The XXZ spin chain at $\Delta = -1/2$: Bethe roots, symmetric functions, and determinants

J. de Gier and M. T. Batchelor
Department of Mathematics, School of Mathematical Sciences,
Australian National University, Canberra ACT 0200, Australia

B. Nienhuis and S. Mitra
Instituut voor Theoretische Fysica, Universiteit van Amsterdam,
1018XE Amsterdam, The Netherlands

(Received 9 October 2001; accepted for publication 20 April 2002)

A number of conjectures have been given recently concerning the connection between the antiferromagnetic XXZ spin chain at $\Delta = -1/2$ and various symmetry classes of alternating sign matrices. Here we use the integrability of the XXZ chain to gain further insight into these developments. In doing so we obtain a number of new results using Baxter’s $Q$ function for the XXZ chain for periodic, twisted and open boundary conditions. These include expressions for the elementary symmetric functions evaluated at the ground state solution of the Bethe roots. In this approach Schur functions play a central role and enable us to derive determinant expressions which appear in certain natural double products over the Bethe roots. When evaluated these give rise to the numbers counting different symmetry classes of alternating sign matrices. © 2002 American Institute of Physics.

[I. INTRODUCTION]

The XXZ Heisenberg spin chain is a central and much studied model in statistical mechanics. It is arguably the best known model solved by means of the Bethe wave function Ansatz. The ground state wave function at the particular anisotropy value $\Delta = -1/2$ has recently been the source of some surprising observations. At this value Razumov and Stroganov observed the appearance of the numbers $A(n) = 1, 2, 7, 42, 429, \ldots$, which count the number of $n \times n$ alternating sign matrices. Here the length of the chain $L$ is odd ($L = 2n + 1$) with periodic boundary conditions imposed. Alternating sign matrices are matrices whose elements are either $-1, 0$ or $1$ such that the elements along each row and each column alternate in sign. Furthermore, the entries in each row and column add up to $+1$. The numbers $A(n)$ are well known to enumerative combinatorialists, having appeared in other distinct problems such as the enumeration of plane partitions and generalizations of determinants. The one-to-one correspondence between vertex configurations of the square lattice ice model with domain wall boundary conditions and ASM’s has been well documented. In particular it led to Kuperberg’s alternate proof of the alternating sign matrix conjecture.

The numbers $A(n)$ appear in the ground state wave function of the XXZ Heisenberg chain at $\Delta = -1/2$ in three ways: (i) as the ratio of the largest and smallest components in the ground state wave function, (ii) in the sum of the components, and (iii) in the sum of the square of the components. These observations at $\Delta = -1/2$ have been extended in a number of directions. Two other known cases are: (i) twisted boundary conditions with $L$ even, and (ii) open boundary conditions with appropriate surface fields (the quantum $U_q[sl(2)]$ invariant chain). For both cases the ground state wave function is complex. Nevertheless, the sums of the wave function components and of their squares are real. The numbers $A(n)$ also appear in the twisted case. However, the open case seems to appear in some symmetry classes. Here appear $A(2n + 1)$, the number of $(2n + 1) \times (2n + 1)$ vertically symmetric alternating sign matrices when $L$
=2n and \( N_8(2n) \), the number of cyclically symmetric transpose complement plane partitions, when \( L=2n-1 \). The number \( N_8(2n) \) is conjectured to be \( A_{VH}(4n+1)/A_V(2n+1) \), where \( A_{VH}(4n+1) \) is the number of \((4n+1)\times(4n+1)\) vertically and horizontally symmetric alternating sign matrices.\(^{14}\) These various numbers also appear in the corresponding \( O(1) \) loop model, for which the ground state wave function is real.\(^{10}\) Further developments include the combinatorial interpretation\(^ {15-17}\) of the elements of the \( O(1) \) loop model wave function and the relation to a one-dimensional Temperley–Lieb stochastic process.\(^ {17}\)

The numerous conjectures made to date for the various ground state wave functions at \( \Delta = -1/2 \) remain to be proved. In earlier work, Stroganov and co-workers\(^ {18-20}\) have found an expression for Baxter’s \( Q \) function\(^ {21}\) in each of the above cases. By definition the zeros of the \( Q \) function are the Bethe roots. However, little if any use has been made of this function. Here we use the \( Q \) function results to obtain closed form expressions for the values of the elementary symmetric functions with the ground state Bethe roots as arguments. This approach also involves the appearance of the Schur function and determinants in a natural way. Ultimately we are led to conjectures for new determinants whose values are related to the alternating sign matrix numbers. These results came from observations on the product of certain combinations of Bethe roots. Although some results can be proved along the way, the evaluation of the determinants, involving the elementary symmetric functions, remains to be done exactly.

