LORENTZIAN GEOMETRY AND PHYSICS IN KASPAROV’S THEORY

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I hereby declare that the contents of this thesis are my own original work, unless otherwise indicated.

Chapters 2 and 3 collect preliminary material regarding unbounded operators (especially spectral triples and Kasparov modules) and pseudo-Riemannian spin geometry, respectively. These chapters are written by me, but their contents are mostly based on the existing literature. The main exception is Section 2.6 (see below).

Chapter 4 is based on joint work [DPR13] with Mario Paschke and Adam Rennie (University of Wollongong).

Section 2.6 and Chapter 5 are based on joint work [DR15] with Adam Rennie (University of Wollongong).

Chapter 6 is based on joint work [BD14] with Jord Boeijink (Radboud University Nijmegen).

Chapter 7 is my own work.

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We study two geometric themes, Lorentzian geometry and gauge theory, from the perspective of Connes’ noncommutative geometry and (the unbounded version of) Kasparov’s $KK$-theory. Lorentzian geometry is the mathematical framework underlying Einstein’s description of gravity. The geometric formulation of a gauge theory (in terms of principal bundles) offers a classical description for the interactions between particles. The underlying motivation is the hope that this noncommutative approach may lead to a unified description of gauge theories coupled with gravity on a Lorentzian manifold.

The main objects in noncommutative geometry are spectral triples, which encompass and generalise Riemannian spin manifolds. A spectral triple defines a class in $K$-homology, via which one can access the topology of the (noncommutative) manifold. In this thesis we present two possible definitions for ‘Lorentian spectral triples’, which offer noncommutative generalisations of Lorentzian manifolds as well. We will prove that both definitions preserve the link with analytic $K$-homology. We will describe under which conditions Lorentzian (or pseudo-Riemannian) manifolds satisfy these definitions. Another main example is the harmonic oscillator, which in particular shows that our framework allows to deal with more than just metrics of indefinite signature.

In the context of noncommutative geometry, the description of a gauge theory can be obtained from so-called almost-commutative manifolds. While the usual approach yields by default a topologically trivial gauge theory (in the sense that the corresponding principal fibre bundle is globally trivial), we show in this thesis that the framework can be adapted, using the internal unbounded Kasparov product, to allow for globally non-trivial gauge theories as well.

Finally, we combine the two themes of Lorentzian geometry and gauge theory, and we define Krein spectral triples, which generalise spectral triples from Hilbert spaces to Krein spaces. We use this definition to construct almost-commutative Lorentzian manifolds. Furthermore, we propose a Lorentzian alternative for the fermionic action, which allows to derive (the fermionic part of) the Lagrangian of a gauge theory. We show that our alternative action recovers exactly the correct physical Lagrangian.
CONTENTS

DECLARATION iii

ACKNOWLEDGEMENTS v

ABSTRACT vii

1 INTRODUCTION 1
  1.1 Noncommutative Lorentzian manifolds 2
    1.1.1 Wick rotations and the link to K-homology 3
    1.1.2 The spectral geometry of pseudo-Riemannian manifolds 4
    1.1.3 The harmonic oscillator 5
  1.2 Gauge theory and the Standard Model 5
    1.2.1 Almost-commutative manifolds and gauge theories 6
    1.2.2 Gravity on almost-commutative manifolds 7
    1.2.3 The noncommutative Standard Model 7
    1.2.4 Non-trivial global structure 8
    1.2.5 Gauge theories on Lorentzian manifolds 8
  1.3 Outline 9

2 SPECTRAL TRIPLES AND KASPAROV MODULES 13
  2.1 Hilbert modules over graded algebras 13
    2.1.1 Interior tensor products 15
    2.1.2 Unbounded operators 16
    2.1.3 Localisations 17
  2.2 Operator spaces 18
    2.2.1 Operator algebras 18
    2.2.2 Operator modules 19
    2.2.3 The Grassmann connection 20
  2.3 Almost (anti-)commuting operators 21
  2.4 Unbounded Kasparov modules 24
    2.4.1 Odd unbounded Kasparov modules 26
    2.4.2 The unbounded Kasparov product 28
  2.5 Smoothly summable spectral triples 30
    2.5.1 Summability 30
    2.5.2 Smoothness 31
    2.5.3 Smooth summability 32
  2.6 Non-symmetric operators 34
    2.6.1 Wick rotation 35
    2.6.2 Reverse Wick rotation 36

3 PSEUDO-RIEMANNIAN SPIN GEOMETRY 39
  3.1 Clifford algebras and spin groups 39
    3.1.1 The Clifford algebra 40
INTRODUCTION

Over the past century, our theoretical understanding of fundamental physics has increased tremendously. There are currently two well-established theories, each describing a different part of the physical realm. The first is Einstein’s theory of General Relativity, which describes gravity. The second is quantum field theory, and in particular the Standard Model, which describes all the elementary particles and their interactions (except for gravity). Both of these theories agree with experiments with an amazing accuracy.

Ever since the formulation of the Standard Model, it has been an outstanding challenge to find a theory of quantum gravity which could combine the theory of gravity with quantum field theory. It is this challenge which provides the underlying motivation for the work contained in this thesis. There are several areas of research which attempt to address this challenge, and most notable among them are string theory and loop quantum gravity. In this thesis we work in the area of noncommutative geometry [Con94], and it will be explained below how this field of mathematics may provide clues for the unification of gravity and quantum field theory.

This thesis combines two geometric themes: Lorentzian geometry and gauge theory. Lorentzian geometry is the mathematical framework underlying Einstein’s description of gravity. The geometric formulation of a gauge theory (in terms of principal bundles) offers a classical description for the interactions between particles. The unifying thread running through this thesis is that both these themes will be approached from the perspective of noncommutative geometry (or more precisely, unbounded KK-theory [Kas80, BJ83]). The ultimate goal is to use these frameworks to obtain a unified description of gauge theories coupled with gravity on a Lorentzian manifold.

Below we will give a brief introduction, from the perspective of noncommutative geometry, to the two themes of Lorentzian geometry and gauge theory. At the same time, we will mention the main contributions to these themes that are made in this thesis. Afterwards, we will give a detailed outline and summary of the contents of this thesis.
1.1 NONCOMMUTATIVE LORENTZIAN MANIFOLDS

The mathematical framework of noncommutative geometry was established by Alain Connes in the 1980’s and 1990’s [Con94]. The main objects in noncommutative geometry are spectral triples, which encompass and generalise Riemannian spin manifolds [Con96, Con13].

Consider a complete Riemannian spin manifold \((M, g)\). We can then canonically construct a Dirac operator \(D\) on the spinor bundle \(S\) over \(M\). This Dirac operator gives rise to a triple \((\mathcal{C}_c^\infty(M), L^2(S), D)\), where the algebra \(\mathcal{C}_c^\infty(M)\) consists of the smooth, compactly supported functions on \(M\), and \(L^2(S)\) denotes the Hilbert space of square-integrable spinors. Since the Dirac operator is essentially self-adjoint and elliptic, this triple satisfies the axioms of a spectral triple (see Section 2.4 for the definition). In fact, for the special case where \(M\) is compact, it was shown by Connes [Con13] that this triple completely characterises the Riemannian spin manifold \((M, g)\). Thus, all the information about the topology and geometry of \(M\) is contained in the triple \((\mathcal{C}_c^\infty(M), L^2(S), D)\). This observation motivates the interpretation that a general spectral triple \((A, \mathcal{H}, \mathcal{D})\), where the algebra \(A\) is in particular allowed to be noncommutative, gives a description of a ‘noncommutative manifold’. For more information on noncommutative geometry and its applications, we refer to the original book by Connes [Con94], or the introductory texts [Lan97, GV01, Vár06].

By imposing certain equivalence relations, a spectral triple \((A, \mathcal{H}, \mathcal{D})\) gives rise to a class \([\langle A, \mathcal{H}, \mathcal{D} \rangle]\) in analytic K-homology [HR00]. Although all geometric information of the (noncommutative) manifold is lost, this K-homology class does give access to topological information of the manifold.

Now suppose we have a Lorentzian spin manifold \((M, g)\). We can again canonically construct a Dirac operator \(D\) on the spinor bundle \(S\) over \(M\) (see Section 3.4), and we again obtain a triple \((\mathcal{C}_c^\infty(M), L^2(S), D)\). The main difference with the Riemannian case is that the Lorentzian Dirac operator is neither symmetric nor elliptic, and hence this triple is not a spectral triple.

In order to allow for Lorentzian manifolds in noncommutative geometry, it is therefore necessary to generalise the notion of a spectral triple. It is not immediately clear what such a generalised definition should be. Indeed, there have been several attempts in the literature to study Lorentzian noncommutative geometry, along several different directions: through studying foliations of spacetime [Haw97, Kop98, KP01, KP02]; by taking a Krein space approach [Sui04, Str06]; by studying the Lorentzian distance function [Mor03, Fra10, Fra14] (see also [RW14]); or by focusing on the causality properties [FE13, FE14, FE15]. Abstract axioms for Lorentzian and globally hyperbolic spectral triples have also been suggested [PS06, PV04].
1.1 Noncommutative Lorentzian Manifolds

1.1.1 Wick rotations and the link to K-homology

As mentioned above, spectral triples provide a way to extend Riemannian geometry to noncommutative spaces, while retaining the connection to the underlying topology via K-homology. The main difference in our approach to Lorentzian versions of spectral triples, compared to the earlier work mentioned above, is our aim to preserve the link with analytic K-homology. Given a triple \((A, \mathcal{H}, \mathcal{D})\), where \(\mathcal{D}\) is allowed to be non-symmetric, we obtain two symmetric operators given by

\[
\mathcal{D}_\pm := \text{Re } \mathcal{D} \pm \text{Im } \mathcal{D} = \frac{1}{2}(\mathcal{D} + \mathcal{D}^*) \mp \frac{i}{2}(\mathcal{D} - \mathcal{D}^*) .
\]

We refer to these operators \(\mathcal{D}_\pm\) as the ‘Wick rotations’ of \(\mathcal{D}\). Our requirement on the definition of a ‘Lorentzian spectral triple’ is then that these Wick rotations should yield two (genuine) spectral triples \((A, \mathcal{H}, \mathcal{D}_\pm)\). This way each ‘Lorentzian spectral triple’ \((A, \mathcal{H}, \mathcal{D})\) gives rise to two K-homology classes \([([A, \mathcal{H}, \mathcal{D}_\pm])]\), thus allowing access to the underlying topology. In this thesis we will provide two different definitions satisfying this requirement.

In Chapter 4 we will present our definition of a pseudo-Riemannian spectral triple \((A, \mathcal{H}, \mathcal{D})\), enabling a noncommutative analogue of pseudo-Riemannian geometry. This definition imposes assumptions on second-order operators constructed from \(\mathcal{D}\). In particular, we assume that \((\mathcal{D})^2 := \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D})\) is essentially self-adjoint and has locally compact resolvent, and that \(\mathcal{R}_\mathcal{D} := -\frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^*\mathcal{D})\) is ‘suitably smooth’ and ‘suitably bounded’ relative to \((\mathcal{D})^2\). These assumptions ensure that the Wick rotated operators \(\mathcal{D}_\pm\) give rise to spectral triples \((A, \mathcal{H}, \mathcal{D}_\pm)\).

Furthermore, we can define notions of smoothness and summability, and under an additional assumption we can prove that these properties are preserved by the procedure of Wick rotation. Thus, under suitable conditions, we can ensure that the Wick rotations of a pseudo-Riemannian spectral triple are smoothly summable, so that we can apply the (non-unital) local index formula [CGRS14].

In Chapter 5 we define indefinite spectral triples. Instead of using the second-order operators \((\mathcal{D})^2\) and \(\mathcal{R}_\mathcal{D}\), we now focus on first-order operators (namely \(\mathcal{D}, \mathcal{D}^*, \text{Re } \mathcal{D}, \text{Im } \mathcal{D}\)), which is more natural. In fact, all our assumptions continue to make sense if we replace the Hilbert space \(\mathcal{H}\) by a Hilbert B-module \(E\), which means that this definition can straightforwardly be generalised to the framework of unbounded KK-theory, and it is in this form that we present the definition and its consequences in Chapter 5.

Another advantage of the definition of indefinite spectral triples is that it does not require any smoothness assumptions. Furthermore, it also allows to reverse the Wick rotation procedure \(\mathcal{D} \mapsto \mathcal{D}_\pm\), which means that we can characterise all pairs of spectral triples that can be obtained from an indefinite spectral triple in this way.
The main technical assumption in the definition of indefinite spectral triples (or indefinite Kasparov modules) is that the real and imaginary parts $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ almost anti-commute, which means that the anti-commutator $\{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\}$ is relatively bounded by $\text{Re} \mathcal{D}$. A theorem by Kaad and Lesch [KL12] (quoted in Theorem 2.22) then allows us to conclude that the Wick rotations $\mathcal{D}_\pm$ are self-adjoint. Subsequently we can prove that these Wick rotations give rise to spectral triples.

1.1.2 The spectral geometry of pseudo-Riemannian manifolds

Let us have a brief look ahead at how pseudo-Riemannian manifolds fit in with our two generalisations of spectral triples described above. Consider an $n$-dimensional time- and space-oriented pseudo-Riemannian spin manifold $(M, g)$ of signature $(t, s)$. We assume that we are given a spacelike reflection $r$ such that the associated Riemannian metric $g_r$ is complete (see Section 3.3 for more details).

As described above, we consider the triple $(C^\infty_c(M), L^2(S), \mathcal{D})$, where the algebra $C^\infty_c(M)$ consists of the smooth, compactly supported functions on $M$, $L^2(S)$ denotes the Hilbert space of square-integrable spinors, and $\mathcal{D}$ is the canonical Dirac operator on the spinor bundle $S \to M$ (see Section 3.4 for details).

The assumption that the Riemannian metric $g_r$ is complete already implies that $\mathcal{D}$ is essentially Krein-self-adjoint, that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ are essentially self-adjoint, and that the Wick rotations of $\mathcal{D}$ yield spectral triples (see Theorem 3.17 and Proposition 3.18). However, this assumption is not enough to obtain a pseudo-Riemannian spectral triple or an indefinite spectral triple from $(M, g)$. In Section 4.3 we find that to obtain a pseudo-Riemannian spectral triple, we furthermore need to assume that the manifold has bounded geometry.

To obtain an indefinite spectral triple from $(M, g)$, we need the operators $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ to almost anti-commute. Unfortunately, the Dirac operator $\mathcal{D}$ on a general pseudo-Riemannian manifold does not satisfy this condition. Indeed, although the anti-commutator $\{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\}$ is a first-order differential operator, it contains in general both spacelike derivatives and timelike derivatives, and thus it is not relatively bounded by $\text{Re} \mathcal{D}$ (which only contains spacelike derivatives). In order to ensure that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ almost anti-commute, we need the timelike part of $\{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\}$ to vanish identically, which places a restriction on the geometry of the pseudo-Riemannian manifold (see below). This asymmetry between the timelike and spacelike parts of $\{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\}$ is artificial, and indicates that it would be desirable to have a more general version of Kaad and Lesch’ theorem (for further discussion, see the Outlook).

To ensure that a Lorentzian manifold does satisfy our definition of an indefinite spectral triple, we impose bounded geometry, and require in addition an assump-
tion of ‘parallel time’ (see Section 5.3.1). The latter assumption ensures that the real and imaginary parts of $D$ almost anti-commute.

1.1.3 The harmonic oscillator

Another recurring example in this thesis is the harmonic oscillator. This example in particular shows that our framework allows to deal with more than just metrics of indefinite signature.

Consider the algebra of Schwartz functions $S(\mathbb{R})$ acting on the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions on $\mathbb{R}$. We define the annihilation operator $a$ and its adjoint, the creation operator $a^*$, as

$$a := x + \frac{d}{dx}, \quad a^* := x - \frac{d}{dx},$$

with initial domain given by the Schwartz functions $S(\mathbb{R})$. These operators satisfy the canonical commutation relations $[a, a] = [a^*, a^*] = 0$ and $[a, a^*] = 2$. The triple $(S(\mathbb{R}), L^2(\mathbb{R}), a)$ can be viewed both as a pseudo-Riemannian spectral triple (Section 4.4) and as an odd indefinite spectral triple (Section 5.4). We can ‘double up’ the Hilbert space $L^2(\mathbb{R})$ to a $\mathbb{Z}_2$-graded Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with grading $\Gamma := 1 \oplus -1$, and define a new operator $D$ on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ by

$$D := \begin{pmatrix} 0 & x - i \frac{d}{dx} \\ x + i \frac{d}{dx} & 0 \end{pmatrix}. $$

The new triple $(S(\mathbb{R}), L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), D)$ is then an even indefinite spectral triple. In Section 5.4.1 we show that this can easily be generalised to obtain an (even) indefinite spectral triple for the harmonic oscillator in arbitrary dimensions.

1.2 Gauge theory and the standard model

As mentioned above, a spectral triple can be interpreted as describing a ‘non-commutative manifold’. Of particular interest to our work is the special case of almost-commutative manifolds, which first appeared in [CL91, DKM89a, DKM89b, DKM90a, DKM90b]. Such almost-commutative manifolds are (usually) given by a product of a Riemannian spin manifold with a finite spectral triple, as we describe in Section 1.2.1 below. Their particular interest lies in the fact that they can be used to derive physical models describing both gravity and (classical) gauge theory, thus providing a first step towards a unified theory. The name almost-commutative manifolds was first coined in [ISS04], their classification started in [Kra98, PS98] and was investigated in further detail in [JS05, JSS05, JSo8, JSo9].
Chamshedine and Connes [CC96, CC97] introduced the spectral action, yielding a formula to calculate the physical Langrangian from an almost-commutative manifold. For a suitably chosen finite spectral triple, they showed that this spectral action recovers the Langrangian of the Standard Model of elementary particle physics (without right-handed neutrinos). Later Chamshedine, Connes, and Marcolli [CCM07] provided a noncommutative-geometric description of the full Standard Model including right-handed neutrinos (with Majorana masses).

1.2.1 Almost-commutative manifolds and gauge theories

Let us now briefly recall how a description of a gauge theory is obtained from an almost-commutative manifold. For a more detailed introduction, the reader may wish to have a look at the original papers [Con96, Con06, CCM07], the book [CM07], the shorter companions [JKSS07, CC10], or the review paper [DS12] (and references therein). We start with a smooth compact 4-dimensional Riemannian spin manifold $M$, which can be described [Con13] in terms of a (real, even) spectral triple $(\mathcal{C}^\infty(M), L^2(S), \mathcal{D}, \Gamma_M, J_M)$, where $\mathcal{D}$ is the Dirac operator on the spinor bundle $S \to M$, $\Gamma_M$ is the grading operator, and the real structure $J_M$ is given by charge conjugation (for a definition of real structures, see Definition 6.13).

Next, we consider a finite spectral triple $(A_F, \mathcal{H}_F, \mathcal{D}_F, \Gamma_F, J_F)$, which we think of as describing the ‘internal degrees of freedom’ of the theory. The finite-dimensional Hilbert space encodes the particle content: each basis element describes a different fermionic particle. The action of the algebra $A_F$ governs the interactions between these particles. The matrix $\mathcal{D}_F$ contains the masses of the particles, while $\Gamma_F$ and $J_F$ distinguish between left- and right-handed chirality, and between particles and anti-particles, respectively. Given such a finite spectral triple, one can consider an almost-commutative manifold given by the product triple

$$M \times F := (\mathcal{C}^\infty(M, A_F), L^2(S) \otimes \mathcal{H}_F, \mathcal{D} \otimes 1 + \Gamma_M \otimes \mathcal{D}_F, \Gamma_M \otimes \Gamma_F, J_M \otimes J_F).$$

For any real spectral triple $T = (A, \mathcal{H}, \mathcal{D}, J)$, we define its gauge group as

$$\mathcal{G}(T) := \{ uJ u^* \mid u \in \mathcal{U}(A) \} \simeq \mathcal{U}(A)/\mathcal{U}(A_J), \quad (1.1)$$

where $A_J$ is the central subalgebra of $A$ consisting of all elements $a \in A$ for which $aJ = JA^*$. Now suppose we have a real even finite spectral triple $F = (A_F, \mathcal{H}_F, \mathcal{D}_F, \Gamma_F, J_F)$ with gauge group $G_F = \mathcal{G}(F)$. Then the product triple $M \times F$ defined above has gauge group $\mathcal{G}(M \times F) \simeq \mathcal{C}^\infty(M, G_F)$ (at least when $M$ is simply connected\(^1\)), which coincides with the ‘classical’ notion of the gauge group of the (globally trivial) principal $G_F$-bundle $P = M \times G_F$.

\(^1\) The isomorphism $\mathcal{G}(M \times F) \simeq \mathcal{C}^\infty(M, G_F)$, stated in [BS11, Proposition 4.3] and [DS12, §2.4.3], is only valid under some additional conditions, and simply-connectedness of $M$ is always sufficient. We shall prove this in general for the globally non-trivial case in Theorem 6.38.
One can show that the inner fluctuations of the operator $\mathcal{D} \otimes 1 + \Gamma_M \otimes D_F$ yield gauge fields (i.e., connection forms on the principal bundle $\mathcal{P}$) as well as scalar fields (which are interpreted as Higgs fields in the noncommutative Standard Model).

### 1.2.2 Gravity on almost-commutative manifolds

To complete the description of a gauge theory, we need to specify the equations of motion. These are typically derived as the Euler-Lagrange equations for some (gauge-invariant) action functional. In the context of noncommutative geometry, this action functional is obtained from two pieces: the spectral action [CC97] and the fermionic action [Con06]. The spectral action for the Dirac operator $\mathcal{D}$ on a compact Riemannian spin manifold $M$ yields the Einstein-Hilbert action $\int_M R \, d\text{vol}_g$ (where $R$ denotes the scalar curvature of the manifold), plus additional higher-order gravitational terms (see Section 6.5.3). The equations of motion derived from this action are precisely Einstein’s field equations, describing the gravitational field.

Thus we can think of the spectral action as a gravitational action functional. Applying this to an almost-commutative manifold, we obtain in addition to the Einstein-Hilbert action also terms describing the gauge field(s) and the Higgs field(s). We can therefore reinterpret the combination of gravitational and gauge interactions on $M$ as simply being described by a gravitational theory on an almost-commutative manifold.

### 1.2.3 The noncommutative Standard Model

The Standard Model describes the elementary particles and all their interactions (except for gravity). To obtain a description of the Standard Model as an almost-commutative manifold, it only remains to specify the choice for the finite spectral triple $F_{\text{SM}}$. First, we take the algebra $A_F := C \oplus H \oplus M_3(C)$, where the three summands (roughly) correspond to the three fundamental interactions (electromagnetic, weak, and strong) of the Standard Model. The Hilbert space $\mathcal{H}_F$ provides the types of elementary particles. The action of the different summands of $A_F$ on these elementary particles encodes the types of gauge interactions that each particle can have. For the precise choices of the representation of $A_F$ and the matrices $D_F, \Gamma_F$, and $J_F$, we refer to Section 7.6.

The gauge group of $F_{\text{SM}}$, as defined in Eq. (1.1), is given (modulo a finite group) by $G(F_{\text{SM}}) = U(1) \times SU(2) \times U(3)$. By imposing a unimodularity condition (see also Remark 6.39 and Section 7.6, and references given there), this group is reduced (again, modulo a finite group) to $U(1) \times SU(2) \times SU(3)$, which is indeed the gauge
group of the Standard Model. The spectral action and the fermionic action recover the full Lagrangian of the Standard Model, including right-handed neutrinos with Majorana masses.

The noncommutative description of the Standard Model, as briefly described above, still faces two main challenges. First, it describes the Standard Model as a classical theory instead of a quantum theory. It is still very much an open question how quantisation should be described in the framework of noncommutative geometry. Second, the description is based on a Riemannian manifold instead of a Lorentzian manifold, which means we do not obtain Einstein’s theory of General Relativity. We will address this second challenge in Section 1.2.5 below.

1.2.4 Non-trivial global structure

We observe that the principal $G_F$-bundle $\mathcal{P} = M \times G_F$, describing the gauge theory of the almost-commutative manifold $M \times F$, is by construction a globally trivial bundle. Thus the question arises whether one can also describe globally non-trivial gauge theories in the framework of noncommutative geometry. This question will be addressed in Chapter 6, where we describe how to extend the construction of a product triple $M \times F$ to a globally non-trivial almost-commutative manifold. The main idea is to replace the finite spectral triple $F$ by an internal space $I^\infty$, which essentially consists of a ‘bundle of finite spectral triples’ over $M$. While the product triple $M \times F$ can be viewed as the external Kasparov product of $M$ with $F$, we show that a globally non-trivial almost-commutative manifold is naturally described as the internal Kasparov product of an internal space $I^\infty$ with the manifold $M$.

Whereas every globally trivial almost-commutative manifold describes a gauge theory, this no longer holds for arbitrary globally non-trivial almost-commutative manifolds. Thus, we focus our attention on those internal spaces that will allow us to obtain a gauge theory, and we define the notion of a principal module, which is an internal space built from a finite spectral triple $F$ and a principal $G_F$-bundle $\mathcal{P}$ over $M$. We show that the algebraic definition of the gauge group of a principal module (defined as in Eq. (1.1)) coincides precisely with the gauge group of $\mathcal{P}$ (i.e. the vertical automorphisms of $\mathcal{P}$), provided that the underlying manifold $M$ is simply connected. The almost-commutative manifold constructed from such a principal module then describes a globally non-trivial gauge theory on $M$.

1.2.5 Gauge theories on Lorentzian manifolds

As mentioned above, almost-commutative manifolds are usually constructed on Riemannian manifolds. However, to correctly describe Einstein’s theory of grav-
ity, we know that we need a Lorentzian manifold. The construction of an almost-commutative manifold still works if the base manifold is Lorentzian (though the result is of course no longer a spectral triple but (one of our versions of) a ‘Lorentzian spectral triple’). However, in order to obtain a gauge theory from an almost-commutative Lorentzian manifold, we furthermore need a recipe to calculate the Lagrangian of the theory. In the Riemannian case this Lagrangian is obtained from two functionals: the spectral action \[CC97\] and the fermionic action \[Con06\].

In Chapter 7 we provide an alternative to the fermionic action for Lorentzian manifolds, which we call the Krein action. This Krein action not only matches the correct signature of physical spacetime, but also avoids the occurrence of the charge conjugation operator in the Lagrangian (which is an unphysical feature of the Riemannian formulation of the fermionic action). We calculate the Krein action for several examples (including the Standard Model), and we show that it recovers exactly the correct physical Lagrangian.

The formulation of the spectral action strongly relies on the self-adjointness of the operator in the spectral triple. Unfortunately, it is entirely unclear if one can find an alternative formulation for ‘Lorentzian spectral triples’. We will discuss this problem in more detail in the Outlook.

1.3 Outline

CHAPTER 2 In this chapter we gather from the literature some necessary preliminaries. In the first two sections we describe the basics of Hilbert modules and operator spaces. Next, we collect the main results regarding almost (anti-)commuting operators, and we prove a few further consequences. In Sections 2.4 and 2.5 we describe unbounded Kasparov modules and smoothly summable spectral triples. Finally, we present in the last section some new material describing our approach to dealing with non-symmetric operators by studying their real and imaginary parts. In particular, we define our notion of ‘Wick rotation’, which maps a (typically non-symmetric) operator to a pair of symmetric operators.

CHAPTER 3 We give an introduction to Dirac operators on pseudo-Riemannian spin manifolds. We will start in the first section with a discussion of finite-dimensional Clifford algebras and spinor modules. In the next section, we recall the basics of the theory of fibre bundles. Subsequently, we show that the constructions of Clifford algebras and spinor modules can be extended to bundles over pseudo-Riemannian spin manifolds. Finally, we construct the Dirac operator and discuss some of its properties in the last section.

CHAPTER 4 This chapter is based on joint work [DPR13] with Mario Paschke and Adam Rennie. We present the definition of pseudo-Riemannian spectral
triples, as well as definitions of smoothness and summability, enabling a non-commutative analogue of pseudo-Riemannian geometry. Subsequently, we prove our main theorem in Section 4.2, showing that we can employ the ‘Wick rotations’ to obtain two spectral triples from a pseudo-Riemannian spectral triple. Under additional assumptions, the process of Wick rotating is shown to preserve spectral dimension, smoothness and integrability, as we define them. In Sections 4.3 to 4.5 we discuss several examples. In the last section we specialise our definition to Lorentz-type spectral triples, and give a simple index-theoretic result.

Chapter 5 This chapter is based on joint work [DR15] with Adam Rennie (University of Wollongong). We start by defining indefinite Kasparov modules as well as pairs of Kasparov modules, and we prove that these definitions are equivalent. We also introduce an odd version of indefinite Kasparov modules in Section 5.2, and we show that these odd modules are characterised by pairs of Kasparov modules for which the two operators are related via a certain unitary equivalence. Next, we describe three examples. We start in Section 5.3 with the main example of the Dirac operator on a pseudo-Riemannian manifold. In Section 5.4 we consider the harmonic oscillator in arbitrary dimensions. Finally, in Section 5.5 we discuss families of spectral triples, and we show that one can naturally associate an indefinite spectral triple to such families.

Chapter 6 This chapter is based on joint work [BD14] with Jord Boeijink (Radboud University Nijmegen). After some preliminaries on (algebra and group) bundles and classical gauge theories, we define globally non-trivial almost-commutative manifolds, and we show that these are naturally given by the internal Kasparov product of an internal space $I$ with the underlying manifold $M$. Next, to obtain a description of a (classical) gauge theory on the underlying base manifold, we introduce in Section 6.3 the notion of a ‘principal module’, which we build from a principal fibre bundle and a finite spectral triple. We prove that the algebraic definition of the gauge group of a principal module coincides precisely with the usual definition of the gauge group, provided that the underlying manifold $M$ is simply connected. In Section 6.4 we define the purely algebraic notion of ‘gauge modules’, and show that this yields a proper subclass of the principal modules. By equipping a principal module with a connection and a ‘mass matrix’, we construct the corresponding principal almost-commutative manifold in Section 6.5, and we describe in detail how this principal almost-commutative manifold describes a gauge theory on $M$. In Section 6.6 we provide two basic but illustrative examples of such gauge theories, namely Yang-Mills theory and electrodynamics.

Chapter 7 We define Krein spectral triples in Section 7.1, in which the notion of spectral triples is generalised from Hilbert spaces to Krein spaces. Subsequently, we define the Krein action as an alternative to the usual formulation of the fermi-
onic action. In Section 7.2 we describe the abstract formulation of gauge theories in terms of Krein spectral triples, and in Section 7.3 we define almost-commutative manifolds in this context. In Sections 7.4 to 7.6 we describe three examples: electrodynamics, the electro-weak theory, and the Standard Model. In particular, we show that the Krein action recovers exactly the correct physical Lagrangians.

**OUTLOOK** Finally, we end this thesis with an Outlook, in which we will describe a few open questions arising from the work in this thesis, and we list a few possible directions for further research.
SPECTRAL TRIPLES AND KASPAROV MODULES

In Sections 2.1 to 2.5 we gather from the literature the necessary preliminaries concerning unbounded operators on Hilbert modules, operator spaces, Kasparov modules, and smoothly summable spectral triples. The reader who is familiar with these topics may wish to skip these sections on a first reading, and use it as reference material. At the end of the chapter, in Section 2.6, we present some new material however, describing our approach to dealing with non-symmetric operators by studying their real and imaginary parts.

2.1 HILBERT MODULES OVER GRADED ALGEBRAS

We assume the reader is familiar with C*-algebras (see e.g. [Mur90, Dav96, Bla06] for an introduction). We will introduce the notion of $\mathbb{Z}_2$-gradings for C*-algebras, largely following [Kas80b, §2]. A C*-algebra $A$ is called $\mathbb{Z}_2$-graded if we have a direct sum decomposition $A = A^0 \oplus A^1$ into closed self-adjoint linear subspaces $A^0$ and $A^1$, such that $A^i \cdot A^j \subset A^{i+j}$, for $i, j \in \mathbb{Z}_2$. A graded homomorphism $\phi: A \to B$ of $\mathbb{Z}_2$-graded C*-algebras is a homomorphism such that $\phi(A^i) \subset B^i$, for $i \in \mathbb{Z}_2$. A C*-algebra $A$ is said to be trivially graded if $A^1 = \{0\}$. An element $a \in A^i$ is called homogeneous with degree $\deg a = i$. On homogeneous elements $a, b \in A$, the graded commutator is defined as

$$[a, b]_{\pm} := ab - (-1)^{\deg a \cdot \deg b} ba,$$

and this definition is extended to all of $A$ by linearity.

We will recall the definition of Hilbert modules (sometimes called C*-modules or Hilbert C*-modules). For a more detailed introduction to Hilbert modules, we refer to [Lan95]. We start with the definition of hermitian modules (sometimes called inner-product modules), where we allow for indefinite inner products.

**Definition 2.1.** Let $A$ be a $\mathbb{Z}_2$-graded $*$-algebra. Let $i, j \in \mathbb{Z}_2$. A $\mathbb{Z}_2$-graded right $A$-module $E$ is a linear space with a linear right action $E^i \times A^j \to E^{i+j}$ (such that $\lambda(ea) = (\lambda e)a = e(\lambda a)$ for all $\lambda \in \mathbb{C}$, $e \in E$, and $a \in A$). A (right) hermitian structure
\((\cdot|\cdot)_A : E^1 \times E^j \to A^{i+j}\) on \(E\) is a sesqui-linear map (anti-linear in the first variable) satisfying
\[
(e_1|e_2)A = (e_1|e_2)_A a; \quad (e_2|e_1)_A = (e_1|e_2)^*_A;
\]
if \((e_1|e_2)_A = 0\) for all \(e_2 \in E_a\) then \(e_1 = 0\),

for \(a \in A, e_1, e_2 \in E\). We also write \((\cdot|\cdot)\) instead of \((\cdot|\cdot)_A\) when no confusion can arise. A module endowed with a hermitian structure is also called a hermitian module.

A hermitian structure is called positive-definite if \((e|e)_A \geq 0\), and if \((e|e)_A = 0\) implies that \(e = 0\) (otherwise, the hermitian structure is called indefinite).

We have an anti-linear map \(\phi : E \to E^* := \text{Hom}_A(E,A)\) given by \(\phi(e) := (e|\cdot)_A\).

The last assumption of the hermitian structure (i.e. if \((e_1|e_2)_A = 0\) for all \(e_2 \in E\), then \(e_1 = 0\)) implies that \(\phi\) is injective. A hermitian structure is called non-degenerate if this map \(\phi\) is also surjective.

Now suppose that \(A\) is a \(C^*\)-algebra, and that \((\cdot|\cdot)_A\) is positive-definite. We then have a norm \(\| \cdot \|\) on \(A\), which induces a norm on \(E\) as \(\|e\|^2 := \|(e|e)_A\|\), and one can ask whether \(E\) is complete in this norm. This leads to the definition of a Hilbert module (sometimes called \(C^*\)-module or Hilbert \(C^*\)-module).

**Definition 2.2.** Let \(A\) be a \(\mathbb{Z}_2\)-graded \(C^*\)-algebra. A \(\mathbb{Z}_2\)-graded right Hilbert \(A\)-module \(E_A\) is a \(\mathbb{Z}_2\)-graded positive-definite hermitian right \(A\)-module \(E\) which is complete in the norm \(\|e\| := \|(e|e)_A\|^{\frac{1}{2}}\).

If no confusion arises, we will usually write \(E\) instead of \(E_A\). If \(A = C\), then \(E\) is just a (\(\mathbb{Z}_2\)-graded) Hilbert space, and we will usually write \(E = \mathcal{H}\).

Contrary to Hilbert spaces, a closed submodule \(F\) of a Hilbert \(A\)-module \(E\) need not be complemented, which means that \(F \oplus F^\perp\) need not be equal to \(E\). Here, we have defined the orthogonal complement \(F^\perp := \{e \in E : (f|e) = 0, \ \forall f \in F\}\). This fact provides an obstacle for generalising the theory of Hilbert spaces to Hilbert modules, and shows that we will need additional assumptions.

A map \(T : E \to E\) is called adjointable if there exists a map \(T^* : E \to E\) such that \((T^*e|f)_A = (e|Tf)_A\) for all \(e, f \in E\). An adjointable map is automatically \(A\)-linear and bounded (see [Lang95, p.8]). We define the endomorphisms \(\text{End}_A(E)\) as the set of adjointable maps \(E \to E\). For any \(e, f \in E\), we define the rank-one endomorphism \(\Theta_{e,f} : E \to E\) as \(\Theta_{e,f}(g) := e(f|g)_A\). Finite linear combinations of such maps are called finite-rank endomorphisms, and we define the compact endomorphisms \(\text{End}_A^0(E) := \overline{\text{span}}(\Theta_{e,f}|e, f \in E)\) as the closure of their linear span. If \(A = C\), so that \(E = \mathcal{H}\) is a Hilbert space, we usually write \(\mathcal{B}(\mathcal{H})\) and \(\mathcal{K}(\mathcal{H})\) for the bounded and compact operators, respectively.

A subset \(Z\) of a Hilbert \(A\)-module \(E\) is called a generating set if the closed submodule generated by \(Z\) equals \(E\). We say that \(E\) is countably generated if \(E\) has a
Consider the separable infinite-dimensional Hilbert space $l^2(\mathbb{N})$. For any $C^*$-algebra $A$ we define the standard module $\mathcal{H}_A := l^2(\mathbb{N}) \otimes A$. Thus, $\mathcal{H}_A$ consists of sequences $(a_j)_{j \in \mathbb{N}}$ in $A$ such that $\sum_{j \in \mathbb{N}} a_j^* a_j$ converges in the norm on $A$. If $A$ is $\sigma$-unital, then $\mathcal{H}_A$ is countably generated.

If a Hilbert $A$-module $E$ is countably generated, then it follows from Kasparov’s Stabilisation Theorem [Kas80a, Theorem 2] that there exists an orthogonal projection $p : \mathcal{H}_A \rightarrow \mathcal{H}_A$ such that $E \simeq p\mathcal{H}_A$. Similarly, a finitely generated $A$-module $E$ is of the form $pA^N$, for some $N \in \mathbb{N}$ and some projection $p \in \mathcal{M}_N(A)$. The restriction of the standard hermitian structure on $A^N$ then gives a non-degenerate hermitian structure on $E$.

**Example 2.3.** Suppose that $A$ is a commutative $C^*$-algebra, so that $A = C_0(X)$ for a locally compact Hausdorff space $X$, by the Gelfand-Naimark theorem (see e.g. [GVF01, Theorem 1.4]). For any separable Hilbert space $\mathcal{H}$, we obtain a countably generated Hilbert module $C_0(X,\mathcal{H})$ of continuous functions $X \rightarrow \mathcal{H}$ which “vanish at infinity”. This Hilbert module is isomorphic to the standard module $\mathcal{H}_A$.

Now suppose that $X$ is compact, and that $E = pC(X)^N$ is a finitely generated module. By the Serre-Swan theorem [Swa62], we have $E = \Gamma(E)$ for some vector bundle $E \rightarrow X$. Then a hermitian structure on $E$ is non-degenerate if and only if it induces an inner product on each fibre of $E$.

As an example of a degenerate positive-definite hermitian structure, consider the globally trivial vector bundle $E = S^1 \times C$ on the base manifold $S^1$ (viewed as a subset of the complex plane) with fibre $C$. Let $f : S^1 \rightarrow [0,\infty)$ be a smooth non-negative function on $S^1$ such that $f(z) = 0$ if and only if $z = 1$. For sections $e_1, e_2 \in \Gamma(E) = C(S^1)$ we define the hermitian structure $(e_1|e_2)(z) := f(z)e_1(z)^* e_2(z)$. Restricted to the fibre $E_0$, this gives a completely degenerate inner product. However, on the sections $\Gamma(E)$, the hermitian structure is positive-definite and satisfies $(e|e) = 0$ if and only if $e = 0$ (indeed, if $(e|e)(z) = 0$ for all $z$, then $e(z) = 0$ for all $z \neq 1$, and hence $e(z) = 0$ for all $z$ by continuity). We point out that $\Gamma(E)$ equipped with this hermitian structure is not a Hilbert $C(S^1)$-module, as it is not complete.

### 2.1.1 Interior tensor products

Let $A$ and $B$ be $C^*$-algebras, and consider a Hilbert $A$-module $E$ and a Hilbert $B$-module $F$. Suppose we have a $*$-homomorphism $\phi : A \rightarrow \text{End}_B(F)$. We can then construct the interior tensor product of $E$ and $F$ (sometimes called the balanced tensor product) as follows. We start by forming the graded algebraic tensor product $E \hat{\otimes}_A F$ over $A$, consisting of finite sums of simple tensors such that $e a \hat{\otimes} f = \phi(e) a \otimes f$.
e \otimes \phi(a)f \text{ for all } e \in E, f \in F, \text{ and } a \in A. 

The gradings on E and F induce a grading on \(E \hat{\otimes}_A F\) such that
\[
(E \hat{\otimes}_A F)^0 = (E^0 \hat{\otimes}_A F^0) \oplus (E^1 \hat{\otimes}_A F^1) \quad \text{and} \quad (E \hat{\otimes}_A F)^1 = (E^0 \hat{\otimes}_A F^0) \oplus (E^1 \hat{\otimes}_A F^1).
\]

This algebraic tensor product is a right \(B\)-module with the right action of \(B\) given by \((e \otimes f)b = e \otimes (fb)\). We can define an inner product on \(E \hat{\otimes}_A F\) as
\[
(e_1 \otimes f_1|e_2 \otimes f_2)_B := (f_1|\phi((e_1|e_2)_A)f_2),
\]
for \(e_1, e_2 \in E\) and \(f_1, f_2 \in F\). We then define the \textit{graded interior tensor product} \(E \otimes_A F\) as the completion of \(E \hat{\otimes}_A F\) with respect to this inner product. For \(T_1 \in \text{End}_A(E)\) and \(T_2 \in \text{End}_B(E)\) such that \(T_2\) commutes with \(\phi(A)\), we have a well-defined operator \(T_1 \otimes T_2\) on \(E \hat{\otimes}_A F\), which is given on homogeneous elements as \((T_1 \otimes T_2)(e \otimes f) := (-1)^{\deg T_2 \cdot \deg e} T_1 e \otimes T_2 f\).

### 2.1.2 Unbounded operators

We consider densely defined operators \(T: \text{Dom } T \subset E \rightarrow E\). The \textit{graph} of an operator \(T\) is the submodule \(G(T) := \{(\psi, T\psi) \mid \psi \in \text{Dom } T\} \subset E \oplus E\). An operator \(T\) is called \textit{closed} if its graph \(G(T)\) is a closed submodule of \(E \oplus E\). As mentioned following \textbf{Definition 2.2}, such a closed submodule need not be complemented. An operator \(T\) is called \textit{closable} if the closure of the graph \(G(T)\) is again the graph of an operator, denoted \(\overline{T}\), and then \(\overline{T}\) is called the closure of \(T\). The domain of the \textit{adjoint} \(T^{*}\) is given by

\[
\text{Dom } T^{*} := \{\xi \in E \mid \exists \eta \in E \text{ such that } \forall \psi \in \text{Dom } T: (\xi|T\psi) = (\eta|\psi)\},
\]

and then we define \(T^{*}\xi = \eta\).

\textbf{Definition 2.4.} Let \(T\) be a densely defined operator on a Hilbert module \(E_A\). We say \(T\) is \textit{semi-regular} if \(T^{*}\) is densely defined. We say \(T\) is \textit{regular} if \(T\) is semi-regular and closed, and \(1 + T^{*}T\) has dense range.

\textbf{Theorem 2.5 ([Lang95, Theorem 9.3])}. If \(T\) is regular, the graph of \(T\) is complemented.

\textbf{Remark 2.6.} The proof of [Lang95, Theorem 9.3] furthermore shows that the operator \(1 + T^{*}T\) is in fact surjective.

\textbf{Lemma 2.7 ([KL12, Lemma 2.1])}. Let \(T\) be semi-regular on \(E_A\). Then \(T\) is \(A\)-linear and closable with closure \(T\), and the adjoint \(T^{*}\) is closed and equal to \((T)^{*}\).

\textbf{Lemma 2.8.} Let \(T\) be a closed, densely defined operator on a Hilbert module \(E_A\). If \(T + i\) is surjective, then \(\text{Ker}(T^{*} - i) = \{0\}\).

\textbf{Proof.} Suppose \(\eta \in \text{Ker}(T^{*} - i)\). Then \((T + i)\xi|\eta) = (\xi|(T^{*} - i)\eta) = 0\) for all \(\xi \in \text{Dom } T\). Since \(T + i\) is surjective, this means \((\psi|\eta) = 0\) for all \(\psi \in E\), and hence \(\eta = 0\). 

\(\square\)
Proposition 2.9 (cf. [Lan95, Lemma 9.8]). Let $T$ be a closed, densely defined, symmetric operator on a Hilbert module $E_A$. Then $T$ is regular and self-adjoint if and only if the operators $T \pm i$ are surjective.

Proof. First, suppose $T$ is regular and self-adjoint. Then $1 + T^2 = (T + i)(T - i) = (T - i)(T + i)$ is surjective (see Remark 2.6), and hence $T + i$ and $T - i$ are also surjective.

Conversely, suppose $T \pm i$ are surjective. To prove self-adjointness, the same proof applies as for Hilbert spaces (see e.g. [RS80, Theorem VIII.3]). Let $\xi \in \text{Dom } T^*$. Since $T - i$ is surjective, there exists an $\eta \in \text{Dom } T$ such that $(T - i)\eta = (T^* - i)\xi$. Since $T$ is symmetric, $T\eta = T^*\eta$ and hence $(T^* - i)(\xi - \eta) = 0$. But $\text{Ker } (T^* - i) = \{0\}$ by Lemma 2.8, so $\xi = \eta \in \text{Dom } T$. Hence $\text{Dom } T^* = \text{Dom } T$ and $T$ is self-adjoint. Since $T \pm i$ are surjective, it then follows that $1 + T^2$ is surjective and hence $T$ is also regular. \qed

2.1.3 Localisations

Here we briefly recall the local-global principle from [KL12]. Let $E_A$ be a right Hilbert $A$-module, and let $\pi$ be a representation of $A$ on a Hilbert space $\mathcal{H}_\pi$. We then get an induced representation $\pi_E$ of $\text{End}_A(E)$ on the interior tensor product $E \hat{\otimes}_A \mathcal{H}_\pi$. This Hilbert space $E \hat{\otimes}_A \mathcal{H}_\pi$ is called the localisation of $E$ with respect to the representation $\pi$.

Now let $T$ be a semi-regular operator on $E_A$. We define the unbounded operator $T_0^\pi$ on $E \hat{\otimes}_A \mathcal{H}_\pi$ as $T_0^\pi(e \otimes h) := (Te) \otimes h$ with domain $\text{Dom } T \hat{\otimes}_A \mathcal{H}_\pi$. Then $T_0^\pi$ is densely defined and closable, and its closure $T^\pi$ is called the localisation of $T$ with respect to $\pi$. If $T$ is symmetric, then so is $T^\pi$. If $\omega$ is a state on $A$, then we denote by $T^\omega$ the localisation of $E$ with respect to the corresponding GNS representation $\pi_\omega$. We refer to [KL12, §2] for more details. We are now ready to quote the following results.

Theorem 2.10 (Local-global principle [KL12, Theorem 4.2]). For a closed, densely defined and symmetric operator $T$ on a Hilbert module $E_A$, the following statements are equivalent:

1) the operator $T$ is self-adjoint and regular;

2) for every representation $(\pi, \mathcal{H}_\pi)$ of $A$ the localisation $T^\pi$ is self-adjoint;

3) for every state $\omega \in S(A)$ the localisation $T^\omega$ is self-adjoint.

Theorem 2.11 ([KL12, Theorems 5.6 and 5.8]). Let $T$ be a closed, densely defined and symmetric operator on a Hilbert module $E_A$. If $A$ is commutative, then $T$ is self-adjoint and regular if and only if for every pure state $\omega \in S(A)$ the localisation $T^\omega$ is self-adjoint.
2.2 OPERATOR SPACES

In this section we recall a few basic notions of operator spaces, which will play a role in the construction of the unbounded Kasparov product in Section 2.4.2. We will not need the theory of operator spaces in its full generality, and therefore we will only describe what we need later. In particular, instead of giving the abstract definitions, we take the concrete picture of algebras represented on a Hilbert space. For a more detailed introduction to operator algebras and their modules we refer to [Pis03, BL04].

We start by recalling the notion of a completely bounded map. First, for a vector space \( V \) we denote by \( M_n(V) \) the set of \( n \times n \)-matrices with entries in \( V \). A matrix \( m \in M_n(V) \) is written as \( (m_{ij}) \), with entries \( m_{ij} \in V \).

Now let \( V \) and \( W \) be normed vector spaces, and let \( L : V \to W \) be a linear map. For each \( n \in \mathbb{N} \), we then have a map \( L_n : M_n(V) \to M_n(W) \) given by \( (m_{ij}) \to (L(m_{ij})) \). Suppose that each matrix space \( M_n(V) \) and \( M_n(W) \) has a norm, and denote by \( \| \cdot \|_n \) the operator norm from \( M_n(V) \) to \( M_n(W) \). We then obtain a norm \( \| \cdot \|_{cb} \) on linear maps \( L : V \to W \) given by

\[
\|L\|_{cb} := \sup_{n \in \mathbb{N}} \|L_n\|_n.
\]

We say that \( L \) is completely bounded if \( \|L\|_{cb} \) is finite.

2.2.1 Operator algebras

Definition 2.12 ([BL04, §2.1]). An operator algebra \( A \) is a closed subalgebra of the bounded operators \( B(\mathcal{H}) \) on a separable Hilbert space \( \mathcal{H} \).

We will consider operator algebras which are also equipped with an involution, but we wish to allow for involutions which are different from taking the adjoint \( T^* \) of an operator \( T \) on \( \mathcal{H} \). We refrain from giving a general definition of involutions on operator algebras; instead, we focus on a special case, following the example of [KL13, §2.3].

Definition 2.13. An operator algebra \( A \subset B(\mathcal{H}) \) is called involutive if there exists a unitary operator \( U \) on \( \mathcal{H} \) such that \( Ua^*U^* \in A \) for all \( a \in A \).

Example 2.14. Let \( (M, g) \) be an oriented Riemannian manifold of dimension \( n \). Denote by \( C^1_0(M) \) the algebra of continuously differentiable complex-valued functions \( f \) on \( M \) such that both \( f \) and its exterior derivative \( df \) “vanish at infinity”. This algebra is naturally equipped with the norm \( \| \cdot \|_1 \) given by

\[
\|f\|_1 := \sup_{x \in M} |f(x)|^2 + \sup_{x \in M} g(df(x), df(x)).
\]
Let $E \to M$ be a vector bundle, and let $D: \Gamma_c^\infty(E) \to L^2(E)$ be a symmetric elliptic first-order differential operator. We denote by $\sigma: T^*M \to \text{End}(E)$ the principal symbol of $D$. Suppose that $D$ has bounded propagation speed, which means that
\[
\sup \{ \|\sigma(\xi)\| \mid \xi \in T^*M, \ g(\xi, \xi) \leq 1 \} < \infty. \tag{2.1}
\]
A function $f \in C^1_0(M)$ acts on $\Gamma_0(E)$ by pointwise multiplication, and we observe that $[D, f] = i\sigma(df)$. The idea of the following construction first appeared in [Mes14]. We consider the map
\[
\iota: C^1_0(M) \to B(L^2(E)^{\oplus 2}), \quad \iota(f) := \begin{pmatrix} f & 0 \\ [D, f] \cdot f \end{pmatrix}.
\]
This map is injective, and we will identify $C^1_0(M)$ with its image under $\iota$, which (by the Leibniz rule) is closed under multiplication and therefore an operator algebra. We also have an involution given by
\[
\iota(f) \mapsto U \iota(f)^* U^* = \iota(\overline{f}), \quad U := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
where $\overline{f}$ denotes the complex conjugate of $f$. Hence $C^1_0(M)$ is an involutive operator algebra. Since $D$ has bounded propagation speed, the operator norm of $\iota(f)$ is bounded by the norm $\|f\|_1$. If furthermore the symbol $\sigma$ of $D$ also satisfies
\[
\inf \{ \|\sigma(\xi)\| \mid \xi \in T^*M, \ g(\xi, \xi) \geq 1 \} > 0, \tag{2.2}
\]
then the operator norm of $\iota(f)$ is equivalent to $\|f\|_1$. In particular, this is true if $D$ is a Dirac-type operator, for which $\|\sigma(\xi)\|^2 = g(\xi, \xi)$. More generally, we see that any symmetric elliptic first-order differential operator $D$ whose symbol satisfies Eqs. (2.1) and (2.2) gives rise to a norm which is equivalent to $\| \cdot \|_1$. We will always consider $C^1_0(M)$ to have the structure of an involutive operator algebra induced by such $D$.

### 2.2.2 Operator modules

Consider the separable infinite-dimensional Hilbert space $l^2(\mathbb{N})$. Given an operator algebra $A$, we define the standard module $\mathcal{H}_A := l^2(\mathbb{N}) \otimes A$. Thus, $\mathcal{H}_A$ consists of sequences $(a_j)_{j \in \mathbb{N}}$ in $A$ such that $\sum_{j \in \mathbb{N}} a_j^* a_j$ converges in the norm on $A$. If the operator algebra is in fact a $C^*$-algebra, this is precisely the definition of the standard module given in Section 2.1.

**Example 2.15.** Consider the operator algebra $C^1_0(M)$ of Example 2.14. We note that any separable Hilbert space $\mathcal{H}$ is isomorphic to $l^2(\mathbb{N})$, and for the standard module over $C^1_0(M)$ we then obtain the isomorphism $\mathcal{H}_{C^1_0(M)} \simeq \mathcal{H} \otimes C^1_0(M)$. Denote
by $C^1_0(M, \mathcal{H})$ the space of continuously differentiable functions $\psi: M \to \mathcal{H}$ such that both $\psi$ and its exterior derivative $d\psi$ “vanish at infinity”. Then $C^1_0(M, \mathcal{H})$ is a Banach space with the norm
\[
\|\psi\|_1 := \sup_{x \in M} \|\psi(x)\| + \sup_{x \in M} \|d\psi(x)\|,
\]
and pointwise multiplication makes $C^1_0(M, \mathcal{H})$ a module over $C^0_0(M)$. Using an orthonormal basis $\{e_j\}$ for $\mathcal{H}$, any element $\psi \in C^1_0(M, \mathcal{H})$ can be written as $\psi = \sum_j \psi_j e_j$, for functions $\psi_j \in C^1_0(M)$. The map $\sum_j \psi_j e_j \mapsto e_j \otimes \psi_j$ then gives an isomorphism $C^1_0(M, \mathcal{H}) \simeq \mathcal{H} \otimes C^1_0(M)$, and therefore $C^1_0(M, \mathcal{H})$ is isomorphic to the standard module $\mathcal{H} \otimes C^1_0(M)$.

**Definition 2.16.** An operator module $\mathcal{E}$ over an involutive operator algebra $\mathcal{A}$ is a direct summand of the standard module $\mathcal{H}_\mathcal{A}$.

We point out that our definition of an operator module (called operator $*$-module in [KL13]) is not the most general definition. The above definition means that there exists a **bounded** projection operator $p: \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{A}$ (satisfying $p^2 = p$ and $p^* = p$) such that $\mathcal{E} \simeq p\mathcal{H}_\mathcal{A}$. There are examples (such as the quantum group $SU_q(2)$ [KS12] and the noncommutative Hopf fibration [BMS13, §6]) for which one needs to use an **unbounded** projection operator $p: \text{Dom } p \to \mathcal{H}_\mathcal{A}$ (note that a projection operator is unbounded if and only if it is not regular [BMS13, Proposition 2.9]). A more general definition of **projective operator modules** can be found in [MR15, §3.1].

2.2.3 *The Grassmann connection*

Denote by $\Omega^1_0(M)$ the continuous one-forms on $M$ vanishing at infinity. The action of the C$^*$-algebra $C_0(M)$ on $\Omega^1_0(M)$ is **essential** (or **non-degenerate**), which means that $C_0(M) \cdot \Omega^1_0(M)$ is dense in $\Omega^1_0(M)$. This implies we have an isomorphism $\mathcal{H} \otimes \Omega^1_0(M) \simeq \mathcal{H} \otimes_{C_0(M)} C_0(M) \Omega^1_0(M)$, where $\mathcal{H} \otimes_{C_0(M)} C_0(M) \simeq C_0(M, \mathcal{H})$ is the standard module over the C$^*$-algebra $C_0(M)$.

Consider the operator module $C^1_0(M, \mathcal{H})$ from **Example 2.15**. Let $E \to M$ be a vector bundle, and let $D: \Gamma_c^\infty(E) \to L^2(E)$ be a symmetric elliptic first-order differential operator with bounded propagation speed. For $f \in C^1_0(M)$, the commutator $[D, f]$ equals $\sigma(df)$, where $\sigma$ denotes the symbol of $D$. We define the **generalised one-forms**
\[
\Omega^1_D(C^1_0(M)) := \left\{ \sum_j f_j[D, g_j] \mid f_j, g_j \in C^1_0(M) \right\} \simeq \sigma(\Omega^1_0(M)) \subset \mathcal{B}(L^2(E)),
\]
where the sums are required to converge in the operator norm on $L^2(E)$. 
Definition 2.17 (cf. [KL13, Definition 4.6]). The operator $1 \otimes \mathcal{D}$ on $\mathcal{H} \otimes L^2(E)$ gives rise to a map
\[
[I \otimes \mathcal{D}, \cdot]: \mathcal{H} \otimes C^1_0(M) \to \mathcal{H} \otimes \Omega^1_D(C^1_0(M)).
\]
Under the isomorphisms
\[
C^1_0(M, \mathcal{H}) \simeq \mathcal{H} \otimes C^1_0(M), \quad \mathcal{H} \otimes \Omega^1_D(C^1_0(M)) \simeq C_0(M, \mathcal{H}) \otimes_{C_0(M)} \Omega^1_D(C^1_0(M)),
\]
we denote this map as
\[
\nabla_{\mathcal{D}}^{Gr}: C^1_0(M, \mathcal{H}) \to C_0(M, \mathcal{H}) \otimes_{C_0(M)} \Omega^1_D(C^1_0(M)),
\]
and we call $\nabla_{\mathcal{D}}^{Gr}$ the Grassmann connection of $\mathcal{D}$.

Proposition 2.18 ([KL13, Proposition 8.2 & Corollary 8.3]). The symbol of $\mathcal{D}$ determines a completely bounded operator $\sigma: \Omega^0_0(M) \to \mathcal{B}(L^2(M, F))$, and the commutator with $\mathcal{D}$ determines a completely bounded map $[\mathcal{D}, \cdot]: C^1_0(M) \to \mathcal{B}(L^2(M, F))$.

2.3 Almost (Anti-)Commuting Operators

Almost (anti-)commuting operators were considered by Mesland in [Mes14], and later generalised by Kaad and Lesch in [KL12], for the construction of the unbounded Kasparov product. They will play an important role in Section 5.1.1 for the proof that the Wick rotations of indefinite Kasparov modules are (genuine) Kasparov modules. In this section, we recall the main results from [KL12], and prove a few further consequences. We write $[\cdot, \cdot]$ for the commutator and $\{\cdot, \cdot\}$ for the anti-commutator.

Definition 2.19 (see [KL12, Assumption 7.1]). Let $S$ and $T$ be regular self-adjoint operators on a Hilbert $A$-module $E$ such that

1) there exists a submodule $E \subset \text{Dom } T$ which is a core for $T$;
2) for each $\xi \in E$ and for all $\mu \in \mathbb{R} \setminus \{0\}$ we have the inclusions
\[
(S - i\mu)^{-1}\xi \in \text{Dom } S \cap \text{Dom } T, \quad T(S - i\mu)^{-1}\xi \in \text{Dom } S.
\]

The pair $(S, T)$ is called an almost commuting pair if in addition

3) the map $[S, T](S - i\mu)^{-1}: E \to E$ extends to a bounded endomorphism in $\text{End}_A(E)$ for all $\mu \in \mathbb{R} \setminus \{0\}$.

Similarly, the pair $(S, T)$ is called an almost anti-commuting pair if instead of 3) we have

3') the map $[S, T](S - i\mu)^{-1}: E \to E$ extends to a bounded endomorphism in $\text{End}_A(E)$ for all $\mu \in \mathbb{R} \setminus \{0\}$.
These conditions are often summarised by simply saying that $[S, T](S - i\mu)^{-1}$ (or $(S, T)(S - i\mu)^{-1}$) is well-defined and bounded.

**Lemma 2.20.** Let $(S, T)$ be a pair of regular self-adjoint operators on a Hilbert module $E$ satisfying 1) and 2) of Definition 2.19. Then $S$ is essentially self-adjoint on the intersection $\text{Dom } S \cap \text{Dom } T$.

**Proof.** By assumption we have $(S \pm i)^{-1}(\xi) \in \text{Dom } S \cap \text{Dom } T$ for all $\xi \in \mathcal{E}$, where $\mathcal{E}$ is dense in $E$. Since $S$ is self-adjoint, the operator $(S \pm i)^{-1}$ is bounded and has range $\text{Dom } S$, which is dense in $E$. Hence $(S \pm i)^{-1} \mathcal{E}$ is also dense in $E$, from which it follows that $\text{Dom } S \cap \text{Dom } T$ is dense in $E$, so the operator $S|_{\text{Dom } S \cap \text{Dom } T}$ is symmetric and densely defined on $\text{Dom } S \cap \text{Dom } T$. Furthermore, the image of $(S \pm i)|_{\text{Dom } S \cap \text{Dom } T}$ contains $\mathcal{E}$ and is therefore also dense, which implies (see Proposition 2.9) that $S|_{\text{Dom } S \cap \text{Dom } T}$ is essentially self-adjoint. \hfill $\Box$

One can easily switch back and forth between almost commuting and almost anti-commuting operators via the following ‘doubling trick’. Given two regular self-adjoint operators $S$ and $T$ on a Hilbert $A$-module $E$, we construct two new operators on $E \oplus E$ given by

$$\tilde{S} := \begin{pmatrix} 0 & iS \\ -iS & 0 \end{pmatrix}, \quad \tilde{T} := \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix},$$

with domains $\text{Dom } \tilde{S} = (\text{Dom } S)^{\oplus 2}$ and $\text{Dom } \tilde{T} = (\text{Dom } T)^{\oplus 2}$. One easily calculates that

$$\{\tilde{S}, \tilde{T}\} = i \begin{pmatrix} [S, T] & 0 \\ 0 & -[S, T] \end{pmatrix}, \quad [\tilde{S}, \tilde{T}] = i \begin{pmatrix} [S, T] & 0 \\ 0 & -(S, T) \end{pmatrix},$$

whenever these operators are defined.

**Lemma 2.21.** Let $S$ and $T$ be regular self-adjoint operators on a Hilbert $A$-module $E$, and let $\tilde{S}$ and $\tilde{T}$ be given as above. Then the following statements hold:

1) if $(S, T)$ is an almost commuting pair, then $(\tilde{S}, \tilde{T})$ is an almost anti-commuting pair;

2) if $(S, T)$ is an almost anti-commuting pair, then $(\tilde{S}, \tilde{T})$ is an almost commuting pair.

**Proof.** We only prove the first statement, as the second statement is similar. So suppose $[S, T](S - i\mu)^{-1}$ is well-defined and bounded. Then

$$[\tilde{S}, \tilde{T}](\tilde{S} - i\mu)^{-1} = i \begin{pmatrix} [S, T] & 0 \\ 0 & -[S, T] \end{pmatrix} \begin{pmatrix} -i\mu & iS \\ -iS & -i\mu \end{pmatrix}^{-1} = i \begin{pmatrix} [S, T](S - i\mu)^{-1} & 0 \\ 0 & -[S, T](S - i\mu)^{-1} \end{pmatrix} \begin{pmatrix} (S - i\mu) & 0 \\ 0 & (S - i\mu) \end{pmatrix} \begin{pmatrix} -i\mu & iS \\ -iS & -i\mu \end{pmatrix}^{-1}.$$
The first matrix is bounded by assumption, and an explicit calculation shows that the product of the remaining two matrices equals
\[
\begin{pmatrix}
(S - i\mu) & 0 \\
0 & (S - i\mu)
\end{pmatrix}
\begin{pmatrix}
-i\mu & iS \\
-iS & -i\mu
\end{pmatrix}^{-1}
= \begin{pmatrix}
i\mu(S + i\mu)^{-1} & iS(S + i\mu)^{-1} \\
-iS(S + i\mu)^{-1} & i\mu(S + i\mu)^{-1}
\end{pmatrix}
\]
and is thus also bounded. Hence \(\tilde{S}, \tilde{T}\)(\(S - i\mu\))\(^{-1}\) is well-defined and bounded.

**Theorem 2.22** ([KL12, Theorem 7.10]). Let \((S, T)\) be an almost commuting pair of regular self-adjoint operators on \(E\). Then the operator
\[
\mathcal{D} := \begin{pmatrix}
0 & S + iT \\
S - iT & 0
\end{pmatrix}
\]
with domain \(\text{Dom } \mathcal{D} := (\text{Dom } S \cap \text{Dom } T)^{\oplus 2}\) is self-adjoint and regular.

**Corollary 2.23.** Let \((S, T)\) be an almost anti-commuting pair of regular self-adjoint operators on \(E\). Then the operators \(S + T\) and \(S - T\) with domain \(\text{Dom } S \pm T = \text{Dom } S \cap \text{Dom } T\) are regular and self-adjoint.

**Proof.** By Lemma 2.21 we know that \((\tilde{S}, \tilde{T})\) is an almost commuting pair, so by Theorem 2.22 it then follows that
\[
\begin{pmatrix}
0 & \tilde{S} + i\tilde{T} \\
\tilde{S} - i\tilde{T} & 0
\end{pmatrix}
\]
is regular and self-adjoint on \((\text{Dom } \tilde{S} \cap \text{Dom } \tilde{T})^{\oplus 2} = (\text{Dom } S \cap \text{Dom } T)^{\oplus 4} \subset E^{\oplus 4}\). Since we can write
\[
\tilde{S} + i\tilde{T} = i \begin{pmatrix}
0 & S + T \\
-S + T & 0
\end{pmatrix}, \quad \tilde{S} - i\tilde{T} = i \begin{pmatrix}
0 & S - T \\
-S - T & 0
\end{pmatrix},
\]
we see that \((S + T)^* = (S + T)\) and \((S - T)^* = S - T\). Furthermore, regularity implies that
\[
1 + \begin{pmatrix}
0 & \tilde{S} + i\tilde{T} \\
\tilde{S} - i\tilde{T} & 0
\end{pmatrix}^2 = \begin{pmatrix}
1 + (S + T)^2 & 0 & 0 & 0 \\
0 & 1 + (S - T)^2 & 0 & 0 \\
0 & 0 & 1 + (S - T)^2 & 0 \\
0 & 0 & 0 & 1 + (S + T)^2
\end{pmatrix}
\]
has dense range in \(E^{\oplus 4}\), and so \(S \pm T\) are also regular.

From the assumption that \((S, T)\) is an almost commuting pair, it does not follow that \(S \pm T\) are self-adjoint on \(\text{Dom } S \cap \text{Dom } T\) (an obvious counter-example would be \(S = \mp T\)). However, it does follow that \(S \pm T\) are essentially self-adjoint on \(\text{Dom } S \cap \text{Dom } T\).
Proposition 2.24. Let \((S, T)\) be an almost commuting pair of regular self-adjoint operators on \(E\). Then the operators \(S \pm T\) are essentially self-adjoint on \(\text{Dom } S \cap \text{Dom } T\).

