On an identity for the volume integral of the square of a vector field

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A proof is given of the vector identity proposed by Gubarev, Stodolsky, and Zakarov that relates the volume integral of the square of a three-vector field to nonlocal integrals of the curl and divergence of the field. The identity is applied to the vector potential and magnetic field of a rotating charged shell. The latter provides a straightforward application of the use of the addition theorem of spherical harmonics. © 2007 American Association of Physics Teachers.
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I. INTRODUCTION

Gubarev *et al.*¹ have reported an interesting identity for a three-vector field $\mathbf{A}(\mathbf{x})$ ($\mathbf{x}=x^1, x^2, x^3$) that is well behaved and vanishes sufficiently rapidly at spatial infinity. The identity is

$$\int d^3x \mathbf{A}^2(\mathbf{x}) = \int d^3x \int d^3x' \frac{\nabla \cdot \mathbf{A}(\mathbf{x}) \nabla' \cdot \mathbf{A}(\mathbf{x}') + \nabla \times \mathbf{A}(\mathbf{x}) \cdot \nabla' \times \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}. \quad (1)$$

The identity has obvious applications to electromagnetic theory. If $\mathbf{A}$ is the magnetic vector potential in the Coulomb gauge with $\nabla \cdot \mathbf{A} = 0$, the first term on the right-hand side of Eq. (1) vanishes, and the right-hand side of Eq. (1) can be expressed solely in terms of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\int d^3x \mathbf{A}^2(\mathbf{x}) = \int d^3x \int d^3x' \frac{\mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}. \quad (2)$$

Gubarev *et al.*¹ point out that Eq. (1) provides a gauge-invariant limitation on $\mathbf{A}(\mathbf{x})$; that is, for a given magnetic field, Eq. (2) gives the minimum value for the volume integral of the square of the vector potential and this minimum value occurs in the Coulomb gauge. The generalization of the identity (1) to four-vectors in order to study the topological structure of gauge field theories was given in Ref. 1.

In Ref. 1 it was claimed that Eq. (1) can be derived by analogy with electrostatics and magnetostatics. However, such a derivation would miss terms that depend on both $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ and vanish when either vanish. Alternatively, it was claimed that Eq. (1) can be derived by extending to position space the identity $(\mathbf{k} \times \mathbf{A})^2 = \mathbf{k}^2 \mathbf{A}^2 - (\mathbf{k} \cdot \mathbf{A})^2$ in momentum space. This argument may be too cryptic for students beginning the study of electromagnetism.

In Sec. II we derive Eq. (1) based on elementary vector analysis and calculus. In Sec. III we note the form of the energy of the electromagnetic field when Eq. (1) is used. We apply Eq. (1) in Sec. IV to the rotating charged spherical shell² and illustrate the use of the addition theorem of spherical harmonics. In the Appendix we show that each of the terms on the right-hand side of Eq. (1) is greater than or equal to zero.

II. DERIVATION OF THE IDENTITY

We start from the theorem of Helmholtz,³⁻⁶ which states that a vector field $\mathbf{A}(\mathbf{x})$ that is well behaved and vanishes sufficiently rapidly at infinity may be expressed in the form

$$\mathbf{A}(\mathbf{x}) = -\nabla f(\mathbf{x}) + \nabla \times \mathbf{F}(\mathbf{x}) = \mathbf{A}_l(\mathbf{x}) + \mathbf{A}_t(\mathbf{x}), \quad (3)$$

where the first and second terms are the longitudinal (l) and transverse (t) terms. The scalar $f(\mathbf{x})$ and vector $\mathbf{F}(\mathbf{x})$ potentials are

$$f(\mathbf{x}) = \int d^3x' \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \int d^3x' \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}, \quad (4)$$

where $\nabla = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z$ is the vector differential operator with respect to $\mathbf{x}$ and ∇' is the operator with respect to $\mathbf{x}'$. We use Eq. (4) to write Eq. (1) as

$$\int d^3x \mathbf{A}^2(\mathbf{x}) = \int d^3x (-\nabla f + \nabla \times \mathbf{F}) \cdot (-\nabla f + \nabla \times \mathbf{F}). \quad (5)$$

If we use the vector identity

$$\nabla \cdot (f \nabla \times \mathbf{F}) = (\nabla \times \mathbf{F}) \cdot \nabla f, \quad (6)$$

we find that the integral of the cross terms in Eq. (5) are zero because they become the integral of a total divergence which, using Gauss's theorem, turns into a surface integral that vanishes if the fields decay sufficiently rapidly at infinity.

