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The dilute A_4 model, the E_7 mass spectrum and the tricritical Ising model^{a)}

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The exact perturbation approach is used to derive the (seven) elementary correlation lengths and related mass gaps of the two-dimensional dilute A_4 lattice model in regime 2⁻ from the Bethe Ansatz solution. This model provides a realization of the integrable $\phi_{(1,2)}$ perturbation of the $c = \frac{7}{10}$ conformal field theory, which is known to describe the off-critical thermal behavior of the tricritical Ising model. The E_7 masses predicted from purely elastic scattering theory follow in the approach to criticality. Universal amplitudes for the tricritical Ising model are calculated. © 2002 American Institute of Physics. [DOI: 10.1063/1.1465515]

I. INTRODUCTION

The deep relationship between conformal field theory and criticality has provided a wealth of detailed information on phase transitions and critical phenomena. Moreover, *perturbed* conformal field theory provides a description of the *approach* to criticality in certain models.¹ One of the most striking examples is the $\phi_{(1,2)}$ perturbation of the minimal unitary conformal field theory $\mathcal{M}_{3,4}$ which is known to describe the scaling limit of the two-dimensional Ising model at $T=T_c$ in a magnetic field. In particular, Zamolodchikov's construction of nontrivial local integrals of motion and thus an integrable quantum field theory led to the remarkable prediction of eight fundamental mass ratios for the magnetic Ising model.² The masses coincide with the components of the Perron–Frobenius vector of the Cartan matrix of the Lie algebra E_8 .

In another development, the exactly solvable dilute A_3 lattice model was discovered³ and (in regime 2 of its four regimes) seen to be in the same universality class as the magnetic Ising model. Most importantly the dilute A_L model^{3,4} admits an off-critical extension in which the Boltzmann weights are parametrized in terms of elliptic theta functions.³ In the dilute A_3 model the elliptic nome plays the role of magnetic field. Its hidden E_8 structure has been revealed by a number of studies.^{5–13} The masses, obtained from the eigenspectrum, may be summarized by the formula^{11,13}

$$m_j \sim \sum_a \sin\left(\frac{a\pi}{g}\right), \quad (1)$$

where index j labels the eight particles, $g=30$ is the Coxeter number for E_8 and the set of allowed a values is given in Table I.

In addition to the correspondence between the dilute A_3 model and E_8 , there are similar correspondences between the dilute A_4 model and E_7 , and the dilute A_6 model and E_6 . In regime 2 these models are lattice realizations of the $\phi_{(1,2)}$ perturbation of the $\mathcal{M}_{4,5}$ and $\mathcal{M}_{6,7}$ minimal

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TABLE I. The integers appearing in (1) and (3) for $L=3$.

j	a
1	1, 11
2	7, 13
3	2, 10, 12
4	6, 10, 14
5	3, 9, 11, 13
6	6, 8, 12, 14
7	4, 8, 10, 12, 14
8	5, 7, 9, 11, 13, 15

unitary conformal field theories, respectively, known to have connection to the other exceptional Lie algebras.¹⁴ Some E-type structures have been observed for these dilute A models.^{15,16}

Based on the results for the eigenspectrum of the dilute A_3 model¹³ and general inversion relations, we proposed^{17,18} that, in the thermodynamic limit and in the appropriate regime, the row transfer matrix eigenvalue excitations

$$r_j(w) = \lim_{N \rightarrow \infty} \frac{\Lambda_j(w)}{\Lambda_0(w)} \quad (2)$$

of the dilute A_3 , A_4 and A_6 models are given by the following general expression.

Proposition: The excitation spectrum of the dilute A_3 , A_4 and A_6 models in regime 2 is given by

$$r_j(w) = \prod_a w \frac{E(-x^{6sa/g}/w, x^{12s}) E(-x^{6s(g-a)/g}/w, x^{12s})}{E(-x^{6sa/g}w, x^{12s}) E(-x^{6s(g-a)/g}w, x^{12s})}. \quad (3)$$

Here the elliptic nome is $p = e^{-\epsilon}$, $w = e^{-2\pi u/\epsilon}$, and $x = e^{-\pi^2/r\epsilon}$. Regime 2 is specified by the range of the spectral parameter: $0 < u < 3\lambda$, and the value of the crossing parameter: $\lambda = \pi s/r$ where $s = L+2$ and $r = 4(L+1)$. For the dilute A_4 model the E_7 Coxeter number is $g = 18$, while for the A_6 model the E_6 Coxeter number is $g = 12$. The standard (conjugate modulus) elliptic function is defined by

$$E(z, q) = \prod_{n=1}^{\infty} (1 - q^{n-1}z)(1 - q^n/z)(1 - q^n).$$

The numbers a appearing in (3) are given in Tables I–III. The integers in these tables have appeared in other contexts in relation to the E-algebras.^{19,20}

In this article we explicitly derive the elementary excitation spectrum of the dilute A_4 model, thereby confirming our Proposition in this case. The result (3) leads to the inverse correlation lengths and mass gaps. Our input to these calculations are the string solutions to the Bethe equations found by Grimm and Nienhuis.^{9,10,21} As discussed later in Sec. IV, our results are

TABLE II. The integers appearing in (3) for $L=4$.

j	a
1	6
2	1, 7
3	4, 8
4	5, 7
5	2, 6, 8
6	4, 6, 8
7	3, 5, 7, 9

TABLE III. The integers appearing in (3) for $L=6$.

j	a
1, $\bar{1}$	4
2	1, 5
3, $\bar{3}$	3, 5
4	2, 4, 6

applicable to the tricritical Ising model which is in the same universality class. In particular, the elliptic nome appearing in the dilute A_4 weights in regime 2 corresponds to the leading thermal off-critical perturbation in the tricritical Ising model. This perturbation is identified with $\phi_{(1,2)}$ ²² and has been shown to exhibit E_7 structures.^{14,23} We are able to obtain exact results for some universal amplitudes of the tricritical Ising model. These results are in agreement with those found recently by other means.^{24,25}

The outline of the article is as follows. The dilute A_L lattice model is defined along with the corresponding Bethe equations in Sec. II. The bulk free energy and the eigenvalue expressions in regime 2 for $L=4$ associated with the seven E_7 masses are derived via the exact perturbation approach in Sec. III (continued in the Appendix). The article concludes in Sec. IV with a discussion of the results and their relevance to universal behavior in the tricritical Ising model.

II. THE DILUTE A_4 MODEL

We here give a short summary of facts about the dilute A_L models^{26,13} which are pertinent to our calculations.

The dilute A_L model is an exactly solvable, L -state restricted solid-on-solid model defined on the square lattice. Its adjacency diagram is the Dynkin diagram of A_L with the additional possibility that a state may be adjacent to itself on the lattice. The model is solvable in four off-critical regimes, with the elliptic nome p of its Boltzmann weights taking the model off-critical. At criticality, the dilute A_L model can be constructed^{3,4} from the dilute $O(n)$ loop model.^{27,28} In regime 2 of the model the central charge is

$$c = 1 - \frac{6}{L(L+1)}.$$

In the majority of exactly solved models the elliptic nome plays the role of temperature.²⁹ In the dilute A_L model the interpretation of the elliptic nome differs according to whether L is even or odd. For L odd the elliptic nome plays the role of a magnetic field,³ and $p>0$ and $p<0$ are related by simple label reversal of the heights. For L even the nome plays a thermal role, and the behavior of the model depends on whether $p>0$ (regime 2⁺) or $p<0$ (regime 2⁻). More specifically, it was shown²⁶ that in regime 2 the nome corresponds to perturbation of the $\mathcal{M}_{L,L+1}$ minimal unitary conformal field theories by the operator $\phi_{(1,2)}$.

