{C}_{|N|,M_s}}{\partial \kappa}$$

If $n_M = M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \kappa} = (1-(M_s-1)\rho)\mathcal{N}_s^{M_s} \frac{\partial \mathcal{C}_{|N|,M_s}}{\partial \kappa}$$

Derivatives for searches with purchases:

If $\max\{i, n_N\} < M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \kappa} = \sum_{m=\max\{i,n_N\}}^{M_s-1} \rho \sum_{k=\max\{i,n_N\}}^{m} \mathcal{P}_s^{i,k} \frac{\partial \mathcal{C}_{|N|,k-1}}{\partial \kappa} + (1-(M_s-1)\rho) \sum_{k=\max\{i,n_N\}}^{M_s} \mathcal{P}_s^{i,k} \frac{\partial \mathcal{C}_{|N|,k-1}}{\partial \kappa}$$

If $\max\{i, n_N\} = M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \kappa} = (1-(M_s-1)\rho)\mathcal{P}_s^{i,M_s} \frac{\partial \mathcal{C}_{|N|,M_s-1}}{\partial \kappa}$$

4. Derivatives with respect to $\mathcal{T}^0$

$$\rho = \frac{\delta}{\mathcal{T}^0 - \omega}$$

$$\frac{\partial \rho}{\partial \mathcal{T}^0} = -\frac{\delta}{(\mathcal{T}^0-\omega)^2}$$

$$\mathcal{T}^k = \mathcal{T}^0 - k\delta, \quad \text{for } k = 1, \dots, M_s$$

$$\frac{\partial \mathcal{T}^k}{\partial \mathcal{T}^0} = 1, \quad \text{for } k = 1, \dots, M_s$$

Derivatives for searches with no purchases:

If $n_N < M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \mathcal{T}^0} = \frac{\partial \rho}{\partial \mathcal{T}^0} \sum_{m=n_N}^{M_s-1} \mathcal{N}_s^m \mathcal{C}_{|N|,m} + \rho \sum_{m=n_N}^{M_s-1} \frac{\partial \mathcal{N}_s^m}{\partial \mathcal{T}^0} \mathcal{C}_{|N|,m}$$

$$-\frac{\partial \rho}{\partial \mathcal{T}^0}(M_s-1)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s} + (1-(M_s-1)\rho)\frac{\partial \mathcal{N}_s^{M_s}}{\partial \mathcal{T}^0}\mathcal{C}_{|N|,M_s}$$

where

$$\mathcal{N}_s^k = \exp(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)), \quad \mathcal{S}_s^k = \sum_{m=1}^{k} \exp(\mathcal{V}_s^m)$$

$$\frac{\partial \mathcal{N}_s^k}{\partial \mathcal{T}^0} = (-\frac{\partial \mathcal{T}^k}{\partial \mathcal{T}^0})(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)) \exp(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)) = \mathcal{S}_s^k \exp(-\mathcal{T}^k) \exp(-\mathcal{S}_s^k \exp(-\mathcal{T}^k))$$

If $n_N = M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \mathcal{T}^0} = -\frac{\partial \rho}{\partial \mathcal{T}^0}(M_s - 1)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s} + (1 - (M_s - 1)\rho)\frac{\partial \mathcal{N}_s^{M_s}}{\partial \mathcal{T}^0}\mathcal{C}_{|N|,M_s}$$

Derivatives for searches with purchases:

If $\max\{i, n_N\} < M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \mathcal{T}^0} = \frac{\partial \rho}{\partial \mathcal{T}^0} \sum_{m=\max\{i,n_N\}}^{M_s-1} \sum_{k=\max\{i,n_N\}}^{m} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} + \sum_{m=\max\{i,n_N\}}^{M_s-1} \rho \sum_{k=\max\{i,n_N\}}^{m} \frac{\partial \mathcal{P}_s^{i,k}}{\partial \mathcal{T}^0}\mathcal{C}_{|N|,k-1}$$

$$-(M_s - 1)\frac{\partial \rho}{\partial \mathcal{T}^0} \sum_{k=\max\{i,n_N\}}^{M_s} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} + (1 - (M_s - 1)\rho) \sum_{k=\max\{i,n_N\}}^{M_s} \frac{\partial \mathcal{P}_s^{i,k}}{\partial \mathcal{T}^0}\mathcal{C}_{|N|,k-1}$$

If $\max\{i, n_N\} = M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \mathcal{T}^0} = (1 - (M_s - 1)\rho)\frac{\partial \mathcal{P}_s^{i,M_s}}{\partial \mathcal{T}^0}\mathcal{C}_{|N|,M_s-1} - (M_s - 1)\frac{\partial \rho}{\partial \mathcal{T}^0}\mathcal{P}_s^{i,M_s}\mathcal{C}_{|N|,M_s-1}$$

To calculate the derivatives of $\mathcal{P}_s^{i,k}$, there are three cases to consider.

