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Citation: [Review of Scientific Instruments](#) **85**, 034501 (2014); doi: 10.1063/1.4866659

View online: <http://dx.doi.org/10.1063/1.4866659>

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State observers and Kalman filtering for high performance vibration isolation systems

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(Received 9 October 2013; accepted 11 February 2014; published online 5 March 2014)

There is a strong scientific case for the study of gravitational waves at or below the lower end of current detection bands. To take advantage of this scientific benefit, future generations of ground based gravitational wave detectors will need to expand the limit of their detection bands towards lower frequencies. Seismic motion presents a major challenge at these frequencies and vibration isolation systems will play a crucial role in achieving the desired low-frequency sensitivity. A compact vibration isolation system designed to isolate in-vacuum optical benches for Advanced Virgo will be introduced and measurements on this system are used to present its performance. All high performance isolation systems employ an active feedback control system to reduce the residual motion of their suspended payloads. The development of novel control schemes is needed to improve the performance beyond what is currently feasible. Here, we present a multi-channel feedback approach that is novel to the field. It utilizes a linear quadratic regulator in combination with a Kalman state observer and is shown to provide effective suppression of residual motion of the suspended payload. The application of state observer based feedback control for vibration isolation will be demonstrated with measurement results from the Advanced Virgo optical bench suspension system. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4866659>]

I. INTRODUCTION

A world wide network of second generation gravitational wave detectors, such as Advanced Virgo and Advanced LIGO, is scheduled to come online within the next few years.^{1,2} In addition, future generation detectors, such as Einstein Telescope, are already in advanced conceptual design phase.³ An important goal of both second and future generation ground based detectors is an expansion of the detection band to lower frequencies: 10 Hz for second generation detectors and 2 Hz for Einstein Telescope. Improving the low-frequency sensitivity of these detectors has a number of important scientific objectives. Spinning asymmetric neutron stars and coalescing compact binaries are considered to be promising sources of the first detectable gravitational waves. The majority of spinning neutron stars discovered using conventional optical techniques are found to revolve at frequencies below 10 Hz.⁴ The coalescence of two compact objects, such as neutron stars and black holes, will sweep frequencies up to roughly 1.5 kHz. The time to coalescence of such systems scales with $f^{-\frac{8}{5}}$ where f is the orbital frequency of the binary system. Decreasing the detection lower bound therefore significantly increases the observed time to coalescence which in turn improves the signal-to-noise ratio and accuracy of subsequent parameter estimation.^{5,6}

A major challenge to low-frequency sensitivity is seismic noise. Current ground based detectors are kilometer scale Michelson interferometers. The vibrations of the interferometer's optical components induced by ground motion easily

exceed the displacement sensitivity required to measure gravitational waves. Custom made high performance vibration isolation systems are therefore employed to reduce seismic noise effects. For example, in the Advanced Virgo detector, the residual motion of the interferometer arm mirrors, recycling mirrors, and the beam-splitter are suppressed by up to 15 orders of magnitude above 10 Hz. This is achieved with 8 m tall suspension systems called superattenuators.⁷ Common to all vibration isolation systems is a feedback control system, with corresponding sensors and actuators, to actively suppress the residual motion of the suspended payload. Tightening requirements on low frequency performance means that conventional control techniques need to be replaced by more optimal solutions. Here, we present a state observer approach to the active control of vibration isolation systems. This will be discussed in the context of a compact vibration isolation system for Advanced Virgo in-vacuum optical benches.

II. VIBRATION ISOLATION OF OPTICAL BENCHES

Auxiliary optical systems distributed around a gravitational wave detector are essential for the control, readout, and alignment of the interferometer. The benches housing these optics also need to be isolated from seismic motion. For example, the angular alignment of the interferometer will be actively controlled with signals from a series of quadrant photodiodes; the unwanted motion of these photodiodes will mimic a misalignment of the cavities and hence introduce control noise. Furthermore, photons that are scattered or diffused off the optical components can re-enter the interferometer. These photons are modulated by the residual motion of the

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TABLE I. Requirements for Advanced Virgo in-vacuum optical benches. Spectral requirements are valid above 10 Hz. All the translational and rotational degrees of freedom are represented by z and θ , respectively.

Bench motion	Requirement
δz	$2.1 \times 10^{-12} \text{ m}/\sqrt{\text{Hz}}$
$\delta \theta$	$3.3 \times 10^{-15} \text{ rad}/\sqrt{\text{Hz}}$
z_{rms}	$1 \times 10^{-6} \text{ m}$
θ_{rms}	$3.1 \times 10^{-8} \text{ rad}$

optics. Any nonlinear behavior of this coupling leads to the up-conversion of low-frequency seismic excitations (<10 Hz) into the detection band (>10 Hz).^{8,9}

Table I lists the isolation requirements for Advanced Virgo in-vacuum optical benches in terms of residual rms motion and spectral displacement above 10 Hz.¹⁰ The translational rms is given by $z_{\text{rms}} = \sqrt{\int_{f_{\text{min}}}^{f_{\text{max}}} P_{zz} df}$ where P_{zz} is the power spectral density of the payload motion and the frequency bounds f_{max} and f_{min} are chosen to span the control band, typically 5 and 0.1 Hz, respectively. Translational rms demands are constrained by the tolerance on the amount of scattered and diffused light, while the rotational and spectral requirements are based on control noise limits.

