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Local existence of quasispherical space–time initial data

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We consider the system of Einstein constraint equations in the special case when the spacelike 3-manifold is assumed to satisfy the quasispherical ansatz of Bartnik. We introduce two different time slicing conditions, under each of which we may view the resulting constraint equations as a parabolic/elliptic/ODE system of partial differential equations. We combine recent existence results for parabolic equations in Sobolev space with an iterative method to prove local existence results for the quasispherical Einstein constraint equations under the two different time slicing conditions. © 2005 American Institute of Physics. [DOI: 10.1063/1.1864250]

I. INTRODUCTION

In general relativity, the Einstein field equations allow us to study the evolution of a spacelike 3-manifold, provided that its metric and extrinsic curvature satisfy a system of geometric constraint equations. The Einstein constraint equations, arise as a consequence of the fact that the 3-manifold in question is necessarily a submanifold of the spacetime its evolution defines. The constraint equations on a 3-manifold Σ with metric h and second fundamental form K are

$$\mathcal{R}_h + (\text{tr}_h K)^2 - \|K\|^2 = 16\pi T_{00}, \quad (1.1)$$

$$D^j K_{ij} - D_i(\text{tr}_h K) = 8\pi T_{0i}, \quad i = 1, 2, 3. \quad (1.2)$$

Here T is the stress-energy tensor and $\mathcal{R}_h$ and D denote the scalar curvature and Levi–Civita connection of the metric h .

Traditionally, solutions of the constraint equations are constructed using the method of Lichnerowicz,¹² which is based on the conformal class of h . This method has been studied extensively, for example in Refs. 7–10. Accordingly, the data set

$$h = \phi^4 \hat{h},$$

$$K = \phi^{-2} \hat{A} + \phi^{-2} l_{\hat{h}}(W) + \frac{1}{3} \tau \phi^4 \hat{h},$$

satisfies the initial data constraints (1.1) and (1.2) if ϕ and W satisfy the semilinear elliptic system

$$8\Delta_{\hat{h}} \phi = \phi \mathcal{R}_{\hat{h}} - \|\hat{A} + l_{\hat{h}}(W)\|^2 \phi^{-7} + \frac{2}{3} \tau^2 \phi^5 - 16\pi \hat{T}_{00}, \quad (1.3)$$

$$\hat{D}_j [l_{\hat{h}}(W)]^{ij} = \frac{2}{3} \phi^6 \hat{D}^i \tau + 8\pi \hat{T}^{0i}, \quad (1.4)$$

where $l_{\hat{h}}$ is the conformal Killing operator; $l_{\hat{h}}(W) = \mathcal{L}_W \hat{h} - \frac{1}{3} \hat{h} \text{tr}_{\hat{h}}(\mathcal{L}_W \hat{h})$.

This conformal method has both advantages and disadvantages; on the one hand, every solution of (1.1) and (1.2) may be obtained in this manner from suitable $\hat{h}$, τ , $\hat{A}$, while on the other

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hand, the elliptic system (1.3) and (1.4) does not always admit solutions, it is difficult to solve numerically in general, and distinct data can give rise to diffeomorphically equivalent solutions of the constraint equations.⁵

More particularly, the conformal method has been shown to be inadequate for the solution of a problem encountered when considering the notion of quasilocal mass in general relativity.³ In Ref. 2 one is confronted with the following extension problem:

Given a bounded data set (Σ_0, h_0, K_0) , find an asymptotically flat data set (Σ, h, K) with boundary $\Gamma = \partial\Sigma \approx \partial\Sigma_0$ such that the complete manifold $\Sigma \cup_\Gamma \Sigma_0$, obtained by gluing Σ and Σ_0 along Γ , forms an initial data set.

Requiring that (Σ, h) have bounded curvature across Γ implies the boundary conditions for the metric h and mean curvature $H_\Gamma = \text{tr}_\Gamma K$,

$$h|_{\Gamma} = h_0|_{\Gamma}, \quad H_\Gamma = H_{\partial\Sigma_0}. \quad (1.5)$$

Given a metric $\hat{h}$ on $\Sigma \setminus \Sigma_0$, for the conformal method, we must solve the semilinear elliptic equation,³

$$\left(\Delta_{\hat{h}} - \frac{1}{8}\mathcal{R}_{\hat{h}}\right)\phi = -\frac{1}{8}\phi^5\mathcal{R}_{\hat{h}},$$

where $h = \phi^4\hat{h}$ is the required 3-metric. The boundary conditions (1.5) imply that $\hat{h}|_{\Gamma} = h_0|_{\Gamma}$ without loss of generality.

Under conformal change of the metric the mean curvature of Γ transforms as

$$H_{\Gamma, h} = 8\phi^{-3}\partial_{\hat{n}}\phi + \phi^{-2}H_{\Gamma, \hat{h}}, \quad (1.6)$$

where $\partial_{\hat{n}}$ is the outward $\hat{h}$ -unit normal derivative.

Equation (1.6) implies that the boundary conditions (1.5) are equivalent to the boundary conditions for ϕ on Γ

$$\phi = 1, \quad \partial_{\hat{n}}\phi = H_{\Gamma, \hat{h}} - H_{\partial\Sigma_0, h_0},$$

where $H_{\Gamma, \hat{h}}$ and $H_{\partial\Sigma_0, h_0}$ are the given mean curvatures. Clearly these boundary conditions are ill-posed.

In this paper we outline an alternative method for constructing solutions to the constraint equations based on the quasispherical (QS) ansatz of Bartnik.⁴ Moreover, we establish local existence results for the constraint equations for a spacelike hypersurface in the QS gauge (Theorems 5.25 and 5.20). We make no mention of the generality of this gauge; the extent to which the QS ansatz applies remains an open problem.

After briefly describing and imposing the quasispherical gauge in Sec. II, we provide an argument to show that the resulting system of constraint equations may be viewed as a coupled system of partial differential equations consisting of a parabolic equation, a first-order elliptic system and (essentially) a system of ordinary differential equations. This system admits several Cauchy problem formulations, depending on which fields are considered as prescribed. In Sec. III we describe two such formulations that arise naturally through geometric considerations. In Sec. IV we reformulate the systems via a complexification of the sphere and introduce an additional constraint equation that allows us to write the systems in a more tractable form. In Sec. V we use a contraction mapping argument, based on an iterative system of partial differential equations, to prove local existence results for the two Einstein constraint systems that were formulated in the preceding sections of the paper.

II. THE QUASISPHERICAL METHOD

We assume that the 3-manifold, Σ , can be foliated by surfaces of constant positive Gauss curvature (i.e., rescaled 2-spheres), such that the area function $4\pi r^2 \in C^\infty(\Sigma)$ can be used as a

global coordinate. Using standard polar coordinates (ϑ, φ) on the 2-spheres, the most general metric compatible with these assumptions may be written as (Ref. 4, Lemma 2.1)

$$ds^2 = u^2 dr^2 + (\beta^1 dr + r d\vartheta)^2 + (\beta^2 dr + r \sin \vartheta d\varphi)^2, \quad (2.1)$$

for functions $u(r, \vartheta, \varphi) > 0$, $\beta^A(r, \vartheta, \varphi)$, $A=1, 2$. We call the metric in these coordinates *quasispherical*, since this naturally generalizes the class of spherically symmetric metrics.

Computing the scalar curvature $\mathcal{R}$ of (2.1) we find that⁴

$$2r\partial_r u - 2\beta_A \nabla_A u = \gamma u^2 \Delta u + (1 + \gamma B)u - \gamma \left(1 - \frac{1}{2} \mathcal{R} r^2\right) u^3, \quad (2.2)$$

where

$$B = \frac{1}{2} |\operatorname{div} \beta|^2 + \frac{1}{2} |\nabla_A \beta_B|^2 - r \partial_r (\operatorname{div} \beta) + \beta_A \nabla_A (\operatorname{div} \beta) - \frac{3}{2} \operatorname{div} \beta,$$

and

$$\gamma = \left(1 - \frac{1}{2} \operatorname{div} \beta\right)^{-1}, \quad \operatorname{div} \beta = \nabla^A \beta_A.$$

Equation (2.2) is a semilinear parabolic equation on S^2 for u , with β^A and $\mathcal{R}$ regarded as prescribed fields. General global existence theorems for solutions of (2.2) have been proven in Ref. 4, under suitable regularity and boundedness assumptions about the prescribed fields. In particular, to ensure that (2.2) is parabolic evolution in the direction of increasing r , β is required to satisfy

$$2 - \operatorname{div} \beta > 0. \quad (2.3)$$

The parabolic equation (2.2) provides a method of solving the Hamiltonian constraint. Using the Hamiltonian constraint (1.1) to *define* the scalar curvature $\mathcal{R}$ in terms of T_{00} , $\|K\|^2$ and $\operatorname{tr}_h K$, specifying β appropriately and solving (2.2) for u yields a quasispherical metric satisfying the Hamiltonian constraint. It is of interest to point out in passing that the quasispherical form of the metric fares much better with the extension problem than the conformal method. If S_r^2 denotes a leaf of the quasispherical foliation, defined by

$$S_r^2 = \{p \in \Sigma : r(p) = r\}$$

then the mean curvature of S_r^2 is

$$H_r = H_{S_r^2} = \frac{1}{ru} (2 - \operatorname{div} \beta).$$

Therefore if β is given and satisfies (2.3), then prescribing $H_{r_0} > 0$ amounts to specifying $u(r_0)$ which may then be taken as initial data for (2.2). Solution of (2.2), with initial data so defined, then yields an extension metric, at least in the case when the mean curvature is prescribed on a standard S^2 . This idea has also been exploited in recent works concerning the connectedness of the space of initial data sets for the Einstein equations¹⁷ and in connection with the positive mass theorem and boundary behaviour of compact manifolds.¹⁶

To solve the momentum constraints (1.2) it is convenient to reparametrize K . We define the quasispherical orthonormal coframe

$$\theta^1 = \beta^1 dr + r d\vartheta,$$

$$\theta^2 = \beta^2 dr + r \sin \vartheta d\varphi,$$

$$\theta^3 = u dr$$

and write

$$K_{ij}\theta^i\theta^j = \left(\eta_{AB} + \frac{1}{2}\mu\delta_{AB}\right)\theta^A\theta^B + \kappa_A(\theta^A\theta^3 + \theta^3\theta^A) + (H - \mu)\theta^3\theta^3, \quad (2.4)$$

where η_{AB} is a symmetric, traceless, 2-tensor on S^2 . The momentum constraints (1.2) may then be written in terms of this quasispherical parametrization as

$$8\pi T_{03}ru = -(r\partial_r\mu - \beta^A\nabla_A\mu + (3 - \operatorname{div}\beta)\mu) + u\operatorname{div}\kappa + \kappa^A\nabla_Au + \eta^{AB}\nabla_{(B}\beta_{A)} + (2 - \operatorname{div}\beta)H, \quad (2.5)$$

$$8\pi T_{0\hat{A}}ru = \nabla^B(u\eta_{AB}) + \mu\nabla_Au + \nabla_A\left(u\left(\frac{1}{2}\mu - H\right)\right) + r\partial_r\kappa_A - \beta^B\nabla_B\kappa_A + ((3 - \operatorname{div}\beta)\delta_{AB} - \nabla_{A\beta_B})\kappa^B. \quad (2.6)$$

Hence if we consider the scalar curvature $\mathcal{R}$ as given in terms of μ , η , κ , and H via (1.1) and (2.4); that is,

$$\mathcal{R} = 16\pi T_{00} + \|\eta\|^2 + 2\|\kappa\|^2 + \frac{3}{2}\mu^2 - 2\mu H,$$

then the system of equations (2.2), (2.5), and (2.6) comprise the spacetime initial data constraints in the quasispherical gauge, and so will be referred to as the *quasispherical Einstein constraint system* (QSECS).

As mentioned already, for Eq. (2.2) to be considered parabolic it is required that β satisfy the condition (2.3). To ensure that this condition is met we *prescribe* β as any element of the set $\{f \in \Gamma^\infty(TS^2 \times [0, \infty)) : 2 - \operatorname{div} f > 0.\}$. This leaves u as the only remaining metric parameter. We therefore cannot prescribe u ; it must be found by solving the QSECS subject to appropriate boundary conditions.

Now that we have established that u cannot be prescribed, we must consider the remaining fields μ , η_{11} , η_{12} , κ_1 , κ_2 , and H . Since we have only four equations in the QSECS, it is obvious that we need to prescribe at least three of these fields and consider the other three as to be determined by the QSECS, subject to suitable boundary conditions. The following question remains: Which of the fields μ , η_{11} , η_{12} , κ_1 , κ_2 , and H should we consider as prescribed? As we will see in a moment, prescribing either H or μ allows us to gain a certain control over the geometrical nature of the initial data. We will thus consider μ or H as prescribed, never both. This leaves us with two remaining fields to prescribe. The only choices are the components of η or the components of κ (it seems implausible to prescribe one component of each). Seeing that κ appears more frequently in the QSECS than does η , we choose to prescribe κ . As we might have hoped, we will see that this choice simplifies the structure of the QSECS, for only a minor cost. We make the comment, in passing, that the case for prescribing η is potentially interesting as well. This case will not be treated here, however.

III. TIME SLICING CONDITIONS

In the conformal method approach to constructing spacetime initial data, prescribing the mean curvature has been made standard practice. This has many advantages both geometrically and physically. Indeed the so-called maximal gauge or maximal time slicing, which amounts to the assumption that the mean curvature is zero, is quite often employed since it greatly simplifies the analysis and may be assumed, without loss of generality, for any asymptotically flat spacetime satisfying an interior condition.¹ These considerations suggest the following boundary value problem.

Problem 1a: QSECS with prescribed mean curvature. Given the prescribed fields $T_{0\alpha}$, β , κ , and H , do there exist solutions u , μ , and η of (2.2), (2.5), and (2.6) satisfying the boundary conditions $(u, \mu)|_{S_1^2} = (u_0, \mu_0)$? Note that $S_1^2 = \{p \in \Sigma : r(p) = 1\}$ is the unit sphere. It may be possible to assign boundary values on a more general 2-surface but in this work we will only treat the case where boundary values are assigned on the unit sphere.

