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THE SEASONAL ADJUSTMENT OF ECONOMIC DATA

BY SPECTRAL METHODS.

by

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Preface.

The author wishes to express his great thanks to his supervisor, Professor E. J. Hannan of the Department of Statistics, Australian National University, for the advice and encouragement given so freely. Professor Hannan suggested the topic and outlined the way it might be treated; this thesis is a development of several of his papers, referred to in the text. Mrs. B. Moore kindly offered to do the typing and did it very expertly, as can easily be seen. Miss M. Campbell and Mr. P. Tindale helped with the computing on the I.B.M. 1620 while the University Administration provided excellent facilities including a scholarship, room, computer time and duplication of the thesis. To all these the author is very grateful.

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CHAPTER 1

The Rationale of the Analysis of Economic Time Series
1.1 The purposes of time series analysis.

When economic data are obtained in the form of time series it is very often required to adjust them for seasonal variation. Before discussing ways of carrying out the adjustment it will be helpful to list some of the reasons why this may be required.

Sometimes one is dealing with only a single series, quite apart from its possible relationships with other series. It may then be required to use, or to publish, a deseasonalized series; this could assist in interpreting past observations, or in adjusting and interpreting new values as they come to hand, or in predicting future values. For example the Commonwealth Bureau of Census and Statistics and the Reserve Bank of Australia are currently considering the publication of many deseasonalized series.

The problem of seasonal adjustment also arises in studies linking two or more series, that is, in econometric models. For instance, yearly series may be too short to estimate the required parameters satisfactorily; quarterly or monthly data are then used but seasonal patterns will reduce the number of effective degrees of freedom, and must be taken into account. Again, it may be desired to allow for seasonal variation as one factor explaining the decisions of businessmen and other
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economic agents; but quite often the agents concerned do not allow for seasonal variation in making their decisions, and if this is so seasonal variation should of course not be used as an explaining factor.

Two interesting examples of the application of seasonal analysis may be quoted. In a study of eggs, cheese and milk Brennan (1959) found that the demand at the Chicago market displayed little seasonality whereas production was adequately represented by a linear trend with an additive seasonal component in the form of a simple sine curve with a period of twelve months; he then discussed the appropriate inventory plan. De Leeuw (1962) analysed the quarterly demand for capital goods by manufacturers and the effective removal of seasonal variation was vital to his conclusions.

It is clear that seasonal adjustment is a problem of considerable importance and that it will be worthwhile to develop the most effective possible means for carrying it out.
1.2 The object of this thesis.

The object of this thesis is to present a formal theory for the seasonal adjustment of a single series in terms of spectral analysis, that is, the analysis of frequency components, together with a practical illustration. Instead of describing the well-known classical procedures we refer the reader to two quite sophisticated examples, the work of the Bank Deutscher Länder (1957) and Shiskin and Eisenpress (1957). It will appear that the spectral method is more powerful for description and analysis than are the classical methods, since it decomposes variables into their fundamental component parts; the estimation of variances and tests of significance are thus greatly facilitated.

During the last fifteen years the mathematical and statistical developments in time series analysis have been considerable but insufficient attention has been given to economic applications. The three main characteristics which distinguish economic time series from those arising in the physical sciences are the limited knowledge of causal relationships, the uncontrollability of most of the factors generating the series and the relative shortness of series of observations which may reasonably be said to be obtained under uniform conditions. The effects of these characteristics will become apparent in what follows.
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The application of the methods to be presented in econometric models may be made straightforwardly to each series. It may alternatively be desirable to construct a multivariate model for seasonal variation in which the seasonal patterns in several series are explained in terms of a few basic seasonal patterns, which may or may not be identifiable as known causal factors. However, it is doubtful whether such a degree of sophistication would often be warranted and the present work will be restricted to univariate series.

The work is presented largely from first principles for two main reasons. Firstly, it may then be read both by economists having little statistics (in particular, spectral theory) and by statisticians having little economics. Secondly, the first principles themselves are always worth careful attention, though most writers on economic time series have passed them by.
There are two approaches to the problem of adjusting an economic time series for seasonal variation.

The first approach is to seek the structure of the observed series in terms of the real-world factors which generate it. Usually the ultimate factors are the growth of population and technology, the movement of the earth around the sun and small chance causes. These basic factors enter into each economic variable in a different and changing way, and the economic variables themselves are inter-related in a complicated and changing manner of which we have quite incomplete knowledge.

In the physical sciences the causal mechanism of variables is often known. In economics, econometric models have explained with varied success the causal structure of some series, either singly or in systems of relations. However, structural models in economics usually refer to annual series for, except in relatively simple cases such as the production of some agricultural products, we have to admit that an entirely structural model for quarterly or monthly series would be far too complicated to formulate, let alone estimate.

We therefore take the second approach which is to attempt to explain the observed series in numerical
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terms; that is, the components are series of numbers which usually have no direct identification with any real variables, though they reflect the inter-actions of all of these.

Knowledge of economic factors affecting the series will still be relevant and a serious analysis could not be carried out entirely mechanically. Thus Brittain (1959) suggested that instead of estimating the seasonal variation in unemployment figures, which may be nearly zero over some periods, it might be more appropriate to estimate the pattern in employment and subtract this from the estimated work force; Nerlove (1962) has however found that the evidence in the United States shows no significant difference between the two methods. Again, Stevens (1963) has suggested that the rate of absorption of school leavers could be estimated independently, thus accounting for one known factor. Similarly any opportunity for disaggregating a given series should be taken if the sub-series (say for each state of Australia) are expected to have different seasonal patterns. If it is found that each sub-series has its own fairly constant seasonal pattern while the proportion which each one bears to the total series is changing, then the problem of a changing seasonal pattern will have been solved in a very satisfactory way.
Nevertheless if many series are to be adjusted and a publication deadline is to be met, a largely mechanical method may be desirable. The thorough analysis given in Chapter 5 is not recommended in all cases but is rather an attempt to gain some insight into the problems.
1.4 The trend, seasonal and residual components.

There are obviously infinitely many ways of representing a given time series in terms of component parts and we can only seek those representations which seem most reasonable. Experience of previous economic series suggests that three factors can be isolated: the trend, seasonal and residual components. We now consider the real meaning of these terms.

The trend, which is here taken to include the business cycle, reflects broad movements in the economy and in particular the gradual increase of population and productivity. It is sometimes thought that the trend is a 'simple' function of time, that is, either expressible by a standard mathematical function or else capable of being graphed with a few sweeps of the hand, together with some type of random disturbance. However since all economic series interact there is no reason to suppose that the trend of any given series should be representable in such a simple manner.

As an example the trend in the number of vacancies registered with the Commonwealth Employment Service in Australia is determined partly by the general movement of the economy and growth of population but also by the trend towards automation, the trend in overseas investment of various types in Australia, the
extent to which the pattern of demand is changing, and so on; all these factors themselves depend on many other things.

An observed series may by chance happen to follow closely a simple trend such as a polynomial of low degree, but this will very often be unsatisfactory, particularly for prediction. Simple trends may be relevant in studies in mathematical economics and occasionally in particular cases such as the logistic curve for sales of a new commodity but usually when the object is to find the way in which the observations were generated statistically we must logically allow the trend as nearly complete freedom of movement as is possible.

The seasonal component reflects the movement of the earth around the sun and occasionally of the moon around the earth. Each economic series has a component due directly to this influence but also contains a component due to its interaction with other variables which also have a seasonal pattern.

For example the series giving the monthly vacancies registered has a seasonal pattern which partly reflects the seasons directly as they affect agricultural and pastoral activity but which also reflects the seasonal nature of production in industries less directly related to the climate due for instance to a seasonal demand for
the product or supply of raw materials. Another factor would be the number of persons leaving school in the different months.

It is clear that the seasonal effect will never be precisely the same from year to year. One of the most important decisions to be made in any particular case is whether the deviations from a precisely repeating seasonal pattern should be considered to be small and unsystematic, or moderately large and systematic.

In the former case we build a model containing a seasonal factor which is stable from year to year whereas in the latter case it may be worthwhile allowing for a seasonal factor which changes slowly over time. Large unsystematic deviations from a stable seasonal pattern, except those assignable to a known cause such as an industrial strike, mean that the model is invalid and should be abandoned, while unsystematic but small deviations would be too difficult to estimate even if it were worthwhile, so they should be included in the residual component.

The residual component is due to a large number of accidental features of the economy, generally though not always small. Since these features will sometimes operate over several successive observations the residual
component is likely to be autocorrelated. Errors of observation are also included in this component since they need be treated separately only when the observations are used to estimate a structural economic model whereas here we are concerned to analyse only the values actually observed.
1.5 The choice of a functional form.

So far we have assumed very little. The representation we have reached may be stated symbolically as

\[ y_t = f(p_t, s_t, x_t), \quad t = 1, 2, \ldots, n \quad (1.5.1) \]

where \( y_t \) is the observed univariate series, \( t \) is the number of the quarter or month or other unit of time and \( y_t \) is stated to depend by an unknown function \( f \) on the trend \( p_t \), the seasonal \( s_t \) which may be stable or changing from year to year and the random term \( x_t \).

The next step is to prescribe a suitable function \( f \). In most economic series \( f \) may be assumed to be approximately multiplicative in its three arguments since large and small values of the trend are likely to be accompanied by respectively large and small values of the seasonal and random terms. The function can hardly be expected to be precisely multiplicative however, so that if \( \hat{f} \) is the assumed functional form then the residual component \( x_t \) must now absorb the error series

\[ e_t = y_t - \hat{f}(p_t, s_t, x_t). \quad (1.5.2) \]

The only other function likely to be suitable is the additive function and in fact this is the only one which leads to convenient methods of
estimation; it can be reached by taking logarithms in the multiplicative model. We therefore always use the additive model

\[ y_t = p_t + s_t + x_t \]  \hspace{1cm} (1.5.3)

whether in the original values or in their logarithms. The flexibility of the additive and multiplicative functions could be judged by constructing series generated by different functions of the three components, but there is unlikely to be any rational basis for choosing such other functions.

If logarithms are used there will be certain effects on the final estimates when antilogarithms are taken. For example the equation

\[ \mathcal{E}(\text{estimate of } \log z) = \log z \]  \hspace{1cm} (1.5.4)

is in general consistent with the inequality

\[ \mathcal{E}(\text{antilog of estimate of } \log z) \neq z \]  \hspace{1cm} (1.5.5)

where \( \mathcal{E} \) means expectation and \( z \) is any parameter. It would be possible to discuss the effect of the transformation on the distribution of the estimates of the original parameters, given the distribution of the residuals, but this is not likely to be necessary since all tests of significance remain valid when carried out
after taking logarithms; that is, if the estimate of the logarithm of a parameter is significant, then so is the corresponding estimate of the original parameter. If some transformation other than the logarithmic one can be found to give an additive seasonal component it could of course be used; this may be suggested by plotting an initial estimate of the seasonal component against trend.

We can now attempt to analyse a given economic series not as a series of mere numbers but with the following ideas which we regard as reasonable:

(i) a trend component may be present
(ii) a seasonal component may be present
(iii) the seasonal component may be changing
(iv) a more or less random disturbance is present
(v) the model is approximately either additive or multiplicative in its components. In the following chapters the statistical consequences of these features will be developed.
CHAPTER 2

An Outline of Statistical Spectral Theory
2.1 Introduction.

The classical method of time series analysis proceeds in terms of the correlogram, that is, the set of lag correlations. This leads to considerable difficulties of interpretation, however. The modern approach involves the Fourier transform of the serial covariances, called the spectral density function. Although there are purposes for which this is not particularly convenient, it will be seen for present purposes to be very convenient mathematically as well as from the point of view of interpretation.

In this chapter an outline of spectral theory and estimation is given, ignoring the many important questions of pure mathematics which arise as these are well covered by Hannan (1960a) and others. This outline is given because the technique is not yet widely known among economic statisticians.
2.2 Formal details of the spectral theory.

Let $u_t$, $t = 1, 2, \ldots$ be any univariate series of real variables. The argument $t$ may in general refer to some spatial or other co-ordinate but here we consider it to be time. Although continuous records are sometimes available in the physical sciences, we here consider observations taken only at discrete and equidistant points of time. The unit of time may be a quarter, month, and so on.

The covariance properties of the series $u_t$ are expressed by the serial covariances $\mathcal{C}(u_s u_{s+t})$, where we have for the moment assumed that

$$\mathcal{C}(u_t) = 0, \quad t = 1, 2, \ldots \quad (2.2.1)$$

A series is called 'covariance stationary' if the serial covariances are independent of time, in which case we write

$$\gamma_t = \mathcal{C}(u_s u_{s+t}), \quad t = \ldots, -1, 0, 1, \ldots; \quad s = 1, 2, \ldots \quad (2.2.2)$$

At the same time we make the very reasonable assumption of a finite variance:

$$0 < \gamma_0 < \infty \quad (2.2.3)$$

For some purposes more strict conditions may be imposed; for instance a series is called 'strictly
stationary' if the distribution of any subset of the variables is completely unaffected by any translation through time. Since we do not deal here with moments of higher order than two we need not assume strict stationarity but will assume covariance stationarity, usually omitting the word 'covariance'.

The assumption of stationarity can hardly be precisely valid but it usually seems to be a good approximation for such series as are met in economic statistics and particularly for the series of residuals $x_t$. It may be necessary to make a prior adjustment for known significant departures from stationarity, for instance as the result of an industrial strike. Spectral methods appear from empirical studies to be quite robust against departures from stationarity although there is a lack of systematic evidence to support this statement. In any case, the stationary model suffices to describe any given finite series, non-stationarity affecting only problems of analysis for prediction of the residual.

Instead of working with the $\gamma_t$ directly we now take their Fourier cosine transform:

$$\mathcal{F}(\lambda) = \lim_{n \to \infty} \frac{1}{n} \sum_{-n}^{n-1} \gamma_t \cos \lambda t$$  \hspace{1cm} (2.2.4)

$$= \lim_{n \to \infty} (\gamma_0 + 2 \sum_{1}^{n-1} \gamma_t \cos \lambda t), \quad -\pi \leq \lambda \leq \pi. \quad (2.2.5)$$
This is called the spectral density function of the process \(u_t\); it is clearly an even function of frequency. The inverse relation is

\[
\gamma_t = \int_{-\pi}^{\pi} f(\lambda) \cos \lambda t \, d\lambda, \quad t = \ldots,-1,0,1,\ldots \quad (2.2.6)
\]

\[
\gamma_t = 2\int_{0}^{\pi} f(\lambda) \cos \lambda t \, d\lambda, \quad t = 0,1,\ldots \quad (2.2.7)
\]

It is sometimes convenient in (2.2.4) and (2.4.6) to replace \(\cos \lambda t\) by the complex harmonics \(e^{i\lambda t}\) and \(e^{-i\lambda t}\) respectively. We can also generalize by replacing \(f(\lambda) d\lambda\) in (2.2.6) and (2.2.7) by \(dF(\lambda)\), giving a Riemann-Stieltjes integral. We call \(F(\lambda)\) the spectral distribution function; instead of calling \(f(\lambda)\) the spectral density we will sometimes call it the spectrum, as is commonly done, although this term properly refers to the set of values over which \(F(\lambda)\) is increasing.

