Generalised Geometries and Lie Algebroid Gauging in String Theory

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Declaration

I declare that the material presented in this thesis is original, except where otherwise stated, and has not been submitted in whole or part for a degree at any university.

K Wright

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To those who inspired it and will not read it.

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Abstract

This thesis investigates the role of algebroid geometries in string theory. Differential geometry provides the framework for general relativity and point particle dynamics. The dynamics of strings and higher dimensional branes are most naturally described by algebroid geometry on vector bundles—with fluxes being incorporated as geometric data describing twisted vector bundles. This thesis contains original results in two areas: Firstly, twisted generalised contact structures and generalised coKähler structures are studied. Secondly, a non-isometric gauging proposal based on Lie algebroids is studied from a geometric perspective. We study generalised contact structures from the point of view of reduced generalised complex structures; naturally incorporating non-coorientable structures as non-trivial fibering. The infinitesimal symmetries are described in detail with a geometric description given in terms of gerbes. As an application of the reduction procedure we define generalised coKähler structures in a way which extends the Kähler/coKähler correspondence. An invariant geometric approach to the Lie algebroid gauging proposal of Kotov and Strobl [114, 99, 89, 90] is presented. The existing literature on Lie algebroid gauging is purely local. We consider global aspects through the integrability of a local algebroid action. The main result is that it is always possible to provide a local non-isometric gauging for any arbitrary background. The necessary and sufficient conditions to gauge with respect to a given choice of vector fields are given. However, requiring a gauge invariant field strength term restricts to Lie groupoids that are locally isomorphic to Lie groups. As an application of this work the proposal of Chatzistavrakidis, Deser, and Jonke for "T-duality without isometry" is studied. We show that this non-isometric T-duality proposal is in fact equivalent to non-abelian T-duality by an appropriate field redefinition.

Contents

Chapter 1

Introduction

This thesis investigates the role of algebroid geometries—with particular focus on symmetries associated to extended objects. Differential geometry provides the framework for general relativity and point particle dynamics. The dynamics of strings and higher dimensional branes are most naturally described by algebroid geometry on vector bundles; fluxes are incorporated as geometric data.

Chapter 2 gives motivation for studing algebroids from both a geometric and physics perspective. The chapter has three aims: Firstly, convince the reader of the ubiquitous nature of algebroids in geometry and physics. Secondly, give an opportunity to present material used throughout the rest of the thesis. Thirdly, the chapter explains how an extension of differential geometric structures from the tangent bundle to more general vector bundles arise naturally in physics models. The chapter bridges the gap between algebroid geometry and the associated physics; hopefully making the thesis accessible to mathematicians and physicists.

Chapter 3 provides the more technical background material for the rest of the thesis. This material is intended to familiarise the reader with various aspects of algebroid geometry. The Chapter provides a review of definitions and theorems on Lie groupoids (and Lie algebroids) which are relevant for the remainder of the thesis. Many standard constructions of differential geometry are generalised to the case of vector bundles endowed with a Leibniz algebroid structure.

Chapter 4 studies generalised contact structures and contains original results. The chapter is closely based on a paper written by the author of this thesis [127]. Generalised contact structures are the odd-dimensional analogue of the well known generalised complex structures of Hitchin and Gualtieri. We study generalised contact structures from the perspective of reduced generalised complex structures. While generalised contact structures have been studied in the literature before there are several new contributions: The spinor description of generalised contact structures is modified to describe the full set of infinitesimal symmetries (Definition 4.14 and the related Theorem 4.15). The symmetries are given a geometric description in terms of gerbes. This interpretation allows us to use twisted algebroids to describe non-coorientable structures as non-trivial fibering. We provide an application of the reduction procedure: generalised coKähler structures are defined in a way which extends the Kähler/coKähler correspondence. It is shown that T-duality maps generalised coKähler structures to other generalised coKähler structures (Proposition 4.19).

Chapter 5 gives an invariant Lie algebroid geometry approach to gauging non-linear sigma models with respect to a Lie groupoid action. The results on T-duality (in Section 5.5) are based on a collaboration with Peter Bouwknegt, Mark Bugden, and Ctirad Klimčík [24]. The results on general Lie algebroid gauging—and discussions of the associated Weinstein Lie groupoid—is an extension of this work containing new results.

A class of non-linear sigma models describe the embedding of a closed string world-sheet in an n-dimensional target space. Non-linear sigma models with isometries are of particular importance; for each isometry there is a conserved quantity. Furthermore, a non-linear sigma model with isometries can be 'gauged'—promoting the global symmetry to a local symmetry. The vector fields generating the isometries describe a Lie algebra. A recent proposal of Kotov and Strobl [114, 99, 89, 90] suggested a generalisation of the gauging procedure. The non-linear sigma model is gauged with respect to a set of vector fields which are not isometries. In general the vector fields define a Lie algebroid. We study a non-isometric gauging proposal based on Lie groupoids. This chapter discusses the integrability of the Lie algebroid action to a Lie groupoid action—something that has not appeared in the physics literature on Lie algebroid gauging.

The main results of Chapter 5 are Theorem 5.4 and Theorem 5.8. The theorems give the necessary and sufficient conditions for carrying out the Lie groupoid procedure with a particular choice of vector fields. Corollary 5.6 states that it is always possible to locally gauge an action non-isometrically. The existence of a gauge invariant field strength term restricts to Lie groupoids that are locally isomorphic to Lie groups. The proposal of Chatzistavrakidis, Deser, and Jonke [31, 33] for "T-duality without isometry" is studied; giving an appliaction of our work. We show that this non-isometric T-duality proposal is equivalent to non-abelian T-duality (Theorem 5.9).

Finally, Chapter 6 provides a brief summary and conclusion of the work contained in this thesis. An outlook on possible future avenues of research is given.

Chapter 2

Motivation

This chapter will provide motivation for the study of algebroids and vector bundle geometry through the study of Lagrangian formulations of particle, string, and higher brane models. Section 2.1 outlines the Lagrangian formulation of physics models; highlighting the role of variational problems in physics. Section 2.2 describes the geometric structure underlying Lagrangian mechanics. We show how Lie algebroids and contact structures arise naturally in the context of Lagrangian mechanics. In Section 2.3 the emergence of a Courant algebroid in the study of non-linear sigma models is demonstrated. Section 2.4 briefly discusses the connection between groupoid actions and symmetries.

2.1 Lagrangian formulation of physical models

Quantum field theory currently provides an excellent description of the physical processes which govern our universe (with the exception of gravity). The most insightful approach is based on Lagrangian mechanics. Central to the path integral formulation of quantum field theory is the Lagrangian approach to classical field theory. Let M be a closed D-dimensional manifold. We wish to describe the dynamics of a field $\varphi \in C^\infty(M)$. The manifold M may represent spacetime or a more general configuration space. The dynamics of φ are encoded in an action

$$S = \int_M L(x, \varphi(x), \partial\varphi(x)) d^D x, \tag{2.1}$$

where x gives a parameterisation of M and $L(x, \varphi(x), \partial\varphi(x))$ is the Lagrangian density.[1] The theory should be independent of the choice of parameterisation of the manifold: $L(x, \varphi(x), \partial\varphi(x))$ must be invariant for any diffeomorphism $x \to x'(x)$.

Of all the possible fields $\varphi \in C^\infty(M)$, it is the set of fields corresponding to station-

[1]The formalism can be extended to $L(x, \varphi(x), \partial\varphi(x), \ldots, \partial^n\varphi(x))$ with little difficulty.

ary points (usually minima) of the action S that are physically realised. The classical physical fields are solutions to the *Euler–Lagrange* equations:

$$\frac{\delta S}{\delta \varphi} = \frac{\partial L}{\partial \varphi} - \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu \varphi)} \right] = 0, \tag{2.2}$$

where $\mu = 0, 1, \ldots, D-1$. Examples will be given in Section 2.1.1. The choice of Lagrangian density is not unique; the addition of a total derivative term, $L \to L + \partial_\mu \Lambda^\mu$, results in the same Euler–Lagrange equations (using the divergence theorem). It is the Euler–Lagrange equations which are fundamental, and not the Lagrangian density itself. However, it is often more convenient to work with the Lagrangian and accepting the ambiguity; rather than deal with Euler–Lagrange equations directly. Noether's theorem gives the relationship between symmetries and conserved quantities. Consider an infinitesimal variation of φ given by

$$\varphi' = \varphi + \delta\varphi = \varphi + \varepsilon_r \Psi_r,$$

where $r = 1, \ldots, k$, $\varepsilon_r \in \mathbb{R}$ and $\Psi_r \in C^\infty(M)$. If this variation induces a variation of the Lagrangian density

$$L' = L + \varepsilon_r \partial_\mu \Lambda_r^\mu,$$

for some $\Lambda_r^\mu \in C^\infty(M)$, then the Euler–Lagrange equations are left unchanged. We say that such a variation generates k symmetries. Associated to these symmetries are conserved *Noether currents*:

$$J_r^\mu = \Lambda_r^\mu - \frac{\partial L}{\partial(\partial_\mu \varphi)} \cdot \Psi_r. \tag{2.3}$$

The Noether currents satisfy $\partial_\mu J_r^\mu = 0$. When there is a symmetry associated to the Lagrangian it is possible to use the conserved quantity to find a relation between the fields φ; reducing the dimension of the problem. If there are enough symmetries it may be possible to solve the system exactly.

Classical field theory Lagrangians are at the heart of the path integral formulation of quantum field theory.[2] The fundamental object in this case is the amplitude

$$\mathcal{A} = \int e^{\frac{i}{\hbar} S[\varphi]} \mathcal{D}\varphi,$$

where $\int \mathcal{D}\varphi$ denotes a path integral measure—the integration is taken over all possible paths (a sum over histories). Each path generates a different phase and the accumu-

[2] To study the full quantum theory the classical Lagrangian must be quantised. There are standard procedures for quantising which will not be covered here.

lation of interfering paths gives the amplitude. The biggest contribution comes when δS is small—paths which are 'close' to the classical paths. Path integrals have been well studied and there are many applications in physics and other fields. A formal introduction to path integrals and their role in physics can be found in [82].

The Lagrangian approach allows a description of classical limits, as well as perturbative analysis about classical limits. Some non-pertubative phenomena such as solitons and instantons can also be captured.

2.1.1 Worldline dynamics

One-dimensional sigma models describe the geodesic flows on (pseudo-)Riemannian manifolds (M, γ). Geodesics are paths which have extremal length in (M, γ). The classical action is given by the pullback of the worldline length in M:

$$S[\tau] = -m \int d\tau \sqrt{-\partial_\tau x^\mu \partial_\tau x^\nu G_{\mu\nu}}, \tag{2.4}$$

where $\mu = 0, 1, \ldots, D-1$ and $x : \mathbb{R} \to M$ describes the embedding of the particle worldline in the target manifold M. The action $S[\tau]$ is the proper time along the worldline.

The physical path is dependent on the choice of the embedded line, but not the choice of parameterisation used to describe the line. The worldline itself should be reparameterisation invariant—the same embedded line in M describes the same physics. Diffeomorphism invariance of the world line means the model described by D fields, x^μ, has $D-1$ degrees of freedom. This diffeomorphism invariance of the action is an example of a symmetry.

It is convenient to make this symmetry manifest by introducing a function η which couples the system to the worldline metric $\eta(d\tau)^2$. For a transformation $\tau' = f(\tau)$, we require that $\eta'(\tau') = (df/d\tau)^{-1}\eta(\tau)$. The action is given by

$$S_\eta[\tau] = -\frac{m}{2} \int G_{\mu\nu} \partial_\tau x^\mu \partial_\tau x^\nu \eta^{-1/2} d\tau - \frac{m}{2} \int \eta^{1/2} d\tau. \tag{2.5}$$

Extremising the field η gives the action (2.4). Alternatively, fixing the field $\eta = 1$ is equivalent to the action

$$S_{\eta=1}[\tau] = -\frac{m}{2} \int G_{\mu\nu} \partial_\tau x^\mu \partial_\tau x^\nu d\tau. \tag{2.6}$$

It is the latter form of the action which is most useful. This form of the action can be quantised using the standard techniques.

A particle worldline can be coupled naturally to a field $A \in \Omega^1(M)$:

$$S[\tau] = e \int A_\mu \partial_\tau x^\mu d\tau = e \int x^* A. \tag{2.7}$$

The associated Euler–Lagrange equation is given by

$$F = dA = 0.$$

The solution to $F = 0$ is not uniquely defined. The field $A' = A + df$ for any $f \in C^\infty(M)$ satisfies $dA' = dA = F$. Transformations $A \to A + df$ are called gauge transformations. These gauge transformation are associated with diffeomorphism invariance. To see how this works consider an infinitesimal diffeomorphism given by a pushforward $x^\mu \to x^\mu + v^\mu \varepsilon$, where $v \in \Gamma(TM)$ and ε a small constant. The corresponding coordinate transformation is $x'^\mu = x^\mu - v^\mu \varepsilon$ (to first order in ε). The induced transformation of A is given by

$$\begin{aligned} A'_\mu(x) \equiv A_\nu(x + v\varepsilon)\frac{\partial x^\nu}{\partial x'^\mu} =& A_\mu(x) + (v^\nu \partial_\nu A_\mu(x) + A_\nu(x)\partial_\mu v^\nu)\varepsilon + \mathcal{O}(\varepsilon^2), \\ =& A_\mu + (\mathcal{L}_v A)_\mu \varepsilon + \mathcal{O}(\varepsilon^2). \end{aligned}$$

To first order we have $A \to A + \varepsilon \mathcal{L}_v A = A + \varepsilon d\iota_v A$, as $dA = 0$. Finally we identify $f = \varepsilon \iota_v A$.

It is also possible to couple the worldline to a non-zero F. Consider

$$F_{\nu\lambda} = \frac{1}{2}\mu \epsilon_{\nu\lambda\kappa}\frac{x^\kappa}{|x|^3}.$$

This has the interpretation of the field of a monopole of magnetic charge μ placed at the origin of $\mathbb{R}^3$. For a closed trajectory one can consider

$$S_{\text{mon}}(x) = e \int_D \tilde{x}^* F,$$

where $\tilde{x} : D \to \mathbb{R}^3/\{0\}$ coincides with x on the boundary of the disk. Two different extensions may give different results. The difference is given by

$$e \int_{\mathsf{S}^2} \tilde{x}^* F, \tag{2.8}$$

with the integral taken over the two-sphere S^2 obtained by gluing the two discs D along the boundaries in opposite orientation. There is no globally defined one-form A which satisfies $dA = F$. Carrying out the integral over the unit sphere gives $4\pi e\mu \in \mathbb{R}$. In quantum mechanics we require the amplitudes $\int e^{iS_{\text{mon}}}\mathcal{D}x$ to be single-valued.

This imposes the constraint $S_{\rm mon} \in 2\pi\mathbb{Z}$; implying that $e\mu \in \frac{1}{2}\mathbb{Z}$. This is the Dirac quantisation condition.

In general we take the curvature to be a representative of $[F] \in H^2(M,\mathbb{Z})$.[3] The field A can be thought of as a choice of connection one-form associated to a principal $\mathsf{U}(1)$-bundle. For closed trajectories a general non-linear sigma model can be written as

$$S = \int_S x^*\gamma + e\int_{S_2} \tilde{x}^* F, \tag{2.9}$$

where S is a manifold with the topology of a circle and S_2 is a manifold with boundary $\partial S_2 = S$. Oriented circle bundles are topologically classified by $[F] \in H^2(M,\mathbb{Z})$; with $[F] \neq 0$ corresponding to topologically non-trivial circle bundles (see for example [38]).

Example: Particle on a group manifold

In general we cannot explicitly solve the Euler–Lagrange equations arising from a generic Lagrangian density. If there is a sufficient amount of symmetry the equations of motion may be exactly solvable. An important example of solvable models comes from the embedding of a worldline into a Lie group $M = \mathsf{G}$. The importance of these models come from fact that there is a natural left (and right) action of the group on itself.

Consider the embedding of a particle worldline into a compact semi-simple Lie group manifold $g : [0,T] \to \mathsf{G}$ into a model of the form

$$S[g] = \frac{-k}{4}\int (g^{-1}\dot{g}, g^{-1}\dot{g})_G d\tau, \tag{2.10}$$

where $\dot{g} := \frac{d}{d\tau}g$, and $(\cdot,\cdot)_G$ is a non-degenerate bilinear form. We have assumed that G is a compact semi-simple Lie group, so that the integral (2.10) is well defined and $(\cdot,\cdot)_G$ exists (the Killing form gives an example). The variation of $S[g]$ under the infinitesimal change δg gives

$$\frac{\delta S}{\delta g} = \frac{k}{2}\int (g^{-1}\delta g, \frac{d}{d\tau}(g^{-1}\dot{g}))_G d\tau.$$

The Euler–Lagrange equation in this case is

$$\frac{d}{d\tau}\left(g^{-1}\frac{d}{d\tau}g\right) = 0.$$

[3] A more sophisticated treatment suggests one should consider Deligne or Cheeger–Simons cohomology classes [20].

The trajectories solving the equations of motion are given by

$$g(\tau) = g_l e^{\tau\lambda/k} g_r^{-1}, \tag{2.11}$$

where g_l and g_r are fixed elements in G and λ may be taken in the Cartan subalgebra[4] $\mathfrak{t} \subset \mathfrak{g}$. The space of solutions form the phase space of the system. The phase space is equipped with a time independent symplectic form

$$\Omega = -id(p, g^{-1}\dot{g})_G, \tag{2.12}$$

where $p(\tau) = \frac{k}{2i} g^{-1}\dot{g}$. The Hamiltonian function is given by

$$H = \frac{1}{k}(p,p)_G = -\frac{k}{4}(g^{-1}\dot{g}, g^{-1}\dot{g})_G. \tag{2.13}$$

The Hamiltonian function defines the Hamiltonian vector fields v_H. The vector fields v_H are defined by the relation $-dH = \iota_{v_H}\Omega$, and satisfy $\mathcal{L}_{v_H}\Omega = 0$.

There are two commuting actions of G on itself, given by

$$h \circlearrowright g(\tau) = hg(\tau), \quad g(\tau) \circlearrowleft h = g(\tau)h^{-1},$$

for $h \in \mathsf{G}$, and $g \in C^\infty([0,T], \mathsf{G})$. Both actions preserve the symplectic structure and Hamiltonian. The left(right) action are generated by flowing along right(left)-invariant vector fields $(X_{R,L})_a \in \Gamma(T\mathsf{G})$, where $a = 1, \dots, \dim(\mathsf{G})$. Taking a basis for the right-invariant vector fields $(X_R)_a \in TG$, we can identify $T_e\mathsf{G} \cong \mathfrak{g}$:

$$[(X_R)_a, (X_R)_b] = C^c{}_{ab}(X_R)_c \quad \Leftrightarrow \quad [T_a, T_b] = C^c{}_{ab}T_c,$$

where $C^c{}_{ab} \in \mathbb{R}$ are the structure constants for the Lie algebra $\mathfrak{g} := \mathsf{Lie}(\mathsf{G})$. We see that a Lie algebra structure naturally emerges in this case. In Section 2.2 we will see how Lie algebroid structures emerge in more general Lagrangian mechanics scenarios.

2.1.2 Worldsheet dynamics

In string theory the fundamental structures are one-dimensional strings. The dynamics of a string in a (pseudo-)Riemannian manifold (M, G) is given by embedding the string worldsheet Σ into a target space M, $X : \Sigma \to M$. The geodesic solutions for the embedding of particle worldlines corresponds to minimising the length. The natural

[4] A *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$ is a maximal subalgebra $\mathfrak{h}$ satisfying $[X,Y] = 0$ for all $X, Y \in \mathfrak{h}$.

analogue for strings are minimal surfaces; given by the pullback of the area

$$S_N[X] = -T \int_\Sigma d\sigma d\tau \sqrt{|\det \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}|},$$

where $\sigma^\alpha = (\tau, \sigma)$ are coordinates on the worldsheet Σ. This is the Nambu–Goto action. The embedding of the surface should be independent of the choice of parameterisation of the the worldsheet.

This action contains a square-root which poses a problem for quantisation. It is more useful to consider the Polyakov action—which is classically equivalent to the Nambu action—but is in a form that is appropriate for path integral quantisation. This is achieved by coupling the worldsheet to a dynamical metric h. The Polyakov action is given by

$$S_P[X] = -\frac{T}{2} \int_\Sigma d\sigma d\tau \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu},$$

where h is the determinant of $h_{\alpha\beta}$. The Euler–Lagrange equations for X^μ give

$$\partial_\alpha(\sqrt{-h} h^{\alpha\beta} \partial_\beta X^\mu) = 0.$$

The variation of the Polyakov action with respect to h gives (see for example [101])

$$\frac{\delta S_P}{\delta h} = -\frac{T}{2} \int_\Sigma d\sigma d\tau \delta h^{\alpha\beta} \sqrt{-h} (\partial_\alpha X^\mu \partial_\beta X^\nu - \tfrac{1}{2} h_{\alpha\beta} h^{\delta\gamma} \partial_\delta X^\mu \partial_\gamma X^\nu) G_{\mu\nu} = 0.$$

The Euler–Lagrange equations are equivalent to the vanishing of the energy-momentum tensor

$$T_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \tfrac{1}{2} h_{\alpha\beta} h^{\delta\gamma} \partial_\delta X^\mu \partial_\gamma X^\nu G_{\mu\nu} = 0.$$

The general solution to $T_{\alpha\beta} = 0$ is given by $h_{\alpha\beta} = f\tilde{h}_{\alpha\beta}$, where $\tilde{h}$ is the induced metric $\tilde{h}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$, and $f \in C^\infty(M)$ is an arbitrary nowhere zero function.

The Polyakov action is invariant under diffeomorphisms generated by

$$X^\mu(\sigma, \tau) \to X'^\mu(\sigma', \tau'), \quad h'_{\alpha\beta}(\sigma', \tau') \to \frac{\partial \sigma^\delta}{\partial \sigma'^\alpha} \frac{\partial \sigma^\gamma}{\partial \sigma'^\beta} h_{\delta\gamma}(\sigma, \tau).$$

The Polyakov action is invariant under an additional transformation; the additional *Weyl transformation* is given by

$$h_{\alpha\beta}(\sigma, \tau) \to \Omega^2(\sigma, \tau) h_{\alpha\beta}(\sigma, \tau) \quad \text{for} \quad \Omega^2(\sigma, \tau) = e^{2\phi(\sigma, \tau)},$$

for some $\phi \in C^\infty(\Sigma)$. Two worldsheet metrics which are related by a combination of a diffeomorphism and a Weyl transformation are considered physically equivalent. The combination of diffeomorphisms and Weyl transformations gives enough gauge freedom to (locally) fix the worldshhet metric to the flat Minkowski metric $h_{\alpha\beta} = \eta_{\alpha\beta}$ (see for example [101]). There is a subgroup of the diffeomorphism and Weyl invariance remaining after fixing the metric; the group of transformations which leave $\eta_{\alpha\beta}$ invariant are

$$X^\mu(\sigma,\tau) \to X'^\mu(\sigma',\tau'), \quad \eta'_{\alpha\beta} \to \Omega^2(\sigma,\tau)\frac{\partial\sigma^\delta}{\partial\sigma'^\alpha}\frac{\partial\sigma^\gamma}{\partial\sigma'^\beta}\eta_{\delta\gamma}.$$

Transformations of this form describe the *conformal group*.

Two-dimensional models exhibiting conformal invariance are extremely important. Conformal invariance in two-dimensions is infinite dimensional and allows one to use the powerful methods of Conformal Field Theory (CFT). The most rigorous mathematical definition of string theory is via CFT.

Example: WZW model

An important example of non-linear sigma models associated to string theory are the Wess–Zumino–Witten (WZW) models. Let $g : \Sigma \to \mathsf{G}$, be the embedding of a string worldsheet into a Lie group G with Lie algebra $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$. Let $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_B$ denote two G-invariant bilinear forms, symmetric and skew-symmetric respectively. The WZW model is described by

$$S_{\mathrm{WZW}}[g] = \frac{1}{2}\int_\Sigma (g^{-1}dg \overset{\wedge}{,} \star\, g^{-1}dg)_G + (g^{-1}dg \overset{\wedge}{,} g^{-1}dg)_B, \tag{2.14}$$

where $g^{-1}dg \in \Omega^1(\Sigma,\mathfrak{g})$ denotes the left-invariant Maurer–Cartan form and

$$(a \overset{\wedge}{,} b)_G(s_1,s_2) := \frac{1}{2}\Big[\big(a(s_1),b(s_2)\big)_G - \big(a(s_2),b(s_1)\big)_G\Big],$$

for $s_1, s_2 \in \Gamma(T\Sigma)$, $a, b \in \Omega^1(\Sigma,\mathfrak{g})$.

This model is studied in detail in Section 5.2. The two-dimensional worldsheet naturally couples to the pullback of a differential form $B \in \Omega^2(\mathsf{G})$. In order to preserve the group action we consider a bi-invariant two-form. In the case that $dB = H = 0$, $B \in \Omega^2_{\mathrm{inv}}(\mathsf{G})$ is defined up to gauge transformations given by a G-invariant closed one-form. A more general model can be given by replacing the $(\cdot,\cdot)_B$ term with

$$\int_{\Sigma_3} H = \frac{1}{6}\int_{\Sigma_3} (g^{-1}dg, [g^{-1}dg, g^{-1}dg]_{\mathfrak{g}})_G,$$

where Σ_3 is a three-dimensional manifold with boundary $\partial\Sigma_3 = \Sigma$. If H is non-trivial in cohomology then B is not a globally defined two-form; instead B forms part of a

$\mathsf{U}(1)$-gerbe structure (see Section 4.1.1). The gerbe structure is naturally associated with a Courant algebroid structure.

WZW models provide a rare example of a string theory which can be solved exactly. A review of the construction of the exact solution of the model from the path integral point of view can be found in the introductory lecture notes by Gawędzki [50].

It is possible to consider a more general non-linear sigma model given by

$$S[X] = \int_\Sigma G_{\mu\nu} dX^\mu \wedge \star dX^\nu + \int_{\Sigma_3} X^* H, \tag{2.15}$$

for a (pseudo-)Riemannian manifold (M, G) and a choice of $H \in \Omega^3_{\mathrm{cl}}(M)$. This model is studied in Chapter 5 from the perspective of Lie algebroid gauging.

2.2 Geometry of Lagrangian Mechanics

In Section 2.1 it was shown that the Lagrangian is a the heart of fundamental physics theories—allowing the calculation of equations of motion and conserved quantities. In this section we look at Lagrangian mechanics from a geometric perspective; it will be shown that symmetries are naturally associated to Lie algebroids. In addition, it will be shown that a Lie algebroid structure underlies the Euler–Lagrange equations.

We will consider the dynamics of a point particle in some manifold, just as we did in Section 2.1. Take $x : [0, T] \to M$, and

$$S = \int_\gamma L(\tau, x, \partial x) d\tau. \tag{2.16}$$

For simplicity we consider Lagrangian densities of the form $L(\tau, x, \partial x) = L(x) \in C^\infty(M)$. In this case we can make the identification

$$L(x(\tau))d\tau = x^*\alpha, \quad \alpha \in \Omega^1(M).$$

The Euler–Lagrange equations for x are

$$\frac{\partial L}{\partial x} = dx^*\alpha = x^* d\alpha = 0.$$

The problem is reduced to

$$S = \int x^*\alpha,$$

and the Euler–Lagrange equations are satisfied by closed α. The physical models we have studied are required to be invariant under diffeomorphisms. Diffeomorphisms are generated infinitesimally by flowing along vector fields. A diffeomorphism—described by $x^\mu \to x^\mu + v^\mu$—induces a transformation on a tensor T; with the infinitesimal transformation given by $T \to T + \mathcal{L}_v T$. A vector field $v \in \Gamma(TM)$ generates a symmetry if $\mathcal{L}_v \alpha = 0$. If a vector field v generates a symmetry on an extremal surface ($d\alpha = 0$) we have

$$\delta S = \int x^*(\mathcal{L}_v \alpha) = 0 = \int x^*(d\iota_v \alpha + \iota_v d\alpha) = \int x^* d\iota_v \alpha.$$

When v generates a symmetry on an extremal surface $x^*(d\iota_v\alpha) = 0$; we conclude that $\iota_v\alpha$ is a closed one form in the image of x (giving a conserved quantity). The Euler–Lagrange equations remain unchanged if an exact term df (where $f \in C^\infty(M)$ is any choice of function) is added to the Lagrangian. In general we require that

$$\mathcal{L}_v \alpha + df = d(\iota_v \alpha + f) = 0, \tag{2.17}$$

giving the conserved quantity $\iota_v\alpha + f$. A pair (v, f) is said to generate a symmetry if Eq. (2.17) holds. Consider the vector bundle $E = TM \oplus \mathbb{R}$, with sections $(v, f) \in \Gamma(TM) \oplus C^\infty(M) \cong \Gamma(E)$. Define a subbundle $C \subset E$ as follows

$$C = \mathrm{graph}(\alpha) := \{(v, f) \in \Gamma(E) : f = -\iota_v \alpha\}.$$

There is a natural flow that preserves C. A pair $(v, f) \in C$ if and only if $f = -\iota_v\alpha$. The flow of a section $(v_1, f_1) \in \Gamma(E)$ preserves the pair $(v, -\iota_v\alpha)$ if it flows to $(v', -\iota_{v'}\alpha)$ for some $v' \in \Gamma(TM)$. The required flow is given by

$$\mathscr{L}_{(v_1,f_1)}(v_2, f_2) = ([v_1, v_2], v_1(f_2) - v_2(f_1)). \tag{2.18}$$

The fact that this flow preserves C is easily verified:

$$\begin{aligned}
\mathscr{L}_{(v_1,f_1)}(v, -\iota_v\alpha) =& ([v_1, v], -v_1(\iota_v\alpha) - v(f_1)) \\
=& ([v_1, v], -\mathcal{L}_{v_1}\iota_v\alpha - \iota_v df_1) \\
=& ([v_1, v], -\iota_{[v_1,v]}\alpha - \iota_v(\mathcal{L}_{v_1}\alpha + df_1)) \\
=& ([v_1, v], -\iota_{[v_1,v]}\alpha),
\end{aligned}$$

where the last line follows from the fact that (v_1, f_1) generates a symmetry and therefore satisfies Eq. (2.17). This flow defines a bracket

$$[(v_1, f_1), (v_2, f_2)] = ([v_1, v_2], v_1(f_2) - v_2(f_1)). \tag{2.19}$$

This bracket defines an Atiyah algebroid on $E = TM \oplus \mathbb{R}$. The Atiyah algebroid provides an example of a Lie algebroid structure (see Definition 3.8). In Section 2.2.2

we shall see that this is associated to a principal G-bundle and contact structure.

Following the discussion of Section 2.1.1 an additional term (representing a monopole) can be added to the action. In this case

$$S = \int_S \alpha + \int_{S_2} F,$$

for some closed two-form $F \in \Omega^2_{\rm cl}(M)$. We can repeat the analysis above to calculate the variation for extremal surfaces—remembering that the variation only has to vanish up to a closed form df:

$$\begin{aligned}\delta S &= \int_S \mathcal{L}_v\alpha + df + \int_{S_2} \mathcal{L}_v F = \int_S d\iota_v\alpha + df + \int_{S_2} d\iota_v F \\ &= \int_S d(\iota_v\alpha + f) + \int_S \iota_v F = 0.\end{aligned}$$

This means that $\iota_v\alpha + f$ is no longer closed but satisfies

$$d(\iota_v\alpha + f) + \iota_v F = 0. \tag{2.20}$$

The modified flow which preserves the subspace C under this condition is

$$[(v_1, f_1), (v_2, f_2)]_F = ([v_1, v_2], v_1(f_2) - v_2(f_1) + \iota_{v_1}\iota_{v_2}F). \tag{2.21}$$

This is verified as before

$$\begin{aligned}\mathscr{L}_{(v_1,f_1)}(v, -\iota_v\alpha) =&([v_1, v], -v_1(\iota_v\alpha) - v(f_1) + \iota_{v_1}\iota_v F)\\ =&([v_1, v], -\mathcal{L}_{v_1}\iota_v\alpha - \iota_v(df_1 + \iota_{v_1}F))\\ =&([v_1, v], -\iota_{[v_1,v]}\alpha - \iota_v(d(\iota_{v_1}\alpha + f_1) + \iota_{v_1}F))\\ =&([v_1, v], -\iota_{[v_1,v]}\alpha).\end{aligned}$$

The bracket (2.21) also defines a Lie algebroid. This is the twisted Atiyah algebroid. In Section 2.2.2 we will see that this is associated to a non-trivial principal $\mathsf{U}(1)$-bundle.

The above argument shows that Lie algebroid structures arises naturally in the context of variational problems. In general we expect that the Lagrangian density is a function of variables on the target M and higher derivatives of the variables $L(\tau, x, \partial x, \ldots, \partial^k x)$. In this case there is still an algebroid structure underlying Lagrangian mechanics. It possible to extend the geometric description by replacing $TM \oplus \mathbb{R}$ with another vector bundle $E \to M$. In particular, it is possible to include k-th order derivative terms by passing to the k-jet bundle. Such a generalisation requires a desciption of Lagrangian mechanics on vector bundles. Briefly outlining this development is the topic of the next section. We will see that a Lie algebroid plays an important role.

2.2.1 Lagrangian mechanics on vector bundles

Lagrangian mechanics can be formulated on a vector bundle $E \to M$ endowed with a Lie algebroid structure. The dynamics can be encoded by geometric structures on the vector bundles TE, T^*E, TE^* and T^*E^*. The Euler–Lagrange equations are constructed using a Lie algebroid structure.[5] Originally this vector bundle construction was done for $E = TM$.

The treatment here is based on references [55] and [80]. The reader is invited to consult these papers for further details. There are two approaches to describing Lagrangian (and Hamiltonian) mechanics geometrically via algebroids. One approach studies algebroids on the tangent space of a vector bundle $E \to M$, while the other involves the prolongation of E. The former approach is more natural for our purposes and will be outlined here. The prolongation approach can be found in [41].

Consider a vector bundle (E, π, M) where $\pi : E \to M$ is a projection map. There is a dual bundle $\pi^* : E^* \to M$. A core concept in the construction is that of a double vector bundle. A double vector bundle is a pair of naturally compatible vector bundles sharing the same total space (the reader is referred to [56] for more details). The relevant vector bundles associated to Lagrangian mechanics are TE, T^*E, TE^* and T^*E^*. There are three natural maps:

$$\tau_E : TE \to E, \quad d\pi : TE \to TM, \quad \chi_E : T^*E \to E.$$

In addition there is another map $v_E : T^*E \to E^*$ that can be interpreted as the vertical derivative. The image under v_E evaluated on an element $e' \in E_{\tau(e)}$ is the derivative of $f \in C^\infty(E, \mathbb{R})$ in the direction of a vertical vector $v_e^{e'} \in V_eE := T_e(E_{\tau(e)}) \subset T_eE$. The *vertical derivative* is defined as follows:

$$\langle v_E(df(e)), e' \rangle = v_e^{e'}(f) := \frac{d}{dt} f(e + te')|_{t=0}, \tag{2.22}$$

where $df(e) \in T_e^*E$ and $\langle \cdot, \cdot \rangle : E \times_M E^* \to \mathbb{R}$ is the canonical pairing. Henceforth we denote the vertical derivative by $d_V f := v_E(df)$.

The spaces T^*E and T^*E^* are canonically isomorphic as double vector bundles. The isomorphism by $R_E : T^*E \to T^*E^*$ can be explicitly constructed. First note that the kernel $K := \{(e, \xi) \in E \times_M E^* : \langle e, \xi \rangle = 0\}$ is a smooth submanifold in $E \times_M E^*$. The isomorphism R_E is defined (through its graph) as the annihilator of

[5] Technically only an almost Lie algebroid structure is required—the Jacobi identity plays no role in the local construction. The Jacobi identity is required when integrating local solutions.

$TK \subset T(E \times_M E^*)$, i.e.

$$\mathrm{graph}(R_E) := \mathrm{Ann}(TK) \subset T^*(E \times_M E^*) \cong T^*E \times_{T^*M} T^*E^*.$$

In fact, $\mathrm{graph}(R_E)$ is a Lagrangian submanifold with respect to the canonical symplectic structure on the cotangent bundle $T^*(E \times_M E^*)$. The isomorphism R_E is also an anti-symplectomorphism; R_E intertwines the 'legs' of double vector bundles T^*E and T^*E^*:

$$v_{E^*} \cdot R_E = \chi_E : T^*E \to E, \quad \text{and} \quad \chi_{E^*} \cdot R_E = v_E : T^*E \to E^*.$$

Take a manifold M with a choice of local coordinates $\{x^\mu\}$, and choose local coordinates

$$(x^\mu, p_\nu) \in T^*M, \quad (x^\mu, y^i) \in E, \quad (x^\mu, \xi_i) \in E^*,$$
$$(x^\mu, y^i, p_\nu, \pi_j) \in T^*E, \quad (x^\mu, \xi_i, p_\nu, \varphi^j) \in T^*E^*.$$

The anti-symplectomorphism R_E can be described locally as

$$R_E : (x^\mu, y^i, p_\nu, \pi_j) \to (x^\mu, \varphi^i, -p_\nu, \xi_j).$$

In order to define Lagrangian (and Hamiltonian) mechanics on the vector bundle $E \to M$ a Lie algebroid structure must be defined on E. If $E = TM$ a Lie algebroid bracket is given by the commutator of vector fields. A Lie algebroid structure on E can be encoded in a bivector $\Lambda_{E^*} : \Gamma(\wedge^2 TE^*)$ (see Section 3.3.3). In the rest of this section we will assume that a Lie algebroid structure—or equivalently Λ_{E^*}—has been specified.

Given a Hamiltonian function $H \in C^\infty(E^*)$ we define the associated Hamiltonian vector field $X_H \in \Gamma(TE^*)$ by

$$X_H := \widetilde{\Lambda}_{E^*}(dH) := \iota_{dH}\Lambda_{E^*}.$$

In the special case that $E = TM$ the bivector Λ_{E^*} is dual to a symplectic form on the phase space.[6] Integral curves of X_H are trajectories of the system. A curve $\xi : [0,T] \to E^*$ is a *Hamiltonian trajectory* if

$$\frac{d}{dt}\xi(t) = \widetilde{\Lambda}_{E^*}(dH(\xi(t))), \quad t \in [0,T].$$

The Lagrangian dynamics on an algebroid can also be understood in the language of natural double vector bundle morphisms. By composing the canonical double vector

[6]Compare (H, Ω) to Equations (2.12) and (2.13).

bundle isomorphism $R_E : T^*E \to T^*E^*$ with $\widetilde{\Lambda}_{E^*} : T^*E^* \to TE^*$ one obtains a double vector bundle morphism $\mathcal{E}_E := \widetilde{\Lambda}_{E^*} \cdot R_E : T^*E \to TE^*$. Explicit expressions for a choice of Λ_{E^*} in local coordinates can be found in [55].

Consider a Lagrangian on $L \in C^\infty(E)$ describing some physical system. The dynamics are encoded in solutions to the Euler–Lagrange equations. The constructions of this section give a geometric formulation of the Lagrangian dynamics. A curve $\gamma : [0,T] \to E$ is a solution of the Euler–Lagrange equations if

$$\frac{d}{d\tau} v_E L(\gamma(\tau)) = \mathcal{E}_E(dL(\gamma(\tau))), \quad \tau \in (0,T). \tag{2.23}$$

This equation should be compared to (2.2).

We see the importance of the vertical derivative (and Lie algebroid structure) in determining the dynamics in Lagrangian mechanics.[7]

Equation (2.23) guarantees that a trajectory $\gamma : [0,T] \to E$ is automatically an admissible curve,

$$\rho(\gamma(\tau)) = \frac{d}{d\tau}\pi(\gamma(\tau)), \quad \forall\tau \in (0,T).$$

This follows from the fact that the morphism $\mathcal{E}_E$ projects to a map $\rho : E \to TM$ under χ_E and $d\pi^*$. The dynamics can be encoded in the following diagram

$$\begin{tikzcd}
T^*E \arrow[rr, "\mathcal{E}_E"] \arrow[rrrr, bend left, "\simeq", "R_E"'] \arrow[d, "\chi_E"] \arrow[rrd, "v_E"] & & TE^* \arrow[d, "\tau_{E^*}"] & & T^*E^* \arrow[ll, "\widetilde{\Lambda}_{E^*}"'] \arrow[d, "\chi_{E^*}"'] \\
E \arrow[u, bend left, "dL"] & & E^* \arrow[rr, "="] & & E^* \arrow[u, bend right, "dH"']
\end{tikzcd}$$

(Lagrangian) (Dynamics) (Hamiltonian)

The phase dynamics are described by the Euler–Lagrange equations on an algebroid and are described by Equation (2.23). The trajectories γ are critical trajectories of a naturally defined action functional.

When the legendre map is a local diffeomorphism the trajectories described by (2.23) are determined by the image of a Lagrangian submanifold $dL(E) \subset T^*E$ under $\mathcal{E}_E$. If the Legendre map is not a local diffeomorphism the conditions on the trajectories $\frac{d}{d\tau}\gamma(\tau)$ are implicit and cannot be written explicitly.

Lagrangian mechanics is described by a geometric structure which is richer than

[7] In addition to the equations of motion, the map v_E gives the Legendre map. The Legendre map $E \to E^*$ is given by $v_E(dL) = d_V L : E \to E^*$.

just Lie algebroids. The various double vector bundle morphisms encode additional structure. It was noted that the Atiyah algebroids (2.19) and (2.21) are associated with principal $\mathsf{U}(1)$-bundles. Principal $\mathsf{U}(1)$-bundles come with a natural contact structure. Expanding on this relationship is the topic of the next section.

2.2.2 Contact geometry and physics

A Lagrangian on a spacetime (or configuration space) M can be viewed as a section of the trivial line bundle $M \times \mathbb{R}$. It is more natural to consider the slightly more general case of a principal G-bundle with Lie group $\mathsf{G} = \mathbb{R}$ or $\mathsf{U}(1)$.[8] The principal G-bundle structure naturally incorporates the ambiguities in the Lagrangian construction and allows for non-trivial topology. There is a gauge freedom associated to the addition of an exact term to the Lagrangian. Principal $\mathsf{U}(1)$-bundles are classified topologically by the first Chern class $c_1 \in H^2(M, \mathbb{Z})$. Given a principal bundle $P(M, \pi, \mathsf{U}(1))$ and a choice of principal connection, $A \in \Omega^1(M, \mathfrak{g})$, the curvature $F = dA$ gives a representative of the first Chern class of P. We will now outline the construction. Let $\mathcal{U} = \{U_\alpha\}$ denote a good cover of M and $\pi^{-1}(U_\alpha) = U_\alpha \times \mathsf{S}^1$ a cover for P. Take local coordinates (x, θ_α), $x \in U_\alpha$, $\theta_\alpha \in \mathsf{S}^1$. We have two sets of coordinates on $\pi^{-1}(U_\alpha \cap U_\beta)$ given by (x, θ_α) and (x, θ_β). The coordinates are related by

$$\theta_\alpha = g_{\alpha\beta}(x)\theta_\beta, \quad x \in U_\alpha \cap U_\beta, \tag{2.24}$$

where $g_{\alpha\beta}(x) \in C^\infty(U_\alpha \cap U_\beta, \mathsf{U}(1))$ are the transition functions. The transition functions are required to satisfy the cocycle conditions

$$\begin{aligned} g_{\alpha\alpha}(x) =& 1 \quad x \in U_\alpha, \\ g_{\alpha\beta}(x) g_{\beta\alpha}(x) =& 1 \quad x \in U_\alpha \cap U_\beta, \\ g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha}(x) =& 1 \quad x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Equation (2.24) implies that

$$d \log \theta_\alpha = d \log g_{\alpha\beta} + d \log \theta_\beta, \quad x \in U_\alpha \cap U_\beta. \tag{2.25}$$

While log is a multi-valued function $d \log$ is single valued. Every term in (2.25) is pure imaginary since $\theta_\alpha, \theta_\beta, g_{\alpha\beta}$ all have absolute value 1.

Consider a connection one-form (given locally by $A_\alpha \in \Omega^1(U_\alpha, \mathbb{R})$) satisfying

$$(\delta A)_{\alpha\beta} \equiv A_\beta - A_\alpha = -i d \log g_{\alpha\beta}. \tag{2.26}$$

[8] When considering quantisation we require $\mathsf{G} = \mathsf{U}(1)$.

It follows from (2.25) and (2.26) that

$$A_\alpha - id\log\theta_\alpha = A_\beta - id\log\theta_\beta,$$

and A is independent of the choice of cover. The curvature $F = dA$ satisfies

$$F = dA_\alpha = dA_\beta, \quad x \in U_\alpha \cap U_\beta,$$

and gives a globally defined two-form $F \in \Omega^2(M,\mathbb{Z})$. The Chern class of P is given by $[F] \in H^2(M,\mathbb{Z})$.

The cocycle conditions on A associated with non-zero F give exactly the correct behaviour to describe a monopole (see Section 2.1.1). We see that the charge can be encoded geometrically.

Gauge transformations

There is an ambiguity in the description of the transition functions $g_{\alpha\beta}$. This ambiguity corresponds to gauge transformations $A \to A + db$, where $b \in C^\infty(M)$. Let $x(t)$ be a curve in M, with $t \in [0,1]$. To each point x we associate a point of the circle $\pi^{-1}(t)$, defining a section. The section is described locally by $\theta_\alpha(x(t))$ for $x(t) \in U_\alpha$. If $x(t) \in U_\alpha \cap U_\beta$, θ_α and θ_β are related by (2.24). The section is called parallel if

$$d\log\theta_\alpha - iA_\alpha(x) = 0,$$

where $A_\alpha(x(t))$ is the restriction of A_α to $x(t)$. Integrating the differential equation gives

$$\theta_\alpha = \exp\left(i\int_x A_\alpha(t)\right). \tag{2.27}$$

Up to a constant factor $\int A_\alpha$ is the phase and the right hand side of (2.27) is the phase factor. For questions of quantisation it is the phase factor that is meaningful and not the phase itself. This leads to gauge transformations

$$A'_\alpha = A_\alpha - id\log h_\alpha, \quad g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}, \quad h_\alpha \in \Omega^0(U_\alpha).$$

Writing $h_\alpha = \exp(2\pi i b_\alpha)$, we have $A'_\alpha = A_\alpha + 2\pi db_\alpha$. The fields A_α and A'_α give the same phase factor when

$$1 = \exp\left(i\int_\gamma (A_\alpha - A'_\alpha)(t)\right) = \exp\left(2\pi i\int_\gamma db_\alpha(t)\right).$$

The requirement that $db_\alpha = 0$ implies that b_α is locally constant and hence $b_\alpha \in \Omega^0(M, \mathbb{Z})$.

Atiyah algebroid and $\mathsf{U}(1)$-bundles

In Section 2.2 it was claimed that the Atiyah algebroid (2.19) is the infinitesimal object associated to $P(M, \pi, \mathsf{U}(1))$. In this section we elaborate on this construction for the $\mathsf{G} = \mathsf{U}(1)$ Atiyah algebroid. For a summary of the general case see Section 3.2. Consider the trivial principal bundle $P(M, \pi, \mathsf{U}(1))$, given by $P = M \times \mathsf{S}^1$, with the $\mathsf{U}(1)$-action being translation on S^1. An Atiyah algebroid has an associated exact sequence of vector bundles

$$0 \longrightarrow P \times \mathbb{R} \longrightarrow TP/\mathsf{U}(1) \xrightarrow{\pi_*} TM \longrightarrow 0 \ ,$$

where $\mathfrak{g} = \mathsf{Lie}(\mathsf{U}(1)) = \mathbb{R}$. This gives an induced map on sections

$$0 \longrightarrow C^\infty(P, \mathbb{R})^{\mathsf{U}(1)} \simeq C^\infty(M) \longrightarrow \mathfrak{X}_{\mathsf{G}}(P) \xrightarrow{\pi_*} \mathfrak{X}(M) \longrightarrow 0 \ ,$$

where $\mathfrak{X}(M) := \Gamma(TM)$ and $\mathfrak{X}_{\mathsf{G}}(P)$ denotes the space of G-invariant vector fields. If $TP/\mathsf{U}(1)$ is a trivial bundle there exists an isomorphism $\mathfrak{X}_{\mathsf{G}}(P) \cong \mathfrak{X}(M) \oplus C^\infty(M)$. To see this explicitly let $\{x\}$ describe coordinates on M, θ a parameterisation of S^1, and $\{x, \theta\}$ gives coordinates for P. The $\mathsf{U}(1)$-action is given by translation on the circle and generated infinitesimally by ∂_θ. Sections $v \in \Gamma(TP)$ are given by

$$v = v^i(x, \theta)\partial_i + f(x, \theta)\partial_\theta, \quad v^i, f \in C^\infty(P),$$

with the bracket on TP given by the commutator of vector fields. Elements of $\mathfrak{X}_{\mathsf{G}}$, the space of $\mathsf{U}(1)$-invariant vector fields, are of the form

$$v_{\mathsf{G}} = v^i(x)\partial_i + f(x)\partial_\theta, \quad v^i, f \in C^\infty(M). \tag{2.28}$$

The bracket on $\mathfrak{X}_{\mathsf{G}}$ is given by

$$[(v_1, f_1), (v_2, f_2)] = ([v_1, v_2], v_1(f_2) - v_2(f_1)), \tag{2.29}$$

where $v_1, v_2 \in \Gamma(TM)$ and $f_1, f_2 \in C^\infty(M)$. In this way we can see that the Atiyah algebroid (2.19) arises naturally as the Lie bracket of $\mathsf{U}(1)$-invariant fields on a $\mathsf{U}(1)$-bundle. It is natural to ask how global information about the principal bundle P can be encoded in the Atiyah algebroid. This information is incorporated in the twisted Atiyah algebroid (2.21), with F being a representative of the first Chern class. A choice of F determines a principal $\mathsf{U}(1)$-bundle up to a choice of flat connection as described in Section 2.2.2.

2.2.3 Contact structures and Lagrangian mechanics

In this section we outline the contact structures underlying Lagrangian mechanics. The following discussion on the connection between contact geometry and Lagrangian mechanics is based on the short survey by Ševera [110]. A description of contact quantisation is given in [65].

A *contact structure* on a manifold M is a field of hyperplanes defining a subbundle of codimension 1, $HM \subset TM$, satisfying a maximal nonintegrability condition. Contact structures are introduced and discussed in Chapter 4. Contact structures can be described by a one-form θ on the line bundle $TM \to HM$, i.e., $\theta \in \Omega^1(TM/HM)$.

A *contact vector field* has a flow which preserves the contact structure. There is a one-to-one correspondence between contact vector fields and sections of the line bundle TM/HM. For any $w \in C^\infty(TM/HM)$ there is a unique v that is equal to w mod HM.

There are two natural examples which are relevant to our discussion of Lagrangian mechanics:

Example 2.1. Consider a symplectic manifold (N, ω). Let $P(N, \pi, \mathsf{G})$ be a principal G-bundle, with $\mathsf{G} = \mathbb{R}$ or $\mathsf{U}(1)$. Choose a connection $A \in \Omega^1(N, \mathfrak{g})$, such that the curvature $dA = \omega$. The horizontal distribution makes P into a contact manifold. The local flow generated by a contact field R preserves the contact structure if and only if it is G-invariant. The contact field is invariant if the contact Hamiltonian h defined by the constraint

$$\iota_R dA = -dh$$

is the pullback of a function on N.

Example 2.2. A classic example of a contact manifold is the space of *contact elements* (CM, M, π) given by hyperplanes in the tangent space of a manifold M. The distribution $H(CM)$ can be described pointwise. Each point $x \in CM$ corresponds to a hyperplane H in $T_{\pi(x)}M$ and $H_x(CM)$ is given by $(d\pi_x)^{-1}(H)$.

Contact geometry on CM gives a geometric interpretation of first order partial differential equations and the Lagrange method of characteristics. Let $E \subset CM$ be a hypersurface representing the equation. A hypersurface on $\Sigma \subset M$ can be lifted to CM via the tangent functor. For a point $x \in \Sigma$ take the hyperplane $T_x\Sigma$ to be a point of the lift $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a Legendre submanifold of CM. The hypersurface Σ solves the equation if $\tilde{\Sigma} \subset E$. Taking $x \in M$ the enveloping cone of the hyperplanes $\pi^{-1}(x) \cap E$ in T_xM defines a field of cones in M. Σ solves the equation if it is tangent to the cones everywhere.

The construction on CM is central to the method of characteristics. The hyperplane field $H(CM)$ cuts a hyperplane field HE. The form θ becomes degenerate when we restrict from $H_x(CM)$ to H_xE. Thus at any $x \in E$ there appears a direction along which θ is degenerate. The integral curves of this direction field are called *characteristics*. In this way we see that Lagrangian mechanics is naturally associated to contact geometry. The solutions of the Euler–Lagrange equations are associated to characteristic curves.

2.3 Higher geometry

In Section 2.2 it was shown that Lagrangian mechanics describing the embedding of worldlines in a target space has a geometric intrepretation in terms of contact structures and Lie algebroids.

The construction of Lagrangian mechanics associated to worldline embeddings can be generalised to describe string dynamics. This involves embedding a two-dimensional worldsheet into a target manifold M, via $X : \Sigma \to M$. We will be particularly interested in non-linear sigma models. An example of such a model was given by (2.15). These models are the subject of Chapter 5. We will see that the structure of a Courant algebroid naturally emerges on the vector bundle $E = TM \oplus T^*M$.

Courant algebroids arise natually in two-dimensional variational problems. Consider the mapping $X : \Sigma \to M$ describing the embedding of a two-dimensional manifold (worldsheet) in a target manifold M. The Lagrangian theory is given by an action

$$S = \int_\Sigma X^*\beta$$

with $\beta \in \Omega^2(M)$.[9] Typically β can describe the tension of the worldsheet or a non-linear sigma model describing the motion of a string in a fixed background. Extremal surfaces are given by $d\beta = 0$.

Suppose we have a vector field $v \in \Gamma(TM)$ and an extremal surface $\beta \in \Omega^2(M)$ satisfying

$$\mathcal{L}_v\beta = d\iota_v\beta = 0.$$

We conclude that $\iota_v\beta$ is closed and hence gives a conserved quantity. Following Section 2.2, the ambiguity of the Lagrangian allows us to add an exact term $d\xi$, where $\xi \in \Omega^1(M)$, satisfying

$$\mathcal{L}_v\beta + d\xi = d(\iota_v\beta + \xi) = 0, \tag{2.30}$$

[9] As in the one-dimensional case we can generalise this construction to the Jet space to incorporate derivatives of fields.

on extremal surfaces (the quantity $\iota_v\beta + \xi$ is conserved). In this case we see that the pair (v, ξ) generates a symmetry. Take $E = TM \oplus T^*M =: \mathbb{T}M$ and consider $e = (v, \xi) \in \Gamma(E) = \Gamma(\mathbb{T}M)$. Let $C \subset E$ be given by

$$C = \mathrm{graph}(\beta) := \{(v, \xi) \in \Gamma(\mathbb{T}M) : \xi = -\iota_v\beta\}.$$

There is a natural flow which preserves C when $\mathcal{L}_v\beta + d\xi = 0$. This flow is given by

$$\mathscr{L}_{(v_1,\xi_1)}(v_2, \xi_2) = (\mathcal{L}_{v_1}v_2, \mathcal{L}_{v_1}\xi_2 - \iota_{v_2}d\xi_1).$$

The preservation of C is easily verified

$$\begin{aligned}\mathscr{L}_{(v_1,\xi_1)}(v_2, \xi_2) =&(\mathcal{L}_{v_1}v_2, \mathcal{L}_{v_1}\xi_2 - \iota_{v_2}d\xi_1)\\ =&([v_1, v_2], -\mathcal{L}_{v_1}\iota_{v_2}\beta - \iota_{v_2}d\xi_1)\\ =&([v_1, v_2], -\iota_{[v_1,v_2]}\beta - \iota_{v_2}(\mathcal{L}_{v_1}\beta + d\xi_1))\\ =&([v_1, v_2], \iota_{[v_1,v_2]}\beta);\end{aligned}$$

where the last line follows as (v_1, ξ_1) generates a symmetry and hence satisfies (2.30). This flow defines a natural product

$$(v_1, \xi_1) \circ (v_2, \xi_2) = ([v_1, v_2], \mathcal{L}_{v_1}\xi_2 - \iota_{v_2}d\xi_1). \tag{2.31}$$

This product combined with the pairing

$$\langle (v_1, \xi_1), (v_2, \xi_2)\rangle = \frac{1}{2}(\iota_{v_1}\xi_2 + \iota_{v_2}\xi_1),$$

defines a Courant algebroid structure (see Definition 3.21).

