Mixed Quantum-Classical Linear Systems Synthesis and Quantum Feedback Control Designs

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Declaration

This thesis contains no material which has been accepted for the award of any other degree of diploma in an university.

I hereby certify that some work embodied in this thesis has been done in collaboration with other researchers, but I am primarily responsible for the contributions.

Shi Wang

Refereed Papers

1. **Shi Wang**, H. I. Nurdin, G. Zhang, and M. R. James, "Quantum optical realization of classical linear stochastic systems," *Automatica*, 49(10), pp. 3090-3096, 2013.

2. **Shi Wang**, M. R. James, "Extended LMI approach to coherent quantum LQG control Design," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 1265-1270, Italy, 2013.

3. **Shi Wang**, H. I. Nurdin, G. Zhang, and M. R. James, "Synthesis and structure of mixed quantum-classical linear systems," in *Proceedings of the 51st IEEE Conference on Decision and Control*, pp. 1093-1098, Maui, Hawaii, USA, 2012.

4. **Shi Wang**, H. I. Nurdin, G. Zhang and M. R. James, "Implementation of classical linear stochastic systems using quantum optical components," in *Proceedings of the 2011 Australian Control Conference (AUCC), Engineers Australia, Australia*, pp. 351-356, 2011.

5. **Shi Wang**, M. R. James, "H^∞ Control of quantum feedback control systems with feedback-loop time delay," submitted to *IEEE Trans. Automat. Contr.*.

6. **Shi Wang**, M. R. James, "Optimal controller design for quantum measurement-based feedback control systems with feedback-loop time delay," submitted to *Automatica.*

7. **Shi Wang**, H. I. Nurdin, G. Zhang, and M. R. James, "Network synthesis problems for mixed quantum-classical linear systems," submitted to *IEEE Trans. Automat. Contr.*.

To my parents

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Abstract

This thesis makes some theoretical contributions towards mixed quantum feedback network synthesis, quantum optical realization of classical linear stochastic systems and quantum feedback control designs.

A mixed quantum-classical feedback network is an interconnected system consisting of a quantum system and a classical system connected by interfaces that convert quantum signals to classical signal (using homodyne detectors), and vice-versa (using electro-optic modulators). In the area of mixed quantum-classical feedback networks, we present a network synthesis theory, which provides a natural framework for analysis and design for mixed linear systems. Physical realizability conditions are derived for linear stochastic differential equations to ensure that mixed systems can correspond to physical systems. The mixed network synthesis theory developed based on physical realizability conditions shows that how a class of mixed quantum-classical systems described by linear stochastic differential equations can be built as an interconnection of linear quantum systems and linear classical systems using quantum optical devices as well as electrical and electric devices.

However, an important practical problem for the implementation of mixed quantum-classical systems is the relatively slow speed of classical parts implemented with standard electrical and electronic devices, since a mixed system will not work correctly unless the electronic processing of classical devices is fast enough. Therefore, another interesting work is to show how classical linear stochastic systems built using electrical and electric devices can be physically implemented using quantum optical components. A complete procedure is proposed for a stable quantum linear stochastic system realizing a given stable classical linear stochastic system. The thesis also explains how it may be possible to realize certain measurement-based feedback control loops fully at the quantum level.

In the area of quantum feedback control design, two numerical procedures based on extended linear matrix inequality (LMI) approach are proposed to de-

sign a coherent quantum controller in this thesis. The extended synthesis linear matrix inequalities are, in addition to new analysis tools, less conservative in comparison to the conventional counterparts since the optimization variables related to the system parameters in extended LMIs are independent of the symmetric Lyapunov matrix. These features may be useful in the optimal design of quantum optical networks. Time delays are frequently encountered in linear quantum feedback control systems such as long transmission lines between quantum plants and linear controllers, which may have an effect on the performance of closed-loop plant-controller systems. Therefore, this thesis investigates the problem of linear quantum measurement-based feedback control systems subject to feedback-loop time delay described by linear stochastic differential equations. Several numerical procedures are proposed to design classical controllers which make quantum measurement-based feedback control systems with time delay stable and also guarantee that their desired control performance specifications are satisfied.

Contents

List of Notation

Notation

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of natural numbers
^{T}	Transpose of a matrix/array
*	The adjoint of a Hilbert space operator
$\bigotimes$	Tensor product
$\langle\cdot\rangle$	Expectation
A^*	The conjugate transpose of a complex matrix A
$X^\#$	If $X = [x_{jk}]$ is a matrix of linear operators or complex numbers, then $X^\# = [x_{jk}^*]$ denotes the operation of taking the adjoint of each element of X
$X^\dagger$	$X^\dagger = [x_{jk}^*]^T$.
$\Re(X)$	$\Re(X) = (X + X^\#)/2$
$\Im(X)$	$\Im(X) = (X - X^\#)/2i$
J_n	$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$
$[\cdot,\cdot]$	If A, B are two Hilbert space operators, the commutator of A, B is defined by $[A, B] = AB - BA$.

i	$i = \sqrt{-1}$
$\mathrm{diag}_n(\cdot, \cdots, \cdot)$	If matrices $M_1, \cdots, M_n$ are square matrices, $\mathrm{diag}_n(M_1, \cdots, M_n)$ denotes a block diagonal matrix with matrices $M_1, \cdots, M_n$ on the diagonal block.
$\mathrm{diag}_n(\cdot)$	$\mathrm{diag}_n(M)$ denotes a block diagonal matrix with a square matrix M appearing n times on the diagonal block.
$\mathrm{Tr}(M)$	Trace of $M \in \mathbb{R}^{n \times n}$
$M > N$	M and N are symmetric and $M - N > 0$.
I_n	I_n denotes the $n \times n$ identity matrix.
$0_{n \times m}$	$0_{n \times m}$ denotes the $n \times m$ zero matrix. n and m can be determined from context when the subscript is omitted.

Chapter 1

Introduction

1.1 Literature review

Generally speaking, quantum control is control for systems whose behavior follows the laws of quantum mechanics that describes and predicts the movement and behavior of particles (such as atoms, electrons, protons, and photons) at the atomic and subatomic levels, [1], [2], [3] and the references therein. Quantum control theory, a combination of quantum mechanics and its related control theory, has more powerful capability than traditional control theory and brings about a bright future for the development of science and technology in control field [4], [5]. Initial developments in quantum control theory can trace back to the early 1980s, [6], [7]. With the development of quantum control theory, it has improved people's understanding of fundamental aspects of quantum mechanics [8], [9]. Quantum control theory has been successfully applied to quantum network [10], [11], [12], [13], [14], [15], physical chemistry [16], quantum optics [17], [18], and quantum filtering [19], [20], [21]. Interested readers may refer to survey papers [22], [23] and the references therein.

1.1.1 Motivation and challenge

In recent years, quantum control systems play a more and more important role in control engineering, [24], [25]. However, as yet, relatively little is known about linear quantum systems, and it is thus natural to push the development of new quantum control techniques via theoretical exploration.

The behavior and movement of particles at these microscopic levels are quite different from anything observed in everyday life [26], [27], which needs a theory like quantum mechanics in order to better describe the natural world. The main difference between quantum and classical mechanics is non-commutativity, which will allow us to highlight the essential features of the quantum problem that distinguish it from classical feedback control [28]. These features lead to the notation of physical realizability [29], which require us to develop new analysis and synthesis methods to exploit and deal with quantum systems. Physical realizability in this thesis means that a linear system described by linear stochastic differential equations should represent the dynamics of a meaningful physical system. This issue of physical realizability has been analyzed in [29], [30], [31]. General physical realizability conditions are given in [32], which render the quantum feedback control problem potentially more complex. These fundamental features of quantum mechanics require a revolution of conventional methods and techniques from control theory when developing the theory of quantum control. Therefore, the major conceptual challenge is to develop the fundamental principles and tools of linear systems theory that take into account the special features of quantum mechanics.

1.1.2 Quantum feedback control

Feedback control of quantum systems is very important in a number of areas of quantum technology, including quantum optical systems, nanomechanical systems, and circuit QED systems [1], [25], [33], [34], [35], [36], [37], [38], [39]. Quantum feedback control can be simply considered as an interconnection of a quantum plant and a controller, where the controller may be a quantum or classical controller [2], [30], [39], [40]. In measurement-based feedback control (measurement is involved in the feedback loop) as shown in Figure 1.1, the plant is a quantum system, while the controller is a classical (i.e. non-quantum) system [3], [5], [41], [42], [43], [44]. The classical controller processes the outcomes of a measurement of an observable of the quantum system (e.g., the quadrature of an optical field) to determine the classical control actions that are applied to control the behavior of the quantum system. It is easy to monitor the feedback information flow and get some information of the system state. The designers can design the control law based on the estimation of the state. However, the weakness of this method is that the measurement causes inevitable back action noise and affects the states of measured quantum systems. If the controller is a quantum system implemented by quantum devices without any interfaces (e.g., homodyne detectors, modulators) involved in the feedback loop as shown Figure 1.2, this is known as coherent quantum feedback control [29], [30], which allows us not to need to take into account the effects of the measurement on the evolution of quantum systems. The quantum controller typically has much higher bandwidth than electronics devices, meaning faster response and processing times and thus would be much faster than classical signal processing. The coherent quantum feedback control can accomplish some tasks which are not possible using classical feedback. Therefore, there is a growing interest in the study of the strategy. The mathematical theory of coherent quantum feedback control has recently been developed for general quantum dynamical systems [10], [11], [13], [15], [38], [45], [46], [47]. There are interesting coherent control schemes for quantum systems proposed in the literature [48], [49], [50], [51], [52], [53], [54], which have shown that coherent feedback control is able to provide better performance. Several studies have also presented the applications of coherent feedback control, such as [2], [12], [45], [55], [56], [57], [58], [59].

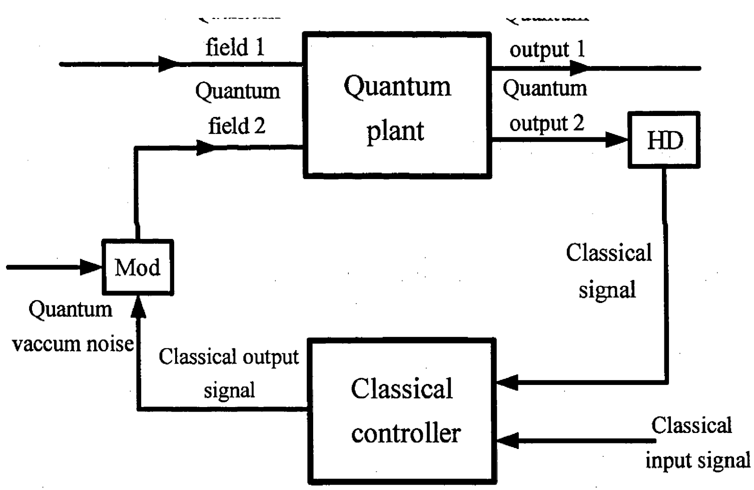

Figure 1.1: A quantum plant with a classical controller.

Classical control theory and technique still play an important role in the quantum technology field and have been successfully applied to a class of linear quantum stochastic systems [14], [22], [29], [30], [31], [50], [54], [60], [61], [62], [63], [64], [65], [66]. Stability, dissipation, passivity and gain are fundamental to analysis and synthesis of feedback systems [67], [68], [69]. The stability of a control system is often very important, which relates to its response to inputs or disturbances. In engineering and stability theory, a square matrix A is called stable matrix (or Hurwitz matrix) if every eigenvalue of A has strictly negative

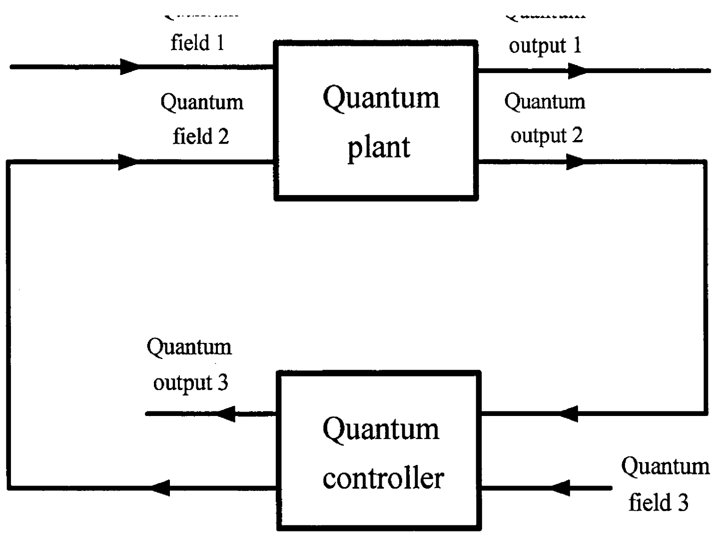

Figure 1.2: A quantum plant with a fully quantum controller.

real part, that is,

$$\mathrm{Re}[\lambda_i] < 0,$$

for each eigenvalue λ_i. A linear system with a stable matrix A is known as a Hurwitz stable system. This stability criterion is still suitable for linear quantum systems; e.g. see Chapter 4. Quantum storage functions are system observables, such as energy, and may be used as Lyapunov functions or Hamiltonian formalism to also determine stability [70], where the total energy of the system is a conserved quantity; e.g. see [71], [72]. Quantum dissipation is the branch of physics that studies the quantum analogues of the process of irreversible loss of energy observed at the classical level. The paper [73] extends theory of dissipative systems to open quantum systems. In [29], a general framework for the quantum H^∞ control is developed, where a quantum version of the Bounded Real Lemma is proposed and applied to derive necessary and sufficient conditions for the H^∞ control of linear quantum stochastic systems. The problem of H^∞ control has also been discussed in [30], [50], [51], [56]. A quantum LQG problem with a classical controller has been solved in [74]. A problem of applying Linear Quadratic Gaussian (LQG) techniques to quantum systems with a quantum controller has been addressed in [54], where a numerical procedure based on standard LMI approach [75] is proposed for finding a quantum controller to achieve desired performance specifications with a given upper bound on the LQG cost. A time-varying coherent quantum LQG control problem has been considered in [76], which seeks a physically realizable quantum controller to minimize the finite-horizon LQG cost, and presents a novel approach towards its solution. The problem of quantum LQG control has also been studied in papers such that [77], [78] and [79]. It is shown that the coherent LQG problem is more challenging than the coherent H^{∞} quantum control since a property of separation of control and physical realizability does not hold and the notion of physical realizability imposes some linear and nonlinear constraints on the system matrices of a physically realizable quantum controller, which may complicate quantum controller designs.

1.1.3 Quantum network synthesis and structure

With the development of quantum technologies, the integration of photonic devices into electronic chips has been the subject of research for more than two decades [2], [5], [80]. Since quantum optical devices have many advantages (e.g. higher bandwidth and faster response and processing times), these devices can solve many physical problems of interconnections, such as precise clock distribution, system synchronization, reduction of power dissipation and so on. Therefore, there has been more and more interest in the structure of quantum systems and mixed quantum-classical systems; e.g., see [81], [82], [83]. The paper [81] prescribes how an arbitrarily complex linear quantum stochastic system can be decomposed into an interconnection of basic building blocks (such as cavities, beam splitters, modulators, phase shifters, amplifiers, fibres, and photon-detectors, etc.) of one degree of freedom open quantum harmonic oscillators and thus be systematically constructed from these building blocks. Synthesis and structure of mixed quantum-classical linear systems have been studied in [32], [84]. Mixed quantum-classical linear systems in this thesis means quantum systems interconnected with classical (non-quantum) devices. In quantum optics, an optical cavity may be part of a mixed quantum-classical system involving photodetectors, electronic amplifiers, piezoelectric actuators, feedback loops and so on. Figure 1.3 illustrates an example of a mixed quantum-classical system, where two Fabry-Perot optical cavities are connected to a classical controller via a homodyne detector (HD) and an electro-optic modulator (MOD), respectively. The classical controller processes the outcomes of a measurement of an observable of the cavity on the left hand side (e.g. the quadrature of an optical field). Modulating the quantum field with the classical controller output by MOD generates another quantum field sent to the cavity on the right. The signals from the classical controller also govern the behavior of the classical system, which can be implemented by electrical and electronic devices.

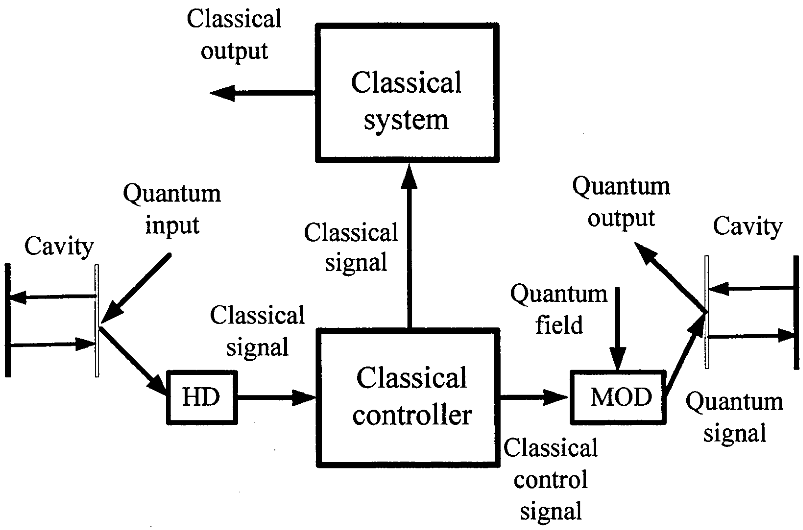

Figure 1.3: A mixed quantum-classical system.

1.1.4 Quantum non-Markovian system with time delay

The concatenated quantum or mixed systems in [32], [81], [82], [83], [84] can be represented by a reduced Markov model, which ignores the effect of time delay. Time delay in classical systems is quite easy to understand and is a generic problem in the control systems, which arises naturally in connection with the system process and information flow for different parts of dynamic system [85], [86]. Time delays are also frequently encountered in quantum feedback control systems such as long transmission lines between quantum plants and classical controllers. The non-Markovian quantum feedback with time delay is first studied in [87], where the feedback controller is a classical controller. [88] and [89] investigate optimal control problem of linear quantum systems with feedback-loop time delay and analyze the effect caused by time delay on control performance. [90] gives a delay-dependent stability criterion for a wide class of nonlinear stochastic systems including quantum spin systems. Quantum non-Markovian models have also been studied in [91], [92].

1.2 Thesis contribution

The contributions of this thesis are as follows.

1. The notions of physical realizability are proposed for mixed quantum-classical linear stochastic systems. Three physical realization constraints are derived for a *standard* form and a general form, respectively. A network theory is developed for synthesizing linear dynamical mixed quantum-classical stochastic systems of the *standard* form in a systematic way. One feedback architecture is presented for this realization.

2. A complete method is proposed in the thesis for a stable quantum linear stochastic system that can realize a given stable classical subsystem, which systematically shows how a classical linear stochastic system can be physically implemented using quantum optical components.

3. Two numerical procedures based on extended LMIs approach are proposed to design quantum LQG controllers, which can provide more parameters for the design of a physically realizable quantum controller and give less conservative solutions to quantum LQG problem since the optimization variables associated with the controller parameters are independent of the symmetric Lyapunov matrix.

4. A numerical procedure is proposed for optimal controller designs for quantum feedback control systems with time delay in the feedback-loop. To this end, a quantum version of delay-independent stability criterion with an upper bound on a cost function is derived.

5. A physical model of a quantum feedback control system with time delay is presented for H^∞ control. Fundamental properties of dissipation, gain and stability for this class of linear models are presented and characterized using linear matrix inequalities (LMIs). A numerical procedure is proposed for feedback controller designs based on a quantum delay version of the Strict Bounded Real Lemma.

1.3 Organization of the thesis

This thesis certainly covers some background materials that predate the research carried out during the candidature; therefore the thesis is organized as follows.

• Chapter 2 collects important preliminaries which will be used regularly in other chapters to facilitate understanding of the main components of this thesis.

• Chapter 3 mainly investigates the synthesis and structure of mixed quantum-classical linear systems. It is shown that a given physically realizable mixed quantum-classical linear stochastic system can be systematically realized as a feedback interconnection of a quantum subsystem and a classical subsystem, together with appropriate interfaces such that modulators and homodyne detectors and so on.

• Chapter 4 mainly studies how a class of classical linear stochastic systems (having a certain form and satisfying certain technical assumptions) can be realized by quantum linear stochastic systems.

• Chapter 5 formulates and solves a quantum LQG problem for quantum coherent feedback control systems based on extended linear matrix inequality (LMI). For comparison, the main results of this chapter is applied to the same example given in [54].

• Chapter 6 investigates a problem of a quantum feedback control system subject to feedback-loop time delay, where the controller is classical. The aim of this chapter is to design dynamic feedback controllers to make not only the closed-loop plant-controller systems stable but also the upper bound of the cost function minimized.

• Chapter 7 formulates and solves the H^∞ control problem of linear quantum measurement-based feedback control systems with time delay in feedback control loop.

Chapter 2

Preliminaries

In this chapter, basic mathematical tools and important preliminaries are collected for use in subsequent chapters. This chapter assumes that the reader is already familiar with various fundamental concepts about quantum mechanics.

2.1 Classical probability and quantum probability

First, let us briefly look at classical probability theory. In the classical theory, a classical probability model is given by $(\Omega, \mathcal{F}, \mathbf{P})$,

1. Ω is the set of possible outcomes of some experiment;
2. $\mathcal{F}$ is a collection of events;
3. $\mathbf{P}$ is a probability measure.

However, classical probability theory is not suitable for quantum mechanics due to non-commutativity. Quantum probability was thus developed in the 1980s as a noncommutative analogue to the Kolmogorovian theory of stochastic processes in which random variables are not assumed to commute; e.g., see [9], [93], [94]. The mathematical ingredients of quantum probability theory derive from the theory of operator algebras.

A quantum probability model $(\mathcal{A}, \rho)$ (or called a quantum probability space) consists of

1. The collection of $\mathcal{A}(\mathfrak{H})$ of projections on a Hilbert space $\mathfrak{H}$;
2. a density operator ρ.

The quantum probability model is a generalization of the classical probability model in Kolmogorovian probability theory, in the sense that every (classical) probability space gives rise to a quantum probability space if $\mathcal{A}$ is chosen as the *-algebra of bounded complex-valued measurable functions on it. The projections $p \subset \mathcal{A}$ are the events in $\mathcal{A}$, and $\rho(p)$ gives the probability of the event p.

2.2 Classical and quantum random variables

Recall that a classical random variable X is *Gaussian* if its probability distribution $\mathbf{P}$ is Gaussian, i.e.

$$\mathbf{P}(a < X < b) = \int_a^b p_X(x)dx, \tag{2.1}$$

where

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{(x-\mu)^2}{2\sigma^2}). \tag{2.2}$$

Here, $\mu = \mathbf{E}[X]$ is the mean, and $\sigma^2 = \mathbf{E}[(X-\mu)^2]$ is the variance.