Using the conjugate variables introduced after (3), and setting $w_j = e^{-2\pi u_j/\epsilon}$, the eigenvalues of the row transfer matrix of the dilute A models (for a periodic strip of width N where for convenience N has been taken as even) can be written⁵

$$\begin{aligned} \Lambda(w) = & \omega \left[\frac{E(x^{4s}/w, x^{2r})E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r})E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^N w_j^{1-2s/r} \frac{E(x^{2s}w/w_j, x^{2r})}{E(x^{2s}w_j/w, x^{2r})} \\ & + \left[\frac{x^{2s}}{w} \frac{E(w, x^{2r})E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r})E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^N w_j \frac{E(w/w_j, x^{2r})E(x^{6s}w_j/w, x^{2r})}{E(x^{2s}w_j/w, x^{2r})E(x^{4s}w_j/w, x^{2r})} \\ & + \omega^{-1} \left[x^{2s} \frac{E(w, x^{2r})E(x^{2s}/w, x^{2r})}{E(x^{4s}, x^{2r})E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^N w_j^{2s/r} \frac{E(x^{8s}w_j/w, x^{2r})}{E(x^{4s}w_j/w, x^{2r})}, \end{aligned} \quad (4)$$

TABLE IV. String positions u_j and corresponding eigenvalue bands for the seven elementary mass excitations m_i of the dilute A_4 model in regime 2^- .²¹ The strings are in units of $\pi/20$.

i	String positions	Band
1	$\pm 2, 10$	w
2	± 7	w^2
3	$\pm 6, 10$	w^2
4	$\pm 3, \pm 9$	w^2
5	$\pm 6, \pm 8$	w^3
6	$\pm 4, \pm 8, 10$	w^3
7	$\pm 5, \pm 7, \pm 9$	w^4

where $\omega = \exp(i\pi\ell/(L+1))$ for $\ell = 1, \dots, L$, and $s = L+2$ and $r = 4(L+1)$ in regime 2. The Bethe equations which give the N roots u_j have the form

$$\omega \left[w_j \frac{E(x^{2s}/w_j, x^{2r})}{E(x^{2s}w_j, x^{2r})} \right]^N = - \prod_{k=1}^N w_k^{2s/r} \frac{E(x^{2s}w_j/w_k, x^{2r})E(x^{4s}w_k/w_j, x^{2r})}{E(x^{2s}w_k/w_j, x^{2r})E(x^{4s}w_j/w_k, x^{2r})}. \quad (5)$$

In the limit $|p| \rightarrow 1$ with u/ϵ fixed, or equivalently $x \rightarrow 0$, the excitations in the eigenspectrum $r_j(w)$, defined in (2), break up into a number of distinct bands labeled by integer powers of w . Numerical investigations of the eigenspectrum^{5,9,10,21,17} have revealed eight and seven thermodynamically significant excitations for $L=3$ and $L=4$, respectively, and provided the data in Table IV.

We previously^{12,13} applied the exact perturbation approach initiated by Baxter³⁰ to calculate the excitations in the eigenspectrum for $L=3$. This involved perturbing away from the strong magnetic field limit at $p \rightarrow 1$; for $L=4$ this limit corresponds to moving far away from the critical temperature. The calculations follow.

III. MASS SPECTRUM

A. Preliminaries

To apply the perturbation technique³⁰ to find the form of the excitations (4), the string structure of the Bethe ansatz roots (5) is required input. The groundstate roots all have u_j pure imaginary, so that $w_j = e^{-2\pi u_j/\epsilon} = a_j$ for $j=1, \dots, N$ with $|a_j|=1$; in this sense they all live on a unit circle. For each excitation i , certain roots acquire a real part $m\pi/20$, as shown in Table IV. (If there are n_i such roots, one says there is an n_i -string associated with the excitation.) For these roots $w_j = b_j x^{2m}$, so that the string entries can be thought of as living on circles of radius x^{2m} with phase b_j , while the other $N - n_i$ roots again lie on the unit circle.

The process of finding the excitations involves using the Bethe equations (5) to set up recurrence relations for auxiliary functions of the unknown roots a_j . As the roots only enter the eigenvalue expression (4) through the auxiliary functions, it just remains to solve the recurrence relations by iteration and to simplify the resulting expressions. The largest eigenvalue Λ_0 , relative to which excitations are measured, was calculated previously in this way¹³ for all L .

The relationship between the excitations (2), the correlation lengths ξ_j and the mass spectrum m_j of the associated field theory is

$$\xi_j^{-1} = -\log r_j = m_j, \quad (6)$$

where we take the isotropic value $u = 3\lambda/2$.

It is convenient to use the notation for products:

$$(z; p_1, \dots, p_k)_\infty = \prod_{n_1, \dots, n_k=0}^{\infty} (1 - p_1^{n_1} \cdots p_k^{n_k} z),$$

$$(z_1, \dots, z_m; p_1, \dots, p_k)_\infty = \prod_{j=1}^m (z_j; p_1, \dots, p_k)_\infty,$$

which satisfy many identities, the ones used repeatedly in what follows being

$$\frac{(z; p)_\infty}{(zp; p)_\infty} = (1 - z),$$

$$\frac{(z; p, q)_\infty}{(zp; p, q)_\infty} = (z; q)_\infty,$$

$$\frac{(zq/p; p, q)_\infty}{(z; p, q)_\infty} = \frac{(zq/p; q)_\infty}{(z; p)_\infty}.$$

The standard elliptic function is thus rewritten as

$$E(z, q) = \prod_{n=1}^{\infty} (1 - q^{n-1}z)(1 - q^n/z)(1 - q^n) = (z, q/z, q; q)_\infty. \quad (7)$$

It also proves convenient to use the shorthand notation $\prod_{j=1}^m a_j = A_m$.

For each m_i , if the associated string of excited roots has length n_i , we define the required auxiliary functions of the as-yet-unknown roots to be

$$F_i(w) = \prod_{j=1}^{N-n_i} (w/a_j; x^{2r})_\infty,$$

$$G_i(1/w) = \prod_{j=1}^{N-n_i} (x^{2r}a_j/w; x^{2r})_\infty. \quad (8)$$

In fact, we actually solve for combinations of these:

$$\mathcal{F}_i(w) = F_i(w)/F_i(x^{16}w) = F_i(w)/F_i(x^{2r-4s}w),$$

$$\mathcal{G}_i(1/w) = G_i(1/w)/G_i(1/x^{16}w) = G_i(1/w)/G_i(1/x^{2r-4s}w), \quad (9)$$

for $i=2,4,5,6,7$ (but for $i=1,3$ slightly different definitions are convenient and are given as required).

So far as possible, we write factors and powers which are common to all eigenvalues (or indeed to all the eigenvalues for other A_L models) in terms of the generic r and s to distinguish them from the particular integers which arise from the input strings. Of course, $r=20$ and $s=6$ throughout. Once the particular string form for the roots has been applied, the calculations are straightforward for all masses except m_1 and m_3 . For this reason, we sketch below the details for the first three masses. The other cases follow similar paths to m_2 or indeed to most of the masses for the dilute A_3 model,¹³ so we relegate them to the appendix. We make some comments concerning m_1 , m_3 and m_6 later on.

B. Mass m_1

We begin the perturbation argument with the structure $w_j = a_j$ for $j = 1, \dots, N-3$ with $w_{N-1} = b_1 x^{-4}$, $w_{N-2} = b_2 x^4$ and $w_N = b_3 x^{20}$, so that the string length is $n_1 = 3$. From the Bethe equations (5) for $j = N-2$, $j = N-1$ and $j = N$ in the limit $x \rightarrow 0$ we can show that $b_1 = b_2 = b_3 = b$. The Bethe equation for the other roots $a_k = a$ is then

$$-\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-3} b^3)^{3/5} \frac{a^2}{b^2} \times \frac{E(x^4 b/a) E(x^{24} b/a) E(x^{28} b/a)}{E(x^4 a/b) E(x^{24} a/b) E(x^{28} a/b)} \prod_{j=1}^{N-3} \frac{E(x^{2s} a/a_j) E(x^{4s} a_j/a)}{E(x^{2s} a_j/a) E(x^{4s} a/a_j)}. \quad (10)$$

In the $x \rightarrow 0$ limit this gives the equation

$$a^{N-2} + \frac{1}{\omega} (A_{N-3} b^3)^{3/5} b^2 = 0, \quad (11)$$

which is an equation of order $(N-2)$, so that there is a missing root on the unit circle, a “hole” which we call a_{N-2} . Since this is an equation for the roots, its left hand side must be equivalent to $\prod_{j=1}^{N-2} (a - a_j)$, and equating the constant terms from these two expressions we obtain

$$\frac{1}{\omega} (A_{N-3} b^3)^{3/5} = A_{N-2} b^2 = A_{N-3} a_{N-2} b^2 \quad (12)$$

(which we later apply to prefactors in Λ_1). The Bethe equations for b , taken together in this limit and combined with (12), give

$$\left[\frac{1}{\omega} (A_{N-3} b^3)^{3/5} \right]^3 = -b^6 (A_{N-3})^2 \Rightarrow A_{N-3} (a_{N-2})^3 = -1.$$