Proof. The statement follows from a straightforward adaptation of the proof of \([KL12, \text{Proposition } 7.7]\). For completeness we will work out the proof for \(S + T\) (the proof for \(S - T\) is similar). We know that \(S + T\) is symmetric on \(\text{Dom } S \cap \text{Dom } T\), so it suffices to prove that \(\text{Dom}(S + T)^* \subset \text{Dom}(S + T)\). Let \(\xi, \in \text{Dom}(S + T)^*\), and define the sequence

\[
\xi_n := (-\frac{i}{n}S + 1)^{-1}\xi \in \text{Dom } S,
\]

which converges in norm to \(\xi\). For \(\eta \in \text{Dom } T\), we can calculate

\[
(\xi_n | \eta) = (\xi | (\frac{i}{n}S + 1)^{-1}T\eta)
\]

\[
= (\xi | T(\frac{i}{n}S + 1)^{-1}\eta) - (\xi | (\frac{i}{n}S + 1)^{-1}[\frac{i}{n}S, T](\frac{i}{n}S + 1)^{-1}\eta)
\]

\[
= (\xi | (S + T)(\frac{i}{n}S + 1)^{-1}\eta) - (\xi | S(\frac{i}{n}S + 1)^{-1}\eta) - (\xi | R_n\eta)
\]

\[
= ((-\frac{i}{n}S + 1)^{-1}(S + T)^*\xi | \eta) - (S(-\frac{i}{n}S + 1)^{-1}\xi | \eta) - (R_n\eta \xi | \eta),
\]

where \(R_n := (\frac{i}{n}S + 1)^{-1}[\frac{i}{n}S, T](\frac{i}{n}S + 1)^{-1}\) is defined as in \([KL12, \text{Lemma } 7.4]\). This proves that \(\xi_n\) is in the domain of \(T^* = T\), and

\[
T\xi_n = (-\frac{i}{n}S + 1)^{-1}(S + T)^*\xi - S\xi_n - R_n\xi.
\]

In \([KL12, \text{Lemma } 7.4]\) it is shown that \(R_n \to 0\) strongly, and hence

\[
(S + T)\xi_n = (-\frac{i}{n}S + 1)^{-1}(S + T)^*\xi - R_n\xi,
\]

converges in norm to \((S + T)^*\xi\), which in particular means that \(\xi = \lim_{n \to \infty} \xi_n\) is an element of \(\text{Dom}(S + T)\).

\[\Box\]

2.4 Unbounded Kasparov Modules

For any two \(\mathbb{Z}_2\)-graded \(C^*\)-algebras \(A\) and \(B\), Kasparov \([Kas80b]\) defined the abelian group \(\mathsf{KK}(A, B)\) as a set of equivalence classes of certain Kasparov \(A\)-\(B\)-modules. In addition, he defined a pairing \(\mathsf{KK}(A, B) \times \mathsf{KK}(B, C) \to \mathsf{KK}(A, C)\), called the \textit{Kasparov product}. More details can be found in e.g. \([Bla98]\).

The elements of these \(\mathsf{KK}\)-groups can be used to access the topology of the \(C^*\)-algebras. In this thesis however, we will be more interested in \textit{geometry} rather than just topology, which is most conveniently described using unbounded operators (first-order differential operators) instead of bounded operators (zeroth-order pseudo-differential operators). Kasparov modules were generalised to the unbounded picture by Baaj and Julg \([BJ83]\). In this thesis we will only focus on this unbounded picture, which we briefly recall below.
Definition 2.25 ([BJ83]). Given $\mathbb{Z}_2$-graded $C^*$-algebras $A$ and $B$, an unbounded Kasparov $A$-$B$-module $(A, \pi E_B, D)$ is given by

- a $\mathbb{Z}_2$-graded, countably generated, right Hilbert $B$-module $E$;
- a $\mathbb{Z}_2$-graded $*$-homomorphism $\pi: A \to \text{End}_B(E)$;
- a separable dense $*$-subalgebra $A \subset A$;
- a closed, regular, odd operator $D: \text{Dom} D \subset E \to E$ such that
  1) there exists a linear subspace $E \subset \text{Dom} D$ which is a core for $D$;
  2) the operator $D$ is essentially self-adjoint on $E$;
  3) we have the inclusion $\pi(a) \cdot E \subset \text{Dom} D$, and the graded commutator $[D, \pi(a)]_\pm$ is bounded on $E$ for each $a \in A$;
  4) the map $\pi(a) \circ \iota: \text{Dom} D \hookrightarrow E \to E$ is compact for each $a \in A$, where $\iota: \text{Dom} D \hookrightarrow E$ denotes the natural inclusion map, and $\text{Dom} D$ is considered as a Hilbert $B$-module with the graph inner product of $D$.

The set of all unbounded Kasparov $A$-$B$-modules is denoted by $\Psi(A, B)$. If no confusion arises, we will usually write $(A, E_B, D)$ instead of $(A, \pi E_B, D)$. If $B = C$ and $A$ is trivially graded, we will write $E = H$ and refer to $(A, H, D)$ as an even spectral triple over $A$ (see [Con94]).

Remark 2.26. We have presented this definition in such a way that it can be more straightforwardly generalised to the indefinite case in Section 5.1. In particular, note that assumption 4) is equivalent to the more commonly used assumption that the resolvent of $D$ is locally compact (which means that $\pi(a)(1 + D^2)^{-\frac{1}{2}}$ is compact for each $a \in A$).

There is a natural map from the unbounded picture to the bounded one. This map is defined by replacing the operator $D$ by $b(D) = D(1 + D^2)^{-\frac{1}{2}}$, where the function $b: \mathbb{R} \to \mathbb{R}$ is given by $b(x) = x(1 + x^2)^{-\frac{1}{2}}$. We refer to this map as the bounded transform.

Theorem 2.27 ([BJ83, Propositions 2.2 & 2.3]). For an unbounded Kasparov module $(A, E_B, D) \in \Psi(A, B)$, the bounded transform yields a class $[(A, E_B, b(D))]$ in $\text{KK}(A, B)$. Moreover, if $A$ is separable and $B$ is $\sigma$-unital, then this map $\Psi(A, B) \to \text{KK}(A, B)$ is surjective.

Remark 2.28. The proofs of these statements, given in [BJ83], are rather succinct. The proofs have been worked out in more detail in [Bla98], but we point out that the proof of [Bla98, Proposition 17.11.3], stating that the bounded transform yields a class in $\text{KK}(A, B)$, is incorrect (or at least incomplete). The proof requires the additional hypothesis (used in [BJ83] but left out in [Bla98]) that $A$ preserves (a core in) the domain of $D$ (see condition 3) in Definition 2.25). For more details, see the discussion in [FMR14], which gives several examples of what goes wrong without this hypothesis.
For an unbounded Kasparov module \((A, E_B, \mathcal{D}) \in \mathcal{V}(A, B)\), we will simply write \([\pi E_B, \mathcal{D}]\) for the class \([\pi E_B, \mathcal{D}]\) \in KK(A, B). The set KK(A, B) forms an abelian group, with addition given by the direct sum. The negative of a class \([\pi E_B, F]\) is given by \([\pi E_B^{\text{op}}, -F]\), where \(E^{\text{op}}\) is defined as the Hilbert module \(E\) with the opposite grading (i.e. \((E^{\text{op}})^0 = E^1\) and \((E^{\text{op}})^1 = E^0\)), and the left action of \(A\) on \(E^{\text{op}}\) is given by \(\pi^{\text{op}}(a) := \pi(a^0) - \pi(a^1)\) for \(a = a^0 + a^1 \in A^0 \oplus A^1 = A\). Similarly, in the unbounded picture, the negative of \([\pi E_B, \mathcal{D}]\) is given by \([\pi E_B^{\text{op}}, -\mathcal{D}]\).

**Remark 2.29.** The elements of KK(A, B) are equivalence classes of (bounded) Kasparov modules. We say that two unbounded Kasparov A-B-modules are equivalent if their bounded transforms represent the same class. This equivalence relation is rather strong; in particular, it does not respect any of the geometric information contained in a Kasparov module. To preserve this geometric information, one should consider only unitary equivalences.

**Definition 2.30.** We say that two unbounded Kasparov A-B-modules \(\pi E_B, \mathcal{D}_1\) and \(\pi E_B, \mathcal{D}_2\) are unitarily equivalent if there exists an even unitary \(U : E \to F\) such that \(\mathcal{D}_2 = U\mathcal{D}_1 U^*\) and \(\phi(a) = U\pi(a)U^*\) for all \(a \in A\).

### 2.4.1 Odd unbounded Kasparov modules

We also introduce an odd version of unbounded Kasparov modules, where all objects are trivially graded.

**Definition 2.31.** Given trivially graded C*-algebras A and B, an odd unbounded Kasparov A-B-module \(\pi E_B, \mathcal{D}\) is given by

- a trivially graded, countably generated, right Hilbert B-module \(E\);
- a \(*\)-homomorphism \(\pi : A \to \text{End}_B(E)\);
- a separable dense \(*\)-subalgebra \(A \subset A\);
- a closed, regular operator \(\mathcal{D} : \text{Dom} \mathcal{D} \subset E \to E\) such that
  1) there exists a linear subspace \(\mathcal{E} \subset \text{Dom} \mathcal{D}\) which is a core for \(\mathcal{D}\);
  2) the operator \(\mathcal{D}\) is essentially self-adjoint on \(\mathcal{E}\);
  3) we have the inclusion \(\pi(A) \cdot \mathcal{E} \subset \text{Dom} \mathcal{D}\), and the commutator \([\mathcal{D}, \pi(a)]\)
      is bounded on \(\mathcal{E}\) for each \(a \in A\);
  4) the map \(\pi(a) \circ \iota : \text{Dom} \mathcal{D} \leftrightarrow E \to E\) is compact for each \(a \in A\), where \(\iota : \text{Dom} \mathcal{D} \leftrightarrow E\) denotes the natural inclusion map, and \(\text{Dom} \mathcal{D}\) is considered as a Hilbert B-module with the graph inner product of \(\mathcal{D}\).

If \(B = C\) and \(A\) is trivially graded, we will write \(E = \mathcal{H}\) and refer to \((A, \mathcal{H}, \mathcal{D})\) as an odd spectral triple over \(A\).
Given an odd Kasparov module \((A, \pi E_B, \mathcal{D})\), it is straightforward to construct an (even) Kasparov module \((\tilde{A}, \tilde{\pi} \tilde{E}_B, \tilde{\mathcal{D}})\) by defining the \(\mathbb{Z}_2\)-graded Hilbert module \(\tilde{E} := E \oplus E\) (where the first summand is considered even and the second summand odd), the *-homomorphism \(\tilde{\pi} := \pi \oplus \pi: A \to \text{End}_B(\tilde{E})\), and the odd operator

\[
\tilde{\mathcal{D}} := \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{pmatrix}.
\]

We observe that \((\tilde{A}, \tilde{\pi} \tilde{E}_B, \tilde{\mathcal{D}})\) is unitarily equivalent to \((A, \pi \text{End}_B E^op, -\mathcal{D})\) via the anti-self-adjoint unitary

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

A converse statement also holds:

**Proposition 2.32.** Let \(A\) and \(B\) be trivially graded \(C^*\)-algebras. Let \((A, \pi E_B, \mathcal{D})\) be an unbounded Kasparov \(A\)-\(B\)-module which is unitarily equivalent to \((A, \pi \text{End}_B E^op, -\mathcal{D})\) via an anti-self-adjoint unitary

\[
\begin{pmatrix} 0 & -U^* \\ U & 0 \end{pmatrix}.
\]

Consider the restrictions \(\pi_0 := \pi|_{E^0}: A \to \text{End}_B(\text{End}_B(E^0))\) and \(\mathcal{D}_0 := \mathcal{D}|_{E^0}: E^0 \to E^1\). Then \((A, \pi_0 E^0_B, U^* \mathcal{D}_0)\) is an odd unbounded Kasparov \(A\)-\(B\)-module.

**Remark 2.33.** The anti-self-adjoint unitary operator given above can be seen as the generator of the Clifford algebra \(\text{Cl}_1\). Since it is odd and anti-commutes with \(\mathcal{D}\), this means that \((A, \pi E_B, \mathcal{D})\) can be seen as an element of \(\Psi(A \otimes \text{Cl}_1, B)\), and thus it represents a class in the odd \(KK\)-theory \(KK^1(A, B) = KK(A \otimes \text{Cl}_1, B)\).

**Proof.** Using the isomorphism \(E^op \simeq E = E^0 \oplus E^1\) as ungraded modules, any even unitary isomorphism \(E \to E^op\) can be written in the form

\[
\begin{pmatrix} 0 & -V^* \\ U & 0 \end{pmatrix},
\]

where \(U\) and \(V\) are unitary isomorphisms \(E^0 \to E^1\). The assumption that this unitary isomorphism is anti-self-adjoint implies that \(U = V\). If we write the self-adjoint operator \(\mathcal{D}\) on \(E^0 \oplus E^1\) as

\[
\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_0^* \\ \mathcal{D}_0 & 0 \end{pmatrix},
\]

the unitary equivalence of \(\mathcal{D}\) and \(-\mathcal{D}\) then yields

\[
\mathcal{D} = -\begin{pmatrix} 0 & -U^* \\ U & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{D}_0^* \\ \mathcal{D}_0 & 0 \end{pmatrix} \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} = \begin{pmatrix} 0 & U^* \mathcal{D}_0 \mathcal{U}^* \\ \mathcal{U}^* \mathcal{D}_0 U & 0 \end{pmatrix}.
\]

Hence it follows that \(\mathcal{D}_0 = \mathcal{U} \mathcal{D}_0^* \mathcal{U}\), which means that \(U^* \mathcal{D}_0\) is self-adjoint.
The algebra $A$ is trivially graded, so its representation on $E$ and $E^\text{op}$ is the same, and therefore the unitary equivalence must commute with the representation of $A$. Writing $a = a_0 \oplus a_1$, this implies that $a_1 = Ua_0U^*$. The boundedness of the commutator $[\mathcal{D}, a]$ then implies that $[U^*\mathcal{D}_0, a]$ is bounded, and since the operator 

$$
(1 + (U^*\mathcal{D}_0)^2)^{-\frac{1}{2}} = (1 + \mathcal{D}_0^*U^*\mathcal{D}_0)^{-\frac{1}{2}} = (1 + \mathcal{D}_0^*\mathcal{D}_0)^{-\frac{1}{2}}
$$

is locally compact by assumption, this completes the proof that $(A, \pi_0 E_B^0, U^*\mathcal{D}_0)$ yields an odd unbounded Kasparov module.

2.4.2 The unbounded Kasparov product

The (internal) Kasparov product is a pairing $\text{KK}(A, B) \times \text{KK}(B, C) \to \text{KK}(A, C)$ defined on equivalence classes of bounded Kasparov modules [Kas80b]. This product also has an unbounded analogue. To be precise, we say that an unbounded Kasparov module $(A, \tilde{E}_C, \mathcal{D})$ represents the (internal) unbounded Kasparov product of $(A, E_B, \mathcal{D}_1)$ and $(B, F_C, \mathcal{D}_2)$ if the class $[(A, \tilde{E}_C, b(1))] \in \text{KK}(A, C)$ is the Kasparov product of $[(A, E_B, b(1))] \in \text{KK}(A, B)$ and $[(B, F_C, b(1))] \in \text{KK}(B, C)$.

This definition of the unbounded Kasparov product relies on a knowledge of the classes represented by the bounded transforms. In practice, it is often more convenient to work only with the unbounded picture. This is made possible by the following theorem of Kucerovsky, which provides sufficient conditions for how one can recognize the Kasparov product of two unbounded Kasparov modules.

**Theorem 2.34 ([Kuc97])**. Let $(A, \pi E_B, \mathcal{D}_1)$ and $(B, \varphi F_C, \mathcal{D}_2)$ be unbounded Kasparov modules. Suppose that $(A, \pi_1 \otimes \text{id}(E \otimes_B F)_C, \mathcal{D})$ is an unbounded Kasparov module such that:

1) for all $e$ in a dense subspace of $\pi(A)E$, the commutators

\[
\begin{pmatrix}
\mathcal{D} & 0 \\
0 & \mathcal{D}_2
\end{pmatrix}
\begin{pmatrix}
0 & T_e \\
T_e^* & 0
\end{pmatrix}
\]

are bounded on $\text{Dom}(|\mathcal{D} \oplus \mathcal{D}_2| \subset (E \otimes_B F) \oplus F$, where $T_e : F \to E \otimes_B F$ is given by $T_e(f) = e \otimes f$;

2) $\text{Dom}(\mathcal{D}) \subset \text{Dom}(|\mathcal{D}_1 \oplus 1|$;

3) there exists $K \in \mathbb{R}$ such that $((\mathcal{D}_1 \otimes 1)x | \mathcal{D}x) + (\mathcal{D}x | (\mathcal{D}_1 \otimes 1)x) \geq K|x|^2$ for all $x \in \text{Dom}(\mathcal{D})$.

Then $(A, \pi \otimes \text{id}(E \otimes_B F)_C, \mathcal{D})$ represents the unbounded Kasparov product of $(A, \pi E_B, \mathcal{D}_1)$ and $(B, \varphi F_C, \mathcal{D}_2)$.

Kucerovsky’s theorem is only useful if one already has a ‘guess’ for what the Kasparov product should be. The next question is then whether one can construct a suitable module representing the Kasparov product. The first result in this
direction was given by Mesland [Mes14], and his hypotheses were subsequently weakened by Kaad and Lesch [KL13].

We will quote the result from [KL13], which provides sufficient conditions for being able to construct the Kasparov product of two odd unbounded Kasparov modules \((A, E_B, \mathcal{D}_1)\) and \((B_1, F_C, \mathcal{D}_2)\). Let \(\mathcal{B}_1\) be an involutive operator algebra which is a dense \(\ast\)-subalgebra of \(B\), such that \([\mathcal{D}_2, b]\) is bounded for all \(b \in \mathcal{B}_1\). As in \textbf{Section 2.2.3}, we define the \textit{generalised one-forms}

\[
\Omega^1_{\mathcal{D}_2}(\mathcal{B}_1) := \left\{ \sum_j a_j [\mathcal{D}_2, b_j] \mid a_j, b_j \in \mathcal{B}_1 \right\} \subset \text{End}_C(F),
\]

where the sums are required to converge in the operator norm on \(F\). We assume that \((\mathcal{B}_1, F_C, \mathcal{D}_2)\) is \textit{essential}, which means that \(B \cdot F\) is dense in \(F\), and \(B \cdot \Omega^1_{\mathcal{D}_2}(\mathcal{B}_1)\) is dense in \(\Omega^1_{\mathcal{D}_2}(\mathcal{B}_1)\). Let \(\mathcal{E}_1 \subset E\) be an operator module over \(\mathcal{B}_1\) which is a dense subspace of \(E\). From [KL13, Definition 4.6] we have the Grassmann connection \(\nabla^{Gr}_{\mathcal{D}_2} : \mathcal{E}_1 \to E \otimes_B \text{End}_C(F)\) (see \textbf{Section 2.2.3} for the special case \(\mathcal{B}_1 = C^0_1(M)\)). We can then define the operator \(1 \otimes \nabla \mathcal{D}_2\) on the interior product \(E \otimes_B F\) as

\[
(1 \otimes \nabla \mathcal{D}_2)(e \otimes f) := e \otimes \mathcal{D}_2 f + (\nabla^{Gr}_{\mathcal{D}_2}(e))f,
\]

for \(e \otimes f\) in the domain \(\text{Dom}(1 \otimes \nabla \mathcal{D}_2) := \mathcal{E}_1 \otimes_{\mathcal{B}_1} \text{Dom} \mathcal{D}_2\). This operator is well-defined, since for \(e \in \mathcal{E}_1, f \in \text{Dom} \mathcal{D}_2,\) and \(b \in \mathcal{B}_1\) we have

\[
(1 \otimes \nabla \mathcal{D}_2)(eb \otimes f) = eb \otimes \mathcal{D}_2 f + (\nabla^{Gr}_{\mathcal{D}_2}(eb))f
\]

\[
= e \otimes b \mathcal{D}_2 f + (\nabla^{Gr}_{\mathcal{D}_2}(e))(bf) + e \otimes [\mathcal{D}_2, b]f
\]

\[
= e \otimes \mathcal{D}_2 (bf) + (\nabla^{Gr}_{\mathcal{D}_2}(e))(bf) = (1 \otimes \nabla \mathcal{D}_2)(e \otimes bf),
\]

where we have used on the second line that the Grassmann connection \(\nabla^{Gr}_{\mathcal{D}_2}\) satisfies the ‘Leibniz rule’

\[
\nabla^{Gr}_{\mathcal{D}_2}(eb) = \nabla^{Gr}_{\mathcal{D}_2}(e)b + e \otimes [\mathcal{D}_2, b],
\]

for all \(e \in \mathcal{E}_1\) and \(b \in \mathcal{B}_1\) [KL13, Proposition 4.7].

\textbf{Definition 2.35 ([KL13, Definition 6.3])}. By a correspondence from \((A, E_B, \mathcal{D}_1)\) to \((\mathcal{B}_1, F_C, \mathcal{D}_2)\) we will understand a pair \((\mathcal{E}_1, \nabla^{Gr}_{\mathcal{D}_2})\) consisting of an operator module \(\mathcal{E}_1\) over a \(\sigma\)-unital involutive operator algebra \(\mathcal{B}_1\) along with the Grassmann connection \(\nabla^{Gr}_{\mathcal{D}_2} : \mathcal{E}_1 \to E \otimes_B \text{End}_C(F)\) such that

1) the operator module \(\mathcal{E}_1 \subset E\) is a dense subspace of \(E\) and the involutive operator algebra \(\mathcal{B}_1 \subset B\) is a dense \(\ast\)-subalgebra of \(B\), and the inclusions are completely bounded and compatible with the module structures and inner products;

2) we have the inclusion \(\mathcal{B}_1 \cdot \text{Dom} \mathcal{D}_2 \subset \text{Dom} \mathcal{D}_2,\) and \([\mathcal{D}_2, \cdot ] : \mathcal{B}_1 \to \text{End}_C(F)\) is a completely bounded derivation;
3) the commutator \([1 \otimes \nabla D_2, a \otimes 1]\): \(\text{Dom}(1 \otimes \nabla D_2) \to E \otimes_B F\) is well-defined and bounded for all \(a\) in the dense \(\ast\)-subalgebra \(A \subset A\);

4) the pair \((D_1 \otimes 1, 1 \otimes \nabla D_2)\) is an almost commuting pair.

**Remark 2.36.** In fact, [KL13] gives a more general definition using \(D_2\)-connections. We have restricted our attention to the special case of the Grassmann connection, as this is all we need for our application of the following theorem in Section 5.5.

**Theorem 2.37 ([KL13, Theorems 6.7 and 7.5]).** Let \((A, E_B, D_1)\) and \((B_1, F_C, D_2)\) be two odd unbounded Kasparov modules. Suppose that there exists a correspondence \((\mathcal{E}_1, \nabla_{D_2}^{Gr})\) from \((A, E_B, D_1)\) to \((B_1, F_C, D_2)\). Then the operator

\[
D_1 \times D_2 := \begin{pmatrix}
0 & D_1 \otimes 1 - i1 \otimes \nabla D_2 \\
D_1 \otimes 1 + i1 \otimes \nabla D_2 & 0
\end{pmatrix}
\]

on the domain \((\text{Dom}(D_1 \otimes 1) \cap \text{Dom}(1 \otimes \nabla D_2))^{\oplus 2} \subset (E \otimes_B F)^{\oplus 2}\) yields an even unbounded Kasparov \(A\)-\(C\)-module \((A, (E \otimes_B F)^{\oplus 2}_C, D_1 \times D_2)\) which represents the Kasparov product of \((A, E_B, D_1)\) and \((B_1, F_C, D_2)\).

2.5 **Smoothly summable spectral triples**

As mentioned in **Definition 2.25**, a spectral triple \((A, \mathcal{H}, D)\) is an unbounded Kasparov \(A\)-\(C\)-module, where \(A\) is the \(C^*\)-closure of \(\tilde{A}\). In this section, we will discuss notions of smoothness and summability for spectral triples. Our discussion is based on [CGRS14], where a general definition of summability in the non-unital/non-compact context was developed.

2.5.1 **Summability**

Consider a densely defined, self-adjoint operator on a Hilbert space \(\mathcal{H}\). For a positive bounded operator \(T\), we define for each \(s > 0\) a weight \(\varphi_s\) by

\[
\varphi_s(T) := \text{Tr} \left( (1 + D^2)^{-\frac{s}{2}} T (1 + D^2)^{-\frac{s}{2}} \right).
\]

**Definition 2.38.** A spectral triple \((A, \mathcal{H}, D)\) is called **finitely summable** if there exists \(s > 0\) such that \(\varphi_s(|a|)\) is finite for all \(a \in A\). In this case, we define the **spectral dimension** \(p\) of \((A, \mathcal{H}, D)\) as the infimum of all such \(s\).

We can also use these weights \(\varphi_s\) to define a notion of ‘summability’ or ‘integrability’ for bounded operators on \(\mathcal{H}\). First, for \(p > 0\) we define the space of **square-integrable** bounded operators \(B_2(D, p)\) as the set of bounded operators \(T \in B(\mathcal{H})\) such that \(\varphi_s(|T|)\) and \(\varphi_s(|T^*|)\) are finite for all \(s > p\), i.e.

\[
B_2(D, p) = \{ T \in B(\mathcal{H}) : \varphi_s(|T|^2), \varphi_s(|T^*|^2) < \infty, \forall s > p \}.
\]
This space $\mathcal{B}_2(\mathcal{D}, p)$ can be equipped with a family of norms $\{Q_n\}_{n=1}^{\infty}$ given by

$$Q_n(T) := \left( \|T\|^2 + \varphi_{p+1, n}(\|T\|^2) + \varphi_{p+1, n}(\|T^*\|^2) \right)^{\frac{1}{2}}.$$

For $T, S \in \mathcal{B}_2(\mathcal{D}, p)$, one can use the operator inequality $S^*T^*TS \leq \|T^*\|S^*S$ to show that $TS$ is again square-integrable, so that $\mathcal{B}_2(\mathcal{D}, p)$ is a $*$-algebra. Equipped with the norms $\{Q_n\}_{n=1}^{\infty}$, the space of square-integrable operators $\mathcal{B}_2(\mathcal{D}, p)$ is in fact a Fréchet algebra [CGRS14, Proposition 2.6].

Given the square-integrable operators, we can define the integrable operators as follows. Consider the finite linear span $\mathcal{B}_2(\mathcal{D}, p)^2$ of products of square-integrable operators $T_{1,j}, T_{2,j} \in \mathcal{B}_2(\mathcal{D}, p)$. For $T \in \mathcal{B}_2(\mathcal{D}, p)^2$, we define the norms [CGRS14, p. 13]

$$\mathcal{P}_n(T) := \inf \left\{ \sum_{j=1}^{k} Q_n(T_{1,j}) Q_n(T_{2,j}) \mid T = \sum_{j=1}^{k} T_{1,j} T_{2,j}, T_{1,j}, T_{2,j} \in \mathcal{B}_2(\mathcal{D}, p) \right\},$$

where the infimum runs over all possible representations of $T$ as a finite linear combination of products of elements in $\mathcal{B}_2(\mathcal{D}, p)$. We now define the integrable operators $\mathcal{B}_1(\mathcal{D}, p)$ as the completion of $\mathcal{B}_2(\mathcal{D}, p)^2$ with respect to the family of norms $\{\mathcal{P}_n\}_{n=1}^{\infty}$.

### 2.5.2 Smoothness

In addition to integrability, we can also ask whether a bounded operator is ‘differentiable’. For this purpose we construct a derivation $\delta$, as follows. Let $\mathcal{D}$ be a densely defined self-adjoint operator on $\mathcal{H}$, and consider the subspaces $\mathcal{H}_k := \text{Dom } \mathcal{D}^k$ and $\mathcal{H}_\infty := \bigcap_{k \geq 0} \mathcal{H}_k$. Given a bounded operator $T$ which preserves the smooth subspace $\mathcal{H}_\infty$, we define

$$\delta(T) := \left[ (1 + \mathcal{D}^2)^{\frac{1}{2}}, T \right].$$

We use the recursive notation $T^{(0)} := T$ and $T^{(k)} := [\mathcal{D}^2, T^{(k-1)}]$ for $k > 0$, and also define maps $L$ and $R$ given by

$$L(T) := (1 + \mathcal{D}^2)^{-\frac{1}{2}} [\mathcal{D}^2, T], \quad R(T) := [\mathcal{D}^2, T] (1 + \mathcal{D}^2)^{-\frac{1}{2}}.$$

It is shown in [CM95, Con95] and [CPRS06a, Proposition 6.5] that

$$\bigcap_{k \geq 0} \text{Dom } L^k = \bigcap_{k \geq 0} \text{Dom } R^k = \bigcap_{k, l \geq 0} \text{Dom } L^k \circ R^l = \bigcap_{k \geq 0} \text{Dom } \delta^k. \quad (2.3)$$

We now define the smooth bounded operators as the set $\text{OP}^{0}(\mathcal{D}) := \bigcap_{k \geq 0} \text{Dom } \delta^k$. This can be generalised to ‘unbounded operators with smooth coefficients’ as follows.
Definition 2.39. Let \( D \) be a densely defined self-adjoint operator on the Hilbert space \( \mathcal{H} \). For \( r \in \mathbb{R} \), the set of \textit{regular order-} \( r \) \textit{pseudo-differential operators} is

\[
\text{OP}^r(D) := (1 + D^2)^{\frac{r}{2}} \left( \bigcap_{k \in \mathbb{N}} \text{Dom} \delta^k \right), \quad \text{OP}^\bullet(D) := \bigcup_{r \in \mathbb{R}} \text{OP}^r(D).
\]

The natural topology of \( \text{OP}^r(D) \) is associated with the family of norms

\[
\sum_{l=0}^{k} \left\| \delta^l((1 + D^2)^{-\frac{r}{2}})T \right\|, \quad k \in \mathbb{N}.
\]

The following lemma will be useful to us later on.

Lemma 2.40. Let \( D \) be a densely defined self-adjoint operator on the Hilbert space \( \mathcal{H} \). Let \( T \in \text{OP}^0(D) \) have bounded inverse. Then \( T^{-1} \in \text{OP}^0(D) \).

Proof. We need to show that \( \delta^n(T^{-1}) \) is bounded for all \( n \geq 1 \). We first check that

\[
\delta(T^{-1}) = -T^{-1}\delta(T)T^{-1}
\]

is given by a product of bounded operators. Iterating this formula shows that there are combinatorial constants \( C_{l,n,k} \) such that

\[
\delta^n(T^{-1}) = \sum_{1 \leq l \leq n} \sum_{1 \leq k_1, \ldots, k_l \leq n} C_{l,n,k} T^{-1}\delta^{k_1}(T)T^{-1}\delta^{k_2}(T)T^{-1} \cdots T^{-1}\delta^{k_l}(T)T^{-1}.
\]

Since \( \delta^k(T) \in \text{OP}^0(D) \) is bounded for all \( k \), we see that \( \delta^n(T^{-1}) \) is bounded for all \( n \). Hence \( T^{-1} \) is indeed an element of \( \text{OP}^0(D) \). \( \square \)

2.5.3 Smooth summability

Finally, we wish to combine the notions of smoothness and integrability. First, we consider the set of integrable operators with `integrable derivatives`.

Definition 2.41. Let \( D \) be a densely defined self-adjoint operator on the Hilbert space \( \mathcal{H} \), and \( p \geq 1 \). Then define for \( k = 0, 1, 2, \ldots \)

\[
\mathcal{B}^k(D, p) := \{ T \in \mathcal{B}(\mathcal{H}) : T : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \text{ and } \delta^l(T) \in \mathcal{B}_1(D, p), \forall l = 0, \ldots, k \},
\]

\[
\mathcal{B}^\infty(D, p) := \bigcap_{k=0}^{\infty} \mathcal{B}^k(D, p).
\]

We equip \( \mathcal{B}^k(D, p) \) (for \( k = 0, 1, 2, \ldots, \infty \)) with the topology determined by the seminorms \( \mathcal{P}_{n,l} \) (where \( n = 1, 2, \ldots \) and \( l = 0, \ldots, k \)) defined by

\[
\mathcal{P}_{n,l}(T) := \sum_{j=0}^{l} \mathcal{P}_n(\delta^j(T)).
\]
We are now ready to define smooth summability for spectral triples.

**Definition 2.42.** Let \((A, \mathcal{H}, D)\) be a spectral triple. Then we say that \((A, \mathcal{H}, D)\) is QC\(k\) **summable** if \((A, \mathcal{H}, D)\) is finitely summable with spectral dimension \(p\) and
\[
A \cup [D, A] \subset B_{\infty}^r(D, p).
\]

We say that \((A, \mathcal{H}, D)\) is **smoothly summable** if it is QC\(k\) summable for all \(k \in \mathbb{N}\) or, equivalently, if
\[
A \cup [D, A] \subset B_{\infty}^\infty(D, p).
\]

If \((A, \mathcal{H}, D)\) is smoothly summable with spectral dimension \(p\), we can equip \(A\) with the topology associated to the family of norms (for \(n = 1, 2, \ldots\) and \(l = 0, 1, 2, \ldots\))
\[
P_{n,1}(A) + P_{n,1}([D, A]),
\]
where \(a \in A\).

For the regular pseudo-differential operators \(\text{OP}^r(D)\) defined above, we can ask whether the coefficients are not only smooth, but also have integrable derivatives. This leads to the following definition.

**Definition 2.43.** Let \(D\) be a densely defined self-adjoint operator on the Hilbert space \(\mathcal{H}\) and \(p \geq 1\). For \(r \in \mathbb{R}\), the set of **order-\(r\) tame pseudo-differential operators** associated with \((\mathcal{H}, D)\) and \(p \geq 1\) is given by
\[
\text{OP}^r_0(D) := (1 + D^2)^{r/2}B_{\infty}^\infty(D, p), \quad \text{OP}^*_{\infty}(D) := \bigcup_{r \in \mathbb{R}} \text{OP}^r_0(D).
\]

We can ask whether the coefficients are integrable derivatives. This leads to the following definition.

**Definition 2.44.** Let \(D\) be a densely defined self-adjoint operator on the Hilbert space \(\mathcal{H}\) and \(p \geq 1\). For \(r \in \mathbb{R}\), the set of **order-\(r\) tame pseudo-differential operators** associated with \((\mathcal{H}, D)\) and \(p \geq 1\) is given by
\[
\text{OP}^r_0(D) := (1 + D^2)^{r/2}B_{\infty}^\infty(D, p), \quad \text{OP}^*_{\infty}(D) := \bigcup_{r \in \mathbb{R}} \text{OP}^r_0(D).
\]

**Proposition 2.44** ([CGRS14, Proposition 2.31]). Let \(D\) be a densely defined self-adjoint operator on the Hilbert space \(\mathcal{H}\) and \(p \geq 1\). For \(z \in \mathbb{C}\) and \(T \in \text{OP}^*_{\infty}(D)\), we define the one-parameter group \(\sigma\) of automorphisms of \(\text{OP}^*_{\infty}(D)\) by
\[
\sigma^z(T) := (1 + D^2)^{r/2} T (1 + D^2)^{-r/2}.
\]

Then \(\sigma\) restricts to a strongly continuous one parameter group on each \(\text{OP}^r(D)\) and \(\text{OP}^r_0(D)\) (for any \(r \in \mathbb{R}\)).
2.6 Non-Symmetric Operators

One of the main recurring themes throughout this thesis is that we will consider unbounded operators which are not symmetric. In this section, we describe our approach to dealing with these non-symmetric operators, namely by studying their real and imaginary parts. We start with a useful lemma regarding the ‘combined graph norm’ of two closed operators on the intersection of their domains.

**Lemma 2.45.** Let $E$ be a right Hilbert $B$-module with inner product $(\cdot | \cdot)$. Let $S$ and $T$ be closed regular operators on $E$ such that $\operatorname{Dom} S \cap \operatorname{Dom} T$ is dense. Then $\operatorname{Dom} S \cap \operatorname{Dom} T$ is a right Hilbert $B$-module with the inner product
\[(\psi | \psi)_{S,T} := (\psi | \psi) + (S\psi | S\psi) + (T\psi | T\psi),\]
and the corresponding norm $\| \psi \|_{S,T} = \|(\psi | \psi)_{S,T}\|_B^{1/2}$.

Furthermore, if a linear subset $E \subset \operatorname{Dom} S \cap \operatorname{Dom} T$ is a core for both $S$ and $T$, then the closure of $E$ in the norm $\| \cdot \|_{S,T}$ is equal to $\operatorname{Dom} S \cap \operatorname{Dom} T$.

**Proof.** We need to show that $\operatorname{Dom} S \cap \operatorname{Dom} T$ is complete in the norm $\| \cdot \|_{S,T}$. Since $S$ is closed, we know that $\operatorname{Dom} S$ is complete for the graph norm $\| \cdot \|_S$ corresponding to the inner product
\[(\psi | \psi)_S := (\psi | \psi) + (S\psi | S\psi),\]
and a similar statement holds for $\operatorname{Dom} T$. The inequalities
\[
\frac{1}{2}(\psi | \psi)_S + \frac{1}{2}(\psi | \psi)_T \leq (\psi | \psi)_{S,T} \leq (\psi | \psi)_S + (\psi | \psi)_T
\]
show that convergence in the norm $\| \cdot \|_{S,T}$ is equivalent to convergence in both graph norms $\| \cdot \|_S$ and $\| \cdot \|_T$. Denote by $W_S$ (respectively $W_T$) the closure of $\operatorname{Dom} S \cap \operatorname{Dom} T$ in the norm $\| \cdot \|_S$ (resp. $\| \cdot \|_T$). The closure of $\operatorname{Dom} S \cap \operatorname{Dom} T$ in the norm $\| \cdot \|_{S,T}$ is then equal to the intersection of $W_S$ and $W_T$. Since $W_S \subset \operatorname{Dom} S$ and $W_T \subset \operatorname{Dom} T$, this intersection $W_S \cap W_T$ is contained in, and hence equal to, $\operatorname{Dom} S \cap \operatorname{Dom} T$, so we conclude that $\operatorname{Dom} S \cap \operatorname{Dom} T$ is complete in the norm $\| \cdot \|_{S,T}$.

If $E \subset \operatorname{Dom} S \cap \operatorname{Dom} T$ is a core for both $S$ and $T$, its closure in the norm $\| \cdot \|_S$ (respectively $\| \cdot \|_T$) is equal to $\operatorname{Dom} S$ (resp. $\operatorname{Dom} T$). Since convergence in the norm $\| \cdot \|_{S,T}$ is equivalent to convergence in both graph norms $\| \cdot \|_S$ and $\| \cdot \|_T$, it follows that the closure of $E$ in the norm $\| \cdot \|_{S,T}$ equals the intersection of $\operatorname{Dom} S$ and $\operatorname{Dom} T$.

In what follows, we will consider a closed regular operator $D$ on a right Hilbert $B$-module $E$, such that $\operatorname{Dom} D \cap \operatorname{Dom} D^*$ is dense in $E$. The above lemma then tells us that $\operatorname{Dom} D \cap \operatorname{Dom} D^*$ is a Hilbert $B$-module with the inner product $(\cdot | \cdot)_{D,D^*}$.\]
Remark 2.46. Suppose that \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \) is dense in \( E \), but not a core for \( \mathcal{D} \). Then we can replace \( \mathcal{D} \) by a new operator \( \tilde{\mathcal{D}} \) given by the closure of the restriction of \( \mathcal{D} \) to \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \), i.e.

\[
\tilde{\mathcal{D}} := \mathcal{D}\big|_{\text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^*},
\]

so that by definition \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \) is a core for \( \tilde{\mathcal{D}} \). We have \( \tilde{\mathcal{D}} \subset \mathcal{D} \) and hence \( \tilde{\mathcal{D}}^* \supset \mathcal{D}^* \), which implies that \( \text{Dom} \tilde{\mathcal{D}} \cap \text{Dom} \tilde{\mathcal{D}}^* \) contains \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \). Thus \( \text{Dom} \tilde{\mathcal{D}} \cap \text{Dom} \tilde{\mathcal{D}}^* \) is a core for \( \tilde{\mathcal{D}} \). For this reason we will usually simply assume that \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \) is a core for \( \mathcal{D} \).

2.6.1 Wick rotation

Before we continue, let us provide some motivation by having a look at the basic example of a Dirac operator on 2-dimensional Minkowski spacetime \( \mathbb{R}^{1,1} \). We consider the Hilbert space \( L^2(\mathbb{R}^2, \mathbb{C}^2) \) of square-integrable spinors, and the Dirac operator

\[
\mathcal{D} := \begin{pmatrix} 0 & \partial_t - \partial_x \\ \partial_t + \partial_x & 0 \end{pmatrix}.
\]

Here \( t \) denotes the time variable, and \( x \) the spatial variable. We observe that we can isolate the spacelike part \( \mathcal{D}_s \) and the timelike part \( \mathcal{D}_t \) as

\[
\mathcal{D}_s := \frac{1}{2}(\mathcal{D} + \mathcal{D}^*) = \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix}, \quad \mathcal{D}_t := \frac{1}{2}(\mathcal{D} - \mathcal{D}^*) = \begin{pmatrix} 0 & \partial_t \\ \partial_t & 0 \end{pmatrix}.
\]

Hence the spacelike part \( \mathcal{D}_s \) is the symmetric part of \( \mathcal{D} \), and the timelike part \( \mathcal{D}_t \) is the anti-symmetric part of \( \mathcal{D} \). If we consider the standard Euclidean metric on \( \mathbb{R}^2 \), the corresponding Dirac operator looks like

\[
\begin{pmatrix} 0 & -i\partial_t - \partial_x \\ -i\partial_t + \partial_x & 0 \end{pmatrix} = \mathcal{D}_s - i\mathcal{D}_t.
\]

Thus, the transition from Minkowski signature to Euclidean signature is implemented at the level of Dirac operators by the map \( \mathcal{D} = \mathcal{D}_s + \mathcal{D}_t \rightarrow \mathcal{D}_s - i\mathcal{D}_t \). Such a transition is often referred to as ‘Wick rotation’ in physics. In the remainder of this section, we extend this procedure of ‘Wick rotation’ to non-symmetric operators on a Hilbert module.

Definition 2.47. Let \( \mathcal{D} \) be a closed regular operator on a right Hilbert \( \mathcal{B} \)-module \( E \), such that \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \) is dense. We define the real and imaginary parts of \( \mathcal{D} \) by setting

\[
\text{Re} \mathcal{D} := \frac{1}{2}(\mathcal{D} + \mathcal{D}^*), \quad \text{Im} \mathcal{D} := -\frac{i}{2}(\mathcal{D} - \mathcal{D}^*),
\]
on the initial domain \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* \). Since these operators are densely defined and symmetric, they are closable, and we denote their closures by \( \text{Re } \mathcal{D} \) and \( \text{Im } \mathcal{D} \) as well. We remark that the closure of \( \text{Re } \mathcal{D} + i \text{Im } \mathcal{D} \) is equal to \( \mathcal{D} \) if and only if \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* \) is a core for \( \mathcal{D} \). Furthermore, we define the ‘Wick rotations’ of \( \mathcal{D} \) by

\[
\mathcal{D}_+ := \text{Re } \mathcal{D} + \text{Im } \mathcal{D}, \quad \mathcal{D}_- := \text{Re } \mathcal{D} - \text{Im } \mathcal{D},
\]
on the initial domain \( \text{Dom } \text{Re } \mathcal{D} \cap \text{Dom } \text{Im } \mathcal{D} \).

**Lemma 2.48.** Let \( \mathcal{D} \) be a closed regular operator on a right Hilbert \( \mathcal{B} \)-module \( E \), such that \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* \) is dense in \( E \). Then the norms \( \| \cdot \|_{\mathcal{D}, \mathcal{D}^*}, \| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}}, \) and \( \| \cdot \|_{\mathcal{D}_+, \mathcal{D}_-} \) (defined as in Lemma 2.45) are equivalent, and

\[
\text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* = \text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D} = \text{Dom } \mathcal{D}_+ \cap \text{Dom } \mathcal{D}_-.
\]

**Proof.** An elementary calculation shows that we have the equalities

\[
(\phi|\psi)_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}} = \frac{1}{2}(\phi|\psi) + \frac{1}{2}(\phi|\psi)_{\mathcal{D}, \mathcal{D}^*}, \quad (\phi|\psi)_{\mathcal{D}_+, \mathcal{D}_-} = (\phi|\psi)_{\mathcal{D}, \mathcal{D}^*},
\]

from which it follows that the three norms \( \| \cdot \|_{\mathcal{D}, \mathcal{D}^*}, \| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}}, \) and \( \| \cdot \|_{\mathcal{D}_+, \mathcal{D}_-} \) are equivalent. Since \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* \) is by construction a core for \( \text{Re } \mathcal{D} \) and \( \text{Im } \mathcal{D} \), it follows by Lemma 2.45 that the closure of \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* \) in the norm \( \| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}} \) is equal to \( \text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D} \). But \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* \) is already closed in the norm \( \| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}} \) (because this norm is equivalent to \( \| \cdot \|_{\mathcal{D}, \mathcal{D}^*} \)), and hence we conclude that \( \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* = \text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D} \). The same argument applies to \( \text{Dom } \mathcal{D}_+ \cap \text{Dom } \mathcal{D}_- \).

2.6.2 **Reverse Wick rotation**

**Definition 2.49.** Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be closed regular symmetric operators on a Hilbert \( \mathcal{B} \)-module \( E \), such that \( \text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2 \) is dense in \( E \). We define the reverse Wick rotation of the pair \( (\mathcal{D}_1, \mathcal{D}_2) \) as the closure of

\[
\mathcal{D} := \frac{1}{2}(\mathcal{D}_1 + \mathcal{D}_2) + i \frac{1}{2}(\mathcal{D}_1 - \mathcal{D}_2)
\]
on the initial domain \( \text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2 \) (note that \( \mathcal{D} \) is closable, because it is the sum of a symmetric and an anti-symmetric operator, which ensures that \( \mathcal{D}^* \) is densely defined).

**Remark 2.50.** We emphasise that the reverse Wick rotation \( \mathcal{D}' \) of the pair \( (\mathcal{D}_2, \mathcal{D}_1) \) is not equal to the reverse Wick rotation \( \mathcal{D} \) of \( (\mathcal{D}_1, \mathcal{D}_2) \), but they are related to each other: \( \mathcal{D}' \) is the closure of the restriction of \( \mathcal{D}^* \) to \( \text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2 \). In other words, \( \mathcal{D}^* \) is a closed extension of the closure of \( \mathcal{D}' \), and they are equal if and only if \( \text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2 \) is a core for \( \mathcal{D}^* \).
Lemma 2.51. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be closed regular symmetric operators on a Hilbert $B$-module $E$, such that $\text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2$ is dense in $E$, and suppose that $\mathcal{D}_1 \pm \mathcal{D}_2$ are regular and essentially self-adjoint on $\text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2$. Let $\mathcal{D}$ be the reverse Wick rotation of $(\mathcal{D}_1, \mathcal{D}_2)$. Then the norms $\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2^*}$, $\| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}}$, and $\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2}$ are all equivalent, and hence

$$\text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^* = \text{Dom } \text{Re } \mathcal{D} \cap \text{Dom } \text{Im } \mathcal{D} = \text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2.$$ 

Proof. Let us write $\mathcal{E} := \text{Dom } \mathcal{D}_1 \cap \text{Dom } \mathcal{D}_2$. The operators $\mathcal{D}_1 \pm \mathcal{D}_2$ are symmetric on $\mathcal{E}$, so the domain of $\mathcal{D}^*$ also contains $\mathcal{E}$ (and in particular $\text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^*$ is dense). For $\psi \in \mathcal{E}$ we can then write

$$\mathcal{D}^* \psi = \frac{1}{2} (\mathcal{D}_1 + \mathcal{D}_2) \psi - \frac{i}{2} (\mathcal{D}_1 - \mathcal{D}_2) \psi.$$ 

Hence on the initial domain $\mathcal{E}$ we can write

$$\mathcal{D}_1 = \mathcal{D}_+, \quad \mathcal{D}_2 = \mathcal{D}_-, \quad \text{Re } \mathcal{D} = \frac{1}{2} (\mathcal{D}_1 + \mathcal{D}_2), \quad \text{Im } \mathcal{D} = \frac{1}{2} (\mathcal{D}_1 - \mathcal{D}_2).$$ 

From Lemma 2.48 it follows that the norms $\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2^*}$, $\| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}}$, and $\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2}$ are equivalent on $\mathcal{E}$, and furthermore $\mathcal{E}$ is closed in these norms. Since the equality $\text{Dom } \text{Re } \mathcal{D} \cap \text{Dom } \text{Im } \mathcal{D} = \text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^*$ was already shown in Lemma 2.48, it remains to show that $\mathcal{E}$ is dense in (and hence equal to) $\text{Dom } \text{Re } \mathcal{D} \cap \text{Dom } \text{Im } \mathcal{D}$.

The real part of $\mathcal{D}$ is by definition the closure of $\frac{1}{2} (\mathcal{D} + \mathcal{D}^*)$ on the initial domain $\text{Dom } \mathcal{D} \cap \text{Dom } \mathcal{D}^*$. By assumption, $\frac{1}{2} (\mathcal{D}_1 + \mathcal{D}_2)$ is essentially self-adjoint. Given that $\text{Re } \mathcal{D}$ is a closed symmetric extension of $\frac{1}{2} (\mathcal{D}_1 + \mathcal{D}_2)$, it must be the unique self-adjoint extension of $\frac{1}{2} (\mathcal{D}_1 + \mathcal{D}_2)$. Similarly, we find that $\text{Im } \mathcal{D}$ is the unique self-adjoint extension of $\frac{1}{2} (\mathcal{D}_1 - \mathcal{D}_2)$. In particular, $\mathcal{E}$ is a core for both $\text{Re } \mathcal{D}$ and $\text{Im } \mathcal{D}$, and since $\mathcal{E}$ is closed in the norm $\| \cdot \|_{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}}$, it must be equal to the intersection $\text{Dom } \text{Re } \mathcal{D} \cap \text{Dom } \text{Im } \mathcal{D}$ (see Lemma 2.45). \hfill $\Box$
In this chapter we give an introduction to Dirac operators on pseudo-Riemannian spin manifolds. Our main reference is Helga Baum’s book [Bau81]. A detailed introduction to finite-dimensional Clifford algebras and spin groups is given in [LM89, Ch. 1]. A short summary of the construction of pseudo-Riemannian Dirac operators can also be found in [BGM05, §2].

We will start in the first section with a discussion of finite-dimensional Clifford algebras and spinor modules. In the next section, we recall the basics of the theory of fibre bundles. Subsequently, we show that the constructions of Clifford algebras and spinor modules can be extended to bundles over pseudo-Riemannian spin manifolds. Finally, we construct the Dirac operator and discuss some of its properties in the last section.

3.1 Clifford Algebras and Spin Groups

Let \( \mathbb{R}^{t,s} \) be the real vector space \( \mathbb{R}^{t+s} \) equipped with the standard basis \( \{e_j\}_{j=1}^{t+s} \) and the non-degenerate symmetric bilinear form

\[
q \left( \sum_{j=1}^{t+s} v_j e_j, \sum_{j=1}^{t+s} w_j e_j \right) := -\sum_{j=1}^{t} v_j w_j + \sum_{j=t+1}^{t+s} v_j w_j.
\]

(3.1)

We think of \( e_j \) as a ‘timelike vector’ if \( j \leq t \) or a ‘spacelike vector’ if \( j > t \). For simplicity, we will assume that the dimension \( n := t + s \) is at least 3 (thus avoiding some subtleties in the lower-dimensional cases). We also assume that the number of time dimensions \( t \) is at most \( \frac{n}{2} \) (otherwise, we can replace \( q \) by \( -q \)).

The pseudo-orthogonal group \( O(t,s) \) is the group of all linear transformations of \( \mathbb{R}^{t,s} \) which preserve \( q \). If \( t = 0 \) we will write \( O(n) := O(0,n) \) for the orthogonal group. A transformation \( T \in O(t,s) \) can be written in matrix form with respect to the decomposition \( \mathbb{R}^{t,s} = \mathbb{R}^t \oplus \mathbb{R}^s \) as

\[
T = \begin{pmatrix}
T_{tt} & T_{ts} \\
T_{st} & T_{ss}
\end{pmatrix}.
\]
The pseudo-orthogonal group has four connected components, determined by whether they preserve time- and or space-orientation, given by

- \( O^{++}(t, s) := \{ A \in O(t, s) : \det(T_{tt}) > 0, \det(T_{ss}) > 0 \} \),
- \( O^{--}(t, s) := \{ A \in O(t, s) : \det(T_{tt}) < 0, \det(T_{ss}) < 0 \} \),
- \( O^{+-}(t, s) := \{ A \in O(t, s) : \det(T_{tt}) > 0, \det(T_{ss}) < 0 \} \),
- \( O^{-+}(t, s) := \{ A \in O(t, s) : \det(T_{tt}) < 0, \det(T_{ss}) > 0 \} \).

If \( t = 0 \) there are only two connected components, namely the special orthogonal group \( SO(n) = O^+(n) = \{ A \in O(n) : \det(A) > 0 \} \), and the component \( O^-(n) = \{ A \in O(n) : \det(A) < 0 \} \). We define two more subgroups of \( O(t, s) \):

- \( SO^+(t, s) := O^{++}(t, s) \),
- \( SO(t, s) := O^{++}(t, s) \cup O^{--}(t, s) = \{ A \in O(t, s) : \det(A) = 1 \} \).

We point out that \( SO^+(t, s) \) is the connected component of the identity in the special pseudo-orthogonal group \( SO(t, s) \).

Given the standard basis \( \{ e_j \}_{j=1}^{t+s} \) of \( \mathbb{R}^{t,s} \), we obtain a direct sum decomposition \( R^{t,s} = R^t \oplus R^s \), where the ‘timelike subspace’ \( R^t \) has the basis \( \{ e_j \}_{j=1}^{t} \) and the ‘spacelike subspace’ \( R^s \) has the basis \( \{ e_j \}_{j=t+1}^{t+s} \). This decomposition is orthogonal with respect to \( q \). We emphasise that the decomposition is not unique, but depends on the initial choice of basis. The pseudo-orthogonal group \( O(t, s) \) does not preserve this decomposition, but its maximal compact subgroup \( K := O(t) \times O(s) \) does. Furthermore, the maximal compact subgroup \( K^+ := K \cap SO^+(t, s) = SO(t) \times SO(s) \) of \( SO^+(t, s) \) also preserves the orientations of \( R^t \) and \( R^s \).

### 3.1.1 The Clifford algebra

The (real) Clifford algebra \( Cl_{t,s} \) associated to \( \mathbb{R}^{t,s} \) is the universal algebra generated by \( v, w \in \mathbb{R}^{t,s} \) subject to the relation

\[
 v \cdot w + w \cdot v = -2q(v, w),
\]

and can be constructed as

\[
 Cl_{t,s} = T(\mathbb{R}^{t,s})/J,
\]

where \( T(\mathbb{R}^{t,s}) = \bigoplus_k T^k(\mathbb{R}^{t,s}) \) is the tensor algebra over \( \mathbb{R}^{t,s} \), and \( J \) is the ideal generated by \( v \cdot v + q(v, v) \). The Clifford algebra is \( \mathbb{Z}_2 \)-graded, with even part \( Cl^0_{t,s} = \bigoplus_k T^{2k}(\mathbb{R}^{t,s})/J \) and odd part \( Cl^1_{t,s} = \bigoplus_k T^{2k+1}(\mathbb{R}^{t,s})/J \). The elements \( 1 \), \( e_{i_1} \cdots e_{i_k} \), for \( 1 \leq k \leq t+s \) and \( 1 \leq i_1 \leq \cdots \leq i_k \leq n \), form a linear basis for the Clifford algebra.
The complexification of the Clifford algebra only depends on the sum \( t + s \) and is denoted \( \text{Cl}_{t+s} := \text{Cl}_{t,s} \otimes \mathbb{R} \mathbb{C} \). This complexified Clifford algebra is given by \( \text{Cl}_{t+s} \simeq M_{2^m}(\mathbb{C}) \) if \( t + s = 2m \) and by \( \text{Cl}_{t+s} \simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) \) if \( t + s = 2m + 1 \), for some \( m \in \mathbb{N} \) [Bau81, Satz 1.3].

### 3.1.2 The spin group and its subgroups

A vector \( v \in \mathbb{R}^{t,s} \) is called a unit vector if \( q(v,v) = \pm 1 \). We define the spin group as the group whose elements are products of an even number of unit vectors:

\[
\text{Spin}(t,s) := \{ v_1 \cdots v_{2k} \in \text{Cl}_{t,s}^0 \mid q(v_j,v_j) = \pm 1, 1 \leq 2k \leq t + s, 1 \leq j \leq 2k \}.
\]

If \( v \in \mathbb{R}^{t,s} \) such that \( q(v,v) = \pm 1 \), then \( v \cdot v = \mp 1 \), and hence \( \text{Spin}(t,s) \) contains \( \pm 1 \). We note that all elements of \( \text{Spin}(t,s) \) are invertible, and the inverse of \( v_1 \cdots v_{2k} \) is given by \( \pm v_{2k} \cdots v_1 \) (the sign depending on the signs of \( q(v_j,v_j) = \pm 1 \)). Let the homomorphism \( \lambda : \text{Spin}(t,s) \to \text{SO}(t,s) \) be given by

\[
\lambda(u)w := u \cdot w \cdot u^{-1},
\]

for \( u \in \text{Spin}(t,s) \) and \( w \in \mathbb{R}^{t,s} \). For \( v \in \mathbb{R}^{t,s} \) we have \( v^{-1} = -q(v,v)^{-1}v \), and one uses the Clifford relation to check that

\[
\lambda(v)w = -w + 2 \frac{q(v,w)v}{q(v,v)}.
\]

Thus \( \lambda(v) \) implements a reflection \( \mathbb{R}^{t,s} \to \mathbb{R}^{t,s} \) (up to a sign). Furthermore, this equality also shows that \( \lambda(v) \) preserves \( q \). Hence \( \lambda(v) \in \text{O}(t,s) \), and since \( \lambda \) is multiplicative it follows that \( \lambda(u) \in \text{O}(t,s) \) for all \( u \in \text{Spin}(t,s) \). As \( u \in \text{Spin}(t,s) \) is a product of an even number of vectors, \( \lambda(u) \) consists of an even number of reflections, and hence \( \lambda(u) \in \text{SO}(t,s) \). Lastly, one can show that the homomorphism \( \lambda : \text{Spin}(t,s) \to \text{SO}(t,s) \) is surjective with kernel \( \{ \pm 1 \} \), and hence the spin group \( \text{Spin}(t,s) \) is a double cover of the special pseudo-orthogonal group \( \text{SO}(t,s) \). For more details, we refer to [LM89, §I.2].

We denote by \( \text{Spin}^+(t,s) \) the pre-image of \( \text{SO}^+(t,s) \) under \( \lambda \), which can be characterised as

\[
\text{Spin}^+(t,s) := \{ v_1 \cdots v_{2k} \in \text{Spin}(t,s) : \prod_{j=1}^{2k} q(v_j,v_j) = 1 \}.
\]

Similarly, we define the maximal compact subgroup \( \tilde{K}^+ \) as the pre-image of the maximal compact subgroup \( K^+ = \text{SO}(t) \times \text{SO}(s) \subset \text{SO}^+(t,s) \), which is given by

\[
\tilde{K}^+ \simeq \begin{cases} 
\text{Spin}(s), & \text{if } t = 1, \\
(\text{Spin}(t) \times \text{Spin}(s))/\mathbb{Z}_2, & \text{if } t > 1,
\end{cases}
\]

where \( \mathbb{Z}_2 \) is the subgroup of \( \text{Spin}(t) \times \text{Spin}(s) \) consisting of the elements \((1, 1)\) and \((-1, -1)\).
3.1.3 The volume element

Let \((e_j)_{j=1}^n\) be a pseudo-orthonormal basis for \(\mathbb{R}^{t,s}\). Then we define the volume element

\[
\omega := i^{n(n+1)/2-t}e_1 \cdots e_n.
\]

This definition is independent of the choice of basis, and it is straightforward to check that \(\omega^2 = 1\) and \(\nu \cdot \omega = (-1)^{n-1} \omega \cdot \nu\). In particular, \(\omega\) is central if \(n\) is odd. If \(n\) is even, then \(\omega \cdot a \cdot \omega = (-1)^{\deg a} a\) for any even or odd element \(a \in C\ell_{\deg a}\) (where \(\deg a = 0, 1\)).

The volume element gives rise to two idempotents \(P_\pm := \frac{1}{2}(1 \pm \omega)\) such that \(P_+ + P_- = 1\) and \(P_+ P_- = P_- P_+ = 0\). If \(n\) is odd, the Clifford algebra \(C\ell_{t,s}\) then decomposes as \(C\ell_{t,s} = C\ell_{t,s}^+ \oplus C\ell_{t,s}^-\), where \(C\ell_{t,s}^\pm := P_\pm C\ell_{t,s}\) are isomorphic subalgebras. If \(n = t + s\) is even, and if \(C\ell_{t,s}\) is represented on a vector space \(V\), then we have a decomposition \(V = V_+ \oplus V_-\) into the \(+1\) and \(-1\) eigenspaces of \(\omega\). In this case, multiplication by a unit vector \(e \in \mathbb{R}^{t,s}\) implements isomorphisms \(e: V_+ \rightarrow V_-\) and \(e: V_- \rightarrow V_+\).

3.1.4 Representations and Clifford multiplication

As mentioned in Section 3.1.1, we have isomorphisms \(\Phi_{t,s}: C\ell_{t+s} \rightarrow M_{2^m}(\mathbb{C})\) if \(t + s = 2m\) and \(\Phi_{t,s}: C\ell_{t+s} \rightarrow M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})\) if \(t + s = 2m + 1\). In the even case, when \(t + s = 2m\), we can use the isomorphism \(\Phi_{t,s}: C\ell_{t+s} \rightarrow M_{2^m}(\mathbb{C})\) to obtain a representation \(\Phi_{t,s}\) of the spin group \(\text{Spin}(t,s)\) on \(\Delta_{t,s} := \mathbb{C}^{2^m}\). We call \(\Delta_{t,s}\) the spinor module and the representation on \(\Delta_{t,s}\) the spinor representation. The spinor module decomposes into the eigenspaces of \(\Phi_{t,s}(e_1 \cdots e_{2m})\) as \(\Delta_{t,s} = \Delta_{t,s}^+ \oplus \Delta_{t,s}^-\). Each of these eigenspaces \(\Delta_{t,s}^\pm\) is invariant under the action of \(\text{Spin}(t,s)\) and forms in fact an irreducible representation of \(\text{Spin}(t,s)\).

In the odd case, when \(t + s = 2m + 1\), we instead obtain two representations \(\Phi^1_{t,s} := \text{proj} \circ \Phi_{t,s}\) on \(\Delta^1_{t,s} := \mathbb{C}^{2^m}\) for \(j = 1, 2\) (corresponding to the two summands \(M_{2^m}(\mathbb{C})\)). In this case, the spinor modules \(\Delta^1_{t,s}\) for \(j = 1, 2\) are two isomorphic irreducible representations of \(\text{Spin}(t,s)\).

Henceforth we shall use the representation on \(\Delta_{t,s}\) if \(t + s = 2m + 1\), and we shall write \(\Phi_{t,s}\) for either \(\Phi_{t,s}\) (if \(t + s = 2m\)) or \(\Phi^1_{t,s}\) (if \(t + s = 2m + 1\)).

In addition to the spinor representation, the spin group \(\text{Spin}(t,s)\) can also be represented on \(\mathbb{R}^{t,s}\) using the covering \(\lambda: \text{Spin}(t,s) \rightarrow SO(t,s)\) from Eq. (3.3). We define the Clifford multiplication \(\mu: \mathbb{R}^{t,s} \times \Delta_{t,s} \rightarrow \Delta_{t,s}\) as \(\mu(x, v) := x \cdot v := \Phi_{t,s}(x)v\). This Clifford multiplication is an intertwiner of representations of \(\text{Spin}(t,s)\):

\[
\mu(\lambda(a)x, \Phi_{t,s}(a)v) = \Phi_{t,s}(a)\mu(x, v)
\]

for all \(a \in \text{Spin}(t,s), x \in \mathbb{R}^{t,s}\), and \(v \in \Delta_{t,s}\).
3.1.5 Inner products

Using the standard basis of $\Delta_{t,s}$ we define a positive-definite inner product

$$\langle v, w \rangle := \sum_{j=1}^{2m} v_j w_j,$$

where $t + s$ equals $2m$ or $2m + 1$. This inner product is invariant under the double cover $\tilde{K}^+$ of the maximal compact subgroup $SO(t) \times SO(s) \subset SO^+(t,s)$ [Bau81, Satz 1.10]. With respect to Clifford multiplication by $x \in \mathbb{R}^n$ it satisfies the relation

$$\langle x \cdot v, w \rangle + \langle v, r(x) \cdot w \rangle = 0,$$

where $r: \mathbb{R}^n \to \mathbb{R}^n$ is the spacelike reflection given by

$$r: \sum_{j=1}^{t+s} x_j e_j \mapsto -\sum_{j=1}^{t} x_j e_j + \sum_{j=t+1}^{t+s} x_j e_j.$$

Consider the element $b := i^{t(t-1)/2} e_1 \cdots e_t$, which commutes with $\tilde{K}^+$, satisfies $b \cdot b = 1$ and is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$. We can then define an indefinite inner product on $\Delta_{t,s}$ by setting

$$\langle v, w \rangle_b := \langle v, bw \rangle.$$

This indefinite inner product is invariant under the (time- and space-)orientation-preservering spin group $\text{Spin}^+(t,s)$ [Bau81, Satz 1.12]. With respect to Clifford multiplication by $x \in \mathbb{R}^n$ it satisfies the relation

$$\langle x \cdot v, w \rangle_b + (-1)^t \langle v, x \cdot w \rangle_b = 0,$$

which means that Clifford multiplication by $i^t x$ is skew-symmetric with respect to the indefinite inner product $\langle \cdot, \cdot \rangle_b$.

3.2 Fibre bundles

In this section we recall some basic notions concerning fibre bundles. All manifolds are assumed to be smooth and all maps between them are also assumed to be smooth. Let $E \to M$ be a smooth fibre bundle. A local trivialisation of $E$ is denoted by $(U, h_U)$, where $U$ is an open neighbourhood in $M$ and the map $h_U: \pi^{-1}(U) \to U \times F$ is a diffeomorphism such that $\text{proj}_1 \circ h_U = \pi$. For two local trivialisations $(U, h_U)$ and $(V, h_V)$ for which $U \cap V \neq \emptyset$, we denote the corresponding transition function by $g_{vu} := h_V \circ h_U^{-1} \in C^\infty(U \cap V, \text{Diff}(F))$. For a more detailed introduction to fibre bundles, we refer to e.g. [KN63].
Definition 3.1. A fibre bundle $\pi: E \to M$ with fibre $V$ is called a vector bundle if $V$ is a vector space and if $h_u|_{\pi^{-1}(x)}: \pi^{-1}(x) \to x \times V$ is a linear isomorphism of vector spaces, for each local trivialisation $(U, h_u)$.

Let $\pi_1: E_1 \to M$ and $\pi_2: E_2 \to M$ be vector bundles. A vector bundle morphism is a map $\phi: E_1 \to E_2$ such that $\pi_2 \circ \phi = \pi_1$, and such that $\phi|_{\pi_1^{-1}(x)}: \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ is a linear map for each $x \in M$.

Let $\pi: E \to M$ be a vector bundle with fibre $V$. A fibre subbundle $\pi': E' \to M$ with fibre $V'$ is a vector subbundle if $V'$ is a vector space and there exist local trivialisations $\{(U, h_u)\}$ for $E$ such that $h_u|_{E'|_{h_u}} \simeq U \times \mathfrak{i}(V')$, where $\mathfrak{i}$ is an injective linear map $V' \to V$.

Definition 3.2. A hermitian metric on a vector bundle $E \to M$ is a fibrewise sesquilinear map $\langle \cdot, \cdot \rangle: E_x \times E_x \to \mathbb{C}$, depending continuously on $x \in M$, which is symmetric (i.e. $\langle e_1|e_2 \rangle = \overline{\langle e_2|e_1 \rangle}$) and non-degenerate (i.e. $\langle e_1|e_2 \rangle = 0$ for all $e_2 \in E_x$ implies that $e_1 = 0 \in E_x$). A hermitian metric is called positive-definite if $\langle e|e \rangle \geq 0$, and if $\langle e|e \rangle = 0$ implies that $e = 0$ (otherwise, the hermitian metric is called indefinite). A vector bundle with a hermitian metric is referred to as a hermitian vector bundle.