Another less familiar but potentially interesting time slicing condition is given by prescribing μ . In general μ is defined as

$$\mu = g^{AB} K_{AB},$$

where g^{AB} , $A, B=1, 2$ refers to the components of the inverse of the induced (spherical) metric.

To illustrate the geometric meaning of prescribing μ we consider the case of spherically symmetric spacetimes with metric given by

$$ds^2 = -X^2 dt^2 + 2Y dr dt + Z^2 dr^2 + r^2 d\sigma^2. \quad (3.1)$$

Calculating μ for the surfaces defined by constant t in this metric, we easily obtain

$$r\mu = 2X^{-1}Y.$$

If it were the case that the metric were written in polar coordinates (for which $Y=0$), we would find that $\mu=0$. For this reason we will refer to the slicing defined by $\mu=0$ as the *polar gauge* and we will call the quantity μ the *polar curvature*.

Next consider a surface defined by $t=f(r)$ in the metric (3.1). The tangent to this surface is given by $\dot{x} = \partial_r + f' \partial_t$. The normal to the surface is given by $n = a\partial_r + b\partial_t$, where a and b are determined by the conditions $g(\dot{x}, n) = 0$ and $g(n, n) = -1$, which lead to

$$a = \frac{X^2 f' - Y}{Z^2 + Y f'} b,$$

$$b^{-2} = X^2 - \frac{2Y(X^2 f' - Y)}{Z^2 + Y f'} - \frac{(X^2 f' - Y)^2}{(Z^2 + Y f')^2}.$$

The polar curvature is given by

$$r\mu = 2a.$$

The polar gauge ($\mu=0$) then implies that $f' = YX^{-2}$ and so the tangent to the surface is given by

$$\dot{x} = \partial_r + YX^{-2} \partial_t,$$

while its normal is

$$n = X^{-1} \partial_t.$$

Calculating the gradient of r , we have

$$\nabla r = g^{\alpha\beta} \partial_\alpha r \partial_\beta = \frac{X^2}{X^2 Z^2 + Y^2} (\partial_r + YX^{-2} \partial_t).$$

Hence, at least in the case of spherically symmetric spacetimes, the $\mu=0$ surfaces are those which have ∇r , the gradient of r , as their tangent vector.

Furthermore, the polar curvature arises naturally upon consideration of the trapped surface condition. If we consider a spacelike 2-surface S , with normal N and second fundamental form $\mathbb{I}$, embedded in a spacelike 3-manifold with normal T and second fundamental form K , the expansion of the null congruences associated with $T \pm N$ are given by

$$\theta_\pm = \text{tr}_S \langle \nabla_{e_A} e_B, T \pm N \rangle, = \text{tr}_S (K_{AB} \pm \mathbb{I}_{AB}) = \mu \pm \text{tr}_S \mathbb{I}.$$

Thus the trapped surface condition is $\mu = \pm \text{tr}_S \mathbb{I}$.

The role the polar curvature plays in more general spacetimes is potentially very interesting but has not as yet been investigated in any detail. However, in anticipation of this investigation we will consider the QSECS with μ a prescribed field. This gives us a second formulation of the QSECS as a boundary value problem.

Problem 2a: QSECS with prescribed polar curvature. Given the prescribed fields $T_{0\alpha}$, β , κ and μ , do there exist solutions u , H , and η of (2.2), (2.5), and (2.6) satisfying the boundary conditions $u|_{S^2_1} = u_0$?

IV. REWRITING THE EQUATIONS

In this section we introduce a complex notation and a coordinate transformation that enables us to write the QSECS in a more tractable form. We define the following complex fields:

$$\beta = \frac{1}{\sqrt{2}}(\beta^1 - i\beta^2),$$

$$\kappa = \frac{1}{\sqrt{2}}(\kappa^1 - i\kappa^2),$$

$$\eta = \eta_{11} - i\eta_{12},$$

$$T_0 = \frac{1}{\sqrt{2}}(T_{01} - iT_{02}),$$

and the subsidiary field

$$\rho = 2 - \text{div } \beta = \gamma^{-1}.$$

We may consider β and κ as S^2 vector fields or as spin-1 sections of the complex line bundle over S^2 .^{6,13} Similarly we may consider η as a symmetric, traceless 2-tensor over S^2 or as a spin-2 section of the complex line bundle over S^2 . The context in which these fields are used should make it clear which representation is being employed.

The complex notation suggests encoding the angular derivatives in terms of the differential operator $\eth$ (a phonetic symbol pronounced “eth”¹³) defined on a spin- s field ψ by

$$\eth\psi = \frac{1}{\sqrt{2}}\sin^s\vartheta(\partial_\vartheta - i\csc\vartheta\partial_\varphi)(\psi\sin^{-s}\vartheta).$$

All the standard differential operators on S^2 may be expressed in terms of $\eth$, for example,

$$\text{div } \beta = \eth\bar{\beta} + \bar{\eth}\beta,$$

$$\text{curl } \beta = i(\eth\bar{\beta} - \bar{\eth}\beta),$$

$$\Delta\psi = (\eth\bar{\eth} + \bar{\eth}\eth)\psi.$$

The spin-weighted spherical harmonics Y_{lm}^s are then the eigenfields of the Laplacian, satisfying

$$\Delta Y_{lm}^s = (s^2 - l(l+1))Y_{lm}^s,$$

with $|s| \leq l$. The eigenfields Y_{lm}^s may be taken proportional to $\eth^s Y_{lm}$ for $s \geq 0$ and $\bar{\eth}^{-s} Y_{lm}$ for $s < 0$, where Y_{lm} are the usual spherical harmonics.¹⁴

An important result concerning $\bar{\eth}$ which will have bearing on our consideration of the QSECS is as follows.^{13,14}

Lemma 4.1: Suppose $s \geq 0$ and let $\mathcal{L}^s$ denote the space of spin- s sections. The mapping $\bar{\partial}: \mathcal{L}^{s+1} \rightarrow \mathcal{L}^s$ has cokernel $\mathcal{C} = \text{span}\{Y_{sm}^s\}$.

We have the obvious corollary.

Corollary 4.2: Suppose $s \geq 0$. The mapping $\bar{\partial}: \mathcal{L}^{s+1} \rightarrow \text{span}\{Y_{lm}^s: l \geq s+1\}$ is a surjection.

Due to the repeated appearance of the operator $r\partial_r - \nabla_\beta$ we will find it convenient to make a change of coordinates that will simplify the structure. We introduce the diffeomorphism of $S^2 \times \mathbb{R}$

$$\Phi: (t, \xi^A) \mapsto (e^t, \Theta^A(t, \xi)) = (r, \zeta^A), \quad (4.1)$$

where $\zeta = (\vartheta, \varphi)$ and $\Theta(t, \xi)$ is defined to be the solution of

$$\frac{\partial}{\partial t} \Theta(t, \xi) = -\beta(e^t, \Theta(t, \xi)), \quad (4.2)$$

subject to suitable boundary conditions, $\beta|_{t=0} = \beta_0$.

Hence, defining $\tilde{f}(t, \xi) = f(e^t, \Theta(t, \xi))$, we then have

$$\partial_{\tilde{t}} \tilde{f}(t, \xi) = e^t \frac{\partial f}{\partial r} + \frac{\partial f}{\partial \zeta^A} \frac{\partial \Theta^A}{\partial t} \bigg|_{(r, \zeta) = (e^t, \Theta(t, \xi))} = \left(r \frac{\partial}{\partial r} - \beta^A \frac{\partial}{\partial \zeta^A} \right) f(e^t, \Theta(t, \xi)) = (r\partial_r - \nabla_\beta) f(e^t, \Theta(t, \xi)).$$

We may then write (2.2), (2.5), and (2.6) in terms of (t, ξ) as

$$2\tilde{\rho} \partial_{\tilde{t}} \tilde{u} = \tilde{u}^2 \tilde{\Delta} \tilde{u} + \tilde{\rho} (1 + \tilde{\gamma} \tilde{B}) \tilde{u} - \left(1 - \frac{1}{2} \tilde{\mathcal{R}} e^{2t} \right) \tilde{u}^3,$$

$$8\pi \tilde{T}_{03} \tilde{u} e^t = -(\partial_{\tilde{t}} \tilde{\mu} + (1 + \tilde{\rho}) \tilde{\mu}) + \tilde{u} (\text{div } \tilde{\kappa}) + (\kappa^A \nabla_A \tilde{u}) + (\eta^{AB} \nabla_{(\beta_A} \tilde{u}) + \tilde{\rho} \tilde{H},$$

$$8\pi \tilde{T}_0 \tilde{u} e^t = \partial_{\tilde{t}} \tilde{\kappa} + (\tilde{\partial}(u\eta) + \mu \tilde{\partial} u + \frac{1}{2} \tilde{\partial}(u\mu) - \tilde{\partial}(uH) + (1 + \rho) \kappa + \bar{\kappa} \tilde{\partial} \beta + \kappa \tilde{\partial} \bar{\beta}).$$

We note that since we have employed the coordinate transformation given by Φ , the Laplacian $\tilde{\Delta}$ appearing in the first of the above equations is with respect to the pulled back metric $\tilde{h} = \Phi^* h$.

As we can see, implementing the coordinate transformation Φ , simplifies the structure of the system in question. It does, however, leave us with a technical hitch. Since the natural angular derivatives will now be in terms of the ξ^A -coordinates, when we come to differentiate the equations, terms involving derivatives of β will arise via (4.2). This is not a problem, as we will assume throughout that β is a smooth, prescribed field and so the terms that arise will be bounded by uniform constants. Moreover, since we assume that β is smooth, the standard regularity theory for ordinary differential equations dependent upon a parameter implies that Φ is a smooth diffeomorphism.

With this observation in mind we may suppress the tilde's. Introducing the notation $\dot{f} = \partial_t f$ and defining the auxiliary field $\hat{\eta} = u\eta$ we then write the QSECS as

$$2\rho \dot{u} = u^2 \Delta u + \rho (1 + \gamma B) u - \left(1 - \frac{1}{2} \mathcal{R} e^{2t} \right) u^3, \quad (4.3)$$

$$8\pi T_{03} u e^t = -(\dot{\mu} + (1 + \rho) \mu) + u \text{div } \kappa + \kappa^A \nabla_A u + \eta^{AB} \nabla_{(\beta_A} u + \rho H, \quad (4.4)$$

$$8\pi T_0 u e^t = \dot{\partial}(\hat{\eta}) + \mu \dot{\partial} u + \frac{1}{2} \dot{\partial}(u\mu) - \dot{\partial}(uH) + \dot{\kappa} + (1 + \rho) \kappa + \bar{\kappa} \dot{\partial} \beta + \kappa \dot{\partial} \bar{\beta}. \quad (4.5)$$

To simplify the discussion and to be in keeping with the usual notions accompanying parabolic equations, we will call $t = \ln r$ "time" and refer to the values of the various fields on the unit sphere as initial values.

Unfortunately a problem arises if we consider (4.5) as giving an equation for $\hat{\eta}$. In making such a consideration we find that (4.5) is of the form $\bar{\partial}f=g$ with $f \in \mathcal{L}^2$. Observing Lemma 4.1, an equation of this form is only solvable if $g \in \text{span}\{Y_{lm}^1 : l \geq 2\}$, that is to say if g is devoid of $l=1$ spherical harmonic components. Hence if we consider (4.5) as giving an equation for $\hat{\eta}$, then we do not have the freedom to prescribe all the fields we have thus far considered prescribed.

Typically the stress-energy tensor will be determined by the matter occupying the spacetime, and we would like to freely specify H or μ for geometric reasons. It is also best to completely specify the metric parameter β so that there is no chance of (2.3) being violated. That leaves us with κ , and we conclude that it is not possible to consider (4.5) as giving an equation for $\hat{\eta}$ and at the same time treat κ as freely specifiable.

We thus look to constrain κ in such a way that (4.5) is solvable. This will produce another equation (the κ constraint) which must then be included as part of the QSECS.

To obtain the required equation we spectrally decompose κ as follows:

$$\kappa = \sum_{m=-1}^1 k^m Y_{1m}^1 + \sum_{l=2}^{\infty} \sum_{m=-l}^l \kappa^{lm} Y_{lm}^1 = k + \sigma,$$

where $\sigma = \sum_{l \geq 2} \sum_{|m| \leq l} \kappa^{lm} Y_{lm}^1$ is devoid of $l=1$ components.

Let $\mathcal{P}_1$ denote the projection onto the subspace spanned by $\{Y_{1m}^1 : m = -1, 0, 1\}$. It is easy to check that if k satisfies

$$-\dot{k} = \mathcal{P}_1 \left\{ \mu \bar{\partial} u + \frac{1}{2} \bar{\partial}(u\mu) - \bar{\partial}(uH) + (1+\rho)\kappa + \bar{\kappa} \bar{\partial} \beta + \kappa \bar{\partial} \bar{\beta} + 8\pi T_{0u} e^t \right\} \quad (4.6)$$

then (4.5) reduces to

$$-\bar{\partial} \hat{\eta} = \dot{\sigma} + (1 - \mathcal{P}_1) \left\{ \mu \bar{\partial} u + \frac{1}{2} \bar{\partial}(u\mu) - \bar{\partial}(uH) + (1+\rho)\kappa + \bar{\kappa} \bar{\partial} \beta + \kappa \bar{\partial} \bar{\beta} + 8\pi T_{0u} e^t \right\}. \quad (4.7)$$

This equation is of the form $\bar{\partial}f=g$ with $g \in \text{span}\{Y_{lm}^1 : l \geq 2\}$, and so is uniquely solvable by Corollary 4.2.

