Spherical Harmonic Expansion of Fisher-Bingham Distribution and 3D Spatial Fading Correlation for Multiple-Antenna Systems

Yibeltal F. Alem, *Student Member, IEEE*, Zubair Khalid, *Member, IEEE* and Rodney A. Kennedy, *Fellow, IEEE*

***Abstract*—This paper considers the 3D spatial fading correlation (SFC) resulting from an angle-of-arrival (AoA) distribution that can be modeled by a mixture of Fisher-Bingham distributions on the sphere. By deriving a closed-form expression for the spherical harmonic transform for the component Fisher-Bingham distributions, with arbitrary parameter values, we obtain a closed-form expression of the 3D-SFC for the mixture case. The 3D-SFC expression is general and can be used in arbitrary multi-antenna array geometries and is demonstrated for the cases of a 2D uniform circular array and a 3D regular dodecahedral array. In computational aspects, we use recursions to compute the spherical harmonic coefficients and give pragmatic guidelines on the truncation size in the series representations to yield machine precision accuracy results. The results are further corroborated through numerical experiments to demonstrate that the closed-form expressions yield the same results as significantly more computationally expensive numerical integration methods.**

***Index Terms*—Fisher-Bingham distribution, spherical harmonic expansion, spatial correlation, MIMO, angle of arrival (AoA).**

I. Introduction

The Fisher-Bingham distribution, also known as the Kent distribution, belongs to the family of spherical distributions in directional statistics [1]. It has been used in applications in a wide range of disciplines for modeling and analyzing directional data. These applications include modeling protein structures [2], 3D beamforming [3], and modeling the distribution of AoA in wireless communication [4]–[6], to name a few.

In this work, we focus our attention to the use of Fisher-Bingham distribution for modeling the distribution of angle of arrival (AoA) in wireless communication and computation of spatial fading correlation (SFC) experienced between elements of multiple-antenna array systems. This is a key factor in evaluating the performance of wireless communication systems with multiple antenna elements [7]–[12]. It has been an active area of research for the past two decades or so and a number of spatial correlation models and closed-form expressions for evaluating the SFC function have been developed, e.g., [4], [7], [13]–[22].

Y. F. Alem and R. A. Kennedy are with the Research School of Engineering, College of Engineering and Computer Science, The Australian National University, Canberra, ACT 2601, Australia (Emails: {yilbeltal.alem,rodney.kennedy}@anu.edu.au). Z. Khalid is with Department of Electrical Engineering, School of Science and Engineering, Lahore University of Management Sciences, Lahore 54792, Pakistan (Email: zubair.khalid@lums.edu.pk). This work was supported by the Australian Research Council's Discovery Projects funding scheme (Project no. DP150101011).

The elliptic (directional) nature of the Fisher-Bingham distribution offers flexibility in modeling the distribution of normalized power or AoA of the multipath components for many practical scenarios. Exploiting this fact, a 3D spatial correlation model has been developed in [6], where the distribution of AoA of the multipath components is modeled by a positive linear sum of Fisher-Bingham distributions, each with different parameters. In that work the SFC was evaluated using numerical integration techniques.

If the spherical harmonic expansion of the Fisher-Bingham distribution is given in a closed-form, the SFC function can be computed analytically using the spherical harmonic expansion of the distribution of AoA of the multipath components [4]. In the current work, we present the spherical harmonic expansion of Fisher-Bingham distribution and a closed-form expression that enables the analytic computation of the spherical harmonic coefficients. We also address the computational considerations required to be taken into account in the evaluation of the proposed closed-form. We also formulate the SFC function experienced between two arbitrary points in 3D-space for the case when the distribution of AoA of the multipath components is modeled by a mixture (positive linear sum) of Fisher-Bingham distributions. Through numerical analysis, we also validate the correctness of the proposed spherical harmonic expansion of Fisher-Bingham distribution and the SFC function. In this paper, our main objective is to employ the proposed spherical harmonic expansion of the Fisher-Bingham distribution for computing the spatial correlation. However, we expect that the proposed spherical harmonic expansion can be useful in various applications where the Fisher-Bingham distribution is used to model and analyze directional data.