Consider the space of smooth compactly supported sections $\Gamma_c^\infty(E)$ of a vector bundle $E$. We will view this space of sections as a right module over the $*$-algebra $C_c^\infty(M)$. If $\phi: E_1 \to E_2$ is a vector bundle morphism, then

$$\phi_*: \Gamma_c^\infty(E_1) \to \Gamma_c^\infty(E_2), \quad (\phi_* s)(x) = \phi(s(x))$$

is a $C_c^\infty(M)$-module morphism.

For ease of notation, we will write in short

$$\mathcal{A} := C_c^\infty(M), \quad \mathcal{E} := \Gamma_c^\infty(E).$$

If $E$ has a hermitian metric $\langle \cdot, \cdot \rangle$, then $E$ has a hermitian structure $\langle \cdot, \cdot \rangle_\mathcal{A}: \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ (as defined in Definition 2.1) given for $e_1, e_2 \in \mathcal{E}$ by

$$\langle e_1|e_2 \rangle_\mathcal{A}(x) := \langle e_1(x)|e_2(x) \rangle.$$

Suppose that $M$ is compact. Then $\Gamma_c^\infty(E) = \Gamma_c^\infty(E)$ is a finitely generated projective $C_c^\infty(M)$-module, with pointwise addition and multiplication by $C_c^\infty(M)$. By the Serre-Swan theorem [Swa62], the assignment $E \mapsto \Gamma_c^\infty(E)$ on objects and the assignment $\phi \mapsto \phi_*$ on morphisms determines an equivalence between the category of smooth vector bundles over $M$ and the category of finitely generated projective modules over $C_c^\infty(M)$.

If $M$ is not compact, one needs to be careful to control the behaviour of the bundle ‘near infinity’, for instance by assuming that the bundle $E \to M$ is a restriction of a vector bundle over some compactification of $M$ (see e.g. the non-unital Serre-Swan theorem [Ren03, Theorem 8]).
We denote by $Ω^1(\mathcal{M})$ the differential one-forms on $\mathcal{M}$.

**Definition 3.3.** Let $E \to \mathcal{M}$ be a vector bundle. A connection $\nabla$ on $E$ is a map $\nabla: E \to E \otimes_A Ω^1(\mathcal{M})$ satisfying the Leibniz rule

$$\nabla(\alpha e) = \nabla(e)\alpha + e \otimes d\alpha,$$

for all $e \in E$ and $\alpha \in A$. If $E \to \mathcal{M}$ is a hermitian vector bundle, then the connection is called hermitian if

$$(\nabla e_1|e_2)_{Ω^1(\mathcal{M})} + (e_1|\nabla e_2)_{Ω^1(\mathcal{M})} = d(e_1|e_2)_A,$$

for all $e_1, e_2 \in E$, where the map $(\cdot|\cdot)_{Ω^1(\mathcal{M})}: E \times (E \otimes A Ω^1(\mathcal{M})) \to Ω^1(\mathcal{M})$ is defined as $(e_1|e_2 \otimes \alpha)_{Ω^1(\mathcal{M})} := (e_1|e_2)_A \alpha$. We then also define (with some abuse of notation) the map $(\cdot|\cdot)_{Ω^1(\mathcal{M})}: (E \otimes A Ω^1(\mathcal{M})) \times E \to Ω^1(\mathcal{M})$ as $e_1 \otimes \alpha|e_2)_{Ω^1(\mathcal{M})} := ((e_2|e_1 \otimes \alpha)_{Ω^1(\mathcal{M})})^*$.

### 3.2.1 Principal fibre bundles

In this section, we briefly recall the definition of a principal fibre bundle, and some basic results. We refer to [KN63, Chapter I] and [Ble81] for more details.

**Definition 3.4.** A principal fibre bundle $\mathcal{P}$ over $\mathcal{M}$ with structure group $G$ (or a principal $G$-bundle for short) consists of a fibre bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ equipped with a smooth right action of $G$ that acts freely and transitively on the fibres, such that for a local trivialisation $(U, h_U)$ of $\mathcal{P}$, the map $h_U$ intertwines the right action of $G$ on $\mathcal{P}|_U$ with the natural right action of $G$ on $U \times G$.

A principal $G$-bundle morphism (over id: $\mathcal{M} \to \mathcal{M}$) between principal $G$-bundles $\mathcal{P}$ and $\mathcal{Q}$ is a smooth map $\phi: \mathcal{P} \to \mathcal{Q}$ such that $\pi_\mathcal{Q}(\phi(p)) = \pi_\mathcal{P}(p)$ and $\phi(pg) = \phi(p)g$ for all $p \in \mathcal{P}$ and $g \in G$. A principal $G$-bundle automorphism is an invertible principal $G$-bundle morphism $\phi: \mathcal{P} \to \mathcal{P}$.

On a compact manifold, one can construct a principal $G$-bundle $\mathcal{P}$ as soon as one knows its (G-valued) transition functions.

**Theorem 3.5** (Reconstruction theorem, [KN63, Chapter I, Proposition 5.2.]). Let $\mathcal{M}$ be a compact manifold, $G$ a Lie-group, and $\{U_i\}_{i \in I}$ an open covering of $\mathcal{M}$. Suppose that for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there is a smooth map $g_{ij}: U_i \cap U_j \to G$ such that $g_{ij}(x)g_{jk}(x)g_{kl}(x) = e$ for all $x \in U_i \cap U_j \cap U_k$. Then there exists a unique principal $G$-bundle $\mathcal{P}$ over $\mathcal{M}$ with the $\{U_i\}$ as trivialising neighbourhoods and the $g_{ij}$ as transition functions.

**Definition 3.6.** Let $\{(U_i, h_i)\}$ be a set of local trivialisations of $\mathcal{P}$ which cover $\mathcal{M}$. A connection $\omega$ on $\mathcal{P}$ is a set of local $g$-valued 1-forms $\omega_i \in Ω^1(U_i, g)$ such that

$$\omega_j = g_{ij}^{-1} d g_{ij} + g_{ij}^{-1} \omega_i g_{ij} \quad (3.7)$$
for i, j such that \( \mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset \).

**Definition 3.7.** Given an action \( \rho \) of \( G \) on a finite-dimensional vector space \( V \), we define the associated vector bundle \( \mathcal{P} \times_{\rho} V \) (or \( \mathcal{P} \times_{G} V \)) as the quotient of the product manifold \( \mathcal{P} \times V \) with respect to the equivalence relation given by \( (pg, v) \sim (p, \rho(g)v) \).

Thus, an element in \( \mathcal{P} \times_{\rho} F \) is given by an equivalence class \([p, v]\), where \( p \in \mathcal{P} \) and \( v \in V \), such that \([pg, v] = [p, \rho(g)v]\) for all \( g \in G \).

**Structure group**

Let \( \mathcal{E} \) be a vector bundle with fibre \( V \). A set of transition functions \( \{(\mathcal{U}_i, g_{ij})\} \) on \( \mathcal{E} \) (where \( \{\mathcal{U}_i\} \) is an open covering of \( M \)) is called a \( G \)-atlas if each transition function takes values in \( G \subset \text{GL}(V) \). If \( \mathcal{E} \) admits a \( G \)-atlas, then we say that \( \mathcal{E} \) has structure group \( G \). Given two \( G \)-atlases \( \{(\mathcal{U}_i, g_{ij})\} \) and \( \{(\mathcal{U}_i, g'_{ij})\} \) (where, after taking a common refinement, we may assume without loss of generality that both atlases are given on the same open covering \( \{\mathcal{U}_i\} \)), we say that they are equivalent if there are functions \( g_i \in C^\infty(\mathcal{U}_i, G) \) such that (for all \( i, j \))

\[
g'_{ij}(x) = g_i(x)^{-1}g_{ij}(x)g_j(x),
\]

for all \( x \in \mathcal{U}_i \cap \mathcal{U}_j \).

Given a \( G \)-atlas \( \{(\mathcal{U}_i, g_{ij})\} \) on \( \mathcal{E} \), Theorem 3.5 constructs a unique principal \( G \)-bundle \( \mathcal{P} \), satisfying \( \mathcal{E} \cong \mathcal{P} \times_{G} V \), which only depends (up to isomorphism) on the equivalence class of the \( G \)-atlas. Conversely, a set of transition functions on \( \mathcal{P} \) uniquely determines an equivalence class of \( G \)-atlases on the associated bundle \( \mathcal{P} \times_{G} V \).

**Example 3.8.** Let \( \mathcal{E} \to M \) be a complex vector bundle with fibre \( C^N \) over a compact manifold \( M \). Then all \( U(N) \)-atlases on \( \mathcal{E} \) are equivalent. Hence there is a unique (up to isomorphism) principal \( U(N) \)-bundle \( \mathcal{P} \) such that \( \mathcal{E} \cong \mathcal{P} \times_{U(N)} C^N \).

**Definition 3.9 (Reduction of structure group).** Let \( \mathcal{P} \to M \) be a principal \( G \)-bundle, and let \( \phi: H \to G \) be a group homomorphism. A principal \( H \)-bundle \( \mathcal{Q} \to M \) is called a reduction of \( \mathcal{P} \) along \( \phi \) if there is a principal bundle morphism \( \tau: \mathcal{Q} \to \mathcal{P} \) (over \( \text{id}: M \to M \)), i.e. a bundle morphism (over \( \text{id}: M \to M \)) such that \( \tau(qh) = \tau(q)\phi(h) \) for all \( q \in \mathcal{Q}, h \in H \). Equivalently, \( \mathcal{Q} \) is a reduction of \( \mathcal{P} \) if

\[
\mathcal{Q} \times_{\phi} G \cong \mathcal{P}
\]
as principal \( G \)-bundles.

If \( \tau: \mathcal{Q} \to \mathcal{P} \) is such a reduction and \( \rho: G \to \text{GL}(V) \) is a finite-dimensional representation, then \( \mathcal{Q} \times_{\rho \circ \phi} V \) is isomorphic to \( \mathcal{P} \times_{\rho} V \). We stress that a reduction need not always exist, and if it exists, it need not be unique.

We emphasise that the group homomorphism \( \phi \) is not required to be injective, and hence the term reduction is somewhat misleading. If \( \phi \) is surjective, then the reduction \( \mathcal{Q} \) is more appropriately called a lift.
3.3 PSEUDO-RIEMANNIAN SPIN MANIFOLDS

In this section we will describe pseudo-Riemannian spin manifolds, and we show how the constructions of Clifford algebras and spinor modules from the first section extend to bundles over such manifolds. Our main reference is the book [Bau81, Ch. 2 & 3], but we have also used [BGM05, §2], and we refer the reader to these references for more details.

Throughout the remainder of this chapter, let \((M, g)\) be an \(n\)-dimensional time- and space-oriented pseudo-Riemannian manifold of signature \((t, s)\), where \(t\) is the number of time dimensions (for which \(g\) is negative-definite) and \(s\) is the number of spatial dimensions (for which \(g\) is positive-definite). We will assume that \(n = t + s \geq 3\).

There exists an orthogonal direct sum decomposition of the tangent bundle \(TM = E_t \oplus E_s\) into a ‘timelike’ subbundle \(E_t\) of dimension \(t\) and a ‘spacelike’ subbundle \(E_s\) of dimension \(s\), such that the metric \(g\) is negative-definite on \(E_t\) and positive-definite on \(E_s\) [Bau81, Satz 0.48]. We emphasise that such a decomposition is far from unique. Given a choice of decomposition \(TM = E_t \oplus E_s\), we have a timelike projection \(T: E_t \oplus E_s \to E_t\) onto the ‘purely timelike’ subbundle. This projection is orthogonal with respect to \(g\), which means that \(g(v - Tv, Tw) = 0\) for all \(v, w \in TM\). We then also have a spacelike reflection \(r := 1 - 2T\) which acts as \((-1) \oplus 1\) on \(E_t \oplus E_s\).

**Definition 3.10** (‘Wick rotation’ of the metric). Given an orthogonal decomposition \(TM = E_t \oplus E_s\) (or, equivalently, given a spacelike reflection \(r\), or a timelike projection \(T\)), we define the ‘Wick rotated’ metric \(g_r\) on \(TM\) by

\[
g_r(v, w) := g(rv, w).
\]

Since \(r(1 - T) = 1 - T\), we readily check that \(T\) is also an orthogonal projection with respect to the new metric \(g_r\). Furthermore, \(g_r\) is positive-definite, and hence \((M, g_r)\) is a Riemannian manifold. Alternatively, we can also write

\[
g(v, w) = g_r(rv, w) = g_r((1 - T)v, (1 - T)w) - g_r(Tv, Tw).
\]

**Example 3.11.** Consider the metric \(q\) on \(\mathbb{R}^{t,s}\) as defined in Eq. (3.1), and let \(r\) be the standard spacelike reflection given in Eq. (3.5). The Wick rotated metric \(q_r\) is then simply given as

\[
q_r \left( \sum_{j=1}^{t+s} v_j e_j, \sum_{j=1}^{t+s} w_j e_j \right) := \sum_{j=1}^{t+s} v_j w_j,
\]

which is just the standard inner product on the Euclidean space \(\mathbb{R}^{t+s}\).
3.3.1 Spin structures

Denote by $\mathcal{F}^+(M)$ the bundle of time- and space-oriented frames over $M$, which is a principal $SO^+(t, s)$-bundle. The tangent bundle $TM$ can be viewed as the associated vector bundle $TM = \mathcal{F}^+(M) \times_\rho \mathbb{R}^{t,s}$, where $\rho$ denotes the standard representation of $SO^+(t, s)$ on $\mathbb{R}^{t,s}$. Thus a vector $v \in TM$ is given by an equivalence class $[f, x]$ where $f \in \mathcal{F}^+(M)$ and $x \in \mathbb{R}^n$, such that $[fT, x] = [f, \rho(T)x]$ for all $T \in SO^+(t, s)$.

Recall from Eq. (3.3) the double cover $\lambda: \text{Spin}^+(t, s) \to SO^+(t, s)$.

**Definition 3.12.** A spin structure $(\text{Spin}^+(M), \Theta)$ on $M$ is given by a principal $\text{Spin}^+(M)$-bundle $\text{Spin}^+(M)$ over $M$ along with a principal bundle morphism $\Theta: \text{Spin}^+(M) \to \mathcal{F}^+(M)$ over $id: M \to M$, so the following diagram commutes.

$$
\begin{array}{ccc}
\text{Spin}^+(M) \times \text{Spin}^+(t, s) & \longrightarrow & \text{Spin}^+(M) \\
\downarrow \Theta \times \lambda & & \downarrow \Theta \\
\mathcal{F}^+(M) \times SO^+(t, s) & \longrightarrow & \mathcal{F}^+(M)
\end{array}
$$

If $M$ is equipped with a given spin structure, then $M$ is called a spin manifold.

A spin structure is a connected double cover of the frame bundle $\mathcal{F}^+(M)$ [Bau81, Satz 2.1]. A time- and space-oriented pseudo-Riemannian manifold is spin if and only if its second Stiefel-Whitney class vanishes [Bau81, Folgerung 2.1]. Hence every time- and space-oriented parallelisable manifold is spin (which includes 4-dimensional globally hyperbolic spacetimes). Furthermore, any open time- and space-oriented 4-dimensional manifold is spin if and only if it is parallelisable. The set of isomorphism classes of spin structures on $M$ is characterised by the first cohomology group $H^1(M, \mathbb{Z}_2)$ of $M$ with coefficients in $\mathbb{Z}_2$ [Bau81, Satz 2.6]. Hence any simply-connected pseudo-Riemannian manifold has at most one spin structure.

We assume that $M$ is a spin manifold with spin structure $\text{Spin}^+(M)$. We then obtain the (complex) spinor bundle as the associated vector bundle

$$S := \text{Spin}^+(M) \times_{\Phi_{t,s}} \Delta_{t,s},$$

where $\Delta_{t,s}$ is the standard representation space of the spin group $\text{Spin}(t, s)$ (see Section 3.1.4).

**Reduction of the spin structure**

Recall from Definition 3.9 the notions of reduction and lift of the structure group of a principal bundle. As mentioned at the start of this section, we have an orthogonal direct sum decomposition $TM = E_t \oplus E_s$, which is invariant under the maximal
compact subgroup $K^+ := \text{SO}(t) \times \text{SO}(s)$ of $\text{SO}^+(t,s)$. This decomposition induces a reduction of the frame bundle $\mathcal{F}^+(M)$ to the principal $K^+$-bundle

$$P := \{(e_1, \ldots, e_n) \in \mathcal{F}^+(M) : (e_1, \ldots, e_l) \in E_t, (e_{l+1}, \ldots, e_n) \in E_s\},$$

and we have $\mathcal{F}^+(M) \simeq P \times_{K^+} \text{SO}^+(t,s)$. Given a spin structure $\text{Spin}^+(M)$ with the double cover $\Theta: \text{Spin}^+(M) \to \mathcal{F}^+(M)$, there exists a lift of $P$ to a principal $K^+$-bundle $0 := \Theta^{-1}(P)$, and this lift $Q$ is a reduction of $\text{Spin}^+(M)$ to the maximal compact subgroup $\tilde{K}^+$: $\text{Spin}^+(M) \simeq Q \times_{\tilde{K}^+} \text{Spin}^+(t,s)$ [Bau81, Lemma 2.1]. We summarise these constructions in the following diagram.

\[\begin{array}{ccc}
P & \xrightarrow{\text{reduction}} & \text{Spin}^+(M) \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\text{lift}} & \mathcal{F}^+(M)
\end{array}\]

All four arrows are compatible with the group actions and preserve fibres over a point in $M$. Using the reduction $Q$ of the spin structure $\text{Spin}^+(M)$, the spinor bundle can then also be written in the form $S = Q \times_{\tilde{K}^+} \Delta_{t,s}$ (see [Bau81, §3.3.1]).

### 3.3.2 The Clifford representation

Recall that the tangent bundle $TM$ can be written as the associated vector bundle $TM = \mathcal{F}^+(M) \times_\rho \mathbb{R}^{t,s}$, where $\mathcal{F}^+(M)$ is the time- and space-oriented frame bundle, and $\rho$ denotes the standard representation of $\text{SO}^+(t,s)$ on $\mathbb{R}^{t,s}$. We define the **Clifford bundle** as the associated bundle

$$\text{Cl}(TM, g) := \mathcal{F}^+(M) \times_\rho \text{Cl}_{t,s},$$

where the representation $\rho$ of $\text{SO}^+(t,s)$ on $\mathbb{R}^{t,s}$ is extended to a representation on $\text{Cl}_{t,s}$ by setting $\rho(T)(v_1 \cdots v_k) := (Tv_1) \cdots (Tv_k)$. Given the spin structure $\text{Spin}^+(M)$, we can write $\mathcal{F}^+(M) = \text{Spin}^+(M) \times_\lambda \text{SO}^+(t,s)$, where $\lambda$ is the double cover $\text{Spin}^+(t,s) \to \text{SO}^+(t,s)$. We can then also view the Clifford bundle as an associated bundle of $\text{Spin}^+(M)$ via

$$\text{Cl}(TM, g) = \text{Spin}^+(M) \times_\lambda \text{SO}^+(t,s) \times_\rho \text{Cl}_{t,s} = \text{Spin}^+(M) \times_{\text{Ad}} \text{Cl}_{t,s},$$

where $\text{Ad} = \rho \circ \lambda$ is given by $\text{Ad}_u(a) = u \cdot a \cdot u^{-1}$ for all $u \in \text{Spin}^+(t,s)$ and $a \in \text{Cl}_{t,s}$. The complexified Clifford bundle is independent of the signature of $g$ and is denoted $\text{Cl}(TM) = \text{Cl}(TM, g) \otimes \mathbb{C} = \mathcal{F}^+(M) \times_\rho \text{Cl}_{t+i,s}$.

Using the natural inclusion $\iota: \mathbb{R}^{t,s} \hookrightarrow \text{Cl}_{t,s}$, we can define the Clifford representation $\gamma: TM \hookrightarrow \text{Cl}(TM, g)$ by

$$\gamma([f, x]) := [f, \iota(x)],$$
where \( f \in \mathcal{F}^+(M) \) and \( x \in \mathbb{R}^{t,s} \) determine \([f,x] \in TM = \mathcal{F}^+(M) \times_\rho \mathbb{R}^{t,s}\). This Clifford representation inherits the relation (3.2) of the Clifford algebra \( \text{Cl}_{t,s} \), so we have \( \gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2g(v,w) \) for all \( v, w \in TM \). We shall denote by \( h \) the map \( T^*M \rightarrow TM \) which sends \( \alpha \in T^*M \) to its dual in \( TM \) with respect to the metric \( g \):

\[
\begin{align*}
\h(\alpha) &= v, & \iff \alpha(w) &= g(v,w) \quad \text{for all } w \in TM.
\end{align*}
\]

We denote by \( c \) the Clifford multiplication \( T^*M \otimes S \rightarrow S \) given by

\[
c(\alpha \otimes \psi) := \gamma(\h(\alpha))\psi.
\]

We view the tangent bundle and the spinor bundle as the associated vector bundles \( TM = \mathcal{F}^+(M) \times_\rho \mathbb{R}^{t,s} \) and \( S = \text{Spin}^+(M) \times_{\hat{\phi}_{t,s}} \Delta_{t,s} \). The Clifford multiplication can then be written as

\[
c(h^{-1}[\Theta(\hat{s}), x] \otimes [\hat{s}, v]) = [\hat{s}, \mu(x,v)],
\]

for \( \hat{s} \in \text{Spin}^+(M) \), \( x \in \mathbb{R}^{t,s} \), and \( v \in \Delta_{t,s} \). Here \( \Theta \colon \text{Spin}^+(M) \rightarrow \mathcal{F}^+(M) \) denotes the double cover, and \( \mu \) denotes the Clifford multiplication \( \mathbb{R}^{t,s} \times \Delta_{t,s} \rightarrow \Delta_{t,s} \) defined in Section 3.1.4.

**Definition 3.13** (‘Wick rotation’ of the Clifford representation). Given an orthogonal decomposition \( TM = E_t \oplus E_s \) (or, equivalently, given a spacelike reflection \( r \), or a timelike projection \( T \)), we define the ‘Wick rotated’ Clifford representations \( \gamma_{\pm} \colon TM \rightarrow \text{Cl}(TM) \) by

\[
\gamma_{\pm}(v) := \pm i\gamma(v_t) + \gamma(v_s)
\]

for any \( v = v_t + v_s \in E_t \oplus E_s = TM \).

Since \( \gamma_{\pm}(v)^2 = -\gamma(v_t)^2 + \gamma(v_s)^2 = g(v_t,v_t) - g(v_s,v_s) = -g_r(v,v) \), we see that both the Wick rotated Clifford representations \( \gamma_{\pm} \) are associated to the Wick rotated metric \( g_r \) from Definition 3.10.

We note that the pseudo-Riemannian Clifford representation \( \gamma \) can also be related to the Riemannian metric \( g_r \). Indeed, we have the equality

\[
\gamma(v)\gamma(w)^* + \gamma(w)^*\gamma(v) = 2g_r(v,w),
\]

which follows from the calculation

\[
\begin{align*}
\gamma(v)\gamma(w)^* &= -\gamma_{\pm}(v - Tv)\gamma_{\pm}(w - Tw) - \gamma_{\pm}(Tv)\gamma_{\pm}(Tw) \\
&\quad \mp i\gamma_{\pm}(v - Tv)\gamma_{\pm}(Tw) \pm i\gamma_{\pm}(Tv)\gamma_{\pm}(w - Tw).
\end{align*}
\]


3.3.3 The spinors

On the vector space \( \Delta_{t,s} \) there exist two hermitian scalar products: the positive-definite \( \langle \cdot, \cdot \rangle \), which is invariant under the action of \( \tilde{K}^+ \), and the indefinite \( \langle \cdot, \cdot \rangle_b \), which is invariant under \( \text{Spin}^+(t,s) \) (see Section 3.1.5). These scalar products give rise to hermitian metrics on the spinor bundle, which are constructed as follows (see [Bau81, §3.3.1] for more details). Recall from Section 3.3.1 that we can write the spinor bundle as the associated vector bundle \( S = 0 \times_{\tilde{K}^+} \Delta_{t,s} \), where \( 0 \) is a reduction of the spin structure to the maximal compact subgroup \( \tilde{K}^+ \) of \( \text{Spin}^+(M) \). The positive-definite scalar product \( \langle \cdot, \cdot \rangle \) on \( \Delta_{t,s} \) induces a positive-definite hermitian metric on the spinor bundle \( S \) by setting

\[
\langle \cdot \rangle_q(x): S_x \times S_x \to \mathbb{C}, \quad (\phi_x | \psi_x)_q(x) := \langle u_x, v_x \rangle,
\]

for \( x \in M \), where \( u_x \) and \( v_x \) are determined by \( \phi_x = [\hat{q}(x), u_x] \) and \( \psi_x = [\hat{q}(x), v_x] \), for some \( \hat{q}(x) \in 0_x \). This hermitian metric is well-defined thanks to the \( \tilde{K}^+ \)-invariance of \( \langle \cdot, \cdot \rangle \), but we emphasize that it depends on the choice of the reduction \( Q \) of the spin structure \( \text{Spin}^+(M) \) (i.e., on the choice of the decomposition \( TM = E_t \oplus E_s \)). The indefinite scalar product \( \langle \cdot, \cdot \rangle_b \) on \( \Delta_{t,s} \) also induces an indefinite hermitian metric on \( S = \text{Spin}^+(M) \times_{\tilde{K}^+} \Delta_{t,s} \) by setting

\[
\langle \cdot \rangle_b(x): S_x \times S_x \to \mathbb{C}, \quad (\phi_x | \psi_x)_b(x) := \langle u_x, v_x \rangle_b,
\]

where \( u_x \) and \( v_x \) are determined by \( \phi_x = [\hat{s}(x), u_x] \) and \( \psi_x = [\hat{s}(x), v_x] \), for some \( \hat{s}(x) \in \text{Spin}^+(M)_x \). We then obtain a positive-definite inner product \( \langle \cdot \rangle_q \) and an indefinite inner product \( \langle \cdot \rangle_b \) on the smooth, compactly supported sections \( \Gamma^\infty_c(S) \) by setting

\[
\langle \psi_1 | \psi_2 \rangle_q := \int_M (\psi_1 | \psi_2)_q(x) d\text{vol}_g, \quad \langle \psi_1 | \psi_2 \rangle_b := \int_M (\psi_1 | \psi_2)_b(x) d\text{vol}_g,
\]

where we have abbreviated \( (\psi_1 | \psi_2)_q(x) = (\psi_1(x) | \psi_2(x))_q(x) \) (and similarly for \( (\psi_1 | \psi_2)_b(x) \)), and where \( d\text{vol}_g \) is the canonical volume form determined by the metric. By [Bau81, Lemma 3.4], the indefinite inner product \( \langle \cdot | \cdot \rangle_b \) on \( \Gamma^\infty_c(S) \) is non-degenerate. For a local vector field \( X \) and local spinors \( \phi, \psi \) we have the relations

\[
\langle \gamma(X) \phi | \psi \rangle_q(x) + (\phi | \gamma(X) \psi)_q(x) = 0,
\]

\[
\langle \gamma(X) \phi | \psi \rangle_b(x) + (-1)^4 (\phi | \gamma(X) \psi)_b(x) = 0,
\]

which follow from Eqs. (3.4) and (3.6). Hence \( \iota^4 \gamma(X) \) is skew-symmetric with respect to the indefinite inner product. The completion of \( \Gamma^\infty_c(S) \) with respect to the positive-definite inner product \( \langle \cdot \rangle_q \) yields the Hilbert space of square integrable sections \( L^2(S)_q \). We will often simply write \( L^2(S) \) instead of \( L^2(S)_q \).
Let \( g_r \) be the Riemannian metric obtained from the pseudo-Riemannian metric \( g \) and the decomposition \( TM = E_t \oplus E_s \) (as in Definition 3.10). This yields a (positive-definite) hermitian metric on the vector bundle \( TM \otimes S \) by setting
\[
\langle X_1 \otimes \psi_1 | X_2 \otimes \psi_2 \rangle_0 := g_r(X_1, X_2) \langle \psi_1 | \psi_2 \rangle_0,
\]
and the completion of \( \Gamma_c^\infty(TM \otimes S) \) with respect to the corresponding inner product is denoted by \( L^2(TM \otimes S)_0 \) (or simply \( L^2(TM \otimes S) \)). The Clifford multiplication \( c: \Gamma_c^\infty(TM \otimes S) \to \Gamma_c^\infty(S) \) is continuous and extends to \( c: L^2(TM \otimes S)_0 \to L^2(S)_0 \).

The two inner products constructed above are related as follows. Consider the operator \( \mathcal{J}_M := 1 \times b \) on \( S = 0 \times \mathbb{R}^+ \Delta_{L,s} \), where \( b = i^{(t-1)/2} e_1 \cdots e_t \) (for the standard basis \( \{ e_j \} \) of \( \mathbb{R}^{1,s} \)) was defined in Section 3.1.5. Since \( b \) commutes with \( \tilde{K}^+ \), this operator \( \mathcal{J}_M \) is well-defined, and locally it is explicitly given by
\[
\mathcal{J}_M := i^{(t-1)/2} \gamma(e_1) \cdots \gamma(e_t),
\]
where \( \gamma \) denotes the Clifford action \( TM \hookrightarrow \text{Cl}(TM) \), and \( \{ e_j \} \) is a local orthonormal frame corresponding to the decomposition \( TM = E_t \oplus E_s \).

**Lemma 3.14** ([Bau81, Lemma 3.5]). The map \( \mathcal{J}_M: \Gamma_c^\infty(S) \to \Gamma_c^\infty(S) \) is bijective, linear, bounded, and formally self-adjoint with respect to \( \langle \cdot | \cdot \rangle_0 \). Furthermore, we have \( \mathcal{J}_M^2 = 1 \) and we have the relation
\[
\langle \phi | \psi \rangle = \langle \phi | \mathcal{J}_M \psi \rangle_0
\]
for all \( \phi, \psi \in \Gamma_c^\infty(S) \).

In addition, note that \( \mathcal{J}_M \) is related to the spacelike reflection \( r \) via
\[
\mathcal{J}_M \gamma(v) \mathcal{J}_M = (-1)^t \gamma(rv).
\]
It follows in particular from the lemma that the indefinite inner product \( \langle \cdot | \cdot \rangle \) is bounded with respect to the norm topology of \( \Gamma_c^\infty(S) \times \Gamma_c^\infty(S) \), and hence extends to \( L^2(S) \times L^2(S) \). Thus we have a bijective, bounded, self-adjoint (w.r.t. \( \langle \cdot | \cdot \rangle_0 \)) linear operator \( \mathcal{J}_M: L^2(S) \to L^2(S) \) such that \( \mathcal{J}_M^2 = 1 \) and \( \langle \phi | \psi \rangle = \langle \phi | \mathcal{J}_M \psi \rangle_0 \) for all \( \phi, \psi \in L^2(S) \). The vector space \( L^2(S) \) can then be viewed as a Krein space equipped with the inner products \( \langle \cdot | \cdot \rangle_0 \) and \( \langle \cdot | \cdot \rangle \) and the fundamental symmetry \( \mathcal{J}_M \) [Bau81, Satz 3.16]. The indefinite inner product \( \langle \cdot | \cdot \rangle \) is often referred to as the Krein inner product, and an operator on \( L^2(S) \) is called Krein-symmetric or Krein-self-adjoint if it is symmetric or self-adjoint with respect to this Krein inner product. For a detailed introduction to Krein spaces, we refer to [Bog74].

### 3.3.4 The grading operator

Given a local orthonormal frame \( \{ e_j \} \), we define the grading operator \( \Gamma_M \) on the Hilbert space \( L^2(S) \) as
\[
\Gamma_M := i^{-t+n(n+1)/2} \gamma(e_1) \cdots \gamma(e_n).
\]
It is straightforward to check that
\[ \Gamma^*_M = \Gamma_M, \quad \partial_M \Gamma_M \partial_M = (-1)^t \Gamma^*_M, \quad \Gamma^2_M = 1. \]

The grading operator \( \Gamma_M \) has the same properties as the volume element \( \omega \) from Section 3.1.3. In particular, if \( n \) is odd, the Clifford bundle \( \text{Cl}(TM, g) \) decomposes as \( \text{Cl}(TM, g)^+ \oplus \text{Cl}(TM, g)^- \), where \( \text{Cl}(TM, g)^\pm \) are isomorphic subbundles. If \( n = t + s \) is even, then we have a decomposition of the spinor bundle \( S = S_+ \oplus S_- \) into the \( +1 \) and \( -1 \) eigenspaces of \( \Gamma_M \), which are isomorphic subbundles.

### 3.3.5 The spin connection

The Levi-Civita connection \( \nabla \) on \((M, g)\) lifts uniquely to a connection \( \nabla^S \) on the spinor bundle \( S = \text{Spin}^+(M) \times \Phi_{\Delta_{t,s}} \). Locally, we can write a spinor \( \psi \in \Gamma^\infty c(S) \) as the equivalence class \([\hat{s}, \sigma]\), where \( \hat{s} \) is a local section of \( \text{Spin}^+(M) \) and \( \sigma \) is a local function with values in \( \Delta_{t,s} \). The double cover \( \Theta: \text{Spin}^+(M) \to F^+(M) \) then yields a local (pseudo-)orthonormal frame \( \Theta(\hat{s}) = (e_1, \ldots, e_n) \), such that
\[
g(e_i, e_j) = \delta_{ij} \kappa(j), \quad \kappa(j) = \begin{cases} -1 & j = 1, \ldots, t; \\ 1 & j = t + 1, \ldots, n. \end{cases}
\]

Using this frame, the spin connection locally takes the form (see \[Bau81, \text{Satz 3.2}\] and \[BGMo5, \text{Eq.}(2.5)\])
\[
\nabla_X^S \psi = \left[ \hat{s}, X(\sigma) + \frac{1}{2} \sum_{j < k} \kappa(j) \kappa(k) g(\nabla_X e_j, e_k) \gamma(e_j) \gamma(e_k) \sigma \right],
\]
for a local vector field \( X = \sum_j X^j e_j \in \Gamma^\infty c(TM) \). If \( n \) is even, then \( \nabla^S \) preserves the decomposition \( S = S_+ \oplus S_- \). With respect to the non-degenerate indefinite inner product \( \langle \cdot | \cdot \rangle \) on \( L^2(S) \) (as defined in Section 3.3.3) this spin connection \( \nabla^S \) satisfies \[Bau81, \text{Lemma 3.6}\]
\[
\langle \nabla^S \psi_1 | \psi_2 \rangle + \langle \psi_1 | \nabla^S \psi_2 \rangle = d \langle \psi_1 | \psi_2 \rangle.
\]

Moreover, with respect to the Clifford representation it satisfies the Leibniz rule
\[
\nabla_X^S (\gamma(Y) \psi) = \gamma(\nabla_X Y) \psi + \gamma(Y) \nabla_X^S \psi, \quad (3.11)
\]
for all vector fields \( X, Y \) and for all spinor fields \( \psi \).
3.4 The Dirac Operator and Its Wick Rotations

We are now ready to define the Dirac operator on a (time- and space-oriented) pseudo-Riemannian spin manifold. This operator is constructed on $\Gamma_c^\infty(S)$ as the composition of the spin connection and the Clifford multiplication:

$$\mathcal{D}: \Gamma_c^\infty(S) \xrightarrow{\nabla^S} \Gamma_c^\infty(T^*M \otimes S) \xrightarrow{\zeta} \Gamma_c^\infty(S).$$

Let $\{\theta^i\}_{i=1}^n$ be the basis of $T^*M$ dual to $\{e_i\}_{i=1}^n$, so that $\theta^i(e_j) = \delta^i_j$. For the dual map $h: T^*M \to TM$ we then see that $h(\theta^i) = \kappa(i)e_i$. In terms of the local frame $\{e_i\}$, we can then write the Dirac operator as

$$\mathcal{D} := c \circ \nabla^S = \sum_{j=1}^n \kappa(j)\gamma(e_j)\nabla^S_{e_j}.$$

We will view the Dirac operator $\mathcal{D} := c \circ \nabla^S$ as an unbounded operator on the Hilbert space $L^2(S)$ with initial domain $\Gamma_c^\infty(S)$.

**Proposition 3.15** ([Bau81, Satz 3.17]). The formal adjoint $\mathcal{D}^\ast$ is locally of the form

$$\mathcal{D}^\ast = \sum_{j=1}^n \gamma(e_j)\partial_M \nabla^S_{e_j} \partial_M = \sum_{j=1}^n \left(\gamma(e_j)\nabla^S_{e_j} + \gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M]\right).$$

Locally we can write $\partial_M = i^{(t-1)/2}\gamma(e_1) \cdots \gamma(e_t)$, and using the relation (3.11) we then find

$$[\nabla^S_{e_j}, \partial_M] = i^{(t-1)/2} \sum_{k=1}^t \gamma(e_1) \cdots \gamma(\nabla_{e_j} e_k) \cdots \gamma(e_t).$$

Each ‘gamma matrix’ $\gamma(e_j)$ has norm 1, and therefore the size of $\gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M]$ is determined by the size of the connection coefficients arising from $\nabla_{e_j} e_k$ (where $1 \leq k \leq t$). While these connection coefficients are smooth, they need not be globally bounded, and hence the term $\gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M]$ could in general be unbounded.

As in Section 2.6.1, we consider the real and imaginary parts $\text{Re}\, \mathcal{D} := \frac{1}{2}(\mathcal{D} + \mathcal{D}^\ast)$ and $\text{Im}\, \mathcal{D} := -\frac{i}{2}(\mathcal{D} - \mathcal{D}^\ast)$, and the ‘Wick rotations’ $\mathcal{D}_\pm := \text{Re}\, \mathcal{D} \pm \text{Im}\, \mathcal{D}$. For the real and imaginary parts of $\mathcal{D}$ we then find

$$\text{Re}\, \mathcal{D} = \sum_{j=t+1}^n \gamma(e_j)\nabla^S_{e_j} + \frac{i}{2} \sum_{j=1}^n \gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M],$$

$$\text{Im}\, \mathcal{D} = i \sum_{j=1}^t \gamma(e_j)\nabla^S_{e_j} + \frac{i}{2} \sum_{j=1}^n \gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M].$$

By construction these operators are symmetric, but let us have a closer look how this symmetry comes about term by term. For each $j$, the term $\gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M]$
is a smooth endomorphism, which is symmetric if \( j \leq t \) and anti-symmetric if \( j > t \). For \( j > t \) however one checks that the sum \( \gamma(e_j)\nabla^S_{e_j} + \frac{1}{2}\gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M] \) is symmetric, and thus \( \text{Re}\,D \) is indeed a symmetric operator (as it should be). The same reasoning applies to \( \text{Im}\,D \). For the Wick rotations of \( D \) we then obtain the local formula

\[
D_\pm = \pm i \sum_{j=1}^{t} \gamma(e_j)\nabla^S_{e_j} + \sum_{j=t+1}^{n} \gamma(e_j)\nabla^S_{e_j} + \frac{1 \pm i}{2} \sum_{j=1}^{n} \gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M]
\]

where we recall the Wick rotated Clifford representation \( \gamma_\pm \) from Definition 3.13. Thus, we see that the principal symbols of the Wick rotations \( D_\pm \) are given by the Wick rotations of the principal symbol of \( D \). We point out however that Wick rotating the operator \( D \) does more than just Wick rotating its principal symbol; it also adds the smooth endomorphism \( \frac{1 \pm i}{2} \sum_{j=1}^{n} \gamma(e_j)\partial_M[\nabla^S_{e_j}, \partial_M] \). We will see in Section 5.3.1 that this endomorphism vanishes identically in the special case of a Lorentzian manifold with 'parallel time'.

In the remainder of this section we will make the following basic assumption:

**Assumption 3.16.** Let \((M, g)\) be a time- and space-oriented pseudo-Riemannian spin manifold of signature \((t, s)\) and dimension \(n = t + s\). Let \( r \) be a spacelike reflection, such that the associated Riemannian metric \( g_r \) is complete.

**Theorem 3.17 ([Bau81, Satz 3.19]).** Let \((M, g, r)\) be as in Assumption 3.16. Then

1) the operators \( \text{Re}\,D \) and \( \text{Im}\,D \) are essentially self-adjoint on \( \Gamma_c^\infty(S) \) with respect to \( \langle \cdot | \cdot \rangle_Q \); and

2) the operator \( i^3D \) is essentially Krein-self-adjoint on \( \Gamma_c^\infty(S) \) with respect to \( \langle \cdot | \cdot \rangle \).

In fact, under the same assumptions it also follows that the Wick rotations \( D_\pm \) give rise to spectral triples over \( M \).

**Proposition 3.18.** Let \((M, g, r)\) be as in Assumption 3.16. Then the Wick rotations of \( D \) yield spectral triples \((C_c^\infty(M), L^2(S), D_\pm)\).

**Proof.** Since \( D_\pm \) are symmetric differential operators with bounded propagation speed, their essential self-adjointness follows from the completeness of the Riemannian metric \( g_r \) (see e.g. [HRoo, Proposition 10.2.11]). Commutators of \( D_\pm \) with functions \( f \in C_c^\infty(M) \) are bounded because \( D_\pm \) is a first-order differential operator, whose coefficients are smooth and hence bounded on any compact set. For a vector \( v \in TM \), the principal symbol of \( D_\pm \) is given by \( i\gamma_\pm(v) \). Since the square of the principal symbol equals the positive-definite metric \( g_r(v, v) \), this implies that \( D_\pm \) is elliptic, and hence it has locally compact resolvent (see e.g. [HRoo, Proposition 10.5.2]). Thus we indeed have spectral triples \((C_c^\infty(M), L^2(S), D_\pm)\). □
Spectral triples provide a way to extend Riemannian geometry to noncommutative spaces [Con94], while retaining the connection to the underlying topology via K-homology [HR00]. In this chapter, which is based on joint work with Mario Paschke and Adam Rennie [DPR13], we provide a definition of pseudo-Riemannian spectral triples, enabling a noncommutative analogue of pseudo-Riemannian geometry.

Our main result is Theorem 4.9, which states that one can associate two spectral triples to a pseudo-Riemannian spectral triple via the procedure of ‘Wick rotation’ (described in Section 2.6.1). Under additional assumptions, the process of Wick rotating is shown to preserve spectral dimension, smoothness and integrability, as we define them. Thus one obtains a K-homology class and the tools to compute index pairings using the local index formula. Since the most important Lorentzian manifolds are non-compact, we have taken care to ensure that our definitions are consistent with the non-unital version of the local index formula, as proved in [CGRS14].

In Section 4.1 we present our definition of pseudo-Riemannian spectral triples, as well as definitions of smoothness and summability. Subsequently, we prove our main theorem in Section 4.2, showing that we can employ the ‘Wick rotations’ to obtain two spectral triples from a pseudo-Riemannian spectral triple. In Sections 4.3 to 4.5 we discuss several examples. In the last section we specialise our definition to Lorentz-type spectral triples, and give a simple index-theoretic result.

4.1 THE DEFINITION

Let \( \mathcal{D} \) be a closed operator on a Hilbert space \( \mathcal{H} \) such that \( \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* \) is dense. As in Section 2.6, we consider its Wick rotations

\[
\mathcal{D}_{\pm} := \text{Re} \mathcal{D} \pm i \text{Im} \mathcal{D} = \frac{1}{2} (\mathcal{D} + \mathcal{D}^*) \mp i \frac{1}{2} (\mathcal{D} - \mathcal{D}^*).
\]

The squares of the Wick rotations are given by

\[
\mathcal{D}_{\pm}^2 = (\text{Re} \mathcal{D})^2 + (\text{Im} \mathcal{D})^2 \pm \{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\} = \frac{1}{2} (\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}) \mp i \frac{1}{2} (\mathcal{D}^2 - \mathcal{D}^*2).
\]
We shall write $\mathcal{D}_\pm^2 = \langle \mathcal{D} \rangle^2 \pm \mathcal{R}_\mathcal{D}$, where we define
\[
\langle \mathcal{D} \rangle^2 := (\text{Re} \mathcal{D})^2 + (\text{Im} \mathcal{D})^2 = \frac{1}{2} (\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}) ,
\]
\[
\mathcal{R}_\mathcal{D} := \{ \text{Re} \mathcal{D}, \text{Im} \mathcal{D} \} = -\frac{i}{2} (\mathcal{D}^2 - \mathcal{D}^{*2}) .
\]

We define a smooth subspace of the Hilbert space by $\mathcal{H}_\infty = \bigcap_{k \geq 0} \text{Dom}\langle \mathcal{D} \rangle^k$. One of our main assumptions in this chapter will be that $\langle \mathcal{D} \rangle^2$ is self-adjoint. We can then consider the set of regular pseudo-differential operators $\text{OP}^*(\langle \mathcal{D} \rangle)$ as defined in Section 2.5.

**Definition 4.1.** A pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by

- a separable Hilbert space $\mathcal{H}$;
- a $*$-algebra $\mathcal{A}$ with a representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$;
- a densely defined, closed operator $\mathcal{D}$: $\text{Dom} \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ such that
  1) $\text{Dom} \mathcal{D}^* \cap \text{Dom} \mathcal{D}^*$ is dense in $\mathcal{H}$, and $\langle \mathcal{D} \rangle^2$ is essentially self-adjoint on this domain;
  2a) $\mathcal{R}_\mathcal{D}: \mathcal{H}_\infty \to \mathcal{H}_\infty$, and $[(\mathcal{D})^2, \mathcal{R}_\mathcal{D}] \in \text{OP}^2(\langle \mathcal{D} \rangle)$;
  2b) $\pi(a)\mathcal{R}_\mathcal{D}(1 + \langle \mathcal{D} \rangle^2)^{-1}$ is compact for all $a \in \mathcal{A}$;
  3) $\pi(a)$ preserves $\text{Dom} \mathcal{D}$ and $\text{Dom} \mathcal{D}^*$, and the commutators $[\mathcal{D}, \pi(a)]$ and $[\mathcal{D}^*, \pi(a)]$ extend to bounded operators on $\mathcal{H}$, for all $a \in \mathcal{A}$;
  4) $\pi(a)(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$.

The pseudo-Riemannian triple is said to be even if there exists $\Gamma \in \mathcal{B}(\mathcal{H})$ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$, and $\Gamma \mathcal{D} + \mathcal{D} \Gamma = 0$. Otherwise the pseudo-Riemannian triple is said to be odd.

The definition implies that in fact $\mathcal{R}_\mathcal{D}$ lies in $\text{OP}^2(\langle \mathcal{D} \rangle)$, as we now show.

**Lemma 4.2.** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Then
\[
(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \mathcal{R}_\mathcal{D}(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \in \text{OP}^0(\langle \mathcal{D} \rangle).
\]

Hence $\mathcal{R}_\mathcal{D} \in \text{OP}^2(\langle \mathcal{D} \rangle)$.

**Proof.** The operators $\mathcal{D}\mathcal{D}^*$ and $\mathcal{D}^*\mathcal{D}$ are positive and bounded by $1 + \langle \mathcal{D} \rangle^2$, from which it follows that
\[
\| \mathcal{D}(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \|^2 = \| (1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \mathcal{D}^*\mathcal{D}(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \| \leq 2.
\]

Hence $\mathcal{D}$ is bounded by $(1 + \langle \mathcal{D} \rangle^2)^{\frac{1}{2}}$ (and the same holds for $\mathcal{D}^*$). Thus we obtain for $\mathcal{R}_\mathcal{D} = -\frac{i}{2} (\mathcal{D}^2 - \mathcal{D}^{*2})$ the norm bound
\[
\| (1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \mathcal{R}_\mathcal{D}(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \| \\
\leq \frac{1}{2} \|(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \mathcal{D}^2(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \| + \frac{1}{2} \|(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \mathcal{D}^*2(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}} \| \leq 2.
\]
Applying \( L^{(D)} = (1 + \langle D \rangle^2)^{\frac{1}{2}} \) repeatedly, and recalling that we have \( R_D^n = [\langle D \rangle^2, R_D^{(n-1)}] \in \text{OP}^{n+1}(\langle D \rangle) \) for all \( n \geq 1 \) from part 2a) of Definition 4.1, we see that

\[
L^n_D((1 + \langle D \rangle^2)^{-\frac{1}{2}} R_D (1 + \langle D \rangle^2)^{-\frac{1}{2}}) = (1 + \langle D \rangle^2)^{-\frac{1}{2}} L^n_D(R_D)(1 + \langle D \rangle^2)^{-\frac{1}{2}}
\]

is bounded for all \( n \geq 1 \). Hence \((1 + \langle D \rangle^2)^{-\frac{1}{2}} R_D (1 + \langle D \rangle^2)^{-\frac{1}{2}} \) is an element of \( \text{Dom} L^n \) for all \( n \in \mathbb{N} \), so by Eq. (2.3) it is an element of \( \text{OP}^0(\langle D \rangle) \).

**Remark 4.3.** In Definition 4.1, both parts of condition 2) are intended to force \( R_D \) to be a ‘first order operator’, regarding \( \langle D \rangle^2 \) as second order. There are two things to control here: the order of the ‘differential operators’ appearing in \( R_D \), and the growth of the ‘coefficients’. All of these quotation marks can be understood quite literally in the classical case of pseudo-Riemannian Dirac operators described in Section 4.3.

If we were to restrict attention, in the classical case, to differential operators with bounded coefficients, we would expect the easiest assumptions to force \( R_D \) to be first order would be

\[
R_D \in \text{OP}^1(\langle D \rangle)
\]

and

\[
a(R_D) \in \text{OP}^1(\langle D \rangle) \quad \text{for all } a \in \mathcal{A}.
\]

In fact, Equation (4.1) actually implies 2a), since \( \text{OP}^1(\langle D \rangle) \subset \text{OP}^2(\langle D \rangle) \), while Equation (4.2) together with 4) implies 2b).

The reason for weakening the assumptions so that \textit{a priori} \( R_D \in \text{OP}^2(\langle D \rangle) \) only, is to allow for unbounded coefficients and also to allow for non-smooth elements in our algebra. (For instance, the condition (4.2) forces \( a \in \mathcal{A} \) to be smooth. The harmonic oscillator, treated in Section 4.4, is an example where unbounded coefficients occur.

### 4.1.1 Smooth summability

Ultimately we will be interested in obtaining \textit{smoothly summable} spectral triples from pseudo-Riemannian spectral triples, so that we can apply the local index formula \cite{CGRS14}. For this reason we introduce the following notion of spectral dimension and smooth summability for pseudo-Riemannian spectral triples.

**Definition 4.4.** A pseudo-Riemannian spectral triple \((A, \mathcal{H}, D)\) is called \textit{finitely summable} if there exists \( s > 0 \) such that \( |a|(1 + \langle D \rangle^2)^{-\frac{s}{2}} \) is trace-class for all \( a \in \mathcal{A} \). In this case, we let

\[
p := \inf \left\{ s > 0 : \forall a \in \mathcal{A}, \, \text{Tr} (|a|(1 + \langle D \rangle^2)^{-\frac{s}{2}}) < \infty \right\},
\]

and call \( p \) the \textit{spectral dimension} of \((A, \mathcal{H}, D)\).
Definition 4.5. Let \((A, \mathcal{H}, D)\) be a pseudo-Riemannian spectral triple. For \(n \geq 1\), we recursively define the sets
\[
S^0 := A \cup [D, A] \cup [D^*, A], \quad S^n := [(D)^2, S^{n-1}] \cup [R_D, S^{n-1}].
\]
Then \((A, \mathcal{H}, D)\) is called QC\(^k\) summable if \((A, \mathcal{H}, D)\) is finitely summable with spectral dimension \(p\) and
\[
S^n \subset B^{k-n}(\langle D \rangle, p)(1 + \langle D \rangle^2)^{\frac{n}{2}}, \quad \forall 0 \leq n \leq k.
\]
We say that \((A, \mathcal{H}, D)\) is smoothly summable if it is QC\(^k\) summable for all \(k \in \mathbb{N}\) or, equivalently, if
\[
S^n \subset B^{0}(\langle D \rangle, p)(1 + \langle D \rangle^2)^{\frac{n}{2}} = \text{OP}_0^n(\langle D \rangle), \quad \forall n \geq 0.
\]

Remark 4.6. If \(D = D^*\), so that \(R_D = 0\) and \(\langle D \rangle^2 = D^2\), then this definition of smooth summability would reduce to \(S^0 \subset \text{OP}_0^0(D)\), which is precisely the usual definition of smooth summability (see Definition 2.42).

4.2 THE WICK ROTATION

In this section we will show that Wick rotations of a pseudo-Riemannian spectral triple give rise to actual spectral triples. We start by quoting two results, which we will need in the proof of our main theorem below. The next lemma appears as [RS80, exercise 28, Chapter X] (see also the proof of [Ber68, Lemma 3]).

Lemma 4.7. Suppose that \(T\) is a symmetric operator on the Hilbert space \(\mathcal{H}\) with \(\text{Dom} \ T^2\) dense in \(\mathcal{H}\). If \(T^2\) is essentially self-adjoint on \(\text{Dom} \ T^2\), then \(T\) is essentially self-adjoint.

Our main technical tool for passing from pseudo-Riemannian spectral triples to spectral triples is the commutator theorem [RS80, Theorem X.36]. We restate a slightly weaker version of this result using the language of pseudo-differential operators from Section 2.5.

Theorem 4.8. Let \(N \geq 1\) be a positive self-adjoint operator on the Hilbert space \(\mathcal{H}\). If \(A \in \text{OP}^2(\mathbb{N}^{\frac{1}{2}})\) is closed and symmetric and furthermore \([N, A] \in \text{OP}^2(\mathbb{N}^{\frac{1}{2}})\) then

1) \(\text{Dom} \ N \subset \text{Dom} \ N + A\), and there is a constant \(C > 0\) such that for all \(\xi \in \text{Dom} \ N\) we have the inequality
\[
\| (N + A)\xi \| \leq C\| N\xi \|;
\]

2) the operator \(N + A\) is essentially self-adjoint on any core for \(N\).

Our main theorem states that the procedure of ‘Wick rotation’ associates to each pseudo-Riemannian spectral triple two bona fide spectral triples.

Theorem 4.9. Let \((A, \mathcal{H}, D)\) be a pseudo-Riemannian spectral triple. Then its Wick rotations \(D_{\pm}\) yield two spectral triples \((A, \mathcal{H}, D_{\pm})\).
Proof. Using assumptions 1) and 2a) of Definition 4.1, it follows from Theorem 4.8 that \( D_{\pm}^2 = \langle D \rangle^2 \pm \mathcal{R}_D \) are essentially self-adjoint on any core for \( \langle D \rangle^2 \). Since the operators \( D_{\pm} \) are symmetric, Lemma 4.7 implies that \( D_{\pm} \) are essentially self-adjoint.

From the definition of pseudo-Riemannian spectral triple, the commutator

\[
[D_{\pm}, a] = \frac{1}{2}i[D, a] + \frac{1}{2}i [D^*, a]
\]

is bounded for all \( a \in \mathcal{A} \). Thus, to conclude that we have a spectral triple, we need to show that \((1 + D_{\pm}^2)^{-\frac{1}{2}}\) is locally compact (see Remark 2.26). We first need to show, for \( a \in \mathcal{A} \), the compactness of

\[
a(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^{-1} = a(1 + \langle D \rangle^2)^{-1} \mp a(1 + \langle D \rangle^2)^{-1} \mathcal{R}_D (1 + \langle D \rangle^2 \pm \mathcal{R}_D)^{-1}.
\]

The first term is compact by condition 4) in the definition of pseudo-Riemannian spectral triple. For the second term we write

\[a(1 + \langle D \rangle^2)^{-1} \mathcal{R}_D = -a(1 + \langle D \rangle^2)^{-1}[\langle D \rangle^2, \mathcal{R}_D](1 + \langle D \rangle^2)^{-1} + a \mathcal{R}_D (1 + \langle D \rangle^2)^{-1}.
\]

Both terms here are compact, the first by 2a) and 4), the second by 2b). Thus the operator \( a(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^{-1} \) is compact. Finally we employ the integral formula for fractional powers to complete the proof that we have a spectral triple. The same reasoning as above gives the compactness of the integrand in

\[a(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} a(1 + \lambda + \langle D \rangle^2 \pm \mathcal{R}_D)^{-1} d\lambda,
\]

and as the integral converges in norm, the left hand side is a compact operator. Thus \((\mathcal{A}, \mathcal{H}, D_{\pm})\) is a spectral triple. \(\square\)

4.2.1 Smooth summability

We now consider the smooth summability of the spectral triple \((\mathcal{A}, \mathcal{H}, D_{\pm})\). While this can be checked directly for each example, as we do for the harmonic oscillator in Section 4.4.1, we first present a sufficient condition guaranteeing the smooth summability of the spectral triple \((\mathcal{A}, \mathcal{H}, D_{\pm})\) given the smooth summability of \((\mathcal{A}, \mathcal{H}, D)\), with the same spectral dimension. Our sufficient condition requires an additional assumption on the boundedness of \((1 + \langle D \rangle^2)(1 + D_{\pm}^2)^{-1}\). The harmonic oscillator (see Section 4.4) shows that this condition is not necessary. We proceed by proving a few lemmas about the structure of pseudo-differential operators associated to \((\mathcal{A}, \mathcal{H}, D)\).

Lemma 4.10. Let \((\mathcal{A}, \mathcal{H}, D)\) be a pseudo-Riemannian spectral triple. Then the operators \((1 + D_{\pm}^2)\) lie in \(\text{OP}^2(\langle D \rangle)\). Furthermore, if \((1 + \langle D \rangle^2)(1 + D_{\pm}^2)^{-1}\) are bounded, then \((1 + D_{\pm}^2)^{-1}\) lie in \(\text{OP}^{-2}(\langle D \rangle)\).
Proof. Since $\mathcal{R}_D \in \text{OP}^2(\langle D \rangle)$ by Lemma 4.2, $1 + D^2_\pm = 1 + \langle D \rangle^2 + \mathcal{R}_D \in \text{OP}^2(\langle D \rangle)$. If furthermore $(1 + \langle D \rangle^2)(1 + D^2_\pm)^{-1}$ is bounded, it follows from Lemma 2.40 that $(1 + \langle D \rangle^2)(1 + D^2_\pm)^{-1} \in \text{OP}^0(\langle D \rangle)$, and hence that $(1 + D^2_\pm)^{-1} \in \text{OP}^{-2}(\langle D \rangle)$. \hfill \square

In the following discussion of smooth summability, the boundedness of the operator $(1 + \langle D \rangle^2)(1 + D^2_\pm)^{-1}$ plays a crucial role. We pause to give a sufficient condition for this boundedness to hold.

**Lemma 4.11.** Let $(A, \mathcal{H}, D)$ be a pseudo-Riemannian spectral triple with

$$\| (1 + \langle D \rangle^2)^{-\frac{1}{2}} \mathcal{R}_D (1 + \langle D \rangle^2)^{-\frac{1}{2}} \| < 1.$$ 

Then $(1 + \langle D \rangle^2)(1 + D^2_\pm)^{-1} \in \text{OP}^0(\langle D \rangle)$, and hence in particular are bounded.

**Proof.** We know that the operator $(1 + \langle D \rangle^2)^{-\frac{1}{2}} \mathcal{R}_D (1 + \langle D \rangle^2)^{-\frac{1}{2}}$ is bounded since $\mathcal{R}_D \in \text{OP}^2(\langle D \rangle)$, and assuming that $\| (1 + \langle D \rangle^2)^{-\frac{1}{2}} \mathcal{R}_D (1 + \langle D \rangle^2)^{-\frac{1}{2}} \| < 1$ ensures that $\pm 1$ is not in the spectrum. Hence the operators

$$(1 \pm (1 + \langle D \rangle^2)^{-\frac{1}{2}} \mathcal{R}_D (1 + \langle D \rangle^2)^{-\frac{1}{2}})^{-1} = (1 + \langle D \rangle^2)^{\frac{1}{2}}(1 + D^2_\pm)^{-1} (1 + \langle D \rangle^2)^{\frac{1}{2}}$$

are bounded. This also implies the boundedness of $(1 + D^2_\pm)^{-\frac{1}{2}}(1 + \langle D \rangle^2)^{\frac{1}{2}}$ and $(1 + \langle D \rangle^2)^{\frac{1}{2}}(1 + D^2_\pm)^{-\frac{1}{2}}$.

From Lemma 4.10 we know that $(1 + \langle D \rangle^2)^{-\frac{1}{2}}(1 + D^2_\pm)(1 + \langle D \rangle^2)^{-\frac{1}{2}}$ are elements of $\text{OP}^0(\langle D \rangle)$, and since these operators have bounded inverses, it follows from Lemma 2.40 that $(1 + \langle D \rangle^2)^{\frac{1}{2}}(1 + D^2_\pm)^{-1}(1 + \langle D \rangle^2)^{\frac{1}{2}}$ lie in $\text{OP}^0(\langle D \rangle)$. This also implies that $(1 + \langle D \rangle^2)(1 + D^2_\pm)^{-1} \in \text{OP}^0(\langle D \rangle)$. \hfill \square

**Lemma 4.12.** Let $(A, \mathcal{H}, D)$ be a pseudo-Riemannian spectral triple such that the operators $(1 + \langle D \rangle^2)(1 + D^2_\pm)^{-1}$ are bounded. Then the ratios

$$(1 + \langle D \rangle^2)^{-s}(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^s, \quad (1 + \langle D \rangle^2)^s(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^{-s},$$

$$(1 + \langle D \rangle^2)^{\frac{s}{2}}(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^{-s}, \quad (1 + \langle D \rangle^2)^{-\frac{s}{2}}(1 + \langle D \rangle^2 \pm \mathcal{R}_D)^s$$

are bounded for all $0 \leq s \in \mathbb{R}$.

**Proof.** Let $\sigma^r(T) := (1 + \langle D \rangle^2)^{r/2} T (1 + \langle D \rangle^2)^{-r/2}$ be the one parameter group associated to $\langle D \rangle$ (see Proposition 2.44). Consider e.g. $(1 + \langle D \rangle^2)^{-s}(1 + D^2_\pm)^{s}$. Then for $s \in \mathbb{R}$ we have

$$(1 + \langle D \rangle^2)^{-s}(1 + D^2_\pm)^{s} = (1 + \langle D \rangle^2)^{-s+1}(1 + \langle D \rangle^2)^{-1}(1 + D^2_\pm)^{s-1}$$

$$= \sigma^{-2s+2}((1 + \langle D \rangle^2)^{-1}(1 + D^2_\pm)) (1 + \langle D \rangle^2)^{-s+1}(1 + D^2_\pm)^{s-1},$$

where $\sigma^{-2s+2}((1 + \langle D \rangle^2)^{-1}(1 + D^2_\pm)) \in \text{OP}^0(\langle D \rangle)$ because $(1 + \langle D \rangle^2)^{-1}(1 + D^2_\pm)$ lies in $\text{OP}^0(\langle D \rangle)$ by Lemma 4.10. Repeating this process shows that we can assume that $0 < s < 1$. Similar arguments hold for the other ratios. Using the boundedness and invertibility (from Lemma 4.10) of the ratios

$$(1 + \langle D \rangle^2)^{-1}(1 + \langle D \rangle^2 \pm \mathcal{R}_D), \quad (1 + \langle D \rangle^2 \pm \mathcal{R}_D)(1 + \langle D \rangle^2)^{-1},$$
we have, for instance, constants $\epsilon, C > 0$ such that

$$0 < \epsilon \leq (1 + \langle \mathcal{D} \rangle^2)^{-1}(1 + \langle \mathcal{D} \rangle^2)^2(1 + \langle \mathcal{D} \rangle^2)^{-1} \leq C^2.$$ 

Inverting and then conjugating by the bounded self-adjoint operator $(1 + \langle \mathcal{D} \rangle^2)^{-1}$ yields

$$\frac{1}{\epsilon}(1 + \langle \mathcal{D} \rangle^2)^{-2} \geq (1 + \langle \mathcal{D} \rangle^2)^{-2} \geq C^{-2}(1 + \langle \mathcal{D} \rangle^2)^{-2}.$$ 

For $0 < s < 1$, the function $t \mapsto t^s$ is operator monotone, so for such an $s$ we find

$$\frac{1}{\epsilon^s}(1 + \langle \mathcal{D} \rangle^2)^{-2s} \geq (1 + \langle \mathcal{D} \rangle^2)^{-2s} \geq C^{-2s}(1 + \langle \mathcal{D} \rangle^2)^{-2s}.$$ 

Inverting and then conjugating by the bounded self-adjoint operator $(1 + \langle \mathcal{D} \rangle^2)^{-s}$ yields

$$0 < \epsilon^s \leq (1 + \langle \mathcal{D} \rangle^2)^{-s}(1 + \langle \mathcal{D} \rangle^2)^2(1 + \langle \mathcal{D} \rangle^2)^{-s} \leq C^{2s}.$$ 

Consequently $(1 + \langle \mathcal{D} \rangle^2)^2(1 + \langle \mathcal{D} \rangle^2)^{-s}$ and $(1 + \langle \mathcal{D} \rangle^2)^{-s}(1 + \langle \mathcal{D} \rangle^2)^2$ are bounded for each $s \in (0, 1)$. The same method applies to obtain the boundedness of the other ratios. 

**Lemma 4.13.** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple such that the operators $(1 + \langle \mathcal{D} \rangle^2)^2(1 + \langle \mathcal{D} \rangle^2)^{-1}$ are bounded. Then $(1 + \langle \mathcal{D} \rangle^2)^2(1 + \langle \mathcal{D} \rangle^2)^{-s}$ lie in $\text{OP}^0(\langle \mathcal{D} \rangle)$ for all $s \geq 0$.

**Proof.** As in the proof of Lemma 4.12 we can reduce the problem to the case $0 < s < 1$. We already know from Lemma 4.12 that $(1 + \langle \mathcal{D} \rangle^2)^2(1 + \langle \mathcal{D} \rangle^2)^{-s}$ is bounded. Let us write $T := (1 + \langle \mathcal{D} \rangle^2)^2$ for brevity. As in the proof of Lemma 2.40, there are (combinatorial) constants $C_{l,n,k}$ such that

$$\delta_{\langle \mathcal{D} \rangle}^n((\lambda + T)^{-1}) = \sum_{1 \leq l \leq n} \sum_{1 \leq k_1, \ldots, k_l \leq n} C_{l,n,k} (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_1}(T)(\lambda + T)^{-1} \cdots (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_l}(T)(\lambda + T)^{-1}.$$ 

Using the integral formula for fractional powers, we can then write

$$(1 + \langle \mathcal{D} \rangle^2)^s \delta_{\langle \mathcal{D} \rangle}^n(T^{-s}) = (1 + \langle \mathcal{D} \rangle^2)^s \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} \delta_{\langle \mathcal{D} \rangle}^n((\lambda + T)^{-1}) d\lambda$$

$$= \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} \sum_{1 \leq l \leq n} \sum_{1 \leq k_1, \ldots, k_l \leq n} C_{l,n,k} (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_1}(T)(\lambda + T)^{-1} \cdots (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_l}(T)(\lambda + T)^{-1} d\lambda.$$ 

Since $\delta_{\langle \mathcal{D} \rangle}^k(T) \in \text{OP}^2(\langle \mathcal{D} \rangle)$, we know that $\delta_{\langle \mathcal{D} \rangle}^k(T)(\lambda + T)^{-1} = \delta_{\langle \mathcal{D} \rangle}^k(T)(1 + \langle \mathcal{D} \rangle^2)^{-1}(1 + \langle \mathcal{D} \rangle^2)^{-1}T(\lambda + T)^{-1}$.
is uniformly bounded in $\lambda$ for each $k$, and hence we see that the integral converges to a bounded operator for all $n$. Thus $(1 + \langle D \rangle^2)^{\frac{k}{2}}(1 + D_\pm^2)^{-k/2} \in \text{OP}^0(\langle D \rangle)$. \qed

**Theorem 4.14.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a smoothly summable pseudo-Riemannian spectral triple such that $(1 + \langle D \rangle^2)(1 + D_\pm^2)^{-1}$ are bounded. Then the corresponding spectral triples $(\mathcal{A}, \mathcal{H}, D_\pm)$ are smoothly summable, with the same spectral dimension.

**Proof.** Since the operators $(1 + D_\pm^2)^{\frac{1}{2}}(1 + \langle D \rangle^2)^{-\frac{1}{2}}$ and $(1 + \langle D \rangle^2)^{\frac{k}{2}}(1 + D_\pm^2)^{-k/2}$ are both bounded (by Lemma 4.12), we see that $|a|(1 + \langle D \rangle^2)^{-\frac{1}{2}}$ is trace-class if and only if $|a|(1 + D_\pm^2)^{-\frac{1}{2}}$ is trace-class. Hence the spectral triples $(\mathcal{A}, \mathcal{H}, D_\pm)$ are finitely summable, with the same spectral dimension.