In quantum mechanics, observables are mathematical representations of physical quantities that can (in principle) be measured, and state vectors ψ summarize the status of physical systems and permit the calculation of expected values of observables. State vectors may be described mathematically as elements of a Hilbert space $\mathfrak{H}$, while observables are self-adjoint operators A on $\mathfrak{H}$. The expected value of an observable A when in pure state ψ is given by the inner product $\langle\psi, A\psi\rangle = \int_{-\infty}^{\infty}\psi(q)^* A\psi(q)dq$. Observables are *quantum random variables.*

A basic example is the quantum harmonic oscillator, a model for a quantum particle in a potential well; see [95, Chapter 14]. The position and momentum of the particle are represented by observables Q and P (also called position quadrature and momentum quadrature), respectively, defined by

$$(Q\psi)(q) = q\psi(q), \quad (P\psi)(q) = -i\frac{d}{dq}\psi(q), \tag{2.3}$$

for $\psi \in \mathfrak{H} = L^2(\mathbf{R})$. Here, $q \in \mathbf{R}$ represents position values. The position and momentum operators do not commute, and in fact satisfy the commutation relation $[Q, P] = i$. In quantum mechanics, such non-commuting observables are referred to as being incompatible. The state vector

$$\psi(q) = (2\pi)^{-\frac{1}{4}}\sigma^{-\frac{1}{2}}\exp(-\frac{(q-\mu)^2}{4\sigma^2}) \tag{2.4}$$

is an instance of what is known as a *Gaussian state.* For this particular *Gaussian state*, the means of P and Q are given by $\int_{-\infty}^{\infty}\psi(q)^* Q\psi(q)dq = \mu$, and $\int_{-\infty}^{\infty}\psi(q)^* P\psi(q)dq = 0$, and similarly the variances are σ^2 and $\frac{1}{4\sigma^2}$, respectively.

2.3 General quantum linear stochastic models

In this section linear quantum stochastic systems are briefly introduced. For a more detailed introduction, please see [29, Theorem 3.4].

Consider an *open quantum harmonic oscillator* consisting of n one degree of freedom open quantum harmonic oscillators coupled to boson fields (e.g. optical beams), [3], [96], [97]. Each oscillator may be represented by position q_j and momentum p_j operators ($j = 1, \dots, n$), while each field channel is described by analogous field operators $w_{q_k}(t)$, $w_{p_k}(t)$, ($k = 1, \dots, m$) [98], [99]. The oscillator variables are *canonical* if they satisfy the canonical commutation relations $[q_j, p_k] = 2i\delta_{jk}$ ($j, k = 1, \dots, n$). In vector form, we write $\xi = [q_1\ p_1\ q_2\ p_2\ \cdots\ q_n\ p_n]^T$, and the commutation relations become

$$\xi\xi^T - (\xi\xi^T)^T = 2i\Theta, \tag{2.5}$$

where in the canonical case, $\Theta = \mathrm{diag}_n(J)$. Similarly, the Ito products for the fields $w = [w_{q_1}\ w_{p_1}\ w_{q_2}\ w_{p_2}\ \cdots\ w_{q_m}\ w_{p_m}]^T$ may be written as

$$dw(t)dw(t)^T = F_w dt, \tag{2.6}$$

where in the canonical case $F_w = I_{2m} + i\mathrm{diag}_m(J)$. Commutation relations for the noise components of w can be defined as:

$$[dw(t), dw(t)^T] = (F_w - F_w^T)dt = 2i\Theta_w dt.$$

The dynamical evolution of an open system is unitary (in the Hilbert space consisting of the system and fields), and in the Heisenberg picture the system variables and output field operators evolve according to equations of the form

$$\begin{aligned} d\xi(t) &= \mathcal{A}\xi(t)dt + \mathcal{B}dw(t), \\ dz(t) &= \mathcal{C}\xi(t)dt + \mathcal{D}dw(t), \end{aligned} \tag{2.7}$$

where $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{B} \in \mathbb{R}^{2n\times n_w}$, $\mathcal{C} \in \mathbb{R}^{n_z\times 2n}$, $\mathcal{D} \in \mathbb{R}^{n_z\times n_w}$. Here, $n_w = 2m$ and n_z is even. We see therefore that in the Heisenberg picture dynamical "state space" equations (2.7) look formally like the familiar state space equations form classical systems and control theory. However, for arbitrary matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, equations (2.7) need not correspond to a canonical open oscillator. The system (2.7) is said to be *physically realizable* if the equations (2.7) correspond to an open quantum harmonic oscillator, [29, Definition 3.3]). The real constant matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfying

$$\mathcal{A}\Theta + \Theta\mathcal{A}^T + \mathcal{B}\Theta_w\mathcal{B}^T = 0, \tag{2.8}$$

$$\mathcal{B}\mathcal{D}^T = \Theta\mathcal{C}^T\Theta_w, \tag{2.9}$$

$$\mathcal{D} = I_{n_w} \quad \text{or} \quad [I_{n_z} \quad 0]. \tag{2.10}$$

As shown in [29], the system (2.7) with $\mathcal{D}$ defined as in (2.10) is physically realizable if and only if the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfy conditions (2.8) and (2.9). In general, we may take the commutation matrix Θ to be skew-symmetric, while the Ito matrix F is non-negative Hermitian. These generalizations, which will be studied in Chapter 3, allow us to consider classical variables, characterized by zero commutation relations, as well as classical noise processes, corresponding to the absence of the imaginary part in the Ito products, [29], [30], [84].

2.4 Quantum network synthesis theory

We briefly review some definitions and results from [81]; see also [82] and [83]. The quantum linear stochastic system (2.7) can be reparametrized in terms of three parameters S, L, H called the scattering, coupling and Hamiltonian operators, respectively. Here S is a complex unitary matrix $S^\dagger S = SS^\dagger = I$, $L = \Lambda x_0$ with $\Lambda \in \mathbb{C}^{n_w \times 2n}$, and $H = \frac{1}{2}x_0^T R x_0$ with $R = R^T \in \mathbb{R}^{2n\times 2n}$. Recall that there is a one-to-one correspondence between the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ in (2.7) and the triplet S, L, H or equivalently the triplet S, Λ, R; see [29] and [81]. Thus, we can represent a quantum linear stochastic system G given by (2.7) with the shorthand notation $G = (S, L, H)$ or $G = (S, \Lambda, R)$ [38]. Given two quantum linear stochastic systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$ with the same number of field channels, the operation of cascading of G_1 and G_2 is represented by the series product $G_2 \lhd G_1$ defined by

$$G_2 \lhd G_1 = \left(S_2S_1, L_2 + S_2L_1, H_1 + H_2 + \frac{1}{2i}(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2)\right).$$

According to [81, Theorem 5.1] a linear quantum stochastic system with n degrees of freedom can be decomposed into an unidirectional connection of n one degree of freedom harmonic oscillators with a direct coupling between two adjacent one degree of freedom quantum harmonic oscillators. Thus an arbitrary quantum linear stochastic system can in principle be synthesized if:

1) Arbitrary one degree of freedom systems of the form (2.7) with n_w input fields and n_w output fields can be synthesized.

2) The bidirectional coupling $H^d = \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} x_k^T \left(R_{jk}^T - \frac{1}{2i}(\Lambda_k^\dagger \Lambda_j - \Lambda_k^T \Lambda_j^\#)\right) x_j$ can be synthesized, where Λ_j denotes the jth row of the complex coupling matrix Λ. The Hamiltonian matrix R is given by $R = \frac{1}{4}P_{2n}^T(-J_n\mathcal{A}+\mathcal{A}^T J_n)P_{2n}$ and the coupling matrix Λ is given by $\Lambda = -\frac{i}{2}\begin{bmatrix} 0_{n_w \times n_w} & I_{n_w} \end{bmatrix} P_{2m}\mathrm{diag}_m(M)P_{2m}^T \mathcal{B}^T J_n P_{2n}$ where $M = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$, P_{2n} denotes a permutation matrix acting on a column vector $f = [f_1\ f_2\ \cdots\ f_{2n}]^T$ as $P_{2n}f = [f_1\ f_{1+n}\ f_2\ f_{2+n}\ \cdots\ f_n\ f_{2n}]^T$.

The work [81] then shows how one degree of freedom systems and the coupling H^d can be approximately implemented using certain linear and nonlinear quantum optical components [34]. Thus in principle any system of the form (2.7) can be constructed using these components.

2.5 Quantum non-demolition condition

The Belavkin's nondemolition principle requires an observable $X(t)$ at a time instant t to be compatible with the past output process $Y(s)$ $(s \le t)$ [100], [101], [102], that is:

$$[X(t), Y(s)^T] = 0, \ \forall\, t \ge s \ge 0. \tag{2.11}$$

Condition (2.11) is known as a *non-demolition condition.* This notation will be used in Chapter 3.

Chapter 3

Network Synthesis for a Class of Mixed Quantum-Classical Linear Stochastic Systems

As introduced in Chapter 1, quantum control systems constructed using quantum optical devices and standard analog or digital electronics have attracted more and more attentions in recent years. This chapter mainly investigates a synthesis problem for a class of linear quantum equations that may describe mixed quantum-classical systems as shown in Figure 1.3. A general model and a standard model for mixed quantum-classical linear stochastic systems are proposed for the design process. Furthermore, a network synthesis theory for a mixed quantum-classical system of the standard form is developed.

3.1 Introduction

In classical engineering, many methods have been developed for designing controllers and electronic systems. The design process begins with some form of specification for the system, and concludes with a physical realization of the system that meets the specifications. Often, mathematical models for the system are used in the design process, such as state space equations for the system. These state space equations may result from a mathematical optimization procedure, such as LQG, or some other procedure. The process of going from such mathematical models to the desired physical systems is a process of *synthesis* or *physical realization*, part of the design methodologies widely used in classical engineering [103]. The nature of the physical components to be used may restrict the range of, say, the state space models that can be used. For instance, capacitors, inductors and resistors cannot by themselves implement non-passive devices like amplifiers. Analogous design issues are beginning to present themselves in quantum technology. For example, linear quantum optics has been proposed as a means of implementing quantum information systems, [35].

Linear quantum optical systems may be described by linear quantum differential equations in the Heisenberg picture of quantum mechanics [3], [38], [96]. These equations look superficially like the classical state space equations familiar to engineering, but in fact are fundamentally different because they are equations for quantum mechanical operators, not numerical variables. The purpose of this chapter is to consider synthesis problems for a class of linear stochastic differential equations that may describe mixed quantum-classical systems. This class of equations is usually presented in a general form given in Subsection 3.2.2 where the quantum-classical nature is captured in the matrices specifying the commutation relations of the system and signal (e.g. boson field) variables. However, the structure of a mixed quantum-classical system is not very clearly presented in a general form and we thus show how a mixed system described in general form can be linearly transformed into a *standard* form defined in Subsection 3.2.3, which reveals in a standard (or canonical) way the internal structure of a mixed quantum-classical system. Furthermore, arbitrary linear stochastic differential equations for a general form or a *standard* form need not correspond to a physical system, and so we derive conditions ensuring that they do; that is, physical realizability. This work generalizes and extends earlier work [29], [84].

This chapter is organized as follows. Section 3.2 proposes two models of mixed quantum-classical linear stochastic systems for the design process and presents a connection between these models. Section 3.3 presents physical realizability definitions and constraints for the two models defined in Section 3.2, respectively. Section 3.4 develops a network synthesis theory for a mixed quantum-classical system of the *standard* form. Then two examples of the network synthesis theory are presented in this section. Finally, Section 3.5 gives the conclusion of this chapter.

3.2 Mixed quantum-classical linear models

In this section, we give two models (or forms) for mixed quantum-classical linear stochastic systems and then derive relations between two models. We allow our models to consider classical inputs and outputs, which are not included in previous models in [29], [84].

3.2.1 Review of mixed quantum-classical linear stochastic systems with quantum inputs and quantum outputs

Before presenting our models of interest, we first review mixed quantum-classical linear stochastic systems with quantum inputs and quantum outputs given in [29], [84].

Now we let x have quantum and classical degrees of freedom, such that $x = [x_q^T \; x_c^T]^T$, where classical variables $x_c(t)$ commute with one another and with the degrees of freedom in quantum variables $x_q(t)$. Thus, the commutation relation for $x(t)$ satisfies

$$xx^T - (xx^T)^T = 2i\Theta_n,$$

where $\Theta_n = \mathrm{diag}(\Theta_{n_q}, 0_{n_c \times n_c})$ with $\Theta_{n_q} = \mathrm{diag}_{n_q}(J)$ is said to be *degenerate canonical* by the terminology of [29].

Consider a mixed quantum-classical linear stochastic system in terms of x given by

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t), \\ dy_q(t) &= C_q x(t)dt + D_q dw(t), \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 2m}$, $C_q \in \mathbb{R}^{2n_{yq}\times n}$ and $D_q \in \mathbb{R}^{2n_{yq}\times 2m}$ $(n = 2n_q + n_c)$. w is a quantum noise defined as $[dw(t), dw(t)^T] = (F_w - F_w^T)dt = 2i\Theta_w dt$. If we are given a component of a vector of classical system variables x_c denoted by x_{c_k}, we may consider x_{c_k} as one of the quadratures of a quantum harmonic oscillator, say the position quadrature q_k. The vector $\tilde{x}_k(t) = \begin{bmatrix} q_k(t) \\ p_k(t) \end{bmatrix}$ is called an *augmentation* of $x_{c_k}(t)$. That is, $x(t)$ can be embedded in a larger vector $\tilde{x}(t) = [x(t)^T \;\; \eta(t)^T]^T$, where any element of $\eta(t) = [\eta_1(t) \; \eta_2(t) \; \cdots \; \eta_{n_c}(t)]^T$ commute with any component of $x_q(t)$, and are conjugate to the components of $x_c(t)$, satisfying $[x_{c,j}(t), \eta_k(t)] = 2i\delta_{jk}$, where δ_{jk} is the Kronecker delta function. Then the commutation relation for $\tilde{x}(t)$ is defined as $\tilde{x}\tilde{x}^T - (\tilde{x}\tilde{x}^T)^T = 2i\widetilde{\Theta}$. So, the *augmented* system of the system (3.1) in terms of $\tilde{x}$ can be defined as:

$$d\tilde{x}(t) = \widetilde{A}\tilde{x}(t)dt + \widetilde{B}dw(t), \tag{3.2}$$

$$d\tilde{y}_q(t) = \widetilde{C}\tilde{x}(t)dt + \widetilde{D}dw(t), \tag{3.3}$$

where $\widetilde{A} = \begin{bmatrix} A & 0 \\ A' & A'' \end{bmatrix}$, $\widetilde{B} = \begin{bmatrix} B \\ B' \end{bmatrix}$, $\widetilde{C} = \begin{bmatrix} C_q & 0 \end{bmatrix}$, $\widetilde{D} = D_q$, $\tilde{y}_q = y_q$, $\widetilde{\Theta} = \begin{bmatrix} \Theta_n & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \begin{bmatrix} 0 & -I \end{bmatrix} & 0 \end{bmatrix}$ is a invertible matrix with $\widetilde{\Theta}\widetilde{\Theta} = -I$ and $\widetilde{\Theta} = -\widetilde{\Theta}^T$. The matrices A', A'' and B' will be given in the proof of Theorem 3.9.

The system (3.1) is said to be physically realizable if its corresponding *augmented* system described by (3.2)-(3.3) can represent the dynamics of an open quantum harmonic oscillator after a suitable relabeling of the components of the variables $\tilde{x}(t)$. Recalling the results of [29], we then have the following theorem.

Theorem 3.1. *A mixed quantum-classical system* (3.1) *with quantum inputs and quantum outputs is physically realizable, where* $D_q = I_{2m}$ *or* $D_q = [I_{2n_{yq}} \quad 0]$, *if and only if* A, B, C_q *and* D_q *satisfy the following conditions*

$$A\Theta_n + \Theta_n A^T + B\Theta_w B^T = 0, \tag{3.4}$$

$$BD_q^T = \Theta_n C_q^T \Theta_w. \tag{3.5}$$

3.2.2 A general form for mixed linear stochastic systems with mixed inputs and mixed outputs

Consider a linear mixed quantum-classical stochastic system in a general form given by

$$\begin{aligned} d\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}dv(t), \\ d\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)dt + \mathbf{D}dv(t), \end{aligned} \tag{3.6}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{n_y\times n}$ and $\mathbf{D} \in \mathbb{R}^{n_y\times n_v}$; $x(t)$ includes quantum and classical system variables satisfying the commutation relation, such that $\mathbf{x}_0\mathbf{x}_0^T - (\mathbf{x}_0\mathbf{x}_0^T)^T = 2i\mathbf{\Theta}_n$ with a skew-symmetric matrix $\mathbf{\Theta}_n$ ($\mathbf{x}(0) = \mathbf{x}_0$); the vector $v(t)$ represents the input signals, which contains quantum and classical noises; $\mathbf{y}(t)$ represents mixed quantum-classical outputs. F_v and $F_\mathbf{y}$ are nonnegative definite Hermitian matrices satisfying $dv(t)dv(t)^T = F_v dt$ and $dy(t)dy(t)^T = F_\mathbf{y} dt$. The transfer function $\Xi_G(s)$ for a system of the form (3.6) is denoted by

$$\Xi_G(s) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right](s) = \mathbf{C}\left(sI_n - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}.$$

3.2.3 A standard form for mixed linear stochastic systems with quantum inputs and mixed outputs

From the *general* form (3.6), it is not obvious which part corresponds to quantum components while which part corresponds to classical components. Therefore, we need to transform the system (3.6) into a form (called *standard* form), which presents a clear structure of a mixed quantum-classical system.

Consider a *standard* form given by

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t), \\ dy(t) &= Cx(t)dt + Ddw(t), \end{aligned} \tag{3.7}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 2m}$, $C \in \mathbb{R}^{n_y\times n}$ and $D \in \mathbb{R}^{n_y\times 2m}$. $y = [y_q^T \quad y_c^T]^T$ with $y_q \in \mathbb{R}^{2n_{yq}}$ and $y_c \in \mathbb{R}^{n_{yc}}$, $w = [w_1^T \quad w_2^T]^T$ with $w_1 \in \mathbb{R}^{2n_{w1}}$ and $w_2 \in \mathbb{R}^{2n_{w2}}$. Here $m = n_{w_1} + n_{w_2}$, $n_y = n_{\mathbf{y}} = 2n_{y_q} + n_{y_c}$. Let initial values $x(0) = x_0$ satisfy the commutation relations $x_0x_0^T - (x_0x_0^T)^T = 2i\Theta_n$. We assume that $\Theta_w = \frac{F_w - F_w^T}{2i}$ with $dw(t)dw(t)^T = F_w dt$ and $\Theta_y = \frac{F_y - F_y^T}{2i}$ with $dy(t)dy(t)^T = F_y dt$. The transfer function for the system of the form (3.7) is given by

$$\Xi_S(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = C\left(sI_n - A\right)^{-1}B + D.$$

Definition 3.1. A mixed quantum-classical linear stochastic system of the form (3.7) is said to be *standard* if the following statements are satisfied:

1. $\Theta_n = \mathrm{diag}(\Theta_{n_q}, 0_{n_c\times n_c})$ with $\Theta_{n_q} = \mathrm{diag}_{n_q}(J)$ and $2n_q + n_c = n$ $(n_c \geq 0)$.

2. $\Theta_w = \mathrm{diag}_m(J)$.

3. $F_y = I_{n_y} + \mathrm{diag}(\Theta_{y_q}, 0_{n_{yc}\times n_{yc}})$, where $n_y = 2n_{y_q} + n_{y_c}$ $(n_{y_q} \leq m)$.

Let the matrices A, B, C, D be partitioned compatibly with partitioning of $x(t)$ into $x_q(t)$ and $x_c(t)$ as $A = \begin{bmatrix} A_{qq} & A_{qc} \\ A_{cq} & A_{cc} \end{bmatrix}$, $B = \begin{bmatrix} B_q \\ B_c \end{bmatrix}$, $C = \begin{bmatrix} C_q \\ C_c \end{bmatrix} = \begin{bmatrix} C_{qq} & C_{qc} \\ C_{cq} & C_{cc} \end{bmatrix}$ and $D = \begin{bmatrix} D_q \\ D_c \end{bmatrix}$. Let $y(t)$ be partitioned into $y_q(t)$ and $y_c(t)$.

Then, the system (3.7) can be rewritten as

$$\begin{aligned} dx_q(t) &= \begin{bmatrix} A_{qq} & A_{qc} \end{bmatrix} x(t)dt + B_q dw(t), & (3.8)\\ dx_c(t) &= \begin{bmatrix} A_{cq} & A_{cc} \end{bmatrix} x(t)dt + B_c dw(t), & (3.9)\\ dy_q(t) &= \begin{bmatrix} C_{qq} & C_{qc} \end{bmatrix} x(t)dt + D_q dw(t), & (3.10)\\ dy_c(t) &= \begin{bmatrix} C_{cq} & C_{cc} \end{bmatrix} x(t)dt + D_c dw(t), & (3.11) \end{aligned}$$

where $A_{qq} \in \mathbb{R}^{2n_q \times 2n_q}$, $A_{qc} \in \mathbb{R}^{2n_q \times n_c}$, $A_{cq} \in \mathbb{R}^{n_c \times 2n_q}$, $A_{cc} \in \mathbb{R}^{n_c \times n_c}$, $B_q \in \mathbb{R}^{2n_q \times 2m}$, $B_c \in \mathbb{R}^{n_c \times 2m}$, $C_{qq} \in \mathbb{R}^{2n_{yq} \times 2n_q}$, $C_{qc} \in \mathbb{R}^{2n_{yq} \times n_c}$, $C_{cq} \in \mathbb{R}^{n_{yc} \times 2n_q}$, $C_{cc} \in \mathbb{R}^{n_{yc} \times n_c}$, $D_q \in \mathbb{R}^{2n_{yq} \times 2m}$, $D_c \in \mathbb{R}^{n_{yc} \times 2m}$.

Remark 3.2. The first item of Definition 3.1 indicates that $x(t)$ has both quantum and classical degrees of freedom, where Θ_{n_q} corresponds to the quantum degrees of freedom x_q while $0_{n_c \times n_c}$ corresponds to the classical degrees of freedom x_c. The second item of Definition 3.1 shows that input signals of the system (3.7) must be fully quantum. The third item of Definition 3.1 implies that

$$\Theta_y = D\Theta_w D^T = \mathrm{diag}(\Theta_{y_q}, 0_{n_{yc} \times n_{yc}}), \tag{3.12}$$

where $\Theta_{y_q} = \mathrm{diag}_{n_{yq}}(J)$ corresponds to quantum outputs while the matrix $0_{n_{yc} \times n_{yc}}$ corresponds to classical outputs, which will be discussed further and proved under suitable hypotheses in Section 3.3. So, the difference between the mixed linear systems (3.1) and (3.7) is that the latter explicitly exhibits classical output signals, and the matrix D has a more general form satisfying condition (3.12), which is equivalent to the following equations:

$$\begin{aligned} D_q \Theta_w D_q^T &= \Theta_{y_q}, & (3.13)\\ D_q \Theta_w D_c^T &= 0, & (3.14)\\ D_c \Theta_w D_c^T &= 0. & (3.15) \end{aligned}$$

3.2.4 Relations between the general and standard forms

The general form (3.6) and the *standard* form (3.7) can be related by Theorem 3.3 and 3.5 below.