A multi-stage vibration isolation system, coined MultiSAS, has been developed and will suspend five of Advanced Virgo's in-vacuum optical benches. Passive isolation is provided by means of a chain of mechanical oscillators that act as second order low-pass filters. A major design challenge was the limited space available at the existing Virgo facility. To satisfy the vibration attenuation demands within a compact design, anti-spring techniques were implemented to achieve low natural frequency ($f_0 < 1$ Hz) oscillators. These techniques include geometric anti-springs (GAS) for vertical^{11,12} and inverted pendulums for horizontal filters.¹³ Geometric anti-springs are realized with a crown of blade springs that are radially compressed to create an anti-spring effect. Inverted pendulums are in balance between the restoring force of their flexures and the anti-spring effect provided by gravity. MultiSAS is equipped with two vertical GAS stages and three horizontal stages consisting of one inverted pendulum and two conventional pendulum stages. In addition, MultiSAS is monitored by two types of motion sensors: linear variable differential transformers (LVDT) that measure displacement, and inertial sensors called geophones. A key feature of the system is active feedback to damp the system's rigid body eigenmodes and to maintain long term position and orientation of the optical bench. This is performed via magnetic voice coil actuators that are co-located with the LVDT sensors. An illustration of MultiSAS is given in Fig. 1 and Fig. 2 shows a schematic layout of the five attenuation stages.

A chain of cascaded mechanical filters provides an attenuation performance above the resonance frequencies proportional to f^{-2n} , where n is the number of filters. At high frequencies this becomes limited by the non-ideal characteristics of real harmonic oscillators, a phenomenon known as the center of percussion effect which is related to the mass distribution of the mechanical filters.¹⁴ As a result, the transfer

FIG. 1. An overview of the MultiSAS design. The vertical LVDT is situated at the center of the top filter and monitors the height of the intermediate filter stage relative to the top stage. A co-located vertical voice coil actuator is used to apply the vertical feedback control forces to the intermediate filter stage.

function levels out to a constant value as f approaches infinity. The performance of MultiSAS is depicted in Fig. 3 where vertical and horizontal ground to optical bench transfer functions are plotted. The modeled transfer functions were derived based on a rigid body system as described below for the vertical model and in Ref. 15 for the horizontal model. The results of the measured vertical transfer function are included in Fig. 3. We see the f^{-4} and f^{-6} decrease in vertical and horizontal transmission, respectively. Above 50 Hz the transfer functions begin to level off and in the measured transfer function we see evidence of the first higher-order modes associated with the non-rigid properties of the system. Above 10 Hz MultiSAS provides roughly 100 dB suppression of vertical vibrations and over 140 dB of horizontal motion. We also note the high peaks at the resonance frequencies of the respective filters. These would spoil the low-frequency performance of

FIG. 2. Sketch of the five attenuation stages of MultiSAS: Two vertical stages (V1, V2) realized by GAS filters and three horizontal stages consisting of one inverted pendulum (H1) and two conventional pendulum stages (H2, H3).

FIG. 3. Vibration isolation performance. Modeled vertical (dashed black curve) and horizontal (solid black curve) transfer functions and the measured vertical transfer function (solid red curve). The f^{-4} and f^{-6} characteristics of the respective vertical and horizontal filter chains are revealed.

the system and are therefore damped using active feedback control in a bandwidth up to roughly 5 Hz.

Being able to control any physical system requires sufficient knowledge of the states of that system. This is achieved by placing sensors at the relevant positions. However, it is not difficult to imagine control problems in which it is not possible or undesirable to place sensors at the required locations, or that the sensors are limited by poor sensitivity. In the light of these difficulties, it is desirable to implement a feedback control system that can compare measurements with *a priori* knowledge of the system dynamics. State observation and the Kalman filter provides the optimal mathematical formalism to do just that. The use of observer based control in the context of seismic attenuation for Advanced Virgo has been demonstrated offline.¹⁶ Here, we present the first implementation of such a system in a realtime environment. Other gravitational wave detector applications of observer based control have also been proposed for different types of suspension systems.^{17,18}

III. SYSTEM MODELING AND OBSERVATION

A. State observer and the Kalman filter

Feedback control utilizes sensors to measure the state of the system under control, which is then fed via a controller to the system input or actuator. A linear, first-order, time-invariant system can be described by the state space equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad (1)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \quad (2)$$

where $\mathbf{x}$ is a vector containing the system's states (for example, the position or velocity of an object within the system). The state matrix $\mathbf{A}$ characterizes the dynamics of the states. External inputs to the system, such as feedback control, are accounted for by the input vector $\mathbf{u}$ via an input matrix $\mathbf{B}$. The measurements of the system $\mathbf{y}$ will be a superposition of the system's states mapped to the sensor outputs via the sensing matrix $\mathbf{C}$. Any direct coupling between input and output signals will be described by the feedthrough matrix $\mathbf{D}$.

A typical feedback approach assumes that each state of the system is accessible or at least could be calculated directly from operations on the available measurements. This is not always the case, particularly for systems with complex internal dynamics or limited sensing capabilities. A strategy to overcome these issues is to estimate the states of the system from a limited number of measurements. This is called a *state observer*.

A state observer models the internal dynamics of the mechanical system under control. It is fed the same input signals $\mathbf{u}$ as the system, and produces an estimate of the corresponding states, denoted by $\hat{\mathbf{x}}$. Its estimates of the output $\hat{\mathbf{y}}$ are compared against the measured output $\mathbf{y}$. The objective is to ensure that the states of the observer will track the states of the system. The estimated states of the observer can then be used as the states of the system, and included in the feedback control scheme. The estimator $\hat{\mathbf{x}}$ has the same dynamics as the system such that

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}. \quad (3)$$

The output estimate of the observer is

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}, \quad (4)$$

assuming, for simplicity, $\mathbf{D} = 0$ in Eq. (2). The error between the measurements of the system and the estimates of the observer is therefore given by

$$\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \mathbf{C}\hat{\mathbf{x}}. \quad (5)$$

An effective observer has a minimal error. In order to minimize $\mathbf{e}$, an additional term is added to Eq. (3) that is proportional to the error. The observer equation then becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}), \quad (6)$$

where $\mathbf{L}$ is the *observer gain matrix* which weighs the observer error. It has as many rows as there are states and as many columns as outputs. Fig. 4 shows a schematic version of Eq. (6). Finding a suitable observer gain matrix is the subject of various state estimation techniques, for example, Ackermann's formula¹⁹ or the Luenberger observer.^{20,21} Naïvely, one would think that designing an observer to respond rapidly to differences between measured and estimated outputs would