To make the significance of these concepts clearer we finally note that (2.2.6) implies (see Hannan (1960a) p. 9) the representation

\[
u_t = \int_{0}^{\pi} \cos \lambda t \, dv(\lambda) + \int_{0}^{\pi} \sin \lambda t \, dw(\lambda), \quad t = 1,2,\ldots \quad (2.2.8)
\]

where \(\varepsilon\left\{dv(\lambda_j)\,dw(\lambda_k)\right\} = 0,\)

\(\varepsilon\left\{dv(\lambda_j)\,dv(\lambda_k)\right\} = \varepsilon\left\{dw(\lambda_j)\,dw(\lambda_k)\right\} = 2 \, dF(\lambda_j) \delta_{\lambda_j,\lambda_k}\)

for \(0 < \lambda_j < \pi, \; 0 < \lambda_k < \pi, \) and
\[ \mathcal{V} \left\{ \text{d}v(0) \right\}^2 = \text{d}F(0), \quad \mathcal{V} \left\{ \text{d}v(\pi) \right\}^2 = \text{d}F(\pi). \]

Here \( \delta_{\lambda_j, \lambda_k} \) is the Kronecker delta.

Again this may be written in the form
\[ u_t = \int_{-\pi}^{\pi} e^{it\lambda} \text{d}z(\lambda), \quad t = 1, 2, \ldots \] (2.2.9)

where \( \text{d}z(\lambda) = \frac{1}{2} \left\{ \text{d}v(\lambda) - i\text{d}w(\lambda) \right\} \)

and \( \mathcal{V} \left\{ |\text{d}z(\lambda)|^2 \right\} = \text{d}F(\lambda), \)

\[ \mathcal{V} \left\{ (\text{d}z(\lambda_j) - \text{d}z(\lambda_k))(\overline{\text{d}z(\lambda_l)} - \overline{\text{d}z(\lambda_m)}) \right\} = 0 \]

for \( \lambda_j \geq \lambda_k > \lambda_l \geq \lambda_m \),

in which the bars denote complex conjugation.
2.3 Interpreting the spectrum.

Either of the representations (2.2.8), (2.2.9) is called the spectral representation of the process $u_t$. In words, $F(\lambda)$ reflects the cumulated expected value of the squared amplitude of the fundamental sine and cosine waves of all frequencies $\lambda$ into which $u_t$ is decomposed. Since the variance, $\gamma$, is equal to $\int_0^\infty dF(\lambda)$ we may also say that $F(\lambda)$ reflects the contribution to variance at all frequencies not greater than $\lambda$ and thus provides in a sense an analysis of variance.

The range of frequencies, that is, oscillations per fixed time unit, at which the spectrum is defined is limited by the interval between successive observations, whereas if a continuous record were available there would be no bound on the frequencies of the spectrum. The discrete observations in economic series lead to the problem known as aliasing, for frequencies greater than $\pi$ will be reflected at certain positions in the interval $[-\pi, \pi]$; this can be seen from a diagram showing sine curves with periods of say two and one time units. This would present a misleading picture if the spectral density were at all great at frequencies exceeding $\pi$. Hence in interpreting the spectrum it is always necessary to consider whether aliasing is likely to be significant. It is probably not significant in monthly economic series,
though it may well be in quarterly series. Hence in quarterly series it will be important to have some prior knowledge of the nature of the seasonal pattern being estimated if it is desired to discuss effects in finer detail, such as the precise location of a seasonal peak, supposing that monthly data are not available.

For all practical purposes \( F(\lambda) \) may be regarded as the sum of a continuous part and a set of discrete jumps. A spectrum consisting only of jumps at certain frequencies corresponds, in view of the spectral representation (2.2.6), to a process \( u_t \) generated as the sum of finite oscillations at those frequencies, whereas a continuous spectrum reflects the integral of an infinite number of oscillations, each of infinitesimal amplitude.

In most economic series a large part of the total 'spectral mass' or 'power' (terms taken from physical science and referring to the spectral density) is concentrated at low frequencies, which correspond to smooth, long-period oscillations in the series, that is, the trend. In particular a non-zero mean is a component at zero frequency. If there were a fixed and precisely repeating seasonal pattern, say monthly, then the spectrum would, as will be seen in more detail later, contain jumps at frequencies \( \pi j/6 \) for \( j = 1, \ldots, 6 \), that is, at periods
12, 6, 4, 3, 2.4, 2 months. In reality since the seasonal pattern never repeats precisely every year these jumps will become peaks of greater or smaller intensity at or near the relevant frequencies. The high frequencies of the spectrum correspond to rapid oscillations and a series of very irregular appearance; the contribution here will usually be quite small in economic series.
2.4 Estimating the spectrum.

A great deal has been written on estimating the spectrum and for further details the reader is referred to Hannan (1960a), Parzen (1957), Blackman and Tukey (1958) and others. It might be thought that if a large body of literature is needed to cover the difficulties of spectral estimation then these difficulties will outweigh the theoretical advantages of the spectral approach; the short length of most economic series is an aggravating factor. Although the outcome depends on the purpose of the analysis and although it is as yet too early to be sure, the balance may well be in favour of spectral methods of seasonal adjustment, for they not only give a neater expression of old ideas but also allow some ideas to be expressed which otherwise could hardly have been thought of.

Suppose we are given a sample \( u_t, \ t = 1, \ldots, n \) from the infinite series described in section 2.2. We begin the search for a desirable estimate of \( f(\lambda) \) by defining the sample periodogram

\[
I_n(\lambda) = \frac{1}{2\pi n} \sum_{s=1}^{n-1} \sum_{t=1}^{n} u_s u_t \cos(s-t)\lambda, \quad -\pi \leq \lambda \leq \pi,
\]

which can be written

\[
I_n(\lambda) = \frac{1}{2\pi}(c_0 + 2 \sum_{t=1}^{n-1} c_t \cos \lambda t), \quad -\pi \leq \lambda \leq \pi, \quad (2.4.1)
\]
where

\[ c_t = \frac{1}{n} \sum_{s=1}^{n-t} u_s u_{s+t}, \quad t = 0, 1, \ldots, n-1. \quad (2.4.2) \]

It would be more natural to use a divisor \((n - t)\) for \(c_t\) as an estimate of \(\gamma_t\), but the divisor \(n\) generally reduces the variance of the final estimates at the cost of a small bias (see Tukey (1961) p.211). It is straightforward to show from (2.2.5) that the periodogram is an asymptotically unbiased estimate of the spectrum, that is, \(\lim_{n \to \infty} \mathbb{E} \left\{ \text{I}_n(\lambda) \right\} = f(\lambda)\).

However, the variance of the periodogram does not approach zero, so that it is not a consistent estimate; it is this fact which has resulted in all the literature mentioned above. The spectrum cannot be estimated consistently at a single point. This result is intuitively reasonable because the spectrum at a single point is only one of infinitely many 'parameters'.

A consistent estimate can be defined for the average value of the spectrum over a range, or band, of frequencies about \(\lambda\), provided that the spectrum there is sufficiently smooth. This estimate is a weighted average of the periodogram values at several frequencies, the weights being concentrated towards the point \(\lambda\). Very lengthy computations would be required to obtain all the \(c_t\) for \(t = 1, \ldots, n\). Since the later covariances are relatively unimportant we are in practice prepared to introduce a small
bias by using only the first $m_n$ covariances. For convenience we drop the subscript $n$ and discuss the choice of $m$ later.

We therefore define

$$I_{m,n}(\lambda) = (1/2\pi)(c_0 + 2 \sum_{t=1}^{m} c_t \cos \lambda t), \quad -\pi \leq \lambda \leq \pi. \quad (2.4.3)$$

Various weights have been suggested but we do not give here the usual comparison of their merits and demerits; we merely present the method known as 'hanning', a commonly used method and the one which will be used in Chapter 5. The estimates are computed at points $\pi j/m$ for $j = 0, 1, \ldots, m$; they are

$$\hat{f}_{m,n}(\pi j/m) = \frac{1}{3} I_{m,n}(\pi(j-1)/m) + \frac{1}{3} I_{m,n}(\pi j/m)$$

$$+ \frac{1}{3} I_{m,n}(\pi(j+1)/m), \quad j = 1, \ldots, m-1$$

$$\hat{f}_{m,n}(0) = \frac{1}{6} I_{m,n}(0) + \frac{1}{3} I_{m,n}(\pi/m)$$

$$\hat{f}_{m,n}(\pi) = \frac{1}{6} I_{m,n}(\pi(m-1)/m) + \frac{1}{3} I_{m,n}(\pi)$$

$$\quad (2.4.4)$$

Alternative expressions for the estimates are

$$\hat{f}_{m,n}(\lambda) = (1/2\pi) \left\{ c_0 + 2 \sum_{t=1}^{m} c_t k(t/m) \cos \lambda t \right\} \quad (2.4.5)$$

where the 'lag window' is

$$k(t/m) = \frac{1}{2}(1 + \cos(\pi t/m)), \quad t = 1, \ldots, m \quad (2.4.6)$$

and by convolution

$$\hat{f}_{m,n}(\lambda) = \int_{-\pi}^{\pi} w_{m}(\theta-\lambda) I_{m,n}(\theta) \, d\theta \quad (2.4.7)$$
where the 'spectral window' is

\[ w_m(\lambda) = \frac{1}{2\pi} \left\{ 1 + 2 \sum_{t=1}^{m} k(t/m) \cos \lambda t \right\} \]

\[ = \left( \frac{(2m+1)}{2\pi} \right) \left\{ \frac{1}{n} \left[ \frac{1}{2}S_{2m+1}(\lambda) + \frac{1}{2}S_{2m+1}(\lambda + \frac{\pi}{m}) + S_{2m+1}(\lambda - \frac{\pi}{m}) \right] \right\} \]

where we introduce the useful notation

\[ S_n^2(\lambda) = \sin \frac{1}{2} \sin \lambda / \sin \frac{1}{2} \lambda, \quad \lambda \neq 0, \quad 2k\pi \]

\[ n, \quad \text{otherwise} \]

(2.4.10)

The various alternative weight functions \( k(t/m) \) or \( w_m(\lambda) \) to be found in the literature can readily be inserted in formulae (2.4.5) and (2.4.7).

If this smoothed periodogram is to produce good estimates of the average spectral density in each band, the true function \( f(\lambda) \) must not vary too much within each band; or if it does vary it should vary about the centre of the band in such a way that the biases on either side cancel out. This is the most vital consideration in the estimation of spectra. The whole problem of seasonal adjustment can be viewed as one of spectral estimation and the smoothness requirement is the reason why we firstly remove the trend and seasonal components since we have prior knowledge of the location of these spectral peaks. We will then estimate the spectrum of the residuals, which we expect to be relatively smooth and which, as we shall see, gives the variances of the seasonal estimates.
Nevertheless, since the seasonal pattern is not precisely located at the points $\pi j/6$, $j = 1, \ldots, 6$, but is spread around them to some extent, the removal by regression of the components at those frequencies may still leave considerable peaks near those frequencies. When the smoothing process is applied for estimation this effect may carry over into the adjacent bands. All this will be relevant in the practical applications in Chapter 5.

The variances of the hanning estimates can be shown to be asymptotically

$$\text{var} \left\{ \hat{f}_{m,n}(\lambda) \right\} = \frac{3}{4} \left( \frac{m}{n} \right) f^2(\lambda)$$

which will be estimated by

$$\text{est var} \left\{ \hat{f}_{m,n}(\lambda) \right\} = \frac{3}{4} \left( \frac{m}{n} \right) \hat{f}^2_{m,n}(\lambda). \quad (2.4.11)$$

The correlation between adjacent estimates is asymptotically $2/3$ and between estimates at points two apart is $1/6$; other correlations approach zero.

Evidently the variance can be reduced by reducing the ratio $m/n$ and by changing the weights $(2.4.6)$ or $(2.4.9)$; however the reduction will eventually be accompanied by an increase in the bias and a loss of 'resolution', that is, the spectrum will be estimated at fewer points and will therefore show less detail. The theory of spectral estimation studies
ways in which the best compromise can be reached according
to various criteria but we do not discuss this further here
except to remark that the variance at any one point of the
spectrum is not likely to be a very relevant quantity and the
theory deals with functions of the mean square error, such as
its integral over the whole spectrum. A suitable ratio $m/n$ for
economic series is likely to be between $1/5$ and $1/10$. In a
general theory $m$ and $n$ would both tend to infinity and their
ratio to zero; they are of course fixed numbers on any one
occasion. The fact that $n$ is small in most economic series
means that the estimates cannot usually be very precise (see
Hannan (1960b) pp.12,13).

Confidence intervals for the estimates can be
obtained on the assumption that the error of the estimate is due
mainly to its variance rather than its bias. We also assume
normality of $\log \left\{ \hat{\phi}_{m,n} (\lambda) \right\}$, which will be reasonable if $n/m$ is
sufficiently large. Stronger statements can be made but these
will not be discussed here. The logarithmic transformation is
made because it stabilizes the variance and this is why spectra
are often plotted on semi-logarithmic paper. The $95\%$ confidence
limits for $\log \left\{ \hat{\phi}_{m,n} (\lambda) \right\}$ when the hanning weights are used are
approximately
\[
\log \left\{ \hat{\phi}_{m,n} (\lambda) \right\} = 1.96 \sqrt{\frac{2}{4} m/n}) \tag{2.4.12}
\]
so that the $95\%$ confidence limits for $\hat{\phi}_{m,n} (\lambda)$ are the anti-
logarithms of the expressions (2.4.12).
31.

CHAPTER 3

The Analysis of a Stable Seasonal Pattern
3.1 Methods of trend removal.

We now consider adjusting a given series \( y_t \) for seasonal variation using the model (1.5.3) which we repeat here:

\[
y_t = p_t + s_t + x_t , \quad t = 1, \ldots, n, \tag{3.1.1}
\]

where the units are either the original ones or their logarithms.

In order to adjust the series for either a constant or an evolving seasonal component it is a natural first step to attempt to remove any trend which may be present. It will later be shown (see section 3.5) and it has already been suggested in section 2.4 that trend removal will in fact usually reduce the variance of the estimates of the seasonal parameters. However, the estimation of a stable seasonal component is not greatly affected by a small amount of trend which may not have been removed, so that very powerful methods of trend removal are not needed in this case.

It should be noted that we wish merely to remove most of the trend, not to estimate it. Sometimes we may also wish to estimate the trend in its own right; then an adjustment will have to be made for the effects of the seasonal pattern. Thus if \( T \) is any operator of the type which estimates trend, we have

\[
Ty_t = Tp_t + Ts_t + Tx_t . \tag{3.1.2}
\]

We may in general assume that \( Tp_t \neq p_t \), so that after
estimating the seasonal component by \( s_t \) we can obtain an estimate of the trend of the form

\[
\hat{p}_t = Ty_t - T_s_t .
\]

There may also be a contribution from the term \( T_x_t \), called the Slutsky-Yule effect; this matter will not be discussed further here, however, as it is not part of the present purpose.

Three approaches are commonly taken to the problem of trend removal. The first is the drawing of a freehand curve and if the trend is all that is of interest there may well be cases when this is the most sensible thing to do. Many procedures for seasonal adjustment use iterated estimates of trend and seasonal components; if the iteration is continued until the resulting trend estimate "looks like the true trend", then it seems that a freehand trend curve might just as well have been drawn in the first place. However, we will need to know the effect of the attempted trend removal on the seasonal and residual components; since this can be known only if a definite rule is followed in removing the trend we will not consider freehand curves further; it should be noted that this is the only good reason why a freehand trend line is not to be recommended.