Replacing (4.5) with (4.6) and (4.7), the constraint system may now be written as

$$2\rho \dot{u} = u^2 \Delta u - u^3 \left(1 - \frac{1}{2} e^{2t} \left(16\pi T_{00} + 2\|\kappa\|^2 + \frac{3}{2} \mu^2 - 2\mu H \right) \right) + \rho \left(1 + \frac{1}{2} \gamma e^{2t} \|\hat{\eta}\|^2 + \gamma B \right) u, \quad (4.8)$$

$$\dot{\mu} = \text{div}(u\kappa) - (1+\rho)\mu + u^{-1} \hat{\eta}^{AB} \nabla_{(B} \beta_{A)} + \rho H - 8\pi T_{03} u e^t, \quad (4.9)$$

$$-\dot{k} = \mathcal{P}_1 \left\{ \mu \bar{\partial} u + \frac{1}{2} \bar{\partial}(u\mu) - \bar{\partial}(uH) + (1+\rho)\kappa + \bar{\kappa} \bar{\partial} \beta + \kappa \bar{\partial} \bar{\beta} + 8\pi T_{0u} e^t \right\}, \quad (4.10)$$

$$-\bar{\partial} \hat{\eta} = \dot{\sigma} + (1 - \mathcal{P}_1) \left\{ \mu \bar{\partial} u + \frac{1}{2} \bar{\partial}(u\mu) - \bar{\partial}(uH) + (1+\rho)\kappa + \bar{\kappa} \bar{\partial} \beta + \kappa \bar{\partial} \bar{\beta} + 8\pi T_{0u} e^t \right\}. \quad (4.11)$$

The problems 1a and 1b may now be stated more correctly as follows.

Problem 1b: QSECS with prescribed mean curvature. Given the prescribed fields T_{0a} , β , σ , and H , do there exist solutions u , μ , $\hat{\eta}$, and k of (4.8)–(4.11) satisfying the initial conditions $(u, \mu, k)|_{S^2 \times \{0\}} = (u_0, \mu_0, k_0)$?

Problem 2b: QSECS with prescribed polar curvature. Given the prescribed fields T_{0a} , β , σ , and μ , do there exist solutions u , H , $\hat{\eta}$, and k of (4.8)–(4.11) satisfying the initial conditions $(u, k)|_{S^2 \times \{0\}} = (u_0, k_0)$?

We have not yet made mention about which function spaces we will be posing Problems 1b and 2b in. Which spaces are to be used will be made clear in the next section after we discuss an iterative, linear system of partial differential equations based on the QSECS. In fact, the existence of solutions to the QSECS will follow from the convergence of the iterative system to a fixed point.

We will begin by considering Problem 1b.

V. LOCAL EXISTENCE

In Ref. 4 the author gives a detailed proof of the global existence and uniqueness of solutions of (2.2) with β and $\mathcal{R}$ prescribed, and satisfying certain conditions. When considering the QSECS with prescribed mean curvature, however, we may not consider $\mathcal{R}$ as completely prescribed, but as determined by the additional fields μ , $\hat{\eta}$, and k , each of which is constrained so as to satisfy (2.5) and (2.6).

To prove the local existence and uniqueness of solutions to the QSECS with prescribed mean curvature, we will not borrow from Bartnik's treatment. Instead we will define a linear system of partial differential equations and produce a sequence of iterates that converge to a solution of the QSECS. In defining the iterative scheme we will see that it is imperative that we have a theory dealing with linear parabolic equations in which the regularity of the coefficients is comparable to that of the solution. Such results were obtained in Ref. 15 where the following general second order parabolic equation was considered:

$$\begin{aligned}\partial_t u + A_t u &= g \quad \text{on } S^2 \times [0, T], \\ u &= u_0 \quad \text{on } S^2 \times \{0\}.\end{aligned}\tag{5.1}$$

Here A_t is a linear differentiable operator of second order on S^2 , which is expressible in divergence form. That is, for a twice differentiable function $u: S^2 \times [0, T] \rightarrow \mathbb{R}$, we may write $A_t u$ in local coordinates as

$$A_t u(x, t) = -\nabla_j(a^{ij}(x, t)\nabla_i u(x, t)) + b^i(x, t)\nabla_i u(x, t) + c(x, t)u(x, t),\tag{5.2}$$

where ∇ denotes the S^2 covariant derivative.

To ensure that (5.1) is a parabolic equation we assume that the operator A_t is an elliptic operator for each value of t , in the sense that for each $\xi \in TS^2$, and each fixed $t \in [0, T]$ we have

$$a^{ij}(x, t)\xi_i \xi_j \geq \theta \bar{g}^{ij}\xi_i \xi_j,$$

for a.e. $x \in S^2$, where $\theta > 0$ is a constant and where $\bar{g}$ is the standard S^2 -metric.

Before stating the existence result for (5.1) we define the parabolic Sobolev spaces in which the solutions reside. Let $f, g: S^2 \times [0, T] \rightarrow \mathbb{R}$ be smooth functions and let

$$\langle f, g \rangle_{LH^s_{\nu, T}} = \int_0^T e^{-2\nu t} \langle f(\cdot, t), g(\cdot, t) \rangle_{H^s(S^2)} dt.$$

We define LH^s_{ν} to be the Hilbert space formed by completion of $C^\infty(S^2 \times [0, T])$ in the corresponding norm. The parabolic Sobolev spaces we require may then be defined as

$$P^m_{\nu, T} = \left\{ f: S^2 \times [0, T] \rightarrow \mathbb{R} : \|\partial_t^i f\|_{LH^{2(m-i)}_{\nu, T}} < \infty, \forall i \leq m \right\}$$

and its associated inner product

$$\langle f, g \rangle_{P^m_{\nu, T}} = \sum_{i \leq m} \langle \partial_t^i f, \partial_t^i g \rangle_{LH^{2(m-i)}_{\nu, T}}.$$

Roughly speaking, a parabolic equation tells us that one time derivative is equivalent to two space derivatives. Hence, in a sense, the Hilbert space $P^m_{\nu, T}$ describes the set of functions that are in total $2m$ times differentiable. We also make the following definition in order to classify functions that have odd total derivative:

$$P_{\nu,T}^{m+1/2} = \left\{ f: S^2 \times [0, T] \rightarrow \mathbb{R} : \|\partial_i^j f\|_{LH_{\nu,T}^{2(m-i)+1}} < \infty, \forall i \leq m \right\}$$

and its associated inner product,

$$\langle f, g \rangle_{P_{\nu,T}^{m+1/2}} = \sum_{i \leq m} \langle \partial_i^j f, \partial_i^j g \rangle_{LH_{\nu,T}^{2(m-i)+1}}.$$

We note that $P_{\nu,T}^0 = LH_{\nu,T}^0$ and $P_{\nu,T}^{1/2} = LH_{\nu,T}^1$. The required existence theorem may now be stated as Ref. 15.

Theorem 5.1: Suppose $m \geq m_0 = 4\frac{1}{2}$, $u_0 \in H^{2m}(S^2)$ and $g \in P_{\nu,T}^{m-1/2}$. There is a unique solution u of (5.1) belonging to the class $P_{\nu,T}^{m+1/2}$ and satisfying the estimate

$$\|u\|_{P_{\nu,T}^{m+1/2}}^2 \leq Q \left(\|u_0\|_{H^{2m}(S^2)}^2 + \|g\|_{P_{\nu,T}^{m-1/2}}^2 \right). \quad (5.3)$$

The constant Q depends only on m and the norms $\|a\|_{P_{\nu,T}^m}$, $\|b\|_{P_{\nu,T}^{m-1/2}}$ and $\|c\|_{P_{\nu,T}^{m-1/2}}$.

In addition to the already mentioned linear parabolic theory, we will need to employ results from linear elliptic equations and linear ordinary differential equations. These will be derived as they are needed.

We begin by proposing an iterative system of partial differential equations before going on to prove that it is well defined.

A. The iteration scheme

Keeping in mind that we are dealing with the QSECS with prescribed mean curvature, we may consider H as some given field. We therefore define the following iterative system for generating the sequence $\{u_n, \mu_n, \hat{\gamma}_n, k_n\}_{n=0}^\infty$:

$$\dot{u}_{n+1} = \hat{\gamma} u_n^2 \Delta u_{n+1} - \hat{\gamma} u_n^3 \left(1 - \frac{1}{2} e^{2t} (16\pi T_{00} + 2\|\kappa_n\|^2 + \frac{3}{2}\mu_n^2 - 2\mu_n H) \right) + \frac{1}{2} \left(1 + \frac{1}{2} \gamma e^{2t} \|\hat{\gamma}_n\|^2 + \gamma B \right) u_n, \quad (5.4)$$

$$\dot{\mu}_{n+1} = \text{div}(u_{n+1} \kappa_{n+1}) - (1 + \rho) \mu_{n+1} + u_n^{-1} \hat{\gamma}_n^{AB} \nabla_{(\beta} \nabla_{A)} \mu_{n+1} + \rho H - 8\pi T_{03} u_{n+1} e^t, \quad (5.5)$$

$$\dot{k}_{n+1} = -\mathcal{P}_1 \left\{ \mu_{n+1} \delta u_{n+1} + \frac{1}{2} \delta(u_{n+1} \mu_n) - \delta(u_{n+1} H) + (1 + \rho) \kappa_n + \bar{\kappa}_n \delta \beta + \kappa_n \delta \bar{\beta} + 8\pi e^t u_{n+1} T_0 \right\}, \quad (5.6)$$

$$-\bar{\delta} \hat{\gamma}_{n+1} = \dot{\sigma} + (1 - \mathcal{P}_1) \left\{ \mu_{n+1} \delta u_{n+1} + \frac{1}{2} \delta(u_{n+1} \mu_{n+1}) - \delta(u_{n+1} H) + (1 + \rho) \kappa_{n+1} + \bar{\kappa}_{n+1} \delta \beta + \kappa_{n+1} \delta \bar{\beta} + 8\pi e^t u_{n+1} T_0 \right\}. \quad (5.7)$$

We have set $\hat{\gamma} = \frac{1}{2} \gamma$ and $\hat{\rho} = \frac{1}{2} \rho$ for convenience. $\hat{\gamma}$ and $\hat{\rho}$ are still just smooth fields depending on β . From here on, unless otherwise stated, we assume that all of the prescribable fields are elements of $C^\infty(S^2 \times [0, \infty))$. Note also that $\kappa_n = k_n + \sigma$.

The proposed iteration scheme is comprised of a hierarchical system of partial differential equations. The basic idea behind using (5.4)–(5.7) to generate a sequence of iterates may be stated informally as follows. Define the zeroth iterate $(u, \mu, k, \hat{\gamma})_{\tilde{0}}$, by making an identification between the initial values u_0 , μ_0 , and k_0 , and the zeroth iterates $u_{\tilde{0}}$, $\mu_{\tilde{0}}$, and $k_{\tilde{0}}$. The zeroth $\hat{\gamma}$ iterate, $\hat{\gamma}_{\tilde{0}}$, is then defined as the solution of (5.7), with $n = -1$.

Equation (5.4) is a linear parabolic equation, and so given appropriate initial data u_0 , we may define its solution u_{n+1} , a function of prescribed regularity, provided that u_n , μ_n , k_n , and $\hat{\gamma}_n$ are known functions of suitable regularity. Equations (5.5) and (5.6) form a coupled system of ordinary differential equations for (μ_{n+1}, k_{n+1}) . Hence if we are given appropriate initial data (μ_0, k_0) , we may define (μ_{n+1}, k_{n+1}) as the (prescribed regularity) solution of (5.5) and (5.6) provided that u_{n+1} , as given by (5.4), and u_n , μ_n , k_n , and $\hat{\gamma}_n$ are functions of suitable regularity.

The last equation (5.7), is a first order elliptic system which, by construction, may be uniquely solved for $\hat{\eta}_{n+1}$ provided that u_{n+1} , μ_{n+1} , and k_{n+1} , as given by (5.4)–(5.6), are suitably regular.

The claim is that by iterating the procedure described above, we can construct a sequence of iterates $(u, \mu, k, \hat{\eta})_n$. We will now make these ideas more precise by showing that the above iteration scheme is consistently defined.

B. Consistency of the iteration scheme

As suggested by the preceding remarks about the iteration scheme, we begin by considering Eq. (5.4). Let us suppose that $u_n, \mu_n, k_n, \hat{\eta}_n \in P_{0,T}^m$, $u_n(\xi, t) \geq \delta_1 > 0$, for all $(\xi, t) \in S^2 \times [0, T]$ and that $u_0 \in H^{2m}(S^2)$ with $u_0(\xi) \geq \delta_2 > 0$, for all $\xi \in S^2$. Note here that the u_0 we have referred to is the initial value, not the zeroth iterate u_0^- which is defined as the time constant extension of $u_0 \in H^{2m}(S^2)$ to $S^2 \times [0, \infty)$, that is

$$u_0^-(\xi, t) = u_0(\xi).$$

Equation (5.4) is a linear parabolic equation for u_{n+1} subject to the initial condition $u_{n+1}|_{S^2 \times \{0\}} = u_0^-$, which is seen to be of the form

$$\dot{u}_{n+1} = \mathcal{A}_t^n u_{n+1} + F_n, \quad u_{n+1}|_{t=0} = u_0^-,$$

where the operator $\mathcal{A}_t^n$ is given as

$$\mathcal{A}_t^n = \hat{\gamma} u_n^2 \Delta.$$

The operator $\mathcal{A}_t^n$ may be written in divergence form (5.2), with

$$a^{ij} = \hat{\gamma} u_n^2 \bar{g}^{ij},$$

$$b^i = -(u_n^2 \nabla_k \hat{\gamma} \bar{g}^{ik} + 2 \hat{\gamma} u_n \nabla_k u_n \bar{g}^{ik}),$$

$$c = 0. \tag{5.8}$$

The source field F_n is defined as

$$F_n = \frac{1}{2} \left(1 + \frac{1}{2} \gamma e^{2t} \|\hat{\eta}_n\|^2 + \gamma B \right) u_n - \hat{\gamma} \left(1 - \frac{1}{2} e^{2t} (16\pi T_{00} + 2\|\kappa_n\|^2 + \frac{3}{2}\mu_n^2 - 2\mu_n H) \right) u_n^3.$$

Recalling Theorem 5.1 we find that if $m \geq m_0$, $u_n, \hat{\eta}_n, k_n \in P_{0,T}^m$ and the prescribable fields are smooth then u_{n+1} is uniquely defined as an element of $P_{0,T}^{m+1/2}$.