We use this, together with the fact that each root a_j , including the hole, must satisfy (11), to show

$$(a_{N-2})^{N-2} = -A_{N-3} a_{N-2} \Rightarrow (a_{N-2})^N = 1. \quad (13)$$

We define the following auxiliary functions of the roots [see (8)]:

$$\mathcal{F}_1(w) = \frac{F_1(w)}{F_1(x^{16}w)} \frac{(x^4 w/b; x^{2r})_\infty}{(x^{12} w/b; x^{2r})_\infty},$$

$$\mathcal{G}_1(1/w) = \frac{G_1(1/w)}{G_1(1/x^{16}w)} \frac{(x^{36} b/w; x^{2r})_\infty}{(x^{24} b/w; x^{2r})_\infty}.$$

They must satisfy recurrence relations arising from (10):

$$\mathcal{F}_1(a) = \left[\frac{(x^{2s} a; x^{2r})_\infty}{(x^{2r-2s} a; x^{2r})_\infty} \right]^N \frac{(x^{24} a/a_{N-2}, x^{28} a/a_{N-2}; x^{2r})_\infty}{(x^{12} a/a_{N-2}, x^{16} a/a_{N-2}; x^{2r})_\infty} \frac{\mathcal{F}_1(x^{2s} a)}{\mathcal{F}_1(x^{4s} a)},$$

$$\mathcal{G}_1(1/a) = \left[\frac{(x^{2r+2s}/a; x^{2r})_\infty}{(x^{6s}/a; x^{2r})_\infty} \right]^N \frac{(x^{36} a_{N-2}/a, x^{40} a_{N-2}/a; x^{2r})_\infty}{(x^{48} a_{N-2}/a, x^{52} a_{N-2}/a; x^{2r})_\infty} \frac{\mathcal{G}_1(x^{2s}/a)}{\mathcal{G}_1(x^{4s}/a)}. \quad (14)$$

Solving these we obtain

$$\begin{aligned}\mathcal{F}_1(a) &= \mathcal{F}_0(a) \frac{(x^{40}a/a_{N-2}; x^{2r})_\infty}{(x^{16}a/a_{N-2}; x^{2r})_\infty} \frac{(x^{36}a/a_{N-2}, x^{48}a/a_{N-2}; x^{12s})_\infty}{(x^{12}a/a_{N-2}, x^{72}a/a_{N-2}; x^{12s})_\infty}, \\ \mathcal{G}_1(1/a) &= \mathcal{G}_0(1/a) \frac{(x^{40}a_{N-2}/a; x^{2r})_\infty}{(x^{64}a_{N-2}/a; x^{2r})_\infty} \frac{(x^{36}a_{N-2}/a, x^{96}a_{N-2}/a; x^{12s})_\infty}{(x^{60}a_{N-2}/a, x^{72}a_{N-2}/a; x^{12s})_\infty}.\end{aligned}\quad (15)$$

Here $\mathcal{F}_0$ and $\mathcal{G}_0$ arise from the square bracketed factors in (14) and give rise to the square bracketed factor in (16). They are related to the groundstate eigenvalue Λ_0 , they are common to the calculation of each mass and we will suppress these factors for $m_2, \dots, m_7$. We now write the eigenvalue expression in terms of the auxiliary functions, the first term being

$$\begin{aligned}\frac{\Lambda_1}{3} &= -\frac{w}{a_{N-2}} \left[\frac{(x^{2r-6s}w, x^{2r-4s}w, x^{4s}/w, x^{6s}/w; x^{2r})_\infty}{(x^{2r-6s}, x^{2r-4s}, x^{4s}, x^{6s}; x^{2r})_\infty} \right]^N \\ &\quad \times \frac{(x^{28}w/a_{N-2}, x^{12}a_{N-2}/w; x^{2r})_\infty}{(x^{12}w/a_{N-2}, x^{28}a_{N-2}/w; x^{2r})_\infty} \mathcal{F}_1(x^{2s}w) \mathcal{G}_1(1/x^{2s}w).\end{aligned}\quad (16)$$

Substituting the solutions (15) gives an expression for the excitation $r_1(w)$ which may be written in elliptic functions (7) as

$$\frac{\Lambda_1}{\Lambda_0} = w \frac{E(-x^{12}/w, x^{12s})E(-x^{48}w, x^{12s})}{E(-x^{12}w, x^{12s})E(-x^{48}/w, x^{12s})}, \quad (17)$$

where we have set $a_{N-2} = -1$. (The other two terms in the eigenvalue always give identical elliptic function expressions to the first, upon simplification.)

The Bethe equations involving b and the ‘‘hole’’ equation, which is (10) with $a = a_{N-2}$, can also be expressed in terms of the auxiliary functions. Application of identities and simplification gives

$$\begin{aligned}E(x^{12}b/a_{N-2}, x^{2r-4s}) &= E(x^{12}a_{N-2}/b, x^{2r-4s}), \\ \left[\frac{E(x^{12}a_{N-2}, x^{12s})E(x^{48}/a_{N-2}, x^{12s})}{E(x^{12}/a_{N-2}, x^{12s})E(x^{48}a_{N-2}, x^{12s})} \right]^N &= (a_{N-2})^N.\end{aligned}$$

Clearly $a_{N-2} = b = -1$ (identified initially from numerical studies) satisfy these conditions; the second reduces to (13) in the $x \rightarrow 0$ limit, and note the similarities with (17).

C. Mass m_2

We begin the perturbation argument with the structure $w_j = a_j$ for $j = 1, \dots, N-2$ with $w_{N-1} = b_1 x^{-14}$ and $w_N = b_2 x^{14}$, so that $n_2 = 2$. From the Bethe equations for $j = N-1$ and $j = N$ we can show that $b_1 = b_2 = b$. The Bethe equation for the other roots $a_k = a$ is then

$$-\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-2}b^2)^{3/5} \frac{a^2}{b^2} \frac{E(x^{10}b/a)E(x^{14}b/a)}{E(x^{10}a/b)E(x^{14}a/b)} \prod_{j=1}^{N-2} \frac{E(x^{2s}a/a_j)E(x^{4s}a_j/a)}{E(x^{2s}a_j/a)E(x^{4s}a/a_j)}. \quad (18)$$

In the $x \rightarrow 0$ limit this gives the equation

$$a^{N-2} + \frac{1}{\omega} (A_{N-2}b^2)^{3/5} / b^2 = 0,$$

which has the same order as the number of unknown roots ($N-2$) so that there is no hole. Equating this with $\prod_{j=1}^{N-2} (a - a_j)$ we obtain

$$\frac{1}{\omega}(A_{N-2}b^2)^{3/5}=A_{N-2}b^2$$

(which we later apply to prefactors in Λ_2). From the other Bethe equations in this limit,

$$\left[\frac{1}{\omega}(A_{N-2}b^2)^{3/5}\right]^2=\frac{(A_{N-2}b^2)^2}{b^{2N}}\Rightarrow b^{2N}=1. \quad (19)$$

Treating the Bethe equation (18) as before gives, in terms of the functions defined in (8) and (9), the recurrences

$$\begin{aligned}\mathcal{F}_2(a) &= \frac{(x^{26}a/b, x^{30}a/b; x^{2r})_\infty}{(x^{10}a/b, x^{14}a/b; x^{2r})_\infty} \frac{\mathcal{F}_2(x^{2s}a)}{\mathcal{F}_2(x^{4s}a)}, \\ \mathcal{G}_2(1/a) &= \frac{(x^{38}b/a, x^{34}b/a; x^{2r})_\infty}{(x^{50}b/a, x^{54}b/a; x^{2r})_\infty} \frac{\mathcal{G}_2(x^{2s}/a)}{\mathcal{G}_2(x^{4s}/a)}.\end{aligned}$$