Theorem 3.3. *Given an arbitrary $n \times n$ real skew-symmetric matrix Θ_n ($n \geq 2$), there always exists a real nonsingular matrix P_n and a block diagonal matrix $\boldsymbol{\Theta}_n = \mathrm{diag}(\Theta_{n_q}, 0_{n_c \times n_c})$ such that*

$$\Theta_n = P_n \boldsymbol{\Theta}_n P_n^T. \tag{3.16}$$

The proof of Theorem 3.3 will use the following lemma.

Lemma 3.4. *Given an arbitrary $n \times n$ real normal matrix Θ_{nn} ($n \geq 2$), there is a real orthogonal matrix $\widetilde{P}$, such that*

$$\widetilde{P}^T \Theta_n \widetilde{P} = O = \mathrm{diag}(O_1, O_2, \ldots, O_j), \tag{3.17}$$

where each O_i is either a real number denoted by λ_i or is a real 2×2 matrix of the form

$$O_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}, \tag{3.18}$$

with $\beta_j > 0$.

The proof of Lemma 3.4 can be found in [104], [105] and hence is omitted here. The proof of Theorem 3.3 is given below.

Proof. Since $\Theta_n \Theta_n^T = (-\Theta_n^T)(-\Theta_n) = \Theta_n^T \Theta_n$, Θ_n is normal. By Lemma 3.4, the relation (3.17) holds. It can be easily verified that O is skew-symmetric, so each $\lambda_i = -\lambda_i$ and each $O_i = -O_i^T$. Then, we can get all $\lambda_i = 0$ and all $\alpha_i = 0$. Θ_n can be written as

$$\Theta_n = \widetilde{P}\mathrm{diag}(O_1, O_2, \ldots, O_k, 0_{(n-2k)\times(n-2k)})\widetilde{P}^T$$

with $O_i = \begin{bmatrix} 0 & \beta_j \\ -\beta_i & 0 \end{bmatrix} = \tilde{S}_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tilde{S}_i^T$, where $\tilde{S}_i = \begin{bmatrix} \sqrt{\beta_i} & 0 \\ 0 & \sqrt{\beta_i} \end{bmatrix}$. Note that $\pm i\beta_i$ are the eigenvalues of Θ_n. Now, we want to construct a $n \times n$ real non-singular diagonal matrix $\tilde{S}$ such that $\tilde{S} = \mathrm{diag}(\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_k, I_{n-2k})$. Then, we get $\mathrm{diag}(O_1, \ldots, O_k, 0_{(n-2k)\times(n-2k)}) = \tilde{S}\mathrm{diag}\left(\Theta_{n_q}, 0_{(n-2k)\times(n-2k)}\right)\tilde{S}^T$. Then, We can obtain $\Theta_n = \widetilde{P}\tilde{S}\Theta_n(\widetilde{P}\tilde{S})^T(P_n^{-1} = \widetilde{P}\tilde{S})$. This completes the proof. □

Theorem 3.5. *Given an arbitrary $m \times m$ nonnegative definite Hermitian matrix F_v, there exists a $2m \times 2m$ matrix $F_w = I_{2m} + i\mathrm{diag}_m(J)$ and a $m \times 2m$ real matrix W such that*

$$F_v = W F_w W^T. \tag{3.19}$$

Proof. Hermitian matrices F_v and F_w can be diagonalized by unitary matrices U_v and U_w, respectively, such that

$$\begin{aligned} F_v &= U_v \Lambda_v U_v^\dagger, && (3.20) \\ F_w &= U_w \Lambda_w U_w^\dagger, && (3.21) \end{aligned}$$

where Λ_w=$\text{diag}_m\left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right)$, Λ_v=diag$(\lambda_1, \lambda_2, \cdots, \lambda_m)$, ($\lambda_j \geq 0$ is an eigenvector of F_v), U_w=$\text{diag}_m\left(\frac{\sqrt{2}}{2}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}\right)$. Since Λ_v and Λ_w are two real diagonal matrices, there exists a $m \times 2m$ complex matrix $Q = [q_1\ q_2\ \cdots\ q_{2m}]$ such that

$$\Lambda_v = Q\Lambda_w Q^\dagger. \tag{3.22}$$

where Λ_v=diag$(\lambda_1, \lambda_2, \cdots, \lambda_m)$, ($\lambda_j \geq 0$ is an eigenvector of F_v). In order to let (3.22) hold, for simplicity we choose

$$\begin{aligned} q_2 &= \begin{bmatrix} \sqrt{\frac{\lambda_1}{2}} & 0 & \cdots & 0 \end{bmatrix}^T, \\ q_4 &= \begin{bmatrix} 0 & \sqrt{\frac{\lambda_2}{2}} & 0 & \cdots & 0 \end{bmatrix}^T, \\ &\vdots \\ q_{2m} &= \begin{bmatrix} 0 & \cdots & 0 & \sqrt{\frac{\lambda_m}{2}} \end{bmatrix}^T \end{aligned}$$

and $q_1, q_3, \cdots, q_{2m-1}$ can be chosen to be arbitrary column vectors of length m and are to be determined later. Combining (3.20), (3.21) and (3.22) gives

$$F_v = U_v Q U_w^\dagger F_w (U Q U_w^\dagger)^\dagger. \tag{3.23}$$

Next, we want to show that Q can be chosen to let $W = U_v Q U_w^\dagger$ be real. First, we get

$$U_v Q = [U_v q_1\ U_v q_2\ \cdots\ U_v q_{2m}]. \tag{3.24}$$

Observing the structure of U_w=$\text{diag}_m\left(\frac{\sqrt{2}}{2}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}\right)$, in order to let W be real, we require that the following relations be satisfied

$$\begin{aligned} q_1 &= -U_v^\dagger U_v^\# q_2, \\ q_3 &= -U_v^\dagger U_v^\# q_4, \\ &\vdots \\ q_{2m-1} &= -U_v^\dagger U_v^\# q_{2m}. \end{aligned}$$

Since $q_2, q_4, \cdots, q_{2m}$, and U_v have already been determined, the matrix Q is hence constructed as

$$Q = \begin{bmatrix} -U_v^\dagger U_v^\# q_2 & q_2 & -U_v^\dagger U_v^\# q_4 & q_4 & \cdots & -U_v^\dagger U_v^\# q_{2m} & q_{2m} \end{bmatrix}.$$

We can get the representation (3.19) with $W = U_v Q U_w^\dagger$. □

Let us look at an example applying Lemma 3.5.

Consider a nonnegative definite Hermitian matrix given by

$$F_v = \begin{bmatrix} 8.9286 & -0.2143+4.8107i & 0.1429+7.2161i \\ -0.2143-4.8107i & 8.3571+0.0000i & 0.4286-2.4054i \\ 0.1429-7.2161i & 0.4286+2.4054i & 8.7143 \end{bmatrix}.$$

It is easily obtained that $F_v = U_v \Lambda_v U^\dagger$ with

$$U_v = \begin{bmatrix} 0.6814 & 0.6814 & 0.2673 \\ -0.1572-0.3922i & -0.1572+0.3922i & 0.8018 \\ 0.1048-0.5883i & 0.1048+0.5883i & -0.5345 \end{bmatrix}$$

and

$$\Lambda_v = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Now following the construction in the proof of Lemma 3.5, we want to find a real matrix W. First, we choose q_2, q_4, q_6 as

$$\begin{aligned} q_2 &= [3 \quad 0 \quad 0]^T, \\ q_4 &= [0 \quad 0 \quad 0]^T, \\ q_6 &= [0 \quad 0 \quad 2]^T. \end{aligned}$$

Then q_1, q_3, q_5 are obtained as

$$\begin{aligned} q_1 &= [0 \quad -3 \quad 0]^T, \\ q_3 &= [0 \quad 0 \quad 0]^T, \\ q_5 &= [0 \quad 0 \quad -2]^T. \end{aligned}$$

So the matrix Q can be constructed as

$$Q = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}.$$

It follows from the above construction that W is obtained as

$$W = \begin{bmatrix} 0 & 2.8909 & 0 & 0 & 0 & 0.7559 \\ -1.6641 & -0.6671 & 0 & 0 & 0 & 2.2678 \\ -2.4962 & 0.4447 & 0 & 0 & 0 & -1.5119 \end{bmatrix}.$$

It is easily checked that $F_v = W F_w W^T$ with $F_w = I_6 + i\mathrm{diag}_3(J)$.

Theorem 3.6. *Given a mixed quantum-classical stochastic system of the general form* (3.6)*, there exists a corresponding* standard *form* (3.7).

Proof. By Lemma 3.3 and 3.5, there exist matrices P_n, W and $P_{\mathbf{y}}$, such that

$$\left.\begin{aligned}\Theta_n &= P_n \mathbf{\Theta}_n P_n^T,\\ \Theta_v &= W\Theta_w W^T,\\ y &= P_{\mathbf{y}}\mathbf{y},\\ \mathrm{diag}(\Theta_{y_q}, 0_{n_{yc}\times n_{yc}}) &= P_{\mathbf{y}}\Theta_{\mathbf{y}}P_{\mathbf{y}}^T,\\ A &= P_n \mathbf{A} P_n^{-1},\\ B &= P_n \mathbf{B} W,\\ C &= P_{\mathbf{y}}\mathbf{C}P_n^{-1},\\ D &= P_{\mathbf{y}}\mathbf{D}W.\end{aligned}\right\} \tag{3.25}$$

Substituting relations (3.25) into (3.6) gives (3.7). Now, we verify the relation between the *standard* $\Xi_S(s)$ and general $\Xi_G(s)$ transfer functions as follows:

$$\begin{aligned}\Xi_S(s) &= C\left(sI_n - A\right)^{-1}B + D\\ &= P_{\mathbf{y}}\mathbf{C}P_n^{-1}\left(sP_nP_n^{-1} - P_n\mathbf{A}P_n^{-1}\right)^{-1}P_n\mathbf{B}W + P_{\mathbf{y}}\mathbf{D}W\\ &= P_{\mathbf{y}}\left(\mathbf{C}\left(sI_n - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right)W\\ &= P_{\mathbf{y}}\Xi_G(s)W.\end{aligned}$$

Thus, the general system (3.6) can be related to its corresponding *standard* system (3.7) by the above linear transformations. □

3.3 Physical realizability of mixed systems

In this section, we will introduce the definition of physical realizability of the *standard* form and a theorem on necessary and sufficient conditions for physical realizability. This is followed by the analogous definition, and necessary and sufficient conditions for the physical realizability of the general form.

3.3.1 Physical realizability for the standard form

The following lemmas will be used for defining the physical realizability of the system (3.7).

Lemma 3.7. *Non-demolition condition* $[\tilde{x}(t), y_q(s)^T] = 0,\ \forall\ t \geq s \geq 0$ *for the* augmented *system* (3.2)-(3.3) *of the system* (3.1) *holds, if and only if*

$$\widetilde{B}\Theta_w\widetilde{D}_q^T = -\widetilde{\Theta}\widetilde{C}_q^T. \tag{3.26}$$

Proof. First, we will argue that $[\tilde{x}(t), y_q^T(s)] = 0$ is equivalent to $[\tilde{x}(t), y_q(t)^T] = 0$, for all $t \geq s \geq 0$. Let $g_s(t) = [\tilde{x}(t), y_q(s)^T]$, for all $t \geq s \geq 0$, where s is fixed. From $[\tilde{x}(t), y_q(t)^T] = 0$ for all $t \geq s \geq 0$, we can infer that $g_s(s) = 0$ and then have

$$\begin{aligned} dg_s(t) &= d[\tilde{x}(t), y_q(s)^T] \\ &= [d\tilde{x}(t), y_q(s)^T] \\ &= \widetilde{A}[\tilde{x}(t), y_q(s)^T]dt \\ &= \widetilde{A}g_s(t)dt. \end{aligned}$$

Solving the above equation gives

$$g_s(t) = \exp\left(\widetilde{A}(t-s)\right) g_s(s) = 0.$$

Therefore, $[\tilde{x}(t), y_q^T(t)] = 0$ implies $[\tilde{x}(t), y_q(s)^T] = 0$, for all $t \geq s \geq 0$. Conversely, it is trivial to verify that $[\tilde{x}(t), y_q(s)^T] = 0$ for all $t \geq s \geq 0$ implies $[\tilde{x}(t), y_q(t)^T] = 0$ for all $t \geq 0$.

Thus, we just need to consider the case where $t = s$. Let $g(t) = [\tilde{x}(t), y_q(t)^T]$ with $g(0) = 0$ and then we have

$$\begin{aligned} dg(t) &= d[\tilde{x}(t), y_q(t)^T] \\ &= [d\tilde{x}(t), y_q(t)^T] + [\tilde{x}(t), dy_q(t)^T] + [d\tilde{x}(t), dy_q(t)^T] \\ &= \widetilde{A}g(t)dt + 2i(\widetilde{\Theta}\widetilde{C}_q^T + \widetilde{B}\Theta_w\widetilde{D}_q^T)dt. \end{aligned}$$

Solving the above equation gives

$$g(t) = \exp(\widetilde{A}t)g(0) + 2i\int_0^t \exp(\widetilde{A}(t-\tau))\left(\widetilde{\Theta}\widetilde{C}_q^T + \widetilde{B}\Theta_w\widetilde{D}_q^T\right) d\tau.$$

From the above equation, it can be easily verified that $g(t) = 0$ for all $t \geq 0$, if and only if

$$\widetilde{\Theta}\widetilde{C}_q^T + \widetilde{B}\Theta_w\widetilde{D}_q^T = 0.$$

□

Lemma 3.8. *Non-demolition condition* $[x(t), y(s)^T] = 0$, $\forall\ t \geq s \geq 0$ *for the system* (3.7) *holds, if and only if*

$$B\Theta_w D^T = -\Theta_n C^T. \tag{3.27}$$

The proof of Lemma 3.8 is similar to that of Lemma 3.7 and is thus omitted.

For better understanding Definition 3.2 and 3.3 below, a discussion regarding the physical realizability of the *standard* form (3.7) will be given first. The system (3.7) can be divided into two parts: one is the system (3.1) with D_q satisfying (3.13), or equivalently described by (3.8)-(3.10); the other is the output equation (3.11). So, the system (3.7) is physically realizable if the two parts are both physically realizable. First, we consider physical realizability conditions of the system (3.1). From the structure of system matrices of the *augmented* system (3.2)-(3.3), it is clear that the dynamics of $x(t)$ of system (3.1) embedded in system (3.2)-(3.3) are not affected by the augmentation, and matrices A', A'', B' in system (3.2)-(3.3) can be chosen to preserve commutation relations for augmented system variables $\tilde{x}$ shown in the proof of Theorem 3.9. Motivated by the results in [29], we want to argue that the system (3.1) with $\widetilde{D} = D_q$ satisfying (3.13) is physically realizable if its *augmented* system (3.2)-(3.3) can be physically realizable. However, the previous definition and theorem of physical realizability in [29] are only suitable for an *augmented* system (3.2)-(3.3) with $\widetilde{D} = I$ or $\widetilde{D} = [I \quad 0]$ (no scattering processes involved). We hence need to transform the *augmented* system (3.2)-(3.3) into a familiar form without scattering processes. Suppose that non-demolition condition $[\tilde{x}(t), y_q(s)^T] = 0$, $\forall\, t \geq s \geq 0$ holds. So, we apply relation (3.26) in Lemma 3.7 to the output (3.3) to give $y_q = \widetilde{D}\bar{y}_q$ with $\bar{y}_q$ defined as $d\bar{y}_q = \bar{C}\tilde{x}(t)dt + dw(t)$, where $\bar{C} = \Theta_w \widetilde{B}^T \widetilde{\Theta}$. Then, a *reduced* system for the *augmented* system (3.2)-(3.3) is defined as

$$\begin{aligned} d\tilde{x}(t) &= \widetilde{A}\tilde{x}(t)dt + \widetilde{B}dw(t), \\ d\bar{y}_q &= \bar{C}\tilde{x}(t)dt + dw(t). \end{aligned} \tag{3.28}$$

It is straightforward to verify that the *reduced* system (3.28) is physically realizable in the sense of Theorem 3.1. The definition of physical realizability of an *augmented* system of the system (3.1) is given below.

Definition 3.2. An *augmented* system (3.2)-(3.3) of the system (3.1) is said to be physically realizable if the following statements hold:

1. The *reduced* system (3.28) is physically realizable in the sense of Theorem 3.1.

2. For the *augmented* system (3.2)-(3.3), non-demolition condition $[\tilde{x}(t), y_q(s)^T] = 0$, $\forall\, t \geq s \geq 0$ holds.

3. $\widetilde{D} = D_q$ is of the form $[I_{n_{y_q}} \ 0]\widetilde{V}$ with $\widetilde{V}$ a symplectic matrix [45] or unitary symplectic [81] such that relation (3.13) holds.

Then we consider physical realizability conditions of the system (3.11). Classical systems are always regarded as being physically realizable since they can be approximately built via digital and analog circuits. Thus, we just need to make sure that output equation (3.11) is classical. Now, we can present a formal definition of physical realizability of the system (3.7).

Definition 3.3. A system of the *standard* form (3.7) is said to be physically realizable if the following statements hold:

1. There exists an *augmented* system (3.2)-(3.3) of the system (3.1) with D_q satisfying (3.13), which is physically realizable in the sense of Definition 3.2.

2. For the system (3.7), non-demolition condition $[x(t), y(s)^T] = 0$, $\forall\, t \geq s \geq 0$ holds.

3. The output (3.11) and system variables x_c both represent classical stochastic processes in the sense of commutation relations

$$[x_c(t), x_c^T(s)] = 0,$$

$$[x_c(t), y_c^T(s)] = 0,$$

and

$$[y_c(t), y_c^T(s)] = 0,$$

for all $t, s \geq 0$, where $[x_c(0), y_c(0)^T] = 0$ and $[y_c(0), y_c(0)^T] = 0$.

Theorem 3.9. *A system of the form* (3.7) *is physically realizable, if and only if matrices A, B, C, and D satisfy the following constraints:*

$$A\Theta_n + \Theta_n A^T + B\Theta_w B^T = 0, \tag{3.29}$$

$$B\Theta_w D^T = -\Theta_n C^T, \tag{3.30}$$

$$D\Theta_w D^T = \mathrm{diag}(\Theta_{y_q}, 0_{n_{y_c} \times n_{y_c}}). \tag{3.31}$$

Proof. First, let conditions (3.29)-(3.31) hold. Multiplying both sides of (3.30) by $\begin{bmatrix} I_{2n_{y_q}} \\ 0 \end{bmatrix}$, we get

$$B\Theta_w D_q^T = -\Theta_n C_q^T. \tag{3.32}$$

It follows by inspection that under conditions (3.29) and (3.32), there exist matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ and $\widetilde{\Theta}$ defined in Subsection 3.2.1 satisfying the following conditions

$$\widetilde{A}\widetilde{\Theta} + \widetilde{\Theta}\widetilde{A}^T + \widetilde{B}\Theta_w\widetilde{B}^T = 0, \tag{3.33}$$

$$\widetilde{C} = \widetilde{D}\Theta_w\widetilde{B}^T\widetilde{\Theta}, \tag{3.34}$$

$$\widetilde{D} = D_q, \tag{3.35}$$

where A', A'', B' satisfy the following relations:

$$B'\Theta_w D_q^T = [0 \quad I]C_q^T, \tag{3.36}$$

$$\begin{bmatrix} 0 & I \end{bmatrix} A'^T - A' \begin{bmatrix} 0 \\ I \end{bmatrix} = B'\Theta_w B'^T, \tag{3.37}$$

$$A'' = \left(A'\Theta_n - [0 \quad I]A^T + B'\Theta_w B^T\right) \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{3.38}$$

From (3.3) and (3.34), we get

$$\bar{C} = \Theta_w\widetilde{B}^T\widetilde{\Theta}. \tag{3.39}$$

So, conditions (3.33) and (3.39) imply the *reduced* system (3.28) is physically realizable in the sense of Theorem 3.1. By Lemma 3.7, condition (3.34) implies that $[\tilde{x}(t), y_q(s)^T] = 0$, $\forall\ t \geq s \geq 0$ holds, which satisfies the second condition of Definition 3.2. Multiplying both sides of (3.31) by $[I \quad 0]$ and $\begin{bmatrix} I \\ 0 \end{bmatrix}$, we can obtain (3.13). Thus, the *augmented* system (3.2)-(3.3) is physically realizable in the sense of Definition 3.2. By Lemma 3.8, condition (3.30) implies that $[x(t), y(s)^T] = 0$, $\forall\ t \geq s \geq 0$ holds, which satisfies the second condition of Definition 3.3. Combining conditions (3.15), (3.30) and using the same approach as shown in the proof of Lemma 3.7, we get

$$d_t[y_c(t), y_c(s)^T] = 0,$$

$$d_t[y_c(s), y_c(t)^T] = 0,$$

$$d[y_c(t), y_c(t)^T] = 0,$$

for all $t \geq s \geq 0$ (here the symbol d_t denotes the forward differential with respect to t), which imply that $[y_c(t), y_c(s)^T] = 0$ holds for all $t, s \geq 0$ under the fact that $[y_c(0), y_c(0)^T] = 0$ given in Definition 3.3. Applying a similar trick, we have

$$[x_c(t), x_c(s)^T] = 0,$$

$$[y_c(t), x_c(s)^T] = 0,$$

for all $t, s \geq 0$. We infer that output (3.11) and x_c are both classical in the sense of the third item of Definition 3.3. Therefore, we conclude that the system (3.7) is physically realizable in the sense of Definition 3.3, which shows that (3.29)-(3.31) are sufficient for realizability.

Conversely, now suppose that a system of the form (3.7) is physically realizable. It follows from Theorem 3.1 and the first item of Definition 4.1 that condition (3.33) holds. Then, reading off the first n rows and columns of both sides of (3.33) gives us condition (3.29). By the second item of Definition 3.3, we have condition (3.30) in the sense of Lemma 3.8. Since the system (3.7) is a *standard* form, it follows from the third item of Definition 3.1 that condition (3.31) holds. Therefore, constraints (3.29)-(3.31) are necessary for realizability. □

3.3.2 Physical realizability for the general form

Without loss of generality, we will also need to give the physical realizability definition and constraints for the *general* form (3.6).

Definition 3.4. A system of the *general* form (3.6) is said to be physically realizable if its corresponding *standard* form (3.7) is physically realizable in the sense of Definition 3.3.