The longitudinal term $\int d^3x \mathbf{A}_l^2(\mathbf{x}) = \int d^3x \nabla f \cdot \nabla f$ is obtained by using the identity

$$\nabla \cdot (f \nabla f) = f \nabla^2 f + \nabla f \cdot \nabla f. \quad (7)$$

From

$$\nabla^2 f(\mathbf{x}) = \int d^3x' \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{4\pi} \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad (8)$$

together with the identity^{3,7}

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \delta(\mathbf{x} - \mathbf{x}'), \quad (9)$$

where δ is the Dirac delta function, we find

$$\nabla^2 f(\mathbf{x}) = -\nabla \cdot \mathbf{A}(\mathbf{x}). \quad (10)$$

Because the integral of the left-hand side of Eq. (7) gives a vanishing surface integral, the integral of the first term on the right-hand side of Eq. (7) becomes

$$\int d^3x A_1^2(\mathbf{x}) = \int d^3x \int d^3x' \frac{\nabla \cdot \mathbf{A}(\mathbf{x}) \nabla' \cdot \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}. \quad (11)$$

The integral of the transverse term is calculated with the help of the identity

$$\begin{aligned} \nabla \cdot [\mathbf{F} \times (\nabla \times \mathbf{F})] \\ = (\nabla \times \mathbf{F}) \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot \nabla \times (\nabla \times \mathbf{F}) \\ = (\nabla \times \mathbf{F}) \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot [\nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}], \end{aligned} \quad (12)$$

thereby giving, due to the vanishing of the volume integral of the left-hand side of Eq. (12),

$$\int d^3x A_1^2(\mathbf{x}) = \int d^3x \mathbf{F} \cdot [\nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}]. \quad (13)$$

The divergence of $\mathbf{F}$,

$$\nabla \cdot \mathbf{F}(\mathbf{x}) = \int d^3x' \nabla \cdot \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}, \quad (14)$$

can be calculated from the identities

$$\nabla' \cdot \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \nabla' \times \mathbf{A}(\mathbf{x}') \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

and

$$\nabla \cdot \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \nabla' \times \mathbf{A}(\mathbf{x}') \cdot \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \quad (15)$$

We find that

$$\nabla \cdot \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = -\nabla' \cdot \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \quad (16)$$

and hence the integrand in Eq. (14) is a perfect divergence that turns into a vanishing surface integral. It follows that $\nabla \cdot \mathbf{F} = 0$ and the first term on the right-hand side of Eq. (13) vanishes.

With the help of Eq. (9) we find

$$\nabla^2 \mathbf{F} = \int d^3x' \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{4\pi} \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -\nabla \times \mathbf{A}(\mathbf{x}), \quad (17)$$

and the second term on the right-hand side of Eq. (13) becomes

$$\int d^3x A_1^2(\mathbf{x})|_2 = \int d^3x \int d^3x' \frac{\nabla \times \mathbf{A}(\mathbf{x}) \cdot \nabla' \times \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}. \quad (18)$$

Hence we have derived Eq. (1). Because Eqs. (3) and (4) are valid for fields that depend on time,^{4,8} Eq. (1) also applies to fields that depend on time.

III. ENERGY OF THE ELECTROMAGNETIC FIELD

If we use Maxwell's equations for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$, the energy of the electric field (in SI units) from Eq. (1) is

$$\begin{aligned} U_E(t) &= \frac{\epsilon_0}{2} \int d^3x \mathbf{E}^2 \\ &= \frac{1}{8\pi\epsilon_0} \int d^3x \int d^3x' \left[\frac{\rho(\mathbf{x},t)\rho(\mathbf{x}',t)}{|\mathbf{x} - \mathbf{x}'|} \right. \\ &\quad \left. + \frac{\epsilon_0^2 \partial \mathbf{B}(\mathbf{x},t)/\partial t \cdot \partial \mathbf{B}(\mathbf{x}',t)/\partial t}{|\mathbf{x} - \mathbf{x}'|} \right]. \end{aligned} \quad (19)$$

The first term describes the instantaneous energy of the charges present, and the second describes the energy in the electric field induced through Faraday's law when $\mathbf{B}$ changes with time. This form generalizes the usual electrostatic result for the energy of a charge distribution.