Case 2: $i < k$

$$\begin{aligned}
\frac{\partial \mathcal{P}_s^{i,k}}{\partial \mathcal{T}^0} &= \frac{\exp(\mathcal{V}_s^i)}{\mathcal{S}_s^k}\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \mathcal{T}^0}\mathcal{S}_s^k \exp(-\mathcal{T}^{k-1}) \exp\Big(-\mathcal{S}_s^k \exp(-\mathcal{T}^{k-1})\Big) \\
&\quad -\frac{\partial \mathcal{T}^k}{\partial \mathcal{T}^0}\mathcal{S}_s^k \exp(-\mathcal{T}^k) \exp\Big(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)\Big)\Big] \\
&= \exp(\mathcal{V}_s^i)\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \mathcal{T}^0} \exp(-\mathcal{T}^{k-1}) \exp\Big(-\mathcal{S}_s^k \exp(-\mathcal{T}^{k-1})\Big) \\
&\quad -\frac{\partial \mathcal{T}_s^k}{\partial \mathcal{T}^0} \exp(-\mathcal{T}^k) \exp\Big(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)\Big)\Big] \\
&= \exp(\mathcal{V}_s^i)\Big[\exp(-\mathcal{T}^{k-1}) \exp\Big(-\mathcal{S}_s^k \exp(-\mathcal{T}^{k-1})\Big) - \exp(-\mathcal{T}^k) \exp\Big(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)\Big)\Big]
\end{aligned}$$

Case 3: $1 < i = k$

$$\frac{\partial \mathcal{P}_s^{k,k}}{\partial \mathcal{T}^0} = \frac{\partial A}{\partial \mathcal{T}^0} - \frac{\partial B}{\partial \mathcal{T}^0} + \frac{\partial C}{\partial \mathcal{T}^0}$$

where

$$\frac{\partial A}{\partial \mathcal{T}^0} = \frac{\partial \mathcal{T}^{k-1}}{\partial \mathcal{T}^0}[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)] \exp(-\mathcal{T}^{k-1}) \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big)$$

$$-\frac{\partial \mathcal{T}^{k}}{\partial \mathcal{T}^0}[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)] \exp(-\mathcal{T}^{k}) \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k})}\Big)$$

$$= [\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)]\Big[\exp(-\mathcal{T}^{k-1}) \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big)$$

$$-\exp(-\mathcal{T}^{k}) \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k})}\Big)\Big]$$

$$\frac{\partial B}{\partial \mathcal{T}^0} = \Big[1 - \frac{\exp(\mathcal{V}_s^k)}{\mathcal{S}_s^k}\Big]\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \mathcal{T}^0}\mathcal{S}_s^k \exp(-\mathcal{T}^{k-1}) \exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big)$$

$$-\frac{\partial \mathcal{T}^{k}}{\partial \mathcal{T}^0}\mathcal{S}_s^k \exp(-\mathcal{T}^{k}) \exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k})}\Big)\Big]$$

$$= \Big[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)\Big]\Big[\exp(-\mathcal{T}^{k-1}) \exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big)$$

$$-\exp(-\mathcal{T}^{k}) \exp\Big(- \frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k})}\Big)\Big]$$

$$\frac{\partial C}{\partial \mathcal{T}^0} = \frac{\partial \mathcal{T}^{k}}{\partial \mathcal{T}^0}\Big[(1 - \exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)))\exp(-\mathcal{T}^k)[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)] \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k})}\Big)$$

$$-\exp(\mathcal{V}_s^k - \mathcal{T}^k)\exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)) \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k})}\Big)\Big]$$

$$= (1 - \exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)))\exp(-\mathcal{T}^k)[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)] \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k})}\Big)$$

$$-\exp(\mathcal{V}_s^k - \mathcal{T}^k)\exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)) \exp\Big(- \frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k})}\Big)$$