A second method is the fitting of a mathematical function by least squares, commonly a polynomial or in particular a set of orthogonal polynomials. However, it may
be difficult to judge the degree of the polynomial, and in any case most trends in economic series would require a polynomial of a degree higher than is computationally practicable; even if a fairly low degree polynomial is satisfactory for the given series it may well be useless when a few more years' data become available. Some reference is made to polynomial trends in Hannan (1960b) but they will not be further mentioned here. It would be possible to represent the trend by low frequency Fourier terms but these are very difficult to estimate (see section 2.4) and it would be hard to know which frequencies to select. Finally, some special mathematical functions such as the logistic curve may occasionally be relevant in theory but are unlikely to be suited to the present practical purpose.

We therefore assume that the third method, a moving average, will be applied; this is a simple and usually effective means of trend elimination. The span and the relative weights should be chosen suitably; the most common ones are the simple moving average, Spencer's 15 point or 21 point formula, and first or second differences. These will be discussed in detail in the next section. The rationale of the moving average method is that it is the result of fitting approximating polynomials of low degree to successive short
sequences of observations. However, the effect of applying a moving average is best expressed in terms of its frequency response function as is done in the next section.
3.2 The frequency response function of a moving average.

The application of a moving average to a series affects the components with different frequencies differently; it leaves certain proportions of each component in the resulting series while it removes the rest. Hence it is called a filter, by analogy with some physical situations. A moving average is in fact a linear filter; the general formula for the transformed values $y'_t$ is

$$y'_t = \sum_{s=-p}^{q} \delta_s y_{t+s}, \quad t = p+1, \ldots, n-q. \quad (3.2.1)$$

The effect of a linear filter on the spectral density of a series can be derived as follows. The new covariance function will be

$$\gamma'_t = \mathbb{E} \left\{ \sum_j \delta_j y_{s+j} \sum_k \delta_k y_{s+t+k} \right\} = \sum \sum \delta_j \delta_k \gamma_{t+k-j} = \sum \sum \delta_j \delta_k \int_{-\pi}^{\pi} e^{i(t+k-j)\lambda} dF(\lambda)$$

$$= \int_{-\pi}^{\pi} e^{it\lambda} | \sum_j \delta_j e^{ij\lambda} |^2 dF(\lambda).$$

By comparison with the exponential form of (2.2.6) it is seen that the spectral density has been multiplied by a factor

$$| h(\lambda) |^2 = \left| \sum_{j=-p}^{q} \delta_j e^{ij\lambda} \right|^2, \quad -\pi \leq \lambda \leq \pi, \quad (3.2.2)$$
called the power transfer function, or squared gain, of the filter (3.2.1). The factor

$$h(\lambda) = \sum_{j=-p}^{q} \delta_j e^{ij\lambda}$$  \hspace{1cm} (3.2.3)

is called the frequency response function, or filter factor; it is in general complex-valued but it is real if the weights \( \delta_j \) are symmetrical when we have

$$h(\lambda) = \delta_0 + 2 \sum_{j=1}^{p} \delta_j \cos \lambda j$$  \hspace{1cm} (3.2.4)

The formulae for the moving average operators mentioned in the previous section are given in Table 1 and their frequency response functions in Table 2. In each case the filter is one which removes, rather than estimates, the trend. The notation for a simple moving average of \( k \) terms is \( \frac{1}{k} [k] \) and in other cases the weights are shown in square brackets. The notation \( S_n(\lambda) \) was introduced in (2.4.10). No values have been assigned for the first \( p \) and last \( q \) points of time. If the observed series is fairly long there will be little harm in simply omitting these end values; if the series is short, however, the least evil may be to assign the end values by eye or by using some arbitrary rule.
Table 1. Moving Average Formulae.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>First differences</td>
<td>([-1, 1])</td>
</tr>
<tr>
<td>Second differences</td>
<td>([-1, 2, -1])</td>
</tr>
<tr>
<td>Centred 12 term</td>
<td>(\frac{1}{24} \left[ 2 \right] \left[ 12 \right])</td>
</tr>
<tr>
<td>Spencer's 15 point</td>
<td>([1] - \frac{1}{320} \left[ 4 \right]^2 \left[ 5 \right] [-3, 3, 4, 3, -3])</td>
</tr>
<tr>
<td>Spencer's 21 point</td>
<td>([1] - \frac{1}{350} \left[ 5 \right]^2 \left[ 7 \right] [-1, 0, 1, 2, 1, 0, -1])</td>
</tr>
</tbody>
</table>

Table 2. Frequency Response Functions.

<table>
<thead>
<tr>
<th>Filter</th>
<th>(h(\lambda))</th>
</tr>
</thead>
<tbody>
<tr>
<td>First differences</td>
<td>(1 - \cos \lambda - i \sin \lambda)</td>
</tr>
<tr>
<td>Second differences</td>
<td>(4 \sin^2 \frac{1}{2} \lambda)</td>
</tr>
<tr>
<td>Centred 12 term</td>
<td>(1 - \frac{1}{24} S_2(\lambda) S_{12}(\lambda))</td>
</tr>
<tr>
<td>Spencer's 15 point</td>
<td>(1 - \frac{1}{320} S_4^2(\lambda) S_5(\lambda) {4 + 6 \cos \lambda - 6 \cos 2\lambda})</td>
</tr>
<tr>
<td>Spencer's 21 point</td>
<td>(1 - \frac{1}{350} S_5^2(\lambda) S_7(\lambda) {2 + 2 \cos \lambda - 2 \cos 3\lambda})</td>
</tr>
</tbody>
</table>

The values of some frequency response functions are given in Table 3. The frequency response function of several linear filters applied successively is obviously the product of the separate frequency response functions, and that for the difference of two filters is the difference of the functions.
### Table 3. Values of Frequency Response Functions.

<table>
<thead>
<tr>
<th>Frequency $\pi j/2^4$</th>
<th>Centred 12 term moving average</th>
<th>Second differences</th>
<th>Spencer's 15 point formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1</td>
<td>0.101</td>
<td>0.017</td>
<td>0.001</td>
</tr>
<tr>
<td>2</td>
<td>0.367</td>
<td>0.068</td>
<td>0.016</td>
</tr>
<tr>
<td>3</td>
<td>0.704</td>
<td>0.152</td>
<td>0.073</td>
</tr>
<tr>
<td>4</td>
<td>1.000</td>
<td>0.268</td>
<td>0.191</td>
</tr>
<tr>
<td>5</td>
<td>1.174</td>
<td>0.413</td>
<td>0.369</td>
</tr>
<tr>
<td>6</td>
<td>1.201</td>
<td>0.568</td>
<td>0.575</td>
</tr>
<tr>
<td>7</td>
<td>1.119</td>
<td>0.782</td>
<td>0.767</td>
</tr>
<tr>
<td>8</td>
<td>1.000</td>
<td>1.000</td>
<td>0.906</td>
</tr>
<tr>
<td>9</td>
<td>0.912</td>
<td>1.235</td>
<td>0.981</td>
</tr>
<tr>
<td>10</td>
<td>0.891</td>
<td>1.482</td>
<td>1.005</td>
</tr>
<tr>
<td>11</td>
<td>0.933</td>
<td>1.739</td>
<td>1.003</td>
</tr>
<tr>
<td>12</td>
<td>1.000</td>
<td>2.000</td>
<td>1.000</td>
</tr>
<tr>
<td>13</td>
<td>1.052</td>
<td>2.261</td>
<td>1.004</td>
</tr>
<tr>
<td>14</td>
<td>1.064</td>
<td>2.518</td>
<td>1.012</td>
</tr>
<tr>
<td>15</td>
<td>1.039</td>
<td>2.765</td>
<td>1.016</td>
</tr>
<tr>
<td>16</td>
<td>1.000</td>
<td>3.000</td>
<td>1.012</td>
</tr>
<tr>
<td>17</td>
<td>0.971</td>
<td>3.218</td>
<td>1.005</td>
</tr>
<tr>
<td>18</td>
<td>0.965</td>
<td>3.414</td>
<td>1.000</td>
</tr>
<tr>
<td>19</td>
<td>0.980</td>
<td>3.587</td>
<td>1.000</td>
</tr>
<tr>
<td>20</td>
<td>1.000</td>
<td>3.732</td>
<td>1.003</td>
</tr>
<tr>
<td>21</td>
<td>1.012</td>
<td>3.848</td>
<td>1.005</td>
</tr>
<tr>
<td>22</td>
<td>1.011</td>
<td>3.932</td>
<td>1.004</td>
</tr>
<tr>
<td>23</td>
<td>1.004</td>
<td>3.983</td>
<td>1.002</td>
</tr>
<tr>
<td>24</td>
<td>1.000</td>
<td>4.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Some power transfer functions are shown in Figure 1. The power transfer function for first differences is $4 \sin^2 \frac{1}{2} \lambda$, while for the other filters it is simply the square of the corresponding frequency response function. The graphs in Figure 1 show the extent to which each filter removes trend (the low frequency components) and its effect at the other frequencies. The neat expression of the effects of a moving average is an
Figure 1 Power Transfer Functions
(Note Scales)

First Differences

Second Differences

12 Term Moving Average

Spencer's 15 Point Formula

Spencer's 21 Point Formula
advantage of the spectral approach.

The first difference operator is not very effective in removing trend and, being asymmetrical, complicates later calculations slightly. On the other hand Spencer's 21 point formula is more effective at very low frequencies than is usually required and loses ten values at each end of the series. The centred 12 term moving average is appropriate for a monthly series, as is a k-term moving average for a series observed k times per year. Second differences and Spencer's 15 point formula are likely to be suitable for most economic series. It may be noted that successive differencing is appropriate if the series fits a model in which the observations are generated as successive sums of a stationary random variable.

Since, as will be seen in the next section, the points are known where a stable seasonal pattern causes jumps in the spectrum, the frequency response function also shows the precise 'filter effect' on a stable seasonal pattern. If the function happens to be unity at all the jump points, as it does for a 12 term moving average and a monthly pattern, then the operator does not affect the seasonal pattern at all; but this is not the general case. Often in practice the filter effects have been removed to some extent by iterative procedures such as those of Shiskin and Eisenpress (1957); it is now clear (see Hannan (1960b)) that iteration is unnecessary.
Various tests are available to determine whether the trend has been eliminated. However, the author does not believe these are ever useful in practice for the present purposes, and suggests instead judgment by eye or estimation of the spectrum of the residuals. Tintner (1952) gives a test for trend (pp.211-215) but in the two examples he gives the result of the test is obvious at sight, so the computations are pointless.
3.3 Least squares estimates of the seasonal constants.

In all practical problems the data should firstly be examined, for instance by plotting the monthly deviations from trend, to see whether it can reasonably be assumed that there is a stable seasonal pattern. Tests using the analysis of variance can be devised for the hypothesis of no seasonal variation but these tests are only approximate and they are not usually necessary.

There are many possibilities. No regular pattern may be apparent at all, in which case it is naturally pointless to estimate one. A stable seasonal pattern may be apparent. Alternatively the pattern may appear to have changed in some way, when the methods of the next chapter may be appropriate. If some unusual event has seriously affected the values for several months they may have to be omitted. An extension of this last idea is sometimes used in which past values are weighted by say geometrically decreasing weights (see I.B.M. Australia (undated)). Such methods may be very sensible on occasions but we do not discuss them further here since they are best applied to particular cases rather than as a general procedure.

We now assume that the seasonal pattern is stable and derive estimates for a monthly series. The analysis is very easily modified for a quarterly, fortnightly or other series so that the loss of generality should be outweighed by a gain
in clarity.

The seasonal component can then be written

$$s_t = \sum_{k=1}^{12} \psi_k c_{k,t}, \quad t = 1, \ldots, n$$

(3.3.1)

where \( c_{k,t} \) is unity when 12 divides \( t-k \) and zero otherwise and we specify

$$\sum_{k=1}^{12} \psi_k = 0.$$ 

(3.3.2)

For the purpose of deriving estimates, however, we write it in an equivalent but more basic form:

$$s_t = \sum_{j=1}^{6} (\alpha_j \cos \lambda_j t + \beta_j \sin \lambda_j t), \quad t = 1, \ldots, n$$

(3.3.3)

where

$$\lambda_j = \frac{2\pi j}{12}.$$ 

(3.3.4)

In either form there are eleven parameters to be estimated; the twelve quantities \( \psi_k \) are constrained by (3.3.2) while in (3.3.3) \( \beta_6 \) is arbitrary and is included only for symmetry. The relations between the two alternative sets of parameters are:

$$\psi_k = \sum_{j=1}^{6} (\alpha_j \cos \lambda_j k + \beta_j \sin \lambda_j k), \quad k = 1, \ldots, 12$$

(3.3.5)

and

$$\begin{align*}
\alpha_j &= \frac{1}{6} \sum_{k=1}^{12} \psi_k \cos \lambda_j k \cdot \left( 1 - \frac{1}{2} \delta_{j,6} \right) \\
\beta_j &= \frac{1}{6} \sum_{k=1}^{12} \psi_k \sin \lambda_j k \\
\end{align*}$$

(3.3.6)

where \( \delta_{j,6} \) is the Kronecker delta.
By comparing (3.3.3) with (2.2.8) and the discussion in section 2.3 it is seen that the case of a stable seasonal pattern reduces to the estimation of jumps of magnitude $\frac{1}{2}(\alpha_j^2 + \beta_j^2)$ at the points $\lambda_j$ of the spectrum. This should be regarded as an approximation to the true spectrum in which these jumps will be replaced by continuous peaks, since the seasonal pattern is not precisely stable. The largest component will usually be that for $j = 1$, since this simple sine curve represents the climatic seasons.

When a suitable moving average has been selected the model (3.1.1) becomes

$$y_t' = p_t' + s_t' + x_t', \quad t = 1+p, \ldots, n-q$$

where the primes indicate that a moving average has been applied, involving the loss of $p$ values at the start and $q$ at the end of the series. We will however continue to write the range as $t = 1, \ldots, n$ without confusion.

We now assume that $p_t' = 0$, that is, that the moving average has in fact removed the trend. An estimate of $p_t'$ could be obtained by applying the original filter to the series $y_t'$ and if the resulting series, say $y_t''$, were not negligible it could be subtracted from $y_t'$ leading to an iterative procedure; however, if the trend has not been removed it is preferable to try to find a better initial filter.
For the transformed seasonal component we obtain:

\[ s_t' = \sum_{j=1}^{6} (\alpha_j' \cos \lambda_j t + \beta_j' \sin \lambda_j t), \quad t = 1, \ldots, n \]  

(3.3.8)

where

\[
\begin{align*}
\alpha_j' &= \alpha_j \sum_j \delta_s \cos \lambda_j s - \beta_j \sum_j \delta_s \sin \lambda_j s \\
\beta_j' &= \alpha_j \sum_j \delta_s \sin \lambda_j s + \beta_j \sum_j \delta_s \cos \lambda_j s
\end{align*}
\]

(3.3.9)

and the \( \delta_s \) are the weights in the moving average.