Solving these we obtain

$$\begin{aligned}\mathcal{F}_2(a) &= \frac{(x^{30}a/b, x^{42}a/b; x^{2r})_\infty}{(x^{14}a/b, x^{26}a/b; x^{2r})_\infty} \frac{(x^{26}a/b, x^{38}a/b, x^{46}a/b, x^{58}a/b; x^{12s})_\infty}{(x^{10}a/b, x^{22}a/b, x^{62}a/b, x^{74}a/b; x^{12s})_\infty}, \\ \mathcal{G}_2(1/a) &= \frac{(x^{38}b/a, x^{50}b/a; x^{2r})_\infty}{(x^{54}b/a, x^{66}b/a; x^{2r})_\infty} \frac{(x^{34}b/a, x^{46}b/a, x^{86}b/a, x^{98}b/a; x^{12s})_\infty}{(x^{50}b/a, x^{62}b/a, x^{70}b/a, x^{82}b/a; x^{12s})_\infty}.\end{aligned}$$

We now substitute these into the eigenvalue expression, the first term of which is

$$\frac{\Lambda_2}{3} = \frac{w^2}{b^2} \frac{(x^{26}w/b, x^{38}w/b, x^{2b}/w, x^{14}b/w; x^{2r})_\infty}{(x^2w/b, x^{14}w/b, x^{26}b/w, x^{38}b/w; x^{2r})_\infty} \mathcal{F}_2(x^{2s}w) \mathcal{G}_2(1/x^{2s}w).$$

This gives an expression for the excitation in elliptic functions (setting $b = -1$):

$$\frac{\Lambda_2}{\Lambda_0} = w^2 \frac{E(-x^2/w, x^{12s})E(-x^{14}/w, x^{12s})E(-x^{38}w, x^{12s})E(-x^{50}w, x^{12s})}{E(-x^2w, x^{12s})E(-x^{14}w, x^{12s})E(-x^{38}/w, x^{12s})E(-x^{50}/w, x^{12s})}. \quad (20)$$

If the product of the six Bethe equations involving b is expressed in terms of the auxiliary functions, the equation for b [generalizing $b^{2N}=1$ seen in the $x \rightarrow 0$ limit in (19)] is clearly satisfied by $b = -1$:

$$\left[\frac{E(x^2b, x^{12s})E(x^{14}b, x^{12s})E(x^{38}/b, x^{12s})E(x^{50}/b, x^{12s})}{E(x^2/b, x^{12s})E(x^{14}/b, x^{12s})E(x^{38}b, x^{12s})E(x^{50}b, x^{12s})}\right]^N = b^{2N}.$$

Compare the pattern of powers of x in this equation with those in (20); this equation has a precise analog for each mass $m_4, \dots, m_7$, which will not be given.

D. Mass m_3

We begin the perturbation argument with the string structure $w_j = a_j$ for $j = 1, \dots, N-3$ with $w_{N-2} = b_1 x^{-12}$, $w_{N-1} = b_2 x^{12}$ and $w_N = b_3 x^{20}$. From the Bethe equations for $j = N-2$ and $j = N-1$ we can show that $b_1 = b_2 = \alpha$, but the Bethe equation for $j = N$ does not link $b_3 = b$ to α in the $x \rightarrow 0$ limit. (This feature was observed also in the $L = 3$ case, for a string of odd length.¹³) The Bethe equation for the other roots $a_k = a$ is then

$$-\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-3} \alpha^2 b)^{3/5} \frac{a^3}{\alpha b^2} \frac{E(x^4 b/a) E(x^8 b/a)}{E(x^4 a/b) E(x^8 a/b)} \\ \times \frac{E(x^{12} \alpha/a) E(x^{16} \alpha/a) E(x^{36} \alpha/a)}{E(x^{12} a/\alpha) E(x^{16} a/\alpha) E(x^{36} a/\alpha)} \prod_{j=1}^{N-3} \frac{E(x^{2s} a/a_j) E(x^{4s} a_j/a)}{E(x^{2s} a_j/a) E(x^{4s} a/a_j)}. \quad (21)$$

In the $x \rightarrow 0$ limit this gives the equation

$$a^{N-3} - \frac{1}{\omega} (A_{N-3} \alpha^2 b)^{3/5} / \alpha b^2 = 0.$$

Equating this as usual with $\prod_{j=1}^{N-3} (a - a_j)$, we obtain

$$\frac{1}{\omega} (A_{N-3} \alpha^2 b)^{3/5} = A_{N-3} \alpha b^2$$

(which we later apply to prefactors in Λ_3). From the other Bethe equations in this limit,

$$\left[\frac{1}{\omega} (A_{N-3} \alpha^2 b)^{3/5} \right]^3 = \frac{(A_{N-3} \alpha b^2)^3}{b^{2N}} \Rightarrow b^{2N} = 1.$$

In this case it is convenient to define

$$\mathcal{F}_3(w) = \frac{F_3(w)}{F_3(x^{16}w)} \frac{(x^{12}w/\alpha; x^{2r})}{(x^4w/\alpha; x^{2r})}, \quad (22) \\ \mathcal{G}_3(1/w) = \frac{G_3(1/w)}{G_3(1/x^{16}w)} \frac{(x^{28}\alpha/w; x^{2r})}{(x^{36}\alpha/w; x^{2r})},$$

because this choice will make it clear that α is a spectator in the solution to the recurrence relation; it does not appear in the eigenvalue expression.

Treating the Bethe equation (21) as before gives the recurrences

$$\mathcal{F}_3(a) = \frac{(x^{32}a/b, x^{36}a/b; x^{2r})_\infty}{(x^4a/b, x^8a/b; x^{2r})_\infty} \frac{\mathcal{F}_3(x^{2s}a)}{\mathcal{F}_3(x^{4s}a)}, \\ \mathcal{G}_3(1/a) = \frac{(x^{56}b/a, x^{60}b/a; x^{2r})_\infty}{(x^{28}b/a, x^{32}b/a; x^{2r})_\infty} \frac{\mathcal{G}_3(x^{2s}/a)}{\mathcal{G}_3(x^{4s}/a)}.$$

Solving these we obtain

$$\mathcal{F}_3(a) = \frac{(x^{36}a/b; x^{2r})_\infty}{(x^{20}a/b; x^{2r})_\infty} \frac{(x^{32}a/b, x^{40}a/b, x^{44}a/b, x^{52}a/b; x^{12s})_\infty}{(x^4a/b, x^8a/b, x^{16}a/b, x^{68}a/b; x^{12s})_\infty}, \\ \mathcal{G}_3(1/a) = \frac{(x^{44}b/a; x^{2r})_\infty}{(x^{60}b/a; x^{2r})_\infty} \frac{(x^{28}b/a, x^{32}b/a, x^{40}b/a, x^{92}b/a; x^{12s})_\infty}{(x^{56}b/a, x^{64}b/a, x^{68}b/a, x^{76}b/a; x^{12s})_\infty}.$$

We now substitute these into the eigenvalue expression, the first term of which is, in terms of the functions (22),

$$\frac{\Lambda_3}{3} = \frac{w^2}{b^2} \frac{(x^{32}w/b, x^8b/w; x^{2r})_\infty}{(x^8w/b, x^{32}b/w; x^{2r})_\infty} \mathcal{F}_3(x^{2s}w) \mathcal{G}_3(1/x^{2s}w).$$

With $b = -1$ this gives the expression in elliptic functions

$$\frac{\Lambda_3}{\Lambda_0} = w^2 \frac{E(-x^8/w, x^{12s})E(-x^{16}/w, x^{12s})E(-x^{44}/w, x^{12s})E(-x^{52}/w, x^{12s})}{E(-x^8/w, x^{12s})E(-x^{16}/w, x^{12s})E(-x^{44}/w, x^{12s})E(-x^{52}/w, x^{12s})}. \quad (23)$$

The Bethe equations involving α and b , also expressed in terms of the auxiliary functions, give

$$E(x^{12}b/\alpha, x^{2r-4s}) = E(x^{12}\alpha/b, x^{2r-4s}),$$

$$\left[\frac{E(x^8b, x^{12s})E(x^{16}b, x^{12s})E(x^{44}/b, x^{12s})E(x^{52}/b, x^{12s})}{E(x^8/b, x^{12s})E(x^{16}/b, x^{12s})E(x^{44}b, x^{12s})E(x^{52}b, x^{12s})} \right]^N = b^{2N}.$$

Notice that $b = -1$ satisfies this second equation, and the (so far missing) link between α and b is provided by the first.

E. Comments on the “odd” strings

To close this rather technical section of the article, we wish to briefly comment on the strings of odd length (see Table IV).