Case 4: $i = k = 1$

$$\frac{\partial \mathcal{P}_s^{1,1}}{\partial \mathcal{T}^0} = -\frac{\partial \mathcal{T}^1}{\partial \mathcal{T}^0}\exp(\mathcal{V}_s^1 - \mathcal{T}^1)\exp(-\exp(\mathcal{V}_s^1 - \mathcal{T}^1)) = -\exp(\mathcal{V}_s^1 - \mathcal{T}^1)\exp(-\exp(\mathcal{V}_s^1 - \mathcal{T}^1))$$

5. Derivatives with respect to δ

$$\rho = \frac{\delta}{\mathcal{T}^0 - \omega}$$

$$\frac{\partial \rho}{\partial \delta} = \frac{1}{\mathcal{T}^0 - \omega}$$

$$\mathcal{T}^k = \mathcal{T}^0 - k\delta, \quad \text{for} \quad k = 1, \ldots, M_s$$

$$\frac{\partial \mathcal{T}^k}{\partial \delta} = -k, \quad \text{for} \quad k = 1, \ldots, M_s$$

Derivatives for searches with no purchases:

If $n_M < M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \delta} = \frac{\partial \rho}{\partial \delta} \sum_{m=n_N}^{M_s-1} \mathcal{N}_s^m \mathcal{C}_{|N|,m} + \rho \sum_{m=n_N}^{M_s-1} \frac{\partial \mathcal{N}_s^m}{\partial \delta} \mathcal{C}_{|N|,m}$$

$$-\frac{\partial \rho}{\partial \delta}(M_s - 1)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s} + (1-(M_s-1)\rho)\frac{\partial \mathcal{N}_s^{M_s}}{\partial \delta}\mathcal{C}_{|N|,M_s}$$

where

$$\mathcal{N}_s^k = \exp(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)), \quad \mathcal{S}_s^k = \sum_{m=1}^{k} \exp(\mathcal{V}_s^m)$$

$$\frac{\partial \mathcal{N}_s^k}{\partial \delta} = (-\frac{\partial \mathcal{T}^k}{\partial \delta})(-\mathcal{S}_s^k \exp(-\mathcal{T}^k))\exp(-\mathcal{S}_s^k \exp(-\mathcal{T}^k)) = -k\mathcal{S}_s^k \exp(-\mathcal{T}^k)\exp(-\mathcal{S}_s^k \exp(-\mathcal{T}^k))$$

If $n_M = M_s$:

$$\frac{\partial \mathbb{P}_s^{0,N}}{\partial \delta} = -\frac{\partial \rho}{\partial \delta}(M_s - 1)\mathcal{N}_s^{M_s}\mathcal{C}_{|N|,M_s} + (1-(M_s-1)\rho)\frac{\partial \mathcal{N}_s^{M_s}}{\partial \delta}\mathcal{C}_{|N|,M_s}$$

Derivatives for searches with purchases:

If $\max\{i, n_N\} < M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \delta} = \frac{\partial \rho}{\partial \delta} \sum_{m=\max\{i,n_N\}}^{M_s-1} \sum_{k=\max\{i,n_N\}}^{m} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} + \sum_{m=\max\{i,n_N\}}^{M_s-1} \rho \sum_{k=\max\{i,n_N\}}^{m} \frac{\partial \mathcal{P}_s^{i,k}}{\partial \delta}\mathcal{C}_{|N|,k-1}$$

$$-(M_s-1)\frac{\partial \rho}{\partial \delta} \sum_{k=\max\{i,n_N\}}^{M_s} \mathcal{P}_s^{i,k}\mathcal{C}_{|N|,k-1} + (1-(M_s-1)\rho) \sum_{k=\max\{i,n_N\}}^{M_s} \frac{\partial \mathcal{P}_s^{i,k}}{\partial \delta}\mathcal{C}_{|N|,k-1}$$

If $\max\{i, n_N\} = M_s$:

$$\frac{\partial \mathbb{P}_s^{i,N}}{\partial \delta} = (1-(M_s-1)\rho)\frac{\partial \mathcal{P}_s^{i,M_s}}{\partial \delta}\mathcal{C}_{|N|,M_s-1} - (M_s-1)\frac{\partial \rho}{\partial \delta}\mathcal{P}_s^{i,M_s}\mathcal{C}_{|N|,M_s-1}$$

To calculate the derivatives of $\mathcal{P}_s^{i,k}$, there are three cases to consider.