If we write the total current $\mathbf{g}$ in terms of the charge transport current $\mathbf{j}$ and the displacement current

$$\mathbf{g}(\mathbf{x},t) = \mathbf{j}(\mathbf{x},t) + \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t}, \quad (20)$$

and Maxwell's equation in the form

$$\nabla \times \mathbf{B}(\mathbf{x},t) = \mu_0 \mathbf{g}(\mathbf{x},t), \quad (21)$$

the energy of the magnetic field may be expressed, using Eq. (1), as

$$\begin{aligned} U_M(t) &= \frac{1}{2\mu_0} \int d^3x \mathbf{B}^2(\mathbf{x},t) \\ &= \frac{\mu_0}{8\pi} \int d^3x \int d^3x' \frac{\mathbf{g}(\mathbf{x},t) \cdot \mathbf{g}(\mathbf{x}',t)}{|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \quad (22)$$

Equation (22) has the same form as the energy of a static current distribution. There are no magnetic charges because $\nabla \cdot \mathbf{B} = 0$, so the term corresponding to the first term on the right-hand side of Eq. (19) is missing. As shown in the Appendix, each of the terms of Eq. (19) is greater than or equal to zero as are both sides of Eq. (22).

IV. APPLICATION TO THE ROTATING CHARGED SPHERICAL SHELL

We discuss a simple model by which the validity of Eq. (1) can be verified for a particular case of the vector potential. The model is a charged spherical shell of radius R rotating with an angular velocity $\boldsymbol{\omega}$ about the z axis through its center.² The shell has a charge Q uniformly distributed over its surface.

In spherical coordinates $\mathbf{x} = (r, \theta, \phi)$, the charge and current densities are

$$\rho(\mathbf{x}) = \frac{Q}{4\pi R^2} \delta(r - R) \quad \text{and} \quad \mathbf{j}(\mathbf{x}) = \frac{Q}{4\pi R^2} \delta(r - R) \boldsymbol{\omega} \times \mathbf{x}, \quad (23)$$

and the magnetic moment of the shell is

$$\mathbf{m} = \int d^3x \mathbf{x} \times \mathbf{j}(\mathbf{x})/2 = \boldsymbol{\omega} QR^2/3. \quad (24)$$

The magnetic fields and potentials are given in Ref. 2. They are

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 \mathbf{m} \times \mathbf{x}}{4\pi R^3} \quad \text{and} \quad \mathbf{B}(\mathbf{x}) = \frac{\mu_0 \mathbf{m}}{2\pi R^3} \quad (0 < r < R) \quad (25)$$

and

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 \mathbf{m} \times \mathbf{x}}{4\pi r^3} \quad \text{and} \quad \mathbf{B}(\mathbf{x}) = \frac{\mu_0 [3\hat{\mathbf{x}}(\mathbf{m} \cdot \hat{\mathbf{x}}) - \mathbf{m}]}{4\pi r^3} \quad (R < r < \infty). \quad (26)$$

The magnetic vector potential $\mathbf{A}$ is continuous at $r=R$ and satisfies the condition $\nabla \cdot \mathbf{A} = 0$ everywhere. Inside the shell the magnetic field $\mathbf{B}$ is uniform. Outside the shell $\mathbf{B}$ has the form given by a magnetic dipole. The normal component of $\mathbf{B}$ is continuous at the surface of the shell.

We first calculate the volume integral of $\mathbf{A}^2$ in the region $0 < r < R$. From

$$(\mathbf{m} \times \mathbf{x}) \cdot (\mathbf{m} \times \mathbf{x}) = \mathbf{m}^2 r^2 - (\mathbf{m} \cdot \mathbf{x})^2 = \mathbf{m}^2 r^2 \sin^2 \theta \quad (27)$$

we obtain

$$\int_{r < R} d^3x \mathbf{A}^2(\mathbf{x}) = \left(\frac{\mu_0}{4\pi} \right)^2 \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^R r^2 dr \frac{\mathbf{m}^2 r^2 \sin^2 \theta}{R^6} = \frac{(\mu_0 \mathbf{m})^2}{30\pi R}, \quad (28)$$

using $\int_0^\pi \sin^3 \theta d\theta = 4/3$. For $R < r < \infty$ we find

$$\int_{R < r} d^3x \mathbf{A}^2(\mathbf{x}) = \left(\frac{\mu_0}{4\pi} \right)^2 \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_R^\infty r^2 dr \frac{\mathbf{m}^2 r^2 \sin^2 \theta}{r^6} = \frac{(\mu_0 \mathbf{m})^2}{6\pi R}. \quad (29)$$

If we add Eqs. (28) and (29), we obtain for the volume integral of $\mathbf{A}^2$ over all space

$$\int d^3x \mathbf{A}^2(\mathbf{x}) = \frac{(\mu_0 \mathbf{m})^2}{5\pi R}. \quad (30)$$