II. Mathematical Preliminaries

A. Signals on the 2-Sphere

We consider complex valued square-integrable functions defined on the 2-sphere, $\mathbb{S}^2$. The set of such functions forms a Hilbert space, denoted by $L^2(\mathbb{S}^2)$, when equipped with an inner product defined for two functions f and g on $\mathbb{S}^2$ as [23]

$$\langle f, g\rangle \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\overline{g(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}), \tag{1}$$

which induces a norm $\|f\| \triangleq \langle f, f\rangle^{1/2}$. Here, $\overline{(\cdot)}$ denotes the complex conjugate operation and unit vector $\widehat{\boldsymbol{x}} \triangleq [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]^T \in \mathbb{S}^2 \subset \mathbb{R}^3$ represents a point

on the 2-sphere, where $[\,\cdot\,]^T$ represents the vector transpose, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ denote the co-latitude and longitude respectively and $ds(\widehat{\boldsymbol{x}}) = \sin\theta\, d\theta\, d\phi$ is the surface measure on the 2-sphere. The functions with finite energy (induced norm) are referred as signals on the sphere.

B. Spherical Harmonics

Spherical harmonics serve as orthonormal basis functions for the representation of functions on the sphere and are defined for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$ as

$$Y_\ell^m(\widehat{\boldsymbol{x}}) \equiv Y_\ell^m(\theta, \phi) \triangleq N_\ell^m P_\ell^m(\cos\theta) e^{im\phi},$$

with normalization factor

$$N_\ell^m \triangleq \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}, \tag{2}$$

such that $\langle Y_\ell^m, Y_p^q \rangle = \delta_{\ell,p}\delta_{m,q}$, where $\delta_{m,q}$ is the Kronecker delta function: $\delta_{m,q} = 1$ for $m = q$ and is zero otherwise. $P_\ell^m(\cdot)$ denotes the associated Legendre polynomial of degree ℓ and order m [23]. We also note the following relation for associated Legendre polynomial

$$P_\ell^m(\cos\theta) = \sqrt{\frac{(\ell+m)!}{(\ell-m)!}}\, d_{m,0}^\ell(\theta), \tag{3}$$

where $d_{m,m'}^\ell(\cdot)$ denotes the Wigner-d function of degree ℓ and orders m and m' [23].

By completeness of spherical harmonics, any finite energy function $f(\widehat{\boldsymbol{x}})$ on the 2-sphere can be expanded as

$$f(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (f)_\ell^m\, Y_\ell^m(\widehat{\boldsymbol{x}}), \tag{4}$$

where $(f)_\ell^m$ is the spherical harmonic coefficient given by

$$(f)_\ell^m = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}). \tag{5}$$

The signal f is said to be band-limited in the spectral domain at degree L if $(f)_\ell^m = 0$ for $\ell > L$.

III. Problem Formulation

A. Fisher-Bingham Distribution on Sphere

Definition 1 (FB5-Distribution): The Fisher-Bingham five-parameter distribution (FB5-Distribution), also known as the Kent distribution, is a distribution on the sphere with probability density function (pdf) defined as [1], [2]

$$g(\widehat{\boldsymbol{x}}; \kappa, \widehat{\boldsymbol{\mu}}, \beta, \mathbf{A}) = \frac{1}{C(\kappa,\beta)} e^{\kappa\widehat{\boldsymbol{\mu}}^T\widehat{\boldsymbol{x}} + \beta\widehat{\boldsymbol{x}}^T\mathbf{A}\widehat{\boldsymbol{x}}}, \tag{6}$$

where $\mathbf{A}$ is a 3×3 symmetric matrix given by $\mathbf{A} = \widehat{\boldsymbol{\eta}}_1\widehat{\boldsymbol{\eta}}_1^T - \widehat{\boldsymbol{\eta}}_2\widehat{\boldsymbol{\eta}}_2^T$. Here $\widehat{\boldsymbol{\mu}}$, $\widehat{\boldsymbol{\eta}}_1$ and $\widehat{\boldsymbol{\eta}}_2$ are orthonormal vectors that denote the mean direction (centre), major axis and minor axis of the distribution, respectively, $\kappa \geq 0$ quantifies the concentration of FB5-Distribution around its mean, and $0 \leq \beta \leq \kappa/2$ quantifies the ellipticity of the distribution. The normalization term is given by

$$C(\kappa,\beta) = 2\pi\sum_{r=0}^{\infty}\frac{\Gamma(r+1/2)}{\Gamma(r+1)}\beta^{2r}(\kappa/2)^{-2r-1/2} I_{2r+1/2}(\kappa),$$

where $I_r(\cdot)$ denotes the modified Bessel function of the first kind of order r.