<u>Case 2</u>: $i < k$

$$\frac{\partial \mathcal{P}_s^{i,k}}{\partial \delta} = \frac{\exp(\mathcal{V}_s^i)}{\mathcal{S}_s^k}\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \delta}\mathcal{S}_s^k \exp(-\mathcal{T}^{k-1})\exp\Big(-\mathcal{S}_s^k\exp(-\mathcal{T}^{k-1})\Big)$$

$$-\frac{\partial \mathcal{T}^{k}}{\partial \delta}\mathcal{S}_s^k \exp(-\mathcal{T}^{k})\exp\Big(-\mathcal{S}_s^k\exp(-\mathcal{T}^{k})\Big)\Big]$$

$$= \exp(\mathcal{V}_s^i)\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \delta}\exp(-\mathcal{T}^{k-1})\exp\Big(-\mathcal{S}_s^k\exp(-\mathcal{T}^{k-1})\Big)$$

$$-\frac{\partial \mathcal{T}_s^{k}}{\partial \delta}\exp(-\mathcal{T}^{k})\exp\Big(-\mathcal{S}_s^k\exp(-\mathcal{T}^{k})\Big)\Big]$$

Case 3: $1 < i = k$

$$\frac{\partial \mathcal{P}_s^{k,k}}{\partial \delta} = \frac{\partial A}{\partial \delta} - \frac{\partial B}{\partial \delta} + \frac{\partial C}{\partial \delta}$$

where

$$\frac{\partial A}{\partial \delta} = [\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)]\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \delta}\exp(-\mathcal{T}^{k-1})\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^{k-1})}\Big)$$
$$-\frac{\partial \mathcal{T}^k}{\partial \delta}\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)\Big]$$
$$\frac{\partial B}{\partial \delta} = \Big[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)\Big]\Big[\frac{\partial \mathcal{T}^{k-1}}{\partial \delta}\exp(-\mathcal{T}^{k-1})\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^{k-1})}\Big)$$
$$-\frac{\partial \mathcal{T}^k}{\partial \delta}\exp(-\mathcal{T}^k)\exp\Big(-\frac{\mathcal{S}_s^k}{\exp(\mathcal{T}^k)}\Big)\Big]$$
$$\frac{\partial C}{\partial \delta} = \frac{\partial \mathcal{T}^k}{\partial \delta}\Big[(1-\exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k)))\exp(-\mathcal{T}^k)[\mathcal{S}_s^k - \exp(\mathcal{V}_s^k]\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)$$
$$-\exp(\mathcal{V}_s^k - \mathcal{T}^k)\exp(-\exp(\mathcal{V}_s^k - \mathcal{T}^k))\exp\Big(-\frac{\mathcal{S}_s^k - \exp(\mathcal{V}_s^k)}{\exp(\mathcal{T}^k)}\Big)\Big]$$

Case 4: $i = k = 1$

$$\frac{\partial \mathcal{P}_s^{1,1}}{\partial \delta} = -\frac{\partial \mathcal{T}^1}{\partial \delta}\exp(\mathcal{V}_s^1 - \mathcal{T}^1)\exp(-\exp(\mathcal{V}_s^1 - \mathcal{T}^1)) = \exp(\mathcal{V}_s^1 - \mathcal{T}^1)\exp(-\exp(\mathcal{V}_s^1 - \mathcal{T}^1))$$

1.3 Likelihood function L^x for the distribution of attributes

Consider an arbitrary observation of attributes x given by $x = (\text{LOGP} = z, \text{STAR} = m, \text{LOC} = n, \text{PRO} = k)$. Taking logarithms of the density function f_x in (4.51) yields the following expression for the log of the likelihood for observation x

$$\log f_x((z,m,n,k)|\lambda) = c(1)\log(q_1) + c(0)\log(1-q_1) + \sum_{(m,n)\in D} d(m,n)\Big(\log(p_{m,n}) + \log h(z; a_{m,n}, b)\Big) \tag{4.64}$$

where $p_{5,5}$ is understood to be given by equation (4.52). Then the partial derivatives with respect to the parameters of the log likelihood for observation x are given as follows:

$$\frac{\partial \log f_x(x;\lambda)}{\partial q_1} = c(1)\frac{1}{q_1} - c(0)\frac{1}{1-q_1}$$