It is important to note that for Theorem 5.1 to apply it is imperative that $\hat{\gamma} u_n^2 \Delta$ be an elliptic operator. The ellipticity of this operator is solely dependent upon u_n 's capacity to stay above zero. Although we have assumed that $u_n(\xi, t) \geq \delta_1 > 0$ for all $(\xi, t) \in S^2 \times [0, T]$, it could happen that u_{n+1} falls below δ_1 after some time $T_* < T$. This possibility means that successive iterates might only be defined as elements of $P_{0,T_n}^{m+1/2}$, with $\{T_n\}$ a decreasing sequence of times. We will say more about this point later on.

Given that u_{n+1} exists and belongs to the class $P_{0,T}^{m+1/2}$ we move on to consider the coupled ordinary differential equation system (5.5) and (5.6).

We note first that, given $t \in [0, \infty)$, $k(\cdot, t)$ is an element of $\text{Span}\{Y_{lm}^1; m = -1, 0, 1\}$ and as such we may equivalently consider $k(\cdot, t)$ simply as an element of $\mathbb{C}^3$. Indeed, we may write

$$k(\xi, t) = k^m(t) Y_{1m}^1(\xi), \quad m = -1, 0, 1,$$

and since the Y_{1m}^1 's are given, we may think of k as mapping $[0, \infty)$ into $\mathbb{C}^3$ as follows:

$$k:[0,\infty) \rightarrow \mathbb{C}^3 t \mapsto (k^{-1}(t), k^0(t), k^1(t)).$$

For simplicity, we will refer to $k(\cdot, t)$ simply as $k(t)$. Note that there will now be an ambiguity surrounding our use of the symbol $k(t)$ since we will use it to denote both a field over S^2 , and the components of that field over $\mathbb{C}^3$. When we say “ $k \in P_{0,T}^m$ ” we mean $k(t) \in \text{Span}\{Y_{1m}^1 : m = -1, 0, 1\}$. Drawing the distinction between this representation and k 's representation over $\mathbb{C}^3$, we have that

$$k = (k^{-1}, k^0, k^1) \in H^m([0, T]; \mathbb{C}^3) \quad \text{if and only if} \quad k = k^m Y_{1m}^1 \in P_{0,T}^m,$$

where $H^m([0, T]; \mathbb{C}^3)$ is the Sobolev space with norm given by

$$\|f\|_{H^m([0, T]; \mathbb{C}^3)}^2 = \sum_{i=1}^m \int_0^T \|\partial_t^i f(s)\|_{\mathbb{C}^3}^2 ds.$$

The context in which the symbol k (or k_n) is used should make it clear which representation we are employing.

Furthermore, for any fixed value of t we have $(\mu(t), k(t)) \in \mathcal{F}(S^2) \times \mathbb{C}^3$, where $\mathcal{F}(S^2)$ is some appropriate function space; we will see that the Sobolev spaces $H^{2j}(S^2)$, with $j \geq m_0$, will suit our needs. Hence we can view the equations (5.5) and (5.6) as a coupled system of ordinary differential equations on $E^j := H^{2j}(S^2) \times \mathbb{C}^3$, and so writing

$$\mathcal{M}(t) := (\mu(t), k(t)),$$

we have

$$\dot{\mathcal{M}}_{n+1} = A_n \mathcal{M}_{n+1} + B_n, \quad (5.9)$$

where

$$A_n \mathcal{M} = (k^m [\delta(u_{n+1} \bar{\delta} Y_{1m}) + \bar{\delta}(u_{n+1} \delta Y_{1m})] - (1 + \rho)\mu - \mathcal{P}_1[\mu \delta u_{n+1}]), \quad (5.10)$$

and

$$B_n = \left(\text{div}(u_{n+1} \sigma) + u_n^{-1} \hat{\gamma}_n^{AB} \nabla_{(\beta} \beta_{\alpha)} + \rho H - 8\pi T_{03} u_{n+1} e^t - \mathcal{P}_1 \left[\frac{1}{2} \delta(u_{n+1} \mu_n) - \delta(u_{n+1} H) + (1 + \rho) \kappa_n + \bar{\kappa}_n \delta \beta + \kappa_n \bar{\delta} \beta - 8\pi T_0 u_{n+1} e^t \right] \right). \quad (5.11)$$

We then have the following result, which asserts that the operator A_n is bounded, provided u_{n+1} satisfies certain conditions. We note also, in passing, that the operator A_n is linear.

Lemma 5.2: Let $u_{n+1}(t) \in H^{2j+1}(S^2)$ for all $t \in [0, T]$, and $\beta \in \Gamma^\infty(TS^2 \times [0, \infty))$. The operator defined above by (5.10) satisfies

$$A_n(t): H^{2j}(S^2) \times \mathbb{C}^3 \rightarrow H^{2j}(S^2) \times \mathbb{C}^3$$

and

$$\|A_n \mathcal{M}(t)\|_{H^{2j}(S^2) \times \mathbb{C}^3} \leq C(1 + \|u_{n+1}(t)\|_{H^{2j+1}(S^2)}) \|\mathcal{M}(t)\|_{H^{2j}(S^2) \times \mathbb{C}^3},$$

for each $t \in [0, T]$ and some constant C depending only on the smooth field β . That is to say that $A_n(t)$ is a uniformly bounded operator for each n , with norm given by

$$\|A_n(t)\|_j = C(1 + \|u_{n+1}(t)\|_{H^{2j+1}(S^2)}).$$

Proof: Let $E^j := H^{2j}(S^2) \times \mathbb{C}^3$ and note that

$$\begin{aligned}\|A_n \mathcal{M}\|_{E^j}^2 &= \|k^m [\delta(u_{n+1} \bar{\delta} Y_{1m}) + \bar{\delta}(u_{n+1} \delta Y_{1m})] - (1 + \rho) \mu\|_{H^{2j}(S^2)}^2 + \|\mathcal{P}_1(\mu \delta u_{n+1})\|_{C^3}^2 \\ &\leq 2\|k^m [\delta(u_{n+1} \bar{\delta} Y_{1m}) + \bar{\delta}(u_{n+1} \delta Y_{1m})]\|_{H^{2j}(S^2)}^2 + 2\|(1 + \rho) \mu\|_{H^{2j}(S^2)}^2 + \|\mathcal{P}_1(\mu \delta u_{n+1})\|_{C^3}^2.\end{aligned}$$

Estimating the first term on the right-hand side we have

$$\begin{aligned}\|k^m [\delta(u_{n+1} \bar{\delta} Y_{1m}) + \bar{\delta}(u_{n+1} \delta Y_{1m})]\|_{H^{2j}(S^2)} &\leq \|k^m \delta u_{n+1} \bar{\delta} Y_{1m}\|_{H^{2j}(S^2)} + \|k^m \bar{\delta} u_{n+1} \delta Y_{1m}\|_{H^{2j}(S^2)} \\ &\quad + \|k^m u_{n+1} \Delta Y_{1m}\|_{H^{2j}(S^2)} \leq C \|u_{n+1}\|_{H^{2j+1}(S^2)} \|k\|_{C^3},\end{aligned}\tag{5.12}$$

while for the second term we have

$$\|(1 + \rho) \mu\|_{H^{2j}(S^2)} \leq C \|\mu\|_{H^{2j}(S^2)}.\tag{5.13}$$

The remaining term is estimated as follows:

$$\|\mathcal{P}_1(\mu \delta u_n)\|_{C^3}^2 = \sum_{m=-1}^1 \left| \oint \mu \delta u_{n+1} Y_{1m}^1 d\Omega^2 \right|^2 \leq C \sum_{m=-1}^1 \left(\oint |\mu \delta u_{n+1}| d\Omega^2 \right)^2 \leq C \|\mu\|_{L^2(S^2)}^2 \|\delta u_{n+1}\|_{L^2(S^2)}^2\tag{5.14}$$

for any $j \geq 0$.

Adding (5.12), (5.13), and (5.14) we obtain

$$\|A_n \mathcal{M}\|_{E^j} \leq C(1 + \|u_{n+1}\|_{H^{2j+1}(S^2)}) (\|\mu\|_{H^{2j}(S^2)} + \|k\|_{C^3}),$$

as required. ■

The boundedness of a linear operator A_n , such as that just established in Lemma 5.2, is the standard property required to infer the existence of solutions of (5.9). The following result is a special case of the standard local existence result for ordinary differential equations in Banach space. See Ref. 11, for example.

Lemma 5.3: Let $E^j = H^{2j}(S^2) \times \mathbb{C}^3$, and suppose $A(t): E^j \rightarrow E^j$ is a uniformly bounded linear operator for each $t \in [0, T]$, with norm $\|A(t)\|_j$. Further suppose that $\mathcal{M}_0 \in E^j$ and $B \in L^1([0, T]; E^j)$. Then there exists a unique $\mathcal{M} \in C^1([0, T]; E^j)$ satisfying

$$\dot{\mathcal{M}} = A\mathcal{M} + B,$$

$$\mathcal{M}(0) = \mathcal{M}_0.$$

The boundedness of the operator A_n is also the foundation for *a priori* estimates for solutions of (5.9). The basic result is as follows.

Lemma 5.4: Let A and B satisfy the hypotheses of Lemma 5.3 and let $\mathcal{M}$ be a solution of

$$\dot{\mathcal{M}} = A\mathcal{M} + B,$$

with initial value $\mathcal{M}_0$. For all $t \in [0, T]$, we have the estimate

$$\|\mathcal{M}(t)\|_{E^j} e^{-t\|A\|_j} \leq \|\mathcal{M}_0\|_{E^j} + \int_0^t \|B(s)\|_{E^j} ds.$$

Proof: We note first that $E^j = H^{2m}(S^2) \times \mathbb{C}^3$ is in fact a Hilbert space and so, fixing $t \in [0, T]$,

$$\frac{d}{dt} \|\mathcal{M}(t)\|_{E^j}^2 = 2\langle \mathcal{M}(t), \dot{\mathcal{M}}(t) \rangle_{E^j} \leq 2\|\mathcal{M}(t)\|_{E^j} \|\dot{\mathcal{M}}(t)\|_{E^j}.$$

Hence we must have

$$\frac{d}{dt}\|\mathcal{M}(t)\|_{E^j} \leq \|\dot{\mathcal{M}}(t)\|_{E^j} \leq \|A\mathcal{M}(t)\|_{E^j} + \|B(t)\|_{E^j} \leq C\|\mathcal{M}(t)\|_{E^j} + \|B(t)\|_{E^j},$$

where it is clear that the smallest such C for which this inequality holds is the norm $\|A\|_j$ of the operator A .

Gronwall's inequality then gives

$$\|\mathcal{M}(t)\|_{E^j} \leq e^{Ct} \left(\|\mathcal{M}_0\|_{E^j} + \int_0^t \|B(s)\|_{E^j} ds \right)$$

as required. $\blacksquare$

Applying Lemmas 5.3 and 5.4 to (5.9), noting Lemma 5.2, we have the following result.

Corollary 5.5: Suppose that $u_n, \mu_n, k_n, \hat{\eta}_n \in P_{0,T}^m$ and $u_{n+1} \in P_{0,T}^{m+1/2}$ is the solution of (5.4), subject to the initial condition $u_0 \in H^{2m}(S^2)$. Equation (5.9), subject to the initial condition $\mathcal{M}_{n+1}|_{t=0} = \mathcal{M}_0 \in E^m$, defines $\mathcal{M}_{n+1} \in C^1([0, T]; E^m)$, which satisfies

$$\|\mathcal{M}_{n+1}(t)\|_{E^m} e^{-t\|A_n\|_m} \leq \|\mathcal{M}_0\|_{E^m} + \int_0^t \|B_n(s)\|_{E^m} ds, \quad (5.15)$$

for all $t \in [0, T]$.

Proof: Since we have already assumed that $m \geq m_0$ we have no problem meeting the conditions $u_{n+1}(t) \in H^{2j+1}(S^2)$ and $B_n \in L^1([0, T]; E^m)$. Lemma 5.3 applies and so $\mathcal{M}_{n+1}$ is uniquely defined as the solution of (5.9) with $\mathcal{M}|_{t=0} = \mathcal{M}_0$. Lemma 5.4 also applies and gives (5.15). $\blacksquare$

To establish the higher temporal regularity of solutions of (5.9) we will consider the following Banach spaces:

$$\mathcal{O}_{0,T}^m := P_{0,T}^m \times H^m([0, T]; \mathbb{C}^3),$$

and their respective norms,

$$\|\mathcal{M}\|_{\mathcal{O}_{0,T}^m}^2 = \sum_{l=0}^m \int_0^T \|\partial_t^l \mathcal{M}(s)\|_{E^{m-l}}^2 ds.$$

We note that upon inspection of (5.5) and (5.6), $\mathcal{O}_{0,T}^m$ is the natural space in which to look for solutions of (5.9) given $u_{n+1} \in P_{0,T}^{m+1/2}$.

To obtain the desired estimate, from which the higher regularity of solutions of (5.9) will follow, we will first need to establish some (essentially trace-type) results concerning the properties of the $P_{0,T}^m$ spaces.

Lemma 5.6: Suppose that $T \geq \epsilon > 0$. Let u be a function belonging to the class $P_{0,T}^{k+1/2}$ with $k \geq 1$. Then

$$\|u(\cdot, t)\|_{H^{2k}(S^2)} \leq C\|u\|_{P_{0,T}^{k+1/2}},$$

for any $t \in [0, T]$. The constant depends only on ϵ .

Proof: We first suppose that $u \in C^\infty(S^2 \times [0, T])$. We have that $u(\cdot, t) \in L^2(S^2)$, for each $t \in \mathbb{R}^+$, we may write u spectrally as follows:

$$u(x, t) = \sum_j c_j(t) \phi_j(x),$$

where the ϕ_j 's form an orthonormal basis for $L^2(S^2)$ satisfying $\Delta \phi_j = -\lambda_j^2 \phi_j$.