The layout of this paper is as follows. In Sec. II we collect some necessary results on the Bethe Ansatz and symmetric polynomials. In Sec. III we give our results for the periodic \( L \text{ odd} \), twisted and reflecting boundary cases. Some detailed working is given in the Appendix. The paper concludes with some remarks in Sec. IV.

II. PRELIMINARIES

A. XXZ spin chain and Bethe’s Ansatz

We consider the periodic anisotropic quantum XXZ spin chain. A spin variable lives on each site of the chain taking either up or down values. The interaction between two neighboring spins is described in terms of Pauli spin matrices by the well known Hamiltonian,

\[
H = -\frac{1}{2} \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z),
\]

where the anisotropy \( \Delta \) is parametrized by

\[
\Delta = -\frac{1}{2} (q + q^{-1}).
\]

We denote the position of the \( i \)th down spin along the chain by \( x_i \). The Hamiltonian (1) is diagonalized via the Ansatz,\(^1\)

\[
\psi(x_1, \ldots, x_n) = \sum_{\pi} A^{\pi_1 \cdots \pi_n} \prod_{j=1}^{n} z_j^{x_j} \prod_{j=1}^{n} \pi_j^{-1}
\]

for the form of its eigenvectors. The sum over \( \pi=(\pi_1, \ldots, \pi_n) \) denotes a sum over all permutations of the numbers 1, \ldots, \( n \). Substituting (3) into the eigenvalue equation for \( H \) one finds the eigenvalues to be given by

\[
E = -L\Delta - \sum_{i=1}^{n} (z_i + z_i^{-1} - 2\Delta).
\]

The amplitudes \( A^{\pi_1 \cdots \pi_n} \) are also expressed in the variables \( z_i \) for which the consistency equations...
where

\[ z_j^L = (-1)^{n-1} \prod_{j=1}^{n} \frac{1 - 2 \Delta z_j + z_j z_{j+1}}{1 - 2 \Delta z_j + z_j z_{j+1}} \]  

are derived.

It will be convenient to make the change of variables

\[ z = \frac{q - w}{qw - 1}, \quad w = \frac{z + q}{qz + 1}, \]  

for which the Bethe equations (5) take the form

\[ \left( \frac{q - w_i}{qw_i - 1} \right)^L + \prod_{j=1}^{n} \frac{w_i - q^2 w_j}{q^2 w_i - w_j} = 0. \]  

Up to a normalization, the amplitudes are given by

\[ A_{\pi_1 \cdots \pi_n} = q^{n(n-1)/2} \prod_{i<j} \frac{1 - 2 \Delta z_{\pi_i} + z_{\pi_i} z_{\pi_j}}{z_{\pi_i} - z_{\pi_j}} \prod_{i<j} \frac{w_{\pi_i} - q^2 w_{\pi_j}}{w_{\pi_i} - w_{\pi_j}}. \]  

Note that the eigenfunctions (3) are symmetric polynomials in the variables \( z_i \). All properties of the XXZ spin chain can therefore be expressed in symmetric functions of these variables. We will review some of the basic properties of symmetric polynomials in the next section.

From (8) we see that the amplitudes can be written in terms of the generalized Vandermonde product,

\[ \det_{\lambda}(w_i^{n-j}) = \det_{\lambda}(w_i^{j-1}) = \prod_{i<j} (w_i + \lambda w_j), \]

where we have introduced the \( \lambda \)-determinant\(^{22} \) which can be defined recursively via Dodgson’s algorithm for evaluating determinants.\(^4 \) If \( X^{(1)} = (x_{ij}^{(1)}) \) is an \( n \times n \) matrix and \( Y^{(1)} \) an \( (n-1) \times (n-1) \) matrix with each element equal to 1, we define new matrices \( X^{(k)} \) and \( Y^{(k)} \) recursively by

\[ x_{ij}^{(k)} = (x_{ij}^{(k-1)}) + \lambda x_{i+1,j+1}^{(k-1)} / y_{ij}^{(k-1)}, \quad i,j = 1,\ldots,n-k+1, \]

\[ y_{ij}^{(k)} = x_{i+1,j+1}^{(k-1)}, \quad i,j = 1,\ldots,n-k. \]