For $(m,n) \in D\backslash\{5,5\}$

$$\frac{\partial \log f_x(x;\lambda)}{\partial p_{m,n}} = d(m,n)\frac{1}{p_{m,n}} - d(5,5)\frac{1}{1-\sum_{(k,l)\in D\backslash\{(5,5)\}} p_{k,l}}$$
$$\frac{\partial \log f_x(x;\lambda)}{\partial b} = \sum_{(m,n)\in D} d(m,n)\frac{\partial}{\partial b}\log h(z; a_{m,n}, b)$$

For $(m,n) \in D$

$$\frac{\partial \log f_x(x;\lambda)}{\partial a_{m,n}} = d(m,n)\frac{\partial}{\partial a_{m,n}} \log h(z; a_{m,n}, b)$$

where

$$\log h(z;a,b) = -\log(b) + \frac{z-a}{b} - \exp(\frac{z-a}{b})$$

So

$$\frac{\partial}{\partial a} \log h(z;a,b) = -\frac{1}{b} + \frac{1}{b}\exp(\frac{z-a}{b})$$

and

$$\frac{\partial}{\partial b} \log h(z;a,b) = -\frac{1}{b} - \frac{z-a}{b^2} + \exp(\frac{z-a}{b})\frac{z-a}{b^2} = -\frac{1}{b}\Big[1 + \frac{z-a}{b}\Big(1 - \exp(\frac{z-a}{b})\Big)\Big]$$

A4.2 Gradient vectors for estimating the variance of the estimator of χ

In Appendix 2, the delta method is used to obtain an estimate of the variance of the estimator of χ, both for the constant expectation and declining expectation models. This section provides expressions for the partial derivatives required to apply the delta method in the two models.

A4.2.1 Constant expectations model

Equation (4.56) determines the relationship between χ and the other parameters. In order to use the delta method to obtain the standard error of the estimate of χ, we use this equation to obtain the derivative of χ with respect to the other parameters. These derivatives are given as follows:

$$\frac{\partial \chi}{\partial q_1} = \sum_{(m,n)\in D} d(m,n) p_{m,n} \Big[\int_{-\infty}^{\infty} \log(1+\exp((z,m,n,1)\beta - \mathcal{T}))h(z;a_{m,n},b)\ \mathrm{d}z$$
$$- \int_{-\infty}^{\infty} \log(1+\exp((z,m,n,0)\beta - \mathcal{T}))h(z;a_{m,n},b)\ \mathrm{d}z\Big]$$

For $(m,n) \in D\backslash\{(5,5)\}$:

$$\frac{\partial \chi}{\partial p_{m,n}} = \sum_{k\in\{0,1\}} c(k) q_k \Big[\int_{-\infty}^{\infty} \log(1+\exp((z,m,n,k)\beta - \mathcal{T}))h(z;a_{m,n},b)\ \mathrm{d}z$$
$$- \int_{-\infty}^{\infty} \log(1+\exp((z,5,5,k)\beta - \mathcal{T}))h(z;a_{5,5},b)\ \mathrm{d}z\Big]$$

For $(m,n) \in D$:

$$\frac{\partial \chi}{\partial a_{m,n}} = \sum_{k\in\{0,1\}} c(k) q_k p_{m,n} \int_{-\infty}^{\infty} \log(1+\exp((z,m,n,k)\beta - \mathcal{T}))\frac{\partial}{\partial a_{m,n}} h(z;a_{m,n},b)\ \mathrm{d}z$$

where

$$\frac{\partial h}{\partial a} = \frac{1}{b}\Big[-\frac{1}{b}\exp(\frac{z-a}{b})\exp(-\exp(\frac{z-a}{b})) - \frac{1}{b}(-\exp(\frac{z-a}{b}))\exp(-\exp(\frac{z-a}{b}))\exp(\frac{z-a}{b})\Big]$$

$$= \frac{1}{b^2}\exp(\frac{z-a}{b})\exp(-\exp(\frac{z-a}{b}))\Big[\exp(\frac{z-a}{b})-1\Big] = \frac{h(z;a,b)}{b}\Big[\exp(\frac{z-a}{b})-1\Big]$$

$$\frac{\partial \chi}{\partial b} = \sum_{k\in\{0,1\}} c(k)q_k \sum_{(m,n)\in D} d(m,n)p_{m,n}\int_{-\infty}^{\infty}\log(1+\exp((z,m,n,k)\cdot\beta - \mathcal{T}))\frac{\partial}{\partial b}h(z;a_{m,n},b)\ \mathrm{d}z$$