In Section 2.2 it was possible to introduce a curvature term F, which results in a twisted Atiyah algebroid. In the two-dimensional case one can add a term $\int_{\Sigma_3} H$ where $\partial\Sigma_3 = \Sigma$ and $H \in \Omega^3(M)$. Section 4.1.1 describes how $H \in \Omega^3(M, \mathbb{Z})$ can be interpreted as the curvature of a $\mathsf{U}(1)$-gerbe structure. Following the argument used for the twisted Atiyah algebroid the twisted Courant algebroid product emerges

$$(v_1, \xi_1) \circ_H (v_2, \xi_2) = ([v_1, v_2], \mathcal{L}_{v_1}\xi_2 - \iota_{v_2}d\xi_1 + \iota_{v_1}\iota_{v_2}H). \tag{2.32}$$

The two-form β is not unique. In fact any closed $B \in \Omega^2(M)$ can be added to β without changing anything; it is $d\beta$ that is important. This is simply the statement that a total derivative term can be added to a Langrangian without changing the Euler–Lagrange equations. So instead of considering C defined by β we should really be looking at an equivalence class of structures given by

$$C_B = \text{graph}(\beta + B) = \{(v, \xi) \in \Gamma(\mathbb{T}M) : \xi = \iota_v \beta + \iota_v B\}$$

for $B \in \Omega^2_{cl}(M)$. This is associated to an automorphism $e^B \in \mathsf{End}(\mathbb{T}M)$ given by

$$e^B(v, \xi) = (v, \xi + \iota_v B).$$

Two structures C_1 and C_2 should be considered equivalent if they are related by such a *B-transformation.* Differential geometry on $TM \oplus T^*M$ is called generalised geometry and naturally incorporates B-transformations. Generalised geometry was introduced in the physics literature and later formalised by Hitchin and Gualtieri [64, 59].

The arguments of this section can be repeated for $\beta_p \in \Omega^p(M)$, with twists $H_{p+1} \in \Omega^{p+1}(M)$, describing the embedding of a p-dimensional worldvolume (a $p-1$-brane) in a target space of dimension $n \geq p$. The calculations are formally the same and give a higher algebroid product defined on $E = TM \oplus \wedge^p T^*M$ with a $\wedge^{p-1}T^*M$ valued pairing $\langle \cdot, \cdot \rangle$.

In this way we can see how variational problems naturally lead to algebroid structures and 'higher' geometry. Geometric structures associated to the vector bundle $E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$ are the subject of Chapter 4.

Example: Loop space model

The twisted Courant algebroid product (2.32) arises naturally in the study of closed strings and sigma models on loop space [3, 46]. Consider the embedding of a closed string worldsheet $\Sigma = \mathsf{S}^1 \times \mathbb{R}$. The phase space can be described locally using local coordinates $X^\mu(\sigma)$ and canonical momenta $p_\mu(\sigma)$. The phase space can be identified with T^*LM, via $(X^\mu, p_\mu) \in T^*LM$, where $LM := \{X : S^1 \to M\}$ is the *loop space.* The system is described by a Hamiltonian

$$H[X, p] = \frac{1}{2} \int_{S^1} G^{\mu\nu} p_\mu p_\nu + G_{\mu\nu} \partial X^\mu \partial X^\nu.$$

The standard symplectic form on T^*LM is given by

$$\omega = \int_{S^1} d\sigma \delta X^\mu \wedge \delta p_\mu,$$

where $\delta : \wedge^\bullet T^*LM \to \wedge^\bullet T^*LM$ is the de Rham differential. This symplectic form can be twisted to give

$$\omega_H = \int_{S^1} d\sigma (\delta X^\mu \wedge \delta p_\mu + \tfrac{1}{2} H_{\mu\nu\lambda} \partial X^\mu \delta X^\nu \wedge \delta X^\lambda).$$

The symplectic structure is preserved under the transformation

$$(X^\mu, p_\mu) \to (X^\mu, p_\mu + B_{\mu\nu}\partial X^\nu)$$

when $dB = 0$. This construction can be identified with geometry on the bundle $E = \mathbb{T}M := TM \oplus T^*M$. A section $(v, \xi) \in \Gamma(\mathbb{T}M)$ is identified with a current $J_\varepsilon \in C^\infty(T^*LM)$:

$$J_\varepsilon(v, \xi) = \int_{S^1} d\sigma \varepsilon (v^\mu p_\mu + \xi_\mu \partial X^\mu),$$

for test function $\varepsilon \in C^\infty(S^1)$. These currents obey a Poisson algebra structure based on the twisted Courant algebroid:

$$\{J_{\varepsilon_1}(v_1, \xi_1), J_{\varepsilon_2}(v_2, \xi_2)\} = -J_{\varepsilon_1\varepsilon_2}((v_1, \xi_1) \circ_H (v_2, \xi_2)) - \int_{S^1} d\sigma \varepsilon_1 \partial \varepsilon_2 \langle (v_1, \xi_1), (v_2, \xi_2)\rangle.$$

For more details the reader is referred to [3, 46] and references within.

String theory inspired examples

The study of target space models in string theory and supergravity lead to Leibniz algebroid structures on vector bundles. Models corresponding to Type II supergravity [39], Heterotic supergravity [48, 10], and M-theory/11-dimensional supergavity [74] can be constructed. A list of supergravity models and the associated algebroid vector bundles is given in Table 2.1.[10] It is also possible to consider quantum corrections in the framework of generalised geometries. First loop α'-corrections have been incorporated in the generalised geometry description of string theory [40].

Vector bundle	Supergravity model
$TM \oplus T^*M$	Type I (& Type II without RR-flux)
$TM \oplus S^\pm \oplus T^*M$	Type II with RR-flux
$TM \oplus \mathrm{ad}\mathsf{G} \oplus T^*M$	Heterotic
$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$	M-theory (11-dim supergravity)

Table 2.1: List of supergravity theories and associated algebroid vector bundles.

2.3.1 Higher Noether's theorem and Poisson–Lie T-duality

Poisson–Lie T-duality, introduced by Klimčík and Ševera [84, 83], describes an equivalence[11] between two non-linear sigma models. There is a Courant algebroid structure

[10] In Table 2.1 $S^\pm$ denote the positive and negative irreducible representations of the group $\mathsf{Spin}(T^*M)$:
$S^+ = \wedge^{ev}T^*M = \sum_{n=0}^{\lfloor \dim(M)/2 \rfloor} \wedge^{2n}T^*M$, and $S^- = \wedge^{odd}T^*M = \sum_{n=0}^{\lfloor \dim(M)/2-1 \rfloor} \wedge^{2n+1}T^*M$.

[11] The equivalence here is a symplectomorphism between the phase spaces of both models.

underlying Poisson–Lie T-duality (a description of Poisson–Lie T-duality from this perspective is given by Ševera in [112]). The non-linear sigma models described by Poisson–Lie T-duality are of the form

$$S[X] = \int dz d\bar{z}(G_{\mu\nu} + B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu =: \int dz d\bar{z} E_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu. \tag{2.33}$$

Suppose that there is a right-action of a Lie group G on the closed target manifold M. Choosing a basis of left-invariant vector fields $v_a \in \Gamma(TM)$, $a = 1, \ldots, \dim(\mathsf{G})$, the Lie algebra $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$ is realised by

$$[v_a, v_b] = C^c{}_{ab} v_c,$$

where $C^c{}_{ab} \in \mathbb{R}$ are structure constants. The associated conserved Noether forms, $J_a \in \Omega^1(\Sigma)$, are given by

$$J_a = v_a^\mu E_{\mu\nu}\bar{\partial} X^\nu d\bar{z} - v_a^\mu E_{\nu\mu}\partial X^\nu dz.$$

We consider the variation $\delta_\varepsilon S$ with respect to $\varepsilon = \varepsilon^a v_a$ for $\varepsilon^a \in C^\infty(\Sigma)$:

$$\begin{aligned}\delta_\varepsilon S = S[X + \varepsilon^a v_a] - S[X] &= \int \varepsilon^a \mathcal{L}_{v_a} E + \int d\varepsilon^a \wedge J_a \\ &= \int \varepsilon^a (\mathcal{L}_{v_a} E - dJ_a).\end{aligned}$$

Requiring that $\delta_\varepsilon S = 0$ for arbitrary $\varepsilon^a \in C^\infty(\Sigma)$ implies $\mathcal{L}_{v_a} E = dJ_a$. When v_a are Killing vectors for E (satisfying $\mathcal{L}_{v_a} E = 0$) we conclude that J_a is closed and hence a Noether current. More generally we may require that J_a satisfies

$$dJ - [J \overset{\wedge}{,} J]_{\tilde{\mathfrak{g}}} = 0, \tag{2.34}$$

where

$$[J \overset{\wedge}{,} J']_{\mathfrak{g}}(s_1, s_2) := \frac{1}{2}\Big([J(s_1), J'(s_2)]_{\tilde{\mathfrak{g}}} - [J(s_2), J'(s_1)]_{\tilde{\mathfrak{g}}}\Big),$$

for $J, J' \in \Omega^1(\Sigma, \mathfrak{g})$ and $s_1, s_2 \in \Gamma(T\Sigma)$. It follows that $J = d\tilde{g}\tilde{g}^{-1}$ is a Maurer–Cartan form for some $\tilde{g} : \Sigma \to \tilde{\mathsf{G}}$ with $\tilde{\mathfrak{g}} = \mathsf{Lie}(\widetilde{\mathsf{G}})$.

There is a pair (v_a, J_a) associated to a non-linear sigma model (M, E) exhibiting Poisson–Lie symmetry. The left-invariant vector fields v_a correspond to the right action of G and J_a are the associated Noetherian currents (as well as right-invariant Maurer–Cartan forms for $\tilde{\mathsf{G}}$). Poisson–Lie T-duality takes a non-linear sigma model (M, E) with (v_a, J_a) and identifies it with a dual non-linear sigma model $(\widetilde{M}, \widetilde{E})$ with $(\tilde{v}_a, \widetilde{J}_a)$. The dual model interchanges the role of G and $\widetilde{\mathsf{G}}$. The vector fields $\tilde{v}_a \in \Gamma(T\widetilde{M})$ correspond to a basis of left-invariant vector fields for $\widetilde{\mathsf{G}}$ with associated Noetherian currents $\widetilde{J}_a$.

If we consider a target manifold $M = \mathsf{G}$ the Poisson–Lie pair is constructed out of a Drinfeld double D. A Drinfeld double is a Lie group D which satisfies $\mathsf{Lie}(\mathsf{D}) =\cong \mathfrak{g}\oplus\tilde{\mathfrak{g}}$.[12] There is an integral surface $f \in C^\infty(\mathsf{D})$ defined via the relations

$$\begin{aligned}\partial f =&\varepsilon^a v_a + \varepsilon^a J_a = \varepsilon^a v_a - E(\cdot, \varepsilon^a v_a),\\ \bar{\partial} f =&\bar{\partial} g + E(\bar{\partial} g, \cdot).\end{aligned}$$

Central to the procedure is the fact that $f \in C^\infty(\mathsf{D})$ admits two decompositions:

$$f(z,\bar{z}) = g(z,\bar{z})\tilde{g}(z,\bar{z}) = \tilde{h}(z,\bar{z})h(z,\bar{z}). \tag{2.35}$$

The dual model requires the definition of a dual tensor $\widetilde{E} = \widetilde{G} + \widetilde{B}$. This is done via an identification of a subspace of $\mathsf{Lie}(\mathsf{D}) \cong \mathfrak{g} \oplus \tilde{\mathfrak{g}}$. Define the graph of E to give a subspace

$$\mathcal{E} := \mathrm{graph}(E) = \{(v,\xi) \in \mathfrak{g} \oplus \mathfrak{g}^* : \xi = E(\cdot, v)\}.$$

Given a choice of dual tensor $\widetilde{E} \in \Gamma(\tilde{\mathfrak{g}}^* \otimes \tilde{\mathfrak{g}}^*)$ there is a second subspace

$$\widetilde{\mathcal{E}} := \mathrm{graph}(\widetilde{E}) = \{(\tilde{v},\tilde{\xi}) \in \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^* : \tilde{\xi} = \widetilde{E}(\cdot, \tilde{v})\}.$$

The Lie algebra associated to a Drinfeld double is endowed with an inner product. The inner product gives an identification $\mathfrak{g} \oplus \mathfrak{g}^* \cong \tilde{\mathfrak{g}}^* \oplus \tilde{\mathfrak{g}}$. Taking the canonical pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ gives an identification $\tilde{\mathfrak{g}} \cong \mathfrak{g}^*$. The tensors E and $\widetilde{E}$ are mutually determined by the requirement that $\mathcal{E} = \widetilde{\mathcal{E}}$.

The difficult part of Poisson–Lie T-duality is determining the necessary and sufficient conditions on the pair $(\mathsf{G}, \widetilde{\mathsf{G}})$ to ensure that all the necessary parts of the construction above exist—in particular the decomposition (2.35). This construction exists and is unique precisely when the pair $(\mathsf{G}, \widetilde{\mathsf{G}})$ defines a Drinfeld double. A pair of Lie groups $(\mathsf{G}, \widetilde{\mathsf{G}})$ form a Drinfeld double if the Lie algebras ($\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$, $\tilde{\mathfrak{g}} = \mathsf{Lie}(\widetilde{\mathsf{G}})$) are compatible:

$$d_{\tilde{\mathfrak{g}}}[\cdot,\cdot]_{\mathfrak{g}} = [d_{\tilde{\mathfrak{g}}}\cdot,\cdot]_{\mathfrak{g}} + [\cdot, d_{\tilde{\mathfrak{g}}}\cdot]_{\mathfrak{g}} \quad \text{(or equivalently} \quad d_{\mathfrak{g}}[\cdot,\cdot]_{\tilde{\mathfrak{g}}} = [d_{\mathfrak{g}}\cdot,\cdot]_{\tilde{\mathfrak{g}}} + [\cdot, d_{\mathfrak{g}}\cdot]_{\tilde{\mathfrak{g}}}),$$

where $d_{\tilde{\mathfrak{g}}}$ is the Chevalley–Eilenberg differential on $\Gamma(\wedge^\bullet \tilde{\mathfrak{g}}^*) \cong \Gamma(\wedge^\bullet \mathfrak{g})$, and $[\cdot,\cdot]_{\mathfrak{g}}$ is the Schouten bracket extension of the Lie bracket on $\mathfrak{g}$ (see Section 3.4.1). The existence of the Drinfeld double structure is enough to guarantee that both (G, E) and $(\widetilde{\mathsf{G}}, \widetilde{E})$ are integrable non-linear sigma models. Poisson–Lie T-duality provides a symplectomorphism between the phase spaces of both models. The relevant Courant algebroid

[12] For more information on Drinfeld doubles and Lie bialgebras see for example [87].

structure associated to Poisson–Lie T-duality is a Lie bialgebroid constructed on the Drinfeld double $\mathsf{Lie(D)}$ (see [112]).

2.4 Classifying Lie algebroids and Lie groupoid gauging

It was seen in Section 2.1 (see also Chapter 5) that WZW models play an important role in the study of non-linear sigma models. Lie group manifolds come with natural left/right actions and geometric structures which are compatible with these actions, e.g., Maurer–Cartan forms and Killing forms. This allows a rather explicit construction of WZW non-linear sigma models for compact semi-simple Lie groups G.

Smooth symmetries are not restricted to group manifolds. Principal G-bundles, defined on a generic base manifold M, have a natural Lie group action. Symmetries are not restricted to group actions. In fact, groupoids provide the most natural setting for discussing symmetries [126]. Smooth symmetries correspond to Lie groupoids. With this in mind, it is of interest to realise geometric structures associated to Lie groupoids in a manner that closely follows that of group manifolds. It turns out that a certain class of transitive Lie algebroids (classifying algebroids) provide a solution to Cartan's realisation problem. The details of the construction are given in [113, 47]. This is an interesting application as it allows us to give an intuitive and explicit method for constructing manifolds with Lie algebroid symmetries.

Classifying Lie algebroids are the infinitesimal generators of a Lie groupoid action. There are invariant Maurer–Cartan forms and vector fields tangent to groupoid orbits. The Lie groupoid symmetry can be generated from flows defining foliations which do not produce a Lie group structure. This allows the construction of non-linear sigma models which are gauged using Lie groupoid actions. Global structure is determined by Lie algebroid cohomology.

The use of Lie groupoid gauging in the study non-linear sigma models is the subject of Chapter 5. It is possible to find an analogue of the WZW model given a Lie groupoid by lifting the geometry to a vector bundle.

Chapter 3

Background

This chapter contains background material required for the rest of the thesis. Lie groupoids, Lie algebroids, and more general vector bundle geometry (based on local Leibniz algebroids) are discussed, along with numerous examples. The chapter does not contain new results and readers familiar with these topics may wish to skim the chapter for notation.

3.1 Lie Groupoids

This section defines Lie groupoids and gives some of their properties. The presentation and examples here are based primarily on the notes by Crainic and Fernandes [43]. A comprehensive introduction to Lie groupoids and Lie algebroids can be found in the textbook by Mackenzie [97].

Definition 3.1. A *groupoid* $(\mathcal{G}, M, s, t, m, u, i)$ consists of a set of arrows, $\mathcal{G}$, a set of objects M, and maps s, t, m, u, i, satisfying the laws of composition, associativity, and inverses:

- The *source* and *target* maps: $s, t : \mathcal{G} \to M$, associating to each arrow h its source object $s(h)$ and target object $t(h)$. We write $h : x \stackrel{h}{\to} y$ for $h \in \mathcal{G}$ satisfying $s(h) = x$ and $t(h) = y$.
- The *set of composable arrows* is denoted by $\mathcal{G}_2$:

$$\mathcal{G}_2 := \{(h_2, h_1) \in \mathcal{G} \times \mathcal{G} : s(h_2) = t(h_1)\}. \tag{3.1}$$

  For a pair of composable arrows (h_2, h_1) the *composition map* $m : \mathcal{G} \to \mathcal{G}$ is the composition $m(h_2, h_1) = h_2 \circ h_1$ (typically denoted $h_2 h_1$ for simplicity).

- The *unit* and *inverse* maps: $u : M \to \mathcal{G}$, $i : \mathcal{G} \to \mathcal{G}$, where u sends $x \in M$ to the identity arrow $1_x \in \mathcal{G}$ at x, and i sends an arrow h to its inverse h^{-1}.

We will often denote a groupoid by $(\mathcal{G}, M)$ or even simply $\mathcal{G}$ if there is no risk of confusion.

Remark. The definition of a groupoid given here can be succinctly phrased is as follows: A groupoid $\mathcal{G}$ is a small category in which every arrow is invertible.

Definition 3.2. A *topological groupoid* is a groupoid $\mathcal{G}$ whose set of arrows and set of objects are both topological spaces, and the structure maps (s, t, u, m, i) are continuous with s and t open.

Definition 3.3. A *Lie groupoid* is a groupoid $\mathcal{G}$ whose set of arrows and set of objects are both manifolds, and the structure maps (s, t, m, u, i) are all smooth with s and t are submersions.

Not all objects in a groupoid need to be composable. Given an object $x \in M$ it is important to keep track of the possible precomposable and postcomposable objects. If $x \in M$ then the sets

$$\mathcal{G}(x, \cdot) = s^{-1}(x), \quad \mathcal{G}(\cdot, x) = t^{-1}(x)$$

are called the *s-fiber* and *t-fiber* at x. The inverse map induces a bijection $i : \mathcal{G}(x, \cdot) \to \mathcal{G}(\cdot, x)$. Multiplication is only defined on the s-fiber, and an arrow $g : x \to y$ induces a bijection

$$R_g : \mathcal{G}(y, \cdot) \to \mathcal{G}(\cdot, x).$$

The set of objects which can be both precomposed and postcomposed with a fixed $x \in M$ define a group; the intersection of the s and t-fiber at $x \in M$

$$\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x) = \mathcal{G}(x, \cdot) \cap \mathcal{G}(\cdot, x)$$

is called the *isotropy group* at x.[1]

We can define an equivalence of two objects $x, y \in M$, denoted $x \sim y$, if there exists an arrow $h \in \mathcal{G}$ whose source is x and target is y. The equivalence class of $x \in M_{\mathcal{G}}$ is called the *orbit through* x

$$\mathcal{O}_x = \{t(h) : h \in s^{-1}(x)\}.$$

The quotient set

$$M/\mathcal{G} := M/\sim_{\mathcal{G}} = \{\mathcal{O}_x : x \in M\}$$

[1]If we consider the transformation groupoid associated to the action of a Lie group G on a manifold M (Example 3.3), the isotropy group defined here coincides with the notion of an isotropy subgroup appearing in the physics literature: For a fixed $x \in M$, the isotropy group $\mathsf{G}_x = \{g \in \mathsf{G} : gx = x\}$.

is called the *orbit set* of $\mathcal{G}$.

Definition 3.4. Given a groupoid $\mathcal{G}$ over M a $\mathcal{G}$*-space* E is defined by a map $\mu : E \to M$, called the *moment map*, together with a map

$$\mathcal{G} \times_M E = \{(g, e) : s(g) = \mu(e)\} \to E, \quad (g, e) \to ge,$$

satisfying the following identities

1. $\mu(ge) = t(g)$;

2. $g(he) = (gh)e$, for all $g, h \in \mathcal{G}$ and $e \in E$ for which the composition is well defined;

3. $1_{\mu(e)}e = e$, for all $e \in E$.

An action of $\mathcal{G}$ on E, with moment map $\mu : E \to M$, associates to each arrow $g : x \to y$ an isomorphism

$$E_x \to E_y, \quad e \to ge,$$

where $E_x = \mu^{-1}(x)$, such that the action identities are satisfied. Each fiber E_x is a representation of the isotropy group $\mathcal{G}_x$.

If we have a topological groupoid $\mathcal{G}$ we might want E to be a topological space in order to describe a continuous action. If M is a manifold we might want E to be a manifold to describe a smooth action.

3.1.1 Lie groupoid examples

In this section we give examples of groupoids. The examples of Lie groupoids are of particular relevance to Lie groupoid gauging discussed in Chapter 5. Many more examples exist in the literature (see for example [125, 63, 97, 43]).

Example 3.1 (Lie group). Every Lie group G can be viewed as a Lie groupoid over a point $(\mathcal{G}, M) = (\mathsf{G}, \mathrm{pt})$.

Example 3.2 (Fundamental Groupoid). Let M be a manifold and let $\Pi_1(M)$ denote the manifold consisting of all homotopy classes (with fixed end points) of curves in M. Then $(\Pi_1(M), M)$ can be endowed with the structure of a Lie groupoid. Let $\gamma : I \to M$ be a curve in M and denote its homotopy class by $[\gamma]$. The source and target maps associate to $[\gamma]$ are its end points i.e., $s([\gamma]) = \gamma(0)$, and $t([\gamma]) = \gamma(1)$. If γ_1 and γ_2 are two curves such that $\gamma_1(1) = \gamma_2(0)$ then we define their product to be concatenation of curves, $[\gamma_2][\gamma_1] = [\gamma_2 \cdot \gamma_1]$. The identity element at a point $x \in M$ is the class of homotopically trivial paths passing through x. The inverse of $[\gamma]$ is the class of $\bar{\gamma} : I \to M$, where $\bar{\gamma}(t) = \gamma(1 - t)$. Note that the orbits of the fundamental groupoid are the connected components of M; the isotropy group at x is the fundamental group of M with base point x.

Example 3.3 (Transformation groupoid). Let G be a group acting on a manifold M. We define the *transformation groupoid* $(\mathcal{G}, M) = (\mathsf{G} \times M, M)$ to be the Lie groupoid whose structure maps are given by

$$s(g,x) = x, \quad t(g,x) = gx, \quad (h,gx)(g,x) = (hg,x), \quad 1_x = (e,x), \quad (g,x)^{-1} = (g^{-1}, gx),$$

where e is the identity of G.

The orbits of the groupoid coincide with those of the action G. The isotropy groups of G coincide with those of the action.

Example 3.4 (Gauge Groupoid). Let $P(M, \pi, \mathsf{G})$ be a principal G-bundle. The *gauge groupoid* of P, denoted $(\mathcal{G}(P), M)$, is defined as

$$\mathcal{G}(P) = \frac{P \times P}{\mathsf{G}},$$

where the quotient refers to the diagonal action of G on $P \times P$ $((p,q) \cdot g = (pg, qg))$. Let us denote by $[p,q]$ the class of (p,q). The structure of $\mathcal{G}(P)$ is given by:

$$\begin{aligned} s[p,q] &= \pi(q), \quad t[p,q] = \pi(p), \quad [p_1,q_1][q_1,p_2] = [p_1,p_2], \\ 1_x &= [p,p] \text{ for some } p \in \pi^{-1}(x), \qquad [p,q]^{-1} = [q,p]. \end{aligned}$$

Example 3.5 (Symplectic groupoid on $T^*\mathsf{G}$). Given a Lie group G, with $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$, we can define a Lie groupoid $(\mathcal{G}, M) = (T^*\mathsf{G}, \mathfrak{g}^*)$, equipped with its canonical symplectic structure. If we identify $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$ the Lie groupoid structure on $T^*\mathsf{G}$ is simply that of the transformation groupoid associated to the coadjoint action of G on $\mathfrak{g}^*$. For more details see [125].

Many more examples of groupoids are known, including manifolds with boundary, convolutions of functions, and more general symplectic groupoids (see for example [43, 63, 125]).

3.1.2 The Lie algebroid of a Lie groupoid

The infinitesimal objects associated to Lie groupoids are Lie algebroids. Lie algebroids are a linearisation of Lie groupoids, retaining most of the structure of the Lie groupoid, but are easier to study.

There are two notable differences between the Lie groupoid case and the Lie group case. Firstly there is a unit for each point in M suggesting a representation on a vector bundle as opposed to a vector space. Secondly, when considering right-invariant sections on $\mathcal{G}$ we must restrict our attention to the sub-bundle $T^s\mathcal{G} = \ker(ds) \subset T\mathcal{G}$—those sections which are tangent to the s-fibers.

Definition 3.5. Given a Lie Groupoid $(\mathcal{G}, M)$ we can define a vector bundle $A \to M$. For each $x \in M$ the fiber A_x coincides with the tangent space at the unit 1_x of the s-fiber at x. In short $A := T^s\mathcal{G}|_M$ and will often denote it simply as $A = \mathsf{Lie}(\mathcal{G})$.

The vector bundle $A \to M$ is endowed with a bracket on the space of right-invariant sections on $\mathcal{G}$. To describe this we need to know the induced action from the composition of arrows $g : x \to y$ and $h : y \to z$. For an arrow $h : y \to z$ we define

$$T^s_h\mathcal{G} := T_h\mathcal{G}(y, \cdot),$$

and any arrow $g : x \to y$ induces a map

$$R_g : T^s_h\mathcal{G} \to T^s_{hg}\mathcal{G},$$

the differential of the right multiplication by g. The space of right-invariant sections on $\mathcal{G}$ is given by the set

$$\mathfrak{X}^s_{\mathrm{inv}}(\mathcal{G}) = \{X \in \Gamma(T^s\mathcal{G}) : X_{hg} = R_g(X_h), \forall (h, g) \in \mathcal{G}_2\},$$

where $\mathcal{G}_2$ is the set of composable arrows defined by (3.1). Given $a \in \Gamma(A)$ a right-invariant section is given by $\tilde{a}_g = R_g(a_{t(g)})$. The invariance of $X \in \mathfrak{X}^s_{\mathrm{inv}}(\mathcal{G})$ shows that X is determined by its values at the points in M

$$X_g = R_g(X_y), \quad \forall g : x \to y.$$

Defining $a := X|_M \in \Gamma(A)$ we have $X = \tilde{a}$. This establishes an isomorphism

$$\Gamma(A) \xrightarrow{\sim} \mathfrak{X}^s_{\mathrm{inv}}(\mathcal{G}), \quad a \to \tilde{a}. \tag{3.2}$$

Definition 3.6. The *Lie algebroid bracket* on A is the bracket obtained from the isomorphism (3.2) and defined as

$$\widetilde{[a_1, a_2]_A} := [\tilde{a}_1, \tilde{a}_2]. \tag{3.3}$$

Definition 3.7. The *anchor map* of A is the bundle map

$$\rho : A \to TM$$

obtained by restricting $dt : T\mathcal{G} \to TM$ to $A \subset T\mathcal{G}$.

The anchor map gives sections of $\Gamma(A)$ a derivation property. For all $a_1, a_2 \in \Gamma(A)$, and all $f \in C^\infty(M)$ the Lie algebroid bracket satisfies

$$[a_1, fa_2]_A = f[a_1, a_2]_A + (\rho(a_1)f)a_2,$$

where $\rho(a_1)f = \mathcal{L}_{\rho(a_1)}f$. This follows from the identification of $\Gamma(A) \cong \mathfrak{X}_{\text{inv}}(\mathcal{G})$ and ρ with dt. Details can be found in [43].

The Lie groupoid action can (partially) be recovered from the infinitesimal Lie algebroid action. This is a generalisation of the relationship between Lie algebras and Lie groups. The action is recovered by flowing along the right-invariant sections $\tilde{a}$. For $x \in M$, we set

$$\phi_a^t(x) := \phi_{\tilde{a}}^t(1_x) \in \mathcal{G},$$

where $\phi_{\tilde{a}}^t$ is *the flow* of the right-invariant section $\tilde{a}$ induced by a. This flow defines an exponential map much like the in the case of Lie algebras:

$$\exp(ta)(x) = \phi_a^t(x).$$

Examples of Lie algebroids are given in Section 3.2.1.

3.2 Lie algebroids

In Section 3.1.2 the Lie algebroid associated to a Lie groupoid was given by taking a bracket structure on right-invariant sections of a vector bundle. The properties of this Lie algebroid can be axiomatised to give a general definition of a Lie algebroid. A natural question arises: Given a Lie algebroid on a vector bundle A, can this be integrated to a Lie groupoid action for some $\mathcal{G}$? In contrast to the case of Lie algebras, this is not true for general Lie algebroids (see Section 3.2.2).

Definition 3.8. A *Lie algebroid* $(A, M, \rho, [\cdot\ ,\cdot]_A)$ consists of a vector bundle A over a manifold M equipped with a bundle map $\rho : A \to TM$ and a bracket $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ satisfying:

$$[a_1, [a_2, a_3]_A]_A = [[a_1, a_2]_A, a_3]_A + [a_2, [a_1, a_3]_A]_A, \tag{3.4a}$$

$$[a_1, fa_2]_A = f[a_1, a_2]_A + ((\rho(a_1)f))a_2, \tag{3.4b}$$

$$[a_1, a_2]_A = -[a_2, a_1]_A, \tag{3.4c}$$

for $a_1, a_2, a_3 \in \Gamma(A)$ and $f \in C^\infty(M)$.

Identities (3.4a) and (3.4c) imply that $[\cdot,\cdot]_A$ is a Lie bracket. The first two identities imply that the anchor map ρ is a bracket homomorphism:

$$\rho([a_1, a_2]_A) = [\rho(a_1), \rho(a_2)]_{TM} \tag{3.5}$$

where $[\cdot\ ,\cdot]_{TM}$ is the bracket defined by the commutator of vector fields.[2]

A *transitive Lie algebroid* is a Lie algebroid with a surjective anchor. we say that A is a *regular Lie algebroid* if the image of the anchor $\mathrm{Im}(\rho_x) \subset T_xM$ has constant rank.

Remark. At any point $x \in M$ the Lie algebra $\mathfrak{g}_x(A) := \ker(\rho_x)$ is the *isotropy Lie algebra* at x. Given a Lie groupoid $\mathcal{G}$ the isotropy Lie algebra $\mathfrak{g}_x$ is isomorphic to the Lie algebra of the isotropy group $\mathcal{G}_x$.

Generically a bundle morphism $\rho : A \to TM$ defines a distribution $\mathrm{Im}(\rho_x) \subset T_xM$. The distribution $\mathrm{Im}(\rho)$ is *smooth* if $\mathrm{Im}(\rho_x)$ is spanned by smooth sections $a(x) \in \mathrm{Im}(\rho_x)$ for all $x \in M$. The rank of a smooth distribution is a lower semi-continuous function. A smooth distribution is called a *regular distribution* if its rank is locally constant.

The anchor homomorphism property (3.5) is quite strong and gives an integrability condition for regular Lie algebroids. A distribution $K \subset TM$ is *integrable* if every point of M is contained in a plaque (a connected, immersed submanifold $\mathcal{O}$ such that $T\mathcal{O} = K|_{\mathcal{O}}$). An integrable distribution K determines a *foliation* $\mathcal{F}$ of M into leaves given by the maximal plaques. The foliation $\mathcal{F}$ is called the *integral foliation* of K. We call K the *tangent distribution* to $\mathcal{F}$, denoted by $T\mathcal{F}$. The Stefan–Sussman theorem asserts that a smooth distribution $K \subset TM$ is integrable if and only if it is involutive[3] and $\mathrm{rank}(K)$ is constant along the flow lines of sections of K [115]. A regular distribution in TM is integrable iff it is involutive. The anchor homomorphism property states that $\mathrm{Im}(\rho)$ is involutive.

In conclusion: If A is a regular Lie algebroid, property (3.5) shows that the resulting distribution is integrable and M is foliated by immersed submanifolds $\mathcal{O}$ defined by orbits $T_x\mathcal{O} = \mathrm{Im}(\rho_x)$ for all $x \in \mathcal{O}$.

Remark. While it is true that every regular Lie algebroid defines a foliation the converse is not expected to be true. To the best of the author's knowledge it is an open question whether or not all foliations can be generated by Lie algebroids (see for example [6] and references within). It has recently been shown that all foliations must by generated by Lie_∞-algebroids [92, 91].

3.2.1 Examples

Here we present some standard examples of Lie algebroids. These examples (along with the associated Lie groupoids) will form the basis of examples of Lie groupoid gauging in Chapter 5.

[2]This is shown in a more general setting in Section 3.4.

[3]A distribution $K \subset TM$ is involutive if $[X, Y] \in \Gamma(K)$ for each $X, Y \in \Gamma(K)$.

Example 3.6 (Lie algebra). A Lie algebra $\mathfrak{g}$ defines a Lie algebroid with $A = \mathfrak{g}$, $M = \mathrm{pt}$, $\rho = 0$. This is the infinitesimal object corresponding to a Lie groupoid $(\mathcal{G}, M) = (\mathsf{G}, \mathrm{pt})$, where $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$.

Example 3.7 (Tangent bundle). Given a manifold M, there is Lie algebroid on TM with the bracket given by commutator of vector fields, and the anchor given by the identity map $\rho = \mathrm{Id}_{TM}$.

Example 3.8 (Foliations). Let $\mathcal{F}$ be a regular foliation on M, so that $T\mathcal{F} \subset TM$ is an involutive distribution of constant rank. The distribution $T\mathcal{F}$ has a Lie algebroid structure, with the bracket given by the commutator of vector fields, and the anchor given by the inclusion $i : T\mathcal{F} \to TM$. The orbits of this Lie algebroid are the leaves of $\mathcal{F}$.

Example 3.9 (Infinitesimal $\mathfrak{g}$-action). Let $\psi : \mathfrak{g} \to \Gamma(TM)$ be an infinitesimal action of a Lie algebra $\mathfrak{g}$ on TM. The *transformation Lie algebroid* is defined on $A = M \times \mathfrak{g}$, where $\rho(x, X) = \psi(X)|_x$ and

$$[X_1, X_2]_{M\times\mathfrak{g}}(x) = [X_1, X_2]_{\mathfrak{g}} + (\psi(X_1) \cdot X_2)(x) - (\psi(X_2) \cdot X_1)(x).$$

This is the Lie algebroid associated to the transformation groupoid (Example 3.3).

Example 3.10 (Atiyah algebroid). Let $P(M, \pi, \mathsf{G})$ be a principal G-bundle. The *Atiyah algebroid* is defined on $A = TP/G$ as part of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (P \times \mathfrak{g})/\mathsf{G} \arrow[r] & TP/\mathsf{G} \arrow[r, "\rho"] & TM \arrow[l, bend left, "\sigma"] \arrow[r] & 0 \end{tikzcd} .$$

Sections of A are identified with right-invariant vector fields on P. The bracket is given by the commutator of right-invariant vector fields; the anchor is $\pi_*|_{TP/G} : A \to TM$ induced by π_*. A short exact sequence of vector bundles gives a short exact sequence of the $C^\infty(M)$-modules of sections:

$$0 \longrightarrow C^\infty(P, \mathfrak{g})^{\mathsf{G}} \longrightarrow \mathfrak{X}_{\mathsf{G}}(P) \xrightarrow{\pi_*} \mathfrak{X} \longrightarrow 0 ,$$

where $C^\infty(P, \mathfrak{g})^{\mathsf{G}}$ is the module of G-equivariant smooth functions from P to $\mathfrak{g}$. If TP/G is a trivial bundle there is an isomorphism $C^\infty(P, \mathfrak{g})^{\mathsf{G}} \cong C^\infty(M, \mathfrak{g})$. A choice of splitting is given by a choice of one-form connection $\sigma \in \Omega^1(M, TP/\mathsf{G})$. The corresponding curvature $F_\sigma \in \Omega^2(M, TP/\mathsf{G})$ is given by

$$F_\sigma(v_1, v_2) = \sigma([v_1, v_2]) - [\sigma(v_1), \sigma(v_2)]_{TP/\mathsf{G}}.$$

Letting $(v,\gamma) \in \Gamma(A) = \mathfrak{X} \oplus C^\infty(M,\mathfrak{g})$ the Atiyah algebroid bracket is given by

$$[(v_1,\gamma_1),(v_2,\gamma_2)]_F = ([v_1,v_2],[\gamma_1,\gamma_2]_{\mathfrak{g}} + \nabla^\sigma_{v_1}\gamma_2 - \nabla^\sigma_{v_2}\gamma_1 - F_\sigma(v_1,v_2)).$$

The Atiyah algebroid is the Lie algebroid associated to the Gauge groupoid (Example 3.4).

The integrability of the Lie algebroid TP/G can be deduced from the properties of the curvature F_σ. See Example 3.15 for a special case and [42, 43] for the general case.

Example 3.11 (Generalised Atiyah sequence). Locally any Lie algebroid can be associated to a generalised Atiyah sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_{\mathcal{O}} \arrow[r] & Q|_{\mathcal{O}} \arrow[r, "\rho"] & T\mathcal{O} \arrow[l, bend left, "\sigma"] \arrow[r] & 0 \end{tikzcd},$$

where $\mathfrak{g}_{\mathcal{O}} = \ker(\rho)$ denotes the isotropy algebra of $\mathcal{O}$ and $T\mathcal{O} = \mathrm{Im}(\rho)$.

Example 3.12 (Poisson cotangent Lie algebroid). Given a Poisson manifold[4] (M,π) there is a *cotangent Lie algebroid*: $A = T^*M$, $\rho(\xi) = \pi(\xi,\cdot)$ for $\xi \in \Gamma(T^*M)$, and the bracket is given by

$$[\xi_1,\xi_2]_{T^*M} = \mathcal{L}_{\pi(\xi_1,\cdot)}\xi_2 - \mathcal{L}_{\pi(\xi_1,\cdot)}\xi_2 - d(\pi(\xi_1,\xi_2)). \tag{3.6}$$

This is the Lie algebroid associated to a symplectic groupoid on T^*M (Example 3.5).

3.2.2 Integrability

All Lie algebras arise as the tangent of some (not necessarily unique) Lie group. However, not all Lie algebroids arise as the tangent of a Lie groupoid.

Definition 3.9. A Lie algebroid A is called *integrable* if it is isomorphic to the Lie algebroid of a Lie groupoid $\mathcal{G}$. We say that $\mathcal{G}$ integrates A.

There are obstructions to the integrability of Lie algebroids. The general conditions for the integrability of Lie algebroids were established by Crainic and Fernandes [42, 43].

Theorem 3.10 (Lie I)**.** *Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid A. There exists a unique s-simply connected*[5] *Lie groupoid $\widetilde{\mathcal{G}}$ whose Lie algebroid is also A.*

Theorem 3.11 (Lie II)**.** *Let $A_1 \to M_1$ and $A_2 \to M_2$ be integrable Lie algebroids. Denote by $\mathcal{G}(A_1)$ the s-simply connected Lie groupoid integrating A_1 and $\mathcal{H}(A_2)$ any Lie groupoid integrating A_2. If $\Phi : A_1 \to A_2$ is a morphism of Lie algebroids*[6] *covering*

[4] A *Poisson structure* is a non-degenerate $\pi \in \Gamma(\wedge^2 TM)$ satisfying $[\pi,\pi]_{\text{Schouten}} = 0$.
[5] We say that $\mathcal{G}$ is s-simply connected if the s-fibers $s^{-1}(x)$ are 1-connected for every $x \in M$.
[6] Lie algebroid/groupoid morphisms are defined in Section 3.3.1.

$\phi : M_1 \to M_2$, then there exists a unique morphism of Lie groupoids $F : \mathcal{G}(A_1) \to \mathcal{H}(A_2)$, also covering ϕ, such that $d_{1_x}F(v) = \Phi(v)$ for all $x \in M_1$ and $v \in T^s_{1_x}\mathcal{G}(A_1)$. In this case we say that F integrates Φ.

In order to state the integrability condition for a general Lie algebroid A, we need to introduce the Weinstein groupoid. The Weinstein groupoid will appear as the groupoid associated to the Lie algebroid gauging procedure discussed in Chapter 5.

Definition 3.12. Fix a Lie algebroid structure on $A \to M$. An *A-path* consists of a pair (a, γ) where $\gamma : [0,1] \to M$ is a path in M, and $a : [0,1] \to A$ is a path in A, where

- a is a path above γ: $a(t) \in A_{\gamma(t)}$ for all $t \in [0,1]$;
- $\rho(a(t)) = \frac{d\gamma}{dt}(t)$ for all $t \in [0,1]$.

The base map can be recovered from a using the anchor ρ, so we will often just refer to a as the *covering map* with the induced base map implied. The *space of paths* associated to a Lie algebroid A will be denoted $P(A)$.

The notion of an A-path is the Lie algebroid generalisation of flow lines associated to vector fields.

An A-*homotopy* between A-paths a_0 and a_1 is a Lie algebroid morphism $T(I \times I) \to A$ which covers a (standard) homotopy with fixed end-points between the base paths $\gamma(a_i(t))$.

Remark. If we start with a Lie groupoid $\mathcal{G}$ and consider a path $g(t) : I \to \mathcal{G}$ which stays within an s-fiber and starts at 1_x, then $a(t) = (dL_{g^{-1}(t)})_{g(t)} \frac{dg(t)}{dt}$ defines an A-path.

Definition 3.13. The *Weinstein groupoid of A* is the topological groupoid

$$\mathcal{G}(A) := P(A)/\sim$$

where $\sim$ denotes the equivalence of A-homotopy paths. The multiplication on $\mathcal{G}(A)$ is given by concatenation of paths. The source and target maps are the projection to the start and end points of the A-paths.

The obstruction to integrating a Lie algebroid is described using the Monodromy groupoid.

Definition 3.14. The *monodromy group of A* at $x \in M$ is the set

$$\mathcal{N}_x(A) = \{v \in Z(\mathfrak{g}_x) : v \text{ is } A\text{-homotopic to the constant zero path}\} \subset A_x,$$

where $\mathfrak{g}_x$ is the isotropy algebra of A at x, and $Z(\mathfrak{g}_x)$ is its center. The monodromy group consists of the subset of $v \in Z(\mathfrak{g}_x)$ which A-homotopic to the trivial path. Each $v \in Z(\mathfrak{g}_x)$ can be viewed as a constant A-path. If there exists a homotopy between the constant A-path and the trivial path we have $v \in \mathcal{N}_x$.

Let $\mathcal{G}(\mathfrak{g}_x) = \mathsf{G}_x(\mathfrak{g})$ denote the simply connected Lie group integrating the Lie algebra $\mathfrak{g}_x$. Define $\widetilde{\mathcal{N}}_x(A)$ as the subgroup of $\mathcal{G}(\mathfrak{g}_x)$ consisting of equivalence classes $[a] \in \mathcal{G}(\mathfrak{g}_x)$ of $\mathfrak{g}_x$-paths that equivalent to the trivial A-path. The group $\widetilde{\mathcal{N}}$ is a subgroup contained in $Z(\mathcal{G}(\mathfrak{g}_x))$, and its intersection with the connected component $Z(\mathcal{G}(\mathfrak{g}_x))^0$ is isomorphic to $\mathcal{N}_x(A)$.

Let L_x be the leaf of A through x. There is a monodromy map $\partial : \pi_2(L_x, x) \to \mathcal{G}(\mathfrak{g}_x)$ whose image is $\widetilde{\mathcal{N}}_x(A)$.

We present two examples which provide tools to discuss integrability of Lie algebroids which could be used in relation to Lie groupoid gauging in Chapter 5.

Example 3.13 (Monodromy Groupoid of a foliation). Let $\mathcal{F}$ be a regular foliation of M. The *monodromy groupoid* of the foliation is the unique Lie groupoid $(\Pi_1(\mathcal{F}), M)$ whose Lie algebroid is $T\mathcal{F}$ and s-fibers are connected and simply connected. It can be described as the space of homotopy classes of curves contained in the leaves of $\mathcal{F}$, where we only allow homotopies which are also contained in the leaves. Multiplication is given by concatenation of paths, the identity elements are given by the class of constant paths, and inversion is given by reversing the direction of the curve. The orbits of the monodromy groupoid of $\mathcal{F}$ are precisely the leafs of the foliation. The isotropy Lie group of a point x is the fundamental group of the leaf through x, with base point x, i.e., $\pi_1(L_x; x)$.

Example 3.14 (Holonomy Groupoid of a foliation). Taking Example 3.13 but replacing homotopy classes of curves with holonomy classes of curves, gives another Lie groupoid, called the *holonomy groupoid* of the foliation $(\mathrm{Hol}(\mathcal{F}), M)$. The associated Lie algebroid is also $T\mathcal{F}$. This Lie groupoid has the property that any Lie groupoid $\mathcal{G}$ whose Lie algebroid is $T\mathcal{F}$ fits into the exact sequence of groupoid covering maps

$$0 \longrightarrow \Pi_1(\mathcal{F}) \longrightarrow \mathcal{G} \longrightarrow \mathrm{Hol}(\mathcal{F}) \longrightarrow 0 \ .$$

In this exact sequence a groupoid covering map is a Lie groupoid morphism whose restriction to each s-fiber is a covering map.

Remark. The previous example mentioned holonomy classes of curves. We give the definition of the holonomy of a curve in a foliated manifold (following [45]). Take some d-dimensional manifold M and a k-dimensional foliation $\mathcal{F} \subset M$. Choose a curve lying in a single leaf of the foliation $\gamma : [0,1] \to L$ with $\gamma(0) = a$ and $\gamma(1) = b$. We can choose small neighbourhoods U_a, U_b of a and b which consist of plaques. Take $(d-k)$-dimensional disks D_a^{d-k}, D_b^{d-k} at a and b which are transversal to the leaves and parameterise the plaques of U_a and U_b. The path γ can be subdivided such that it consists of paths which lie within the neighbourhood of plaques U_i (where $i = 0, \ldots, n$, $U_0 = U_a$, and $U_n = U_b$) which intersect uniquely with U_{i-1} and U_{i+1}. This chain of plaques defines a diffeomorphism of a neighbourhood D_a^{d-k} to a neighbourhood D_b^{d-k}.

This diffeomorphism gives the holonomy of the curve. The holonomy of the curve is independent of choices of U_i.

Theorem 3.15 (Lie III [42])**.** *Let $A \to M$ be a Lie algebroid. The following statements are equivalent:*

- *A is integrable.*
- *The Weinstein groupoid $\mathcal{G}(A)$ is a Lie groupoid.*
- *The monodromy groups are uniformly discrete.*

It is useful to consider several corollaries of this theorem:

- Any transitive Lie algebroid on a contractible base is integrable.
- Any Lie algebroid with an injective anchor is integrable.
- Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^*$ be its dual Lie–Poisson manifold. The cotangent Lie algebroid is integrable.

These integrability conditions will be sufficient for most examples of interest in Chapter 5 when considering Lie algebroid gauging of sigma models.

Example 3.15 (Integrability). Consider the Lie algebroid A, defined by the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & A \arrow[r, "\rho"] & TM \arrow[r] \arrow[l, bend left, "\sigma"] & 0 \end{tikzcd} ,$$

where $L \to M$ is a line bundle. A choice of splitting can be encoded in a choice of connection one-form $\sigma \in \Omega^1(M, A)$. Let $F \in \Omega^2_{\mathrm{cl}}(M)$ be the curvature of the connection σ. A choice of splitting induces a local map on modules $\Gamma(A) \cong \Gamma(TM) \oplus C^\infty(M)$. The bracket is defined on sections $(v, f) \in \Gamma(TM) \oplus C^\infty(M)$:

$$[(v_1, f_1), (v_2, f_2)]_F = ([v_1, v_2]_{TM}, \mathcal{L}_{v_1} f_2 - \mathcal{L}_{v_2} f_1 - F(v_1, v_2)).$$

If M is simply connected and integral, $[F/2\pi] \in H^2(M, \mathbb{Z})$, and we have a prequantisation principal $\mathsf{U}(1)$-bundle. In this case the Lie algebroid integrates to the gauge groupoid $\mathcal{G}(P)$ associated to $P(M, \pi, \mathsf{U}(1))$ (Example 3.4). The groupoid $\mathcal{G}(P)$ is transitive and the isotropy Lie groups are all isomorphic to each other and to $\mathbb{R}/\Gamma_F$, where Γ_F is the group of spherical periods of F:

$$\Gamma_F = \left\{ \int_\gamma F : \ \gamma \in \pi_2(M) \right\} \subset \mathbb{R}.$$

The group Γ_F is the monodromy group. When F is not integral, we don't have a prequantization bundle, and the construction of the groupoid fails. It is still possible for the Lie algebroid A to integrate to a simply connected Lie groupoid if the isotropy groups are isomorphic to $\mathbb{R}/\Gamma_F$. In this case A is integrable if $\Gamma_F \subset \mathbb{R}$ is a discreet subgroup. Let us take, for example, $M = \mathsf{S}^2 \times \mathsf{S}^2$ with $F = dS \oplus \lambda dS$, where dS is the standard area form on S^2. Taking λ to be irrational the group of spherical periods is $\Gamma_F = \mathbb{Z} \oplus \lambda\mathbb{Z}$, so that $\mathbb{R}/\Gamma_F$ is non-discrete, and the corresponding Lie algebroid is non-integrable.

3.3 Lie algebroid geometry

In the previous section we introduced Lie algebroids and saw that they formed a generalisation of Lie algebras. A standard example of a Lie algebroid comes from the tangent bundle of a manifold (Example 3.7). The tangent bundle comes equipped with additonal structure of interest in differential geometry ,e.g, a Lie derivative $\mathcal{L}$ defining the flow of tensors along a vector field, and the de Rham differential d defining de Rham cohomology on the complex $\Omega^\bullet(M)$. The presence of a Lie algebroid $A \to M$ allows the study of differential geometry on the vector bundle A. This section outlines Lie algebroid geometry.

Given a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ one can define $d_A : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$:

$$\begin{aligned}(d_A\omega)(a_0, a_1, \dots, a_k) = &\sum_{i=0}^{k}(-1)^i \rho(a_i)(\omega(a_0, a_1, \cdots, \hat{a}_i, \dots, a_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([a_i, a_j]_A, a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k),\end{aligned} \tag{3.7}$$

where $a_i \in \Gamma(A)$, $\omega \in \Gamma(\wedge^k A^*)$, and $\hat{a}_i$ denotes omission. A lengthy but straightforward calculation shows that the operator d_A satisfies $d_A^2 = 0$ if and only if the Lie algebroid axioms are satisfied. This can be recovered from the more general expression for $\mathbb{d}_E^2$ (3.37) given in Section 3.4.3.

There is a one-to-one correspondence between Lie algebroids $(A, [\cdot,\cdot]_A, \rho)$ and Lie algebroid differentials d_A. It is clear from the definition of d_A that a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ uniquely defines a differential d_A. Conversely, given a differential d_A, the Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ can be recovered as follows: The anchor is given by the action of differentiation on a function $\rho(a)f = (d_A f)(a)$; the bracket can then be recovered by considering the differential on $\alpha \in \Gamma(\wedge^1 A^*)$:

$$(d_A\alpha)(a_1, a_2) = \rho(a_1)\alpha(a_2) - \rho(a_2)\alpha(a_1) - \alpha([a_1, a_2]_A).$$

The *Lie algebroid cohomology of A* is the cohomology $H^\bullet(A)$ groups associated to the complex $(d_A, \Gamma(\wedge^\bullet A^*))$.

Example 3.16 (de Rham cohomology). Take $A = TM$, and the Tangent Lie algebroid. In this case 3.7 defines the de Rham differential, d, and $\Gamma(\wedge^\bullet A^*) = \Gamma(\wedge^\bullet T^*M)$.

Example 3.17 (Lie algebra cohomology). Take $A = \mathfrak{g}$, with the Lie algebroid defined by the Lie algebra $[\cdot,\cdot]_{\mathfrak{g}}$. In this case 3.7 defines the Chevalley–Eilenberg differential $d_{\mathfrak{g}}$ on the complex $\wedge^\bullet \mathfrak{g}^*$.

Example 3.18 (Foliated cohomology). Take a manifold M and foliation $\mathcal{F} \subset M$. The foliated Lie algebroid (Example 3.8) defines foliated cohomology with $(d_{\mathcal{F}}, \Gamma(\wedge^\bullet(T^*\mathcal{F})))$.

Given a Lie algebroid $A \to M$, and the canonical pairing $\langle \cdot,\cdot\rangle : A \times A^* \to \mathbb{R}$, a Lie derivative can be defined which satisfies

$$\mathscr{L}_{a_1}\langle a_2, \alpha\rangle = \langle \mathscr{L}_{a_1} a_2, \alpha\rangle + \langle a_2, \mathscr{L}_{a_1}\alpha\rangle.$$

Defining $\mathscr{L}_a f := \rho(a) f$ for $f \in C^\infty(M)$ and $\mathscr{L}_{a_1} a_2 := [a_1, a_2]_A$, we define

$$\langle \mathscr{L}_{a_1}\alpha, a_2\rangle := \rho(a_1)\langle \alpha, a_2\rangle - \langle [a_1, a_2]_A, \alpha\rangle, \tag{3.8}$$

for any $\alpha \in \Gamma(A^*)$. This definition can be extended in the natural way to general tensors $\mathsf{T} \in \Gamma((\otimes^p A) \otimes (\otimes^q A^*))$.[7] This gives a notion of flowing tensors along sections of a vector bundle endowed with a Lie algebroid. These definitions satisfy the Cartan calculus on $\Gamma(\wedge^\bullet A^*)$:

$$[d_A, d_A] = 2d_A^2 = 0; \tag{3.9a}$$

$$[d_A, \iota_a] = \iota_a d_A + d_A \iota_a = \mathscr{L}_a; \tag{3.9b}$$

$$[d_A, \mathscr{L}_a] = d\mathscr{L}_a - \mathscr{L}_a d = 0; \tag{3.9c}$$

$$[\mathscr{L}_{a_1}, \mathscr{L}_{a_2}] = \mathscr{L}_{a_1}\mathscr{L}_{a_2} - \mathscr{L}_{a_2}\mathscr{L}_{a_1} = \mathscr{L}_{[a_1,a_2]_A}; \tag{3.9d}$$

$$[\mathscr{L}_{a_1}, \iota_{a_2}] = \mathscr{L}_{a_1}\iota_{a_2} - \iota_{a_2}\mathscr{L}_{a_1} = \iota_{[a_1,a_2]_A}; \tag{3.9e}$$

$$[\iota_{a_1}, \iota_{a_2}] = \iota_{a_1}\iota_{a_2} + \iota_{a_2}\iota_{a_1} = 0. \tag{3.9f}$$

Equations (3.9) use the graded bracket $[d_1, d_2] = d_1 d_2 - (-1)^{|d_1||d_2|} d_2 d_1$ with weights $|\iota_a| = -1$, $|d_A| = 1$, and $|\mathcal{L}_a| = 0$.

Additional geometric structure can be defined using the notion of a Lie algebroid connection.

Definition 3.16. Given a Lie algebroid $A \to M$, and a vector bundle $E \to M$, an *A-connection on E* is a bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ which satisfies the

[7] Taking $\mathscr{L}_{a_1}$ to act as a derivation on T.

following conditions:

- $\nabla_{fa}e = f\nabla_a e$;
- $\nabla_a fe = f\nabla_a e + (\rho(a)f)e$;

where $a \in \Gamma(A)$, $e \in \Gamma(E)$, and $f \in C^\infty(M)$.

Example 3.19 (Affine connection). When $A = E = TM$ we recover the notion of an affine connection.

Example 3.20 (Vector bundle connection). Given a vector bundle $E \to M$, and the Tangent Lie algebroid on TM, a TM-connection on E coincides with the definition of a vector bundle connection.

Example 3.21. Given a vector bundle connection ∇ (TM-connection on A), it is possible to construct a number of Lie algebroid connections:

1. An A-connection on A: $\overline{\nabla}_{a_1} a_2 := \nabla_{\rho(a_1)} a_2$.
2. An A-connection on A, called the *adjoint connection*:

$$ {}^A\nabla_{a_1} a_2 := \nabla_{\rho(a_2)} a_1 + [a_1, a_2]_A. \tag{3.10} $$

3. An A-connection on TM: $\nabla_{a_1}\rho(a_2) := \rho({}^A\nabla_{a_1} a_2)$.

The adjoint connection plays an important role in the integrability of path groupoids, and is central in understanding Lie groupoid gauging (Chapter 5).