FIG. 4. Schematic of a state observer. The system can be described by the state space matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. The observer has as input both $\mathbf{u}$ and the measurements from the system $\mathbf{y}$, which it compares with its estimated outputs $\hat{\mathbf{y}}$. The error is then weighed with the observer gain matrix $\mathbf{L}$. The observer's output is the state estimate $\hat{\mathbf{x}}$. The integral sign $\int$ represents the computation of $\hat{\mathbf{x}}$ from $\dot{\hat{\mathbf{x}}}$.

be most effective. However, this strategy makes the state estimates more sensitive to uncorrelated noise in the system, such as measurement noise at the system output or the uncertainties in the input response. Given the statistical properties of the various noise sources and knowledge of the internal dynamics of the system an “optimal” observer can be devised that minimizes the mean square difference between the measured and estimated states. Such a state observer is known as a *Kalman filter*.^{22,23} It is a recursive process that updates its gain matrix based on the complete history of measurements, and can adapt to changes in the uncertainty of the measurements or estimates. These noise terms are assumed to be zero mean, independent, white Gaussian random processes.

B. Linear quadratic regulator (LQR)

In general, a feedback control system is used to force a mechanical system to obey some prescribed motion. In many applications (in particular the one described here), the aim of the control system is simply to achieve the greatest possible reduction of residual motion. Optimal control strategies exist that attempt to minimize a performance index or cost function that is proportional to the measure of the mechanical system’s response.

The LQR provides an optimal feedback controller for a *linear* system based on a *quadratic* cost function. The cost function is used to balance the relation between feedback signals $\mathbf{u}$ and system states $\mathbf{x}$ and can be tuned to reduce deviations from their desired values. For a continuous-time system described by Eq. (1), the LQR cost function^{24,25} is given by

$$J_{\text{LQR}} = \int_0^{\infty} (\mathbf{x}^T Q_{\text{LQR}} \mathbf{x} + \mathbf{u}^T R_{\text{LQR}} \mathbf{u}) dt. \quad (7)$$

The positive-definite symmetric matrices Q_{LQR} and R_{LQR} are used to weigh the system state and control power, respectively. These are defined by the control system designer based on specified control criteria. This usually involves a sequence of trials to tune the weighing factors to achieve the desired results. It can be shown that the feedback controller that minimizes the cost function is

$$\mathbf{u} = -K\mathbf{x}, \quad (8)$$

where K is the LQR gain matrix given by $K = R_{\text{LQR}}^{-1} B^T P$ and P is found by solving the continuous time algebraic Riccati equation,

$$A^T P + PA - PBR_{\text{LQR}}^{-1} B^T P + Q_{\text{LQR}} = 0. \quad (9)$$

This optimal solution does assume that all of the states are available for feedback. It has been argued above that this is not always the case. Instead, an observer can be implemented to estimate all the states from a limited number of measurements. It can be shown that the characteristics of the observer can be chosen independently of those for the state feedback system.²⁴ Effectively, this means that the LQR parameters will remain the same, regardless of the type or configuration of the observer being used. This is known as the *separation principle*. The combination of the LQR controller with a Kalman filter as an observer, is referred to as a linear quadratic

Gaussian (LQG) controller, due to the Gaussian characteristics of the Kalman filter noise terms.

IV. MULTISAS SYSTEM IDENTIFICATION

Modeling of the attenuation system is an important part of understanding its dynamic characteristics and a useful tool for the design of control strategies. Here, MultiSAS is modeled as a rigid body system. This means that it is assumed that each component of the system has no internal vibrational modes and we concentrate on the modes only associated with the coupled vibrations of the rigid components. This assumption is generally accurate when studying the low frequency (<10 Hz) dynamics of such systems and corresponds to the frequency range in which active feedback will be implemented. Furthermore, we assume that the vertical degrees of freedom are uncoupled from all other modes. In this way, we can divide the modeling problem into separate models, each one responsible for describing mutually uncoupled modes. Here, only the vertical model will be presented, but a general approach will be applied.

The vertical dynamics can be reduced to the double mass spring system shown in Fig. 5. The assumption is made that the masses of the springs are negligible with respect to the suspended masses. The generalized coordinates describing the degrees of freedom are the vertical displacement of the intermediate filter of mass m_1 given by y_1 , and the vertical displacement of the bench payload with mass m_2 denoted by y_2 . The GAS filters are characterized by the corresponding spring constants k_1 and k_2 , with viscous damping coefficients γ_1 and γ_2 . The actuation force f_y for the active control system is applied to the intermediate stage via the vertical voice coil actuator on the top stage. It is convenient to define $f_y = k_1 u_y$, where u_y is a virtual control displacement. Finally, the top stage vertical motion is given by y_0 and can be considered equivalent to the vertical ground motion.

The Euler-Lagrange approach is used to solve the equations of motion. Given the potential energy U and the

FIG. 5. Schematic model showing the vertical displacements of the ground (top stage), intermediate stage (m_1), and payload (m_2), given by y_0 , y_1 , and y_2 , respectively. The top and intermediate filter spring constants are given by k_1 and k_2 with corresponding viscous damping coefficients γ_1 and γ_2 . The feedback and actuation force f_y is applied to the intermediate mass. Sensor positions for the LVDT and geophone are also indicated.

kinetic energy T and defining the Lagrangian $L = T - U$, the equations of motion will follow from the Euler-Lagrange equation²⁶

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}_i} \right) - \frac{\partial L}{\partial y_i} = Q_i. \quad (10)$$

Here, each of the generalized coordinates y_1 and y_2 are accounted for by $i = [1, 2]$ and external forces enter into the equation through the generalized forces Q_i . Non-conservative forces such as friction and other sources of dissipation or damping require special treatment in the Euler-Lagrange method. In the special case of viscous damping, it is useful to introduce the *Rayleigh dissipation function* R and describe the generalized forces in terms of their conservative contribution only, denoted here by Q_i^* .²⁷ Equation (10) then becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}_i} \right) - \frac{\partial L}{\partial y_i} + \frac{\partial R}{\partial \dot{y}_i} = Q_i^*. \quad (11)$$