The FB5-Distribution belongs to the family of spherical distributions in directional statistics and is the analogue of the Euclidean domain bivariate normal distribution with unconstrained covariance matrix [1], [2].

B. 3D Spatial Fading Correlation (SFC)

In MIMO systems, the 3D multipath channel impulse response for a signal arriving at antenna array is characterized by the steering vector of the antenna array. For an antenna array consisting of M antenna elements placed at $\boldsymbol{z}_p \in \mathbb{R}^3$, $p = 1, 2, \ldots, M$, the steering vector, denoted by $\boldsymbol{\alpha}(\widehat{\boldsymbol{x}})$, is given by

$$\boldsymbol{\alpha}(\widehat{\boldsymbol{x}}) = \big[\alpha_1(\widehat{\boldsymbol{x}}), \alpha_2(\widehat{\boldsymbol{x}}), \ldots, \alpha_M(\widehat{\boldsymbol{x}})\big],\ \alpha_p(\widehat{\boldsymbol{x}}) \triangleq e^{ik\boldsymbol{z}_p\cdot\widehat{\boldsymbol{x}}}, \tag{7}$$

where $\widehat{\boldsymbol{x}} \in \mathbb{R}^3$ denotes a unit vector pointing in the direction of wave propagation and $k = 2\pi/\lambda$ with λ denoting the wavelength of the arriving signal. For any $h(\widehat{\boldsymbol{x}})$ representing the pdf of the angles of arrival (AoA) of the multipath components or the unit-normalized power of a signal received from the direction $\widehat{\boldsymbol{x}}$, the 3D SFC function between the p-th and the q-th antenna elements, located at $\boldsymbol{z}_p$ and $\boldsymbol{z}_q$, respectively, with an assumption that signals arriving at the antenna elements are narrowband, is given by [4]

$$\begin{aligned}\rho(\boldsymbol{z}_p, \boldsymbol{z}_q) &\triangleq \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,\alpha_p(\widehat{\boldsymbol{x}})\,\overline{\alpha_q(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}) \\ &= \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\, e^{ik(\boldsymbol{z}_p-\boldsymbol{z}_q)\cdot\widehat{\boldsymbol{x}}}\, ds(\widehat{\boldsymbol{x}}) \equiv \rho(\boldsymbol{z}_p - \boldsymbol{z}_q),\end{aligned} \tag{8}$$

which indicates that the SFC only depends on $\boldsymbol{z}_p - \boldsymbol{z}_q$ and is, therefore, spatially wide-sense stationary.

C. FB5-Distribution Based Spatial Correlation Model

The FB5-Distribution offers the flexibility, due to its directional nature, to model the distribution of normalized power or AoA of the multipath components for many practical scenarios. Utilizing this capability of FB5-Distribution, a 3D spatial correlation model for the mixture of FB5-Distributions defining the AoA distribution has been developed in [6]. Here the mixture refers to the positive linear sum of a number of FB5-Distributions, each with different parameters. We defer the formulation of FB5 based correlation model until Section V. Although the correlation model proposed in [6] is general in a sense that the SFC function can be computed for any arbitrary antenna geometry, the formulation of SFC function involves the computation of integrals that can only be carried out using numerical integration techniques. If the spherical harmonic expansion of the FB5-Distribution is given in closed-form, the SFC function can be computed analytically following the approach used in [4].

D. Problems Under Consideration

In this paper, we derive a *closed-form expression to compute the spherical harmonic expansion of the FB5-Distribution*. Using the spherical harmonic expansion of FB5-Distribution, we also formulate a *closed-form SFC function for the mixture of FB5-Distributions*. We also give computational considerations when using our expressions.

IV. Spherical Harmonic Expansion of the FB5-Distribution

Two of the five parameters in the FB5-Distribution relate to its shape: the concentration, κ, and the ellipticity, β. The other three vectors, $\widehat{\boldsymbol{\mu}}$, $\widehat{\boldsymbol{\eta}}_1$ and $\widehat{\boldsymbol{\eta}}_2$, have an interpretation as the orientation but can also be characterized by three Euler angles, φ, ϑ and ω, as follows. Define the rotation matrix

$$\begin{aligned}\boldsymbol{R} &\triangleq [\,\widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\mu}}\,] \\ &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_z(\omega),\end{aligned} \tag{9}$$

where

$$\boldsymbol{R}_y(\vartheta) \triangleq \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix}, \ \boldsymbol{R}_z(\varphi) \triangleq \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