In the case of a filter with symmetrical weights the sine terms in (3.3.9) vanish in pairs, giving

\[ \frac{\alpha_j'}{\alpha_j} = \frac{\beta_j'}{\beta_j} = h(\lambda_j), \quad j = 1, \ldots, 6, \]  

(3.3.10)

where \( h(\lambda) \) is the frequency response function of the filter, defined in (3.2.4). Thus these equations reflect the way in which the spectrum has been changed at the points \( \lambda_j \). We note that the inverse relation to (3.3.9) is

\[
\begin{align*}
\alpha_j &= (\alpha_j' \sum \delta_s \cos \lambda_j s + \beta_j' \sum \delta_s \sin \lambda_j s) |h(\lambda_j)|^2 \\
\beta_j &= (-\alpha_j' \sum \delta_s \sin \lambda_j s + \beta_j' \sum \delta_s \cos \lambda_j s) |h(\lambda_j)|^2
\end{align*}
\]

(3.3.11)

while in the symmetrical case the inverse becomes

\[ \frac{\alpha_j}{\alpha_j'} = \frac{\beta_j}{\beta_j'} = 1/h(\lambda_j), \quad j = 1, \ldots, 6. \]  

(3.3.12)

The model in its most basic form is then

\[ y_t' = \sum_{j=1}^{6} (\alpha_j' \cos \lambda_j t + \beta_j' \sin \lambda_j t) + x_t', \quad t = 1, \ldots, n. \]  

(3.3.13)
It has been shown by Grenander (1954) that the least squares procedure in this situation is asymptotically efficient; it is not the maximum likelihood procedure since even if the residuals were normally distributed they are not in general independent, but no better procedure is obtainable by that method. Least squares is in any case the natural and common procedure even for small $n$. The least squares estimates are:

\[ \begin{align*}
\hat{\theta}_j' &= \frac{2}{n} \sum_{t=1}^{n} y_t' \cos \lambda_j t. \left( 1 - \frac{1}{2} \delta_j, \delta_j' \right) \\
\hat{\beta}_j' &= \frac{2}{n} \sum_{t=1}^{n} y_t' \sin \lambda_j t
\end{align*} \]

We have assumed here that the number of observations is a multiple of 12; this should always be arranged by truncating the trend-free series if necessary, for otherwise strict least squares estimation will involve the inversion of a matrix, although the above formulae could still be used with little error.

Since the sines and cosines have period 12 these estimates can be written more simply, from a computational point of view, in terms of the monthly averages of the trend-free series, of which the traditional methods essentially consist. Let

\[ \hat{\psi}_k' = \frac{1}{d} \sum_{s=1}^{d} y_{12s+k}' \;
\]

where $d$ is the number of years of data. The $\hat{\psi}_k'$ should be adjusted to sum to zero in view of (3.3.2). We then have
Unbiased estimates $\hat{\alpha}_j$, $\hat{\beta}_j$ of the original parameters are then obtained by substituting the estimates (3.3.14) or (3.3.16) in (3.3.11) or by simply dividing by $h(\lambda_j)$ as in (3.3.12) in the symmetrical case, which we write out:

$$
\hat{\alpha}_j = \frac{2}{n} \sum_{t} y'_t \cos \lambda_j t \cdot (1 - \frac{1}{2} \delta_{j,6}) / h(\lambda_j) \\
\hat{\beta}_j = \frac{2}{n} \sum_{t} y'_t \sin \lambda_j t / h(\lambda_j)
$$

In the symmetrical case, the estimated monthly seasonal adjustment factors $\hat{\psi}_k$ in the original form (3.3.1) are then found by substituting the estimates $\hat{\alpha}_j$, $\hat{\beta}_j$ in (3.3.5); they are:

$$
\hat{\psi}_k = \sum_{s=1}^{12} \hat{\psi}'_s v_{s-k}, \quad k = 1, \ldots, 12,
$$

where

$$
\nu_{\ell} = \frac{1}{6} \sum_{j=1}^{12} \{ (\cos \lambda_j \ell \sum_{r=-p}^{q} \delta_r \cos \lambda_j r + \sin \lambda_j \ell \sum_{r=-p}^{q} \delta_r \sin \lambda_j r) \\
\quad \times \left(1 - \frac{1}{2} \delta_{j,6} \right) / |h(\lambda_j)|^2 \} \\
\quad \times \frac{1}{12} \sum_{j=1}^{11} \left\{ e^{i\lambda_j \ell} / h(\lambda_j) \right\}, \quad \ell = -11, \ldots, 11,
$$

or in the symmetrical case

$$
\nu_{\ell} = \frac{1}{6} \sum_{j=1}^{6} \left\{ \cos \lambda_j \ell \cdot (1 - \frac{1}{2} \delta_{j,6}) / h(\lambda_j) \right\}, \quad \ell = -11, \ldots, 11.
$$

... (3.3.20)
The values $v_\ell$ can be computed in advance for any given moving average and the calculation of the estimates is then extremely simple using (3.3.15) and (3.3.18). The values of $v_\ell$ for second differences and for Spencer's 15 point formula are given in Table 4. An alternative and possibly simpler procedure is available when second (or other) differences are used: this is to take successive sums of the unadjusted estimates, which clearly inverts the effect of differencing (see Hannan (1963) p. 39). For a 12 term moving average, as for any filter whose frequency response is unity at the points $\lambda_j$, no adjustment is to be made, that is, $\hat{\psi}_k = \hat{\psi}'_k$, $k = 1, \ldots, 12$.

<table>
<thead>
<tr>
<th>$t \ell$</th>
<th>Second Differences</th>
<th>Spencer's 15 Point Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>143</td>
<td>1.638</td>
</tr>
<tr>
<td>1,11</td>
<td>77</td>
<td>0.538</td>
</tr>
<tr>
<td>2,10</td>
<td>-1</td>
<td>0.262</td>
</tr>
<tr>
<td>3,9</td>
<td>-19</td>
<td>-0.103</td>
</tr>
<tr>
<td>4,8</td>
<td>-49</td>
<td>-0.444</td>
</tr>
<tr>
<td>5,7</td>
<td>-67</td>
<td>-0.686</td>
</tr>
<tr>
<td>6</td>
<td>-73</td>
<td>-0.774</td>
</tr>
</tbody>
</table>

In summary, then, we apply a smoothing operator to the series, estimate the seasonal pattern in the trend-free series by taking monthly averages $\hat{\psi}'_k$, and then adjust by means of (3.3.18) for the effect, if any, which the smoothing operation
had on the seasonal pattern. In the past, when this adjustment has been made at all it has usually been made by iteration (see for example Shiskin and Eisenpress (1957)) which does not normally reduce the bias very rapidly (see Hannan (1963) p.38) and is, in view of the above results, unnecessary.

Durbin (1963) has shown that the computation of the $\hat{\Psi}_k$ reduces to the averaging of the monthly means without trend removal, followed by a simple correction based on the end values of the series. However, compressing the analysis in this way gains little since with a computer available the time saving is trivial, and it will often be desired to compute the trend in any case, for it was pointed out at the beginning of this section that the deviations from trend should be plotted before estimating any seasonal component; otherwise we are voluntarily working in the dark.
3.4 Variances and covariances of the estimates.

To derive the variances and covariances we work in terms of the $\alpha_j$ and $\beta_j$ since these are more easily manipulated than the forms given for easy computation. We repeat the relevant formulae from the previous section:

\[ y'_t = \sum_{j=1}^{6} (\alpha'_j \cos \lambda_j t + \beta'_j \sin \lambda_j t) + x'_t, \quad t = 1, \ldots, n \]  

(3.4.1)

\[ \hat{\alpha}'_j = \frac{2}{n} \sum_{t=1}^{n} y'_t \cos \lambda_j t \left( 1 - \frac{1}{2 \delta_j} \right) \]

\[ \hat{\beta}'_j = \frac{2}{n} \sum_{t=1}^{n} y'_t \sin \lambda_j t \]

(3.4.2)

The $\hat{\alpha}'_j$ and $\hat{\beta}'_j$ are related to the primed estimates by (3.3.11) or (3.3.12) but as a first step we derive the variances and covariances of the primed estimates, that is, before adjusting for the filter effects.

Since we are assuming (see section 2.2) that $\varepsilon(x_t) = 0$ and that $\gamma_t = \varepsilon(x_s x_{s+t})$ is independent of $s$, we have

\[ \text{var}(\hat{\alpha}'_j) = \varepsilon \left\{ \frac{2}{n} \sum_{t} y'_t \cos \lambda_j t \left( 1 - \frac{1}{2 \delta_j} \right) - \alpha'_j \right\}^2 \]

\[ = \varepsilon \left\{ \left[ \frac{2}{n} \sum_{t} x'_t \cos \lambda_j t \left( 1 - \frac{1}{2 \delta_j} \right) \right]^2 \right\} \]

\[ = \frac{1}{2} \sum_{s} \sum_{t} \gamma'_{t-s} \cos \lambda_j s \cos \lambda_j t \left( 1 - \frac{1}{2 \delta_j} \right)^2 \]

\[ = \frac{8}{n^2} \int_{0}^{\pi} \sum_{s} \sum_{t} \cos (t-s) \lambda \cos \lambda_j s \cos \lambda_j t \ dF'(\lambda) \left( 1 - \frac{1}{2 \delta_j} \right)^2 \]

where $\gamma'$ and $F'$ are the covariance and spectral distribution.
functions of $x'_t$; in fact from section 3.2 we have $F'/(\lambda) = |h(\lambda)|^2 F(\lambda)$, where $F(\lambda)$ is the spectral distribution function of the residuals $x_t$.

By writing the product of cosines as the real part of a sum of complex exponentials and noting that

$$\sum_{t=1}^{n} e^{i\lambda t} = S_n(\lambda) \exp\left\{ \frac{1}{2} i(n+1)\lambda \right\}, \quad (3.4.3)$$

where $S_n(\lambda)$ was defined in (2.4.10), we obtain $\text{var}(\hat{\alpha}'_j)$ and similarly $\text{var}(\hat{\beta}'_j)$. We finally substitute these expressions in (3.3.11) or (3.3.12) to obtain $\text{var}(\hat{\alpha}_j)$ and $\text{var}(\hat{\beta}_j)$. We give the formulae below only in the case of a symmetrical filter:

$$\text{var}(\hat{\alpha}_j) = \frac{2}{n^2} \int_{0}^{\pi} \left\{ S_n^2(\lambda+\lambda_j) + S_n^2(\lambda-\lambda_j) + 2S_n(\lambda+\lambda_j)S_n(\lambda-\lambda_j) \cos \lambda_j \right\} \left\{ \frac{h^2(\lambda)}{h^2(\lambda_j)} \right\} dF(\lambda) (1 - \frac{1}{2\delta_j,6})^2, \quad j = 1, \ldots, 6. \quad (3.4.4)$$

For $\text{var}(\hat{\beta}_j)$ it is only necessary to change the sign of the cross-product term. For the covariances we find

$$\text{cov}(\hat{\alpha}_j, \hat{\alpha}_k) = \frac{(-1)(j+k)}{d(2/n^2)} \int_{0}^{\pi} \left\{ S_n(\lambda+\lambda_j) S_n(\lambda+\lambda_k) + S_n(\lambda-\lambda_j) S_n(\lambda-\lambda_k) \cos \frac{1}{2}(\lambda_j-\lambda_k) + S_n(\lambda+\lambda_j) S_n(\lambda-\lambda_k) \cos \frac{1}{2}(\lambda_j+\lambda_k) \right\} \left\{ \frac{h^2(\lambda)}{h(\lambda_j)h(\lambda_k)} \right\} dF(\lambda) (1 - \frac{1}{2\delta_j,6})(1 - \frac{1}{2\delta_k,6}), \quad j, k = 1, \ldots, 6. \quad (3.4.5)$$
For $\text{cov}(\hat{\beta}_j, \hat{\beta}_k)$ the sign before the second square bracket is changed, while for $\text{cov}(\hat{\alpha}_j, \hat{\beta}_k)$ the two cosine factors become sines. We do not require higher order moments.

It will be seen in section 3.6 that the variances and covariances of the $\hat{\alpha}_j$ and $\hat{\beta}_j$ have very convenient asymptotic properties. The precise variances and covariances of the estimates of the original seasonal constants $\hat{\nu}_k$ follow on substitution into (3.3.5); they are long expressions and we give their asymptotic values in section 3.6.
3.5 The effect of trend removal on the variance.

In this section we analyse in detail the variance of the estimates of the stable seasonal parameters. Since the component \( j = 1 \) is likely to be the most important one, and since results for the other components are analogous, we restrict attention to \( \text{var}(\hat{\alpha}_1) \). We also take a five-year monthly series, that is, \( n = 60 \). This is a small value of \( n \) and it is the minimum value likely to occur for economic series, so the results will be helpful in judging the relevance of the asymptotic formulae to be obtained in the next section.

If no trend removing operator were applied the variance, from (3.4.4), would be

\[
\text{var}(\hat{\alpha}_1) = \frac{2}{3600} \int_0^{\pi} \left\{ S_{60}^2(\lambda+\lambda_1) + S_{60}^2(\lambda-\lambda_1) + \right.
\]

\[
\sqrt{3} S_{60}(\lambda+\lambda_1) S_{60}(\lambda-\lambda_1) \left\} \, dF(\lambda)
\]

\[
\approx 2 \int_0^{\pi} K_{60}(\lambda) \, dF(\lambda), \quad (3.5.1)
\]

where \( F(\lambda) \) is the spectral distribution function of the residuals after removing by regression the components at the seasonal frequencies, but not the trend component. The weight, or 'kernel', \( K_n(\lambda) \), is heavily concentrated around the point \( \lambda = \lambda_1 = \pi/6 \) since \( S_n^2(\lambda) \) has a peak of \( n^2 \) at the origin which becomes rapidly sharper as \( n \) increases. The first and third terms in \( K_n(\lambda) \) therefore contribute very little at any frequency
The weights $K_{60}(\lambda)$ are given at some selected frequencies in the second column of Table 5.

It is already clear that the variance will be greater on account of the presence of trend if $f(\lambda)$ is so concentrated near the origin as to outweigh the weight $K_n(\lambda)$ which is small near the origin. The purpose of the remainder of this section is to make this issue clearer by quantifying it to some extent and also to consider what effects trend removal may have at other frequencies, particularly those near $\pi/6$ where $K_n(\lambda)$ is concentrated.

It is now required to specify the spectrum. We could use the estimated spectrum of an actual series but it would be convenient to have one which can be described in terms of a few parameters. We therefore consider an autoregressive process with parameter near to one.

Let the autoregressive relation be

$$z_t = \rho z_{t-1} + \epsilon_t, \quad t = 1, 2, \ldots$$

where $|\rho| < 1$ and the $\epsilon_t$ are uncorrelated random variables with zero mean and variance $\sigma^2$ and therefore have a constant spectral density function $\sigma^2/2\pi$. This model is often useful in economics and elsewhere as it expresses the current observation as a fraction of the preceding value, plus an independent 'shock' variable. By the method of section 3.2 we obtain for
the spectrum \( f(\lambda) \) of \( z_t \) the relation

\[
|1 - \rho e^{-i\lambda}|^2 f(\lambda) = \sigma^2/2\pi
\]

whence

\[
f(\lambda) = \sigma^2/2\pi(1 - 2\rho \cos \lambda + \rho^2)
\]  \hspace{1cm} (3.5.3)

The values of this function, apart from the constant factor, are given in the third column of Table 5 for \( \rho = .99 \). As is to be expected, the spectrum shows high concentration at very low frequencies which means that the series \( z_t \) will be quite smooth. The spectra of different economic series may of course be more or less concentrated at the origin than the autoregressive one, but the autoregressive process with \( \rho = .99 \) will be satisfactory for present purposes, as it will be easy to see what would happen in other cases.

It is likely that there will be small peaks remaining in the spectrum near the frequencies \( \pi j/6 \) even after the regression has been carried out, particularly as each regression takes place at only one point whereas the corresponding peak will be spread out to some extent. Thus the whole peak will not be removed but only the centre of it. We do not make any adjustment in \( f(\lambda) \) for this, but its possible effects will be easy to see.