Given an A-connection on E, denoted by ∇, there is a natural definition of curvature $R_\nabla \in \Gamma(\wedge^2 A^* \otimes \mathsf{End}(E))$:

$$ R_\nabla(a_1, a_2)(e) = \nabla_{a_1}\nabla_{a_2} e - \nabla_{a_2}\nabla_{a_1} e - \nabla_{[a_1,a_2]_A} e, \tag{3.11} $$

for $a_1, a_2 \in \Gamma(A)$ and $e \in \Gamma(e)$.

Given an A-connection on A, denoted ∇, we can define the Lie algebroid torsion $T_\nabla \in \Gamma(\wedge^2 A^* \otimes A)$:

$$ T_\nabla(a_1, a_2) = \nabla_{a_1} a_2 - \nabla_{a_2} a_1 - [a_1, a_2]_A. \tag{3.12} $$

Given an A-connection on E, and Lie algebroid differential d_A, we can define a modified differential $d_A^\nabla : \Gamma(\wedge^k A^* \otimes E) \to \Gamma(\wedge^{k+1} A^* \otimes E)$ given by:

$$ (d_A^\nabla \omega)(a_0, a_1, \ldots, a_k) = \sum_{i=0}^{k} (-1)^i \nabla_{a_i}(\omega(a_0, a_1, \cdots, \hat{a}_i, \ldots, a_k)) \tag{3.13} $$

$$+\sum_{i<j}(-1)^{i+j}\omega([a_i,a_j]_A,a_0,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_k),$$

where $\omega \in \Gamma(\wedge^k A^* \otimes E)$. A straightforward calculation gives $(d_A^\nabla)^2\omega = R_\nabla \wedge \omega$ and hence d_A^∇ defines a differential if and only if ∇ is flat.

Definition 3.17. A *representation* of a Lie algebroid A over M on a vector bundle $E \to M$ is a choice of flat connection $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$.

Given a representation of a Lie algebroid (a choice of flat connection $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$) we can construct a representation of the associated groupoid. The correct groupoid is the Weinstein groupoid and is constructed using an A-path. Choose an A-path $a : [0,1] \to A$ covering $\gamma : [0,1] \to M$. The derivative of a path $e : [0,1] \to E$ (over γ) along a, denoted $\nabla_a e$, is defined as follows: choose a time dependent section $\xi^t \in \Gamma(E)$ such that $\xi^t(t,\gamma(t)) = e(t)$, then

$$\nabla_a e(t) = \nabla_a \xi^t(x) + \frac{d\xi^t}{dt}(x), \quad x = \gamma(t).$$

The derivative is independent of the choice of ξ^t used to parameterise e.

3.3.1 Lie algebroid morphisms

Let $A \to M_1$ and $B \to M_2$ be two vector bundles, and endow each with a Lie algebroid structure. Consider a bundle map

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d] & B \arrow[d] \\ M_1 \arrow[r, "\phi"] & M_2 \end{tikzcd}.$$

There is an induced pullback map $\Phi^* : \Gamma(B^*) \to \Gamma(A^*)$ defined by

$$\langle (\Phi^*\beta)_x, a_x \rangle = \langle (\beta \circ \phi)(x), \Phi(a) \rangle,$$

where $x \in M_1$, $\beta \in \Gamma(B^*)$, and $a \in \Gamma(A)$. This maps extends to $\Phi^* : \Gamma(\wedge^\bullet B^*) \to \Gamma(\wedge^\bullet A^*)$ (we define $\Phi^*(f) = f \circ \phi$ for $f \in C^\infty(M_2)$).

Definition 3.18. A bundle map (Φ, ϕ) from $A \to M_1$ to $B \to M_2$ is a *Lie algebroid morphism* if the map

$$\Phi^* : (\Gamma(\wedge^\bullet B^*), d_B) \to (\Gamma(\wedge^\bullet A^*), d_A) \tag{3.14}$$

is a chain map.

Let $A \to M$ and $B \to M$ be vector bundles over the same base M. A morphism $\Phi : A \to B$ induces a map of smooth sections $\Gamma(A) \to \Gamma(B)$, given by $a \to \Phi \circ a$, which is a linear map of $C^\infty(M)$-modules. In this case the notion of a Lie algebroid morphism is equivalent to the existence of a morphism Φ between the vector bundles preserving the bracket

$$\Phi \circ [a_1, a_2]_A = [\Phi \circ a_1, \Phi \circ a_2]_B,$$

for $a_1, a_2 \in \Gamma(A)$. However, for vector bundles $A \to M_1$ and $B \to M_2$ over different bases, a morphism $\Phi : A \to B$ and $\phi : M_1 \to M_2$, does not induce a map between modules of sections. It is necessary to consider the pullback bundle ϕ^*B. Sections $a \in \Gamma(A)$ can be pushed forward to sections $\Phi(a) \in \Gamma(\phi^*B)$, and sections $b \in \Gamma(B)$ can be pulled back to $\phi^*b \in \Gamma(\phi^*B)$. Given a section $a \in \Gamma(A)$, there is a decomposition

$$\Phi^*(a) = f^i \otimes b_i,$$

for some suitable $f^i \in C^\infty(M_1)$, and $b_i \in \Gamma(B)$. However this decomposition is not unique. Choose a connection $\nabla : \Gamma(TM_2) \times \Gamma(B) \to \Gamma(B)$ and define $\bar{\nabla} : \Gamma(TM_1) \times \Gamma(\phi^*B) \to \Gamma(\phi^*B)$ by

$$\bar{\nabla}_v(f_i \otimes b^i) = v(f_i) \otimes b^i + f_i \otimes \nabla_{T(\phi)(v)} b^i, \tag{3.15}$$

where $v \in \Gamma(TM_1)$, $f_i \in C^\infty(M_1)$, and $b^i \in \Gamma(B)$. The definition of $\bar{\nabla}$ is not dependent on the choice of decomposition. Define the torsion of the pullback connection $\bar{\nabla}$ by

$$T_{\bar{\nabla}}(f_i \otimes b^i, f'_j \otimes b'^j) := f_i f'_j \phi^* T_\nabla(b_i, b_j),$$

and curvature $F_\Phi \in \Gamma(\wedge^2 A^* \otimes \phi^*B)$ by

$$F_\Phi(a_1, a_2) = \bar{\nabla}_{\rho(a_1)}\Phi(a_2) - \bar{\nabla}_{\rho(a_2)}\Phi(a_1) - \Phi([a_1, a_2]_A) - T_{\bar{\nabla}}(\Phi(a_1), \Phi(a_2)). \tag{3.16}$$

The definition of F_Φ is independent of the choice of connection ∇. The pair (Φ, ϕ) defines a Lie algebroid morphism if and only if $F_\Phi \equiv 0$. This flatness condition is closely related to the original definition of a Lie algebroid morphism [63]. The definition given in this thesis was shown to be equivalent by Vaĭntrob [117].

There is a convenient description of certain Lie algebroid morphisms in terms of a Maurer–Cartan form. This interpretation is due to Fernandes and Struchiner [47].

Consider a Lie algebroid morphism (Φ, ϕ) where $\Phi : TM_1 \to B$, and $\phi : M_1 \to M_2$.

The key observation is to consider $\Phi : \Gamma(TM_1) \to \Gamma(B)$ as $\eta \in \Gamma(T^*M_1 \otimes B)$:

$$\eta(v) := \Phi^*(v),$$

for $v \in \Gamma(TM_1)$. The maps (Φ, ϕ) define a Lie algebroid morphism if and only if η can be interpreted as a Maurer–Cartan form.

The setup can be described with the following commutative diagrams:

$$\begin{tikzcd} TM_1 \arrow[r, "\eta"] \arrow[d] & B \arrow[d] \\ M_1 \arrow[r, "\phi"] & M_2 \end{tikzcd} \qquad \text{such that} \qquad \begin{tikzcd} TM_1 \arrow[r, "\eta"] \arrow[dr, "\phi_*"'] & B \arrow[d, "\rho_2"] \\ & TM_2 \end{tikzcd},$$

where the second diagram expresses the anchor compatibility condition. Given a choice of TM_1-connection on B, denoted ∇, define

$$(d_\nabla \eta)(v_1, v_2) := \nabla_{v_1}\eta(v_2) - \nabla_{v_2}\eta(v_1) - \eta([v_1, v_2]_{TM_1}), \tag{3.17}$$

$$[\eta \overset{\wedge}{,} \eta']_\nabla(v_1, v_2) = \frac{1}{2}\left(T_\nabla(\eta(v_1), \eta'(v_2)) + T_\nabla(\eta(v_2), \eta'(v_1))\right), \tag{3.18}$$

where $v_1, v_2 \in \Gamma(TM_1)$ and $\eta, \eta' \in \Gamma(T^*M_1 \otimes B)$. The expression

$$F_\eta(v_1, v_2) = (d_\nabla \eta - [\eta \overset{\wedge}{,} \eta]_\nabla)(v_1, v_2), \tag{3.19}$$

defines an element $F_\eta \in \Gamma(\wedge^2 T^*M_1 \otimes B)$ which is independent of the choice of ∇.

An element $\eta \in \Gamma(T^*M_1 \otimes B) \cong \Omega^1(M_1, B)$ satisfies $F_\eta \equiv 0$ if and only if it defines a Lie algebroid morphism between TM_1 and B. In this case η can be interpreted as a Maurer–Cartan form as follows: Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid B. Left translation by an element $h \in \mathcal{G}$ is a diffeomorphism between s-fibres $L_h : s^{-1}(s(h)) \to s^{-1}(t(h))$. A *left-invariant one-form* on $\mathcal{G}$ is an s-foliated one-form η on $\mathcal{G}$ such that for all $h \in \mathcal{G}$

$$\eta(X) = \eta(d_g L_h(X)), \quad \forall g \in s^{-1}(s(h)),\ X \in T^s_g\mathcal{G}.$$

This is also denoted $(L_h)^*\eta = \eta$. A *Maurer-Cartan form on a Lie groupoid* $\mathcal{G}$ is the B-valued s-foliated left-invariant one-form defined by

$$\eta(X) = (dL_{h^{-1}})_h(X),$$

for $X \in T^s_h\mathcal{G}$. The Maurer–Cartan form $\eta : T^s\mathcal{G} \to B$ covers the target map $t : \mathcal{G} \to M$.

Theorem 3.19 ([47]). *Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid B and let η be its left invariant Maurer–Cartan form. If $\eta' : TM_1 \to B$ is a solution of the Maurer–Cartan equation covering a map $\phi : M_1 \to M_2$, then for each $x \in M_1$ and $h \in \mathcal{G}$ such that $\phi(x) = s(h)$, there exists a unique locally defined diffeomorphism $\phi' : M_1 \to s^{-1}(\phi(x))$ satisfying:*

$$\phi'(x) = h, \quad \phi'^{*}\eta = \eta'.$$

Example 3.22. Choose a manifold M and a Lie algebra $\mathfrak{g}$. A Lie algebroid morphism $TM \to \mathfrak{g}$ is the same thing as a one-form $\eta \in \Omega^1(M, \mathfrak{g})$ satisfying the Maurer–Cartan equation $d\eta - [\eta \overset{\wedge}{,} \eta]_{\mathfrak{g}} = 0$.

Example: Pullback of a Lie algebroid

In this subsection we consider the pullback of a Lie algebroid structure. This will be of particular interest in Section 5.3 when discussing Lie algebroid gauging of non-linear sigma models.

Consider a Lie algebroid $B \to M_2$, and a smooth map $\phi : M_1 \to M_2$. There is no natural induced Lie algebroid on $\phi^* B$, due to the fact that a vector bundle morphism does not induce a map between the modules of sections. However, there may be an induced Lie algebroid structure. This construction is due Higgins and Mackenzie [63].

Consider the following bundle map:

$$\begin{tikzcd} & B \arrow[d, "\rho_2"] \\ TM_1 \arrow[r, "\phi_*"] & TM_2 \end{tikzcd}$$

The aim is to construct a Lie algebroid structure on $A \to M_1$, using the bundle maps

$$\begin{tikzcd} A \arrow[r, "\Phi^*"] \arrow[d, "\rho_1"'] & \phi^* B \arrow[d, "\phi^*\rho_2"] \\ TM_1 \arrow[r, "(\phi_*)^*"] & \phi^* TM_2 \end{tikzcd}$$

Sections of A are of the form $v \oplus \beta$, where $v \in \Gamma(TM_1)$ and $\beta \in \Gamma(\phi^* B)$. The induced Lie algebroid $\phi^{**}B$ exists whenever ϕ is a surjective submersion, or B is transitive, or if $T(\phi)$ and ρ_2 are transversal. In these cases there is a Lie algebroid structure on A

described as follows: define $\rho_1(v \oplus \beta) = v$ and

$$[a_1, a_2]_A = [v_1 \oplus \beta_1, v_2 \oplus \beta_2]_A := [v_1, v_2] \oplus \Big(\bar{\nabla}_{v_1}\beta_2 - \bar{\nabla}_{v_2}\beta_1 - T_{\bar{\nabla}}(\beta_1, \beta_2)\Big), \qquad (3.20)$$

for $v_1, v_2 \in \Gamma(TM_1)$ and $\beta_1, \beta_2 \in \Gamma(\phi^* B)$.

Example 3.23. Take M_2 to be a point, and hence $B = \mathfrak{g}$ is a Lie algebra. The inverse image connection in $M_1 \times \mathfrak{g}$ is the standard flat connection $\check{\nabla}_v(X) = v(X)$, and the above formula reduces to the standard expression of the bracket in $TM_1 \oplus (M_1 \times \mathfrak{g})$.

It is of interest to consider the existence of the groupoid $\phi^{**}\mathcal{G}$. Given a Lie groupoid $\mathcal{G}$ on M_2 and a smooth map $\phi : M_1 \to M_2$, such that $\phi \times \phi : M_1 \times M_1 \to M_2 \times M_2$ and $(s,t) : \mathcal{G} \to M_2 \times M_2$ are transversal, the pullback

$$\begin{tikzcd}
\phi^{**}\mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "{(s,t)}"] \\
M_1 \times M_1 \arrow[r, "\phi \times \phi"'] & M_2 \times M_2
\end{tikzcd}$$

is a manifold and has a groupoid structure. However, it is not necessarily true that the source or target maps are submersions, meaning $\phi^{**}\mathcal{G}$ may not be a Lie groupoid. If the composition

$$\phi^*\mathcal{G} \longrightarrow \mathcal{G} \xrightarrow{t} M_2,$$

is a submersion then $\phi^{**}\mathcal{G}$ will be a Lie groupoid. Here $\phi^*\mathcal{G}$ denotes the pullback of ϕ and s:

$$\phi^*\mathcal{G} := \{(X, x) \in \mathcal{G} \times M_1 : s(X) = \phi(x)\}.$$

The double pullback construction will be essential in defining true Lie algebroid gauging in Chapter 5.

3.3.2 Superspace description of Lie algebroids

There is a description of Lie algebroids given by a homological vector field Q on a super manifold $\mathcal{M}$. The description is due to Vaĭntrob [117]. The correspondence is reasonably straightforward. Take a supermanifold $\mathcal{M}$ described locally by coordinates (x^μ, ζ^i). There is an isomorphism between the supermanifold $\mathcal{M}$ and a vector bundle $A \to M$. The even coordinates x^μ give local coordinates on the base manifold M. The odd coordinates ζ^i give a local basis for the fibers A^*, which can then be identified with a basis for A. The degree-1 homological vector field is given by

$$Q = C^k{}_{ij}\zeta^i\zeta^j\partial_{\zeta^k} + \rho_i^\mu\zeta^i\partial_{x^\mu},$$

where ρ_i^μ are identified as the components of the anchor $\rho : A \to TM$, and $C^k{}_{ij}$ are identified with the structure functions given by $[e_i, e_j]_A = C^k{}_{ij}e_k$ (taking e_i to be a basis for A dual to ζ^i). Homological vector fields are those which satisfy

$$[Q, Q] = 2Q^2 = 0.$$

The condition $Q^2 = 0$ is equivalent to the axioms of a Lie algebroid:

$$\begin{aligned} Q^2 =& (2\rho^\nu_{[a|}\partial_\nu\rho^\mu_{|b]} - C^c{}_{ab}\rho^\mu_c)\zeta^a\zeta^b\partial_\mu \\ &+ (2C^d{}_{[a|k}C^k{}_{b]c} + 2\rho^\mu_{[a|}\partial_\mu C^d_{b]c} - C^k{}_{ab}C^d{}_{kc} + \rho^\mu_c\partial_\mu C^d_{ab})\zeta^a\zeta^b\zeta^c\partial_{\zeta_d}. \end{aligned}$$

It is clear that the operator d_A and homological vector fields Q are in a direct correspondence. Throughout this thesis we will concentrate on the description in terms of the operator d_A. Equivalent statements in terms of homological vector fields will be left unstated.

3.3.3 Bivector description of almost Lie algebroids

A Lie algebroid structure $(A \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_A)$ can be equivalently characterised as a bivector $\Lambda_{A^*} \in \Gamma(\wedge^2 TA^*)$. The construction presented here is from [54], and we refer the reader to this paper for details. The Lie algebroid anchor and bracket are recovered from the bivector as follows:

$$\Lambda_{A^*}(d_A\iota_a, (\pi^*)^*d_Af) := (\pi^*)^*(\rho(a)f), \quad \Lambda_{A^*}(d_A\iota_a, d_A\iota_{a'}) := \iota_{[a,a']_A}, \tag{3.21}$$

where $\iota_a, \iota_{a'}$ and $\iota_{[a,a']_A}$ are linear functionals on A^* induced by sections $a, a', [a, a']_A \in \Gamma(A)$, and $\langle a, \alpha\rangle = \iota_a\alpha$.

3.4 Leibniz algebroid geometry

The concept of a Lie algebroid naturally unifies Lie algebras and differential geometry. There are further useful generalisations that can be made. There are two core properties in the definition of a Lie algebroid. The first is the derivation property for $[a, \cdot]_A$, giving the Jacobi identity (3.4a). The second is the anchor homomorphism property (3.5). It is possible to relax one or other of these requirements and get interesting structures.

A *pre-Lie algebroid* structure is a quadruple $(A, M, \rho, [\cdot,\cdot]_A)$ satisfying (3.4b)-(3.5).

Pre-Lie algebroids are relevant in the study of local structures where integrability is not guaranteed. Integrability of pre-Lie structures are intimately connected to the Jacobi identity (3.4a).

It is possible to incorporate the derivation property and the anchor homomorphism property—whilst still generalising Lie algebroids—by considering brackets which are not skew-symmetric. This leads us to a Leibniz algebroid. Of particular interest are L_∞-algebroids, giving a natural generalisation of closure up to homotopy (see for example [111, 124, 16]). Leibniz algebroids (in particular L_∞-algebroids) form the basis of higher gauge theory (see for example [8, 106, 81]).

3.4.1 Lie bialgebroids and Courant algebroids

A concrete motivation for studying higher algebroids are Lie bialgebroids and Courant algebroids. A Lie bialgebroid is a notion of doubling for Lie algebroids, and was first introduced by Mackenzie and Xu [96]. A Lie bialgebroid consists of a Lie algebroid (A, d_A) and another Lie algebroid (A^*, d_{A^*}) satisfying a compatibility condition described below by (3.22).

In order to describe the compatibility condition we need to extend the Lie bracket $[\cdot,\cdot]_A$ from a map $\Gamma(A)\otimes\Gamma(A)\to\Gamma(A)$ to a map $\Gamma(\bigwedge^i A)\otimes\Gamma(\bigwedge^j A)\to\Gamma(\bigwedge^{i+j-1}A)$. The *generalised Schouten bracket* is the unique extension $[\cdot,\cdot]_A$ of the Lie bracket on $\Gamma(A)$ satisfying the following conditions:

1. $[a, f]_A = (\rho(a)f)$ for $a\in\Gamma(A)$, $f\in C^\infty(M)$;
2. For $a\in\Gamma(\bigwedge^i A)$ $[a,\cdot]_A$ is a derivation of degree $i-1$ on $\Gamma(\bigwedge^\bullet A)$;
3. $[a_1, a_2]_A = -(-1)^{(i-1)(j-1)}[a_2, a_1]_A$ for $a_1\in\Gamma(\bigwedge^i A)$ and $a_2\in\Gamma(\bigwedge^j A)$.

The Lie algebroid differential d_A has a natural action on the complex $\Gamma(\bigwedge^\bullet A^*)$ and d_{A^*} has a natural action on $\Gamma(\bigwedge^\bullet A)$.

A *Lie bialgebroid* is a pair of Lie algebroids (d_A, d_{A^*}) satisfying a compatibility relation. Compatibility requires that d_{A^*} is a derivation of the Schouten bracket $[\cdot,\cdot]_A$ on $\Gamma(\bigwedge^\bullet A)$:

$$d_{A^*}[a_1, a_2]_A = [d_{A^*}a_1, a_2]_A + [a_1, d_{A^*}a_2]_A, \tag{3.22}$$

for $a_1, a_2\in\Gamma(A)$.

Remark. We note that it is possible to generalise this by considering 'twisted' differentials $d_H := d_A + H\wedge$, and $d_R := d_{A^*} + R\wedge$, for $H\in\Gamma(\wedge^3 A^*)$, and $R\in\Gamma(\wedge^3 A)$ (for details see [85]).

Proposition 3.20 ([96]). *If the pair (d_A, d_{A^*}) is a Lie bialgebroid, then (d_{A^*}, d_A) is also a Lie bialgebroid ,i.e., If d_A is a derivation of $[\cdot,\cdot]_{A^*}$, then d_{A^*} is a derivation of $[\cdot,\cdot]_A$.*

Definition 3.21. A *Courant algebroid* is a quadruple $(E, \circ, \langle\cdot,\cdot\rangle, \rho)$ where $E \to M$ is a vector bundle, $\circ : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a Dorfman product, $\langle\cdot,\cdot\rangle$ a non-degenerate bilinear form, and $\rho : \Gamma(E) \to \Gamma(TM)$ is an anchor satisfying:

$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3); \tag{3.23a}$$

$$\rho(e_3)\langle e_1, e_2\rangle = \langle e_3 \circ e_1, e_2\rangle + \langle e_1, e_3 \circ e_2\rangle; \tag{3.23b}$$

$$e_1 \circ e_1 = \tfrac{1}{2}\mathbf{d}\langle e_1, e_1\rangle; \tag{3.23c}$$

for $e_i \in \Gamma(E)$ and $\mathbf{d}$ is defined by $\langle \mathbf{d}f, e_1\rangle = \rho(e_1)f$.

The notion of a Lie bialgebroid led to the axiomatic definition of Courant algebroids [94]. Given a Lie bialgebroid (d_A, d_{A^*}) there is a Courant algebroid on $E = A \oplus A^*$ defined as follows:

$$(a_1, \alpha_1) \circ (a_2, \alpha_2) = \Big([a_1, a_2]_A + \mathscr{L}^{A^*}_{\alpha_1} a_2 - \iota_{\alpha_2} d_{A^*} a_1, \tag{3.24a}$$
$$[\alpha_1, \alpha_2]_{A^*} + \mathscr{L}^{A}_{a_1}\alpha_2 - \iota_{a_2} d_A \alpha_1\Big),$$

$$\langle (a_1, \alpha_1), (a_2, \alpha_2)\rangle = \tfrac{1}{2}(\iota_{a_1}\alpha_2 + \iota_{a_2}\alpha_1), \tag{3.24b}$$

$$\rho_E(a, \alpha) = \rho_A(a) + \rho_{A^*}(\alpha), \tag{3.24c}$$

where $a_1, a_2 \in \Gamma(A)$, and $\alpha_1, \alpha_2 \in \Gamma(A^*)$.

The Lie bialgebroid (3.24) satisfies the Courant algebroid axioms if and only if (d_A, d_{A^*}) form a Lie bialgebroid. Conversely, a Courant algebroid defined on $E = L_1 \oplus L_2$, where L_1, L_2 are Dirac structures (maximal subbundles which are isotropic[8] and involutive with respect to $\circ$), defines a Lie bialgebroid with the Lie algebroid structures coming from the restriction of the fields to sections of L_1 and L_2.

Example 3.24 (Lie bialgebra). Take $(d_A, d_{A^*}) = (d_{\mathfrak{g}}, d_{\mathfrak{g}^*})$ where $d_{\mathfrak{g}}$ is the Chevalley–Eilenberg differential associated to a Lie algebra $\mathfrak{g}$. The Lie algebroid compatibility condition (3.22) is satisfied if and only if $\mathfrak{g}$ and $\mathfrak{g}^*$ form a Lie bialgebra. The reader is referred to [87] for an introduction to Lie bialgebras.

Example 3.25 (Standard Courant bracket). Take $(A, d_A, d_{A^*}) = (TM, d, 0)$, where d is the de Rham differential. This defines a Lie bialgebroid given by the exact Courant algebroid. The Courant algebroid is given explicitly by (4.2) (in this case $H = 0$).

Example 3.26 (Poisson bialgebroid). Take $(A, d_A, d_{A^*}) = (TM, d, d_\pi)$, where d is the de Rham differential, and d_π is the Poisson algebroid differential associated to the Poisson cotangent Lie algebroid (Example 3.12). The compatibility condition is equivalent to $[\pi, \pi]_{TM} = 0$ (for the Schouten extension of $[\cdot,\cdot]_{TM}$) which is satisfied by definition for any Poisson structure π.

[8] A subbundle L is isotropic if $\langle a_1, a_2\rangle = 0$ for all $a_1, a_2 \in \Gamma(L)$.

3.4.2 Leibniz algebroids

A *Leibniz algebroid* on a vector bundle $E \to M$ is a triple $(E, \circ, \rho)$, where the product $\circ : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and anchor $\rho : \Gamma(E) \to \Gamma(TM)$ satisfy

$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3), \tag{3.25a}$$

$$e_1 \circ f e_2 = f(e_1 \circ e_2) + (\rho(e_1) f) e_2. \tag{3.25b}$$

The two axioms state that $e_1 \circ \cdot$ acts as a derivation on $f \in C^\infty(M)$ and $e_i \in \Gamma(E)$.

The lack of skew-symmetry doesn't restrict the first entry to behave as a first order differential operator or even be local. We will be interested in local Leibniz algebroids which have nice geometric structure.

A *local Leibniz algebroid* $(E, \circ, \rho)$ is a Leibniz algebroid which satisfies

$$(f e_1) \circ e_2 = f(e_1 \circ e_2) - (\rho(e_2) f) e_1 + L(\mathbf{d} f, e_1, e_2), \tag{3.26}$$

where L is viewed as a $C^\infty(M)$-trilinear map $L : \Gamma(E^*) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and $\mathbf{d} : C^\infty(M) \to \Gamma(E^*)$ is defined by $\langle \mathbf{d} f, e \rangle = \rho(e) f$. The map L measures the failure of the product to be skew-symmetric. For all $f \in C^\infty(M)$, $e_1, e_2, e_3 \in \Gamma(E)$ we have

$$L(\mathbf{d} f, e_1, e_2) \circ e_3 = (\rho(e_3) f)(e_1 \circ e_2 + e_2 \circ e_1) - L(\mathbf{d} f, e_3, e_1 \circ e_2 + e_2 \circ e_1),$$

and in particular $L(\mathbf{d} f, e_1, e_2) \circ e_3 = L(\mathbf{d} f, e_2, e_1) \circ e_3$.

Example 3.27 (Lie algebroid). A Lie algebroid $(A, [\cdot, \cdot]_A, \rho)$ is a Leibniz algebroid with $L(\mathbf{d} f, e_1, e_2) \equiv 0$.

Example 3.28 (Courant algebroid). A Courant algebroid $(E, \circ, \langle \cdot, \cdot \rangle, \rho)$ is a Leibniz algebroid with $L(\mathbf{d} f, e_1, e_2) = \mathbf{d} f \langle e_1, e_2 \rangle$.

Example 3.29 (Higher Courant algebroid). Consider a vector bundle $E \to M$ with $E = TM \oplus \wedge^p T^* M$, for some $p > 1$. Denoting sections $(v, \xi) \in \Gamma(TM) \oplus \Gamma(\wedge^p T^* M)$ the Higher Courant algebroid[9] is given by:

$$(v_1, \xi_1) \circ (v_2, \xi_2) = ([v_1, v_2], \mathcal{L}_{v_1} \xi_2 - \iota_{v_2} d\xi_1), \quad \rho(v, \xi) = v.$$

In this case $L(\mathbf{d} f, e_1, e_2) = \mathbf{d} f \wedge (\iota_{v_1} \xi_2 + \iota_{v_2} \xi_1)$.

Example 3.30 (E_6 algebroid). Consider a vector bundle $E \to M$, with $E = TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M$. Denoting sections $(v, \sigma, \tau) \in \Gamma(TM) \oplus \Gamma(\wedge^2 T^* M) \oplus \Gamma(\wedge^5 T^* M)$, the

[9] The Higher Courant algebroid structure is *not* a Courant algebroid, as there is no compatible pairing $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$.

E_6 algebroid is given by:

$$\begin{aligned}(v_1,\sigma_1,\tau_1)\circ(v_2,\sigma_2,\tau_2) =&([v_1,v_2],\mathcal{L}_{v_1}\sigma_2-\iota_{v_2}d\sigma_1,\mathcal{L}_{v_1}\tau_2-\iota_{v_2}d\tau_1+d\sigma_1\wedge\sigma_2),\\ \rho(v,\sigma,\tau) =&v.\end{aligned}$$

In this case $L(\mathbf{d}f,e_1,e_2)=\mathbf{d}f\wedge(\iota_{v_1}\sigma_2+\iota_{v_2}\sigma_1)+\mathbf{d}f\wedge(\iota_{v_1}\tau_2+\iota_{v_2}\tau_1)$.

Remark. The E_6 algebroid is related to 11-dimensional supergravity [74, 9]. The name comes from the fact that the symmetries of this algebroid can be identified the Lie group E_6 when $\dim(M)=6$ (see [9] for details).

Example 3.31 (Closed form Leibniz algebroid). A local Leibniz algebroid structure can be constructed on a vector bundle $E\to M$ of the form

$$E=TM\bigoplus_{i=1}^{n}(V_i\otimes\wedge^{i-1}T^*M),$$

where V_i is a graded vector space of weight $-i$. The construction is due to Baraglia and can be found in [9]. The algebroid is constructed as a derived bracket using the de Rham differential.

For sections $v\in\Gamma(TM)$ and $\xi\in\Gamma(\bigoplus_{i=1}^{n}(V_i\otimes\wedge^{i-1}T^*M))$ the Leibniz algebroid is given by

$$(v_1,\xi_1)\circ(v_2,\xi_2) =([v_1,v_2]_{TM},\mathcal{L}_{v_1}\xi_2-\iota_{v_2}d\xi_1+[d\xi_1,\xi_2]), \tag{3.27a}$$

$$\rho(v,\xi) =v, \tag{3.27b}$$

where we refer the reader to [9] for the definition of $\mathcal{L}_v\xi, d\xi$ and $[d\xi_1,\xi_2]$.

Remark. The derived bracket construction can be used to show that the local closed form Leibniz algebroids are L_∞-algebroids. This is a result based on the work of Getzler [51], and further details can be found in [9].

The axioms of a Leibniz algebroid imply that the anchor homomorphism property holds:

$$\rho(e_1\circ e_2)=[\rho(e_1),\rho(e_2)]_{TM}. \tag{3.28}$$

To prove this identity we expand $(e_1\circ e_2)\circ fe_3$ in two ways: First we note that

$$(e_1\circ e_2)\circ fe_3 =e_1\circ(e_2\circ fe_3)-e_2\circ(e_1\circ fe_3).$$

Alternatively, we have

$$(e_1\circ e_2)\circ fe_3=f((e_1\circ e_2)\circ e_3)+\rho((e_1\circ e_2)f)e_3.$$

We compare these expansions to $e_1 \circ (e_2 \circ fe_3)$, which is given by

$$\begin{aligned} e_1 \circ (e_2 \circ fe_3) =& e_1 \circ (f(e_2 \circ e_3) + (\rho(e_2)f)e_3) \\ =& f(e_1 \circ (e_2 \circ e_3)) + \rho(e_1)(\rho(e_2)f)e_3 + (\rho(e_1)f)e_2 \circ e_3 + (\rho(e_2)f)e_1 \circ e_3, \end{aligned}$$

and the result follows from (3.25a). As a consequence we find that the discussion following Equation (3.5) for regular Lie algebroids (Section 3.2) holds for Leibniz algebroids as well. If the anchor of a Leibniz algebroid is regular then M is foliated by immersed submanifolds $\mathcal{O}$, which are defined by orbits $T_x\mathcal{O} = \mathrm{Im}(\rho_x)$, for all $x \in \mathcal{O}$.

We note that for Local Leibniz algebroids the anchor homomorphism property implies that

$$\rho(L(\mathbf{d}f, e_1, e_2)) = 0, \qquad \forall e_1, e_2 \in \Gamma(E),\ f \in C^\infty(M). \tag{3.29}$$

To see this we expand $\rho((fe_1) \circ e_2)$ and use the bracket homomorphism property:

$$\begin{aligned} \rho((fe_1) \circ e_2) =& f\rho((e_1 \circ e_2)) - (\rho(e_2)f)\rho(e_1) + \rho(L(\mathbf{d}f, e_1, e_2)), \\ =& f[\rho(e_1), \rho(e_2)]_{TM} - (\rho(e_2)f)\rho(e_1) + \rho(L(\mathbf{d}f, e_1, e_2)), \end{aligned}$$

and note that $[\rho(fe_1), \rho(e_2)]_{TM} = f[\rho(e_1), \rho(e_2)]_{TM} - (\rho(e_2)f)\rho(e_1)$. The result follows by equating the two expressions.

Given an TM-connection on a vector bundle $V \to M$, denoted by ∇, and a local Leibniz algebroid $(E, \circ, \rho)$, there is a natural definition of curvature $R_\nabla \in \Gamma(E^* \otimes E^* \otimes \mathsf{End}(V))$:

$$R_\nabla(e_1, e_2)(\nu) = \nabla_{\rho(e_1)}\nabla_{\rho(e_2)}\nu - \nabla_{\rho(e_2)}\nabla_{\rho(e_1)}\nu - \nabla_{\rho(e_1 \circ e_2)}\nu, \tag{3.30}$$

for $e_1, e_2 \in \Gamma(E)$ and $\nu \in \Gamma(V)$. This definition of the curvature differs from the form of the Lie algebroid curvature (3.11), due to the presence of the anchor ρ. This is required in order to ensure that R_∇ is a tensor: The condition

$$R_\nabla(fe_1, e_2) = R_\nabla(e_1, fe_2) = fR_\nabla(e_1, e_2),$$

holds because $\rho(L(\mathbf{d}f, e_1, e_2)) = 0$.

The curvature R_∇ is not skew-symmetric. It is possible to define $R'_\nabla \in \Gamma(\wedge^2 E^* \otimes \mathsf{End}(V))$ by

$$R'_\nabla(e_1, e_2)(\nu) = \nabla_{\rho(e_1)}\nabla_{\rho(e_2)}\nu - \nabla_{\rho(e_2)}\nabla_{\rho(e_1)}\nu - \nabla_{\rho([\![e_1, e_2]\!])}\nu, \tag{3.31}$$

where $[\![e_1, e_2]\!] = \frac{1}{2}(e_1 \circ e_2 - e_2 \circ e_1) = e_1 \circ e_2 - \frac{1}{2}\mathbf{d}\langle e_1, e_2\rangle$. In the case of closed form

Leibniz algebroids $\rho([\![e_1, e_2]\!]) = \rho(e_1 \circ e_2)$ and the two definitions agree.

There is another notion of *generalised curvature* that can be defined for a Leibniz algebroid. Take an E-connection on E, denoted ∇, and define $GR_\nabla : \Gamma(E) \times \Gamma(E) \to \mathsf{End}(E)$:

$$GR_\nabla(e_1, e_2) = \nabla_{e_1}\nabla_{e_2} - \nabla_{e_2}\nabla_{e_1} - \nabla_{e_1 \circ e_2}. \tag{3.32}$$

A straightforward calculation shows that

$$GR_\nabla(e_1, fe_2) = fGR_\nabla(e_1, e_2), \quad GR_\nabla(fe_1, e_2) = fGR_\nabla(e_1, e_2) + \nabla_{L(\mathbf{d}f, e_1, e_2)}.$$

It is clear that GR_∇ is not a tensor for all $e_1, e_2 \in \Gamma(E)$. The generalised curvature is a tensor when e_1 and e_2 are chosen from a subspace such that $L(\mathbf{d}f, e_1, e_2) = 0$. The generalised curvature was introduced by Gualtieri for the special case of the standard Courant algebroid on $\mathbb{T}M$ [60]. This provides the Courant analogue of curvature when $e_1 \in \Gamma(C_\pm)$ and $e_2 \in \Gamma(C_\mp)$ (where $C_\pm$ are defined by a choice of generalised metric (4.7)).

Given a Courant algebroid $(E, \rho, \circ, \langle\cdot,\cdot\rangle)$ and an E-connection on E, denoted by ∇, the *Gualtieri torsion* $GT_\nabla \in \Gamma(\wedge^3 E^*)$ is given by

$$GT_\nabla(e_1, e_2, e_3) = \langle \nabla_{e_1} e_2 - \nabla_{e_2} e_1 - [\![e_1, e_2]\!], e_3\rangle + \langle \nabla_{e_3} e_1, e_2\rangle - \langle \nabla_{e_3} e_2, e_1\rangle. \tag{3.33}$$

This modified notion of torsion is valid for all $e_1, e_2, e_3 \in \Gamma(E)$.

The construction of a generalised Lie derivative on a vector bundle $A \to M$ endowed with a Lie algebroid, discussed in Section 3.3, can be generalised to the case of a local Leibniz algebroid: Given a local Leibniz algebroid $(E, \circ, \rho)$, and a non-degenerate pairing $\langle\cdot,\cdot\rangle : E \times E^* \to \mathbb{R}$, a Lie derivative can be constructed. Define $\mathscr{L}_e f := \rho(e)f$ for $f \in C^\infty(M)$, and $\mathscr{L}_{e_1} e_2 := e_1 \circ e_2$, and

$$\langle \mathscr{L}_{e_1}\epsilon, e_2\rangle := \rho(e_1)\langle\epsilon, e_2\rangle - \langle e_1 \circ e_2, \epsilon\rangle, \tag{3.34}$$

where $\epsilon \in \Gamma(E^*)$. This definition can be extended in the natural way to general tensors $\mathsf{T} \in \Gamma((\otimes^p E) \otimes (\otimes^q E^*))$. This gives a notion flowing tensors along sections of a vector bundle endowed with a Leibniz algebroid.

Example 3.32 (Generalised Killing equation). Consider the vector bundle $E = TM \oplus T^*M$, with a Courant algebroid product

$$(v_1, \xi_1) \circ (v_2, \xi_2) = ([v_1, v_2], \mathcal{L}_{v_1}\xi_2 - \iota_{v_2} d\xi_1 - \iota_{v_1}\iota_{v_2} H), \quad \rho(v, \xi) = v,$$

where $(v,\xi) \in \Gamma(TM \oplus T^*M)$ and $H \in \Omega^3_{\mathrm{cl}}(M)$. Consider a generalised metric[10] $\mathbb{G} \in \mathsf{End}(TM \oplus T^*M)$, defined by

$$\mathbb{G}(\mathbb{X}_1, \mathbb{X}_2) = G(v_1, v_2) + G^{-1}(\xi_1 + B(v_1), \xi_2 - B(v_2)),$$

where $\mathbb{X} = (v,\xi) \in \Gamma(TM \oplus T^*M)$, G is a Riemannian metric, and $B \in \Omega^2(M)$. Identifying $\mathbb{G} \in \Gamma(E^* \otimes E^*)$, we can consider the *generalised Killing equation*:

$$(\mathscr{L}_{\mathbb{X}}\mathbb{G})(\mathbb{X}_1, \mathbb{X}_2) = \rho(\mathbb{X})(\mathbb{G}(\mathbb{X}_1, \mathbb{X}_2)) - \mathbb{G}(\mathscr{L}_{\mathbb{X}}\mathbb{X}_1, \mathbb{X}_2) - \mathbb{G}(\mathbb{X}_1, \mathscr{L}_{\mathbb{X}}\mathbb{X}_2) = 0.$$

Letting $\mathbb{X}_1 = (v_1, 0)$, and $\mathbb{X}_2 = (0, \xi_2)$ we can calculate the component $\mathscr{L}_{\mathbb{X}}\mathbb{G}((v_1,0),(0,\xi_2))$ and get the result:

$$\begin{aligned}
&\mathscr{L}_{\mathbb{X}}\mathbb{G}((v_1,0),(0,\xi_2)) \\
&= v(G^{-1}(B(v_1),\xi_2)) - G^{-1}(B(v_1), \mathcal{L}_v\xi_2) - G^{-1}(\iota_{[v,v_1]}B - \iota_{v_1}d\xi - \iota_v\iota_{v_1}H, \xi_2) \\
&= v(G^{-1}(B(v_1),\xi_2)) - G^{-1}(B(v_1), \mathcal{L}_v\xi_2) - G^{-1}(\mathcal{L}_v\iota_{v_1}B - \iota_{v_1}\mathcal{L}_vB - \iota_{v_1}d\xi - \iota_v\iota_{v_1}H, \xi_2) \\
&= v(G^{-1}(B(v_1),\xi_2)) - G^{-1}(\mathcal{L}_vB(v_1), \xi_2) - G^{-1}(B(v_1), \mathcal{L}_v\xi_2) \\
&\quad + G^{-1}(\iota_{v_1}(\mathcal{L}_vB + d\xi - \iota_vH), \xi_2) \\
&= (\mathcal{L}_vG^{-1})(B(v_1),\xi_2) + G^{-1}(\iota_{v_1}(\mathcal{L}_vB + d\xi - \iota_vH), \xi_2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\mathscr{L}_{\mathbb{X}}\mathbb{G}((0,\xi_1),(0,\xi_2)) =& (\mathcal{L}_vG^{-1})(\xi_1,\xi_2), \\
\mathscr{L}_{\mathbb{X}}\mathbb{G}((0,\xi_1),(v_2,0)) =& -(\mathcal{L}_vG^{-1})(\xi_1, B(v_2)) - G^{-1}(\iota_{v_2}(\mathcal{L}_vB + d\xi - \iota_vH), \xi_1), \\
\mathscr{L}_{\mathbb{X}}\mathbb{G}((v_1,0),(v_2,0)) =& (\mathcal{L}_vG)(v_1,v_2) - (\mathcal{L}_vG^{-1})(B(v_1), B(v_2)) \\
&+ G^{-1}(\iota_{v_1}(\mathcal{L}_vB + d\xi - \iota_vH), B(v_2)) \\
&+ G^{-1}(B(v_1), \iota_{v_2}(\mathcal{L}_vB + d\xi - \iota_vH)).
\end{aligned}$$

It follows that

$$\mathscr{L}_{\mathbb{X}}\mathbb{G} = 0 \quad \Leftrightarrow \quad \mathcal{L}_vG = 0, \quad \iota_vH = \mathcal{L}_vB + d\xi. \tag{3.35}$$

The generalised Killing equation for $\mathbb{G}$ reproduces the gauging constraints associated to non-linear sigma models describing string theory (studied in Chapter 5). In the generalised geometry picture the fields B and H are naturally incorporated in the geometric data describing the vector bundle $E = TM \oplus T^*M$ (for details see Section 4.1.1).

[10] Generalised metrics on $TM \oplus T^*M$ are defined and discussed in Section 4.1.2.

3.4.3 Leibniz algebroids: $\mathbb{d}_E$ and $(\mathsf{D}^\bullet, \pitchfork)$

In the previous section we have seen how a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ can be equivalently described by a differential operator d_A. There is a similar definition of a derivation for general Leibniz algebroids described in [58]. The Leibniz algebroid is encoded in $\mathbb{d}_E$ acting on a complex $(\mathsf{D}^\bullet, \pitchfork)$: Consider an operator $\mathbb{d}_E$ defined on $\omega \in \mathrm{Lin}^p(\Gamma(E), C^\infty(M))$ as follows

$$\begin{aligned}\mathbb{d}_E\omega(e_0, e_1, \dots, e_p) = &\sum_{i=0}^{k}(-1)^i \rho(e_i)\omega(e_0, \dots, \hat{e}_i, \dots, e_p) \\ &- \sum_{i<j}(-1)^i \omega(e_0, \dots, \hat{e}_i, \dots, \overbrace{e_i \circ e_j}^{(j)}, \dots, e_p),\end{aligned} \tag{3.36}$$

where $\hat{e}$ denotes omission, and $e \in \Gamma(E)$. This satisfies

$$\begin{aligned}&\mathbb{d}_E^2\omega(e_0, \dots, e_{p+1}) = \\ &\sum_{i<j}(-1)^{i+j+1}([\rho(e_i), \rho(e_j)] - \rho(e_i \circ e_j))\omega(e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{p+1}) + \\ &\sum_{i<j<k}(-1)^{i+j}\omega(e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_i \circ (e_j \circ e_k) - e_j \circ (e_i \circ e_k) - (e_i \circ e_j) \circ e_k, \dots, e_{p+1}).\end{aligned} \tag{3.37}$$

We see that $\mathbb{d}_E^2 \equiv 0$ if the bracket $(E, \circ)$ satisfies the axioms of a Leibniz algebroid.

Taking $\xi \in \Gamma(E^*)$ we have $\mathbb{d}_E\xi(e_0, e_1) + \mathbb{d}_E\xi(e_1, e_0) = -\xi(e_0 \circ e_1 + e_1 \circ e_0) \neq 0$ in general. For a local Leibniz algebroid[11] $\mathbb{d}_E\xi(e_0, fe_1) = f\mathbb{d}_E\xi(e_0, e_1)$ and $\mathbb{d}_E\xi(fe_0, e_1) = f\mathbb{d}_E\xi(e_0, e_1) - \xi(L(\mathbf{d}f, e_0, e_1))$. It is clear that $\mathbb{d}_E$ is not a chain map for $\Gamma(\bigwedge^\bullet E^*)$ or even $\Gamma(\otimes^\bullet E^*)$. We should consider $\mathrm{Lin}^\bullet(\Gamma(E), C^\infty(M))$. In particular we consider

$$\mathcal{D}^k(E) := \mathcal{D}^k(\Gamma(E), C^\infty(M)) \subset \mathrm{Lin}^k(\Gamma(E), C^\infty(M)),$$

consisting of all multidifferential operators of total order at most k.

As a special case we can consider $\mathsf{D}^k(E) \subset \mathcal{D}^k(E)$ the subset of multidifferential operators $\mathcal{D}^k(E)$ which are order 0 with respect to the last variable, with total degree $\leq k-1$, and set $\mathsf{D}^\bullet(E) = \bigoplus_{k=0}^\infty \mathsf{D}^k(E)$. We define $\mathsf{D}^0(E) = C^\infty(M)$.

A simple calculation shows that $\mathbb{d}_E : \mathsf{D}^k(E) \to \mathsf{D}^{k+1}(E)$, and we see that $\mathsf{D}^\bullet(E) := \bigoplus_{i=0}^\infty \mathsf{D}^i(E)$ is the appropriate complex when dealing with the Leibniz differential $\mathbb{d}_E$.

Definition 3.22. For any $\omega \in \mathcal{D}^p(E)$ and $\omega' \in \mathcal{D}^q(E)$, $p, q \in \mathbb{N}$, we denote the *shuffle*

[11] If $(E, \circ, \rho)$ is not a Local Leibniz algebroid then we do not necessarily have a locally defined complex.

product

$$\omega \pitchfork \omega'(e_1,\dots,e_{p+q}) := \sum_{\sigma\in \mathrm{sh}(p,q)} \mathrm{sign}(\sigma)\omega(e_{\sigma(1)},\dots,e_{\sigma(p)})\omega'(e_{\sigma(p+1)},\dots,e_{\sigma(q)}),$$

where $\mathrm{sh}(p,q)$ is the subset of the symmetric group made up of all (p,q)-shuffles.

The complex $\mathsf{D}^\bullet(E)$, with the shuffle product $\pitchfork$, is a graded algebra, the *reduced shuffle algebra*.

If ω, ω' are skew-symmetric this coincides with the wedge product. The shuffle product is graded symmetric $\omega \pitchfork \omega' = (-1)^{|\omega||\omega'|}\omega' \pitchfork \omega$ for $\omega, \omega' \in \mathsf{D}^\bullet(E)$.

The differential $\mathbb{d}_E$ defines a derivation of degree +1 on the reduced shuffle algebra. We can also define a degree -1 derivation given by the interior product,

$$\iota_e\omega(e_1,\dots,e_k) = \omega(e,e_1,\dots,e_k)$$

where $e_i \in \Gamma(E)$, $\omega \in \mathsf{D}^{k+1}$ and $\iota_e\omega \in \mathsf{D}^k$. The derivation property means we have the following identities

$$\mathbb{d}_E(\omega \pitchfork \omega') = \mathbb{d}_E\omega \pitchfork \omega' + (-1)^k\omega \pitchfork \mathbb{d}_E\omega', \qquad \iota_e(\omega \pitchfork \omega') = \iota_e\omega \pitchfork \omega' + (-1)^k\omega \pitchfork \iota_e\omega'.$$

Using our +1 and -1 derivations we can define Cartan relations on $\mathsf{D}^\bullet(E)$ using graded commutators:

$$[\mathbb{d},\mathbb{d}] = 2\mathbb{d}^2 = 0; \tag{3.38a}$$

$$[\mathbb{d},\iota_e] = \mathbb{d}\iota_e + \iota_e\mathbb{d} = \mathfrak{L}_e; \tag{3.38b}$$

$$[\mathbb{d},\mathfrak{L}_e] = \mathbb{d}\mathfrak{L}_e - \mathfrak{L}_e\mathbb{d} = 0; \tag{3.38c}$$

$$[\mathfrak{L}_{e_1},\mathfrak{L}_{e_2}] = \mathfrak{L}_{e_1}\mathfrak{L}_{e_2} - \mathfrak{L}_{e_2}\mathfrak{L}_{e_1} = \mathfrak{L}_{e_1\circ e_2}; \tag{3.38d}$$

$$[\mathfrak{L}_{e_1},\iota_{e_2}] = \mathfrak{L}_{e_1}\iota_{e_2} - \iota_{e_2}\mathfrak{L}_{e_1} = \iota_{e_1\circ e_2}; \tag{3.38e}$$

$$[\iota_{e_1},\iota_{e_2}] = \iota_{e_1}\iota_{e_2} + \iota_{e_2}\iota_{e_1} \neq 0. \tag{3.38f}$$

We conclude with an alternative characterisation of Leibniz algebroids:

Theorem 3.23 ([57]). *Let E be a vector bundle. There exists a one-to-one correspondence between equivalence classes of differentials*

$$\mathbb{d}_E \in \mathsf{Der}_1(\mathsf{D}^\bullet(E),\pitchfork),\ \mathbb{d}_E^2 = 0$$

and Leibniz algebroid structures on E.

There is a superpace inspired description of Courant algebroids using a homological vector field Q which plays the same role as the operator $\mathbb{d}_E$. A nice description can be found in Roytenberg's thesis [104]. More generally, L_∞-algebroids (a subset of local Leibniz algebroids) can be associated to a homological vector field Q, defined on certain $\mathbb{N}$-graded vector bundles referred to as NQ manifolds (see for example [4, 111, 105]).

Chapter 4

Generalised contact geometry

This chapter describes generalised contact geometry, an extension of contact geometry, associated to a Courant algebroid structure on the vector bundle

$$E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M.$$

Generalised geometry on the generalised tangent bundle $E \cong TM \oplus T^*M$ is reviewed in Section 4.1. Generalised geometry, introduced by Hitchin [64] and developed by Gualtieri [59], has proven to be a very successful extension of differential geometry on the generalised tangent bundle $TM \oplus T^*M$. Generalised complex structures (defined on even dimensional manifolds) unify and interpolate between symplectic and complex structures. Much of the interest in generalised geometry—particularly in relation to T-duality in string theory—is due to the enlarged symmetry group of the structures on $TM \oplus T^*M$.

Section 4.2 introduces generalised contact geometry, the odd-dimensional counterpart to generalised complex geometry. Generalised contact geometry is defined on the vector bundle $E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$. While generalised contact geometry has been studied before it is not as well developed as generalised complex geometry (see [118, 119, 102]). The description of the symmetries associated to generalised contact structures was incomplete. This chapter presents generalised contact structures as S^1-invariant reductions of generalised complex structures. A geometric interpretation of twisted generalised contact structures is given through bundle gerbes classifying the splitting of the vector bundle $E \to M$.

This chapter modifies the definition of generalised contact structures given in [78] (as well as the mixed pair description [57]) to include the full set of symmetries. Generalised contact structures are described as S^1-invariant reductions of generalised complex structures in Section 4.3. The extended κ-symmetries noted by Sekiya [108] correspond to reductions of non-trivial S^1-bundles. In Section 4.4 twisted generalised coKähler structures are described as reductions of generalised Kähler structures. This gives a generalised analogue of the correspondence between coKähler structures on M and Kähler structures on a principal circle bundle $\mathsf{S}^1 \hookrightarrow P \to M$. The role of the extended symmetries are discussed in the context of T-duality in Section 4.5. The main result being that generalised coKähler structures are mapped to other generalised coKähler structures under T-duality. Finally, in Section 4.6, the relationship between the twisted contact structures in this chapter and geometry on the generalised derivation bundle $\mathbb{D}L \cong \mathfrak{D}L \oplus \mathfrak{J}^1 L$ (introduced in [123]) is given.

The twisted coKähler structures may be of interest to physics when conisdering the Kaluza–Klein reduction of generalised complex structures with respect to non-trivial circle bundles.

4.1 Generalised tangent spaces and Courant algebroids

Generalised geometry is the study of geometric structures on a vector bundle equipped with an algebroid structure. Courant algebroids underly the generalised geometries associated with both generalised complex structures and generalised contact structures.

Standard generalised geometry is the study of geometric structures on the generalised tangent bundle $E \to M$ given by the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & T^*M \arrow[r, "\rho^*"] & E \arrow[r, "\rho"] \arrow[l, bend left, "s^*"] & TM \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd} . \tag{4.1}$$

Courant algebroids on E, specified by (4.1), are called *exact Courant algebroids*. Every exact Courant algebroid admits a splitting $s : TM \to E$, which satisfies $\rho s = \mathrm{Id}$, and is isotropic ($\langle s(v_1), s(v_2)\rangle = 0$ for all $v_1, v_2 \in \Gamma(TM)$). Two exact Courant algebroids are equivalent if they differ by a choice of isotropic splitting.

A choice of splitting defines an isomorphism $E \cong s(TM) \oplus \rho^*(T^*M) := \mathbb{T}M$. Using the identification $\Gamma(e) = s(v) + \rho^*(\xi) := (v, \xi)$, for $v \in \Gamma(TM)$ and $\xi \in \Gamma(T^*M)$ the standard Courant algebroid is given by

$$(v_1, \xi_1) \circ_H (v_2, \xi_2) = \Big([v_1, v_2], \mathcal{L}_{v_1}\xi_2 - \iota_{v_2}\xi_1 - \iota_{v_1}\iota_{v_2}H\Big); \tag{4.2a}$$

$$\langle (v_1, \xi_1), (v_2, \xi_2)\rangle = \tfrac{1}{2}(\iota_{v_1}\xi_2 + \iota_{v_2}\xi_1); \tag{4.2b}$$

$$\rho(v, \xi) = v; \tag{4.2c}$$

where $H \in \Omega^3_{\mathrm{cl}}(M)$, is given by

$$H(v_1, v_2) = s^*(s(v_1) \circ s(v_2)).$$

For further details see [20]. The Leibniz identity for $\circ_H$ (4.2a) gives the Bianchi identity $dH = 0$. The Dorfman product, $\circ_H$, is natural in the sense that it is the derived bracket of $d_H := d + H\wedge$ (with d the de Rham differential) acting on $\Gamma(\wedge^\bullet T^*M)$ [86].

Consider splitting (4.1) with two different isotropic splittings $s_i : TM \to E$, $i = 1, 2$, satisfying $\rho(s_1 - s_2) = 0$. Exactness implies that there exists a unique $B \in \Omega^2(M)$ satisfying $s_1(v) - s_2(v) = \rho^*(B(v))$ for all $v \in \Gamma(TM)$. It can be shown that

$$H_1 - H_2 = dB.$$

Exact Courant algebroids are classified by $[H] \in H^3(M, \mathbb{R})$; a point first noted by Ševera [109].

The equivalence of the exact Courant algebroid under isotropic splittings (corresponding to some $B \in \Omega^2(M)$) is closely related to the concepts of symmetries in generalised geometry structures.

4.1.1 Courant algebroid symmetries

Perhaps the most interesting aspect of generalised geometry is the enhanced symmetry group. The symmetry group of the Tangent Lie algebroid (given by the commutator of vector fields on TM) is $\mathsf{Diff}(M)$. Exact Courant algebroids have a symmetry group given by $\mathsf{Diff}(M) \ltimes \Omega^2_{\mathrm{cl}}(M)$ if $H = 0$.