The number \( X^{(n)} \) thus defined is called the \( \lambda \)-determinant of \( X^{(1)} \). For the special value \( \lambda = -1 \) this procedure evaluates the ordinary determinant \( \det X^{(1)} \). Just as the determinant can be written as a sum over the set of permutation matrices, the \( \lambda \)-determinant can be written as a sum over the set of alternating sign matrices,\(^22 \)

\[ \det_{\lambda} M = \sum_{A \in \mathcal{A}_n} \lambda^{\mathcal{I}(A)} (1 + \lambda^{-1})^{N(A)} \prod_{i,j=1}^{n} m_{ij}^{(n)} \]  

where \( \mathcal{A}_n \) is the set of \( n \times n \) alternating sign matrices, \( \mathcal{I}(A) \) is the inversion number of \( A \) and \( N(A) \) the number of \(-1\)’s in \( A \) (see e.g., Ref. 4 for the definition of \( \mathcal{I} \)). The total number of terms in this sum, or equivalently the number of \( n \times n \) alternating sign matrices is given by

\[ A(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \prod_{i=1}^{n} \frac{n+i+j-1}{2i+j-1}. \]

We will see later that this and related numbers surprisingly show up in certain combinations of the variables \( z_i \) when evaluated at a particular solution of the Bethe equations (5) for \( q = e^{i\pi/3} \).
B. Symmetric polynomials

This section is included for the convenience of the reader. The results collected herein can be found in standard textbooks such as Refs. 4 or 23.

A partition \( \mu \) of \( k \) is a nonincreasing set of integers \( \mu_1, \ldots, \mu_l \) such that \( \mu_1 + \cdots + \mu_l = k \). We denote this by \( \mu \vdash k \). Partitions define the shape of a semistandard tableau, which is a two-dimensional array of integers with the restriction that these integers are nondecreasing across each row of length \( \mu_j \) and strictly increasing down columns. An example of a semistandard tableau of shape \( (4,4,3,1,1) \) is

\[
\begin{array}{ccccc}
1 & 1 & 2 & 3 & \\
2 & 3 & 4 & 4 & \\
4 & 4 & 5 & & . \\
5 & & & & \\
6 & & & & \\
\end{array}
\]

A conjugate partition \( \mu' \) or conjugate shape is the shape obtained by reflecting the semistandard tableau of shape \( \mu \) in its main diagonal. For example, the conjugate partition of \( (4,4,3,1,1) \) is \( (5,3,3,2) \). The integers in the semistandard tableau may be interpreted as indices of variables, and thus to every semistandard tableau is associated a monomial in which the power of each variable is the number of times its index occurs in the tableau, e.g., for the example above the monomial is given by

\[
w_1^2 w_2^2 w_3^2 w_4^4 w_6.
\]

Given a tableau \( T \) the corresponding monomial is denoted by \( w^T \). In this way one may associate with every tableau of shape \( \mu \) a polynomial \( s_\mu \), called the Schur polynomial, by the definition

\[
s_\mu(w_1, \ldots, w_n) = \sum_T w^T,
\]

where the sum is over all semistandard tableaux of shape \( \mu \) with entries chosen from \( \{1, \ldots, n\} \). If \( \mu \vdash k \), \( s_\mu \) is a polynomial of degree \( k \). We will see later that the Schur function is a symmetric function.

The monomial symmetric function of degree \( k \) is defined by

\[
m_\mu = \sum_\pi \prod_{j=1}^n w_{\pi_j}^\mu,
\]

where the sum is over all \textit{distinct} permutations,

\[
\pi = (\pi_1, \ldots, \pi_n) \quad \text{of the numbers} \quad (\mu_1, \ldots, \mu_k, 0, \ldots, 0).
\]

The elementary symmetric function of degree \( k \) in \( n \) variables is defined as the monomial symmetric function that corresponds to the partition with \( k \)'s,

\[
e_0 = 1
\]

\[
e_1 = w_1 + \cdots + w_n,
\]

\[
e_2 = w_1 w_2 + w_1 w_3 + \cdots + w_{n-1} w_n
\]

\[
\vdots
\]

\[
e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} w_{i_1} \cdots w_{i_k}.
\]
Given a partition \( \mu \), this definition is extended to

\[
e_\mu = e_{(\mu_1, \ldots, \mu_l)} = e_{\mu_1} \cdots e_{\mu_l},
\]

(17)

Finally, the complete symmetric function \( h_k \) is defined to be the sum over all monomial symmetric functions of degree \( k \), i.e.,

\[
h_k = \sum_{\mu \vdash k} m_\mu.
\]

(18)

In a similar way as above this is extended to

\[
h_\mu = h_{(\mu_1, \ldots, \mu_l)} = h_{\mu_1} \cdots h_{\mu_l},
\]

(19)

The following facts are well known concerning the various functions defined above:

(i) The elementary and complete symmetric functions are dual to each other in the sense,

\[
e_k = \det(h_{1-i+j})_{i,j=1}^k,
\]

(20)

\[
h_k = \det(e_{1-i+j})_{i,j=1}^k,
\]

(21)

where we put by convention \( e_{-k} = h_{-k} = 0 \) for \( k > 0 \). Their generating functions are given by

\[
\sum_{i=0}^\infty e_i(w_1, \ldots, w_n) t^i = \prod_{j=1}^n \frac{1}{1-w_j t},
\]

(22)

\[
\sum_{i=0}^\infty h_i(w_1, \ldots, w_n) t^i = \prod_{j=1}^n \frac{1}{1-w_j t}.
\]

(23)

(ii) The Jacobi–Trudi identity.