Theorem 3.10. *A system of the* general *form* (3.6) *is physically realizable, if and only if the following constraints are satisfied:*

$$\mathbf{A}\boldsymbol{\Theta}_n + \boldsymbol{\Theta}_n\mathbf{A}^T + \mathbf{B}\Theta_v\mathbf{B}^T = 0, \tag{3.40}$$

$$\mathbf{B}\Theta_v\mathbf{D}^T = -\boldsymbol{\Theta}_n\mathbf{C}^T, \tag{3.41}$$

$$\mathbf{D}\Theta_v\mathbf{D}^T = \Theta_{\mathbf{y}}. \tag{3.42}$$

Proof. Suppose that equations (3.40)-(3.42) hold. It follows from Theorem 3.6 that the general system (3.6) can be transformed to its corresponding *standard* system (3.7). Using relations (3.25) and equations (3.40)-(3.42), we get constraints (3.29)-(3.31). The corresponding *standard* system (3.7) is physically realizable in the sense of Theorem 3.9. Therefore, we conclude that (3.40)-(3.42) are sufficient for physical realizability.

Conversely, suppose that a system of the general form (3.6) is physically realizable. It follows from Definition 3.4 and Theorem 3.9 that constraints (3.29)-(3.31) hold. Conditions (3.40)-(3.42) can be obtained from constraints (3.29)-(3.31) by direct substitution using relations (3.25). Thus, constraints (3.40)-(3.42) are necessary for realizability. □

3.4 Systematic synthesis of mixed systems

In this section, we will present our main results of this chapter, which are illustrated with two examples.

3.4.1 Main synthesis theorem

By Definition 3.4 and Theorem 3.6, we know that a system of the general form (3.6) can be physically realized, if its corresponding *standard* form (3.7) is physically realizable. Therefore, our purpose in this section is to develop a network synthesis theory only for a mixed quantum-classical system of the *standard* form (3.7) that generalizes the results in [84].

Lemma 3.11. *The mixed quantum-classical linear stochastic system* (3.7) *is physically realizable if and only if conditions* (3.13), (3.14), (3.15) *and the constraints below are all satisfied*

$$A_{qq}\Theta_{n_q} + \Theta_{n_q}A_{qq}^T + B_q\Theta_w B_q^T = 0, \tag{3.43}$$
$$A_{cq}\Theta_{n_q} + B_c\Theta_w B_q^T = 0, \tag{3.44}$$
$$B_c\Theta_w B_c^T = 0, \tag{3.45}$$
$$B_c\Theta_w D_q^T = 0, \tag{3.46}$$
$$B_q\Theta_w D_q^T = -\Theta_{n_q}C_{qq}^T, \tag{3.47}$$
$$B_c\Theta_w D_c^T = 0, \tag{3.48}$$
$$B_q\Theta_w D_c^T = -\Theta_{n_q}C_{cq}^T. \tag{3.49}$$

Proof. By Theorem 3.9, it is easily checked that conditions (3.13)-(3.15) are equivalent to (3.31) while (3.43)-(3.49) are equivalent to (3.29)-(3.30). □

Lemma 3.12. *If a matrix D_q satisfies the following condition,*

$$D_q\Theta_w D_q^T \quad = \quad \Theta_{y_q}, \tag{3.50}$$

then there exists a matrix N such that $\begin{bmatrix} D_q \\ N \end{bmatrix}\Theta_w \begin{bmatrix} D_q \\ N \end{bmatrix}^T = \Theta_w$, *so that D_q can be embedded into a symplectic matrix* $\widetilde{V} = \begin{bmatrix} D_q^T & N^T \end{bmatrix}^T$.

Proof. The matrix D_q can be written in the form of

$$D_q = \begin{bmatrix} I & 0_{2n_{yq}\times(2m-2n_{yq})} \end{bmatrix} \begin{bmatrix} D_q \\ N \end{bmatrix}, \tag{3.51}$$

where N is a $(2m - 2n_{y_q}) \times 2m$ matrix. Let the rows of D_q be denoted by $d_1, d_2, \cdots, d_{2n_{yq}}$. Let $P(a|b_1, b_2, \cdots, b_k)$ denote the orthogonal projection of the row vector a onto the subspace spanned by the row vectors $b_1, b_2, \cdots, b_k$.

Now, we want to build a $(2m - 2n_{y_q}) \times 2m$ matrix N, following analogously the construction of the matrix V defined in [84, Lemma 6].

First, choose a row vector $v_1^{(1)} \in \mathbb{R}^{2m}$ linearly independent of $d_1, d_2, \cdots, d_{2n_{yq}}$, and set

$$v_1^{(2)} = v_1^{(1)} - P(v_1^{(1)}|d_1, d_2, \cdots, d_{2n_{yq}})$$

and

$$v_1 = v_1^{(2)}\Theta_w.$$

Next, choose a row vector $v_2^{(1)} \in \mathbb{R}^{2m}$ linearly independent of $d_1, d_2, \cdots, d_{2n_{yq}}$ and set

$$v_2^{(2)} = v_2^{(1)} - P(v_2^{(1)}|d_1, d_2, \cdots, d_{2n_{yq}})$$

and

$$v_2 = v_2^{(2)}\Theta_w.$$

Repeat this procedure analogously for $k = 3, \cdots, m - n_{y_q}$ to obtain vectors $v_3, v_4, \cdots, v_{m-n_{yq}}$.

Then, we choose a row vector $w_1^{(1)} \in \mathbb{R}^{2m}$ that is linearly independent of $d_1, d_2, \cdots, d_{2n_{yq}}$ and $v_2, v_3, \cdots, v_{m-n_{yq}}$ such that

$$(w_1^{(1)} - P(w_1^{(1)}|d_1, d_2, \cdots, d_{2n_{yq}}, v_2, v_3, \cdots, v_{m-n_{yq}}))v_1^T \neq 0.$$

We set

$$w_1^{(2)} = w_1^{(1)} - P(w_1^{(1)}|d_1, d_2, \cdots, d_{2n_{yq}}, v_2, v_3, \cdots, v_{m-n_{yq}})$$

and

$$w_1 = w_1^{(2)}\Theta_w/(v_1 w_1^{(2)T}).$$

Next, we choose $w_2^{(1)} \in \mathbb{R}^{2m}$ that is linearly independent of $d_1, d_2, \cdots, d_{2n_{yq}}$ and $v_1, w_1, v_3, v_4, \cdots, v_{m-n_{yq}}$, such that

$$(w_1^{(1)} - P(w_1^{(1)}|d_1, d_2, \cdots, d_{2n_{yq}}, v_1, w_1, v_3, v_4, \cdots, v_{m-n_{yq}}))v_2^T \neq 0.$$

Set

$$w_2^{(2)} = w_2^{(1)} - P(w_2^{(1)}|d_1, d_2, \cdots, d_{2n_{yq}}, v_1, w_1, v_3, v_4, \cdots, v_{m-n_{yq}})$$

and

$$w_2 = w_2^{(2)}\Theta_w/(v_2 w_2^{(2)T}).$$

Repeat the procedure in an analogous manner to construct $w_3, w_4, \cdots, w_{m-n_{yq}}$. So the matrix N is defined as

$$N = [v_1^T \; w_1^T \; v_2^T \; w_2^T \; \cdots \; v_{m-n_{yq}}^T \; w_{m-n_{yq}}^T]^T. \tag{3.52}$$

By the above construction, we readily verify that the $2m \times 2m$ matrix $\widetilde{V} = \begin{bmatrix} D_q^T & N^T \end{bmatrix}^T$ is a symplectic matrix ($\widetilde{V}\Theta_w\widetilde{V}^T = \Theta_w$) using equations (3.50) and (3.52). $\square$

Suppose that the system (3.7) is physically realizable. We are now in a position to explain how to realize the system (3.7) as an interconnection of a quantum system G_1 and a classical system G_2. We define G_1 to be a fully quantum system given by

$$\begin{aligned} dx_q(t) &= A_{qq}x_q(t)dt + B_q dw'(t) + Eu(t)dt, &(3.53)\\ dy_q(t) &= C_{qq}x_q(t)dt + D_q dw'(t), &(3.54)\\ dy'_q(t) &= C'_{qq}x_q(t)dt + D'_q dw'(t), &(3.55)\end{aligned}$$

where x_q, y_q, A_{qq}, B_q, C_{qq}, D_q are defined as in (3.8) and (3.10). Here $D'_q = N$ and $C'_{qq} = D'_q\Theta_w B_q^T\Theta_{n_q}$, where N can be obtained from D_q using Lemma 3.12. Note in particular that $D'_q\Theta_w(D'_q)^T = \Theta_{y'_q}$ and $\begin{bmatrix} D_q \\ D'_q \end{bmatrix}\Theta_w\begin{bmatrix} D_q \\ D'_q \end{bmatrix}^\top = \Theta_w$. Here $w'(t) = \begin{bmatrix} w'_1(t) \\ w'_2(t) \end{bmatrix}$, where $w'_1(t)$ and $w'_2(t)$ are two vectors of independent vacuum boson fields and will be defined later. The Hamiltonian of G_1 is given by $H_q = \frac{1}{2}x_q^T R_q x_q + x_q^T K_q u(t)$ with a real matrix $K_q = -\Theta_{n_q}E$; $u(t)$ a vector of real locally square integrable functions, representing a classical control signal; see [12], [46] for how to implement the linear part $x_q^T K_q u$ using classical devices. We then define G_2 be a classical system given by

$$\begin{aligned} dx_c(t) &= A'_{cc}x_c(t)dt + B'_c du_c(t), &(3.56)\\ dy_c(t) &= C'_{cc}x_c(t)dt + D'_c du_c(t), &(3.57)\\ y'_{c1}(t) &= C'_{c1}x_c(t), &(3.58)\\ y'_{c2}(t) &= C'_{c2}x_c(t), &(3.59)\end{aligned}$$

where x_c and y_c are defined as in (3.9) and (3.11). Here $u_c(t)$ is real locally square-integrable classical signal satisfying

$$du_c(t) = \alpha_c(t)dt + dw_c(t),$$

where $w_c(t)$ is a vector of independent standard classical Wiener processes, and $\alpha_c(t)$ is a vector of real stochastic processes of locally bounded variation.

The rest undefined system matrices, input and output signals appearing in (3.53)-(3.59) can be found in the following theorem, which presents a feedback architecture for the realization of the system (3.7).

Theorem 3.13. *Assume that the system* (3.7) *is physically realizable and its system matrices are all already known. Let* $C'_c = \begin{bmatrix} C'_{c1} \\ C'_{c2} \end{bmatrix}$ *and there exist matrices* C'_c, $\mathbf{G}$, B'_c, *and* D'_c, *such that*

$$\begin{aligned} D_q C'_c &= C_{qc}, && (3.60)\\ B'_c \mathbf{G} C'_{qq} &= A_{cq}, && (3.61)\\ B'_c \mathbf{G} D'_q &= B_c, && (3.62)\\ D'_c \mathbf{G} C'_{qq} &= C_{cq}, && (3.63)\\ D'_c \mathbf{G} D'_q &= D_c. && (3.64) \end{aligned}$$

Then the feedback network shown in Figure 3.1, with the identification

$$\begin{aligned} u(t) &= x_c(t),\\ du_c(t) &= \mathbf{G} dy'_q(t),\\ w'_1(t) &= y'_{c1}(0) + \int_0^t y'_{c1}(s)ds + w_1(t),\\ w'_2(t) &= y'_{c2}(0) + \int_0^t y'_{c2}(s)ds + w_2(t),\\ E &= A_{qc} - B_q C'_c,\\ A'_{cc} &= A_{cc} - B'_c \mathbf{G} D'_q C'_c,\\ C'_{cc} &= C_{cc} - D'_c \mathbf{G} D'_q C'_c, \end{aligned}$$

is a physical realization of the system (3.7) *consisting of a quantum system* G_1 *described by* (3.53)-(3.55) *and a classical system* G_2 *described by* (3.56)-(3.59), *where the network* G *can realize the matrix* $\mathbf{G} = KV$ *to produce classical signals* $u_c = \mathbf{G} y'_q(t)$ *satisfying*

$$[u_c(t), u_c(s)^T] = 0,$$

$\forall t, s \geq 0$, where $K = [k_1^T \; k_2^T \; \cdots k_r^T]^T$ ($k_j = [0 \; 0 \; \cdots \; 0 \; 1 \; 0 \; \cdots \; 0] \in \mathbb{R}^{1\times(n_c+n_{y_c})}$ with the 1 in the (2j-1)-th position) and V is a symplectic matrix; the network S realizes a symplectic transformation $\begin{bmatrix} D_q \\ D_q' \end{bmatrix}$.

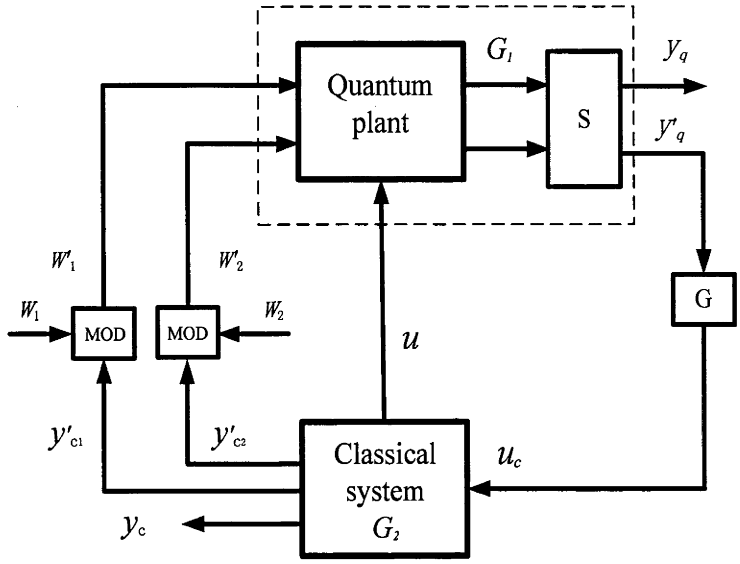

Figure 3.1: Feedback interconnection of a quantum system G_1 and a classical system G_2. The two sets of modulators (MODs) displace the vectors of vacuum quantum fields w_1 and w_2 to produce the quantum signals $w_1'(t)$ and $w_2'(t)$ by the classical vector signals $y_{c_1}'(t)$ and $y_{c_2}'(t)$, respectively. The network G corresponds to measurement processes.

Proof. First, we will show that under physical realization constraints (3.13)-(3.15) and (3.43)-(3.49), we can build matrices C_c', $\mathbf{G}$, B_c', and D_c' to satisfy (3.60)-(3.64). It follows from equations (3.13) with invertible Θ_{y_q} that the wide matrix D_q has full row rank and $\text{rank}(D_q) = \text{rank}\left(\begin{bmatrix} D_q & C_{qc} \end{bmatrix}\right)$. So, the solution of equation (3.60) is written as

$$C_c' = D_q^T(D_qD_q^T)^{-1}C_{qc} + N(D_q),$$

where $N(D_q)$ denotes a matrix of the same dimension as C_c' whose columns are in the kernel space of D_q. Let $\bar{B}_c = B_c'\mathbf{G}$ and $\bar{D}_c = D_c'\mathbf{G}$. Now we will show that the equation $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix} D_q' = \begin{bmatrix} B_c \\ D_c \end{bmatrix}$ has solutions for $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}$. Combining equations (3.14) and (3.46) gives

$$\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w (D'_q)^T = \begin{bmatrix} 0_{2n_{yq}\times(2m-2n_{yq})} \\ \Theta_{y'_q} \end{bmatrix}, \tag{3.65}$$

$$\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix} = \begin{bmatrix} 0_{2n_{yq}\times(n_c+n_{yc})} \\ D'_q\Theta_w \begin{bmatrix} B_c^T \; D_c^T \end{bmatrix} \end{bmatrix}, \tag{3.66}$$

where $\Theta_{y'_q} = \mathrm{diag}_{(m-n_{yq})}(J)$. From equations (3.65) and (3.66), we can infer that

$$\mathrm{rank}\left((D'_q)^T\right) = \mathrm{rank}\left(\Theta_{y'_q}\right),$$

and

$$\mathrm{rank}\left(\begin{bmatrix} B_c^T & D_c^T \end{bmatrix}\right) = \mathrm{rank}\left(D'_q\Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix}\right).$$

Given that $\Theta_{y'_q}$ has full row rank, we can conclude that

$$\mathrm{rank}\left(\Theta_{y'_q}\right) = \mathrm{rank}\left(\begin{bmatrix} \Theta_{y'_q} & D'_q\Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix} \end{bmatrix}\right),$$

which implies that $\mathrm{rank}\left((D'_q)^T\right) = \mathrm{rank}\left(\begin{bmatrix} (D'_q)^T & B_c^T & D_c^T \end{bmatrix}\right)$. So, there exist $\bar{B}_c$ and $\bar{D}_c$ satisfying (3.62) and (3.64), respectively. From constraints (3.44), (3.49), and $C'_{qq} = D'_q\Theta_w B_q^T \Theta_{n_q}$, we conclude that equation (3.62) implies (3.61), and (3.64) implies (3.63), respectively.

Then it is straightforward to verify from (3.60)-(3.64) that interconnecting the system G_1 and the system G_2 gives the *standard* form (3.7), or equivalently described by (3.8)-(3.11). Now let us check that the system G_1 is a physically realizable fully quantum system. It follows from conditions (3.13) and (3.43) that the system G_1 satisfies constraints (3.29) and (3.31) in the sense of Theorem 3.9 with matrices A, B, D, Θ_n and $\mathrm{diag}(\Theta_{y_q}, 0_{n_{yc}\times n_{yc}})$ replaced by corresponding matrices A_{qq}, B_q, D_q, Θ_{n_q} and Θ_{y_q}, respectively. The system G_1 also satisfies constraint (3.30) with its matrices replaced by corresponding matrices in equations (3.56)-(3.59) with the proof below:

$$\begin{aligned} -\Theta_{n_q}\left(\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w B_q^T \Theta_{n_q}\right)^T &= -\Theta_{n_q}\Theta_{n_q}^T B_q \Theta_w^T \begin{bmatrix} D_q \\ D'_q \end{bmatrix}^T \\ &= B_q\Theta_w \begin{bmatrix} D_q \\ D'_q \end{bmatrix}^T. \end{aligned}$$

So, the system G_1 is a physically realizable quantum system, where y'_q is the input to the network G. Given that $D'_q\Theta_w(D'_q)^T = \Theta_{y'_q} = \mathrm{diag}_{(m-n_{yq})}(J)$, we get from equations (3.15), (3.45), (3.48), (3.62) and (3.64) that

$$\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix} \Theta_{y'_q} \begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}^T = 0. \tag{3.67}$$

From equation (3.67), we know that the matrix $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}$ with rank $\left(\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}\right) = r$ can be decomposed as

$$\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix} = PZKV = \begin{bmatrix} P_1 Z \\ P_2 Z \end{bmatrix} KV$$

(see [84, Lemma 6] for details), where $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is a permutation matrix; Z is a matrix of the form $Z = \begin{bmatrix} I_r \\ X \end{bmatrix}$ if $r < n_c + n_{y_c}$, where X is some $(n_c + n_{y_c} - r) \times r$ matrix, and $Z = I_{(n_c+n_{y_c})}$ if $r = n_c + n_{y_c}$, and V is a symplectic matrix. So, we can define

$$\begin{aligned} \mathbf{G} &= KV, \\ B'_c &= P_1 Z, \\ D'_c &= P_2 Z, \end{aligned}$$

and the symplectic transformation V can be realized as a suitable static quantum optical network. Applying K to $Vy'_q(t)$ is to measure the first r amplitude quadrature components of $Vy'_q(t)$ to obtain the measurement result $u_c(t) = KVy'_q(t)$. So, $\mathbf{G}$ represents measurement processes [84]. Then we can show that $\forall t, s \geq 0$,

$$\begin{aligned} [u_c(t), u_c(s)^T] &= \mathbf{G}[y'_q(t), y'_q(s)^T]\mathbf{G}^T \\ &= \delta_{ts}\mathbf{G}\mathrm{diag}_{n_{y'_q}/2}(J)\mathbf{G}^T \\ &= \delta_{ts} \times 0 = 0, \end{aligned}$$

which implies that u_c is a classical signal. Thus G_2 described by (3.56)-(3.59) is a classical system, where the classical vector signals $y'_{c_1}(t)$ and $y'_{c_2}(t)$ are used to produce the quantum signals $w'_1(t)$ and $w'_2(t)$ injected into G_1. □

3.4.2 Examples

In this subsection, two examples are presented to check our main results.