We define the function $\tilde{u}(\cdot, t) = u(\cdot, t_0 + t)\chi(t)$, where χ is a smooth function satisfying

$$\chi(s) = \begin{cases} 1, & s < \frac{T}{4}, \\ 0, & s > \frac{3T}{4}, \end{cases}$$

so that $\tilde{u}(\cdot, T) = 0$ and $\tilde{u}(\cdot, 0) = u(\cdot, t_0)$. Then given that $k \geq 1$, we have

$$\begin{aligned} \|u(\cdot, t_0)\|_{H^{2k}(S^2)}^2 &= \sum_j |c_j(t_0)|^2 |\lambda_j|^{2k} \\ &= \sum_j |\tilde{c}_j(0)|^2 |\lambda_j|^{2k} \\ &= \sum_j |\lambda_j|^{2k} \int_0^T \frac{d}{ds} |\tilde{c}_j(s)|^2 ds \\ &\leq \sum_j \int_0^T |\tilde{c}_j(s)|^2 |\lambda_j|^{2k+1} + |\dot{\tilde{c}}_j(s)|^2 |\lambda_j|^{2k-1} ds \leq \|\tilde{u}\|_{P_{0,T}^{k+1/2}}^2 \leq C \|u\|_{P_{0,T}^{k+1/2}}^2. \end{aligned}$$

Note that the constant depends only on the cutoff function χ . With the result now established for smooth functions it is an easy matter to get the result for $u \in P_{0,T}^{k+1/2}$; it follows, after mollifying u , by a standard approximation argument. ■

Corollary 5.7: Suppose that $T \geq \epsilon > 0$. Let u belong to the class $P_{0,T}^{m+1/2}$, then

$$\|\partial_t^j u(\cdot, t)\|_{H^{2(m-j)}(S^2)} \leq C \|u\|_{P_{0,T}^{m+1/2}},$$

for all $j \leq m$ and $t \in [0, T]$. The constant depends only on ϵ .

Proof: We note that $u \in P_{0,T}^{m+1/2}$ means that $\partial_t^j u \in P_{0,T}^{m-j+1/2}$ and so applying Lemma 5.6 with $k = m-j$ we have the result. ■

In the above trace-type results we saw that we may control the spatial norms of time derivatives of functions, by a constant multiple of the $P_{0,T}^{m+1/2}$ -norm. Unfortunately the constant's dependence upon ϵ means that we lose that control as ϵ approaches zero. The following result shows how, with the loss of one time derivative of regularity, we may control the spatial norms of time derivatives by a constant which approaches zero as we consider smaller and smaller time intervals.

Lemma 5.8: Let $u \in P_{0,T}^{k+1}$, with $k \geq 0$, then

$$\|u(t)\|_{H^{2k}(S^2)}^2 \leq \|u_0\|_{H^{2k}(S^2)}^2 + CT \|u\|_{P_{0,T}^{k+1}}^2,$$

for any $t \in [0, T]$. The constant C depends only on k .

Proof: Fixing $\xi \in S^2$, we have

$$u(\xi, t) = u(\xi, 0) + \int_0^t \partial_t u(\xi, s) ds.$$

It follows, using the Hölder inequality, that

$$|\nabla^{2j}(u(\xi, t) - u_0(\xi))| \leq \sqrt{T} \left(\int_0^T |\partial_t \nabla^{2j} u(\xi, s)|^2 ds \right)^{1/2}$$

for any $0 \leq j \leq m$ and $t \in [0, T]$. Integrating this result gives

$$\begin{aligned} \|u(\cdot, t) - u_0\|_{H^{2k}(S^2)}^2 &\leq C \oint |\nabla^{2k}(u - u_0)|^2 + |u - u_0|^2 \, d\mu_g \\ &\leq \oint T \int_0^T |\partial_t u|^2 + |\partial_t \nabla^{2k} u|^2 \, dt \, d\mu_g \leq T \|u\|_{P_{0,T}^{k+1}}^2. \end{aligned}$$

The result follows after observing the elementary inequality $\|x\| - \|y\| \leq \|x - y\|$. ■

Remark: Applying Lemma 5.8 to $\partial_t^j u$ gives

$$\|\partial_t^j u(t)\|_{H^{2(k-l)}(S^2)}^2 \leq \|\partial_t^j u(0)\|_{H^{2(k-l)}(S^2)}^2 + CT \|u\|_{P_{0,T}^{k+1}}^2.$$

If it happened that u was also the solution of say a parabolic equation, then we could reduce the above estimate to

$$\|\partial_t^j u(t)\|_{H^{2(k-l)}(S^2)}^2 \leq C \left(\|u_0\|_{H^{2k}(S^2)}^2 + T \|u\|_{P_{0,T}^{k+1}}^2 \right),$$

where the constant C now depends on the coefficients and source functions appearing in the parabolic equation.

We now proceed to derive the basic $\mathcal{O}_{0,T}^m$ *a priori* estimates for $\mathcal{M}_{n+1}$, the solutions of (5.9).

Proposition 5.9: Suppose $u_{n+1} \in P_{0,T}^{m+1/2}$ and let $\mathcal{M}_{n+1}$ be the solution of (5.9). Then $\mathcal{M}_{n+1}$ satisfies the inequality,

$$\|\mathcal{M}_{n+1}\|_{\mathcal{O}_{0,T}^m}^2 \leq TC \left\{ 1 + \|B_n\|_{\mathcal{O}_{0,T}^m}^2 \right\}. \quad (5.16)$$

The constant C depends on $\|u_{n+1}\|_{P_{0,T}^{m+1/2}}$, $\|u_0\|_{H^{2m}(S^2)}$ and $\|\mathcal{M}_0\|_{E^m}$.

Proof: We begin by noting that if we square, then integrate (5.15), we obtain

$$\int_0^T \|\mathcal{M}_{n+1}(t)\|_{E^m}^2 \, dt \leq 2Te^{TC} \left(\|\mathcal{M}_0\|_{E^m}^2 + \|B_n\|_{\mathcal{O}_{0,T}^m}^2 \right), \quad (5.17)$$

where the constant C depends on $\|u_{n+1}\|_{P_{0,T}^{m+1/2}}$. This gives us a bound for the first term in the sum

$$\|\mathcal{M}_{n+1}\|_{\mathcal{O}_{0,T}^m}^2 = \sum_{l=0}^m \int_0^T \|\partial_t^l \mathcal{M}_{n+1}\|_{E^{m-l}}^2 \, dt. \quad (5.18)$$

Considering the $(j+1)^{\text{th}}$ term in (5.18) we have, upon differentiating (5.24),

$$\int_0^T \|\partial_t^j \mathcal{M}_{n+1}\|_{E^{m-j}}^2 \, dt \leq \int_0^T C_j \sum_{l=0}^{j-1} \|\partial_t^l A_n \partial_t^{j-l-1} \mathcal{M}_{n+1}\|_{E^{m-j}}^2 + \|\partial_t^{j-1} B_n\|_{E^{m-j}}^2 \, dt, \quad (5.19)$$

where C_j is a constant dependent upon j . The expression $\partial_t^j A_n \partial_t^{j-l-1} \mathcal{M}_{n+1}$ represents the outcome of the operator $\partial_t^j A_n$, given by (5.10) with u_{n+1} replaced by $\partial_t^l u_{n+1}$, acting on $\partial_t^{j-l-1} \mathcal{M}_{n+1}$. It is obvious that results analogous to Lemma 5.2 exist for $\partial_t^j A_n$ as well. Hence

$$\|\partial_t^j A_n \partial_t^{j-l-1} \mathcal{M}_{n+1}\|_{E^{m-j}}^2 \leq C_l \|\partial_t^{j-l-1} \mathcal{M}_{n+1}\|_{E^{m-j}}^2,$$

where, by virtue of Lemma 5.8, we see that

$$C_l \leq C \left(\|u_0\|_{H^{2m}(S^2)}^2 + T \|u_{n+1}\|_{P_{0,T}^{m+1/2}}^2 \right),$$

where we have used the fact that u_{n+1} is the solution of (4.8) to control the initial values of the time derivatives of u_{n+1} .

Observing that Lemma 5.8 may easily be extended to apply to the E^{m-j} and $\mathcal{O}_{0,T}^m$ spaces, the second term on the right-hand side of (5.19) satisfies

$$\|\partial_t^{j-1} B_n(\cdot, s)\|_{E^{m-j}}^2 \leq C \left(1 + T \|B_n\|_{\mathcal{O}_{0,T}^m}^2\right),$$

for all $s \leq T$. The constant C depends on $\|\partial_t^{j-1} B_n(\cdot, 0)\|_{E^{m-j}}$. This quantity can be expressed in terms of E^{m-j} -norms of the $(j-1)$ th time derivatives of u_{n+1} , μ_n , k_n , and $\hat{\eta}_n$ evaluated at $t=0$. Since these functions are themselves solutions of their respective equations, we may in turn express time derivatives of u_{n+1} , μ_n , k_n , and $\hat{\eta}_n$, evaluated at $t=0$, in terms of the initial values u_0 , μ_0 , and k_0 . We therefore have

$$\begin{aligned} \int_0^T \|\partial_t^j \mathcal{M}_{n+1}\|_{E^{m-j}}^2 dt &\leq C \sum_{l=0}^{j-1} \int_0^T \|\partial_t^{j-l-1} \mathcal{M}_{n+1}\|_{E^{m-j}}^2 + T \|B_n\|_{\mathcal{O}_{0,T}^m}^2 + 1 dt \\ &\leq CT \left(1 + T \|B_n\|_{\mathcal{O}_{0,T}^m}^2\right) + C \sum_{l=0}^{j-1} \int_0^T \|\partial_t^{j-l-1} \mathcal{M}_{n+1}\|_{E^{m-j}}^2 dt, \end{aligned} \quad (5.20)$$

where C depends on $\|u_{n+1}\|_{P_{0,T}^{m+1/2}}$, $\|u_0\|_{H^{2m-1}(S^2)}$, $\|\mu_0\|_{H^{2(m-1)}(S^2)}$, and $\|k_0\|_{C^3}$.

The estimate (5.16) is then obtained by using (5.20) to iterate from $j=1$ to m starting from (5.17). $\blacksquare$

Thus in light of Lemma 5.3 and Proposition 5.9, we find that if $u_0 \in H^{2m}(S^2)$, $\mathcal{M}_0 \in E^m$, $u_n \in P_{0,T}^{m+1/2}$, $u_{n+1} \in P_{0,T}^{m+1/2}$, and $\mathcal{M}_n \in \mathcal{O}_{0,T}^m$ then $\mathcal{M}_{n+1}$ exists and belongs to $\mathcal{O}_{0,T}^m$.

Moving now to the final equation in the iteration scheme, the elliptic system for $\hat{\eta}_{n+1}$. We note again that by construction (5.7) is guaranteed to have a solution. Hence $\hat{\eta}_{n+1}$ exists and the following results establish that $\hat{\eta}_{n+1} \in P_{0,T}^m$ provided $(u_{n+1}, \mathcal{M}_{n+1}) \in P_{0,T}^{m+1/2} \times \mathcal{O}_{0,T}^m$.

Lemma 5.10: Suppose $g \in L^2(\text{span}\{Y_{lm}^1 : l \geq 2\})$ and let $\hat{\eta}$ be the solution of

$$\bar{\partial} \hat{\eta} = g. \quad (5.21)$$

We have the estimate

$$\|\hat{\eta}\|_{H^1(S^2)}^2 \leq C \|g\|_{L^2(S^2)}^2,$$

where C is a constant.

Proof: The result follows easily from the observation that since $\bar{\partial} \hat{\eta}$ is devoid of $l=1$ spherical harmonics, we may solve (5.21). Moreover, we are able to write $\hat{\eta}$ explicitly, using spectral decomposition as

$$\hat{\eta} = \sum_{l=2}^{\infty} \sum_{m=-l}^l \hat{\eta}_{lm} Y_{lm}^2.$$

It follows that we may write

$$\bar{\partial} \hat{\eta} = \sum_{l=2}^{\infty} \sum_{m=-l}^l C_l \hat{\eta}_{lm} Y_{lm}^1,$$

where C_l , $l=0, 2, 3, \dots$ are constants. Checking the norms in question we then find

$$\|\bar{\partial} \hat{\eta}\|_{L^2(S^2)}^2 = \oint \left| \sum_{l,m} C_l \hat{\eta}_{lm} Y_{lm}^1 \right|^2 d\Omega^2 \geq \sum_{l,m} C_l^2 \hat{\eta}_{lm}^2,$$

and

$$\|\hat{\eta}\|_{H^1(S^2)}^2 = \oint |\nabla \hat{\eta}|^2 + |\hat{\eta}|^2 d\Omega^2 \leq \oint 2|\bar{\partial} \hat{\eta}|^2 + |\hat{\eta}|^2 d\Omega^2 \leq C \sum_{l,m} C_l^2 \hat{\eta}_{lm}^2.$$

Thus we have

$$\|\hat{\eta}\|_{H^1(S^2)}^2 \leq C \sum_{l,m} C_l^2 \hat{\eta}_{lm}^2 \leq C \|\bar{\partial} \hat{\eta}\|_{L^2(S^2)}^2,$$

and the result follows from Eq. (5.21). ■

Lemma 5.11: Suppose $\hat{\eta}$ is a solution of (5.21), where now $g \in H^k(\text{span}\{Y_{lm}^1 : l \geq 2\})$. We have the estimate

$$\|\hat{\eta}\|_{H^{k+1}(S^2)}^2 \leq C_k \|g\|_{H^k(S^2)}^2, \quad (5.22)$$

where C_k is a constant depending only on k .