Let \( \mu = (\mu_1, \ldots, \mu_n) \) be a partition into at most \( n \) part, then

\[
s_\mu(w_1, \ldots, w_n) = \det(h_{\mu_i+j-i})_{i,j=1}^n.
\]

(24)

(iii) The Nägelsbach–Kostka identity.

Let \( \mu' \) be the partition conjugate to \( \mu \) and \( k \) the number of parts in \( \mu' \), then

\[
s_\mu = \det(e_{\mu'_i+j-i})_{i,j=1}^k.
\]

(25)

(iv) The Schur function is equal to a ratio of Vandermonde determinants,

\[
s_{(\mu_1, \ldots, \mu_n)}(w_1, \ldots, w_n) = \frac{\det w_i^{n-j+\mu_j}}{\det w_i^{n-j}}.
\]

(26)

Usually (26) is taken as the definition of the Schur function.

III. RESULTS FOR \( q = e^{i\pi/3} \)

We now turn to the XXZ spin chain in the following cases: (i) periodic boundaries and \( L = 2n + 1 \), (ii) twisted boundaries with twist angle \( \pi/3 \) and \( L = 2n \), and (iii) reflecting boundaries. In these cases the XXZ chain has a trivial ground state energy. It is to be understood that in this section we take \( q = e^{i\pi/3} \).

A. Periodic boundaries

Consider the function \( Q_n(w) \), of which the zeros are the solutions of the Bethe equations (7),

\[
Q_n(w) = \prod_{i=1}^n (w - w_i) = \sum_{l=0}^n (-1)^l w^{n-l} e_l,
\]

(27)
where \( e_i \) are the elementary symmetric functions (16) with the variables \( w_j \) as arguments. Stroganov showed\(^2\) using Baxter’s T–Q relation\(^2\) that \( Q_n(w) \) can be calculated analytically in the case where the \( w_j \) are the solution of (7) with \( L = 2n + 1 \) corresponding to the lowest value of the energy (4). The answer is given as a rational function in \( w \). Using a binomial coefficient identity, the explicit polynomial form of \( Q_n(w) \) is calculated in Appendix A. Comparing expression (A7) now with the formal expansion of \( Q_n(w) \) in (27) one can read off the values of the elementary symmetric functions at these particular values of \( w \). We find

\[
e_i = \binom{n}{n}^{1/3} \sum_{p=0}^n \left[ \frac{2n - 3p + l}{2n} \binom{n - 1/3}{n - p} \frac{n + 1/3}{p} \right] - \left( \frac{2n - 3p + l - 1}{2n} \binom{n - 1/3}{n - p} \frac{n + 1/3}{p} \right).
\]

(28)

The series in (27) with the coefficients as in (28) in general does not appear to be summable, i.e., cannot be written as a simple product, but it can be verified without too much difficulty that it satisfies the recursion relation

\[
(w + 1)^2(3n + 2)Q_{n+1}(w) = 3(w^3 - 1)(2n + 1)Q_n(w)
\]

\[
-(w^2 - w + 1)^2(3n + 1)Q_{n-1}(w).
\]

(29)

For special values of \( w \), \( Q_n(w) \) simplifies dramatically. The results

\[
q^{2n}Q_n(q^{-1}) = 2^n \prod_{j=1}^n \frac{2j - 1}{3j - 1}, \quad Q_n(0) = (-)^n,
\]

(30)

can for example be calculated easily from (29) when \( w = q^{-1} \) and \( w = 0 \). From this we conclude that

\[
\prod_{j=1}^n (1 + z_j + z_j^2) = \left( \frac{3}{4} \right) \prod_{j=1}^n \frac{(3j - 1)}{(2j - 1)}^2.
\]

(31)

As another example to derive closed form expressions for symmetric combinations of Bethe roots, we consider the product

\[
\prod_{i \neq j}^n (1 + z_i + z_i z_j) = \prod_{i \neq j}^n \frac{i\sqrt{3}(q^2w_j - w_i)}{(qw_i - 1)(qw_j - 1)}
\]

\[
= \prod_{i=1}^n \frac{\sqrt{3}q^{-1}}{(w_i - q^{-1})^2} \prod_{i < j}^n \frac{w_i^3 - w_j^3}{w_i - w_j},
\]