For this model strings of odd length appear for the first, third and sixth masses. In the first case, the odd string of excited roots is accompanied by a “hole” among the roots on the unit circle; it is only this hole a_{N-2} which appears in the eigenvalue expression. In the case of the third mass, the phase b of the string entry $m = 10$ appears alone in the eigenvalue expression, and the other entries of the string seem to have a spectator role. (For the sixth mass, there was nothing special about the calculation.) In the calculations for dilute A_3 (Ref. 13) odd strings were involved for masses 4 and 6, where again the calculation of the associated excitation was less straightforward than for even strings. In one case, both the coefficient of the “odd” entry *and* a hole appeared in the eigenvalue, while the other calculation resembles that of m_3 in this article. In general the string entries come in pairs $\pm m$, except for $m = r/2$, which stands alone if it occurs, due to the period of the original elliptic functions in u_j . This is the only source of strings of odd length; we can only conclude that when such an entry occurs, it in some sense dominates calculations following the exact perturbation technique. For strings of even length, all the excited roots seem to contribute in a more equal fashion to the calculation and to the resulting eigenvalue expression.

IV. DISCUSSION

In this article we have made use of the Bethe Ansatz string solutions found by Grimm and Nienhuis to derive the excitation spectrum of the dilute A_4 model via the exact perturbation approach. Our expressions for the seven thermodynamically significant excitations for the dilute A_4 eigenspectrum in regime 2^- are given in (17), (20), (23), (A2), (A4), (A6) and (A7). In this way we have verified for a second case the Proposition given by (3).

It is perhaps unsatisfying that an elegant closed form expression such as (3) has been confirmed in the A_4 case by relying on numerical data for the strings (Table IV). Indeed, as described for the A_3 case in Ref. 13, and in the detailed study of Ref. 10, tracing the strings from $p = 0$ (criticality) to the position they take in the scaling (massive) limit reveals complicated structure (reported with one difference by two groups of authors^{5,10}). Fortunately, (3) was conjectured^{17,18} on the basis of general properties of the dilute A models and of the E-type algebras, known to be linked by their common connection to the $\phi_{(1,2)}$ perturbation of the minimal unitary series; the (scaling limit) string data used here has not contradicted it, and (admittedly limited) numerical studies agreed with the lower eigenvalues.¹⁷ A forthcoming paper³¹ should shed some new light, from the perspective of Coxeter geometry, on the excitations (3) and, hence, among other things, on the string conjectures to which they are related as demonstrated here in the $L = 4$ case.

Recall that the central charge for dilute A_4 in regime 2 is $c = \frac{7}{10}$. There are several other known manifestations of the $c = \frac{7}{10}$ theory. The Blume–Capel model³² is related to the Blume–Emery–Griffiths model,³³ a classical spin-1 Ising model with lattice Hamiltonian

TABLE V. The four perturbations of the tricritical Ising model, and the objects from statistical mechanics to which they are related in the scaling limit.

Perturbation	Field	Weight	IRF model	$\mathcal{H}_{\text{BEG}}$
Leading magnetic	$\phi_{(2,2)}$	$\frac{3}{80}$	Not integrable	H
Leading thermal	$\phi_{(1,2)}$	$\frac{1}{10}$	Dilute A_4 , regime 2	$1/J$
Subleading magnetic	$\phi_{(2,1)}$	$\frac{7}{16}$	Dilute A_3 , regime 1	H_3
Subleading thermal	$\phi_{(1,3)}$	$\frac{3}{5}$	ABF A_4 , regime III	D

$$\mathcal{H}_{\text{BEG}} = -J \sum_{\langle i,j \rangle} S_i S_j - D \sum_i (1 - S_i^2) - H \sum_i S_i - H_3 \sum_{\langle i,j \rangle} S_i S_j (S_i + S_j), \quad (24)$$

where J is the nearest-neighbor interaction, D is a crystal field, H a magnetic field term, and H_3 is a staggered magnetic field. The phase diagrams of these models exhibit a tricritical point, as had been observed in physical systems.³⁴ The critical exponents, known from renormalization group studies, are related to the Kac table of the $c = \frac{7}{10}$ conformal field theory.¹

After the Ising critical point, the universality class of the tricritical Ising model corresponds to the second simplest unitary conformal field theory in two dimensions. It is also the first of the super-conformal minimal models. It can be perturbed by its four relevant scaling fields, shown in Table V ordered according to the associated conformal weight. The leading magnetic perturbation is believed to be nonintegrable,²² and each of the other three perturbations give integrable quantum field theories. In the scaling limit these can each be associated with a solvable interaction round a face (IRF) model [or to the terms in (24)]. The ABF A_4 model in regime III^{35,36} realizes the subleading thermal perturbation. A lattice realization of the subleading magnetic perturbation is given by the dilute A_3 model in regime 1,³⁷ and the scaling limit of the leading thermal perturbation corresponds to the dilute A_4 model as considered in this article.

The leading thermal perturbation is known to be integrable and massive, the masses being described by E_7 Toda field theory.^{14,23} Numerical results from a finite-size analysis in the spin-chain formulation,³⁸ and from field theory via the truncated conformal space approach²² demonstrated the first few masses.

These are

$$\begin{aligned}
 m_1 &= 1 \quad \text{odd,} \\
 m_2 &= 2 \cos \frac{5\pi}{18} = 1.285\,575\dots \quad \text{even,} \\
 m_3 &= 2 \cos \frac{\pi}{9} = 1.879\,385\dots \quad \text{odd,} \\
 m_4 &= 2 \cos \frac{\pi}{18} = 1.969\,615\dots \quad \text{even,} \\
 m_5 &= 4 \cos \frac{\pi}{18} \cos \frac{5\pi}{18} = 2.532\,088\dots \quad \text{even,} \\
 m_6 &= 4 \cos \frac{\pi}{9} \cos \frac{2\pi}{9} = 2.879\,385\dots \quad \text{odd,} \\
 m_7 &= 4 \cos \frac{\pi}{18} \cos \frac{\pi}{9} = 3.701\,666\dots \quad \text{even.}
 \end{aligned} \quad (25)$$

The mass spectrum can be classified²² into even and odd states [as indicated in (25)] corresponding to the Z_2 symmetry of the affine E_7 Dynkin diagram. Each of the above seven masses appears in the high-temperature phase of the tricritical Ising model. However, only the even subset appears in the low-temperature phase. This is consistent with the numerical observations on the eigenspectrum of the dilute A_4 model.^{21,17} For regime 2^+ , in a study of the low-lying excitations, the first and third were absent. As we have demonstrated, all seven excitations are present in regime 2^- which (through a quirk in labelling) corresponds to $T > T_c$.

Our expression (3) gives the correlation lengths and related masses (6), expressed in terms of standard elliptic functions and the original nome p , as

$$m_j = \xi_j^{-1} = 2 \sum_a \log \frac{\vartheta_4(a\pi/36 + \pi/4, p^{5/9})}{\vartheta_4(a\pi/36 - \pi/4, p^{5/9})}. \quad (26)$$

In the critical limit $p \rightarrow 0$ the leading order behavior is

$$m_j \sim 8p^{5/9} \sum_a \sin \frac{a\pi}{18}. \quad (27)$$

Substituting the integers of Table II, applying trigonometric identities and taking mass ratios it was demonstrated¹⁷ that the E_7 mass spectrum (25) is recovered.

The ground states of the tricritical Ising model (in zero magnetic field) have been identified.^{32,22} For $T < T_c$, the system is in a two-phase region of spontaneously broken spin reversal symmetry, with two degenerate ground states in the thermodynamic limit. For $T > T_c$ there is one ground state. This ground state picture is also consistent with that of the dilute A_4 model³⁹ as $|p| \rightarrow 1$. In regime 2^+ there are two possible ferromagnetic ground states, while in regime 2^- there is a single disordered ground state. (It is the presence of such disordered states for L even which complicates the calculation of order parameters for this half of the dilute A_L hierarchy.)