Similarly, it is straightforward to show that $\mathcal{B}_2(D_\pm, p)$ coincides with $\mathcal{B}_2(\langle D \rangle, p)$, and so also that $\mathcal{B}_1(D_\pm, p) = \mathcal{B}_1(\langle D \rangle, p)$. Recall from Definition 4.5 the recursive definitions $S^0 := A \cup [D, A] \cup [D^*, A]$ and $S^n := [(\langle D \rangle^2, S^{n-1}] \cup [R_{\mathcal{D}}, S^{n-1}]$. To prove smooth summability for $(\mathcal{A}, \mathcal{H}, D_\pm)$, we need to show that $A \cup [D_\pm, A]$ is contained in $\mathcal{B}_n(\langle D \rangle, p)$. Since $A \cup [D_\pm, A]$ consists of linear combinations of elements of $S^0$, it suffices to show that $S^0 \subset \mathcal{B}_n(\langle D \rangle, p)$.

So suppose that $T \in S^0$, which by assumption is contained in $\text{OP}^0(\langle D \rangle)$. Let us write $T^{(0)} := T$ and $T^{(n)} := [D^{2n}, T^{(n-1)}] = [(\langle D \rangle^2, T^{(n-1)}] + [R_{\mathcal{D}}, T^{(n-1)}]$. It then follows that $T^{(n)}$ is a finite linear combination of elements of $S^n$, which by assumption is contained in $\text{OP}^n(\langle D \rangle)$. Because we can write

$$R_{D_\pm}^k(T) = T^{(k)}(1 + D_\pm^2)^{-k/2} = T^{(k)}(1 + \langle D \rangle^2)^{-k/2}(1 + \langle D \rangle^2)^{k/2}(1 + D_\pm^2)^{-k/2},$$

and because $(1 + \langle D \rangle^2)^{k/2}(1 + D_\pm^2)^{-k/2} \in \text{OP}^0(\langle D \rangle)$ by Lemma 4.13, we see that $R_{D_\pm}^k(T)$ is an element of $\text{OP}^n(\langle D \rangle)$, so in particular $R_{D_\pm}^k(T)$ lies in $\mathcal{B}_n(\langle D \rangle, p)$ for all $k$. Therefore $T$ is an element of $\mathcal{B}_n(\langle D \rangle, p)$, and hence $S^0 \subset \mathcal{B}_n(\langle D \rangle, p)$. \qed

Given a pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we have constructed two spectral triples $(\mathcal{A}, \mathcal{H}, D_\pm)$, and this construction preserves the property of smooth summability needed for the local index formula. This means that each smoothly summable pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, D)$ yields two K-homology classes $[(\mathcal{A}, \mathcal{H}, D_+)]$ and $[(\mathcal{A}, \mathcal{H}, D_-)]$ in $K^*(\mathcal{A})$ (where $\mathcal{A}$ is the norm completion of $\mathcal{A}$), and we can compute the pairing of these classes with $K_*(\mathcal{A})$ using the (non-unital) local index formula [CGRS14]. In Section 4.6 we will see a refinement of the definition of pseudo-Riemannian spectral triples (from [PS06]), which guarantees that these two K-homology classes are in fact negatives of each other.

### 4.3 PSEUDO-RIEMANNIAN MANIFOLDS

In Section 3.4 we have constructed the Dirac operator $\mathcal{D}$ on the spinor bundle $\mathcal{S}$ over a pseudo-Riemannian manifold $(M, g)$. In this section we will show un-
der which conditions the triple \((C_c^\infty(M), L^2(S), \mathcal{D})\) can be viewed as a pseudo-Riemannian spectral triple.

We will again require our basic Assumption 3.16. Thus, let \((M, g)\) be an \(n\)-dimensional time- and space-oriented pseudo-Riemannian spin manifold of signature \((t, s)\), and let \(r\) be a spacelike reflection, such that the associated Riemannian metric \(g_r\) is complete. We have already seen in Proposition 3.18 that this assumption ensures that the Wick rotations of \(\mathcal{D}\) give rise to spectral triples \((C_c^\infty(M), L^2(S), \mathcal{D}_\pm)\). However, the basic Assumption 3.16 is not enough to obtain a pseudo-Riemannian spectral triple from \((M, g)\). Instead, we will see below that we furthermore need to assume that the manifold has bounded geometry.

As in Section 4.1, we consider the operators

\[
\langle \mathcal{D} \rangle^2 := \frac{1}{2} (\mathcal{D} \mathcal{D}^* + \mathcal{D}^* \mathcal{D}), \quad \mathcal{R}_\mathcal{D} := -\frac{i}{2} \left( \mathcal{D}^2 - \mathcal{D}^* \mathcal{D}^2 \right).
\]

We will first show that \(\langle \mathcal{D} \rangle^2\) is a ‘Laplace-type’ operator.

**Lemma 4.15.** Let \((M, g, r)\) satisfy Assumption 3.16. The operator \(\langle \mathcal{D} \rangle^2\) defined on the domain \(\text{Dom}(\mathcal{D})^2 := \text{Dom} \mathcal{D} \mathcal{D}^* \cap \text{Dom} \mathcal{D}^* \mathcal{D}\) is essentially self-adjoint, elliptic and commutes with the fundamental symmetry \(\partial_M\). Hence it is also essentially Krein-self-adjoint.

**Proof.** The operator \(\langle \mathcal{D} \rangle^2\) has principal symbol \(g_r(\xi, \xi)\) (which easily follows using Eq. (3.8)) and is therefore elliptic. Consider the operator

\[T : \text{dom} \mathcal{D} \cap \text{dom} \mathcal{D}^* \rightarrow L^2(S) \oplus L^2(S), \quad T \psi := (\mathcal{D} \psi, \mathcal{D}^* \psi).\]

The graph norm of \(T\) equals the ‘combined graph norm’ \(\| \cdot \|_{\mathcal{D}, \mathcal{D}^*}\), and we then know from Lemma 2.45 that \(T\) is closed. Thus we have a self-adjoint operator \(T^* T\) which is an extension of \(\mathcal{D} \mathcal{D}^* + \mathcal{D}^* \mathcal{D} = 2\langle \mathcal{D} \rangle^2\). In particular, the restriction of \(T^* T\) to \(C_c^\infty(S)\) equals \(2\langle \mathcal{D} \rangle^2\). Since the metric \(g_r\) is complete, it follows from [BMS02, Corollary 2.10] (noting that the metric \(g^{TM}\) defined therein is equal to \(2g_r\)) that \(T^* T\) is essentially self-adjoint on \(C_c^\infty(S)\), which proves that \(\langle \mathcal{D} \rangle^2\) is essentially self-adjoint. Finally, by Krein-selfadjointness of \(i^* \mathcal{D}\) (Theorem 3.17) we know that \(\mathcal{D}^* = (-1)^i \partial_M \mathcal{D} \partial_M\), from which it easily follows that \(\langle \mathcal{D} \rangle^2, \partial_M \rangle = 0\).

We have seen in Proposition 3.18 that the Wick rotations \(\mathcal{D}_\pm\) yield spectral triples. It has been shown in [CGRS14, Proposition 5.9] that these triples are smoothly summable (with spectral dimension given by the dimension of the underlying manifold) if the Riemannian manifold \((M, g_r)\) is complete and of bounded geometry. To obtain smooth summability for the triple \((C_c^\infty(M), L^2(S), \mathcal{D})\), we will now introduce similar assumptions for the pseudo-Riemannian manifold \((M, g)\).

Recall that the injectivity radius \(r_{\text{inj}} \in [0, \infty)\) of the Riemannian manifold \((M, g_r)\) is defined as \(r_{\text{inj}} := \inf_{x \in M} \sup \{r_x > 0\}\), where \(r_x\) is such that the exponential map \(\exp_x\) (defined w.r.t. the Riemannian metric \(g_r\)) is a diffeomorphism from the ball
B(0, r_x) ⊂ T_x M to an open neighborhood U_x of x ∈ M. We observe that there is a related notion of injectivity radius for Lorentzian manifolds which is adapted to the pair of metrics \((g, g_r)\); see [CL08].

**Definition 4.16.** Let \((M, g, r)\) be as in Assumption 3.16. We say that \((M, g, r)\) has bounded geometry if \((M, g_r)\) has strictly positive injectivity radius, and all the covariant derivatives of the (pseudo-Riemannian) curvature tensor of \((M, g)\) are bounded (w.r.t. \(g_r\)) on \(M\). A Dirac bundle \(S \to M\) is said to have bounded geometry if in addition all the covariant derivatives of \(\Omega^5\), the curvature tensor of the connection \(\nabla^5\), are bounded (w.r.t. \(g_r\)) on \(M\). For brevity, we simply say that \((M, g, r, S)\) has bounded geometry.

A differential operator is said to have uniform \(C^\infty\)-bounded coefficients, if for any atlas consisting of charts of normal coordinates, the derivatives of all orders of the coefficients are bounded on the chart domain and the bounds are uniform on the atlas. It is shown in [Roe88, Propositions 5.4 & 5.5] that the assumption of bounded geometry is equivalent to the existence of a good coordinate ball, i.e. a ball \(B\) with center 0 in \(\mathbb{R}^n\) which is the domain of a normal coordinate system at every point of \(M\), such that the Christoffel symbols of \(\nabla\) and \(\nabla^5\) lie in a bounded subset of the Fréchet space \(C^\infty(B)\). Thus bounded geometry implies that the Dirac operator \(\mathcal{D}\) has uniform \(C^\infty\)-bounded coefficients.

**Proposition 4.17.** Let \((M, g, r)\) be as in Assumption 3.16. Suppose that \((M, g, r, S)\) has bounded geometry. Then the triple \((C^\infty_c(M), L^2(S), \mathcal{D})\) is a pseudo-Riemannian spectral triple.

**Proof.** We need to check that the triple satisfies the conditions of Definition 4.1. From Lemma 4.15 we know that \(\langle \mathcal{D} \rangle^2\) is essentially self-adjoint, which proves 1), and elliptic, which proves 4) (see e.g. [Hroo, Proposition 10.5.2]). Since \(\mathcal{D}\) and \(\mathcal{D}^*\) are first-order differential operators, 3) is immediate. Consider the operator \(R_{\mathcal{D}} := -\frac{1}{2}(\mathcal{D}^2 - \mathcal{D}^*\mathcal{D})\), initially defined on the dense subset of compactly supported smooth sections \(\Gamma_c(S)\). Since \(\mathcal{D}^*\mathcal{D} = \mathcal{D}_M\mathcal{D}_M^2\) by Theorem 3.17, we have \(\mathcal{D}^2 - \mathcal{D}^*\mathcal{D} = [\mathcal{D}, \mathcal{D}_M]\mathcal{D}_M\). As \(\mathcal{D}^2\) is a second-order differential operator whose principal symbol commutes with \(\mathcal{D}_M\), we see that \(R_{\mathcal{D}}\) is a first-order differential operator. Under the assumption of bounded geometry, all covariant derivatives of the coefficients of \(R_{\mathcal{D}}\) are uniformly bounded, which implies 2a). In particular, \(R_{\mathcal{D}}(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}}\) is a bounded operator. Ellipticity of \(\langle \mathcal{D} \rangle^2\) then implies 2b). \(\square\)

**Remark 4.18.** Note that, while condition 2a) of Definition 4.1 only requires that \(R_{\mathcal{D}}\) is second-order (i.e. in \(OP^2(\langle \mathcal{D} \rangle)\)), we have in fact that \(R_{\mathcal{D}}\) is first-order (i.e. in \(OP^1(\langle \mathcal{D} \rangle)\)).

To obtain finite summability for the triple \((C^\infty_c(M), L^2(S), \mathcal{D})\), we observe that the bounded geometry hypothesis ensures that \(\text{Tr}(a(1 + \langle \mathcal{D} \rangle^2)^{-\frac{1}{2}})\) is finite for
\[ s > n = \dim M, \]  
where \( a \in C_c^\infty(M) \) is a compactly supported smooth function. Hence the spectral dimension \( p \) is equal to the metric dimension \( n \); see [CGRS14, Proposition 5.9].

For smooth summability, we also need \( \text{Tr}(\gamma(e_j)a(1 + (D)^2)^{-\frac{1}{2}}) \) to be finite, which holds since \( a \) is compactly supported, and so \( \gamma(e_j)a \) is also compactly supported and bounded. Furthermore, we observe that \( \langle D \rangle^2 \) is a uniformly elliptic second-order differential operator with scalar principal symbol (given by the Riemannian metric). Hence \( \langle D \rangle^2 \) determines the usual order of compactly supported pseudo-differential operators.

As mentioned in Remark 4.18, the operator \( \mathcal{R}_D = -\frac{i}{2}(D^2 - D^*D) \) is a first-order operator in \( \text{OP}^1(\langle D \rangle) \). Hence \( [\mathcal{R}_D, \text{OP}^\alpha_\delta(\langle D \rangle)] \subset \text{OP}^{\alpha+1}_\delta(\langle D \rangle) \). It thus follows that the triple \( (C_c^\infty(M), L^2(S), D) \) is a smoothly summable pseudo-Riemannian spectral triple (as defined in Definition 4.5).

The operator \( D_\pm^2 \) is self-adjoint and positive, since \( D_\pm \) is self-adjoint by Proposition 3.18. Since \( \mathcal{R}_D \) is a first-order differential operator and \( \langle D \rangle^2 \) is second-order and elliptic, it follows that \( D_\pm^2 = \langle D \rangle^2 \pm \mathcal{R}_D \) is also elliptic. This means that \( (1 + D_\pm^2)^{-1} \) is a pseudo-differential operator of order \(-2\), and therefore the operator \( (1 + \langle D \rangle^2)(1 + D_\pm^2)^{-1} \) is bounded. Theorem 4.14 then tells us that the spectral triples \( (C_c^\infty(M), L^2(S), D_\pm) \) are smoothly summable.

### 4.4 THE ONE-DIMENSIONAL HARMONIC OSCILLATOR

Let \( \mathcal{H} = L^2(\mathbb{R}) \), and consider the annihilation operator \( a \), and its adjoint, the creation operator \( a^* \), given by

\[
    a = x + \frac{d}{dx}, \quad a^* = x - \frac{d}{dx},
\]

defined on the initial domain of Schwartz functions \( S(\mathbb{R}) \subset L^2(\mathbb{R}) \). The real and imaginary parts of \( a \) and its Wick rotations are given by

\[
    \text{Re} a = x, \quad \text{Im} a = -i \frac{d}{dx}, \quad a_\pm = x \mp i \frac{d}{dx}.
\]

We let \( C_c^\infty(\mathbb{R}) \) be the smooth functions all of whose derivatives are integrable on \( \mathbb{R} \). To show that we obtain a pseudo-Riemannian spectral triple \( (C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), a) \), we first observe that the operator

\[
    \langle a \rangle^2 = -\frac{d^2}{dx^2} + x^2
\]

is well-known to be essentially self-adjoint and to have compact resolvent. For all \( f \in C_c^\infty(\mathbb{R}) \), the commutators \([a, f]\) and \([a^*, f]\) are bounded. The other items to check involve the operator

\[
    \mathcal{R}_a := -\frac{i}{2} (a^2 - a^*a) = -i \left( 1 + 2x \frac{d}{dx} \right),
\]
which is first-order as a differential operator, but has an unbounded coefficient.

**Lemma 4.19.** For all \( n \geq 0 \) the operator \( \mathcal{R}_a^{(n)} \) is an element of \( \operatorname{OP}^2(\langle a \rangle) \), and for all \( f \in C_1^\infty(\mathbb{R}) \) the operator \( f\mathcal{R}_a(1 + \langle a \rangle)^{-1} \) is compact.

**Proof.** Straightforward calculations show that

\[
\mathcal{R}_a^{(1)} = \langle a \rangle^2; \mathcal{R}_a = -4i\langle a \rangle^2 + 8ix^2, \quad \mathcal{R}_a^{(2)} = 16\mathcal{R}_a,
\]

and so it suffices to check the first claim for \( n = 0, 1 \). We begin by observing that

\[
0 \leq x^2 \leq x^2 - \frac{d^2}{dx^2} + 1 \implies \left\| x \left( 1 + x^2 - \frac{d^2}{dx^2} \right)^{-\frac{1}{2}} \right\|^2 = \left\| \left( 1 + x^2 - \frac{d^2}{dx^2} \right)^{-\frac{1}{2}} x \right\|^2 \leq 1. \quad (4.3)
\]

Similarly \( \left\| \frac{d}{dx} \left( 1 + x^2 - \frac{d^2}{dx^2} \right)^{-\frac{1}{2}} \right\| \leq 1 \). Therefore

\[
(1 + \langle a \rangle)^{-\frac{1}{2}} \mathcal{R}_a (1 + \langle a \rangle)^{-\frac{1}{2}} \quad \text{and} \quad (1 + \langle a \rangle)^{-\frac{1}{2}} \mathcal{R}_a^{(1)} (1 + \langle a \rangle)^{-\frac{1}{2}}
\]

are bounded operators. Hence \( \mathcal{R}_a \) and \( \mathcal{R}_a^{(1)} \) lie in \( \operatorname{OP}^2(\langle a \rangle) \), which proves the first statement. For the second statement, if \( f \) is a bounded integrable function, the product \( fx \) is bounded, and so by the compactness of \( (1 + \langle a \rangle)^{-\frac{1}{2}} \) we see that \( f\mathcal{R}_a(1 + \langle a \rangle)^{-1} \) is compact. \( \square \)

Thus the harmonic oscillator gives rise to a pseudo-Riemannian spectral triple.

**Proposition 4.20.** The pseudo-Riemannian spectral triple \( (C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), a) \) is smoothly summable with spectral dimension 1.

**Proof.** Let \( f : \mathbb{R} \to [0, \infty) \) be a smooth integrable function, and let \( (x, y) \mapsto k_t(x, y) \) be the integral kernel of \( e^{-t\langle a \rangle^2} \) for \( a = \frac{d}{dx} + x \). Mehler’s formula gives

\[
k_t(x, y) = \frac{1}{\sqrt{2\pi \sinh(2t)}} e^{-\frac{1}{2} \coth(2t)(x^2+y^2)+\coth(2t)xy}.
\]

Then for \( s > 2 \)

\[
\operatorname{Tr}(f(1 + \langle a \rangle)^{-\frac{s}{2}}) = \frac{1}{\Gamma(s/2)} \int_{\mathbb{R}} f(x) \int_0^\infty t^{s/2-1} e^{-t} k_t(x, x) \, dt \, dx \\
\leq \frac{1}{\Gamma(s/2)} \int_{\mathbb{R}} f(x) \, dx \int_0^\infty t^{s/2-1} e^{-t} \frac{1}{\sqrt{2\pi \sinh(2t)}} \, dt
\]

and this remains finite for \( s > 1 \). Thus the spectral dimension is at most 1. To see that the spectral dimension is also at least 1, and so is precisely 1, one computes this trace for the function \( f : x \mapsto e^{-x^2} \).

For \( f \in C_1^\infty(\mathbb{R}) \), the commutators \([a, f]\) and \([a^*, f]\) are again elements of \( C_1^\infty(\mathbb{R}) \). With the notation of **Definition 4.5**, we thus find that \( S^0 := A \cup [a, A] \cup [a^*, A] = A \).

The above computations now allow us to see that \( S^0 \) lies in \( B_1(\langle a \rangle, 1) \).
Finally, we need to check that $S^n \subset OP_{0}^{n}(\langle a \rangle)$ for all $n$. First, by Equation (4.3), and the relations

\[
\left[ (\alpha)^2, \frac{d}{dx} \right] = 2x, \quad \left[ (\alpha)^2, x \right] = -2 \frac{d}{dx} \quad \left[ R_{\alpha}, x^n \frac{d^m}{dx^m} \right] = 2i(n-m)x^n \frac{d^m}{dx^m},
\]

we can see that both multiplication by $x$ and $\frac{d}{dx}$ lie in $OP_{1}(\langle a \rangle)$. For $f \in C_{\infty}(R)$, we use the computation

\[
\left[ (\alpha)^2, f \right] = -f'' - 2f' \frac{d}{dx}
\]

and a simple induction to see that $f \in OP_{0}^{\infty}(\langle a \rangle)$. Hence $C_{\infty}^{\infty}(R) \subset OP_{0}^{\infty}(\langle a \rangle)$. It is then straightforward to check that any element $T \in S^n$ can be written in the form

\[
T = \sum_{k+1 \leq n} f_{k,l} x^k \frac{d^l}{dx^l}
\]

for functions $f_{k,l} \in C_{\infty}^{\infty}(R)$. This is obviously true for $n = 0$. Assuming it holds for all $T \in S^n$ for some $n$, one shows it also holds for $n + 1$ by explicitly calculating the commutators $[(\alpha)^2, T]$ and $[R_{\alpha}, T]$. So it follows by induction that indeed we have $T = \sum_{k+1 \leq n} f_{k,l} x^k \frac{d^l}{dx^l}$ for all $T \in S^n$, for all $n$. Since $f_{k,l} \in OP_{0}^{\infty}(\langle a \rangle)$, $x^k \in OP_{k}(\langle a \rangle)$, and $\frac{d^l}{dx^l} \in OP_{l}(\langle a \rangle)$, it follows that $S^n \subset OP_{0}^{\infty}(\langle a \rangle)$ for all $n$. Therefore we conclude that the triple $(C_{\infty}^{\infty}(R), L^2(R), \frac{d}{dx} + x)$ is a smoothly summable pseudo-Riemannian spectral triple. \( \square \)

### 4.4.1 The K-homology class of the harmonic oscillator

Next we consider the K-homology classes of the spectral triples obtained from the harmonic oscillator, given by the Wick rotations

\[
a_{\pm} = x \mp i \frac{d}{dx}.
\]

We have already seen in Proposition 4.20 that the pseudo-Riemannian spectral triple $(C_{\infty}^{\infty}(R), L^2(R), \frac{d}{dx} + x)$ is smoothly summable, with spectral dimension $p$ equal to 1. In order to conclude from Theorem 4.14 that the Wick rotated spectral triple $(C_{\infty}^{\infty}(R), L^2(R), a_{\pm})$ is also smoothly summable, we would need to check that the operator $(1 + (a)^2)(1 + a_{\pm}^2)^{-1}$ is bounded. We have been unable to prove this, and at present have no reason to believe it is true.

On the other hand, we can simply check directly that $C_{\infty}^{\infty}(R) \subset B_{\infty}^{\infty}(a_{\pm}, 1)$. This can be shown in the same way as in the proof of Proposition 4.20. The integral kernels of $e^{-t a_{\pm}^2}$ and $(1 + a_{\pm}^2)^{-\frac{1}{2}}$ are given by

\[
k_{t}(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/4t \mp i(x^2-y^2)/2},
\]

\[
k_{s}(x, y) = \frac{1}{\Gamma(\frac{s}{2})} \int_{0}^{\infty} t^{\frac{s}{2} - 1} e^{-t} k_{t}(x, y) dt.
\]
Then for any \( f \in C_c^\infty(\mathbb{R}) \) one finds that
\[
\text{Tr}(f(1 + a_\pm^2)^{-s/2}) = \frac{\Gamma(s - \frac{1}{2})}{2\sqrt{\pi}(s/2)} \int_\mathbb{R} f(x) \, dx.
\]
Thus the spectral dimension is at least \( 1 \), and taking \( f(x) = e^{-x^2} \) shows that the spectral dimension is precisely \( 1 \). The smoothness is an easy check, using the same computations as in the proof of Proposition 4.20. To show that the spectral triple is smoothly summable, we observe that for all \( f \in C_c^\infty(\mathbb{R}) \) we have \([a_\pm^2, f] = -f'' + 2if' a_\pm \). Then a straightforward induction shows that \( C_c^\infty(\mathbb{R}) \subset B_1^\infty(a_\pm, 1) \). Hence the spectral triple \((C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), a_\pm)\) is also smoothly summable with spectral dimension \( 1 \), and we can apply the local index formula of [CGRS14, Theorem 4.33].

For a unitary \( u \) in the unitization of \( C_c^\infty(\mathbb{R}) \), the local index formula computes the pairing of the class of the spectral triple with the \( K \)-theory class of \( u \) as
\[
\langle [u], [(C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), a_\pm)] \rangle = -\lim_{s \to \frac{1}{2}} (s - \frac{1}{2}) \text{Tr} (u^*[a_\pm, u](1 + a_\pm^2)^{-s}).
\]
The odd \( K \)-theory of the real line is \( K_1(C_0(\mathbb{R})) = \mathbb{Z} \). For \( m \in \mathbb{Z} \) we choose representatives of these classes to be \( u = e^{2im \tan^{-1}(x)} \), and this gives \( u^*[a_\pm, u] = \pm \frac{2m}{1+x^2} =: f_m(x) \). Using the trace calculations above we have
\[
\lim_{s \to \frac{1}{2}} \frac{s - \frac{1}{2}}{2} \text{Tr} (u^*[a_\pm, u](1 + a_\pm^2)^{-s})
\]
\[
= \lim_{s \to \frac{1}{2}} \frac{1}{2\sqrt{\pi}(s/2)} \int_\mathbb{R} f_m(x) \int_0^\infty \frac{s - \frac{1}{2}}{2} t^{s/2 - \frac{1}{2} - 1} e^{-t} \, dt \, dx
\]
\[
= \lim_{s \to \frac{1}{2}} \frac{1}{2\sqrt{\pi}(s/2)} \int_\mathbb{R} f_m(x) \int_0^\infty \frac{d}{dt} (t^{s/2 - \frac{1}{2}}) e^{-t} \, dt \, dx
\]
\[
= \pm \frac{m}{\pi} \int_\mathbb{R} \frac{1}{1+x^2} \, dx = \pm m.
\]
Hence the spectral triple \((C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), a_\pm)\) has a non-trivial \( K \)-homology class, and it coincides with the class of \((C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), \mp i \frac{d}{dx})\), see [HRo0, page 298]. From the perspective of principal symbols this is not surprising, however the unboundedness of the perturbation means that the result is not immediate. Lastly, we observe that the classes of \( a_+ \) and \( a_- \) are negatives of each other.

### 4.5 Other Examples

#### 4.5.1 Finite Geometries

Just as there are virtually no constraints to the existence of a spectral triple for a finite-dimensional algebra, pseudo-Riemannian spectral triples are easily constructed in this case. So let \( A \) be a finite-dimensional complex algebra, i.e. a direct sum
of simple matrix algebras. Choose two representations of \( A \) on finite dimensional Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). Let \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), and \( \Gamma = 1 \oplus \{-1\} \) with respect to this decomposition. Choose any linear map \( B : \mathcal{H}_1 \to \mathcal{H}_2 \), and set
\[
\mathcal{D} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.
\]
Then the definition of an even pseudo-Riemannian spectral triple is trivially fulfilled. Likewise it is trivially smoothly summable. For the Wick rotations, we find
\[
\mathcal{D}_\pm = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 0 & \pm iB^* \\ B & 0 \end{pmatrix}.
\]

4.5.2 First order differential operators

We consider a constant coefficient first-order differential operator of the form
\[
\mathcal{D} = \sum_{j=1}^{n} M_j \frac{\partial}{\partial x_j} + K,
\]
where \( K, M_j \in \text{M}_d(\mathbb{C}) \). The operator \( \mathcal{D} \), acting on the smooth compactly supported sections in \( L^2(\mathbb{R}^n, \mathbb{C}^d) \), extends to a closed and densely defined operator. One may check that \( (\mathcal{C}_0^\infty(\mathbb{R}^n), L^2(\mathbb{R}^n, \mathbb{C}^d), \mathcal{D}) \) yields a pseudo-Riemannian spectral triple provided that for all \( j, k = 1, \ldots, n \) and for all \( 0 \neq \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) the following three conditions hold:
\[
\sum_{j,k=1}^{n} (M_j^* M_k + M_j M_k^*) \xi_j \xi_k \in \text{GL}_d(\mathbb{C}), \quad (4.4)
\]
\[
\{M_j, M_k\} = \{M_j^*, M_k^*\}, \quad (4.5)
\]
\[
\left[ \sum_{j,k=1}^{n} (M_j^* M_k + M_j M_k^*) \xi_j \xi_k, \sum_{j=1}^{n} \left( \{M_j, K\} + \{M_j^*, K^*\} \right) \xi_j \right] = 0. \quad (4.6)
\]
Provided that in addition
\[
[M_j, M_k M_j^* + M_k^* M_j] = 0, \quad \text{and} \quad [K, M_k M_j^* + M_k^* M_j] = 0, \quad (4.7)
\]
the pseudo-Riemannian spectral triple is in fact smoothly summable.

One can think of these conditions roughly as follows. The condition \( (4.4) \) ensures that \( \langle \mathcal{D} \rangle^2 \) is elliptic. The condition \( (4.5) \) is a reality condition ensuring that the principal symbols of \( \mathcal{D}^2 \) and \( \mathcal{D}^{*2} \) are equal (and if \( \mathcal{D} \) were of Dirac-type, this would simply mean that the Riemannian metric is real), which implies that \( \mathcal{R}_\mathcal{D} \) is a first-order differential operator. Lastly, the conditions \( (4.6) \) and \( (4.7) \) ensure that the principal symbol of \( \langle \mathcal{D} \rangle^2 \) is central.
For the Wick rotations of $\mathcal{D}$, we find
\[
\mathcal{D}_\pm = \sum_j \tilde{M}_j \frac{\partial}{\partial x^j} + \frac{i}{2} (K + K^*) \mp \frac{i}{2} (K - K^*)
\]
where $\tilde{M}_j = M_j$ if $M_j^* = -M_j$ and $\tilde{M}_j = iM_j$ if $M_j = M_j^*$. Since $\mathcal{R}_\mathcal{D}$ is a first-order differential operator in these examples, and $\mathcal{D}_\pm^2$ are second-order and uniformly elliptic, one can show that $\mathcal{R}_\mathcal{D} (1 + \mathcal{D}_\pm^2)^{-1}$ are bounded, and Theorem 4.14 gives us smoothly summable spectral triples.

4.5.3 Semifinite examples

There is a notion of semifinite spectral triple [BF06, CPRS04, CPRS06a, CPRS06b], where $(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}), \text{Tr})$ are replaced by $(\mathcal{N}, \mathcal{K}(\mathcal{N}, \tau), \tau)$ where $\mathcal{N}$ is an arbitrary semifinite von Neumann algebra, $\mathcal{K}(\mathcal{N}, \tau)$ is the ideal of $\tau$-compact operators in $\mathcal{N}$, and $\tau$ is a faithful, semifinite, normal trace. Thus we require $\mathcal{D}$ affiliated to $\mathcal{N}$, and the compact resolvent condition is relative to $\mathcal{K}(\mathcal{N}, \tau)$. Examples of semifinite spectral triples arising from graph and k-graph $C^*$-algebras were described in [PR06, PRS08]. These were constructed using the natural action of the torus $T^k$ on a k-graph algebra, by ‘pushing forward’ the Dirac operator on the torus. More sophisticated examples coming from covering spaces of manifolds of bounded geometry were considered in [CGRS14] also.

For the k-graph algebras, $k \geq 2$, we may of course take a Lorentzian Dirac operator (or more generally pseudo-Riemannian Dirac operator) and push this forward instead. This gives rise to a ‘semifinite pseudo-Riemannian spectral triple’, but as the details would take us too far afield, we leave this to the reader to explore.

4.6 Lorentz-type spectral triples

In this section we specialise our definition of pseudo-Riemannian spectral triples to Lorentz-type spectral triples (closely related to the Lorentzian spectral triples of [PS06]). This specialisation implies that the K-homology classes of the two Wick rotations are negatives of one another, which leads to an index-theoretic result.

**Definition 4.21.** A Lorentz-type spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ is given by

- a separable Hilbert space $\mathcal{H}$;
- a $*$-algebra $\mathcal{A}$ with a representation $\mathcal{A} \to \mathcal{B}(\mathcal{H})$;
- a self-adjoint, unitary operator $\mathcal{J}$ on $\mathcal{H}$ which commutes with $\mathcal{A}$;
- a densely defined, closed operator $\mathcal{D}$: $\text{Dom} \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ such that
  - o) $\text{Dom} \mathcal{D}^* \cap \text{Dom} \mathcal{D}^*$ is dense in $\mathcal{H}$ and $(\mathcal{D})^2$ is essentially self-adjoint on this domain;
1) \( i\mathcal{D} \) is essentially self-adjoint on \( \text{Dom} \mathcal{D}^* \cap \text{Dom} \mathcal{D} \);

2a) \( [\mathcal{J} \mathcal{D}^2, \mathcal{J}] : \mathcal{H}_\infty \to \mathcal{H}_\infty \) and \( \{ (\mathcal{D})^2, \mathcal{J}[\mathcal{D}^2, \mathcal{J}] \} \in \text{OP}^2((\mathcal{D})) \);

2b) \( \mathcal{A}[\mathcal{D}^2, \mathcal{J}](1 + (\mathcal{D})^2)^{-1} \in \mathcal{K}(\mathcal{H}) \) for all \( \mathcal{A} \in \mathcal{A} \);

3) \( \mathcal{J} \) preserves \( \text{Dom} \mathcal{D} \) and \( \text{Dom} \mathcal{D}^* \), and the commutators \( [\mathcal{D}, \mathcal{J}] \) and \( [\mathcal{D}^*, \mathcal{J}] \) extend to bounded operators on \( \mathcal{H} \), for all \( \mathcal{A} \in \mathcal{A} \);

4) \( \mathcal{A}(1 + (\mathcal{D})^2)^{-1/2} \in \mathcal{K}(\mathcal{H}) \) for all \( \mathcal{A} \in \mathcal{A} \).

The triple is said to be even if there exists \( \mathcal{A} \in \mathcal{B}(\mathcal{H}) \) such that \( \mathcal{G} = \mathcal{G}^*, \mathcal{G}^2 = 1 \), \( \mathcal{G} \mathcal{A} = \mathcal{A} \mathcal{G} \) for all \( \mathcal{A} \in \mathcal{A} \), \( \mathcal{G} \mathcal{G}^* + \mathcal{G}^* \mathcal{G} = 0 \), and \( \mathcal{G} \mathcal{D} + \mathcal{D} \mathcal{G} = 0 \). Otherwise the triple is said to be odd.

Since the operator \( \mathcal{J} \) is self-adjoint and unitary, it can be viewed as a fundamental symmetry which turns \( \mathcal{H} \) into a Krein space (we refer to [Bog74] for an introduction to Krein spaces). The essential self-adjointness of \( i\mathcal{D} \) implies that \( \mathcal{D}^* = -\mathcal{J} \mathcal{D} \) on \( \text{Dom} \mathcal{D}^* \cap \text{Dom} \mathcal{D} \), and that \( \mathcal{J} \) preserves this domain. Thus, \( i\mathcal{D} \) is Krein-self-adjoint on the Krein space \( \mathcal{H} \) with fundamental symmetry \( \mathcal{J} \). We then observe that we can rewrite

\[
\mathcal{J}[\mathcal{D}^2, \mathcal{J}] = \mathcal{J}\mathcal{D}^2 \mathcal{J} - \mathcal{D}^2 = \mathcal{D}^* \mathcal{D} - \mathcal{D}^2 = -2\mathcal{J} \mathcal{R} \mathcal{D}.
\]

Thus, a Lorentz-type spectral triple is a special case of a pseudo-Riemannian spectral triple.

As noted in [PS06, page 5], the condition \( \mathcal{G} \mathcal{G}^* + \mathcal{G}^* \mathcal{G} = 0 \) is zero. This can be refined using a real structure.

**Lemma 4.22.** Let \( (\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}) \) be a Lorentz-type spectral triple. Then the K-homology classes arising from the Wick rotations \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) are negatives of one another.

**Proof.** Using \( \mathcal{D}^* = -\mathcal{J} \mathcal{D} \), one simply computes that \( \mathcal{J} \mathcal{D}_+ \mathcal{J} = -\mathcal{D}_- \), which shows that \( (\mathcal{A}, \mathcal{H}, \mathcal{D}_+) \) is unitarily equivalent to \( (\mathcal{A}, \mathcal{H}, -\mathcal{D}_-) \). \( \square \)

**Definition 4.23.** By analogy with the classical case, we say that the fundamental symmetry \( \mathcal{J} \) is Lorentz-harmonic if it commutes with \( \mathcal{D}^2 \).

**Proposition 4.24.** Let \( (\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}, \mathcal{G}) \) be an even Lorentz-type spectral triple with \( \mathcal{A} \) unital. Assume that there exists at least one even continuous function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(0) \neq 0 \) and \( f(\mathcal{D}_\pm) \) trace class. If \( \mathcal{J} \) is Lorentz-harmonic, then

\[
\text{Index}\left(\frac{1 - \mathcal{G}}{2} \mathcal{D}_\pm \frac{1 + \mathcal{G}}{2}\right) = 0.
\]

That is, the pairing of \( [(\mathcal{A}, \mathcal{H}, \mathcal{D}_\pm)] \in \mathcal{K}_0(\mathcal{A}) \) with the class \( [1] \in \mathcal{K}_0(\mathcal{A}) \) is zero.

**Remark 4.25.** The hypothesis on the summability of \( \mathcal{D}_\pm \) (i.e., the existence of such a function \( f \)) is of course implied by \( \theta \)-summability or finite summability.
Proof. First, observe that $J$ commutes with $(\mathcal{D})^2$. Since by assumption $D^*2 - D^2 = J[D^2, J] = 0$, we find that $D^2 = (\mathcal{D})^2$. Hence $J$ commutes with $D^2$ and also with any function of $D^2$. The index can be computed using the McKean-Singer formula, and we refer to [CPRS06b] for this version. For any even function $f$ not vanishing at 0 and such that $f(D^\pm)$ is trace class we have

$$\text{Index} \left( \frac{1 - \Gamma}{2} D_\pm \pm \frac{1 + \Gamma}{2} \right) = \frac{1}{f(0)} \text{Tr}(\Gamma f(D^\pm)).$$

Then using $J^2 = 1$, $J\Gamma + \Gamma J = 0$, $JD^2 = D^2 J$, and cyclicity of the trace, it is straightforward to show that $\text{Tr}(\Gamma f(D^\pm))$ must vanish identically. Hence the index vanishes, and the proof is complete. \qed
The framework of Connes’ noncommutative geometry [Con94], as well as the more general framework of unbounded KK-theory [Kas80, BJ83], deals with noncommutative generalisations of elliptic, self-adjoint differential operators. As such, these frameworks are particularly suited to describe Riemannian manifolds. In this chapter, which is based on joint work with Adam Rennie [DR15], we aim to extend these frameworks to allow for non-elliptic and non-symmetric operators, and in particular (normally) hyperbolic operators. Our motivating example is the Dirac operator on a pseudo-Riemannian manifold, i.e. a manifold equipped with an indefinite (but non-degenerate) metric. It is precisely this example that has inspired the terminology for the indefinite Kasparov modules we introduce in Definition 5.1.

This chapter is a continuation in the spirit of Chapter 4, where we defined pseudo-Riemannian spectral triples \((A, \mathcal{H}, \mathcal{D})\) as a generalisation of spectral triples, and we showed that their Wick rotations \(\mathcal{D}_\pm\) yield spectral triples. Although the motivation for the present chapter is the same, there are nonetheless several significant differences. First, we work more generally with Kasparov modules instead of spectral triples. Second, while the definition of pseudo-Riemannian spectral triples requires assumptions on the second-order operators \(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}\) and \(\mathcal{D}^2 - \mathcal{D}^{*2}\), the definition of indefinite Kasparov modules focuses more on first-order operators (namely \(\mathcal{D}, \mathcal{D}^*, \text{Re}\mathcal{D},\) and \(\text{Im}\mathcal{D}\)), which is more natural. Third, the definition of indefinite Kasparov modules has the advantage that it does not require any smoothness properties (using the OP-spaces defined in Section 2.5). And fourth, it allows to reverse the Wick rotation procedure \(\mathcal{D} \mapsto \mathcal{D}_\pm\), which means that we can characterise all pairs of unbounded Kasparov modules that can be obtained from an indefinite Kasparov module in this way.

Given an indefinite Kasparov module \((A, E_B, \mathcal{D})\), the main technical challenge is to obtain self-adjointness for \(\mathcal{D}_\pm\). In Chapter 4, this is achieved by assuming that \(\langle \mathcal{D} \rangle^2 := (\text{Re}\mathcal{D})^2 + (\text{Im}\mathcal{D})^2\) is self-adjoint, and that the anti-commutator \(\{\text{Re}\mathcal{D},\text{Im}\mathcal{D}\}\) is ‘suitably bounded’ relative to \(\langle \mathcal{D} \rangle^2\). In this chapter, we prefer to avoid assumptions on the second-order operator \(\langle \mathcal{D} \rangle^2\). Instead, we now impose the
condition that the real and imaginary parts of $D$ almost anti-commute, which means that the anti-commutator $\{\text{Re } D, \text{Im } D\}$ is relatively bounded by $\text{Re } D$. A theorem by Kaad and Lesch [KL12] (quoted in Theorem 2.22) then allows us to conclude that $D_\pm$ are self-adjoint. Unfortunately, our main motivating example, namely the Dirac operator $D$ on a pseudo-Riemannian manifold, does not satisfy this condition (see Remark 5.19). Indeed, although the anti-commutator $\{\text{Re } D, \text{Im } D\}$ is a first-order differential operator, it contains in general both spacelike derivatives and timelike derivatives, and thus it is not relatively bounded by $\text{Re } D$ (which only contains spacelike derivatives). In order to ensure that $\text{Re } D$ and $\text{Im } D$ almost anti-commute, we need the timelike part of $\{\text{Re } D, \text{Im } D\}$ to vanish identically, which places a restriction on the geometry of the pseudo-Riemannian manifold (see Section 5.3.1).

For this reason, we would like to replace the almost anti-commuting condition by a weaker condition, which should simply ensure that $\{\text{Re } D, \text{Im } D\}$ is ‘first-order’. Unfortunately, it is currently unclear whether such a weaker condition could still suffice to prove self-adjointness of $D_\pm$ (see the Outlook for further discussion).

Let us give a brief overview of this chapter. In Section 5.1 we define indefinite Kasparov modules as well as pairs of Kasparov modules, and we prove that these definitions are equivalent.

We introduce an odd version of indefinite Kasparov modules in Section 5.2. As for usual Kasparov modules, it is straightforward to turn an odd indefinite Kasparov module into an even one by ‘doubling it up’. We then prove that these odd modules are characterised by pairs of Kasparov modules for which the two operators are related via a certain unitary equivalence.

Next, we describe several examples. The main motivating example, namely the Dirac operator on a pseudo-Riemannian spin manifold, will be discussed in Section 5.3. In Section 5.4 we consider the harmonic oscillator in arbitrary dimensions. This example in particular shows that manifolds with indefinite metrics are not the only examples of our framework. Finally, in Section 5.5 we discuss families of spectral triples (building upon work by Kaad and Lesch [KL13]), and we show that one can naturally associate an indefinite Kasparov module to such families. Our work on families of spectral triples was initially motivated by the study of foliations of spacetime from the perspective of noncommutative geometry.

## 5.1 Indefinite Kasparov Modules

For a closed operator $D$ on a Hilbert $B$-module $E$ such that $\text{Dom } D \cap \text{Dom } D^*$ is dense, we recall from Lemma 2.45 that $\text{Dom } D \cap \text{Dom } D^*$ can be viewed as a Hilbert module with the ‘combined graph inner product’ $(\cdot | \cdot)_{D, D^*}$. The definition we give below relies on the use of this inner product and the corresponding ‘combined
graph norm $\| \cdot \|_{D, D^*}$. We also recall from Section 2.3 that a pair of self-adjoint operators $(S, T)$ is an almost anti-commuting pair if the anti-commutator $\{S, T\}$ is relatively bounded by $S$.

**Definition 5.1.** Given $\mathbb{Z}_2$-graded $C^*$-algebras $A$ and $B$, an *indefinite* unbounded Kasparov $A$-$B$-module $(A, \pi E_B, D)$ is given by

- a $\mathbb{Z}_2$-graded, countably generated, right Hilbert $B$-module $E$;
- a $\mathbb{Z}_2$-graded $*$-homomorphism $\pi: A \to \text{End}_B(E)$;
- a separable dense $*$-subalgebra $A \subset \mathcal{A}$;
- a closed, regular, odd operator $D: \text{Dom} D \subset E \to E$ such that
  1. there exists a linear subspace $E \subset \text{Dom} D \cap \text{Dom} D^*$ which is dense with respect to $\| \cdot \|_{D, D^*}$, and which is a core for $D$;
  2. the operators $\text{Re} D$ and $\text{Im} D$ are regular and essentially self-adjoint on $E$;
  3. the pair $(\text{Re} D, \text{Im} D)$ is an *almost anti-commuting pair*;
  4. we have the inclusion $\pi(A) \cdot E \subset \text{Dom} D \cap \text{Dom} D^*$, and the graded commutators $[D, \pi(a)]_\pm$ and $[D^*, \pi(a)]_\pm$ are bounded on $E$ for each $a \in A$;
  5. the map $\pi(a) \circ \iota: \text{Dom} D \cap \text{Dom} D^* \hookrightarrow E \to E$ is compact for each $a \in A$, where $\iota: \text{Dom} D \cap \text{Dom} D^* \hookrightarrow E$ denotes the natural inclusion map, and $\text{Dom} D \cap \text{Dom} D^*$ is considered as a Hilbert $B$-module with the inner product $\langle \cdot, \cdot \rangle_{D, D^*}$.

If $B = C$ and $A$ is trivially graded, we will write $E = \mathcal{H}$ and refer to $(A, \mathcal{H}, D)$ as an even *indefinite spectral triple* over $A$.

If $D$ is self-adjoint, this is just the usual definition (see Definition 2.25) of an unbounded Kasparov $A$-$B$-module (or spectral triple if $B = C$).

Next, we will show that the linear subspace $E$ in the above definition can always be replaced by $\text{Dom} D \cap \text{Dom} D^*$. The trickiest part turns out to be condition 4), for which we prove a separate lemma first.

**Lemma 5.2** (cf. [FMR14, Proposition 2.1]). Let $D$ be a closed regular operator on a Hilbert $B$-module $E$ such that $\text{Dom} D \cap \text{Dom} D^*$ is dense. Let $E \subset \text{Dom} D \cap \text{Dom} D^*$ be dense with respect to the norm $\| \cdot \|_{D, D^*}$, and let $A$ be a $*$-subalgebra of $\text{End}_B(E)$. Suppose that we have $A \cdot E \subset \text{Dom} D \cap \text{Dom} D^*$, and that for each $a \in A$ the operators $[D, a]$ and $[D^*, a]$ are bounded on $E$. Then $A$ also preserves $\text{Dom} D \cap \text{Dom} D^*$, and $[D, a]$ and $[D^*, a]$, initially defined on $\text{Dom} D \cap \text{Dom} D^*$, extend to bounded endomorphisms on $E$, for all $a \in A$.

**Proof.** The proof is a straightforward adaptation of [FMR14, Proposition 2.1], which proves the statement for the case of self-adjoint operators on a Hilbert space. For completeness we will work out the details here.
Let $\psi \in \text{Dom} \, D \cap \text{Dom} \, D^*$. By assumption there exists a sequence $\psi_n \in \mathcal{E}$ such that $\psi_n \rightarrow \psi$ in the norm $\| D \psi_n - D \psi_m \|$, which is equivalent to $\psi_n \rightarrow \psi$, $D \psi_n \rightarrow D \psi$, and $D^* \psi_n \rightarrow D^* \psi$, in the usual norm. The sequence $D a \psi_n$ is Cauchy (in the usual norm), since

$$
\| D a \psi_n - D a \psi_m \| = \| a D \psi_n - a D \psi_m + [D, a] \psi_n - [D, a] \psi_m \| \\
\leq \| a \| \| D \psi_n - D \psi_m \| + \| [D, a] \| \| \psi_n - \psi_m \|,
$$

and similarly $D^* a \psi_n$ is also Cauchy. Hence the sequence $a \psi_n \in \text{Dom} \, D \cap \text{Dom} \, D^*$ is Cauchy in the norm $\| \cdot \|_D, D^*$, so there exists a $\xi \in \text{Dom} \, D \cap \text{Dom} \, D^*$ such that $a \psi_n \rightarrow \xi$ in the norm $\| \cdot \|_D, D^*$. But this implies that $a \psi_n \rightarrow \xi$ in the usual norm, and since we already know that $a \psi_n \rightarrow a \psi$ in the usual norm, we conclude that $\xi = a \psi$, and hence $a \psi \in \text{Dom} \, D \cap \text{Dom} \, D^*$. Thus we have shown that $a$ preserves $\text{Dom} \, D \cap \text{Dom} \, D^*$.

To conclude that the commutator $[D, a]$ (and similarly $[D^*, a]$), initially defined on $\text{Dom} \, D \cap \text{Dom} \, D^*$, extends to a bounded endomorphism, it suffices to show that its adjoint is densely defined, since then it is closable, and $[D, a]^* \supset [D, a]$, which is everywhere defined and bounded. For $\psi \in \text{Dom} \, D$ and $\eta \in \mathcal{E}$, we have

$$
([D, a] \psi | \eta) = (D a \psi | \eta) - (a D \psi | \eta) = (\psi | a^* D^* \eta) - (\psi | D^* a^* \eta) = (\psi | -[D^*, a^*] \eta),
$$

which is well-defined because $a^* \in A$ maps $\mathcal{E}$ to $\text{Dom} \, D \cap \text{Dom} \, D^*$. Hence the domain of $[D, a]^*$ contains the dense subset $\mathcal{E}$, which implies that $[D, a]$ is closable.

The same argument applies to $[D^*, a]$.

**Proposition 5.3.** If $(A, E_B, D)$ is an indefinite unbounded Kasparov $A$-$B$-module, then the subset $\mathcal{E}$ in Definition 5.1 can be replaced by $\text{Dom} \, D \cap \text{Dom} \, D^*$.

**Proof.** If $\mathcal{E} \subset \text{Dom} \, D \cap \text{Dom} \, D^*$ is a core for $D$, then so is $\text{Dom} \, D \cap \text{Dom} \, D^*$. Lemma 2.48 implies that $\text{Dom} \, D \cap \text{Dom} \, D^*$ is contained in the domains of $\text{Re} \, D$ and $\text{Im} \, D$, so the operators $\text{Re} \, D$ and $\text{Im} \, D$ are also essentially self-adjoint on $\text{Dom} \, D \cap \text{Dom} \, D^*$. Using Lemma 5.2 then concludes the proof.

### 5.1.1 Pairs of Kasparov modules

**Definition 5.4.** We say $(A, E_B, D_1, D_2)$ is a pair of unbounded Kasparov $A$-$B$-modules if $(A, E_B, D_1)$ and $(A, E_B, D_2)$ are unbounded Kasparov $A$-$B$-modules such that:

1) there exists a linear subspace $\mathcal{E} \subset \text{Dom} \, D_1 \cap \text{Dom} \, D_2$ which is a common core for $D_1$ and $D_2$;

2) the operators $D_1 + D_2$ and $D_1 - D_2$ are regular and essentially self-adjoint on $\mathcal{E}$;
3) the operator \((D_1^2 - D_2^2)(D_1 + D_2 - i\mu)^{-1}\) is a well-defined and bounded endomorphism for all \(\mu \in \mathbb{R}\setminus\{0\}\).

If \(B = C\) and \(A\) is trivially graded, we will write \(E = \mathcal{H}\) and refer to \((A, \mathcal{H}, D_1, D_2)\) as an even pair of spectral triples over \(A\).

**Remark 5.5.** The third assumption shows that the pair \((D_1 + D_2, D_1 - D_2)\) is an almost anti-commuting pair, and it then follows from Corollary 2.23 that in fact \(\text{Dom } D_1 = \text{Dom } D_2\). Similarly to Proposition 5.3, we can then replace \(E\) by \(\text{Dom } D_1 = \text{Dom } D_2\).

Recall from Section 2.6.1 the Wick rotations \(D_{\pm} := \text{Re } D \pm i\text{Im } D\) of a closed operator \(D\). The definition of pairs of Kasparov modules is motivated by the fact that they arise as the Wick rotations of indefinite Kasparov modules, which we now show.

**Proposition 5.6** (Wick rotation). Let \((A, E_B, D)\) be an indefinite unbounded Kasparov \(A\)-\(B\)-module. Then the Wick rotations \(D_{\pm}\) and \(D_{\mp}\) form a pair of unbounded Kasparov \(A\)-\(B\)-modules \((A, E_B, D_{\pm}, D_{\mp})\).

**Proof.** By assumption, the operators \(\text{Re } D\) and \(\text{Im } D\) form an almost anti-commuting pair \((\text{Re } D, \text{Im } D)\). By Corollary 2.23 it then follows that \(D_{\pm} = \text{Re } D \pm \text{Im } D\) are self-adjoint on the domain \(\text{Dom } \text{Re } D \cap \text{Dom } \text{Im } D\). In particular, the domain \(\text{Dom } \text{Re } D \cap \text{Dom } \text{Im } D\) is closed with respect to each of the graph norms of \(D_{\pm}\), so by Lemma 2.48 we have \(\text{Dom } D_{\pm} = \text{Dom } \text{Re } D \cap \text{Dom } \text{Im } D\) which shows property 1) for \(E := \text{Dom } D_{\pm}\). By assumption \(D_{\mp} = 2 \text{Re } D\) and \(D_{\mp} = 2 \text{Im } D\) are essentially self-adjoint on \(\text{Dom } D_{\pm}\), which gives property 2). Since \(D_{\mp}^2 - D_{\pm}^2 = 2[\text{Re } D, \text{Im } D]\) we also have property 3).

To complete the proof that \(D_{\pm}\) yield unbounded Kasparov modules, first observe that \([\text{Re } D, a]\) and \([\text{Im } D, a]\) are bounded on \(\text{Dom } D_{\pm} = \text{Dom } D \cap \text{Dom } D^*\), and hence it follows that \([\text{Re } D \pm \text{Im } D, a]\), initially defined on \(\text{Dom } D \cap \text{Dom } D^*\), extend to bounded endomorphisms on \(E\).

Finally, we know from Lemma 2.48 that \(\text{Dom } D_{\mp} = \text{Dom } \text{Re } D \cap \text{Dom } \text{Im } D\) is equal to \(\text{Dom } D \cap \text{Dom } D^*\) (with the same norm-topology), and by assumption the map \(\pi(a) \circ \iota : \text{Dom } D \cap \text{Dom } D^* \rightarrow E\) is compact. Thus the Wick rotations \((A, E_B, D_{\pm})\) are indeed unbounded Kasparov modules.

Given two symmetric operators \(D_1\) and \(D_2\), we recall from Section 2.6.2 the reverse Wick rotation \(D := \frac{1}{2}(D_1 + D_2) + \frac{i}{2}(D_1 - D_2)\). We now prove a converse to the above proposition.

**Proposition 5.7** (reverse Wick rotation). Let \((A, E_B, D_1, D_2)\) be a pair of unbounded Kasparov modules, and let \(D\) be the reverse Wick rotation of \((D_1, D_2)\). Then \((A, E_B, D)\) is an indefinite unbounded Kasparov \(A\)-\(B\)-module.
Proof. As mentioned in Remark 5.5, we can pick $\mathcal{E} = \text{Dom} \mathcal{D}_1 = \text{Dom} \mathcal{D}_2$. By Lemma 2.51 we have the equalities $\mathcal{E} = \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^* = \text{Dom} \text{Re} \mathcal{D} \cap \text{Dom} \text{Im} \mathcal{D}$ (where $\mathcal{E}$ is equipped with the norm $\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2}$), and this domain is by definition a core for the reverse Wick rotation $\mathcal{D}$. On this domain we can write
\[ \text{Re} \mathcal{D} \psi = \frac{1}{2} (\mathcal{D}_1 + \mathcal{D}_2) \psi, \quad \text{Im} \mathcal{D} \psi = \frac{1}{2} (\mathcal{D}_1 - \mathcal{D}_2) \psi. \]
Thus by assumption the operators $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ are essentially self-adjoint on $\mathcal{E}$.

The operator
\[ \{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\} (\text{Re} \mathcal{D} - i \mu)^{-1} = (\mathcal{D}_1^2 - \mathcal{D}_2^2)(\mathcal{D}_1 + \mathcal{D}_2 - 2i \mu)^{-1} \]
is well-defined and bounded by assumption. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ have bounded commutators with $\mathcal{A}$, it follows immediately that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ also have bounded commutators with $\mathcal{A}$. The identity map $(\mathcal{E}, \| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2}) \rightarrow (\text{Dom} \mathcal{D}_1, \| \cdot \|_{\mathcal{D}_1})$ is continuous, because the graph norm of $\mathcal{D}_1$ is bounded by the norm $\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2}$ on $\mathcal{E}$ (and similarly for $\mathcal{D}_2$). Since $\mathcal{D}_1$ (or $\mathcal{D}_2$) has locally compact resolvent, it then follows that the map $\pi(a) \circ \iota : \mathcal{E} \leftrightarrow \mathcal{E}$ is compact for each $a \in \mathcal{A}$. \hfill $\square$

Remark 5.8. 1) Let $(\mathcal{A}, \mathcal{E}, \mathcal{D})$ be an indefinite unbounded Kasparov module with Wick rotations $\mathcal{D}_+$ and $\mathcal{D}_-$. Denote by $\tilde{\mathcal{D}}$ the reverse Wick rotation of $(\mathcal{D}_+, \mathcal{D}_-)$. By construction we then have
\[ \tilde{\mathcal{D}} = \mathcal{D} |_{\text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^*}. \]

In other words, the reverse Wick rotation of the Wick rotations of $\mathcal{D}$ is precisely the closure of the restriction of $\mathcal{D}$ discussed in Remark 2.46. Since we have assumed that $\text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^*$ is a core for $\mathcal{D}$, it follows that $\tilde{\mathcal{D}} = \mathcal{D}$. Thus, this assumption ensures that our procedure of Wick rotation is reversible.

2) We can consider unitary equivalences of indefinite Kasparov modules or pairs of Kasparov modules as in Definition 2.30, and one easily sees that Wick rotations and reverse Wick rotations respect such unitary equivalences.

Combining these observations with Propositions 5.6 and 5.7, we can summarise our results as follows:

**Theorem 5.9.** The procedure of (reverse) Wick rotation implements a bijection between:

- the set of indefinite unbounded Kasparov $\mathcal{A}$-$\mathcal{B}$-modules $(\mathcal{A}, \mathcal{E}, \mathcal{D})$; and
- the set of pairs of unbounded Kasparov $\mathcal{A}$-$\mathcal{B}$-modules $(\mathcal{A}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2)$.

This bijection also descends to the corresponding sets of unitary equivalence classes.
We introduce an odd version of indefinite Kasparov modules, where all gradings are trivial, and the operator $D$ is (of course) no longer assumed to be odd.

**Definition 5.10.** Given trivially graded $C^*$-algebras $A$ and $B$, an *odd indefinite* unbounded Kasparov $A$-$B$-module $(A, \pi E_B, D)$ is given by

- a trivially graded, countably generated, right Hilbert $B$-module $E$;
- a $*$-homomorphism $\pi: A \to \text{End}_B(E)$;
- a separable dense $*$-subalgebra $A \subset A$;
- a closed, regular operator $D: \text{Dom } D \subset E \to E$ such that
  1) there exists a linear subspace $E \subset \text{Dom } D \cap \text{Dom } D^*$ which is dense in the norm $\| \cdot \|_{D, D^*}$ and which is a core for $D$;
  2) the operators $\text{Re } D$ and $\text{Im } D$ are regular and essentially self-adjoint on $E$;
  3) the pair $(\text{Re } D, \text{Im } D)$ is an *almost commuting pair*;
  4) we have the inclusion $\pi(A) \cdot E \subset \text{Dom } D \cap \text{Dom } D^*$, and the commutators $[D, \pi(a)]$ and $[D^*, \pi(a)]$ are bounded on $E$ for each $a \in A$;
  5) the map $\pi(a) \circ \iota: \text{Dom } D \cap \text{Dom } D^* \hookrightarrow E \to E$ is compact for each $a \in A$,

where $\iota: \text{Dom } D \cap \text{Dom } D^* \hookrightarrow E$ denotes the natural inclusion map, and $\text{Dom } D$ is considered as a Hilbert module with the inner product $(\cdot | \cdot)_{D, D^*}$.

If $B = C$, we will write $E = \mathcal{H}$ and refer to $(A, \mathcal{H}, D)$ as an *odd indefinite spectral triple* over $A$.

**Remark 5.11.**

- We emphasise that, in the odd case, the pair $(\text{Re } D, \text{Im } D)$ is assumed to almost *commute* (instead of almost anti-commute). This assumption can be reinterpreted as saying that the commutator $[D, D^*]$ is relatively bounded by the sum $D + D^*$; in this sense $D$ is ‘almost normal’.
- It follows from 3) and *Theorem 2.22* that in fact we have $\text{Dom } D = \text{Dom } D^*$, and as in *Proposition 5.3* we can then always replace $E$ by $\text{Dom } D$.

Given an odd indefinite unbounded Kasparov module $(A, E_B, D)$, we can again consider its Wick rotations

$$D_+ := \text{Re } D + \text{Im } D, \quad D_- := \text{Re } D - \text{Im } D,$$

on the initial domain $\text{Dom } \text{Re } D \cap \text{Dom } \text{Im } D = \text{Dom } D \cap \text{Dom } D^* = \text{Dom } D$ (see *Lemma 2.48*). The following example shows that these Wick rotations are not as well-behaved as in the $\mathbb{Z}_2$-graded case.
Example 5.12. Let \((A, E_B, D)\) be an odd unbounded Kasparov module, and consider the operator \(\tilde{D} := (1 + i)D\). Then \((A, E_B, \tilde{D})\) is an odd indefinite unbounded Kasparov module, and its Wick rotations are \(\tilde{D}_+ = 2\tilde{D}\) and \(\tilde{D}_- = 0\). The problematic one is obviously \(\tilde{D}_-\), as it is not closed on \(\text{Dom}\tilde{D}\), and it does not have locally compact resolvent.

Hence the assumptions of an odd indefinite unbounded Kasparov module do not imply that the Wick rotations yield odd unbounded Kasparov modules. However, by Proposition 2.24 we do know that \(\tilde{D}_+\) and \(\tilde{D}_-\) are essentially self-adjoint, and we will denote their self-adjoint closures by \(D_+\) and \(D_-\) as well.

Given an odd unbounded Kasparov module \((A, E_B, D)\), we can construct an (even) unbounded Kasparov module \((A, (E \oplus E)_B, \tilde{D})\) (as discussed in Section 2.4), where in \(E \oplus E\) the first summand is considered even and the second summand odd. The following theorem gives a similar ‘doubling trick’ for the indefinite case.

Theorem 5.13. Given trivially graded \(C^*\)-algebras \(A\) and \(B\), let \(E\) be a trivially graded, countably generated right Hilbert \(B\)-module with a \(*\)-homomorphism \(\pi: A \to \text{End}_B(E)\), let \(A \subset A\) be a separable dense \(*\)-subalgebra, and let \(D: \text{Dom}\tilde{D} \to E\) be a closed, regular operator. Consider (the closures of) the operators

\[
D_+ := \text{Re} D + \text{Im} D, \quad D_- := \text{Re} D - \text{Im} D, \\
\tilde{D}_+ := \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}, \quad \tilde{D}_- := \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.
\]

Then the following are equivalent:

1) \((A, E_B, D)\) is an odd indefinite unbounded Kasparov \(A\)-\(B\)-module;
2) \((A, (E \oplus E)_B, \tilde{D})\) is an indefinite unbounded Kasparov \(A\)-\(B\)-module;
3) \((A, (E \oplus E)_B, \tilde{D}_+, \tilde{D}_-\) is a pair of unbounded Kasparov \(A\)-\(B\)-modules.

Proof. One easily sees that the reverse Wick rotation of \((\tilde{D}_+, \tilde{D}_-)\) equals \(\tilde{D}\), and the equivalence of 2) and 3) then follows from Theorem 5.9. Hence it suffices to prove the equivalence of 1) and 3).

1) \(\Rightarrow\) 3): Let \((A, E_B, D)\) be an odd indefinite unbounded Kasparov \(A\)-\(B\)-module. We have \(\text{Dom} D = \text{Dom} D^\ast = \text{Dom} \text{Re} D \cap \text{Dom} \text{Im} D\) (the first equality holds by Remark 5.11, the second follows from Lemma 2.48), and we can write \(D = \text{Re} D + i \text{Im} D\) and \(D^\ast = \text{Re} D - i \text{Im} D\). Thus the operators

\[
\tilde{D}_+ = \begin{pmatrix} 0 & D^\ast \\ D & 0 \end{pmatrix}, \quad \tilde{D}_- = \begin{pmatrix} 0 & D \\ D^\ast & 0 \end{pmatrix}
\]

are self-adjoint on \((\text{Dom} \text{Re} D \cap \text{Dom} \text{Im} D)^{\oplus 2}\). For all \(a \in A\), we know that \([D, a]\) and \([D^\ast, a]\) are bounded, and therefore \([\tilde{D}_+, a]\) and \([\tilde{D}_-, a]\) are also
bounded. Furthermore, the inclusion of \((\text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D})^{\oplus 2}\) in \(E \oplus E\) is locally compact, because \(\text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D} = \text{Dom } \mathcal{D} \hookrightarrow E\) is locally compact by assumption. Thus both \((A, (E \oplus E)_B, \mathcal{D}_\pm)\) are unbounded Kasparov \(A-B\)-modules.

The operators \(\mathcal{D}_+ + \mathcal{D}_-\) and \(\mathcal{D}_+ - \mathcal{D}_-\) are essentially self-adjoint on the domain \((\text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D})^{\oplus 2}\), because \(\text{Re } \mathcal{D}\) and \(\text{Im } \mathcal{D}\) are essentially self-adjoint on \(\text{Dom Re } \mathcal{D} \cap \text{Dom Im } \mathcal{D}\). The difference between the squares of \(\mathcal{D}_+\) and \(\mathcal{D}_-\) equals

\[
(\mathcal{D}_+)^2 - (\mathcal{D}_-)^2 = \begin{pmatrix}
2i[\text{Re } \mathcal{D}, \text{Im } \mathcal{D}] & 0 \\
0 & -2i[\text{Re } \mathcal{D}, \text{Im } \mathcal{D}]
\end{pmatrix},
\]

and since each of the corners is relatively bounded by \(\text{Re } \mathcal{D}\), it follows that \((A, (E \oplus E)_B, \mathcal{D}_+, \mathcal{D}_-)\) is indeed a pair of unbounded Kasparov \(A-B\)-modules.

3) \(\Rightarrow\) 1): Suppose \((A, (E \oplus E)_B, \mathcal{D}_+, \mathcal{D}_-)\) is a pair of unbounded Kasparov \(A-B\)-modules. The property \(\text{Dom } \mathcal{D}_+ = \text{Dom } \mathcal{D}_-\) (see Remark 5.5) then implies that \(\text{Dom } \mathcal{D} = \text{Dom } \mathcal{D}^*\). Since \(\mathcal{D}_+ \pm \mathcal{D}_-\) are essentially self-adjoint it follows that \(\text{Re } \mathcal{D}\) and \(\text{Im } \mathcal{D}\) are essentially self-adjoint on \(\text{Dom } \mathcal{D}\). From Eq. (5.1) and the boundedness of \(((\mathcal{D}_+)^2 - (\mathcal{D}_-)^2)(\mathcal{D}_+ + \mathcal{D}_- - i\mu)^{-1}\) we conclude that \((\text{Re } \mathcal{D}, \text{Im } \mathcal{D})\) is an almost commuting pair. For all \(a \in A\), we know that \([\mathcal{D}_+, a]\) and \([\mathcal{D}_-, a]\) are bounded, and therefore \([\mathcal{D}, a]\) and \([\mathcal{D}^*, a]\) are also bounded. Finally, since the inclusion \((\text{Dom } \mathcal{D})^{\oplus 2} \hookrightarrow E \oplus E\) is locally compact, it follows that the inclusion \(\text{Dom } \mathcal{D} \hookrightarrow E\) is also locally compact. Thus \((A, E_B, \mathcal{D})\) is indeed an odd indefinite unbounded Kasparov \(A-B\)-module.