(32)

where in the last step we recognize the Schur function in the ratio of the two Vandermonde determinants [see Eq. (26)]. Thus we find, using (30),

\[
\prod_{i \neq j}^n (1 + z_i + z_i z_j) = \left( \prod_{j=1}^n \frac{\sqrt{3}}{4} \left( \frac{3j - 1}{2j - 1} \right)^2 \right)^{n^{-1}}
\]

\[
\times s_{(2(n-1), 2(n-2), \ldots, 2, 0)}(w_1, \ldots, w_n).
\]

(33)

This can be rewritten using the fact that a Schur function can be written as a determinant over the elementary symmetric functions (see (25)). We now have

\[
s_{(2(n-1), 2(n-2), \ldots, 2, 0)}(w_1, \ldots, w_n) = \det(e_{n-[i+(i+1)/2]-j})_{j=1}^{2(n-1)}.
\]

(34)
with \( e_i \) given by (28). We thus have derived a closed form expression for the product (32). When evaluated explicitly for small values of \( n \) we find

\[
\prod_{i \neq j}^{n} (1 + z_i + z_j) = A(n)^3,
\]

where \( A(n) \) is the number of \( n \times n \) alternating-sign matrices (13). Although we have not been able to evaluate the determinant in (34) analytically, we conjecture that (35) is true for all values of \( n \).

The smallest and largest component of the groundstate wave function are given by \( \psi(1,2,\ldots,n) \) and \( \psi(1,3,\ldots,2n-1) \) respectively. In Ref. 2 it was conjectured that their ratio is equal to \( A(n) \). Using the definitions of Sec. II A, we find by numerical calculation for small \( n \) that the smallest and largest component are given by

\[
\sum_{i<j} \prod_{l<j}^{n} \frac{1 + z_{\pi_l}^{-1} + z_{\pi_l}}{z_{\pi_l} - z_{\pi_l}} = \sum_{i<j} \prod_{l<j}^{n} q^{-1} \frac{qw_{\pi_l} - 1}{qw_{\pi_l}} \frac{w_{\pi_l} - q^2 w_{\pi_l}}{w_{\pi_l} - w_{\pi_l}} = A(n),
\]

\[
\sum_{i<j} \prod_{l<j}^{n} \frac{z_{\pi_l}^{-1}(1 + z_{\pi_l}^{-1} + z_{\pi_l})}{z_{\pi_l} - z_{\pi_l}} = \sum_{i<j} \prod_{l<j}^{n} q^{-1} \left( \frac{qw_{\pi_l} - 1}{qw_{\pi_l}} \right)^2 \frac{w_{\pi_l} - q^2 w_{\pi_l}}{w_{\pi_l} - w_{\pi_l}} = A(n)^2.
\]

We conjecture that these equations are true for arbitrary values of \( n \). We see that indeed the ratio of these two components is \( A(n) \). In fact, using the natural, but otherwise arbitrary, normalization of the amplitudes (8), we find that the smallest component itself is normalized to \( A(n) \). Since both functions above are (up to a common factor) symmetric polynomials, we hope that these conjectures can be proven by making use of (28).

### B. Twisted boundaries

For twisted boundary conditions,

\[
\sigma_{L+1}^z = (\sigma_{L+1}^x \pm i\sigma_{L+1}^y) = e^{\pm 2i\phi} \sigma_1^z,
\]

the eigenvectors and eigenvalues of \( H \) are still given by (3) and (4). The equations for \( z_i \) or \( w_i \) however are modified and for the special case \( \phi = \pi/3 \) become

\[
\left( \frac{q - w_i}{qw_i - 1} \right)^L + q^{-2} \prod_{j=1}^{n} \frac{w_i - q^2 w_j}{qw_i - w_j} = 0.
\]

Also for this case \( Q_n(w) \) can be calculated analytically when the solution of (39) with \( L = 2n \) corresponds to the ground state.\(^{18}\) We find that the elementary symmetric functions now have the values

\[
e_i = \binom{n-1/3}{n} \sum_{p=0}^{1/3} \binom{2n-3p+l-1}{2n-1} \binom{n-1/3}{n-p} \binom{n-2/3}{p} - \binom{2n-3p+l+1}{2n-1} \binom{n-1/3}{n-2/3} \binom{n-2/3}{p-1} \binom{n-2/3}{n-p}.
\]

In a similar way as was done above for the periodic boundaries we can use this expression to evaluate symmetric polynomials in the variables \( z_i \). For example, we find that
When evaluated for small values of $n$ we see this is equal to

$$\prod_{i \neq j}^{n} (1 + z_i + z_j z_{ji}) = e^{-(n-1)i\pi/3} A(n) A_{HT}(2n-1),$$

where

$$A_{HT}(2n+1) = A(n)^2 \prod_{k=1}^{n} \frac{3}{4} \left( \frac{3k-1}{2k-1} \right)^2$$

is the number of $(2n+1) \times (2n+1)$ half-turn symmetric alternating sign matrices. We conjecture that (42) is true for arbitrary values of $n$.