Example 3.1

Consider a mixed quantum-classical system of the *standard* form with A, B, C, D satisfying conditions (3.29)-(3.31), such that

$$A = \begin{bmatrix} 0 & 0.1 & -1 \\ -0.1 & 0 & -3 \\ 0.4 & 0 & -5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & -0.2 \\ 4 & 0 & 4-5 & 0 & 3 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 4.8 & 0 & -42 \\ 0 & 0 & 0.35 \\ 0 & 0 & -2 \\ 0 & 0 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 0 & 10 & 0 & 6 & 0 \\ 0 & 0.04 & 0 & 0.05 & 0 & 0.03 \\ 3 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix}.$$

Following the construction in the proof of Theorem 3.13, we have the quantum system G_1 given by

$$dx_q(t) = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} du(t),$$

$$dy_q(t) = \begin{bmatrix} 4.8 & 0 \\ 0 & 0 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 8 & 0 & 10 & 0 & 6 & 0 \\ 0 & 0.04 & 0 & 0.05 & 0 & 0.03 \end{bmatrix} \times \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix},$$

$$dy'_q(t) = \begin{bmatrix} 0.04 & 0 \\ 0 & 0 \\ -0.2 & 0 \\ 0 & 0 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0.4 & 0 & -0.5 & 0 & 0.3 & 0 \\ 0 & 0.8 & 0 & -1 & 0 & 0.6 \\ 3 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix} \times \begin{bmatrix} dw'_1(t) \\ dw'_2(t) \end{bmatrix}$$

and the classical system G_2 described by

$$
\begin{aligned}
dx_c(t) &= -34x_c(t) + [0 \quad 10]du_c(t), \\
dy_c(t) &= \begin{bmatrix} -5 \\ 5.6 \end{bmatrix} x_c(t)dt + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} du_c(t), \\
y'_{c1}(t) &= \begin{bmatrix} 1 & 3 \end{bmatrix}^T x_c(t), \\
y'_{c2}(t) &= \begin{bmatrix} -5 & 1 & 0 & 6 \end{bmatrix}^T x_c(t).
\end{aligned}
$$

It can be easily checked that the closed-loop system described by (3.7) with the above matrices A, B, C, D is obtained by making the identification

$$
\begin{aligned}
u(t) &= x_c(t), \\
du_c(t) &= \begin{bmatrix} 0.04 & 0 \\ 0 & 0 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0.4 & 0 & -0.5 & 0 & 0.3 & 0 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix} \times \\
&\quad \begin{bmatrix} dw'_1(t) \\ dw'_2(t) \end{bmatrix}, \\
dw'_1(t) &= \begin{bmatrix} 1 & 3 \end{bmatrix}^T x_c(t)dt + dw_1(t), \\
dw'_2(t) &= \begin{bmatrix} -5 & 1 & 0 & 6 \end{bmatrix}^T x_c(t) + dw_2(t),
\end{aligned}
$$

where $\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Example 3.2

Consider a mixed quantum-classical system of the *standard* form with A, B, C, D satisfying conditions (3.29)-(3.31), such that

$$
A = \begin{bmatrix} -9 & -3 & -1 \\ 1 & -7 & -3 \\ -0.72 & -0.6 & -12 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 1 & 2 & -7 & 0 & -3 & 5 \\ 2 & 5 & 1 & -3 & 6 & -8 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix},
$$

$$
C = \begin{bmatrix} 38 & 46 & -42 \\ 0.31 & 0.4 & 0.35 \\ 4.2 & -6 & 5 \end{bmatrix},
$$

$$D = \begin{bmatrix} 8 & 0 & 10 & 0 & 6 & 0 \\ 0 & 0.04 & 0 & 0.05 & 0 & 0.03 \\ 0 & 0.8 & 0 & -1 & 0 & 0.6 \end{bmatrix}.$$

Following the construction in the proof of Theorem 3.13, we have the quantum system G_1 given by

$$dx_q(t) = \begin{bmatrix} -9 & -3 \\ 1 & -7 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 1 & 2 & -7 & 0 & -3 & 5 \\ 2 & 5 & 1 & -3 & 6 & -8 \end{bmatrix} \begin{bmatrix} dw_1'(t) \\ dw_2'(t) \end{bmatrix} + \begin{bmatrix} -30.4 \\ 22.2 \end{bmatrix} du(t),$$

$$dy_q(t) = \begin{bmatrix} 38 & 46 \\ 0.31 & 0.4 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 8 & 0 & 10 & 0 & 6 & 0 \\ 0 & 0.04 & 0 & 0.05 & 0 & 0.03 \end{bmatrix} \begin{bmatrix} dw_1'(t) \\ dw_2'(t) \end{bmatrix},$$

$$dy_q'(t) = \begin{bmatrix} dy_{q_1}'(t) \\ dy_{q_2}'(t) \end{bmatrix} = \begin{bmatrix} -1.1 & 2.3 \\ 4.2 & -6 \\ -47 & -14 \\ -0.72 & -0.6 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0.4 & 0 & -0.5 & 0 & 0.3 & 0 \\ 0 & 0.8 & 0 & -1 & 0 & 0.6 \\ 3 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix} \times \begin{bmatrix} dw_1'(t) \\ dw_2'(t) \end{bmatrix}$$

and the classical system G_2 described by

$$\begin{aligned} dx_c(t) &= -12x_c(t) + [3.6836 \quad -0.4345]du_c(t), \\ dy_c(t) &= 12x_c(t)dt + [-0.2065 \quad 1.2388]du_c(t), \\ y_{c1}'(t) &= 0, \\ y_{c2}'(t) &= \begin{bmatrix} y_{c21}'(t) \\ y_{c22}'(t) \end{bmatrix} = \begin{bmatrix} -4.2 & 7 & 0 & 0 \end{bmatrix}^T x_c(t). \end{aligned}$$

It can be easily checked that the closed-loop system described by (3.7) with the above matrices A, B, C, D is obtained by making the identification

$$u(t) = x_c(t),$$

$$du_c(t) = \begin{bmatrix} 0.2086 & -0.7489 \\ 3.4253 & -4.9684 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0 & 0.1109 & 0 & -0.0971 & 0 & 0.014 \\ 0 & 0.6643 & 0 & -0.8235 & 0 & 0.4867 \end{bmatrix} \times \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix},$$

$$dw_1'(t) = dw_1(t),$$

$$dw_2'(t) = \begin{bmatrix} dw_{21}'(t) \\ dw_{22}'(t) \end{bmatrix} = \begin{bmatrix} -4.2 & 7 & 0 & 0 \end{bmatrix}^T x_c(t) + \begin{bmatrix} dw_{21}(t) \\ dw_{22}(t) \end{bmatrix},$$

where $du_c(t) = \begin{bmatrix} du_{c_1} \\ du_{c_2} \end{bmatrix}$, $dw_2 = \begin{bmatrix} dw_{21}(t) \\ dw_{22}(t) \end{bmatrix}$ and $\mathbf{G} = \begin{bmatrix} 0 & 0.0971 & 0 & 0.2769 \\ 0 & 0.8235 & 0 & 0.0462 \end{bmatrix}$.
The realization of this mixed system is shown in Figure 3.2.

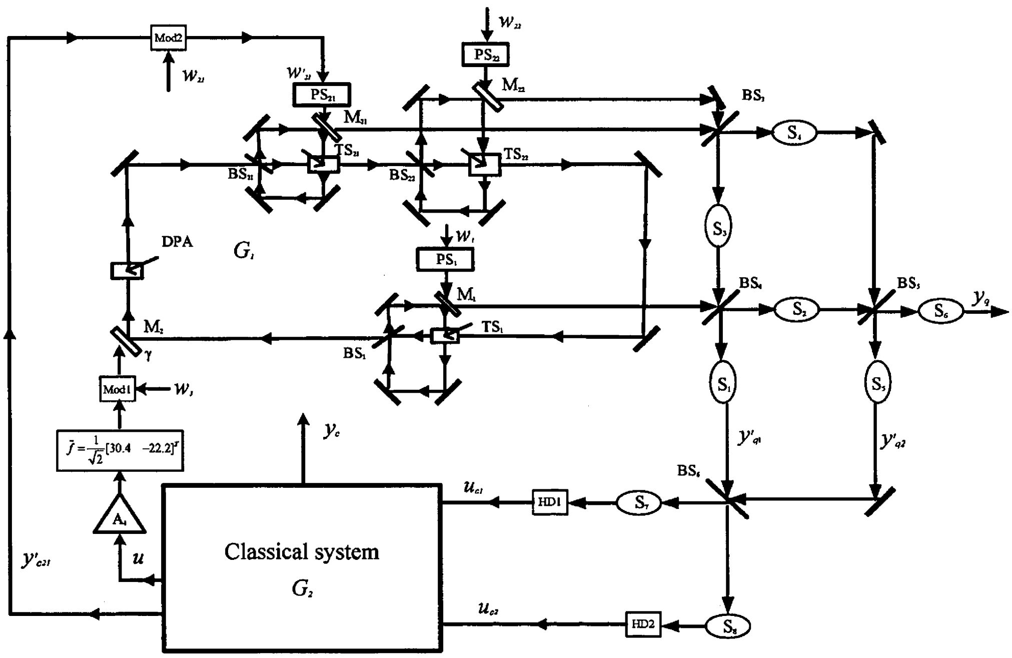

Figure 3.2: A realization of the mixed quantum-classical system. Black rectangles denote fully reflecting mirrors. $\mathrm{M}_1, \mathrm{M}_{21}, \mathrm{M}_{22}$ and M_3 represent transmitting mirrors with coupling constants $\kappa_1, \kappa_{21}, \kappa_{22}$ and γ, respectively ($\gamma \ll 1, \gamma \ll \kappa_1, \kappa_{21}, \kappa_{22}$); $\mathrm{BS}_1, \mathrm{BS}_{21}, \mathrm{BS}_{22}, \mathrm{BS}_3, \mathrm{BS}_4, \mathrm{BS}_5$ and BS_6 represent beam splitters; $\mathrm{TS}_1, \mathrm{TS}_{21}$ and TS_{22} represent two-mode squeezers; PS_1, PS_{21}, PS_{22} represent phase shifters; $\mathrm{S_i}$ ($i = 1, 2 \cdots, 8$) represents a squeezer; DPA is short for degenerate parametric amplifier; Modi ($i = 1, 2, 3, 4$) represents a modulator; HDi ($i = 1, 2$) represents a homodyne detector; A_1 is an amplifier with gain $\frac{1}{\sqrt{\gamma}}$. $\tilde{f}$ can be realized using a computer. w_1, w_{21}, w_{22}, w_3 are vacuum noises and the contribution of w_3 to quantum system noise is negligible compared to that of other vacuum noises.

3.5 Concluding remarks

In this chapter, two forms (a general form and a *standard* form) are presented for the physical realization of the mixed quantum-classical linear stochastic system.

We have shown the relation between these two forms. Three physical realization constraints are derived for the *standard* form and the general form, respectively. A network theory is developed for synthesizing linear dynamical mixed quantum-classical stochastic systems of the *standard* form in a systematic way, and we then propose one feedback architecture for this realization. Our results are illustrated with several examples.

Chapter 4

Quantum Optical Realization of Classical Linear Stochastic Systems

In Chapter 3, we have presented how a physically realizable mixed quantum-classical linear stochastic system can be realized as a feedback interconnection circuit consisting of a fully quantum linear system implemented by quantum optical devices, and a classical linear system implemented by standard electrical and electronic devices, together with modulators and homodyne detectors. However, classical systems have the relatively slow processing speed while quantum optical systems typically have much higher bandwidth than electronic devices, meaning faster response and processing times, and have a potential for providing better performance than classical systems. So, the performance of a mixed quantum-classical system will be affected if the classical part is not fast enough. To solve this problem, in this chapter, we develop a method that show how classical linear stochastic systems can be physically implemented using quantum optical components. A complete procedure is proposed for a stable quantum linear stochastic system realizing a given stable classical linear stochastic system.

4.1 Introduction and motivation

Linear systems are of basic importance to control engineering, and also arise in the modeling and control of quantum systems; see [3] and [27]. As is known to all, the state space representation provides a convenient and compact way to model and analyze systems with multiple inputs and outputs. A classical linear system described by the state space representation can be approximately realized using electrical and electronic components by linear electrical network synthesis theory, see [106]. For example, consider a classical system given by

$$\begin{aligned} \dot{\xi}(t) &= -\xi(t) + v_1(t), \\ y(t) &= \xi(t) + v_2(t), \end{aligned} \tag{4.1}$$

where $\xi(t)$ is the state, $v_1(t)$ and $v_2(t)$ are inputs, and $y(t)$ is the output. Implementation of the system (4.1) at the hardware level is shown in Figure 4.1. Analogously to the electrical network synthesis theory of how to synthesize linear analog circuits from basic electrical components, [81] have proposed a quantum network synthesis theory (briefly introduced in Section 2.4 of Chapter 2), which details how to realize a quantum system described by state space representations using quantum optical devices.

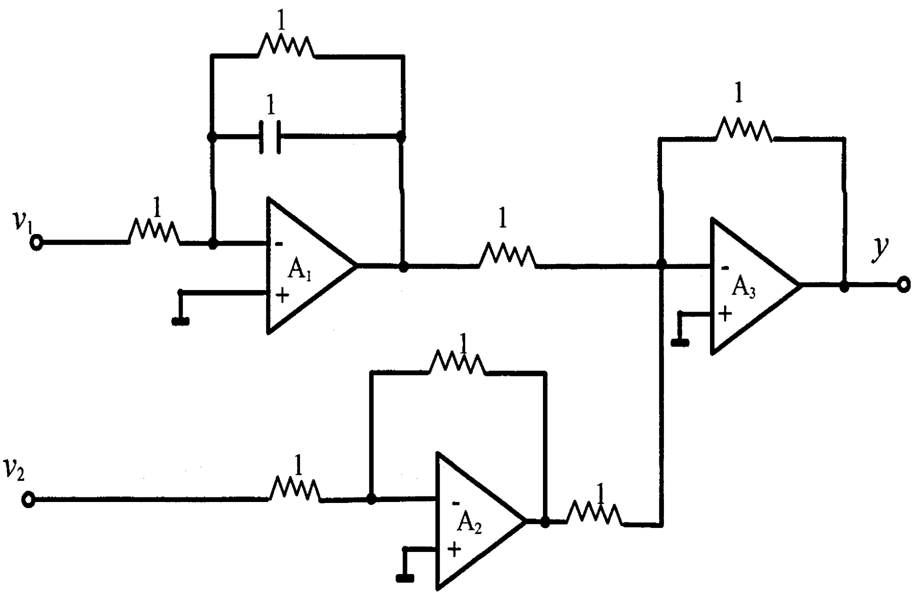

Figure 4.1: Classical hardware implementation of the system (4.1).

The purpose of this chapter is to address this issue of quantum physical realization for a class of linear systems. For example, the quantum physical realization of the system (4.1) is shown in Figure 4.2 (see **Example 4.1** for more details). The essential quantum optical components used in Figure 4.2 include optical cavities, degenerate parametric amplifiers (DPA), phase shifters, beam

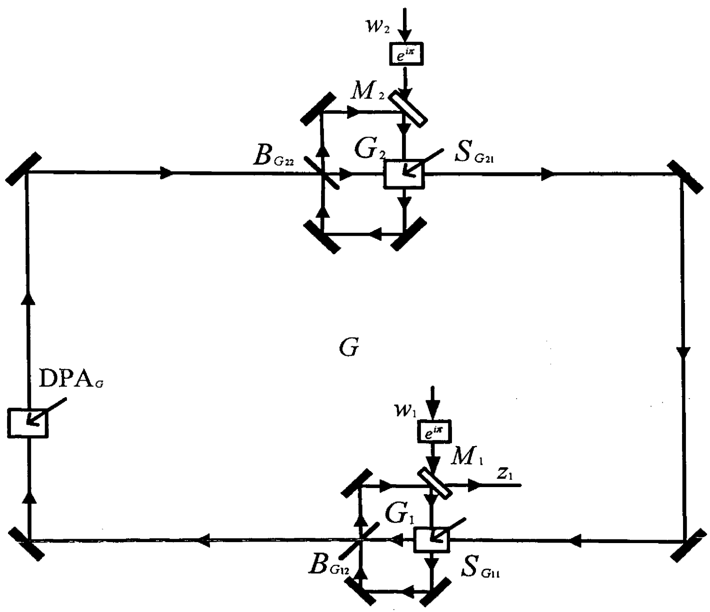

Figure 4.2: Quantum hardware realization of the system (4.1).

splitters, and squeezers, etc; Interested readers may refer to [81], [107] for a more detailed introduction to these optical devices. From the above introduction, it can be easily seen that this issue of quantum physical realization can be solved by seeking state space equations of a quantum linear system to replace (or realize) state space equations of the classical linear system, where the classical and quantum systems should satisfy some relation discussed in Section 4.3. However, [29] and [32] point out that a linear system with arbitrary system matrices does not correspond to a physically meaningful quantum system. Therefore, we propose Theorem 4.2 to show how to construct state space representation matrices of a physically meaningful quantum system according to state space equations of a given classical linear system. The motivation of this work will be discussed below.

Compared with classical systems typically implemented using standard analog or digital electronics, quantum mechanical systems can have better physical properties, which may provide a bandwidth much higher than that of conventional electronics and thus increase processing times. For instance, quantum optical systems can have frequencies up to 10^{14} Hz or higher. Furthermore, it is becoming feasible to implement quantum networks in semiconductor materials, for example, photonic crystals are periodic optical nanostructures that are designed to affect the motion of photons in a similar way that periodicity of a semiconductor crystal affects the motion of electrons, and it may be desirable to implement control networks on the same chip (rather than interfacing to a separate system); see [33], [80].

This chapter is organized as follows. Section 4.2 formulates a problem of quantum physical realization to be solved in this chapter. Section 4.3 presents the main results of this chapter. Section 4.4 presents a potential application of the main results of this chapter to measurement-based feedback control of quantum systems. Some examples are also presented to illustrate the application of the main results in this section. Section 4.5 study if a quantum realization of a classical controller can improve the overall closed-loop control performance. Finally, Section 4.6 gives the conclusion of this chapter.

4.2 Problem formulation

4.2.1 Classical linear systems

Consider a class of *classical* linear systems of the form

$$\begin{aligned}\dot{\xi}(t) =& A\xi(t) + Bv_1(t),\\ y(t) =& C\xi(t) + Dv_2(t),\end{aligned} \tag{4.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_{v_1}}$, $C \in \mathbb{R}^{n_y\times n}$ and $D \in \mathbb{R}^{n_y\times n_{v_2}}$ are real constant matrices, $v_1(t)$ and $v_2(t)$ are input signals and independent. The initial condition $\xi(0) = \xi_0$ is Gaussian, while $y(0) = 0$. The transfer function $\Xi_C(s)$ from the noise input channel v to the output channel y for the classical system (4.2) is denoted by

$$\begin{aligned}\Xi_C(s) =& \left[\begin{array}{c|c} A & \begin{bmatrix} B & 0_{n\times n_{v_2}} \end{bmatrix} \\ \hline C & \begin{bmatrix} 0_{n_{v_2}\times n_{v_2}} & D \end{bmatrix} \end{array}\right](s)\\ =& \begin{bmatrix} C\left(sI_n - A\right)^{-1}B, & D \end{bmatrix}.\end{aligned} \tag{4.3}$$

4.2.2 Quantum linear stochastic systems

Consider a *quantum* linear stochastic system of the form (see e.g. [3], [27], [30], [54] and [108])

$$\begin{aligned}dx(t) =& \tilde{A}x(t)dt + \tilde{B}dw(t),\\ dz(t) =& \tilde{C}x(t)dt + \tilde{D}dw(t),\end{aligned} \tag{4.4}$$

where $\tilde{A} \in \mathbb{R}^{2n\times 2n}$, $\tilde{B} \in \mathbb{R}^{2n\times n_w}$, $\tilde{C} \in \mathbb{R}^{n_z\times 2n}$ and $\tilde{D} \in \mathbb{R}^{n_z\times n_w}$ are real constant matrices. We assume that n_w and n_z are even (see [29, Section II] for details). We refer to n as the *degrees of freedom* of systems of the form (4.4). Equation (4.4) is a *quantum stochastic differential equation* (QSDE) [27], [109], [110] and [111]. In equation (4.4), $x(t)$ is a vector of self-adjoint possibly non-commuting operators, with the initial value $x(0) = x_0$ satisfying the commutation relations

$$x_{0j}x_{0k} - x_{0k}x_{0j} = 2i\tilde{\Theta}_{jk}, \tag{4.5}$$

where $\tilde{\Theta}$ is a skew-symmetric real matrix. The matrix $\tilde{\Theta}$ is said to be *canonical* if it is the form $\tilde{\Theta} = J_n$. The components of the vector $w(t)$ are quantum stochastic processes with the non-zero Ito products

$$dw_j(t)dw_k(t) = F_{jk}dt, \tag{4.6}$$

where F is a non-negative definite Hermitian matrix. The matrix F is said to be *canonical* if it is the form $F = I_{n_w} + iJ_{\frac{n_w}{2}}$. In this chapter we will take $\tilde{\Theta}$ and F to be canonical. The transfer function for the quantum linear stochastic system (4.4) is given by

$$\Xi_Q(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right](s) = \tilde{C}\left(sI_{2n} - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D}. \tag{4.7}$$

Here we mention that while the equations (4.4) look formally like the classical equations (4.2), they are not classical equations, and in fact give the Heisenberg dynamics of a system of coupled open quantum harmonic oscillators. The variables $x(t)$, $w(t)$ and $z(t)$ are in fact vectors of quantum observables (self-adjoint non-commuting operators, or quantum stochastic processes).

The quantum system (4.4) is *(canonically) physically realizable* (cf.[32]), if and only if the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ satisfy the following conditions:

$$\tilde{A}J_n + J_n\tilde{A}^T + \tilde{B}J_{\frac{n_w}{2}}\tilde{B}^T = 0, \tag{4.8}$$

$$\tilde{B}J_{\frac{n_w}{2}}\tilde{D}^T = -J_n\tilde{C}^T, \tag{4.9}$$

$$\tilde{D}J_{\frac{n_w}{2}}\tilde{D}^T = J_{\frac{n_z}{2}}, \tag{4.10}$$

where $n_w \geq n_z$. In fact, under these conditions the quantum linear stochastic system (4.4) corresponds to an open quantum harmonic oscillator [29, Theorem 3.4] consisting of n oscillators (satisfying canonical commutation relations) coupled to n_w fields (with canonical Ito products and commutation relations). In particular, in the canonical case, $x_0 = (q_1\ q_2\ \dots\ q_n\ p_1\ p_2\ \dots\ p_n)^T$, where q_j and p_j are the position and momentum operators of the oscillator j (which constitutes the jth of degree of freedom of the system) that satisfy the commutation relations $[q_j, p_k] = 2i\delta_{jk}$, $[q_j, q_k] = [p_j, p_k] = 0$ in accordance with (4.5). Hence by the results of [81] the system can be implemented using standard quantum optics components. It is also possible to consider other quantum physical implementations.

4.2.3 Quantum physical realization

We have briefly reviewed some notations and concepts about classical probability and quantum probability as well as classical and quantum random variables introduced in Chapter 2. If we are given a classical vector-valued random variable

$$\tilde{X} = [X_1\ X_2\ \cdots\ X_n]^T,$$

we may *realize* (or represent) it using a quantum vector-valued random variable $\check{X}_Q$ with associated state ψ in a suitable Hilbert space in the sense that the distribution of $\tilde{X}$ is the same as the distribution of $\check{X}_Q$ with respect to the state ψ. For instance, if the variable $\tilde{X}$ have a multivariate Gaussian distribution with its probability density function given by

$$f(\tilde{x}) = (2\pi)^{-\frac{n}{2}}|\Sigma|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(\tilde{x}-\tilde{\mu})^T\Sigma^{-1}(\tilde{x}-\tilde{\mu})\right) \tag{4.11}$$

with mean $\tilde{\mu} \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$, we may realize this classical random variable $\tilde{X}$ using an open harmonic oscillator. Indeed, we can take the realization to be the position quadrature $\check{X}_Q = [Q_1^T\ Q_2^T\ \cdots\ Q_n^T]^T$ (for example), with the state ψ selected so that $(\tilde{\mu}, \Sigma^2) = (\tilde{\mu}_Q, \Sigma_Q^2)$. So statistically $\tilde{X} \equiv \check{X}_Q$. The quantum vector $\check{X} = [\check{X}_Q^T\ \check{X}_P^T]^T$ is called an *augmentation* of $\tilde{X}$, where $\check{X}_P = [P_1^T\ P_2^T\ \cdots\ P_n^T]^T$ is the momentum quadrature. The quantum realization of the classical random variable may be expressed as

$$\tilde{X} \equiv \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \check{X}_Q \\ \check{X}_P \end{bmatrix}. \tag{4.12}$$

As is well known, for a linear system, its state space representation can be associated to a unique transfer function representation. Then, we will show how the transfer function matrix $\Xi_C(s)$ can be realized (in a sense to be defined more precisely below) using linear *quantum* components. In general, the dimension of vectors in (4.4) is greater than the vector dimension in (4.2), and so to obtain a quantum realization of the classical system (4.2) using the quantum system (4.4) we require that the transfer functions be related by

$$\Xi_C(s) = M_o \Xi_Q(s) M_i, \tag{4.13}$$

as illustrated in Figure 4.3. Here, the matrix M_i and M_o correspond to operation of selecting elements of the input vector $w(t)$ and the output vector $z(t)$ of the quantum realization that correspond to quantum representation of $v(t)$ and $y(t)$, respectively (as discussed in Section 4.2). In Figure 4.3, the unlabeled box on the left indicates that $v(t)$ is represented as an element of $w(t)$ (e.g. modulation*) [49], whereas the unlabeled box on the right indicates that $y(t)$ corresponds to some element of $z(t)$ (quadrature measurement).