denote rotation about the y and z axes, respectively. Then from (9), $\vartheta = \cos^{-1}(R_{3,3})$, where $R_{a,b}$ denotes the entry at a-th row and b-th column of the matrix $\boldsymbol{R}$ given in (9). Similarly $\varphi \in [0, 2\pi)$ and $\omega \in [0, 2\pi)$ can be found by a four-quadrant search satisfying

$$\sin\varphi = \frac{R_{2,3}}{\sqrt{1-(R_{3,3})^2}}, \quad \cos\varphi = \frac{R_{1,3}}{\sqrt{1-(R_{3,3})^2}}, \tag{10}$$

$$\sin\omega = \frac{R_{3,2}}{\sqrt{1-(R_{3,3})^2}}, \quad \cos\omega = \frac{-R_{3,1}}{\sqrt{1-(R_{3,3})^2}}, \tag{11}$$

respectively.

With mean (centre) $\widehat{\boldsymbol{\mu}}^0$ located along the z-axis, major axis $\widehat{\boldsymbol{\eta}}_1^0$ and minor axis $\widehat{\boldsymbol{\eta}}_2^0$ aligned along the x-axis and y-axis, respectively then $\boldsymbol{R} = \boldsymbol{I}$. This defines the *standard* Fisher-Bingham distribution (FB-Distribution)

$$f(\widehat{\boldsymbol{x}}; \kappa, \beta) \triangleq \frac{1}{C(\kappa,\beta)} e^{\kappa\cos\theta + \beta\sin^2\theta\cos 2\phi}, \tag{12}$$

and the more general FB5-Distribution can be recovered through

$$g(\widehat{\boldsymbol{x}}; \kappa, \beta, \widehat{\boldsymbol{\mu}}, \mathbf{A}) = f\big(\boldsymbol{R}^{-1}\widehat{\boldsymbol{x}}; \kappa, \beta\big). \tag{13}$$

Therefore, our approach is to first compute the spherical harmonic coefficients of the FB-Distribution and then determine by transformation the coefficients of the FB5-Distribution.

A. Spherical Harmonic Expansion of FB-Distribution

Our first task is to derive the spherical harmonic coefficients of the standard FB-Distribution, (12), which can be written as

$$(f)_\ell^m = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}}; \kappa, \beta)\overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}). \tag{14}$$

Since $f(\widehat{\boldsymbol{x}}; \kappa, \beta)$ is a real-valued function, it suffices to compute the coefficients for positive orders $0 \le m \le \ell$ for each degree $\ell \ge 0$, and use $(f)_\ell^{-m} = (-1)^m\overline{(f)_\ell^m}$ to infer those of negative orders [23].

To derive a closed-form expression for computing the spherical harmonic coefficients, we rewrite (14) explicitly as

$$\begin{aligned}(f)_\ell^m &= \frac{N_\ell^m}{C(\kappa,\beta)} \int_0^\pi e^{\kappa\cos\theta} P_\ell^m(\cos\theta)\sin\theta\, d\theta \\ &\qquad \times \int_0^{2\pi} e^{-im\phi} e^{\beta\sin^2\theta\cos 2\phi} d\phi \\ &= \frac{N_\ell^m}{C(\kappa,\beta)} \int_0^\pi e^{\kappa\cos\theta}\, F_m(\theta)\, P_\ell^m(\cos\theta)\sin\theta\, d\theta,\end{aligned} \tag{15}$$

where $F_m(\theta) \triangleq \int_0^{2\pi} e^{\beta\sin^2\theta\cos 2\phi - im\phi} d\phi$, using Mathematica, reduces to

$$F_m(\theta) = \begin{cases} 2\pi I_{m/2}(\beta\sin^2\theta) & m \in 0, 2, 4, \ldots \\ 0 & m \in 1, 3, 5, \ldots \end{cases}. \tag{16}$$