C. Reflecting boundaries

For the open chain with diagonal, or spin conserving, boundaries, the Hamiltonian is given by

$$H = -\frac{1}{2} \sum_{j=1}^{L-1} (\sigma_{j}^{x} \sigma_{j+1}^{x} + \sigma_{j}^{y} \sigma_{j+1}^{y}) + \Delta \sigma_{j}^{z} \sigma_{j+1}^{z} + \frac{1}{2}(q - q^{-1})(\sigma_{j}^{z} - \sigma_{j+1}^{z}).$$

The eigenvectors are now

$$\psi(x_1, \ldots, x_n) = \sum_{\pi, \sigma} A_{\pi_1 \cdots \pi_n} \prod_{j=1}^{n} z_{\sigma_j}^{x_j},$$

where the sum runs over all permutations $\pi = (\pi_1, \ldots, \pi_n)$ of the numbers $1, \ldots, n$ and all signs $\sigma_j = \pm 1$. In this case the energy is given by

$$E = -\frac{1}{2}(L-1)\Delta - \sum_{i=1}^{n} (z_i + z_i^{-1} - 2\Delta).$$

The Bethe equations become

$$\left( \frac{q - w_i}{q w_i - 1} \right)^{2L} \prod_{j \neq i}^{n} \left( \frac{q^2 w_j - w_i}{w_j - q^2 w_i} \right) (q^2 - w_i w_j) = 0,$$

where $z$ and $w$ are again related by (6). The amplitudes are up to a normalization given by

$$A_{\pi_1 \cdots \pi_n} \prod_{i=1}^{n} z_{\sigma_i}^{x_{\pi_i}} (q + q^{-1}) \prod_{i \neq j}^{n} (w_{\pi_i} w_{\pi_j}) (1 - w_{\pi_i} w_{\pi_j})^{-1}.$$
In this case the function \( Q \) is defined by

\[
Q_n(w) = \prod_{i=1}^{n} (\bar{w} - \bar{w}_i) = \sum_{j=0}^{n} (-1)^j \bar{w}^{n-j} e_j(\bar{w}_1, \ldots, \bar{w}_n), \quad \bar{w} = w + w^{-1}.
\]  

(49)

Also for this case \( Q_n(w) \) can be given analytically when the solution of (47) corresponds to the lowest value of the energy (46).\(^{18}\) The explicit polynomial form of \( Q_n(w) \) is given in the Appendix, from which the \( e_i \) can be read off.

In analogy with the periodic and twisted cases we consider the product

\[
\prod_{i=1}^{2n} \prod_{j=1}^{2n} \frac{(1+z_i+z_jz_i)}{w_i \neq w_j} = \prod_{i=1}^{2n} \prod_{j=1}^{2n} \frac{(q-q^{-1})(q^2w_j-w_i)}{(qw_j-1)(qw_i-1)}
\]

\[
= \prod_{i=1}^{n} \left( \frac{q-q^{-1}}{\bar{w}_i - q - q^{-1}} \right)^{4(n-1)} \prod_{i\neq j} \frac{w_i^3 + w_i^{-3} - w_j^3 - w_j^{-3}}{\bar{w}_i - \bar{w}_j}
\]

\[
= \prod_{i=1}^{n} \left( \frac{q-q^{-1}}{\bar{w}_i - q - q^{-1}} \right)^{4(n-1)} \left( \frac{\det(w_i^{3(n-j)+w_i^{-3(n-j)}})}{\det(w_i^{n-j}+w_i^{-n+j})} \right)^2.
\]  

(50)

Here we use the convention \( w_{i+n} = w_i^{-1} \). The ratio of determinants in (50) is up to a factor a symmetric polynomial in \( \bar{w}_1, \ldots, \bar{w}_n \). Unfortunately we have not succeeded in expressing it in the known basis of elementary symmetric functions. Nevertheless, numerical evaluation of (50) leads us to conjecture that

\[
\prod_{i=1}^{2n} \prod_{j=1}^{2n} \frac{(1+z_i+z_jz_i)}{z_i \neq z_j} = A_V(2n+1)^2 N_S(2n)^4,
\]

(51)

where \( A_V(2n+1) \) is the number of \((2n+1) \times (2n+1)\) vertically symmetric alternating sign matrices (Ref. 8, Theorem 2) given by

\[
A_V(2n+1) = \prod_{j=0}^{n-1} \frac{(3j+2)(2j+1)(6j+3)}{(4j+2)(4j+1)(4j+3)}
\]