We point out that the indefinite Kasparov module \((A, (E \oplus E)_B, \mathcal{D})\), given (as defined above) by the operator

\[
\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_+ \\ \mathcal{D}_- & 0 \end{pmatrix},
\]

is a very special type of indefinite Kasparov module. For instance, its entries \(\mathcal{D}_+\) and \(\mathcal{D}_-\) are both essentially self-adjoint, and they have a common core (namely \(\text{Dom } \mathcal{D}\)). The special nature of such \(\mathcal{D}\) is reflected by the following property of their Wick rotations \(\mathcal{D}_\pm\).

**Proposition 5.14.** Let \((A, E_B, \mathcal{D})\) be an odd indefinite unbounded Kasparov module, and consider the corresponding pair of unbounded Kasparov modules \((A, (E \oplus E)_B, \mathcal{D}_+, \mathcal{D}_-)(\text{as in Theorem 5.13})\). Then \((A, (E \oplus E)_B, \mathcal{D}_+)\) and \((A, (E \oplus E)_B^\text{op}, -\mathcal{D}_-)\) are unitarily equivalent, and hence we have \([[(A, (E \oplus E)_B, \mathcal{D}_+)] = -[[A, (E \oplus E)_B, \mathcal{D}_-]] \in \text{KK}(A, B)\).
Proof. First, the operators $\tilde{D}_+$ and $-\tilde{D}_-$ are unitarily equivalent:

$$\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
D^* & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -D^* \\
-D^* & 0
\end{pmatrix}.$$  

However, we also find that

$$\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
-1 \ 0 \\
0 & 1
\end{pmatrix},$$

so under this unitary equivalence the $\mathbb{Z}_2$-grading becomes the opposite. An element $a \in A$ acts as $a \oplus a$ on $E \oplus E$, and therefore it remains unchanged. Thus, we have the unitary equivalence $(A, (E \oplus E)_B, \tilde{D}_+) \sim (A, (E \oplus E)_B^{\text{op}}, -\tilde{D}_-)$. For the last statement, we recall that the class of $(A, (E \oplus E)_B^{\text{op}}, -\tilde{D}_-)$ is the negative of the class of $(A, (E \oplus E)_B, \tilde{D}_-)$. \hfill $\square$

We would like to characterise the types of indefinite Kasparov modules that are obtained from odd indefinite Kasparov modules, and for this purpose we prove a converse to the above proposition, which extends Proposition 2.32 to the indefinite case.

**Proposition 5.15.** Let $A$ and $B$ be trivially graded C*-algebras. Let $(A, E_B, D_1, D_2)$ be a pair of unbounded Kasparov $A$-$B$-modules such that $(A, E_B, D_1)$ is unitarily equivalent to $(A, E_B^{\text{op}}, -D_2)$ via an anti-self-adjoint unitary

$$\begin{pmatrix}
0 & -U^* \\
U & 0
\end{pmatrix},$$

where $U$ is a unitary isomorphism $E^0 \to E^1$ and we identify $E^{\text{op}} \simeq E = E^0 \oplus E^1$ as ungraded modules. Then $(A, E_B^0, U^* D_1|_{E^0})$ is an odd indefinite unbounded Kasparov $A$-$B$-module.

**Remark 5.16.** Suppose that $D_1 = D_2$, so we just have an unbounded Kasparov module $(A, E_B, D_1)$. As mentioned in Remark 2.33, we then have an unbounded Kasparov module $(A \otimes \text{Cl}_1, E_B, D_1)$ which represents a class in the odd KK-theory $KK^1(A, B) = KK(A \otimes \text{Cl}_1, B)$. If $D_1 \neq D_2$ however, the anti-self-adjoint unitary does not anti-commute with $D_1$ (or $D_2$), so the pair of Kasparov $A$-$B$-modules does not extend to a pair of Kasparov $A \otimes \text{Cl}_1$-$B$-modules.

**Proof.** If we write the self-adjoint operator $D_1$ on $E^0 \oplus E^1$ as

$$D_1 = \begin{pmatrix}
0 & D_0^* \\
D_0 & 0
\end{pmatrix},$$

...
the unitary equivalence of $\mathcal{D}_1$ and $-\mathcal{D}_2$ then yields
\[
\mathcal{D}_2 = -\begin{pmatrix} 0 & -U^* \\ U & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{D}_0^* \\ \mathcal{D}_0 & 0 \end{pmatrix} \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} = \begin{pmatrix} 0 & U^* \mathcal{D}_0 U^* \\ \mathcal{U}^* \mathcal{D}_0 U & 0 \end{pmatrix}.
\]

The algebra $\mathcal{A}$ is trivially graded, so its representation on $E$ and $E^\text{op}$ is the same. Writing $a = a_0 \oplus a_1$, we find that $a_1 = Ua_0 U^*$. Hence the representation of $\mathcal{A}$ on $E$ is determined by its representation on $E^0$. Using the identification $E^1 = UE^0$, we can rewrite $a$, $\mathcal{D}_1$, and $\mathcal{D}_2$ as operators on $E^0 \oplus E^0$ as
\[
a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}, \quad \mathcal{D}_1 = \begin{pmatrix} 0 & \mathcal{D}_0^* U \\ U^* \mathcal{D}_0 & 0 \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} 0 & U^* \mathcal{D}_0 \\ \mathcal{D}_0^* U & 0 \end{pmatrix}.
\]

By defining $\mathcal{D} := U^* \mathcal{D}_0$: $\text{Dom} \mathcal{D}_0 \to E^0$, this can be rewritten as
\[
\mathcal{D}_1 = \begin{pmatrix} 0 & \mathcal{D}_0^* \\ \mathcal{D}_0 & 0 \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}_0^* & 0 \end{pmatrix}.
\]

Hence it follows from Theorem 5.13 that $(\mathcal{A}, E^0_B, \mathcal{D})$ is an odd indefinite unbounded Kasparov module. \qed

We point out that our constructions are well-defined and reversible up to unitary equivalence (where we need to allow for unitary equivalence because of the freedom in the unitary isomorphism $U$: $E^0 \to E^1$). Combining the previous two propositions with Theorem 5.13, we thus obtain:

**Theorem 5.17.** Let $\mathcal{A}$ and $\mathcal{B}$ be trivially graded $C^*$-algebras. The constructions of Propositions 5.14 and 5.15 implement a bijection between:

1) the set of unitary equivalence classes of odd indefinite unbounded Kasparov $\mathcal{A}$-$\mathcal{B}$-modules $(\mathcal{A}, E^0_B, \mathcal{D})$; and

2) the set of unitary equivalence classes of pairs of unbounded Kasparov $\mathcal{A}$-$\mathcal{B}$-modules $(\mathcal{A}, \tilde{E}_B, \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2)$ such that $(\mathcal{A}, \tilde{E}_B, \tilde{\mathcal{D}}_1)$ is unitarily equivalent to $(\mathcal{A}, \tilde{E}^\text{op}_B, -\tilde{\mathcal{D}}_2)$ via an anti-self-adjoint unitary.

### 5.3 Pseudo-Riemannian Manifolds

In Section 3.4 we have constructed the Dirac operator $\mathcal{D}$ on the spinor bundle $\mathcal{S}$ over a pseudo-Riemannian manifold $(M, g)$. In this section we will see what conditions we need to impose on a spin manifold $(M, g)$ to ensure that the Dirac operator $\mathcal{D}$ yields an indefinite spectral triple.

We will again require our basic Assumption 3.16. Thus, let $(M, g)$ be an $n$-dimensional time- and space-oriented pseudo-Riemannian spin manifold of signature $(t, s)$, and let $r$ be a spacelike reflection, such that the associated Riemannian
metric $g_r$ is complete. We have already seen that this assumption ensures that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ are essentially self-adjoint (Theorem 3.17), and that the Wick rotations of $\mathcal{D}$ give rise to spectral triples $(C_0^\infty(M), L^2(S), \mathcal{D}_\pm)$ (Proposition 3.18). The remaining question is whether these Wick rotations in fact form a pair of spectral triples (see Definition 5.4). Assumption 3.16 already brings us quite close.

**Proposition 5.18.** Let $(M, g)$ be as in Assumption 3.16, and consider the spectral triples $(C_0^\infty(M), L^2(S), \mathcal{D}_\pm)$. Then $\text{Dom} \mathcal{D}_+ \cap \text{Dom} \mathcal{D}_-$ is a common core for $\mathcal{D}_+$ and $\mathcal{D}_-$, and $\mathcal{D}_+ \pm \mathcal{D}_-$ is essentially self-adjoint on this domain.

**Proof.** From Lemma 2.48 we have for the domains of the Wick rotations $\mathcal{D}_\pm$ the equality $\text{Dom} \mathcal{D}_+ \cap \text{Dom} \mathcal{D}_- = \text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^\dagger$. Since this domain contains $\Gamma_0^\infty(S)$, it is a core for both $\mathcal{D}_+$ and $\mathcal{D}_-$. The operators $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ are essentially self-adjoint on $\Gamma_0^\infty(S)$ by Theorem 3.17, and since they can be extended to symmetric operators on $\text{Dom} \mathcal{D} \cap \text{Dom} \mathcal{D}^\star$, it follows that these symmetric extensions are also essentially self-adjoint. Thus $\text{Re} \mathcal{D} = \frac{1}{2}(\mathcal{D}_+ + \mathcal{D}_-)$ and $\text{Im} \mathcal{D} = \frac{1}{2}(\mathcal{D}_+ - \mathcal{D}_-)$ are essentially self-adjoint on $\text{Dom} \mathcal{D}_+ \cap \text{Dom} \mathcal{D}_-$.

**Remark 5.19.** The above proposition shows that (under only mild assumptions) a pseudo-Riemannian spin manifold gives rise to two spectral triples satisfying conditions 1) and 2) in Definition 5.4. From the reverse Wick rotation of Proposition 5.7, we then almost obtain an indefinite spectral triple $(C_0^\infty(M), L^2(S), \mathcal{D})$, except that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ do not almost anti-commute. Indeed, although the anti-commutator $\{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\}$ is a first-order differential operator, it contains in general both spacelike derivatives and timelike derivatives, and thus it is not relatively bounded by $\text{Re} \mathcal{D}$. In order to ensure that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ almost anti-commute, we need the timelike part of $\{\text{Re} \mathcal{D}, \text{Im} \mathcal{D}\}$ to vanish identically. In the next subsection, we will provide sufficient conditions on a Lorentzian manifold to ensure that $\text{Re} \mathcal{D}$ and $\text{Im} \mathcal{D}$ almost anti-commute.

The main reason for imposing this almost anti-commuting condition on indefinite spectral triples is so that we can prove the self-adjointness of the Wick rotations $\mathcal{D}_\pm$. For the Dirac operator $\mathcal{D}$ however, we can simply prove the self-adjointness of the Wick rotations $\mathcal{D}_\pm$ directly (as we did in Proposition 5.18), and this condition is therefore not necessary for describing pseudo-Riemannian manifolds. Hence, as also mentioned at the start of Chapter 5, it would be desirable to be able to weaken the almost anti-commuting condition in the definition of an indefinite spectral triple. We will discuss this in more detail in the Outlook.

5.3.1 **Lorentzian manifolds with parallel time**

**Assumption 5.20.** Let $(M, g, r)$ be an even-dimensional time- and space-oriented Lorentzian spin manifold of signature $(1, n - 1)$, with a given spinor bundle $S$. 

Let \( r \) be a spacelike reflection, such that the associated Riemannian metric \( g_r \) is complete. Assume furthermore that \((M, g, r, S)\) has bounded geometry (as defined in Definition 4.16). Lastly, we assume that the spacelike reflection \( r \) is parallel (i.e. the unit timelike vector field \( e_0 \)). The assumption that \( e_0 \) is parallel then implies that \([\nabla^S, \gamma(e_0)] = 0\). The expressions for the real and imaginary parts of \( \mathcal{D} \) and its Wick rotations then simplify to:

\[
\text{Re } \mathcal{D} = \sum_{j=1}^{n-1} \gamma(e_j) \nabla^S_{e_j}, \quad \text{Im } \mathcal{D} = i \gamma(e_0) \nabla^S_{e_0}, \quad \mathcal{D}_\pm = \sum_{k=0}^{n-1} \gamma_\pm(e_k) \nabla^S_{e_k},
\]

where we recall from Definition 3.13 the Wick rotated Clifford representations \( \gamma_\pm(v) := \pm i \gamma(v_t) + \gamma(v_s) \) (for \( v = v_t + v_s \in E_t \oplus E_s = TM \)).

**Lemma 5.21.** Let \((M, g, r, S)\) satisfy Assumption 5.20. The real and imaginary parts of \( \mathcal{D} \) yield an almost anti-commuting pair \((\text{Re } \mathcal{D}, \text{Im } \mathcal{D})\).

**Proof.** Since \( \gamma(e_0) \) commutes with \( \nabla^S \) and anti-commutes with \( \gamma(e_j) \) (for \( j \neq 0 \)), we calculate

\[
(\text{Re } \mathcal{D}, \text{Im } \mathcal{D}) = i \sum_{j=1}^{n-1} \left( \gamma(e_j) \nabla^S_{e_j} \gamma(e_0) \nabla^S_{e_0} + \gamma(e_0) \nabla^S_{e_0} \gamma(e_j) \nabla^S_{e_j} \right)
\]

\[
= i \sum_{j=1}^{n-1} \left( \gamma(e_j) \gamma(e_0) \nabla^S_{e_j} \nabla^S_{e_0} + \gamma(e_0) \gamma(e_j) \nabla^S_{e_0} \nabla^S_{e_j} + \gamma(e_0) [\nabla^S_{e_0}, \gamma(e_j)] \nabla^S_{e_j} \right)
\]

\[
= i \sum_{j=1}^{n-1} \left( \gamma(e_0) \gamma(e_j) [\nabla^S_{e_0}, \nabla^S_{e_j}] + \gamma(e_0) [\nabla^S_{e_0}, \gamma(e_j)] \nabla^S_{e_j} \right)
\]

\[
= i \sum_{j=1}^{n-1} \left( \gamma(e_0) \gamma(e_j) \nabla^S_{[e_0,e_j]} + \gamma(e_0) \gamma(e_j) \Omega^S(e_0, e_j) \right)
\]

(5.2)

The curvature \( \Omega^S(e_0, e_j) \) and the commutator \([\nabla^S_{e_0}, \gamma(e_j)]\) are bounded by the assumption of bounded geometry, and hence the second term (on the last line) in Eq. (5.2) is bounded and the third term is relatively bounded by \( \text{Re } \mathcal{D} \). Since \( e_0 \) is parallel, we have \([e_0, e_j] = \nabla_{e_0} e_j - \nabla_{e_j} e_0 = \nabla_{e_0} e_j \). Since \( e_0 \) and \( e_j \) are orthogonal (for \( j > 0 \)), we find that

\[
g(\nabla_{e_0} e_j, e_0) = -g(e_j, \nabla_{e_0} e_0) + e_0(g(e_j, e_0)) = 0,
\]
and hence \([e_0, e_j] = \nabla_{e_i} e_j \in E_s\). This means that the first term in Eq. (5.2) also only has spacelike derivatives, and is therefore relatively bounded by \(\text{Re} D\) as well. \(\square\)

Combining this with Proposition 5.18, and applying the reverse Wick rotation of Proposition 5.7, we obtain the following.

**Corollary 5.22.** Let \((M, g, r, S)\) satisfy Assumption 5.20. The Wick rotations \(D_{\pm}\) yield an even pair of spectral triples \((C^\infty_c(M), L^2(S), D_+, D_-)\), and hence \((C^\infty_c(M), L^2(S), D)\) is an even indefinite spectral triple.

Finally, we relate the Wick rotations \(D_{\pm}\) of the Lorentzian Dirac operator on \((M, g)\) to the canonical Dirac operator on the Riemannian manifold \((M, g_r)\). Since \(M\) is even-dimensional, recall from Section 3.3.4 that the spinor bundle \(S\) is \(\mathbb{Z}_2\)-graded with the grading operator given by

\[
\Gamma_M := i^{-1 + \frac{n(n+1)}{2}} \gamma(e_0) \cdots \gamma(e_{n-1}),
\]

where for the Lorentzian signature we of course have \(t = 1\).

**Proposition 5.23.** The Wick rotations \(D_{\pm}\) of the Dirac operator \(D\) on the Lorentzian manifold \((M, g, r, S)\) are the two canonical Dirac operators on the Wick rotated Riemannian spin manifold \((M, g_r, S)\) corresponding to the two possible choices of orientation \(\Gamma_M\) on \(S\). In other words, the following diagram commutes.

\[
\begin{array}{ccc}
(M, g, r, S , \Gamma_M) & \xrightarrow{\text{Wick rotate}} & (M, g_r, S, \Gamma_M^-) \\
\Downarrow & & \Downarrow \\
D & \xrightarrow{\text{Wick rotate}} & D_{\pm}
\end{array}
\]

**Proof.** In Definition 3.13 we have given two Clifford representations \(\gamma_{\pm}\) corresponding to the Riemannian metric \(g_r\). The grading operators corresponding to these Clifford representations are given by

\[
\Gamma_M := i^{-1 + \frac{n(n+1)}{2}} \gamma_{\pm}(e_0) \cdots \gamma_{\pm}(e_{n-1}) = \pm i^{1 + \frac{n(n+1)}{2}} \gamma(e_0) \cdots \gamma(e_{n-1})
\]

\[
= \mp i^{-1 + \frac{n(n+1)}{2}} \gamma(e_0) \cdots \gamma(e_{n-1}) = \mp \Gamma_M,
\]

where \(\Gamma_M\) is given by Eq. (5.3). Hence the choice of sign for the Wick rotation of \(\gamma\) corresponds to the choice of orientation for the spinor bundle \(S\) (in the terminology of [Ply86, §2.7], the choice \((S, \Gamma_M^-)\) is the reverse spin structure of \((S, \Gamma_M^+)\)). Next, the assumption that the spacelike reflection \(r\) is parallel implies that the Levi-Civita connection \(\nabla\) of \(g\) is also the Levi-Civita connection for the Riemannian metric \(g_r\). Hence the canonical Dirac operators corresponding to each of the orientations are given by \(D_{\pm} = \sum_{j=0}^{n-1} \gamma_{\pm}(e_j) \nabla^{S}_{e_j^r}\), which are precisely the Wick rotations of the Lorentzian Dirac operator \(D\). \(\square\)

The above proposition motivates our use of the term ‘Wick rotations’ for \(D_{\pm}\), as they are precisely the Dirac operators corresponding to the ‘Wick rotated’ metric \(g_r\).
5.4 The Harmonic Oscillator

Let us first discuss the case of the 1-dimensional harmonic oscillator. As in Section 4.4, we consider the annihilation and creation operators

\[ a = x + \frac{d}{dx}, \quad a^* = x - \frac{d}{dx}, \]

defined on the initial domain of Schwartz functions \( S(\mathbb{R}) \subset L^2(\mathbb{R}) \).

**Proposition 5.24.** We have an odd indefinite spectral triple \((S(\mathbb{R}), L^2(\mathbb{R}), a = x + \frac{d}{dx})\).

**Proof.** On the initial domain \( S(\mathbb{R}) \), we have

\[ a^* = x - \frac{d}{dx}, \quad \text{Re } a = x, \quad \text{Im } a = -i \frac{d}{dx}, \quad [\text{Re } a, \text{Im } a] = i. \]

The operators \( \text{Re } a = x \) and \( \text{Im } a = -i \frac{d}{dx} \) are essentially self-adjoint on \( S(\mathbb{R}) \). Because \((x, -i \frac{d}{dx})\) is an almost commuting pair (the commutator is in fact bounded), it follows from Theorem 2.22 that \( \text{Dom } a = \text{Dom } a^* = \text{Dom } x \cap \text{Dom } \frac{d}{dx} \), and that \( a^* = x - \frac{d}{dx} \). The closure of \( S(\mathbb{R}) \) in the norm \( \| \cdot \|_{\text{Re } a, \text{Im } a} \) is equal to \( \text{Dom } x \cap \text{Dom } \frac{d}{dx} \). Since the graph norm of \( a \) is bounded by \( \| \cdot \|_{\text{Re } a, \text{Im } a} \), the closure of \( S(\mathbb{R}) \) in the graph norm of \( a \) includes \( \text{Dom } x \cap \text{Dom } \frac{d}{dx} \), and is therefore equal to \( \text{Dom } x \cap \text{Dom } \frac{d}{dx} \). Thus \( S(\mathbb{R}) \) is indeed a core for \( a \).

The algebra \( S(\mathbb{R}) \) preserves the initial domain \( S(\mathbb{R}) \subset \text{Dom } a = \text{Dom } a^* \), and the commutators \([a, f] = \frac{df}{dx}\) and \([a^*, f] = -\frac{df}{dx}\) are bounded for all \( f \in S(\mathbb{R}) \). Since \(-i \frac{d}{dx}\) has locally compact resolvent, we know that the inclusion \( \text{Dom } \frac{d}{dx} \hookrightarrow L^2(\mathbb{R}) \) is locally compact, and hence the inclusion \( \text{Dom } a = \text{Dom } x \cap \text{Dom } \frac{d}{dx} \hookrightarrow L^2(\mathbb{R}) \) is also locally compact.

Following the construction in Section 5.2, we enlarge the Hilbert space \( L^2(\mathbb{R}) \) to a \( \mathbb{Z}_2 \)-graded Hilbert space \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) with grading \( \Gamma := 1 \oplus -1 \), and define a new operator \( \mathcal{D} \) on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) by

\[
\mathcal{D} := \begin{pmatrix} 0 & x - i \frac{d}{dx} \\ x + i \frac{d}{dx} & 0 \end{pmatrix}.
\]

From Theorem 5.13 we then immediately obtain:

**Corollary 5.25.** The harmonic oscillator yields an even indefinite spectral triple

\[
\left( S(\mathbb{R}), \begin{pmatrix} L^2(\mathbb{R}) \\ L^2(\mathbb{R}) \end{pmatrix}, \begin{pmatrix} 0 & x - i \frac{d}{dx} \\ x + i \frac{d}{dx} & 0 \end{pmatrix} \right),
\]

and its Wick rotations form an even pair of spectral triples given by

\[
\begin{pmatrix} S(\mathbb{R}), \begin{pmatrix} L^2(\mathbb{R}) \\ L^2(\mathbb{R}) \end{pmatrix}, \begin{pmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{pmatrix} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & x + \frac{d}{dx} \\ x - \frac{d}{dx} & 0 \end{pmatrix}.
\]
It follows from Proposition 5.14 that the classes given by the Wick rotations are negatives of each other, and indeed we have already explicitly checked this in Section 4.4.1.

5.4.1 Arbitrary dimensions

Let us compare the above example of the 1-dimensional harmonic oscillator with the d-dimensional harmonic oscillator as discussed in [GW13, §2.1] (cf. [Wul10]). In [GW13], the harmonic oscillator is ‘deformed’ to obtain a description of the spectral geometry of the (noncommutative) Moyal plane with harmonic propagation. Here, we only consider the classical (commutative) case. In the notation of [GW13], we have for the 1-dimensional case the operators

\[ a := x + \frac{d}{dx}, \quad a^* = x - \frac{d}{dx}, \quad b := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

acting on the Hilbert space \( \tilde{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2 \), satisfying the (anti-)commutation relations

\[ \{a, a\} = \{a^*, a^*\} = 0, \quad \{a, a^*\} = 2, \quad \{b, b\} = \{b^*, b^*\} = 0, \quad \{b, b^*\} = 1. \]

These operators give rise to two self-adjoint operators

\[ D_1 := a \otimes b^* + a^* \otimes b = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}, \quad D_2 := a \otimes b + a^* \otimes b^* = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}, \]

which are precisely the two operators in the even pair of spectral triples in Corollary 5.25.

**Remark 5.26.** Please note that in [GW13] the second operator is defined as \( D_2 := ia \otimes b - ia^* \otimes b^* \) instead. However, our and their definitions yield the same square

\[ D_2^2 = (a \otimes b + a^* \otimes b^*)^2 = (ia \otimes b - ia^* \otimes b^*)^2 = [a \otimes b, a^* \otimes b^*]. \]

Following [GW13], we now generalise this example to dimension d. So, on \( L^2(\mathbb{R}^d) \) we consider the bosonic annihilation and creation operators with canonical commutation relations:

\[ a_\mu := \omega x_\mu + \partial_\mu, \quad a^*_\mu := \omega x_\mu - \partial_\mu, \quad [a_\mu, a_\nu] = [a^*_\mu, a^*_\nu] = 0, \quad [a_\mu, a^*_\nu] = 2\omega \delta_{\mu\nu}, \]

for \( \mu, \nu = 1, \ldots, d \). Here we have also introduced a frequency parameter \( \omega > 0 \). On the exterior algebra \( \Lambda(\mathbb{C}^d) \), we introduce the fermionic partners \( b_\mu, b^*_\mu \) satisfying the anti-commutation relations

\[ \{b_\mu, b_\nu\} = \{b^*_\mu, b^*_\nu\} = 0, \quad \{b_\mu, b^*_\nu\} = \delta_{\mu\nu}. \]
Denote by $|0\rangle_{\ell}$ the fermionic vacuum satisfying $b_\mu |0\rangle_{\ell} = 0$ for all $\mu$. By repeated application of the creation operators $b^*_\mu$, one constructs out of this vacuum the 2-dimensional Hilbert space $\Lambda (C^d) \simeq C^{2d}$, yielding the standard orthonormal basis elements $(b^*_1)^{s_1} \cdots (b^*_d)^{s_d} |0\rangle_{\ell}$ (where $s_\mu \in \{0,1\}$). The fermionic number operator $N_\ell := \sum_{\mu=1}^d b^*_\mu b_\mu$ naturally defines an $\mathbb{N}$-grading $\Lambda (C^d) = \bigoplus_{p=0}^d \Lambda^p (C^d)$ such that $b_\mu: \Lambda^p (C^d) \to \Lambda^{p-1} (C^d)$ and $b^*_\mu: \Lambda^p (C^d) \to \Lambda^{p+1} (C^d)$. The induced $\mathbb{Z}_2$-grading $\Gamma$ on $\Lambda (C^d)$ then satisfies

$$\Gamma = (-1)^{N_\ell}, \quad \Gamma^2 = 1, \quad \Gamma^* = \Gamma, \quad \Gamma b_\mu = -b_\mu \Gamma, \quad \Gamma b^*_\mu = -b^*_\mu \Gamma.$$ 

Thus we obtain a $\mathbb{Z}_2$-grading on the Hilbert space $L^2 (\mathbb{R}^d) \otimes \Lambda (C^d)$ given by $1 \otimes \Gamma$, which we will also simply denote by $\Gamma$. On $L^2 (\mathbb{R}^d) \otimes \Lambda (C^d)$ we then consider the odd operators

$$\mathcal{D}_1 := \sum_{\mu=1}^d (a_\mu \otimes b^*_\mu + a^*_\mu \otimes b_\mu), \quad \mathcal{D}_2 := \sum_{\mu=1}^d (a_\mu \otimes b_\mu + a^*_\mu \otimes b^*_\mu).$$

Their squares are of the form

$$\mathcal{D}_1^2 = H \otimes 1 + \omega \otimes \Sigma, \quad \mathcal{D}_2^2 = H \otimes 1 - \omega \otimes \Sigma,$$

where the Hamiltonian $H$ and the spin matrix $\Sigma$ are defined as

$$H := \sum_{\mu=1}^d (\omega^2 x^2_\mu - \partial^2_\mu), \quad \Sigma := \sum_{\mu=1}^d [b^*_\mu, b_\mu].$$

**Proposition 5.27.** $(\mathcal{S} (\mathbb{R}^d), L^2 (\mathbb{R}^d) \otimes \Lambda (C^d), \mathcal{D}_1, \mathcal{D}_2)$ is an even pair of spectral triples.

**Proof.** The operator $H$ is well-known to be essentially self-adjoint on $\mathcal{S} (\mathbb{R}^d)$ and to have compact resolvent. Since $\omega \otimes \Sigma$ is only a bounded perturbation of $H \otimes 1$, it follows that $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$ are essentially self-adjoint on $\mathcal{S} (\mathbb{R}^d) \otimes \Lambda (C^d)$ and also have compact resolvent. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are symmetric and their squares are essentially self-adjoint, it follows from Lemma 4.7 that $\mathcal{D}_1$ and $\mathcal{D}_2$ are also essentially self-adjoint. Likewise, compactness of their resolvents follows from the compactness of the resolvents of their squares. Furthermore, commutators of $\mathcal{D}_1$ and $\mathcal{D}_2$ with Schwartz functions are bounded. Hence $\mathcal{D}_1$ and $\mathcal{D}_2$ indeed yield even spectral triples.

To show that these spectral triples in fact form an even pair, we need to check the axioms in Definition 5.4. Since $\mathcal{D}_1^2 - \mathcal{D}_2^2$ is bounded, it follows that $\text{Dom} \mathcal{D}_1 = \text{Dom} \mathcal{D}_2$, and that $(\mathcal{D}_1^2 - \mathcal{D}_2^2)(\mathcal{D}_1 + \mathcal{D}_2 - i\lambda)^{-1}$ is bounded for all $\lambda \in \mathbb{R} \setminus \{0\}$. Lastly, the operators

$$\mathcal{D}_1 + \mathcal{D}_2 = \sum_{\mu=1}^d (a_\mu + a^*_\mu) \otimes (b_\mu + b^*_\mu) = \sum_{\mu=1}^d 2\omega x_\mu \otimes (b_\mu + b^*_\mu),$$

$$\mathcal{D}_1 - \mathcal{D}_2 = \sum_{\mu=1}^d (a^*_\mu - a_\mu) \otimes (b_\mu - b^*_\mu) = \sum_{\mu=1}^d -2\partial_\mu \otimes (b_\mu - b^*_\mu),$$
are essentially self-adjoint on \(\mathcal{S}(\mathbb{R}^d) \otimes \Lambda(\mathbb{C}^d) \subset \text{Dom} \mathcal{D}_1 = \text{Dom} \mathcal{D}_2\). Since the graph norm of \(\mathcal{D}_1 \pm \mathcal{D}_2\) is bounded by the norm \(\| \cdot \|_{\mathcal{D}_1, \mathcal{D}_2}\) (cf. Lemma 2.51), it follows that the domain of the closure of \(\mathcal{D}_1 \pm \mathcal{D}_2\) contains \(\text{Dom} \mathcal{D}_1 \cap \text{Dom} \mathcal{D}_2\), so that \(\mathcal{D}_1 \pm \mathcal{D}_2\) is also essentially self-adjoint on \(\text{Dom} \mathcal{D}_1 = \text{Dom} \mathcal{D}_2\).\(\square\)

We can now take the reverse Wick rotation of the pair \((\mathcal{D}_1, \mathcal{D}_2)\), and from Proposition 5.7 we then obtain the following.

**Corollary 5.28.** The operator

\[
\mathcal{D} := \frac{1}{2}(\mathcal{D}_1 + \mathcal{D}_2) + \frac{i}{2}(\mathcal{D}_1 - \mathcal{D}_2) = \sum_{\mu=1}^{d} (\omega x_\mu \otimes (b_\mu + b_\mu^*) - i\partial_\mu \otimes (b_\mu - b_\mu^*))
\]

yields an even indefinite spectral triple \((\mathcal{S}(\mathbb{R}^d), \mathcal{L}^2(\mathbb{R}^d) \otimes \Lambda(\mathbb{C}^d), \mathcal{D})\).

We remark that this operator \(\mathcal{D}\) still encodes all the information of the \(d\)-dimensional harmonic oscillator. In particular, the Hamiltonian \(H\) and the spin matrix \(\Sigma\) can be recovered via

\[
\frac{1}{2}(\mathcal{D}^* \mathcal{D} + \mathcal{D} \mathcal{D}^*) = \frac{1}{2}(\mathcal{D}_1^2 + \mathcal{D}_2^2) = H \otimes 1, \quad -\frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^2) = \frac{1}{2}(\mathcal{D}_1^2 - \mathcal{D}_2^2) = \omega \otimes \Sigma.
\]

### 5.5 Families of Spectral Triples

In this section we study families of spectral triples \(\{(\mathcal{A}_x, \pi_x, \mathcal{H}, \mathcal{D}_1(x))\}_{x \in M}\) which are parametrised by a Riemannian manifold \(M\). We use these families to construct examples of pairs of spectral triples and thus of indefinite spectral triples. Our approach is largely based on and inspired by work of Kaad and Lesch [KL13, §8], who studied the spectral flow of a family of operators \(\{\mathcal{D}_1(x)\}_{x \in M}\).

#### 5.5.1 The parameter space

Let \(M\) be a complete oriented Riemannian manifold of dimension \(m\), and let \(\mathcal{H}\) be a separable Hilbert space. Consider the operator module \(\mathcal{C}_0^1(M, \mathcal{H})\) over the involutive operator algebra \(\mathcal{C}_0^1(M)\), as described in Examples 2.14 and 2.15. The completeness of \(M\) ensures that \(\mathcal{C}_0^1(M)\) is \(\sigma\)-unital (see [KL13, Note 2.9]).

Let \(\mathcal{D}_2: \Gamma_c^\infty(M, \mathcal{F}) \to \Gamma_c^\infty(M, \mathcal{F})\) be a first-order symmetric elliptic differential operator on a hermitian vector bundle \(\mathcal{F} \to M\), which has bounded propagation speed, i.e. the principal symbol \(\sigma: T^*M \to \text{End}(\mathcal{F})\) of \(\mathcal{D}_2\) satisfies

\[
\sup \left\{ \| \sigma(x, \xi) \| \mid (x, \xi) \in T^*M, \; g(\xi, \xi) \leq 1 \right\} < \infty.
\]

**Proposition 5.29** (cf. [KL13, §8]). The operator \(\mathcal{D}_2\) defined above yields an odd spectral triple \((\mathcal{C}_0^1(M), \mathcal{L}^2(M, \mathcal{F}), \mathcal{D}_2)\).
Proof. The completeness of $M$ and the bounded propagation speed ensure the essential self-adjointness of $D_2$ on $\Gamma_0^\infty(M, F)$ (see e.g. [HRoo, Proposition 10.2.11]). The derivatives of functions in $C^1_0(M)$ by assumption vanish at infinity. Since $D_2$ is a first-order differential operator, the commutators with these functions are bounded. Lastly, ellipticity of $D_2$ ensures that its resolvent is locally compact (see e.g. [HRoo, Proposition 10.5.2]). □

Recall from Definition 2.17 the Grassmann connection
\[
\nabla_{D_2}^{\text{Gr}} = (D_2, ) : C^1_0(M, \mathcal{H}) \to C^0_0(M, \mathcal{H}) \otimes C^0_0(M) \otimes L^2(M, F).
\]
As in Section 2.4.2, we define $1 \otimes \nabla D_2$ on $C^0_0(M, \mathcal{H}) \otimes C^0_0(M) \otimes L^2(M, F)$ as
\[
(1 \otimes \nabla D_2)(\psi \otimes f) := \psi \otimes D_2 f + (\nabla_{D_2}^{\text{Gr}} \psi)f,
\]
for $\psi \in C^1_0(M, \mathcal{H})$ and $f \in \text{Dom} D_2$. Using $C^0_0(M, \mathcal{H}) \simeq \mathcal{H} \otimes C^0_0(M)$, we also have the isomorphism $C^0_0(M, \mathcal{H}) \otimes C^0_0(M) \otimes L^2(M, F) \simeq \mathcal{H} \otimes L^2(M, F)$, and under this identification we can write
\[
1 \otimes \nabla D_2 = 1 \otimes D_2,
\]
and we will simply write $D_2$ for this operator.

5.5.2 The family of spectral triples

Let us start with a brief discussion of families of operators parametrised by the manifold $M$.

Definition 5.30. A map $S(\cdot) : M \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $x \mapsto S(x)$, is said to have a uniformly bounded weak derivative if the map is weakly differentiable (i.e. the map $x \mapsto (S(x)\xi, \eta)$ is differentiable for each $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$), the weak derivative $dS(x) : \mathcal{H}_1 \to \mathcal{H}_2 \otimes T^*_x(M)$ is bounded for all $x \in M$, and the supremum $\sup_{x \in M} \|dS(x)\|$ is finite.

We gather a few statements from [KL13, §8] into the following lemma.

Lemma 5.31. Let $S(\cdot) : M \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ have a uniformly bounded weak derivative. Then the following statements hold.
1) If $x, y \in M$ lie in the same coordinate chart, then
\[
\|S(x) - S(y)\| \leq \sup_{z \in M} \|dS(z)\| \cdot \text{dist}(x, y),
\]
where $\text{dist}(x, y)$ denotes the geodesic distance between $x$ and $y$.
2) The map $S(\cdot)$ yields a well-defined operator $C^0_0(M, \mathcal{H}_1) \to C^0_0(M, \mathcal{H}_2)$ by setting
\[
(S(\cdot)\psi)(x) := S(x)\psi(x),
\]
for $\psi \in C^0_0(M, \mathcal{H}_1)$. 

\[

\]
3) Suppose that $\mathcal{H}_1$ is densely embedded into $\mathcal{H}_2$, so we can think of each $S(x)$ as an unbounded operator on $\mathcal{H}_2$ with domain $\mathcal{H}_1$. If $S(x)$ is self-adjoint for each $x \in M$, then $S(\cdot): C_0(M, \mathcal{H}_1) \to C_0(M, \mathcal{H}_2)$ is self-adjoint and regular.

Proof. We refer to [KL13, Remark 8.4, 2.] for a short proof of 1). For 2) we need to check that $S(x)\psi(x)$ is continuous in $x$. We have the inequality

$$||S(x)\psi(x) - S(y)\psi(y)|| \leq ||S(x) - S(y)|| \psi(x)|| + ||S(y)|| \psi(x) - \psi(y)||.$$ 

As $y \to x$, each of these terms approaches zero; the first term by the first statement of this lemma, the second by continuity of $\psi$. To prove 3), first recall that the pure states $\epsilon_x$ of $C_0(M)$ are given by evaluation at $x \in M$. The localisation $S(\cdot)\epsilon_x$ of $S(\cdot)$ is given by the closure of $S(x)$ on the domain $C_0(M, \mathcal{H}_1) \otimes C_0(M)$, which is just $S(x)$. Since $S(x)$ is self-adjoint by assumption, it follows from the local-global principle for commutative $C^*$-algebras that $S(\cdot)$ is self-adjoint and regular (see Theorem 2.11). \qed

Definition 5.32. A weakly differentiable family of spectral triples $\{(A_x, \pi_x, \mathcal{H}, D_1(x))\}_{x \in M}$ parametrised by the manifold $M$ is a family of spectral triples such that the following conditions are satisfied:

- there exists another Hilbert space $W$ which is continuously and densely embedded in $\mathcal{H}$ such that the inclusion map $\iota: W \to \mathcal{H}$ is locally compact, i.e. the composition $\pi_x(a) \circ \iota$ is compact for each $x \in M$ and $a \in A$;
- the domain of $D_1(x)$ is independent of $x$ and equals $W$, and the graph norm of $D_1(x)$ is uniformly equivalent to the norm of $W$ (i.e. there exist constants $C_1, C_2 > 0$ such that $C_1\|\xi\|_W \leq \|\xi\|_{D_1(x)} \leq C_2\|\xi\|_W$ for all $\xi \in W$ and all $x \in M$);
- for each $a \in A$, the maps $D_1(\cdot): M \to B(W, \mathcal{H})$ and $\pi(\cdot): M \to B(\mathcal{H})$ have uniformly bounded weak derivatives, and the map $[D_1(\cdot), \pi(\cdot)]: M \to B(\mathcal{H})$ is continuous.

Remark 5.33. 1) The unbounded operator $D_1(x): \text{Dom} \ D_1(x) \to \mathcal{H}$ is viewed as a bounded operator $W \to \mathcal{H}$, where $W = \text{Dom} \ D_1(x)$ is a Hilbert space with respect to the graph inner product of $D_1(x)$. Since the graph norms of $D_1(x)$ are equivalent for all $x \in M$, it follows that the bound on the operator norm of $dD_1(x)$ is thus a relative bound with respect to $D_1(y)$, for any $y \in M$.

2) The requirement that the graph norm of $D_1(x)$ is uniformly equivalent to the norm of $W$ implies that $\sup_{x \in M} \|D_1(x)\|$ is finite.

3) The case where $A = C$ and $\pi_x$ is scalar multiplication brings us back to the case of a family of operators $\{D_1(x)\}$ as studied in [KL13, §8].
Consider the Hilbert $C_0(M)$-module $C_0(M, \mathcal{H})$. The family of representations $\pi_x : A \to \mathcal{B}(\mathcal{H})$ determines a representation

$$\pi : A \otimes C_0(M) \simeq C_0(M, A) \to C_0(M, \mathcal{B}(\mathcal{H})) \simeq \text{End}_{C_0(M)}(C_0(M, \mathcal{H}))$$

by setting

$$(\pi(a)\psi)(x) := \pi_x(a(x))\psi(x),$$

for $\psi \in C_0(M, \mathcal{H})$ and $a \in C_0(M, A)$. The family of operators $\{\mathcal{D}_1(x)\}$ on the Hilbert space $\mathcal{H}$ defines a new operator $\mathcal{D}_1(\cdot)$ on the $C_0(M)$-module $C_0(M, \mathcal{H})$ with domain $C_0(M, W)$ by setting

$$(\mathcal{D}_1(\cdot)\psi)(x) := \mathcal{D}_1(x)\psi(x).$$

The assumption of weak differentiability is more than sufficient to ensure that $\pi$ and $\mathcal{D}_1(\cdot)$ are well-defined (see the second statement of Lemma 5.31). The operator $\mathcal{D}_1(\cdot) : C_0(M, W) \to C_0(M, \mathcal{H})$ is densely defined and symmetric.

**Remark 5.34.** In [KL13, §8] the family $\{\mathcal{D}_1(x)\}_{x \in M}$ is used to construct a class in the odd K-theory $K_1(C_0(M)) = KK^1(C, C_0(M))$ of $C_0(M)$. In order to ensure that $\mathcal{D}_1$ has compact resolvent (as an operator on the $C_0(M)$-module $C_0(M, \mathcal{H})$), it is then necessary to replace $\mathcal{D}_1$ by $f^{-1}\mathcal{D}_1$, for a strictly positive function $f \in C_0^\infty(M)$. In our approach we aim to construct instead a class in $KK^1(C_0(M, A), C_0(M))$, for which introducing this function $f$ is not necessary, as now we only need the resolvent to be *locally* compact (for the left action by $C_0(M, A)$).

**Proposition 5.35** (cf. [KL13, Proposition 8.7]). If $\{(A, \pi_x, \mathcal{H}, \mathcal{D}_1(x))\}_{x \in M}$ is a weakly differentiable family of spectral triples, then $(C_0(M, A), C_0(M, \mathcal{H})_{C_0(M)}, \mathcal{D}_1(\cdot))$ is an odd unbounded Kasparov $C_0(M, A)$-$C_0(M)$-module.

**Proof.** The operator $\mathcal{D}_1(\cdot)$ is self-adjoint and regular, because $\mathcal{D}_1(x)$ is self-adjoint for each $x \in M$ (see Lemma 5.31, part 3). The algebraic tensor product $A \otimes C_0^\infty(M)$ is dense in $C_0(M, A)$, and for $a \otimes f \in A \otimes C_0^\infty(M)$ the commutators

$$[\mathcal{D}_1(\cdot), \pi(a \otimes f)](x) = f(x)[\mathcal{D}_1(x), \pi_x(a)]$$

are bounded for each $x$. By assumption such commutators are continuous, and the compact support of $f$ then ensures that they are globally bounded.

It remains to show that $\pi(a \otimes f)(\mathcal{D}_1(\cdot) \pm i)^{-1}$ is compact (as an operator on the $C_0(M)$-module $C_0(M, \mathcal{H})$) for each $a \in A$ and $f \in C_0(M)$. The compact operators on $C_0(M, \mathcal{H})$ are given by $C_0(M, \mathcal{H})_0$. Since $(A, \pi_x, \mathcal{H}, \mathcal{D}_1(x))$ is a spectral triple, the operator $\pi_x(a)(\mathcal{D}_1(x) \pm i)^{-1}$ is compact and bounded by $\|a\|$ for each $x \in M$. Furthermore, $(\mathcal{D}_1(x) \pm i)^{-1}$ depends continuously on $x$, since by the resolvent identity and the first statement of Lemma 5.31 we have

$$\|(\mathcal{D}_1(x) \pm i)^{-1} - (\mathcal{D}_1(y) \pm i)^{-1}\| = \|(\mathcal{D}_1(x) \pm i)^{-1}(\mathcal{D}_1(y) - \mathcal{D}_1(x)))(\mathcal{D}_1(y) \pm i)^{-1}\|$$
We would now like to ‘glue together’ our family of spectral triples by taking the domain \( \text{Dom}(D) \) we consider the operators on the Schrödinger operator \( H \). Hence the map \( M \to \mathcal{K} ( \mathcal{H} ) \) given by \( x \mapsto \pi_x ( a ) ( D_1 ( x ) \pm i )^{-1} \) is continuous and globally bounded by \( \| a \| \). If we then also multiply by \( f \in C_0 ( M ) \), we find that \( \pi ( a \otimes f ) ( D_1 ( \cdot ) \pm i )^{-1} \in C_0 ( M, \mathcal{K} ( \mathcal{H} ) ) \). \( \square \)

5.5.3 The Kasparov product

We would now like to ‘glue together’ our family of spectral triples by taking the odd unbounded Kasparov product of \( ( C_0 ( M, A ), C_0 ( M, \mathcal{H} )_0, D_1 ( \cdot ) ) \) with \( ( C_0^1 ( M ), L^2 ( M, F ), \mathcal{D} ) \). On the internal tensor product of the Hilbert modules

\[
C_0 ( M, \mathcal{K} ) \otimes C_0 ( M ) L^2 ( M, \mathcal{F} ) \simeq L^2 ( M, \mathcal{K} \otimes \mathcal{F} ),
\]

we consider the operators \( D_1 ( \cdot ) \otimes 1 = D_1 ( \cdot ) \) and \( 1 \otimes \mathcal{D}_2 = \mathcal{D}_2 \).

**Proposition 5.36** (cf. [KL13, Proposition 8.11]). Let \( \nabla^G_{\mathcal{D}_2} \) be the Grassmann connection of Eq. (5.4). Then the pair \( ( C_0^1 ( M, \mathcal{H} ), \nabla^G_{\mathcal{D}_2} ) \) is a correspondence from the Kasparov module \( ( C_0 ( M, A ), C_0 ( M, \mathcal{H} ) )_0, D_1 ( \cdot ) ) \) to the spectral triple \( ( C_0^1 ( M ), L^2 ( M, F ), \mathcal{D} ) \).

**Proof.** Recall from Definition 2.35 the definition of a correspondence. For a family of operators, this has been shown in [KL13, Proposition 8.11]. For a family of spectral triples, the only difference is that we now consider a left action by \( C_0 ( M, A ) \) (instead of \( C ) \) on \( C_0 ( M, \mathcal{H} ) \). Thus we need to check the third condition in Definition 2.35, which requires that the commutator

\[
[D_2, \pi ( a \otimes f ) \otimes 1] : \text{Dom}(\mathcal{D}_2) \to C_0 ( M, \mathcal{H} ) \otimes C_0 ( M ) L^2 ( M, \mathcal{F} ) \simeq L^2 ( M, \mathcal{H} \otimes \mathcal{F} )
\]

is well-defined and bounded for all \( a \otimes f \in A \otimes C_0^1 ( M ) \). The commutator with \( f \in C_0^1 ( M ) \) simply yields \( [D_2, f] = i \sigma ( df ) \), which is bounded because \( f \in C_0^1 ( M ) \) implies that \( df \) is bounded, and because \( \sigma \) is completely bounded by Proposition 2.18. Similarly, the commutator \( [D_2, \pi_x ( a )] = i \sigma ( d ( \pi_x ( a ) ) ) \) is bounded, because by assumption the weak derivative of \( \pi_x ( a ) \) is uniformly bounded. Thus we indeed have a correspondence. \( \square \)

**Definition 5.37.** Given the operators \( D_1 ( \cdot ) \) and \( D_2 \) as above, we define the Dirac-Schrödinger operator on \( L^2 ( M, \mathcal{H} \otimes \mathcal{F} )^{\otimes 2} \) as

\[
D_1 \times D_2 := \begin{pmatrix}
0 & D_1 ( \cdot ) - iD_2 \\
D_1 ( \cdot ) + iD_2 & 0
\end{pmatrix}
\]

with the domain \( ( \text{Dom}(D_1 ( \cdot )) \cap \text{Dom}(D_2) )^{\otimes 2} \).
Theorem 5.38. Let $M$ be a complete oriented Riemannian manifold of dimension $m$, and let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{D}_2$ be a closed first-order symmetric elliptic differential operator on a hermitian vector bundle $F \rightarrow M$, which has bounded propagation speed. Let $\{ (A_\tau, \pi_\tau, \mathcal{H}, \mathcal{D}_1(x)) \}_{x \in M}$ be a weakly differentiable family of spectral triples. Then the following statements hold:

1) the triple $(A \odot C^1_0(M), \mathcal{H}(\mathcal{H} \otimes F)^{\otimes 2}, \mathcal{D}_1 \times \mathcal{D}_2)$ is an even spectral triple which represents the odd unbounded Kasparov product

$$(C_0(M, A), C_0(M, \mathcal{H})_{\mathcal{C}_0(M)}, \mathcal{D}_1(\cdot)) \otimes_{\mathcal{C}_0(M)} (C_0^1(M), \mathcal{H}(\mathcal{H} \otimes F), \mathcal{D}_2);$$

2) the triple $(A \odot C^1_0(M), \mathcal{H}(\mathcal{H} \otimes F), \mathcal{D}_1(\cdot) + i\mathcal{D}_2)$ is an odd indefinite spectral triple.

Proof. Using Proposition 5.36, the first statement follows from Theorem 2.37. For the second statement, consider the operator $\mathcal{D} := \mathcal{D}_1(\cdot) + i\mathcal{D}_2$ on the domain $\text{Dom} \mathcal{D} = \text{Dom} \mathcal{D}_1(\cdot) \cap \text{Dom} \mathcal{D}_2$. We know from Proposition 5.36 that $(\mathcal{D}_1(\cdot), \mathcal{D}_2)$ is an almost commuting pair, so it follows from Theorem 2.22 that $\mathcal{D}^* = \mathcal{D}_1(\cdot) - i\mathcal{D}_2$ on $\text{Dom} \mathcal{D}^* = \text{Dom} \mathcal{D}_1(\cdot) \cap \text{Dom} \mathcal{D}_2$, and therefore we have $\text{Re} \mathcal{D} = \mathcal{D}_1(\cdot)$ and $\text{Im} \mathcal{D} = \mathcal{D}_2$ on this domain.

The operators $\mathcal{D}_1(\cdot)$ and $\mathcal{D}_2$ are both essentially self-adjoint on the domain $\text{Dom} \mathcal{D}_1(\cdot) \cap \text{Dom} \mathcal{D}_2$ (for $\mathcal{D}_1(\cdot)$ this follows from Lemma 2.20, and for $\mathcal{D}_2$ this follows from the completeness of the Riemannian manifold). The algebra $A \odot C^1_0(M)$ preserves $\text{Dom} \mathcal{D}_1(\cdot) \cap \text{Dom} \mathcal{D}_2$, and both $\mathcal{D}_1(\cdot)$ and $\mathcal{D}_2$ have bounded commutators with $A \odot C^1_0(M)$. Lastly, $\iota : \text{Dom} \mathcal{D}_1(\cdot) \cap \text{Dom} \mathcal{D}_2 \hookrightarrow \mathcal{H}(\mathcal{H} \otimes F)$ is locally compact because (by the first statement) $\mathcal{D}_1 \times \mathcal{D}_2$ has locally compact resolvent. \(\square\)

Remark 5.39. In the construction of the Dirac-Schrödinger operator $\mathcal{D}_1 \times \mathcal{D}_2$ we may replace $\mathcal{D}_2$ by $-\mathcal{D}_2$, without affecting the first statement of the above theorem. We thus obtain two different spectral triples with the operators

$$\begin{pmatrix} 0 & \mathcal{D}_1(\cdot) - i\mathcal{D}_2 \\ \mathcal{D}_1(\cdot) + i\mathcal{D}_2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \mathcal{D}_1(\cdot) + i\mathcal{D}_2 \\ \mathcal{D}_1(\cdot) - i\mathcal{D}_2 & 0 \end{pmatrix}.$$ 

The second statement of the theorem could have been proved alternatively by showing that these two spectral triples form a pair of spectral triples (as defined in Definition 5.4). It then follows from Theorem 5.13 that $\mathcal{D} = \mathcal{D}_1(\cdot) + i\mathcal{D}_2$ yields an odd indefinite spectral triple.

5.5.4 Generalised Lorentzian cylinders

Dirac operators on generalised pseudo-Riemannian cylinders have been studied in [BGM05]. Here we will specialise to the Lorentzian case, and we will show that
this provides an example of a family of spectral triples parametrised by the real line \( \mathbb{R} \). In fact, this example provided our initial motivation to consider families of spectral triples.

Let \( \Sigma \) be an \((n-1)\)-dimensional smooth spin manifold, and let \( g_t \) be a smooth family of complete Riemannian metrics on \( \Sigma \) parametrised by \( t \in \mathbb{R} \). Consider the generalised Lorentzian cylinder \((M,g) := (\Sigma \times \mathbb{R}, g_t - dt^2)\), and define the hypersurfaces \( \Sigma_t := (\Sigma \times \{t\}, g_t) \). The vector field \( \nu := \partial_t \) is a unit timelike vector field which is orthogonal to \( \Sigma_t \). As in [BGM05, §4], we find \( \nabla_\nu \nu = 0 \), which means that the integral curves of \( \nu \) are geodesics.

Since each hypersurface \( \Sigma_t \) is a complete Riemannian spin manifold, we obtain for each \( t \in \mathbb{R} \) a spectral triple

\[
\left( C^\infty_c(\Sigma), L^2(\Sigma_t, S_t), \mathcal{D}(t) \right),
\]

where \( S_t \) is the spinor bundle over \( \Sigma_t \), and \( \mathcal{D}(t) = \gamma_t \circ \nabla^{S_t} \) is the canonical Dirac operator on \( \Sigma_t \).

For \( x \in \Sigma \) and \( t_0, t_1 \in \mathbb{R} \), parallel transport along the curve \( t \mapsto (t, x) \in M \) (i.e. an integral curve of the vector field \( \nu \)) yields a linear isometry \( \tau^1_{t_0} : (S_{t_0})_x \to (S_{t_1})_x \). The Hilbert spaces \( \mathcal{H}_t := L^2(\Sigma_t, S_t) \) of square-integrable spinors on \( \Sigma_t \) can be identified via this parallel transport, and we shall write \( \mathcal{H} := \mathcal{H}_0 \). Under this identification, the action of \( C^\infty_c(\Sigma) \) on \( \mathcal{H}_t \simeq \mathcal{H} \) (given by pointwise multiplication) does not depend on \( t \).

A local orthonormal frame \( \{e_1, \ldots, e_{n-1}\} \) on \( \Sigma_0 \) can be extended to a local orthonormal frame \( \{\nu, e_1, \ldots, e_{n-1}\} \) on \( M \), and this extended frame then satisfies \( \nabla_\nu e_j = 0 \). Consequently, the Clifford multiplication \( \gamma \) on \((M, g)\) satisfies

\[
[\nabla_\nu, \gamma(e_j)] = \gamma(\nabla_\nu e_j) = 0,
\]

so \( \gamma \) is parallel along the vector field \( \nu \). Under the identification \( \tau^0_t : \mathcal{H}_t \to \mathcal{H}_0 \), the Clifford multiplication \( \gamma_t \) on \( \mathcal{H}_t \) is mapped to \( \tau^0_t \circ \gamma_t(\tau^1_0 X) \circ \tau^1_0 = \gamma_0(X) \) on \( \mathcal{H}_0 \) (see also [BGM05, §5]). Thus, upon identifying \( \mathcal{H}_t \simeq \mathcal{H}_0 \), the Clifford multiplication becomes independent of \( t \).

**Proposition 5.40.** Let \((M, g)\) be an even-dimensional generalised Lorentzian cylinder as constructed above. Suppose that the smooth family of metrics \( g_t \) has derivatives of all orders (both in \( t \) and along \( \Sigma \)) which are globally bounded. Then the spectral triples \( \left( C^\infty_c(\Sigma), L^2(\Sigma_t, S_t), \mathcal{D}(t) \right) \) form a weakly differentiable family of spectral triples (as in Definition 5.32) parametrised by the real line \( M = \mathbb{R} \).

**Proof.** We define the Hilbert space \( W := \text{Dom} \mathcal{D}(0) \) equipped with the graph inner product of \( \mathcal{D}(0) \). Then \( W \) is continuously and densely embedded in \( \mathcal{H} := L^2(\Sigma_0, S_0) \). Since \( \mathcal{D}(0) \) is elliptic, this embedding is locally compact.
Using the fact that $\gamma_t$ is independent of $t$ under the identification $L^2(\Sigma_t, S_t) \simeq H$, we can write $\mathcal{D}(t) - \mathcal{D}(0) = \gamma_0 \circ (\nabla^g_t - \nabla^g_0)$, which is a smooth endomorphism on $S_0$. The assumption that $g_t$ has globally bounded derivatives ensures that the difference $\mathcal{D}(t) - \mathcal{D}(0)$ is globally bounded, and therefore the graph norms of $\mathcal{D}(t)$ are uniformly equivalent.

For $f \in C_c^\infty(\Sigma)$, the commutator $[\mathcal{D}(t), f]$ is given by Clifford multiplication with $df$. Hence, under the identification $L^2(\Sigma_t, S_t) \simeq H$, both $f$ and $[\mathcal{D}(t), f]$ are independent of $t$. Lastly, since $g_t$ has globally bounded derivatives, it follows from [BGM05, Theorem 5.1] that the time-derivative of $\mathcal{D}(t)$ is relatively bounded by $\mathcal{D}(t)$ (and hence by $\mathcal{D}(0)$).

By considering $\mathcal{D}_2 = -i\partial_x$ on $L^2(\mathbb{R})$, Theorem 5.38 then yields the odd indefinite spectral triple

$$(C_c^\infty(\Sigma \times \mathbb{R}), L^2(\mathbb{R}, H), \mathcal{D}(\cdot) + \partial_t),$$

describing the Dirac operator on the foliated spacetime $\Sigma \times \mathbb{R}$. 
GLOBALLY NON-TRIVIAL ALMOST-COMMUTATIVE MANIFOLDS

In this chapter, which is based on joint work [BD14] with Jord Boeijink (Radboud University Nijmegen), we define and study globally non-trivial (or topologically non-trivial) almost-commutative manifolds.

In the previous chapters, our main focus was the noncommutative description of pseudo-Riemannian manifolds. In this chapter we will jump to the description of gauge theories in terms of almost-commutative manifolds. As this description has only been obtained for the Riemannian case, we will now restrict our attention to Riemannian manifolds. In the next chapter, we will bring back the pseudo-Riemannian signature, and discuss the fermionic action.

The framework of Connes’ noncommutative geometry [Con94] provides a generalisation of ordinary Riemannian spin manifolds to noncommutative manifolds. In this chapter, we will focus on the special case of so-called almost-commutative manifolds. The main reason for studying this special case is that it can be used to describe gauge theories, and therefore to obtain models of particle physics (as described in Section 1.2.1). As explained in Section 1.2.3, for a suitably chosen almost-commutative manifold, one obtains the full Standard Model of high energy physics, including the Higgs mechanism and neutrino mixing [Con96, Con06, CCM07].

The standard construction of almost-commutative manifolds (detailed in Section 1.2.1) leads to topologically trivial gauge theories (in the sense that the corresponding principal bundles are globally trivial bundles). The aim of this chapter is to adapt the framework in order to allow for globally non-trivial gauge theories as well. Such a generalisation has previously been obtained for the special case of Yang-Mills theory [BS11].

We assume throughout this chapter that the base manifold $M$ is compact. A large part of this chapter continues to make sense in the non-compact case. In particular, the definition of a (globally non-trivial) almost-commutative manifold, and its description as a Kasparov product, remain valid. However, in the non-compact case, obtaining a gauge theory yields a few additional challenges. First of all, the gauge group is defined using the unitary elements of the algebra $A$,
and thus relies on $A$ being unital. In the non-unital (i.e. non-compact) case, we are therefore required to consider a unitisation of $A$. Second, the Serre-Swan theorem, which gives an equivalence between smooth vector bundles over $M$ and modules over $C^\infty(M)$, is only valid for compact manifolds. In the non-compact case, one needs to ensure that the bundles are ‘well-behaved near infinity’, which can be done by assuming they extend to vector bundles over a compactification of $M$ (corresponding to a unitisation of the $C^*$-algebra $C_0(M)$). Third, the spectral action principle, which we use to obtain the Lagrangian of the gauge theory, only applies to the unital case. For a non-compact manifold, it then becomes necessary to restrict to compact neighbourhoods and obtain a ‘local’ Lagrangian.

This chapter is organised as follows. In Section 6.2 we describe the generalisation of the product triples $M \times F$ to (in general globally non-trivial) almost-commutative manifolds. We show that these almost-commutative manifolds are naturally given by the internal Kasparov product of an internal space $I$ (replacing the finite spectral triple $F$) with the underlying manifold $M$.

While every globally trivial almost-commutative manifold describes a gauge theory, this no longer holds for arbitrary globally non-trivial almost-commutative manifolds. In Section 6.3 we therefore focus our attention on those internal spaces that will allow us to obtain a gauge theory. After briefly recalling the classification of finite spectral triples, we define the notion of a principal module, which is an internal space built from a finite spectral triple $F$ and a principal $G_F$-bundle $P$ over $M$. We show that the algebraic definition of the gauge group of a principal module (defined similarly to Eq. (1.1)) coincides precisely with the usual definition of the gauge group of $P$ (i.e. the vertical automorphisms of $P$), provided that the underlying manifold $M$ is simply connected.

One of the main ideas in the development of noncommutative geometry has been the translation of geometric data into (operator-)algebraic data. Whereas principal modules are constructed from geometric objects (namely principal fibre bundles), we devote Section 6.4 to the purely algebraic notion of what we call a gauge module. We prove that these gauge modules form a proper subclass of the principal modules, which are characterised by a lift of $P$ to a principal $\mathcal{U}(A_F)$-bundle (where $A_F$ is the algebra of the finite spectral triple $F$).

By equipping a principal module with a connection and a ‘mass matrix’, we construct the corresponding principal almost-commutative manifold in Section 6.5. The remainder of this section is used to establish the main goal of this chapter; namely, we describe in detail how this principal almost-commutative manifold describes a gauge theory on $M$.

In Section 6.6 we provide two basic but illustrative examples of such gauge theories, namely Yang-Mills theory and electrodynamics. The Yang-Mills example in particular shows that not every principal module is a gauge module. However,
we also show that the Yang-Mills example is a gauge module when the underlying manifold is simply-connected and 4-dimensional. Hence on such manifolds we have no example of a principal module which is not a gauge module.

6.1 Fibre bundles and gauge theories

The definitions concerning fibre bundles in this chapter may differ from the definitions in some other literature, including [BS11], so that we find it necessary to include a list of the definitions we use. We have already introduced vector bundles and principal fibre bundles in Section 3.2. In this section we will also need algebra bundles and group bundles, which we combine into the following general definition.

Definition 6.1. Let $\mathcal{C}$ be some subcategory of the category of smooth manifolds, with objects $\text{Ob}_\mathcal{C}$ and morphisms $\text{Mor}_\mathcal{C}(A, B)$ for any $A, B \in \text{Ob}_\mathcal{C}$. Let $M$ be a smooth manifold. A fibre bundle $\pi: E \to M$ with fibre $F$ is called a $\mathcal{C}$-bundle if $F \in \text{Ob}_\mathcal{C}$ and if $h_u|_{\pi^{-1}(x)}: \pi^{-1}(x) \to x \times F$ is an isomorphism in $\text{Mor}_\mathcal{C}(\pi^{-1}(x), F)$, for each local trivialisation $(U, h_u)$.

Let $\pi_1: E_1 \to M$ and $\pi_2: E_2 \to M$ be fibre bundles. A bundle morphism is a smooth map $\phi: E_1 \to E_2$ such that $\pi_2 \circ \phi = \pi_1$. If $E_1$ and $E_2$ are $\mathcal{C}$-bundles, then $\phi$ is called a $\mathcal{C}$-bundle morphism if $\phi|_{\pi_1^{-1}(x)}: \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ is an element of $\text{Mor}_\mathcal{C}(\pi_1^{-1}(x), \pi_2^{-1}(x))$ for each $x \in M$.

Let $\pi: E \to M$ be a $\mathcal{C}$-bundle with fibre $F$. A fibre subbundle $\pi': E' \to M$ with fibre $F'$ is a $\mathcal{C}$-subbundle if $F' \in \text{Ob}_\mathcal{C}$ and there exist local trivialisations $([U, h_u])$ for $E$ such that $h_u|_{E'|_U} \simeq U \times i(F')$, where $i$ is an injective morphism in $\text{Mor}_\mathcal{C}(F', F)$.

If $\mathcal{C}$ is the category of finite-dimensional vector spaces, finite-dimensional ($\ast$-)algebras, or Lie groups, then $\mathcal{C}$-bundles are referred to as vector bundles, ($\ast$-)algebra bundles, or group bundles (respectively). We point out that this definition of vector bundles agrees with Definition 3.1.

Remark 6.2. Note that according to Definition 6.1 a ($\ast$-)algebra bundle is always locally trivial, in contrast with the definition of ($\ast$-)algebra bundle in [BS11] (where the bundle is only assumed to be locally trivial as a vector bundle). The weaker notion given in [BS11] will here be referred to as weak ($\ast$-)algebra bundle, following terminology of [Cać12]. A weak algebra bundle is thus a vector bundle $B \to M$, equipped with a fibrewise product $B_x \times B_x \to B_x$ which depends continuously on $x \in M$. This definition does not guarantee that the fibres $B_x$ are isomorphic (as algebras).

For example, consider the globally trivial vector bundle $B = S^1 \times \mathbb{C}$ on the base manifold $S^1$ (viewed as a subset of the complex plane) with fibre $\mathbb{C}$. Consider a smooth non-negative function $f: S^1 \to [0, \infty)$ such that $f(z) = 0$ if and only if $z = 1$. 
For elements \( a, b \) in the fibre \( B_z \) over \( z \in S^1 \) we define the product \( a \cdot z b := f(z)ab \). This equips the vector bundle \( B \) with a smooth product on the fibres, and hence \( B \) is a \emph{weak} algebra bundle. However, if \( z \neq 0 \), then the algebras \( B_z \) and \( B_0 \) are not isomorphic.

**Example 6.3** (Unitary group bundle). If \( B \) is a unital \(*\)-algebra bundle, we define the \emph{unitary group bundle} of \( B \) as \( \mathcal{U}(B) := \{ b \in B \mid bb^* = b^*b = 1 \} \). Then \( \mathcal{U}(B) \) is a fibre subbundle of \( B \), which forms a group bundle with group multiplication of \( \mathcal{U}(B)_x = \mathcal{U}(B_x) \) inherited from the algebra multiplication of \( B_x \), and group inverse given by the involution \(*\). The sections of the unitary group bundle are equal to the unitary sections of the algebra bundle: \( \Gamma^\infty(\mathcal{U}(B)) = \mathcal{U}(\Gamma^\infty(B)) \).

**Example 6.4** (Endomorphism bundle). Let \( E \to M \) be a (hermitian) vector bundle with fibre \( V \) and local trivialisations \( \{ h^E \} \). Then the bundle of endomorphisms \( \text{End}(E) \) is a unital \((*)\)-algebra bundle over \( M \) with fibre \( \text{End}(V) \), and its local trivialisations \( \{ h^E \text{End}(E) \} \) are induced from \( \{ h^E \} \).

**Theorem 6.5** ([BS11, Theorem 3.8]). Let \( M \) be a compact manifold. There is an equivalence between the category of (unital) weak \((*)\)-algebra bundles over \( M \) and the category of (unital) \((*)\)-algebra bundles that are finitely generated projective as \( C^\infty(M)\)-modules.