The vertical kinetic energy, potential energy, and dissipation are described by

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2, \\ U &= \frac{1}{2} k_1 (y_1 - y_0)^2 + \frac{1}{2} k_2 (y_2 - y_1)^2, \\ R &= \frac{1}{2} \gamma_1 (\dot{y}_1 - \dot{y}_0)^2 + \frac{1}{2} \gamma_2 (\dot{y}_2 - \dot{y}_1)^2, \end{aligned} \quad (12)$$

and the conservative contributions of the external forces become $Q_1^* = f_y$ and $Q_2^* = 0$. We then arrive at the following equations of motion for y_1 and y_2 , respectively:

$$\begin{aligned} m_1 \ddot{y}_1 &= -k_1 (y_1 - y_0) + k_2 (y_2 - y_1) \\ &\quad - \gamma_1 (\dot{y}_1 - \dot{y}_0) + \gamma_2 (\dot{y}_2 - \dot{y}_1) + f_y, \end{aligned} \quad (13)$$

$$m_2 \ddot{y}_2 = -k_2 (y_2 - y_1) - \gamma_2 (\dot{y}_2 - \dot{y}_1). \quad (14)$$

To facilitate a transformation to a state space description, the additional first order differential equations $v_1 = \dot{y}_1$ and $v_2 = \dot{y}_2$ are defined. It is now possible to reduce the equations of motion from two second order to four first order differential equations. These can be rewritten into a state space representation given by

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}}_{\dot{\mathbf{x}}_p} &= \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & \frac{-(\gamma_1+\gamma_2)}{m_1} & \frac{\gamma_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \frac{\gamma_2}{m_2} & \frac{-\gamma_2}{m_2} \end{bmatrix}}_{A_p} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix}}_{\mathbf{x}_p} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{k_1}{m_1} \\ 0 \end{bmatrix}}_{B_p} u_y + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k_1}{m_1} & \frac{\gamma_1}{m_1} \\ 0 & 0 \end{bmatrix}}_{B_{pn}} \underbrace{\begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}}_{w_d}, \end{aligned} \quad (15)$$

$$\underbrace{\begin{bmatrix} y_{\text{lvdt}} \\ y_{\text{geo}} \end{bmatrix}}_{\mathbf{y}_p} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C_p} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix}}_{\mathbf{x}_p}. \quad (16)$$

The dynamics of the vertical system can now be described by the state space matrices $[A_p, B_p, C_p, D_p = 0]$, state vector $\mathbf{x}_p = [y_1 \ y_2 \ v_1 \ v_2]^T$, and input u_y . Disturbances to the system from ground motion are contained in $B_{pn} w_d$. The state space model also provides transfer function predictions of input signals to output measurements given by

$$H(s) = \frac{Y(s)}{U(s)} = C_p (s\mathbb{I} - A_p)^{-1} B_p + D_p, \quad (17)$$

where Y and U are the Fourier transforms of the output and input signals, respectively, and s is the Laplace parameter. Measurements of the MultiSAS vertical transfer function were made by applying low-pass filtered white noise to the input signal u_y . The low-pass corner frequency was set at 5 Hz. It is assumed that the forced displacement is much larger than the external (seismic) displacement ($u_y \gg w_d$) such that the last term in Eq. (15) can be neglected. The displacement y_1 was recorded with the LVDT and the velocity v_2 with the geophone on the payload. Note that the raw LVDT measurement is in fact $y_1 - y_0$; here y_{lvdt} refers to the *sensor corrected* LVDT signal that has been compensated for ground motion, by using a witness seismometer on the floor. In this case, we can approximate $y_1 \approx y_{\text{lvdt}}$. The measurement results and model predictions are shown in Fig. 6. The parameters used in the model were chosen to best fit the results given the constraint that the known total mass is 420 kg. They are given in Table II.

Two eigenmodes are visible in Fig. 6. The lowest, at 200 mHz, corresponds to the common mode where both intermediate and bench masses move in phase. The second, at around 750 mHz, is associated with the differential mode in which the two masses move in antiphase. There is agreement between the modeled and measured transfer functions. The noise seen above 5 Hz is from the loss of correlation between the excitation noise and the measured signals. At around 620 mHz a small glitch is visible. This is due to the coupling of the payload pitch and roll modes to the vertical motion of the top keystone. Damping properties also vary between measured and modeled results. This is evident in the sharper modeled peaks and steeper modeled phase transitions, and emphasized by the insets showing the ratio between measured and modeled transfer function magnitudes. This can be attributed to additional structural damping in the GAS blades, where the model only regards viscous damping. Structural damping is more prevalent for lower resonance frequencies. It can partially be accounted for in the model by adding a small complex term in the spring constant such that $k' = k(1 + i\phi)$. This effectively adds a damping term that is proportional to the displacement but in phase with the velocity. For the MultiSAS vertical model $\phi \approx 0.05$ rad is sufficient to account for the structural damping. However, to simplify subsequent computations and control schemes this term is further neglected.

FIG. 6. Comparison between measured and modeled input to output transfer functions for the vertical degrees of freedom. Model predictions are given with no structural damping ($\phi = 0$, dashed black curve) and with structural damping ($\phi = 0.05$, dotted dashed cyan curve). The insets show the ratio between modeled and measured transfer functions in the range 0.1–1 Hz. (a) Transfer function of the intermediate stage motion measured by the LVDT (y_{lvdt}/u_y). (b) Bench velocity over forced displacement (y_{geo}/u_y).

Fig. 6 also demonstrates the attenuation performance of MultiSAS. At 5 Hz the top filter already provides 40 dB of vibration suppression while the transmission to the velocity of the bench is -60 dB. Translated to bench displacement this is equal to an attenuation of roughly 90 dB.

V. VERTICAL CONTROL AND THE STATE OBSERVER

The control system for MultiSAS foresees the use of displacement sensors (LVDT) and inertial sensors (geophone).

TABLE II. Parameters for the vertical state space model.