Thus as a first step in demonstrating the relevant issues we plot in Figure 2 the ordinates in the integral (3.5.1) with \( f(\lambda) \) as in (3.5.3) and \( \rho = .99 \); the values are given in the fourth column of Table 5. It is seen that the effect of the
presence of trend is small compared with the contribution near \( \pi/6 \), but it must be remembered that many actual series will have much greater spectral mass near the origin than has the autoregressive series. It is not easy to find a way to characterize a spectral density which would have this greater concentration, so we simply note that the peak near the origin may often be greater than in Figure 2, though it will sometimes be even less.

When a (symmetric) trend removing operator is applied the weights in (3.5.1) are multiplied by \( h^2(\lambda)/h^2(\lambda_1) \) where \( h(\lambda) \) is the frequency response function defined in (3.2.4) and tabulated for three filters in Table 3; see also Figure 1. The multipliers are shown in more detail in the last three columns of Table 5 for the same three filters, and the results of the multiplication of the ordinate values by these quantities are graphed in Figure 2.
Table 5. The Effect of Trend Removal on $\text{var}(\hat{\lambda}_1)$.

<table>
<thead>
<tr>
<th>$\lambda$ (degrees)</th>
<th>$k_{60}(\lambda)$</th>
<th>$2\pi f(\lambda)$</th>
<th>Ordinate in $\text{var}(\hat{\lambda}_1)$</th>
<th>$h^2(\lambda)/h^2(\lambda_{1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2) x (3)</td>
<td>12 mth moving avge</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0 8</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0001</td>
<td>5580</td>
<td>0.42</td>
<td>$2\times10^{-8}$</td>
</tr>
<tr>
<td>1</td>
<td>0.0003</td>
<td>2519</td>
<td>0.71</td>
<td>$3\times10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>0.0009</td>
<td>765</td>
<td>0.68</td>
<td>$5\times10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>0.0013</td>
<td>356</td>
<td>0.46</td>
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</tr>
<tr>
<td>4</td>
<td>0.0011</td>
<td>203</td>
<td>0.22</td>
<td>0.0009</td>
</tr>
<tr>
<td>5</td>
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<td>0.05</td>
<td>0.0021</td>
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</tr>
<tr>
<td>9</td>
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<td>41</td>
<td>0.12</td>
<td>0.13</td>
</tr>
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</tr>
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<td>15</td>
<td>0.13</td>
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<td>5.00</td>
<td>0.75</td>
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<tr>
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<td>4.31</td>
<td>2.77</td>
<td>0.93</td>
</tr>
<tr>
<td>26</td>
<td>0.8842</td>
<td>4.03</td>
<td>3.56</td>
<td>0.97</td>
</tr>
<tr>
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<td>0.9652</td>
<td>3.89</td>
<td>3.75</td>
<td>1.23</td>
</tr>
<tr>
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<td>1</td>
<td>3.77</td>
<td>3.77</td>
<td>1.03</td>
</tr>
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<td>3.65</td>
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<td>0.04</td>
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<td>1.11</td>
<td>0.01</td>
<td>1.09</td>
</tr>
<tr>
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<td>0</td>
<td>0.92</td>
</tr>
<tr>
<td>51</td>
<td>0.0063</td>
<td>0.93</td>
<td>0.01</td>
<td>0.92</td>
</tr>
<tr>
<td>54</td>
<td>0.0049</td>
<td>0.79</td>
<td>0.00</td>
<td>0.81</td>
</tr>
<tr>
<td>57</td>
<td>0</td>
<td>0.51</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>60</td>
<td>0.0049</td>
<td>0.79</td>
<td>0.00</td>
<td>0.93</td>
</tr>
<tr>
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<td>0.50</td>
<td>0</td>
<td>0.93</td>
</tr>
<tr>
<td>66</td>
<td>0</td>
<td>0.25</td>
<td>0</td>
<td>1</td>
</tr>
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<tr>
<td>72</td>
<td>0</td>
<td>0.25</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The meaning of each column of the table is given in the text.
Before filtering.

After applying Spencer's 15 point formula.

After applying a 12 month moving average.

Second differences: intermediate result, not shown.

Figure 2. $\text{Var}(\alpha_i)$ Before and After Filtering.
Each filter removes the contribution from trend almost completely. However, the effect of applying the filters at some other frequencies, particularly those near and above $\pi/6$, is in some cases disadvantageous, due to the shape of the power transfer functions at those frequencies. The effect near $\pi/6$ may be greater than that appearing in Figure 2 because of a possible peak remaining in $f(\lambda)$, as mentioned above.

The comparison between the three filters in the present case favours the 12 month moving average, since its comparative lack of power to remove trend is not important while its small effect at other frequencies is an advantage. The high efficiency of Spencer's 15 point formula in removing trend is not needed here while its effects at other frequencies are unfavourable. Second differencing gives results intermediate between these two but the curve is not drawn in Figure 2 to keep the diagram clear; note that although this operator magnifies the high frequencies greatly the weight $K_n(\lambda)$ is extremely small there, as is the spectrum. Similar analyses could be given in any problem of seasonal estimation. One would of course never go to such lengths in practice, but would consider the matter more briefly.

The conclusion from the study in this section is, then, that if a given series has a large trend component it will certainly be wise to eliminate it, but that if it does not have a large trend component then it may even be harmful to apply a moving
average as a matter of routine, since the small gain through removal of the contribution to $\text{var}(\hat{\lambda}_j)$ at low frequencies may be more than offset by a loss due to the shape of the power transfer function at other frequencies, particularly those near $\lambda_j$. These effects are, however, only likely to occur when $n$ is small.
3.6 The asymptotic variances and covariances.

In this section we obtain asymptotic formulae for the variances and covariances in the case of a symmetrical filter as the number of monthly observations increases. We proceed to find the asymptotic covariance of $\hat{\alpha}_j$ with $\hat{\alpha}_k$ from which the other asymptotic results will follow.

We write (3.4.5) as the integral from $-\pi$ to $\pi$ of half the value shown. Now as $n \to \infty$ the factor $S_n(\lambda)$ becomes increasingly concentrated at the origin, and since the peaks in the residual spectrum $h^2(\lambda)f(\lambda)$ will be dominated by the peak in $S_n(\lambda)$ it follows that

$$\text{cov}(\hat{\alpha}_j, \hat{\alpha}_k) \approx (-1)^{(j+k)} (1/n^2) \int_{-\pi}^{\pi} \left\{ S_n(\lambda+\lambda_j)S_n(\lambda+\lambda_k) \cos \frac{1}{2} (\lambda_j - \lambda_k) \right. $$

$$+ S_n(\lambda+\lambda_j)S_n(\lambda-\lambda_k) \cos \frac{1}{2} (\lambda_j + \lambda_k) \left. \right\} d\lambda$$

$$+ \left\{ h^2(\lambda_j)f(\lambda_j) + h^2(\lambda_k)f(\lambda_k) \right\} \left\{ 1/h(\lambda_j)h(\lambda_k) \right\}$$

$$(1 - \frac{1}{2\delta_j}) (1 - \frac{1}{2\delta_k}).$$

But the integral is easily evaluated using (3.4.3); thus

$$\int_{-\pi}^{\pi} S_n(\lambda+\lambda_j)S_n(\lambda+\lambda_k) d\lambda = \exp \left\{ -\frac{1}{2} i(n+1)(\lambda_j + \lambda_k) \right\}$$

$$\sum_{s=1}^{n} \sum_{t=1}^{n} \exp \left\{ i(\lambda_j s + \lambda_k t) \right\} \int_{-\pi}^{\pi} \exp \left\{ i\lambda(s+t-n-1) \right\} d\lambda$$

$$= 2\pi \exp \left\{ -\frac{1}{2} i(n+1)(\lambda_j + \lambda_k) \right\} \sum_{s=1}^{n} \exp \left\{ \lambda_j s + \lambda_k (n+1-s) \right\}$$

$$= 2\pi S_n(\lambda_j + \lambda_k)$$

$$= 2\pi n \delta_{j,k}.$$
Hence
\[
\text{var}(\hat{\alpha}_j) \approx \text{var}(\hat{\beta}_j) \approx \left(\frac{4\pi}{n}\right) f(\lambda_j)(1 - \frac{1}{2\delta_j} \cdot 6)^2, \quad j = 1, \ldots, 6, \quad (3.6.1)
\]
while all covariances, including \(\text{cov}(\hat{\alpha}_j, \hat{\beta}_j)\), tend to zero. These are the simple properties mentioned in section 3.4.

The asymptotic variances and covariances of the estimates of the original seasonal constants \(\hat{\psi}_k\) follow by virtue of (3.3.5); they are
\[
\text{var}(\hat{\psi}_k) \approx \left(\frac{4\pi}{n}\right) \sum_{j=1}^{6} f(\lambda_j)(1 - \frac{1}{2\delta_j} \cdot 6)^2 \]
\[
\text{cov}(\hat{\psi}_k, \hat{\psi}_\ell) \approx \left(\frac{4\pi}{n}\right) \sum_{j=1}^{6} f(\lambda_j) \cos \lambda_j(k-\ell) \cdot (1 - \frac{1}{2\delta_j} \cdot 6) \]
\[
\ldots \quad (3.6.2)
\]

An idea of their size can be obtained by setting
\[
f(\lambda_j)/\bar{f}(\lambda) = 100, 10, 3, 2, 1, 0 \quad \text{for} \quad j = 1, 2, 3, 4, 5, 6, \quad \text{where} \quad \bar{f}(\lambda) \quad \text{is any constant; these values probably give a fair indication of the results to be expected.}
\]
We obtain
\[
\text{var}(\hat{\psi}_k) = \left(\frac{4\pi}{n}\right) \cdot 116 \bar{f}(\lambda)
\]
\[
\text{cov}(\hat{\psi}_k, \hat{\psi}_\ell) = \left(\frac{4\pi}{n}\right) \bar{f}(\lambda) \{90, 42, -8, -54, -82, -92\}
\]
for months \(k, \ell\) differing by 1, \ldots, 6. Thus neighbouring \(\hat{\psi}_k\) are highly positively correlated while those far apart are highly negatively correlated, in contrast to the asymptotically uncorrelated estimates \(\hat{\alpha}_j, \hat{\beta}_j\).

The asymptotic results will almost always be usable, in
view of the very rapid concentration of the factor $S_n(\lambda)$; even with only sixty observations, as in the previous section, the terms neglected in the asymptotic results in fact made very small contributions at most frequencies, though we do not tabulate the values of the separate terms. A count of squares in Figure 2 shows about 380 squares, giving a 'true' variance of

$$2 \times 380 \times \frac{1}{15} \times \frac{1}{180} \pi = 0.28\pi$$

compared with the asymptotic formula which gives

$$4 \times \frac{1}{60} \times 3.77\pi = 0.25\pi.$$
3.7 Estimating the variances and testing for significance.

Estimating the variances amounts to estimating the spectrum \( f(\lambda) \) of the residuals in the original model (3.1.1), which may be in logarithms. As explained in section 2.4 we do this by first estimating the spectrum of the residuals after removing the trend and seasonal components and then adjust for the effects of those operations. In section 3.2 we saw that the removal of trend by a moving average simply multiplied \( f(\lambda) \) by the function \( |h(\lambda)|^2 \) defined by (3.2.2), so division by this factor is all that is required. The adjustment for the effect of regressing out the components at frequencies \( \lambda_j, j = 1, \ldots, 6, \) requires more analysis.

Neglecting trend removal for the present we have from (3.3.13) and (3.3.14) that

\[
\hat{x}_t = x_t - \sum_{j=1}^{6} \left\{ \frac{2}{n} \sum_{s=1}^{n} x_s \cos \lambda_j (s-t) \left( 1 - \frac{1}{2} \delta_j, 6 \right) \right\}, \quad t = 1, \ldots, n.
\]

Writing the periodogram in exponential form we then have

\[
I_n(\lambda) = \frac{1}{2\pi n} \sum_{s=1}^{n} \sum_{t=1}^{n} \left\{ \left[ x_s - \sum_{u=1}^{n} \frac{2}{n} \sum_{j=1}^{6} x_u \cos \lambda_j (s-u) \left( 1 - \frac{1}{2} \delta_j, 6 \right) \right] e^{i(s-t)\lambda} \right\}
\]

\[
\left[ x_t - \sum_{v=1}^{n} \frac{2}{n} \sum_{k=1}^{6} x_v \cos \lambda_k (v-t) \left( 1 - \frac{1}{2} \delta_k, 6 \right) \right] e^{i(s-t)\lambda}
\]

\[
\ldots \quad (3.7.1)
\]

Clearly the expected value of the first of the four terms in
(3.7.1) approaches \( f(\lambda) \). Using (2.2.6) the expected value of the second term becomes

\[
- \left( \frac{1}{2n} \right) \frac{2}{n} \int_{-\pi}^{\pi} f(\theta) \sum_{s=1}^{n} e^{i(\lambda - \theta)s} \left( \sum_{k=1}^{6} \sum_{t=1}^{n} \cos \lambda_k(v-t) e^{iv\theta} e^{-it\lambda} d\theta \right) (1 - \frac{1}{2} \delta_k, \epsilon)
\]

\[
\approx -\left( \frac{1}{\pi \sqrt{n}} \right) f(\lambda) \sum_{s=1}^{n} e^{i\lambda s} \int_{-\pi}^{\pi} e^{i\theta(v-s)} d\theta \left( \sum_{k=1}^{6} \sum_{t=1}^{n} \cos \lambda_k(v-t) e^{-it\lambda} (1 - \frac{1}{2} \delta_k, \epsilon) \right).
\]

But \( \int_{-\pi}^{\pi} e^{i\theta(v-s)} d\theta = 2\pi \delta_{v,s} \).

Therefore the second term is asymptotically

\[
-\left( \frac{2}{n^2} \right) f(\lambda) \sum_{k=1}^{6} \sum_{s=1}^{n} \sum_{t=1}^{n} \cos \lambda_k(s-t) e^{i\lambda(s-t)} (1 - \frac{1}{2} \delta_k, \epsilon)
\]

\[
= -\left( \frac{1}{n^2} \right) f(\lambda) \sum_{k=1}^{6} \left( S_n^2(\lambda + \lambda_k) + S_n^2(\lambda - \lambda_k) \right) (1 - \frac{1}{2} \delta_k, \epsilon).
\]

The third term in (3.7.1) is the same as the second while the fourth is similarly shown to have the same modulus but opposite sign. Hence

\[
\mathbb{E} \left\{ \text{I}_n(\lambda) \right\} \approx f(\lambda) \left[ 1 - \left( \frac{1}{n^2} \right) \sum_{k=1}^{6} \left( S_n^2(\lambda + \lambda_k) + S_n^2(\lambda - \lambda_k) \right) (1 - \frac{1}{2} \delta_k, \epsilon) \right]
\]

\[
\text{...} \quad (3.7.2)
\]

In view of the smoothing process (2.4.7) we therefore adjust the estimate of \( f(\lambda) \) obtained from the observed residuals.
by dividing by $|h(\lambda)|^2$ on account of the trend removal and by $k_{m,n}(\lambda)$ on account of the removal of the seasonal components, where

$$k_{m,n}(\lambda) = 1 - \left(\frac{1}{n^2}\right) \sum_{k=1}^{\delta} \int_{-\pi}^{\pi} \left\{ s_n^2(\theta + \lambda_k) + s_n^2(\theta - \lambda_k) \right\} \frac{1}{\pi} \sin(\theta) d\theta.$$

(3.7.3)

Since $s_n^2(\lambda)$ is concentrated near the origin we have

$$k_{m,n}(\lambda) = 1 - \left(\frac{1}{n^2}\right) \sum_{k=1}^{\delta} \left\{ w_m(\lambda_k + \lambda) + w_m(\lambda_k - \lambda) \right\} (1 - \frac{1}{2}) \int_{-\pi}^{\pi} s_n^2(\theta) d\theta$$

$$= 1 - \left(\frac{2\pi}{n}\right) \sum_{k=1}^{\delta} \left\{ w_m(\lambda_k + \lambda) + w_m(\lambda_k - \lambda) \right\} (1 - \frac{1}{2}) \int_{-\pi}^{\pi} s_n^2(\theta) d\theta. \quad (3.7.4)$$

In the particular case when $w_m(\lambda)$ is given by (2.4.9) we find that

$$k_{m,n}(q\pi/6m) = \begin{cases} 1 - (m/n), & q = jm, j = 1, \ldots, 5 \\ 1 - (m/2n), & q = 6m; \quad jm = l, \quad j = 1, \ldots, 5 \\ 1 - (m/4n), & q = 6(m-l) \\ 1, & \text{all other } q. \end{cases} \quad (3.7.5)$$

The resulting estimates of $f(\lambda)$ may then be inserted in (3.6.1) and (3.6.2) to give estimates of the variance of the $\hat{\alpha}_j$, $\hat{\beta}_j$, and $\hat{\gamma}_k$. Obviously the factor $k_{m,n}(\lambda)$ only inverts the regression effect approximately; in fact the series of residuals
is easily seen to be not even stationary, though it is nearly so.