We again emphasise the difference between algebra bundles and weak algebra bundles as mentioned in Remark 6.2. We are grateful to Eli Hawkins who pointed out to us that a weak algebra bundle is locally trivial (and so is an algebra bundle) if and only if there exists a connection \( \nabla \) satisfying the Leibniz rule

\[
\nabla(ab) = (\nabla a)b + a(\nabla b).
\]

In the continuous case, if \( B = \Gamma(B) \simeq \hat{p}C(M)^N \) for some projection \( \hat{p} \in M_N(C(M)) \), there always exists a norm-homotopic projection \( p \in M_N(C^\infty(M)) \) (see [LRV12, Lemma 2.3]), which yields the smooth submodule \( \mathcal{B} := pC^\infty(M)^N \). The bundle \( \mathcal{B} \) is then locally trivial if and only if there exists a connection on the smooth submodule \( \mathcal{B} \) satisfying the above Leibniz rule.

### 6.1.1 Classical gauge theory

**Definition 6.6.** Consider a principal \( G \)-bundle \( P \to M \). Given an action \( \rho \) of \( G \) on a smooth manifold \( F \), we define the \emph{associated bundle} \( P \times_\rho F \) (or \( P \times G F \)) as the quotient of the product manifold \( P \times F \) with respect to the equivalence relation given by \( (pg, f) \sim (p, \rho(g)f) \). If \( F \in \text{Ob}_\mathcal{E} \) and \( \rho(g) \in \text{Mor}_\mathcal{E}(F, F) \) for all \( g \in G \), then \( P \times_\rho F \) is a \( \mathcal{E} \)-bundle.

**Example 6.7.** The \emph{adjoint bundle} \( \text{Ad}P \) is defined as the associated bundle \( P \times \text{Ad}G \) with respect to the adjoint action \( \text{Ad}(g)h := ghg^{-1} \), \( (g, h \in G) \). The adjoint bundle
is a group bundle with fibres isomorphic to $G$, and its sections $\Gamma^{\infty}(\text{Ad} P)$ then form a group with fibre-wise multiplication.

**Definition 6.8.** A gauge transformation of a principle $G$-bundle $\pi: P \to M$ is a principal $G$-bundle automorphism of $P$ (over $\text{id}: M \to M$), i.e. a smooth map $\phi: P \to P$ such that $\pi(\phi(p)) = \pi(p)$ and $\phi(pg) = \phi(p)g$ for all $p \in P$ and $g \in G$. The set of all gauge transformations is called the gauge group $\mathcal{G}(P)$ of $P$, where the group multiplication is given by composition.

**Theorem 6.9** (see e.g. [Ble81, Ch. 3]). The gauge group $\mathcal{G}(P)$ is isomorphic to the group $\Gamma^{\infty}(\text{Ad} P)$.

**Definition 6.10.** Let $M$ be a manifold and $G$ a Lie group. A classical $G$-gauge theory over $M$ is a principal fibre bundle $P$ with structure group $G$. Connections $\omega$ on $P$ are also called gauge potentials.

More precisely, the bundle $P$ forms the setting for a classical gauge theory. The particle fields can be described as sections of associated bundles of $P$. The description of the gauge theory is completed by specifying the equations of motion. These are typically derived as the Euler-Lagrange equations for some action functional, which depends on the connection and on the particle fields, and which is invariant under the action of the gauge group.

### 6.1.2 Conjugate modules and vector bundles

In the construction of gauge modules in Section 6.4 we will make explicit use of the notion of a conjugate module. For completeness, we recall the definition of conjugate modules and vector bundles here. We will consider hermitian modules as defined in Definition 2.1.

**Definition 6.11.** Let $E$ be an $A$-$B$-bimodule with a (right) $B$-valued hermitian structure $(\cdot | \cdot)_B$. Its conjugate module $\overline{E}$ is equal to $E$ itself as an additive group. It can naturally be endowed with a $B$-$A$-bimodule structure and a (left) $B$-valued hermitian structure $\overline{B} (\cdot | \cdot)$ by setting

$$b\overline{e} := \overline{eb}^*, \quad \overline{ea} := a^*\overline{e}, \quad \overline{B} (\overline{e_1} | \overline{e_2}) := (e_1 | e_2)_B,$$

for all $a \in A$, $b \in B$, $e, e_1, e_2 \in E$.

If $E = \Gamma^{\infty}_c(E)$ is the $\mathcal{C}^\infty_c(M)$-module of sections of some (hermitian) vector bundle $\mathcal{E}$, then the conjugate module $\overline{E}$ is equal to the $\mathcal{C}^\infty_c(M)$-module of sections of the conjugate vector bundle $\overline{\mathcal{E}}$ which is defined as:

**Definition 6.12.** Let $E \to M$ be a complex vector bundle. Take $\overline{E}$ to be equal to $E$ as fibre bundles over $M$, and write $\overline{e}$ for the element in $\overline{E}$ that corresponds to $e \in E$.
under this identification. The bundle $\mathcal{E}$ is turned into a vector bundle over $M$ by defining the vector space structure in $\mathcal{E}_x$ by

$$(\varepsilon_1, \varepsilon_2) \mapsto \varepsilon_1 + \varepsilon_2, \quad \lambda \cdot \varepsilon = \overline{\lambda} \varepsilon,$$

for all $\lambda \in \mathbb{C}$, $\varepsilon, \varepsilon_1, \varepsilon_2 \in \mathcal{E}_x$. The vector bundle $\mathcal{E} \to M$ is called the **conjugate vector bundle** of $\mathcal{E}$.

The identification $\mathcal{E} \ni \varepsilon \mapsto \overline{\varepsilon} \in \mathcal{E}$ in the above definition is an anti-linear isomorphism of vector bundles.

Let the vector space $V$ denote the fibre of $\mathcal{E}$. A local trivialisation $(U, h)$ of $\mathcal{E}$ induces a local trivialisation of $\mathcal{E}$ given by the map

$$\overline{h}: \pi_{\mathcal{E}}^{-1}(U) \ni \varepsilon \mapsto \overline{h}(\varepsilon) \in U \times \nabla,$$

where $\overline{(x, \nabla)} := (x, \overline{\nabla}) \in U \times \nabla$. If $g_{ij}$ is a transition function between two local trivialisations $(U_i, h_i)$ and $(U_j, h_j)$ of $\mathcal{E}$, then the transition function $\overline{g}_{ij}$ between the corresponding local trivialisations $(U_i, \overline{h}_i)$ and $(U_j, \overline{h}_j)$ is equal to

$$\overline{g}_{ij}(x, \overline{\nabla}) = \overline{h}_i \circ \overline{h}_j^{-1}(x, \overline{\nabla}) = h_i h_j^{-1}(x, \overline{\nabla}) = (x, \overline{g_{ij}(x) \nabla}) = (x, \overline{\nabla} \cdot g_{ij}(x)^*). \quad (6.1)$$

From here on, we consider a vector bundle $\mathcal{E} \to M$, and set $\mathcal{A} := \mathcal{C}_c^\infty(M)$ and $\mathcal{E} := \Gamma_\mathcal{E}^\infty(\mathcal{E})$. Recall the definition of a connection $\nabla: \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega^1(M)$ from **Definition 3.3**. The conjugate connection $\overline{\nabla}: \overline{\mathcal{E}} \to \Omega^1(M) \otimes_\mathcal{A} \overline{\mathcal{E}}$ is given by

$$\overline{\nabla} \varepsilon = \overline{\nabla} \varepsilon, \quad (\varepsilon \in \mathcal{E}),$$

where $\overline{\varepsilon \otimes \omega} = \omega^* \otimes \overline{\varepsilon}$ for all $\varepsilon \otimes \omega \in \mathcal{E} \otimes_\mathcal{A} \Omega^1(M)$. Here $*: \Omega^1(M) \to \Omega^1(M)$ is defined as $(f dg)^* = f^* (dg)^*$. It then follows that $\overline{\nabla}$ is also hermitian for the map $\Omega^1(M) \ni \cdot | \cdot : (\Omega^1(M) \otimes_\mathcal{A} \overline{\mathcal{E}}) \times \overline{\mathcal{E}} \to \Omega^1(M)$ defined by setting $\Omega^1(M) \ni (\alpha \otimes \overline{e_1} | \overline{e_2}) := (e_1 \otimes \alpha^* | e_2)_{\Omega^1(M)} = \alpha(e_1 | e_2)_A$.

For a commutative algebra $\mathcal{A} = \mathcal{C}_c^\infty(M)$ (or $\mathcal{A} = \mathcal{C}_0(M)$) the notion of left and right modules are equivalent. If $\mathcal{E}$ is a left $\mathcal{A}$-module with (left) $\mathcal{A}$-valued hermitian structure $\mathcal{A} \ni (\cdot, \cdot)$, then $(e_1, e_2)_A := \mathcal{A}(e_2, e_1)$ defines a right $\mathcal{A}$-valued hermitian structure on $\mathcal{E}$ when it is seen as a right $\mathcal{A}$-module. Whenever $\mathcal{A}$ is commutative, we will freely use this identification.

### 6.2 Almost-Commutative Manifolds

In **Section 1.2.1** we have described almost-commutative manifolds $M \times F$. In this section we will describe their generalisation to the globally non-trivial case. Before we continue, let us first recall the definition of a real structure on a spectral triple.
Table 1: The signs $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ depending on the KO-dimension $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon'$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon''$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Definition 6.13.** Let $(A, \mathcal{H}, D)$ be an even or odd spectral triple, and in the even case let $\Gamma$ be the grading operator (i.e. a self-adjoint unitary operator implementing the $\mathbb{Z}_2$-grading of the Hilbert space $\mathcal{H}$).

The spectral triple is called real if there exists an anti-unitary isomorphism $J : \mathcal{H} \to \mathcal{H}$, called a real structure, satisfying

\[ J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad J\Gamma = \varepsilon'' \Gamma J \text{ (if } \Gamma \text{ exists)}, \]
\[ [a, Jb^*] = 0, \quad [[D, a], Jb^*] = 0, \quad \forall a, b \in A. \]

The signs $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ determine the KO-dimension $n$ modulo 8 of the real spectral triple, according to Table 1. We will refer to the conditions $[a, Jb^*] = 0$ and $[[D, a], Jb^*] = 0$ as the zeroth- and first-order conditions, respectively. A real, even spectral triple is usually denoted by the data $(A, \mathcal{H}, D, \Gamma, J)$.

Given an algebra $A$, we define the opposite algebra as the vector space $A^\text{op} := \{ a^\text{op} \mid a \in A \}$ with the opposite product $a^\text{op}b^\text{op} = (ba)^\text{op}$. For a real spectral triple, we therefore have a linear representation of $A^\text{op}$ on $\mathcal{H}$ given by $a^\text{op} \mapsto Ja^*J^*.$

Let $(M, g)$ be a smooth compact even-dimensional Riemannian spin manifold (with a fixed spin structure). We assume (throughout this section) that $M$ has dimension 4. By the reconstruction theorem [Con13], the manifold $M$ can be completely characterised by the real even spectral triple

\[(C^\infty(M), L^2(S), D, \Gamma_M, J_M),\]

which is often referred to as the canonical spectral triple for $M$. Here $S$ is the spinor bundle with grading $\Gamma_M$ (see Section 3.3.4), and $D = c \circ \nabla^5$ is the canonical Dirac operator (constructed in Section 3.4) corresponding to the Riemannian metric $g$. The anti-linear operator $J_M$ is called charge conjugation. It commutes with the Clifford multiplication by a real vector field, and it equips the spectral triple for $M$ with a real structure of KO-dimension 4.

Given a real even finite spectral triple $(A_F, \mathcal{H}_F, D_F, \Gamma_F, J_F)$ (with $\dim \mathcal{H}_F < \infty$), we can construct the product triple

\[ M \times F := (C^\infty(M), A_F, L^2(S) \otimes \mathcal{H}_F, D \otimes 1 + \Gamma_M \otimes D_F, \Gamma_M \otimes \Gamma_F, J_M \otimes J_F). \]
Defining the (globally trivial) algebra bundle \( B = M \times A_F \) and the (globally trivial) vector bundle \( E = M \times H_F \), we can rewrite \( \mathcal{C}^\infty(M, A_F) \simeq \Gamma^\infty(B) \) and \( L^2(S) \otimes H_F \simeq L^2(S \otimes E) \). The purpose of this section is to generalise the construction of \( M \times F \) to \textit{globally non-trivial} bundles over \( M \). At the same time, we will put this generalised construction in the context of the Kasparov product between unbounded Kasparov modules. The globally non-trivial case was first considered in [BS11] for the case of algebra bundles with fibre \( M_N(C) \) (describing Yang-Mills theory), and has also been studied more generally in [Cač12].

6.2.1 The internal space

**Definition 6.14.** A (smooth) internal space \( \mathcal{I}^\infty \) over a compact manifold \( M \) is given by the data

\[
\mathcal{I}^\infty := (\Gamma^\infty(B), \Gamma^\infty(E), D_1),
\]

where \( E \) is a hermitian vector bundle over \( M \), \( B \) is a unital \( \ast \)-algebra subbundle of \( \text{End}(E) \), and \( D_1 \) is a hermitian element of \( \Gamma^\infty(\text{End}(E)) \simeq \text{End}_{C^\infty(M)}(\Gamma^\infty(E)) \).

An internal space is called \textit{even} if there is a grading \( \Gamma_1 \), i.e. an endomorphism \( \Gamma_1 \in \Gamma^\infty(\text{End}(E)) \) such that

\[
\begin{align*}
\Gamma_1^* &= \Gamma_1, \\
\Gamma_1^2 &= 1, \\
\Gamma_1 D_1 &= -D_1 \Gamma_1, \\
\Gamma_1 a &= a \Gamma_1 \quad \forall a \in \Gamma^\infty(B).
\end{align*}
\]

An even internal space is called \textit{real} if there is a real structure \( J_1 \), i.e. an anti-unitary endomorphism \( J_1 \) on \( E \) such that

\[
\begin{align*}
\begin{array}{ll}
J_1^2 &= \varepsilon, \\
J_1 D_1 &= \varepsilon' D_1 J_1, \\
J_1 \Gamma_1 &= \varepsilon'' \Gamma_1 J_1, \\
[a, Jb^* J^*] &= 0, \\
[[D_1, a], Jb^* J^*] &= 0, \\
&\forall a, b \in \Gamma^\infty(B),
\end{array}
\end{align*}
\]

where the signs \( \varepsilon, \varepsilon', \varepsilon'' = \pm 1 \) determine the KO-dimension of the internal space according to Table 1.

We shall write \( A = C^\infty(M) \), \( B = \Gamma^\infty(B) \), and \( E = \Gamma^\infty(E) \). Their respective \( C^\ast \)-closures are denoted by \( A = C(M) \), \( B = \Gamma(B) \), and \( E = \Gamma(E) \).

For a non-compact manifold \( M \), the above definition still makes sense if we replace the smooth sections of \( B \) and \( E \) by smooth \textit{compactly supported} sections.

**Remark 6.15.** The endomorphism \( D_1 \) will be interpreted as a mass matrix describing the masses of the elementary particles. We would like to point out a few things about this mass matrix.

1) On a local trivialisation (say, around a point \( x \in M \)) we can view the endomorphism \( D_1 \) as a matrix-valued function \( D_1(x) \), but the precise form of this matrix \( D_1(x) \) depends on the choice of local trivialisation. However, since the
transition functions are unitary, two different choices of local trivialisations yield two unitarily equivalent mass matrices, and hence the eigenvalues of the matrix $D_1(x)$ (i.e. the masses of the particles) are independent of the choice of local trivialisation.

2) These eigenvalues of $D_1(x)$ are (by default) allowed to vary as a function of $x \in M$. In the standard (globally trivial) approach one can also make the (ad hoc) decision to promote the mass parameters to functions (although this is usually not done). However, this would be unnatural from the perspective that a (globally trivial) almost-commutative manifold is the (external) Kasparov product of a Riemannian spin manifold with a finite spectral triple. Instead, varying mass parameters are more naturally described by replacing the finite spectral triple by an internal space (which works equally well in the globally trivial case) and replacing the external by the internal Kasparov product. As such, the promotion of the mass parameters to functions becomes a natural attribute of our framework.

3) One could ask whether it is always possible to choose these mass parameters to be globally constant (as in the usual approach). We expect that this might not always be possible in the general globally non-trivial case, but it is unclear what the precise topological obstructions would be.

**Proposition 6.16.** An even internal space $I^\infty = (\Gamma^\infty(B), \Gamma^\infty(E), D_1)$ yields an unbounded Kasparov $B$-$A$-module $I = (B, \Gamma(E)_A, D_1)$.

**Proof.** By assumption, the grading $\Gamma_1$ commutes with $A$ and $B$, and hence their $C^*$-closures $A$ and $B$ are trivially graded $C^*$-algebras. The module $E = \Gamma(E)$ is a $\mathbb{Z}_2$-graded, finitely generated projective, right Hilbert $A$-module, with a left action of $B$ that commutes with the (right) action of $A$. The properties of $\Gamma_1$ guarantee that all conditions with respect to the grading are satisfied. For instance, the condition $(E^i|E^j) \subset A^{i+j}$, where $i, j \in \mathbb{Z}_2$, is satisfied, since the self-adjointness of $\Gamma_1$ implies that $(s|t) = 0$ as soon as one of the arguments is odd and the other is even. The operator $D_1$ is a bounded, self-adjoint, odd operator by definition (and hence it is automatically regular). The boundedness of $D_1$ implies that $[D_1, b]$ is also bounded for all $b \in B$.

For a compact manifold $M$ the compact endomorphisms of the $C(M)$-module $\Gamma(E)$ are exactly the sections of the endomorphism bundle: $\text{End}_{C(M)}^0(\Gamma(E)) = \Gamma(\text{End}(E))$ (since $\Gamma(\text{End}(E))$ is already unital, the compact endomorphisms of $\Gamma(E)$ are actually all the bounded endomorphisms, see e.g. [GVF01, Proposition 3.9]). Thus, $b(1 + D_1^2)^{-\frac{1}{2}}$ is compact for all $b \in B$, because both $(1 + D_1^2)^{-\frac{1}{2}}$ and $b$ are compact. Hence $(B, \Gamma(E)_A, D_1)$ has all the properties mentioned in Definition 2.25. \qed
6.2.2 The product space

We are now ready to define almost-commutative manifolds as the product of an internal space with the canonical triple over the base manifold. This definition can be given for arbitrary dimensions, but for simplicity we will only give the explicit formula for the case of dimension 4.

**Definition 6.17.** Let $I^\infty := (\Gamma^\infty(B), \Gamma^\infty(E), D_I, \Gamma_1, J_1)$ be a real even internal space over $M$, with $M$ a compact 4-dimensional Riemannian spin manifold. Let $\nabla^I$ be a hermitian connection on $E$. We define a real even almost-commutative manifold to be

$$I^\infty \times_{\nabla} M := (\Gamma^\infty(B), L^2(E \otimes S), D_E + D_I \otimes \Gamma_M, \Gamma_1 \otimes \Gamma_M, J_1 \otimes J_M),$$

where $L^2(E \otimes S) \simeq \Gamma(E) \otimes_{C(M)} L^2(S)$ are the $L^2$-sections of the twisted spinor bundle $E \otimes S$, and $D_E$ is the twisted Dirac operator

$$D_E := 1 \otimes \nabla := 1 \otimes D + (1 \otimes c) \circ (\nabla^I \otimes 1).$$

We note that our definition of almost-commutative manifolds fits within the slightly more general definition of almost-commutative spectral triples given in [Čač12, Definition 2.3].

The order of $I^\infty$ and $M$ in the notation $I^\infty \times_{\nabla} M$ is reversed in comparison with the order of $F$ and $M$ in $M \times F$. The reason is that the order $I^\infty \times_{\nabla} M$ is more natural for its description as a Kasparov product (see Section 6.2.3), whereas the notation $M \times F$ for the globally trivial case is quite standard in the literature. The operator $\mathcal{D} := D_E + D_I \otimes \Gamma_M$ has been defined to match the existing literature. Given that we have reversed the order of the product, the more natural definition (in terms of graded tensor products) would have been $(\Gamma_1 \otimes 1)D_E + D_I \otimes 1$. However, our definition of $\mathcal{D}$ is unitarily equivalent to this more natural definition (by the unitary operator which equals $-1$ on $\Gamma(E) \otimes_{C(M)} L^2(S) \otimes 1$ and 1 elsewhere), and hence there is no harm in using the operator $\mathcal{D}$ which matches the literature.

In the remainder of this section, we show in detail that an almost-commutative manifold $I^\infty \times_{\nabla} M$ determines an unbounded Kasparov B-C-module (i.e. a spectral triple over $B$) which represents the Kasparov product between the KK-classes of the internal space $I^\infty$ and the canonical spectral triple for $M$.

**Proposition 6.18.** Let $I^\infty = (\Gamma^\infty(B), \Gamma^\infty(E), D_I, \Gamma_1, J_1)$ be a real even internal space of even KO-dimension $k$ over a compact Riemannian spin manifold $M$. Let $\nabla^I$ be a hermitian connection on $E$ that commutes with the grading $\Gamma_1$, satisfies $\nabla^I_\mu J_1 = J_1 \nabla^I_\mu$, and is such that the induced connection $[\nabla^I, \cdot]$ on $\text{End} E$ restricts to a connection on $B$. Then the real even almost-commutative manifold $I^\infty \times_{\nabla} M$ is a real even spectral triple of KO-dimension $4 + k \pmod 8$. 
Proof. Let us write \( \mathcal{D} := \mathcal{D}_E + \mathcal{D}_I \otimes \Gamma_M \). We need to show that \([\mathcal{D}, a]\) is bounded for all \( a \in \Gamma^\infty(B)\). Since \( \mathcal{D}_I \) is bounded itself, we need only check this for the twisted Dirac operator \( \mathcal{D}_E \), and we find

\[
[\mathcal{D}_E, a] = c([\nabla^1, a]),
\]

where, with some abuse of notation, we set \( c(T \otimes \alpha) = T \otimes c(\alpha) \) for \( T \in \Gamma^\infty(\text{End} \ E) \) and \( \alpha \in \Omega^1(M) \). Hence for smooth \( a \) the commutator \([\mathcal{D}_E, a]\) indeed acts as a bounded operator on \( L^2(E \otimes S) \). Furthermore we need to show that \( \mathcal{D} \) has compact resolvent, and (as \( M \) is compact) for this it is sufficient to show that \( \mathcal{D}^2 \) (and hence \( \mathcal{D} \)) is elliptic. The Lichnerowicz-Weitzenböck formula shows that the square of the twisted Dirac operator \( \mathcal{D}_E \) is a generalised Laplacian, and hence is elliptic. The bounded (zeroth-order) perturbation \( \mathcal{D}_E \to \mathcal{D}_E + \mathcal{D}_I \otimes \Gamma_M \) does not affect this ellipticity. Hence \( \Gamma^\infty \times \nabla M \) is indeed a spectral triple.

Given the grading operators \( \Gamma_I \) and \( \Gamma_M \), it is straightforward to check that we have \( \mathcal{D}(\Gamma_I \otimes \Gamma_M) = -(\Gamma_I \otimes \Gamma_M)\mathcal{D} \), provided that \([\nabla^1, \Gamma_I] = 0\).

Given the real structures \( J_I \) and \( J_M \), the operator \( J_I \otimes J_M \) is anti-unitary and satisfies

\[
(J_I \otimes J_M)^2 = -\epsilon, \quad \mathcal{D}(J_I \otimes J_M) = (J_I \otimes J_M)\mathcal{D},
\]

\[
(J_I \otimes J_M)(\Gamma_I \otimes \Gamma_M) = \epsilon''(\Gamma_I \otimes \Gamma_M)(J_I \otimes J_M), \tag{6.2}
\]

where the signs \( \epsilon, \epsilon'' \) are determined by the KO-dimension \( k \) of \( J_I \). The first equality in Eq. (6.2) is immediate from \( J_M^2 = -1 \) and \( J_I^2 = \epsilon \). Using the relations

\[
J_M \mathcal{D} = \mathcal{D} J_M, \quad \gamma^\mu J_I = J_I \gamma^\mu, \quad \Gamma_M J_M = J_M \Gamma_M, \quad J_I \mathcal{D}_I = \mathcal{D}_I J_I, \quad \nabla^\mu_I J_I = J_I \nabla^\mu_I,
\]

the second equality in Eq. (6.2) is checked by a local calculation (where we write \((1 \otimes c) \circ (\nabla^1 \otimes 1) = \nabla^1_\mu \otimes \gamma^\mu\)):

\[
\mathcal{D}(J_I \otimes J_M)(s \otimes \psi) = (J_I s) \otimes (\mathcal{D} J_M \psi) + (\nabla^1_\mu J_I s) \otimes (\gamma^\mu J_M \psi) + (\mathcal{D}_I J_I s) \otimes (\Gamma_M J_M \psi) = (J_I s) \otimes (J_M \mathcal{D} \psi) + (J_I \nabla^1_\mu s) \otimes (J_M \gamma^\mu \psi) + (J_I \mathcal{D}_I s) \otimes (J_M \Gamma_M \psi) = (J_I \otimes J_M) \mathcal{D}(s \otimes \psi).
\]

The third equality in Eq. (6.2) immediately follows from \([J_M, \Gamma_M] = 0 \) and \( J_I \Gamma_I = \epsilon'' \Gamma_I J_I \). From the values of \(-\epsilon \) and \( \epsilon'' \) it is immediate that the KO-dimension of \( \Gamma^\infty \times \nabla M \) should be \( 4 + k \) (mod 8) (see Table 1).

The zeroth-order condition on \( \Gamma^\infty \times \nabla M \) is immediate from the zeroth-order condition on \( \Gamma^\infty \). Moreover,

\[
[[\mathcal{D}_E, a], JbJ^*] = [c(\nabla^1, a), JbJ^*] = c([\nabla^1, a], JbJ^*) = 0,
\]

because, by assumption, \([\nabla^1, a] \in \Gamma^\infty(B) \otimes \Omega^1(M) \), which commutes with \( JbJ^* \). Together with the first-order condition on \( \mathcal{D}_I \), this implies that \( \mathcal{D} \) satisfies the first-order condition. \( \square \)
For a unital real spectral triple $T = (A, \mathcal{H}, D, J)$, the gauge group is defined in [DS12, Definition 2.5] as

$$
\mathcal{G}(T) := \{ uJ^*u | u \in \mathcal{U}(A) \} \simeq \mathcal{U}(A)/\mathcal{U}(A_J),
$$

where the central subalgebra $A_J$ is defined as $A_J := \{ a \in A \mid aJ = Ja^* \}$. For an almost-commutative manifold (constructed from a compact manifold $M$), we therefore obtain the gauge group

$$
\mathcal{G}(\mathbb{I} \times J^* M) = \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J)
$$

for the real structure $J = J_1 \otimes J_M$. However, since $\mathcal{B}_1 \simeq \mathbb{B}_1$, we find that the gauge group of the almost-commutative manifold is completely determined by the internal space, and we write

$$
\mathcal{G}(\mathbb{I} \times J^* M) \simeq \mathcal{G}(\mathbb{I}^\infty) := \{ uJ_1^*u | u \in \mathcal{U}(\mathcal{B}) \}.
$$

6.2.3 The Kasparov product

We now show that the product $\mathbb{I} \times J^* M$ is an unbounded representative for the Kasparov product of the KK-classes of $\mathbb{I}^\infty$ and the canonical spectral triple for $M$. We first prove this for the cases where $\mathcal{D}_1 = 0$, and then show that the presence of $\mathcal{D}_1$ is irrelevant at the level of KK-classes.

Let $\mathbb{I}^\infty$ be an internal space over $M$, where $\mathcal{D}_1 = 0$, and consider the unbounded Kasparov module $I := (\mathcal{B}, E_A, 0)$, where $E = \Gamma(E)$. We know from Proposition 6.18 that $\mathbb{I} \times J^* M = (\mathcal{B}, L^2(E \otimes S), D)$ is a spectral triple, which thus yields an unbounded Kasparov module $I \times J^* M = (\mathcal{B}, L^2(E \otimes S)_{\mathcal{C}}, D) \in \Psi(E, \mathbb{C})$ (Definition 2.25).

**Proposition 6.19.** The unbounded Kasparov module $I \times J^* M$ represents the Kasparov product of (the classes of) $I \in \Psi(E, A)$ and $(A, L^2(S)_{\mathcal{C}}, D) \in \Psi(A, \mathbb{C})$.

**Proof.** It suffices to check the conditions of Theorem 2.34. Since $\mathcal{D}_1 = 0$, conditions 2) and 3) are trivial, and we only need to check condition 1). For all $e$ in a dense subspace of $BE = E$, we need to check boundedness of

$$
\mathcal{D}T_e - T_e\mathcal{D} \quad \text{on } \text{Dom}(\mathcal{D}) \subset L^2(S),
$$

$$
\mathcal{D}T^*_e - T^*e\mathcal{D} \quad \text{on } \text{Dom}(\mathcal{D}) \subset E \otimes_A L^2(S) \simeq L^2(E \otimes S),
$$

where $\mathcal{D} = \mathcal{D}_E = (1 \otimes c) \circ (1 \otimes \nabla^5 + \nabla^1 \otimes 1)$. For $\psi \in \text{Dom}(\mathcal{D})$ we obtain

$$
(\mathcal{D}T_e - T_e\mathcal{D})\psi = (1 \otimes c) \circ (1 \otimes \nabla^5 + \nabla^1 \otimes 1)e \otimes \psi - e \otimes \mathcal{D}\psi = c(\nabla^1 e) \otimes \psi,
$$

which is indeed bounded for all $e$ in the dense subspace $\mathcal{E} = \Gamma^\infty(E)$. Next, for $f \otimes \psi \in \Gamma^\infty(B \otimes S) \subset \text{Dom}(\mathcal{D})$ we obtain

$$
(\mathcal{D}T^*_e - T^*e\mathcal{D})(f \otimes \psi) = \mathcal{D}(e|f)\psi - (e|f)D\psi - (e|c(\nabla^1 f))\psi = c(\nabla^1 e|f)\psi,
$$

as desired.
where we have used the compatibility of the connection $\nabla^1$ with the hermitian form $\langle \cdot, \cdot \rangle$, and so $\mathcal{D}T^*_e - T^*_e \mathcal{D}$ is a zeroth-order differential operator for smooth $e$.

To prove a similar result for the case where $\mathcal{D}_1 \neq 0$, we use the following two lemmas.

**Lemma 6.20.** If $E_A$ is a finitely generated projective Hilbert $A$-module, then for any self-adjoint, odd endomorphism $F \in \text{End}_A(E)$, the unbounded Kasparov $B$-$A$-modules $(\mathcal{B}, E_A, F)$ and $(\mathcal{B}, E_A, 0)$ represent the same class in $\text{KK}(B, A)$.

**Proof.** Since $E$ is a finitely generated projective $A$-module, all bounded endomorphisms are in fact compact, i.e. $\text{End}_A(E) = \text{End}_A^0(E)$. The equivalence of the compact operators $0$ and $b(F) = F(I + F^2)^{-\frac{1}{2}}$ is then simply obtained via the operator homotopy $t \mapsto tb(F)$, for $t \in [0, 1]$. Hence the modules $(\mathcal{B}, E_A, b(F))$ and $(\mathcal{B}, E_A, 0)$ are equivalent bounded Kasparov $B$-$A$-modules.

**Lemma 6.21** (see also [Kuc97, Corollary 17]). Let $(\mathcal{B}, \pi E_A, \mathcal{D}) \in \Psi(B, A)$ and let $T \in \text{End}_A(E)$ be self-adjoint and odd. Then

1) $(\mathcal{B}, \pi E_A, \mathcal{D} + T)$ is also an unbounded Kasparov module in $\Psi(B, A)$; and

2) $(\mathcal{B}, \pi E_A, \mathcal{D} + T)$ and $(\mathcal{B}, \pi E_A, \mathcal{D})$ represent the same class in $\text{KK}(B, A)$.

**Proof.** 1) Since $T$ is bounded and self-adjoint, it follows from the Kato-Rellich theorem for Hilbert modules (see [KL12, Theorem 4.3]) that the sum $\mathcal{D} + T$ remains self-adjoint and regular. The only non-trivial thing to prove is that $\mathcal{D} + T$ has locally compact resolvent, i.e. $\pi(b)(1 + (\mathcal{D} + T)^2)^{-\frac{1}{2}} \in \text{End}_A^0(E)$ for all $b \in \mathcal{B} \subset B$. This is equivalent to showing that $\pi(b)(\pm i + \mathcal{D} + T)^{-1}$ is compact. The operator $(\pm i + \mathcal{D} + T)^{-1}$ maps $E$ into $\text{Dom}(\mathcal{D} + T) = \text{Dom} \mathcal{D}$, so that $(\pm i + \mathcal{D})(\pm i + \mathcal{D} + T)^{-1}$ is a well-defined bounded operator on $E$. From

$$\pi(b)(\pm i + \mathcal{D} + T)^{-1} = \pi(b)(\pm i + \mathcal{D})^{-1}(\pm i + \mathcal{D})(\pm i + \mathcal{D} + T)^{-1}$$

we then see that $\pi(b)(\pm i + \mathcal{D} + T)^{-1}$ is compact.

2) The idea is to prove that $(\mathcal{B}, \pi E_A, \mathcal{D} + T) \in \Psi(B, A)$ represents the Kasparov product $[(\mathcal{B}, \pi E_A, \mathcal{D})] \otimes_A [(A, A_A, 0)]$. It is enough to show that all the conditions in Theorem 2.34 are satisfied. First of all,

$$A \ni a \mapsto (\mathcal{D} + T)t_e(a) = ((\mathcal{D} + T)e)a,$$

$$f \otimes a \mapsto T^*_e(\mathcal{D} + T)(f \otimes a) = ((\mathcal{D} + T)e, f)_A a,$$

are both clearly bounded on $A$ and $\text{Dom}(\mathcal{D} + T) = \text{Dom} \mathcal{D}$, respectively, for all $e \in \text{Dom} \mathcal{D}$. In particular, this holds for all $e \in \pi(\mathcal{B}) \text{Dom} \mathcal{D}$, which is a dense subset of $\pi(\mathcal{B})E$. This proves that condition 1) in Theorem 2.34 is
satisfied. Since \( \text{Dom } D = \text{Dom } (D + T) \), condition 2) is also satisfied. For the
final condition, a small calculation shows that

\[
((D + T)e|De) + (De|(D + T)e) = ((D + T)e|D + T)e) - (D + T)e|Te) + (De|(D + T)e) \\
= ((D + T)e|D + T)e) - (Te|Te) + (De|De) \geq -|T|^2(e|e),
\]

for all \( e \in \text{Dom } D \), since \( ((D + T)e|D + T)e) \) and \( (De|De) \) are positive. \( \square \)

**Corollary 6.22.** The unbounded Kasparov module \( I \times \nabla M = (B, L^2(E \otimes_S)C, D) \) represents the Kasparov product of \( I = (B, E_A, D_1) \) with \( (A, L^2(S)_C, D) \).

**Proof.** By Lemma 6.20 we know that \( (B, E_A, D_1) \) and \( (B, E_A, 0) \) represent the same Kasparov class. From Proposition 6.19 it then follows that \( (B, E \otimes_A L^2(S)_C, D_E) \) also represents the Kasparov product of \( (B, E_A, D_1) \) with \( (A, L^2(S)_C, D) \). According to Lemma 6.21 the cycle \( I \times \nabla M = (B, E \otimes_A L^2(S)_C, D_E + D_1 \otimes \Gamma_M) \) represents the same Kasparov class as \( (B, E \otimes_A L^2(S)_C, D_E) \), so it also represents this Kasparov product. \( \square \)

**Remark 6.23.**

1) The construction of \( I \times \nabla M \) via Kasparov products fits naturally in the framework of Mesland’s category of spectral triples [Mes14], where the internal space \( \Gamma^\infty \) with the connection \( \nabla \) can be seen as (a representative of) a morphism from the canonical triple for \( M \) to the almost-commutative manifold \( \Gamma^\infty \times \nabla M \). The construction also gives an example of the framework for gauge theories using factorisation in unbounded KK-theory as proposed in [BMS13].

2) As is clear from the above discussion, the presence of the operator \( D_1 \) (or \( D_1 \otimes \Gamma_M \)) is completely irrelevant on the level of KK-classes. In this sense the KK-equivalence is too strong for our purposes, because in the models under consideration the presence of the operator \( D_1 \) certainly does matter. We will describe in Section 6.5 how this operator plays the role of a ‘mass matrix’ for the elementary fermions of the gauge theory, and gives rise to the Higgs field in the noncommutative Standard Model (see also Section 6.6.2 for a concrete example of \( D_1 \) as a mass matrix). Hence, if one wants to retain the physical information, one should only consider unitary equivalences (see also Remark 2.29).

### 6.3 Principal Modules

We would like to describe a classical gauge theory on a manifold \( M \) by considering an almost-commutative manifold \( \Gamma^\infty \times \nabla M \). For this purpose we now restrict our attention to a special case of internal spaces, which we call principal modules.
In Section 6.3.1 we first recall (part of) the classification of finite-dimensional real spectral triples due to Krajewski [Kra98] and Paschke and Sitarz [PS98]. In Section 6.3.2 we then define the notion of principal modules, and we show that, when the base manifold (which is of arbitrary dimension) is simply connected, the gauge group of a principal module (as defined for internal spaces in Eq. (6.4)) is isomorphic to the classical notion of the gauge group of a principal fibre bundle (as defined in Definition 6.8).

6.3.1 Real finite spectral triples

Finite-dimensional real spectral triples have been classified for the case of KO-dimension 0 [Kra98, PS98]. With similar arguments, this can be generalised to arbitrary KO-dimension [Sui14]. In the following theorem we give the result for complex algebras, while also setting the matrix $\mathcal{D}_F = 0$. Below c.c. denotes complex conjugation of the coefficients with respect to the standard basis of $\mathbb{C}^{m_{ij}}$.

**Theorem 6.24** ([Kra98, PS98]). Let $F := (A_F, \mathcal{H}_F, 0, J_F)$ be a real finite spectral triple over a complex $*$-algebra $A_F$. Up to unitary equivalence, this triple is of the form

$$A_F = \bigoplus_{i=1}^1 M_{N_i}(\mathbb{C}), \quad \mathcal{H}_F = \bigoplus_{i,j=1}^1 \mathcal{H}_{ij}, \quad \mathcal{H}_{ij} := M_{N_i,N_j}(\mathbb{C}) \otimes \mathbb{C}^{m_{ij}},$$

such that $m_{ij} = m_{ji}$, and the inner product on each copy of $M_{N_i,N_j}(\mathbb{C})$ is given by $\langle t_1, t_2 \rangle = \text{Tr}(t_1^*t_2).$ If $J_F^2 = \varepsilon$, then $J_F$ acts on $\mathcal{H}_{ii} \oplus \mathcal{H}_{jj}$, $(i < j)$, as

$$\left( \begin{array}{cc} 0 & \varepsilon(-)^* \\ (-)^* & 0 \end{array} \right) \otimes (\text{Id}_{m_{ij}} \circ \text{c.c.}).$$

If $J_F^2 = 1$, the real structure $J_F$ acts on $\mathcal{H}_{ii} \simeq M_{N_i}(\mathbb{C}) \otimes \mathbb{C}^{m_{ii}}$ as

$$(-)^* \otimes (\text{Id}_{m_{ii}} \circ \text{c.c.}),$$

If $J_F^2 = -1$, then $m_{ii}$ is even and $J_F$ acts on $(M_{N_i}(\mathbb{C}) \oplus M_{N_i}(\mathbb{C})) \otimes \mathbb{C}^{\frac{m_{ii}}{2}}$ as

$$\left( \begin{array}{cc} 0 & (-)^* \\ (-)^* & 0 \end{array} \right) \otimes (\text{Id}_{\frac{m_{ii}}{2}} \circ \text{c.c.}).$$

The different copies of $M_{N_i,N_j}(\mathbb{C})$ (with respect to the above decomposition) in $\mathcal{H}_{ij}$ are denoted by $\mathcal{H}_{ij}^\alpha$, where $1 \leq \alpha \leq m_{ij}$.

**Remark 6.25.** For finite-dimensional complex vector spaces $V$ and $W$, consider the linear isomorphism

$$L: V \otimes W \rightarrow \text{Hom}(W,V), \quad v \otimes w \mapsto (w' \mapsto v\langle w, w' \rangle), \quad v \in V, \ w, w' \in W,$$
where \( \overline{W} \) denotes the conjugate vector space. For \( i \in I := \{1, \ldots, l\} \), we write \( V_i = \mathbb{C}^{N_i} \), endowed with the standard inner product. Then the finite-dimensional Hilbert space \( \mathcal{H}_{ij} \) can also be put in the form

\[
\mathcal{H}_F = \bigoplus_{(i,j) \in K} V_i \otimes \overline{V}_j,
\]

endowed with its standard inner product. Here \( K \) is a multiset consisting of pairs in \( 1 \times 1 \) such that the multiplicity of \( (i, j) \) is equal to that of \( (j, i) \) and such that the projection \( K \rightarrow I \) on either of the factors is surjective (this last condition is equivalent to the faithfulness of the action of \( A_F \) on \( \mathcal{H}(F) \)). The algebra \( A_F \otimes A_F^{op} \) acts on a summand \( V_i \otimes \overline{V}_j \) as

\[
(a, b^{op})(v \otimes w) = a_i v \otimes b_j^* \overline{w},
\]

and the corresponding real structure on \( V_i \otimes \overline{V}_j \rightarrow V_j \otimes \overline{V}_i \) is simply given by

\[
J_F(v \otimes w) = \pm w \otimes v,
\]

where the signs are determined by the KO-dimension of \( F \). We will use this form of the real finite spectral triple in Section 6.4.

From now on we assume that every real finite spectral triple (with \( D_F = 0 \)) is of the form as mentioned in Theorem 6.24. Later on, the algebra \( (A_F)_F \) will also be of interest, so we conclude this subsection by determining its precise form.

Recall that, in general, for any real spectral triple \( (A, \mathcal{H}, D, J) \), the complex central subalgebra \( A_I \) is defined as \( A_I = \{ a \in A : a^* = J a \} \).

**Proposition 6.26.** With notation as above, we have

\[
(A_F)_F = \left\{ a = \bigoplus_{i \in I} \lambda_i \, \text{id}_{N_i} \in A_F \mid \lambda_i \in \mathbb{C}; \lambda_i = \lambda_j \text{ if } \mathcal{H}_{ij} \neq \{0\} \right\}.
\]

**Proof.** We can assume that \( J \) is in standard form. Write \( A_F = \bigoplus_{i \in I} M_{N_i}(\mathbb{C}) \) and consider an element \( a = \bigoplus_{i \in I} a_i \in A_F \). If \( t \in \mathcal{H}^M_{ij} \) \((1 \leq \alpha \leq m_{ij})\), then

\[
a(J_F t) = \pm a_i t^* \quad \text{and} \quad J_F(a^* t) = \pm t^* a_i.
\]

For \( 1 \leq k \leq N_j \) and \( 1 \leq l \leq N_i \), choose \( t^* = e_{k1} \) (a standard basis vector of \( M_{N_j,N_i} \)). Then

\[
(a_i e_{k1})_{\gamma \beta} = (a_i)_{\gamma k} \delta_{\beta 1}, \quad \text{and} \quad (e_{k1} a_i)_{\gamma \beta} = \delta_{\gamma 1} (a_i)_{l \beta},
\]

Therefore, \( a | J_F a | = J_F a^* \) if and only if

\[
(a_i)_{\gamma k} \delta_{\beta 1} = (a_i)_{l \beta} \delta_{\gamma k},
\]

for all \( 1 \leq k, \gamma \leq N_j \) and \( 1 \leq l, \beta \leq N_i \). It follows that \( a_i \), \( a_j \) are diagonal and \( (a_i)_{kk} = (a_i)_{l l} \) for all \( 1 \leq k \leq N_j \) and \( 1 \leq l \leq N_i \). Hence, \( a \in (A_F)_F \) if and only if each \( a_i = \lambda_i \text{id}_{N_i} \) and \( \lambda_i = \lambda_j \) if \( \mathcal{H}_{ij} \neq \{0\} \). \( \square \)
We now want to find spectral triples for gauge theories that are globally non-trivial. Take any real finite spectral triple \( F \) and a certain compatibility requirements (which we will specify later in Definition 6).

Recall from Definition 6.3.1 we define an equivalence relation on \( I \) as follows. For \( i \neq j \in I \) we set \( i \sim j \) if there exists a sequence \( i = i_0, \ldots, i_k = j \) such that \( \mathcal{H}_{i_m,i_{m+1}} \neq \{0\} \) for all \( 0 \leq m < k \). If \( i \sim j \) we say that \( i \) is connected to \( j \).

Proposition 6.26 in particular shows that \( C \subset \{ A_F \}_{ij} \subset Z(A_F) \).

Corollary 6.28. We have the isomorphism \( \{ A_F \}_{ij} \simeq \bigoplus_{\{ i \in I \}} C \). In particular, the two extreme cases are:

- \( \{ A_F \}_{ij} = Z(A_F) \) if and only if \( \mathcal{H}_{ij} = 0 \) for all \( i \neq j \) (that is, \( I/\sim \simeq 1 \));
- \( \{ A_F \}_{ij} = C \) if and only if \( i \) is connected to \( j \) for all \( i,j \in I \) (that is, \( I/\sim \simeq \{ 1 \} \)).

6.3.2 Principal modules

We now want to find spectral triples for gauge theories that are globally non-trivial. Recall from Definition 6.10 that a general gauge theory with structure group \( G_F \) on a manifold \( M \) is given by a principal \( G_F \)-bundle \( P \) over \( M \) (along with a prescribed action functional or Lagrangian).

If \( \{ A_F, \mathcal{H}_F, D_F, J_F \} \) is a finite-dimensional real spectral triple, then the corresponding gauge group \( G_F \) is given by (see also Eq. (6.3))

\[
G_F := \{ uJ_F u^* F | u \in U(A_F) \} \simeq U(A_F)/U(\{ A_F \}_{ij}^1).
\]

Such finite spectral triples can be used to describe globally trivial gauge theories over \( M \) (see the Introduction). Any finite spectral triple \( F \) automatically yields an internal space

\[
I_F^\infty = (\Gamma^\infty(M \times A_F), \Gamma^\infty(M \times \mathcal{H}(F), D_F, J_F)),
\]

where now \( D_F \) and \( J_F \) are seen as constant bundle endomorphisms acting on the fibre \( \mathcal{H}_F \). We now want to generalise this construction in order to describe globally non-trivial gauge theories. Of course, fibre-wise we want to obtain the finite-dimensional situation that has been explained in Section 6.3.1.

The most straightforward way to obtain (examples of) globally non-trivial gauge theories over \( M \) would then be as follows (see also [Cač12, Lemma 2.5] and [BS11]). Take any real finite spectral triple \( F := \{ A_F, \mathcal{H}_F, D_F, J_F \} \) with gauge group \( G_F \), and let \( M \) be a smooth compact 4-dimensional Riemannian spin manifold. Take any principal \( G_F \)-bundle \( P \to M \). We construct the globally non-trivial triple of the form

\[
P \times_{G_F} F := (\Gamma^\infty(P \times_{G_F} A_F), \Gamma^\infty(P \times_{G_F} \mathcal{H}(F), D_P, 1 \times J_F)).
\]

Here \( D_P \) is an endomorphism acting on the vector bundle \( P \times_{G_F} \mathcal{H}_F \) satisfying certain compatibility requirements (which we will specify later in Definition 6.45).
Remark 6.29. Note that (in contrast to [Cać12]) we do not require $D_P$ to be of the form $1 \times D_F$, where $D_F$ is a $G_F$-invariant operator on $\mathcal{H}_F$, as such an assumption is too strong for our purposes. In particular, in specific examples (such as the noncommutative Standard Model) that requirement would prevent the appearance of a scalar (Higgs-like) field through inner fluctuations.

For the remainder of this section we ignore the endomorphism $D_P$, since it is not relevant for the definition of the gauge group.

Definition 6.30. Let $F := (A_F, \mathcal{H}_F, 0, J_F)$ be a real finite spectral triple of the same form as in Theorem 6.24. Write $G_F$ for the corresponding gauge group. Let $M$ be a smooth compact Riemannian spin manifold and let $P \to M$ be any principal $G_F$-bundle. A triplet of the form

$$P \times_{G_F} F := (\Gamma^\infty(P \times_{G_F} A_F), \Gamma^\infty(P \times_{G_F} \mathcal{H}_F), 1 \times J_F),$$

is called a principal $G_F$-module over $M$ (or $C^\infty(M)$) with fibre $F$. For brevity, we introduce the notation $B := P \times_{G_F} A_F$, $E := P \times_{G_F} \mathcal{H}_F$, $B := \Gamma^\infty(B)$, $E := \Gamma^\infty(E)$, and $J := 1 \times J_F$.

Remark 6.31. The principal fibre bundle $P$ is an explicit ingredient in the definition of a principal module. From $P$ we constructed the associated vector bundle $E = P \times_{G_F} \mathcal{H}_F$, and (as discussed in Section 3.2.1) $P$ equips $E$ with a unique equivalence class of $G_F$-atlases. Whenever we consider transition functions of $E$, we therefore assume that they form a $G_F$-atlas in the equivalence class obtained from $P$. Given a $G_F$-atlas, the vector bundle $E$ inherits a hermitian structure from the inner product on $\mathcal{H}_F$, which is well-defined because the action of $G_F$ on $\mathcal{H}_F$ is unitary. For two equivalent $G_F$-atlases, the corresponding hermitian structures are isometric.

We stress that, in order to reconstruct the principal $G_F$-bundle $P$ from a vector bundle $E$ (with structure group $G_F$) using Theorem 3.5, it is not sufficient to know only the bundle $E$; in addition, we also need to know the corresponding equivalence class of $G_F$-atlases.

Proposition 6.32. A principal module $P \times_{G_F} F = (\Gamma^\infty(B), \Gamma^\infty(E), 1 \times J_F)$ yields a real internal space $(\Gamma^\infty(B), \Gamma^\infty(E), 0, 1 \times J_F)$ over $M$.

Proof. The action of $G_F$ on $A_F$ is given by conjugation when $A_F$ is considered as a $*$-subalgebra of $\text{End}(\mathcal{H}_F)$. Consequently, the fibre-wise action of the $*$-algebra bundle $B = P \times_{G_F} A_F$ on $E$ is well defined, and hence $B$ is a unital $*$-algebra subbundle of $\text{End}(E)$. The operator $D_I = 0$ is trivially a hermitian endomorphism. Since the operator $J_F$ commutes with $G_F$, it induces a real structure $J_x$ on each fibre of $E$. The operator $J = 1 \times J_F$ denotes the anti-linear operator on $E$ that is induced by these real structures $J_x$ on the fibres. □

Remark 6.33. Because $(u_J u_F^*) a (J_F u^*) = u a u^*$ for all $a \in A_F$, $u \in \mathcal{U}(A_F)$, we see that the given action of an element $u_J u_F^* \in G_F$ on $A_F$ coincides with
the usual conjugation of the element \( u \in \mathcal{U}(A_F) \). Since \((A_F)_I \subset Z(A_F)\), the map 
\( \tau: G_F \ni uF \mapsto \text{Ad}(u)F \) does not depend on the choice of \( u \). Thus, the surjective map \( \tau: G_F \to \mathcal{U}(A_F) \) is induced by the usual map \( \mathcal{U}(A_F) \to \text{Inn}(A_F) \) (recall that \( G_F \) is the quotient \( \mathcal{U}(A_F)/\mathcal{U}((A_F)_I) \)).

6.3.3 The gauge group

Consider a principal module \( P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J) \) over \( M \). Using the classification of \( A_F \) and \( \mathcal{H}_F \), as given in Section 6.3.1, we can decompose the bundles \( \mathcal{B} = P \times_{G_F} A_F \) and \( E = P \times_{G_F} \mathcal{H}_F \) in a similar way:

\[
\mathcal{B} = \bigoplus_{i \in I} \mathcal{B}_i, \quad \mathcal{B}_i = P \times_{G_F} M_{N_i}(C), \\
\mathcal{E} = \bigoplus_{i,j \in I} \mathcal{E}_{ij}, \quad \mathcal{E}_{ij} = P \times_{G_F} \mathcal{H}_{(ij)}.
\]

Each vector bundle \( \mathcal{E}_{ij} \) carries the obvious action by \( \mathcal{B} \otimes \mathcal{B}^{op} \). Note, however, that even though \( \mathcal{H}_{ij} = C^{N_i} \otimes C^{N_i} \otimes C^{m_{ij}} \), the bundle \( \mathcal{E}_{ij} \) is not necessarily of the form \( \mathcal{E}_i \otimes \mathcal{E}_j \otimes C^{m_{ij}} \) for some vector bundles \( \mathcal{E}_i \) and \( \mathcal{E}_j \) (see Section 6.6.1 for an example).

Note that, for the case \( i = j \), the bundle \( \mathcal{E}_{ii} \) is necessarily isomorphic to (a number of copies of) \( \mathcal{B}_i \). Indeed, the \( G_F \)-valued transition functions act on the fibres of \( \mathcal{E}_{ii} \), which are isomorphic to (copies of) \( M_{N_i}(C) \), by conjugation with an element \( u \in \mathcal{U}(N_i) \), and are therefore inner automorphisms of the algebra \( M_{N_i}(C) \). By Remark 6.33 these transition functions are equal to those for the \( * \)-algebra bundle \( \mathcal{B}_i \).

Denote by \([i]\) the equivalence class of all \( j \in I \) that are connected to \( i \) (see Definition 6.27). Write

\[
\mathcal{B}_{[i]} = \bigoplus_{s \in [i]} \mathcal{B}_s,
\]

and write \( \mathcal{B}_{[i]} \) for the projection of an element \( b \) onto \( \mathcal{B}_{[i]} \). As \( \{\mathcal{B}_{[i]}\}_I = C^\infty(M) \) we obtain (see also Corollary 6.28)

\[
\mathcal{B}_J = \bigoplus_{[i] \in I/\sim} C^\infty(M).
\]

The gauge group of the principal module \( P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J) \) is defined as (see Eq. (6.4))

\[
\mathcal{G}(P \times_{G_F} F) := \{uFJ^* \mid u \in \mathcal{U}(\mathcal{B})\} \sim \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J).
\]

At the same time, a principal \( G_F \)-bundle \( P \to M \) is equipped with the gauge group \( \mathcal{G}(P) = \Gamma^\infty(\text{Ad}P) \) (see Section 6.1.1). We now aim at showing that for a
principal module $P \times_{G_F} F$, the gauge groups $\mathcal{G}(P \times_{G_F} F)$ and $\mathcal{G}(P)$ coincide, provided that $M$ is simply connected. Consider the group bundle map
\[
\phi: \mathcal{U}(B) \simeq P \times_{G_F} \mathcal{U}(A_F) \rightarrow P \times_{G_F} \mathcal{U}(H_F), \quad u_x \mapsto u_x J_x u_x J_x^*.
\]
The image $\phi(\mathcal{U}(B))$ is a group subbundle of $P \times_{G_F} \mathcal{U}(H_F)$, with fibres isomorphic to $G_F$. In fact, this subbundle is isomorphic to the group bundle $AdP$. The induced map $\phi_*$ on the sections $\mathcal{U}(B) \simeq \mathcal{U}(\Gamma^\infty(B)) \rightarrow \mathcal{U}(\Gamma^\infty(\text{End}(E)))$ is precisely the map $u \mapsto u J u^*$, $u \in \mathcal{U}(B)$. Thus, $\phi_*$ maps $\mathcal{U}(B)$ into $\Gamma^\infty(AdP)$. However, even though $\phi: \mathcal{U}(B) \rightarrow AdP$ is surjective, this does not imply that $\phi_*$ is also surjective. We will proceed by showing that in our case, under the assumption that $M$ is simply connected, we do have surjectivity. First, we give a basic result which yields a sufficient condition for when sections of a quotient group bundle can be lifted.

*Lifting sections of quotient group bundles*

Given a surjective group bundle morphism $\phi: H \rightarrow G$, we would like to know whether the induced map $\phi_*: \Gamma^\infty(H) \rightarrow \Gamma^\infty(G)$ is also surjective. This need not always be the case, as the following example shows.

**Example 6.34.** Take $M = SO(3)$ and consider the *globally trivial* group bundles $\mathcal{H} = M \times \mathcal{U}(2)$ and $\mathcal{G} = M \times \mathcal{PSU}(2)$, with the obvious group bundle morphism $\phi: \mathcal{H} \rightarrow \mathcal{G}$ given by the quotient $\mathcal{U}(2) \rightarrow \mathcal{PSU}(2)$. Since $\mathcal{H}$ and $\mathcal{G}$ are globally trivial, we can make the identifications $\Gamma^\infty(\mathcal{H}) \simeq C^\infty(SO(3), \mathcal{U}(2))$ and $\Gamma^\infty(\mathcal{G}) \simeq C^\infty(SO(3), \mathcal{PSU}(2))$. Consider the map $f: SO(3) \rightarrow \mathcal{PSU}(2)$ given by the identification of $PSU(2)$ with $SO(3)$, i.e. $f = \text{id}$ on $SO(3)$. If there exists a lift $\tilde{f}: SO(3) \rightarrow \mathcal{U}(2)$ such that $f = \phi \circ \tilde{f}$, then $\tilde{f}$ is nothing but a global section of the $\mathcal{U}(1)$-principal bundle $\pi: \mathcal{U}(2) \rightarrow SO(3)$. However, such a section does not exist, as this bundle is not globally trivial (the fundamental group of $\mathcal{U}(2)$ is $\mathbb{Z}$, whereas the fundamental group of $SO(3) \times \mathcal{U}(1)$ is $\mathbb{Z}_2 \times \mathbb{Z}$). Hence the map $f$, seen as a section in $\Gamma^\infty(G)$, is not contained in the image of $\phi_*$. In other words, we would like to have sufficient conditions to ensure that for any section $s: M \rightarrow G$ there exists a lift $\tilde{s}: M \rightarrow H$ such that $\phi_*(\tilde{s}) = s$. Though the existence of lifts for covering maps has been well-studied, we will typically be dealing with more general fibrations $\phi: H \rightarrow G$, for which the problem of existence of lifts is more complicated. We avoid this problem by reducing it to the case of covering maps, as follows.

**Lemma 6.35.** Let $p: E \rightarrow B$ be a fibration, and consider some map $f: M \rightarrow B$. Suppose there exists a submanifold $C \subset E$ such that $p|_C: C \rightarrow B$ is a covering space, satisfying $f_*(\pi_1(M, m)) \subset p_*(\pi_1(C, c))$, where $m \in M$ and $c \in C$ are such that $f(m) = p(c)$. Then there exists a lift $\tilde{f}: M \rightarrow E$ satisfying $p \circ \tilde{f} = f$ and $\tilde{f}(m) = c$. 


Proof. Consider the following diagram.

\[ \begin{array}{ccc}
M & \xrightarrow{f} & B \\
\downarrow{p} & & \downarrow{\pi} \\
\Gamma & \xrightarrow{\phi} & C
\end{array} \]

The assumption \( f_* (\pi_1 (M, m)) \subset p_* (\pi_1 (C, c)) \) is precisely the criterion for the existence of a lift \( \tilde{f} : M \to C \) satisfying \( \tilde{f}' (m) = c \) (see e.g. [Hato2, Proposition 1.33]), and then we can simply define \( \tilde{f} : M \to E \) as the composition \( M \xrightarrow{\tilde{f}'} C \hookrightarrow E \).

We now translate the above lemma into the setting of group bundles.

**Corollary 6.36.** Let \( M \) be a simply connected manifold, and let \( G, H \) be group bundles over \( M \). If \( G \) is covered by a subbundle \( U \) of \( H \) via a group bundle morphism \( \phi : H \to G \), then the map \( \phi_* : \Gamma^\infty (H) \to \Gamma^\infty (G) \), given by \( s \mapsto \phi \circ s \), is surjective.

**Proof.** By assumption, \( \phi|_U : U \to G \) is a covering space. Since \( \pi_1 (M, m) \) is trivial (by definition of simply-connectedness) it follows from Lemma 6.35 that each section \( s : M \to G \) can be lifted to a section \( \tilde{s} : M \to U \subset H \) such that \( \phi_* (\tilde{s}) = s \).

The isomorphism of gauge groups

We will now continue to prove that the map \( \phi_* : \Upsilon (B) \to \Gamma^\infty (\text{Ad} \mathcal{P}) \) is surjective. In order to be able to apply Corollary 6.36, we need to construct a subbundle of \( \Upsilon (B) \) which covers \( \text{Ad} \mathcal{P} \).

**Proposition 6.37.** Let \( \mathcal{P} \times_{G_f} F \) be a principal module over \( M \). There exists a group subbundle \( U \subset \Upsilon (B) \) such that the restriction \( \phi : U \to \text{Ad} \mathcal{P} \) is a covering map.

**Proof.** Consider the subbundle \( E_{[i]} := B_{[i]} \cdot E \) (i.e. the subbundle on which \( B_{[i]} \) acts non-trivially). Define the group subbundle

\[ U := \{ u \in \Upsilon (B) \mid \det u_{[i]} = 1 \text{ for all } [i] \}, \]

where \( \det u_{[i]} \) denotes the fibrewise determinant of \( u_{[i]} \) seen as an element of the bundle \( \text{End} E_{[i]} \). Denote the rank of \( E_{[i]} \) by \( N_{[i]} \). Since any element \( u \in \Upsilon (B) \) can be written as \( u = vw \), where \( v \in U \) and \( w \in \Upsilon (B_f) \) (just take \( w_{[i]} = (\det u_{[i]})^{-1} \text{id}_{N_{[i]}} \) and \( v = uw^{-1} \)), we see that the image \( \phi (U) \) is equal to the image \( \phi (\Upsilon (B)) = \text{Ad} \mathcal{P} \).

Let us calculate the kernel \( \phi_x : \Upsilon_x \to (\text{Ad} \mathcal{P})_x \). Choose \( u \in \Upsilon_x \cap \ker \phi_x \). Since \( u \in \ker \phi_x \), each \( u_{[i]} \) is diagonal. Because \( \det u_{[i]} = 1 \), we obtain that \( u_{[i]} = \lambda_{[i]} \text{id}_{N_{[i]}} \), where \( \lambda_{[i]} \) is an \( N_{[i]} \)-th root of unity. Since there are only finitely many equivalence classes \([i] \), the group \( \Upsilon_x \cap \ker \phi_x \) is finite.

The condition for a map to be a covering map is of a local nature, so we can assume that all bundles are globally trivial. In that case, it follows from the fact that \( \Upsilon_x \cap \ker \phi_x \) is finite, that \( U \to \text{Ad} \mathcal{P} \) is a covering map.
Combining Proposition 6.37 with Corollary 6.36 and Theorem 6.9 immediately yields the desired result:

**Theorem 6.38.** Let \( P \times_{G_F} F \) be a principal module over \( M \). If \( M \) is simply connected, then

\[
\mathcal{G}(P \times_{G_F} F) \simeq \Gamma^\infty(\text{Ad} P) \simeq \mathcal{G}(P).
\]

**Remark 6.39.** It follows from the above that for each element \( g \) of the gauge group \( \mathcal{G}(P \times_{G_F} F) \), there exists a unitary section \( u \in \mathcal{U}(\mathcal{B}) \) with (fibre-wise) determinant equal to 1, such that \( g = uJu^* \). (As we will see in Lemma 6.46, we similarly have that a gauge field is determined by an anti-hermitian section in \( u(\mathcal{B}) \) with (fibre-wise) trace equal to 0.) In this sense, the gauge group is unimodular by default. This only holds for complex algebras \( \mathcal{B} \). For real algebras (including the one describing the noncommutative Standard Model [Con96, Con06, CCM07]) one needs to impose unimodularity by hand (see also [LS01] and references therein).

### 6.4 Gauge Modules

In Section 6.3.2 we introduced the notion of principal modules, and we observe that these have an entirely geometric nature. In this section we introduce so-called gauge modules, which are of a purely algebraic nature. We show that each gauge module is in fact also a principal module, but unfortunately not all principal modules can be obtained from gauge modules.

Inspired by the standard form of finite spectral triples as described in Theorem 6.24 and Remark 6.25, we introduce the following definition, which might be considered an extension of Krajewski diagrams to the globally non-trivial case.

**Definition 6.40.** Let \( \mathcal{A} \) := \( C^\infty(M) \). Suppose we are given a finite set of non-degenerate hermitian finitely generated projective \( \mathcal{A} \)-modules \( \mathcal{E}_i \) (for \( i \in I = \{1, \ldots, l\} \)), and define the module algebras \( \mathcal{B}_i := \text{End}_\mathcal{A}(\mathcal{E}_i) \). Take a multiset \( \mathcal{K} \) consisting of pairs in \( I \times I \) such that the multiplicity of \( (i,j) \) is equal to the multiplicity of \( (j,i) \), and such that the projection \( \mathcal{K} \to I \) on either of the factors is surjective. Denote the multiplicity of the pair \((i,j)\) by \( m_{ij} \) and write \((i_\alpha, j_\alpha) \) \((1 \leq \alpha \leq m_{ij})\) to distinguish the pairs in \( \mathcal{K} \) that occur more than once (see also Theorem 6.24 for this notation).

A gauge module \( (\mathcal{B}, \mathcal{E}, J) \) is of the form

\[
\mathcal{B} := \bigoplus_{i \in I} \mathcal{B}_i, \quad \mathcal{E} := \bigoplus_{(i, j) \in \mathcal{K}} \mathcal{E}_i \otimes_\mathcal{A} \overline{\mathcal{E}_j}, \quad J: \mathcal{E}_i \otimes_\mathcal{A} \overline{\mathcal{E}_j} \to \mathcal{E}_j \otimes_\mathcal{A} \overline{\mathcal{E}_i},
\]

where \( J \) is of the same standard form as the finite operator \( J_F \) in Theorem 6.24 (and which depends on the value of \( J^2 = \varepsilon = \pm 1 \), e.g. \( J_{ij}(e_{i_\alpha} \otimes \overline{e_{j_\alpha}}) = \varepsilon e_{j_\alpha} \otimes \overline{e_{i_\alpha}} \) for \( e_{i_\alpha} \otimes \overline{e_{j_\alpha}} \in \mathcal{E}_{i_\alpha} \otimes \overline{\mathcal{E}_{j_\alpha}} \) if \( j < i \)).
The local triviality of $B_G$ vector bundle $E$ hermitian structure on module $E$ of $B_G$ that $G$ class of $E$ bundles $U$ that for gauge modules it is possible to uniquely reconstruct the corresponding principal $G$-bundle. The main distinctive feature of gauge modules is that the vector bundle $E$ decomposes as a direct sum of tensor products of hermitian vector bundles. For each $E_i$ there uniquely (up to isomorphism) corresponds a principal $G_i$-bundle. To each $E_i$ there uniquely (up to isomorphism) corresponds a principal $G_i$-bundle.

As mentioned in Remark 6.31, given a principal module $P \times G_i \mathcal{F} = (B, E, J)$ (but not $P$ itself), it is not possible to reconstruct $P$, unless we are given the equivalence class of $G$-atlases on the vector bundle $E = P \times G_i \mathcal{F}$. However, we will show below that for gauge modules it is possible to uniquely reconstruct the corresponding principal $G$-bundle. The gauge module $E_i$ then follows from the local triviality of $E_i$, which means that $B_i$ is in fact a unital $*$-algebra bundle.

**Proposition 6.41.** Let $(B, E, J)$ be a gauge module. Then:

1) there exist a real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$ and a principal $U(A_F)$-bundle $Q$ such that $(B, E, J) = Q \times_{U(A_F)} F$;

2) there exists a principal $G$-bundle $P$ such that $(B, E, J) = P \times_{G} F$.