(52)

and \( N_S(2n) \) is the number of cyclically symmetric transpose complement plane partitions\(^{4,22}\) given by

\[
N_S(2n) = \prod_{j=1}^{n-1} \frac{(3j+1)(2j)(6j)!}{(4j+1)!}
\]

(53)

IV. CONCLUDING REMARKS

We have made a first step towards proving the appearance of certain numbers related to alternating sign matrices in the ground state eigenvector of the XXZ spin chain. Many of the combinatorial results concerning alternating sign matrices have been obtained using the connection with the integrable six-vertex model.\(^ {7,8}\) The XXZ spin chain is closely related to the six-vertex model and the methods used in this paper provide a different application of integrability. The eigenvectors of the XXZ Hamiltonian are given in the form of Bethe’s Ansatz as a result of the integrability of the spin chain for each of the different boundary conditions under consideration. In fact, the normalization of the amplitudes (8) ensures that all eigenvectors are symmetric polynomials in the Bethe roots. As is well known, a basis for the space of symmetric polynomials is given by the elementary symmetric functions. Using the results of Stroganov and his collaborators\(^ {18-20}\)
for Baxter’s $Q$-function, we were able to derive explicitly the values that the elementary symmetric functions take at the ground state solution of the Bethe roots. See, e.g., Eqs. (28) and (40) for the periodic and twisted cases. Although in principle possible, we were not able to re-express the ground state in terms of the elementary symmetric functions. However, we could show that certain natural double products over the Bethe roots can be rewritten in this way. Using some results from the theory of symmetric functions we could then derive determinant expressions that when evaluated give rise to the numbers counting different symmetry classes of alternating sign matrices. Our key results (35), (42), and (51) thus remain as conjectures. It is yet to be seen if such products over Bethe roots have any direct combinatorial meaning.

ACKNOWLEDGMENTS

We thank Vladimir Mangazeev for stimulating discussions. This work has been supported by The Australian Research Council and by the Dutch foundation “Fundamenteel Onderzoek der Materie (FOM).”

APPENDIX: DETAILS

1. Periodic boundaries

Stroganov’s result for \( Q_n(w) \) in the case of periodic boundary conditions and odd system size is

\[
Q_n(w) = \prod_{i=1}^{n} (w - w_i) = \left( \binom{n - 1/3}{n} \right)^{-1} \sum_{k=0}^{n} (-1)^k \binom{n - 1/3}{k} \times \left( \frac{n + 1/3}{n - k} \right)^{w \cdot \frac{k}{1}} \left( \frac{n - 6k - 1}{1 + w} \right)^{n + 1/3}.
\]

(A1)

We would like to rewrite this in the form

\[
Q_n(w) = \sum_{i=0}^{n} (-1)^i w^{n-i} e_i,
\]

(A2)

where the \( e_i \) are the elementary symmetric functions with arguments \( w_1, \ldots, w_n \). The right-hand side of (A1) can be expanded using

\[
\sum_{m=0}^{\infty} (-1)^m \binom{2n+m}{m} w^{-2n-m-1} = (1 + w)^{-2n-1}.
\]

(A3)

It follows that \( Q \) can be written as

\[
Q_n(w) = \left( \binom{n - 1/3}{n} \right)^{-1} \sum_{k=0}^{n} \sum_{m=0}^{\infty} (-1)^{k+m} w^{n-3k-m} \left[ \binom{2n+m}{m} \binom{n - 1/3}{n - k} \binom{n + 1/3}{k} \right]
\]

\[
= \left( \frac{2n+m-1}{m-1} \right) \left( \binom{n - 1/3}{n - k} \binom{n + 1/3}{k} \right).
\]

(A4)

To show that (A4) reduces to a finite sum, we first collect terms of the form \( m = 3j \), \( m = 3j + 1 \) and \( m = 3j + 2 \). Then we use the following identity to rewrite (A4):

\[
\sum_{k=0}^{n} \sum_{j=0}^{\infty} a_{k,j} = \sum_{l=0}^{n} \sum_{p=0}^{l} a_{l-p,p} + \sum_{l=n+1}^{\infty} \sum_{p=0}^{n} a_{p,l-p}.
\]

(A5)

To proceed we need the result
\[
\sum_{p=0}^{n} \binom{3p-n+s}{2n} \binom{n-1/3}{p} \binom{n+1/3}{n-p} = \sum_{p=0}^{n} \binom{3p-n+s-1}{2n} \binom{n-1/3}{p} \binom{n+1/3}{n-p}.
\]

(A6)