Confidence intervals and tests of significance may be constructed on the assumption that the estimates are normally distributed. In fact under reasonable conditions \( \hat{\alpha}_j \) and \( \hat{\beta}_j \) will be asymptotically normally distributed as Hannan (1961) has shown. Tests on \( \hat{\alpha}_j \) or \( \hat{\beta}_j \) separately are not likely to be required since their values depend on the phasing of the estimates; the squared amplitudes \( \hat{\alpha}_j^2 + \hat{\beta}_j^2 \) may however be tested by the quantity \( (\hat{\alpha}_j^2 + \hat{\beta}_j^2)/\text{var}(\hat{\alpha}_j) \) which has the \( \chi^2 \) distribution with two degrees of freedom. Naturally we replace \( \text{var}(\hat{\alpha}_j) \) here by its estimated value, and this does not affect the test asymptotically (see Cramer (1947) p. 254). Tests for any hypothesis involving the \( \hat{\psi}_k \) can be obtained from (3.6.2).

The estimates are used in practice in a straightforward way. For instance new values are adjusted by the given month's seasonal factor before publication (or better, the original and adjusted values are both published); or if the observed value falls outside the tolerance limits set then one may conclude that a new factor, such as the beginning of an economic recession, has probably entered, provided that the assumption of a stable seasonal pattern was a reasonable one.
CHAPTER 4.

The Analysis of an Evolving

Seasonal Pattern
4.1 The nature of an evolving seasonal pattern.

When estimating a stable seasonal pattern in the previous chapter the nature of the pattern was of little importance. As soon as the assumption of stability is relaxed, however, the best procedure will depend very much on the nature of the evolving pattern. There are many possibilities but the main division is according to whether the change is sudden or gradual.

The simplest case is a single abrupt change which divides the observed series into two fairly uniform parts. This may occur when there is an important technical innovation in some industry, or even in an industry supplementary or complementary to a given one. This might be a case of a break in the trend, in which case a moving average should not be applied to the whole series but to each part separately. Lange (1957, p. 64) gives the example of coal consumption being reduced on the setting up of a hydro or atomic power station. Another example concerning the hog cycle and using harmonic analysis is given by Abel (1962). In such cases one should obviously estimate a separate seasonal pattern for each part of the series.

Sometimes the phasing of the pattern remains constant while only the amplitude changes. For instance there will always be an increase in shopping at Christmas, but the amount of the increase may vary from year to year according to the
prosperity of the economy. For some commodities this effect will be caused by changes in the weather from year to year, one summer being hotter than another. In these cases Wald's technique is appropriate (see Tintner (1952) pp. 227-233). Smaller effects of the same type are caused by the changes from year to year in the number of working days in each month, and so on.

The most common case occurs when both the amplitude and the phase of the pattern change gradually. The economies of most countries are evolving in many directions. The ultimate causes of the evolution are usually, as we have said, technical progress and population growth. Each series is dependent upon many other series, all of which are playing changing roles in its determination. Now if one or two of these generating series are known to be causing most of the evolution then the observed seasonal component may be estimated as a weighted sum, with changing weights, of the stable seasonal components in those series. An example of this would be the disaggregation by areas mentioned in section 1.3.

Another example in which an apparently evolving seasonal pattern may become stable if the correct model is used is given by Lange (1957, p.77). Railway freight traffic in Poland is divided into transport of industrial and agricultural produce. In Poland's six-year plan for 1949-55 industrial output expanded greatly while agricultural output remained nearly constant. Now the volume of industrial transportation has no significant
seasonal pattern but agricultural transportation has a marked one. A constant absolute seasonal pattern therefore represented a falling relative one; in other words, by recognising this as a case for an additive rather than a multiplicative model, a stable seasonal component would be obtained.

It is not always easy in practice to decide when the assumption of a stable seasonal pattern can be made and when it can not. It is not hard to devise tests for the stability of the pattern; for instance one could test the goodness of fit of the observed deviations to the estimated stable seasonal pattern. Alternatively one could test the deviations from trend in all the Januaries for homogeneity, and all the Februaries, and so on, in an analysis of variance. These results could then be incorporated in an over-all test. However such a test would be rather unwieldy and would be likely to conceal some effects within the series which are evident to the eye. Personal judgment on the nature of the deviations from trend is therefore probably the best way to decide, though to distinguish genuine changes from mere chance fluctuations is not easy even in principle. In any case if an estimated 'evolving' pattern turns out to show little evolution one can then decide to treat it as stable.
4.2 Some methods for estimating a slowly evolving pattern and their disadvantages.

In this section we consider methods of estimating a seasonal pattern which evolves slowly and is therefore not one of the more obvious cases discussed in the previous section. The model (3.1.1) is still used but $s_t$ need no longer be precisely periodic, nor need its amplitude be constant. Although we will compare the methods outlined in this section unfavourably with that in the next section, this attitude is to some extent tentative; there is probably no such thing as a final conclusion here because conditions, experience and problems change.

Few generalizations can be made about procedures used in practice since they are not often described in detail, but it can probably be said that most published seasonally adjusted series have a changing component removed and that the commonest method for estimating this is to recalculate the seasonal component, using a stable model, each time a new observation or a new year's observations are available. This may be combined with a system of weighting down past observations, as mentioned in section 3.3. This is a very simple procedure but it does not seem very satisfactory to base the evolving part of the seasonal component solely on the difference between the current month's value, which is
included, and the corresponding value say ten years ago, which is excluded. A misleading impression of accuracy is likely to result.

A second common method is to plot the deviations from trend for each month; instead of the Januaries, Februaries, and so on, being represented by their average they are represented by a straight line or other function. This method is also simple at first sight, but on closer inspection it is seen to be not so simple. The trouble is that the 'filter effect' (see section 3.2) of the moving average, from which the deviations are taken, is very difficult to allow for, since the operations are from this point of view quite complicated. Even the centred 12 term moving average will affect an evolving seasonal component. The difficulties involved have been set out by Hannan (1963) and will not be repeated here. One fears that in practice the estimates of this type are sometimes biased since no attempt is made to allow for the filter effect. Sometimes iterative methods are used in this situation, but it is hard to evaluate their effectiveness.

A third and, from an analytical point of view, a better method than the above ones would be to introduce a polynomial (or sinusoidal) factor in the model (3.3.3) which would then become, in the linear case,
\[ s_t = \sum_{j=1}^{6} \left\{ (\alpha_{1,j} + \alpha_{2,j} t) \cos \lambda_j t + (\beta_{1,j} + \beta_{2,j} t) \sin \lambda_j t \right\}. \]

The details will not be given here as they have again been given by Hannan (1960b). The disadvantages here are the computational difficulty and the fact that a linear factor will not generally be adequate, while even a quadratic one would require a very complex analysis.

A fourth possible method, obviously, would be to estimate by least squares regression the components at some points in the range of frequencies covered by the seasonal peaks. In fact we can write quite generally

\[ s_t = a_t b_t \]

where \( b_t \) is a stable seasonal component:

\[ b_t = \sum_{j=1}^{6} (\alpha_j \cos \lambda_j t + \beta_j \sin \lambda_j t) \]

and \( a_t \) is a slowly evolving component which modifies \( b_t \):

\[ a_t = \sum_{k=1}^{r} (\eta_k \cos \theta_k t + \xi_k \sin \theta_k t) \]

where the \( \theta_k \) are various low frequencies. (It may be noted that if \( a_t \) depends only on the year and not on the month we get a simple case which is covered by Wald's method.) The product \( s_t \) is then expressed as a sum of components at frequencies \( \lambda_j \pm \theta_k \) for all the values of \( j \) and \( k \); specifically
\[ s_t = \sum_j \sum_k \left\{ A_{j,k} \cos(\lambda_j + \theta_k) t + B_{j,k} \sin(\lambda_j + \theta_k) t \right\} + Y_{j,k} \cos(\lambda_j - \theta_k) t + Z_{j,k} \sin(\lambda_j - \theta_k) t \],

where

\[ A_{j,k} = \frac{1}{2}(\alpha_j \eta_k - \beta_j \xi_k), \quad B_{j,k} = \frac{1}{2}(\beta_j \eta_k + \alpha_j \xi_k), \]

\[ Y_{j,k} = \frac{1}{2}(\alpha_j \eta_k + \beta_j \xi_k), \quad Z_{j,k} = \frac{1}{2}(\beta_j \eta_k - \alpha_j \xi_k). \]

The practical difficulty here would clearly be to decide which frequencies \( \theta_k \) to specify, or in other words at which frequencies to make the estimates. Also the regressors will not in general be orthogonal, though they will be in the special case

\[ \theta_k = \frac{\pi(k - 1)}{6d}, \quad k = 1, \ldots, r, \]

where \( d \) is the number of years of data; \( r \) would here be small, say 4. In very long series it would perhaps be feasible but, as we have said, this is not usually the case with economic data.

A fifth method is to devise a filter with the property that its response function is small except in the neighbourhood of each of the points \( \lambda_j \), where it has a peak which is as nearly as possible rectangular. Applying such a filter would yield the estimated evolving seasonal pattern. There is however no way of knowing how wide the peaks should be and even if this could be guessed it is not at all easy to construct such a filter.
4.3 A suggested method for estimating an evolving pattern.

A method which avoids the difficulties of the methods mentioned in the previous section is now presented. It appears to be more flexible and more convenient to formulate, estimate and interpret than the other methods. It is a direct generalization of the method given in the previous chapter for a stable pattern.

We use the model (3.1.1), with (3.3.3) replaced by

\[ s_t = \sum_{j=1}^{6} (\alpha_{j,t} \cos \lambda_j t + \beta_{j,t} \sin \lambda_j t), \quad t = 1, \ldots, n. \quad (4.3.1) \]

The idea is to try to find the way in which the fundamental components \( \alpha_{j,t} \) and \( \beta_{j,t} \) vary over the course of the series by estimating them from successive small sequences of observations. The resulting estimates \( \hat{\alpha}_{j,t} \) and \( \hat{\beta}_{j,t} \) are then smoothed and recombined to give the estimated seasonal component.

Naturally the first step is still to remove the trend as well as is possible or convenient, after which the model becomes

\[ y'_t = \sum_{j=1}^{6} (\alpha'_{j,t} \cos \lambda_j t + \beta'_{j,t} \sin \lambda_j t) + x'_t, \quad t = 1, \ldots, n. \quad (4.3.2) \]

The \( \alpha'_{j,t} \) and \( \beta'_{j,t} \) can conveniently be estimated for each \( t \) from a number of consecutive observations over which the
parameters can be assumed to be roughly constant, that is, independent of \( t \). This reflects the assumption that the seasonal pattern is evolving slowly. We will consider the case when this is assumed of successive sets of twelve observations. (A multiple of twelve is convenient as the regressors are then precisely orthogonal.) Thus we put

\[ \alpha_{j,t+s} = \alpha_{j,t}, \quad \beta_{j,t+s} = \beta_{j,t}, \quad s = -6, -5, \ldots, 5. \quad (4.3.3) \]

Note that this assumption of (approximate) equality is made for each successive set of twelve values, so that (4.3.3) does not imply that the whole series \( \alpha_{j,t} \) is constant; the series may vary, but we assume that the variation is so slow as to be negligible over any twelve successive values. Twelve being an even number we will however 'centre' the estimates, so that using least squares we have

\[
\begin{align*}
\hat{\alpha}'_{j,t} &= \frac{2}{12} \sum_{s=-6}^{6} y_{t+s}' \cos \lambda_{j}(t+s) \cdot (1-\frac{1}{6}|s|_{6}) \cdot (1-\frac{1}{6} |s|_{6}) \\
\hat{\beta}'_{j,t} &= \frac{2}{12} \sum_{s=-6}^{6} y_{t+s}' \sin \lambda_{j}(t+s) \cdot (1-\frac{1}{6}|s|_{6}) \cdot (1-\frac{1}{6} |s|_{6}) \\
\end{align*}
\]

\[ j = 1, \ldots, 6. \quad (4.3.4) \]

These estimates are next adjusted for any filter effects in the same way as was done in section 3.3; for a symmetrical filter we therefore have

\[
\hat{\alpha}'_{j,t}/\hat{\alpha}_{j,t} = \hat{\beta}'_{j,t}/\hat{\beta}_{j,t} = h(\lambda_{j}). \quad (4.3.5)
\]
The values $\hat{\alpha}_{j,t}$ and $\hat{\beta}_{j,t}$ are then graphed and examined. (The reader may see Figure 6 of the next chapter at this point). If they show little variation then the stable model should be used. If however some of them show a marked evolution they may be smoothed by means of a polynomial or a moving average or even by hand; the smoothed values are then inserted in (4.3.1) to give the estimated evolving seasonal component. (Inserting the unsmoothed $\hat{\alpha}_{j,t}$ and $\hat{\beta}_{j,t}$ would of course be useless, being equivalent to adding the deviations from trend back in again.) It may also be helpful to plot the successive squared amplitudes $(\hat{\alpha}_{j,t}^2 + \hat{\beta}_{j,t}^2)$ of each parameter and their phase angles $\tan^{-1}(\hat{\beta}_{j,t}/\hat{\alpha}_{j,t})$, to throw more light on the nature of the evolution present.

Since there are eleven series of estimates to be examined this might appear to be a long job; but usually, as we have remarked, the terms with $j = 1$ will dominate the situation while the terms with $j = 3,4,5,6$ will usually be negligible and could certainly be replaced by their estimated value in the stable model.

In practice the best procedure might be firstly to plot the deviations from trend; even if it is then not certain that the pattern is stable one could obtain the estimates $\hat{\alpha}_{j},\hat{\beta}_{j}$ assuming stability, in order to see which values of $j$ are important and which need no further
investigation. (The possibility of wide variation about a small average value should not however be neglected.) For those values of $j$ which need further investigation the $\hat{\alpha}_{j,t}$ and $\hat{\beta}_{j,t}$ are computed and graphed. Of these, some may be quite constant and will be represented by their mean or by the corresponding stable estimate; the remaining ones, if any remain, will be smoothed appropriately before reassembling the estimates $\hat{s}_t$.