We now evaluate the nonlocal integral on the right-hand side of Eq. (2). We write $1/|\mathbf{x} - \mathbf{x}'|$ as⁷

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} P_l(\cos \gamma) \frac{r_{<}^l}{r_{>}^{l+1}}, \quad (31)$$

where P_l is a Legendre polynomial, $r_{<}$ and $r_{>}$ are the lesser and greater parts of the magnitude of $\mathbf{x}$ and $\mathbf{x}'$, and γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$ with $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$. The addition theorem for the spherical harmonics $Y_{lm}(\theta, \phi)$,⁷

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi), \quad (32)$$

gives

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \quad (33)$$

The orthogonality condition for the spherical harmonics is

$$\int d\Omega Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{l'l} \delta_{m'm}, \quad (34)$$

where Ω is the solid angle corresponding to θ, ϕ . If the terms that we integrate are independent of ϕ , we deduce that only the $m=0$ terms of Eq. (33) survive the angular integrations, and Eq. (33) becomes

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} \Rightarrow \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{l0}(\theta') Y_{l0}(\theta). \quad (35)$$

The double integration of the right-hand side of Eq. (2) is composed of inside-inside, outside-outside, and inside-outside contributions. The first contribution is the simplest because the magnetic field is uniform over the volume inside the sphere. The integral of the right-hand side of Eq. (2) for the inside-inside region is

$$\frac{\mathbf{B}^2}{4\pi} \sum_{l=0}^{\infty} \int d\Omega Y_{l0}(\theta) \int d\Omega' Y_{l0}(\theta') \int_0^R dr r^2 \int_0^R dr' r'^2 \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}}, \quad (36)$$

where the volume element $d^3x = d\Omega r^2 dr$. We write $1 = 2\pi^{1/2} Y_{00}$ in the angular integrals⁷ and, from Eq. (34), the integral vanishes unless $l=0$. Equation (36) then becomes

$$4\pi \mathbf{B}^2 \int_0^R dr r^2 \int_0^R dr' r'^2 \frac{1}{r_{>}} \quad (37)$$

or

$$4\pi \mathbf{B}^2 \int_0^R dr r^2 \left[\int_0^r dr' r'^2 \frac{1}{r} + \int_r^R dr' r'^2 \frac{1}{r'} \right] = 4\pi \mathbf{B}^2 \frac{2R^5}{15}, \quad (38)$$

where the first integral is for the region $0 < r' < r$ and the second for $r < r' < R$. The integral in Eq. (38) is easily evaluated to give $8\pi \mathbf{B}^2 R^3/15$ or $2(\mu_0 \mathbf{m})^2/15\pi R$.

From Eqs. (25) and (26) the integral of the right-hand side of Eq. (2) for the inside-outside region contains terms that depend on θ and θ' with $0 < r' < R$ and $R < r < \infty$ in the form

$$\begin{aligned} \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}') &= 2 \left(\frac{\mu_0}{4\pi} \right)^2 \frac{3(\mathbf{m} \cdot \hat{\mathbf{x}}) - \mathbf{m}^2}{R^3 r^3} \\ &= 2 \left(\frac{\mu_0 \mathbf{m}}{4\pi} \right)^2 \frac{3 \cos^2 \theta - 1}{R^3 r^3}. \end{aligned} \quad (39)$$

We note that Eq. (39) is proportional to the spherical harmonic $Y_{20} = \frac{1}{4} \sqrt{5/\pi} (3 \cos^2 \theta - 1)$. Hence, the integrals over the angles are proportional to

$$\sum_{l,m} \int d\Omega Y_{lm}(\theta, \phi) Y_{20}(\theta) \int d\Omega' Y_{lm}(\theta', \phi'). \quad (40)$$

From Eq. (34) it is required that $l=2$ from the θ integral and $l=0$ from the θ' integral for Eq. (40) to be nonvanishing. Because it is impossible to satisfy both of these conditions simultaneously, the inside-outside integral vanishes.