Definition 4.1. A *Courant algebroid symmetry* is a bundle automorphism $S : E \to E$ such that

$$\langle S(e_1), S(e_2)\rangle = \langle e_1, e_2\rangle, \quad S(e_1) \circ S(e_2) = S(e_1 \circ e_2). \tag{4.3}$$

Given a diffeomorphism $f : M \to M$, the induced action on a section $(v, \xi) \in \Gamma(\mathbb{T}M)$ is given by

$$v_p + \xi_p \to (T_p f)(v_p) + (T_{f(p)} f^{-1})^*(\xi_p).$$

There is an infinitesimal action of $B \in \Omega^2(M)$ on $(v, \xi) \in \Gamma(\mathbb{T}M)$, given by $B(v, \xi) = (0, \iota_v B)$. The corresponding finite action is called a *B-transformation*:

$$e^B(v, \xi) = (v, \xi + \iota_v B).$$

A B-transformation satisfies

$$\begin{aligned} \langle e^B(v_1, \xi_1), e^B(v_2, \xi_2)\rangle =& \langle (v_1, \xi_1), (v_2, \xi_2)\rangle; \\ e^B(v_1, \xi_1) \circ_H e^B(v_2, \xi_2) =& e^B((v_1, \xi_1) \circ_{H+dB} (v_2, \xi_2)). \end{aligned}$$

When $H = 0$ the vector bundle $\mathbb{T}M$ is trivial and a B-transformation is a Courant algebroid symmetry if and only if $B \in \Omega^2_{\rm cl}(M)$. The Lie group composition of closed two forms is

$$e^{B'}e^{B''} = e^{B'+B''}, \quad B' \cdot B'' = B' + B''.$$

The H-twisted exact Courant algebroid and the B-transformations have a close connection to $\mathsf{U}(1)$-gerbes. If one requires that H has integral periods, $[H/2\pi] \in H^3(M,\mathbb{Z})$, there is a gerbe describing the patching of T^*M to TM constructing $\mathbb{T}M$ [64].

Consider a good cover of M, denoted $\mathcal{U} = \cup_\alpha U_\alpha$, with $U_{\alpha\beta\ldots\gamma} := U_\alpha \cap U_\beta \cdots \cap U_\gamma$. A gerbe is described by the cocycle $g_{\alpha\beta\gamma} = \exp(i\Lambda_{\alpha\beta\gamma}) \in \mathsf{U}(1)$, a connection given by $A_{\alpha\beta} \in \Omega^1(U_{\alpha\beta})$, and $B_\alpha \in \Omega^2(U_\alpha)$ satisfying

$$\begin{aligned} B_\beta - B_\alpha =& dA_{\alpha\beta} \quad \text{on } U_{\alpha\beta}, \\ A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} =& d\Lambda_{\alpha\beta\gamma} \quad \text{on } U_{\alpha\beta\gamma}. \end{aligned}$$

$H = dB_\alpha = dB_\beta$ on $U_{\alpha\beta}$ is independent of the cover and is a globally defined 3-form.

Given a representative of a class $[H] \in H^3(M,\mathbb{R})$ it is possible to reconstruct the bundle E. Choose an open cover $\mathcal{U}$ and a representative $H \in H^3_{\rm dR}(M,\mathbb{R})$, which consists of a 4-tuple $(\Lambda_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha, H)$ in the Čech–de Rham complex (over $\mathbb{R}$). The bundle E is constructed by the clutching construction

$$E = \bigsqcup_\alpha (TU_\alpha \oplus T^*U_\alpha)/\sim,$$

identifying $(v_\alpha, \xi_\alpha) \in \Gamma(TU_\alpha) \oplus \Gamma(T^*U_\alpha)$ with $(v_\beta, \xi_\beta) \in \Gamma(TU_\alpha) \oplus \Gamma(T^*U_\alpha)$ on overlaps if and only if $v_\beta = v_\alpha$ and $\xi_\beta = \xi_\alpha + \iota_{v_\alpha} dA_{\alpha\beta}$. Consistency on triple overlaps $U_{\alpha\beta\gamma}$ follows from $dA_{\beta\gamma} - dA_{\alpha\gamma} + dA_{\alpha\beta} = (d\delta A)_{\alpha\beta\gamma} = (d^2\Lambda)_{\alpha\beta\gamma} = 0$, where δ is the Čech differential. On $TM|_{U_\alpha}$ the splitting is given by

$$s|_{U_\alpha} : v_\alpha \to v_\alpha + \iota_{v_\alpha} B_\alpha,$$

and consistency on overlaps follows from $B_\beta - B_\alpha = dA_{\alpha\beta}$.

A closed B-field such that $B/2\pi$ has integral periods should be considered a gauge transformation. This means that the notion of equivalence of generalised structures should not be just the diffeomorphisms connected to the identity, but extended by $\Omega^2_{\rm cl}(M)$.

The importance of the H-twist appearing in the exact Courant algebroid (4.2a) is in identifying it as $H(v_1, v_2) = s^*(s(v_1), s(v_2))$ for some non-trivial bundle $E \to M$.

Remark. In string theory applications the requirement that $[H/2\pi] \in H^3(M,\mathbb{Z})$ arises naturally as the requirement ensuring a single valued path-integral. The three-form H

giving a representative $[H/2\pi] \in H^3(M,\mathbb{Z})$ is interpreted as the Neveu–Schwarz flux, arising from a local two-form potential B satisfying $H|_{U_\alpha} = dB_\alpha$. B-transformations in $\Omega^2_{\mathrm{cl}}(M)$ are viewed as gauge transformations and should be quotiented out when considering physically distinct states. Generalised geometry can be seen as a way of encoding flux geometrically.

4.1.2 Generalised geometric structures

Almost all differential geometry structures have a counterpart on the vector bundle $\mathbb{T}M$ in generalised geometry. Of most interest are generalised complex structures, generalised metric structures, and generalised Kähler structures. The standard introductory reference being Gualtieri's Ph.D. thesis [59].

The generalised tangent bundle admits a Clifford action of sections $(v,\xi) \in \Gamma(\mathbb{T}M)$ on differential forms $\varphi \in \Omega^\bullet(M)$ given by

$$(v,\xi)\cdot\varphi = \iota_v\varphi + \xi\wedge\varphi. \tag{4.4}$$

The Clifford action satisfies $(v,\xi)^2\cdot\varphi = \langle (v,\xi),(v,\xi)\rangle\varphi := ||(v,\xi)||^2\varphi$. Forms $\varphi \in \Gamma((\wedge^m T^*)^{\frac{1}{2}} \otimes \Omega^{\mathrm{od/ev}}(M))$, where $m = \dim(M)$, describe spinors [59]. Associated to each spinor is the annihilator bundle

$$L_\varphi := \mathrm{Ann}(\varphi) = \{(v,\xi) \in \Gamma(\mathbb{T}M) : (v,\xi)\cdot\varphi = 0\}.$$

A *complex pure spinor* is a $\varphi \in \Gamma(\Omega^{\mathrm{od/ev}}(M)\otimes\mathbb{C})$ that is non-degenerate with respect to the Mukai pairing, that is $(\varphi,\varphi)_M \neq 0$ where

$$(\varphi_1,\varphi_2)_M = (\alpha(\varphi_1)\wedge\varphi_2)_m, \tag{4.5}$$

α is the Clifford anti-automorphism $\alpha(dx^1\otimes dx^2\otimes\cdots\otimes dx^k) = dx^k\otimes dx^{k-1}\otimes\cdots\otimes dx^1$, $m = \dim(M)$, and $(\cdot)_m$ denotes the projection onto $\Omega^m(M)$. The non-degeneracy condition $(\varphi,\varphi)_M \neq 0$ implies that L_φ is a maximal isotropic subbundle. Given a pure spinor φ and a function $f \in C^\infty(M)$ we have

$$(v,\xi)\cdot f\varphi = f(v,\xi)\cdot\varphi, \quad (f\varphi, f\varphi)_M = f^2(\varphi,\varphi)_M.$$

If f is nowhere zero, then $f\varphi$ describes the same maximal isotropic subspace as φ. The class $\varphi \sim f\varphi$ is called a *conformal class*. There is a local one-to-one correspondence between maximally isotropic subspaces of $\mathbb{T}M$ and conformal classes of pure spinors.

Definition 4.2. A *generalised almost complex structure* on $\mathbb{T}M$ is given by $\mathbb{J} \in \mathsf{End}(\mathbb{T}M)$ satisfying $\mathbb{J}^* = -\mathbb{J}$ and $\mathbb{J}^2 = -\mathrm{Id}$.

A generalised almost complex structure can be equivalently described by a maximal isotropic complex subbundle $L_{\mathbb{J}} \subset \mathbb{T}M \otimes \mathbb{C}$, satisfying $L_{\mathbb{J}} \cap \bar{L}_{\mathbb{J}} = \{0\}$, for

$$L_{\mathbb{J}} = \{e \in \Gamma(\mathbb{T}M \otimes \mathbb{C}) : \mathbb{J}(e) = ie\}.$$

There is a local one-to-one correspondence between generalised almost complex structures and conformal classes of complex pure spinors (where a complex pure spinor satisfies the non-degeneracy condition $(\bar{\varphi}, \varphi)_M \neq 0$).

A generalised almost complex structure $\mathbb{J}$, is *H-involutive* if all sections e of the $+i$-eigenbundle $L_{\mathbb{J}}$ are involutive with respect to $\circ_H$: $e_1, e_2 \in L_{\mathbb{J}} \Rightarrow e_1 \circ_H e_2 \in L_{\mathbb{J}}$.

Definition 4.3. A *generalised complex structure* is an H-involutive generalised almost complex structure.

The H-involutive property of a maximal isotropic subbundle L_φ can be encoded as a constraint on the pure spinor φ: A pure spinor φ is H-involutive if and only if there exists some $e \in \Gamma(\mathbb{T}M \otimes \mathbb{C})$ such that

$$d\varphi + H \wedge \varphi = e \cdot \varphi. \tag{4.6}$$

A proof of this fact can be found in [59]. Given an H-involutive pure spinor φ and a nowhere zero complex function $f \in C^\infty(M)$ we have

$$df\varphi + H \wedge f\varphi = f(d\varphi + H \wedge \varphi) + df \wedge \varphi = fe \cdot \varphi + df \wedge \varphi := e' \cdot f\varphi,$$

where $e' = e + f^{-1}df$.

There is a local one-to-one correspondence between Generalised complex structures and conformal classes of H-involutive pure spinors.

Example 4.1. Given an (almost) symplectic structure $\omega \in \Omega^2(M)$, we can define a generalised (almost) complex structure with the spinor $\varphi_\omega = e^{i\omega}$. The symplectic structure is non-degenerate so $(\varphi_\omega, \bar{\varphi}_\omega)_M = \omega^{m/2} \neq 0$. The $+i$-eigenbundle is given by

$$L_\omega = \{(X, \xi) \in \Gamma(\mathbb{T}M \otimes \mathbb{C}) : \xi = i\omega(X, \cdot)\}.$$

Example 4.2. Given an (almost) complex structure $J \in \mathsf{End}(TM)$ (satisfying, $J^* = -J$ and $J^2 = -\mathrm{Id}$) we can define a generalised (almost) complex structure with the spinor $\varphi_J = \Omega$, where $\Omega \in \Omega^{(m,0)}(M)$ is a locally defined generator of the $(m, 0)$-forms for the complex structure J. The non-degeneracy condition is $(\varphi_J, \bar{\varphi}_J)_M = \Omega \wedge \bar{\Omega} \neq 0$. The $+i$-eigenbundle is given by

$$L_\Omega = T^{(0,1)}M \oplus T^{*(1,0)}M$$

The tangent bundle has a structure group $\mathsf{GL}(m)$. A reduction of the structure group $\mathsf{GL}(m)$ to its maximal compact subgroup $\mathsf{O}(m)$ defines a choice of Riemannian metric. The generalised tangent bundle $\mathbb{T}M$ equipped with metric $\langle\cdot,\cdot\rangle$ has structure group $\mathsf{O}(m,m)$.

Definition 4.4. A *generalised metric* $\mathbb{G}$ is a positive definite metric on $\mathbb{T}M$, corresponding to a choice of reduction of the structure group from $\mathsf{O}(m,m)$ to $\mathsf{O}(m)\times\mathsf{O}(m)$.

The inner product $\langle\cdot,\cdot\rangle$ determines a splitting $\mathbb{T}M = C_+ \oplus C_-$, where C_+ is positive definite with respect to $\langle\cdot,\cdot\rangle$ and C_- is negative definite. The generalised metric structure is defined by

$$\mathbb{G}(e_1,e_2) := \langle e_1,e_2\rangle|_{C_+} - \langle e_1,e_2\rangle|_{C_-}. \tag{4.7}$$

Using the metric $\langle\cdot,\cdot\rangle$ to identify $\mathbb{T}M$ with $\mathbb{T}^*M$, a generalised metric can be identified with $\mathbb{G}\in\mathsf{End}(\mathbb{T}M)$ satisfying $\mathbb{G}^*=\mathbb{G}$, and $\mathbb{G}^2=\mathrm{Id}$. It follows from (4.7) that $C_\pm$ correspond to the ±1-eigenbundles of $\mathbb{G}$.

Given a Riemannian metric G a generalised metric $\mathbb{G}$ can be defined by the identification

$$C_\pm = \{(v,\xi)\in\mathbb{T}M : \xi = \pm G(v,\cdot)\}.$$

Definition 4.5. A *generalised almost Kähler structure* is a pair of almost generalised structures satisfying $\mathbb{J}_1\mathbb{J}_2=\mathbb{J}_2\mathbb{J}_1$ and $-\mathbb{J}_1\mathbb{J}_2=\mathbb{G}$ for some generalised metric $\mathbb{G}$.

The role of extended symmetry is important as it provides deformations of generalised metric and generalised complex structures. This provides a way to generate examples and gives a notion of equivalence which goes beyond diffeomorphisms.

Example 4.3 (Twisted generalised metric). A generalised metric, defined by a Riemannian metric G can be twisted by a 2-form B to give another generalised metric

$$C_\pm^B = \{(v,\xi)\in\mathbb{T}M : \xi = \pm G(v,\cdot)+B(v,\cdot)\}.$$

Example 4.4 (Twisted generalised complex structure). Take a pure spinor φ defining an (almost) complex structure. We can define $\varphi^B = e^{-B}\wedge\varphi$, for some $B\in\Omega^2(M)$, satisfying

$$(\varphi^B,\bar\varphi^B)_M = (\varphi,\bar\varphi)_M, \quad d_{H'}(e^{-B}\wedge\varphi) - e'\cdot(e^{-B}\wedge\varphi) = e^{-B}(d_H\varphi - e\cdot\varphi),$$

where $H' = H + dB$ and $e' = e^B\cdot e = e^B(v,\xi) = (v,\xi+\iota_v B)$.

A generalised H-involutive complex structure φ can be deformed to a $(H+dB)$-involutive complex structure $e^{-B}\wedge\varphi$.

A generalised (almost) complex structure is said to be of *geometric type-k*, if $t_L(x) := \operatorname{codim}_{\mathbb{C}}(\rho(L_x)) = k$. Generically the type can change at each point $x \in M$.

Example 4.5. Locally every generalised (almost) complex structure of type-k can be associated (non-canonically) to a pure spinor $\varphi_J = \Omega \wedge e^{B+i\omega}$, where $\omega \in \Omega^2(M)$, Ω is a complex decomposable form of degree k, and $\omega^{m/2-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$.

In Section 4.4 generalised coKähler structures will be defined in a way that mirrors the definition in terms of Kähler structures. This Section concludes with the definition of generalised Calabi–Yau structures and hyperKähler structures.

Definition 4.6. A *generalised almost Calabi–Yau structure* consists of two pure spinors (φ_1, φ_2) which describing two generalised almost complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ which define a generalised almost Kähler structure. In addition, the lengths of these sections are related by a constant

$$(\varphi_1, \bar{\varphi}_1)_M = c(\varphi_2, \bar{\varphi}_2)_M,$$

where $c \in \mathbb{R}$ can be scaled to either $+1$ or -1 by rescaling φ_1.

A generalised Calabi–Yau structure is a generalised almost Calabi–Yau structure where (φ_1, φ_2) are both H-involutive.

Example 4.6 (Calabi–Yau). A Calabi–Yau manifold is a Kähler manifold of complex dimension m with symplectic form ω and holomorphic volume form Ω satisfying $\omega^m = 2^{-m} i^m m! \Omega \wedge \bar{\Omega}$. This gives a generalised Calabi–Yau structure with $\varphi_1 = e^{i\omega}$ and $\varphi_2 = \Omega$ satisfying

$$(e^{i\omega}, e^{-i\omega})_M = (-1)^{\frac{m(m-1)}{2}} (\Omega, \bar{\Omega})_M.$$

Example 4.7 (hyperKähler). Given a hyperKähler structure (M, g, I, J, K) a generalised Kähler structure can be constructed:

$$\varphi_1 = e^{B+i\omega_1}, \quad \varphi_2 = e^{-B+i\omega_2},$$

where $B = \omega_K,\ \omega_1 = \omega_I - \omega_J,\ \omega_2 = \omega_I + \omega_J$.

4.2 Generalised contact geometry

The exact Courant algebroid and Dirac structures play a fundamental role in generalised complex geometry. The corresponding objects in generalised contact geometry are the contact Courant algebroid and contact Dirac structures.

For sections $V = (v, f, g, \xi) \in \Gamma(TM) \oplus C^\infty(M) \oplus C^\infty(M) \oplus \Gamma(T^*M)$, the *contact Courant algebroid* is given by [20]:

$$\begin{aligned} V_1 \circ_{H_3,H_2,F} V_2 = \Big([v_1, v_2], v_1(f_2) - v_2(f_1) - \iota_{v_1}\iota_{v_2}F, v_1(g_2) - v_2(g_1) - \iota_{v_1}\iota_{v_2}H_2,& \\ \mathcal{L}_{v_1}\xi_2 - \iota_{v_2}d\xi_1 - \iota_{v_1}\iota_{v_2}H_3 + g_2 df_1 + f_2 dg_1& \qquad (4.8a) \\ + f_1\iota_{v_2}H_2 - f_2\iota_{v_1}H_2 + g_1\iota_{v_2}F - g_2\iota_{v_1}F\Big);& \end{aligned}$$

$$\langle V_1, V_2\rangle = \tfrac{1}{2}(\iota_{v_1}\xi_2 + \iota_{v_2}\xi_1 + f_1 g_2 + g_1 f_2); \qquad (4.8b)$$

$$\rho(V) = \rho((v, f, g, \xi)) = v; \qquad (4.8c)$$

where the twists $(H_3, H_2, F) \in \Omega^3(M) \oplus \Omega^2(M) \oplus \Omega^2(M)$ are globally defined forms required to satisfy the Bianchi identities:

$$dH_3 + H_2 \wedge F = 0, \quad dH_2 = 0, \; dF = 0. \qquad (4.9)$$

This is a twisted version of the contact Courant algebroid that has appeared previously in the generalised contact literature [78, 79, 57]. The twists (H_3, H_2, F) play an essential role in describing symmetries and deformations of generalised contact structures.

First consider the case that $H_3 = H_2 = F = 0$. There is an action of $B \in \Omega^2(M)$, $a, b \in \Omega^1(M)$ on $V = (v, f, g, \xi) \in \Gamma(E)$:

$$e^{(B,b,a)}(v, f, g, \xi) = \Big(v, f + 2\langle v, a\rangle, g + 2\langle v, b\rangle, \xi + \iota_v B - fb - ga - \langle v, a\rangle b - \langle v, b\rangle a\Big). \qquad (4.10)$$

This action satisfies

$$\langle e^{(B,b,a)}V_1, e^{(B,b,a)}V_2\rangle = \langle V_1, V_2\rangle, \quad e^{(B,b,a)}V_1 \circ_{(0,0,0)} e^{(B,b,a)}V_2 = e^{(B,b,a)}(V_1 \circ_{H_3', H_2', F'} V_2),$$

where

$$H_3' = dB + \tfrac{1}{2}(da \wedge b + a \wedge db), \quad H_2' = db, \quad F' = da.$$

We conclude that when $H_3 = H_2 = F = 0$ the bracket (4.8a) has the symmetry group

$$\mathsf{Diff}(M) \ltimes \Omega^2_{\text{cl}}(M) \oplus \Omega^1_{\text{cl}}(M) \oplus \Omega^1_{\text{cl}}(M).$$

The group action is generated by the algebra action

$$(B, b, a) \cdot (v, f, g, \xi) = (0, \iota_v a, \iota_v b, \iota_v B - fb - ga).$$

The algebra composition is given by

$$(B_2, b_2, a_2) \cdot (B_1, b_1, a_1) = (B_1 + B_2 - \tfrac{1}{2}(b_1 \wedge a_2 + a_1 \wedge b_2), b_1 + b_2, a_1 + a_2).$$

The group action can be recovered from the algebra action by exponentiation

$$e^{(B,b,a)}V_1 = V_1 + (B, b, a) \cdot V_1 + \frac{1}{2!}(B, b, a)^2 \cdot V_1 + \dots.$$

For non-trivial (H_3, H_2, F) the symmetries are described by a gerbe structure. The construction follows from Baraglia's general argument for twisting closed form Leibniz algebroids [9]. The twists are constructed from

$$(H_3, H_2, F) = (dB_\alpha - \tfrac{1}{2}a_\alpha \wedge db_\alpha - \tfrac{1}{2}da_\alpha \wedge b_\alpha, db_\alpha, da_\alpha), \tag{4.11}$$

where $(B_\alpha, b_\alpha, a_\alpha) \in \Omega^2(U_\alpha) \oplus \Omega^1(U_\alpha) \oplus \Omega^1(U_\alpha)$ are required to satisfy (4.9) and the cocycle conditions

$$(B_{\alpha\beta}, b_{\alpha\beta}, a_{\alpha\beta}) \cdot (B_{\beta\gamma}, b_{\beta\gamma}, a_{\beta\gamma}) \cdot (B_{\gamma\alpha}, b_{\gamma\alpha}, a_{\gamma\alpha}) = 0 \quad \text{on } U_{\alpha\beta\gamma}, \tag{4.12}$$

where $(B_{\alpha\beta}, b_{\alpha\beta}, a_{\alpha\beta}) = (B_\alpha - B_\beta + \tfrac{1}{2}b_\alpha \wedge a_\beta + \tfrac{1}{2}a_\alpha \wedge b_\beta, b_\alpha - b_\beta, a_\alpha - a_\beta)$. The gerbe structure defines a twisted bundle with sections patched together on $U_{\alpha\beta}$ using

$$(v_\alpha, f_\alpha, g_\alpha, \xi_\alpha) = e^{(B_{\alpha\beta}, b_{\alpha\beta}, a_{\alpha\beta})}(v_\beta, f_\beta, g_\beta, \xi_\beta)$$

to define a global section $(v, f, g, \xi) \in \Gamma(E)$. The twists do not define (B, b, a) uniquely and it is possible to make a different choice (B', b', a') as long as $(B_{\alpha\beta}, b_{\alpha\beta}, a_{\alpha\beta}) = (B'_{\alpha\beta}, b'_{\alpha\beta}, a'_{\alpha\beta})$. This gives the relation

$$(B'_\alpha, b'_\alpha, a'_\alpha) = (B_\alpha + B'' - \tfrac{1}{2}b_\alpha \wedge a'' - \tfrac{1}{2}a_\alpha \wedge b'', b_\alpha + b'', a_\alpha + a''), \tag{4.13}$$

where (B'', b'', a'') are globally defined forms satisfying

$$(dB'' + H_3 - \tfrac{1}{2}da'' \wedge b'' - \tfrac{1}{2}a'' \wedge db'', db'' + H_2, da'' + F_2) = 0. \tag{4.14}$$

Definition 4.7. Consider a choice of twists $(H_3, H_2, F) \in \Omega^3(M) \oplus \Omega^2(M) \oplus \Omega^2(M)$ satisfying (4.9). A (B, b, a)-*transformation* corresponds to a choice of triple $(B_\alpha, b_\alpha, a_\alpha) \in \Omega^2(U_\alpha) \oplus \Omega^1(U_\alpha) \oplus \Omega^1(U_\alpha)$ which generate the twists (H_3, H_2, F), i.e., satisfying conditions (4.11) and (4.12). The choice of (B, b, a)-transformation is not unique. Two transformations (B, b, a) and (B', b', a') will produce the same twists (H_3, H_2, F) if they are related by a set $(B'', b'', a'') \in \Omega^2(M) \oplus \Omega^1(M) \oplus \Omega^1(M)$ satisfying (4.14). A transformation (B'', b'', a'') satisfying (4.14) defines a (B'', b'', a'')-*gauge transformation.*

The description above shows that the twisted contact Courant algebroid can be seen as a bracket on a twisted vector bundle E via a clutching function construction. The insight here is to interpret the contact Courant algebroid bracket as an S^1-reduction of the standard twisted Courant algebroid. The identification is made as follows: Consider a Courant algebroid associated to the vector bundle E given as

$$\begin{tikzcd} 0 \arrow[r] & T^*P \arrow[r, "\rho_B^*"] & E \arrow[l, bend left, "(s_B)^*"] \arrow[r, "\rho_B"] & TP \arrow[l, bend left, "s_B"] \arrow[r] & 0, \end{tikzcd}$$

with the standard H-twisted Courant algebroid structure (4.2), identifying

$$H(X_1, X_2) = s_B^*(s_B(X_1), s_B(X_2)), \quad X_1, X_2 \in \Gamma(TP).$$

If $P(M, \pi, \mathsf{U}(1))$ is a principal $\mathsf{U}(1)$-bundle, then there are Atiyah algebroids associated to TP and $(TP)^* \cong T^*P$:

$$\begin{tikzcd} 0 \arrow[r] & P \times \mathbb{R} \arrow[r, "r_a"] & TP/\mathsf{U}(1) \arrow[l, bend left, "t_a"] \arrow[r, "\pi_*"] & TM \arrow[l, bend left, "s_a"] \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & P^* \times \mathbb{R} \arrow[r, "t_b"] & T^*P/\mathsf{U}(1) \arrow[l, bend left, "(r_b)^*"] \arrow[r, "s_b"] & T^*M \arrow[l, bend left, "(\pi_b)^*"] \arrow[r] & 0. \end{tikzcd}$$

The Atiyah algebroid constructed from the reduction of $P(M, \pi, \mathsf{U}(1))$ is described in detail in Section 2.2.2 (page 19) and the general case of $P(M, \pi, \mathsf{G})$ is described in Example 3.10. The reduction gives a decomposition of the three-form $H \in \Omega^3(P)$. First recall that the splitting $s_a : TM \to TP/\mathsf{U}(1)$ can be defined by a choice of principal connection on P, which we denote A, with curvature $F = dA \in \Omega^2(M, \mathbb{Z})$. The decomposition of H is given by

$$H = \pi^* H_3 + H_2 \wedge A,$$

where $H_3 \in \Omega^3(M, \mathbb{Z})$ and $H_2 \in \Omega^2(M, \mathbb{Z})$. The reader is referred to [25] for more details on Courant algebroid reduction. Equivalently we can identify

$$H_2(v_1, v_2) = t_b(s_b(v_1), s_b(v_2)), \quad F(v_1, v_2) = t_a(s_a(v_1), s_a(v_2)),$$

for $v_1, v_2 \in \Gamma(TM)$. The induced map on sections is

$$0 \longrightarrow C^\infty(P,\mathbb{R})^{\mathsf{U}(1)} \simeq C^\infty(M) \xrightarrow{r} \mathfrak{X}_{\mathsf{G}}(P) \simeq \mathfrak{X}(M) \oplus C^\infty(M) \xrightarrow{\pi_*} \mathfrak{X}(M) \longrightarrow 0 ,$$

allowing the local identification

$$\Gamma(E) \cong \Gamma(TP/\mathsf{U}(1)) \oplus \Gamma((TP/\mathsf{U}(1))^*) \cong \Gamma(TM) \oplus C^\infty(M) \oplus C^\infty(M) \oplus \Gamma(T^*M).$$

The reduction of the $\mathsf{U}(1)$-invariant exact Courant algebroid on P gives (4.8). In this way the twisted contact Courant algebroid can be identified with

$$0 \longrightarrow (TP/\mathsf{U}(1))^* \longrightarrow E \longrightarrow TP/\mathsf{U}(1) \longrightarrow 0 .$$

The twisted contact Courant algebroid is constructed out of (H_3, H_2, F) and makes no reference to $(B_\alpha, b_\alpha, a_\alpha)$. Another choice $(B'_\alpha, b'_\alpha, a'_\alpha)$ giving the same (H_3, H_2, F) will give an isomorphic twisted contact Courant algebroid. Any two choices $(B_\alpha, b_\alpha, a_\alpha)$ and $(B'_\alpha, b'_\alpha, a'_\alpha)$ give the same twists if and only if they are related by a (B'', b'', a'')-gauge transformation. The notion of equivalence of generalised contact geometry should be extended to include to the full set of symmetries—diffeomorphisms and (B'', b'', a'')-gauge transformations. Geometrically the gauge transformations can be interpreted as a change in splitting of the bundle $E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$. Geometry on the bundle E should not be dependent on the choice of splitting.

Remark. Sekiya note κ-symmetries when studying generalised contact structures associated to the trivial line bundle $L = M \times \mathbb{R}$ [108]. This corresponds to $(0, b, a)$-transformations for globally defined forms not subject to periodicity constraints. This clarifies the geometric origin of Sekiya's κ-symmetries. The $(0, b, a)$-transformations correspond to a choice of connection for a circle/line-bundle $P \to M$. The non-abelian composition law for B with $(0, b, a)$ reflects the fact that there is choice in which order one splits the sequences.

4.3 Generalised contact structures

This section describes the mixed pair description of generalised contact structures, the odd-dimensional analogue of the pure spinor description in generalised complex geometry. Aldi and Grandini [2] gave a proposal for mixed pairs which were compatible with B-transformations; but mixed pairs were not compatible with the full set of (B, b, a)-transformations. The original definition cannot incorporate non-coorientable structures.

There is a Clifford action of sections $(v, f, g, \xi) \in \Gamma(TM) \oplus C^\infty(M) \oplus C^\infty(M) \oplus$

$\Gamma(T^*M)$ on pairs of differential forms $(\varphi, \psi) \in \Omega^\bullet(M)$:

$$(v, f, g, \xi) \cdot (\varphi, \psi) = ((v, \xi) \cdot \varphi + f\psi, g\varphi - (v, \xi) \cdot \psi), \tag{4.15}$$

where $(v, \xi) \cdot \varphi = \iota_v \varphi + \xi \wedge \varphi$ is the Clifford product on $\mathbb{T}M$. This product satisfies

$$\begin{aligned}(v, f, g, \xi)^2 \cdot (\varphi, \psi) =& (\iota_v \xi + fg)(\varphi, \psi) = \langle (v, f, g, \xi), (v, f, g, \xi) \rangle (\varphi, \psi) \\ =& ||(v, f, g, \xi)||^2 (\varphi, \psi).\end{aligned}$$

It is interesting to consider the annihilator bundles of a pair (φ, ψ):

$$\mathrm{Ann}((\varphi, \psi)) := \{(v, f, g, \xi) \in E \otimes \mathbb{C} : (v, f, g, \xi) \cdot (\varphi, \psi) = 0\}.$$

When $f = g = 0$, $(v, 0, 0, \xi) \cdot (\varphi, \psi) = 0$ implies that $(v, \xi) \cdot \varphi = 0$ and $(v, \xi) \cdot \psi = 0$, the same annihilator condition as Section 4.1. For some pairs (φ, ψ), there may be solutions for non-zero f or g. In this case

$$f\psi = -(v, \xi) \cdot \varphi, \quad g\varphi = (v, \xi) \cdot \psi,$$

indicating that there exist sections $(v, \xi) \in \mathbb{T}M$ which relate φ and ψ. Pure spinors play an important role in describing Dirac structures in $\mathbb{T}M \otimes \mathbb{C}$; mixed pairs describe the odd-dimensional analogue of Dirac structures.

Definition 4.8. Let M be an odd-dimensional manifold ($m = \dim(M)$). A *contact Dirac structure* is a decomposition of a vector bundle $E \otimes \mathbb{C}$ into isotropic subspaces

$$E \otimes \mathbb{C} = L \oplus \bar{L} \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2, \qquad L \cap \bar{L} = 0,$$

where $\dim_{\mathbb{R}}(L) = m - 1$, and $e_1, e_2 \in \Gamma(\mathbb{T}M)$. A contact Dirac structure is specified by a triple (L, e_1, e_2).

The pairing $(\cdot\, , \cdot)_M$ for two pairs of differential forms (φ_i, ψ_i) $(i = 1, 2)$ is given by

$$((\varphi_1, \psi_1), (\varphi_2, \psi_2))_M := (-1)^{|\varphi_1|} (\alpha(\varphi_1) \wedge \psi_2)_{m-1} + (-1)^{|\psi_1|} (\alpha(\psi_1) \wedge \varphi_2)_{m-1}, \tag{4.16}$$

where $\alpha(dx^1 \otimes dx^2 \otimes \cdots \otimes dx^k) = dx^k \otimes dx^{k-1} \otimes \cdots \otimes dx^1$, $m = \dim(M)$, $|\varphi| = k$ for $\varphi \in \Omega^k(M)$, and $(\cdot\,)_{m-1}$ is the projection to $\Omega^{m-1}(M)$.

Definition 4.9. A *Dirac pair* consists of two differential forms $\varphi, \psi \in \Gamma(\Omega^{\mathrm{ev/od}}(M) \otimes \mathbb{C})$ satisfying

$$\begin{aligned}(\alpha(\varphi) \wedge \bar{\varphi})_{m-1} \neq 0, \quad (\alpha(\psi) \wedge \bar{\psi})_{m-1} \neq& 0, \quad ((\varphi, \psi), (\bar{\varphi}, \bar{\psi}))_M \neq 0 \\ e_1 \cdot \varphi = \psi,& \quad e_2 \cdot \psi = \varphi,\end{aligned}$$

for some $e_1, e_2 \in \Gamma(\mathbb{T}M)$.

The second condition implies that $\varphi \in \Gamma(\Omega^{\mathrm{ev}}(M) \otimes \mathbb{C})$ and $\psi \in \Gamma(\Omega^{\mathrm{odd}}(M) \otimes \mathbb{C})$, or $\varphi \in \Gamma(\Omega^{\mathrm{odd}}(M) \otimes \mathbb{C})$ and $\psi \in \Gamma(\Omega^{\mathrm{ev}}(M) \otimes \mathbb{C})$.

Given two nowhere-zero functions $f_1, f_2 \in C^\infty(M)$ and a Dirac pair (φ, ψ), the pair $(f_1\varphi, f_2\psi)$ satisfy the non-degeneracy condition and

$$e_1 \cdot f_1\varphi - f_2\psi = f_2(e_1' \cdot \varphi - \psi) = 0, \quad e_2 \cdot f_2\psi - f_1\varphi = f_1(e_2' \cdot \psi - \varphi) = 0,$$

for some $e_1' = f_2/f_1 e_1$ and $e_2' = f_1/f_2 e_2$. Thus (φ, ψ) and $(f_1\varphi, f_2\psi)$ describe the same contact Dirac structure (L, e_1, e_2).

To motivate the definition of generalised contact structures it is helpful to briefly consider the relationship between contact structures and symplectic structures. A contact structure is a maximally non-integrable codimension-1 hyperplane distribution $D \subset TM$. This can be described by the line bundle TM/D. Letting η be a TM/D-valued one-form; the distribution is given by $D = \ker(\eta)$. The non-integrability condition can be given as $\eta \wedge (d\eta)^m \neq 0$, where $\dim(M) = 2m + 1$. There is a transverse symplectic structure on D: $\omega_D = d\eta$. In addition, there is another symplectic structure associated with the manifold $N := M \times \mathbb{R}_t$. Take $\alpha = dt + \eta$ and set $\omega_t = d(e^t\alpha)$. When TM/D is a non-trivial line bundle there is no globally defined contact form η. It is possible to consider the same construction with $\mathsf{S}^1 \hookrightarrow P' \to M$. In this case there is an Atiyah algebroid structure and the contact structure can be associated with an S^1-invariant reduction. In the non-trivial case η is no longer globally defined but describes a connection one-form with a globally defined curvature two-form $F = d\eta$.

The ability to construct two symplectic structures from a contact structure is the guiding principle of generalised contact structures. A generalised contact structure should be able to be viewed as a (possibly non-trivial) S^1-reduction of a generalised complex structure (see Examples 4.8 and 4.9). In addition the definition should be compatible with (B, b, a)-transformations.

Remark. Contact structures are usually associated to symplectic structures defined on line bundles TM/D and $N = M \times \mathbb{R}$. Throughout this chapter S^1-bundles will be considered primarily. The motivation for this is the fact that $[H_2/2\pi], [F/2\pi] \in H^2(M; \mathbb{Z})$ have a nice interpretation in terms of gerbes (as outlined in Section 4.2). The corresponding Courant algebroid description applicable to non-trivial line bundles was given by Vitagliano and Wade [123] and is briefly described in Section 4.6.

Generalised contact structures have been studied in a number of papers [79, 102, 2]. However, the $(0, b, a)$-twists (which allow the description of non-coorientable structures when $H_2, F \neq 0$) have received little attention.

Definition 4.10 ([108])**.** A *Sekiya quadruple* on an odd-dimensional manifold M is given by the quadruple $(\Phi, e_1, e_2, \lambda) \in \mathsf{End}(\mathbb{T}M) \oplus \Gamma(\mathbb{T}M) \oplus \Gamma(\mathbb{T}M) \oplus C^\infty(M)$, satisfying the following conditions:

$$\langle e_1, e_1 \rangle = 0 = \langle e_2, e_2 \rangle, \ \langle e_1, e_2 \rangle = \tfrac{1}{2}; \tag{4.17a}$$

$$\Phi^* = - \Phi; \tag{4.17b}$$

$$\Phi(e_1) = \lambda e_1, \ \Phi(e_2) = -\lambda e_2; \tag{4.17c}$$

$$\Phi^2(e) = - e + 2(1 + \lambda^2)(\langle e, e_2 \rangle e_1 + \langle e, e_1 \rangle e_2), \text{ for } e \in \Gamma(\mathbb{T}M). \tag{4.17d}$$

Generalised contact structures coming from Sekiya quadruples with $\lambda = 0$ have been well studied and are often referred to as Poon–Wade triples [102]. The importance of considering $\lambda \neq 0$ is the inclusion of the (B, b, a)-symmetries, which should be considered on an equal footing to B-transformations, a fundamental part of the theory.

Definition 4.11. Let M be an odd-dimensional manifold of dimension m. A *generalised almost contact structure* is a quadruple (L, e_1, e_2, λ), where $L \subset \mathbb{T}M \otimes \mathbb{C}$ is a maximal isotropic subspace $\dim_{\mathbb{R}}(L) = m - 1$, $e_1, e_2 \in \Gamma(\mathbb{T}M)$, satisfy

$$\langle e_1, e_1 \rangle = 0, \quad \langle e_2, e_2 \rangle = 0, \quad \langle e_1, e_2 \rangle = \tfrac{1}{2},$$

and $\lambda \in C^\infty(M)$.

A Sekiya quadruple can be associated to a generalised almost contact structure: Let L represent the $+i$-eigenbundle of Φ, and e_1, e_2 specify the $\pm\lambda$ eigenbundles respectively.

It is clear that the pairs (e_1, e_2, λ) and $(e_2, e_1, -\lambda)$ describe the same generalised almost contact structure. When $\lambda = 0$ it follows that $\dim(\ker(\Phi)) = 2$ and there is a $\mathsf{O}(1, 1)$ freedom in the choice of e_1, e_2.

A generalised almost contact structure on M can be constructed from an S^1-invariant generalised almost complex structure on a principal circle bundle $P(M, \pi, \mathsf{U}(1))$. Consider a principal bundle $P(M, \pi, \mathsf{U}(1))$ over an odd-dimensional manifold M. Let $\mathcal{U} = \{U_\alpha\}$ denote a good cover of M, and $\pi^{-1}(U_\alpha) = U_\alpha \times \mathsf{S}^1$ a cover for P. Take local coordinates (x, t_α), $x \in U_\alpha$, $t_\alpha \in \mathsf{S}^1$. We have two set of coordinates on $\pi^{-1}(U_\alpha \cap U_\beta)$ denoted (x, t_α) and (x, t_β). The coordinates are related by $t_\alpha = g_{\alpha\beta} t_\beta$, where $g_{\alpha\beta} \in C^\infty(M, \mathsf{U}(1))$ are transition functions. Choose an S^1-invariant connection $\mathcal{A}$, given locally by $\mathcal{A}_\alpha = dt_\alpha + A(x)$, where $A \in \Omega^1(M)$. On $x \in U_{\alpha\beta}$ we have $\mathcal{A}_\alpha = \mathcal{A}_\beta - id \log g_{\alpha\beta}$. Assume that there is an S^1-invariant generalised almost complex structure $\mathbb{J}_{\text{inv}} \in \mathsf{End}(\mathbb{T}P)$. A choice of connection induces a decomposition of S^1-invariant sections $\Gamma(\mathbb{T}P) = \Gamma(\mathbb{T}M) \oplus C^\infty(M) \oplus C^\infty(M)$: $v + \xi + f\partial_t + g\mathcal{A}$, for

$v \in \Gamma(TM)$, $f, g \in C^\infty(M)$, and $\xi \in \Gamma(T^*M)$. This gives the decomposition:

$$\mathbb{J}_{\text{inv}} = \begin{pmatrix} \Phi & \mu e_1 & \mu e_2 \\ -2\mu\langle e_2, \cdot\rangle & -\lambda & 0 \\ -2\mu\langle e_1, \cdot\rangle & 0 & \lambda \end{pmatrix}, \tag{4.18}$$

where $\Phi \in \mathsf{End}(\mathbb{T}M)$, $\mu = \sqrt{1+\lambda^2}$, $\lambda \in C^\infty(M)$, and $e_1, e_2 \in \Gamma(\mathbb{T}M)$. The properties of a Sekiya quadruple $(\Phi, e_1, e_2, \lambda)$ follow from $\mathbb{J}_{\text{inv}}^2 = -\text{Id}$ and $\mathbb{J}_{\text{inv}}^* = -\mathbb{J}_{\text{inv}}$. The clutching construction for global sections (X, f, g, ξ) involve transition functions $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathsf{U}(1))$. The global sections can be viewed as a (B, b, a)-transformation with $B = b = 0$, $a = \mathcal{A}$, generating twists $H_3 = H_2 = 0$, and $F = d\mathcal{A}$. The choice of transition functions $g_{\alpha\beta}$ are not unique; $g_{\alpha\beta}$ can be replaced with $g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}$, for any $h_\alpha \in C^\infty(U_\alpha, \mathsf{U}(1))$. Replacing $g_{\alpha\beta}$ with $g'_{\alpha\beta}$ corresponds to a $(0,0,a)$-gauge transformation, describing the decomposition with respect a connection $\mathcal{A}' = \mathcal{A} + a$ satisfying $da = 0$.

A generalised almost contact structure (L, e_1, e_2, λ) is (H_3, H_2, F)-*involutive* if all sections V of the $+i$-eigenbundle $L_{\mathbb{J}_{\text{inv}}}$ (defined by (4.18)) are involutive with respect to $\circ_{H_3,H_2,F}$: $V_1, V_2 \in L_{\mathbb{J}_{\text{inv}}} \Rightarrow V_1 \circ_{H_3,H_2,F} V_2 \in L_{\mathbb{J}_{\text{inv}}}$.

Definition 4.12. An (H_3, H_2, F)-*generalised contact structure* is an (H_3, H_2, F)-involutive generalised almost contact structure.

Generalised (almost) contact structures can be encoded in differential forms in a relationship analogous to that of pure spinors and generalised (almost) complex structures.

Definition 4.13 ([2])**.** A *mixed pair* $(\varphi, \psi, e_1, e_2)$ consists of two differential forms $\varphi, \psi \in \Gamma(\Omega^{\text{ev/od}}(M) \otimes \mathbb{C})$ and a choice of two sections $e_1, e_2 \in \Gamma(\mathbb{T}M)$ satisfying

$$(\varphi, \bar{\varphi})_{m-1} \neq 0, \quad (\psi, \bar{\psi})_{m-1} \neq 0, \quad ((\varphi, \psi), (\bar{\varphi}, \bar{\psi}))_{m-1} \neq 0, \tag{4.19a}$$

$$e_1 \cdot \psi = 0, \quad \mu e_1 \cdot \varphi = (1+i\lambda)\psi, \quad e_2 \cdot \varphi = 0, \quad \mu e_2 \cdot \psi = (1-i\lambda)\varphi, \tag{4.19b}$$

where $\mu = \sqrt{1+\lambda^2}$, and $\lambda \in C^\infty(M)$.

Remark. The definition of mixed pair given here differs slightly from that given in [2], which is valid for $\lambda = 0$ only.

Given a nowhere zero function $f \in C^\infty(M)$ the pair $(f\varphi, f\psi)$ satisfies the equations (4.19) for fixed (e_1, e_2, λ).

A generalised almost contact structure can be described using a mixed pair. Fix a generalised almost contact structure (L, e_1, e_2, λ), and identify $\text{Ann}(\psi) = L \oplus \mathbb{C}e_1$ and $\text{Ann}(\varphi) = L \oplus \mathbb{C}e_2$. The isotropic subbundle L can be recovered as the intersection of the annihilator bundles of (φ, ψ).

There is a local correspondence between generalised almost contact structures given by (L, e_1, e_2, λ) and a conformal class of mixed pairs (φ, ψ).

The definition of mixed pairs is motivated by the decomposition of a pure spinor $\rho_{\mathbb{J}}$ (associated to an S^1-invariant generalised complex structure $\mathbb{J}_{\text{inv}}$ on $M \times \mathsf{S}^1$ see [2]) into a mixed pair (φ, ψ) associated to a Sekiya quadruple on M: $\rho_{\mathbb{J}} \to \varphi + idt \wedge \psi$. The pure spinor condition $(\rho_{\mathbb{J}}, \bar{\rho}_{\mathbb{J}})_{M\times\mathsf{S}^1} \neq 0$ gives (4.19a). Note that $\mathbb{J}|_{\mathsf{S}^1}(e_j) = ie_{(j)}$ $(j = 1, 2)$ for $e_{(1)} = (\mu e_1, i - \lambda, 0)$, $e_{(2)} = (\mu e_2, 0, i + \lambda)$—this implies that $e_{(j)} \cdot \rho_{\mathbb{J}}|_{\mathsf{S}^1} = 0$ and gives (4.19b).

Definition 4.14. A mixed pair (φ, ψ) is said to be (H_3, H_2, F)-*involutive* if there exists a $V = (v, f, g, \xi) \in \Gamma(E)$ such that

$$d_{H_3,H_2,F}(\varphi, \psi) = V \cdot (\varphi, \psi), \tag{4.20}$$

where

$$\begin{aligned} d_{H_3,H_2,F}(\varphi, \psi) :=&(d\varphi + H_3 \wedge \varphi + F \wedge \psi, H_2 \wedge \varphi - d\psi - H_3 \wedge \psi), \\ V \cdot (\varphi, \psi) =&(v, f, g, \xi) \cdot (\varphi, \psi) = (\iota_v\varphi + \xi \wedge \varphi + f\psi, g\varphi - \iota_v\psi - \xi \wedge \psi), \end{aligned}$$

and (H_3, H_2, F) satisfy the Bianchi identities (4.9).

Remark. The terminology (H_3, H_2, F)-involutive is justified by Theorem 4.15.

Given a non-zero function $h \in C^\infty(M)$ and mixed pair $(\varphi, \psi, e_1, e_2)$, satisfying $d_{H_3,H_2,F}(\varphi, \psi) = V \cdot (\varphi, \psi)$, the quadruple $(h\varphi, h\psi, e_1, e_2)$ satisfies

$$d_{H_3,H_2,F}(h\varphi, h\psi) = V' \cdot (h\varphi, h\psi), \quad V' = (v, f, g, \xi - h^{-1}dh).$$

Thus the (H_3, H_2, F)-involutive property is not dependent on the choice of (φ, ψ) chosen to represent the almost contact structure (L, e_1, e_2, λ).

Let us briefly recall the Clifford product on $U \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ on the generalised tangent bundle $\mathbb{T}M$:

$$(v, \xi) \cdot \rho = \iota_v\rho + \xi \wedge \rho, \quad (v, \xi)\cdot : \Gamma(\wedge^{\text{ev/odd}}T^*M \otimes \mathbb{C}) \to \Gamma(\wedge^{\text{odd/ev}}T^*M \otimes \mathbb{C}),$$

for $(v, \xi) \in \Gamma(TM) \oplus \Gamma(T^*M)$, and $\rho \in \Gamma(U)$, where $U \subset \wedge^\bullet T^*M \otimes \mathbb{C}$. By Clifford multiplication on U we obtain filtrations of the even and odd exterior forms (here $2n$ is the real dimension of the manifold):

$$\begin{aligned} U =&U_0 < U_2 < \cdots < U_{2n} = \wedge^{\text{ev/odd}}T^*M \otimes \mathbb{C}, \\ L^* \cdot U =&U_1 < U_3 < \cdots < U_{2n-1} = \wedge^{\text{odd/ev}}T^*M \otimes \mathbb{C}, \end{aligned}$$

where ev/odd is chosen according to the parity of U itself, U_k is defined as $CL^k \cdot U$, and CL^k is spanned by products of not more than k elements of $\mathbb{T}M$ [59]. Note that we have the canonical isomorphism $L^* \otimes U = ((TM \oplus T^*M) \otimes \mathbb{C})/L$ (using the inner product) and hence U_1 is isomorphic to $L^* \otimes U_0$. Theorem 3.38 of [59] shows that an almost Dirac structure defined by $\mathrm{Ann}(\rho)$ is Courant involutive if and only if $d(C^\infty(U_0)) \subset C^\infty(U_1)$, i.e., $d\rho = \iota_v \rho + \xi \wedge \rho$ for some $(v, \xi) \in \Gamma(TM) \oplus \Gamma(T^*M)$.

A similar statement holds in the almost contact case. Consider a mixed pair (φ, ψ). The definition requires that $(\varphi, \psi) \in \Gamma(\wedge^{\mathrm{ev/odd}} T^*M \otimes \mathbb{C}) \oplus \Gamma(\wedge^{\mathrm{odd/ev}} T^*M \otimes \mathbb{C})$.

$$\begin{aligned} (v, f, g, \xi) \cdot (\varphi, \psi) &= (\iota_v \varphi + \xi \wedge \varphi + f\psi, g\varphi - \iota_v \psi - \xi \wedge \psi), \\ (v, f, g, \xi) \cdot :& \Gamma((\wedge^{\mathrm{ev/odd}} T^*M \otimes \mathbb{C}) \oplus (\wedge^{\mathrm{odd/ev}} T^*M \otimes \mathbb{C})) \\ &\to \Gamma((\wedge^{\mathrm{odd/ev}} T^*M \otimes \mathbb{C}) \oplus (\wedge^{\mathrm{ev/odd}} T^*M \otimes \mathbb{C})). \end{aligned}$$

This gives a filtration:

$$\begin{aligned} W =& W_0 < W_2 < \cdots < W_{m+1} = (\wedge^{\mathrm{ev/odd}} T^*M \otimes \mathbb{C}) \oplus (\wedge^{\mathrm{odd/ev}} T^*M \otimes \mathbb{C}), \\ L^* \cdot W =& W_1 < W_3 < \cdots < W_m = (\wedge^{\mathrm{odd/ev}} T^*M \otimes \mathbb{C}) \oplus (\wedge^{\mathrm{ev/odd}} T^*M \otimes \mathbb{C}), \end{aligned}$$

where $\dim(M) = m$ is odd-dimensional and φ, ψ are being viewed as pure spinors on a local trivialisation of $M \times \mathsf{S}^1$.

Theorem 4.15. *The annihilator bundle $Ann(\varphi, \psi)$ of a (H_3, H_2, F)-involutive pair (φ, ψ) is involutive under the (H_3, H_2, F)-contact Courant algebroid product.*

Proof. Let (L, e_1, e_2, λ) be the generalised almost contact structure and let (φ, ψ) be a trivialisation of representative of (L, e_1, e_2, λ) over some open set.

We show below that

$$V_1 \circ_{H_3,H_2,F} V_2 \cdot (\varphi, \psi) = -V_2 \cdot V_1 \cdot d_{H_3,H_2,F}(\varphi, \psi),$$

for any sections $V_1, V_2 \in \mathrm{Ann}(\varphi, \psi)$. The subbundle $\mathrm{Ann}(\varphi, \psi)$ is involutive if and only if for any $V_1, V_2 \in \mathrm{Ann}(\varphi, \psi)$ the condition $V_1 \cdot V_2 \cdot d_{H_3,H_2,F}(\varphi, \psi) = 0$ holds. This condition holds if and only if $d_{H_3,H_2,F}(\varphi, \psi)$ is in $C^\infty(W_1)$ (elements of W_k are precisely those which are annihilated by $k+1$ elements in $\mathrm{Ann}(\varphi, \psi)$).

A section $V = (v, f, g, \xi) \in \mathrm{Ann}(\varphi, \psi)$ satisfies

$$(v, f, g, \xi) \cdot (\varphi, \psi) = (\iota_v \varphi + \xi \wedge \varphi + f\psi, g\varphi - \iota_v \psi - \xi \wedge \psi) = 0.$$

Rearranging we have $\iota_v \varphi = -\xi \wedge \varphi - f\psi, \quad \iota_v \psi = g\varphi - \xi \wedge \psi$.

$$
\begin{aligned}
\iota_{[v_1,v_2]}\varphi =& [\mathcal{L}_{v_1}, \iota_{v_2}]\varphi = \mathcal{L}_{v_1}\iota_{v_2}\varphi - \iota_{v_2} d\iota_{v_1}\varphi - \iota_{v_2}\iota_{v_1} d\varphi \\
=& \mathcal{L}_{v_1}(-\xi_2 \wedge \varphi - f_2\psi) - \iota_{v_2} d(-\xi_1 \wedge \varphi - f_1\psi) - \iota_{v_2}\iota_{v_1} d\varphi \\
=& -\mathcal{L}_{v_1}\xi_2 \wedge \varphi - \xi_2 \wedge d(-\xi_1 \wedge \varphi - f_1\psi) - \xi_2 \wedge \iota_{v_1} d\varphi - (\mathcal{L}_{v_1} f_2)\psi \\
& - f_2 \mathcal{L}_{v_1}\psi - \iota_{v_2}(-d\xi_1 \wedge \varphi + \xi_1 \wedge d\varphi - df_1 \wedge \psi - f_1 d\psi) - \iota_{v_2}\iota_{v_1} d\varphi \\
=& (-\mathcal{L}_{v_1}\xi_2 + \iota_{v_2} d\xi_1 + \xi_2 \wedge d\xi_1) \wedge \varphi - \xi_2 \wedge (\iota_{v_1} + \xi_1 \wedge) d\varphi \\
& + (\xi_2 \wedge df_1 - \iota_{v_1} df_2 + \iota_{v_2} df_1) \wedge \psi + f_1 \xi_2 \wedge d\psi - f_2 d(g_1\varphi - \xi_1 \wedge \psi) \\
& - f_2 d(g_1 \varphi - \xi_1 \wedge \psi) \\
& - \iota_{v_2}(\xi_1 \wedge d\varphi + \iota_{v_1} d\varphi) - df_1 \wedge (g_2\varphi - \xi_2 \wedge \psi) + f_1 \iota_{v_2} d\psi \\
=& -(\mathcal{L}_{v_1}\xi_2 - \iota_{v_2} d\xi_1 + g_2 df_1 - f_2 dg_1) \wedge \varphi - (\iota_{v_1} df_2 - \iota_{v_2} df_1) \wedge \psi \\
& + f_1(\iota_{v_2} + \xi_2 \wedge) d\psi + f_2(\xi_1 \wedge d\psi + g_1 d\varphi) \\
& - (\iota_{v_2} + \xi_2) \wedge (\iota_{v1} + \xi_1 \wedge) d\varphi.
\end{aligned}
$$

A similar calculation holds for $\iota_{[v_1,v_2]}\psi$. Combining the results gives

$$
\begin{aligned}
& \iota_{V_1 \circ V_2}(\varphi, \psi) = \\
& \quad \big(-(\iota_{v_2} + \xi_2 \wedge)(\iota_{v_1} + \xi_1 \wedge) d\varphi - f_2 g_1 d\varphi - f_2(\iota_{v_1} + \xi_1 \wedge) d\psi + f_1(\iota_{v_2} + \xi_2 \wedge) d\psi, \\
& \quad (\iota_{v_2} + \xi_2 \wedge)(\iota_{v_1} + \xi_1 \wedge) d\psi + g_2 f_1 d\psi + g_1(\iota_{v_2} + \xi_2 \wedge) d\varphi - g_2(\iota_{v_1} + \xi_1 \wedge) d\varphi \big) \\
& = -(v_2, f_2, g_2, \xi_2) \cdot ((\iota_{v_1} + \xi_1 \wedge) d\varphi - f_1 d\psi, g_1 d\varphi + (\iota_{v_1} + \xi_1 \wedge) d\psi) \\
& = -(v_2, f_2, g_2, \xi_2) \cdot (v_1, f_1, g_1, \xi_1) \cdot (d\varphi, -d\psi) = -V_2 \cdot V_1 \cdot d(\varphi, \psi).
\end{aligned}
$$

So the annihilator bundle $\mathrm{Ann}(\varphi, \psi)$ corresponds to an involutive subbundle. A similar argument holds for the twisted case $\iota_{V_1 \circ_{H_3,H_2,F} V_2}(\varphi, \psi) = -V_2 \cdot V_1 \cdot d_{H_3,H_2,F}(\varphi, \psi)$. □

Let us consider a generalised almost contact structure generated by a cosymplectic structure on M and examine the integrability condition. An almost cosymplectic structure is a pair $(\theta, \eta) \in \Omega^2(M) \oplus \Omega^1(M)$ satisfying $\eta \wedge \theta^n \neq 0$. From standard results in contact geometry there exists a Reeb vector field $R \in \Gamma(TM)$ such that $\iota_R \eta = 1$ and $\iota_R \theta = 0$. A mixed pair $(\varphi, \psi, e_1, e_2)$ can be given by

$$\varphi = e^{i\theta}, \quad \psi = \eta \wedge e^{i\theta}, \quad e_1 = \eta, \quad e_2 = R.$$

We have $d\varphi = id\theta \wedge \varphi$, and $d\psi = d\eta \wedge \varphi + id\theta \wedge \psi$. If $d\theta = d\eta = 0$ (a cosymplectic structure) then $d_{0,0,0}(\varphi, \psi) = 0$. If $d\eta = \theta$ (a contact 1-form η) then $d_{0,d\eta,0}(\varphi, \psi) = 0$. In fact the pair (θ, η) will form a $(0, d\eta, 0)$-generalised contact structure if $d\theta = 0$. We conclude the following:

- A cosymplectic structure defines a $(0, 0, 0)$-generalised contact structure.
- A contact 1-form η defines a $(0, d\eta, 0)$-generalised contact structure.