Very recently, an array of universal ratios for the critical amplitudes of the tricritical Ising model have been calculated^{24,25} by field theoretic methods. Not all of these quantities appear to be accessible via the dilute A_4 model. However, one such ratio involves the correlation length prefactors $\xi_0^\pm$, above and below the critical temperature. Our results and observations on the eigenspectrum of dilute A_4 give this same value:

$$\frac{\xi_0^+}{\xi_0^-} = \frac{\xi_1}{\xi_2} = 2 \cos \frac{5\pi}{18}.$$

We previously¹⁷ derived the amplitude

$$f_s \xi_1^2 = \frac{1}{8\sqrt{3} \cos(2\pi/9)} = 0.09420\dots, \quad (28)$$

where f_s is the singular part of the free energy. This agrees with the determination of this quantity for the $\phi_{(1,2)}$ perturbation of the $c = \frac{7}{10}$ field theory.⁴⁰ A related universal quantity is the amplitude ratio associated with the correlation length⁴¹

$$R_\xi^\pm = A^{1/2} \xi_0^\pm,$$

where A/α is the amplitude of the specific heat and α is the related critical exponent. Our expressions for these quantities are

$$R_\xi^+ = \left[\frac{10}{9^3 \sqrt{3} \cos(2\pi/9)} \right]^{1/2} = 0.101678\dots,$$

$$R_{\xi}^{-} = \left[\frac{5}{2^3 9^2 \sqrt{3} \cos(5\pi/18) \sin(5\pi/9)} \right]^{1/2} = 0.083\,889\dots,$$

which agree with the numerical values of Ref. 25 (allowing for a difference in definition by a factor $\alpha^{1/2}$). As remarked,^{24,25} such values may be observed in experimental systems within the tricritical Ising universality class.

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APPENDIX: FURTHER MASS CALCULATIONS

1. Mass m_4

We begin the perturbation argument with the structure $w_j = a_j$ for $j = 1, \dots, N-4$ with $w_{N-3} = b_1 x^{-18}$, $w_{N-2} = b_2 x^{18}$, $w_{N-1} = b_3 x^{-6}$ and $w_N = b_4 x^6$. From the Bethe equations for $j = N-3, \dots, N$ we can show that $b_1 = b_2 = b_3 = b_4 = b$. The Bethe equation for the other roots is

$$-\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-4} b^4)^{3/5} \frac{a^4}{b^4} \frac{E(x^2 b/a) E(x^6 b/a)}{E(x^2 a/b) E(x^6 a/b)} \prod_{j=1}^{N-4} \frac{E(x^{2s} a_j/a) E(x^{4s} a_j/a)}{E(x^{2s} a_j/a) E(x^{4s} a_j/a)}. \quad (\text{A1})$$

In the $x \rightarrow 0$ limit this gives the equation

$$a^{N-4} + \frac{1}{\omega} (A_{N-4} b^4)^{3/5} / b^4 = 0,$$

so that as usual we find an expression involving the prefactors

$$\frac{1}{\omega} (A_{N-4} b^4)^{3/5} = A_{N-4} b^4.$$

Using this with the other Bethe equations in the $x \rightarrow 0$ limit we obtain

$$\left[\frac{1}{\omega} (A_{N-4} b^4)^{3/5} \right]^4 = \frac{(A_{N-4} b^4)^4}{b^{4N}} \Rightarrow b^{4N} = 1.$$

From (A1) come the recurrences

$$\mathcal{F}_4(a) = \frac{(x^{34} a/b, x^{38} a/b; x^{2r})_{\infty}}{(x^2 a/b, x^6 a/b; x^{2r})_{\infty}} \frac{\mathcal{F}_4(x^{2s} a)}{\mathcal{F}_4(x^{4s} a)},$$

$$\mathcal{G}_4(1/a) = \frac{(x^{30} b/a, x^{26} b/a; x^{2r})_{\infty}}{(x^{58} b/a, x^{62} b/a; x^{2r})_{\infty}} \frac{\mathcal{G}_4(x^{2s}/a)}{\mathcal{G}_4(x^{4s}/a)}.$$

The solutions are

$$\mathcal{F}_4(a) = \frac{(x^{38} a/b, x^{42} a/b, x^{50} a/b, x^{54} a/b; x^{2r})_{\infty}}{(x^2 a/b, x^6 a/b, x^{14} a/b, x^{18} a/b; x^{2r})_{\infty}} \frac{(x^{34} a/b, x^{38} a/b, x^{46} a/b, x^{50} a/b; x^{12s})_{\infty}}{(x^{70} a/b, x^{74} a/b, x^{82} a/b, x^{86} a/b; x^{12s})_{\infty}},$$

$$\mathcal{G}_4(1/a) = \frac{(x^{26}b/a, x^{30}b/a, x^{38}b/a, x^{42}b/a; x^{2r})_\infty}{(x^{62}b/a, x^{66}b/a, x^{74}b/a, x^{78}b/a; x^{2r})_\infty} \frac{(x^{94}b/a, x^{98}b/a, x^{106}b/a, x^{110}b/a; x^{12s})_\infty}{(x^{58}b/a, x^{62}b/a, x^{70}b/a, x^{74}b/a; x^{12s})_\infty}.$$

In terms of these functions the eigenvalue may be represented as

$$\frac{\Lambda_4}{3} = \frac{w^2}{b^2} \frac{(x^{18}w/b, x^{30}w/b, x^{10}b/w, x^{22}b/w; x^{2r})_\infty}{(x^{10}w/b, x^{22}w/b, x^{30}b/w, x^{18}b/w; x^{2r})_\infty} \mathcal{F}_4(x^{2s}w) \mathcal{G}_4(1/x^{2s}w).$$

Thus, application of the perturbation argument yields the excitation to be

$$\frac{\Lambda_4}{\Lambda_0} = w^2 \frac{E(-x^{10}/w, x^{12s})E(-x^{14}/w, x^{12s})E(-x^{46}w, x^{12s})E(-x^{50}w, x^{12s})}{E(-x^{10}w, x^{12s})E(-x^{14}w, x^{12s})E(-x^{46}/w, x^{12s})E(-x^{50}/w, x^{12s})}, \quad (A2)$$

where we have put $b = -1$.

2. Mass m_5

We begin the perturbation argument with $w_j = a_j$ for $j = 1, \dots, N-4$ and $w_{N-3} = b_1 x^{-16}$, $w_{N-2} = b_2 x^{16}$, $w_{N-1} = b_3 x^{-12}$, $w_N = b_4 x^{12}$. We can show that the b_i are equal, and we call them b . The Bethe equation for the other roots is

$$-\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-4} b^4)^{3/5} \frac{a^4}{b^4} \frac{E(x^8 b/a) E^2(x^{12} b/a)}{E(x^8 a/b) E^2(x^{12} a/b)} \\ \times \frac{E(x^{16} b/a)}{E(x^{16} a/b)} \prod_{j=1}^{N-4} \frac{E(x^{2s} a/a_j) E(x^{4s} a_j/a)}{E(x^{2s} a_j/a) E(x^{4s} a/a_j)}. \quad (A3)$$

In the $x \rightarrow 0$ limit this gives the equation

$$a^{N-4} + \frac{1}{\omega} (A_{N-4} b^4)^{3/5} / b^4 = 0,$$

which leads in the usual way to a prefactor expression

$$\frac{1}{\omega} (A_{N-4} b^4)^{3/5} = A_{N-4} b^4.$$

From this and the other Bethe equations

$$\left[\frac{1}{\omega} (A_{N-4} b^4)^{3/5} \right]^4 = \frac{(A_{N-4} b^4)^4}{b^{5N}} \Rightarrow b^{5N} = 1.$$

Rearranging (A3), the auxiliary functions obey the recurrences

$$\mathcal{F}_5(a) = \frac{(x^{24}a/b, x^{28}a/b, x^{28}a/b, x^{32}a/b; x^{2r})_\infty}{(x^8a/b, x^{12}a/b, x^{12}a/b, x^{16}a/b; x^{2r})_\infty} \frac{\mathcal{F}_5(x^{2s}a)}{\mathcal{F}_5(x^{4s}a)}, \\ \mathcal{G}_5(1/a) = \frac{(x^{32}b/a, x^{36}b/a, x^{36}b/a, x^{40}b/a; x^{2r})_\infty}{(x^{48}b/a, x^{52}b/a, x^{52}b/a, x^{56}b/a; x^{2r})_\infty} \frac{\mathcal{G}_5(x^{2s}/a)}{\mathcal{G}_5(x^{4s}/a)}.$$