By expanding the modified Bessel function $I_{m/2}(\beta\sin^2\theta)$ as

$$I_{m/2}(\beta\sin^2\theta) = \sum_{t=0}^{\infty} \left(\frac{\beta}{2}\right)^{2t+m/2} \frac{(\sin\theta)^{4t+m}}{t!\,(t+m/2)!}, \tag{17}$$

the exponential $e^{\kappa\cos\theta}$ as [5]

$$e^{\kappa\cos\theta} = \sqrt{\frac{\pi}{2\kappa}} \sum_{n=0}^{\infty} (2n+1) I_{n+1/2}(\kappa) P_n^0(\cos\theta), \tag{18}$$

and using (3), and the expansion of the Wigner-d function

$$d_{m,n}^\ell(\theta) = i^{n-m} \sum_{u=-\ell}^{\ell} d_{u,m}^\ell(\pi/2)\, d_{u,n}^\ell(\pi/2)\, e^{iu\theta}, \tag{19}$$

we obtain a closed-form expression for the spherical harmonic coefficients of non-negative even order:

$$\begin{aligned}(f)_\ell^m = &\frac{\pi i^{-m}}{C(\kappa,\beta)} \sqrt{\frac{2\ell+1}{2\kappa}} \sum_{n=0}^{\infty} (2n+1) I_{n+1/2}(\kappa) \times \\ &\sum_{u=-n}^{n} \big(d_{u,0}^n(\pi/2)\big)^2 \sum_{t=0}^{\infty} \frac{(\beta/2)^{2t+m/2}}{\Gamma(t+1)\Gamma(t+m/2+1)} \times \\ &\sum_{u'=-\ell}^{\ell} d_{u',0}^\ell(\pi/2)\, d_{u',m}^\ell(\pi/2)\, G(4t+m+1, u+u'), \\ &\qquad \text{for } m = 0, 2, 4, \ldots,\end{aligned} \tag{20}$$

where $\Gamma(\cdot)$ denotes the Gamma function and we have used

$$\begin{aligned}G(p,q) &\triangleq \int_0^\pi (\sin\theta)^p e^{iq\theta} d\theta, \\ &= \frac{\pi e^{iq\pi/2}\Gamma(p+2)}{2^p(p+1)\Gamma\big((p+q+2)/2\big)\Gamma\big((p-q+2)/2\big)},\end{aligned} \tag{21}$$

which is an identity given in [24, Sec. 3.892].

Finally, by $(f)_\ell^{-m} = (-1)^m\overline{(f)_\ell^m}$ and (16), $(f)_\ell^m = 0$ for order m odd, and $(f)_\ell^m = (f)_\ell^{-m}$, that is, (20) can be used for order m negative even.

B. Spherical Harmonic Expansion of FB5-Distribution

In the spatial domain, the FB5-Distribution, g, can be related to the standard FB-Distribution, f, through a rotation, denoted by operator $\mathcal{D}(\varphi, \vartheta, \omega)$, with Euler angles $(\varphi, \vartheta, \omega)$. Then it is a standard result to relate the spherical harmonic coefficients for the two distributions. The desired spherical harmonic expansion of FB5-Distribution is then

$$\begin{aligned}(g)_\ell^m \triangleq \big\langle g, Y_\ell^m \big\rangle &= \big\langle \mathcal{D}(\varphi,\vartheta,\omega) f, Y_\ell^m \big\rangle \\ &= \sum_{m'=-\ell}^{\ell} D_{m,m'}^\ell(\varphi,\vartheta,\omega)(f)_\ell^{m'}\end{aligned} \tag{22}$$

where $(f)_\ell^m$ is given in (20) and the

$$D_{m,m'}^\ell(\varphi,\vartheta,\omega) = e^{-im\varphi} d_{m,m'}^\ell(\vartheta) e^{-im'\omega} \tag{23}$$

are the Wigner D-functions [23].

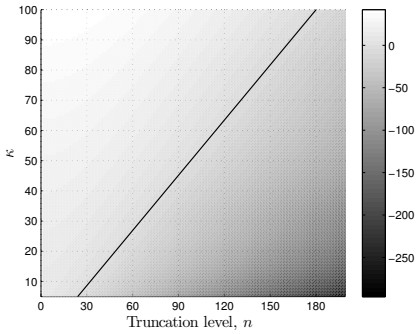

Fig. 1. Modified Bessel function $I_{n+1/2}(\kappa)$ vs concentration κ and truncation level n from (18). $I_{n+1/2}(\kappa) < 10^{-16}$ is approximated by region below line (24).

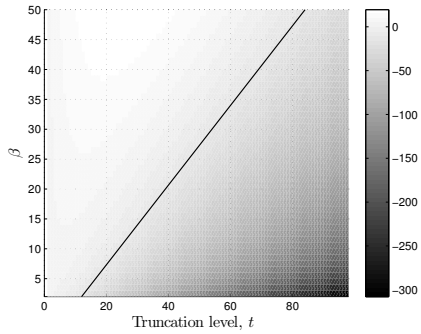

Fig. 2. Maximal summand in (17), for $m = 0$ and $\theta = \pi/2$, $S(\beta, t)$ vs ellipticity β and truncation t. $S(\beta, t) < 10^{-16}$ is approximated by region below line (25).