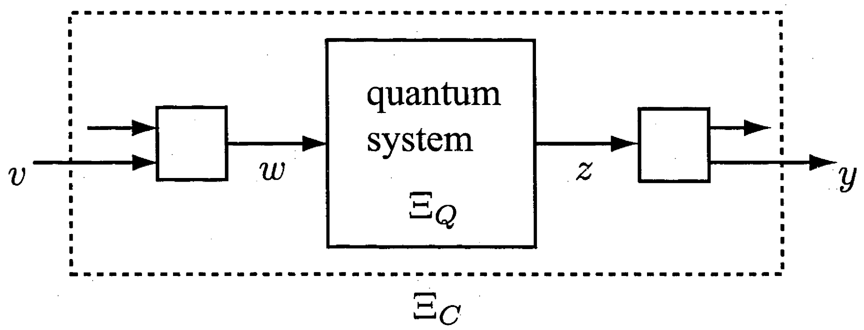

Figure 4.3: Quantum realization of classical system $\Xi_C : v \mapsto y$.

Then we have the following definition.

Definition 4.1. The classical linear stochastic system (4.2) is said to be canonically *realized* by the quantum linear stochastic system (4.4) provided:

1. The dimension of the quantum vectors $x(t)$, $w(t)$ and $z(t)$ are twice the lengths of the corresponding classical vectors $\xi(t)$, $v(t) = [v_1(t)^T \ v_2(t)^T]^T$ and $y(t)$, where $w(t) = [v_1(t)^T \ v_2(t)^T \ u_1(t)^T \ u_2(t)^T]^T$, $x(t){=}[\xi(t)^T \ \theta(t)^T]^T$ with $\xi(t) = [q_1(t) \ q_2(t) \ \cdots \ q_n(t)]^T$ and $\theta(t) = [p_1(t) \ p_2(t) \ \cdots \ p_n(t)]^T$.

2. The classical $\Xi_C(s)$ and quantum $\Xi_Q(s)$ transfer functions are related by equation (4.13) for the choice

$$M_o = \begin{bmatrix} I_{n_y} & 0_{n_y \times n_y} \end{bmatrix},$$

*Modulation is the process of merging two signals to form a third signal with desirable characteristics of both in a manner suitable for transmission.

and

$$M_i = \left[\begin{array}{cc} I_{n_v} & 0_{n_v \times n_v} \end{array} \right]^T.$$

3. The quantum linear stochastic system (4.4) is canonically physically realizable (as described in Subsection 4.2.2) with the system matrices $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ having the special structure:

$$\begin{aligned} \tilde{A} &= \left[\begin{array}{cc} A_0 & 0_{n\times n} \\ A_1 & A_2 \end{array} \right], \qquad (4.14) \\ \tilde{B} &= \left[\begin{array}{cccc} B_0 & 0_{n\times n_{v_2}} & 0_{n\times n_{v_1}} & 0_{n\times n_{v_2}}, \\ B_1 & B_2 & B_3 & B_4 \end{array} \right], \\ \tilde{C} &= \left[\begin{array}{cc} C_0 & 0_{n_w\times n} \\ C_1 & C_2 \end{array} \right], \\ \tilde{D} &= \left[\begin{array}{cccc} 0_{n_y\times n_{v_1}} & D_0 & 0_{n_y\times n_{v_1}} & 0_{n_y\times n_{v_2}} \\ D_1 & D_2 & D_3 & D_4 \end{array} \right] \qquad (4.15) \end{aligned}$$

with $A_0 \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times n_{v_1}}$, $B_1 \in \mathbb{R}^{n\times n_{v_1}}$, $B_2 \in \mathbb{R}^{n\times n_{v_2}}$, $B_3 \in \mathbb{R}^{n\times n_{v_1}}$, $B_4 \in \mathbb{R}^{n\times n_{v_2}}$, $C_0 \in \mathbb{R}^{n_y\times n}$, $C_1 \in \mathbb{R}^{n_y\times n}$, $C_2 \in \mathbb{R}^{n_y\times n}$, $D_0 \in \mathbb{R}^{n_y\times n_{v_2}}$, $D_1 \in \mathbb{R}^{n_y\times n_{v_1}}$, $D_2 \in \mathbb{R}^{n_y\times n_{v_2}}$, $D_3 \in \mathbb{R}^{n_y\times n_{v_1}}$, and $D_4 \in \mathbb{R}^{n_y\times n_{v_2}}$.

Remark 4.1. According to the structure of matrices $\tilde{A}$ and $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, it can be verified directly that commutation relations for $\xi(t), \theta(t)$ satisfy $[\xi(t), \xi(s)^T] = 0$, $[\xi(t), \theta(s)^T] \neq 0$ and $[\theta(t), \theta(t)^T] \neq 0$ (for all t, s), which mean that the subvector $\xi(t)$ in $x(t)$ acts with respect to $\xi(t)$ itself like a classical random vector while the subvector $\xi(t)$ in $x(t)$ behaves with respect to the vector $x(t)$ as a quantum random vector. So, the quantum realization of the classical variable $\xi(t)$ may be expressed as $\xi(t) = \left[\begin{array}{cc} I & 0 \end{array} \right] x(t) = \left[\begin{array}{cc} I & 0 \end{array} \right] \left[\begin{array}{c} \xi(t) \\ \theta(t) \end{array} \right]$. The structures of the matrices $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ in the above definition ensure that the classical system (4.2) can be embedded as an invariant commutative subsystem of the quantum system (4.4), as discussed in [29], [32] and [38]. Here, the classical variables and the classical signals are represented within an invariant commutative subspace of the full quantum feedback system, and the additional quantum degrees of freedom introduced in the quantum controller have no influence on the behavior of the feedback system; see [29] for details. In fact, $\tilde{D}$ represents static Bogoliubov transformations or symplectic transformations, which can be realized as a suitable static quantum optical network (eg. ideal squeezers), [45], [81], [84].

4.3 Main results

In this section we will present our results concerning the quantum physical realization of classical linear systems.

In what follows we restrict our attention to stable classical systems, since it may not be desirable to attempt to implement an unstable quantum system. By a *stable* quantum system (4.4) we mean that the $\tilde{A}$ is Hurwitz. We will seek stable quantum realizations. Furthermore, given the quantum physical realizability conditions (4.8)-(4.10), we cannot do the quantum realizations for an arbitrary classical system (4.2). For these reasons we make the following assumptions regarding the classical linear stochastic system (4.2).

Assumption 4.3.1. Assume the following conditions hold:

1. The matrix A is a Hurwitz matrix.
2. The pair $(-A, B)$ is stabilizable.
3. The matrix D is of full row rank.

Theorem 4.2. *Under Assumption 4.3.1, there exists a stable quantum linear stochastic system* (4.4) *realizing the given classical linear stochastic system* (4.2) *in the sense of Definition 4.1, where the matrices $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ can be constructed according to the following steps:*

1. *$A_0 = A$, $B_0 = B$, $C_0 = C$ and $D_0 = D$, with A B, C and D as given in* (4.2).
2. *B_1, B_2 are arbitrary matrices of suitable dimensions.*
3. *The matrices A_2 and B_3 can be fixed simultaneously by*
$$A_2 = -A^T - B_3 B^T, \tag{4.16}$$
where B_3 is chosen to let A_2 be a Hurwitz matrix.
4. *The matrices B_4 and D_4 are given by*
$$B_4 = -C^T(DD^T)^{-1}D + N_1(D)^T, \tag{4.17}$$
$$D_4 = (DD^T)^{-1}D + N_2(D)^T, \tag{4.18}$$
where $N_1(D)$ (resp., $N_2(D)$) denotes a matrix of the same dimension as B_4^T (resp., D_4^T) whose columns are in the kernel space of D.

5. *For a given D_4, there always exist matrices D_1, D_2, D_3 satisfying*

$$-D_3D_1^T - D_4D_2^T + D_1D_3^T + D_2D_4^T = 0. \tag{4.19}$$

The simplest choice is $D_1 = 0$, $D_2 = 0$, and $D_3 = 0$.

6. *The remaining matrices can be constructed as*

$$C_2 = -D_3B^T, \tag{4.20}$$

$$C_1 = D_4B_2^T + D_3B_1^T - D_2B_4^T - D_1B_3^T, \tag{4.21}$$

$$A_1 = \Xi + \frac{1}{2}(B_3B_1^T - B_1B_3^T - B_2B_4^T + B_4B_2^T), \tag{4.22}$$

where Ξ is an arbitrary $n \times n$ real symmetric matrix.

Proof. The idea of the proof is to represent the classical stochastic processes $\xi(t)$ and $v(t)$ as quadratures of quantum stochastic processes $x(t)$ and $w(t)$ respectively, and then determine the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ in such a way that the requirements of Definition 4.1 and the Hurwitz property of $\tilde{A}$ are fulfilled. To this end, we set the number of oscillators to be $n = n_c$, the number of field channels as $n_w = 2n_v = 2(n_{v_1} + n_{v_2})$ and the number of output field channels as $n_z = 2n_y$.

Equations (4.16)-(4.22) can be obtained from the physical realizability constraints (4.8)-(4.10). According to the second assumption of Assumption 4.3.1, we can choose B_3 such that $A_2 = -A^T - B_3B^T$ is a Hurwitz matrix. From the first assumption of Assumption 4.3.1, we can conclude that $\tilde{A}$ is a Hurwitz matrix, which means the quantum linear stochastic system (4.4) is stable. Using M_i and M_o as defined in Definition 4.1 and then combining these with equations (4.14)-(4.22), we can verify the relation between the classical $\Xi_C(s)$ and quantum $\Xi_Q(s)$ transfer functions, such that

$$\begin{aligned}
&M_o\Xi_Q(s)M_i \\
=&\begin{bmatrix} I_{n_y} & 0_{n_y\times n_y}\end{bmatrix}\left\{\begin{bmatrix} C & 0_{n_y\times n} \\ C_1 & C_2\end{bmatrix}\left(sI_{2n}-\begin{bmatrix} A & 0_{n\times n} \\ A_1 & A_2\end{bmatrix}\right)^{-1}\begin{bmatrix} B & 0_{n\times n_{v_2}} & 0_{n\times n_{v_1}} & 0_{n\times n_{v_2}} \\ B_1 & B_2 & B_3 & B_4\end{bmatrix}+\tilde{D}\right\} \\
&\times\begin{bmatrix} I_{n_v} & 0_{n_v\times n_v}\end{bmatrix}^T \\
=&\begin{bmatrix} C & 0_{n_y\times n}\end{bmatrix}\begin{bmatrix} (sI_n - A)^{-1} & 0_{n\times n} \\ -(sI_n - A_2)^{-1}A_1(sI_n - A)^{-1} & (sI_n - A_2)^{-1}\end{bmatrix}\begin{bmatrix} B & 0_{n\times n_{v_2}} \\ B_1 & B_2\end{bmatrix}+ \\
&\begin{bmatrix} 0_{n_y\times n_{v_1}} & D\end{bmatrix} \\
=&\begin{bmatrix} C\,(sI_n - A)^{-1}B & D\end{bmatrix} \\
=&\Xi_C(s).
\end{aligned}$$

This completes the proof. □

4.4 Application and examples

4.4.1 Application

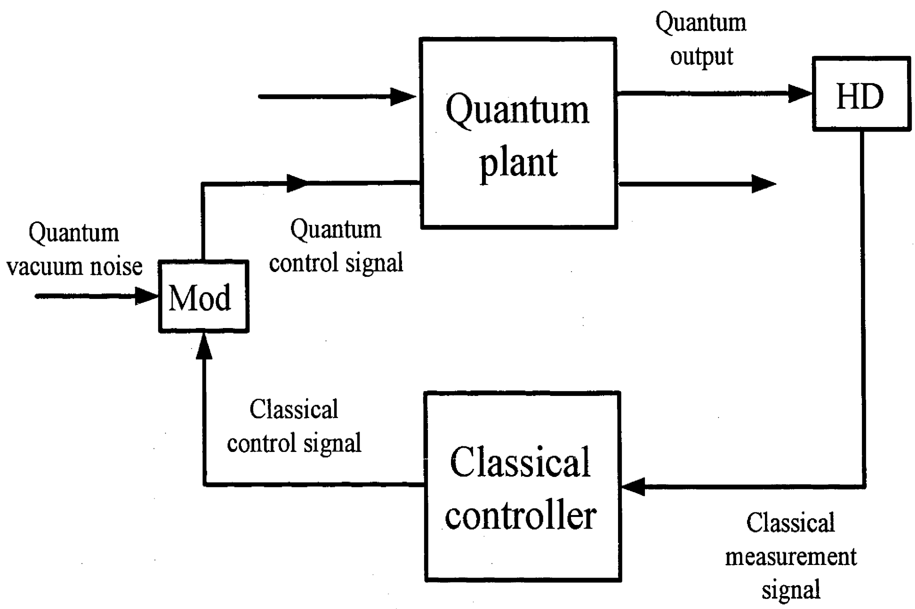

Figure 4.4: Measurement-based feedback control of a quantum system, where HD represents the homodyne detector and Mod represents the optical modulator.

The main results of this chapter may have a practical application in measurement-based feedback control of quantum systems, which is important in a number of areas of quantum technology, including quantum optical systems, nanomechanical systems, and circuit QED systems; see [3], [4], [5], [25]. In measurement-based feedback control, the plant is a quantum system, while the controller is a classical (i.e. non-quantum) system [3]. The classical controller processes the outcomes of a measurement of an observable of the quantum system (e.g. the quadrature of an optical field) to determine the classical control actions that are applied to control the behavior of the quantum system. The closed-loop system involves both quantum and classical components, such as an electronic device for measuring a quantum signal, as shown in Figure 4.4. However, the state of quantum systems is easily affected by interaction with measurement devices, which causes the loss of quantum information. This thus motivates the replacement of the classical controller in measurement-based feedback control system as shown in Figure 4.4 by a coherent quantum controller, which is directly interconnected with a quan-

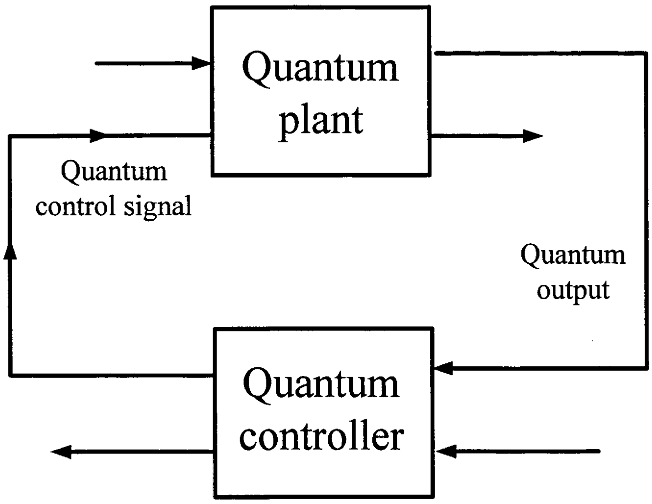

Figure 4.5: Quantum realization of a measurement-based feedback control system.

tum plant without any interfaces (eg. homodyne detectors, modulators) involved [5], [35].

According to the main results of Section 4.3, it may be possible to realize the measurement-based feedback loop illustrated in Figure 4.4 fully at the quantum level. For instance, if the plant is a quantum optical system where the classical control is a signal modulating a laser beam, and if the measurement of the plant output (a quantum field) is a quadrature measurement (implemented by a homodyne detection scheme), then the closed-loop system might be implemented fully using quantum optics, Figure 4.5. The functions of the modulator and the measurement device are built into the couplings between the quantum controller device (a quantum system) and the quantum fields used to carry signals in the feedback loop. In other words, the role of the quantum controller in the feedback loop is equivalent to that of a combination of the classical controller, the modulator and the measurement devices in the feedback loop as shown in Figure 4.6. In Subsection 4.4.2, **Example 4.3** will be provided to illustrate the application of our main results to the measurement-based feedback control of quantum systems.

4.4.2 Examples

In this subsection, we will provide three explicit examples of the application of the main result to the implementation of classical systems as quantum systems, where we will also detail the construction of the quantum system G using various

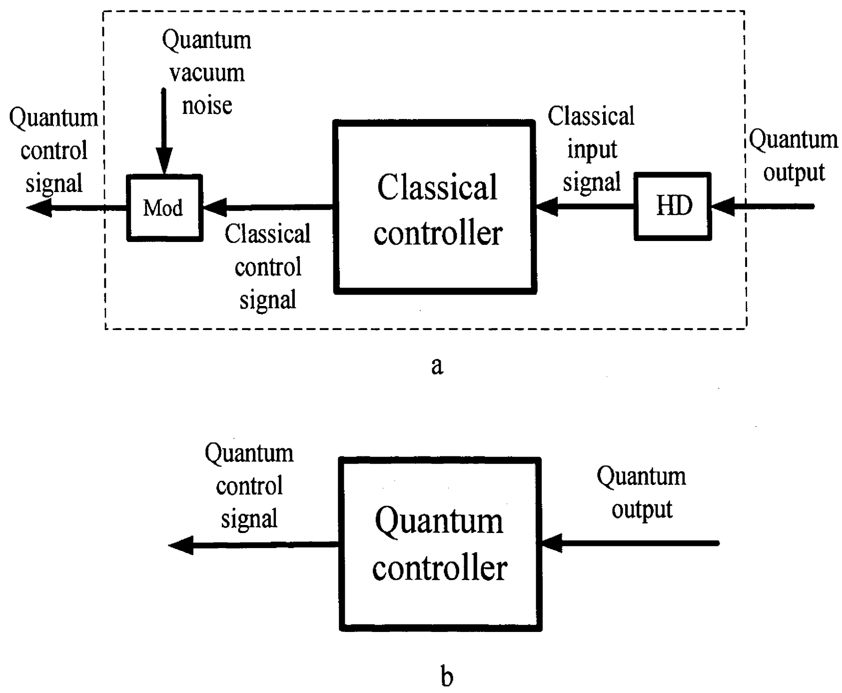

Figure 4.6: Relations between classical and quantum controllers. The function of the quantum controller shown in Figure 4.6b is equivalent to that of the combination of the classical controller together with the homodyne detector and optical modulator shown in Figure 4.6a.

linear and nonlinear quantum optical devices. Before continuing to read, please briefly review some definitions and results introduced in Section 2.4 of Chapter 2. In the following we will use the construction proposed in [81] to realize systems of the form (4.4) without further comment. The details of the construction can be found in [81] and the references therein.

Example 4.1

Let us realize the classical system (4.1) introduced in Section 4.1. The classical transfer function is $\Xi_C(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \end{bmatrix}$. By Theorem 4.2, we can construct a quantum system G given by

$$\begin{aligned} dx_1 &= -x_1dt + dv_1, \\ dx_2 &= -x_2dt + 2du_1 - du_2, \\ dz_1 &= x_1dt + dv_2, \end{aligned}$$

$$dz_2 = du_2. \tag{4.23}$$

The quantum transfer function is given by

$$\Xi_Q(s) = \begin{bmatrix} \frac{1}{s+1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since in this case $M_o = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $M_i = \begin{bmatrix} I_2 & 0_{2\times 2} \end{bmatrix}^T$, we see that $\Xi_C(s) = M_o \Xi_Q(s) M_i$. The commutative subsystem $dx_1 = -x_1 dt + dv_1$, $dz_1 = x_1 dt + dv_2$ can clearly be seen in these equations, with the identifications $y = z_1$, $\xi = x_1$. It can be seen that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ satisfy the physically realizable constraints (4.8) and (4.9).

Let us realize this classical system. The parameter R for G is given by $R = 0$, which means no Degenerate Parametric Amplifier (DPA) is required to implement R; see [81, section 6.1.2]. The coupling matrix Λ for G is given by

$$\begin{aligned} \Lambda &= \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -0.5i \\ 0.5 & 0 \end{bmatrix}. \end{aligned}$$

From the above equation, we can get $\Lambda_1 = \begin{bmatrix} -1 & -0.5i \end{bmatrix}$ and $\Lambda_2 = \begin{bmatrix} 0.5 & 0 \end{bmatrix}$. The coupling matrix $L_1 = \Lambda_1 x_0$ for G is given by

$$\begin{aligned} L_1 &= \Lambda_1 \begin{bmatrix} q \\ p \end{bmatrix} \\ &= \Lambda_1 \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \\ &= -1.5a - 0.5a^*, \end{aligned}$$

where $a = \frac{1}{2}(q + ip)$ is the oscillator annihilation operator and $a^* = \frac{1}{2}(q - ip)$ is the creation operator of the system G with position and momentum operators q and p, respectively. L_1 can be approximately realized by the combination of a two-mode squeezer $\Upsilon_{G_{11}}$, a beam splitter $B_{G_{12}}$, and an auxiliary cavity G_1. If the dynamics of G_1 evolve on a much faster time scale than that of G then the coupling operator L_1 is approximately given by $L_1 = \frac{1}{\sqrt{\gamma_1}}(-\epsilon_{12}^* a + \epsilon_{11} a^*)$, where γ_1 is the coupling coefficient of the only partially transmitting mirror of G_1, ϵ_{11} is the effective pump intensity of $\Upsilon_{G_{11}}$ and ϵ_{12} is the coefficient of the effective Hamiltonian for $B_{G_{12}}$ given by $\epsilon_{12} = 2\Theta_{12} e^{-i\Phi_{12}}$, where Θ_{12} is the mixing angle of

$B_{G_{12}}$ and Φ_{12} is the relative phase between the input fields introduced by $B_{G_{12}}$; see [81]. To be a good approximation for adiabatic elimination, we require that $\sqrt{\gamma_1}, |\epsilon_{11}|, |\epsilon_{12}|$ be sufficiently large. So assuming that the coupling coefficient of the mirror M_1 is $\gamma_1 = 100$, we then can get $\epsilon_{11} = -5$, $\epsilon_{12} = 15$, $\Phi_{12} = 0$ and $\Theta_{12} = 7.5$. The scattering matrix for G_1 is $e^{i\pi} = -1$ and all other parameters are set to be 0. In a similar way, the coupling operator $L_2 = \Lambda_2 x_0$ can be realized by the combination of $\Upsilon_{G_{21}}$, $B_{G_{22}}$, and G_2. In this case, if we set the coupling coefficient of the partially transmitting mirror M_2 of G_2 to $\gamma_2 = 100$, we find the effective pump intensity ϵ_{21} of $\Upsilon_{G_{21}}$ given by $\epsilon_{21} = 5$, the relative phase Φ_{22} of $B_{G_{22}}$ given by $\Phi_{22} = \pi$, the mixing angle Θ_{22} of $B_{G_{22}}$ given by $\Theta_{22} = 2.5$, the scattering matrix for G_2 to be $e^{i\pi} = -1$. All other parameters are set to be 0. The implementation of the quantum system G is shown in Figure 4.2.