The solutions are

$$\mathcal{F}_5(a) = \frac{(x^{32}a/b, x^{40}a/b, x^{44}a/b; x^{2r})_\infty}{(x^{12}a/b, x^{16}a/b, x^{24}a/b; x^{2r})_\infty} \frac{(x^{28}a/b, x^{36}a/b, x^{40}a/b, x^{44}a/b, x^{48}a/b, x^{56}a/b; x^{12s})_\infty}{(x^8a/b, x^{12}a/b, x^{20}a/b, x^{64}a/b, x^{72}a/b, x^{76}a/b; x^{12s})_\infty},$$

$$\mathcal{G}_5(1/a) = \frac{(x^{40}b/a, x^{36}b/a, x^{48}b/a; x^{2r})_\infty}{(x^{56}b/a, x^{64}b/a, x^{68}b/a; x^{2r})_\infty} \frac{(x^{32}b/a, x^{36}b/a, x^{44}b/a, x^{88}b/a, x^{96}b/a, x^{100}b/a; x^{12s})_\infty}{(x^{52}b/a, x^{60}b/a, x^{64}b/a, x^{68}b/a, x^{72}b/a, x^{80}b/a; x^{12s})_\infty},$$

which we next substitute into the eigenvalue expression

$$\frac{\Lambda_5}{3} = -\frac{w^3}{b^3} \frac{(x^{24}w/b, x^{28}w/b, x^{36}w/b, x^{44}w/b, x^{12}b/w, x^{16}b/w; x^{2r})_\infty}{(x^4w/b, x^{12}w/b, x^{16}w/b, x^{24}w/b, x^{28}w/b, x^{36}w/b; x^{2r})_\infty} \mathcal{F}_5(x^{2s}w) \mathcal{G}_5(1/x^{2s}w),$$

to obtain (with $b = -1$) an expression in elliptic functions of nome x^{12s} ,

$$\frac{\Lambda_5}{\Lambda_0} = w^3 \frac{E(-x^4/w)E(-x^{12}/w)E(-x^{16}/w)E(-x^{40}/w)E(-x^{48}/w)E(-x^{52}/w)}{E(-x^4w)E(-x^{12}w)E(-x^{16}w)E(-x^{40}w)E(-x^{48}w)E(-x^{52}w)}. \quad (\text{A4})$$

3. Mass m_6

We begin the perturbation argument with $w_j = a_j$ for $j = 1, \dots, N-5$ and $w_{N-4} = b_1 x^{20}$, $w_{N-3} = b_2 x^{-16}$, $w_{N-2} = b_3 x^{16}$, $w_{N-1} = b_4 x^{-8}$, $w_N = b_5 x^8$. We can show that the b_i are equal, and we call them b . The Bethe equation for the other roots is

$$\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-5} b^5)^{3/5} \frac{a^5}{b^5} \frac{E(x^4b/a)E(x^8b/a)}{E(x^4a/b)E(x^8a/b)} \times \frac{E(x^{12}b/a)E(x^{16}b/a)}{E(x^{12}a/b)E(x^{16}a/b)} \prod_{j=1}^{N-5} \frac{E(x^{2s}a_j/a)E(x^{4s}a_j/a)}{E(x^{2s}a_j/a)E(x^{4s}a_j/a)}. \quad (\text{A5})$$

In the $x \rightarrow 0$ limit this gives the equation

$$a^{N-5} - \frac{1}{\omega} (A_{N-5} b^5)^{3/5} / b^5 = 0,$$

which leads in the usual way to the expression

$$\frac{1}{\omega} (A_{N-5} b^5)^{3/5} = A_{N-5} b^5.$$

From this and the other Bethe equations

$$\left[\frac{1}{\omega} (A_{N-5} b^5)^{3/5} \right]^5 = \frac{(A_{N-5} b^5)^5}{b^{5N}} \Rightarrow b^{5N} = 1.$$

After rearranging (A5), the auxiliary functions obey the recurrences

$$\mathcal{F}_6(a) = \frac{(x^{24}a/b, x^{28}a/b, x^{32}a/b, x^{36}a/b; x^{2r})_\infty}{(x^4a/b, x^8a/b, x^{12}a/b, x^{16}a/b; x^{2r})_\infty} \frac{\mathcal{F}_6(x^{2s}a)}{\mathcal{F}_6(x^{4s}a)},$$

$$\mathcal{G}_6(1/a) = \frac{(x^{28}b/a, x^{32}b/a, x^{36}b/a, x^{40}b/a; x^{2r})_\infty}{(x^{48}b/a, x^{52}b/a, x^{56}b/a, x^{60}b/a; x^{2r})_\infty} \frac{\mathcal{G}_6(x^{2s}/a)}{\mathcal{G}_6(x^{4s}/a)}.$$

The solutions are

$$\mathcal{F}_6(a) = \frac{(x^{36}a/b, x^{40}a/b; x^{2r})_\infty}{(x^{16}a/b, x^{20}a/b; x^{2r})_\infty} \frac{(x^{32}a/b, x^{36}a/b, x^{40}a/b, x^{44}a/b, x^{48}a/b, x^{52}a/b; x^{12s})_\infty}{(x^4a/b, x^8a/b, x^{12}a/b, x^{16}a/b, x^{20}a/b, x^{24}a/b; x^{12s})_\infty},$$

$$\mathcal{G}_6(1/a) = \frac{(x^{40}b/a, x^{44}b/a; x^{2r})_\infty (x^{28}b/a, x^{32}b/a, x^{36}b/a, x^{40}b/a, x^{92}b/a, x^{96}b/a; x^{12s})_\infty}{(x^{60}b/a, x^{64}b/a; x^{2r})_\infty (x^{56}b/a, x^{60}b/a, x^{64}b/a, x^{68}b/a, x^{72}b/a, x^{76}b/a; x^{12s})_\infty},$$

which we next substitute into the eigenvalue expression

$$\frac{\Lambda_6}{3} = -\frac{w^3 (x^{28}w/b, x^{32}w/b, x^8b/w, x^{12}b/w; x^{2r})_\infty}{b^3 (x^8w/b, x^{12}w/b, x^{28}b/w, x^{32}b/w; x^{2r})_\infty} \mathcal{F}_6(x^{2s}w) \mathcal{G}_6(1/x^{2s}w),$$

to obtain (with $b = -1$) an expression in elliptic functions of nome x^{12s} ,

$$\frac{\Lambda_6}{\Lambda_0} = w^3 \frac{E(-x^8/w)E(-x^{12}/w)E(-x^{16}/w)E(-x^{44}w)E(-x^{48}w)E(-x^{52}w)}{E(-x^8w)E(-x^{12}w)E(-x^{16}w)E(-x^{44}/w)E(-x^{48}/w)E(-x^{52}/w)}. \quad (\text{A6})$$

4. Mass m_7

We begin with $w_j = a_j$ for $j = 1, \dots, N-6$ and $w_{N-5} = b_1 x^{-18}$, $w_{N-4} = b_2 x^{18}$, $w_{N-3} = b_3 x^{-14}$, $w_{N-2} = b_4 x^{14}$, $w_{N-1} = b_5 x^{-10}$, $w_N = b_6 x^{10}$. Once again the $b_i (=b)$ are all equal. The Bethe equation for the other roots is

$$-\omega \left[a \frac{E(x^{2s}/a)}{E(x^{2s}a)} \right]^N = (A_{N-6} b^6)^{3/5} \frac{a^6 E(x^6 b/a) E^2(x^{10} b/a)}{b^6 E(x^6 a/b) E^2(x^{10} a/b)} \\ \times \frac{E^2(x^{14} b/a) E(x^{18} b/a)}{E^2(x^{14} a/b) E(x^{18} a/b)} \prod_{j=1}^{N-6} \frac{E(x^{2s} a/a_j) E(x^{4s} a_j/a)}{E(x^{2s} a_j/a) E(x^{4s} a/a_j)}.$$

In the $x \rightarrow 0$ limit this gives

$$a^{N-6} + \frac{1}{\omega} (A_{N-6} b^6)^{3/5} b^6 = 0,$$

which leads to the expression in the various coefficients

$$\frac{1}{\omega} (A_{N-6} b^6)^{3/5} = A_{N-6} b^6,$$

and from the six Bethe equations involving b ,

$$\left[\frac{1}{\omega} (A_{N-6} b^6)^{3/5} \right]^6 = \frac{(A_{N-6} b^6)^6}{b^{4N}} \Rightarrow b^{4N} = 1.$$