C. Computational Considerations

Here we discuss the computation of Wigner-d functions, at a fixed argument of $\pi/2$, which are essentially required for the computation of spherical harmonic expansion of standard FB or FB5-Distribution using the proposed formulation (20). Another computational consideration that is addressed here is the evaluation of infinite summations over t and n involved in the computation of (20).

1) Computation of Wigner-d functions: To compute (20), we are required to evaluate the Wigner-d functions at a fixed argument of $\pi/2$, that is, $d^{\ell}_{u,m}(\pi/2)$, for each $|u|, |m| \leq \ell$. Let $\mathbf{D}_\ell$ denote the matrix of size $(2\ell+1)\times(2\ell+1)$ with entries $d^{\ell}_{u,m}(\pi/2)$ for $|u|, |m| \leq \ell$. The matrix $\mathbf{D}_\ell$ can be computed for each $\ell = 1, 2, \ldots,$ using the relation given in [25] that recursively computes $\mathbf{D}_\ell$ from $\mathbf{D}_{\ell-1}$.

2) Truncation over n: The expansion of the exponential function in (18) introduces an infinite sum over n. We propose to truncate the index n to level N such that the discarded terms are less than 10^{-16} and do not impact the summation. The truncation N will depend on the concentration κ.

To characterize the summand in (18), we note that it is dominated by modified Bessel function $I_{n+1/2}(\kappa)$ because $|P^0_n(\cdot)| \leq 1$. In Fig. 1 we plot $I_{n+1/2}(\kappa)$ vs truncation n and concentration $0 \leq \kappa \leq 100$. We approximate the relationship between such a truncation level n and the concentration κ, by the affine truncation law, also shown in Fig. 1,

$$N = (3/2)\,\kappa + 24, \tag{24}$$

This truncation level is found to give truncation error less than the machine precision level for the values of concentration κ in the range $0 \leq \kappa \leq 100$ (used in practice [5], [6]).

3) Truncation over t: The expansion of the modified Bessel functions in (17) introduces an infinite sum over t. The decaying summand in (17) with the increasing t makes it possible to truncate the infinite sum depending on the ellipticity β.

We plot the *maximal* summand in (17), which occurs for $m = 0$ and $\theta = \pi/2$, that is, $S(\beta, t) \triangleq \beta^{2t}/(t!\,t!\,2^{2t})$, for different values of the ellipticity β in Fig. 2. We again propose to truncate to level T such that the discarded terms are less than 10^{-16} and do not impact the summation. We approximate the relationship between such a truncation level T and the ellipticity β, by the affine truncation law, also shown in Fig. 2,

$$T = (36/25)\,\beta + 12. \tag{25}$$

and this is valid for all positive orders m and all θ.

V. 3D Spatial Fading Correlation for Mixture of Fisher-Bingham Distributions

In this section, we derive a closed-form expression to compute the SFC function for the spatial correlation model based on a mixture of FB5-Distributions defining the distribution of AoA of multipath components [5], [6]. Let $h(\widehat{\boldsymbol{x}})$ denote the positive linear sum of W FB5-Distributions given by

$$h(\widehat{\boldsymbol{x}}) = \sum_{w=1}^{W} K_w\, g(\widehat{\boldsymbol{x}}; \kappa_w, \widehat{\boldsymbol{\mu}}_w, \beta_w, \mathbf{A}_w), \tag{26}$$

where K_w denotes the positive weight of w-th FB5-Distribution with parameters $\kappa_w, \widehat{\boldsymbol{\mu}}_w, \beta_w$ and $\mathbf{A}_w$. We assume the K_w are such that h is normalized. For the distribution with pdf defined in (26), the SFC function, given in (8) has been formulated in [6] and studied for a uniform circular antenna array using non closed-form numerical integrals.