**Proof.**

1) The gauge module $(B, E, J)$ is constructed from a given set of hermitian vector bundles $E_i$ of rank $N_i$ and the index (multi)sets $I$ and $K$. By assumption $B_i = \text{End}(E_i)$, and so $B_i$ has typical fibre $M_{N_i}(C)$. We define

$$A_F := \bigoplus_{i \in I} M_{N_i}(C), \quad \mathcal{H}_F := \bigoplus_{(i,j) \in K} C^{N_i} \otimes \overline{C^{N_j}}.$$ 

For each $E_i$, there is a principal $U(N_i)$-bundle $Q_i$ (which is unique up to isomorphism) such that $E_i \simeq Q_i \times_{U(N_i)} C^{N_i}$ (see Example 3.8). Let $(U, u_{ij})$ be a $U(N_i)$-atlas on $E_i$ corresponding to local trivialisations of $Q_i$. The transition functions $u_{ij}^i$ of $E_i$ are given by the right action of $(u_{ij})^*$ on $C^{N_i}$ (see Eq. (6.1)), which is implemented as $(v_i \otimes \overline{w_j})(u_{ij}^i)^* = ju_{ij}^i(v_i \otimes \overline{w_j})$. Hence we obtain transition functions for $E$ of the form

$$u_{ij}^i \otimes (u_{ij}^j)^{\text{op}} = \bigoplus_{(i,j) \in K} u_{ij}^i u_{ij}^j.$$ 

Writing $u_{ij} = \bigoplus_{i \in I} u_{ij}^i \in C^\infty(\cup \cap V, U(A_F))$, we see that $g_{ij} = u_{ij}^i u_{ij}^j$, which lies in $C^\infty(\cup \cap V, G_F)$. Since the $u_{ij}^i$ are transition functions of $Q_i$, we see that the $u_{ij}^i$ are the transition functions of the principal $U(A_F)$-bundle

$$Q := Q_1 \times_M \cdots \times_M Q_l := \{(q_1, \ldots, q_l) \in Q_1 \times \cdots \times Q_l : \pi_1(q_1) = \cdots = \pi_l(q_l)\}.$$
Since the action of $u_{uv}$ on $\mathcal{H}_F$ is given by $g_{uv} = u_{uv}J u_{uv} J^*$, we see that $E \simeq Q \times U(A_F) J \mathcal{H}_F$ as hermitian vector bundles. As conjugation by $u_{uv}$ coincides with conjugation by $g_{uv}$ on the algebra $A_F$, we also have $B \simeq Q \times U(A_F) A_F$. It is straightforward to check that $J$ is invariant under conjugation by a transition function $g_{uv}$, and hence it is simply of the form $J = 1 \times J_F$. Since $J$ is an anti-unitary operator satisfying $J^2 = \varepsilon$ and the order-zero condition, it follows that $J_F$ is a real structure on $\mathcal{H}_F$.

2) Given the principal $\mathcal{U}(A_F)$-bundle $Q$ from the first part of this lemma, we simply construct a principal $G_F$-bundle as

$$P := Q \times \mathcal{U}(A_F) G_F,$$

where $u \in \mathcal{U}(A_F)$ acts on $G_F$ as left multiplication by the element $u J_F u J_F^*$. The transition functions of $P$ are given by $g_{uv} = u_{uv} J u_{uv} J^* \in C^\infty(U \cap V, G_F)$. It then straightforwardly follows that

$$P \times G_F \mathcal{H}_F \simeq (Q \times \mathcal{U}(A_F) G_F) \times G_F \mathcal{H}_F \simeq Q \times \mathcal{U}(A_F) \mathcal{H}_F \simeq E,$$

and similarly we obtain $P \times G_F A_F \simeq B$. \hfill \Box

The above proposition shows that each gauge module is in fact a principal module $P \times G_F F$ (where we can uniquely reconstruct $F$ and $P$), such that $P$ can be lifted to a principal $\mathcal{U}(A_F)$-bundle $Q$ (which is unique up to isomorphism). We now show the converse, namely that a principal module $P \times G_F F$ with a lift $\tau \colon Q \to P$ uniquely corresponds to a gauge module.

**Proposition 6.42.** Let $P \times G_F F = (B, \mathcal{E}, J)$ be a principal module, and suppose we have a principal $\mathcal{U}(A_F)$-bundle $Q$ that lifts $P$. Then $Q$ naturally induces a gauge module structure on $(B, \mathcal{E}, J)$.

**Proof.** As in Section 6.3.1, the real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$ has a decomposition of the form

$$A_F = \bigoplus_{i \in I} M_{N_i}(C), \quad \mathcal{H}_F = \bigoplus_{(i, j) \in K} C^{N_i} \otimes \overline{C^{N_j}}.$$

Thus we have $\mathcal{U}(A_F) = \chi_{i \in I} \mathcal{U}(N_i)$, and the principal $\mathcal{U}(A_F)$-bundle $Q$ then decomposes as $Q_1 \times_M \cdots \times_M Q_L$, where each $Q_l$ is a principal $\mathcal{U}(N_l)$-bundle given by $Q_l := Q \times \mathcal{U}(A_F) \mathcal{U}(N_l)$. We then construct

$$B_l := Q \times \mathcal{U}(A_F) M_{N_l}(C) \simeq Q_l \times \mathcal{U}(N_l) M_{N_l}(C),$$

$$E_l := Q \times \mathcal{U}(A_F) C^{N_l} \simeq Q_l \times \mathcal{U}(N_l) C^{N_l},$$

where $\mathcal{U}(A_F) = \chi_{i \in I} \mathcal{U}(N_l)$ acts on $C^{N_l}$ as left multiplication by the factor $\mathcal{U}(N_l)$, and on $M_{N_l}(C)$ as conjugation by $\mathcal{U}(N_l)$. The bundle $E_l$ naturally inherits a hermitian structure from the standard inner product on $C^{N_l}$. Because $Q$ lifts $P$, the
bundles $B$ and $E$ corresponding to the principal module $P \times_{G_F} F$ are in fact of the form

$$B := Q \times_{\mathcal{U}(A_F)} A_F = \bigoplus_{i \in I} B_i,$$

$$E := Q \times_{\mathcal{U}(A_F)} \mathcal{H}_F = \bigoplus_{(i,j) \in K} E_i \otimes E_j.$$  

Furthermore, as the transition functions of $B_i$ are given by conjugation by the transition functions of $E_i$, and as its fibre equals $M_{N_i}(C) = \text{End}(C^{N_i})$, it follows that $B_i = \text{End}(E_i)$ and $B_i$ acts as such on $E_i$. Hence we have shown that the principal module $P \times_{G_F} F$ is equal to the gauge module given by the modules $E_i := \Gamma^\infty(E_i)$ and the real structure $J = 1 \times J_F$. 

The previous two propositions then lead us to the main result of this section.

**Theorem 6.43.** A gauge module is characterised uniquely (up to isomorphism) by a principal module $P \times_{G_F} F$ for which there exists a principal $\mathcal{U}(A_F)$-bundle $Q$ that lifts $P$.

**Proof.** Given a gauge module, we have shown in Proposition 6.41 that we can uniquely construct a real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$, a principal $G_F$-bundle $P$, and a principal $\mathcal{U}(A_F)$-bundle $Q$ that lifts $P$. Conversely, given such $F$, $P$, and $Q$, Proposition 6.42 shows that $P \times_{G_F} F$ is in fact given by a gauge module. These constructions are inverse to each other. 

**Remark 6.44.**

1) If there exists a principal $\mathcal{U}(A_F)$-bundle $Q$ that lifts $P$, then $Q$ is unique up to isomorphism, because each principal $\mathcal{U}(N_i)$-bundle $Q_i$ is unique up to isomorphism (cf. Example 3.8). (If $M$ is non-compact however, the principal bundles $Q_i$ need not be unique, and different lifts $Q$ then give rise to different gauge modules.)

2) Every globally trivial principal module, constructed from a finite spectral triple $F$ and the principal bundle $P = M \times G_F$, is in fact a gauge module, with the lift $Q = M \times \mathcal{U}(A_F)$.

3) An example of a principal module that is (in general) not a gauge module (except when for instance the underlying manifold is simply connected and 4-dimensional) is described in Section 6.6.1.

6.5 **Gauge Theory**

In this section we show how principal modules describe gauge theories on 4-dimensional compact spin manifolds. First we will introduce a ‘mass matrix’. Viewing the (now massive) principal module as an internal space and endowing it with a (suitable) connection, we can then use it to construct an almost-commutative manifold. Subsequently, we determine the inner fluctuations and provide an explicit formula for the spectral action of this almost-commutative manifold. We end
this section by stating our main result, namely that such an almost-commutative manifold indeed describes a gauge theory in the sense of Definition 6.10.

6.5.1 Principal almost-commutative manifolds

**Definition 6.45.** Consider a principal module \( P \times_{G_F} E = (\mathcal{B}, \mathcal{E}, J_1) \) (from here on we include a subscript \( I \) in order to differentiate between the different operators occurring). In order to be able to describe massive gauge theories, we now introduce a ‘mass matrix’

\[ D_I \in \Gamma^n(\text{End}(E)) \simeq \text{End}_A(E), \]

satisfying

\[ \mathcal{D}_I = \mathcal{D}_I^\prime, \quad \mathcal{D}_I J_1 = \varepsilon' J_1 \mathcal{D}_I, \quad [[\mathcal{D}_I, a], J_b J^*] = 0 \quad \forall a, b \in \mathcal{B}, \]

where the sign \( \varepsilon' \) (along with the signs \( \varepsilon, \varepsilon'' \) obtained through the finite spectral triple \( E \)) is determined by the KO-dimension according to Table 1 of Definition 6.13.

We then call \( I^\mathcal{B}_6 := (\mathcal{B}, \mathcal{E}, \mathcal{D}_I, J_1) \) a massive principal module over \( M \). We say \( I^\mathcal{B}_6 \) is even if there exists a grading operator \( \Gamma_1 \) on \( \mathcal{E} \) such that

\[ \mathcal{D}_I \Gamma_1 = -\Gamma_1 \mathcal{D}_I, \quad \Gamma_1 J_1 = \varepsilon'' J_1 \Gamma_1 \]

and \( a \Gamma_1 = \Gamma_1 a \) for all \( a \in \mathcal{B} \).

It is an immediate consequence of the definition that a massive principal module over \( M \) is a real internal space over \( M \). If \( (\mathcal{B}, \mathcal{E}, J_1) \) is in fact a gauge module, we shall call \( (\mathcal{B}, \mathcal{E}, \mathcal{D}_I, J_1) \) a massive gauge module.

Let \( P \times_{G_F} E \) be a principal module. Denote by \( g_F \) the Lie algebra of the structure group \( G_F \). Take a connection on \( P \), i.e. for each local trivialisation \( (U_i, h_i) \) of \( P \) we have a (local) \( g_F \)-valued 1-form \( \omega_i \in \Omega^1(U_i, g_F) \) such that

\[ \omega_j = g^{-1}_{ij} dg_{ij} + g^{-1}_{ij} \omega_i g_{ij} \]

for all \( i, j \) such that \( U_i \cap U_j \neq \emptyset \) (see Definition 3.6). These connection one-forms yield a connection \( \nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1(M) \) by defining locally (i.e. on local trivialisations \( (U_i, h_i) \) of \( E \) that are induced by those of \( P \) ) the expression

\[ \nabla|_{U_i} := h_i^{-1} \circ (d + \omega_i) \circ h_i, \]

where \( d \) is the exterior derivative acting on the components of the local trivialisation. The transformation property of \( \omega_i \) ensures that \( \nabla \) is globally well-defined. Connections on \( \mathcal{E} \) of this form are also referred to as \( G_F \)-compatible connections, or simply \( G_F \)-connections.

Consider the associated vector bundle \( \text{ad} P := P \times_{ad} g_F \), where \( ad \) is the adjoint action of \( G_F \) on \( g_F \). Since \( g_F \) is (isomorphic to) the image of \( u(A_F) \) in \( u(J_F) \) under the map \( t \mapsto t + J_F t J^*_F \), the bundle \( ad P \) is (isomorphic to) the image of \( u(B) \) in \( u(E) \).
under the map \( \tau: t \mapsto t + J_1 t J_1^* \). The kernel of this map is equal to the set of all elements \( t \in u(B) \) satisfying \( t = -J_1 t J_1^* J_1 = t^* J_1^* J_1^* \), or equivalently,

\[
\ker \tau = \{ t \in u(B) : t J_1 = J_1 t^* \} = u(B_1).
\]

Hence we see that \( \text{ad} \ P \) is isomorphic to \( u(B)/u(B_1) \), and \( g_F = u(A_F)/u((A_F)_{J_1}) \).

**Lemma 6.46.** The induced map \( \tau: u(B) \to \Gamma^\infty(\text{ad} \ P) \) is surjective, and

\[
\Gamma^\infty(\text{ad} \ P) \simeq u(B)/u(B_1).
\]

Moreover, \( \text{ad} \ P \) is isomorphic to the subbundle

\[
u = \{ t \in u(B) \mid \text{Tr} \ t_{[i]} = 0 \text{ for all } [i]\}
\]

of \( u(B) \), where \( \text{Tr} \ t_{[i]} \) denotes the fibrewise trace of \( t_{[i]} \), seen as an element of the bundle \( \text{End} \ E_{[i]} \), and \( u(B) = \ker \tau \oplus \nu \), with \( \ker \tau = u(B_1) \).

**Proof.** Though the first two statements follow immediately from the exactness of the Serre-Swan equivalence functor \( \Gamma^\infty \), we prove them directly by showing that \( \text{ad} \ P \) is isomorphic to the subbundle \( \nu \) (compare also Proposition 6.37). Indeed, every \( t \in u(B) \) can be written as \( s + q \), where \( s \in u \) and \( q \in u(B_1) \) (just take \( q_{[i]} = \frac{1}{N_{[i]}} \text{Tr} \ t_{[i]} \cdot \text{id}_{[i]} \) and \( s = t - q \)). Hence \( \tau|_\nu \) is surjective.

Suppose now that \( t \in \ker \tau|_\nu \). Because \( t \in \ker \tau \), we obtain \( t_{[i]} = \lambda_{[i]} \text{id}_{N_{[i]}} \), where \( \lambda_{[i]} \in i\mathbb{R} \) (see Proposition 6.26). Since \( t \in u \), each of the \( t_{[i]} \) is traceless. Hence each of the \( \lambda_{[i]} \) is zero, and consequently, the kernel of \( \tau|_\nu \) is trivial. \( \square \)

**Lemma 6.47.** Let \( P \times_G F = (B, \mathcal{E}, J_1, \Gamma_1) \) be an even principal module. Then any \( G_F \)-compatible connection \( \nabla \) on \( \mathcal{E} \) commutes with the real structure \( J_1 \) (in the sense that \( \nabla \mu J_1 = J_1 \nabla \mu \)) and the grading \( \Gamma_1 \).

**Proof.** It is sufficient to show that \( J_F \) and \( \Gamma_F \) commute with elements in \( g_F \). Any element in \( g_F \) is of the form \( t + J_F t J_F^* \), with \( t \in u(A_F) \). In particular, \( J_F \) commutes with these elements. Since \( \Gamma_F \) commutes with elements in \( A_F \), and (anti-)commutes with \( J_F \), the grading \( \Gamma_F \) commutes with elements in \( g_F \), too. \( \square \)

If the principal module is obtained from a gauge module \( (B, \mathcal{E}, J_1) \), we can construct such a \( G_F \)-connection explicitly as follows. Consider the decomposition \( \mathcal{E} = \bigoplus_{[i,j]} \mathcal{E}_i \otimes_A \mathcal{E}_j^* \), and choose a hermitian connection \( \nabla^1 \) on each \( \mathcal{E}_i \). We define

\[
\nabla := \bigoplus_{(i,j)} \left( \nabla^1 \otimes 1 + 1 \otimes \overline{\nabla^1} \right),
\]

where the conjugate connection \( \overline{\nabla^1} \) is defined in Section 6.1.2. In order to see that \( \nabla \) corresponds to a connection on the principal bundle \( P \), we first need to check that its local connection one-forms take values in the Lie algebra \( g_F \). If \( (U, h_U^1) \) are
local trivializations of $\mathcal{E}_i$, we can write $\nabla^i_{\mathcal{U}} = (h^i_u)^{-1} \circ (d + \omega_i^u) \circ h^i_u$ for some local connection one-forms $\omega_i^u \in \Omega^1(\mathcal{U}, u(N_i))$. The connection $\nabla$ then locally has the connection 1-form

$$\omega_u := \bigoplus_{(i,j)} (\omega_u^i \otimes 1 + 1 \otimes (\omega_u^j)^{op}) \in \Omega^1(\mathcal{U}, \mathcal{A}_F \otimes \mathcal{A}_F^{op}).$$

This ensures that $[\nabla, \cdot]$ yields a connection on $\mathcal{B} \otimes \mathcal{B}^{op}$, which preserves $\mathcal{B}$ and $\mathcal{B}^{op}$. Writing $t_u = \bigoplus_{t \in I} \omega_u^i$, we can write $\omega_u = t_u + J_F t_u J_F^* \in \Omega^1(\mathcal{U}, \mathcal{G}_F)$. To verify that $\omega_u$ defines a connection on the principal $\mathcal{G}_F$-bundle $\mathcal{P}$ we need to show that $\omega_u$ transforms correctly under the $\mathcal{G}_F$-valued transition functions.

So, consider two neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, and let $u = \chi \cdot u_i \in C^\infty(\mathcal{U} \cap \mathcal{V}, (\mathcal{U}, \mathcal{A}_F))$ be a transition function for the principal $\mathcal{U}(\mathcal{A}_F)$-bundle $\mathcal{Q}$. The corresponding transition function for the principal $\mathcal{G}_F$-bundle $\mathcal{P}$ is $g := u_j F u_j^*$. Since the $\omega_u^i$ are connection forms on $\mathcal{E}_i$, $t_u$ transforms as

$$t_v = \bigoplus_{i \in I} \omega_v^i = \bigoplus_{i \in I} (u_i^* \omega_u^i u_i + u_i^* d u_i) = u^* t_u u + u^* d u.$$

We then see that

$$\omega_v = t_v + J_F t_u J_F^* = u^* t_u u + u^* d u + J_F (u^* t_u u + u^* d u) J_F^*$$

$$= u^* J_F u^* J_F^* t_u u_j F u_j J_F^* + J_F (u^* J_F u^* J_F^* t_u u_j F u_j J_F^*)$$

$$+ u^* J_F u^* J_F^* (d u) J_F u_j J_F^* + u^* J_F u^* J_F^* (d u) J_F u_j J_F^*$$

$$= g^{-1} (t_u + J_F t_u J_F^*) g + g^{-1} d g = g^{-1} \omega_u g + g^{-1} d g.$$

Thus, $\mathcal{U} \mapsto \omega_u$ indeed defines a $\mathcal{G}_F$-connection.

**Proposition 6.48.** Let $(\mathcal{B}, \mathcal{E}_i, \mathcal{J})$ be a gauge module. A connection on $\mathcal{E}$ is of the form

$$\bigoplus_{(i,j)} \left( \nabla^i \otimes 1 + 1 \otimes \nabla^j \right)$$

if and only if it induces a connection on the principal $\mathcal{U}(\mathcal{A}_F)$-bundle $\mathcal{Q}$ from Proposition 6.41.

**Proof.** Consider a local trivialisation $(\mathcal{U}, h_u)$ of $\mathcal{P}$, and let $\omega_u \in \Omega^1(\mathcal{U}, u(A_F))$ be a local connection form on $\mathcal{Q}$, yielding a connection $\nabla$ on $\mathcal{E} = \mathcal{Q} \times_{\mathcal{U}(A_F)} \mathcal{J}_F$. Since the decomposition $u(A_F) = \bigoplus_{i \in I} u(N_i)$ is preserved by the action of $u(A_F)$, we can write $\omega_u = \bigoplus_{i \in I} \omega_i$, where each $\omega_i \in \Omega^1(\mathcal{U}, u(N_i))$ yields a connection $\nabla^i$ on $\mathcal{E}_i$. For $x \in \mathcal{U}$, the connection form $\omega_u$ acts on $[E_i \otimes E_j]_x \simeq C^{N_i \otimes \overline{N_j}}$ as

$$\omega_u(v_i \otimes \overline{w_j}) = \omega_i(v_i \otimes \overline{w_j}) + v_i \otimes \overline{w_j} \omega^*_j,$$

from which it follows that $\nabla = \bigoplus_{(i,j)} \left( \nabla^i \otimes 1 + 1 \otimes \nabla^j \right)$.

For the converse, consider a connection $\nabla = \bigoplus_{(i,j)} \left( \nabla^i \otimes 1 + 1 \otimes \nabla^j \right)$ on $\mathcal{E}$. On a local trivialisation $(\mathcal{U}, h_u)$ of $\mathcal{E}_i$, each connection $\nabla^i$ yields a local connection form $\omega_i \in \Omega^1(\mathcal{U}, u(N_i))$. Then $\omega_u := \bigoplus_{i \in I} \omega_i \in \Omega^1(\mathcal{U}, u(A_F))$ is a connection form on $\mathcal{Q}$ that induces $\nabla$. \qed
Definition 6.49. Let $I^\infty_p = (B, E, D, I_1)$ be a massive principal module of KO-dimension $k$ over $M$, where $M$ now has dimension $4$. Let $\nabla$ be a $G_F$-compatible connection on $E$. We construct the real almost-commutative manifold $I^\infty_p \times_\nabla M$ as in Definition 6.17. Since $I^\infty_p$ is now a massive principal module (instead of a more general internal space), we will refer to $I^\infty_p \times_\nabla M$ as a principal almost-commutative manifold.

If $I^\infty_p$ is even with grading $\Gamma_1$, we obtain a real even almost-commutative manifold $I^\infty_p \times_\nabla M$. Since the connection $\nabla$ is $G_F$-compatible, it automatically commutes with $J_1$ and $\Gamma_1$ (see Lemma 6.47). Moreover, the same condition implies that the induced connection $[\nabla, \cdot]$ on $\text{End} E$ restricts to $B$. It then follows from Proposition 6.18 that $I^\infty_p \times_\nabla M$ is a real even spectral triple of KO-dimension $4 + k \pmod{8}$.

We continue in the remainder of this section, as in the usual approach for globally trivial almost-commutative manifolds (see [Con96, Con06, CCM07] or the review [DS12]), by generating the gauge fields and Higgs fields via inner fluctuations, and subsequently calculating the spectral action.

6.5.2 Inner fluctuations

Let $(B, H, D)$ be a spectral triple. We consider the generalised one-forms given by

$$\Omega^1_D(B) := \left\{ \sum_j a_j [D, b_j] \mid a_j, b_j \in B \right\},$$

where the sums must converge in norm. For the canonical triple $(A, L^2(S), D)$ of a spin manifold $M$, the generalised one-forms $\Omega^1_D(A)$ are simply given by the Clifford multiplication $c$ of the usual one-forms $\Omega^1(M)$. To be precise, for smooth functions $f_1, f_2 \in A$, we obtain $f_1 [D, f_2] = f_1 c(df_2)$.

Definition 6.50. Let $(B, H, D, J)$ be a real spectral triple. An inner fluctuation of the operator $D$ is a self-adjoint element $A = A^* \in \Omega^1_D(B)$. Such an inner fluctuation yields the fluctuated operator

$$D_A := D + A + \epsilon' JAJ^*,$$

where the sign $\epsilon' = \pm 1$ is determined by the KO-dimension of the spectral triple (see Definition 6.13).

For the remainder of this chapter, we again assume that the dimension of $M$ is equal to 4. We would like to show that, for a principal almost-commutative manifold, these inner fluctuations yield gauge fields and scalar fields (the latter are interpreted as Higgs fields in the noncommutative Standard Model). The inner
fluctuations of the twisted Dirac operator \( \mathcal{D}_E := 1 \otimes_{\nabla} \mathcal{D} \) are (sums of) elements of the form
\[
a[\mathcal{D}_E, b] = (1 \otimes c) \circ (a[\nabla, b] \otimes 1),
\]
for \( a, b \in \mathcal{B} \), where \( c \) denotes Clifford multiplication. The fact that \( \nabla \) is a \( G_F \)-compatible connection ensures that \( a[\nabla, b] \in \mathcal{B} \otimes \mathcal{A} \Omega^1(M) \simeq \Omega^1(M, \mathcal{B}) \). Requiring that \( a[\mathcal{D}_E, b] \) is self-adjoint then implies that \( a[\nabla, b] \in \Omega^1(M, u(\mathcal{B})) \), where \( u(\mathcal{B}) \) contains the anti-hermitian elements of \( \mathcal{B} \). An arbitrary inner fluctuation of \( \mathcal{D}_E \) is thus given by
\[
\alpha := \sum_j a_j[\nabla, b_j] \in \Omega^1(M, u(\mathcal{B})).
\]
We can then write \( Ja[\mathcal{D}_E, b]J^* = (1 \otimes c) \circ (J_1\alpha J_1^* \otimes 1) \), and consequently we have
\[
a[\mathcal{D}_E, b] + Ja[\mathcal{D}_E, b]J^* = (1 \otimes c) \circ ((\alpha + J_1\alpha J_1^*) \otimes 1).
\]

The inner fluctuations of the operator \( \mathcal{D}_I \otimes \Gamma_M \) are of the form \( \phi \otimes \Gamma_M \), where
\[
\phi = \phi^* := \sum_j a_j[\mathcal{D}_I, b_j] \in \Gamma^\infty(\text{End}(\mathcal{E})).
\]

**Proposition 6.51.** The fluctuated Dirac operator \( \mathcal{D}_A := \mathcal{D} + A + JA^* \) for a real even almost-commutative manifold is of the form
\[
\mathcal{D}_A = 1 \otimes_{\nabla'} \mathcal{D} + \Phi \otimes \Gamma_M,
\]
for a new connection \( \nabla' := \nabla + \beta \) (where \( \beta \in \Omega^1(M, \text{ad} \mathcal{P}) \)) and for the ‘Higgs field’ \( \Phi = \Phi^* := \mathcal{D}_I + \phi + J_1\phi J_1^* \in \Gamma^\infty(\text{End}(\mathcal{E})) \) (where \( \phi = \phi^* := \sum_j a_j[\mathcal{D}_I, b_j] \)).

**Proof.** The element \( \beta = \alpha + J_1\alpha J_1^* \) is an \( \text{ad} \mathcal{P} \)-valued 1-form on \( M \) (see Lemma 6.46). Noting that \( \varepsilon' = 1 \) by assumption, the statement follows straightforwardly. \( \Box \)

The construction of \( \Gamma^\infty_{\mathcal{P}^*} \times_{\nabla} M \) explicitly uses the choice of a connection \( \nabla \). However, we now show that this choice is irrelevant once we take the inner fluctuations into account. We need the following lemma.

**Lemma 6.52.** Let \( \mathcal{B} \to M \) be a unital \( \ast \)-algebra bundle, and let \( \mathcal{D} \) be a connection on \( \mathcal{B} = \Gamma^\infty(\mathcal{B}) \) such that \( \mathcal{D}(1) = 0 \), where 1 denotes the identity section. Write \( \mathcal{A} = \mathcal{C}^\infty(M) \). Then
\[
\left\{ \sum_j a_j \mathcal{D}(b_j) \mid a_j, b_j \in \mathcal{B} \right\} = \mathcal{B} \otimes \mathcal{A} \Omega^1(M) \simeq \Omega^1(M, \mathcal{B}). \tag{6.5}
\]

Consequently, the anti-hermitian elements in \( \left\{ \sum_j a_j \mathcal{D}(b_j) \mid a_j, b_j \in \mathcal{B} \right\} \) form the space of \( u(\mathcal{B}) \)-valued one-forms \( \Omega^1(M, u(\mathcal{B})) \).
Proof. Since \( \tilde{\nabla}(b) \in \mathcal{B} \otimes \mathcal{A} \Omega^1(M) \), the left hand side of Eq. (6.5) is clearly contained in the right hand side of Eq. (6.5). For the converse inclusion, first suppose that both \( a_j \) and \( b_j \) are in \( \mathcal{A} \subset \mathcal{Z}(\mathcal{B}) \). In that case,

\[
\left\{ \sum_j f_j \tilde{\nabla}(g_j \text{Id}_B) \mid f_j, g_j \in \mathcal{A} \right\} \simeq \left\{ \sum_j f_j dg_j \mid f_j, g_j \in \mathcal{A} \right\} = \Omega^1(M).
\]

It follows from this that

\[
\left\{ \sum_j a_j \tilde{\nabla}(g_j 1) \mid a_j \in \mathcal{B}, g_j \in \mathcal{A} \right\} = \mathcal{B} \otimes \mathcal{A} \Omega^1(M).
\]

Of course, the left-hand side is contained in \( \left\{ \sum_j a_j \tilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \right\} \), which proves the other inclusion. \( \square \)

**Proposition 6.53.** Let \( P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_1) \) be a principal module over \( M \) (for simplicity we consider here the massless case \( D_1 = 0 \)) with two \( (G_F\text{-compatible}) \) connections \( \nabla \) and \( \nabla' \). Then \( 1 \otimes \nabla \cdot \mathcal{D} \) is obtained as an inner fluctuation of \( 1 \otimes \nabla \cdot \mathcal{D} \).

**Proof.** The difference between the two connections \( \beta := \nabla' - \nabla \) is an element in \( \Omega^1(M, \text{ad} P) \). By Lemma 6.46 we know that there exists a (unique) element \( \alpha \in \Omega^1(M, u \subset \Omega^1(M, u(\mathcal{B}))) \) such that \( \beta = \alpha + J_1 \alpha J_1^* \). The connection \( \tilde{\nabla} = [\nabla, \cdot] \) on \( \text{End}(\mathcal{E}) \) restricts to a connection on \( \mathcal{B} \), and satisfies \( \tilde{\nabla}(1) = 0 \). Lemma 6.52 now implies that \( \beta \) is obtained as an inner fluctuation. \( \square \)

**Remark 6.54.** We have seen that considering inner fluctuations of the Dirac operator essentially replaces the \( G_F \)-connection \( \nabla \) (chosen in the construction of the almost-commutative manifold \( I_\infty \times_{\nabla} M \)) by a different (arbitrary) \( G_F \)-connection \( \nabla' \). Therefore, after taking into account the inner fluctuations, our construction of principal almost-commutative manifolds becomes independent of the initial choice of the connection \( \nabla \). However, we also note that the endomorphisms \( \Phi \) obtained through inner fluctuations in general remain dependent on the initial choice of \( \mathcal{D}_1 \).

### 6.5.3 The spectral action

As mentioned immediately below Definition 6.10, the dynamics of a gauge theory can be obtained from a gauge-invariant action functional. In the case of almost-commutative manifolds, such an action functional can be formulated in terms of the spectral triple.

Let us first recall the definitions of the bosonic and fermionic action functionals for an arbitrary spectral triple \( \mathcal{T} = (\mathcal{A}, \mathcal{H}, \mathcal{D}) \). The bosonic part of the action functional is given by the spectral action \([CC97]\), defined as

\[
S_b(\mathcal{T}) := \text{Tr} \left( f \left( \frac{\mathcal{D}_{\mathcal{A}}}{\Lambda} \right) \right).
\]
Here $\text{Tr}$ denotes the operator trace on $\mathcal{B}(\mathcal{H})$, $\mathcal{D}_\Lambda$ is the fluctuated Dirac operator, $f: \mathbb{R} \to \mathbb{R}$ is some positive even function, and $\Lambda \in \mathbb{R}$ is a (large) cut-off parameter. The function $f$ is assumed to decay sufficiently rapidly at infinity so that the trace of $f(\mathcal{D}_\Lambda/\Lambda)$ exists. In particular, $f$ could be considered as a smooth approximation to a cut-off function (and as such it counts the number of eigenvalues of $\mathcal{D}_\Lambda$ whose absolute values are smaller than $\Lambda$), but this viewpoint is not necessary for the following.

If the spectral triple is even (with grading $\Gamma$) and has a real structure $J$ of KO-dimension 2, the fermionic action [Con06] is defined as

$$S_f(T) := \frac{1}{2} \langle J\check{\xi} | \mathcal{D}_\Lambda \check{\xi} \rangle,$$

where $\check{\xi}$ is the Grassmann variable corresponding to a vector $\xi \in \mathcal{H}^+$ (i.e. $\Gamma \xi = \xi$).

We quote the following well-known result.

**Proposition 6.55** (see e.g. [DS12, §2.6.1]). For a real spectral triple $T = (A, \mathcal{H}, \mathcal{D}, J, \Gamma)$, the action functionals $S_b(T)$ and $S_f(T)$ (the latter only defined in KO-dimension 2) are invariant under the action of the gauge group $\mathcal{G}(T)$.

We now provide explicit formulas for the spectral action of principal almost-commutative manifolds (formulas for the fermionic action will only be given for the example of electrodynamics in Section 6.6.2). The spectral action was calculated in [Con06, CCM07] for the product triple $M \times F$, where $F$ was chosen in order to describe the full Standard Model of elementary particle physics. In the remainder of this section we largely follow the notation of [DS12], where also detailed derivations of the formulas provided here can be found.

For the canonical triple $(C^\infty(M), L^2(S), \mathcal{D})$ of a smooth compact 4-dimensional Riemannian spin manifold $M$, the spectral action yields the asymptotic formula

$$S_b(M) \sim_{\Lambda \to \infty} \int_M \mathcal{L}_M(g_{\mu\nu}) \sqrt{|g|} d^4x + O(\Lambda^{-1}),$$

where $g$ is the Riemannian metric on $M$. The Lagrangian $\mathcal{L}_M$ is given by

$$\mathcal{L}_M(g_{\mu\nu}) := \frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} s + \frac{f(0)}{16\pi^2} \left( \frac{1}{30} \Delta s - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{11}{360} R^* \ast R^* \right). \quad (6.6)$$

Here $s$ denotes the scalar curvature of $M$, $\Delta$ is the scalar Laplacian, $C$ is the Weyl curvature, and $R^* \ast R^*$ is a topological term, which integrates to (a multiple of) the Euler characteristic (see e.g. [Nak03, §11.4]). The coefficients $f_k$ (for $k > 0$) are the moments of $f$, defined as

$$f_k := \int_0^\infty f(t) t^{k-1} dt.$$

We now provide the spectral action for a principal almost-commutative manifold. As all calculations are local, the result is exactly the same as for the spectral
action of a product triple $M \times F$, and we refer to [DS12] for the detailed calculations.

In Proposition 6.51 we saw that the fluctuated Dirac operator is determined by a connection $\nabla' = \nabla + \beta$ and an endomorphism $\Phi$ on $E$. From here on we shall work on a local trivialisation $(U, h_U)$, where we can write $\nabla|_U = h_U^{-1} \circ (\mu + \omega_U) \circ h_U$, and define the local $g_U$-valued 1-form $B := \omega_U + h_U \circ \beta|_U \circ h_U^{-1} \in \Omega^1(U, g_U)$ (for ease of notation we do not make the dependence of $B$ on the local chart $U$ explicit). Thus $B$ is the local connection form for $\nabla'$. Using a local coordinate basis $\delta_{\mu}$, we define $B_\mu := B(\delta_{\mu}) \in C^\infty(U, g_F)$. We omit the local trivialisation $h_U$ from our notation, so we write e.g. $\nabla'_\mu = \delta_{\mu} + B_\mu$. Furthermore, we introduce the notation

$$D_\mu \Phi := [\nabla'_\mu, \Phi] = \partial_\mu \Phi + [B_\mu, \Phi], \quad \Gamma_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu].$$

**Proposition 6.56.** For a principal almost-commutative manifold $\Pi^\infty_\mathbb{F} \times \nabla M$, the spectral action is asymptotically given by the local formula

$$S_b(\Pi^\infty_\mathbb{F} \times \nabla M) \sim_\Lambda \int_M \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) \sqrt{|g|} \, d^4x + o(\Lambda^{-1}),$$

for

$$\mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) := N\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_B(g_{\mu\nu}, B_\mu) + \mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi).$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is given in Eq. (6.6), and $N$ is the rank of $E$. The Lagrangian $\mathcal{L}_B$ gives the kinetic term of the gauge field and equals

$$\mathcal{L}_B(g_{\mu\nu}, B_\mu) := \frac{f(0)}{24\pi^2} \text{tr}(\Gamma_{\mu\nu} F^{\mu\nu}),$$

where $\text{tr}$ denotes the fibre-wise trace for endomorphisms on the bundle $E \otimes S$. Finally, we have the Higgs Lagrangian $\mathcal{L}_H$ given by

$$\mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2 \Lambda^2}{4\pi^2} \text{tr}(\Phi^2) + \frac{f(0)}{8\pi^2} \left( \text{tr}(\Phi^4) + \frac{1}{3} \Delta(\text{tr}(\Phi^2)) + \frac{1}{6} \text{tr}(\Phi^2) + \text{tr}((D_\mu \Phi)(D^\mu \Phi)) \right),$$

where the first two terms form the Higgs potential, the third is a boundary term, the fourth couples the Higgs field to the scalar curvature, and finally we have the kinetic term including interactions with the gauge field.

**Remark 6.57.** Although the above explicit formulas for the spectral action are exactly the same as for a product triple $M \times F$, there can nonetheless be a significant difference, because the constant matrix $\mathcal{D}_F$ is replaced by a global endomorphism $\mathcal{D}_F$. For a product triple $M \times F$, the inner fluctuations of $\Gamma_M \otimes \mathcal{D}_F$ also lead to global endomorphisms of the form $\Gamma_M \otimes \Phi$, where $\Phi \in \Gamma^\infty(\text{End}(E))$ (though this $\Phi$ would be more restricted than in our construction). However, there may be
components of $\mathcal{D}_F$ that are not affected by inner fluctuations, and hence remain constant (this occurs for instance for the Majorana masses of right-handed neutrinos in the case of the noncommutative Standard Model [CCM07]). In the case of a principal almost-commutative manifold, these components could be non-constant from the start. Hence, compared to the case of product triples, derivatives of the field $\Phi$ might contain additional terms. This difference is not yet visible in the general formulas above, but it may have consequences once we look at concrete examples (see Remark 6.62).

6.5.4 Gauge theory

The results of this section can be summarised into the main result of this chapter.

**Theorem 6.58.** Let $M$ be a smooth compact 4-dimensional Riemannian spin manifold. Consider a massive even principal module $I^\infty_F = (B, E, D, I_1, I_1)$ of KO-dimension $k$ over $M$. Let $\nabla$ be a $G_F$-compatible connection on $E$. If $M$ is simply connected, then the principal almost-commutative manifold $I^\infty_F \times_\nabla M$ of KO-dimension $4 + k \mod 8$ describes a classical gauge theory over $M$ with gauge group $G(I^\infty_F \times_\nabla M)$.

**Proof.** The principal module $I^\infty_F$ is constructed from a principal $G_F$-bundle $P$ over $M$, such that $B$ and $E$ are given by smooth sections of bundles associated to $P$. By assumption $M$ is simply connected, so it follows from Theorem 6.38 that we have the isomorphism $G(I^\infty_F \times_\nabla M) \simeq G(P)$. We have seen in Section 6.5.2 that the inner fluctuations transform a $G_F$-compatible connection on $E$ to another $G_F$-compatible connection, which hence corresponds to a connection on $P$ (and by Proposition 6.53 any connection on $P$ can be obtained in this way). Finally, the spectral action and the fermionic action provide a gauge-invariant action functional (see Proposition 6.55). Thus the principal almost-commutative manifold $I^\infty_F \times_\nabla M$ provides all the necessary ingredients for a classical gauge theory over $M$, as described in Definition 6.10.

6.6 Examples

In this section we adapt two simple examples of (globally trivial) gauge theories in the context of noncommutative geometry to the globally non-trivial case. In each example, we assume (as before) that the underlying manifold $M$ is a smooth compact 4-dimensional Riemannian spin manifold.

In Section 6.6.1 we describe the Yang-Mills case that was studied in [BS11], and provided the motivation for this work. In particular, we show that the Yang-Mills case provides examples of principal modules that cannot be described by gauge modules. In Section 6.6.2 we discuss the abelian gauge theory of electrodynamics,
based on the (globally trivial) description in [DS13]. We will describe the resulting (globally non-trivial) gauge theory, and provide explicit formulas for both the spectral action and the fermionic action.

6.6.1 Yang-Mills

Globally trivial Yang-Mills theory was already studied in the setting of spectral triples by Chamseddine and Connes [CC97]. It is described by the (real, even) finite spectral triple

$$F_{YM} := (M_N(\mathbb{C}), M_N(\mathbb{C}), D_F = 0, J_F = (\cdot)^*, \Gamma_F = \text{id}),$$

where the algebra $M_N(\mathbb{C})$ acts on the Hilbert space $M_N(\mathbb{C})$ by left-multiplication. The KO-dimension of this spectral triple is 0 and the structure group $G_F$ is equal to $\text{PSU}(N)$.

This has been generalised to the globally non-trivial case in [BS11]. Let $B \to M$ be an arbitrary $*$-algebra bundle with fibre $M_N(\mathbb{C})$, and let $B = \Gamma^\infty(B)$ be its unital, involutive $C^\infty(M)$-module algebra of sections. We consider the real even internal space

$$I^\infty_{YM} := (B, B, D_I = 0, J_I = (\cdot)^*, \Gamma_I = \text{id}).$$

For a general principal module $P \times_{G_F} F$ we do not know how to reconstruct the principal bundle $P$ from the module. However, in the Yang-Mills case we do.

**Lemma 6.59.** There exists a principal $\text{PSU}(N)$-bundle $P \to M$ (unique up to isomorphism) such that $I^\infty_{YM} \simeq P \times_{\text{PSU}(N)} F_{YM}$.

**Proof.** The transition functions of the $*$-algebra bundle $B$ take values in the group $\text{Aut}(M_N(\mathbb{C})) \simeq \text{PSU}(N)$ (where $\text{PSU}(N)$ acts on $M_N(\mathbb{C})$ by conjugation). Hence by Theorem 3.5 we can reconstruct a principal $\text{PSU}(N)$-bundle $P$ such that $B \simeq P \times_{\text{PSU}(N)} M_N(\mathbb{C})$. Since $\text{PSU}(N)$ is the full automorphism group of the fibre, the bundle $P$ is uniquely defined. □

**Remark 6.60.** Note that $I^\infty_{YM}$ will in general not be a gauge module. If this were the case, the structure group $\text{PSU}(N)$ of $B$ could be lifted to $\mathcal{U}(N)$ by Proposition 6.41. This is only possible if the Dixmier-Douady class $\delta(B) \in \hat{H}^3(M, \mathbb{Z})$ is identically zero (see e.g. [RW98, Ch.5] or [Scho99] for more details on Dixmier-Douady classes), which is equivalent to saying that $B$ is an endomorphism bundle (note that this is consistent with the condition $B_i = \text{End}(E_i)$ in Definition 6.40). However, not every $*$-algebra bundle with fibre $M_N(\mathbb{C})$ has zero Dixmier-Douady class (see e.g. [Scho99]), so this example shows that there exist principal modules that are not gauge modules. However, in our description of gauge theories in Section 6.5 we have restricted our attention to simply connected, 4-dimensional manifolds, and it
turns out that in this case the Dixmier-Douady class always vanishes (as we will prove next). It is unclear if there exist other examples of principal modules that are not gauge modules.

**Proposition 6.61.** Let $B$ be a $*$-algebra bundle with fibre $M_N(C)$ over a simply connected, 4-dimensional, oriented, compact manifold $M$. Then the Dixmier-Douady class of $B$ is identically zero.

**Proof.** Since $M$ is simply connected, its fundamental group is trivial, and hence (see e.g. [Hat02, Theorem 2.A.1]) the first singular homology group $H_1(M, \mathbb{Z})$ is trivial. By Poincaré duality (see e.g. [Hat02, Proposition 3.25 & Theorem 3.30]) it then follows that the third cohomology group $H^3(M, \mathbb{Z})$ is also trivial. The Dixmier-Douady class by definition takes values in the third Čech cohomology group $\check{H}^3(M, \mathbb{Z})$. Since for compact manifolds these cohomology groups are equal, it follows that $\check{H}^3(M, \mathbb{Z})$ is trivial and hence that the Dixmier-Douady class of $B$ must vanish. \qed

A connection $\nabla: B \rightarrow B \otimes_A \Omega^1(M)$ is $PSU(N)$-compatible (cf. Section 6.5.1) if and only if it satisfies the algebraic identities (see [BS11, §3.2])

$$\nabla(ab) = \nabla(a)b + a\nabla(b), \quad (\nabla a)^* = \nabla(a^*), \quad \forall a, b \in B.$$  

Such a connection thus corresponds to a connection form $\omega$ on $P$. If we pick any such connection, we can then consider the (principal) almost-commutative manifold

$$I_\infty^{\text{YM}} \times \nabla M := (\Gamma^{\infty}(B), L^2(B \otimes S), D_B, I_I \otimes I_M, \Gamma_I \otimes \Gamma_M).$$

If $M$ is simply connected, $\mathcal{G}(I_\infty^{\text{YM}} \times \nabla M)$ is isomorphic to $\mathcal{G}(P)$, and $I_\infty^{\text{YM}} \times \nabla M$ describes a $PSU(N)$ gauge theory $(P, \omega)$ over $M$. We denote the local connection form of $\nabla$ by $B_{\mu}$, and its curvature tensor by $F_{\mu \nu}$. From Proposition 6.56 we find that the spectral action yields the Lagrangian

$$\mathcal{L}(g_{\mu \nu}, B_{\mu}) = N^2 \mathcal{L}_M(g_{\mu \nu}) + \mathcal{L}_{YM}(g_{\mu \nu}, B_{\mu}),$$

where the Yang-Mills Lagrangian is given (up to a normalisation constant) by the usual expression:

$$\mathcal{L}_{YM}(g_{\mu \nu}, B_{\mu}) := \frac{f(0)}{24\pi^2} \text{tr}(F_{\mu \nu}F^{\mu \nu}).$$

6.6.2 **Electrodynamics**

The example of (globally trivial) electrodynamics in the context of noncommutative geometry appeared in [DS13]. Here we describe its generalisation to the
globally non-trivial case. The finite spectral triple for electrodynamics is given by [DS13]
\[ \Gamma_{\mathbb{E}D} := (C^2, C^4, \mathcal{D}_F, \Gamma_F, J_F). \]
We shall generalise this finite triple to a massive even gauge module \( I_{\mathbb{E}D}^\infty \) over \( M \). First, we set the algebra \( \mathcal{B} \) to be of the form
\[ \mathcal{B} := A \oplus A = C^\infty(M) \oplus C^\infty(M). \]
Let \( L \) be a complex line bundle over \( M \), with a given hermitian structure, so that its structure group is \( U(1) \). We shall take two identical copies of this line bundle, which we denote by \( E_L \) and \( E_R \), with smooth sections \( \mathcal{E}_L = \Gamma^\infty(E_L) \) and \( \mathcal{E}_R = \Gamma^\infty(E_R) \). Then the \( \mathcal{B} \)-\( A \)-bimodule \( \mathcal{E} \) is defined as
\[ \mathcal{E} := (\mathcal{E}_L \oplus \mathcal{E}_R) \oplus (\overline{\mathcal{E}_L} \oplus \overline{\mathcal{E}_R}), \]
where the first component of \( \mathcal{B} \) acts on \( \mathcal{E}_L \oplus \mathcal{E}_R \), and the second component acts on its conjugate. The grading is defined as \( \Gamma_I := 1 \oplus (-1) \oplus (-1) \oplus 1 \) with respect to this decomposition. The real structure \( J_I \) is the anti-linear map \( \mathcal{E}_L, R \mapsto \mathcal{E}_L, R \) and \( \mathcal{E}_L, R \mapsto \mathcal{E}_L, R \) of KO-dimension 6 (see Definition 6.13). We then have the subalgebra \( \mathcal{B}_I \simeq A \subset \mathcal{B} \), where the inclusion is given by \( a \mapsto a \oplus a \). Imposing all conditions in Definition 6.45, the ‘mass matrix’ \( \mathcal{D}_I \) is restricted to be of the form
\[ \mathcal{D}_I := \begin{pmatrix} 0 & d & 0 & 0 \\ \overline{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d} \\ 0 & d & 0 & 0 \end{pmatrix}, \]
where \( d \in C^\infty(M) \) (see [DS13, §4.1.1]).

**Remark 6.62.** In order to interpret \( d \) as a mass parameter, it would have to be given by a single real-valued parameter. For this reason we restrict ourselves to the case \( d = -\text{im} \) (see [DS13, Remark 4.4]). We stress here that in general (as mentioned in Remark 6.15) the mass \( m \) is not a fixed parameter, but a function on \( M \) (although it can be chosen to be constant). In other words, our framework allows the mass of a particle to vary from point to point in \( M \), so essentially the Yukawa mass parameter is replaced by a Yukawa field. This could of course have significant physical implications, which we intend to study in future work.

The module \( I_{\mathbb{E}D}^\infty = (\mathcal{B}, \mathcal{E}, \mathcal{D}_I, \Gamma_I, J_I) \) defined in this way is in fact a massive even gauge module. To be precise, if we write \( \mathcal{E}_1 := \Gamma^\infty(L) = \mathcal{E}_L = \mathcal{E}_R \) and \( \mathcal{E}_2 := A \), then we have \( \mathcal{B}_1 = \text{End}_A(\mathcal{E}_1) = \Gamma^\infty(L \otimes L^*) \simeq A \) and also \( \mathcal{B}_2 \simeq A \). Furthermore, the module \( \mathcal{E} \) can be written as
\[ \mathcal{E} \simeq \bigoplus_{(i,j) \in K} \mathcal{E}_i \otimes_A \overline{\mathcal{E}_j}, \quad K := \{(1,2), (1,2), (2,1), (2,1)\}. \]
The hermitian structure on $L$ determines a class of transition functions of $L$ taking values in $U(1)$, so using Theorem 3.5 we can uniquely reconstruct a principal $U(1)$-bundle $P$, and we have $I^\infty_{rD} \cong P \times U(1) F_{rD}$ as massless modules (i.e. ignoring the mass matrices $D_F$ and $D_1$).

**Proposition 6.63.** The gauge group is given by
\[ \mathcal{G}(I^\infty_{rD}) \cong U(B)/U(B_I) \cong \Gamma^\infty(Ad P) \cong C^\infty(M, U(1)). \]

**Proof.** Note that the group bundle $Ad P \cong M \times U(1)$ is globally trivial, because the structure group $U(1)$ is abelian. As in Section 6.3.3, the main thing to prove is the surjectivity of the map $\phi_* : U(B) \to \Gamma^\infty(Ad P)$, which is given by $\phi_*(u) = uJu^*$. But for $u = (u_1, u_2) \in U(B)$, this map is given by
\[ (u_1, u_2) \mapsto \begin{pmatrix} u_1 u_2^* & 0 \\ 0 & u_2 u_1^* \end{pmatrix}, \]
so $\phi_*(u_1, u_2)$ can be identified with $u_1 u_2^*$. Hence each section $v \in \Gamma^\infty(Ad P) \cong C^\infty(M, U(1))$ is the image of $(v, 1) \in U(B)$.

**Remark 6.64.** Note that in this particular example it is not necessary to require that $M$ is simply connected, as we did in the general case (see Theorem 6.38).

An element $\lambda \in \mathcal{G}(I^\infty_{rD})$ acts on $\mathcal{E}_L \oplus \mathcal{E}_R$ as multiplication by $\lambda$, and acts on $\overline{\mathcal{E}}_L \oplus \overline{\mathcal{E}}_R$ as multiplication by $\overline{\lambda}$.

Pick a connection $\nabla^L$ on $L$, and let the connection $\nabla$ on $\mathcal{E}$ be given by
\[ \nabla := \nabla^L \oplus \nabla^k \oplus \overline{\nabla}^L \oplus \overline{\nabla}^L. \]
On a local trivialisation (say on a neighbourhood $U$), the connection $\nabla^L$ is determined by a local connection form $\omega^k_U \in \Omega^1(U, i\mathbb{R})$, where $i\mathbb{R}$ is the Lie algebra of $U(1)$. For the connection $\nabla$ on $\mathcal{E}$ this yields the connection form
\[ \omega_U = \omega^k_U \oplus \omega_\mu_U \oplus \overline{\omega}^\mu_U \oplus \overline{\omega}^k_U = \omega^k_U (1 \oplus 1 \oplus (-1) \oplus (-1)), \]
where the last equality follows because the action of $\overline{\omega}^\mu_U$ is given by (right) multiplication with $\omega^\mu_U$.

Now consider the almost-commutative manifold $I^\infty_{rD} \times \nabla M$ of KO-dimension 2, which (by Theorem 6.58) describes a $U(1)$-gauge theory over $M$. Taking inner fluctuations simply amounts to choosing a different connection $\nabla^L$ (see Proposition 6.53), while there is no Higgs field (because $D_1$ commutes with $B$). Hence we ignore these inner fluctuations, and simply consider the local gauge field $A_\mu := \omega^\mu_U(\partial_\mu)$, on some coordinate basis $\partial_\mu$. Its curvature is defined as $\mathcal{F}_{\mu \nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. From Proposition 6.56 (see also [DS13, Proposition 4.2]) we find that the spectral action for $I^\infty_{rD} \times \nabla M$ is asymptotically given by the local formula
\[ S_B(I^\infty_{rD} \times \nabla M) \sim \Lambda \to \int_M \mathcal{L}(g_{\mu \nu}, A_\mu, \Lambda) \sqrt{|g|} d^4 x + O(\Lambda^{-1}), \]
for
\[ L(g_{\mu\nu}, A_\mu, m) := 4L_M(g_{\mu\nu}) + L_A(g_{\mu\nu}, A_\mu) + L_H(g_{\mu\nu}, m). \]

Here \( L_M(g_{\mu\nu}) \) is the Lagrangian (6.6), and \( L_H(g_{\mu\nu}, m) \) yields additional terms depending on the mass \( m \) and the scalar curvature \( s \):
\[ L_H(g_{\mu\nu}, m) := -\frac{2f_2\Lambda^2 m^2}{\pi^2} + \frac{f(0)m^4}{2\pi^2} + \frac{f(0)m^2 s}{12\pi^2}. \]

The Lagrangian for the gauge field is given by
\[ L_A(g_{\mu\nu}, A_\mu) := \frac{f(0)}{6\pi^2} F_{\mu\nu} F^{\mu\nu}. \]

The interaction of the \( U(1) \) gauge field with the fermions is described by the fermionic action, and is given by (see [DS13, Proposition 4.3 and Theorem 4.5])
\[ S_f(I^\infty_{ED} \times \nabla M) = \int_M L_f(g_{\mu\nu}, A_\mu, m) \sqrt{|g|} d^4x, \]
for the Lagrangian
\[ L_f(g_{\mu\nu}, A_\mu, m) := (J^M_M \tilde{\chi} | (\gamma^\mu(\nabla^S_{\mu} - A_\mu) + im) \tilde{\psi}), \]
where \( \chi \) and \( \psi \) are two Dirac spinors in \( L^2(S) \). We summarise this as follows:

**Proposition 6.65.** The total Lagrangian for \( I^\infty_{ED} \times \nabla M \) is given by a gravitational part
\[ L_{\text{grav}}(g_{\mu\nu}, m) := 4L_M(g_{\mu\nu}) + L_H(g_{\mu\nu}, m), \]
and a part for electrodynamics
\[ L_{\text{ED}}(g_{\mu\nu}, A_\mu, m) := (J^M_M \tilde{\chi} | (\gamma^\mu(\nabla^S_{\mu} - A_\mu) + im) \tilde{\psi}) + \frac{f(0)}{6\pi^2} F_{\mu\nu} F^{\mu\nu}. \]

**Remark 6.66.** The fermionic part \( L_f \) of the Lagrangian does not exactly correspond to the correct physical Lagrangian for electrodynamics (as given in e.g. [PS95, §4.1]). First of all, we have assumed throughout this chapter that \( M \) is a Riemannian manifold, while in reality space-time is a Lorentzian manifold. Second, the fermionic action uses the charge conjugation operator \( J_M \), which in reality should not be there. We will propose an alternative for the fermionic action in Chapter 7, and we show in Section 7.4 that for electrodynamics this alternative does give exactly the correct (fermionic part of the) Lagrangian.
In this chapter we define *Krein spectral triples*, in which the notion of spectral triples is generalised from Hilbert spaces to Krein spaces. The main motivation is the fact that the spinors on a pseudo-Riemannian manifold naturally give rise to a Krein space (as discussed in Section 3.3.3). Thus, to obtain a physical description of spinor fields on space-time, it is more natural to work with Krein spaces instead of Hilbert spaces.

As we described in Section 1.2.3, so-called *almost-commutative manifolds* can be used to describe gauge theories, and therefore to obtain models of particle physics. For a suitably chosen almost-commutative manifold, one obtains the full Standard Model of high energy physics, including the Higgs mechanism and neutrino mixing [Con96, Con06, CCM07]. This description of gauge theories relies on two action functionals which allow to derive the Lagrangian of the theory: the *spectral action* [CC97] and the *fermionic action* [Con06]. As described in Section 6.5.3, the spectral action yields the bosonic part of the Lagrangian, while the fermionic action yields (of course) the fermionic part.

Both of these action functionals however rely on the Riemannian signature of the manifold. In this chapter we will describe a *Lorentzian* alternative to the fermionic action, which we call the *Krein action*. While the Lagrangian obtained from the original formulation of the fermionic action closely resembles (term by term) the physical Lagrangian, a comparison of the fermionic action of [Con06] with the physical Lagrangian still shows three discrepancies:

1) the fermionic action is given in Riemannian signature, while physical space-time has Lorentzian signature;

2) the fermionic action is defined by using the real structure (the ‘charge conjugation operator’), but (except for possible Majorana mass terms) the charge conjugation operator should not be present in the physical Lagrangian;

3) the fermionic action is not automatically real-valued.

Our alternative Krein action is closer in spirit to [Bar07], where the emphasis is also put on the Lorentzian signature. We will show that our Krein action resolves
the above problems, and recovers exactly the correct physical Lagrangian of the model under consideration. We will calculate this Lagrangian explicitly for the cases of electrodynamics, electro-weak theory, and the Standard Model.

7.1 KEIN SPECTRAL TRIPLES

Let $\mathcal{H}$ be a Krein space with the indefinite inner product $\langle \cdot | \cdot \rangle$. A fundamental symmetry $J$ is a self-adjoint unitary operator $J: \mathcal{H} \rightarrow \mathcal{H}$ such that $(1 + J)\mathcal{H}$ is positive-definite and $(1 - J)\mathcal{H}$ is negative-definite. We denote by $\mathcal{H}_J$ the corresponding Hilbert space for the positive-definite inner product $\langle \cdot | \cdot \rangle_J := \langle J \cdot | \cdot \rangle$.

For an operator $T$, we will denote by $T^+$ the Krein-adjoint (i.e., the adjoint operator with respect to the Krein inner product $\langle \cdot | \cdot \rangle$). By the adjoint $T^*$ we will mean the usual adjoint in the Hilbert space $\mathcal{H}_J$ (i.e., with respect to the positive-definite inner product $\langle \cdot | \cdot \rangle_J$). These adjoints are related via $T^+ = J T^* J$.

Definition 7.1. A Krein space $\mathcal{H}$ with fundamental symmetry $J$ is called $\mathbb{Z}_2$-graded if $\mathcal{H}_J$ is $\mathbb{Z}_2$-graded and $J$ is homogeneous.

The assumption that $\mathcal{H}_J$ is $\mathbb{Z}_2$-graded (see also Section 2.1) means we have a decomposition $\mathcal{H}_J^0 \oplus \mathcal{H}_J^1$, and that this decomposition is respected by the positive-definite inner product $\langle \cdot | \cdot \rangle_J$ (which means that $\langle \psi_0 | \psi_1 \rangle_J = 0$ for all $\psi_0 \in \mathcal{H}_J^0$ and $\psi_1 \in \mathcal{H}_J^1$). The bounded operators $B(\mathcal{H})$ then also decompose into even operators $B^0(\mathcal{H})$ and odd operators $B^1(\mathcal{H})$. The assumption that the fundamental symmetry $J$ is homogeneous means that $J$ is either even or odd. If $J$ is odd, it implements a unitary isomorphism $\mathcal{H}_J^0 \simeq \mathcal{H}_J^1$. Given the decomposition $\mathcal{H}_J^0 \oplus \mathcal{H}_J^1$, we have a (self-adjoint, unitary) grading operator $\Gamma$ which acts as $(-1)^j$ on $\mathcal{H}_J^j$ (for $j \in \mathbb{Z}_2$). If $J$ is odd, we note that $\Gamma$ is Krein-anti-self-adjoint (indeed, $\Gamma^+ = J \Gamma J = -\Gamma J^2 = -\Gamma$).

The following definition aims to adapt the notion of spectral triple to Krein spaces. Similar approaches for such an adaptation have been given in [PS06, Str06].

Definition 7.2. An even Krein spectral triple $(A, \pi \mathcal{H}, D, J)$ consists of

- a $\mathbb{Z}_2$-graded Krein space $\mathcal{H}$;
7.1 Krein spectral triples

- a trivially graded $\ast$-algebra $\mathcal{A}$ along with an even $\ast$-algebra representation $\pi: \mathcal{A} \to B^0(\mathcal{H})$;
- a fundamental symmetry $\mathcal{J}$ (satisfying $\mathcal{J}^* = \mathcal{J}$ and $\mathcal{J}^2 = 1$) which commutes with the algebra $\mathcal{A}$ and which is either even or odd;
- a densely defined, closed, odd operator $\mathcal{D}: \text{Dom} \mathcal{D} \to \mathcal{H}$ such that:
  1) there exists a linear subspace $\mathcal{E} \subset \text{Dom} \mathcal{D} \cap \mathcal{J} \cdot \text{Dom} \mathcal{D}$ which is dense with respect to $\| \cdot \|_{\mathcal{D},\mathcal{J},\mathcal{D}}$, and which is a core for $\mathcal{D}$;
  2) the operator $\mathcal{D}$ is essentially Krein-self-adjoint on $\mathcal{E}$ (or, equivalently, $\mathcal{J} \mathcal{D}: \text{Dom} \mathcal{D} \to \mathcal{H}_\mathcal{J}$ is essentially self-adjoint);
  3) we have the inclusion $\pi(\mathcal{A}) \cdot \mathcal{E} \subset \text{Dom} \mathcal{D} \cap \mathcal{J} \cdot \text{Dom} \mathcal{D}$, and the commutator $[\mathcal{D}, \pi(a)]$ is bounded on $\mathcal{E}$ for each $a \in \mathcal{A}$;
  4) the map $\pi(a) \circ i: \text{Dom} \mathcal{D} \cap \mathcal{J} \cdot \text{Dom} \mathcal{D} \to \mathcal{H} \to \mathcal{H}$ is compact for each $a \in \mathcal{A}$, where $i: \text{Dom} \mathcal{D} \cap \mathcal{J} \cdot \text{Dom} \mathcal{D} \to \mathcal{H}$ denotes the natural inclusion map, and $\text{Dom} \mathcal{D} \cap \mathcal{J} \cdot \text{Dom} \mathcal{D}$ is considered as a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{D},\mathcal{J},\mathcal{D}}$.

We say an even Krein spectral triple $(\mathcal{A}, \pi \mathcal{H}, \mathcal{D}, \mathcal{J})$ is of Lorentz-type when $\mathcal{J}$ is odd.

As follows from the next proposition, the assumption that $\mathcal{J}$ is odd actually just captures the fact that the number $t$ of time dimensions is odd; it does not necessarily imply that $t = 1$ (as was noted already in [PS06, page 5]).

**Proposition 7.3.** Let $(\mathcal{M}, g)$ be an even-dimensional time- and space-oriented pseudo-Riemannian spin manifold of signature $(t, s)$, and let $\tau$ be a spacelike reflection, such that the associated Riemannian metric $g_\tau$ is complete (as in Assumption 3.16). Then we obtain an even Krein spectral triple 

$$(C_c^\infty(\mathcal{M}), L^2(\mathcal{S}), i^1 \mathcal{D}, \mathcal{J}_\mathcal{M})$$

with grading operator $\Gamma_\mathcal{M}$. If $t$ is odd, the triple is a Lorentz-type spectral triple.

**Proof.** We know from Proposition 5.18 that the triple $(C_c^\infty(\mathcal{M}), L^2(\mathcal{S}), i^1 \mathcal{D})$ satisfies conditions 1), 3), and 4) of Definition 7.2. Condition 2) follows from Theorem 3.17. Since $\mathcal{M}$ is even-dimensional, we have from Section 3.3.4 a grading operator $\Gamma_\mathcal{M}$ which satisfies the relation $\Gamma_\mathcal{M} \mathcal{J}_\mathcal{M} = (-1)^t \mathcal{J}_\mathcal{M} \Gamma_\mathcal{M}$. This implies that we have a Lorentz-type spectral triple if and only if $t$ is odd. $\square$

7.1.1 The Krein action

A quadratic form on a Krein space $\mathcal{H}$ is a sesquilinear map $q: \text{Dom} q \times \text{Dom} q \to \mathbb{C}$ (conjugate-linear in the first variable and linear in the second variable), where the form domain $\text{Dom} q$ is a dense linear subspace of $\mathcal{H}$. If $q(\psi_1, \psi_2) = q(\psi_2, \psi_1)$ for all $\psi_1, \psi_2 \in \mathcal{H}$ we say that $q$ is symmetric. If $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ is $\mathbb{Z}_2$-graded (and we think
of $\mathbb{C}$ as being trivially graded), then we say that $q$ is $\mathbb{Z}_2$-graded if $q(\psi_0, \psi_1) = 0$ for any $\psi_0 \in \mathcal{H}^0 \cap \text{Dom} \ q$ and $\psi_1 \in \mathcal{H}^1 \cap \text{Dom} \ q$.

**Proposition 7.4.** Let $(A, \mathcal{H}, D, J)$ be an even Krein spectral triple. Then

$$\mathcal{F}(\psi_1, \psi_2) := \langle \psi_1 | D \psi_2 \rangle = \langle J \psi_1 | D \psi_2 \rangle$$

defines a symmetric quadratic form $\mathcal{F}$ with form domain $\text{Dom} \mathcal{F} = \text{Dom} \ D$. Moreover, if the Krein spectral triple is of Lorentz-type, then $\mathcal{F}$ is $\mathbb{Z}_2$-graded.

**Proof.** Sesquilinearity is immediate from the definition, and using the Krein-self-adjointness of $D$ we also find symmetry:

$$\langle \psi_1 | D \psi_2 \rangle = \langle D \psi_2 | \psi_1 \rangle = \langle \psi_2 | D \psi_1 \rangle.$$

If the triple is of Lorentz-type, then the grading operator $\Gamma$ is Krein-anti-self-adjoint. For $\psi_0 \in \mathcal{H}^0$ and $\psi_1 \in \mathcal{H}^1$ we then find that

$$\langle \psi_0 | D \psi_1 \rangle = \langle \Gamma \psi_0 | D \psi_1 \rangle = -\langle \psi_0 | \Gamma D \psi_1 \rangle = \langle \psi_0 | D \Gamma \psi_1 \rangle = -\langle \psi_0 | D \psi_1 \rangle. \quad \Box$$

**Definition 7.5 (Krein action).** Let $(A, \mathcal{H}, D, J)$ be a Lorentz-type spectral triple. We define the **Krein action** $S_\mathcal{K}$ to be the functional

$$S_\mathcal{K}[\psi] := \mathcal{F}(\psi, \psi), \quad \psi \in \mathcal{H}^0.$$  

We immediately see that the Krein action addresses the three discrepancies of the fermionic action mentioned at the start of this chapter. Indeed, our Krein space approach obviously allows for Lorentzian signature; the charge conjugation operator does not make its appearance; and since $\mathcal{F}$ is a symmetric quadratic form, $S_\mathcal{K}[\psi]$ is automatically real-valued. Furthermore, we will show in Sections 7.4 to 7.6 that this Krein action recovers the correct (fermionic part of the) Lagrangians for electrodynamics, electro-weak theory, and the Standard Model.

### 7.2 Gauge Theory

In this section, we develop the abstract formalism for a description of gauge theories using Krein spectral triples. Later, we will apply this to the case of almost-commutative manifolds.

Let $A$ be a trivially graded unital $*$-algebra. Denote by $A^{\text{op}} := \{ a^{\text{op}} | a \in A \}$ the *opposite algebra* of $A$, which equals $A$ as a vector space but has the opposite product $a^{\text{op}} b^{\text{op}} = (ba)^{\text{op}}$. Let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded Krein space with fundamental symmetry $J$, and suppose we have two commuting even representations $\pi: A \to \mathcal{B}(\mathcal{H})$ and $\pi^{\text{op}}: A^{\text{op}} \to \mathcal{B}(\mathcal{H})$. For ease of notation, we will often simply write $a$ instead of $\pi(a)$ and $a^{\text{op}}$ instead of $\pi^{\text{op}}(a^{\text{op}})$.
We obtain a representation of the algebraic tensor product $A \otimes A^\text{op}$ on $\mathcal{H}$ by setting $ar{\pi}(a \otimes b^\text{op}) := \pi(a)\pi^\text{op}(b^\text{op})$. Now suppose that $(A \otimes A^\text{op}, \pi\mathcal{H}, D, J)$ is a Krein spectral triple. We say that this triple satisfies the order-one condition if

$$[[\pi(a), [D, \pi^\text{op}(b^\text{op})]]] = 0$$

(7.1)

for all $a, b \in A$. In the remainder of this section we consider an even Krein spectral triple $(A \otimes A^\text{op}, \pi\mathcal{H}, D, J)$. We will assume that this triple is unital, which means that $A$ is unital and that $\bar{\pi}$ is unital. We develop the abstract formalism below without assuming that this triple satisfies the order-one condition.