It is possible to describe a non-coorientable contact structure arising from a TM/D-valued 1-form η. In the case of a non-trivial line bundle TM/D, η is not globally defined. Local trivialisations η_α and η_β are related using transition functions $g_{\alpha\beta}$. If the line bundle on a compact manifold is of the form $TM/D \cong T\mathsf{S}^1$, for an S^1-foliation then η satisfies the conditions of a $(0, 0, \eta)$-transformation ((4.11) and (4.12)) with twists $(0, 0, d\eta)$.

Remark. In [102] an almost generalised contact structure (defined with $e_1 \in \Gamma(TM)$, $e_2 \in \Gamma(T^*M)$, $\lambda = 0$) is called a generalised contact structure if $L \oplus \mathbb{C}e_1$ is involutive. A *strong* generalised contact structure is a generalised contact structure where $L \oplus \mathbb{C}e_2$ is involutive. In [2] a generalised normal contact structure is a generalised contact structure arising from an invariant generalised complex structure $\mathbb{J}$ on $M \times \mathbb{R}$. In both cases a contact form η with $d\eta \neq 0$ does not give a strong generalised contact structure.

An essential property of the definition of a generalised (almost) contact structure is that it is compatible with the (B, b, a)-symmetries. The action on a Dirac pair (φ, ψ) is:

$$e^{(B,b,a)}(\varphi, \psi) = \left(e^{-B}\varphi + ae^{-B}\psi - \tfrac{1}{2}abe^{-B}\varphi, e^{-B}\psi - be^{-B}\varphi - \tfrac{1}{2}bae^{-B}\psi\right), \tag{4.21}$$

where $\wedge$ has been omitted. The action preserves the pairing,

$$(e^{(B,b,a)}(\varphi_1, \psi_1), e^{(B,b,a)}(\varphi_2, \psi_2))_M = ((\varphi_1, \psi_1), (\varphi_2, \psi_2))_M,$$

and satisfies

$$d_{H_3',H_2',F'}(e^{(B,b,a)}(\varphi, \psi)) - V' \cdot (e^{(B,b,a)}(\varphi, \psi)) = e^{(B,b,a)}(d_{H_3,H_2,F}(\varphi, \psi) - V \cdot (\varphi, \psi)), \tag{4.22}$$

where

$$\begin{aligned} H_3' &= H_3 + dB + a \wedge H_2 + b \wedge F + \frac{1}{2}(da \wedge b + a \wedge db), \\ H_2' &= H_2 + db, \quad F' = F + da, \quad V' = e^{(B,b,a)}V. \end{aligned}$$

This shows that given a (H_3, H_2, F)-involutive mixed pair (φ, ψ) there exists a (H_3', H_2', F')-involutive Dirac pair $e^{(B,b,a)}(\varphi, \psi)$.

Below two standard examples of generalised contact structures (first appearing in [79] for $M \times \mathbb{R}$) are presented from the perspective of reduced generalised complex structures on $\mathsf{S}^1 \hookrightarrow P \to M$.

Example 4.8 (Almost symplectic to almost cosymplectic structure). Let $\omega \in \Gamma(\wedge^2 T^*P)$ be a symplectic form, where $\mathsf{S}^1 \hookrightarrow P \to M$, and $\dim(M) = m = 2n+1$. Consider a connection specified by locally by $\mathcal{A}_\alpha = dt_\alpha + A_\alpha \in \Omega^1(P_\alpha)$ (with $\pi^{-1}(U_\alpha) = U_\alpha \times \mathsf{S}^1$ defining a cover of P) with curvature $F = d\mathcal{A} \in H^2(M, \mathbb{Z})$. The symplectic form is S^1-invariant if it admits the decomposition

$$\omega_\alpha = \theta_\alpha + \mathcal{A}_\alpha \wedge \eta_\alpha \in \Omega^2(P_\alpha),$$

where ω is globally defined, but η is not if the bundle is not trivial (η satisfies the usual cocycle conditions).

$$0 \neq \omega^{n+1} = (\theta + \mathcal{A} \wedge \eta)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \theta^{n+1-k} (\mathcal{A} \wedge \eta)^k = (n+1)\theta^n \wedge \mathcal{A} \wedge \eta$$

giving $\eta \wedge \theta^n \neq 0$. There exists a Reeb vector field $R \in \Gamma(TM)$ such that $\iota_R \eta = 1$ and $\iota_R \theta = 0$. θ is non-degenerate on $\ker(\eta)$. The generalised complex structure $\mathbb{J}_\omega$ is reduced to a generalised contact structure $(\Phi, e_1, e_2, \lambda)$:

$$\mathbb{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \Rightarrow \quad \Phi = \begin{pmatrix} 0 & -\theta^{-1} \\ \theta & 0 \end{pmatrix}, \quad e_1 = \eta, \quad e_2 = R, \quad \lambda = 0,$$

$$\rho_{\mathbb{J}_\omega} = e^{i\omega} \quad \Rightarrow \quad \varphi = e^{i\theta}, \quad \psi = \eta \wedge e^{i\theta},$$

where $\mathbb{J}_\omega$ is written in the splitting $TP \oplus T^*P$, Φ in the splitting $\ker(\eta) \oplus \mathrm{Ann}(R)$, and $\rho_{\mathbb{J}_\omega}$ is a pure spinor associated to $\mathbb{J}_\omega$.

Let us look at the integrability conditions:

$$d\varphi = id\theta \wedge \varphi, \quad d\psi = d\eta \wedge \varphi + id\theta \wedge \psi.$$

If $d\theta = 0$ then $d_{0,d\eta,0}(\varphi, \psi) = 0$. Noting that $\omega = \theta + \mathcal{A} \wedge \eta$, we see that $d\omega = d\theta + d\mathcal{A} \wedge \eta + \mathcal{A} \wedge d\eta$ is not necessarily zero. So a generalised $(0, d\eta, 0)$-contact structure can arise from a pre-symplectic structure.

Example 4.9 (almost complex to almost contact structure). Consider an almost complex structure $J \in \mathsf{End}(TP)$ on $\mathsf{S}^1 \hookrightarrow P \to M$, where $\dim M = 2n+1$, and the S^1-bundle is specified by the choice of connection $\mathcal{A}$ (given locally by $\mathcal{A}_\alpha = dt_\alpha + A_\alpha \in \Omega^1(P_\alpha)$). Given local coordinates x for M and t for S^1, the almost complex structure is S^1-

invariant if there exists a decomposition

$$J^I_K \partial_I \otimes dx^K = (\phi'^i_k - R^i A_k)\partial_i \otimes dx^k + R^i \partial_i \otimes \mathcal{A} + \alpha_k \kappa \otimes dx^k.$$

Setting $\phi := \phi' - R \otimes A$ the conditions $J^* = -J$ and $J^2 = -1$ give

$$\iota_R \alpha = 1, \quad \Phi(R) = 0 = \phi^*(\alpha), \quad \phi^2(v) = -v + (\iota_v \alpha) R.$$

The generalised almost complex structure $\mathbb{J}_J$ reduces to a generalised almost contact structure $(\Phi, e_1, e_2, \lambda)$:

$$\mathbb{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \quad \Rightarrow \quad \Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}, \quad e_1 = \alpha, \quad e_2 = R, \quad \lambda = 0,$$
$$\rho_J = \Omega_J \quad \Rightarrow \quad \varphi = \Omega_\phi, \quad \psi = \alpha \wedge \Omega_\phi,$$

where $\Omega_J \in \Omega^{2n+2,0}(P)$ is the decomposable top form giving the pure spinor describing $\mathbb{J}_J$ and $\alpha \wedge \Omega_\phi := \Omega_J$.

Let us look at the integrability conditions:

$$d\varphi = d\Omega_\phi, \quad d\psi = d\alpha \wedge \varphi + \alpha \wedge d\Omega_\phi.$$

We require that $d\Omega_\phi = 0$. In this case we have $d_{0,d\alpha,0}(\varphi, \psi) = 0$.

4.3.1 Deformations of generalised contact structures

The (B, b, a)-transformations described in Section 4.1.1 provide deformations of generalised contact structures. While B-transformations have been studied before, the (b, a)-transformations have not been incorporated. The $\mathcal{K}_\pm(\kappa)$ ($\kappa \in \Gamma(T^*M)$) symmetries introduced by Sekiya in [108] are equivalent to the more geometrically natural (b, a)-transformations.

A Sekiya quadruple $(\Phi, e_1, e_2, \lambda)$, can be deformed to give another Sekiya quadruple $(\Phi', e_1', e_2', \lambda')$:

$$\begin{aligned} \Phi'(e) =& e^B \Phi e^{-B}(e) - \langle e, a\rangle \Phi(b) - \langle e, b\rangle \Phi(a) + 2\mu\langle e, a\rangle e_1 + 2\mu \langle e, b\rangle e_2 \\ &+ \langle e^B \Phi(a), e\rangle b - 2\mu\langle e_1, a\rangle\langle e, a\rangle b - 2\mu\langle e_2, a\rangle\langle e, b\rangle b - 2\mu\langle e^B e_2, e\rangle b \qquad (4.23a) \\ &+ \langle e^B \Phi(b), e\rangle a - 2\mu\langle e_2, b\rangle\langle e, b\rangle a - 2\mu\langle e_1, b\rangle\langle e, a\rangle a - 2\mu\langle e^B e_1, e\rangle a. \\ &+ \langle a, \Phi(b)\rangle\langle a, e\rangle b + \langle b, \Phi(a)\rangle\langle b, e\rangle a. \end{aligned}$$

$$\mu' e_1' = \mu e^B e_1 - \lambda' b + \mu\langle e_1, b\rangle a + \mu\langle e_1, a\rangle b - e^B \Phi(b) \qquad (4.23b)$$

$$\mu' e_2' = \mu e^B e_2 + \lambda' a + \mu\langle e_2, a\rangle b + \mu\langle e_2, b\rangle a - e^B \Phi(a) \qquad (4.23c)$$

$$\lambda' = \lambda + 2\langle b, \Phi(a)\rangle + 2\mu\langle e_1, a\rangle - 2\mu\langle e_2, b\rangle \tag{4.23d}$$

where $e = (v, \xi) \in \Gamma(\mathbb{T}M)$.

This follows immediately from considering $e^{(B,b,a)}$ transformation on $\mathbb{J}_{\mathrm{inv}}$, via

$$e^{(B,b,a)}\mathbb{J}_{\mathrm{inv}}e^{(-B,-b,-a)} = \begin{pmatrix} \Phi' & \mu' e_1' & \mu' e_2' \\ -2\mu'\langle e_2', \cdot\rangle & -\lambda' & 0 \\ -2\mu'\langle e_1', \cdot\rangle & 0 & \lambda' \end{pmatrix},$$

where $\mathbb{J}_{\mathrm{inv}}$ is the S^1-invariant generalised complex structure associated to $(\Phi, e_1, e_2, \lambda)$ by (4.18), noting that $e^{(B,b,a)}e^{(-B,-b,-a)}V = V$.

Proposition 4.16. *The transformation $e^{(B,b,a)}$ given by* (4.10) *maps a (H_3, H_2, F)-generalised contact structure $(\Phi, e_1, e_2, \lambda)$ to a (H_3', H_2', F')-generalised contact structure $(\Phi', e_1', e_2', \lambda')$ given by* (4.23)*, where*

$$H_3' = H_3 + dB + a \wedge H_2 + b \wedge F + \tfrac{1}{2}(da \wedge b + a \wedge db),\ H_2' = H_2 + db,\ F' = F + da.$$

Proof. This follows directly from the mixed pair description and the fact that the transformation $e^{(B,b,a)}$ preserves the pairing and (4.22). □

Example 4.10 (Symmetries). Any generalised almost contact structure gives a family of generalised almost contact structures using (B, b, a)-transformations. Deforming Example 4.8 gives

$$\begin{aligned}
&\varphi' = \big(1 + a \wedge \eta - \tfrac{1}{2}a \wedge b\big) \wedge e^{-B+i\theta}, \quad \psi' = \big(\eta - b - \tfrac{1}{2}b \wedge a \wedge \eta\big) \wedge e^{-B+i\theta} \\
&\mu' e_1' = e^B \wedge \eta - \lambda' b + \iota_{\rho^*(b)}\theta^{-1} + \iota_{\iota_{\rho^*(b)}\theta^{-1}}B \\
&\mu' e_2' = R + \iota_R B + \lambda' a + \tfrac{1}{2}(\iota_R a)b + \tfrac{1}{2}(\iota_R b)a + \iota_{\rho(a)}\theta^{-1} + \iota_{\iota_{\rho(a)}\theta^{-1}}B, \\
&\lambda' = \iota_R b - \iota_{\rho^*(b)}\iota_{\rho^*(a)}\theta^{-1},
\end{aligned}$$

where $\rho^*(a) : T^*M \to \mathrm{Ann}(R)$ is the dual anchor combined projection onto the Annihilator of R.

Deforming Example 4.9 we get

$$\begin{aligned}
&\varphi' = \big(1 + a \wedge \alpha - \tfrac{1}{2}a \wedge b\big) \wedge e^{-B} \wedge \Omega_\phi, \quad \psi' = \big(\alpha - b - \tfrac{1}{2}b \wedge a\big) \wedge e^{-B} \wedge \Omega_\phi, \\
&\mu' e_1' = \alpha + \lambda' a + \phi^*(\rho(a)), \\
&\mu' e_2' = R + \iota_R B - \lambda' b + \tfrac{1}{2}(\iota_R b)a + \tfrac{1}{2}(\iota_R a)b + \phi^*(\rho(b)), \\
&\lambda' = \iota_R a.
\end{aligned}$$

Remark. These examples show that (b, a)-transformations can change λ. The (b, a)-

transformation can be interpreted geometrically as twisting the S^1-bundle. The correspondence between a-transformations and twisting comes from the discussion preceding Definition 4.10. The description following Defintion 4.7 shows that the splitting of invariant sections of TP and T^*P correspond to an a-transformation and a dual b-transformation.

Example 4.11 (Products [52]). Let $M = M_1 \times M_2$ with projections $pr_i : M \to M_i$. If (L_1, e_1, e_2) is a generalised almost contact structure on M_1 and L_2 is a generalised almost complex structure on M_2, then $(pr_1^* L_1 \oplus pr_2^* L_2, pr_1^* e_1, pr_1^* e_2)$ is a generalised almost contact structure on M.

There are manifolds which admit generalised contact structures but not contact structures. A class of examples come from S^1-bundles of nilmanifolds. A nilmanifold is a homogeneous space $M = \mathsf{G}/\Gamma$, where G is a simply connected nilpotent real Lie group and Γ is a lattice of maximal rank in G. For the associated generalised complex structures on nilmanifolds see [29]. The structure of a particular nilpotent Lie algebra can be given by specified by listing exterior derivatives of the elements of a Malcev basis, as an n-tuple of two-forms $d\epsilon_k = \sum c_k^{ij} \epsilon_i \epsilon_j$, (henceforth $\wedge$ is omitted, so that $\epsilon_i \wedge \epsilon_j = \epsilon_i \epsilon_j$).

Example 4.12 ($(0,0,12,13,14+23,34+52) \times \mathsf{S}^1$). Specify a 6-dimensional nilmanifold via the coframe $\{\epsilon_i\}$, $i = 1, \ldots, 6$ satisfying:

$$d\epsilon_1 = 0,\ d\epsilon_2 = 0,\ d\epsilon_3 = \epsilon_1\epsilon_2,\ d\epsilon_4 = \epsilon_1\epsilon_3,\ d\epsilon_5 = \epsilon_1\epsilon_4 + \epsilon_2\epsilon_3,\ d\epsilon_6 = \epsilon_3\epsilon_4 + \epsilon_5\epsilon_2.$$

Let $E = M \times \mathsf{S}^1$, where M is the nilmanifold specified by $(0,0,12,13,14+23,34+52)$, and S^1 is parameterised by t. The one-form dt gives a flat connection on S^1. Define $\eta = \pi^* dt$ where $\pi : M \times \mathsf{S}^1 \to M$ is the projection. Let $R = \pi^* \partial_t$ be the corresponding Reeb vector field. The generalised almost contact structure is given by

$$\begin{aligned}
\Omega =& \epsilon_1 + i\epsilon_2,\\
B =& \epsilon_2\epsilon_6 - \epsilon_3\epsilon_5 + \epsilon_3\epsilon_6 - \epsilon_4\epsilon_5,\\
\omega =& \epsilon_3\epsilon_6 + \epsilon_4\epsilon_5,\\
\varphi =& e^{B+i\omega}\Omega, \quad \psi = \eta e^{B+i\omega}\Omega, \quad e_1 = \eta,\ e_2 = R,\ \lambda = 0.
\end{aligned}$$

Example 4.13 (S^1-bundles on nilmanifolds). There are manifolds which have no symplectic or complex structures but do have generalised complex structures. In [29] generalised complex structures are constructed on nilmanifolds which do not admit symplectic or complex structures. Each of these examples define a generalised complex structure via a pure spinor $\rho = \Omega \wedge e^{B+i\omega}$. This construction can be modified to find generalised contact structures which do not admit contact structures. Take

$\mathsf{S}^1 \hookrightarrow E \to M$. Choose an S^1-invariant connection $\mathcal{A}$. Define a vector field R such that $\iota_R \mathcal{A} = 1$. Take the generalised complex structure described by the pure spinor $\rho = \Omega \wedge e^{i\omega + B}$. The corresponding mixed pair is

$$\varphi = e^{B+i\omega}\Omega, \quad \psi = \mathcal{A}e^{B+i\omega}\Omega, \quad e_1 = \mathcal{A},\ e_2 = \kappa,\ \lambda = 0.$$

Example 4.14. Consider $\mathbb{R}^5$, described using coordinates $\{t, z_1, z_2\}$ where z_1, z_2 are standard coordinates in $\mathbb{C}^2 \cong \mathbb{R}^4$. A generalised complex structure is defined by the pure spinor $\rho = z_1 + dz_1 dz_2$. When $z_1 = 0$, $\varphi = dz_1 dz_2$ defines a standard complex structure, whereas $z_1 \neq 0$, φ defines a B-symplectic structure since $\rho = z_1 \exp(dz_1 dz_2 / z_1)$. A generalised contact structure is given by

$$\varphi = z_1 + dz_1 dz_2, \quad \psi = dt(z_1 + dz_1 dz_2), \quad e_1 = dt,\ e_2 = \partial_t.$$

4.4 Generalised coKähler geometry

Generalised geometric structures are of great interest in string theory due to the fact that T-duality is associated to $\mathfrak{so}(T \oplus T^*) = \mathsf{End}(T) \oplus \wedge^2 T^* \oplus \wedge^2 T$. The generalised metric incorporates the Riemannian metric G and B-field associated with the Neveu–Schwarz flux H in the bosonic sector of supergravity. Generalised Kähler structures are equivalent to bi-hermitian structures and are the most general geometry associated to two-dimensional target space models with $\mathcal{N} = (2, 2)$ supersymmetry [49].

CoKähler structures are the odd-dimensional counterpart to Kähler structures. The relationship between Kähler and coKähler structures is described in [93, 11]. Li gave a structure result for compact coKähler manifolds stating that such a manifold is always a Kähler mapping torus. The coKähler structure on an odd-dimensional manifold M can be associated to a Kähler structure on an S^1-bundle (using a symplectomorphism) [93]. Further results on coKähler structures were given in [11].

Generalised coKähler structures have appeared in the literature before [53]. The definition given in [53] deals with generalised Kähler structures on $M_1 \times M_2$, and the definition is compatible with B-transformations. In this section we will consider the case where $M_2 = \mathsf{S}^1$, but will not restrict to product manifolds, instead considering principal circle bundles. The definition is compatible with the full (B, b, a)-transformations. Generalised coKähler structures will be presented as S^1-invariant reductions of generalised Kähler structures.

Remark. Generalised Kähler structures play an important role in string theory. In [49] generalised Kähler structures (written as a bi-Hermitian structures) appear in the study of $\mathcal{N} = (2, 2)$ non-linear sigma models with torsion. The torsion arises from the connections $\nabla^{\pm} = \nabla^{\mathrm{LC}} \pm g^{-1}H$, where ∇^{LC} is the Levi-Civita connection. Abelian

T-duality can be carried out when the metric has an S^1-isometry, and the T-duality procedure involves Kaluza–Klein reduction. T-duality is most interesting when the S^1-isometry corresponds to a topologically non-trivial S^1-bundle. In this case there is an interesting relationship between topology and H-flux [17, 18]. The study of S^1-reductions of generalised Kähler structures is interesting in this context.

4.4.1 Generalised metric structure

The inner product (4.8b) is non-degenerate and a generalised contact metric can be constructed using maximally isotropic subspaces

$$\mathbb{G}(V_1, V_2) = \langle V_1, V_2\rangle|_{C_+} - \langle V_1, V_2\rangle|_{C_-},$$

where $V_1, V_2 \in \Gamma(E) \cong \Gamma(TM) \oplus C^\infty(M) \oplus C^\infty(M) \oplus \Gamma(T^*M)$; mirroring the case of generalised geometry on $\mathbb{T}M$ described in Section 4.1.2. In the present case we have

$$C_\pm = \{(v, f, g, \xi) \in \Gamma(E) : g = \pm fh^2,\ \xi = \pm G(v, \cdot)\} \tag{4.24}$$

for some $h \in C^\infty(M)$ and Riemannian metric G. This satisfies

$$\langle (v, f, \pm fh^2, \pm G(v, \cdot)), (v, f, \pm fh^2, \pm G(v, \cdot))\rangle = \pm G(v, v) \pm f^2h^2,$$

verifying that $C_\pm$ describe the maximal positive/negative definite subbundles.

As $\langle\cdot,\cdot\rangle$ is invariant under (B, b, a)-transformations the subbundles $C_\pm$, defining a generalised metric $\mathbb{G}$, can be transformed to $e^{(B,b,a)}C_\pm$ defining a generalised metric $\mathbb{G}' = e^{(B,b,a)}\mathbb{G}e^{-(B,b,a)}$. The maximal subspaces are given by

$$\begin{aligned} C_\pm = \{(v, \xi, f, g) :& \xi = \pm G(v, \cdot) + B(v, \cdot) - fb - fh^2a - 2\langle v, b\rangle a, \\ & g = \pm fh^2 + 2\langle v, b\rangle + 2h^2\langle v, a\rangle + \langle v, b\rangle\langle v, a\rangle\}. \end{aligned} \tag{4.25}$$

All subspaces $C_\pm$ can be described in the form (4.25) for some choice of (G, h, B, b, a).

Definition 4.17. A *generalised coKähler structure* on an odd-dimensional manifold M, consists of two generalised (H_3, H_2, F)-contact structures $(L_1, e_1^{(1)}, e_2^{(1)}, \lambda_1)$ and $(L_2, e_1^{(2)}, e_2^{(2)}, \lambda_2)$ whose associated Sekiya quadruples $\mathbb{J}_1 = \{\Phi_1, e_1^{(1)}, e_2^{(1)}, \lambda_1\}$ and $\mathbb{J}_2 = \{\Phi_2, e_1^{(2)}, e_2^{(2)}, \lambda_2\}$ give a generalised Kähler structure.

The commuting condition $\mathbb{J}_1\mathbb{J}_2 = \mathbb{J}_2\mathbb{J}_1$ places the restrictions on the Sekiya quadruples $(\Phi_1, e_1^{(1)}, e_2^{(1)}, \lambda_1)$ and $(\Phi_2, e_1^{(2)}, e_2^{(2)}, \lambda_2)$. The Sekiya quadruples are required to satisfy

$$\mathbb{C}e_1^{(1)} \oplus \mathbb{C}e_2^{(1)} = \mathbb{C}e_1^{(2)} \oplus \mathbb{C}e_2^{(2)} := \mathcal{E};$$

$$\Phi_1\Phi_2(v) = \Phi_2\Phi_1(v) \quad \forall v \in \mathcal{E}^\perp$$
$$e_1^{(1)} = e_1^{(2)},\ e_2^{(1)} = e_2^{(2)},\ \text{when } \lambda_1 = \lambda_2 \quad \text{or} \quad e_1^{(1)} = e_2^{(2)},\ e_2^{(1)} = e_1^{(2)},\ \forall \lambda_1, \lambda_2 \in \mathbb{R}.$$

Remark. If $\lambda_1 = \lambda_2 = 0$ then there is a $\mathsf{O}(1,1)$-freedom in the description and it is possible to choose $e_1^{(1)} = e_2^{(2)}$ and $e_2^{(1)} = e_1^{(2)}$.

Example 4.15 (coKähler). A *coKähler structure* on an odd-dimensional manifold M, is given by the quadruple (J, R, η, G), where (J, R, η) is an almost contact structure and G is a Riemannian metric satisfying $G(Jv_1, Jv_2) = G(v_1, v_2) - \eta(v_1)\eta(v_2)$ for all $v_1, v_2 \in \Gamma(TM)$. The integrability conditions are $[J, J] = 0$, and $d\omega = d\eta = 0$, where $\omega(v_1, v_2) := G(Jv_1, v_2) \in \Omega^2(M)$. This defines a generalised coKähler structure (with $\lambda = H_3 = H_2 = F = 0$):

$$\varphi_1 = e^{i\omega},\ \psi_1 = \eta \wedge e^{i\omega},\ e_1^{(1)} = \eta,\ e_2^{(1)} = R, \quad \varphi_2 = \Omega_J,\ \psi_2 = \eta \wedge \Omega_J,\ e_1^{(2)} = \eta,\ e_2^{(2)} = R.$$

Example 4.16 (Generalised Kähler to generalised coKähler). We will reduce a generalised Kähler structure to produce a generalised coKähler structure. It was shown in Example 4.8 (Example 4.9) that the reduction of a symplectic (complex) structure (over the same S^1-bundle) gives

$$\mathbb{J}_{\omega_{\mathrm{inv}}} = \begin{pmatrix} 0 & -\omega^{-1} & 0 & \eta \\ \omega & 0 & R & 0 \\ 0 & -2\langle \eta, \cdot \rangle & 0 & 0 \\ -2\langle R, \cdot \rangle & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{J}_{J_{\mathrm{inv}}} = \begin{pmatrix} -J & 0 & -R & 0 \\ 0 & J^* & 0 & \eta \\ 2\langle R, \cdot \rangle & 0 & 0 & 0 \\ 0 & -2\langle \eta, \cdot \rangle & 0 & 0 \end{pmatrix},$$

where ω and J are non-degenerate on D. The condition that $-\mathbb{J}_{\omega_{\mathrm{inv}}}\mathbb{J}_{J_{\mathrm{inv}}} = \mathbb{G}$ for

$$\mathbb{G} = \begin{pmatrix} 0 & G^{-1} & 0 & 0 \\ G & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

requires that $\omega(Jv_1, v_2) = G(v_1, v_2)$, giving a transverse Kähler structure, $\langle R, \eta \rangle = \frac{1}{2}$, $\langle R, R \rangle = 0 = \langle \eta, \eta \rangle$.

The almost generalised complex structures $\mathbb{J}_{\omega_{\mathrm{inv}}}$ and $\mathbb{J}_{J_{\mathrm{inv}}}$ will define a generalised coKähler structure when $H_2 = d\alpha = d\eta$ (see Examples 4.8 and 4.9 for notation).

Example 4.17 (Twisted generalised coKähler). It is clear from Example 4.16 that the reduction of a generalised Kähler structure can produce a generalised coKähler structure. It is possible to deform any generalised coKähler structure $(\mathbb{J}_{\omega_{\mathrm{inv}}}, \mathbb{J}_{\Omega_{\mathrm{inv}}}, \mathbb{G})$ to get another:

$$(e^{(B,b,a)}\mathbb{J}_{J_{\mathrm{inv}}}e^{-(B,b,a)}, e^{(B,b,a)}\mathbb{J}_{\omega_{\mathrm{inv}}}e^{-(B,b,a)}, e^{(B,b,a)}\mathbb{G}e^{-(B,b,a)}).$$

Definition 4.18. A *generalised almost coKähler–Einstein structure* on an odd-dimensional manifold M (with $m = \dim(M)$) is described by two mixed pairs (φ_1, ψ_1) and (φ_2, ψ_2) satisfying

$$((\varphi_1, \psi_1), (\bar{\varphi}_1, \bar{\psi}_1))_M = c((\varphi_2, \psi_2), (\bar{\varphi}_2, \bar{\psi}_2))_M,$$

where $c \in \mathbb{R}$ can be scaled to $+1$ or -1 by scaling (φ_1, ψ_1). A generalised almost coKähler–Einstein structure is a generalised coKähler–Einstein structure if $(\varphi_1, \psi_1, e_1^{(1)}, e_2^{(1)})$ and $(\varphi_2, \psi_2, e_1^{(2)}, e_2^{(2)})$ are generalised (H_3, H_2, F)-contact structures.

Example 4.18 (S^1-invariant generalised Calabi–Yau). Let $N = M \times \mathsf{S}^1$ be an even dimensional manifold with an S^1-invariant generalised Calabi–Yau structure (ρ_1, ρ_2). The decompositions $\rho_j = \varphi_j + idt\psi_j$ $(j = 1, 2)$ defines a generalised coKähler–Einstein structure: $(\varphi_1, \psi_1, e_1^{(1)} = \partial_t, e_2^{(1)} = dt)$ and $(\varphi_2, \psi_2, e_1^{(2)} = \partial_t, e_2^{(2)} = dt)$, where $\lambda = H_3 = H_2 = F = 0$.

Example 4.19 (coKähler–Einstein). A coKähler–Einstein structure on an odd-dimensional manifold M is a Ricci-flat coKähler structure. A coKähler structure has an associated cosymplectic structure (η, θ). Consider $N = M \times \mathsf{S}^1$, with S^1 parameterised by t, and $pr_1(N) = M$. Let $\omega = dt \wedge pr_1^*\eta + pr_1^*\theta$, and $G_N = pr_1^*G + (dt)^2$. This defines a Calabi–Yau structure on N. A Calabi–Yau structure defines a generalised Calabi–Yau structure (Example 4.6). Using the reduction procedure (Example 4.18) we get a generalised $(0, 0, 0)$-coKähler–Einstein structure.

Example 4.20. A (B, b, a)-transformation maps an involutive mixed pair to another involutive mixed pair, preserving the length. It follows that a generalised coKähler(–Einstein) structure, $((\varphi_1, \psi_1), (\varphi_2, \psi_2))$, is mapped to another coKähler(-Einstein) structure by a (B, b, a)-transformation, $((e^{(B,b,a)}\varphi_1, e^{(B,b,a)}\psi_1), (e^{(B,b,a)}\varphi_2, e^{(B,b,a)}\psi_2))$. A generalised (H_3, H_2, F)-contact structure is mapped to generalised $(H_3 + dB, H_2 + db, F + da)$-contact structure.

4.5 T-duality

T-duality provides an isomorphism between Courant algebroids defined on two torus bundles $\mathsf{T}^k \hookrightarrow E \to M$ and $\widehat{\mathsf{T}}^k \hookrightarrow \widehat{E} \to M$. The topological description of T-duality is described in [17, 18]. The isomorphism of Courant algebroid structures in [30]. The situation is described by the following diagram:

$$\begin{tikzcd}
 & E \times_M \widehat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\
E \arrow[dr, "\pi"'] & & \widehat{E} \arrow[dl, "\hat{\pi}"] \\
 & M &
\end{tikzcd}$$

where

$$\begin{tikzcd}
\mathsf{T}^k \arrow[r] & E \arrow[d, "\pi"] \\
 & M
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\widehat{\mathsf{T}}^k \arrow[r] & \widehat{E} \arrow[d, "\hat{\pi}"] \\
 & M
\end{tikzcd} .$$

A torus T^k can be viewed as an abelian group with lie algebra $\mathfrak{t}$. Principal torus bundles $E(M, \pi, \mathsf{T}^k)$ are classified by $H^2(M, \mathbb{Z}^k)$. A representative can be found by choosing a principal torus connection $\mathcal{A} \in \Omega^1(M, \mathfrak{t})$ and taking the curvature $F = d\mathcal{A} \in \Omega^2(M, \mathfrak{t})$. A choice of connection determines an isomorphism of T^k invariant vector fields on E: $TE/\mathsf{T}^k \cong TM \oplus \mathfrak{t}$ (see Example 3.10). The isomorphism allows us to identify the curvature with a T^k invariant two-form $F \in \Omega^2_{\mathsf{T}^k}(E)$ (in fact $F \in \Omega^2(M)$). We can similarly identify $\widehat{F} \in \Omega^2_{\widehat{\mathsf{T}}^k}(\widehat{E})$. The relevant fluxes are $H \in \Omega^3_{\mathsf{T}^k}(E)$ and $\widehat{H} \in \Omega^3_{\widehat{\mathsf{T}}^k}(\widehat{E})$.

The bundles E and $\widehat{E}$ are *T-dual* if

$$\pi_* H = \widehat{F}, \quad \hat{\pi}_* \widehat{H} = F, \quad p^* H - \hat{p}^* \widehat{H} = d\mathcal{F},$$

for some T^{2k}-invariant 2-form on the correspondence space $\mathcal{F} \in \Omega^2(E \times_M \widehat{E})$ such that $\mathcal{F} : \mathfrak{t} \otimes \hat{\mathfrak{t}} \to \mathbb{R}$ is non-degenerate.

H is *admissible* if it satisfies [30]:

$$H(X_1, X_2, \cdot\,) = 0, \quad \forall X_1, X_2 \in \mathfrak{t} \in E.$$

The requirement that H is admissible ensures that the T-dual bundle $\widehat{E}$ is in fact a principal torus bundle. If H is not admissible the T-dual is not a principal torus bundle, although it may still admit an interpretation in terms of a non-commutative/non-associative space [98, 19]. We will assume that H and $\widehat{H}$ are admissible.

If we have admissible T-dual pairs (H, F) and $(\widehat{H}, \widehat{F})$ we have the decomposition

$$H = \widehat{F} \wedge \mathcal{A} + h_3, \quad \widehat{H} = F \wedge \widehat{\mathcal{A}} + h_3,$$

where $h_3 \in \Omega^3(M)$.

The T-duality map $\tau_{\mathcal{F}} : (\Omega^\bullet_{\mathsf{T}^k}(M), d_H) \to (\Omega^\bullet_{\widehat{\mathsf{T}}^k}(\widehat{M}), d_{\widehat{H}})$ gives an isomorphism of chain complexes and is defined by the formula

$$\tau_{\mathcal{F}}\omega := \int_{\widehat{\mathsf{T}}^k} e^{\mathcal{F}} p^* \omega,$$

for T^k-invariant differential forms $\omega \in \Omega^\bullet_{\mathsf{T}^k}(M)$. In that case that $k = 1$, the T-duality map gives an isomorphism of mixed pairs:

$$\tau_{\mathcal{F}}(\varphi + i\mathcal{A}\psi) = \int_{\widehat{\mathsf{S}}^1} e^{\mathcal{F}} p^*(\varphi + i\mathcal{A}\psi) = \hat{\varphi} + i\widehat{\mathcal{A}}\hat{\psi},$$

where (φ, ψ) is a mixed pair, and $\mathcal{F} = -\mathcal{A}\widehat{\mathcal{A}}$. The map $\tau_{\mathcal{F}}$ can be seen as the combination of a pullback from E to the correspondence space $E \times_M \widehat{E}$, followed by a B-transformation $e^{\mathcal{F}}$, and then the pushforward to $\widehat{E}$. This can be viewed as a type of geometric Fourier transform.

The complexes $(\Omega^\bullet_{\mathsf{T}^k}(M), d_H)$ and $(\Omega^\bullet_{\widehat{\mathsf{T}}^k}(\widehat{M}), d_{\widehat{H}})$ determine the T-dual Courant algebroids $\circ_H$ and $\circ_{\widehat{H}}$. The exact Courant algebroid $\circ_H$ can be viewed as a derived bracket on $\mathbb{T}M$ generated by the twisted differential $d_H := d + H\wedge$ (for details see [86, 9]). The T-dual Courant algebroid $\circ_{\widehat{H}}$ is a derived bracket on $\mathbb{T}\widehat{M}$ generated by $d_{\widehat{H}}$.

The description of T-duality for generalised (almost) contact structures on the trivial bundle $E = M \times \mathbb{R}$ is given in [2].

Given the interpretation of generalised contact structures as S^1-reduced generalised complex structures, T^k-duality of generalised contact structures is T^{k+1}-duality of the corresponding generalised complex structure. T-duality for a circle bundle is considered as an example in [30]. The killing vector generates an S^1-foliation and—considering S^1-invariant fields—the Courant bracket (4.2a) is reduced to (4.8a). T-duality corresponds to the interchange $(F, f) \leftrightarrow (H_2, g)$. Contact geometry corresponds to an extra S^1-invariant reduction without the interchange and pushforward.

T-duality in the cone direction, denoted t, is considered in [2] . In this case the mixed pair (φ, ψ) is mapped to the mixed pair (ψ, φ). A b-transformation is interpreted as a change in connection for the S^1-bundle defining the generalised contact structure. An a-transformation corresponds to a choice of connection in the T-dual direction.

Proposition 4.19. *T-duality maps a generalised coKähler(–Einstein) structure to another generalised coKähler(–Einstein) structure.*

Proof. T-duality preserves the pairing, and maps a mixed pair to another mixed pair. □

4.6 Contact line bundles versus reduction

It has recently been shown that generalised contact geometry has a conceptually nice description as generalised geometry on the generalised derivation bundle $\mathbb{D}L \cong \mathfrak{D}L \oplus \mathfrak{J}^1 L$ for a (possibly non-trivial) line bundle L [122, 123]. This section briefly outlines the description and relates this to the current chapter. A generalised contact structure viewed as a reduced generalised complex structure $\mathcal{J}|_{\mathsf{S}^1}$ is the S^1-bundle version of the generalised complex structure $\mathcal{I} \in \mathsf{End}(\mathbb{D}L)$. A mixed pair (φ, ψ) is associated to a pure spinor $\varpi \in \Gamma(\wedge^\bullet \mathfrak{J}^1 L, L)$.

Many interesting examples of contact structures are in fact non-coorientable and not defined by a globally defined contact one-form. Contact structures are determined by a line bundle $L = TM/D$ (as described in Section 4.3). It is of interest to have a formalism that allows the description of non-trivial line bundles while making the symmetries explicit.

The description of generalised contact bundles is given via the Atiyah (or gauge) algebroid, defined on $\mathbb{D}L = \mathfrak{D}L \oplus \mathfrak{J}^1 L$, where sections of $\mathfrak{D}L$ are derivations of L, and $\mathfrak{J}^1 L$ is the first jet bundle of L. A derivation $\nabla \in \mathfrak{D}E$ has a unique *s*ymbol $\sigma : \mathfrak{D}E \to TM$ such that, for $f \in C^\infty(M)$ and $\lambda \in \Gamma(E)$,

$$\nabla(f\lambda) = (\sigma\nabla)(f)\lambda + f\nabla\lambda = X(f)\lambda + f\nabla\lambda,$$

where $X = \sigma(\nabla)$. This makes it clear that $\mathfrak{D}E$ is part of the exact sequence

$$0 \longrightarrow \mathfrak{gl}(E) \longrightarrow \mathfrak{D}E \xrightarrow{\sigma} TM \longrightarrow 0 .$$

There is a natural Lie algebroid structure associated with $\mathfrak{D}E$: the Lie bracket is given by the commutator of derivations and the anchor given by σ. In the case of a line bundle the induced map on sections gives:

$$0 \longrightarrow \Gamma(\mathfrak{gl}(L)) \cong C^\infty(M) \longrightarrow \Gamma(\mathfrak{D}L) \xrightarrow{\sigma} \Gamma(TM) \longrightarrow 0 . \tag{4.26}$$

If the bundle $\mathfrak{D}L$ is trivial there is an isomorphism $\Gamma(\mathfrak{D}L) \cong \Gamma(TM) \oplus C^\infty(M)$. If $\mathfrak{D}$ is a non-trivial bundle there are local isomorphisms $\Gamma(\mathfrak{D}L|_{U_\alpha}) \cong \Gamma(TU_\alpha) \oplus C^\infty(U_\alpha)$ patched globally using transition function (in a manner analogous $\mathsf{U}(1)$-bundles described in Section 2.2.2). Sections $(v, f) \in \Gamma(TM) \oplus C^\infty(M)$ are the line bundle version of the S^1-invariant sections of TP for $\mathsf{S}^1 \hookrightarrow P \to M$.

The 1-jet bundle $\mathfrak{J}^1 E$ can be defined, at a point $p \in M$, by the equivalence relation in $\Gamma(E)$:

$$e_1 \sim e_2 \leftrightarrow e_1(p) = e_2(p), \quad d\langle e_1, \zeta\rangle = d\langle e_2, \zeta\rangle, \quad \forall \zeta \in \Gamma(E^*).$$

There exists $\mathbb{p} : \mathfrak{J}^1 E \to E$, such that $\ker(\mathbb{p}) \cong \mathsf{Hom}(TM, E)$, giving

$$0 \longrightarrow \mathsf{Hom}(TM, E) \longrightarrow \mathfrak{J}^1 E \xrightarrow{\mathbb{p}} E \longrightarrow 0 \ .$$

In the case of a line bundle the induced map on sections gives:

$$0 \longrightarrow \Gamma(\mathsf{Hom}(TM, L)) \cong \Gamma(T^*M) \longrightarrow \Gamma(\mathfrak{J}^1 L) \longrightarrow \Gamma(L) \cong C^\infty(M) \longrightarrow 0 \ . \tag{4.27}$$

If the bundle $\mathfrak{J}^1 L$ is trivial there is an isomorphism $\Gamma(\mathfrak{J}^1 L) \cong \Gamma(T^*M) \oplus C^\infty(M)$. Sections $(\xi, g) \in \Gamma(T^*M) \oplus C^\infty(M)$ are the line bundle version of the S^1-invariant sections of T^*P.

It is shown in [36] that $\mathfrak{D}E$ is E-dual to $\mathfrak{J}^1 E$, there is a non-degenerate E-valued pairing $\langle\cdot,\cdot\rangle_E : \mathfrak{D}E \times \mathfrak{J}^1 E \to E$. For sections $\nabla \in \Gamma(\mathfrak{D}E)$ and $\chi = \sum f j^1 e \in \Gamma(\mathfrak{J}^1 E)$ the pairing is given by $\langle \nabla, \chi\rangle_E = \sum f \nabla(e)$. The pairing between $\mathfrak{D}E$ and $\mathfrak{J}^1 E$ has a geometric interpretation: $\langle \nabla, \chi\rangle_E$ can be viewed as the covariant derivation of χ with respect to ∇.

Given the E-valued pairing between $\mathfrak{D}E$ and $\mathfrak{J}^1 E$ there is a natural E-Courant (more specifically an omni-Lie) algebroid defined on the generalised derivation bundle $\mathbb{D}E$ given by:

$$0 \longrightarrow \mathfrak{J}^1 E \longrightarrow \mathbb{D}E \xrightarrow{\rho} \mathfrak{D}E \longrightarrow 0 \ .$$

The definition and properties of omni-Lie and E-Courant algebroids can be found in [36] and [37] respectively. The E-Courant algebroid can be viewed as a derived bracket for the differential $d_{\mathfrak{D}E}$ acting on the complex $\Omega^k_E := \Gamma(\wedge^k \mathfrak{J}^1 E, E)$:

$$\begin{aligned} d_{\mathfrak{D}E}\varpi(\nabla_0, \nabla_1, \ldots, \nabla_k) =& \sum_{i=0}^{k} (-1)^i \nabla_i(\varpi(\nabla_0, \ldots, \hat{\nabla}_i, \ldots, \nabla_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \varpi([\nabla_i, \nabla_j], \nabla_0, \ldots, \hat{\nabla}_i, \ldots, \hat{\nabla}_j, \ldots, \nabla_k), \end{aligned}$$

for $\nabla_i \in \Gamma(\mathfrak{D}E)$, $\varpi \in \Omega^k_E$, and $\hat{\cdot}$ denoting omission. The action of $\mathfrak{L}_\nabla \varpi := d_{\mathfrak{D}E}\iota_\nabla \varpi + \iota_\nabla d_{\mathfrak{D}E}\varpi$ gives a Lie derivative on $\Gamma(\wedge^\bullet \mathfrak{J}^1 E, E)$ satisfying an analogue of the Cartan relations.

From this construction the omni-Lie algebroid on a line bundle $L \to M$ is given as:

$$
\begin{aligned}
(\nabla_1, \psi_1) \circ^L (\nabla_2, \psi_2) =& \Big([\nabla_1, \nabla_2], \mathfrak{L}_{\nabla_1} \psi_2 - \iota_{\nabla_2} d_{\mathfrak{D}L} \psi_1 \Big); && (4.28a) \\
\langle\langle (\nabla_1, \psi_1), (\nabla_2, \psi_2) \rangle\rangle =& \langle \nabla_1, \psi_2 \rangle_L + \langle \nabla_2, \psi_1 \rangle_L; && (4.28b) \\
\rho(\nabla, \psi) =& \nabla, && (4.28c)
\end{aligned}
$$

for $\nabla \in \Gamma(\mathfrak{D}L)$ and $\psi \in \Gamma(\mathfrak{J}^1 L)$. The bracket (4.28a) can be identified with (4.8a) when $H_3 = 0$. In the case of a trivial line bundle, $H_2 = F = 0$, this has already been noted [122]. If the line bundle is non-trivial then $F = H_2$ is given by the curvature of a connection specifying the bundle.

Having identified the Courant algebroids (4.28a) with (4.8a) the identification of generalised contact structures as generalised complex structures is straightforward. The generalised complex structure $\mathcal{J}|_{\mathsf{S}^1} \in \mathsf{End}(TP \oplus T^*P)$ can be identified with $\mathcal{I} \in \mathsf{End}(\mathbb{D}L)$ (satisfying $\mathcal{I}^2 = -\mathrm{Id}$ and $\mathcal{I}^* = -\mathcal{I}$) by splitting the sequences (4.26) and (4.27). The generalised complex structure $\mathcal{I}$ is identified with a Dirac structure $L_{\mathcal{I}} \subset \mathbb{D}L$ and described by a pure spinor $\varpi \in \Gamma(\wedge^{\bullet} \mathfrak{J}^1 L, L)$. A choice of decomposition $\Gamma(\mathfrak{J}^1 L) = C^{\infty}(M) \oplus \Gamma(T^*M)$ coming from (4.27), induces a decomposition $\varpi \in \Gamma(\wedge^{\bullet} \mathfrak{J}^1 L, L)$ into a mixed pair $(\varphi, \psi) \in \Gamma(\wedge^{\bullet} T^*M) \oplus \Gamma(\wedge^{\bullet-1} T^*M)$.

Chapter 5

Lie algebroid gauging of non-linear sigma models

This chapter describes the underlying Lie algebroid geometry associated to the non-isometric gauging proposal by Kotov and Strobl [114, 88, 89, 99]. The main results are Theorem 5.4 and Theorem 5.8 which give the necessary and sufficient conditions for carrying out the non-isometric gauging procedure with a particular choice of vector fields. Corollary 5.6 states that it is always possible to locally gauge an action non-isometrically. Lie algebroid structures underpin the non-isometric gauging procedure. This chapter discusses the integrability of the Lie algebroid action to a Lie groupoid action—something that has not appeared in the Lie algebroid gauging literature. In contrast to the isometric case, non-isometric gauging does not necessarily represent an underlying symmetry with a Noether charge. Applications of gauging (such as T-duality or Yang–Mills theory) involve adding a field strength term to the action. The existence of a field strength with desirable gauge transformation properties provides an obstruction to gauging.

An interesting application of Lie algebroid gauging is non-isometric T-duality. A proposal for non-isometric T-duality was given by Chatzistavrakidis, Deser, and Jonke (CDJ) [31, 33]. The existence of a gauge invariant field strength term in the action gives Theorem 5.9: The non-isometric T-duality proposal is in fact equivalent to the standard non-abelian T-duality procedure through a change of Lie algebroid frame.

5.1 Non-linear sigma models

Non-linear sigma models play an important role in the study of physical theories. This chapter will consider non-linear sigma models which describe the motion of closed strings in a fixed background. A study of the massless sector of bosonic string theory gives two rank two-tensors, G which is symmetric, and B which is skew-symmetric;

there is an additional scalar field called the dilaton. The field G is a Riemannian metric describing the geometry of the fixed background in which the string propagates. The field B is the Kalb–Ramond field and is the stringy analogue of electromagnetic potential. The dilaton will play no role in Lie algebroid gauging and will be omitted for convenience.

A two-dimensional non-linear sigma model consists of the data $(X, \Sigma, h, M, G, B, S[X])$: where $X : \Sigma \to M$ describes the embedding of a two-dimensional (pseudo-)Riemannian surface (Σ, h) (the string worldsheet) in an n-dimensional (pseudo-)Riemannian manifold (M, G) (the target space). The dynamics of the string are encoded in an action

$$S[X] = \frac{1}{2} \int_{\Sigma} X^* G_{\mu\nu} dX^\mu \wedge \star dX^\nu + X^* B_{\mu\nu} dX^\mu \wedge dX^\nu,$$

where $\star$ is the Hodge star on the worldsheet, and $B \in \Omega^2(M)$.[1] The equations of motion for the string are given by the Euler–Lagrange equations specifying the stationary points of $S[X]$.

Symmetries of the physical theory are encoded in symmetries of the action $S[X]$. A non-linear sigma model has a smooth symmetry group G (a Lie group) if the action of $h \in \mathsf{G}$ on the fields X satisfy $S[h \cdot X] = S[X]$.[2] The continuous symmetries are diffeomorphisms and the infinitesimal symmetries are generated by vector fields. There exist a set of right-invariant vector fields which can be associated to $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$. The set of right-invariant vector fields form a Lie algebra $\mathfrak{g}$, with the bracket given by the commutator of vector fields, and a choice of a linearly independent spanning set of right-invariant vector fields give a frame for the Lie algebra $\mathfrak{g}$.

Symmetries play an important role in the study of physical theories. For every continuous symmetry Noether's theorem tells us there is a conserved quantity. Suppose we have some $S[X]$ which is invariant under the action of a fixed group element $h \in \mathsf{G}$. It is possible to introduce a field $A \in \Omega^1(\Sigma, \mathfrak{g})$ (which transforms in a particular way) to produce an action $S[X, A]$ which is invariant under any position dependent element $h \in C^\infty(\Sigma, \mathsf{G})$. This procedure is called 'gauging' the action and will be described in detail for the case of a group manifold in Section 5.2. The gauging procedure originated in particle physics to describe Yang–Mills theories. The field A is called a gauge field and describes the mediation of forces between particles. In the case of electromagnetism, A is the four-potential. From the mathematical perspective the gauging procedure is associated to lifting the action $S[X]$ to the total space of the associated principal G-bundle (see for example [12]).

The focus of this Chapter is generalising the gauging of non-linear sigma model

[1] Technically B is not required to be a globally defined two-form, and should really be thought of as a $\mathsf{U}(1)$-gerbe satisfying $dB = H \in H^3(M, \mathbb{Z})$ as described in Section 4.1.1

[2] It is enough that $S[h \cdot X] = S[X] + \int d\xi$, for $\xi \in \Omega^1(M)$, as discussed in Section 2.1

actions to include Lie groupoid actions. Kotov and Strobl have given a local description of Lie algebroid gauging in [114, 88, 89, 99]. The description given there is only valid for Lie algebroids which are bundles of Lie algebras (Theorem 5.1). An invariant geometric description applicable to general Lie algebroids—as well as a discussion of associated groupoid actions—is new. Conceptually the gauging process is captured in the following diagram:

$$\begin{array}{ccccc}
\text{Global action} & \longrightarrow & \text{Local action} & \longrightarrow & \text{Identify gauge transformations} \\
 & & & & \\
S[h\cdot X]=S[X] & \longrightarrow & S[X,A] & \longrightarrow & S[X,A]\cong S[h\cdot X, h\circlearrowleft A] \\
 & & & & \\
\text{fixed } h\in \mathsf{G} & & h\in C^\infty(\Sigma,\mathsf{G}) & & \text{quotient } \mathsf{G}/\sim
\end{array}$$

An action $S[X]$ with a global symmetry (corresponding to a fixed element $h \in \mathsf{G}$) is gauged to produce $S[X, A]$; invariant under the action of $h \in C^\infty(\Sigma, \mathsf{G})$. Any two fields related by a gauge transformation are considered physically equivalent. The space of physically distinct fields is given by identifying gauge equivalent fields $C^\infty(\Sigma, \mathsf{G})/\sim$.

The Lie group action on $S[X]$ is generated by the right-invariant vector fields $\rho_a = \rho_a^\mu \partial_\mu \in \Gamma(TM)$ $(a = 1, \ldots, \dim(\mathsf{G}))$ satisfying

$$[\rho_a, \rho_b] = C^c{}_{ab}\rho_c,$$

where $C^c{}_{ab} \in \mathbb{R}$ are the structure constants for the Lie algebra $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$. The induced infinitesimal action on the fields X is generated by

$$\delta_\varepsilon X^\mu := \rho(\varepsilon)(X^\mu) = \varepsilon^a \rho_a^\nu \partial_\nu X^\mu = \varepsilon^a \rho_a^\mu.$$

The infinitesimal variation $\delta_\varepsilon S[X]$ (for constant ε) is required to vanish; giving the constraint

$$\delta_\varepsilon S[X] = \frac{1}{2}\int_\Sigma \varepsilon^a \Big((\mathcal{L}_{\rho_a} G)_{\mu\nu} dX^\mu \wedge \star dX^\nu + (\mathcal{L}_{\rho_a} B)_{\mu\nu} dX^\mu \wedge dX^\nu \Big) = 0,$$

which is satisfied by[3]

$$\mathcal{L}_{\rho_a} G = 0, \quad \mathcal{L}_{\rho_a} B = 0.$$

This establishes a correspondence between Lie group symmetries and isometries. The

[3]It is sufficient that $\mathcal{L}_{\rho_a} B = d\xi$ for some locally defined 1-form ξ. This is included in the general considerations in Section 5.3, but does not play an important role in the current discussion.

above discussion shows that the set of right-invariant vector fields, associated to a Lie algebra $\mathfrak{g}$, are required to generate isometries. Geometrically this corresponds to the existence of flowlines—integrating the vector fields—along which the metric G is preserved. Conversely, associated to a smooth one-parameter group of diffeomorphisms preserving a metric is a Killing vector field given by differentiation.

The existence of isometries is a rather special property; a generic metric would not be expected to have any isometries. Section 5.3 describes a proposal of Kotov and Strobl to gauge non-linear sigma models with a Lie algebroid action. The formalism presented there suggests it is sometimes possible to gauge models without isometries. A rather exciting proposal indeed! The new results contained in this chapter show that all metrics can be gauged non-isometrically (at least locally). The precise statement is given by Theorem 5.4. The real constraint of the gauging procedure comes from attempting to construct a field strength which imposes a flatness condition on the gauge fields. In the case of non-isometric T-duality the gauge transformation of the field strength imposes a constraint so strong that we conclude that non-isometric T-duality is locally equivalent to non-abelian T-duality (Theorem 5.9).