We follow the approach introduced in [4] to derive a closed-form 3D SFC function, which relies on the spherical harmonic expansion of a plane wave

$$e^{ik\boldsymbol{z}_p\cdot\widehat{\boldsymbol{x}}} = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell\big(k\|\boldsymbol{z}_p\|\big) \sum_{m=-\ell}^{\ell} Y^m_\ell\big(\boldsymbol{z}_p/\|\boldsymbol{z}_p\|\big)\overline{Y^m_\ell(\widehat{\boldsymbol{x}})},$$

and expanding the mixture distribution $h(\widehat{\boldsymbol{x}})$ in (26), following (4), and employing the orthonormality of spherical harmonics, we write the SFC function in (8) as [4]

$$\rho(\boldsymbol{z}_p - \boldsymbol{z}_q) = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell\big(k\|\boldsymbol{z}_p - \boldsymbol{z}_q\|\big) \sum_{m=-\ell}^{\ell} (h)^m_\ell Y^m_\ell\Big(\frac{\boldsymbol{z}_p - \boldsymbol{z}_q}{\|\boldsymbol{z}_p - \boldsymbol{z}_q\|}\Big), \tag{27}$$

where

$$(h)^m_\ell = \sum_{w=1}^{W} K_w \sum_{m'=-\ell}^{\ell} D^{\ell}_{m,m'}\,(\varphi_w, \vartheta_w, \omega_w)(f)^{m'}_\ell, \tag{28}$$

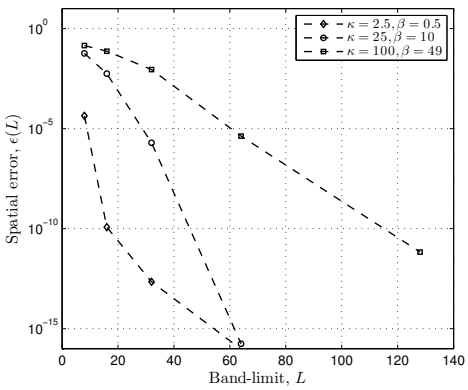

Fig. 3. Spatial error $\epsilon(L)$, given in (29), between the standard FB-Distribution given in (12) and the reconstructed standard FB-Distribution from its coefficients up to the degree $L-1$.

which is obtained by combining (22) and (26). Here, $(f)_\ell^{m'}$ denotes the spherical harmonic coefficient of the standard FB-Distribution and the Euler angles $(\varphi_w, \vartheta_w, \omega_w)$ relate the w-th FB5-Distribution $g(\widehat{\boldsymbol{x}}; \kappa_w, \widehat{\boldsymbol{\mu}}_w, \beta_w, \mathbf{A}_w)$ of the mixture and the standard FB distribution $f(\widehat{\boldsymbol{x}})$ through (13). In the computation of the SFC using the proposed formulation, given in (27), we note that the summation for ℓ over first few terms yields sufficient accuracy as higher order Bessel functions decay rapidly to zero for points near each other in space, as indicated in [4], [26]. We conclude this section with a note that the SFC function can be analytically computed using the proposed formulation for an arbitrary antenna array geometry and the distribution of AoA modeled by a mixture of FB5-Distributions, each with different parameters. In the next section, we evaluate the proposed SFC function for uniform circular array and a 3D regular dodecahedron array.

VI. Numerical Study

We conduct numerical experiments to validate the correctness of the proposed analytic expressions, formulated in (20)–(21) and (27)–(28). For computing the spherical harmonic transform and discretization on the sphere, we employ the optimal-dimensionality sampling scheme on the sphere [27].

A. Accuracy Analysis – Standard FB-Distribution

In order to analyze the accuracy of the proposed closed-form expression, for the computation of spherical harmonic coefficients of the standard FB-Distribution (20), we define the spatial error as

$$\epsilon(L) = \frac{1}{L^2} \sum_{\widehat{\boldsymbol{x}}_p} \left| f(\widehat{\boldsymbol{x}}_p) - \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\widehat{\boldsymbol{x}}_p) \right|^2, \quad (29)$$

which quantifies the error between the standard FB-Distribution (12) and its reconstruction from its coefficients (20), up to the degree $L-1$. The summation in (29) is averaged over L^2 number of samples of the sampling scheme [27]. We plot the spatial error $\epsilon(L)$ against band-limit L for different parameters of the standard FB-Distr bution in Fig. 3, where it is evident that the spatial error converges to zero (machine precision) as L increases. Consequently, the standard FB-Distribution reconstructed from its coefficients converges to the formulation of standard FB-Distribution in spatial domain and thus corroborates the correctness of (20).