	Parameter	Value
Mass [kg]	m_1	105
	m_2	315
Spring constant [N m^{-1}]	k_1	1036
	k_2	1046
Damping coefficient [N s m^{-1}]	γ_1	20
	γ_2	12

There is a strong case for the use of both types of sensors in the control of the system. The LVDT has an excellent low frequency performance with a resolution of a few $\text{nm}/\sqrt{\text{Hz}}$ across a broad frequency range. It provides error signals for DC and low frequency (<0.1 Hz) control. Geophones, on the other hand, typically lose sensitivity below 0.1 Hz, yet have superior performance in comparison to the LVDTs at frequencies above roughly 0.5 Hz. In addition, inertial sensors are not limited by noise from a reference frame and will therefore not re-inject seismic noise when used in closed loop feedback. A standard technique, in a situation where LVDT and geophone measurements of the *same* object are available, is sensor blending.²⁸ In the MultiSAS vertical motion case at hand, the LVDT registers displacements between the top and intermediate stages, while the geophone monitors the bench motion. The dynamics between the two are governed by Eq. (15). Standard sensor blending techniques can therefore not be implemented. A state observer can however be generated based on Eq. (15) with both LVDT and geophone measurements as inputs.

A. Kalman filter with non-white noise

We developed an optimal estimator based on a state space model and known input and measurement noises. Standard Kalman filter methods assume that the noise terms are zero mean, white Gaussian distributed and mutually uncorrelated. In reality, it is often the case, as it is with MultiSAS, that the relevant noise sources are non-white. Here, a method is discussed that extends the state space model to include the colored properties of the disturbance and measurement noises. The method utilizes shaping filters that produce colored noise from a white Gaussian noise input.²⁹ The extended state space model can then be manipulated, with the usual tools, to generate a Kalman state observer K_{est} and subsequent feedback controllers.^{25,30–32} A schematic of the vertical state observer with shaping filters is shown in Fig. 7.

The measurement noise of the LVDT and geophone are readily determined. By locking the LVDT firmly in its zero position while recording the demodulated output, its noise

FIG. 7. Schematic of a vertical state observer for MultiSAS. The Kalman state observer is represented by K_{est} while the system dynamics are described by the system P . The shaping filters W_L , W_g , and W_d account for the colored frequency response of the LVDT, geophone, and disturbance noises, respectively. The feedback control signal is denoted by u_y and the displacements measured by the LVDT and geophone by y_{lvdt} and y_{geo} , respectively.

FIG. 8. The modeled shaping filters for the measurement and disturbance noises. The curves show their frequency response to a zero mean white noise input. The respective measured spectra are also plotted.

floor could be measured. The LVDT noise is flat with a low frequency $1/f$ characteristic (due to electronic noise) with a corner frequency at 0.03 Hz. A single order shaping filter W_L is sufficient to model the LVDT noise and is described by a state space system with states $\mathbf{x}_L$, zero mean flat input noise n_1 , and output w_L , such that

$$\begin{aligned}\dot{\mathbf{x}}_L &= A_L \mathbf{x}_L + B_L n_1, \\ w_L &= C_L \mathbf{x}_L + D_L n_1.\end{aligned}\quad (18)$$

At frequencies below a few hundred Hz, the geophones are limited by the Johnson noise of their sensing coils.³³ This noise can be measured by shorting the preamplifier input with a resistor of the same resistance as the geophone coil. The resulting geophone noise can be described by a second order shaping filter W_g represented by $[A_g, B_g, C_g, D_g]$ with states $\mathbf{x}_g$, zero mean flat input noise n_2 , and output w_g . The measured and modeled LVDT and geophone noises are plotted in Fig. 8.

Note that the geophone noise tapers down at low frequencies. This is not a property of the geophone sensitivity but simply the effect of the high-pass filters used to eliminate large offsets as a result of dividing out the geophone sensitivity which approaches zero as ω does. It is clear that, based on these noise properties, an ideal combination of the LVDT and geophone data would require the fusion of the two signals with a cross over frequency around 0.3 Hz.

The disturbance noise w_d enters at the input of the system u_y . In the case of MultiSAS seismic motion of the top stage acts on the system by applying additional forces to the intermediate stage through the top filter. Neglecting other input disturbances such as DAC noise or voice coil nonlinearities we can approximate w_d by the measured seismic noise in the laboratory. This noise will be modeled by a sixth order shaping filter W_d with a corresponding state space representation given by $[A_d, B_d, C_d, D_d]$ with states $\mathbf{x}_d$, zero mean white input noise d , and output w_d . The disturbance noise shaping filter and measured seismic noise are plotted in Fig. 8.

The state space equations governing the dynamics of the vertical stages of MultiSAS, given by Eqs. (15) and (16), can

now be extended to include the shaping filters.³⁰ The state variables for the colored noise processes now appear as extra state variables of the physical system and all noise inputs are represented by white processes as required. The extended state space equations can then be written as

$$\begin{aligned}\underbrace{\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_d \\ \dot{\mathbf{x}}_L \\ \dot{\mathbf{x}}_g \end{bmatrix}}_{\dot{\mathbf{x}}} &= \underbrace{\begin{bmatrix} A_p & B_p C_d & \emptyset & \emptyset \\ \emptyset & A_d & \emptyset & \emptyset \\ \emptyset & \emptyset & A_L & \emptyset \\ \emptyset & \emptyset & \emptyset & A_g \end{bmatrix}}_A \underbrace{\begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_d \\ \mathbf{x}_L \\ \mathbf{x}_g \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} B_p \\ \emptyset \\ \emptyset \\ \emptyset \end{bmatrix}}_{B_u} u_y \\ &+ \underbrace{\begin{bmatrix} B_p D_d & \emptyset & \emptyset \\ B_d & \emptyset & \emptyset \\ \emptyset & B_L & \emptyset \\ \emptyset & \emptyset & B_g \end{bmatrix}}_{B_n} \underbrace{\begin{bmatrix} d \\ n_1 \\ n_2 \end{bmatrix}}_{\mathbf{n}},\end{aligned}\quad (19)$$