5.2 G manifolds and the WZW model

Before considering the local Lie algebroid gauging procedure it is instructive to detail the gauging procedure for a non-linear sigma model on a Lie group manifold. This case provides the motivation and intuition for the more general case of Lie algebroid gauging.

A standard example of a non-linear sigma model is given by the Wess–Zumino–Witten (WZW) model. In this case the target manifold is a Lie group, G, with the group action coming from composition. We denote the corresponding Lie algebra $\mathfrak{g} := \mathsf{Lie}(\mathsf{G})$.

Let $g : \Sigma \to \mathsf{G}$, be the embedding of a string worldsheet into a Lie group G. The non-linear sigma model is given by the action

$$S_{\mathrm{WZW}}[g] = \frac{1}{2}\int_\Sigma (g^{-1}dg \overset{\wedge}{,} \star\, g^{-1}dg)_G + (g^{-1}dg \overset{\wedge}{,} g^{-1}dg)_B, \tag{5.1}$$

where $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_B$ denote two G-invariant bilinear forms, symmetric and skew-symmetric respectively; $g^{-1}dg \in \Omega^1(\Sigma,\mathfrak{g})$ denotes the left-invariant Maurer–Cartan form, $\star$ is the worldsheet Hodge star, and

$$(a \overset{\wedge}{,} b)_G(s_1,s_2) := \frac{1}{2}\Big(\big(a(s_1),b(s_2)\big)_G - \big(a(s_2),b(s_1)\big)_G\Big),$$

for $s_1,s_2 \in \Gamma(T\Sigma)$, $a,b \in \Omega^1(\Sigma,\mathfrak{g})$.

The action (5.1) is manifestly invariant under the left action of a constant $h \in \mathsf{G}$:

$$(hg)^{-1}d(hg) = g^{-1}h^{-1}hdg = g^{-1}dg.$$

It is possible to promote the left-invariant symmetry for a constant $h \in \mathsf{G}$ to $h \in C^\infty(\Sigma, \mathsf{G})$ through the introduction of a gauge field $A \in \Omega^1(\Sigma, \mathfrak{g})$. The gauged action is given by

$$S_{\mathrm{WZW}}[g, A] = \frac{1}{2}\int_\Sigma (g^{-1}Dg \overset{\wedge}{,} \star g^{-1}Dg)_G + (g^{-1}Dg \overset{\wedge}{,} g^{-1}Dg)_B, \tag{5.2}$$

where $g^{-1}Dg = g^{-1}dg - g^{-1}Ag$. The left action of $h \in C^\infty(\Sigma, \mathsf{G})$ is defined to be

$$h \circlearrowright (g, A) = ({}^h g, {}^h A) := (hg, hAh^{-1} + dhh^{-1}).$$

This left group action leaves $g^{-1}Dg$ invariant:

$$\begin{aligned} h \circlearrowright g^{-1}Dg =& (hg)^{-1}d(hg) - (hg)^{-1}hAh^{-1}(hg) - (hg)^{-1}dhh^{-1}(hg) \\ =& g^{-1}h^{-1}(dh)g + g^{-1}dg - g^{-1}Ag - g^{-1}h^{-1}(dh)g \\ =& g^{-1}dg - g^{-1}Ag = g^{-1}Dg. \end{aligned}$$

It follows immediately that $S_{\mathrm{WZW}}[g, A]$ is invariant under the action of $h \in C^\infty(\Sigma, \mathsf{G})$.

The original action $S_{\mathrm{WZW}}[g]$ can be recovered from the gauged action $S_{\mathrm{WZW}}[g, A]$ if there exists a global gauge transformation which sets $A = 0$:

$${}^h A = 0 = hAh^{-1} + dhh^{-1} \quad \Rightarrow \quad A = -h^{-1}dh, \text{ for some } h \in C^\infty(\Sigma, \mathsf{G}).$$

The gauge field $A \in \Omega^1(\Sigma, \mathfrak{g})$ satisfies $A = -h^{-1}dh$ for some $h \in C^\infty(\Sigma, \mathsf{G})$ if and only if A is equal to the left-invariant Maurer–Cartan form (up to diffeomorphism). It is well known that the Maurer–Cartan form is locally the unique solution (up to diffeomorphism) to the zero field strength condition $F = 0$, where

$$F = dA - [A \overset{\wedge}{,} A]_{\mathfrak{g}}, \tag{5.3}$$

and

$$[A \overset{\wedge}{,} A']_{\mathfrak{g}}(s_1, s_2) := \frac{1}{2}\Big([A(s_1), A'(s_2)]_{\mathfrak{g}} - [A(s_2), A'(s_a)]_{\mathfrak{g}}\Big),$$

for $A, A' \in \Omega^1(\Sigma, \mathfrak{g})$ and $s_1, s_2 \in \Gamma(T\Sigma)$. The flatness condition can be imposed on the model in two ways: One option is to impose the condition on solutions 'by hand' as an extra constraint. Alternatively, it is possible to introduce an extra term to Lagrangian such that the constraint follows from the Euler–Lagrange equations. Taking the latter option the gauged Lagrangian is:

$$S_{\mathrm{WZW}}[g, A, \hat{X}] = \frac{1}{2}\int_\Sigma (g^{-1}Dg \overset{\wedge}{,} \star g^{-1}Dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}Dg \overset{\wedge}{,} g^{-1}Dg)_B + \int_\Sigma \langle \widehat{X}, F\rangle, \tag{5.4}$$

where $\widehat{X} \in \mathfrak{g}^*$ is a *Lagrange multiplier*, and $\langle \cdot\ , \cdot\rangle$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Imposing the flatness condition via a Lagrange multiplier is necessary for T-duality and will be discussed in Section 5.5. For the purposes of general gauging the flatness condition shall be viewed as an extra constraint.

Finally, it is worth noting that in the physics literature the gauging is often described by an infinitesimal Lie algebra action generated by $\varepsilon \in C^\infty(\Sigma, \mathfrak{g})$:

$$\delta_\varepsilon(g, A) = (\varepsilon g, d\varepsilon + [\varepsilon, A]_{\mathfrak{g}}). \tag{5.5}$$

The infinitesimal action can be generated from the group action $\delta_\varepsilon A := \frac{d}{dt}({}^{\exp(t\varepsilon)}A)|_{t=0}$.

$$\begin{aligned}
\delta_\varepsilon A =& \frac{d}{dt}\Big(\exp(t\varepsilon)A\exp(-t\varepsilon) + (d\exp(t\varepsilon))\exp(-t\varepsilon)\Big)|_{t=0} \\
=& [\varepsilon, A]_{\mathfrak{g}} + \frac{d}{dt}(td\varepsilon\exp(t\varepsilon)\exp(-t\varepsilon))|_{t=0} \\
=& d\varepsilon + [\varepsilon, A]_{\mathfrak{g}}.
\end{aligned}$$

5.2.1 Geometric interpretation

The above discussion of gauging the WZW model was largely algebraic. There is an associated geometric description which generalises to the Lie algebroid case and will serve as intuition for the geometric interpretation of Lie algebroid gauging in Section 5.4.

A Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$ can be identified with the set of right-invariant vector fields on G; which we denote $\mathfrak{X}_{\mathrm{inv}}(\mathsf{G})$. The vector space $\mathfrak{g}$ can be identified with the tangent space at the identity $\mathfrak{g} \equiv T_e\mathsf{G}$. A basis for the Lie algebra T_a ($a = 1, \ldots, \dim(\mathfrak{g})$) can be constructed from evaluating a chosen basis of right-invariant vector fields $v_a \in \mathfrak{X}_{\mathrm{inv}}(\mathsf{G})$ at the identity. The Lie bracket is given by the commutator of vector fields

$$[v_a, v_b] = C^c{}_{ab}v_c \quad \Leftrightarrow \quad [T_a, T_b]_{\mathfrak{g}} := C^c{}_{ab}T_c.$$

This gives a well defined Lie algebra: the space of right-invariant vector fields is closed under the usual Lie bracket of vector fields, $[\mathfrak{X}_{\mathrm{inv}}(\mathsf{G}), \mathfrak{X}_{\mathrm{inv}}(\mathsf{G})] \subset \mathfrak{X}_{\mathrm{inv}}(\mathsf{G})$, and the commutator bracket automatically satisfies the Jacobi identity.

The structure constants $C^a{}_{bc}$ corresponding to a Lie algebra $\mathfrak{g}$ can be associated to an orthonormal coframe $\eta^a \in \Gamma(T^*\mathsf{G})$. The coframe is defined by the relation

$$d\eta^a = \frac{1}{2} C^a{}_{bc} \eta^b \wedge \eta^c.$$

The coframe $\{\eta^a\}$ is dual to the frame specified by a basis of right-invariant vector fields $\{v_a\}$ (with the pairing satisfying $\langle v_a, \eta^b \rangle = \delta^b_a$). The Lie algebra $\mathfrak{g}$ specifies the coframe at $T^*_e\mathsf{G}$. Lie Groups are in fact parallelisable, and the frame defined at the identity can be extended to give a global trivialisation. The frame is globalised by pushing forward the frame at the identity by the group action $(R_h)_* T_a$, for $h \in \mathsf{G}$. Geometrically the frame is transported by flowing along the right-invariant vector fields.

Using the coframe $\{\eta^a\}$ it is possible to construct $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_B$ which are invariant under the left action, as required for gauging. Define $E = E_{ab}\eta^a \otimes \eta^b$, where $E_{ab} \in \mathbb{R}$. This is manifestly left-invariant. Symmetrising E_{ab} defines $(\cdot,\cdot)_G$ (if it is non-degenerate); skew-symmetrising defines $(\cdot,\cdot)_B$.

There is a similar construction for Lie groupoids $\mathcal{G}$. The notation here follows the introduction to Lie groupoids and Lie algebroids given in Section 3.1. Locally a Lie algebroid defined on a vector bundle $Q \to M$ can be identified with right-invariant sections at the unit 1_x in $Q = T^s\mathcal{G}$ (see Section 3.1.2). It is possible to construct an orthonormal coframe and left-invariant Riemannian metric using the left-invariant Lie algebroid generalisation of the Maurer–Cartan form (Section 3.3.1). In general this construction cannot be extended to the entire manifold as there are obstructions to integrating a Lie algebroid (see for example [43]). Explicitly, the fibre of $T^s(\mathcal{G})$ at an arrow $g : y \to z$ is

$$T^s_g\mathcal{G} := T_g\mathcal{G}(y,\cdot),$$

where $\mathcal{G}(x,\cdot) = s^{-1}(x)$, and $T^s\mathcal{G} = \ker(ds) \subset T\mathcal{G}$. The left action by an arrow $h : x \to y$ is only defined on the s-fibre at y and induces a bijection

$$R_h : \mathcal{G}(y,\cdot) \to \mathcal{G}(x,\cdot), \qquad \text{and} \qquad (R_h)_* : T^s_g\mathcal{G} \to T^s_{gh}\mathcal{G}.$$

The set of right-invariant sections on $\mathcal{G}$ is defined as:

$$\mathfrak{X}^s_{\mathrm{inv}}(\mathcal{G}) = \{\mathfrak{X} \in \Gamma(T^s\mathcal{G}) : \mathfrak{X}_{hg} = (R_h)_*(\mathfrak{X}_g), \forall (g,h) \in \mathcal{G}_2\}.$$

The Lie algebroid (groupoid) proposal, described in this chapter, has a geometric interpretation. The action of the Lie algebroid is described in terms of the right-invariant sections $\mathfrak{X}^s_{\mathrm{inv}}$. The connections ${}^Q\nabla^{\pm}$ (Given by Equation 5.29) define representations of Lie algebroids on Q. If the Lie algebroids are integrable $Q^{\pm} = T^s\mathcal{G}^{\pm}$ (where Q^+

and Q^- are isomorphic as vector bundles and may be simply denoted Q). The Lie algebroid actions are generated infinitesimally by the right-invariant sections $\mathfrak{X}^s_{\text{inv}}(Q^\pm)$. The Lie groupoid actions can be partially recovered from sections $\mathfrak{X}^s_{\text{inv}}(Q^\pm)$ using the flow $\phi^t_{\mathfrak{X}^s_{\text{inv}}(Q^\pm)}$ (see Section 3.1.2). This flow defines the $Q^\pm$-paths and the associated Weinstein groupoids (described in Section 3.2.2). The Lie groupoid actions associated to Lie algebroid gauging are given by the flows $\phi^t_{\mathfrak{X}^s_{\text{inv}}(Q^\pm)}$.

5.2.2 Comments on the gauge algebra and integrability

Closure of the gauge algebra (and later gauge algebroid) is of fundamental importance. The set of physical fields in a non-linear sigma model is given by a quotient: the set of smooth fields identified by an equivalence relation where any two fields related by a gauge transformation are identified. In order for the quotient to be well defined the set of gauge transformations must form a group (or groupoid). In particular, the composition of any two gauge transformations must itself be a gauge transformation:

$$({}^{h_2}({}^{h_1}g), {}^{h_2}({}^{h_1}A)) = ({}^{h_3}g, {}^{h_3}A), \tag{5.6}$$

where h_3 is a gauge transformation generated from h_1 and h_2. The corresponding infinitesimal version of this constraint is

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}](g, A) = \delta_{\varepsilon_3}(g, A), \tag{5.7}$$

where ε_3 is a Lie algebra (algebroid) element generated from ε_1 and ε_2. These identities will not hold for arbitrary field transformations. In the case of WZW model the group action can be verified directly:

$$\begin{aligned}
({}^{h_2}({}^{h_1}g), {}^{h_2}({}^{h_1}A)) =& (h_2h_1g, h_2(h_1Ah_1^{-1} + dh_1h_1^{-1})h_2^{-1} + dh_2h_2^{-1}) \\
=& (h_2h_1g, h_2h_1A(h_2h_1)^{-1} + h_2dh_1(h_2h_1)^{-1} + dh_2h_1h_1^{-1}h_2^{-1}) \\
=& (h_2h_1g, h_2h_1A(h_2h_1)^{-1} + d(h_2h_1)(h_2h_1)^{-1}) \\
=& ({}^{h_2h_1}g, {}^{h_2h_1}A),
\end{aligned}$$

giving $h_3 = h_2h_1$ the group composition of h_2 with h_1. The infinitesimal action (5.5) was shown to be generated from the group action. It follows automatically that (5.7) holds and $\varepsilon_3 = [\varepsilon_2, \varepsilon_1]_{\mathfrak{g}}$ is generated by the Lie algebra bracket on $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$.

The Lie algebroid gauging proposal of Kotov and Strobl involves 'exotic' gauge transformations which do not correspond to infinitesimal algebroid actions in general. Closure of the gauge algebroid will impose an important constraint on Lie algebroid gauging; ultimately leading to flat connections which define representations of the Lie algebroids generating the gauging action.

Not all gauge transformations must be composable when considering Lie Groupoid gauging. For example the dimension of $\mathrm{Im}(\rho)$, and the associated leaf space, may change at different points of the manifold M. In this case different leaves will have different dimensions and there is clearly no well defined composition of gauge transformations for all points. However, when the composition of two gauge transformations is well defined the composition must be a gauge transformation. If the gauge transformations do generate a Lie groupoid $\mathcal{G}$ the groupoid action corresponds to flowing along smooth paths in the Weinstein groupoid. It is important to note that in order to have a well defined notion of a gauge groupoid it is necessary for the gauge algebroid to close at all points on M.

In the physics literature non-linear sigma models are often analysed locally using an infinitesimal action δ_ε. Given a Lie group action, denoted G there is a unique Lie algebra given by $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$. However, an infinitesimal action generated by a Lie algebra $\mathfrak{g}$ does not correspond to a unique Lie group. Lie groups that are related by covering maps or quotients of discrete subgroups have isomorphic algebras and cannot be distinguished by local considerations alone. Local considerations of infinitesimal algebra actions are not enough to determine the topology.[4]

The Lie algebroid proposal of Kotov and Strobl is a local gauging theory—global issues have not considered. An integrable Lie algebroid is isomorphic to the tangent Lie algebroid of some s-connected Lie groupoid. However, several Lie groupoids may produce the same tangent Lie algebroid. The situation is complicated by the fact that not all Lie algebroids are integrable. For a discussion of Lie algebroid integrability see Section 3.2.2 and references within. In this thesis we will consider examples of integrable Lie algebroids and consider the associated Lie groupoids.

5.3 Kotov–Strobl Lie algebroid gauging

This section outlines the local coordinate description of Lie algebroid gauging developed by Kotov and Strobl [114, 88], Mayer and Strobl [99], and further studied with Chatzistavrakidis, Deser, and Jonke (CDJ) [32].

The general proposal for Lie algebroid gauging can be found in [32]. Consider a map $X : \Sigma \to M$, embedding a string worldsheet into a target space M. This map can be described locally by X^μ, for $\mu = 1, \ldots, \dim(M)$. The key generalisation associated to Lie algebroid gauging is the ability to gauge with respect to a set of involutive vector

[4] Non-abelian T-duality is described by a local gauging procedure and an associated topological description is still unknown [5, 116].

fields $\rho_a \in TM$, $a = 1, \ldots, d$ satisfying[5]

$$[\rho_a, \rho_b] = C^c{}_{ab}(X)\rho_c, \quad C^c{}_{ab}(X) \in C^\infty(M), \tag{5.8}$$

defining a Lie algebroid. A Lie algebroid structure can be defined as follows: Let $Q \to M$ be a vector bundle, specified locally by a frame $\{e_a\}$, $a = 1, \ldots, d = \dim(Q)$, satisfying

$$[e_a, e_b]_Q := C^c{}_{ab}(X)e_c. \tag{5.9}$$

The anchor $\rho : Q \to TM$ is defined by $\rho(e_a) := \rho_a$. The description is not invariant; given a change of frame $\tilde{e}_a = K^b{}_a e_b$, $K^b{}_a \in C^\infty(M)$, the description becomes

$$[\tilde{e}_a, \tilde{e}_b]_Q = \widetilde{C}^c{}_{ab}(X)\tilde{e}_c,$$

where

$$\widetilde{C}^c{}_{ab} = (K^{-1})^c{}_d(K^e{}_a K^f{}_b C^d{}_{ef} + K^e{}_a \rho^\mu_e \partial_\mu K^d{}_b - K^e{}_b \rho^\mu_e \partial_\mu K^d{}_a). \tag{5.10}$$

The choice of frame has a drastic effect on the coefficients $\widetilde{C}^c{}_{ab}$, and it may happen that there exists a frame in which they particularly nice (or nasty). An invariant description is clearly preferred to avoid confusion arising from a poor choice of frame for Q.

The gauged action given in [32] is

$$S_{\mathrm{KS}}[X, A] = \frac{1}{2}\int_\Sigma G_{\mu\nu} DX^\mu \wedge \star DX^\nu + \int_{\Sigma_3} H + \int_\Sigma (A^a \wedge \alpha_a + \frac{1}{2}\gamma_{ab} A^a \wedge A^b), \tag{5.11}$$

where $DX^\mu := dX^\mu - \rho^\mu_a A^a$, $H \in \Omega^3(M)$, Σ_3 is a three manifold with boundary Σ, $A \in \Omega^1(\Sigma, X^*Q)$, $\alpha \in \Gamma(Q^*)$, and $\gamma \in \Gamma(\wedge^2 Q^*)$.

The infinitesimal gauge transformations are of the form

$$\delta_\varepsilon X^\mu = \rho^\mu_a(X)\varepsilon^a \tag{5.12a}$$

$$\delta_\varepsilon A^a = d\varepsilon^a + C^a{}_{bc}(X)A^b\varepsilon^c + \omega^a{}_{\mu b}(X)\varepsilon^b DX^\mu + \phi^a{}_{\mu b}(X)\varepsilon^b \star DX^\mu, \tag{5.12b}$$

where $\omega^a{}_{\mu b}, \phi^a{}_{\mu b} \in C^\infty(M)$ are a priori undetermined fields, and $\star$ denotes the Hodge star on the worldsheet. It is noted that there are implied pullbacks in the action and (5.12). Closure of the gauge algebroid requires that the Lie algebroid structure can be pulled back. In general this is not possible. This represents a serious restriction on the allowable Lie algebroids described by this method (see Section 5.3.1).

[5] The integer d need not be equal $\dim(M)$.

Under a change of frame $\tilde{e}_a = K^b{}_a e_b$ the fields $\omega^a{}_{\mu b}$ and $\phi^a{}_{\mu b}$ transform as

$$\tilde{\omega}^a{}_{\mu b} = (K^{-1})^a{}_c \omega^c{}_{\mu d} K^d{}_b - K^c{}_b \partial_\mu (K^{-1})^a{}_c, \quad \tilde{\phi}^a{}_{\mu b} = (K^{-1})^a{}_c \phi^c{}_{\mu d} K^d{}_b. \tag{5.13}$$

Thus $\omega : \Gamma(Q) \to \Gamma(Q \otimes T^*M)$ defines a TM-connection on Q, and $\phi \in \Omega^1(M, \mathsf{End}(Q))$.

The action $S_{\mathrm{KS}}[X, A]$ is invariant under the gauge transformations (5.12) if the following constraints hold:

$$\mathcal{L}_{\rho_a} G = \omega^b{}_a \vee \iota_{\rho_b} G + \phi^b{}_a \vee \alpha_b, \tag{5.14a}$$

$$\mathcal{L}_{\rho_a} H = d\alpha_a - \omega^b{}_a \wedge \alpha_b \pm \phi^b{}_a \iota_{\rho_b} G, \tag{5.14b}$$

$$\gamma_{ab} = \iota_{\rho_a} \alpha_b, \tag{5.14c}$$

$$\mathcal{L}_{\rho_a} \alpha_b = C^c{}_{ab} \alpha_c + \iota_{\rho_b}(d\alpha_a - \iota_{\rho_a} H), \tag{5.14d}$$

where $(\omega^b{}_a \vee \iota_{\rho_b} G)_{\mu\nu} = \omega^b{}_{\mu a} \rho_b^\lambda G_{\lambda\nu} + \omega^b{}_{\nu a} \rho_b^\lambda G_{\mu\lambda}$, and the choice $\pm$ is given by the choice of Lorentzian ($\star^2 = 1$) or Euclidean ($\star^2 = -1$) signature on the worldsheet.

There are two natural questions:

1. For a given choice of G and H, do there exist $(\rho_a, \alpha_a, \omega^a{}_{\mu b}, \phi^a{}_{\mu b})$ satisfying the constraints (5.14)?

2. If $(\rho_a, \alpha_a, \omega^a{}_{\mu b}, \phi^a{}_{\mu b})$ satisfying (5.14) can be found is the choice unique?

An answer to the existence question is given for special cases in [32, 35]. The results of this thesis give a more complete answer: Corollary 5.6 states that for any choice of G and H there exist $(\rho_a, \alpha_a, \omega^a{}_{\mu b}, \phi^a{}_{\mu b})$ which satisfy the constraints (5.14) for some $U \subset M$. Necessary and sufficient conditions to gauge with respect to a chosen set of vector fields $\rho_a \in \Gamma(TM)$ are determined (Theorem 5.4 and Theorem 5.8). If $\rho_a \in \Gamma(TM)$ do satisfy the necessary and sufficient conditions a (not necessarily unique) solution for $(\alpha_a, \omega^a{}_{\mu b}, \phi^a{}_{\mu b})$ is given.

5.3.1 Pullback constraint of Kotov–Strobl gauging

The variation of the gauge fields for the Kotov–Strobl gauging proposal is given by (5.12). The closure of the gauge algebroid requires that

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] X^\mu = \delta_{\varepsilon_3} X^\mu, \quad [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] A^a = \delta_{\varepsilon_3} A^a,$$

for some $\varepsilon_3 = \sigma(\varepsilon_1, \varepsilon_2) = -\sigma(\varepsilon_2, \varepsilon_1) \in \Gamma(X^*Q)$. The field $\varepsilon_3 \in \Gamma(X^*Q)$ can be written using the pullback of a basis for Q: $\varepsilon_3 = \varepsilon_3^a X^* e_a \in \Gamma(X^*Q)$. The following expression for ε_3^a is given in the literature (Eq. (10) in [99]):

$$\varepsilon_3^a = \varepsilon_1^b \varepsilon_2^c C^a{}_{bc}. \tag{5.15}$$

Consider the Lie algebroid structure $(Q, [\cdot,\cdot]_Q, \rho)$ restricted to the image $(X(\Sigma), Q|_{X(\Sigma)}) \subset (M, Q)$. Denote the restricted algebroid structure $(Q|_{X(\Sigma)}, [\cdot,\cdot]_{X(\Sigma)}, \rho_{X(\Sigma)})$. Take a change of frame on $Q|_{X(\Sigma)}$ given by

$$\tilde{e}_a = K(X(\sigma))^b{}_a e_b.$$

Invariance of sections gives the transformation of the coefficients:

$$\varepsilon = \varepsilon^a X^* e_a = \tilde{\varepsilon}^a X^* \tilde{e}_a = \tilde{\varepsilon}^a K(\sigma)^b{}_a X^* e_b, \quad \Rightarrow \quad \tilde{\varepsilon}^a = (K^{-1})^a{}_b \varepsilon^b.$$

This gives a constraint on the transformation of the structure functions on $X(\Sigma)$:

$$\varepsilon_3 = \varepsilon_1^b \varepsilon_2^c C^a{}_{bc} X^* e_a = \tilde{\varepsilon}_1^b \tilde{\varepsilon}_2^c \widetilde{C}^a{}_{bc} X^* \tilde{e}_a = \varepsilon_1^y \varepsilon_2^z (K^{-1})^b{}_y (K^{-1})^c{}_z \widetilde{C}^x_{yz} K^a{}_x X^* e_a,$$

so that

$$\widetilde{C}^a{}_{bc} = (K^{-1})^a{}_x C^x{}_{yz} K^y{}_b K^z{}_c. \tag{5.16}$$

However, it follows from (5.10) that the structure functions (restricted to $X(\Sigma)$) transform as

$$\widetilde{C}^c{}_{ab} = (K^{-1})^c{}_d (K^e{}_a K^f{}_b C^d{}_{ef} + K^e{}_a (X_*^{-1}\rho)_e (K^d{}_b) - K^e{}_b (X_*^{-1}\rho)_e (K^d{}_a)), \tag{5.17}$$

where $(X_*^{-1}\rho)_a$ denotes the pushforward of the map X^{-1} (which exists as X is a diffeomorphism when restricted to $X(\Sigma) \subset M$). It is clear that the requirement (5.16) places a tight constraint on the allowable Lie algebroids for gauging. In particular, the requirement that (5.16) and (5.17) hold simultaneously, mean that the Lie algebroid bracket $[\cdot,\cdot]_{X(\Sigma)}$ is $C^\infty(X(\Sigma))$ linear.

Theorem 5.1. *The Lie algebroid gauging procedure outlined by Kotov, Mayer, Strobl and CDJ [114, 99, 88, 89, 32] is only valid when $(Q|_{X(\Sigma)}, [\cdot,\cdot]_{X(\Sigma)}, \rho_{X(\Sigma)})$—the restriction of $(Q, [\cdot,\cdot]_Q, \rho)$ to the image of X—is a bundle of Lie algebras.*

Remark. The vector fields ρ_a must vanish on $T(X(\Sigma))$ if we require that the definition of ε_3 be invariant under arbitrary invertible changes of frame $K(X(\sigma))^a{}_b \in C^\infty(X(\Sigma))$. In this case the gauged action is equivalent to the ungauged action over $X(\Sigma)$. The usual procedure of Lie algebra gauging takes a set of non-zero vector fields $v_a \in \Gamma(TM)$ satisfying $[v_a, v_b] = C^c{}_{ab} v_c$ for $C^c{}_{ab} \in \mathbb{R}$ defining a Lie algebra. The Lie algebra structure is defined on the $\mathbb{R}$-span of the vectors $\mathrm{span}_{\mathbb{R}}\{v_1, \ldots, v_d\}$ (a Lie algebroid is defined on $\mathrm{span}_{C^\infty(M)}\{v_1, \ldots, v_d\}$). In this case the allowable frame transformations are given by $K^a{}_b \in \mathbb{R}$, and (5.16) is equal to (5.17). In all frames the structure functions are constant, and there is really only a Lie algebra structure on Q.

5.4 General Lie algebroid sigma models

This section describes a general construction for considering non-linear sigma models which are gauged with respect to a Lie algebroid action. This construction is valid for any integrable Lie algebroid, $Q \cong \mathsf{Lie}(\mathcal{G})$.

The Lie algebroid gauged model takes a non-linear sigma model $(X, \Sigma, h, M, G, S[X])$ (defined on page 94) and constructs a 'gauged' action $S_Q[X, A]$ using ρ_a and $(\Omega^{\pm})^a{}_{\mu b}$, where the vector fields $\rho_a \in \Gamma(TM)$ define a Lie algebroid

$$[\rho_a, \rho_b] =: C^c{}_{ab}\rho_c, \quad C^c{}_{ab} \in C^\infty(M),$$

and $(\Omega^{\pm})^a{}_{\mu b}$ define TM-connections on a vector bundle Q. The TM-connections on Q will be denoted $\nabla^{\pm}$. Let $\{e_a\}$ be a local frame for Q. The connections $\nabla^{\pm}$ are defined as

$$\nabla^{\pm} e_a := (\Omega^{\pm})^b{}_a \otimes e_b. \tag{5.18}$$

The fields $(\rho_a, (\Omega^{\pm})^a{}_{\mu b})$ determine $(\check{\rho}_a, \check{\Omega}^a{}_{\alpha b})$ which are defined in Section 5.4.1.

Let $\{\sigma^\alpha\}$ be local coordinates on the worldsheet Σ. Consider the following action (for $\star^2 = 1$ Lorentzian worldsheet):

$$S_Q[X, A] = \int_\Sigma (X^*E)_{\alpha\beta} D_-\sigma^\alpha \wedge D_+\sigma^\beta - \int_\Sigma X^*C + \int_{\Sigma_3} X^*H, \tag{5.19}$$

where, $E = G + C$ for some $C \in \Omega^2(M)$, $D\sigma = d\sigma - \check{\rho}(A)$, $A \in \Omega^1(\Sigma, X^{**}Q)$, and $D_\pm\sigma = \frac{1}{2}(D\sigma \pm \star D\sigma)$.

The infinitesimal variation is given by

$$\delta_\varepsilon X = \check{\rho}(\varepsilon)(X), \tag{5.20a}$$

$$\delta_\varepsilon A^i = d\varepsilon^i + \check{C}^i{}_{jk} A^j \varepsilon^k + (\check{\Omega}^+)^i{}_{\alpha j}\varepsilon^j D_+\sigma^\alpha + (\check{\Omega}^-)^i{}_{\alpha j}\varepsilon^j D_-\sigma^\alpha, \tag{5.20b}$$

and $\delta_\varepsilon S_Q[X, A] = 0$ if the following conditions are met:

$$(\mathcal{L}_{\rho_a} E)_{\mu\nu} = E_{\mu\lambda}\rho_b^\lambda (\Omega^+)^b{}_{\nu a} + E_{\lambda\nu}\rho_b^\lambda (\Omega^-)^b{}_{\mu a}, \tag{5.21a}$$

$$\mathcal{L}_{\rho(\varepsilon)} C = \iota_{\rho(\varepsilon)} H. \tag{5.21b}$$

We will refer to (5.21a) as the *generalised Killing equation.*

When $\star^2 = -1$ (Euclidean worldsheet), the corresponding action is

$$S_Q[X, A] = \int_\Sigma (X^*E)_{\alpha\beta} D_+\sigma^\alpha \wedge D_-\sigma^\beta - \int_\Sigma X^*C + \int_{\Sigma_3} X^*H, \tag{5.22}$$

where, $E = iG + C$, $D\sigma = d\sigma - \check{\rho}(A)$, and $D_{\pm}\sigma = \frac{1}{2}(D\sigma \pm i \star D\sigma)$. In this case the infinitesimal variation is given by

$$\delta_\varepsilon X = \check{\rho}(\varepsilon)(X), \tag{5.23a}$$

$$\delta_\varepsilon A^i = d\varepsilon^i + \check{C}^i{}_{jk} A^j \varepsilon^k + (\check{\Omega}^-)^i{}_{\alpha j} \varepsilon^j D_+ \sigma^\alpha + (\check{\Omega}^+)^i{}_{\alpha j} \varepsilon^j D_- \sigma^\alpha. \tag{5.23b}$$

and $\delta_\varepsilon S_Q[X, A] = 0$ if (5.21) holds.

5.4.1 Non-isometric gauge algebroid

It was shown in Section 5.3.1 that taking $\varepsilon \in \Gamma(X^*Q)$ and setting

$$\delta_\varepsilon X^\mu = X^*\rho(\varepsilon)(X^\mu)$$

leads to a strong restriction on the class of Lie algebroids that can be used to gauge. This is a consequence of the fact that a Lie algebroid structure does not naturally pullback in general. However, there may be a natural Lie algebroid structure defined on $X^{**}Q$. The construction of the double pullback algebroid is due to Higgins and Mackenzie [63] and reviewed in Section 3.3.1. The induced Lie algebroid exists whenever the map $\phi : M_1 \to M_2$ is a surjective submersion. The induced Lie algebroid $X^{**}Q$ always exists when considering Lie algebroid gauging as $X : \Sigma \to M$ is an embedding. If there is a Lie groupoid $\mathcal{G}(Q)$ such that $Q = \mathsf{Lie}(\mathcal{G})$, it follows that $\mathcal{G}(X^{**}Q)$ will give a well defined Lie groupoid.

The Lie algebroid bracket $[\cdot,\cdot]_{X^{**}Q}$ is induced from $[\cdot,\cdot]_Q$ (defined via (3.20)) and in a local frame $\{\check{e}_a\}$ for $X^{**}Q$ determines the Lie algebroid structure functions $\check{C}^i{}_{jk} \in C^\infty(\Sigma)$:

$$[\check{e}_i, \check{e}_j]_{X^{**}Q} := \check{C}^k{}_{ij} \check{e}_k,$$

where $i, j, k = 1, \ldots, \dim(X^{**}Q)$. Sections of $X^{**}Q$ are given by $\varepsilon = (\sigma, \epsilon) \in \Gamma(T\Sigma \oplus X^*Q)$. The anchor $\check{\rho} : X^{**}Q \to T\Sigma$ is given by

$$\check{\rho}(\varepsilon) = \check{\rho}(\sigma, \epsilon) := \sigma.$$

The variation $\delta_\varepsilon X = \check{\rho}(\varepsilon)(X)$ is now a well defined quantity, and

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}](X) = \delta_{[\varepsilon_1, \varepsilon_2]}(X) \quad \Rightarrow \quad [\check{\rho}(\varepsilon_1), \check{\rho}(\varepsilon_2)]_{T\Sigma}(X) = \check{\rho}([\varepsilon_1, \varepsilon_2]_{X^{**}Q})(X).$$

We conclude that $[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}](X) = \delta_{[\varepsilon_1, \varepsilon_2]}(X)$ is equivalent to the anchor homomorphism property of the Lie algebroid $(X^{**}Q, [\cdot,\cdot]_{X^{**}Q}, \check{\rho})$. The variation $\delta_\varepsilon X$ is generated by infinitesimal diffeomorphisms on Σ generated by $\check{\rho}(\varepsilon) \in \Gamma(T\Sigma)$.

The fields $(\Omega^{\pm})^a{}_{\mu b}$ ($a, b = 1, \ldots, \dim(Q)$ and $\mu = 1, \ldots, \dim(M)$) define connections via $\nabla^{\pm} e_a := (\Omega^{\pm})^b{}_a \otimes e_b$. We define $(X^*\Omega^{\pm})^a{}_{\alpha b}$ by writing the connections $X^*\nabla^{\pm}$ (defined via (3.15)) in the basis $\{X^* e_a\}$. We define a $T\Sigma$-connection on $X^{**}Q$ as follows:

$$\check{\nabla}^{\pm}_{\sigma'} \sigma \oplus \epsilon := \rho(\sigma')(\sigma) \oplus \epsilon + \sigma \oplus X^* \nabla^{\pm}_{\sigma'} \epsilon. \tag{5.24}$$

In the basis $\check{e}_i = \partial_\alpha \oplus X^* e_a$, we get the connection coefficients:

$$\check{\nabla}^{\pm} \check{e}_i = (\check{\Omega}^{\pm})^j{}_i \otimes \check{e}_j.$$

With these definitions the variation of the gauge fields $A \in \Omega^1(\Sigma, X^{**}Q)$ is a well defined quantity.

Remark. Having established the correct gauge algebroid structure, we will henceforth omit the $\check{}$ with the relevant double pullback maps implied. This will allow a more direct comparison to the formulae appearing in the literature on Lie algebroid gauging.

5.4.2 Examples

The action $S_Q[X, A]$ includes the non-linear sigma models described by $S_{\mathrm{KS}}[X, A]$ and $S_{\mathrm{WZW}}[X, A]$ as well as allowing the possibility of gauging the standard non-linear sigma model with any integrable Lie algebroid.

Example 5.1 (WZW). $S_{\mathrm{WZW}}[X, A]$ can be described using $S_Q[X, A]$ by taking $M = \mathsf{G}$, $Q = T\mathsf{G} = \mathsf{G} \times \mathfrak{g}$. Choose the coframe $dX = \eta = g^{-1}dg$ and identify

$$(E, \rho, C, H) = ((G + B)_{\mu\nu} \eta^\mu \otimes \eta^\nu, \mathrm{Ad}_{g^{-1}}, B, H),$$

where $H = (g^{-1}dg, [g^{-1}dg, g^{-1}dg]_{\mathfrak{g}})_{\mathsf{G}}$ is given by $H = d_{\mathfrak{g}} B$.

Example 5.2 (Poisson sigma model). Define $B_{\mu\nu} := A_\mu (\pi(A, \cdot))^{-1}_\nu$ where π is a Poisson bivector; the action is given by

$$S[X, A] = \frac{1}{2} \int_\Sigma G_{\mu\nu} DX^\mu \wedge \star DX^\nu + \int_\Sigma A_\mu \wedge dX^\mu + \frac{1}{2} \pi^{\mu\nu} A_\mu \wedge A_\nu,$$

where $DX = dX - \pi(A, \cdot)$. The Lie algebroid structure on $Q = T^*M$ is given by the Poisson cotangent Lie algebroid (Example 3.12). Integrable Poisson Lie algebroids are associated to symplectic groupoids [125, 28].

Example 5.3 (Universal). The 'universal' action of Kotov–Strobl [35] is given by

$$S_{\mathrm{univ}}[X, V, W] = \int_\Sigma \Big(\frac{1}{2} g_{\mu\nu} DX^\mu \wedge \star DX^\nu + W_\mu \wedge (dX^\mu - \frac{1}{2} V^\mu) \Big) + \int_{\Sigma_3} H,$$

where $DX^\mu = dX^\mu - V^\mu$, $V \in \Omega^1(\Sigma, X^{**}T\Sigma)$ and $W \in \Omega^1(\Sigma, X^{**}T^*\Sigma)$. This is equivalent to $S_Q[X, A]$ through the identification $V^\mu = \check{\rho}^\mu_a A^a$ and $W_\mu = \check{C}_{\mu\nu}\check{\rho}^\nu_a A^a$.

The Kotov–Strobl action and Poisson sigma model are special cases of the universal action. The Kotov–Strobl action is found by identifying $V^\mu = \rho^\mu_a A^a$, and $W_\mu = \alpha_{\mu a} A^a$. The Poisson sigma model (for $Q = T^*M$) is identified via $V^\mu = \pi^{\nu\mu} A_\nu$, and $W_\mu = A_\mu$. The universal action naturally interpolates between the Poisson sigma model and the Kotov–Strobl model. The Lie algebroid is found by restricting the Lie bialgebroid constructed from $[\cdot,\cdot]$, $[\cdot,\cdot]_\pi$ to a (small) Dirac[6] structure (see Section 3.4.1 for a description of Lie bialgebroids).

The universal title refers to the fact each different choice of small Dirac structure on $TM \oplus T^*M$ gives a different model. This gives a class of examples, but not all Lie algebroids arise as subbundles of $TM \oplus T^*M$.

Example 5.4 (Foliation). Let $Q \subset TM$ be an involutive subbundle (a constant rank subbundle closed under the Lie bracket). This defines a Lie algebroid (Example 3.8).

An involutive linearly independent set of vector fields v_a will satisfy (by definition)

$$[v_a, v_b] = C^c{}_{ab} v_c, \quad C^c{}_{ab} \in C^\infty(M),$$

but in general $C^c{}_{ab}$ will not be constant. Involutive foliated vector fields $Q = T\mathcal{F} \subset TM$ are not generated by group actions in general. The integrability of smooth distributions is given by the Steffan–Sussmann conditions, see [115].

Example 5.5 (Lie Groupoid). Take any Lie groupoid $(\mathcal{G}, M)$, and take the Lie algebroid $Q = \mathsf{Lie}(\mathcal{G})$. The action $S_Q[X, A]$ provides a sigma model action.

It is possible to consider sigma models for all the examples of Lie algebroids considered in Section 3.2.1 (Examples (3.6)–(3.12)). Integrability is guaranteed when you start with a Lie groupoid. Examples of Lie groupoids can be found in Section 3.1.1 (Examples (3.1)–(3.5)).

Remark. We can consider topologically non-trivial examples by considering the gauge groupoid $\mathcal{G}(P)$ (Example 3.4) corresponding to a topologically non-trivial principal bundle $P(M, \pi, \mathsf{G})$.

Remark. It is entirely consistent to consider a Lie algebroid with an anchor with a non-trivial kernel, $\ker(\rho) \neq 0$. Such an algebroid has an associated isotropy Lie algebra at each point $x \in M$ which corresponds to a gauge symmetry that is not generated by a vector field in TM. This possibility was considered in [34]. This would correspond to symmetries not generated by vector fields on M. In string theory M is spacetime and the dimension is fixed for quantisation consistency. A non-zero isotropy Lie algebra would correspond to introducing extra degrees of freedom to gauge theories on M.

[6] A small Dirac structure is an involutive and isotropic subbundle. A Dirac structure is a small Dirac structure of maximal dimension.

It is clear that many natural examples of non-linear sigma models can be described using the action $S_Q[X, A]$. The task remaining is to understand when such models can be gauged.

5.4.3 Closure constraint

As emphasised in Section 5.2.2 it is necessary for the gauge algebroid to be closed. There is another way to see the importance of closure of the gauge algebroid from the perspective of gauged sigma models. The gauged model will be equivalent to the original model if there exists a finite gauge transformation which sets the gauge fields to zero. The infinitesimal gauge transformations (5.20) (or (5.23)) will only integrate to finite gauge transformations when the gauge algebroid is closed. Closure of the gauge algebroid does not follow automatically from (5.20) (or (5.23)). We will see that closure of the gauge algebroid imposes important constraints. We note that the formulas in this section have been calculated using the Lie algebroid $([\cdot,\cdot]_{X^{**}Q}, \check{\rho})$; we have left the pullback maps implied throughout in order to better connect to the literature.

The first constraint gives

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]X = \delta_{[\varepsilon_1,\varepsilon_2]_Q}X \quad \Leftrightarrow \quad [\rho(\varepsilon_1), \rho(\varepsilon_2)]X = \rho([\varepsilon_1, \varepsilon_2]_Q)X,$$

the anchor homomorphism property of a Lie algebroid. Closure on the gauge field A is more involved. It is convenient to define $D_\pm X := \frac{1}{2}(DX \pm \star DX)$. A straightforward computation gives

$$\begin{aligned}
\delta_\varepsilon D_\pm X^\mu =&\varepsilon^a(\partial_\nu \rho^\mu_a - \rho^\mu_b(\Omega^\pm)^b{}_{\nu a})D_\pm X^\nu, \quad \text{if } \star^2 = 1\\
\delta_\varepsilon D_\pm X^\mu =&\varepsilon^a(\partial_\nu \rho^\mu_a - \rho^\mu_b(\Omega^\mp)^b{}_{\nu a})D_\pm X^\nu, \quad \text{if } \star^2 = -1.
\end{aligned}$$

A lengthy calculation for the $\star^2 = 1$ case gives:

$$\begin{aligned}
([\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] - \delta_{[\varepsilon_1,\varepsilon_2]_Q})A^a =&\varepsilon_1^b\varepsilon_2^c(-\nabla^-_\mu(T_{\nabla^-})^a_{bc} + 2\rho^\nu_{[b|}(R_{\nabla^-})^a{}_{\nu\mu|c]})D_-X^\mu\\
&+\varepsilon_1^b\varepsilon_2^c(-\nabla^+_\mu(T_{\nabla^+})^a_{bc} + 2\rho^\nu_{[b|}(R_{\nabla^+})^a{}_{\nu\mu|c]})D_+X^\mu,
\end{aligned}$$

and for the $\star^2 = -1$ case:

$$\begin{aligned}
([\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] - \delta_{[\varepsilon_1,\varepsilon_2]_Q})A^a =&\varepsilon_1^b\varepsilon_2^c(-\nabla^-_\mu(T_{\nabla^-})^a_{bc} + 2\rho^\nu_{[b|}(R_{\nabla^-})^a{}_{\nu\mu|c]})D_+X^\mu\\
&+\varepsilon_1^b\varepsilon_2^c(-\nabla^+_\mu(T_{\nabla^+})^a_{bc} + 2\rho^\nu_{[b|}(R_{\nabla^+})^a{}_{\nu\mu|c]})D_-X^\mu,
\end{aligned}$$

where

$$(T_{\nabla^\pm})^a{}_{bc} =\rho^\mu_b(\Omega^\pm)^a{}_{\mu c} - \rho^\mu_c(\Omega^\pm)^a{}_{\mu b} - C^a{}_{bc}; \tag{5.25}$$

$$\begin{aligned}(R_{\nabla^\pm})^a{}_{\mu\nu b} =& 2\partial_{[\mu}(\Omega^\pm)^a{}_{\nu]b} + 2(\Omega^\pm)^a{}_{[\mu|c}(\Omega^\pm)^c{}_{|\nu]b}; \\ (\nabla^\pm_\mu T_\nabla)^a_{bc} =& \partial_\mu (T_\nabla)^a_{bc} + (T_\nabla)^d_{bc}(\Omega^\pm)^a{}_{\mu d} - (T_\nabla)^a_{dc}(\Omega^\pm)^d{}_{\mu b} - (T_\nabla)^a_{bd}(\Omega^\pm)^d{}_{\mu c}.\end{aligned} \tag{5.26}$$

The closure constraints for $\phi = 0$ (or equivalently $\Omega^+ = \Omega^-$) have appeared in the literature before [99].[7]

Closure of the gauge algebroid requires that either $D_\pm X \equiv 0$ or $\nabla^\pm_\mu (T_{\nabla^\pm})^a_{bc} = 2\rho^\nu_{[b|}(R_{\nabla^\pm})^a{}_{\nu\mu|c]}$. If $D_\pm X \equiv 0$ then the gauged terms disappear from the Lagrangian. While it may be possible to consider $D_\pm X = 0$ as an 'on-shell' condition it is not clear that this is a natural constraint physically. In addition, quantisation requires 'off-shell' closure of the gauge algebroid. It makes sense to require closure for all $X : \Sigma \to M$, including $D_\pm X \neq 0$. With this assumption it follows that

$$([\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] - \delta_{[\varepsilon_1,\varepsilon_2]_Q})A^a = 0 \quad \Leftrightarrow \quad \nabla^\pm_\mu (T_{\nabla^\pm})^a{}_{bc} = 2\rho^\nu_{[b|}(R_{\nabla^\pm})^a{}_{\nu\mu|c]}. \tag{5.27}$$

The closure constraint on the curvatures $\nabla^\pm$, given by (5.27), is a curious constraint and is not readily interpreted. When $\nabla^\pm$ are flat connections, the closure of the gauge algebra acquires a nice interpretation: $\nabla^\pm (T_{\nabla^\pm})^a{}_{bc} = 0$, which is equivalent to the fact that covariantly constant sections are closed on $[\cdot\,,\cdot]_Q$:

$$(\nabla^\pm q_1 = 0 \;\&\; \nabla^\pm q_2 = 0 \;\Rightarrow \nabla^\pm [q_1, q_2]_Q = 0) \quad \Leftrightarrow \quad \nabla^\pm_\mu (T_{\nabla^\pm})^a{}_{bc} = 0,$$

for $q_1, q_2 \in \Gamma(Q)$. The constraint for $R_{\nabla^\pm} \neq 0$ is best understood by 'lifting' the TM-connections on Q to Q-connections on Q using the adjoint connections (3.10):

$${}^Q\nabla^\pm_{q_1} q_2 := \nabla^\pm_{\rho(q_2)} q_1 + [q_1, q_2]_Q. \tag{5.28}$$

The adjoint connection is well known in the mathematics literature and plays an important role in integrating Q-paths in a Lie groupoid [43]. In the local frame specified by $\{e_a\}$ the components of the adjoint connections are given by

$$({}^Q\Omega^\pm)^a{}_{bc} := \rho^\mu_c (\Omega^\pm)^a{}_{\mu b} + C^a{}_{bc}. \tag{5.29}$$

The torsion $T_{{}^Q\nabla^\pm}$ and curvature $R_{{}^Q\nabla^\pm}$ can be calculated in terms of the induced connections $\overline{\nabla}^\pm_{q_1} q_2 := \nabla^\pm_{\rho(q_1)} q_2$, using (5.28), with the following results:

$$T_{{}^Q\nabla^\pm}(q_1, q_2) = - T_{\overline{\nabla}^\pm}(q_1, q_2), \tag{5.30}$$

$$\begin{aligned}R_{{}^Q\nabla^\pm}(q_1, q_2)(q_3) =& R_{\overline{\nabla}^\pm}(q_1, q_3) q_2 - R_{\overline{\nabla}^\pm}(q_2, q_3) q_1 \\ &+ \overline{\nabla}^\pm_{q_3} T_{\nabla^\pm}(q_1, q_2) - T_{\overline{\nabla}^\pm}(\overline{\nabla}^\pm_{q_3} q_1, q_2) - T_{\overline{\nabla}^\pm}(q_1, \overline{\nabla}^\pm_{q_3} q_2).\end{aligned} \tag{5.31}$$

[7] We replace the Lie bracket $[\varepsilon_1, \varepsilon_2]_{X^*Q}$ with the Lie algebroid bracket $[\varepsilon_1, \varepsilon_2]_{X^{**}Q}$ but the expression is formally the same.

Calculating $R_{^Q\nabla^\pm}$ in local coordinates produces the closure constraint (5.27). In conclusion:

$$([\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] - \delta_{[\varepsilon_1,\varepsilon_2]_Q})A = 0 \quad \Rightarrow \quad R_{^Q\nabla^\pm} = 0. \tag{5.32}$$

If ρ has a non-trivial kernel then $R_{^Q\nabla^\pm} = 0$ does not necessarily imply closure of the gauge algebroid. If $\{\rho_a\}$ are linearly independent we can define $Q \cong T\mathcal{F} \subset TM$ to be the span of $\{\rho_a\}$. In this case ρ has no kernel and $R_{^Q\nabla^\pm} = 0$ implies the closure of the gauge algebroid.

Remark. In the case that $\phi = 0$ ($\Omega^+ = \Omega^-$) it has been noted in [89] that closure of the gauge algebra is equivalent to the Q-flat condition.

Closure of the gauge algebra implies the existence of two flat Q-connections on Q. These connections define representations of Lie algebroids on Q. The Lie algebroids define Q-paths associated to the Weinstein groupoid (as outlined in Section 3.2.2). The Q-paths define the orbits describing our gauge symmetry. It follows from Theorem 3.19 that ${}^{X^{**}Q}\nabla^\pm$ will define representations of Lie algebroids corresponding to Lie groupoids $\mathcal{G}^\pm(X^{**}Q)$ if ${}^Q\nabla^\pm$ define a representation of $\mathcal{G}^\pm(Q)$. Thus it is sufficient to study the flatness of the connections ${}^Q\nabla^\pm$. Alternatively we come to the same conclusion by noting that $R_{^Q\nabla^\pm}$ are tensors, so $R_{^{X^{**}Q}\nabla^\pm} = X^{**}R_{^Q\nabla^\pm}$, and it is sufficient to check the flatness of ${}^Q\nabla^\pm$.

The flatness condition $R_{^Q\nabla^\pm} = 0$ implies that ${}^Q\Omega^\pm$ are Maurer–Cartan forms for the frame bundle $\mathcal{B}(Q)$ over M. It follows that

$${}^Q\Omega^\pm = K_\pm^{-1} d_Q K_\pm, \quad ({}^Q\Omega^\pm)^a{}_{bc} = (K_\pm^{-1})^a{}_d \rho^\mu_b \partial_\mu (K_\pm)^d{}_c, \tag{5.33}$$

for some $K_\pm \in C^\infty(M, \mathsf{GL}(d))$ where $d = \dim(Q)$.

Remark. Another way to see that ${}^Q\Omega^\pm$ have the form (5.33) is by looking at the transformation properties: Take

$${}^Q\nabla^\pm_{q_1} q_2 = (q_1^b \rho_b^\mu \partial_\mu q_2^a + q_1^b q_2^c ({}^Q\Omega^\pm)^a{}_{bc}) e_a = (\tilde{q}_1^b \tilde{\rho}_b^\mu \partial_\mu \tilde{q}_2^a + \tilde{q}_1^b \tilde{q}_2^c ({}^Q\widetilde{\Omega}^\pm)^a{}_{bc}) \tilde{e}_a, \tag{5.34}$$

where $\{e_a\}$ is a choice of frame for Q, and $\tilde{e}_a = (K_\pm)^b{}_a e_b$. The change of frame induces the transformations $\tilde{q}^a = (K_\pm^{-1})^a{}_b q^b$, and $\tilde{\rho}_a = (K_\pm)^b{}_a \rho_b$. This induces the transformation

$$({}^Q\widetilde{\Omega}^\pm)^a{}_{bc} = (K_\pm^{-1})^a{}_x \tilde{\rho}_b^\mu \partial_\mu (K_\pm)^x{}_c + (K_\pm^{-1})^a{}_x ({}^Q\Omega^\pm)^x{}_{yz} (K_\pm)^y{}_b (K_\pm)^z{}_c. \tag{5.35}$$

A flat connection is one that can be set to zero by an appropriate choice of frame. If ${}^Q\Omega^\pm$ is zero is the frame $\{e_a\}$ then it follows from (5.35) that in a generic frame $\{\tilde{e}_a\}$, the connection coefficients are of the form (5.33).

5.4.4 Solving the Gauge constraints

Closure of the gauge algebra implies that $R_{Q\nabla^{\pm}} = 0$. This constraint is much easier understood than (5.27) and allows an explicit construction of solutions to the gauging constraints (5.21). Equation (5.21b) can be used to determine a choice of field $C \in \Omega^2(M)$ and does not pose a constraint to gauging.

Our first task is to find the general solution to the generalised Killing equation (5.21a). Written in matrix form the generalised Killing condition is given by a system of matrix equations

$$\mathcal{L}_{\rho_a} E = E\rho\Omega_a^+ + E^T\rho\Omega_a^-. \tag{5.36}$$

For each $a = 1, \ldots, k$ this is a linear matrix equation and a solution for Ω_a^+ exists if and only if

$$\mathcal{L}_{\rho_a} E - E^T\rho\Omega_a^- = E\rho(E\rho)^+(\mathcal{L}_{\rho_a} E - E^T\rho\Omega_a^-), \tag{5.37}$$

for each a, where $(E\rho)^+$ denotes the Moore–Penrose pseudo-inverse of $E\rho$. If this constraint is satisfied the solution Ω_a^+ is given by

$$\Omega_a^+ = (E\rho)^+(\mathcal{L}_{\rho_a} E - E^T\rho\Omega_a^-) + (I - (E\rho)^+E\rho)X_a, \tag{5.38}$$

where I is the $k \times k$ identity matrix, and $X_a \in C^\infty(M, \mathbb{R}^{k\times n})$ is arbitrary.[8] In explicit index notation we have

$$\begin{aligned}(\Omega_a^+)^b{}_\mu =& ((E\rho)^+)^b{}_\lambda(\mathcal{L}_{\rho_a} E)^\lambda{}_\mu - ((E\rho)^+)^b{}_\lambda E_{\kappa\lambda}\rho_c^\kappa(\Omega^-)^c{}_{\mu a} \\ &+ (\delta^b{}_c - ((E\rho)^+)^b{}_\lambda(E\rho)_c^\lambda)(X_a)^c{}_\mu.\end{aligned}$$

Theorem 5.2. *Given a field $E \in \Gamma(T^*M \otimes T^*M)$ and a choice of involutive vector fields $\rho_a \in \Gamma(TM)$ $(a = 1, \ldots, k)$ defining a Lie algebroid $[\rho_a, \rho_b] = C^c{}_{ab}\rho_c$, the generalised Killing equation*

$$(\mathcal{L}_{\rho_a} E)_{\mu\nu} = E_{\mu\lambda}\rho_b^\lambda(\Omega^+)^b{}_{\nu a} + E_{\lambda\nu}\rho_b^\lambda(\Omega^-)^b{}_{\mu a},$$

has solutions for $\Omega^\pm \in \Gamma(T^*M \otimes Q)$ if and only if

$$\mathcal{L}_{\rho_a} E - E^T\rho\Omega_a^- = E\rho(E\rho)^+(\mathcal{L}_{\rho_a} E - E^T\rho\Omega_a^-) \tag{5.39}$$

holds for some set $\{\Omega_a^-\}$. If there exists a set $\{\rho_a, \Omega_a^-\}$, satisfying the stated conditions,

[8] Remember that $a, b, c = 1, \ldots, k$ and $\mu, \nu, \lambda = 1, \ldots, n = \dim(M)$.

then solutions are given by

$$\Omega_a^+ =(E\rho)^+(\mathcal{L}_{\rho_a}E - E^T\rho\Omega_a^-) + (I-(E\rho)^+E\rho)X_a, \tag{5.40}$$

where $X_a \in C^\infty(M,\mathbb{R}^{k\times n})$ is arbitrary. In addition, any solution Ω_a^+ satisfies (5.40).