Example 4.2

Consider a classical system of the form (4.2) with matrices

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$
$$C = \begin{bmatrix} 2 & 5 \end{bmatrix}, \quad D = 1.$$

By Theorem 4.2, we can construct a quantum system G, an augmentation of the above classical system, with the following matrices

$$\tilde{A} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 1 & 0 & 3 & -2 \\ 0 & 1 & 8 & -4 \end{bmatrix},$$
$$\tilde{B} = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 3 & -5 \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} 0 & -3 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $M_o = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, $M_i = \begin{bmatrix} I_2 & 0_{2\times 2} \end{bmatrix}^T$, it can be easily checked that $\Xi_C(s) = M_o \Xi_Q(s) M_i$.

The system G has two degrees of freedom, which can be realized as an interconnection of two one degree of freedom systems G_1 and G_2. The parameter R for G is given by

$$R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix},$$

where R_1 is for the subsystem G_1, R_2 is for the subsystem G_2, and R_{12} is for a direct interaction between G_1 and G_2. We can realize R_1 and R_2 as two DPAs with detuning parameters Δ_{G_1} and Δ_{G_2} and effective pump intensities ϵ_{G_1} and ϵ_{G_2}, respectively. These parameters are $\Delta_{G_1} = -0.5$, $\epsilon_{G_1} = -3 + 0.5i$, $\Delta_{G_2} = -0.5$, and $\epsilon_{G_2} = 1 + 0.5i$.

The coupling matrix Λ_G for the quantum system G is given by

$$\Lambda_G = \begin{bmatrix} \Lambda_{G_1} \mid \Lambda_{G_2} \end{bmatrix},$$

where $\Lambda_{G_1} = \begin{bmatrix} \Lambda_{G_{1,1}} \\ \Lambda_{G_{1,2}} \end{bmatrix}$ for the subsystem G_1, $\Lambda_{G_2} = \begin{bmatrix} \Lambda_{G_{2,1}} \\ \Lambda_{G_{2,2}} \end{bmatrix}$ for the subsystem G_2.

Defining the coupling operators $L_{G_{j,k}} = \Lambda_{G_{j,k}}[q_j, p_j]^T$ for $j, k = 1, 2$, we can realize these coupling operators with the construction employed in **Example 4.1**. That is, for the coupling operator $L_{G_{j,k}}$ we will need an auxiliary cavity mode $G_{j,k}$, a two-mode squeezer with effective pump intensity $\epsilon_{G_{j,k},1}$ and a beam splitter with mixing angle $\Theta_{G_{j,k}}$ and relative phase $\Psi_{G_{j,k}}$, giving the coefficient $\epsilon_{G_{j,k},2} = 2\Theta_{G_{j,k}} e^{-i\Psi_{G_{j,k}}}$, and a partially transmitting mirror of $G_{j,k}$ with coupling coefficient $\gamma_{G_{j,k}}$. These values are given by

$$\begin{aligned} \epsilon_{G_{1,1},1} &= -15; & \Theta_{G_{1,1}} &= -7.5; \\ \Psi_{G_{1,1}} &= 0; & \epsilon_{G_{1,1},2} &= -15; \\ \gamma_{G_{1,1}} &= 100; & \epsilon_{G_{1,2},1} &= 10; \\ \Theta_{G_{1,2}} &= -5; & \Psi_{G_{1,2}} &= 0; \end{aligned}$$

$$\begin{aligned} \epsilon_{G_{1,2},2} &= -10; & \gamma_{G_{1,2}} &= 100; \\ \epsilon_{G_{2,1},1} &= -5; & \Theta_{G_{2,1}} &= 12.5; \\ \Psi_{G_{2,1}} &= 0; & \epsilon_{G_{2,1},2} &= 25; \\ \gamma_{G_{2,1}} &= 100; & \epsilon_{G_{2,2},1} &= 25; \\ \Theta_{G_{2,2}} &= -12.5; & \Psi_{G_{2,2}} &= 0; \\ \epsilon_{G_{2,2},2} &= -25; & \gamma_{G_{2,2}} &= 100. \end{aligned}$$

We now consider the implementation of the direct interaction Hamiltonian $H^d_{G_1G_2}$ between G_1 and G_2 given by

$$\begin{aligned} H^d_{G_1G_2} &= x_2^T\left(R_{12}^T - \frac{1}{2i}(\Lambda^\dagger_{G_2}\Lambda_{G_1} - \Lambda^T_{G_2}\Lambda^\#_{G_1})\right)x_1 \\ &= 0.5a_1a_2^* - 1.5ia_1a_2 + 1.5ia_1^*a_2^* - 0.5ia_1^*a_2 \\ &= \epsilon^*_{G_1G_{2,1}}a_1a_2 + \epsilon^*_{G_1G_{2,2}}a_1a_2^* + \epsilon_{G_1G_{2,2}}a_1^*a_2 + \epsilon_{G_1G_{2,1}}a_1^*a_2^*. \end{aligned}$$

Define $H^d_{G_1G_{2,1}} = -1.5ia_1a_2 + 1.5ia_1^*a_2^*$ and $H^d_{G_1G_{2,2}} = 0.5ia_1a_2^* - 0.5ia_1^*a_2$ so that $H^d_{G_1G_2} = H^d_{G_1G_{2,1}} + H^d_{G_1G_{2,2}}$. The first part $H^d_{G_1G_{2,1}}$ can be implemented by a two-mode squeezer $\Upsilon_{G_1G_{2,1}}$. The two modes a_1 of G_1 and a_2 of G_2 interact in a suitable χ^2 nonlinear crystal with a classical pump beam of effective intensity $\epsilon_{G_1G_{2,1}} = 3$. On the other hand, the second part $H^d_{G_1G_{2,2}}$ can be simply implemented as a beam splitter $B_{G_1G_{2,2}}$ with a mixing angle $\Theta_{G_1G_{2,2}} = -0.5$. All other parameters are set to be 0. The implementation of G is shown in Figure 4.7.

Example 4.3

Consider a closed-loop system which consists of a quantum plant G and a real classical controller K shown in Figure 4.4. The quantum plant G, an optical cavity, is of the form (4.4) and is given in quadrature form by the equations

$$dq = (-\frac{\gamma}{2}q + \omega p)dt - \sqrt{\gamma}\,dw_1, \tag{4.24}$$

$$dp = (-\frac{\gamma}{2}p - \omega q)dt - \sqrt{\gamma}\,dw_2, \tag{4.25}$$

$$d\eta_1 = \sqrt{\gamma}\,qdt + dw_1, \tag{4.26}$$

$$d\eta_2 = \sqrt{\gamma}\,pdt + dw_2, \tag{4.27}$$

where ω is the detuning parameter, and γ is a coupling constant. The output of the homodyne detector (Figure 4.4) is $\zeta = \eta_1$. The quantum control signal (w_1, w_2) is the output of a modulator corresponding to the equations $dw_1 = \xi dt + d\tilde{w}_1$, $dw_2 = d\tilde{w}_2$, where $(\tilde{w}_1, \tilde{w}_2)$ is a quantum Wiener process, and ξ is a

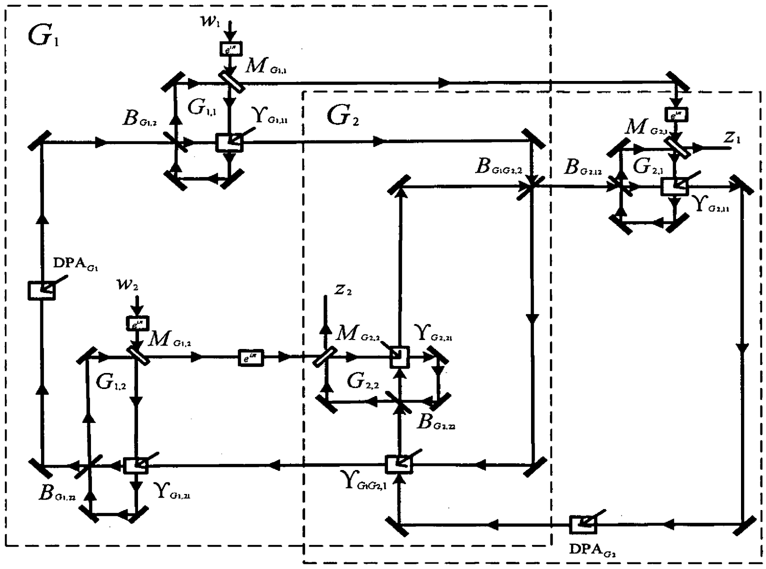

Figure 4.7: Realization of G.

classical state variable associated with the classical controller K, with dynamics $d\xi = -\xi dt + d\zeta$. The combined hybrid quantum-classical system G-K is given by the equations

$$\begin{aligned} dq &= (-\frac{\gamma}{2}q + \omega p - \sqrt{\gamma}\,\xi)dt - \sqrt{\gamma}\,d\tilde{w}_1, \\ dp &= (-\frac{\gamma}{2}p - \omega q)dt - \sqrt{\gamma}\,d\tilde{w}_2, \\ d\xi &= \sqrt{\gamma}qdt + d\tilde{w}_1, \\ d\zeta &= (\sqrt{\gamma}\,q + \xi)dt + d\tilde{w}_1. \end{aligned} \tag{4.28}$$

Note that this hybrid system is an open system, and consequently the equations are driven by quantum noise. The quantum realization of the system $d\xi = -\xi dt + d\zeta$, $dw_1 = \xi dt + d\tilde{w}_1$, denoted here by K_Q is, from **Example 4.1**, given by equations (4.23) (with the appropriate notational correspondences). The combined quantum plant and quantum controller system G-K_Q is specified by Figure 4.5, with corresponding closed-loop equations

$$\begin{aligned} dq &= (-\frac{\gamma}{2}q + \omega p - \sqrt{\gamma}\,x_1)dt - \sqrt{\gamma}\,dv_2, \\ dp &= (-\frac{\gamma}{2}p - \omega q)dt - \sqrt{\gamma}\,du_2, \\ dx_1 &= \sqrt{\gamma}\,qdt + dv_2, \end{aligned}$$

$$dx_2 \quad = \quad (-x_2 + 2\sqrt{\gamma}\, p)dt + du_2. \tag{4.29}$$

The hybrid dynamics (4.28) can be seen in these equations (with x_1, v_2 and u_2 replacing ξ, $\tilde{w}_1$ and $\tilde{w}_2$, respectively). By the structure of the equations, joint expectations involving variables in the hybrid quantum plant-classical controller system equal the corresponding expectations for the combined quantum plant and quantum controller. For example, $E[q(t)\xi(t)] = E[q(t)x_1(t)]$. A physical implementation of the new closed-loop quantum feedback system is shown in Figure 4.8.

We consider now the conditional dynamics for the cavity, [3], [9]. Let $\hat{q}(t)$ and $\hat{p}(t)$ denote the conditional expectations of $q(t)$ and $p(t)$ given the classical quantities $\zeta(s), \xi(s),\ 0 \leq s \leq t$. Then

$$\begin{aligned} d\hat{q} &= (-\frac{\gamma}{2}\hat{q} + \omega\hat{p} - \sqrt{\gamma}\xi)dt + K_q d\nu, && (4.30)\\ d\hat{p} &= (-\frac{\gamma}{2}\hat{p} - \omega\hat{q})dt + K_p d\nu, && (4.31)\end{aligned}$$

where $K_q = \widehat{q^2} - (\hat{q})^2 + 1$ and $K_p = \widehat{qp} - \hat{q}\hat{p}$ are the Kalman gains for the two quadratures, and ν is the measurement noise (the innovations process, itself a Wiener process). The output also has the representation

$$d\zeta = (\sqrt{\gamma}\,\hat{q} + \xi)dt + d\nu.$$

The conditional cavity dynamics combined with the classical controller dynamics leads to the feedback equations

$$\begin{aligned} d\hat{q} &= (-\frac{\gamma}{2}\hat{q} + \omega\hat{p} - \sqrt{\gamma}\xi)dt + K_q d\nu, && (4.32)\\ d\hat{p} &= (-\frac{\gamma}{2}\hat{p} - \omega\hat{q})dt + K_p d\nu, && (4.33)\\ d\xi &= \sqrt{\gamma}\hat{q}dt + d\nu, && (4.34)\\ d\zeta &= (\sqrt{\gamma}\,\hat{q} + \xi)dt + d\nu. && (4.35)\end{aligned}$$

Here we can see the measurement noise $\nu(t)$ explicitly in the feedback equations. By properties of conditional expectation, we can relate expectations involving the conditional closed-loop system with the hybrid quantum plant classical controller system, e.g. $E[\hat{q}(t)\xi(t)] = E[q(t)\xi(t)]$. We therefore see that the expectations involving the hybrid system, the conditional system, and the quantum plant quantum controller system are all consistent.

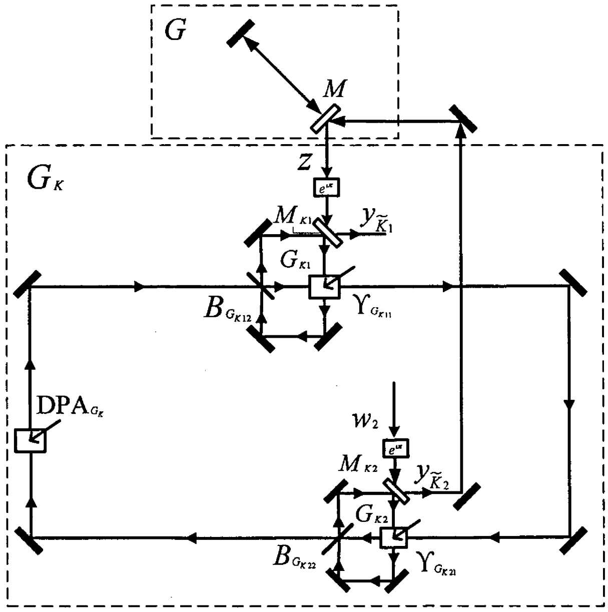

Figure 4.8: Quantum realization of the closed-loop system shown in Figure 4.5.

4.5 Performance analysis

As discussed in the previous sections, we have shown that a class of classical linear stochastic systems (having a certain form and satisfying certain technical assumptions) can be realized by quantum linear stochastic systems. However, the structure of the closed-loop system has changed, which may affect the closed-loop control performance. Therefore, the purpose of this section is to investigate conditions under which such quantum realization can preserve the original closed-loop control performance, such as LQG performance. Given a quantum plant and some control performance specifications, we can first design a classical controller by means of measurement-based quantum feedback methods. Then the classical controller is realized using quantum optical devices so that a quantum controller is obtained.

Consider a quantum plant G to be controlled described by non-commutative stochastic models of the following form

$$\begin{aligned} dx_p(t) =& A_p x_p(t)dt + B_{pu}du(t) + B_{pw}dw(t), \\ dy(t) =& C_p x_p(t)dt + D_{pw}dw(t), \end{aligned} \tag{4.36}$$

where x_p represents a vector of plant variables satisfying

$$x_p x_p^T - (x_p x_p^T)^T = 2i\Theta_n.$$

w is a quantum noise satisfying

$$[dw(t), dw(t)^T] = F_w - F_w^T = 2i\Theta_w dt.$$

u is the control signal satisfying

$$[du(t), du(t)^T] = F_u - F_u^T = 2i\Theta_u dt.$$

Quantum output y satisfies

$$[dy(t), dy(t)^T] = F_y - F_y^T = 2i\Theta_y dt,$$

which can be represented by position y_q and momentum y_p.

Construct a classical controller for the quantum plant (4.36) as

$$\begin{aligned} d\xi(t) =& A\xi(t)dt + Bdv_1(t), \\ y_c(t) =& C\xi(t), \end{aligned} \tag{4.37}$$

where y_c is a classical output. Here v_1 is the measurement signal of the position quadrature of the output y of the quantum plant, such that

$$v_1 = [I \quad 0]y.$$

Note that our controller is not exactly of the form (4.2), but we can relate it to another system which is of the form (4.4). So, consider Figure 4.4. Note that the output $y_K(t)$ of the classical controller K feeds a modulator. Since $\xi(t)$ is real, modulation here means that the signal $y_K(t)$ modulates a quantum noise with quadratures (v_2, u_2) to produce a quantum signal $y^Q(t)$ at the output of the modulator of the form

$$dy^Q(t) = \begin{bmatrix} C\xi(t) \\ 0 \end{bmatrix} dt + \begin{bmatrix} dv_2 \\ du_2 \end{bmatrix}.$$

Note that the first component of $dy^Q(t)$ is $C\xi(t)dt + dv_2(t)$. Now we define another classical system $\tilde{K}$ that extends the controller K with equations given by

$$\begin{aligned} d\xi(t) =& A\xi(t)dt + Bdv_1(t), \\ y_c(t) =& C\xi(t) + Ddv_2(t). \end{aligned} \tag{4.38}$$

Note that the output of $\tilde{K}$ is the first component of $y^Q(t)$, and it is a classical system of the form (4.2). We may consider this as the system that we will realize using a quantum system, say, $G_{\tilde{K}}$. By the construction, this quantum realization of $\tilde{K}$ will have as output the quantum signal $y^Q(t)$ and thus will replace the combination of K together with the homodyne detector (that measures $v_1(t)$) and the modulator that will produce $y^Q(t)$.

The plant and controller can be connected by modulators and homodyne detectors. $y^Q(t)$ can be considered as to be produced by displacing the vectors of vacuum quantum fields $w_2 = \begin{bmatrix} v_2 \\ u_2 \end{bmatrix}$ via modulators. If the quantum output signals $y(t)$ are measured by homodyne detectors (HD), classical signals $v_1(t) = \mathbf{G}y(t)$ are produced during measurement processes $\mathbf{G}$ defined as in Chapter 3 that satisfies the condition

$$\mathbf{G}\Theta_y\mathbf{G}^T = 0 \tag{4.39}$$

with $\text{rank}(\mathbf{G}) \leq \frac{n_y}{2}$. Thus interconnecting systems (4.36) and (4.38) by setting $u(t) = y^Q(t)$ and $v_1(t) = \mathbf{G}y(t)$ gives

$$dx_{cl}(t) \;=\; A_{cl}x_{cl}(t)dt + B_{cl}dw_{cl}(t) \tag{4.40}$$

where $x_{cl} = \begin{bmatrix} x_p \\ \xi \end{bmatrix}$, $w_{cl} = \begin{bmatrix} w_p \\ w_2 \end{bmatrix}$ with $w_2 = [v_2^T\ u_2^T]^T$, $A_{cl}=\begin{bmatrix} A_p & \begin{bmatrix} B_{pu}C \\ 0 \end{bmatrix} \\ BGC & A \end{bmatrix}$, $B_{cl} = \begin{bmatrix} B_{cl1} & B_{cl2} \end{bmatrix}$ with $B_{cl1} = \begin{bmatrix} B_{pw} \\ B\mathbf{G}D_{pw} \end{bmatrix}$ and $B_{cl2} = \begin{bmatrix} B_{pu} \\ 0 \end{bmatrix}$.

Define a LQG performance variable for (4.40) as

$$z_l(t) = C_{zl}x_p(t) + D_{zl}\beta(t) = \tilde{C}_{zl}x_{cl}(t), \tag{4.41}$$

where $\beta(t) = \begin{bmatrix} C \\ 0 \end{bmatrix} \xi(t)$ is the signal part of $u = y^Q$; $\tilde{C}_{zl} = \begin{bmatrix} C_{zl} & D_{zl}\begin{bmatrix} C \\ 0 \end{bmatrix} \end{bmatrix}$.

Along the line of [54], the infinite-horizon LQG cost for (4.40) can be defined as

$$\begin{aligned} J_c \;&=\; \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \langle z_l(s)^T z_l(s)\rangle ds \\ &=\; \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \text{Tr}(\tilde{C}_{zl}^T\tilde{C}_{zl}\tilde{S}(t))ds \\ &=\; \text{Tr}(\tilde{C}_{zl}^T\tilde{C}_{zl}\tilde{S}), \end{aligned} \tag{4.42}$$

where $\tilde{S} = \lim_{t\to+\infty}\tilde{S}(t)$ and the symmetric matrix $\tilde{S} = \int_0^t e^{A_{cl}\tau}B_{cl}B_{cl}^T e^{A_{cl}^T\tau}d\tau$ solves the following Lyapunov equation

$$A_{cl}\tilde{S} + \tilde{S}A_{cl}^T + B_{cl}B_{cl}^T = 0, \tag{4.43}$$

and the solution is unique.

By Theorem 4.2, we can construct a quantum realization of the classical controller (4.37) as

$$\begin{aligned} dx_k(t) =& \tilde{A}x_k(t)dt + \tilde{B}\begin{bmatrix} dw_1(t)^T & dw_2(t)^T \end{bmatrix}^T, \\ dy_k(t) =& \tilde{C}x_k(t)dt + \tilde{D}dw_2(t), \end{aligned} \tag{4.44}$$

where $x_k(t) = [\xi(t)^T \quad \theta(t)^T]^T$, $w_1 = [u_1^T \quad v_1^T]^T$, $w_2 = [u_2^T \quad v_2^T]^T$, $\tilde{A} = \begin{bmatrix} A & 0_{n\times n} \\ A_1 & A_2 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix}$ with $\tilde{B}_1 = \begin{bmatrix} B & 0 \\ B_1 & B_3 \end{bmatrix}$ and $\tilde{B}_2 = \begin{bmatrix} 0 & 0 \\ B_2 & B_4 \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} C & 0 \\ C_1 & C_2 \end{bmatrix}$, $\tilde{D} = \begin{bmatrix} 0 & 0 & D & 0 \\ D_1 & D_3 & D_2 & D_4 \end{bmatrix}$.