We consider now just what the series $\hat{\alpha}_{j,t}$ and $\hat{\beta}_{j,t}$ represent. This is important, for otherwise they may be interpreted wrongly. Their average value will of course be roughly the same as $\hat{\alpha}_j, \hat{\beta}_j$ in the stable model (unless there is a phase change, that is, a different month taken for $t = 1$, in which case the average amplitudes would still be equal). The variation of the estimates around their average will depend on the validity of the assumption of a slow enough evolution for the seasonal component in successive sets of twelve or another number of observations to be roughly constant.

Although we are assuming a slow evolution, it is of interest to find the implications for the estimates of a failure of this assumption to hold. The effect of a sudden change in say $\alpha_{1,t}$ can be judged by specifying

$$\alpha_{1,t} = \begin{cases} 0, & t = 1, \ldots, 12, \\ 48, & t = 13, 14, \ldots, \\ \text{all other } \alpha_{j,t}, \beta_{j,t} = 0. \end{cases}$$
This implies that

\[ y_t = \begin{cases} 0 & , \ t = 1, \ldots, 12 \\ 48 \cos \lambda_1 t & , \ t = 13, 14, \ldots \end{cases} \]

This jump in the true parameter will appear as a smooth change, since the estimates are based on the assumption of smoothness. The estimates are easily found to be as given in Table 6.

| Table 6. Estimation of a Seasonal Parameter Containing a Jump. |
|-----------------|-----------------|-----------------|
| \( t \) | \( \alpha_{1,t} \) | \( \hat{\alpha}_{1,t} \) |
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |
| 4 | 0 | 0 |
| 5 | 0 | 0 |
| 6 | 0 | 0 |
| 7 | 0 | 3 |
| 8 | 0 | 7 |
| 9 | 0 | 8 |
| 10 | 0 | 9 |
| 11 | 0 | 13 |
| 12 | 0 | 20 |
| 13 | 48 | 27 |
| 14 | 48 | 31 |
| 15 | 48 | 32 |
| 16 | 48 | 33 |
| 17 | 48 | 37 |
| 18 | 48 | 44 |
| 19 | 48 | 48 |
| 20 | 48 | 48 |
| 21 | 48 | 48 |
Illustrations such as that given in Table 6 assist in interpreting the series of estimates and in deciding how to smooth them. The variances and covariances (before smoothing) can be obtained exactly as in Chapter 3 with \( n = 12 \), on the assumption of constancy over successive sets of twelve observations and these will also help to determine which fluctuations in the \( \hat{\alpha}_{j,t} \) and \( \hat{\beta}_{j,t} \) are significant and which are of a random nature and should therefore be smoothed out. In this regard it may also be helpful to repeat the estimation using several different filters for trend removal, as some fluctuations may be due merely to the particular filter used.

As in the stable model we can now examine the effect of trend removal on the estimates. By the same method as was used in section 3.4 it is easy to show that

\[
\text{cov}(\hat{\alpha}_{j,t}, \hat{\alpha}_{j,t+\tau}) = \frac{1}{2}(2/12)^2 \int \left\{ \sum_{i=1}^{12} \left( \frac{S_{j}^{2}(\lambda+\lambda_{j})}{2} \cos^{2} \left( \frac{\lambda+\lambda_{j}}{2} \right) \cos \tau (\lambda+\lambda_{j}) \right) \\
+ \sum_{i=1}^{12} \left( \frac{S_{j}^{2}(\lambda-\lambda_{j})}{2} \cos^{2} \left( \frac{\lambda-\lambda_{j}}{2} \right) \cos \tau (\lambda-\lambda_{j}) \right) \\
+ 2 \sum_{i=1}^{12} \left( \frac{S_{j}^{2}(\lambda+\lambda_{j})}{2} S_{j}^{2}(\lambda-\lambda_{j}) \cos (2\tau) \lambda_{j} \cos \tau \lambda \right) \right\} \left\{ \frac{h^2(\lambda)}{h^2(\lambda_j)} \right\} f(\lambda) d\lambda \left( 1 - \frac{1}{26} \delta_{j,6} \right)^2
\]

(4.3.6)

where \( f(\lambda) \) is the spectrum of the residuals. Since the covariance (4.3.6) thus depends on \( t \) and not only on \( \tau \), the series \( \hat{\alpha}_{j,t} \) is not stationary; however it is approximately
stationary as the only term depending on \( t \) is the cross product one, which is relatively small (compare the discussion in section 3.6). Conclusions similar to those in section 3.5 can be obtained when \( \tau \) is set equal to zero; when \( f(\lambda) \) has been estimated the serial covariances of the series \( \hat{q}_{j,t} \) can be estimated and this obviously helps to interpret the series of estimates. In fact the (approximate) spectral density of this series follows by comparing (4.3.6) with (2.2.7); it is

\[
g_j(\lambda) = \frac{1}{4}(2/12)^2 \frac{\pi^2}{12}(\lambda) \cos^2 \frac{1}{2}\lambda \left\{ f(\lambda+\lambda_j)h^2(\lambda+\lambda_j) + f(\lambda-\lambda_j)h^2(\lambda-\lambda_j) \right\} (1/h^2(\lambda_j)) \left(1-\frac{1}{2}\delta_j,6\right)^2.
\]

(4.3.7)

For the series \( \beta_{j,t} \) the sign before the second term in braces is changed.

An alternative view of the procedure described in this section is instructive. This regards the whole procedure as a single combined filter applied to the initial series to produce the observed residuals (compare the last method considered in the previous section). This filter would combine the moving average which eliminated trend with the removal of the component \( s_t \) which is itself the result of the least squares filter, the simple filter which adjusts for the effect of trend removal, and those filters chosen to smooth the \( \hat{q}_{j,t} \) and \( \beta_{j,t} \). The resulting filter would have a frequency response function with troughs of certain widths around the origin and the seasonal frequencies, whereas in
the stable case the residuals were produced by a filter with a trough at the origin and extremely narrow spikes extending downward at the points $\lambda_j$. Any method of adjusting for an evolving seasonal component could in principle be described in the same way. However in the present case the resulting expressions are long and will not be given.

The estimates are again used in a straightforward manner. The estimated evolving pattern could be tested against the observed deviations from trend for goodness of fit, if there were any doubt about this. The seasonal pattern for the most recent year can then be used for the adjustment of new data.
CHAPTER 5

A Practical Illustration
5.1 Trading bank advances.

In this chapter we illustrate the theory of the previous chapters by analysing an actual series. The analysis given is not necessarily the best for any particular practical purpose but it is carried out instead in order to obtain as much insight as possible into the various problems.

We use the series giving the loans, advances and bills discounted by the major Australian trading banks monthly from January 1950 to February 1963. The series is given by the Reserve Bank of Australia (see bibliography); the raw data are listed in Table 7. We do not use the series before 1950 as a labour government was then in power and the attitude to advances and loans was different. For many purposes (see for example Lydall (1962)) it would be required to deflate the series by a price index; (it may be noted that deflation could conceivably stabilize an apparently evolving seasonal component; compare the discussion of section 4.1).

The first thing to note is the effect of the Korean War boom in 1951; as a result of this we will confine the analysis to the period 1953 to 1963. If one happened to be working say in 1951, one would simply not attempt to adjust current figures for seasonal variation: it would be all that one could do to keep up with the trend.
Table 7. Loans, Advances and Bills Discounted by Major Australian Trading Banks.
(Monthly averages, £ million)

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>447.6</td>
<td>546.7</td>
<td>719.8</td>
<td>678.8</td>
<td>725.7</td>
<td>1951</td>
<td>446.4</td>
<td>560.5</td>
<td>736.3</td>
<td>665.7</td>
<td>727.6</td>
<td>1952</td>
<td>451.8</td>
<td>571.2</td>
<td>744.7</td>
<td>652.1</td>
<td>736.2</td>
<td>1953</td>
<td>463.8</td>
<td>566.1</td>
<td>745.8</td>
<td>648.0</td>
<td>756.8</td>
<td>1954</td>
<td>464.0</td>
<td>563.2</td>
</tr>
</tbody>
</table>

The next step is to obtain some evidence on which to decide between the additive and multiplicative models. Quarterly values were used for this purpose to save time. A centred four term moving average was taken to represent the trend and the deviations from it are given in Table 8. It is easily seen from Table 7 that the general trend of the series is upward; since the deviations from trend also tend to increase in absolute value over the given period we use the multiplicative model. Common logarithms of the monthly data were therefore taken.
Table 8. Absolute Deviations from Moving Average.

(£100 m.)

<table>
<thead>
<tr>
<th>Quarter</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td></td>
<td>-6</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1951</td>
<td>1</td>
<td>-4</td>
<td>-4</td>
<td>4</td>
</tr>
<tr>
<td>1952</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>1953</td>
<td>-6</td>
<td>-2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>1954</td>
<td>-5</td>
<td>-4</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>1955</td>
<td>-5</td>
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<td>7</td>
<td>5</td>
</tr>
<tr>
<td>1956</td>
<td>-6</td>
<td>-1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1957</td>
<td>-7</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1958</td>
<td>-10</td>
<td>4</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>1959</td>
<td>-7</td>
<td>-3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1960</td>
<td>-11</td>
<td>-2</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>1961</td>
<td>-3</td>
<td>-3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1962</td>
<td>-11</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average: -5.4  -1.5  3.3  3.6

The next step is to find a filter which eliminates the trend satisfactorily. For comparison the following work was carried out three times, using the 12 term moving average, second differences and Spencer's 15 point formula. The deviations from each of these estimated trends are shown in Figure 3 for each year from 1950 to 1962. The deviations were also plotted month by month for each filter; we do not show all these graphs but give as an example those for the April deviations in Figure 4. The boom years stand out sharply as expected and will not be considered further; but it is clear from Figure 3 that it will be meaningful to estimate a stable seasonal component for the period 1953 to 1962. It is hardly necessary to remark that there is no
Figure 3. Deviations from Trend

(Series of logarithms in £100 m; $\frac{1}{50} = 0.005$

<table>
<thead>
<tr>
<th>Year</th>
<th>12 Term Moving Average</th>
<th>Second Differences</th>
<th>Spencer's 15 Pt Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>'50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'51</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>'52</td>
<td></td>
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<tr>
<td>'53</td>
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<td></td>
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<td>'54</td>
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<td></td>
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<tr>
<td>'55</td>
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<td></td>
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<td>'56</td>
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<tr>
<td>'58</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>'59</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'60</td>
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<td></td>
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<tr>
<td>'61</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'62</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4. April Deviations from Trend, 1950-62.

Scale: 1" = 100 x logarithm of original value in £100m.

12 Term Moving Average

Second Differences

Spencer's 15 Point Formula

50 51 52 53 54 55 56 57 58 59 60 61 62
point in testing the hypothesis that there is no seasonal component, as we could not possibly believe that this hypothesis is true.

Only three months stand clearly apart from the usual pattern in Figure 3: May, June and July of 1956. This happened to be due to the delayed effects of the Little Budget of March 1956 and associated causes; see Commonwealth Bank of Australia Annual Report (1956) and Arndt (1960) Chapter III. In particular the banks were not as ready as usual to make loans for the taxation requirements which arise in June and July.

One could now compute the stable seasonal estimates $\hat{\psi}_k$ using the simple procedure given by (3.3.18). However we obtain firstly the $\hat{a}_j$ and $\hat{b}_j$ for $j = 1, \ldots, 6$ using formulae (3.3.14) and (3.3.17); the results are given in Table 9 together with the amplitudes and phase angles. The phasing of the $\hat{a}_j$ and $\hat{b}_j$ is such that $t = 1$ corresponds to February in each case. As was expected the component with $j = 1$ is the principal one, and there is very little difference between the results for the three filters.
Table 9. Estimated Stable Seasonal Parameters.

(Unit: 10,000 x common logarithm of original value in £100 m.; phase angle in degrees)

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{\alpha}_j)</td>
<td>-65.1</td>
<td>1.5</td>
<td>-7.1</td>
<td>17.0</td>
<td>-11.8</td>
<td>-0.5</td>
</tr>
<tr>
<td>(\hat{\beta}_j)</td>
<td>-79.2</td>
<td>-39.4</td>
<td>-11.4</td>
<td>12.7</td>
<td>-3.4</td>
<td>-</td>
</tr>
<tr>
<td>(\sqrt{(\hat{\alpha}^2_j + \hat{\beta}^2_j)})</td>
<td>102.5</td>
<td>39.5</td>
<td>13.5</td>
<td>21.2</td>
<td>12.2</td>
<td>0.5</td>
</tr>
<tr>
<td>(\tan^{-1}(\hat{\beta}_j/\hat{\alpha}_j))</td>
<td>230.6</td>
<td>272.1</td>
<td>238.1</td>
<td>36.8</td>
<td>196.2</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Second differences

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{\alpha}_j)</td>
<td>-58.1</td>
<td>2.4</td>
<td>-6.4</td>
<td>15.6</td>
<td>-12.0</td>
<td>0.2</td>
</tr>
<tr>
<td>(\hat{\beta}_j)</td>
<td>-82.4</td>
<td>-35.9</td>
<td>-9.7</td>
<td>11.4</td>
<td>-2.9</td>
<td>-</td>
</tr>
<tr>
<td>(\sqrt{(\hat{\alpha}^2_j + \hat{\beta}^2_j)})</td>
<td>105.3</td>
<td>36.0</td>
<td>11.6</td>
<td>19.3</td>
<td>12.3</td>
<td>0.2</td>
</tr>
<tr>
<td>(\tan^{-1}(\hat{\beta}_j/\hat{\alpha}_j))</td>
<td>234.8</td>
<td>273.8</td>
<td>236.7</td>
<td>36.1</td>
<td>193.5</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Spencer's 15 point formula

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{\alpha}_j)</td>
<td>-53.4</td>
<td>1.5</td>
<td>-6.7</td>
<td>14.6</td>
<td>-12.3</td>
<td>0.0</td>
</tr>
<tr>
<td>(\hat{\beta}_j)</td>
<td>-87.9</td>
<td>-36.1</td>
<td>-8.3</td>
<td>11.8</td>
<td>-3.6</td>
<td>-</td>
</tr>
<tr>
<td>(\sqrt{(\hat{\alpha}^2_j + \hat{\beta}^2_j)})</td>
<td>102.8</td>
<td>36.2</td>
<td>10.7</td>
<td>18.7</td>
<td>12.8</td>
<td>0.0</td>
</tr>
<tr>
<td>(\tan^{-1}(\hat{\beta}_j/\hat{\alpha}_j))</td>
<td>238.7</td>
<td>272.4</td>
<td>231.1</td>
<td>38.6</td>
<td>196.2</td>
<td>0.0</td>
</tr>
</tbody>
</table>

The \(\hat{\psi}_k\) follow from (3.3.5) and from these we obtain the factors \(D_k\) by which the original values are to be divided for seasonal adjustment:

\[D_k = \left\{\text{antilog} \left(\frac{\hat{\psi}_k}{10,000}\right)\right\} \times 100\%, \quad k = 1, \ldots, 12.\]

The values of \(D_k\) are given in Table 10.
Table 10. Estimated Seasonal Adjustment

Factors $D_k$ Using Stable Model.