Remark. Theorem 5.2 is a new result. The necessary and sufficient conditions for the generalised Killing equation to have solutions, as well as the form of the general solution has not appeared before. In [32] a solution was found in the special case that $(\alpha^*+\rho)^{-1}$ exists.[9] In particular Eq. (3.17) and Eq. (3.18) of [32] give

$$\begin{aligned}
\omega^\mu{}_{\nu\lambda} =&\Gamma^\mu{}_{\nu\lambda} - \phi^\mu{}_{\nu\lambda} + [(\alpha^*-\rho)^{-1}]^\mu{}_\kappa\left(\nabla^{LC}_\nu(\alpha^*+\rho)^\kappa{}_\lambda - \frac{1}{2}\rho^\tau_\nu H_{\tau\lambda}{}^\kappa\right),\\
\phi^\mu{}_{\nu\lambda} =&[(\alpha^*-\rho)^{-1}]^\mu{}_\kappa\left(\nabla^{LC}_\lambda\rho^\kappa_\nu - \rho^\kappa_\tau[(\alpha^*-\rho)^{-1}]^\tau{}_\eta\left(\nabla^{LC}_\nu(\alpha^*+\rho)^\eta{}_\lambda - \frac{1}{2}\rho^\tau_\nu H_{\tau\lambda}{}^\iota\right)\right),
\end{aligned}$$

where ∇^{LC} denotes the Levi–Civita connection of G, with connection coefficients $\Gamma^\mu{}_{\nu\lambda}$. It follows from Theorem 5.2 that a family of solutions exist in this case, with the above being a particular solution. No statement was made about the closure of the gauge algebroid for this choice, nor the existence of solutions when $(\alpha^*+\rho)^{-1}$ does not exist.

Having found the general solution to the generalised Killing equation (5.21a) we now turn our attention to those which result in a closed gauge algebroid. These are gaugings which are at least locally integrable. As shown in Section 5.4.3 the task now is to find $\Omega_a^\pm$ such that ${}^Q\Omega_a^\pm$ define flat connections ${}^Q\nabla^\pm$.

First we will consider the case of gauging with respect to a set of linearly independent independent vector fields $\{\rho_a\}$, $a=1,\ldots,k$ where $k\leq \dim(M)$. In such a case $\mathrm{rank}(\mathrm{Im}(\rho))=k$ and corresponds to gauging with respect to a regular Lie algebroid. The general case, allowing $\mathrm{rank}(\mathrm{Im}(\rho))$ to vary at different points in M, is treated separately. An example of the non-isometric gauging, demonstrating the results of this chapter, is given in Example 5.8.

Gauging with linearly independent vector fields

Here we consider the case of gauging with respect to a set of linearly independent vector fields $\{\rho_a\}$ where $a=1,\ldots,k$ and $k\leq\dim(M)$. This allows us to describe gauging with respect to regular Lie algebroids $[\rho_a,\rho_b]=C^c{}_{ab}\rho_c$. The components $\rho^\mu{}_a$ define a $n\times k$ matrix. As a consequence of the linear independence of $\{\rho_a\}$, the columns of ρ are linearly independent and $\rho^T\rho$ is invertible. In this case we have an explicit formula

[9] The notation follows Section 5.3, and we refer the reader to the paper [32] for details.

for the pseudo-inverse

$$\rho^+ = (\rho^T\rho)^{-1}\rho^T.$$

It follows that ρ^+ is a *left-inverse* for ρ: explicitly $\rho^+\rho = I$ the $k \times k$ identity matrix.

Throughout this chapter we assume that E is invertible (this is true for almost all physical models) and hence satisfies $\operatorname{rank}(E) = n$. As E defines an endomorphism it follows that $\operatorname{rank}(E\rho) = \operatorname{rank}(\rho)$. $E\rho$ is a $n \times k$ matrix, and has linearly independent columns

$$(E\rho)^+ = (\rho^T E^T E\rho)^{-1}\rho^T E^T$$

and $(E\rho)^+$ is a *left-inverse* for $E\rho$

$$(E\rho)^+ E\rho = I. \tag{5.41}$$

Assuming that $\{\rho_a\}$ are linearly independent simplifies the general solution to the generalised Killing equation (5.21a):

$$\Omega^+_a =(E\rho)^+(\mathcal{L}_{\rho_a}E - E^T\rho\Omega^-_a). \tag{5.42}$$

When condition (5.37) holds, the generalised Killing equation has solutions, and they are given by (5.42).

In order to impose the Q-flatness condition on ${}^Q\nabla^\pm$ it is useful to get an expression for the components $({}^Q\Omega^+_a)^b{}_c$. Using (5.29) we have:

$$\begin{aligned}
({}^Q\Omega^+_a)^b{}_c =&(\Omega^+_a)^b{}_\mu\rho^\mu_c + C^b_{ac}\\
=&(E\rho)^+(\mathcal{L}_{\rho_a}E - E^T\rho\Omega^-_a)^b{}_\mu\rho^\mu_c + C^b{}_{ac}\\
=&((E\rho)^+)^b{}_\lambda(\mathcal{L}_{\rho_a}E)_{\lambda\mu}\rho^\mu_c - ((E\rho)^+)^b{}_\lambda E_{\kappa\lambda}\rho^\kappa_d(\Omega^-_a)^d{}_\mu\rho^\mu_c + C^b{}_{ac}\\
=&((E\rho)^+)^b{}_\lambda(\mathcal{L}_{\rho_a}E)_{\lambda\mu}\rho^\mu_c - ((E\rho)^+)^b{}_\lambda E_{\kappa\lambda}\rho^\kappa_d(({}^Q\Omega^-)^d{}_{ac} - C^d{}_{ac}) + C^b{}_{ac}.
\end{aligned}$$

This expression can be simplified.

Lemma 5.3. *The following identity holds:*

$$(\mathcal{L}_{\rho_a}E)_{\lambda\mu}\rho^\mu_c = (\rho^+)^d_\lambda\rho^\kappa_a\partial_\kappa(\rho^\nu_d E_{\nu\mu}\rho^\mu_c) - (\rho^+)^c_\lambda C^d{}_{ac}\rho^\kappa_d E_{\kappa\mu}\rho^\mu_c - E_{\lambda\mu}C^d{}_{ac}\rho^\mu_d. \tag{5.43}$$

Proof. Recall that $[\rho_a, \rho_b] := C^c{}_{ab}\rho_c$. By direct computation

$$\begin{aligned}
(\mathcal{L}_{\rho_a}E)_{\lambda\mu}\rho^\mu_c =&\rho^\kappa_a(\partial_\kappa E_{\lambda\mu})\rho^\mu_c + (\partial_\lambda\rho^\kappa_a)E_{\kappa\mu}\rho^\mu_c + (\partial_\mu\rho^\kappa_a)E_{\lambda\kappa}\rho^\mu_c\\
=&\rho^\kappa_a(\partial_\kappa E_{\lambda\mu}\rho^\mu_c) - E_{\lambda\mu}(\rho^\kappa_a\partial_\kappa\rho^\mu_c - \rho^\kappa_c\partial_\kappa\rho^\mu_a) + (\partial_\lambda\rho^\kappa_a)E_{\kappa\mu}\rho^\mu_c
\end{aligned}$$

$$
\begin{aligned}
&= \rho_a^\kappa(\partial_\kappa E_{\lambda\mu}\rho_c^\mu) - E_{\lambda\mu}([\rho_a,\rho_c])^\mu + (\partial_\lambda \rho_a^\kappa)E_{\kappa\mu}\rho_c^\mu \\
&= \rho_a^\kappa(\partial_\kappa E_{\lambda\mu}\rho_c^\mu) - E_{\lambda\mu}C^d{}_{ac}\rho_d^\mu + (\partial_\lambda \rho_a^\kappa)E_{\kappa\mu}\rho_c^\mu. \qquad (5.44)
\end{aligned}
$$

Next we use the fact that $\rho^+\rho = I$,

$$
\rho_a^\kappa\partial_\kappa(E_{\lambda\mu}\rho_c^\mu) = \rho_a^\kappa\partial_\kappa((\rho^+)^d_\lambda\rho_d^\nu E_{\nu\mu}\rho_c^\mu) = \rho_a^\kappa\partial_\kappa((\rho^+)^d_\lambda)\rho_d^\nu E_{\nu\mu}\rho_c^\mu + \rho_a^\kappa(\rho^+)^d_\lambda\partial_\kappa(\rho_d^\nu E_{\nu\mu}\rho_c^\mu). \qquad (5.45)
$$

Using $d(\rho^+\rho) = dI = 0$ to conclude that $(d\rho^+)\rho = -\rho^+ d\rho$, we have

$$
\begin{aligned}
\rho_a^\kappa\partial_\kappa((\rho^+)^d_\lambda)\rho_d^\nu E_{\nu\mu}\rho_c^\mu &= -(\rho^+)^d_\lambda\rho_a^\kappa\partial_\kappa\rho_d^\nu E_{\nu\mu}\rho_c^\mu \\
&= (\rho^+)^d_\lambda C^e{}_{da}\rho_e^\nu E_{\nu\mu}\rho_c^\mu - (\rho^+)^d_\lambda\rho_d^\kappa\partial_\kappa\rho_a^\nu E_{\nu\mu}\rho_c^\mu \\
&= (\rho^+)^c_\lambda C^d{}_{ca}\rho_d^\kappa E_{\kappa\mu}\rho_c^\mu - (\partial_\lambda\rho_a^\nu)E_{\nu\mu}\rho_c^\mu. \qquad (5.46)
\end{aligned}
$$

Combining (5.44), (5.45) and (5.46) gives the desired result (5.43). □

Substituting (5.43) in the expression for ${}^Q\Omega_a^+$ gives

$$
\begin{aligned}
({}^Q\Omega_a^+)^b{}_c =& ((E\rho)^+)^b{}_\lambda(\rho^+)^d_\lambda\rho_a^\kappa\partial_\kappa(\rho_d^\nu E_{\nu\mu}\rho_c^\mu) + (\delta^b{}_d - ((E\rho)^+E\rho)^b{}_d)C^d{}_{ac} \\
&- ((E\rho)^+)^b{}_\lambda E_{\kappa\lambda}\rho_d^\kappa({}^Q\Omega^-)^d{}_{ac} \\
=& ((\rho^T E\rho)^+)^b{}_\lambda\rho_a^\kappa\partial_\kappa(\rho^T E\rho)^\lambda{}_c - (E\rho)^+E^T\rho({}^Q\Omega_a^-)^b{}_c.
\end{aligned}
$$

This can be written more succinctly as

$$
{}^Q\Omega_a^+ = (\rho^T E\rho)^+\rho_a(\rho^T E\rho) - (E\rho)^+E^T\rho\,{}^Q\Omega_a^-. \qquad (5.47)
$$

Whenever (5.37) holds there exist solutions to the gauging constraint (5.21a); the general solution is given by (5.47).

Closure of the gauge algebroid imposes the Q-flatness condition $R_{{}^Q\nabla^\pm} = 0$. If ${}^Q\nabla^-$ is flat there exists a choice of frame $\tilde{e}_b$ such that $({}^Q\widetilde{\Omega}^-)^a{}_{bc} = 0$. Take $M \in \mathsf{End}(Q)$, such that $\tilde{e}_b = M^a{}_b e_a$. Now $\tilde{\rho}_b = M^a{}_b\rho_a$ and in this frame

$$
{}^Q\widetilde{\Omega}_a^+ = (\tilde{\rho}^T E\tilde{\rho})^+\tilde{\rho}_a(\tilde{\rho}^T E\tilde{\rho}). \qquad (5.48)
$$

Recall that the coefficients ${}^Q\Omega_a^\pm$ define flat connections ${}^Q\nabla^\pm$ if and only if they are of the form

$$
{}^Q\Omega_a^\pm = (K_\pm)^{-1}\rho_a(K_\pm) = (K_\pm)^{-1}\rho_a^\mu\partial_\mu K_\pm, \qquad (5.49)
$$

for some $K_\pm \in C^\infty(M, \mathsf{GL}(k))$. We require that

$$^Q\widetilde{\Omega}^+_a = (\widetilde{K}_+)^{-1}\tilde{\rho}_a(\widetilde{K}_+) = (\tilde{\rho}^T E\tilde{\rho})^+\tilde{\rho}_a(\tilde{\rho}^T E\tilde{\rho}),$$

which is satisfied if $\widetilde{K}_+ = \tilde{\rho}^T E\tilde{\rho}$ and $\tilde{\rho}^T E\tilde{\rho}$ is invertible. This provides a necessary condition for $^Q\widetilde{\Omega}^+_a$ to be flat. Furthermore we note that

$$\mathrm{rank}(\tilde{\rho}^T E\tilde{\rho}) = \mathrm{rank}(\rho^T E\rho),$$

as $\tilde{\rho}_a$ and ρ_b are related by $M \in \mathsf{End}(Q)$. If $^Q\nabla^+$ is flat then $(\rho^T E\rho)$ is invertible, and $(\rho^T E\rho)^+ = (\rho^T E\rho)^{-1}$. Returning to (5.47) we find

$$^Q\Omega^+_a = (\rho^T E\rho)^{-1}\rho_a(\rho^T E\rho) - (E\rho)^+E^T\rho^Q\Omega^-_a, \tag{5.50}$$

and have a solution for $^Q\Omega^-_a = 0$, $^Q\Omega^+ = (K_+)^{-1}\rho^\mu_a\partial_\mu K_+$ with $K_+ = \rho^T E\rho$. In local coordinates

$$(^Q\Omega^-)^a_{bc} = 0 = (\Omega^-)^a{}_{\mu b}\rho^\mu_c + C^a{}_{bc}, \quad \Rightarrow (\Omega^-)^a{}_{\mu b} = -(\rho^+)^c_\mu C^a{}_{bc}.$$

Let us define $\Psi_a \in \Gamma(T^*M \otimes T^*M)$

$$(\Psi_a)_{\mu\nu} := (\mathcal{L}_{\rho_a}E)_{\mu\nu} - E_{\lambda\nu}\rho^\lambda_b C^b{}_{ac}(\rho^+)^c_\mu. \tag{5.51}$$

Taking $^Q\Omega^-_a = 0$ modifies Eq. (5.38) (the necessary condition for the existence of Ω^+_a) so that consistency requires

$$\Psi_a = E\rho(E\rho)^+\Psi_a. \tag{5.52}$$

We are now ready to state the main result for gauging with respect to linearly independent vector fields.

Theorem 5.4. *Given an invertible field $E \in \Gamma(T^*M\otimes T^*M)$ and a choice of involutive linearly independent vector fields $\rho_a \in \Gamma(TM)$, $a = 1, \ldots, k$, defining a regular Lie algebroid $[\rho_a, \rho_b] = C^c{}_{ab}\rho_c$, the generalised Killing equation*

$$(\mathcal{L}_{\rho_a}E)_{\mu\nu} = E_{\mu\lambda}\rho^\lambda_b(\Omega^+)^b{}_{\nu a} + E_{\lambda\nu}\rho^\lambda_b(\Omega^-)^b{}_{\mu a},$$

has solutions for $\Omega^\pm_a \in \Gamma(T^*M \otimes Q)$ if and only if

$$\Psi_a = E\rho(E\rho)^+\Psi_a, \quad \det(\rho^T E\rho) \neq 0, \tag{5.53}$$

where

$$(\Psi_a)_{\mu\nu} := (\mathcal{L}_{\rho_a} E)_{\mu\nu} - E_{\lambda\nu}\rho^\lambda_b C^b{}_{ac}(\rho^+)^c_\mu, \quad (E\rho)^+ = (\rho^T E^T E\rho)^{-1}\rho^T E^T. \tag{5.54}$$

If these conditions are satisfied then a solution is given by

$$(\Omega^-)^a{}_{\mu b} = -(\rho^+)^c_\mu C^a{}_{bc}, \qquad (\Omega^+)^a{}_{\mu b} = ((E\rho)^+)^a{}_\lambda(\Psi_b)_{\lambda\mu}, \tag{5.55}$$

where $(\Omega^\pm)^a{}_{\mu b}$ *are connection coefficients defining* $\nabla^\pm$*. The corresponding flat adjoint connections* ${}^Q\nabla^\pm$ *are defined by*

$${}^Q\Omega^-_a = 0, \qquad {}^Q\Omega^+_a = (\rho^T E\rho)^{-1}\rho^\lambda_a\partial_\lambda(\rho^T E\rho). \tag{5.56}$$

Definition 5.5. A non-linear sigma model $(X, \Sigma, h, E, H, S_Q[X])$ can be *locally gauged* when there exists some $U \subset M$ such that $\rho_a \in \Gamma(TU)$ and ${}^Q\Omega^\pm \in \Gamma(Q^*_U \otimes \mathsf{End}(Q_U))$ satisfy the requirements of Theorem 5.4. The local non-linear sigma model is given by the restriction of $(X, \Sigma, h, E, H, S_Q[X])$ to $U \subset M$. The adjoint connections ${}^Q\nabla^\pm$ give representations of local algebroid actions. If $U = M$, and the Lie algebroid actions can be integrated to Lie groupoid actions $\mathcal{G}({}^Q\nabla^\pm)$, we say that the non-linear sigma model can be *gauged*.

Corollary 5.6. *All non-linear sigma models* $(X, \Sigma, h, E, H, S_Q[X])$ *can be locally gauged when* $E \in \Gamma(T^*M \otimes T^*M)$ *is invertible.*

Proof. Choose a local trivialisation of M, denoted $U \subset \mathbb{R}^n$, and a set of linearly independent $\{\rho_a\} \in \Gamma(TM)$, $a = 1, \ldots, n = \dim(M)$ (such a set of vector fields exists see for Example 5.6). These vector fields define a local frame for $Q_U \cong TU$ and define a Lie algebroid $[\rho_a, \rho_b] = C^c{}_{ab}\rho_c$. The associated matrix ρ is square and invertible, giving $\rho^+ = \rho^{-1}$. If E is invertible then $E\rho$ and $\rho^T E\rho$ have rank(n) and $(E\rho)^+ = (E\rho)^{-1}$. The compatibility conditions (5.53) are satisfied. The construction of Theorem 5.4 gives a choice of $\nabla^\pm$ (or equivalently ${}^{Q_U}\nabla^\pm$). □

Example 5.6 (Local gauging). Take a manifold M, and some $U \subset M$ with local coordinates $\{x_1, x_2, \ldots, x_n\}$. It is always possible to take $Q_U \cong TU$ and $\{\rho_a\} = \{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ giving $\rho = I_{n\times n}$.

Corollary 5.7 (Global gauging)**.** *If there exists a global frame (the manifold* M *is parallelisable) the non-linear sigma model* $(X, \Sigma, h, E, H, S_Q[X])$ *can be gauged.*

Proof. Take a set of globally defined non-zero linearly independent vector fields which span TM. These exist by the definition of a parallelisable manifold. The construction of Corollary 5.6 can be extended globally using this choice of vector fields. □

Examples of parallelisable manifolds include Lie groups, S^1, S^3, S^7, and all oriented three-manifolds.

We see that the notion of local gauging allows us to get a local understanding of gauging with respect to regular smooth distributions. A regular smooth distribution has locally constant rank by definition. Fix some $U \subset M$ such that $\dim(\mathrm{Im}(\rho)) = k$ is constant. We can construct a local gauging using the procedure described above (by definition there is a local set of linearly independent vector fields spanning the distribution). The local gaugings may be of different dimensions for different choices of $U \subset M$.

Consider rescaling a set of vectorfields $\{\rho_a\}$ by $f \in C^\infty(M)$ to give $\{f\rho_a\}$. Suppose that $\{\rho_a\}$ satisfies the generalised Killing condition

$$(\mathcal{L}_{\rho_a} E)_{\mu\nu} = E_{\mu\lambda}\rho_b^\lambda (\Omega^+)^b{}_{\nu a} + E_{\lambda\nu}\rho_b^\lambda (\Omega^-)^b{}_{\mu a}.$$

It follows that

$$(\mathcal{L}_{f\rho_a} E)_{\mu\nu} = E_{\mu\lambda} f\rho_b^\lambda (\Omega'^+)^b{}_{\nu a} + E_{\lambda\nu} f\rho_b^\lambda (\Omega'^-)^b{}_{\mu a},$$

for $(\Omega'^\pm)^b{}_{\mu a} = (\Omega^\pm)^b{}_{\mu a} + \delta^b{}_a f^{-1}\partial_\mu f$. If $f > 0$ it is possible rescale any set $\{\rho_a\}$ of vector fields whilst still satisfying the generalised Killing equations (5.21a) by modifying the connection. The generalised Killing equation is satisfied trivially if the vector fields $\{\rho_a\}$ are identically zero. We note that we cannot use a partition of unity type construction to globalise the vector fields used in the local gauged solutions by setting $\{\rho_a\}$ to zero outside of U. It is clear that in the limit $f \to 0$ we have $f^{-1}\partial_\mu f = \partial_\mu \ln(f) \to \infty$ and $\Omega'^\pm$ is unbounded. There is no straightforward way to extend local gaugings to the entire manifold.

Gauging general vector fields

In this section we consider Lie algebroid gauging for a general set of involutive vector fields $\{\rho_a\}$—dropping the linear independence requirement. A different perspective is taken from the last section. If the generalised Killing equation holds, then a projected version of the equation holds (5.57). This projected form can be 'lifted' to an equation on the vector bundle Q using Lie algebroid geometry. An expression defining the flat adjoint connections ${}^Q\nabla^\pm$ can be found in terms of objects on the vector bundle Q.

A general set of involutive vector fields may become linearly dependent for some set of points in M. There are a fixed number of vector fields $k \leq \dim(M)$, but the image of the distribution spanned by the vector fields is allowed to change dimension. A set of involutive vector fields describes a generalised distribution and defines a singular foliation. Not all singular foliations are generated by vector fields in this way and we restrict ourselves to those which do. As a concrete example of the type of vector fields we are interested in consider the following:

Example 5.7 (Generalised distribution). Let $M = \mathbb{R}^2$ with coordinates $\{x, y\}$. Consider the vector fields

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial x} + f(x)\frac{\partial}{\partial y},$$

where $f(x) \in C^\infty(M)$ satisfies $f(x) = 0$ for $x \leq 0$, and $f(x) > 0$ for $x > 0$. Let Δ be the distribution given by the span of X and Y. At each point $x \in M$, $\Delta_x \subseteq TM_x$ is the vector space given by the linear span of $X(x)$ and $Y(x)$. For $x > 0$ $\dim(\Delta) = 2$, and for $x \leq 0$ $\dim(\Delta) = 1$.

$$[X, Y] = f'(x)\partial_y.$$

At each point $x_0 \in M$ we have that $[X, Y](x_0)$ is in the span of $\{X(x_0), Y(x_0)\}$ and the distribution is involutive. To be explicit, when $x > 0$ $[X, Y](x_0) = \ln(f)'Y(x_0) - \ln(f)'X(x_0)$ and when $x \leq 0$ $[X, Y] = 0$. By the Stefan–Sussman theorem the distribution Δ describes a singular foliation, with Δ giving the partition into leaves. For an enlightening discussion of this example see [115].

Theorem 5.2 does not assume that $\{\rho_a\}$ are linearly independent and still applies as a necessary condition on solving the generalised Killing equation (5.21a) in the general case. The issue remaining is the possible construction of flat connections ${}^Q\nabla^\pm$ to ensure closure of the gauge algebroid. If a set of vector fields $\{\rho_a\}$ satisfy the generalised Killing equation

$$(\mathcal{L}_{\rho_a} E)_{\mu\nu} = E_{\mu\lambda}\rho^\lambda_b(\Omega^+)^b{}_{\nu a} + E_{\lambda\nu}\rho^\lambda_b(\Omega^-)^b{}_{\mu a},$$

it follows that

$$\rho^\mu_d\rho^\nu_c(\mathcal{L}_{\rho_a} E)_{\mu\nu} = \rho^\mu_d\rho^\nu_c E_{\mu\lambda}\rho^\lambda_b(\Omega^+)^b{}_{\nu a} + \rho^\mu_d\rho^\nu_c E_{\lambda\nu}\rho^\lambda_b(\Omega^-)^b{}_{\mu a}. \tag{5.57}$$

The converse is not true in general, Eq. (5.57) does not imply that the generalised Killing equation is satisfied. It is instructive to 'lift' Equation 5.57 to the vector bundle Q. By 'lift' we mean replacing $\Omega^\pm$ (which define the TM-connections on Q) with ${}^Q\Omega^\pm$ (which define the Q-connections on Q), and replacing the field $E \in \Gamma(TM \otimes TM)$ with $\mathsf{E} : \Gamma(Q \otimes Q)$; where

$$\mathsf{E}(\cdot, \cdot) := E(\rho(\cdot), \rho(\cdot)).$$

Noting that (5.29) gives $\rho^\mu_c(\Omega^\pm)^a{}_{\mu b} = ({}^Q\Omega^\pm)^a{}_{bc} - C^a{}_{bc}$, it follows that (5.57) is equivalent to

$$\begin{aligned}\rho^\mu_d\rho^\nu_c(\mathcal{L}_{\rho_a}E)_{\mu\nu} =&\rho^\mu_d E_{\mu\lambda}\rho^\lambda_b(({}^Q\Omega^+)^b{}_{ac} - C^b{}_{ac}) + \rho^\lambda_b E_{\lambda\nu}\rho^\nu_c(({}^Q\Omega^-)^b{}_{ad} - C^b{}_{ad}),\\ =&\mathsf{E}_{db}(({}^Q\Omega^+)^b{}_{ac} - C^b{}_{ac}) + \mathsf{E}_{bc}(({}^Q\Omega^-)^b{}_{ad} - C^b{}_{ad}).\end{aligned} \tag{5.58}$$

The Lie algebroid defined by $[\rho_a, \rho_b] = C^c{}_{ab}\rho_c$ can be lifted to a Lie algebroid on Q as follows: Choose a frame for Q, denoted $\{e_a\}$, and define a Lie algebroid by $[e_a, e_b]_Q := C^c{}_{ab}e_c$, and $\rho(e_a) := \rho_a$.

Recall from Section 3.3 that we can define a Lie derivative on Q,

$$(\mathscr{L}_q\mathsf{E})(q_1, q_2) = \rho(q)(\mathsf{E}(q_1, q_2)) - \mathsf{E}([q, q_1]_Q, q_2) - \mathsf{E}(q_1, [q, q_2]_Q),$$

where $q, q_1, q_2 \in \Gamma(Q)$ and $\mathsf{E} \in \Gamma(Q^* \otimes Q^*)$. The expression in the local basis $\{e_a\}$ is

$$(\mathscr{L}_q\mathsf{E})(q_1, q_2) = q_1^a q_2^b(q^c\rho^\mu_c\partial_\mu\mathsf{E}_{ab} - C^d{}_{ca}\mathsf{E}_{db}q^c - C^d{}_{cb}\mathsf{E}_{ad}q^c + \mathsf{E}_{cb}\rho^\mu_a\partial_\mu q^c + \mathsf{E}_{ac}\rho^\mu_b\partial_\mu q^c),$$

and it follows that

$$(\mathscr{L}_{e_a}\mathsf{E})_{dc} = \rho^\mu_a\partial_\mu\mathsf{E}_{dc} - C^b{}_{ad}\mathsf{E}_{bc} - C^b{}_{ac}\mathsf{E}_{db}. \tag{5.59}$$

The Lie algebroid Lie derivative acting on E can be related to the standard Lie derivative on E:

$$\begin{aligned}(\mathscr{L}_{e_a}\mathsf{E})_{dc} =&\rho^\lambda_a\partial_\lambda(E_{\mu\nu}\rho^\mu_d\rho^\nu_c) - C^b{}_{ad}\rho^\mu_b E_{\mu\nu}\rho^\nu_c - C^b{}_{ac}\rho^\mu_d E_{\mu\nu}\rho^\nu_b,\\ =&\rho^\lambda_a\partial_\lambda(E_{\mu\nu})\rho^\mu_d\rho^\nu_c + E_{\mu\nu}(\rho^\lambda_a\partial_\lambda\rho^\mu_d)\rho^\nu_c + E_{\mu\nu}\rho^\mu_d\rho^\lambda_a\partial_\lambda\rho^\nu_c\\ &- ([\rho_a, \rho_d])^\mu E_{\mu\nu}\rho^\nu_c - ([\rho_a, \rho_c])^\nu\rho^\mu_d E_{\mu\nu}\\ =&\rho^\lambda_a\partial_\lambda(E_{\mu\nu})\rho^\mu_d\rho^\nu_c + (\rho^\lambda_d\partial_\lambda\rho^\mu_a)E_{\mu\nu}\rho^\nu_c + (\rho^\lambda_c\partial_\lambda\rho^\nu_a)\rho^\mu_d E_{\mu\nu}\\ =&\rho^\mu_d\rho^\nu_c(\mathcal{L}_{\rho_a}E)_{\mu\nu}.\end{aligned}$$

Combining (5.58) and (5.59) gives

$$\rho^\mu_a\partial_\mu\mathsf{E}_{dc} = \mathsf{E}_{db}({}^Q\Omega^+)^b{}_{ac} + \mathsf{E}_{bc}({}^Q\Omega^-)^b{}_{ad}. \tag{5.60}$$

We conclude that (5.60) holds if $\{\rho_a\}$ satisfy the generalised Killing condition (5.21a).

We would like to choose a frame in which $({}^Q\Omega^-)^b{}_{ad} = 0$ (such a frame exists as ${}^Q\Omega^-$ is flat). In this case (5.60) simplifies to

$$\rho^\mu_a\partial_\mu\mathsf{E}_{dc} = \mathsf{E}_{db}({}^Q\Omega^+)^b{}_{ac}.$$

As ${}^Q\Omega^+$ is flat we can use (5.33) to conclude that

$$d_Q\mathsf{E} = \mathsf{E}K^{-1}d_QK, \quad K \in C^\infty(M, \mathsf{GL}(k)). \tag{5.61}$$

Now we wish to construct a K satisfying Equation (5.61). If rank$(\mathsf{E}) = k$ we can simply choose $K = \mathsf{E}$. If rank$(\mathsf{E}) = j < k$, we can still construct K. There exists a frame where E is in the form

$$\mathsf{E} = \begin{pmatrix} \mathsf{E}' & X' \\ 0 & 0 \end{pmatrix}, \tag{5.62}$$

where $\mathsf{E}' \in C^\infty(M, \mathbb{R}^{j\times j})$ has rank j, and $X' \in C^\infty(M, \mathbb{R}^{j\times(k-j)})$. Let us denote such a frame by $\{\tilde{e}_a\}$ where $\tilde{e}_a = N^b{}_a e_b$ for some $N \in C^\infty(M, \mathsf{GL}(k))$ and $\{e_a\}$ is the original frame defined by $\rho(e_a) = \rho_a$. We can now take

$$K = \begin{pmatrix} \mathsf{E}' & X' \\ 0 & I \end{pmatrix} \quad \Rightarrow \quad {}^Q\Omega^+ = \begin{pmatrix} \mathsf{E}'^{-1}d'_Q\mathsf{E} & \mathsf{E}'^{-1}d_QX' \\ 0 & 0 \end{pmatrix}.$$

A straightforward calculation shows that this choice satisfies (5.61). After this choice is made it is possible to change back to the original frame.

The construction given above will produce a valid choice ${}^Q\Omega^+$ with ${}^Q\Omega = 0$ (when Theorem 5.2 holds) in the frame where E is of the form (5.62). Transforming back to the original frame will give $({}^Q\Omega)^a{}_{bc} = N^a{}_x\rho^\mu_b\partial_\mu(N^{-1})^x{}_c$. So we need to find a set of vector fields $\{\rho_a\}$ which satisfy Theorem 5.2 when Ω^- satisfies $\rho^\mu_c(\Omega^-)^a{}_{\mu b} = C^a{}_{cb} + {}^Q\Omega^a{}_{bc}$. We are now ready to state the gauging theorem for arbitrary involutive vector fields.

Theorem 5.8. *Given an invertible field $E \in \Gamma(T^*M \otimes T^*M)$ and arbitrary involutive vector fields $\rho_a \in \Gamma(TM)$, $a = 1, \ldots, k$, defining a Lie algebroid $[\rho_a, \rho_b] = C^c{}_{ab}\rho_c$, the generalised Killing equation*

$$(\mathcal{L}_{\rho_a}E)_{\mu\nu} = E_{\mu\lambda}\rho^\lambda_b(\Omega^+)^b{}_{\nu a} + E_{\lambda\nu}\rho^\lambda_b(\Omega^-)^b{}_{\mu a},$$

has solutions defining flat adjoint connections ${}^Q\nabla^\pm$ if

$$\Psi'_a = E\rho(E\rho)^+\Psi'_a, \tag{5.63}$$

where

$$(\Psi'_a)_{\mu\nu} := (\mathcal{L}_{\rho_a}E)_{\mu\nu} - E_{\lambda\nu}\rho^\lambda_b(\Omega^-)^b_{\mu a}, \tag{5.64}$$

for some $(\Omega^-)^b_{\mu a}$ which satisfies $\rho^\mu_c(\Omega^-)^b_{\mu a} = C^b{}_{ca} + ({}^Q\Omega)^b{}_{ac}$ (where ${}^Q\Omega^b{}_{ac}$ is given by (5.65)). Let $N \in C^\infty(M, \mathsf{GL}(k))$ be such that the frame $\tilde{e}_a = N^b{}_a e_a$ brings E to the *form* (5.62)*. If these conditions are satisfied then there exist* ${}^Q\nabla^{\pm}$ *defined by*

$$({}^Q\Omega^-)^a{}_{bc} = N^a{}_x \rho^\mu_b \partial_\mu (N^{-1})^x{}_c, \qquad ({}^Q\Omega^+)^a{}_{bc} = ((KN^{-1})^{-1})^a{}_x \partial^\mu_b (KN^{-1})^x{}_c, \quad (5.65)$$

where $K \in C^\infty(M, \mathsf{GL}(k))$ *is constructed following the discussion preceding this theorem.*

Remark. It appears that lifting the field E, and the generalised Killing equation, from the tangent bundle TM to the vector bundle Q has not been considered before. Instead the projection of geometric structures on the base has been considered in [90]. Algebroid geometry is usually more natural when considered on the vector bundle and becomes less natural when the geometry is projected onto the base. This is familiar in generalised geometry, where the definition of generalised Kähler structures is more natural than the equivalent definition of bi-hermitian structures.

Non-isometric gauging example

We conclude the discussion of non-isometric gauging with an original example. The results of this chapter give a methodical procedure for determining whether a set of vector fields can be used to gauge an action—as well as constructing a choice of connection coefficients when gauging is possible.

Example 5.8. Let $M = \mathbb{R}^3$ with coordinates $\{x, y, z\}$. Take

$$G = (dx)^2 + (dy)^2 + (1 + x^2)(dz)^2, \quad C = 2x dx \wedge dz,$$

giving

$$E = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & 1 + x^2 \end{pmatrix}, \quad H = dC = 0.$$

The field $E \in \Gamma(T^*M \otimes T^*M)$ is invertible as $\det(E) = 1 + 2x^2 \neq 0$.

Consider the possibility of gauging with respect to $\{\rho_a\} = \{\partial_x, \partial_y\}$.

$$\mathcal{L}_{\partial_x} E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2x \end{pmatrix}, \quad \mathcal{L}_{\partial_y} E = 0, \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We note that $\mathcal{L}_{\partial_x} E \neq 0$ so ∂_x is not a Killing vector field. The vector fields $\{\partial_x, \partial_y\}$ are linearly independent—so we apply Theorem 5.4. In this case the necessary consistency condition (5.52) is not satisfied:

$$(E\rho)(E\rho)^{+} = \frac{1}{1+x+x^2}\begin{pmatrix} x & 0 & 1-2x^2 \\ 0 & 0 & 0 \\ -x & 0 & 2x^3-1 \end{pmatrix} \quad \Rightarrow \quad \mathcal{L}_{\partial_x}E \neq E\rho(E\rho)^{+}\mathcal{L}_{\partial_x}E.$$

We conclude that it is not possible to gauge E non-isometrically using the vector fields $\{\partial_x, \partial_y\}$.

Now consider the possibility of gauging with respect to $\{\rho'_a\} = \{\partial_x, \partial_z\}$.

$$\mathcal{L}_{\partial_x}E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2x \end{pmatrix}, \quad \mathcal{L}_{\partial_z}E = 0, \quad \rho' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho'^{+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case the necessary consistency conditions are satisfied, with the only non-trivial part being

$$\mathcal{L}_{\partial_x}E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2x \end{pmatrix} = E\rho'(E\rho')^{+}\mathcal{L}_{\partial_x}E.$$

We conclude that we can gauge with respect to $\{\partial_x, \partial_z\}$. Explictly, using (5.55), we find

$$\Omega^{-} = 0, \quad \Omega^{+} = \frac{1}{1+2x^2}\begin{pmatrix} xdx + (1-x^2)dz & 0 \\ -dx + 3xdz & 0 \end{pmatrix} \quad \Rightarrow \quad {}^{Q}\Omega^{-} = 0, \quad {}^{Q}\Omega^{+} = \mathsf{E}^{-1}\partial_x\mathsf{E}dx,$$

where

$$\mathsf{E} = \rho^T E\rho = \begin{pmatrix} 1 & x \\ -x & 1+x^2 \end{pmatrix}, \quad \det(\mathsf{E}) = 1 + 2x^2 \neq 0.$$

Example 5.8 provides us with an explicit example of gauging where $R_{\nabla^+} \neq 0$ but the gauge algebroid closes ($R_{{}^Q\nabla^{\pm}} = 0$). A direct calculation gives

$$R_{\nabla^+} = \frac{1}{2(1+2x^2)^2}\begin{pmatrix} -2x(5+x^2) & 0 \\ 5-8x^2 & 0 \end{pmatrix} dx \wedge dz.$$

In this case ∇^+ doesn't define a representation of a Lie algebroid on $Q \cong T\mathcal{F}$ (where the leaves of $\mathcal{F}$ are the xz-planes). The local gauging data is given by $\{\rho_a, \Omega^{\pm}\}$. However, from a geometric perspective the gauging symmetry should not be viewed as arising from $\nabla^{\pm}$. The vector fields $\{\rho_a\}$ generate the action of the gauging symmetry on TM. The connections ${}^Q\Omega^{\pm}$ describe the lifted action on sections of Q. The gauging is associated to the flow of the Lie algebroid actions defined by ${}^Q\nabla^{\pm}$. The flat connections ${}^Q\nabla^{\pm}$ define representations of two Lie algebroids $(Q, {}^Q\nabla^{\pm})$. The infinitesimal action is generated by $(Q, {}^Q\nabla^{\pm})$. The finite groupoid action comes from the Weinstein Lie groupoids $\mathcal{G}({}^Q\nabla^{\pm})$.

The choice of gauging in Example 5.8 is not unique. Example 5.8 can also be gauged with respect to the vector fields $\{\rho''_a\} = \{\partial_x, \partial_y, \partial_z\}$. In this case one can take ${}^Q\Omega^- = 0$ and ${}^Q\Omega^+ = E^{-1}\partial_x E dx$.

5.4.5 Alternative choice of gauging: Poisson–Lie

The necessary and sufficient conditions for gauging $S_Q[X]$, given a choice of $E \in \Gamma(T^*M \otimes T^*M)$ and vector fields $\{\rho_a\}$, were described in the previous section. An explicit construction of ${}^Q\Omega^{\pm}$ was given when $\{\rho_a\}$ satisfy the consistency conditions. The choice of connection coefficients $({}^Q\Omega^{\pm})^a{}_{bc}$ is not unique. It is possible to consider gaugings where ${}^Q\Omega^-$ is not set to zero in the $\{e_a\}$ frame if one desires. If the choice of vector fields $\{\rho_a\}$ correspond to isometries of E, then for that particular choice of vector fields we have $\Omega^{\pm} = 0$. Even if E admits no isometries there may still be multiple gaugings. In this section we show if E satisfies the conditions for Poisson–Lie T-duality then there is a non-isometric gauging which differs from that described in Section 5.4.4.

Poisson–Lie T-duality is a proposal for non-isometric T-duality in the presence of a Lie bialgebra structure. Poisson–Lie T-duality was introduced by Klimčík and Ševera [84, 83] and we refer the reader to those papers for details. An overview of Poisson–Lie T-duality is given in Section 2.3.1, and we refer the reader there for notation. Poisson–Lie data consists of

$$(E, \rho_a, J_a) \in \Gamma(T^*M \otimes T^*M) \oplus \Gamma(TM) \oplus \Gamma(T^*\Sigma),$$

where $a = 1, \dots, d = \dim(\mathfrak{g})$, $\{\rho_a\}$ define a Lie algebra $\mathfrak{g}$, and $\{J_a\}$ define a dual Lie algebra $\mathfrak{g}^*$, specified locally by:

$$[\rho_a, \rho_b] = C^c{}_{ab}\rho_c, \qquad dJ_a = \frac{1}{2}\widetilde{C}^{bc}{}_a J_b \wedge J_c, \qquad C^c{}_{ab}, \widetilde{C}^{bc}{}_a \in \mathbb{R}.$$

The Lie algebras are required to form a Lie bialgebra (see Section 3.4.1) satisfying the compatibility conditions:

$$2C^d{}_{[a|g|}\widetilde{C}^{gl}{}_{b]} + 2\widetilde{C}^{dg}{}_{[b}C^f{}_{a]g} - C^g{}_{ab}\widetilde{C}^{dl}{}_g = 0.$$

The metric compatibility condition is given by

$$(\mathcal{L}_{\rho_a}E)_{\mu\nu} = \widetilde{C}^{kl}{}_a \rho^\lambda_k \rho^\tau_l E_{\lambda\nu}E_{\mu\tau}. \tag{5.66}$$

We claim that a sigma model satisfying the Poisson–Lie gauging conditions can be gauged for a choice ${}^Q\Omega^\pm$ that differs from that of Section 5.4.4. This highlights the lack of uniqueness of the Lie groupoid gauging procedure.

Choose the frame where $\rho_a := \rho(e_a)$ coincides with the vector fields for Poisson–Lie gauging. Take

$$({}^Q\Omega^+)^a{}_{bc} = C^a{}_{bc}, \qquad ({}^Q\Omega^-)^a{}_{bc} = \widetilde{C}^{ad}{}_b \mathsf{E}_{cd} + C^a{}_{bc},$$

where $\mathsf{E} = E(\rho(\cdot), \rho(\cdot))$. The Lie algebroid gauging condition (5.21a) can be directly verified. Calculating the Q-curvatures we find:

$$(R^{{}^Q\nabla^+})_{abc} = \rho^\mu_a \partial_\mu C^d{}_{bc} e_d; \tag{5.67a}$$

$$\begin{aligned}(R^{{}^Q\nabla^-})_{abc} =& \Big(\rho^\mu_a \partial_\mu C^d{}_{bc} + 2\rho^\mu_{[a}\partial_{|\mu|}(\widetilde{C}^{dl}{}_{b]}\mathsf{E}_{cl}) + 2(C^d{}_{[a|g|}\widetilde{C}^{gl}{}_{b]} - C^g{}_{ab}\widetilde{C}^{dl}{}_g)\mathsf{E}_{cl} \\ &+ 2\widetilde{C}^{dl}{}_{[a}C^g{}_{b]c}\mathsf{E}_{gl} + 2\widetilde{C}^{dk}{}_{[a|}\mathsf{E}_{gk}\widetilde{C}^{gl}{}_{|b]}\mathsf{E}_{cl}\Big) e_d.\end{aligned} \tag{5.67b}$$

It is clear that $R_{{}^Q\nabla^+} = 0$, as $C^a{}_{bc} \in \mathbb{R}$. To see that $R_{{}^Q\nabla^-}$ vanishes requires a little bit more work. Using the definition of the Lie derivative, and (5.66), we have

$$\rho^\lambda_a \partial_\lambda E_{\mu\nu} = \widetilde{C}^{kl}_a \rho^\lambda_k \rho^\tau_l E_{\lambda\nu} E_{\mu\tau} - (\partial_\mu \rho^\lambda_a) E_{\lambda\nu} - (\partial_\nu \rho^\lambda_a) E_{\mu\lambda}.$$

Multiplying by $\rho^\mu_b \rho^\nu_c$ (which are invertible):

$$\begin{aligned}\rho^\mu_b \rho^\nu_c \rho^\lambda_a \partial_\lambda E_{\mu\nu} =& \widetilde{C}^{kl}_a \mathsf{E}_{kc}\mathsf{E}_{bl} - (\rho^\mu_b \partial_\mu \rho^\lambda_a) E_{\lambda\nu}\rho^\nu_c - (\rho^\nu_c \partial_\nu \rho^\lambda_a) E_{\mu\lambda}\rho^\mu_b \\ =& \widetilde{C}^{kl}_a \mathsf{E}_{kc}\mathsf{E}_{bl} + C^d{}_{ab}\rho^\lambda_d E_{\lambda\nu}\rho^\nu_c + (\rho^\mu_a \partial_\mu \rho^\lambda_b) E_{\lambda\nu}\rho^\nu_c \\ &+ C^d{}_{ac}\rho^\lambda_d E_{\mu\lambda}\rho^\mu_b + (\rho^\nu_a \partial_\nu \rho^\lambda_c) E_{\mu\lambda}\rho^\mu_b,\end{aligned}$$

so that $\rho^\mu_a \partial_\mu \mathsf{E}_{bc} = \widetilde{C}^{kl}_a \mathsf{E}_{kc} E_{bl} + C^d{}_{ab}\mathsf{E}_{dc} + C^d{}_{ac}\mathsf{E}_{bd}$. Substituting this into (5.67b), gives

$$(R^{{}^Q\nabla^-})_{abc} = (2C^d{}_{[a|g|}\widetilde{C}^{gl}{}_{b]} + 2\widetilde{C}^{dg}{}_{[b}C^f{}_{a]g} - C^g{}_{ab}\widetilde{C}^{dl}{}_g)\mathsf{E}_{cl}e_d.$$

We conclude that $R_{{}^Q\nabla_-} = 0$ as $C^a{}_{bc}$ and $\widetilde{C}^{ab}{}_c$ define a Lie bialgebra.

5.5 Application: T-duality

As an application of non-isometric Lie algebroid gauging we consider non-isometric T-duality. A proposal for non-isometric T-duality was given by CDJ in [31, 33]. The constraints for non-isometric T-duality are slightly different than those of Lie algebroid gauging. In order to clarify the difference we briefly review the case of non-abelian T-duality for a WZW model.

Recall from Section 5.2 that a gauged WZW model is described by

$$S_{WZW}[g, A, \widehat{X}] = \frac{1}{2}\int_\Sigma (g^{-1}Dg \overset{\wedge}{,} \star g^{-1}Dg)_G + \int_\Sigma (g^{-1}Dg \overset{\wedge}{,} g^{-1}Dg)_B + \int_\Sigma \langle \widehat{X}, F\rangle,$$

where $\widehat{X}$ is a Lagrange multiplier whose equations of motion impose the flatness condition $F = 0$. When $F = 0$ there exists a gauge transformation which allows us to set $A = 0$ and recover the ungauged action.

The first two terms were shown to be invariant in Section 5.2. The left group action on the field strength gives

$$\begin{aligned} h \circlearrowleft F =& d(hAh^{-1} + dhh^{-1}) - [(hAh^{-1} + dhh^{-1}) \overset{\wedge}{,} (hAh^{-1} + dhh^{-1})]_{\mathfrak{g}} \\ =& h(dA - [A \overset{\wedge}{,} A]_{\mathfrak{g}})h^{-1} = \mathrm{Ad}_h F, \end{aligned}$$

where we have used the identities $dhh^{-1} = -hdh^{-1}$ (which follows from $d(hh^{-1}) = d(e) = 0$) and $[A \overset{\wedge}{,} A']_{\mathfrak{g}} = [A' \overset{\wedge}{,} A]_{\mathfrak{g}}$.

The final term in the Lagrangian will be invariant under the left action of $h \in C^\infty(\Sigma, \mathsf{G})$ if and only if

$$h \circlearrowleft \widehat{X} = \mathrm{Ad}^*_{h^{-1}}\widehat{X},$$

where $\mathrm{Ad}^*_{h^{-1}}$ is the *coadjoint action* $\langle \mathrm{Ad}^*_h \widehat{X}, X\rangle := \langle \hat{X}, \mathrm{Ad}_h X\rangle$for $\widehat{X} \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. The importance of this standard calculation is that the invariance of the action with the additional field strength term does not follow automatically from invariance of the other terms.

The full left gauging action is given by

$$h \circlearrowleft (g, A, \widehat{X}) = (hg, hAh^{-1} + dhh^{-1}, \mathrm{Ad}^*_{h^{-1}}\widehat{X}). \tag{5.68}$$

The introduction of the curvature term is an essential part of the non-abelian T-duality procedure. For each invariant action $S[X]$ (invariant under the action of a Lie algebra $\mathfrak{g}$) a gauged action $S[X, A, \widehat{X}]$ can be constructed. The original action can be recovered by solving the equations of motion for $\widehat{X}$ and fixing the gauge fields $A = 0$. Alternatively, solving the equations of motion for A and gauge fixing X gives an action $\widehat{S}[\widehat{X}]$ describing a dual model. In the dual model the fields $\widehat{X}$ are interpreted as local coordinates on some dual manifold $\widehat{M}$.

The procedure can be summarised by the following diagram:[10]

$$
\begin{tikzcd}
 & S[X, A, \widehat{X}] \arrow[dl, bend right, "\text{integrate } \widehat{X}"', "\text{gauge fix } A"] \arrow[dr, bend left, "\text{integrate } A", "\text{gauge fix } X"'] & \\
S[X] \arrow[ur, bend right, "\text{gauge isometries}"'] & & \widehat{S}[\widehat{X}]
\end{tikzcd}
$$

The construction of the dual model was given in [103, 44] and is described in Section 5.5.4. We note that the T-dual pair $(S[X], \widehat{S}[\widehat{X}])$ are uniquely determined by the choice of gauged action $S[X, A, \widehat{X}]$. If two gauging proposals produce the same gauged action they must describe the same T-duality pair.

5.5.1 Non-isometric T-duality proposal

A proposal to carry out T-duality with respect to a set of vector fields which do not generate isometries has recently appeared in the literature [31, 33]. This 'non-isometric T-duality' is based on the formalism of Lie algebroid gauging developed in [32, 99, 88]. The non-linear sigma model is specified by the data $(X, \Sigma, h, M, G, H, S[X])$, where $X : \Sigma \to M$ is a map embedding a (pseudo-)Riemannian string worldsheet (Σ, h) into a (pseudo-)Riemannian manifold (M, G), and $H \in \Gamma(\wedge^3 T^* M)$. The action is given by (Eq. 2.1 in [31]):

$$S[X] = \frac{1}{2} \int_\Sigma G_{\mu\nu} dX^\mu \wedge \star dX^\nu + \frac{1}{6} \int_{\Sigma_3} H_{\mu\nu\lambda} dX^\mu \wedge dX^\nu \wedge dX^\lambda, \tag{5.69}$$

where Σ_3 is a three manifold with boundary $\partial\Sigma_3 = \Sigma$.

The action $S[X]$ is gauged with respect to a set of vector fields $\rho_a \in \Gamma(TM)$, satisfying $[\rho_a, \rho_b] = C^c{}_{ab} \rho_c$, for $C^a{}_{bc} \in C^\infty(M)$.

The gauged action is (Eq 2.8 in [31]):

$$\begin{aligned} S_{\text{gauged}} = \frac{1}{2} \int_\Sigma G_{\mu\nu} DX^\mu \wedge \star DX^\nu + \frac{1}{6} \int_{\Sigma_3} H_{\mu\nu\lambda} dX^\mu \wedge dX^\nu \wedge dX^\lambda \\ - \int_\Sigma (\theta_a + d\widehat{X}_a) \wedge A^a + \frac{1}{2} \int_\Sigma (\iota_{\rho_{[a}} \theta_{b]} + C^a{}_{bc} \widehat{X}_a) A^b \wedge A^c - \int_\Sigma \omega^a{}_{\mu b} \widehat{X}_a A^b \wedge DX^\mu, \end{aligned} \tag{5.70}$$

where $DX^\mu = dX^\mu - \rho_a^\mu A^a$ and $\theta_a = \theta_{a\mu} dX^\mu$.

The infinitesimal gauge transformations are of the following form:

$$\delta_\varepsilon X^\mu = \varepsilon^a \rho_a^\mu, \tag{5.71a}$$

[10] This Ti*k*Z picture is modified from a presentation by Mark Bugden.

$$\delta_\varepsilon A^a = d\varepsilon^a + C^a{}_{bc} A^b \varepsilon^c + \omega^a_{\mu b}(dX^\mu - \rho^\mu_c A^c), \tag{5.71b}$$

$$\delta_\varepsilon \widehat{X}_a = -\iota_{\rho_{(a}} \theta_{b)} \varepsilon^b - C^c{}_{ba} \varepsilon^b \widehat{X}_c + \rho^\mu_a \omega^d{}_{\mu b} \widehat{X}_d \varepsilon^b. \tag{5.71c}$$

The gauged action is invariant under the gauge transformations if the following *non-isometric gauging conditions* hold:

$$\mathcal{L}_{\rho_a} G = \omega^b{}_a \vee \iota_{\rho_b} G, \tag{5.72a}$$

$$\iota_{\rho_a} H = d\theta_a + \theta_b \wedge \omega^b{}_a - \widehat{X}_b R^b{}_a. \tag{5.72b}$$

There are additional consistency conditions required to carry out non-isometric T-duality:

$$\mathcal{L}_{\rho_{[a}} \theta_{b]} = C^d{}_{ab} \theta_d - \iota_{\rho_d} \theta_{[a} \omega^d{}_{b]} - \iota_{\rho_{[a}} \omega^d{}_{b]} \theta_d - D^c{}_{ba} \widehat{X}_c, \tag{5.73a}$$

$$\frac{1}{3} \iota_{\rho_a} \iota_{\rho_b} \iota_{\rho_c} H = \iota_{\rho_{[a}} C^d{}_{bc]} \theta_d - 2\iota_{\rho_{[a}} \omega^d{}_b \iota_{\rho_{c]}} \theta_d - 2\widetilde{D}^e{}_{abc} \widehat{X}_e, \tag{5.73b}$$

where

$$D^e{}_{ab} = dC^e{}_{ab} + C^c{}_{ab} \omega^e{}_c + 2C^e{}_{d[a} \omega^d_{b]} + 2\iota_{\rho_d} \omega^e{}_{[b} \omega^d{}_{a]} + 2\mathcal{L}_{\rho_{[b}} \omega^e{}_{a]} + \iota_{\rho_{[a}} R^e{}_{b]}, \tag{5.73c}$$

$$\widetilde{D}^e{}_{abc} = \iota_{\rho_{[a}} \iota_{\rho_b} R^e{}_{c]}, \tag{5.73d}$$

$$0 = C^d{}_{[ab} C^e{}_{c]d} + \rho^\mu_{[c} \partial_\mu C^e{}_{ab]}. \tag{5.73e}$$

In local coordinates these form a formidable set of equations. It is not clear when these equations have non-trivial solutions.

The gauging procedure can be immediately recognised as Lie algebroid gauging (described in Section 5.4 with $\phi = 0$) with the action:

$$S_{\text{gauged}} = \int_\Sigma E_{\mu\nu} D_- X^\mu D_+ X^\nu - \int_\Sigma X^* C + \int_{\Sigma_3} X^* H + \int_\Sigma \langle \widehat{X}, F_{\nabla^\omega} \rangle,$$

where $E = G + C$, C is defined by $\mathcal{L}_{\rho(\varepsilon)} C = \iota_{\rho(\varepsilon)} H$, $\theta := \iota_\rho C$, and

$$(F_{\nabla^\omega})^a e_a = (dA^a + \tfrac{1}{2} C^a_{bc} A^b \wedge A^c + \omega^a_{\mu b} A^b \wedge DX^\mu) e_a. \tag{5.74}$$

Equation (5.74) is the field strength of ∇^ω, a TM-connection on Q, defined locally by $\nabla^\omega e_a := \omega^b{}_a \otimes e_b$.

The constraints (5.73) are solved if and only if we have a Lie algebroid $(\rho, [\cdot,\cdot]_Q, Q)$ coming from a (small) Dirac structure of the H-twisted standard Courant algebroid. For details see Example 5.3. It is the existence of Lie algebroid that is of importance to the procedure not a Courant algebroid.