From Eq. (26) the integral of the right-hand side of Eq. (2) for the outside-outside region contains terms that depend on θ and θ' in the form

$$\begin{aligned}\mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}') &= \left(\frac{\mu_0}{4\pi} \right)^2 \frac{\mathbf{m}^2 - 3(\mathbf{m} \cdot \hat{\mathbf{x}})^2 - 3(\mathbf{m} \cdot \hat{\mathbf{x}}')^2 + 9\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}'(\mathbf{m} \cdot \hat{\mathbf{x}})(\mathbf{m} \cdot \hat{\mathbf{x}}')}{r^3 r'^3} \\ &= \left(\frac{\mu_0 \mathbf{m}}{4\pi} \right)^2 \frac{1 - 3(\cos^2 \theta + \cos^2 \theta') + 9 \cos \gamma \cos \theta \cos \theta'}{r^3 r'^3}.\end{aligned}\quad (41)$$

It can be verified, for example by using symbolic manipulation software, that the numerator of Eq. (41) can be expressed as

$$\begin{aligned}&1 - 3(\cos^2 \theta + \cos^2 \theta') + 9 \cos \gamma \cos \theta \cos \theta' \\ &= \frac{16\pi}{5} \left\{ Y_{20}(\theta) Y_{20}(\theta') + \frac{3}{4} [Y_{21}^*(\theta, \phi) Y_{21}(\theta', \phi') \right. \\ &\quad \left. + Y_{2-1}^*(\theta, \phi) Y_{2-1}(\theta', \phi')] \right\},\end{aligned}\quad (42)$$

where $Y_{2\pm 1}(\theta, \phi) = \mp \sqrt{15/8\pi} \cos \theta \sin \theta e^{\pm i\phi}$. If we use Eq. (34), the integrals over the angles of the spherical harmonics in Eqs. (33) and (42) produce a factor of 8π . The integration for this region therefore becomes

$$\frac{8\pi}{5} \left(\frac{\mu_0 \mathbf{m}}{4\pi} \right)^2 \int_R^\infty dr r^2 \int_R^\infty dr' r'^2 \frac{1}{r^3 r'^3} \frac{r_{<}^2}{r_{>}^3}, \quad (43)$$

with the $(2l+1)$ factor in Eq. (33) for $l=2$ giving a factor of 5 in the denominator. The explicit integrations over r and r' yield (44)

$$\begin{aligned}&\frac{(\mu_0 \mathbf{m})^2}{10\pi} \int_R^\infty \frac{dr}{r} \left[\int_R^r \frac{dr'}{r'} \frac{r'^2}{r^3} + \int_r^\infty \frac{dr'}{r'} \frac{r^2}{r'^3} \right] \\ &= \frac{(\mu_0 \mathbf{m})^2}{10\pi} \frac{2}{3R} = \frac{(\mu_0 \mathbf{m})^2}{15\pi R},\end{aligned}\quad (44)$$

where the first integral is for the region $R < r' < r$ and the second for $r < r' < \infty$. If we add the result of Eq. (44) to the inside-inside value of $2(\mu_0 \mathbf{m})^2/15\pi R$, we obtain the total contribution of $(\mu_0 \mathbf{m})^2/5\pi R$, in agreement with the value of $(\mu_0 \mathbf{m})^2/5\pi R$ obtained from the volume integral of $\mathbf{A}^2$. Hence, we have verified the identity of Ref. 1 for the rotating charged shell in the Coulomb gauge.

V. SUMMARY

A new identity for three-vector fields¹ was derived using vector analysis. The identity is of interest in electromagnetism because it shows how, for a given distribution of magnetic field, the integrated value of the square of the vector potential has a minimum value when the Coulomb gauge condition holds. In this sense the Coulomb gauge can be said to be a minimal gauge.

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APPENDIX: PROOF OF AN INEQUALITY

To show that both terms in Eq. (1) are greater than or equal to zero we need to show that

$$I = \int d^3x \int d^3x' \frac{f(\mathbf{x})f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \geq 0 \quad (A1)$$

for any well behaved scalar function $f(\mathbf{x})$. We do so by substituting into Eq. (A1) the interaction term in the form⁷

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{2\pi^2} \int d^3q \frac{e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')}}{q^2} \quad (A2)$$

to obtain

$$I = \frac{1}{2\pi^2} \int d^3q \frac{1}{q^2} \int d^3x \int d^3x' f(\mathbf{x})f(\mathbf{x}') e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')}.\quad (A3)$$

Equation (A3) can be written as

$$I = 4\pi \int d^3q \frac{g(\mathbf{q})g(-\mathbf{q})}{q^2}, \quad (A4)$$

where $g(\mathbf{q})$ is the Fourier transform of $f(\mathbf{x})$:

$$g(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-i\mathbf{q} \cdot \mathbf{x}} f(\mathbf{x}). \quad (A5)$$

If $f(\mathbf{x})$ is real, then $g(-\mathbf{q}) = g^*(\mathbf{q})$, and thus from Eq. (A4) $I \geq 0$.

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