$$\frac{\partial h}{\partial b} = -\frac{1}{b^2}\exp(\frac{z-a}{b})\exp(-\exp(\frac{z-a}{b})) + \frac{1}{b}\Big[-\frac{z-a}{b^2}\exp(\frac{z-a}{b})\exp(-\exp(\frac{z-a}{b}))$$

$$-\frac{z-a}{b^2}(-\exp(\frac{z-a}{b}))\exp(-\exp(\frac{z-a}{b}))\exp(\frac{z-a}{b})\Big]$$

$$= \frac{1}{b^2}\exp(\frac{z-a}{b})\exp(-\exp(\frac{z-a}{b}))\Big[\frac{z-a}{b}\exp(\frac{z-a}{b}) - \frac{z-a}{b} - 1\Big]$$

$$= \frac{h(z;a,b)}{b}\Big[\frac{z-a}{b}\exp(\frac{z-a}{b}) - \frac{z-a}{b} - 1\Big]$$

For $j = 1, \dots, J$

$$\frac{\partial \chi}{\partial \beta_j} = \sum_{k\in\{0,1\}} c(k)q_k \sum_{(m,n)\in D} d(m,n)p_{m,n}\int_{-\infty}^{\infty}\frac{x^j\exp((z,m,n,k)\beta - \mathcal{T})}{1+\exp((z,m,n,k)\beta-\mathcal{T})}h(z;a_{m,n},b)\ \mathrm{d}z$$

(where $J = 4, x^1 = z, x^2 = m, x^3 = n, x^4 = k$).

$$\frac{\partial \chi}{\partial \mathcal{T}} = \sum_{k\in\{0,1\}} c(k)q_k \sum_{(m,n)\in D} d(m,n)p_{m,n}\int_{-\infty}^{\infty}\frac{-\exp((z,m,n,k)\beta - \mathcal{T})}{1+\exp((z,m,n,k)\beta-\mathcal{T})}h(z;a_{m,n},b)\ \mathrm{d}z$$

A4.2.2 Declining expectations model

The threshold equation for the declining expectations is obtained by replacing $-\mathcal{T}$ in the threshold equation for the constant expectation model (4.56) by $16\delta + \gamma - \mathcal{T}^0$. The gradient of the threshold equation is obtained by taking the derivative of χ in this equation with respect to the parameters $\mathcal{T}_0, \delta, \lambda$ and β. These derivatives are the same as in the constant expectations model, except that, in the expressions for the derivatives (1) $\frac{\partial\chi}{\partial\mathcal{T}}$ is replaced by $\frac{\partial\chi}{\partial\mathcal{T}^0}$ (2) instances of $-\mathcal{T}$ are replaced by $16\delta+\gamma-\mathcal{T}^0$. Also we need to include an expression for $\frac{\partial\chi}{\partial\delta}$, which is given as follows:

$$\frac{\partial \chi}{\partial \delta} = \sum_{k\in\{0,1\}} c(k)q_k \sum_{(m,n)\in D} d(m,n)p_{m,n}\int_{-\infty}^{\infty}\frac{16\exp((z,m,n,k)\beta + 16\delta + \gamma - \mathcal{T}^0)}{1+\exp((z,m,n,k)\beta+16\delta+\gamma-\mathcal{T}^0)}h(z;a_{m,n},b)\ \mathrm{d}z$$

Appendix 5: Destinations

Thus far we have omitted from the analysis any effect of destination on observed choices. For a given destination, it may be that DMs who select that destination tend to have a systematically different preferences than those who select a distinct destination. Given the potential for self-selection, we cannot simply include destination dummies alongside the other attributes in the random utility model. The simplest way of incorporating destination into the analysis is to estimate different models for different destinations, or at least different classes of destinations. The results in our paper were obtained from a dataset comprising the top ten destinations (the ten destinations with the most searches). Table A2 compares these results to the results of estimating the model for (i) a dataset for the top five destinations and (ii) a dataset for the most popular destination.