The recurrences to be solved for the auxiliary functions are

$$\mathcal{F}_7(a) = \frac{(x^{22}a/b, x^{26}a/b, x^{26}a/b, x^{30}a/b, x^{30}a/b, x^{34}a/b; x^{2r})_\infty \mathcal{F}_7(x^{2s}a)}{(x^6a/b, x^{10}a/b, x^{10}a/b, x^{14}a/b, x^{14}a/b, x^{18}a/b; x^{2r})_\infty \mathcal{F}_7(x^{4s}a)}, \\ \mathcal{G}_7(1/a) = \frac{(x^{30}b/a, x^{34}b/a, x^{34}b/a, x^{38}b/a, x^{38}b/a, x^{42}b/a; x^{2r})_\infty \mathcal{G}_7(x^{2s}/a)}{(x^{46}b/a, x^{50}b/a, x^{50}b/a, x^{54}b/a, x^{54}b/a, x^{58}b/a; x^{2r})_\infty \mathcal{G}_7(x^{4s}/a)},$$

which have solution

$$\mathcal{F}_7(a) = \frac{(x^{34}a/b, x^{38}a/b, x^{42}a/b, x^{46}a/b; x^{2r})_\infty}{(x^{10}a/b, x^{14}a/b, x^{18}a/b, x^{22}a/b; x^{2r})_\infty} \\ \times \frac{(x^{30}a/b, x^{34}a/b, x^{38}a/b, x^{42}a/b, x^{46}a/b, x^{50}a/b, x^{54}a/b; x^{12s})_\infty}{(x^6a/b, x^{10}a/b, x^{14}a/b, x^{18}a/b, x^{22}a/b, x^{26}a/b, x^{30}a/b; x^{12s})_\infty},$$

$$\mathcal{G}_7(1/a) = \frac{(x^{34}b/a, x^{38}b/a, x^{42}b/a, x^{46}b/a; x^{2r})_\infty}{(x^{58}b/a, x^{62}b/a, x^{66}b/a, x^{70}b/a; x^{2r})_\infty} \\ \times \frac{(x^{30}b/a, x^{34}b/a, x^{38}b/a, x^{42}b/a, x^{46}b/a, x^{50}b/a, x^{54}b/a; x^{12s})_\infty}{(x^{54}b/a, x^{58}b/a, x^{62}b/a, x^{66}b/a, x^{70}b/a, x^{74}b/a, x^{78}b/a; x^{12s})_\infty}.$$

Substitution into

$$\frac{\Lambda_7}{3} = \frac{w^4 (x^{22}w/b, x^{26}w/b, x^{30}w/b, x^{34}w/b, x^{38}w/b, x^{42}w/b, x^{46}w/b; x^{2r})_\infty}{b^4 (x^6w/b, x^{10}w/b, x^{14}w/b, x^{18}w/b, x^{22}w/b, x^{26}w/b, x^{30}w/b; x^{2r})_\infty} \\ \times \mathcal{F}_7(x^{2s}w) \mathcal{G}_7(1/x^{2s}w)$$

yields the result (with $b = -1$ and elliptic nome x^{12s})

$$\frac{\Lambda_7}{\Lambda_0} = w^4 \frac{E(-x^6/w)E(-x^{10}/w)E(-x^{14}/w)E(-x^{18}/w)}{E(-x^6w)E(-x^{10}w)E(-x^{14}w)E(-x^{18}w)} \\ \times \frac{E(-x^{42}w)E(-x^{46}w)E(-x^{50}w)E(-x^{54}w)}{E(-x^{42}/w)E(-x^{46}/w)E(-x^{50}/w)E(-x^{54}/w)}. \quad (\text{A7})$$

¹See, e.g., M. Henkel, *Conformal Invariance and Critical Phenomena* (Springer, Heidelberg, 1999) and references therein.

²A. B. Zamolodchikov, *Int. J. Mod. Phys. A* **4**, 4235 (1989); *Adv. Stud. Pure Math.* **19**, 641 (1989).

³S. O. Warnaar, B. Nienhuis, and K. A. Seaton, *Phys. Rev. Lett.* **69**, 710 (1992); *Int. J. Mod. Phys. B* **7**, 3727 (1993).

⁴P. Roche, *Phys. Lett. B* **285**, 49 (1992).

⁵V. V. Bazhanov, B. Nienhuis, and S. O. Warnaar, *Phys. Lett. B* **322**, 198 (1994).

⁶S. O. Warnaar and P. A. Pearce, *J. Phys. A* **27**, L891 (1994).

⁷J. Suzuki, *Nucl. Phys. B* **528**, 683 (1998).

⁸Y. Hara, M. Jimbo, H. Konno, S. Odake, and J. Shiraishi, *J. Math. Phys.* **40**, 3791 (1999).

⁹U. Grimm and B. Nienhuis, in *Statistical Models, Yang-Baxter Equation and Related Topics*, edited by M. L. Ge and F. Y. Wu (World Scientific, Singapore, 1996), pp. 384–393.

¹⁰U. Grimm and B. Nienhuis, *Phys. Rev. E* **55**, 5011 (1997).

¹¹B. M. McCoy and W. P. Orrick, *Phys. Lett. A* **230**, 24 (1997).

¹²M. T. Batchelor and K. A. Seaton, *J. Phys. A* **30**, L479 (1997).

¹³M. T. Batchelor and K. A. Seaton, *Nucl. Phys. B* **520**, 697 (1998).

¹⁴V. A. Fateev and A. B. Zamolodchikov, *Int. J. Mod. Phys. A* **5**, 1025 (1990).

¹⁵S. O. Warnaar and P. A. Pearce, *Int. J. Mod. Phys. A* **11**, 291 (1996).

¹⁶J. Suzuki, in *Physical Combinatorics*, edited by M. Kashiwara and T. Miwa (Birkhäuser, Boston, 2000), pp. 217–247.

¹⁷M. T. Batchelor and K. A. Seaton, *Eur. Phys. J. B* **5**, 719 (1998).

¹⁸K. A. Seaton and M. T. Batchelor, in *Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics*, edited by S. P. Corney, R. Delbourgo, and P. D. Jarvis (International, Cambridge, MA, 1998), pp. 274–278.

¹⁹B. Kostant, *Proc. Natl. Acad. Sci. USA* **81**, 5275 (1984).

²⁰H. W. Braden, E. Corrigan, P. E. Dorey, and R. Sasaki, *Nucl. Phys. B* **338**, 689 (1990).

²¹U. Grimm and B. Nienhuis (private communication).

²²M. Lässig, G. Mussardo, and J. L. Cardy, *Nucl. Phys. B* **348**, 591 (1991).

²³P. Christe and G. Mussardo, *Nucl. Phys. B* **330**, 465 (1990).

²⁴D. Fioravanti, G. Mussardo, and P. Simon, *Phys. Rev. Lett.* **85**, 126 (2000).

²⁵D. Fioravanti, G. Mussardo, and P. Simon, *Phys. Rev. E* **63**, 016103 (2001).

²⁶S. O. Warnaar, P. A. Pearce, K. A. Seaton, and B. Nienhuis, *J. Stat. Phys.* **74**, 469 (1994).

²⁷B. Nienhuis, *Int. J. Mod. Phys. B* **4**, 929 (1990).

²⁸S. O. Warnaar and B. Nienhuis, *J. Phys. A* **26**, 2301 (1993).

²⁹R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).

³⁰R. J. Baxter, *Ann. Phys. (N.Y.)* **70**, 193 (1972).

- ³¹C. Korff and K. A. Seaton (unpublished).
- ³²H. Capel, *Physica A* **32**, 966 (1966); M. Blume, *Phys. Rev.* **141**, 517 (1966).
- ³³M. Blume, V. J. Emery, and R. B. Griffiths, *Phys. Rev. A* **4**, 1071 (1971).
- ³⁴I. D. Lawrie and S. Sarbach, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic, New York, 1984), Vol. 9, pp. 1–161.
- ³⁵G. E. Andrews, R. J. Baxter, and P. J. Forrester, *J. Stat. Phys.* **35**, 193 (1984).
- ³⁶D. A. Huse, *Phys. Rev. B* **30**, 3908 (1984).
- ³⁷K. A. Seaton, *J. Phys. A* **35**, 1597 (2002).
- ³⁸G. von Gehlen, *Nucl. Phys. B* **330**, 741 (1990).
- ³⁹S. O. Warnaar, B. Nienhuis, and K. A. Seaton (unpublished).
- ⁴⁰V. A. Fateev, *Phys. Lett. B* **324**, 45 (1994).
- ⁴¹V. Privman, P. C. Hohenberg, and A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic, New York, 1991), Vol. 14, pp. 4–134.