Proof: Estimating the right-hand side of (5.22) we have

$$\begin{aligned} \|\hat{\eta}\|_{H^{k+1}(S^2)}^2 &= \|\nabla^{k+1} \eta\|_{L^2(S^2)}^2 + \|\hat{\eta}\|_{H^k(S^2)}^2 \\ &\leq \|\nabla^k \hat{\eta}\|_{H^1(S^2)}^2 + C_k \|\bar{\partial} \hat{\eta}\|_{H^{k-1}(S^2)}^2 \\ &\leq C \|\bar{\partial} \nabla^k \eta\|_{L^2(S^2)}^2 + C_k \|\bar{\partial} \hat{\eta}\|_{H^k(S^2)}^2 \\ &= C \|\nabla^k \bar{\partial} \hat{\eta}\|_{L^2(S^2)}^2 + C'_k \|\bar{\partial} \hat{\eta}\|_{H^k(S^2)}^2 \leq C \|\bar{\partial} \hat{\eta}\|_{H^k(S^2)}^2 + C'_k \|\bar{\partial} \hat{\eta}\|_{H^k(S^2)}^2 \leq C''_k \|\bar{\partial} \hat{\eta}\|_{H^k(S^2)}^2. \end{aligned}$$

The result follows from Eq. (5.21). ■

The following result is now immediate from the definition of $P_{0,T}^k$.

Corollary 5.12: Let g , belonging to the class $P_{0,T}^k$, be devoid of $l=1$ spherical harmonic components so that $\hat{\eta}$ exists as a solution of (5.21). Then

$$\|\hat{\eta}\|_{P_{0,T}^{k+1/2}} \leq C_k \|g\|_{P_{0,T}^k}.$$

C_k is a constant depending only on k .

The foregoing results have shown that if,

$$(u_n, \mu_n, k_n, \hat{\eta}_n) \in P_{0,T}^{m+1/2} \times \mathcal{O}_{0,T}^m \times P_{0,T}^m$$

and

$$(u_0, \mu_0, k_0) \in H^{2m+1}(S^2) \times E^{m+1/2}$$

then

$$(u_{n+1}, \mu_{n+1}, k_{n+1}, \hat{\eta}_{n+1}) \in P_{0,T}^{m+1/2} \times \mathcal{O}_{0,T}^m \times P_{0,T}^m$$

also.

For the sake of brevity we consider the four iterates as a single entity residing in the appropriate Banach space. In particular we define the four-tuple,

$$\mathbf{U}_n := (u_n, \mu_n, k_n, \hat{\eta}_n),$$

residing in the Banach space,

$$\Psi_{0,T}^m := P_{0,T}^{m+1/2} \times P_{0,T}^m \times H^m([0, T]; \mathbb{C}^3) \times P_{0,T}^m.$$

To summarize, we have proven the following result.

Theorem 5.13: Let $m \geq m_0$ and let $u_0 \in H^{2m+1}(S^2)$, $\mu_0 \in H^{2m}(S^2)$, and $k_0 \in \mathbb{C}^3$. There exists a nonincreasing, but possibly decreasing sequence $\{T_n\}_{n=1}^\infty$ with $T_n > 0$, and a sequence of iterates $\mathbf{U}_n \in \Psi_{0,T_n}^m$ which satisfy (5.4)–(5.7).

Theorem 5.13 is not ideal. It tells us that the iterates exist as elements of Ψ_{0,T_n}^m , with the possibility of $T_n \rightarrow 0$. To be able to prove local existence for the QSECS however, we will need the

iterates to be defined over some nonzero time interval. This means that we will need to show that there exists an $\epsilon > 0$ such that $T_n \geq \epsilon$, for all n . This fact will follow from the uniform estimates we derive for the iterates in the next section.

C. Uniform estimates for the iterates

In this section we will show that all of the iterates satisfy uniform bounds, at least locally in time. We consider the iteration scheme proposed by (5.4)–(5.7), which we may write more simply as

$$\dot{u}_{n+1} = \hat{\gamma} u_n^2 \Delta u_{n+1} + F_n, \quad (5.23)$$

$$\dot{\mathcal{M}}_{n+1} = A_n \mathcal{M}_{n+1} + B_n, \quad (5.24)$$

$$\bar{\delta} \hat{\gamma}_{n+1} = \mathcal{L}(u_{n+1}, \mathcal{M}_{n+1}) - \dot{\sigma}, \quad (5.25)$$

where A_n and B_n are defined by (5.10) and (5.11), respectively, and where

$$F_n = \frac{1}{2} \left(1 + \frac{1}{2} \gamma e^{2t} \|\hat{\gamma}_n\|^2 + \gamma B \right) u_n - \hat{\gamma} \left(1 - \frac{1}{2} e^{2t} (16\pi T_{00} + 2\|\kappa_n\|^2 + \frac{3}{2} \mu_n^2 - 2\mu_n H) \right) u_n^3 \quad (5.26)$$

and

$$\mathcal{L}(u, \mathcal{M}) = (1 - \mathcal{P}_1) \left\{ \mu \delta u + \frac{1}{2} \delta(u\mu) - \delta(uH) + (1 + \rho) \kappa + \bar{\kappa} \bar{\delta} \beta + \kappa \bar{\delta} \bar{\beta} + 8\pi T_0 u e^t \right\}. \quad (5.27)$$

The key result needed to prove that the iterates satisfy uniform bounds is the following simple inequality.

Lemma 5.14: Suppose u belongs to the localized parabolic space $P_{0,T}^1$. Then

$$\|u\|_{P_{0,T}^0}^2 \leq 2T \left(\|u_0\|_{L^2(S^2)}^2 + \|u\|_{P_{0,T}^1}^2 \right).$$

Proof: From the definition of $P_{0,T}^0$ we have

$$\|u - u_0\|_{P_{0,T}^0}^2 \leq \int_0^T \oint |\partial_t u(\xi, s)|^2 ds d\mu_{\bar{g}} dt \leq \int_0^T (T-s) \oint |\partial_t u(\xi, s)|^2 d\mu_{\bar{g}} ds \leq T \|\partial_t u\|_{P_{0,T}^0}^2.$$

The result then follows since $\|u_0\|_{P_{0,T}^0}^2 = T \|u_0\|_{L^2(S^2)}^2$.

Corollary 5.15: Any $u \in P_{0,T}^{m+1}$ satisfies the estimate

$$\|u\|_{P_{0,T}^m}^2 \leq CT \left(1 + \|u\|_{P_{0,T}^{m+1}}^2 \right). \quad (5.28)$$

The constant C depends on m and $\|\partial_t^j u(\cdot, 0)\|_{H^{2(m-j)}(S^2)}$, $j=0, \dots, m$.

Proof: We need only check terms of the form $\int_0^T \|\partial_t^j u\|_{H^{2(m-j)}(S^2)}^2 dt$. We have

$$\begin{aligned} \int_0^T \|\partial_t^j u\|_{H^{2(m-j)}(S^2)}^2 dt &\leq C \int_0^T \|\partial_t^j \nabla^{2(m-j)} u\|_{L^2(S^2)}^2 + \|\partial_t^j u\|_{L^2(S^2)}^2 dt \\ &= C \left(\|\partial_t^j \nabla^{2(m-j)} u\|_{P_{0,T}^0}^2 + \|\partial_t^j u\|_{P_{0,T}^0}^2 \right) \leq 4CT \left(\|\partial_t^j u(\cdot, 0)\|_{H^{2(m-j)}(S^2)}^2 + \|u\|_{P_{0,T}^{m+1}}^2 \right). \end{aligned}$$

The estimate (5.28) is then obtained by summing the terms above from $j=0$ to m . ■

Remark: If we also know that u is a solution of a parabolic equation we may express the constant C in terms of the initial values of u , the coefficients and source function. In particular, if u_n is the solution of (5.4) we have

$$\|u_n\|_{P_{0,T}^m}^2 \leq T\mathcal{C}\left(1 + \|u_n\|_{P_{0,T}^{m+1}}^2\right), \quad (5.29)$$

where the constant $\mathcal{C}$ depends on m , $\|u_0\|_{H^{2m}(S^2)}$, $\|\mu_0\|_{H^{2m}(S^2)}$, and $\|k_0\|_{\mathbb{C}^3}$.

We now prove that given suitable bounds on the initial data (which constitute $\mathbf{U}_0$), all the iterates $\{\mathbf{U}_n\}_{n=0}^\infty$ may be confined within a ball in $\Psi_{0,T}^m$, for some $T > 0$.

Proposition 5.16: Let $m \geq m_0$ and suppose $u_0 \in H^{2m+1}(S^2)$, $\mu_0 \in H^{2m}(S^2)$, and $k_0 \in \mathbb{C}^3$. There exists $T^* > 0$ and $N > 0$, depending only on m , the initial data and the prescribable fields, such that

$$\|\mathbf{U}_n\|_{\Psi_{0,T^*}^m} \leq N$$

for all $n \in \mathbb{N}$.

Proof: Since $u_0 \in H^{2m+1}(S^2)$, $\mu_0 \in H^{2m}(S^2)$, and $k_0 \in \mathbb{C}^3$, there exist numbers N_1 , N_2 , N_3 , and N_4 such that

$$\|u_0\|_{H^{2m+1}(S^2)} \leq N_1, \quad \|\mu_0\|_{H^{2m}(S^2)} \leq N_2, \quad \|k_0\|_{\mathbb{C}^3} \leq N_3, \quad \|\hat{\eta}_0\|_{H^{2m}(S^2)} \leq N_4. \quad (5.30)$$

Defining the zeroth iterates as the time constant extensions of these initial values and assuming, without loss of generality, that $T \leq 1$ we find that (5.30) implies

$$\|u_{\tilde{0}}\|_{P_{0,T}^{m+1/2}} \leq N_1, \quad \|\mu_{\tilde{0}}\|_{P_{0,T}^m} \leq N_2, \quad \|k_{\tilde{0}}\|_{H^m([0,T];\mathbb{C}^3)} \leq N_3, \quad \|\hat{\eta}_{\tilde{0}}\|_{P_{0,T}^m} \leq N_4,$$

where we have used the subscript $\tilde{0}$ to distinguish the zeroth iterates $u_{\tilde{0}}$, $\mu_{\tilde{0}}$, $k_{\tilde{0}}$, and $\hat{\eta}_{\tilde{0}}$ from the initial values u_0 , μ_0 , k_0 , and $\hat{\eta}_0$. Hence Proposition 5.16 holds for $n = \tilde{0}$, with $N = \sum_{i=1}^4 N_i$.

Hypothesizing then, that

$$\|u_j\|_{P_{0,T}^{m+1/2}} \leq N_1, \quad \|\mu_j\|_{P_{0,T}^m} \leq N_2, \quad \|k_j\|_{H^m([0,T];\mathbb{C}^3)} \leq N_3, \quad \|\hat{\eta}_j\|_{P_{0,T}^m} \leq N_4,$$

for all $j \in [0, n]_{\mathbb{N}}$ and some $T > 0$, we aim to show the same is true for the $(n+1)$ th iterates.

We consider (5.23), which is a linear parabolic equation for u_{n+1} . Since $m \geq m_0$, (5.3) gives

$$\|u_{n+1}\|_{P_{0,T_1}^{m+1/2}}^2 \leq \mathcal{Q}\left(\|u_0\|_{H^{2m}(S^2)}^2 + \|F_n\|_{P_{0,T_1}^{m-1/2}}^2\right).$$

We note that the constant $\mathcal{Q}$ depends on the P_{0,T_1}^m norm of the coefficients, which in this case, observing (5.8), amount to u_n . Hence

$$\mathcal{Q} = \mathcal{Q}(\|u_n\|_{P_{0,T_1}^m}).$$

However, since u_n is itself the solution of a linear parabolic equation, (5.29) applies and we have

$$\|u_n\|_{P_{0,T_1}^m}^2 \leq T_1\mathcal{C}\left(1 + \|u_n\|_{P_{0,T_1}^{m+1}}^2\right). \quad (5.31)$$

Moreover, by inspection of the proof of (Ref. 15, Theorem 4.6, p. 132), it can be seen that the constant $\mathcal{Q}$ depends on $\|u_n\|_{P_{0,T_1}^m}$ in such a way that, if $\|u_n\|_{P_{0,T_1}^m}$ is bounded as in (5.31), then $\mathcal{Q}(\|u_n\|_{P_{0,T_1}^m})$ is also bounded and satisfies the inequality

$$\mathcal{Q}(\|u_n\|_{P_{0,T_1}^m}) \leq T_1\tilde{\mathcal{Q}}, \quad (5.32)$$

where the constant $\tilde{\mathcal{Q}}$ depends on $\|u_n\|_{P_{0,T_1}^{m+1}}$.

The P_{0,T_1}^{m+1} norm of u_n can also be estimated using (5.3),

$$\|u_n\|_{P_{0,T_1}^{m+1}}^2 \leq \mathcal{Q}^* \left(\|u_0\|_{H^{2m+1}(S^2)}^2 + \|F_n\|_{P_{0,T_1}^m}^2 \right). \quad (5.33)$$

The constant $\mathcal{Q}^*$ depends only on $\|u_{n-1}\|_{P_{0,T_1}^{m+1/2}}$ which by hypothesis is less than N_1 . Similarly, $\|F_n\|_{P_{0,T}^m}^2$ can be controlled by a constant depending on N_1, N_2, N_3, N_4 and the prescribable fields. Hence $\|u_n\|_{P_{0,T_1}^{m+1}}$ is bounded by a constant that depends only on N_1, N_2, N_3, N_4 and the prescribable fields also. This in turn implies that the constant $\tilde{\mathcal{Q}}$ is also bounded by a term involving only N_1, N_2, N_3, N_4 and the norms of the prescribable fields.