and

$$\begin{aligned}\underbrace{\begin{bmatrix} y_{\text{lvdt}} \\ y_{\text{geo}} \end{bmatrix}}_{\mathbf{y}} &= \underbrace{\begin{bmatrix} C_{p1} & \emptyset & C_L & \emptyset \\ C_{p2} & \emptyset & \emptyset & C_g \end{bmatrix}}_C \underbrace{\begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_d \\ \mathbf{x}_L \\ \mathbf{x}_g \end{bmatrix}}_{\mathbf{x}} \\ &+ \underbrace{\begin{bmatrix} \emptyset & D_L & \emptyset \\ \emptyset & \emptyset & D_g \end{bmatrix}}_{D_n} \underbrace{\begin{bmatrix} d \\ n_1 \\ n_2 \end{bmatrix}}_{\mathbf{n}},\end{aligned}\quad (20)$$

where C_{p1} and C_{p2} are the rows of C_p corresponding to the y_{lvdt} and y_{geo} outputs, respectively. The symbol $\emptyset$ denotes zero matrices of the relevant sizes.

To be sure the system under consideration is in fact observable and controllable, a verification of the observability and controllability criteria can be performed on the state space model. This corresponds to determining the rank of the respective observability and controllability matrices.³² The rank must equal the order of the system and in the case of observability implies that each state of the system can be inferred from the outputs. The dual concept of controllability implies that each state can be moved from an initial to a final position by the system's inputs. The system model given by $[A_p, B_p, C_p, D_p = 0]$ is both observable and controllable.

The estimate of system's state $\mathbf{x}$, denoted by $\hat{\mathbf{x}} = [\hat{y}_1 \ \hat{y}_2 \ \hat{v}_1 \ \hat{v}_2]^T$ will be obtained here with a Kalman filter. Equations (19) and (20) can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + B_u u_y + B_n \mathbf{n}_p,$$

$$\begin{bmatrix} y_{\text{lvdt}} \\ y_{\text{geo}} \end{bmatrix} = \mathbf{C} \mathbf{x} + \mathbf{n}_m,$$

where $\mathbf{n}_p = [d \ n_1 \ n_2]^T$ is defined to be the process noise, and the measurement noise is given by $\mathbf{n}_m = D_n \mathbf{n}_p = [D_L n_1 \ D_g n_2]^T$. Of importance for the Kalman filter equations are the process and measurement noise covariance matrices $\mathbf{Q}_n$, $\mathbf{R}_n$,

and N_n defined as

$$\begin{aligned} Q_n &\equiv E[\mathbf{n}_p \mathbf{n}_p^T] = \mathbb{I}_{3 \times 3}, \\ R_n &\equiv E[\mathbf{n}_m \mathbf{n}_m^T] = \begin{bmatrix} D_L^2 & 0 \\ 0 & D_g^2 \end{bmatrix}, \\ N_n &\equiv E[\mathbf{n}_p \mathbf{n}_m^T] = \begin{bmatrix} 0 & 0 \\ D_L & 0 \\ 0 & D_g \end{bmatrix}. \end{aligned} \quad (21)$$

The process covariance matrix Q_n is an identity matrix because the input noises are defined to have unity variance. The Kalman observer gain matrix is given by $L = (PC^T + B_n N_n) R_n^{-1}$, where the covariance matrix P is derived by solving the algebraic Riccati equation

$$\begin{aligned} [A - B_n N_n R_n^{-1} C] P + P [A - B_n N_n R_n^{-1} C]^T \\ - PC^T R_n^{-1} C P + B_n [Q_n - N_n R_n^{-1} N_n^T] B_n^T = 0. \end{aligned} \quad (22)$$

The resulting observer equations are given by

$$\begin{aligned} \hat{\mathbf{x}} &= A\hat{\mathbf{x}} + B_u u_y + B_n \mathbf{n}_p + L(\mathbf{y} - \hat{\mathbf{y}}), \\ \hat{\mathbf{y}} &= \begin{bmatrix} \hat{y}_{\text{lvd}} \\ \hat{y}_{\text{geo}} \end{bmatrix} = C\hat{\mathbf{x}} + \mathbf{n}_m. \end{aligned} \quad (23)$$

The state observer relates measurements $\mathbf{y}$ and input u_y to estimates of the states $\hat{\mathbf{x}}$ and outputs $\hat{\mathbf{y}}$ in an optimal way based on the respective noise levels of the measurements and input.

Of interest in the vertical control problem for MultiSAS is the state observer for $\hat{y}_1$: the displacement of the intermediate stage. This is also the point where a control force will be applied via the top stage vertical voice coil actuator. The frequency response of the Kalman filter for $\hat{y}_1$ is shown in Fig. 9 and illustrates how the measurements will be blended to obtain an estimate of the state y_1 . At low frequencies $\hat{y}_1$ is based solely on the LVDT signal. This is expected due to the superior sensitivity of the LVDT with respect to the geophone at these frequencies. Towards increasing frequency the geophone signal is blended with that of the LVDT and

FIG. 9. Frequency response of the Kalman filters for the y_1 estimator. The LVDT signal contributes the most at low frequencies while the geophone contribution dominates above 0.5 Hz. The contribution from input u_y is small due to the high level of disturbance noise.

FIG. 10. Time domain results of the vertical Kalman filter for MultiSAS. State estimates $\hat{\mathbf{x}}$ (dashed blue curves) are shown in order from top to bottom: y_1 , v_1 , y_2 , and v_2 . Measured $\mathbf{y}$ (solid red curves) and estimated $\hat{\mathbf{y}}$ (dashed dotted green curves) outputs are shown for the LVDT and geophone signals in the top and bottom panels, respectively.

from 0.5 Hz starts to dominate the y_1 estimate. This corresponds to the cross over in sensor noise as seen in Fig. 8. The shape of the geophone filter reflects the dynamics of the system by gaining magnitude towards higher frequencies where the bench motion is attenuated by the system's mechanics. This geophone contribution decreases again above 4 Hz as the signal-to-noise ratio in the geophone becomes too small. The estimate of y_1 based on the input u_y is largely neglected due to the high level of disturbance noise introduced by seismic motion.