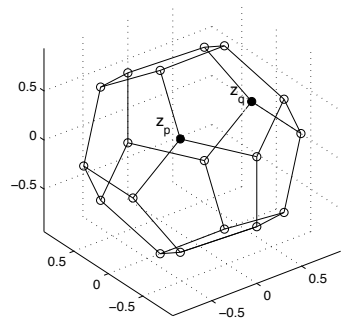

Fig. 4. Regular dodecahedron array (RDA) – 20 antenna elements positioned at the vertices of a regular dodecahedron inscribed in a sphere of radius $R = 1$.

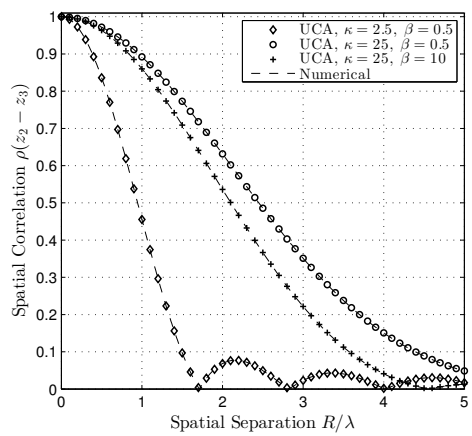

Fig. 5. Magnitude of the SFC function $\rho(\boldsymbol{z}_2 - \boldsymbol{z}_3)$ for 16-element UCA of radius R.

B. SFC Function Numerical Study

Here, we corroborate the validity of the proposed closed-form expression for the SFC function through numerical experiments. In our study, we consider both 2D and 3D antenna array geometries in the form of uniform circular array (UCA) and regular dodecahedron array (RDA), respectively. The antenna elements of M-element UCA are placed at the following spatial positions

$$\boldsymbol{z}_p = \left[R\cos(2\pi p/M),\ R\sin(2\pi p/M),\ 0 \right]^T \in \mathbb{R}^3, \quad (30)$$

where R is the radius. For the RDA, array elements are positioned at the vertices of a regular dodecahedron inscribed in a sphere of radius R, as shown in Fig. 4. We assume that the signal AoA follows a standard FB-Distribution.

Using the proposed closed-form expression in (27), we determine the SFC between the second and third UCA antenna elements and plot the magnitude of the SFC function $\rho(\boldsymbol{z}_2 - \boldsymbol{z}_3)$ in Fig. 5 against the normalized radius R/λ. In the same figure, we also plot the numerically evaluated SFC function, formulated in (8) and originally proposed in [6],

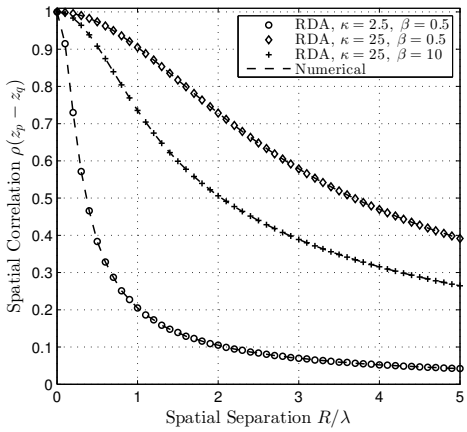

Fig. 6. Magnitude of the SFC function $\rho(\boldsymbol{z}_p - \boldsymbol{z}_q)$ between RDA antenna elements shaded in black in Fig. 4.

which matches with the proposed closed-form expression for the SFC function. We emphasize that numerical evaluation of the integrals employs computationally intensive techniques to obtain sufficiently accurate results. Similarly, we compute the SFC between two antenna elements positioned at $\boldsymbol{z}_p$ and $\boldsymbol{z}_q$ on RDA, which are indicated in Fig. 4, and plot the magnitude $\rho(\boldsymbol{z}_p - \boldsymbol{z}_q)$ Fig. 6, which again matches with the numerically evaluated SFC function given in (8) and thus corroborates the correctness of proposed SFC function.

VII. Conclusions

In this paper, the spherical harmonic expansion of the Fisher-Bingham distribution has been presented in closed-form. Thereby, we have been able to provide a closed-form expression for the spatial fading correlation (SFC) when the angle-of-arrival (AoA) distribution can be modeled by a mixture of Fisher-Bingham distributions. Furthermore, we have corroborated the correctness of proposed closed-form expressions through experiments with a 2D uniform circular array and a 3D regular dodecahedral array. We have focussed on the use of Fisher-Bingham distribution for the computation of SFC function. However, we believe that the proposed spherical harmonic expansion of the Fisher-Bingham distribution has a great potential of applicability in various applications for directional statistics and data analysis on the sphere.

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