The above argument implies that, given the induction hypotheses, if we take $T_1 \leq T$ then

$$\|u_{n+1}\|_{P_{0,T_1}^{m+1/2}}^2 \leq T_1 C_1, \quad (5.34)$$

where C_1 is a constant depending on the prescribable fields, N_1, N_2, N_3 , and N_4 . Thus provided that

$$T_1 \leq \min \left\{ T, \frac{N_1^2}{C_1} \right\}$$

we have

$$\|u_{n+1}\|_{P_{0,T_1}^{m+1/2}} \leq N_1. \quad (5.35)$$

Turning our attention now to (5.24), Proposition 5.9 gives

$$\|\mathcal{M}_{n+1}\|_{\mathcal{O}_{0,T_2}^m}^2 \leq CT_2 \left(1 + \|B_n\|_{\mathcal{O}_{0,t_2}^m}^2 \right)$$

provided that $T_2 \leq T_1$. By hypothesis again, we can control the $\mathcal{O}_{0,T_2}^m$ norm of B_n by a constant depending on N_1, N_2, N_3 , and N_4 . That is,

$$\|\mathcal{M}_{n+1}\|_{\mathcal{O}_{0,T_2}^m}^2 \leq T_2 C_2. \quad (5.36)$$

Therefore, so long as

$$T_2 \leq \min \left\{ T_1, \frac{N_2^2}{C_2}, \frac{N_3^2}{C_2} \right\},$$

we have

$$\|\mu_{n+1}\|_{P_{0,T_2}^m} \leq N_2, \quad \text{and} \quad \|k_{n+1}\|_{H^m([0,T_2];\mathbb{C}^3)} \leq N_3. \quad (5.37)$$

Finally, from Corollary 5.12, we have

$$\|\hat{\eta}_{n+1}\|_{P_{0,T_3}^m} \leq C_m \|\bar{\delta} \hat{\eta}_{n+1}\|_{P_{0,T_3}^{m-1/2}} \leq C_m \|\mathcal{L}(u_{n+1}, \mathcal{M}_{n+1}) - \dot{\sigma}\|_{P_{0,T_3}^{m-1/2}}.$$

Observing (5.27), we thus find that

$$\|\hat{\eta}_{n+1}\|_{P_{0,T_3}^m}^2 \leq C \left(\|u_{n+1}\|_{P_{0,T_3}^m}^2 + \|\mu_{n+1}\|_{P_{0,T_3}^m}^2 + \|k_{n+1}\|_{H^m([0,T_3];\mathbb{C}^3)}^2 + \|\sigma\|_{P_{0,T_3}^{m+1/2}}^2 \right) \leq T_3 C_3 + C_4 \|\sigma\|_{P_{0,T}^{m+1/2}}^2,$$

provided that $T_3 \leq T_2$, so that our use of (5.34) and (5.36) is justified. The constant C_3 depends only on the prescribable fields, N_1, N_2 , and N_3 and the constant C_4 depends only on the prescribable fields.

Hence, given that

$$T_3 \leq \min \left\{ T_2, \frac{N_4^2 - C_4 \|\sigma\|_{P_{0,T}^{m+1/2}}^2}{C_3} \right\},$$

we find that

$$\|\hat{\eta}_{n+1}\|_{P_{0,T_3}^m} \leq N_4. \quad (5.38)$$

Note that we require $N_4^2 > C_4 \|\sigma\|_{P_{0,T}^{m+1/2}}^2$, but this is easily arranged.

Hence, setting $T^* = T_3$, (5.35), (5.37), and (5.38) imply $\|u_{n+1}\|_{P_{0,T^*}^{m+1/2}} \leq N_1$, $\|\mu_{n+1}\|_{P_{0,T^*}^m} \leq N_2$, $\|k_{n+1}\|_{H^m([0,T^*];\mathbb{C}^3)} \leq N_3$, and $\|\hat{\eta}_{n+1}\|_{P_{0,T^*}^m} \leq N_4$. The result follows by induction with $N = \sum_{i=1}^4 N_i$. ■

This last result has bearing on the problem encountered in Theorem 5.13. There we found that, due to the possibility that u_n might become negative after some ever decreasing interval of time, the iterates were possibly only defined on the interval $[0, T_n]$ with $T_n \rightarrow 0$. Now that we have uniform bounds for all of the iterates in a Sobolev space of high enough order, however, we may conclude that the supremum of $|\partial_t u_n|$ over $S^2 \times [0, T^*]$ is no greater than N_1 . This leads to the following result.

Theorem 5.17: Let $m \geq m_0$ and let $u_0 \in H^{2m+1}(S^2)$ satisfy $\inf_{S^2} u_0 \geq 2\delta > 0$. Also suppose $\mu_0 \in H^{2m}(S^2)$ and $k_0 \in \mathbb{C}^3$. There exists a $T > 0$, depending on the initial values and prescribable fields, such that u_n , as defined by (5.4), satisfies

$$\inf_{S^2 \times [0, T]} u_n \geq \delta.$$

Hence there exists a sequence of iterates $\{\mathbf{U}_n\}_{n=0}^\infty \subset \Psi_{0,T}^m$ which satisfy (5.4)–(5.7) and $\|\mathbf{U}_n\|_{\Psi_{0,T}^m} \leq N$.

Proof: We already mentioned above that $\sup_{S^2 \times [0, T]} |\partial_t u_n| \leq N_1$. This fact implies that $\inf_{S^2 \times [0, \delta/N_1]} u_n \geq \delta$, since

$$u_n(\xi, t) = u_n(\xi, 0) + \int_0^t \partial_t u_n(\xi, s) ds \geq 2\delta - T \sup_{S^2 \times [0, T]} |\partial_t u_n| \geq 2\delta - TN_1 \geq \delta$$

if $T \leq \delta/N_1$. ■

D. Convergence of the iteration scheme

The uniform bounds obtained in Proposition 5.16 will now be used to prove that $\{\mathbf{U}_n\}_{n=0}^\infty$ is a Cauchy sequence in an appropriate Banach space. The Banach space we will aim to show convergence in $\Psi_{0,T}^1$, for some appropriate $T > 0$. This will be enough to infer the existence of strong local solutions to the quasispherical constraint system; the higher regularity of these solutions will follow from Proposition 5.16. In what follows we will use the shorthand notation $f(t) = f(\cdot, t)$.

If we define

$$w_{n+1} = u_{n+1} - u_n,$$

then w_{n+1} satisfies the following equation:

$$\dot{w}_{n+1} = \gamma u_n^2 \Delta w_{n+1} + \gamma w_n \bar{w}_n \Delta u_n + F_n - F_{n-1}, \quad (5.39)$$

where $\bar{w}_n = u_n + u_{n-1}$ and where F_n is given by (5.26).

Estimating the $H^3(S^2)$ -norm of w_{n+1} we have

$$\frac{1}{2} \partial_t \|\nabla^3 w_{n+1}\|_{L^2(S^2)}^2 = \oint \nabla^3 w_{n+1} \nabla^3 \dot{w}_{n+1} d\Omega^2 = \oint \nabla^3 w_{n+1} \nabla^3 (\gamma u_n^2 \Delta w_{n+1} + \gamma w_n \bar{w}_n \Delta u_n + F_n - F_{n-1}) d\Omega^2.$$

It is apparent upon inspection of this expression, that the highest order derivatives are fourth order (integrating by parts wherever necessary). Using the parabolic Sobolev imbedding (Ref. 14, Lemma 3.3.5), Proposition 5.16 gives us pointwise bounds for all such derivatives, and it is an easy matter to obtain an expression of the form

$$\partial_t \|\nabla^3 w_{n+1}(t)\|_{L^2(S^2)}^2 \leq C_1 \|w_{n+1}(t)\|_{H^3(S^2)}^2 + C_2 \|w_n(t)\|_{H^3(S^2)}^2.$$

This expression should only be considered true if $t \in [0, T^*]$. The constants C_1 and C_2 depend on the prescribable fields and N .

Estimating $\partial_t \|\nabla^2 w_{n+1}(t)\|_{L^2(S^2)}^2$, $\partial_t \|\nabla w_{n+1}(t)\|_{L^2(S^2)}^2$, and $\partial_t \|w_{n+1}(t)\|_{L^2(S^2)}^2$ in the same way, we find

$$\partial_t \|w_{n+1}(t)\|_{H^3(S^2)}^2 \leq C'_1 (\|w_{n+1}(t)\|_{H^3(S^2)}^2 + \|w_n(t)\|_{H^3(S^2)}^2), \quad (5.40)$$

for all $t \in [0, T^*]$.

To obtain a similar sort of control over the other fields let us also define the following differences:

$$v_{n+1} = \mu_{n+1} - \mu_n,$$

$$q_{n+1} = k_{n+1} - k_n,$$

$$\chi_{n+1} = \hat{\eta}_{n+1} - \hat{\eta}_n.$$

Then since the iterative system defining these quantities is linear it is a simple matter to show that v_{n+1} , q_{n+1} , and χ_{n+1} satisfy the following equations:

$$\begin{aligned} \dot{v}_{n+1} = & \operatorname{div}(u_{n+1} q_{n+1}) + \operatorname{div}(w_{n+1} k_n) + \operatorname{div}(w_{n+1} \sigma) - (1 + \rho) v_{n+1} - w_n u_n^{-1} u_{n-1}^{-1} \hat{\eta}_n^{AB} \nabla_{(B} \beta_A) + u_n^{-1} \chi_n \nabla_{(B} \beta_A) \\ & - 8\pi T_{03} e^t w_{n+1}, \end{aligned} \quad (5.41)$$

$$\begin{aligned} \dot{q}_{n+1} = & -\mathcal{P}_1 \{ \mu_{n+1} \delta w_{n+1} + v_{n+1} \delta u_n + \frac{1}{2} \delta(u_{n+1} v_n) + \frac{1}{2} \delta(w_{n+1} \mu_{n-1}) - \delta(w_{n+1} H) + (1 + \rho) q_n + \bar{q}_n \delta \beta \\ & + q_n \delta \bar{\beta} + 8\pi T_0 e^t w_{n+1} \}, \end{aligned} \quad (5.42)$$

$$\begin{aligned} \bar{\delta} \chi_{n+1} = & (1 - \mathcal{P}_1) \{ v_{n+1} \delta u_{n+1} + \mu_n \delta w_{n+1} + \frac{1}{2} \delta(u_{n+1} v_{n+1}) + \frac{1}{2} \delta(u_n v_{n+1}) - \delta(w_{n+1} H) + (1 + \rho) q_{n+1} \\ & + \bar{q}_{n+1} \delta \beta + q_{n+1} \delta \bar{\beta} + 8\pi T_0 e^t w_{n+1} \}. \end{aligned} \quad (5.43)$$

Focusing on (5.41), we may again use Proposition 5.16 to infer pointwise control over spatial derivatives of up to third order. Estimating the $H^2(S^2)$ -norm of $v_{n+1}(t)$, we then have

$$\begin{aligned} \partial_t \|\nabla^2 v_{n+1}(t)\|_{L^2(S^2)}^2 &= 2 \oint \nabla^2 v_{n+1}(t) \nabla^2 \dot{v}_{n+1}(t) d\Omega^2 \\ &\leq C_1 \|v_{n+1}(t)\|_{H^2(S^2)}^2 + C_2 \|w_{n+1}(t)\|_{H^3(S^2)}^2 \\ &\quad + C_3 \|q_{n+1}(t)\|_{C^3}^2 + C_4 \|\chi_n(t)\|_{H^2(S^2)}^2 + C_5 \|w_n(t)\|_{H^3(S^2)}^2, \end{aligned}$$

for all $t \in [0, T^*]$. The constants $C_1, \dots, C_5$ depend only on the prescribable fields and N . Similar estimates for $\partial_t \|\nabla v_{n+1}(t)\|_{L^2(S^2)}^2$ and $\partial_t \|v_{n+1}(t)\|_{L^2(S^2)}^2$ combine to give

$$\partial_t \|v_{n+1}(t)\|_{H^2(S^2)}^2 \leq C'_2 \left(\|v_{n+1}(t)\|_{H^2(S^2)}^2 + \|w_{n+1}(t)\|_{H^3(S^2)}^2 + \|q_{n+1}(t)\|_{C^3}^2 + \|\chi_n(t)\|_{H^2(S^2)}^2 + \|w_n(t)\|_{H^3(S^2)}^2 \right), \quad (5.44)$$

for all $t \in [0, T^*]$. In a similar fashion we may obtain

$$\partial_t \|q_{n+1}(t)\|_{C^3}^2 \leq C'_3 \left(\|q_{n+1}(t)\|_{C^3}^2 + \|v_{n+1}(t)\|_{H^2(S^2)}^2 + \|w_{n+1}(t)\|_{H^3(S^2)}^2 + \|q_n(t)\|_{C^3}^2 + \|v_n(t)\|_{H^2(S^2)}^2 \right), \quad (5.45)$$

for all $t \in [0, T^*]$, from (5.42).

Adding (5.40), (5.44), and (5.45) and integrating via Gronwall's inequality, we find

$$\begin{aligned} \|w_{n+1}(t)\|_{H^3(S^2)}^2 + \|v_{n+1}(t)\|_{H^2(S^2)}^2 + \|q_{n+1}(t)\|_{C^3}^2 &\leq C \int_0^t \|w_n(s)\|_{H^3(S^2)}^2 + \|v_n(s)\|_{H^2(S^2)}^2 + \|q_n(s)\|_{C^3}^2 \\ &\quad + \|\chi_n(s)\|_{H^2(S^2)}^2 \, ds, \end{aligned} \quad (5.46)$$

for all $t < T^*$.

Finally from the elliptic estimate (5.22) and (5.43) we have

$$\|\chi_{n+1}\|_{H^2(S^2)}^2 \leq C_4 \left(\|w_{n+1}\|_{H^3(S^2)}^2 + \|v_{n+1}\|_{H^2(S^2)}^2 + \|q_{n+1}\|_{C^3}^2 \right). \quad (5.47)$$

For convenience we define

$$\mathcal{H}^j := H^{j+1}(S^2) \times H^j(S^2) \times C^3 \times H^j(S^2),$$

so that upon combining (5.46) and (5.47) and integrating from 0 to $T < T^*$, we have

$$\int_0^T \|(w_{n+1}, v_{n+1}, q_{n+1}, \chi_{n+1})\|_{\mathcal{H}^2}^2 \, dt \leq CT \int_0^T \|(w_n, v_n, q_n, \chi_n)\|_{\mathcal{H}^2}^2 \, dt. \quad (5.48)$$

The constant C depends on the prescribable fields and the number N encountered in Proposition 5.16.

Returning to Eqs. (5.39) and (5.41)–(5.43) we may use (5.48) to obtain

$$\int_0^T \|(\dot{w}_{n+1}, \dot{v}_{n+1}, \dot{q}_{n+1}, \dot{\chi}_{n+1})\|_{\mathcal{H}^0}^2 \, dt \leq CT \int_0^T \|(w_n, v_n, q_n, \chi_n)\|_{\mathcal{H}^2}^2 \, dt,$$

with the constant C depending on the same quantities as the constant in (5.48).