The Kalman filter's ability to observe all states is demonstrated with time domain data in Fig. 10. MultiSAS was excited vertically with a sinusoidal input signal u_y with an amplitude of $10 \mu\text{m}$ and a frequency of 0.1 Hz. The (sensor corrected) LVDT, geophone, and input signals were collected and the estimates $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ were calculated offline. The output estimates $\hat{\mathbf{y}}$ will include the dynamics of the measurement noise shaping filters, as these are included in the measurement matrix C of Eq. (20), while the estimates $\hat{\mathbf{x}}$ are state observations governed by the system and disturbance noise dynamics only. The top panel of Fig. 10 shows the LVDT signal, the estimated LVDT output, and the estimate of state y_1 . The system takes some time to respond to the input signal but after roughly 30 s the sinusoidal excitation is clearly visible. The estimated output $\hat{y}_{\text{lvdt}}$ and state $\hat{y}_1$ are very similar because the LVDT noise is not significant at these levels of displacement. The difference between the measured LVDT output y_{lvdt} and the estimates is the contribution from the geophone at higher frequencies where the geophone has better sensitivity. The low frequency components, in particular the 0.1 Hz excitation

response, are contributed mainly by the measured LVDT signal. The two central panels present data for the estimates of the unmeasured states, v_1 and y_2 . Finally, the measured and estimated geophone signals and corresponding estimate of state v_2 are shown in the bottom panel of Fig. 10. The measured y_{geo} and estimated $\hat{y}_{\text{geo}}$ outputs are very similar owing to the filter's belief that the LVDT will not be able to contribute to the estimated output: at low frequencies the LVDT may have better sensitivity but the geophone output (and therefore its estimate) will be dominated by sensor noise from the geophone anyway. In the state estimate $\hat{v}_2$, on the other hand, the filter rejects low frequency geophone signal in favor of (low frequency) LVDT measurements.

B. Vertical linear quadratic Gaussian control

Linear quadratic Gaussian control combines the linear quadratic regulator with the Kalman state observer discussed above.^{18,31,32} The LQR assumes that all the states of the system are known and produces a gain matrix K that optimizes a quadratic cost function (see Eq. (7)). However, in real control problems all of the states are not always easily accessible. The Kalman observer provides an optimal means to estimate all the states based on the measurements that are available, and based on knowledge of the system dynamics and noise sources. Finally, the separation principle makes it possible to compose an optimal LQG controller by first determining the LQR and Kalman observer separately before combining them. LQG control for MultiSAS utilizes multiple measurement inputs to produce a single feedback signal. It is therefore considered to be a multiple-input single-output (MISO) regulator. A schematic of a LQG controlled system is shown in Fig. 11.

A LQG regulator can be tuned by adjusting the weighing matrices Q_{LQR} and R_{LQR} of the LQR, and Q_n and R_n of the Kalman filter. Generally, one of the matrices in each pair is set to unity and the other scaled accordingly. In the current MultiSAS example, $R_{\text{LQR}} = 1$ and $Q_{\text{LQR}} = \text{Diag}[Q, 0, 0, Q, 0, \dots, 0]$, where Q can be tuned and weighs the y_1 and v_2 state estimates evenly. The trailing zeros account for the additional shaping filter states that can be rejected in further calculations.

FIG. 11. Schematic of the MultiSAS vertical control scheme with a MISO regulator consisting of a state observer K_{est} and LQR gain matrix K . The MultiSAS dynamics are described by the system P . The shaping filters W_L , W_g , and W_d account for the colored frequency response of the LVDT, geophone, and disturbance noises, respectively.

Increasing Q places more emphasis on minimizing the system states, rather than the input signal. This has a similar effect to increasing the overall gain of the controller.

The Kalman filter weighing matrices are given in Eqs. (21). To facilitate the ability to tune the disturbance noise with respect to the measurement noise, the process covariance matrix is redefined here as

$$Q_n = \begin{bmatrix} Q_d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

where Q_d is the variance of the white noise input to the disturbance noise shaping filter. By increasing Q_d with respect to R_n and the other diagonal elements of Q_n , the Kalman filter places a stronger belief in the measurements, rather than in the system model.

C. Vertical proportional integral derivative (PID) control

To compare the LQG control with conventional techniques, we briefly discuss the design of a PID controller. Here, the sensor corrected vertical LVDT signal is used in the feedback loop and the actuation signal is again applied to the co-located vertical voice coil actuator. This is the traditional approach to resonance frequency damping and will function as a performance base line against which to compare the LQG control. The vertical PID controller can be described by

$$C(s) = G \left(G_D s + G_P + \frac{G_I}{s} \right), \quad (25)$$

where G_D , G_P , and G_I are the derivative, proportional, and integral gains, respectively, with G representing an overall gain. For the vertical PID controller, the values $G_D = 1$, $G_P = 0.5$, and $G_I = 0.05$ were chosen. At low frequencies ($s < 0.05$ rad/s), the integration term dominates and produces a feedback signal proportional to the integrated value of the vertical position. As the frequency increases the proportional term takes over applying a force proportional to the measured displacement. This is useful for keeping the vertical position around zero (or an arbitrary set position) at frequencies below the resonance. At increasing frequencies ($s > 0.5$ rad/s), the derivative term dominates and produces a feedback signal proportional to the vertical velocity, essentially implementing viscous damping, a technique effective in attenuating resonance frequencies.

VI. RESULTS

The effectiveness of the control scheme can be tested by injecting white noise at the input, i.e., at the voice coil, while at the same time implementing the feedback control and monitoring the system response. In this way, the signal levels are increased above any sensor noise floors. In addition, the results can be plotted as forced displacement transfer functions. It should be noted that the measurement results presented here are from in-loop sensors and, as such, can only provide an indication of the true performance.