<table>
<thead>
<tr>
<th>Month</th>
<th>By present method</th>
<th>See text</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12 month moving average</td>
<td>Second differences</td>
</tr>
<tr>
<td>Jan</td>
<td>98.5</td>
<td>98.7</td>
</tr>
<tr>
<td>Feb</td>
<td>97.1</td>
<td>97.3</td>
</tr>
<tr>
<td>Mch</td>
<td>96.6</td>
<td>96.7</td>
</tr>
<tr>
<td>Apl</td>
<td>98.7</td>
<td>98.6</td>
</tr>
<tr>
<td>May</td>
<td>100.0</td>
<td>99.8</td>
</tr>
<tr>
<td>Jun</td>
<td>100.2</td>
<td>100.1</td>
</tr>
<tr>
<td>Jul</td>
<td>102.4</td>
<td>102.2</td>
</tr>
<tr>
<td>Aug</td>
<td>101.6</td>
<td>101.5</td>
</tr>
<tr>
<td>Sep</td>
<td>101.0</td>
<td>101.1</td>
</tr>
<tr>
<td>Oct</td>
<td>102.0</td>
<td>102.1</td>
</tr>
<tr>
<td>Nov</td>
<td>101.6</td>
<td>101.7</td>
</tr>
<tr>
<td>Dec</td>
<td>100.5</td>
<td>100.6</td>
</tr>
</tbody>
</table>

There is a little interest in comparing the estimates with those given by Hall (1961, p. 38) and Hannan (1963, p. 40) which are also shown in Table 10. Hall's estimates were obtained by the traditional method of averaging the deviations from a twelve months moving average. Hall used a different series which covered all cheque-paying banks, but the major trading banks represent about 90% of the total business. In addition Hall's estimates were apparently based only on the data for 1959-1961. Nevertheless the agreement is obviously very close. Hannan's estimates are based on the period August 1958 to February 1962.
and these also agree closely with the present estimates. Since the seasonal pattern is very nearly constant over the period used and since the trend is not difficult to remove, it is only natural that all these results are similar; other series might show more interesting differences between the results of different methods, but the present series at least serves to demonstrate the spectral method of analysis.

The next step was to estimate the spectrum of the residuals in order to obtain the standard errors of the estimated seasonal parameters. As stated in section 2.4 the weights used were those called hanning and the computations were carried out using 12 and 24 bands, so that the formula used was (2.4.4) with \( n = 120 \) and \( m = 12, 24 \). Since the results for the three filters are so close they are given only for second differences. Table 11 gives the autocorrelation coefficients while Table 12 and Figure 5 show the estimates of the spectrum.

Several comments may be made on the estimates of the spectrum. No estimate is obtainable at frequency zero since the second differencing removes the entire component there. The spectrum of the residuals has a large peak near the origin and this would lead to poor estimates, but of course this does not apply since the spectrum was quite
Table 11. Autocorrelation Coefficients \( \left( c_t/c_0 \right) \) for Series of Residuals

<table>
<thead>
<tr>
<th>Lag in months</th>
<th>Autocorrelation coefficient</th>
<th>Lag in months</th>
<th>Autocorrelation coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
<td>13</td>
<td>-0.003</td>
</tr>
<tr>
<td>1</td>
<td>-0.348</td>
<td>14</td>
<td>-0.026</td>
</tr>
<tr>
<td>2</td>
<td>-0.028</td>
<td>15</td>
<td>0.148</td>
</tr>
<tr>
<td>3</td>
<td>0.022</td>
<td>16</td>
<td>-0.199</td>
</tr>
<tr>
<td>4</td>
<td>-0.120</td>
<td>17</td>
<td>-0.006</td>
</tr>
<tr>
<td>5</td>
<td>0.162</td>
<td>18</td>
<td>0.038</td>
</tr>
<tr>
<td>6</td>
<td>-0.021</td>
<td>19</td>
<td>-0.025</td>
</tr>
<tr>
<td>7</td>
<td>-0.087</td>
<td>20</td>
<td>0.092</td>
</tr>
<tr>
<td>8</td>
<td>-0.112</td>
<td>21</td>
<td>0.025</td>
</tr>
<tr>
<td>9</td>
<td>0.096</td>
<td>22</td>
<td>-0.231</td>
</tr>
<tr>
<td>10</td>
<td>0.029</td>
<td>23</td>
<td>0.125</td>
</tr>
<tr>
<td>11</td>
<td>-0.078</td>
<td>24</td>
<td>0.028</td>
</tr>
</tbody>
</table>

Table 12. Estimated Spectral Density of Residuals

<table>
<thead>
<tr>
<th>Band centred at ( \pi j/24 )</th>
<th>( 2\pi \hat{f}(\lambda) ) at 24 bands</th>
<th>( 2\pi \hat{f}(\lambda) ) at 12 bands</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 1 )</td>
<td>1,087,030</td>
<td>99,084</td>
</tr>
<tr>
<td>( j = 2 )</td>
<td>112,458</td>
<td></td>
</tr>
<tr>
<td>( j = 3 )</td>
<td>31,425</td>
<td></td>
</tr>
<tr>
<td>( j = 4 )</td>
<td>13,109</td>
<td>9,749</td>
</tr>
<tr>
<td>( j = 5 )</td>
<td>4,040</td>
<td></td>
</tr>
<tr>
<td>( j = 6 )</td>
<td>1,436</td>
<td>2,152</td>
</tr>
<tr>
<td>( j = 7 )</td>
<td>1,470</td>
<td></td>
</tr>
<tr>
<td>( j = 8 )</td>
<td>1,350</td>
<td>1,372</td>
</tr>
<tr>
<td>( j = 9 )</td>
<td>1,198</td>
<td></td>
</tr>
<tr>
<td>( j = 10 )</td>
<td>1,045</td>
<td>809</td>
</tr>
<tr>
<td>( j = 11 )</td>
<td>659</td>
<td></td>
</tr>
<tr>
<td>( j = 12 )</td>
<td>309</td>
<td>366</td>
</tr>
<tr>
<td>( j = 13 )</td>
<td>179</td>
<td></td>
</tr>
<tr>
<td>( j = 14 )</td>
<td>170</td>
<td>240</td>
</tr>
<tr>
<td>( j = 15 )</td>
<td>272</td>
<td></td>
</tr>
<tr>
<td>( j = 16 )</td>
<td>410</td>
<td>287</td>
</tr>
<tr>
<td>( j = 17 )</td>
<td>286</td>
<td></td>
</tr>
<tr>
<td>( j = 18 )</td>
<td>205</td>
<td>259</td>
</tr>
<tr>
<td>( j = 19 )</td>
<td>288</td>
<td></td>
</tr>
<tr>
<td>( j = 20 )</td>
<td>295</td>
<td>232</td>
</tr>
<tr>
<td>( j = 21 )</td>
<td>196</td>
<td></td>
</tr>
<tr>
<td>( j = 22 )</td>
<td>156</td>
<td>159</td>
</tr>
<tr>
<td>( j = 23 )</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>( j = 24 )</td>
<td>94</td>
<td>123</td>
</tr>
</tbody>
</table>
Figure 5. Estimated Spectral Density of Residuals.

- 12 bands
- 24 bands
flat before the adjustment for the filter factor (see section 3.2) was made. The increased resolution (see section 2.4) obtained when \( m \) is increased from 12 to 24 is valuable, particularly at low frequencies where the resolution is always relatively poor since the bands are of constant absolute width throughout the spectrum. The coefficients of variation of the estimates follow from (2.4.11); they are 0.39 for the 24 band estimates and 0.27 for 12 bands. A convenient check on the computations is obtained by finding the area under the rectangles in Figure 5; putting \( t = 0 \) in (2.2.7) this area should be close to \( 2\pi \) times the estimated variance \( \hat{\gamma}_0 \) which in this case was 1586; the check is satisfied.

The estimated standard errors of the \( \hat{\alpha}_j, \hat{\beta}_j \) and \( \hat{\gamma}_k \) are given by (3.6.1) and (3.6.2) and are found to be as follows, using the 24 band estimates:

\[
\text{est var } (\hat{\gamma}_k) = 16.07, \quad k = 1, \ldots, 12
\]
\[
\text{est var } (\hat{\alpha}_j) = \text{est var } (\hat{\beta}_j) = 14.78, 4.74, 2.28, 2.61, 2.21 \text{ and } 1.26 \text{ for } j = 1, \ldots, 6.
\]

The standard errors of the adjustment divisors \( D_k \) are thus found to be approximately 0.4 for each \( k \). This very low standard error is to be expected since the number of observations, 120, is not very small. The ease with which
the standard errors are estimated is an advantage of the spectral method. Confidence limits follow directly (see section 3.7) but are not tabulated here.

The evidence indicates that the seasonal pattern in the present series is fairly stable. Nevertheless, as an illustration of the method given in section 4.3 no harm will be done by obtaining the estimates \( \hat{\alpha}_{j,t} \) and \( \hat{\beta}_{j,t} \) defined by (4.3.4) and (4.3.5) using the same phasing as for \( \hat{\alpha}_j \) and \( \hat{\beta}_j \); these need be obtained only for \( j = 1 \) and 2, the other components being very small. In fact all the coefficients were obtained, still using only the second differenced series; the machine time on an I.B.M. 1620 for this job was 2\( \frac{1}{2} \) hours. Again the years 1950-1952 were clearly of a different character but in the remaining ten-year series the only coefficient which showed an appreciable evolution was \( \hat{\beta}_{1,t} \). The other coefficients were therefore represented by their values in the stable model (which amounts to smoothing them with a horizontal straight line) while a cubic was fitted by least squares to \( \hat{\beta}_{1,t} \); this is shown in Figure 6 together with the graph of \( \hat{\alpha}_{1,t} \). The choice of a
Figure 6. Estimated Evolving Seasonal Parameters.
cubic is of course somewhat arbitrary but it does reflect what seem to be the main features of the series: a rise from 1953 to 1958, a fall to 1960 and a rise to 1962. The fact that $\hat{\beta}_{1,t}$ evolves slowly while $\hat{\alpha}_{1,t}$ is quite stable shows that both the amplitude and phase of the oscillations are evolving slowly.

The appearance of marked oscillations with a period of about eight months in the series $\hat{\alpha}_{1,t}$ calls for an explanation and this is easily obtained by considering the spectral density of the series, given by (4.3.7). The estimated spectrum is given in Table 13 where it is seen that the combined effect of the shape of the spectrum of the residuals and the weights in the second column is to create a peak near the frequency $\pi/4$, or period 8 months, while there is very little spectral mass at high frequencies, the remainder occurring at low frequencies.
Table 13. Estimated Spectral Density of $\hat{\psi}_{1,t}$.  
(See formula (4.3.7))

<table>
<thead>
<tr>
<th>Frequency $\pi j/24$</th>
<th>$S_{12}^2(\lambda)$ $\cos^2 \frac{\lambda}{2}$</th>
<th>$2\pi(\hat{\varphi}(\lambda+\lambda_1)h^2(\lambda+\lambda_1) + \hat{\varphi}(\lambda-\lambda_1)h^2(\lambda-\lambda_1))$</th>
<th>$0.0718 \frac{2\pi}{144} \hat{\psi}_j(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>144</td>
<td>1.882</td>
<td>271,008</td>
</tr>
<tr>
<td>1</td>
<td>116.431</td>
<td>1.419</td>
<td>165,216</td>
</tr>
<tr>
<td>2</td>
<td>57.696</td>
<td>1.015</td>
<td>58,561</td>
</tr>
<tr>
<td>3</td>
<td>12.636</td>
<td>1.218</td>
<td>15,392</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1.544</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>4.339</td>
<td>2.143</td>
<td>9,298</td>
</tr>
<tr>
<td>6</td>
<td>5.828</td>
<td>2.818</td>
<td>16,423</td>
</tr>
<tr>
<td>7</td>
<td>2.056</td>
<td>2.720</td>
<td>5,592</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>2.177</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1.119</td>
<td>1.604</td>
<td>1,795</td>
</tr>
<tr>
<td>10</td>
<td>1.698</td>
<td>1.569</td>
<td>2,664</td>
</tr>
<tr>
<td>11</td>
<td>.650</td>
<td>2.979</td>
<td>1,936</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>5.038</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>.384</td>
<td>4.784</td>
<td>1,837</td>
</tr>
<tr>
<td>14</td>
<td>.588</td>
<td>4.691</td>
<td>2,758</td>
</tr>
<tr>
<td>15</td>
<td>.223</td>
<td>5.691</td>
<td>1,269</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>5.352</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>.122</td>
<td>3.816</td>
<td>466</td>
</tr>
<tr>
<td>18</td>
<td>.172</td>
<td>3.495</td>
<td>601</td>
</tr>
<tr>
<td>19</td>
<td>.058</td>
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<td>5.195</td>
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<td>2.959</td>
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<td>22</td>
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<td>2.395</td>
<td>41</td>
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<td>23</td>
<td>.002</td>
<td>3.700</td>
<td>7</td>
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<tr>
<td>24</td>
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<td>4.116</td>
<td>0</td>
</tr>
</tbody>
</table>

The model (4.3.1) was reassembled to give the estimated evolving component. The effect of allowing for the evolution was naturally quite small, the $\hat{\psi}_{k,t}$ differing from the stable $\hat{\psi}_k$ by an amount typically of the order of 10% and the $D_k$ varying by only 1% or 2%. When the latest available adjustment factors are applied to new and recent observations the results
are as shown in Table 14. The adjusted series over the last year and a half shows a very steady expansion of bank advances which was to some extent hidden in the original series by the seasonal variation.

Table 14. Seasonal Adjustment of New and Recent Data.

(Adjustment divisors are those for August 1961 to July 1962, using second differences.)

<table>
<thead>
<tr>
<th>Month</th>
<th>Observed value, £m.</th>
<th>Adjustment divisor, %</th>
<th>Adjusted value, £m.</th>
</tr>
</thead>
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<tr>
<td>1962</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Jan</td>
<td>976.7</td>
<td>98.1</td>
<td>995.6</td>
</tr>
<tr>
<td>Feb</td>
<td>965.2</td>
<td>96.8</td>
<td>997.1</td>
</tr>
<tr>
<td>Mch</td>
<td>972.0</td>
<td>96.6</td>
<td>1006.2</td>
</tr>
<tr>
<td>Apl</td>
<td>1008.8</td>
<td>99.0</td>
<td>1019.0</td>
</tr>
<tr>
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<td>1022.0</td>
<td>100.7</td>
<td>1014.9</td>
</tr>
<tr>
<td>Jun</td>
<td>1031.4</td>
<td>101.1</td>
<td>1020.2</td>
</tr>
<tr>
<td>Jul</td>
<td>1060.9</td>
<td>103.2</td>
<td>1028.0</td>
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<tr>
<td>Aug</td>
<td>1052.1</td>
<td>101.7</td>
<td>1034.5</td>
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<td>1032.2</td>
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<tr>
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<tr>
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<tr>
<td>Jun</td>
<td>1101.2</td>
<td>101.1</td>
<td>1089.2</td>
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</table>

A final comment concerns the use of an electronic computer. The programming for all the computations required for the spectral methods used is not difficult and was done by the author. Although the machine time required is not
very great there is one slightly inconvenient feature with present computers: the work must be done in several steps between which human judgment is used (for instance to decide which, if any, of the $\hat{\alpha}_{j,t}$ and $\hat{\beta}_{j,t}$ to estimate, and how to smooth them) and this means that some output becomes input in the next stage and the machine has to be set up several times.

As mentioned in section 4.2 there is little point in speaking of a final conclusion on the problem dealt with in this thesis. Nevertheless it can be said that while the spectral methods presented may be slightly more sophisticated than is required when a large number of series are to be adjusted for publication, they are virtually bound to be useful in detailed studies of economic series, since they deal with their fundamental statistical components.
105.

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