Let $\delta \mathbf{U}_{n+1} = \mathbf{U}_{n+1} - \mathbf{U}_n$ denote the difference between two successive iterates, then choosing $T < 1/C$ we have proven the following result.

Proposition 5.18: Let the initial data satisfy the hypotheses of Proposition 5.16. There exists a $T_ > 0$, depending on the prescribable fields and the initial data, such that*

$$\|\delta \mathbf{U}_{n+1}\|_{\Psi_{0,T_*}^1} \leq \alpha \|\delta \mathbf{U}_n\|_{\Psi_{0,T_*}^1},$$

with $\alpha < 1$.

Proposition 5.18 asserts that the iteration scheme defined by (5.4)–(5.7), when considered over $S^2 \times [0, T_*]$, defines a mapping $\Gamma: \mathbf{U}_n \mapsto \mathbf{U}_{n+1}$ which is a contraction with respect to the Ψ_{0,T_*}^1 norm. This fact gives us the following corollary.

Corollary 5.19: Let the initial data satisfy the hypotheses of Proposition 5.16. There exists a $T_ > 0$, depending on the prescribable fields and the initial data, such that the sequence $\{\mathbf{U}_n\}_{n=0}^\infty$ is a Cauchy sequence with respect to the Ψ_{0,T_*}^1 norm.*

Proof: From Proposition 5.18 it is clear, upon iterating the result, that

$$\|\delta \mathbf{U}_{n+1}\|_{\Psi_{0,T_*}^1} \leq \alpha^n \|\delta \mathbf{U}_1\|_{\Psi_{0,T_*}^1}.$$

Since $\alpha < 1$ it is obvious that $\|\delta \mathbf{U}_{n+1}\|_{\Psi_{0,T_*}^1} \rightarrow 0$ as $n \rightarrow \infty$. Clearly this means that $\{\mathbf{U}_n\}_{n=0}^\infty$ is a Cauchy sequence in Ψ_{0,T_*}^1 . ■

Theorem 5.20: *Let the prescribable fields $\beta, H, T_{0\alpha}, \sigma \in C^\infty(S^2 \times \mathbb{R}^+)$ and let the initial data satisfy $u_0 \in H^{2m+1}(S^2)$, $\mu_0 \in H^{2m}(S^2)$, and $k_0 \in \mathbb{C}^3$, with $m > m_0$. There exists a $T > 0$ depending on the prescribable fields and the initial data such that the system of equations (4.8)–(4.11) with the above data, has a unique solution*

$$\mathbf{U} = (u, \mu, k, \hat{\eta}) \in \Psi_{0,T}^m.$$

Proof: Given the hypotheses of the theorem, Corollary 5.19 applies and so there exists a $T > 0$ such that the sequence $\{\mathbf{U}_n\}_{n=0}^\infty$ is Cauchy in $\Psi_{0,T}^1$. Now $\Psi_{0,T}^1$ is a complete space, and so $\mathbf{U}_n$ converges to $\mathbf{U} \in \Psi_{0,T}^1$. The limit $\mathbf{U}$ is the unique fixed point of the iteration and so clearly uniquely satisfies the system (4.8)–(4.11), with the given data. We note that since $\mathbf{U} \in \Psi_{0,T}^1$, $\mathbf{U}$ is regular enough to satisfy the system in the strong sense. Moreover, Proposition 5.16 states that $\|\mathbf{U}_n\|_{\Psi_{0,T}^m} \leq N$ and so there is a subsequence $\{\mathbf{U}_{n_j}\}_{j=0}^\infty \subset \{\mathbf{U}_n\}_{n=0}^\infty$ and $\mathbf{V} \in \Psi_{0,T}^m$ such that $\mathbf{U}_{n_j} \rightarrow \mathbf{V}$ in $\Psi_{0,T}^m$. However, since the fixed point $\mathbf{U}$ is unique, we conclude that $\mathbf{U} = \mathbf{V} \in \Psi_{0,T}^m$. ■

This last result gives us a partial solution to Problem 1b. It says that the QSECS with prescribed mean curvature is uniquely solvable at least on some finite interval of time. We will now derive a similar result concerning Problem 2b, the QSECS with prescribed polar curvature.

E. Prescribing the polar curvature

In Sec. III we saw that we could ascribe a potentially interesting geometric nature to our space–time initial data if we were to allow for the possibility of prescribing the polar curvature. Our aim in this section then, is to prove a theorem analogous to Theorem 5.20, where now μ is taken as prescribed instead of H . Our method will again be to look at an associated linear iterative system of partial differential equations, and infer the convergence of this system to a unique fixed point. To this end we will consider the following iterative system of equations:

$$\dot{u}_{n+1} = \hat{\gamma} u_n^2 \Delta u_{n+1} - \hat{\gamma} u_n^3 \left(1 - \frac{1}{2} e^{2t} (16\pi T_{00} + 2\|\kappa_n\|^2 + \frac{3}{2}\mu^2 - 2\mu H_n) \right) + \frac{1}{2} (1 + \hat{\gamma} e^{2t} \|\hat{\eta}_n\|^2 + \gamma B) u_n, \quad (5.49)$$

$$\dot{k}_{n+1} = -\mathcal{P}_1 \left\{ \mu \delta u_{n+1} + \frac{1}{2} \delta(\mu u_{n+1}) - \delta(u_{n+1} H_n) + (1 + \rho) \kappa_{n+1} + \bar{\kappa}_n \delta \beta + \kappa_n \delta \bar{\beta} + 8\pi e^t u_{n+1} T_0 \right\}, \quad (5.50)$$

$$-\bar{\delta} \hat{\eta}_{n+1} = \dot{\sigma} + (1 - \mathcal{P}_1) \left\{ \mu \delta u_{n+1} + \frac{1}{2} \delta(\mu u_{n+1}) - \delta(u_{n+1} H_n) + (1 + \rho) \kappa_{n+1} + \bar{\kappa}_{n+1} \delta \beta + \kappa_{n+1} \delta \bar{\beta} + 8\pi e^t u_{n+1} T_0 \right\}, \quad (5.51)$$

$$\rho H_{n+1} = \dot{\mu} - \text{div}(u_{n+1} \kappa_{n+1}) + (1 + \rho) \mu + u_{n+1}^{-1} \hat{\eta}_{n+1}^{AB} \nabla_{(\beta_A)} + 8\pi T_{03} u_{n+1} e^t. \quad (5.52)$$

Informally, we implement the scheme as follows.

Define the zeroth iterate $(u, \hat{\eta}, k, H)_0$ by first defining u_0 and k_0 as the time constant extensions of the initial data u_0 and k_0 , respectively. We may then use Eq. (5.50) with $n = -1$, to define H_{-1} . Note that this is possible since (5.50) may be considered as an algebraic equation for H_n since $\mathcal{P}_1 \delta$ is just a multiplier. This is the only time we will consider (5.50) as an equation for H_n ; from now on we will consider it as an equation for k_n . Moreover, we may dispense with H_{-1} after we substitute it into (5.51) with $n = -1$, and we define $\hat{\eta}_0$ as the solution of the resulting equation. With u_0, k_0 , and $\hat{\eta}_0$ all defined, we may take $n = -1$ and define H_0 as the solution of the algebraic equation (5.52).

If we take $u_0 \in H^{2m+1}(S^2)$ and $k_0 \in \mathbb{C}^3$ then it is clear that $(u, \hat{\eta}, k, H)_0 \in \Psi_{0,T}^m$.

Using Theorem 5.1, it is an easy matter to see that if $(u, \hat{\eta}, k, H)_n \in \Psi_{0,T}^m$ then $u_{n+1} \in P_{0,T}^{m+1/2}$ exists as the solution to (5.49) with data u_0 , for any $T \in [0, \infty)$.

Moving on to (5.50), we may pose this equation equivalently as the following ordinary differential equation over $\mathbb{C}^3$:

$$\dot{k}_{n+1} = Ak_{n+1} + B_n, \quad (5.53)$$

where now

$$Ak = -\mathcal{P}_1\{(1 + \rho)k\} \quad (5.54)$$

and

$$B_n = -\mathcal{P}_1\left\{\mu\partial u_{n+1} + \frac{1}{2}\partial(\mu u_{n+1}) - \partial(u_{n+1}H_n) + (1 + \rho)\sigma + \bar{\kappa}_n\partial\beta + \kappa_n\partial\bar{\beta} + 8\pi e^t u_{n+1}T_0\right\}.$$

The following result, which is analogous to Lemma 5.2, is a simple consequence of the definition (5.54)

Lemma 5.21: Let $\beta \in \Gamma^\infty(TS^2 \times [0, \infty))$. The operator $A: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ defined by (5.54) is a bounded linear operator.

Proof: Linearity is obvious from the definition as is the fact that

$$\|Ak\|_{\mathbb{C}^3}^2 \leq C_\rho \|k\|_{\mathbb{C}^3}^2.$$

We may use Lemma 5.3, substituting $\mathbb{C}^3$ for E^j , to infer the unique existence of $k_{n+1} \in C^1([0, T]; \mathbb{C}^3)$, the solution of (5.53) subject to the initial value $k_0 \in \mathbb{C}^3$, provided that $u_{n+1} \in P_{0,T}^{m+1/2}$, $k_n \in H^m([0, T]; \mathbb{C}^3)$, and $H_n \in P_{0,T}^m$. Similarly we may use Lemma 5.4 to obtain the estimate

$$\|k_{n+1}(t)\|_{\mathbb{C}^3} e^{-C_\rho t} \leq \|k_0\|_{\mathbb{C}^3} + \int_0^t \|B_n(s)\|_{\mathbb{C}^3} ds,$$

for all $t \in [0, T]$.

It is an easy matter to amend the proof of Proposition 5.9 to give the following result which establishes the higher regularity of solutions to (5.53).

Lemma 5.22: Let $k_0 \in \mathbb{C}^3$ and suppose $u_{n+1} \in P_{0,T}^{m+1/2}$ and $(H_n, k_n) \in \mathcal{O}_{0,T}^m$. Then k_{n+1} , the solution of (5.53) satisfies

$$\|k_{n+1}\|_{H^m([0, T]; \mathbb{C}^3)}^2 \leq TC \left(1 + \|B_n\|_{H^m([0, T]; \mathbb{C}^3)}^2\right).$$

The constant C depends on β , $\|u_0\|_{H^{2m}(S^2)}$ and $\|k_0\|_{\mathbb{C}^3}$.

Proof: The result is just Proposition 5.9 with k_{n+1} replacing $\mathcal{M}_{n+1}$ and the much simpler equation (5.53) replacing (5.9). Note in particular that there is no dependence of the constant on u_n , only β . Other than these minor details, the proof is identical. ■

In light of this last result we see that if $u_0 \in H^{2m}(S^2)$, $k_0 \in \mathbb{C}^3$, $u_{n+1} \in P_{0,T}^{m+1/2}$, and $(H_n, k_n) \in \mathcal{O}_{0,T}^m$ then k_{n+1} exists and belongs to $H^m([0, T]; \mathbb{C}^3)$.

As was the case with (5.7), (5.51) is solvable by construction, with the higher regularity of solutions being governed by Corollary 5.12. The equation for H_{n+1} , (5.52), is algebraic and so it is easy to see that $\hat{\eta}_{n+1}$, $H_{n+1} \in P_{0,T}^m$ provided that $u_{n+1} \in P_{0,T}^{m+1/2}$ and $k_{n+1} \in H^m([0, T]; \mathbb{C}^3)$.

For brevity we introduce the following notation:

$$\tilde{U}_n = (u, \hat{\eta}, k, H)_n.$$

Despite being rather heuristic in nature, the foregoing discussion shows how the results proven in the preceding section can be amended to give analogous results concerning the iteration scheme

presently under consideration. In particular, we have the following analog of Theorem 5.13.

Theorem 5.23: *Let $m \geq m_0$ and let $u_0 \in H^{2m+1}(S^2)$ and $k_0 \in \mathbb{C}^3$. There exists a nonincreasing, but possibly decreasing sequence $\{T_n\}_{n=1}^\infty$ with $T_n \geq 0$, and a sequence of iterates $\tilde{U}_n \in \Psi_{0,T_n}^m$ which satisfy (5.4)–(5.7).*

Proposition 5.16 also has an analogous result concerning the iteration scheme presently under consideration. In fact, using the same ideas as those found in the proof of Proposition 5.16 it is a fairly simple matter to obtain.

Proposition 5.24: *Let $m \geq m_0$ and suppose $u_0 \in H^{2m+1}(S^2)$ and $k_0 \in \mathbb{C}^3$. There exists $T^* > 0$ and $\tilde{N} > 0$, depending only on m , the initial data and the prescribable fields, such that*

$$\|\tilde{U}_n\|_{\Psi_{0,T^*}^m} \leq \tilde{N}$$

for all $n \in \mathbb{N}$.

As in Theorem 5.17, we can use Proposition 5.24 to deduce that the proposed iteration scheme (5.49)–(5.52) is consistently defined and produces a sequence of iterates $\{\tilde{U}_n\}_{n=0}^\infty \subset \Psi_{0,T^*}^m$. Moreover, results analogous to Proposition 5.18 and Corollary 5.19 are obtained with practically identical proofs and the counterpart to Theorem 5.20 follows easily.

Theorem 5.25: *Let the prescribable fields $\beta, \mu, T_{0\alpha}, \sigma \in C^\infty(S^2 \times \mathbb{R}^+)$ and let the initial data satisfy $u_0 \in H^{2m+1}(S^2)$ and $k_0 \in \mathbb{C}^3$, with $m \geq m_0$. There exists a $T > 0$ depending on the prescribable fields and the initial data such that the system of equations (4.8)–(4.11), with the above data, has a unique solution*

$$\tilde{U} = (u, \hat{\eta}, k, H) \in \Psi_{0,T}^m.$$

This gives us the sought after result concerning Problem 2b.

It is clear that by construction we may transform the solutions given by Theorems 5.20 and 5.25 under the inverse of (4.1) and thereby construct quasispherical initial data which locally satisfies the Einstein constraint equations.^{7–10}

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