FIG. 12. Vertical transfer functions in open and closed loop to test the performance of the PID controller. The gain is varied from 0 (open loop) to 2. Note that, without noise, y_{geo} corresponds to the bench velocity (see Eq. (16)).

Fig. 12 shows the transfer function results of the vertical PID controller for various values of the overall controller gain. In the top panel, the transmission of input noise to LVDT displacement is shown. The transmission to the bench velocity measured by the geophone is given in the bottom panel. We see that the control is successful in damping the resonance frequencies as seen at both the top stage LVDT and the bench geophone. However, due to a notch in the top stage response, only a small signal is measured by the LVDT and hence a weak feedback force is applied around 300 mHz. As a result, the bench motion is left largely intact at these frequencies. This limits the effectiveness of the PID in reducing the rms motion of the payload. A certain amount of noise injection is evident above 4 Hz. Increasing the gain further aggravates this issue.

Transfer function results with LQG control are shown in Fig. 13 for various values of Q and Q_d . The LQG regulator is successfully able to damp the vertical resonances. Examining the bench motion as observed by the geophone, we see a significant improvement over the PID control. Because the LQG controller is designed to minimize both LVDT and geophone signal, the bench motion is actively suppressed thanks to the blended contributions of the additional geophone information. There is evidence of extra control noise injected in the frequency range from 1 to 5 Hz. This effect can be somewhat reduced by increasing Q_d . As Q_d becomes large the injected noise decreases, suggesting that a stronger belief in the measurement (with respect to the model) reduces noise injection. Forced displacement measurements however, increase signal levels well above their noise floors making an increased Q_d

FIG. 13. Vertical transfer functions in open and closed loop to test the performance of the LQG controller. The parameters Q and Q_d are varied. Note that, without noise, y_{geo} corresponds to the bench velocity (see Eq. (16)).

a plausible solution. Under normal operating conditions, this may not be the case.

For the controlled MultiSAS to meet its requirements, the control system must suppress rms motion without interfering with the passive isolation performance above 10 Hz. Therefore, the injection of noise in the 1–5 Hz range is not troublesome. In fact, the improved reduction of low frequency bench motion by up to 40 dB will be highly advantageous.

The performance of the control system was tested with MultiSAS in its free running state with only environmental disturbances, i.e., without forced excitation. These results are shown in Fig. 14. In the top panel, the LVDT displacements are shown for the open and closed loop configurations alongside the ground motion as measured by the Trillium 240 seismometer. In the bottom panel, the projection of the rms motion of the bench from the geophone measurements is presented. The resonance peaks are clearly visible in the open loop results (solid red curves). These are effectively damped by both the PID (dashed blue curve) and LQG (dotted green curve) control schemes. In general, the LQG control method outperforms the PID control below 1 Hz. Fine tuning the PID gain values may lead to improved performance, however, due to the notch in the intermediate filter response, the bench motion at these frequencies cannot be suppressed below the open loop response by using the LVDT signal alone.

A critical measure for control performance is the low frequency rms motion. Observing the results in Fig. 14(b) at 0.1 Hz the PID control is seen to reduce rms motion by a factor of 3 in comparison to the open loop system. The LQG controller improves on this by an additional factor of 2, bringing the rms motion down to $0.5 \mu\text{m}$. This is within the

FIG. 14. MultiSAS vertical control results with environmental disturbances only. Closed loop performance plotted for the PID controller with a gain $G = 2$ and the LQG controller designed with $Q = 100$ and $Q_d = 10$. (a) The LVDT displacement signal and the vertical ground displacement measured by the seismometer. (b) The bench displacement projected from the geophone signal and the equivalent rms motion. At frequencies below 0.1 Hz the signal is dominated by measurement noise and above 2 Hz by acoustic noise coupling to the suspended mass.

translational requirements for the optical bench of $1 \mu\text{m}$. Geophone results below 0.1 Hz are limited by the sensor's sensitivity. Measuring above a few Hz was not feasible with the current out-of-vacuum setup as acoustic coupling acting directly on the payload dominates its residual motion at these frequencies.

VII. SUMMARY

High performance vibration isolation will play a crucial role in improving the low-frequency sensitivity of next generation gravitational wave detectors. Improvements to the active feedback control of these systems are needed to respond to the tightening seismic noise requirements. We have demonstrated the use of a novel to this field, observer based control scheme for the vertical degrees of freedom of a compact, high performance vibration isolation system designed to suspend in-vacuum optical benches in Advanced Virgo. Performance results coincided with modeled predictions of over 100 dB

suppression of vertical ground motion above 10 Hz, while horizontal isolation in excess of 140 dB is predicted above 10 Hz.

It was shown that the system could be effectively modeled with Lagrangian mechanics in order to produce accurate state space representations. Measured transfer function results were shown to coincide with modeled predictions. Shaping filters were used to extend the state space model to include the colored properties of the measurement and disturbances noise sources.

A Kalman state observer was implemented to estimate all the states of the system, including those states that could not directly be measured. In addition, the state observer was able to effectively combine signals from the LVDT and geophone, situated at different locations within the system, in order to more accurately estimate the states based on the noise characteristics of the respective sensors, and the dynamic properties of the system. Finally, the estimated states were combined with a linear quadratic regulator in order to implement a linear quadratic Gaussian feedback control scheme.

This approach proved effective in damping resonance frequencies. Over 40 dB active suppression was demonstrated with forced displacement transfer functions and a factor of six reduction in low-frequency rms motion was achieved with a freely running system. In comparison with conventional control techniques, the observer based control outperformed a proportional integral derivative based controller in reducing rms motion below 1 Hz.

ACKNOWLEDGMENTS

The authors would like to thank Fred Schimmel for his instrumentation contributions for the performance measurements and Benoit Mours and colleagues from LAPP-Anecy for the DAQ and computing support. The design and mechanical knowledge provided by Martin Doets and Frans Mul was also greatly appreciated. This work is part of the research program of the Foundation for Fundamental Research on Matter (FOM), which is part of The Netherlands Organisation for Scientific Research (NWO).

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