This identity is proven by first showing that both the left and right hand sides satisfy the same recursion relation in \(n\), i.e., both sums for \(n=m+3\) can be expressed in the same sums for \(n=m+2, m+1\) and \(n=m\). One then shows explicitly that the identity holds for \(n=0, 1, 2\). The recursion relation for (A6) can be easily derived using the Paule and Schorn Mathematica implementation of an algorithm of Zeilberger's.\(^{25}\)

From (A6) it then follows that the infinite sum in (A5) vanishes and after recollecting terms again we can finally write

\[
Q_n(w) = \binom{n-1/3}{n}^{-1} \sum_{l=0}^{[l/3]} \sum_{p=0}^{n} (-1)^{w^n-l} \binom{2n-3p+l}{2n} \binom{n-1/3}{p} \binom{n+1/3}{n-p} \\
\quad - \binom{2n-3p+l-1}{2n} \binom{n-1/3}{n-p} \binom{n+1/3}{n-p},
\]

(A7)

the desired result.

2. Twisted boundaries

In this case the result of Fridkin et al.\(^{18}\) for \(Q_n(w)\) is

\[
Q_n(w) = \binom{n-1/3}{n}^{-1} \sum_{k=0}^{n} (-1)^{n-2/3} \binom{n}{n-k} \times \binom{n-1/3}{k} \binom{n+1/3}{k-1} w^{3k} \binom{n-1/3}{k} \binom{n+1/3}{k-1} w^{3n-3k+2}.
\]

(A8)

In analogy with the periodic case we need to rewrite \(Q_n(w)\) in powers of \(w\). This can be done along the lines of the previous subsection with the help of the result

\[
\sum_{p=0}^{n} \binom{3p-n+s}{2n} \binom{n-1/3}{p} \binom{n-2/3}{n-p} = \sum_{p=0}^{n} \binom{3p-n+s+2}{2n} \binom{n-1/3}{p} \binom{n+1/3}{n-p-1}.
\]

(A9)

As in the case of (A6), this identity can be proven by showing that both the left and right-hand sides satisfy the same recursion relation in \(n\). We then find

\[
Q_n(w) = \binom{n-1/3}{n}^{-1} \sum_{l=0}^{[l/3]+1} (-1)^{w^n-l} \times \sum_{p=0}^{[l/3]+1} \binom{2n-3p+l-1}{2n-1} \binom{n-1/3}{n-p} \binom{n-2/3}{p} - \binom{2n-3p+l+1}{2n} \binom{n-1/3}{n-p} \binom{n-2/3}{p-1}.
\]

(A10)

3. Reflecting boundaries

Following Fridkin et al.\(^{18}\) one can prove that \(Q\) is given by

\[
\binom{2n-2/3}{2n}^{-1} \sum_{k=-n}^{n} (-1)^{n+k} \binom{2n+2/3}{n+k} \binom{2n-2/3}{n-k} w^{3k+1} w^{3k-1} (w-3w^{-1}(2+w+w^{-1})^{2n}.
\]

(A11)

Using (A3) and
the expression for $Q$ can be rewritten so that we can read off the values of the elementary symmetric functions with arguments $\tilde{w}_1, \ldots, \tilde{w}_p$, where $\tilde{w} = w + w^{-1}$. Namely,

$$Q_n(w) = 2^{-3n} \left( \frac{2n - 2/3}{2n} \right)^n \prod_{p=0}^{n} \left( -2 \right)^p (n-p-1)^{n-2p} \left( \frac{n-p}{n} \right)^{2n+2/3} \left( \frac{n-p-1}{n} \right)^{2n-2/3}$$

$$+ \sum_{m=0}^{n} \sum_{k=1}^{n} \left( -1 \right)^{\frac{1}{2}(k+m+1) + m} \frac{3n-p-m-1}{2n-1} \left( \frac{n+k}{n} \right)^{2n+2/3} \left( \frac{n-k}{n} \right)^{2n-2/3}$$

$$\times \left( \frac{n+k}{n} \right)^{\frac{1}{2}(k+m+1) + m} \left( \frac{n-k}{n} \right)^{\frac{1}{2}(k+m+1) + m}$$

$$+ \sum_{m=0}^{n} \sum_{k=1}^{n} \left( -1 \right)^{\frac{1}{2}(k+m+1) + m} \frac{3n-p-m-1}{2n-1} \left( \frac{n+k}{n} \right)^{2n+2/3} \left( \frac{n-k}{n} \right)^{2n-2/3}$$

$$\times \left( \frac{n+k}{n} \right)^{\frac{1}{2}(k+m+1) + m} \left( \frac{n-k}{n} \right)^{\frac{1}{2}(k+m+1) + m}.$$ (A13)