The first three terms of S_{gauged} are invariant under the gauge transformations (5.20).

The curvature term is invariant under the action δ_ε if and only if

$$\delta_\varepsilon\langle \widehat{X}, F_{\nabla^\omega}\rangle = 0 = \langle \delta_\varepsilon \widehat{X}, F_{\nabla^\omega}\rangle + \langle \widehat{X}, \delta_\varepsilon F_{\nabla^\omega}\rangle. \tag{5.75}$$

As $\delta_\varepsilon \widehat{X}$ must be independent of A it follows that $\delta_\varepsilon F_{\nabla^\omega} \propto F_{\nabla^\omega}$ is required for (5.75) to hold.

The infinitesimal variation can be calculated directly:

$$\delta_\varepsilon(F_{\nabla^\omega})^a = (C^a{}_{bc} - \rho^\mu_b \omega^a{}_{c\mu})\varepsilon^c (F_{\nabla^\omega})^b + (R_{\nabla^\omega})^a{}_{b\mu\nu}\varepsilon^b DX^\mu \wedge DX^\nu + D^a{}_{bc\mu}\varepsilon^c DX^\mu \wedge A^b.$$

It follows immediately that we require $R_{\nabla^\omega} = D^a{}_{bc\mu} = 0$. A calculation shows that

$$D^a{}_{bc\mu} = (\nabla^\omega_\mu T_{\nabla^\omega})^a{}_{bc} - 2\rho^\nu_{[b}(R_{\nabla^\omega})^a{}_{c]\nu\mu},$$

so $D^a{}_{bc\mu} = 0$ is simply the closure of the gauge algebroid. The flatness condition $R_{\nabla^\omega} = 0$ implies that ω is a Maurer–Cartan form for the frame bundle $\mathcal{B}(M)$ (viewing $\mathcal{B}(M)$ as a principal $\mathsf{GL}(M)$ bundle over M). Hence

$$\omega = K^{-1}dK, \quad \omega^a{}_{\mu b} = (K^{-1})^a{}_c \partial_\mu (K)^c{}_b,$$

for some $K \in C^\infty(M, \mathsf{GL}(d))$, where there are d vector fields $\{\rho_a\}$.

There exists a field redefinition such that $\tilde{\omega} = 0$. Take $\tilde{e}_a = (K^{-1})^b{}_a e_b$ and use (5.13):

$$\begin{aligned}
\tilde{\omega}^a{}_{\mu b} =& K^a{}_c \omega^c{}_{\mu d}(K^{-1})^d{}_b - (K^{-1})^c{}_b \partial_\mu K^a{}_c \\
=& K^a{}_c (K^{-1})^c{}_e \partial_\mu K^e{}_d (K^{-1})^d{}_b - (K^{-1})^c{}_b \partial_\mu K^a{}_c \\
=& 0.
\end{aligned}$$

In this frame the non-isometric gauging conditions (5.72) and the infinitesimal gauge transformations (5.71) become:

$$\begin{aligned}
\mathcal{L}_{\tilde{\rho}_a} G =& 0, \qquad \iota_{\tilde{\rho}_a} H = d\tilde{\theta}_a, \\
\delta_\varepsilon(X^\mu, \widetilde{A}^a, \widetilde{\widehat{X}}_a) =& (\varepsilon^a \tilde{\rho}^\mu_a, d\varepsilon^a + \widetilde{C}^a{}_{bc}\widetilde{A}^b \varepsilon^c, -\widetilde{C}^a{}_{bc}\varepsilon^b \widetilde{\widehat{X}}_c).
\end{aligned}$$

In this frame the gauged action is given by

$$\begin{aligned}
S_{\text{gauged}} =& \frac{1}{2}\int_\Sigma DX^\mu \wedge (G_{\mu\nu} \star DX^\nu + C_{\mu\nu} DX^\nu) + \frac{1}{6}\int_{\Sigma_3} H_{\mu\nu\lambda} dX^\mu \wedge dX^\nu \wedge dX^\lambda \\
& - \frac{1}{2}\int_\Sigma C_{\mu\nu} dX^\mu \wedge dX^\nu + \int_\Sigma \widetilde{\widehat{X}}_a (d\widetilde{A}^a + \widetilde{C}^a{}_{bc}\widetilde{A}^b \widetilde{A}^c).
\end{aligned}$$

The closure of gauge algebroid gives the constraint

$$\widetilde{\nabla}^{\tilde{\omega}}_{\mu}(T_{\widetilde{\nabla}^{\tilde{\omega}}})^{a}{}_{bc} = 0 = \partial_{\mu}\widetilde{C}^{a}{}_{bc},$$

indicating that $\widetilde{C}^{a}{}_{bc} \in \mathbb{R}$. Thus,

$$[\tilde{\rho}_a, \tilde{\rho}_b] = \widetilde{C}^{c}{}_{ab}\tilde{\rho}_c, \quad \widetilde{C}^{c}{}_{ab} \in \mathbb{R},$$

defines a Lie algebra $\widetilde{\mathfrak{g}}$. It is clear that in this frame we recover the gauged action of non-abelian T-duality (in the presence of a WZW term) corresponding to the Roček–Verlinde intermediate gauge theory [103, 44].

Theorem 5.9 ([24])**.** *Infinitesimal gauge invariance of the non-isometrically gauged Lie algebroid sigma model* (5.70) *implies that* ∇^{ω} *is flat, and there exists a field redefinition (given by changing the Lie algebroid frame) such that* $\mathfrak{g}(Q,\omega)$ *has constant structure functions* $C^{a}{}_{bc}$ *and defines a Lie algebra.*

Corollary 5.10. *The Non-isometric T-duality proposal of CDJ is equivalent to non-abelian T-duality.*

It was noted in [89] that the flat condition on ω implied that there existed a frame such that $\tilde{\omega} = 0$. The necessity of the flatness condition for T-duality—and the fact that it renders non-isometric T-duality equivalent to non-abelian T-duality—seems not to have been realised before the publication of [24].

Remark. The change of Lie algebroid frame can be interpreted simply as an alternative choice of vector fields $\{\tilde{\rho}_a\}$. The above result can then be stated as follows: Whenever a non-isometric T-dual gauging exists for vector fields $\{\rho_a\}$, it is always possible to choose a set of Killing vector fields $\{\tilde{\rho}_a\}$ which give an equivalent gauged action (and hence T-duality pair).

The above result implies that the non-isometric proposal of CDJ is *locally* equivalent to non-abelian T-duality. It is possible that there may exist admissible Lie algebroids (Q, ∇^{ω}) for which the corresponding Lie algebra $\mathfrak{g}(Q,\omega)$ acts locally on the target manifold M but does not integrate to a global action. A possible scenario in which this may occur is given by the quotient of a Lie group by its discreet subgroups. The physical interpretation could be that of strings winding around non-contractible cycles of M.

A model with winding modes wrapped around non-contractible cycles of M would correspond to a Lie algebroid action. The integrability of such an algebroid would be determined by the associated monodromy and holonomy groupoid (see Examples 3.13 and 3.14 for the monodromy and holonomy groupoids).

5.5.2 Field strength F_{∇^ω}

The field strength term associated with non-isometric T-duality (5.74) can be written invariantly in the following form:[11]

$$F_{\nabla^\omega} = d_{\nabla^\omega} A - [A \overset{\wedge}{,} A]_{\nabla^\omega}.$$

This can be verified by an explicit calculation in local coordinates:

$$\begin{aligned}(d_{\nabla^\omega} A)(v_1, v_2) =& \nabla^\omega_{v_1}(A(v_2)) - \nabla^\omega_{v_2}(A(v_1)) - A([v_1, v_2]_{TM}) \\ =& v_1^\mu v_2^\nu 2(\partial_{[\mu} A^a_{\nu]} + \omega^a{}_{[\mu|b} A^b_{|\nu]}) e_a, \\ [A \wedge A]_{\nabla^\omega}(v_1, v_2) =& \tfrac{1}{2}(T_{\nabla^\omega}(A(v_1), A(v_2)) + T_{\nabla^\omega}(A(v_2), A(v_1))) \\ =& v_1^\mu v_2^\nu A^b_{[\mu} A^c_{\nu]} (2\rho^\lambda_{[b|} \omega^a{}_{\mu|c]} - C^a{}_{bc}) e_a.\end{aligned}$$

Taking $v_1^\mu = dX^\mu$ and $v_2^\nu = dX^\nu$, we have

$$\begin{aligned}F_{\nabla^\omega} =& (dA^a + \omega^a{}_{\mu b} A^b \wedge dX^\mu - \omega^a_{\mu b} A^b \wedge \rho^\mu_c A^c + \tfrac{1}{2} C^a{}_{bc} A^b \wedge A^c) e_a \\ =& (dA^a + \omega^a{}_{\mu b} A^b \wedge DX^\mu + \tfrac{1}{2} C^a{}_{bc} A^b \wedge A^c) e_a,\end{aligned}$$

which matches the local coordinate formula given in [99].

The condition $F_{\nabla^\omega} = 0$ is equivalent to the statement that $A : T\Sigma \to X^{**}Q$ is a Lie algebroid morphism (see Section 3.3.1). In particular, we can interpret $A \in \Omega(\Sigma, X^{**}Q)$ as a Maurer–Cartan form for the Lie groupoid $\mathcal{G}(X^{**}Q^\omega)$ (where Q^ω denotes the Lie algebroid defined by the representation ∇^ω). The groupoid is specified by flowing along Q-paths defined on the Lie algebroid Q^ω. It follows from Theorem 5.9 that $\mathcal{G}(Q^\omega)$ is isomorphic to a Lie group G. In the frame where $\omega^a{}_{\mu b} = 0$ the Q-paths coincide with the flowlines along the right-invariant sections $\mathfrak{X}_{\rm inv}(\mathsf{G})$.

The invariant curvature formula can also be used to provide a nice description of the variation of A:

$$\delta_\varepsilon A = d_{\nabla^\omega} \varepsilon + [\varepsilon, A]_{\nabla^\omega}. \tag{5.76}$$

To understand and justify this statement recall that $\phi = 0$, so that $\Omega^\pm = \omega$ and $A_\pm = A$. We explicitly calculate the expression in local coordinates:

$$(d_{\nabla^\omega} \varepsilon)(v) = \nabla^\omega_v \varepsilon = (v^\mu \partial_\mu(\varepsilon^a) + v^\mu \varepsilon^b \omega^a{}_{\mu b}) e_a = (d\varepsilon^a + \omega^a{}_{\mu b} \varepsilon^b dX^\mu) e_a,$$

[11] Here we ignore issues regarding pullbacks (discussed in Section 5.4.1) to compare with the expression given in the literature.

if $v^\mu = dX^\mu$.

$$\begin{aligned}[\varepsilon, A(v)]_{\nabla^\omega} =& T_{\nabla^\omega}(\varepsilon, A(v)) = \varepsilon^a A^b_\mu v^\mu(\rho^\mu_b \omega^a{}_{\mu c} - \rho^\mu_c \omega^a{}_{\mu b} - C^a{}_{bc})e_a \\ =& (C^a{}_{bc}A^b\varepsilon^c + \rho(\varepsilon)^\mu \omega^a{}_{\mu c}A^c - \omega^a{}_{\mu b}\varepsilon^b \rho(A)^\mu)e_a.\end{aligned}$$

$$\begin{aligned}d_{\nabla^\omega}\varepsilon + [\varepsilon, A]_{\nabla^\omega} =& (d\varepsilon^a + C^a{}_{bc}A^b\varepsilon^c + \omega^a{}_{\mu b}\varepsilon^b(dX^\mu - \rho^\mu_c A^c) + \rho(\varepsilon)^\mu \omega^a{}_{\mu c}A^c)e_a \\ =& \delta_\varepsilon A^a e_a + A^b(\rho(\varepsilon)^\mu \omega^a{}_{\mu b}e_a) = \delta_\varepsilon A^a e_a + A^b(\nabla_{\rho(\varepsilon)}e_b) \\ =& \delta_\varepsilon A^a e_a + A^a \delta_\varepsilon e_a = \delta_\varepsilon(A^a e_a) = \delta_\varepsilon A.\end{aligned}$$

This gives the correct coordinate variation (5.71b)

$$\delta_\varepsilon A^a = d\varepsilon^a + C^a{}_{bc}A^b\varepsilon^c + \omega^a{}_{\mu b}\varepsilon^b(dX^\mu - \rho^\mu_c A^c),$$

as well as naturally incorporating the variation of the basis $\delta_\varepsilon e_a := \nabla_{\rho(\varepsilon)}e_a$. The variation $\delta_\varepsilon A$ involves comparing the value of the field A at some point $x \in M$ with the value at some point $x' = x + \pi(\rho(\varepsilon))$ found by flowing along the vector field $\rho(\varepsilon)$. In order to compare $A|_x$ to $A|_{x'}$ it is necessary to parallel transport $e_a|_x$ to $e_a|_{x'}$ using $e_a|_{x'} = \nabla_{\rho(\varepsilon)}e_a|_x$.

T-duality and $R_{\nabla^\omega} \neq 0$

It is not possible to perform T-duality for non-flat ω. The action $S_{\text{gauged}}[X, A, \widehat{X}]$ is not invariant under the gauge transformations (5.71) if $R_{\nabla^\omega} \neq 0$.

One may wonder if it is possible to introduce a modified curvature, which would transform covariantly without requiring $R_{\nabla^\omega} = 0$. In [89] Kotov and Strobl introduced a different curvature

$$(G_{\nabla^\omega})^a = dA^a + \frac{1}{2}C^a{}_{bc}A^b \wedge A^c + \omega^a{}_{\mu b}DX^\mu \wedge A^b + \frac{1}{2}B^a{}_{\mu\nu}DX^\mu \wedge DX^\nu, \tag{5.77}$$

which satisfies

$$\delta_\varepsilon(G_{\nabla^\omega})^a = (C^a{}_{bc} - \rho^\mu_b \omega^a{}_{\mu c})(G_{\nabla^\omega})^b\varepsilon^c,$$

under the assumption that the gauge algebroid closes and

$$(R_{\nabla^\omega})^a_b + \mathcal{L}_{\rho_b}B^a - \omega^c{}_b \wedge \iota_{\rho_c}B^a + \iota_{\rho_b}\omega^a{}_c B^c + (T_{\nabla^\omega})^a_{bc}B^c = 0.$$

This curvature is not appropriate for T-duality. The condition $G_{\nabla^\omega} = 0$ does not imply $A = 0$, and there is an extra field $B \in \Gamma(\wedge^2 T^*M \otimes Q)$ which has no clear interpretation in the context of T-duality. The point of introducing the curvature term

F_{∇^ω} in the standard gauging is to ensure that $A = 0$ can achieved globally, and the ungauged action can recovered from the original. The curvature G_{∇^ω} was introduced for a different purpose in [89].

Remark. The curvature G_{∇^ω} may be related to the 'fake-curvature' associated to two-connections and higher gauge theory (see for example [107]). However, this possible interpretation was not discussed in [89].

The curvature term F_{∇^ω} has the interpretation as the field strength for a Maurer–Cartan field A. The gauge field A can be interpreted as a Lie algebroid morphism. This suggests that a curvature term for Lie groupoid gauging would be based on gauge fields $\mathcal{A}^\pm \in \Gamma((X^{**}Q)^* \otimes Q)$ describing a Lie algebroid morphism between Lie algebroids on the vector bundles $X^{**}Q$ and Q. The flat connections ${}^Q\nabla^\pm$ define Lie algebroids, however the gauging procedure does not define any fields which could be interpreted as the required Maurer–Cartan forms $\mathcal{A}^\pm$. This suggests that it is not possible to add a gauge invariant curvature term to the action when considering Lie algebroid gauging. This limits the usefulness of the Lie algebroid gauging procedure.

5.5.3 Example of non-isometric T-duality

We present an example of non-isometric gauging in order to better understand the connection between 'non-isometric T-duality' and non-abelian T-duality. We show how an explicit change of frame gives the standard non-abelian gauged action. Consider a group manifold G. The WZW action S_{WZW} is invariant under the left action, but not under the right action. It is possible to gauge S_{WZW} non-isometrically with respect to the right action. We will see that the non-isometric gauging with respect to the right action is in fact isomorphic to the standard gauging of the left action.

Recall a WZW group target model $g : \Sigma \to \mathsf{G}$ given by,

$$S_{\mathrm{WZW}}[g] = \frac{1}{2}\int_\Sigma (g^{-1}dg \overset{\wedge}{,} \star\, g^{-1}dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}dg \overset{\wedge}{,} g^{-1}dg)_B,$$

repeated here for convenience. The Lie algebroid Q is the tangent bundle $T\mathsf{G}$. The standard non-abelian gauging (described in Section 5.2) written in the CDJ formalism corresponds to defining the connection ∇^ω by declaring that the right-invariant vector fields on G are covariantly constant.

The purpose of this example is to consider non-isometric gauging with respect to a basis of left-invariant vector fields. The right action of the group G on the left-invariant Maurer–Cartan form gives:

$$g^{-1}dg \circlearrowleft h = (gh)^{-1}d(gh) = h^{-1}g^{-1}dgh = \mathrm{Ad}_{h^{-1}}g^{-1}dg,$$

and

$$S_{\mathrm{WZW}}[gh] = \frac{1}{2}\int_\Sigma (\mathrm{Ad}_{h^{-1}} g^{-1}dg \overset{\wedge}{,} \star \mathrm{Ad}_{h^{-1}} g^{-1}dg)_G + (\mathrm{Ad}_{h^{-1}} g^{-1}dg \overset{\wedge}{,} \mathrm{Ad}_{h^{-1}} g^{-1}dg)_B.$$

The action is not invariant under the right action.

There is a non-isometric gauging of the WZW action with respect to the right action:

$$\begin{aligned} S_{WZW}[g, A, \widehat{X}] =& \frac{1}{2}\int_\Sigma (g^{-1}Dg \wedge \star g^{-1}Dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}Dg \wedge g^{-1}Dg)_B \\ &+ \int_\Sigma \langle \widehat{X}, dA + A \wedge A + g^{-1}Dg \wedge A + A \wedge g^{-1}Dg \rangle, \end{aligned} \tag{5.78}$$

where $g^{-1}Dg = g^{-1}dg - A$. The associated gauge transformations are given by

$$(g, A, \widehat{X}) \circlearrowleft h := (g, A, \widehat{X})^h = (gh, A - g^{-1}dg + (gh)^{-1}d(gh), \widehat{X}).$$

In order to check that this corresponds to a well defined groupoid action it is sufficient (in this case) to check that

$$((g, A, \widehat{X})^{h_1})^{h_2} = (g, A, \widehat{X})^{h_1 h_2}.$$

This follows by direct calculation:

$$\begin{aligned} &((g,A, \widehat{X})^{h_1})^{h_2} = (gh_1, A - g^{-1}dg + (gh_1)^{-1}d(gh_1), \widehat{X})^{h_2} \\ &=(gh_1h_2, A - g^{-1}dg + (gh_1)^{-1}d(gh_1) - (gh_1)^{-1}d(gh_1) + (gh_1h_2)^{-1}d(gh_1h_2), \widehat{X}) \\ &=(gh_1h_2, A - g^{-1}dg + (gh_1h_2)^{-1}d(gh_1h_2), \widehat{X}) = (g, A, \widehat{X})^{h_1h_2}. \end{aligned}$$

To connect this to the CDJ proposal it is necessary to consider the corresponding infinitesimal gauge transformations

$$\delta_\varepsilon(g, A, \widehat{X}) = (g\varepsilon, d\varepsilon + [A, \varepsilon]_{\mathfrak{g}} + \mathrm{Ad}_{g^{-1}Dg}\varepsilon, 0).$$

It follows immediately that $\omega = \mathrm{ad}_{g^{-1}dg}$.

We wish to show that the non-isometric gauging with respect to the right action is equivalent to the standard left action gauging. It is enough to find the change of frame K which gives the field redefinition relating them. To convert the right action to a left action take $K = \mathrm{Ad}_g$ giving the field redefinitions

$$\tilde{h} = ghg^{-1}, \quad \widetilde{A} = gAg^{-1}, \quad \widetilde{\widehat{X}} = \mathrm{Ad}_g^* \widehat{X}.$$

Taking these field redefinitions we have the action

$$S_{WZW}[g,\widetilde{A},\widetilde{\widehat{X}}] = \frac{1}{2}\int_\Sigma (g^{-1}Dg \stackrel{\wedge}{,} \star g^{-1}Dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}Dg \stackrel{\wedge}{,} g^{-1}Dg)_B + \int_\Sigma \langle \widetilde{\widehat{X}}, d\widetilde{A} - [\widetilde{A} \stackrel{\wedge}{,} \widetilde{A}]_{\mathfrak{g}} \rangle,$$

where $g^{-1}Dg = g^{-1}dg - g^{-1}\widetilde{A}g$ and gauge transformations are given by

$$(g,\widetilde{A},\widetilde{\widehat{X}}) \to (hg, h\widetilde{A}h^{-1} + dhh^{-1}, \mathrm{Ad}^*_{h^{-1}}\widetilde{\widehat{X}}).$$

This describes the standard gauged action of non-abelian T-duality.

We conclude that whenever a Lie group manifold can be gauged non-isometrically, there must exist a choice of frame for Q which gives a standard isometric action. This is in accordance with Theorem 5.9. This example gives the intuition of the general case: whenever there is a non-isometric action (generated by a set of vector fields ρ_a) there exists another set of vector fields $\tilde{\rho}_a$ which generate a Lie algebra describing an equivalent gauged action.

5.5.4 Constructing the Non-Abelian T-dual background

In the final section of this chapter we will describe how to construct the T-dual background $\widehat{E}$ given a non-linear sigma model E with a group of isometries associated to a compact semi-simple Lie group G.

Consider a principal G-bundle $P(M,\pi,\mathsf{G})$ and a non-linear sigma model $S[X]$ gauged with a set of vector fields $\{\rho_a\}$, $a = 1,\dots,d = \dim(\mathsf{G})$, satisfying $[\rho_a,\rho_b] = C^c{}_{ab}\rho_c$ where $C^c{}_{ab}$ are the structure constants of $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$. We consider the action, defined on a Euclidean worldsheet[12] Σ, given by

$$S[X] = \frac{1}{4}\int_\Sigma X^*G + iX^*B = \int_\Sigma dzd\bar{z}(E_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu),$$

where $\partial X = \partial_z X$, $\bar{\partial} X = \partial_{\bar{z}} X$, and $\eta = \eta_+ dz + \eta_- d\bar{z}$. The set $\{\rho_a\}$ are assumed to be isometries of E:

$$\mathcal{L}_{\rho_a} E = 0.$$

Locally the decomposition $P \cong M \times \mathsf{G}$ induces a decomposition of $E \in P^* \otimes P^*$: Choose the set of coordinates described by the frame $\{dX^i\} = \{dX^1,\dots,dX^{n-d},\eta^1,\dots,\eta^d\}$.

[12] Here we choose the flat worldsheet metric $h = (d\sigma)^2 + (d\tau)^2$.

Let $\mu = 1, \ldots, n-d$. The field E decomposes, in this frame, as

$$E = E_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + E_{a\nu}\eta_+^a \bar{\partial} X^\nu + E_{\mu b}\partial X^\mu \eta_-^b + E_{ab}\eta_+^a \eta_-^b.$$

We can gauge the action $S[X]$ with gauge fields $A^a = A_+^a dz + A_-^a d\bar{z}$. The covariant derivatives are defined by

$$DX^\mu = \partial X^\mu, \quad \bar{D}X^\mu = \bar{\partial} X^\mu, \quad D\eta_\pm^a = \eta_\pm^a - A_\pm^a.$$

The gauged action becomes

$$\begin{aligned} S = \int_\Sigma dz d\bar{z} \Big(& E_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + E_{a\nu}(\eta_+^a - A_+^a)\bar{\partial} X^\nu + E_{\mu b}\partial X^\mu (\eta_-^b - A_-^b) \\ & + E_{ab}(\eta_+^a - A_+^a)(\eta_-^b - A_-^b) + \widehat{X}_a(\partial A_-^a - \bar{\partial} A_+^a + C^a{}_{bc}A_+^b A_-^c) \Big). \end{aligned} \tag{5.79}$$

In order to recover the original metric $S[X]$ we integrate out $\widehat{X}$ and gauge fix $A = 0$. We are interested in finding the dual background by integrating out the gauge field A and gauge fixing η^a.

Denoting the Lagrangian density by L ($S := \int_\Sigma dz d\bar{z} L$) the equations of motion for A split into $A_\pm$:

$$\begin{aligned} \frac{\delta S}{\delta A_+} =& \frac{\partial L}{\partial A_+} - \partial\left(\frac{\partial L}{\partial(\partial A_+)}\right) - \bar{\partial}\left(\frac{\partial L}{\partial(\bar{\partial} A_+)}\right) \\ =& C^b{}_{ac}\widehat{X}_b A_-^c - E_{a\nu}\bar{\partial} X^\nu - E_{ab}(\eta_-^b - A_-^b) + \bar{\partial}\hat{X}_a = 0, \end{aligned}$$

and

$$\frac{\delta S}{\delta A_-} = C^b{}_{ab}\widehat{X}_b A_+^c - E_{\mu b}\partial X^\mu - E_{ab}(\eta_+^a - A_+^a) - \partial\widehat{X}_a = 0.$$

This system of first order equations can be solved for $A_\pm$ giving

$$\begin{aligned} A_+^a =& (E_{ab} + C^c{}_{ab}\widehat{X}_c)^{-1}(E_{\mu b}\partial X^\mu + E_{bc}\eta_+^c + \partial\widehat{X}_b), \\ A_-^a =& (E_{ab} + C^c{}_{ab}\widehat{X}_c)^{-1}(E_{b\nu}\bar{\partial} X^\nu + E_{bc}\eta_-^c - \bar{\partial}\widehat{X}_b). \end{aligned}$$

Substituting into the action (5.79) and using $\int \widehat{X}_a(\partial A_-^a - \bar{\partial} A_+^a) = \int \bar{\partial}\widehat{X}_a A_+^a - \partial\widehat{X}^a A_-^a$ we have

$$\begin{aligned} S = \int_\Sigma dz d\bar{z} \Big(& \widehat{E}_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + \widehat{E}_{a\nu}\partial\widehat{X}_a \bar{\partial} X^\nu + \widehat{E}_{\mu b}\partial X^\mu \bar{\partial}\widehat{X}_b + \widehat{E}_{ab}\partial\widehat{X}_a \bar{\partial}\widehat{X}_b \\ & + N_{a\nu}\eta_+^a \bar{\partial} X^\nu + N_{\mu a}\partial X^\mu \eta_-^a + N_{ab}\eta_+^a \eta_-^b + N_{ab}^+\eta_+^a \bar{\partial}\widehat{X}_b + N_{ab}^-\partial\widehat{X}_a \eta_-^b \Big). \end{aligned} \tag{5.80}$$

where $\widehat{E}$ is given by the *non-abelian Buscher rules*:

$$\begin{pmatrix} \widehat{E}_{\mu\nu} & \widehat{E}_{\mu a} \\ \widehat{E}_{a\nu} & \widehat{E}_{ab} \end{pmatrix} = \begin{pmatrix} E_{\mu\nu} - E_{\mu b}(E_{bc} + C^d{}_{bc}\widehat{X}_d)^{-1}E_{cv} & E_{\mu b}(E_{ba} + C^c{}_{ba}\widehat{X}_c)^{-1} \\ -(E_{ab} + C^c{}_{ab}\widehat{X}_c)^{-1}E_{b\nu} & (E_{ab} + C^c{}_{ab}\widehat{X}_c)^{-1} \end{pmatrix}, \quad (5.81)$$

and $M_{ab} := E_{ab} + C^c{}_{ab}\widehat{X}_c$,

$$N_{a\nu} = E_{a\nu} - E_{ab}M^{-1}_{bc}E_{c\nu}, \quad N_{\mu a} = E_{\mu a} - E_{\mu b}M^{-1}_{bc}E_{ca}, \quad N_{ab} = E_{ab} - E_{ac}M^{-1}_{cd}E_{db},$$
$$N^{+}_{ac} = E_{ac}M^{-1}_{cb}, \quad N^{-}_{ab} = -M^{-1}_{ac}E_{cb}.$$

The first line of (5.80) describes a dual ungauged sigma model. The second line contains terms involving $\eta = \eta_+ dz + \eta_- d\bar{z}$. Remember that η is the left-invariant Maurer–Cartan form for G and can be set to zero by an appropriate choice of gauge. Choosing a gauge where $\eta = 0$ we produce the dual sigma model

$$\widehat{S} = \int_\Sigma dz d\bar{z}\Big(\widehat{E}_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + \widehat{E}_{a\nu}\partial \widehat{X}_a \bar{\partial} X^\nu + \widehat{E}_{\mu b}\partial X^\mu \bar{\partial}\widehat{X}_b + \widehat{E}_{ab}\partial\widehat{X}_a\bar{\partial}\widehat{X}_b\Big). \quad (5.82)$$

Given E with an isometry group G the T-dual metric $\widehat{E}$ is given by the non-abelian Buscher rules. In this dual model the coordinates associated to the coframe η^a are integrated out and the Lagrange multipliers $\widehat{X}_a$ play the role of dual coordinates. The dual field $\widehat{E}$ is defined on a space with local coordinates $\{X^\mu, \widehat{X}^a\} \in U \subset \mathbb{R}^n$. The topology of the non-abelian T-dual space is an open problem. A discussion on the difficulties in establishing the topology of the non-abelian T-dual can be found in [5, 116].

Chapter 6

Conclusion and Outlook

This thesis highlights the role of algebroid geometry in mathematics and physics. Lie algebroids and associated geometric structures (such as contact structures) form the basis of the mathematical structure describing the dynamics of point particles. The dynamics of strings and higher dimensional branes are most naturally described by more general Leibniz algebroid geometry.

Symmetry plays a central role in mathematics and physics. In differential geometry two objects related by a diffeomorphism are considered equivalent. In the presence of a Lie group symmetry all structures should be invariant under the group action and the geometry is governed by an Atiyah algebroid. Differential geometry provides the appropriate mathematical framework for general relativity and principal G-bundles play a crucial role in describing Yang–Mills gauge theory.

Symmetry in string theory leads to a different notion of equivalence resulting in exotic 'stringy geometry'. Abelian T-duality is an exact symmetry of string theory [26, 27] which establishes an equivalence of string theories described on geometrically (and even topologically) different target spaces. Consider a compact manifold defined by a principal torus bundle with fibers T^d. Take a model with d Killing vectors (corresponding to the torus action). The appropriate framework for invariant structures is not based on the group $\mathsf{O}(d)$, but rather the T-duality group $\mathsf{O}(d,d)$.[1] The most general backgrounds will be constructed out of spaces called T-folds. T-folds are described using charts $\mathcal{U} = \bigsqcup_\alpha U_\alpha$ where $U_\alpha \subset \mathbb{R}^d$ are patched together using $\mathsf{O}(d,d)$ valued transition functions.

The notion of extended symmetry motivates much of this thesis. In Chapter 4 the extended symmetry group of the generalised contact algebroid—generated by (B,b,a)-gauge transformations—was taken to be the starting point of generalised contact geometry. The drive to construct geometric structures which are invariant under this

[1] The T-duality group is $\mathsf{O}(d,d,\mathbb{Z})$ but when considering the associated geometry we take the symmetry group $\mathsf{O}(d,d,\mathbb{R})$.

group led to a modification of the definitions previously appearing in the literature on generalised contact structures. This modification allows the use of twisted algebroids to describe non-coorientable structures. All structures were defined in a way that is compatible with an extended notion of equivalence. Thiese structures are suitable to describe a geometric subgroup of string geometry on odd-dimensional manifolds.

Extended symmetry, in the guise of gauge symmetry, was studied in Chapter 5. The notion of gauging an action was extended from symmetries arising from Lie groups to those arising from Lie groupoids. Elucidating the underlying Lie groupoid structure was crucial to understanding the global properties of the local proposal appearing in the literature. The invariant Lie algebroid geometry was key to establishing the equivalence of 'non-isometric T-duality' and non-abelian T-duality; as well as the necessary and sufficient conditions to undertake the general Lie algebroid gauging proposal.

We close this thesis with a few remarks on extensions of this work and possible future directions of research. Hitchin's generalised geometry provides a mathematical background to describe a parabolic subgroup of the T-duality group $\mathsf{O}(d,d)$—the so called 'geometric subgroup'. In the physics literature there is evidence of 'non-geometric' backgrounds, which display non-commutative and non-associative behaviour [72, 73, 75, 7, 95]. Double field theory (DFT) [76, 66, 1] is a proposal which makes T-duality manifest and is based on doubling the degrees of freedom (physically interpreted as adding winding modes). For a geometric perspective on DFT we refer the reader to [120]. A section constraint projects the theory to some physical subspace which halves the doubled degrees of freedom. In this framework non-geometric backgrounds correspond to a choice of the section constraint which describes a dual space dependent on the winding modes. Exceptional Field Theory (EFT) [68, 69, 70] is a closely related theory which aims to make U-duality manifest. EFT is an extension of exceptional generalised geometry relevant to M-theory based on a Leibniz algebroid with a symmetry group related to the exceptional lie groups.

Global aspects of DFT (and EFT) have been considered [67, 77, 100]. Questions relating to integrability and global properties on the doubled space remain. The approaches are very computational and an invariant approach especially in the non-geometric frames would be conceptually important.

Spherical T-duality is a topological theory for $\mathsf{S}^3 \hookrightarrow P \to M$ bundles (viewed as principal $\mathsf{SU}(2)$ bundles) [21, 22, 23]. Spherical T-duality gives a well defined topological duality (at least when the base space has $\dim(M) \leq 4$) but the local geometric structure is missing. Fluxes appearing in spherical T-duality can be associated with twisted Leibniz algebroids related to exceptional generalised geometry. However, an appropriate geometric version remains elusive.

I believe the best prospect to explore global aspects of DFT is through Poisson–Lie T-duality. There exists a generalisation of DFT which is based on doubling Lie group manifolds [121, 13, 14, 61]. Recently Poisson–Lie T-duality has been described in this doubled formalism [62]. In Poisson–Lie T-duality the doubled space is given by a Drinfeld double. These Lie groups are well studied in the mathematical literature, and provide a well defined example of a doubled space. The existence and uniqueness of the Poisson–Lie dual relies crucially on the special properties of Drinfeld doubles. Looking at integrability properties of the dual models in DFT for group manifolds is likely to impose tight constraints on the structure of general doubled group spaces. The original Poisson–Lie construction is not based on a Roček–Verlinde intermediate gauge theory; the doubled group proposal suggests would suggest that such a description might exist. I expect that if a gauging construction exists for Poisson–Lie T-duality it will be based on Courant algebroid gauging. It would be interesting to see if this is possible. More generally one could consider L_∞-algebroid gauging proposals, which may have applications in string theory [71, 15].

Bibliography

[1] G. Aldazabal, D. Marqués, and C. Núñez, *Double Field Theory: A Pedagogical Review*, Class.Quant.Grav. **30** (2013) 163001, [arXiv:1305.1907].

[2] M. Aldi, and D. Grandini, *Generalized contact geometry and T-duality*, J. Geom. Phys. **92** (2015) 78–93, [arXiv:1312.7471].

[3] A. Alekseev, and T. Strobl, *Current Algebras and Differential Geometry*, JHEP **03** (2005) 035, [arXiv:hep-th/0410183].

[4] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory*, Int. J. Modern Phys. **A12** (1997) 1405–1429, [arXiv:arXiv:hep-th/9502010].

[5] E. Álvarez, L. Álvarez–Gaumé, J.L.F. Barbón, and Y. Lozano, *Some Global Aspects of Duality in String Theory*, Nucl. Phys. **B415** (1994) 71–100, [arXiv:hep-th/9309039].

[6] I. Androulidakis, and M. Zambon, *Smoothness of holonomy covers for singular foliations and essential isotropy*, M. Math. Z. **275** (2013) 921–951, [arXiv:1111.1327].

[7] P. Aschieri, and R.J. Szabo, *Triproducts, nonassociative star products and geometry of R-flux string compactifications*, Journal of Physics: Conference Series **634** (2015) 012004, [arXiv:1504.03915].

[8] J. Baez, and J. Huerta, *An invitation to Higher Gauge Theory*, General Relativity and Gravitation **43** (2011) 2335–2392, [arXiv:1003.4485].

[9] D. Baraglia, *Leibniz algebroids, twistings and generalized geometry*, J. Geom. Phys. **62** (2012) 903–934, [arXiv:1101.0856].

[10] D. Baraglia, and P. Hekmati, *Transitive Courant algebroids, String structures and T-duality*, Adv. Theor. Math. Phys. **19(3)** (2015) 613–672, [arXiv:1308.5159].

[11] G. Bazzoni, and J. Oprea, *On the structure of co-Kähler manifolds*, J. Geom Dedicata **170** (2014), [arXiv:1209.3373].

[12] D. Bleeker, *Gauge Theory and Variational Principles*, Addison–Wesley (1981), (Global Analysis, Pure and Applied, No. 1; Series Advanced Graduate Level Text).

[13] R. Blumenhagen, F. Hassler, D. Lüst, *Double field theory on group manifolds*, JHEP **02** (2015) 001, [arXiv:1410.6374].

[14] R. Blumenhagen, P. du Bosque, F. Hassler, and D. Lüst, *Generalized metric formulation of Double field theory on Group manifolds*, JHEP **08** (2015) 056, [arXiv:1502.02428].

[15] R. Blumenhagen, I. Brunner, V. Kupriyanov, and D. Lüst, *Bootstrapping Non-commutative Gauge Theories from L_∞ algebras*, JHEP **97** (2018) [arXiv:1803.00732].

[16] G. Bonavolontà, and N. Poncin, *On the category of Lie n-algebroids*, J. Geom. Phys. **73** (2013) 70–90, [arXiv:1207.3590].

[17] P. Bouwknegt, J. Evslin, and V. Mathai, *T-duality: Topology change from H-flux*, Comm. Math. Phys. **249** (2004) 383–415, [arXiv:hep-th/0306062].

[18] P. Bouwknegt, J. Evslin, and V. Mathai, *Topology and H-flux of T-dual manifolds*, Phys. Rev. Lett. **98** (2004) 181601, [arXiv:hep-th/0312052].

[19] P. Bouwknegt, K. Hannabuss, and V. Mathai, *T-duality for principal torus bundles and dimensionally reduced Gysin sequences*, Adv. Theor. Math. Phys. **9** (2005) 749–773, [arXiv:hep-th/0412268].

[20] P. Bouwknegt, *Lectures on Cohomology, T-duality and Generalized Geometry*, in the proceedings of the the Summer School "New Paths Towards Quantum Gravity", Holbæk, Denmark, 10-16 May 2008, Editors: B. Booss–Bavnbek, G. Esposito and M. Lesch, Lecture Notes in Physics **807** (2010) 261–311.

[21] P. Bouwknegt, J. Evslin, and V. Mathai, *Spherical T-duality*, Commun. Math. Phys. **337** (2015) 909–954, [arXiv:1405.5844].

[22] P. Bouwknegt, J. Evslin, and V. Mathai, *Spherical T-duality II: An infinity of spherical T-duals for non-principal SU(2)-bundles*, J. Geom. Phys. **92** (2015) 46–54, [arXiv:1409.1296].

[23] P. Bouwknegt, J. Evslin, and V. Mathai, *Spherical T-duality and the spherical Fourier–Mukai transform*, J. Geom. Phys. **133** (2018) 303–314, [arXiv:1502.04444].

[24] P. Bouwknegt, M. Bugden, C. Klimčík, and K. Wright, *Hidden isometry of "T-duality without isometry"*, JHEP **08** (2017) 116, [arXiv:1705.09254].

[25] H. Bursztyn, G.R. Cavalcanti, and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Adv. Math. **211(2)** (2007) 726–765, [arXiv:math/0509640].

[26] T. Buscher, *A symmetry of the string background field equations*, Phys. Lett. B. **194** (1987) 59–62.

[27] T. Buscher, *Path-integral derivation of quantum duality in nonlinear sigma models*, Phys. Lett. B **201** (1988) 466–472.

[28] A.S. Cattaneo, and G. Felder, *Poisson sigma models and symplectic groupoids*, Quantization of Singular Symplectic Quotients, (ed. N. P. Landsman, M. Pflaum, M. Schlichenmeier), Progress in Mathematics **198** (2001), 61–93, [arXiv:math/0003023].

[29] G. Cavalcanti, and M. Gualtieri, *Generalized complex manifolds on nilmanifolds*, J. Symplectic Geom. **3** (2004) 393–410, [arXiv:math/0404451].

[30] G. Cavalcanti, and M. Gualtieri, *Generalized complex geometry and T-duality*, A Celebration of the Mathematical Legacy of Raoul Bott (CRM Proceedings & Lecture Notes), American Mathematical Society (2010) 341–366, [arXiv:1106.1747].

[31] A. Chatzistavrakidis, A. Deser, and L. Jonke, *T-duality without isometry via extended gauge symmetries of 2D sigma models*, JHEP **01** (2016) 154, [arXiv:1509.01829].

[32] A. Chatzistavrakidis, A. Deser, L. Jonke, and T. Strobl, *Beyond the standard gauging: gauge symmetries of Dirac Sigma Models*, JHEP **08** (2016) 172, [arXiv:1607.00342].

[33] A. Chatzistavrakidis, *Non-isometric T-duality from gauged sigma models*, PoS CORFU2015 (2016) 085, [arXiv:1604.03739].

[34] A. Chatzistavrakidis, A. Deser, L. Jonke, and T. Strobl, *Strings in singular space-times and their universal gauge theory*, Ann. Henri Poincaré **18** (2017) 2641, [arXiv:1608.03250].

[35] A. Chatzistavrakidis, A. Deser, L. Jonke, and T. Strobl, *Gauging as constraining: the universal generalised geometry in two dimensions*, Proceedings of Science CORFU2016 (2016) 087, [arXiv:1705.05007].

[36] Z. Chen, and Z.–J. Liu, *Omni-Lie algebroids*, J. Geom. Phys. **60** (2010) 799–808, [arXiv:0710.1923].

[37] Z. Chen, Z. Liu, and Y. Sheng, *E-Courant algebroids*, Int. Math. Res. Not. **22** (2010) 4334–4376, [arXiv:0805.4093].

[38] S. Chern, *Circle bundles*, In: Palis J., do Carmo M. (eds) Geometry and Topology. Lecture Notes in Mathematics, vol 597. Springer, Berlin, Heidelberg (1977).

[39] A. Coimbra, C. Strickland–Constable, and D. Waldram, *Supergravity as Generalised Geometry I: Type II Theories*, JHEP **11** (2011) 091, [arXiv:1107.1733].

[40] A. Coimbra, R. Minasian, H. Triendl, and D. Waldram, *Generalised geometry for string corrections*, JHEP **11** (2014) 160, [arXiv:1407.7542].

[41] J. Cortéz, M. De León, J. C. Marrero, D. de Diego, and E. Martínez, *A survey of Lagrangian mechanics and control theory on Lie algebroids and groupoids*, Int. J. Geom. Meth. Mod. Phys. **3** (2006), no. 3, 509—558, [arXiv:math-ph/0511009].

[42] M. Crainic, and R.L. Fernandes, *On integrability of Lie brackets*, Annals of Mathematics **157** (2003) 575–620, [arXiv:math/0105033].

[43] M. Crainic, and R.L. Fernandes, *Lectures on the integrability of Lie brackets*, [arXiv:0611259].

[44] X.C. de La Ossa, and F. Quevedo, *Duality symmetries from non-Abelian isometries in string theory*, Nucl. Phys. **B403** (1993) 377–394, [arXiv:hep-th/9210021].

[45] C. Ehresmann, *Structures Feuilletées*, Proc. 5th Canadian Math. Congress, Univ. of Toronto Press 1963, (1961) 109–172.

[46] J. Ekstrand, and M. Zabzine, *Courant-like brackets and loop spaces*, JHEP **03** (2011) 074, [arXiv:0903.3215].

[47] R.L. Fernandes, and I. Struchiner, *Lie algebroids and classification problems in geometry*, Sao Paulo Journal of mathematical sciences **2** (2008) 263–283, [arXiv:0712.3198].

[48] M. Garcia–Fernandez, *Torsion-free generalized connections and Heterotic Supergravity*, M. Commun. Math. Phys. **332** (2014) 89–115, [arXiv:1304.4294].

[49] S.J. Gates, C.M. Hull, and M. Roček, *Twisted multiplets and new supersymmetric non-linear σ-models*, Nuclear Physics **B248**(1) (1984) 157–186.

[50] K. Gawędzki, *Conformal field theory: a case study*, [arXiv:hep-th/9904145].

[51] E. Getzler, *Higher derived brackets*, [arXiv:1010.5859].

[52] R. Gomez, and J. Talvacchia, *On products of generalized geometries*, Geometriae Dedicata **175** (2015) 211–218, [arXiv:1311.0381].

[53] R. Gomez, and J. Talvacchia, *Generalized coKähler Geometry and Application to Generalized Kähler structures*, J. Geom. Phys. **98** (2015) 493–503, [arXiv:1502.07046].

[54] J. Grabowski, and P. Urbański, *Algebroids—general differential calculi on vector bundles*, J. Geom. Phys. **31** (1999) 111–141, [arXiv:math/9909174].

[55] K. Grabowska, J. Grabowski, and P. Urbański, *Geometrical Mechanics on algebroids*, Int. J. Geom. Meth. Mod. Phys. 3 (2006) 559–575, [arXiv:math-ph/0509063].

[56] J. Grabowski, and M. Rotkiewicz, *Higher vector bundles and multi-graded symplectic manifolds*, J. Geom. Phys. **59** (2009) 1285–1305, [arXiv:math/0702772].

[57] J. Grabowski, *Graded contact bundles and contact Courant algebroids*, J. Geom. Phys. **68** (2013) 27–58, [arXiv:1112.0759].

[58] J. Grabowski, D. Khudaverdyan, and N. Poincin, *The Supergeometry of Loday Algebroids*, J. Geom. Mech. **5(2)** (2013) 185–213, [arXiv:1103.5852].

[59] M. Gualtieri, *Generalized complex geometry*, Ann. Math. **174** (2011) 75–123, [arXiv:math/0401221].

[60] M. Gualtieri, *branes on Poisson varieties*, The Many Facets of Geometry: A Tribute to Nigel Hitchin (SSJ–P. Bourguignon, O. Garcia–Prada, ed.), [arXiv:0710.2719].

[61] F. Hassler, *The topology of Double Field Theory*, JHEP **04** (2018) 128 [arXiv:1611.07978].

[62] F. Hassler, *Poisson–Lie T-duality in Double Field Theory*, [arXiv:1707.08624].

[63] P.J Higgins, and K. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, J. Algebra **129** (1990) 194–230 .

[64] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math **54** (2003) 281–308, [arXiv:math/0209099].

[65] G. Herczeg, E. Latini, and A. Waldron, *Contact Quantization: Quantum Mechanics = Parallel Transport*, [arXiv:1805.11731].

[66] O. Hohm, C. Hull, and B. Zwiebach, *Generalized metric formulation of double field theory*, JHEP **08** (2010) 008, [arXiv:1006.4823].

[67] O. Hohm, and B. Zwiebach, *Large gauge transformations in Double Field theory*, JHEP **02** (2013) 075, [arXiv:1207.4198].

[68] O. Hohm, and H. Samtleben, *Exceptional Field Theory I: $E_{6(6)}$ covariant Form of M-Theory and Type IIB*, Phy. Rev. D. **89** 066016, [arXiv:1312.0614].

[69] O. Hohm, and H. Samtleben, *Exceptional Field Theory II: $E_{7(7)}$*, Phy. Rev. D. **89** 066017, [arXiv:1312.4542].

[70] O. Hohm, and H. Samtleben, *Exceptional Field Theory III: $E_{8(8)}$*, Phy. Rev. D. **90** 066002, [arXiv:1406.3348].

[71] O. Hohm, B. Zwiebach, *L_∞ Algebras and Field Theory*, Fortsch.Phys. **65** (2017) no.3–4, 1700014, [arXiv:1701.08824].

[72] C. Hull, *A geometry for non-geometric string backgrounds*, JHEP **10** (2005) 065, [arXiv:hep-th/0406102].

[73] C. Hull, *Double Geometry and T-folds*, JHEP **07** (2007) 080, [arXiv:hep-th/0605149].

[74] C. Hull, *Generalised geometry for M-theory*, JHEP **07** (2007) 079, [arXiv:hep-th/0701203].

[75] C. Hull and R. Reid-Edwards, *Non-geometric backgrounds, doubled geometry and generalised T-duality*, JHEP **09** (2009) 014, [arXiv:0902.4032]

[76] C. Hull, and B. Zwiebach, *Double field theory*, JHEP **09** (2009) 099, [arXiv:0904.4664].

[77] C.M. Hull, *Finite gauge transformations and geometry in double field theory*, JHEP **04** (2015) 109, [arXiv:1406.7794].

[78] D. Iglesias, and J.C. Marrero, *Lie algebroid foliations and $\mathcal{E}^1(M)$-Dirac structures*, J. Phys. A:Math.Gen. **35** (2002), [arXiv:math/0106086].

[79] D. Iglesias, and A. Wade, *Contact manifolds and generalized complex structures*, J. Geom. Phys. **53** (2005) 249–258, [arXiv:math/0404519].

[80] M. Jóźwikowski, *Prolongations vs. Tulczyjew triples in Geometric Mechanics*, [arXiv:1712.09858].

[81] B. Jurčo, C. Sämann, and M. Wolf, *Higher Groupoid bundles, Higher Spaces, and Self-Dual Tensor Field Equations*, Fortschr. Phys. **64** (2016) 674–717, [arXiv:1604.01639].

[82] H. Kleinert, *Path integrals in quantum mechanics, statistics, polymer physics, and financial markets*, World Scientific (2004).

[83] C. Klimčík, *Poisson-Lie T-duality*, Nucl. Phys. Proc. Suppl. **46** (1996) 116–121, [arXiv:hep-th/9509095].

[84] C. Klimčík, and P. Ševera, *Dual Non-Abelian Duality and Drinfeld Double*, Phys. Lett. **B351** (1995) 455–462, [arXiv:hep-th/9502122].

[85] Y. Kosmann–Schwarzbach, *Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory*, Marsden J.E., Ratiu T.S. (eds) The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol 232. Birkhäuser Boston, [arXiv:math/0310359].

[86] Y. Kosmann–Schwarzbach, *Derived Brackets*, Lett. Math. Phys. **69** (2004) 61–87, [arXiv:math/0312524].

[87] Y. Kosmann–Schwarzbach, *Lie bialgebras, poisson Lie groups and dressing transformations*, Integrability of Nonlinear Systems. Lecture Notes in Physics, 495 Springer, Berlin, Heidelberg (2007) 104–170.

[88] A. Kotov, and T. Strobl, *Gauging without initial symmetry*, J. Geom. Phys. **99** (2016) 184–189, [arXiv:1403.8119].

[89] A. Kotov, and T. Strobl, *Curving Yang–Mills–Higgs gauge theories*, Phys. Rev. D. **92** (2015) 085032 [arXiv:1510.07654].

[90] A. Kotov, and T. Strobl, *Geometry on Lie Algbroids I: Compatible Geometric Structure on the Base*, [arXiv:1603.04490].

[91] C. Laurent–Gengoux, S. Lavau, and T. Strobl, *The universal Lie ∞-algebroid of a singular foliation*, [arXiv:1806.00475].

[92] S. Lavau, *Lie ∞-algebroids and singular foliations*, PhD thesis, Université Claude Bernard Lyon 1 (2016), [arXiv:1703.07404]

[93] H. Li, *Topology of Co-symplectic/Co-Kähler manifolds*, Asian J. Math. **12(4)** (2008) 527-–544.

[94] Z.–J. Liu, A. Weinstein, and P. Xu, *Manin triples for Lie bialgebroids*, J. Diff. Geom. **45(3)** (1997), 547–574, [arXiv:dg-ga/9508013].

[95] D. Lüst, E. Malek, and R.J. Szabo, *Non-geometric Kaluza–Klein monopoles and magnetic duals of M-theory R-flux backgrounds*, JHEP **144** (2017) , [arXiv:1705.09639].

[96] K. Mackenzie, and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. **73** (1994) 415–452.

[97] K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, (London Mathematical Society Lecture Note Series). Cambridge: Cambridge University Press. (2005).

[98] V. Mathai, and J. Rosenberg, *T-Duality for Torus Bundles with H-fluxes via Non-commutative Topology*, Comm. Math. Phys. **253(3)** (2005) 705–721, [arXiv:hep-th/0401168].

[99] C. Mayer, and T. Strobl, *Lie Algebroid Yang-Mills with Matter fields*, J. Geom. Phys. **59** (2009) 1613–1623, [arXiv:0908.3161].

[100] G. Papadopoulos, *C-spaces, generalized geometry and double field theory*, JHEP **29** (2015) , [arXiv:1412.1146].

[101] J. Polchinski, *String Theory: Volume 1. An introduction to the Bosonic String*, Cambridge Monographs on Mathematical Physics.

[102] Y.S. Poon, and A. Wade, *Generalized Contact Structures*, J. London Math. Soc. (2011) 333–352, [arXiv:0912.5314].

[103] M. Roček, and E. Verlinde, *Duality, quotients, and currents*, Nucl. Phys. **B373** (1992) 640–646, [arXiv:hep-th/9110053].

[104] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis, University of California Berkeley (1999), [arXiv:math/9910078].

[105] D. Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, In Quantization, Poisson brackets and beyond (Manch- ester, 2001), volume 315 of Contemp. Math., pages 169—185. Amer. Math. Soc., Providence, RI, 2002, [arXiv:math/0203110].

[106] H. Sati, U. Schreiber, and J. Stasheff, *Twisted differential String and Fivebrane structures*, Commun. Math. Phys. **315** (2012) 169–213 , [arXiv:0910.4001].

[107] U. Schreiber, and K. Waldorf, *Connections on non-abelian Gerbes and their Holonomy*, Theory Appl. Categ. **28** (2013) 476–540, [arXiv:0808.1923].

[108] K. Sekiya, *Generalized almost contact structures and generalized Sasakian structures*, Osaka J. Math. **52** (2015), [arXiv:1212.6064].

[109] P. Ševera, *Letters to Alan Weinstein about Courant algebroids*, [arXiv:1707.00265].

[110] P. Ševera, *Contact geometry in Lagrangean mechanics*, [arXiv:physics/9802046].

[111] P. Ševera, *Some title containing the words "homotopy" and "symplectic", e.g. this one*, Travaux mathématiques. Fasc. XVI, 121-137, Trav. Math., XVI, Univ. Luxemb., Luxembourg, (2005), [arXiv:math/0105080].

[112] P. Ševera, *Poisson-Lie T-duality and Courant algebroids*, Lett. Math. Phys. **105** (2015) 1689–1701, [arXiv:1502.04517].

[113] I. Struchiner, *The classifying Lie algebroid of a geometric structure*, PhD thesis, University of Campinas — UNICAMP (2009).

[114] T. Strobl, *Algebroid Yang–Mills Theories*, Phys. Rev. Lett. **93** (2004) 211601, [arXiv:hep-th/0406215].

[115] H. Sussman, *Orbits of Families of Vector Fields and Integrability of Distributions*, Trans. Am. Math. **189** (1973) 171–188.

[116] D. Thompson, *T-duality invariant Approaches to String Theory*, Ph.D. thesis, Queen Mary University of London (2010), [arXiv:1012.4393].

[117] A. Yu. Vaĭntrob, *Lie algebroids and homological vector fields*, Russ. Math. Surv. **52** (1997) 428.

[118] I. Vaisman, *Dirac structures and generalized complex structures on $TM \times \mathbb{R}^h$*, Adv. Geom. **7** (2007) 453–474, [arXiv:math/0607216].

[119] I. Vaisman, *Generalized CRF-structures*, Geom. Dedicata **133** (2008) 129–154, [arXiv:0705.3934].

[120] I. Vaisman, *On the geometry of double field theory*, J. Math. Phys. **53** (2012) 033509, [arXiv:1203.0836].

[121] I. Vaisman, *Towards a double field theory on para-Hermitian manifolds*, J. Math. Phys. **54** (2013) 123507, [arXiv:1209.0152].

[122] L. Vitagliano, *Dirac-Jacobi bundles*, J. Sympl. Geom **16** (2018) 485–561, [arXiv:1502.05420].

[123] L. Vitagliano, and A. Wade, *Generalized contact bundles*, Comptes Rendus Mathematique **354** (2016) 313–317, [arXiv:1507.03973].

[124] T. Voronov, *Higher derived brackets and homotopy algebras*, J. Pure Appl. Algebra **202** (2005) 133–153, [arXiv:math/0304038].

[125] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. (New Series) Amer. Math. Soc. **16** (1987), 101–104.

[126] A. Weinstein, *Groupoids: unifying internal and external symmetry*, Notices of the AMS **43(7)** (1996) 744–752, [arXiv:math/9602220].

[127] K. Wright, *Generalised contact geometry as reduced generalised complex geometry*, J. Geom. Phys. **130** (2018) 331–348 , [arXiv:1708.09550].