7.2.1 Inner perturbations

We will now introduce fluctuations of the operator $D$. Later, we will see how these fluctuations give rise to gauge fields as well as scalar fields (the latter being interpreted as the Higgs field in the case of electroweak theory and the Standard Model, see Sections 7.5 and 7.6).

We adapt the approach described in [CCS13] to our Krein spectral triples. Let $A$ be a trivially graded unital $*$-algebra. For an element $\Lambda = \sum_j a_j \otimes b_j^\text{op} \in A \otimes A^\text{op}$ we define its adjoint as $\Lambda^* := \sum_j b_j^* \otimes a_j^\text{op}$, and we say that $\Lambda$ is normalised if $\sum_j a_j b_j = 1 \in A$. For $\Lambda = \sum_j a_j \otimes b_j^\text{op} \in A \otimes A^\text{op}$ and $\Lambda' = \sum_j a_j' \otimes b_j^\text{op} \in A \otimes A^\text{op}$ we have

$$(\Lambda \Lambda')^* = \sum_{j,k} (a_j a_k' \otimes (b_k' b_j)^\text{op})^* = \sum_{j,k} (b_k' b_j)^* \otimes (a_j a_k')^\text{op}$$

$$= \left( \sum_j b_j^* \otimes a_j^\text{op} \right) \left( \sum_k b_k'^* \otimes a_k'^\text{op} \right) = \Lambda^* \Lambda'^*,$$

and in particular, if $\Lambda$ and $\Lambda'$ are self-adjoint, then $\Lambda \Lambda'$ is also self-adjoint. Furthermore, if $\Lambda$ and $\Lambda'$ are normalised, we find

$$\sum_{j,k} a_j a_k' b_k b_j = \sum_j a_j \left( \sum_k a_k' b_k \right) b_j = \sum_j a_j b_j = 1,$$

so the product $\Lambda \Lambda'$ is also normalised. Hence the properties of self-adjointness and normalisation are preserved by multiplication, and we can define the following.

Definition 7.6 ([CCS13, §III]). We define the perturbation semi-group $\text{Pert}(A)$ as the set of self-adjoint normalised elements in $A \otimes A^\text{op}$, with the multiplication inherited from the algebra $A \otimes A^\text{op}$.

Let $(\mathcal{B}, \mathcal{H}, D, J)$ be a Krein spectral triple. As in Section 6.5.2, we define the generalised one-forms as

$$\Omega^1_{[D]}(\mathcal{B}) := \left\{ \sum_j a_j [D, b_j] \mid a_j, b_j \in \mathcal{B} \right\},$$
where the sums must converge in norm.

**Definition 7.7.** We define a map \( \eta_D : \text{Pert}(A) \to \Omega_D^1(A \odot A^\text{op}) \subset \mathcal{B}(\mathcal{H}) \) by

\[
\eta_D \left( \sum_j a_j \otimes b_j^\text{op} \right) := \sum_{j,k} \hat{\pi}(a_j \otimes (a_k^*)^\text{op}) \left[ D, \hat{\pi}(b_j \otimes (b_k^*)^\text{op}) \right] .
\]

If the order-one condition is satisfied, we see that the expression for \( \eta_D \) can be simplified.

**Proposition 7.8 ([CCS13]).** Suppose that \((A \odot A^\text{op}, \mathcal{H}, D, \beta)\) satisfies the order-one condition \((7.1)\). Then for \( A = \sum_j a_j \otimes b_j^\text{op} \in \text{Pert}(A) \) we obtain the expression

\[
\eta_D(A) = \sum_j a_j[D, b_j] + \sum_j a_j^\text{op}[D, b_j^\text{op}].
\]

**Proof.** We will suppress \( \pi \) and \( \pi^\text{op} \) from our notation. Since \( \hat{\pi}(a \otimes b^\text{op}) = ab^\text{op} \) for all \( a, b \in A \), we can calculate

\[
\eta_D(A) = \sum_{j,k} a_j a_k^\text{op}[D, b_j b_k^\text{op}]
\]

\[
= \sum_{j,k} a_j a_k^\text{op}[D, b_j] b_k^\text{op} + \sum_{j,k} a_j a_k^\text{op} b_j [D, b_k^\text{op}]
\]

\[
= \sum_{j,k} a_j[D, b_j] a_k^\text{op} b_k^\text{op} + \sum_{j,k} a_j b_j a_k^\text{op}[D, b_k^\text{op}]
\]

\[
= \sum_j a_j[D, b_j] + \sum_{j,k} a_j^\text{op}[D, b_j^\text{op}],
\]

where in the third step we used the order-one condition, and in the last step we used the normalisation of \( A \).

**Lemma 7.9 ([CCS13, Lemma 4.(ii)]).** The map \( \eta_D : \text{Pert}(A) \to \Omega_D^1(A \odot A^\text{op}) \) is involutive (i.e. \( \eta_D(A^*) = \eta_D(A)^+ \)).

**Proof.** Using Krein-self-adjointness of \( D \) and the fact that the fundamental symmetry commutes with \( A \) (so that \( a^* = a^+ \) for \( a \in A \)), we find

\[
\eta_D \left( \sum_j b_j^* \otimes (a_j^*)^\text{op} \right) = \sum_{j,k} \hat{\pi}(b_j^* \otimes (b_k^*)^\text{op}) \left[ D, \hat{\pi}(a_j^* \otimes (a_k^*)^\text{op}) \right] ^+
\]

\[
= \left( \sum_{j,k} - \left[ D, \hat{\pi}(a_j \otimes (a_k^*)^\text{op}) \right] \hat{\pi}(b_j \otimes (b_k^*)^\text{op}) \right)^+ 
\]

\[
= \left( \sum_{j,k} - \left[ D, 1 \right] + \hat{\pi}(a_j \otimes (a_k^*)^\text{op}) \left[ D, \hat{\pi}(b_j \otimes (b_k^*)^\text{op}) \right] \right)^+ 
\]

\[
= \left( \sum_{j,k} \hat{\pi}(a_j \otimes (a_k^*)^\text{op}) \left[ D, \hat{\pi}(b_j \otimes (b_k^*)^\text{op}) \right] \right)^+ ,
\]
where in the third step we used the normalisation
\[
\sum_{j,k} \hat{\pi}(a_j \otimes (a_k^*)^{\text{op}}) \hat{\pi}(b_j \otimes (b_k^*)^{\text{op}}) = \hat{\pi} \left( \sum_j a_j b_j \otimes \sum_k ((a_k b_k^*)^*)^{\text{op}} \right) = 1. \quad \square
\]

**Definition 7.10.** By the fluctuation of \( \mathcal{D} \) by \( A \in \text{Pert}(\mathcal{A}) \) we mean the map
\[
\mathcal{D} \mapsto \mathcal{D}_A := \mathcal{D} + \eta_D(A),
\]
and we refer to \( \mathcal{D}_A \) as the fluctuated Dirac operator.

We point out that the map \( \eta_D \) is not multiplicative and therefore it does not yield a representation of the semi-group \( \text{Pert}(\mathcal{A}) \) on \( \mathcal{H} \). Instead, we obtain an action of \( \text{Pert}(\mathcal{A}) \) on the space of fluctuated Dirac operators.

**Proposition 7.11 ([CCS13, Proposition 5.(ii)].)** A fluctuation of a fluctuated Dirac operator is again a fluctuated Dirac operator. To be precise: \( (\mathcal{D}_A)_{A'} = \mathcal{D}_{A'A} \) for all perturbations \( A, A' \in \text{Pert}(\mathcal{A}) \).

**Proof.** Let \( A, A' \in \text{Pert}(\mathcal{A}) \) be given by \( A = \sum_j a_j \otimes b_j^* \) and \( A' = \sum_j a_j' \otimes (b_j')^* \). First, for \( A'A = \sum_{j,k} a_k' a_j \otimes (b_k b_j')^* \) we find
\[
\eta_D(A'A) = \sum_{j,k,l,m} a_k' a_j (a_m^*)^{\text{op}} [\mathcal{D}, b_l b_k (b_m b_j')^*]^{\text{op}}
= \sum_{k,l} a_k' (a_l^*)^{\text{op}} \eta_D(A) b_k (b_l')^{\text{op}} + \eta_D(A')
= \eta_D(A) + \sum_{k,l} a_k' (a_l^*)^{\text{op}} [\eta_D(A), b_k (b_l')^{\text{op}}] + \eta_D(A')
\]

Second, we calculate
\[
\eta_{D+\eta_D(A)}(A') = \sum_{k,l} a_k' (a_l^*)^{\text{op}} [\mathcal{D} + \eta_D(A), b_k (b_l')^{\text{op}}]
= \eta_D(A') + \sum_{k,l} a_k' (a_l^*)^{\text{op}} [\eta_D(A), b_k (b_l')^{\text{op}}].
\]

Combining these two expressions we find \( \eta_D(A'A) = \eta_D(A) + \eta_{D+\eta_D(A)}(A') \), which implies that indeed \( (\mathcal{D}_A)_{A'} = \mathcal{D}_{A'A} \). \quad \square

### 7.2 Gauge action

The unitary group \( \mathcal{U}(\mathcal{A}) \) can be embedded into \( \text{Pert}(\mathcal{A}) \) via the semi-group homomorphism \( \Delta: \mathcal{U}(\mathcal{A}) \to \text{Pert}(\mathcal{A}) \) given by \( u \mapsto u \otimes (u^*)^{\text{op}} \). This then yields an obvious action of \( u \in \mathcal{U}(\mathcal{A}) \) on \( \text{Pert}(\mathcal{A}) \) given by multiplication with \( \Delta(u) \). To be precise, for \( A = \sum_j a_j \otimes b_j^* \in \text{Pert}(\mathcal{A}) \), the action of \( u \in \mathcal{U}(\mathcal{A}) \) is given by
\[
\Delta(u)A = \sum_j u a_j \otimes (u^*)^{\text{op}} b_j^* \otimes = \sum_j u a_j \otimes (b_j u^*)^{\text{op}}.
\]
Since \( \text{Pert}(A) \subset A \otimes A^{\text{op}} \), we can compose \( \Delta \) with the *-algebra representation \( \hat{\pi} \) to obtain a group representation
\[
\rho := \hat{\pi} \circ \Delta : \mathcal{U}(A) \to \mathcal{B}(\mathcal{H}).
\]

**Definition 7.12.** We define the gauge group as
\[
\mathcal{G}(A) := \{ \rho(u) \mid u \in \mathcal{U}(A) \} \simeq \mathcal{U}(A)/\text{Ker} \rho.
\]

We also consider an action \( \gamma \) of the unitary group \( \mathcal{U}(A) \) on \( \Omega_D^1(A \otimes A^{\text{op}}) \). For \( T \in \Omega_D^1(A \otimes A^{\text{op}}) \) and \( u \in \mathcal{U}(A) \), this action is given by
\[
\gamma_u(T) := \rho(u) T \rho(u^*) + \eta_D \circ \Delta(u) = \rho(u) T \rho(u^*) + \rho(u) [\mathcal{D}, \rho(u^*)].
\]

We point out that this is the usual transformation of gauge potentials under the gauge group \( \mathcal{G}(A) \).

**Lemma 7.13** ([CCS13, Lemma 4.(iii)]). The map \( \eta_D : \text{Pert}(A) \to \Omega_D^1(A \otimes A^{\text{op}}) \) is covariant with respect to the actions by \( \mathcal{U}(A) \). To be precise, \( \gamma_u \circ \eta_D(A) = \eta_D(\Delta(u)A) \) for all \( u \in \mathcal{U}(A) \).

**Proof.** For \( A = \sum_a a_j \otimes b_j^{\text{op}} \), we calculate
\[
\eta_D \left( \sum_j u a_j \otimes (b_j u^*)^{\text{op}} \right) = \sum_{j,k} u a_j (u a_k)^{\text{op}} [\mathcal{D}, b_j u^* (u b_k)^{\text{op}}] \\
= \sum_{j,k} \rho(u) a_j a_k^{\text{op}} [\mathcal{D}, b_j b_k^{\text{op}}] \rho(u^*) + \rho(u) [\mathcal{D}, \rho(u^*)] \\
= \gamma_u (a_j a_k^{\text{op}} [\mathcal{D}, b_j b_k^{\text{op}}]) \\
= \gamma_u \circ \eta_D \left( \sum_j a_j \otimes b_j^{\text{op}} \right). \quad \square
\]

As mentioned above, we have an action of \( \mathcal{G}(A) \) on \( \mathcal{H} \). We can also define an action of \( \mathcal{G}(A) \) on the space of fluctuated Dirac operators. For \( \rho(u) \in \mathcal{G}(A) \), this action is given by \( \mathcal{D}_A \mapsto \mathcal{D}_{\Delta(u)A} \). If \( u \in \text{Ker} \rho \), then \( \eta_D(\Delta(u)A) = \eta_D(A) \) and hence \( \mathcal{D}_{\Delta(u)A} = \mathcal{D}_A \). Therefore this action of \( \rho(u) \) on \( \mathcal{D}_A \) is independent of the choice of representative \( u \in \mathcal{U}(A) \) for \( \rho(u) \in \mathcal{G}(A) \).

**Proposition 7.14.** The Krein action \( S_K[\psi, A] := \langle \psi | \mathcal{D}_A \psi \rangle \) of the fluctuated Dirac operator \( \mathcal{D}_A \) is invariant under the action of the gauge group given by \( \psi \mapsto \rho(u) \psi \) and \( A \mapsto \Delta(u)A \).

**Proof.** Using Lemma 7.13, we find
\[
\mathcal{D}_{\Delta(u)A} = \mathcal{D} + \eta_D(\Delta(u)A) = \mathcal{D} + \gamma_u \circ \eta_D(A) \\
= \mathcal{D} + \rho(u) \eta_D(A) \rho(u^*) + \rho(u) [\mathcal{D}, \rho(u^*)] \\
= \rho(u) (\mathcal{D} + \eta_D(A)) \rho(u^*) = \rho(u) \mathcal{D}_A \rho(u^*).
Since the unitary $\rho(u)$ commutes with $\mathcal{J}$ (so $\rho(u)$ is also unitary for the Krein inner product), we find
\[
S_{\mathcal{X}}[\rho(u)|\psi,\Delta(u)|A] = \langle \rho(u)|\psi|\mathcal{D}_{\Delta(u)}A\rho(u)|\psi \rangle = \langle \rho(u)|\rho(u)|\mathcal{D}_{A}\psi \rangle = \langle \psi|\mathcal{D}_{A}\psi \rangle = S_{\mathcal{X}}[\psi|A].
\]

\section{Almost-Commutative Manifolds}

In Proposition 7.3 we have seen that a pseudo-Riemannian spin manifold $(M, g)$ of signature $(t, s)$ gives rise to the Krein spectral triple $(C_0^\infty(M), L^2(S), i^1\mathcal{D}, \mathcal{J}_M)$. We will now introduce the notion of a finite space $F$, and construct the corresponding almost-commutative manifold as the product $F \times M$. Contrary to Chapter 6, we do not consider the globally non-trivial case. Our purpose in the remainder of this chapter is to explicitly calculate the Krein action for several examples of almost-commutative manifolds, and since these calculations are local, the global structure is irrelevant to our present purpose. Therefore, we only give the globally trivial construction (as described in Section 1.2.3), adapted to the Krein spectral triples of this chapter.

\begin{definition}
A finite space $F := (A_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F)$ with $\mathbb{Z}_2$-grading $\Gamma_F$ is an even Krein spectral triple such that $\mathcal{J}_F$ is finite-dimensional.
\end{definition}

\begin{definition}
Let $(M, g)$ be an even-dimensional pseudo-Riemannian spin manifold as in Proposition 7.3. An almost-commutative pseudo-Riemannian manifold $F \times M$ is the product of a finite space $F$ with the manifold $M$, given by
\[
\left( C_0^\infty(M, A_F), \mathcal{H}_F \otimes L^2(S), i^{\deg \mathcal{J}_F} \otimes i^1\mathcal{D} + i^{\deg \mathcal{J}_M}\mathcal{D}_F \otimes 1, i^{\deg \mathcal{J}_F \cdot \deg \mathcal{J}_M} \mathcal{J}_F \otimes \mathcal{J}_M \right),
\]
equipped with the grading operator $\Gamma_F \otimes \Gamma_M$.

As we already discussed in Section 6.2.2, we construct almost-commutative manifolds as $F \times M$ instead of $M \times F$ (which is more common in the literature). Here we have also written the product in terms of the graded tensor product $\otimes$, thus avoiding explicit use of the grading operators.

\begin{proposition}
An almost-commutative pseudo-Riemannian manifold is an even Krein spectral triple.
\end{proposition}

\begin{proof}
We shall write
\[
\begin{align*}
A &:= C_0^\infty(M, A_F), & \mathcal{H} &:= \mathcal{H}_F \otimes L^2(S), & \Gamma &:= \Gamma_F \otimes \Gamma_M, \\
\mathcal{D} &:= i^{\deg \mathcal{J}_F} \otimes i^1\mathcal{D} + i^{\deg \mathcal{J}_M}\mathcal{D}_F \otimes 1, & \mathcal{J} &:= i^{\deg \mathcal{J}_F \cdot \deg \mathcal{J}_M} \mathcal{J}_F \otimes \mathcal{J}_M.
\end{align*}
\]

First we check that $\mathcal{J}$ is a fundamental symmetry for the $\mathbb{Z}_2$-graded Krein space $\mathcal{H}$. The adjoint of the graded tensor product $\mathcal{J}_F \otimes \mathcal{J}_M$ is given by
\[
(\mathcal{J}_F \otimes \mathcal{J}_M)^* = (-1)^{\deg \mathcal{J}_F \cdot \deg \mathcal{J}_M} \mathcal{J}_F^* \otimes \mathcal{J}_M^* = (-1)^{\deg \mathcal{J}_F \cdot \deg \mathcal{J}_M} \mathcal{J}_F \otimes \mathcal{J}_M.
\]

\end{proof}
so that $\mathcal{J}$ is indeed self-adjoint. Its square equals

$$\mathcal{J}^2 = (-1)^{\deg \mathcal{J}} \mathcal{J}$$(\mathcal{J} \otimes \mathcal{J}_M)(\mathcal{J}_F \otimes \mathcal{J}_M) = \mathcal{J}_F^2 \otimes \mathcal{J}_M^2 = 1 \otimes 1,$$

so $\mathcal{J}$ is also unitary. Furthermore, $\mathcal{J}$ commutes with the algebra $\mathcal{A}$ because $\mathcal{J}_F$ and $\mathcal{J}_M$ both commute with the algebra. Since $\mathcal{J}_F$ and $\mathcal{J}_M$ are homogeneous, $\mathcal{J}$ is also homogeneous with degree $\deg \mathcal{J} = \deg \mathcal{J}_F + \deg \mathcal{J}_M$. We continue to check the properties for $\mathcal{D}$ in \textbf{Definition 7.2}.

1) There exists a linear subspace $\mathcal{E}_M \subset \text{Dom } \mathcal{D} \cap \mathcal{J}_M \cdot \text{Dom } \mathcal{D}$ which is dense with respect to $\| \cdot \|_{\mathcal{D},\mathcal{J}_M,\mathcal{J}_M,\mathcal{D}}$, and which is a core for $\mathcal{D}$. For the operator $\mathcal{D}$ we can then simply consider the dense linear subspace $\mathcal{E} := \mathcal{H}_F \otimes \mathcal{E}_M$.

2) The factors $i^{\deg \mathcal{J}_F}$ and $i^{\deg \mathcal{J}_M}$ before $1 \otimes i^t \mathcal{D}$ and $\mathcal{D}_F \otimes 1$ (respectively) are chosen to ensure that $\mathcal{D}$ is again Krein-symmetric. Indeed, on the domain $\text{Dom } \mathcal{D} \cap \mathcal{J} \cdot \text{Dom } \mathcal{D}$ we can write

$$\mathcal{J} \mathcal{D}^* = i^{\deg \mathcal{J}_F} \otimes i^{\deg \mathcal{J}_M} (\mathcal{J}_F \otimes \mathcal{J}_M)(1) \mathcal{D}^* + (-1)^{\deg \mathcal{J}} \mathcal{J}_M \mathcal{D}_F^* \otimes 1)$$

$$= i^{\deg \mathcal{J}_F} \otimes i^{\deg \mathcal{J}_M} (\mathcal{J}_F \otimes i^t \mathcal{D}_M)(i^{\deg \mathcal{J}_M} \mathcal{D}_F^* \otimes 1)$$

$$= i^{\deg \mathcal{J}_F} \otimes i^{\deg \mathcal{J}_M} (\mathcal{J}_F \otimes i^t \mathcal{D} \mathcal{J}_M + i^{\deg \mathcal{J}_M} \mathcal{D}_F \mathcal{J}_F \otimes \mathcal{J}_M)$$

$$= i^{\deg \mathcal{J}_F} \otimes i^{\deg \mathcal{J}_M} (\mathcal{J}_F \otimes i^t \mathcal{D} \mathcal{J}_M + i^{\deg \mathcal{J}_M} \mathcal{D}_F \mathcal{J}_F \otimes \mathcal{J}_M)$$

3) From the properties of $\mathcal{D}$, it is clear that $\mathcal{A}$ preserves $\mathcal{D}$, and that the commutator $[\mathcal{D}, \pi(a)]$ is bounded on $\mathcal{E}$ for each $a \in \mathcal{A}$.

4) The map $\pi(a) \circ : \text{Dom } \mathcal{D} \cap \mathcal{J} \cdot \text{Dom } \mathcal{D} \hookrightarrow \mathcal{D}_F \rightarrow \mathcal{H}$ is compact for each $a \in \mathcal{A}$, because $\mathcal{D}$ has this property and $\mathcal{H}_F$ is only finite-dimensional. \hfill $\square$

Our typical example will be an almost-commutative manifold constructed from an even-dimensional Lorentzian manifold, which is of course of Lorentz-type (for which $\mathcal{J}_M$ is odd). In order to be able to apply the Krein action, we need this almost-commutative manifold to be of Lorentz-type as well, which means that the finite space should not be of Lorentz-type. Hence we impose the restriction that $\mathcal{J}_F$ is even. The almost-commutative manifold is then of the form

$$F \times M := \left( C^\infty_c(M, A_F), \mathcal{H}_F \otimes L^2(S), 1 \otimes i^t \mathcal{D} + i \mathcal{D}_F \otimes 1, \mathcal{J}_F \otimes \mathcal{J}_M \right). \quad (7.2)$$

### 7.4 Electrodynamics

As a first example, we will calculate the Krein action for electrodynamics. The model of electrodynamics was first studied in the context of noncommutative geometry in [DS13]. Here, we take a slightly different approach, since we have no
We denote the standard basis of \( H \) where we have used that \( U \frac{\rho}{Ker} \). We have

\[
\text{Proof.}
\]

for \((\lambda, \mu) \in C \oplus C\), which gives the representation \( \hat{\pi}((\lambda, \mu) \otimes (\lambda', \mu')) = \lambda \mu 1_2 \) of \( A_F \otimes A_F^{\text{op}} \) on \( H_F \). We also note that these representations obviously satisfy the order-one condition (7.1). Since we have set \( \mathcal{J}_F = 1 \), this finite space is in fact an ordinary finite spectral triple, and hence also a Krein spectral triple which is not of Lorentz-type.

**Proposition 7.18.** The gauge group of the finite space \( F_{\text{ED}} \) equals \( \mathcal{G}(F_{\text{ED}}) = U(1) \).

**Proof.** We have \( U(A_F) = U(1) \times U(1) \), and the kernel of \( \rho: U(A_F) \to B(H_F) \) equals \( \text{Ker} \rho = \{(\lambda, \mu) \in U(A_F) \mid \lambda \in U(1)\} \cong U(1) \), which yields for the quotient \( \mathcal{G}(F_{\text{ED}}) = U(A_F)/\text{Ker} \rho \cong U(1) \).

Let \((M, g)\) be an even-dimensional pseudo-Riemannian spin manifold as in Proposition 7.3, for which \( t \) is odd, and consider the corresponding almost-commutative manifold as in Eq. (7.2):

\[
F_{\text{ED}} \times M := \left(C^\infty_c(M, A_F \otimes A_F^{\text{op}}), H_F \otimes L^2(S), 1 \otimes i^1 D + iD_F \otimes 1, 1 \otimes J_M \right).
\]

We consider a perturbation \( A \in \text{Pert}(C^\infty_c(M, A_F)) \). Since \( A := C^\infty_c(M, A_F) \) is commutative, \( A^{\text{op}} \cong A \) and we can write \( A = \sum a_j \otimes b_j \) for \( a_j = (\lambda_j, \mu_j) \) and \( b_j = (\lambda'_j, \mu'_j) \) in \( A \). Given such \( A \) we obtain from Proposition 7.8 the expression

\[
\eta_D(A) = \sum_j \lambda_j [i^1 D, \lambda'_j] + \sum_j \mu_j [i^1 D, \mu'_j] =: A_\mu \otimes i^1 \gamma_{\mu},
\]

where we have used that \( D_F \) commutes with the algebra elements, and we have defined \( A_\mu := \sum_j (\lambda_j \partial_\mu \lambda'_j + \mu_j \partial_\mu \mu'_j) \in C^\infty_c(M) \). Since \( A \) is self-adjoint, we know from Lemma 7.9 that \( \eta_D(A) \) is also self-adjoint. Since \( i^1 \gamma^\mu \) is Krein-anti-symmetric, \( A_\mu \) must also be Krein-anti-symmetric, and hence \( A_\mu \in C^\infty_c(M, i\mathbb{R}) \). We consider the corresponding fluctuated Dirac operator given by

\[
\mathcal{D}_A := 1 \otimes i^1 D + iD_F \otimes 1 + A_\mu \otimes i^1 \gamma^\mu.
\]
The almost-commutative manifold \( F_{ED} \times M \) is of Lorentz-type, and hence we can apply Definition 7.5 for the Krein action. An arbitrary vector \( \xi \in \mathcal{H}^0 = \mathcal{H}^0_F \otimes L^2(S)^0 \oplus \mathcal{H}^1_F \otimes L^2(S)^1 \) can be written uniquely as

\[
\xi = e_R \hat{\otimes} \psi_R + e_L \hat{\otimes} \psi_L,
\]

for Weyl spinors \( \psi_L \in L^2(S)^0 \) and \( \psi_R \in L^2(S)^1 \). Note that the vector \( \xi \in \mathcal{H}^0 \) is therefore completely determined by one Dirac spinor \( \psi := \psi_L + \psi_R \).

**Proposition 7.19.** The Krein action for \( F_{ED} \times M \) is given by

\[
S_{ED}[\psi, A] = \langle \psi | (iD + \gamma^\mu A_\mu) - m \rangle \psi.
\]

**Proof.** We need to calculate the inner product \( \langle \mathcal{J} \xi | \mathcal{D}_A \xi \rangle \), where \( \xi \), is given as in Eq. (7.4). First, for \( \mathcal{J} = 1 \otimes \mathcal{J}_M \) we calculate

\[
\mathcal{J} \xi = -e_R \hat{\otimes} i \mathcal{D}_M \psi_R + e_L \hat{\otimes} i \mathcal{D}_M \psi_L.
\]

For the fluctuated Dirac operator \( \mathcal{D}_A \) of Eq. (7.3) we find

\[
\mathcal{D}_A \xi = -e_R \hat{\otimes} i \mathcal{D}_M \psi_R + e_L \hat{\otimes} i \mathcal{D}_M \psi_L - m e_L \hat{\otimes} \psi_R + m e_R \hat{\otimes} \psi_L
\]

\[
- A_\mu e_R \hat{\otimes} i \gamma^\mu \psi_R + A_\mu e_L \hat{\otimes} i \gamma^\mu \psi_L
\]

\[
= -e_R \hat{\otimes} (i \mathcal{D}_M \psi_R - m \psi_R + i \gamma^\mu A_\mu \psi_R) + e_L \hat{\otimes} (i \mathcal{D}_M \psi_L - m \psi_R + i \gamma^\mu A_\mu \psi_L).
\]

Taking the inner product of \( \mathcal{J} \xi \) with \( \mathcal{D}_A \xi \) and using the orthogonality of \( L^2(S)^0 \) and \( L^2(S)^1 \), we obtain

\[
\langle \mathcal{J} \xi | \mathcal{D}_A \xi \rangle = \langle \mathcal{J}_M \psi | i \mathcal{D}_M \psi - m \psi + i \gamma^\mu A_\mu \psi \rangle_{\mathcal{J}_M}.
\]

**Remark 7.20.** Let us consider the above result for the usual case of a 4-dimensional Lorentzian manifold of signature \((1,3)\). We have \( \mathcal{J}_M = \gamma^0 \), and (using standard physics notation) we will write the (indefinite) inner product as \( \langle \psi | \phi \rangle = \int_M \overline{\psi} \phi \, d\text{vol}_g \), where \( \overline{\psi} = \psi^\dagger \gamma^0 \) is the Dirac adjoint of \( \psi \). We can then rewrite the Krein action for electrodynamics as

\[
S_{ED}[\psi, A] = \int_M \mathcal{L}_{ED}[\psi, A] \, d\text{vol}_g,
\]

where the Lagrangian for electrodynamics is given by

\[
\mathcal{L}_{ED}[\psi, A] := \overline{\psi} (i \gamma^\mu (\nabla_\mu + A_\mu) - m) \psi.
\]

This is indeed precisely the usual (fermionic part of the) Lagrangian for electrodynamics (compare, for instance, [PS95, §4.1]).
7.5 THE ELECTRO-WEAK THEORY

In this section we will describe the electro-weak interactions between leptons (i.e., neutrinos and electrons). The description given here is largely an adaptation of [DS12, §5].

Consider the finite-dimensional Hilbert space \( \mathcal{H}_F := \mathcal{H}_R \oplus \mathcal{H}_L \), where \( \mathcal{H}_R = \mathcal{H}_L = \mathbb{C}^2 \). This Hilbert space is \( \mathbb{Z}_2 \)-graded with even part \( \mathcal{H}_F^0 = \mathcal{H}_L \) and odd part \( \mathcal{H}_F^1 = \mathcal{H}_R \). We denote the basis of \( \mathcal{H}_F \) by \( \{ \nu_R, e_R, \nu_L, e_L \} \), where the elements \( \nu_R, \nu_L \) describe the (right- and left-handed) neutrinos, and \( e_R, e_L \) describe the electrons.

We consider the (real) algebra \( \mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \), along with two even (real-linear) representations \( \pi: \mathcal{A}_F \rightarrow \mathcal{B}(\mathcal{H}_R) \oplus \mathcal{B}(\mathcal{H}_L) \) and \( \pi^{op}: \mathcal{A}_F^{op} \rightarrow \mathcal{B}(\mathcal{H}_R) \oplus \mathcal{B}(\mathcal{H}_L) \) given by

\[
\pi(\lambda, q) := q\lambda + q := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \pi^{op}(\lambda, q)^{op} := \lambda \oplus q,
\]

for \( \lambda \in \mathbb{C} \) and \( q = \alpha + \beta j \in \mathbb{H} \). The representation \( \hat{\pi} := \pi \otimes \pi^{op} \) of \( \mathcal{A}_F \otimes \mathcal{A}_F^{op} \) on \( \mathcal{H}_R \oplus \mathcal{H}_L \) is then given by

\[
\hat{\pi}(\lambda, q) \otimes (\lambda', q')^{op} = \lambda'q\lambda + \lambda'q.
\]

We define the mass matrix on the basis \( \{ \nu_R, e_R, \nu_L, e_L \} \) as

\[
\mathcal{D}_F := \begin{pmatrix} 0 & 0 & -im_\nu & 0 \\ 0 & 0 & 0 & -im_e \\ im_\nu & 0 & 0 & 0 \\ 0 & im_e & 0 & 0 \end{pmatrix}.
\]

We then consider the even finite space \( \mathcal{F}_{EW} := (\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{D}_F) = 1 \).

**Proposition 7.21.** The gauge group of \( \mathcal{F}_{EW} \) equals \( \mathcal{S}(\mathcal{F}_{EW}) = (U(1) \times SU(2))/\mathbb{Z}_2 \).

**Proof.** We have \( \mathcal{U}(\mathcal{A}_F) = U(1) \times SU(2) \). The kernel of \( \rho = \pi \circ \Delta: \mathcal{U}(\mathcal{A}_F) \rightarrow \mathcal{B}(\mathcal{H}_F) \) equals \( \text{Ker} \rho = \{ \pm 1, \pm 1 \} \subset \mathcal{U}(\mathcal{A}_F) \) \( \cong \mathbb{Z}_2 \). The quotient \( \mathcal{S}(\mathcal{F}_{ED}) = \mathcal{U}(\mathcal{A}_F)/\text{Ker} \rho \) is thus given by \( (U(1) \times SU(2))/\mathbb{Z}_2 \). \( \square \)

The representations \( \pi \) and \( \pi^{op} \) obviously extend to representations of \( C^\infty_c(M, \mathcal{A}_F) \) and \( C^\infty_c(M, \mathcal{A}_F^{op}) \) on \( \mathcal{H}_F \otimes L^2(S) \), and it is easy to see that these representations satisfy the order-one condition (7.1). We consider the almost-commutative manifold

\[
\mathcal{F}_{EW} \times M := (C^\infty_c(M, \mathcal{A}_F \otimes \mathcal{A}_F^{op}), \mathcal{H}_F \otimes L^2(S) \otimes \mathcal{D}_F \otimes 1 \otimes \mathcal{D}_M) \).
\]
Proposition 7.22. The fluctuation of $D := 1 \otimes i^1 + iD_F \otimes 1$ by $A \in \text{Pert}(C_c^\infty(M, A_F))$ is of the form

$$D_A = D + \eta_D(A) = 1 \otimes i^1 + A_\mu \otimes i^1 \gamma^\mu + (iD_F + \phi) \otimes 1,$$

where the gauge field $A_\mu$ and the Higgs field $\phi$ are given by

$$A_\mu = \begin{pmatrix} 0 & 0 \\ 0 & -2\Lambda_\mu \\ Q_\mu - \Lambda_\mu 1_2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & m_\nu \Phi_1 & m_\nu \Phi_2 \\ 0 & 0 & -m_e \phi_2 & m_e \phi_1 \\ -m_\nu \phi_1 & m_e \Phi_2 & 0 & 0 \\ -m_\nu \phi_2 & -m_e \Phi_1 & 0 & 0 \end{pmatrix},$$

for the gauge fields $(\Lambda_\mu, Q_\mu) \in C_c^\infty(M, i\mathbb{R} \oplus \mathfrak{su}(2))$ and the Higgs field $(\phi_1, \phi_2) \in C_c^\infty(M, \mathbb{C}^2)$.

Proof. Write $A = \sum_j a_j \otimes b_j^{op} = \sum_j (\lambda_j, q_j) \otimes (\lambda'_j, q'_j)^{op} \in \text{Pert}(C_c^\infty(M, A_F))$. By Proposition 7.8 the fluctuation looks like

$$\eta_D(A) = a_j [D, b_j] + a_j^{op} [D, b_j^{op}],$$

where $D := 1 \otimes i^1 + iD_F \otimes 1$. The below calculations are similar to those in [DS12, §5.2.2], and therefore we shall be brief. From the commutators with $iD_F$, we obtain the Higgs field

$$\phi := \sum_j a_j [iD_F, b_j] = \begin{pmatrix} 0 & 0 & m_\nu \phi_1' & m_\nu \phi_2' \\ 0 & 0 & -m_e \Phi_2' & m_e \Phi_1' \\ -m_\nu \phi_1 & m_e \Phi_2 & 0 & 0 \\ -m_\nu \phi_2 & -m_e \Phi_1 & 0 & 0 \end{pmatrix},$$

where we define

$$\phi_1 = \sum_j \alpha_j (\lambda'_j - \alpha_j') + \beta_j \beta'_j, \quad \phi_2 = \sum_j \alpha_j \beta'_j - \beta_j (\lambda'_j - \alpha_j'),$$

$$\phi_1' = \sum_j \lambda_j (\alpha_j - \lambda'_j), \quad \phi_2' = \sum_j \lambda_j \beta'_j.$$

We also observe that $D_F$ commutes with $\pi^{op}$, so $a_j^{op} [iD_F, b_j^{op}] = 0$. We note that self-adjointness of the perturbation $A$ ensures Krein-self-adjointness of $\phi$, and therefore self-adjointness of $i\phi$ (since $\phi$ anti-commutes with $J$). Hence we must have $\phi_1' = \Phi_1$ and $\phi_2' = \Phi_2$. Furthermore, we can write

$$\sum_j a_j [i^1 D, b_j] = \begin{pmatrix} \Lambda_\mu & 0 \\ 0 & -\Lambda_\mu \\ Q_\mu \end{pmatrix} \otimes i^1 \gamma^\mu, \quad \sum_j a_j^{op} [i^1 D, b_j^{op}] = -\Lambda_\mu 1_4 \otimes i^1 \gamma^\mu.$$
where the gauge field $\Lambda_\mu$ is given by
\[
\Lambda_\mu := \begin{pmatrix}
0 & 0 \\
0 & -2\Lambda_\mu \\
Q_\mu & -\Lambda_\mu 1_2
\end{pmatrix}.
\]

The almost-commutative manifold $F_{\text{EW}} \times M$ is of Lorentz-type, and hence we can apply Definition 7.5 for the Krein action. An arbitrary vector $\xi \in \mathcal{H}^0 = \mathcal{H}_L \otimes L^2(S)^0 \oplus \mathcal{H}_R \otimes L^2(S)^1$ can be written uniquely as
\[
\xi = \nu_R \otimes \psi_R^\nu + e_R \otimes \psi_R^e + \nu_L \otimes \psi_L^\nu + e_L \otimes \psi_L^e,
\]
for Weyl spinors $\psi_L^\nu, \psi_L^e \in L^2(S)^0$ and $\psi_R^\nu, \psi_R^e \in L^2(S)^1$. We observe that this vector $\xi \in \mathcal{H}^0$ is now completely determined by two Dirac spinors $\psi^\nu := \psi_L^\nu + \psi_R^\nu$ (describing the neutrino) and $\psi^e := \psi_L^e + \psi_R^e$ (describing the electron). We combine these spinors into the doublets of Weyl spinors
\[
\Psi_L := \begin{pmatrix}
\psi_L^\nu \\
\psi_L^e
\end{pmatrix} \in L^2(S)^0 \otimes \mathbb{C}^2,
\Psi_R := \begin{pmatrix}
\psi_R^\nu \\
\psi_R^e
\end{pmatrix} \in L^2(S)^1 \otimes \mathbb{C}^2,
\]
and the corresponding doublet of Dirac spinors $\Psi := \Psi_L + \Psi_R \in L^2(S) \otimes \mathbb{C}^2$.

**Proposition 7.23.** The Krein action for $F_{\text{EW}} \times M$ is given by
\[
S_{\text{EW}}[\Psi, A] = \langle \Psi | i^1D\Psi \rangle + \langle \psi_R^\nu | -2i^1\gamma^\mu A_\mu \psi_R^\nu \rangle + \langle \Psi_L | i^1\gamma^\mu (Q_\mu - A_\mu) \Psi_L \rangle + \langle \Psi_R | \Phi \Psi_L \rangle + \langle \Psi_L | \Phi^* \Psi_R \rangle,
\]
where the gauge fields $A_\mu$ and $Q_\mu$ and the Higgs field $(\phi_1, \phi_2)$ are given in Proposition 7.22, and the Higgs field $(\phi_1, \phi_2)$ acts via
\[
\Phi := \begin{pmatrix}
-m_\nu(\phi_1 + 1) & -m_\nu \phi_2 \\
m_\nu \phi_2 & -m_e(\phi_1 + 1)
\end{pmatrix}.
\]

**Proof.** We need to calculate the inner product $\langle J\xi, D_A \xi \rangle_J$, where $\xi$ is given as in Eq. (7.5). First, for $J = 1 \otimes J_M$ we find
\[
J\xi = -\nu_R \otimes J_M \psi_R^\nu - e_R \otimes J_M \psi_R^e + \nu_L \otimes J_M \psi_L^\nu + e_L \otimes J_M \psi_L^e.
\]

For the fluctuated Dirac operator $D_A$ of Proposition 7.22 we find
\[
D_A \xi = -\nu_R \otimes i^1D\psi_R^\nu - e_R \otimes i^1D\psi_R^e + \nu_L \otimes i^1D\psi_L^\nu + e_L \otimes i^1D\psi_L^e.
\]
\[+ 2\lambda_\mu e_R \otimes i^\gamma^\mu \psi_R^\xi + i^\gamma^\mu (Q_\mu - \Lambda_\mu) (\nu_L \otimes \psi_R^\xi, e_L \otimes \psi_L^\xi) \]

\[-\nu_L \otimes (m_\nu \phi_1 + 1) \phi_2 \psi_R^\xi - e_L \otimes (m_\nu \phi_2 \psi_R^\xi) - e_R \otimes (m_\nu (\phi_1 + 1) \psi_R^\xi)\]

\[+ \nu_R \otimes (m_\nu \phi_1 + 1) \phi_2 \psi_L^\xi - e_R \otimes (m_\nu \phi_2 \psi_L^\xi)\].

Taking the inner product of \( \hat{J} \xi \) with \( \mathcal{D}_A \xi \), and using the notation \( \Psi_L, \Psi_R, \) and \( \Phi \), we obtain

\[\langle \hat{J} \xi | \mathcal{D}_A \xi \rangle = \langle \hat{J}_M \psi_R^\xi | i^\gamma^\mu \psi_R^\xi \rangle_{\hat{J}_M} + \langle \hat{J}_M \psi_L^\xi | i^\gamma^\mu \psi_L^\xi \rangle_{\hat{J}_M}\]

\[+ \langle \hat{J}_M \psi_R^\xi | i^\gamma^\mu \psi_L^\xi \rangle_{\hat{J}_M} + \langle \hat{J}_M \psi_L^\xi | i^\gamma^\mu \psi_R^\xi \rangle_{\hat{J}_M}\]

The desired expression for \( S_{\text{EW}}[\Psi, A] = \langle \xi | \mathcal{D}_A \xi \rangle = \langle \hat{J} \xi | \mathcal{D}_A \xi \rangle \) then follows by using the orthogonality of \( L^2(S)^0 \) and \( L^2(S)^1 \) and the symmetry of \( \langle \cdot | \cdot \rangle \).

We observe that the Lagrangian calculated above is precisely the (fermionic part of) the usual Lagrangian for the lepton sector of the Glashow-Weinberg-Salam theory of electroweak interactions, including right-handed neutrinos (but without Majorana masses). For instance, the term \( \langle \Psi_L | \Phi^* \Psi_R \rangle \) can be rewritten in the form

\[-m_\nu \begin{pmatrix} \phi_1 + 1 \\ \phi_2 \end{pmatrix} \psi_R^\xi - m_e \begin{pmatrix} -\phi_2 \\ \phi_1 + 1 \end{pmatrix} \psi_L^\xi,\]

which is of the same form as [PS95, Eq.(20.101)] (though there it is given for quarks instead of leptons). If we substitute the vacuum expectation value for the Higgs field, setting \( \phi_1 + 1 = \nu/\sqrt{2} \) and \( \phi_2 = 0 \), we obtain

\[-\frac{1}{\sqrt{2}} \nu \left(m_\nu \langle \psi_L^\xi | \psi_R^\xi \rangle + m_e \langle \psi_L^\xi | \psi_R^\xi \rangle\right),\]

which are indeed the standard mass terms for the neutrino and the electron.

### 7.5.1 Majorana masses

Let us briefly discuss how one can add Majorana masses into the model as well. For this purpose, we double up the Hilbert space, and introduce a real structure (see Definition 6.13). Given the Hilbert space \( \mathcal{H}_F \) with basis \( \{v_R, e_R, v_L, e_L\} \), we create an identical copy \( \mathcal{H}_F \) on which we denote the basis as \( \{\overline{v_R}, \overline{e_R}, \overline{v_L}, \overline{e}_L\} \), and we interpret this new copy as describing the anti-particles. We then consider the new Hilbert space \( \mathcal{H}_F := \mathcal{H}_F \oplus \mathcal{H}_F \) along with a mass matrix \( \mathcal{D}_F \), a ‘fundamental symmetry’ \( \hat{J}_F \), a grading \( \hat{f}_F \), and the real structure \( \hat{J}_F \) given by

\[\mathcal{D}_F := \begin{pmatrix} \mathcal{D}_F & -\mathcal{D}_F^* \\ \mathcal{D}_F & \mathcal{D}_F \end{pmatrix}, \quad \hat{J}_F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{f}_F := \begin{pmatrix} \Gamma_F & 0 \\ 0 & -\Gamma_F \end{pmatrix}, \quad \hat{J}_F := \begin{pmatrix} 0 & \text{c.c.} \\ \text{c.c.} & 0 \end{pmatrix},\]

where \( \text{c.c.} \) denotes the complex conjugate.
where c.c. denotes complex conjugation (with respect to the standard basis), and \( \overline{D}_F \) is the complex conjugate of the mass matrix \( D_F \). The map \( D_M : \mathcal{H}_F \rightarrow \mathcal{H}_F \) is given as \( D_M \psi_R := i m_R \overline{\psi}_R \), where \( m_R \in \mathbb{R} \) is the Majorana mass of the right-handed neutrino, and \( D_M \psi_L = D_M \psi_L = 0 \). We point out that \( \hat{D}_F \) is Krein-self-adjoint, and that \( \hat{J}_F \) anti-commutes with both \( \hat{J}_F \) and \( \hat{F}_F \).

Recalling the representations \( \pi : A_F \rightarrow B(\mathcal{H}_F) \) and \( \pi \circ \pi : A_F \circ A_F \rightarrow B(\mathcal{H}_F \otimes \mathcal{H}_F) \), we obtain representations \( \hat{\pi} : A_F \rightarrow B(\mathcal{H}_F \oplus \mathcal{H}_F) \) and \( \hat{\pi} \circ \hat{\pi} : A_F \circ A_F \rightarrow B(\mathcal{H}_F \oplus \mathcal{H}_F) \) given by

\[
\hat{\pi}(a) := \pi(a) \oplus \pi(a^\dagger), \quad \hat{\pi}(a) := \hat{J}_F \hat{\pi}(a^\ast) \hat{J}_F,
\]

where \( a^\dagger \) denotes the matrix transpose of \( a \). With these definitions, we obtain a new finite space \( \hat{F} := (A_F \circ A_F, \hat{\mathcal{H}}_F, \hat{D}_F) \) with grading operator \( \hat{F}_F \) and with a real structure \( \hat{J}_F \).

Now consider a 4-dimensional Lorentzian spin manifold \( M \). We also equip the Krein spectral triple over \( M \) (given in Proposition 7.3) with a real structure, given by the **charge conjugation operator** \( J_M \) on the spinor bundle. The charge conjugation operator commutes with the Clifford representation and with the Dirac operator \( D \), anti-commutes with the grading operator \( \Gamma_M \), and satisfies \( \hat{J}_M = -1 \) [Bau94, Proposition 3].

Next, we can consider the almost-commutative manifold \( \hat{F} \times M \), which we equip with the real structure \( J := \hat{J}_F \otimes J_M \). Since we have doubled the finite-dimensional Hilbert space, we have introduced too many degrees of freedom. To correct this, we consider vectors \( \eta \in \mathcal{H}_F \) which (in addition to \( \Gamma \eta = \eta \)) also satisfy \( J_\eta = \eta \) (as is also assumed in [Baro75]). Since \( J^2 = 1 \), this assumption makes sense, and it means we can write \( \eta = \xi + J_\xi \), where \( \xi \) is an element of \( (\mathcal{H}_F \otimes L^2(S))^0 \) as given in Eq. (7.5). The fermionic action is then of the form

\[
\langle \eta | D A \eta \rangle \beta = \langle \xi | D A \xi \rangle \beta + \langle \xi | D A \xi \rangle \beta + \langle \xi | D A J \xi \rangle \beta + \langle \xi | D A J \xi \rangle \beta.
\]

Since \( J \) commutes with \( D_A \) and with \( \beta \), we find that \( \langle \beta | D A J \xi \rangle \beta = \langle \beta | D A J \xi \rangle \beta \). Hence the only new contributions to the fermionic action come from \( \langle \beta | D A J \xi \rangle \beta \). The subspaces \( \mathcal{H}_F \otimes L^2(S) \) and \( \mathcal{H}_F \otimes L^2(S) \) are orthogonal, so we only need to consider the part of \( D_A \) which mixes particles and anti-particles, which is precisely just the Majorana mass matrix \( D_M \). For the vector \( \xi_R^\gamma := \nu_R \otimes \psi_R^\gamma \) representing the right-handed neutrino, we calculate

\[
J_\xi_R^\gamma = -\hat{J}_F \nu_R \otimes J_M \psi_R^\gamma = -\overline{\nu}_R \otimes J_M \psi_R^\gamma,
\]

\[
J \xi_R^\gamma = -\hat{J} (\nu_R \otimes J_M \psi_R^\gamma) = -\hat{J}_F \nu_R \otimes J_M \psi_R^\gamma = \overline{\nu}_R \otimes J_M \psi_R^\gamma,
\]

\[
(i D_M \otimes 1) \xi_R^\gamma = -m_R \overline{\nu}_R \otimes \psi_R^\gamma,
\]

\[
(-i D_M \otimes 1) \xi_R^\gamma = m_R \nu_R \otimes J_M \psi_R^\gamma.
\]
This gives
\[
\langle \beta \xi | D_A \xi \rangle_{\beta} = \langle -v_R \otimes J_M \psi_R^\gamma | m_R v_R \otimes J_M \psi_R^\gamma \rangle_{\beta} = -m_R \langle J_M \psi_R^\gamma | J_M \psi_R^\gamma \rangle_{\beta},
\]
\[
\langle \beta | D_A \xi \rangle_{\beta} = \langle v_R \otimes J_M \psi_R^\gamma | -m_R v_R \otimes \psi_R^\gamma \rangle_{\beta} = -m_R \langle J_M | \psi_R^\gamma | \psi_R^\gamma \rangle_{\beta}.
\]
Summarising, we can extend the electro-weak theory to include Majorana masses for right-handed neutrinos, and we obtain the new action \( S_{\text{EW}+M} \) given by
\[
S_{\text{EW}+M}[\Psi, A] = 2S_{\text{EW}}[\Psi, A] - m_R \langle \psi_R^\gamma | J_M \psi_R^\gamma \rangle - m_R \langle J_M \psi_R^\gamma | \psi_R^\gamma \rangle,
\]
where \( S_{\text{EW}} \) is given in Proposition 7.23.

7.6 The Standard Model

Given the description of the electro-weak theory of the previous section, it is fairly straightforward to extend this theory to the full Standard Model as described in [Cono6, CCM07]. This extension is basically obtained by including a summand \( M_3(C) \) in the algebra \( A_F \) to describe the strong interactions, and by enlarging the Hilbert space \( \mathcal{H}_F \) to incorporate the quarks. Moreover, the Hilbert space is then enlarged three-fold to include three generations of all elementary particles. Since most of the details are similar to the electro-weak theory, and since there is already plenty of literature available on the noncommutative description of the Standard Model (see e.g. [Con96, Con06, CCM07, CM07, JKSS07, CC10, DS12]), we shall be rather brief in this section.

Thus, we take the algebra \( A_F = C \oplus \mathcal{H} \oplus M_3(C) \), which is represented on the finite-dimensional Hilbert space \( \mathcal{H}_F := (\mathcal{H}_R \oplus \mathcal{H}_L) \otimes C^3 \). The factor \( C^3 \) describes the fact that there are three generations of elementary particles. The righthanded particles \( \mathcal{H}_R \) and the left-handed particles \( \mathcal{H}_L \) are both given by \( C^2 \oplus (C^2 \otimes C^3) \). Here the first summand \( C^2 \) describes the two leptons \( \nu \) and \( e \), and the second summand \( C^2 \otimes C^3 \) describes the quarks \( u^c \) and \( d^c \) (which occur in three colours \( c = r, q, b \)).

We will consider the commuting representations \( \pi: A_F \rightarrow B((\mathcal{H}_R \oplus \mathcal{H}_L) \otimes C^3) \) and \( \pi^\text{op}: A_F^{\text{op}} \rightarrow B((\mathcal{H}_R \oplus \mathcal{H}_L) \otimes C^3) \) given by
\[
\pi(\lambda, q, b) := (\{ q \lambda \oplus (q \lambda \otimes 1_3) \} \oplus \{ q \otimes (q \otimes 1_3) \}) \otimes 1_3,
\]
\[
\pi^\text{op}(\lambda, q, b) := ((\lambda 1_2 \oplus (1_2 \otimes b^t)) \oplus (\lambda 1_2 \oplus (1_2 \otimes b^t))) \otimes 1_3,
\]
where \( 1_N \) denotes the identity matrix acting on \( C^N \), and \( b^t \) denotes the matrix transpose of \( b \). The representation \( \hat{\pi} := \pi \otimes \pi^\text{op}: A_F \otimes A_F^{\text{op}} \rightarrow B(\mathcal{H}_F) \) is then given by
\[
\hat{\pi}((\lambda, q, b) \otimes (\lambda', q', b')^{\text{op}}) = ((\lambda' q \lambda \oplus (q \lambda \otimes b'^t)) \oplus (\lambda' q \otimes (q \otimes b'^t))) \otimes 1_3.
\]
We consider the even finite space $F_{SM} := (A_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F = 1)$, where the mass matrix is given by

$$\mathcal{D}_F := \begin{pmatrix} 0 & 0 & -iY_v & 0 \\ 0 & 0 & 0 & -iY_e \\ iY_v & 0 & 0 & 0 \\ iY_e & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & -iY_u & 0 \\ 0 & 0 & 0 & -iY_d \\ iY_u & 0 & 0 & 0 \\ 0 & iY_d & 0 & 0 \end{pmatrix} \otimes 1_3.$$ 

Here each $Y_\mu$ is a hermitian $3 \times 3$-matrix corresponding to the three generations of each type of particle.

Similarly to Proposition 7.21, the gauge group of the finite space $F_{SM}$ is given by $G(F_{SM}) = (\mathbb{U}(1) \times \mathbb{SU}(2) \times \mathbb{SU}(3))/\mathbb{Z}_2$. This gauge group does not match the gauge group of the Standard Model (even modulo finite groups), since we have a factor $\mathbb{U}(3)$ instead of $\mathbb{SU}(3)$. As in [CCM07, §2.5] (see also [DS12, §6.2.1]), we will therefore impose the unimodularity condition $\det|_{\mathcal{J}_F}(\rho(u)) = 1$, which yields the subgroup

$$SG(F_{SM}) = \{ \rho(u) \in G(F_{SM}) : u = (\lambda, q, b) \in U(A_F), (\lambda \det b)^{12} = 1 \}.$$ 

The effect of the unimodularity condition is that the determinant of $b \in U(3)$ is identified (modulo the finite group $\mu_{12}$ of 12th-roots of unity) to $\bar{x} \in U(1)$. In other words, imposing the unimodularity condition provides us, modulo some finite abelian group, with the gauge group $U(1) \times SU(2) \times SU(3)$ of the Standard Model.

The calculations for the inner fluctuations and the fermionic action of the Standard Model are similar to the case of the electro-weak theory (although somewhat more cumbersome). Below we will simply give the results.

**Proposition 7.24.** The fluctuation of $\mathcal{D}$ by $A \in \text{Pert}(C^\infty(M, A_F))$ is of the form

$$\mathcal{D}_A = \mathcal{D} + \eta_2(A) = 1 \otimes \mathcal{D} + A_\mu \otimes i^4 \gamma^\mu + (i\mathcal{D}_F + \phi) \otimes 1,$$

where the gauge field $A_\mu$ and the Higgs field $\phi$ are given by

$$A_\mu = \begin{pmatrix} 0 & 0 & \Lambda_{2\mu} \\ 0 & -2\Lambda_{1\mu} \\ Q_{1\mu} - \Lambda_{1\mu} \end{pmatrix} \oplus \begin{pmatrix} \frac{4}{3}\Lambda_{1\mu}1_3 \\ 0 \\ -\frac{2}{3}\Lambda_{1\mu}1_3 \\ (Q_\mu + \frac{1}{3}\Lambda_{1\mu}) \otimes 1_3 \end{pmatrix} + 1_4 \otimes \gamma_\mu,$$

$$\phi = \begin{pmatrix} 0 & 0 & Y_v\phi_1 & Y_v\phi_2 \\ 0 & 0 & -Y_e\phi_2 & Y_e\phi_1 \\ -Y_v\phi_1 & Y_e\phi_2 & 0 & 0 \\ -Y_v\phi_2 & -Y_e\phi_1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & Y_u\phi_1 & Y_u\phi_2 \\ 0 & 0 & -Y_d\phi_2 & Y_d\phi_1 \\ -Y_u\phi_1 & Y_d\phi_2 & 0 & 0 \\ -Y_u\phi_2 & -Y_d\phi_1 & 0 & 0 \end{pmatrix} \otimes 1_3.$$
for the gauge fields \((\Lambda_\mu, Q_\mu, V_\mu) \in C^\infty_c(M, iR \oplus su(2) \oplus su(3))\) and the Higgs field \((\phi_1, \phi_2) \in C^\infty_c(M, C^2)\).

Similarly to Eq. (7.5), an arbitrary vector \(\xi \in \mathcal{H}^0 = \mathcal{H}_L \otimes L^2(S)^0 \oplus \mathcal{H}_R \otimes L^2(S)^1\) is uniquely determined by Dirac spinors \(\psi^\nu\) (describing the three neutrinos), \(\psi^e\) (describing the electron, muon, and tau-particle), \(\psi^u\) (describing the up, charm, and top quarks in three colours) and \(\psi^d\) (describing the down, strange, and bottom quarks in three colours), where we have omitted the generational index from our notation. We group these spinors together into the multiplets \(\Psi^l \in L^2(S) \otimes C^2 \otimes C^3\) (describing the leptons) and \(\Psi^q \in L^2(S) \otimes C^2 \otimes C^3 \otimes C^3\) (describing the quarks).

**Proposition 7.25.** The Krein action for \(F_{SM} \times M\) is given by

\[
S_{EW}[\Psi, A] = \langle \psi^l | i^t \mathcal{D} \psi^l \rangle + \langle \psi^q | i^t \mathcal{D} \psi^q \rangle \\
+ \langle \psi^e_R | -2i^t \gamma^\mu \Lambda_\mu \psi^e_L \rangle + \langle \psi^u_R | \frac{2}{3} i^t \gamma^\mu \Lambda_\mu \psi^u_L \rangle + \langle \psi^d_R | -\frac{2}{3} i^t \gamma^\mu \Lambda_\mu \psi^d_L \rangle \\
+ \langle \psi^l_L | i^t \gamma^\mu (Q_\mu - \Lambda_\mu) \psi^l_R \rangle + \langle \psi^q_L | i^t \gamma^\mu (Q_\mu - \Lambda_\mu) \psi^q_R \rangle + \langle \psi^q | V_\mu \psi^q \rangle \\
+ \langle \psi^l_R | \Phi^l \psi^l \rangle + \langle \psi^q_L | \Phi^q \psi^q \rangle + \langle \psi^q | \Phi^q \psi^q \rangle + \langle \psi^l | (\Phi^l)^\dagger \psi^l \rangle,
\]

where the gauge fields \(\Lambda_\mu, Q_\mu, \text{ and } V_\mu\) and the Higgs field \((\phi_1, \phi_2)\) are given in Proposition 7.24, and the Higgs field acts via

\[
\Phi^l := \begin{pmatrix} -Y_\nu (\phi_1 + 1) & -Y_\nu \phi_2 \\ Y_e \phi_2 & -Y_e (\phi_1 + 1) \end{pmatrix}, \quad \Phi^q := \begin{pmatrix} -Y_u (\phi_1 + 1) & -Y_u \phi_2 \\ Y_d \phi_2 & -Y_d (\phi_1 + 1) \end{pmatrix}.
\]

We observe that the Lagrangian calculated above is precisely the (fermionic part of) the usual Lagrangian for the Standard Model, including right-handed neutrinos (but without Majorana masses). For the possible inclusion of Majorana masses for the right-handed neutrinos, the procedure is the same as in Section 7.5.1, and we shall not repeat it here.
OUTLOOK

In this Outlook we will describe a few open questions arising from the work in this thesis, and we list a few possible directions for further research.

8.1 almost anti-commuting operators

Our definition of an indefinite Kasparov module \((A, E_B, D)\) in Section 5.1 uses the assumption that \(\text{Re } D\) and \(\text{Im } D\) almost anti-commute. The main reason for requiring this assumption is that it allows us to use a theorem by Kaad and Lesch [KL12] (quoted in Theorem 2.22) to conclude that the Wick rotations \(D_\pm = \text{Re } D \pm \text{Im } D\) are self-adjoint.

Unfortunately, as we have seen in Section 5.3, the canonical Dirac operator \(\mathcal{D}\) on a pseudo-Riemannian manifold in general does not satisfy this definition. Indeed, although the anti-commutator \(\{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}\}\) is a first-order differential operator, it contains in general both spacelike derivatives and timelike derivatives, and thus it is not relatively bounded by \(\text{Re } \mathcal{D}\). The failure of this condition for the Dirac operator \(\mathcal{D}\) however does not prevent us from proving that the Wick rotations \(\mathcal{D}_\pm\) are self-adjoint. Indeed, we can simply prove this directly, as we did in Proposition 5.18.

In order to ensure that \(\text{Re } \mathcal{D}\) and \(\text{Im } \mathcal{D}\) almost anti-commute, we need the timelike part of \(\{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}\}\) to vanish identically. This asymmetry between the timelike and spacelike parts of \(\{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}\}\) is artificial, and indicates that it would be desirable to have a more general version of Kaad and Lesch’ theorem. A more natural condition would be to assume that the anti-commutator \(\{\text{Re } \mathcal{D}, \text{Im } \mathcal{D}\}\) is ‘first-order’ (as compared to \(\text{Re } \mathcal{D}\) and \(\text{Im } \mathcal{D}\)). It is currently unclear whether such a weaker condition could still suffice to prove self-adjointness of the Wick rotations \(\mathcal{D}_\pm\).

We hope to address this issue in a future work. Apart from applying this to our framework of indefinite Kasparov modules and their Wick rotations, such a generalisation of Kaad and Lesch’ theorem could also play an important role in the construction of the internal unbounded Kasparov product.
8.2 Families of Spectral Triples and Foliations of Spacetime

In Section 5.5 we have described the construction of an indefinite spectral triple from a family of spectral triples parametrised by the real line. Subsequently, we have shown in Section 5.5.4 that a generalised Lorentzian cylinder provides an example of this construction. There, we described a family of Dirac operators \( \{\mathcal{D}(t)\}_{t \in \mathbb{R}} \) on a Riemannian spin manifold \( \Sigma \), giving rise to a Lorentzian Dirac operator on the Lorentzian manifold \( \Sigma \times \mathbb{R} \). We have seen in the proof of Proposition 5.40 that \( \mathcal{D}(t) - \mathcal{D}(0) \) is a bounded operator. Our definition of families of spectral triples however allows to deal with relatively bounded operators. This suggests that it should be possible to describe more general foliations of Lorentzian spacetimes as families of spectral triples.

8.3 The Lorentzian Spectral Action

As we have seen in Chapters 6 and 7, almost-commutative manifolds can be used to describe gauge theories and thus models in particle physics. The construction of almost-commutative manifolds still works if the underlying manifold is Lorentzian instead of Riemannian. The main challenge for obtaining a complete description of a gauge theory on a Lorentzian manifold in this way, is the formulation of a suitable action functional, from which we can obtain the Lagrangian (and hence the equations of motion) of the theory.

In the Riemannian case, this Lagrangian is obtained from the spectral action [CC97] and the fermionic action [Cono6]. We have already shown in Chapter 7 how the fermionic action should be adapted to the Lorentzian setting. The formulation of a Lorentzian version of the spectral action appears to be more complicated.

The spectral action (or in particular, the derivation of the Standard Model Lagrangian from the spectral action) relies heavily on heat kernel techniques. We expect that in the Lorentzian case it might be better to replace the heat kernel of a Laplace-type operator by the ‘Schrödinger kernel’ of a normally hyperbolic operator. A formal expansion for this kernel has already been used by Schwinger [Sch51] and DeWitt [DeW65] to study the renormalisation of the Feynman propagator. The coefficients appearing in this expansion, called the Hadamard coefficients, formally have the same expressions as the heat kernel coefficients of a Laplace-type operator, and it therefore seems plausible that the Standard Model Lagrangian (but now in Lorentzian signature) can also be derived from the Hadamard coefficients.

These Hadamard coefficients also appear in the asymptotic expansion of the fundamental solutions of a normally hyperbolic operator [BGP07]. In a joint research project with Michal Eckstein (Jagiellonian University) and Christoph Stephan (University of Potsdam), we hope to employ these Hadamard coefficients to derive the
8.4 Manifolds with Boundary

In order for a Riemannian spin manifold to give rise to a spectral triple, the manifold needs to be complete, which in particular means it has no boundary. In order for a Lorentzian manifold to satisfy the definitions of ‘Lorentzian spectral triples’ we have given in Chapters 4 and 5, we needed to impose a similar assumption: the Riemannian metric obtained via Wick rotation needs to be complete.

However, not every physically reasonable spacetime has a complete Riemannian metric associated to it by our Wick rotation procedure. Hence, to deal with general Lorentzian manifolds, one would have to be able to deal with Riemannian manifolds with boundary. This requires careful consideration of appropriate boundary conditions, and we refer to [BDT89] for a comprehensive discussion of boundary conditions in K-homology and [IL11] for a definition of ‘spectral triple with boundary’.

8.5 Globally Non-Trivial Almost-Commutative Manifolds

One of the main ideas in the development of noncommutative geometry has been the translation of geometric data into (operator-)algebraic data. In this light, it is somewhat unsatisfactory that our definition of principal modules (see Section 6.3.2) relies entirely on the geometric notion of a principal bundle. Our discussion of gauge modules is an attempt to provide a purely algebraic approach, but as we have shown, these gauge modules only yield a proper subclass of principal modules. It is still an open question how arbitrary principal modules should be described algebraically, that is, what algebraic structure on a triplet $(\mathcal{B}, \mathcal{E}, J)$ would completely characterise the properties of a principal module. The decompositions $\mathcal{E} = \bigoplus_{i,j \in I} \mathcal{E}_{ij}$ and $\mathcal{B} = \bigoplus \mathcal{B}_i$ (as described in Section 6.3.3) are not yet enough to ensure that $(\mathcal{B}, \mathcal{E}, J)$ is a principal module. On the other hand, the condition that $\mathcal{E}_{ij} = \mathcal{E}_i \otimes A \mathcal{E}_j$ (modulo multiplicities) along with $\mathcal{B}_i = \text{End}(\mathcal{E}_i)$, as for gauge modules, is in fact too strong.

In Section 6.6 we described two basic examples, namely Yang-Mills theory and electrodynamics. It would of course be more interesting to also put the description of the noncommutative Standard Model [CCM07] into our globally non-trivial framework. This should certainly be possible, though it would require some small modifications to accommodate real algebras (in Chapter 6 we have always as-
sumed that our algebras are complex). In particular, for real algebras the resulting gauge group would not automatically be unimodular (see also Remark 6.39), and one would have to impose unimodularity by hand (as in Section 7.6). More importantly, as we also mentioned in Remarks 6.15 and 6.62, in our framework the mass parameters (i.e. the Yukawa couplings and the Majorana terms) of the theory are not restricted to be constant, but they are allowed to vary on spacetime. Such variation of the Majorana mass then naturally leads to a new scalar field $\sigma$, which was used in [CC12] to restore the consistency of the noncommutative Standard Model with the experimental value of the Higgs mass. In addition however, the variation of the Yukawa couplings will also have its effect on the physical theory. We hope to provide a more detailed study of these physical implications in a future work.
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169


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