Table A2: Estimates by destination class

	Top Ten	Standard Error	Top Five	Standard Error	Most Popular	Standard Error
<u>number of</u>:						
searches	6044		3774		1299	
observations	176263		111378		38475	
purchases	304		228		101	
<u>estimate of</u>:						
initial threshold, $\mathcal{T}^0$	6.5946**	0.5106	6.2427**	0.5946	5.5750**	0.9041
expectation fall, δ	0.02299**	0.006360	0.02358**	0.00710	0.01783	0.01036
click probability, κ	0.01076**	0.000164	0.01151 **	0.000220	0.01352**	0.000402
lower bound, ω	5.6753**	0.4958	5.2747**	0.5671	4.6656**	0.8958
information cost, χ	0.000793**	0.0000696	0.000976**	0.0000982	0.00134**	0.000371
<u>coefficient of</u>:						
log price, β_1	-0.8700**	0.1137	-0.8060**	0.1251	-0.9504**	0.2092
stars, β_2	0.4920**	0.0921	0.3936**	0.1092	0.4590*	0.2133
location, β_3	0.2057**	0.0507	0.1716**	0.0618	0.0647	0.1191
promotion, β_4	0.4647**	0.1225	0.4097**	0.1422	0.7182**	0.2519

(* $p < 0.05$, ** $p < 0.01$)

The signs of the parameters are aligned across the three sets of results. Further, like the estimates for the top-ten dataset, the estimates for the top-five dataset are significantly different from zero at a one percent level. When we turn to the dataset for the most popular destination, however, the standard errors are notably higher than those for the top-ten dataset. This is because the number of searches for the most popular destination is less than a quarter of that in the top-ten dataset; and the number of purchases is less than a third of that in the top-ten dataset. For the dataset for the most popular destination, the estimates of β_3 and δ are no longer significant at a five per cent level. The situation is worse for other destinations; for example, moving from the dataset for the most popular destination to the second most popular, the number of searches falls from 1299 to 872, and the number of purchases from 101 to 35. Accordingly, the model becomes insufficiently precise when applied to individual destinations. Nevertheless, our paper can be interpreted as illustrating how we could model online choice for individual destinations if there were more data available for those destinations.

Appendix 6: Data cleaning

The steps in the data cleaning process were as follows:

(1) We follow Ursu (2018) regarding the removal of data based on price and tax. There are some very low and very high prices. There are also prices inconsistent with observed gross payments made to a hotel. So we eliminate searches with a hotel whose price exceeds US\$1000 or is less than US\$10. We also eliminate searches with an observation with implausibly low or implausibly high implied hotel taxes. That is, we eliminate searches with observations such that either: (i) 0.3 $<$ implied tax rate = [gross bookings /(price $\times$ length of stay $\times$ number of rooms)] $-$ 1; or (ii) US\$1 $>$ implied tax on price = [gross bookings/(length of stay $\times$ number of rooms)] $-$ price.

(2) We then remove all searches with at least one of the following properties.

(i) The search includes an "opaque position" 5, 11, 17, 23. These positions include "opaque offers" which, typically, are discounted offers for which the customer is not provided with the name of the hotel, but is given some information about it. For a discussion of opaque positions, see Ursu (2018, p.540). In our paper, the "i^{th}" position strictly speaking means the i^{th} non-opaque position.

(ii) The search is not "consecutive". A search is said to be "consecutive" if it (a) includes an observation in position 1 and (b) includes an observation at position $k-1$ if it includes an observation in position $k > 1$ (where the "i^{th}" position means the i^{th} non-opaque position).

(iii) The search includes an observation where the star rating is 0 or 1. Star ratings of 0 are removed because this value may be assigned when the rating is unknown or cannot be publicised. Star ratings of 1 are removed because only a small proportion of observations have a star rating of 1, so removing these data points simplifies the estimation of the attribute distribution in Appendix 2.

(iv) The search includes an observation where the review score is NULL.

(3) We then remove all searches where the first page of search results is unusually long or short. That is, we eliminate all searches where the final position on the search page is (i) 9 or less (fewer than 10 non-opaque observations), or (ii) 34 or more (more than 33 non-opaque observations).

(4) We then remove all searches with non-random rankings.

(5) We discretise the location score, assigning it a value of 1 for values less than 2, 5 for values more than 4, and 3 otherwise. This simplifies the estimation of the attribute distribution in Appendix 2.

(6) Using the data resulting from steps (1)-(5) , we ascertain, for each destination, how many searches there are for that destination. We then eliminate all searches except those for the ten "most popular" destinations - that is, the ten destinations with the most searches.

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