The new closed-loop plant-controller system can be obtained by setting $u(t) = y_k(t)$ and $w_1(t) = y(t)$. So interconnecting systems (4.36) and (4.44) gives

$$d\hat{x}_{cl}(t) = \hat{A}_{cl}\hat{x}_{cl}(t)dt + \hat{B}_{cl}d\hat{w}_{cl}(t), \tag{4.45}$$

where $\hat{x}_{cl} = [x^T \quad x_k^T]^T$, $\hat{w}_{cl} = [w_p^T \quad w_2^T]^T$. The matrices $\hat{A}_{cl} = \begin{bmatrix} A_p & B_{pu}\tilde{C} \\ \tilde{B}_1 C_p & \tilde{A} \end{bmatrix} =$

$$\begin{bmatrix} A & B_{pu}\begin{bmatrix} C \\ C_1 \end{bmatrix} & B_{pu}\begin{bmatrix} 0 \\ C_2 \end{bmatrix} \\ [B \quad 0]C_p & A & 0 \\ [B_1 \quad B_3]C_p & A_1 & A_2 \end{bmatrix}, \quad \hat{B}_{cl} = \begin{bmatrix} B_{pw} & B_{pu} \\ [B \quad 0]D_{pw} & 0 \\ [B_1 \quad B_3]D_{pw} & [B_2 \quad B_4] \end{bmatrix}.$$

Define a LQG performance variable for (4.45) as

$$\hat{z}_l(t) = \hat{C}_{zl}\hat{x}_{cl}(t), \tag{4.46}$$

where $\hat{C}_{zl} = \begin{bmatrix} C_{zl} & D_{zl}\begin{bmatrix} C \\ C_1 \end{bmatrix} & D_{zl}\begin{bmatrix} 0 \\ C_2 \end{bmatrix} \end{bmatrix}$.

We also associate a infinite-horizon LQG cost for (4.45), such that

$$\begin{aligned} J_q &= \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \langle \hat{z}(s)^T\hat{z}_l(s)\rangle ds \\ &= \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \mathrm{Tr}(\hat{C}_{zl}^T\hat{C}_{zl}\hat{S}(t))ds \\ &= \mathrm{Tr}(\hat{C}_{zl}^T\hat{C}_{zl}\hat{S}), \end{aligned} \tag{4.47}$$

where the symmetric matrix $\hat{S} = \lim_{t\to+\infty} \hat{S}(t) = \int_0^t e^{\hat{A}_{cl}\tau}\hat{B}_{cl}\hat{B}_{cl}^T e^{\hat{A}_{cl}^T\tau} d\tau$ solves the Lyapunov equation

$$\hat{A}_{cl}\hat{S} + \hat{S}\hat{A}_{cl}^T + \hat{B}_{cl}\hat{B}_{cl}^T = 0, \tag{4.48}$$

and the solution is unique.

Theorem 4.3. *Assume that* $D_1 = 0$, $D_2 = 0$, $D_3 = 0$, $B_2 = 0$, $\mathbf{G} = [I \quad 0]$, *and* A_{cl}, $\hat{A}_{cl}$ *are both Hurwitz matrices. The closed-loop system* (4.45) *with the quantum realization controller* (4.44) *can provide the same LQG performance specifications as the closed-loop system* (4.40) *with the classical controller* (4.37) *for the plant* (4.36) *to be controlled.*

Proof. By Theorem 4.2, if $D_1 = 0$, $D_2 = 0$, $D_3 = 0$, $B_2 = 0$, then $C_1 = 0$ and $C_2 = 0$. So $\hat{A}_{cl}$, $\hat{B}_{cl}$, $\hat{C}_{zl}$ can be rewritten as

$$\hat{A}_{cl} = \begin{bmatrix} A_{cl} & 0 \\ \bar{A}_1 & A_2 \end{bmatrix}, \tag{4.49}$$

$$\hat{B}_{cl} = \begin{bmatrix} B_{cl} \\ \bar{B} \end{bmatrix}, \tag{4.50}$$

$$\hat{C}_{zl} = \begin{bmatrix} C_{zl} & D_{zl}\begin{bmatrix} C \\ 0 \end{bmatrix} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{C}_{zl} & 0 \end{bmatrix}, \tag{4.51}$$

where A_{cl} and B_{cl} are defined as before with $\mathbf{G} = [I \quad 0]$, $\bar{A}_1 = [[B_1\ B_3]C_p \quad A_1]$ and $\bar{B} = [B_1 D_{pw} \quad B_3 D_{pw} \quad B_2 \quad B_4]$.

Now we will show these two closed-loop systems have the same LQG performance. Under assumptions of Theorem 4.3, we then have

$$\begin{aligned}
J_q &= \limsup_{t_f\to+\infty} \frac{1}{t_f}\int_0^{t_f} \mathrm{Tr}\left(\hat{C}_{zl}\hat{S}(\tau)\hat{C}_{zl}^T\right) d\tau \\
&= \mathrm{Tr}\left(\hat{C}_{zl}\hat{S}\hat{C}_{zl}^T\right) \\
&= \mathrm{Tr}\left(\int_0^\infty \hat{C}_{zl}e^{\hat{A}_{cl}\tau}\hat{B}_{cl}\hat{B}_{cl}^T e^{\hat{A}_{cl}^T\tau}\hat{C}_{zl}^T d\tau\right) \\
&= \mathrm{Tr}\left(\int_0^\infty \hat{C}_{zl}\sum_{k=0}^\infty \left(\frac{1}{k!}(\hat{A}_{cl}\tau)^k\hat{B}_{cl}\right)\left(\frac{1}{k!}(\hat{A}_{cl}\tau)^k\hat{B}_{cl}\right)^T \hat{C}_{zl}^T d\tau\right) \\
&= \mathrm{Tr}\left(\int_0^\infty \sum_{k=0}^\infty \left([\tilde{C}_{zl} \quad 0]\frac{1}{k!}\left(\begin{bmatrix} A_{cl} & 0 \\ \bar{A}_1 & A_2 \end{bmatrix}\tau\right)^k \begin{bmatrix} B_{cl} \\ \bar{B} \end{bmatrix}\right)\right. \times \\
&\left.\sum_{k=0}^\infty \left([\tilde{C}_{zl} \quad 0]\frac{1}{k!}\left(\begin{bmatrix} A_{cl} & 0 \\ \bar{A}_1 & A_2 \end{bmatrix}\tau\right)^k \begin{bmatrix} B_{cl} \\ \bar{B}_{cl} \end{bmatrix}\right)^T d\tau\right)
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{Tr}\left(\int_0^\infty \sum_{k=0}^\infty \left(\frac{1}{k!}\begin{bmatrix} \tilde{C}_{zl}(A_{cl}\tau)^k & 0 \end{bmatrix}\begin{bmatrix} B_{cl} \\ \bar{B} \end{bmatrix}\right) \times \right. \\
&\quad \left. \sum_{k=0}^\infty \left(\frac{1}{k!}\begin{bmatrix} \tilde{C}_{zl}(A_{cl}\tau)^k & 0 \end{bmatrix}\begin{bmatrix} B_{cl} \\ \bar{B} \end{bmatrix}\right)^T d\tau\right) \\
&= \mathrm{Tr}\left(\int_0^\infty \sum_{k=0}^\infty \left(\frac{1}{k!}\tilde{C}_{zl}(A_{cl}\tau)^k B_{cl}\right)\sum_{k=0}^\infty \left(\frac{1}{k!}\tilde{C}_{zl}(A_{cl}\tau)^k B_{cl}\right)^T d\tau\right) \\
&= \mathrm{Tr}\left(\int_0^\infty \tilde{C}_{zl}e^{A_{cl}\tau}B_{cl}\left(\tilde{C}_{zl}e^{A_{cl}\tau}B_{cl}\right)^T d\tau\right) \\
&= \mathrm{Tr}\left(\tilde{C}_{zl}\tilde{S}\tilde{C}_{zl}^T\right) = J_c.
\end{aligned}$$

This completes the proof. □

4.6 Concluding remarks

In this chapter, we have developed theories and methods of how to construct quantum optical systems equivalent to classical systems. Our results are illustrated with examples from quantum optics. We also study if a quantum realization of a classical controller can improve the overall closed-loop control performance. It is hoped that the main results of the work will help the implementation of classical linear systems using quantum optical devices.

Chapter 5

Extended LMI Approach to Coherent Quantum LQG Control Design

As mentioned in Chapter 1, in quantum coherent feedback control loop, a designed linear controller is itself a quantum system that is required to be physically realizable. Thus, additional non-linear and linear constraints must be imposed on the coefficients of a physically realizable quantum controller, which differs the quantum Linear Quadratic Gaussian (LQG) design from the standard LQG problem. This chapter proposes numerical procedures based on extended linear matrix inequality (LMI) approach and new physical realizability conditions proposed in [32] to design a coherent quantum controller. The extended LMI approach is not only a new analysis tool but also less conservative in comparison to the conventional counterpart, which may be useful in the optimal design of quantum optical networks.

5.1 Introduction

In previous works [113], [114], [115], extended LMI technique has been applied to designs of classical controllers, which characterize stability and performance specifications. Recalling some knowledge about extended LMIs approach: A linear system with system matrices A, B, C (assume that $D = 0$) is Hurwitz stable and the squared H_2-norm of its transfer function T satisfies $\|T\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Tr}(T(iw)^T(T(iw)) < \gamma$ if and only if there exists a general matrix F, symmetric matrices $P > 0$ and Q such that

$$\begin{bmatrix} FA + A^T F^T & P - F + A^T F^T & FB \\ P - F^T + FA & -F - F^T & FB \\ B^T F^T & B^T F^T & -I \end{bmatrix} < 0, \tag{5.1}$$

$$\begin{bmatrix} P & C^T \\ C & Q \end{bmatrix} > 0, \tag{5.2}$$

$$\mathrm{Tr}(Q) < \gamma. \tag{5.3}$$

It can be seen from (5.1) that the extra instrumental variable F introduced in extended LMIs gives a suitable structure in which the system matrices are completely independent from the Lyapunov matrix and provides a more positive impact on the design of quantum controllers compared with standard LMI conditions used in [54], [75]. In a significant way, the problem of minimizing the norm on one channel, subject to some moderate H^∞ performance requirement on another channel can be addressed with employing different Lyapunov matrices to test all the objectives, which gives us less conservative solutions.

Therefore, the purpose of this chapter is to propose two new numerical procedures based on extended LMI approach and new physical realizability conditions presented in [32] to design quantum controllers. We may optimize over extra parameters in extended LMIs and new physical realizability conditions to improve the LQG control performance of a closed-loop plant-controller system.

This chapter is organized as follows. Section 5.2 formulates the set-up of a closed-loop quantum system with a physically realizable quantum controller, and then we present a quantum LQG problem to be solved in this section. Section 5.3 proposes two numerical procedures based on extended LMIs approach to solve the quantum LQG problem. Section 5.4 applies the numerical procedures proposed in Section 5.3 to the same example given in [54] for comparison. Finally, Section 5.5 gives the conclusion of this chapter.

5.2 Problem formulation

Consider a quantum plant described by non-commutative stochastic models of the following form

$$\begin{aligned} dx_p(t) &= A_p x_p(t)dt + B_{pw} dw_p(t) + B_{pu} du(t), \\ dy(t) &= C_p x_p(t)dt + D_{pw} dw_p(t), \\ z(t) &= C_{pz} x_p(t) + D_{pz} \beta_u(t), \end{aligned} \tag{5.4}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_{pw} \in \mathbb{R}^{n\times n_{w_p}}$, $B_{pu} \in \mathbb{R}^{n\times n_{w_p}}$, $C_p \in \mathbb{R}^{n_{y_p}\times n}$, $D_{pw} \in \mathbb{R}^{n_{y_p}\times n_{w_p}}$ (n, n_{w_p} are even). x_p represents a vector of plant variables and w_p is a quantum noise. u is a control input and $\beta_u(t) = C_c x_c(t)$ is the signal part of $u(t)$. $z(t)$ is the performance output. Let initial values $x_p(0) = x_{p0}$ satisfy the commutation relations $x_{p0}x_{p0}^T - (x_{p0}x_{p0}^T)^T = 2i\Theta_{n_p}$. We assume that $\Theta_{w_p} = \frac{F_{w_p}-F_{w_p}^T}{2i}$ with $dw_p(t)dw_p(t)^T = F_{w_p}dt$.

Construct a quantum controller given by

$$
\begin{aligned}
dx_c(t) &= A_c x_c(t)dt + B_{c_1}dw_{c_1}(t) \\
&\quad + B_{c_2}dw_{c_2}(t) + B_{c_3}dy_p(t), &(5.5)\\
du(t) &= C_c x_c(t)dt + D_c dw_{c_1}(t), &(5.6)
\end{aligned}
$$

where $A_c \in \mathbb{R}^{n\times n}$, $B_{c_1} \in \mathbb{R}^{n\times n_{w_{c1}}}$, $B_{c_2} \in \mathbb{R}^{n\times n_{w_{c2}}}$, $B_{c_3} \in \mathbb{R}^{n\times n_{w_{c3}}}$, $C_c \in \mathbb{R}^{n_u\times n}$, $D_c \in \mathbb{R}^{n_u\times n_{w_{c1}}}$ ($n_{w_{c_1}} = n_u$, $n_{w_{c_2}} = n$, $n_{w_{c_3}} = n_{y_p}$ are even). x_c represents a vector of controller variables of the same order as $x_p(t)$. The commutation relation for $x_c(t)$ satisfies

$$x_c x_c^T - (x_c x_c^T)^T = 2i\Theta_{n_c},$$

where Θ_{n_c} is an arbitrary anti-symmetric matrix. The quantum Wiener disturbance vectors w_{c_1}, w_{c_2}, w_p are independent of each other and satisfy the following relations

$$
\begin{aligned}
[dw_{c_1}(t), dw_{c_1}(t)^T] &= (F_{w_{c_1}} - F_{w_{c_1}}^T)dt = 2i\Theta_{w_{c_1}}dt,\\
[dw_{c_2}(t), dw_{c_2}(t)^T] &= (F_{w_{c_2}} - F_{w_{c_2}}^T)dt = 2i\Theta_{w_{c_2}}dt,\\
[dw_{c_3}(t), dw_{c_1}(t)^T] &= (F_{w_{c_3}} - F_{w_{c_3}}^T)dt = 2i\Theta_{w_{c_3}}dt,
\end{aligned}
$$

where $F_{w_{c_1}}, F_{w_{c_2}}, F_{w_{c_3}}$ are nonnegative definite Hermitian matrices and their corresponding $\Theta_{w_{c_1}}, \Theta_{w_{c_2}}, \Theta_{w_{c_3}}$ are skew-symmetric matrices. A physically realizable quantum controller (5.5) should require its system matrices $A_c, B_{c_1}, B_{c_2}, B_{c_3}, C_c$ to satisfy the following conditions:

$$
\begin{aligned}
&A_c\Theta_{n_c} + \Theta_{n_c}A_c^T + B_{c_1}\Theta_{w_{c_1}}B_{c_1}^T + B_{c_2}\Theta_{w_{c_2}}B_{c_2}^T + B_{c_3}\Theta_{w_{c_3}}B_{c_3}^T = 0, &(5.7)\\
&B_{c_1}\Theta_{w_{c_1}}D_c^T = -\Theta_{n_c}C_c^T, &(5.8)\\
&D_c\Theta_{w_{c_1}}D_c^T = \Theta_{w_{c_1}}. &(5.9)
\end{aligned}
$$

Interconnecting systems (5.4) and (5.5) gives

$$
\begin{aligned}
dx(t) &= Ax(t)dt + Bdw(t),\\
z(t) &= Cx(t), &(5.10)
\end{aligned}
$$

where $x = [x_p^T \quad x_c^T]^T, w = [w_p \quad w_{c_1}^T \quad w_{c_2}^T]^T$, $A = \begin{bmatrix} A_p & B_{pu}C_c \\ B_{c_3}C_p & A_c \end{bmatrix}$, $B = \begin{bmatrix} B_{pw} & B_{pu}D_c & 0 \\ B_{c_3}D_{pw} & B_{c_1} & B_{c_2} \end{bmatrix}$, $C = \begin{bmatrix} C_{pz} & D_{pz}C_c \end{bmatrix}$. Along the line of [54], the infinite-horizon LQG cost can be defined as

$$\begin{aligned} J_\infty &= \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \langle z(s)^T z(s)\rangle ds \\ &= \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \mathrm{Tr}(C^T C S(t)) ds \\ &= \mathrm{Tr}(C^T C S), \end{aligned} \tag{5.11}$$

where the symmetric matrix S solves the following Lyapunov equation and the solution is unique.

$$AS + SA^T + BB^T = 0, \tag{5.12}$$

where $S < P^{-1}$ and the symmetric matrix P is shown in (5.1)-(5.2).

In the next section, we will focus our attention to solve the following problem:

Problem 5.1: Given a cost bound parameter $\gamma > 0$, and design a quantum controller of the form (5.5) satisfying the following statements:

1. There exist symmetric matrices $P > 0$ and Q as well as a general matrix F satisfying (5.1)-(5.3).

2. $J_\infty < \gamma$.

3. The conditions (5.7)-(5.9) should be satisfied.

5.3 Quantum controller synthesis

In this section, we will propose numerical procedures based on extended LMI approach to design coherent quantum controllers, which can solve **Problem 5.1**.

5.3.1 Controller parametrization

In order to fit **Problem 5.1** into extended LMIs frame, let us redefine our plant below without changing the structure of the closed-loop system (5.10)

$$\begin{aligned} dx_p(t) &= A_p x_p(t)dt + B_{pu}\beta_u(t) + \bar{B}_{pw} d\bar{w}_p(t), \\ d\bar{y}(t) &= \bar{C}_p x_p(t)dt + \bar{D}_{pw} d\bar{w}_p(t), \\ z(t) &= C_{pz} x_p(t)dt + D_{pz}\beta_u(t), \end{aligned} \tag{5.13}$$

where $\bar{w}_p = w$, $\bar{B}_{pw} = [B_{pw} \quad B_{pu}D_c \quad 0]$, $\bar{C}_p = [0 \quad 0 \quad C_p^T]^T$, and $\bar{D}_{pw} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ D_{pw} & 0 & 0 \end{bmatrix}$.

Let us redefine our controller as

$$\begin{aligned} dx_c(t) &= A_c x_c(t)dt + B_c d\bar{y}(t), \\ \beta_u(t) &= C_c x_c(t), \end{aligned} \tag{5.14}$$

where $B_c = [B_{c_1} \quad B_{c_2} \quad B_{c_3}]$.

To remove the nonlinear terms in (5.1) and (5.2), we now follow the ideas of [114], [115] by introducing $n \times n$ general matrices $\mathbf{X}, \mathbf{Y}, \mathbf{U}, \mathbf{V}$.

Let $K = \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$, $F = \begin{bmatrix} \mathbf{X} & * \\ \mathbf{U} & * \end{bmatrix}$ and $F^{-1} = \begin{bmatrix} \mathbf{Y}^T & * \\ \mathbf{V}^T & * \end{bmatrix}$. $F \begin{bmatrix} \mathbf{Y}^T \\ \mathbf{V}^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $F \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{U} \end{bmatrix}$ can be inferred from $F^{-1}F = I$. Then, we have

$$FT_1 = T_2,$$

where $T_1 = \begin{bmatrix} I & \mathbf{Y}^T \\ 0 & \mathbf{V}^T \end{bmatrix}$ and $T_2 = \begin{bmatrix} \mathbf{X} & I \\ \mathbf{U} & 0 \end{bmatrix}$. Define the following nonlinear transformation

$$\begin{aligned} \begin{bmatrix} \mathbf{A_c} & \mathbf{B_c} \\ \mathbf{C_c} & 0 \end{bmatrix} &= \begin{bmatrix} \mathbf{V} & \mathbf{Y}B_{pu} \\ 0 & I \end{bmatrix} K \begin{bmatrix} \mathbf{U} & 0 \\ \bar{C}_p\mathbf{X} & I \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{Y} \\ 0 \end{bmatrix} A_p \begin{bmatrix} \mathbf{X} & 0 \end{bmatrix}, \end{aligned} \tag{5.15}$$

$$\begin{bmatrix} \mathbf{N} & \mathbf{J} \\ \mathbf{J}^T & \mathbf{H} \end{bmatrix} = T_1^T P T_1, \tag{5.16}$$

$$\mathbf{S} = \mathbf{YX} + \mathbf{VU}, \tag{5.17}$$

where $\mathbf{A_c} = \mathbf{Y}A_p\mathbf{X} + \mathbf{B_c}\bar{C}_p\mathbf{X} + \mathbf{Y}B_{pu}C_c\mathbf{U} + \mathbf{V}A_c\mathbf{U}, \mathbf{B_c} = \mathbf{V}B_c, \mathbf{C_c} = C_c\mathbf{U}$. $\mathbf{N}, \mathbf{H}$ are symmetric matrices and $\mathbf{J}$ is a general $n \times n$ matrix. Performing congruence transformations on inequalities (5.1) and (5.2) with $\mathrm{diag}(T_1, T_1, I_n)$ and $\mathrm{diag}(T_1, I_n)$, respectively, we obtain new inequalities

$$\begin{bmatrix} A_p\mathbf{X}+B_{pu}\mathbf{C_c} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{Y}A_p+\mathbf{B_c}\bar{C}_p & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ 0 & A_p-\mathbf{S}^T+\mathbf{J} & -\mathbf{X}-\mathbf{X}^T & (\cdot)^T & (\cdot)^T \\ 0 & \mathbf{Y}A_p+\mathbf{B_c}\bar{C}_p-\mathbf{Y}^T+\mathbf{H} & -I_n-\mathbf{S} & -\mathbf{Y} & (\cdot)^T \\ 0 & (\mathbf{Y}\bar{B}_{pw}+\mathbf{B_c}\bar{D}_{pw})^T & \bar{B}_{pw}^T & (\mathbf{Y}\bar{B}_{pw}+\mathbf{B_c}\bar{D}_{pw})^T & 0 \end{bmatrix}+$$

$$\begin{bmatrix} (A_p\mathbf{X}+B_{pu}\mathbf{C_c})^T & (\cdot)^T & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ \mathbf{A_c}+A_p^T & (\mathbf{Y}A_p+\mathbf{B_c}\bar{C}_p)^T & 0 & 0 & 0 \\ A_p\mathbf{X}+B_{pu}\mathbf{C_c}-\mathbf{X}^T+\mathbf{N} & 0 & -\mathbf{X}-\mathbf{X}^T & 0 & 0 \\ \mathbf{A_c}-I_n+\mathbf{J}^T & 0 & 0 & -\mathbf{Y}^T & 0 \\ \bar{B}_{pw}^T & 0 & 0 & 0 & -I_n \end{bmatrix} < 0, \quad (5.18)$$

$$\begin{bmatrix} \mathbf{N} & (\cdot)^T & (\cdot)^T \\ \mathbf{J}^T & \mathbf{H} & (\cdot)^T \\ C_{pz}\mathbf{X}+D_{pz}\mathbf{C_c} & C_{pz} & Q \end{bmatrix} > 0, \quad (5.19)$$

$$\mathrm{Tr}(Q) < \gamma. \quad (5.20)$$

Multiplying both sides of the left hand side of (5.7) with $\mathbf{V}$ and $\mathbf{V}^T$ produces new variables $\mathbf{\Theta_{n_c}} = \mathbf{V}\Theta_{n_c}$, $\mathbf{B_{c_i}} = \mathbf{V}B_{c_i}$ $(i = 1, 2, 3)$. Then, conditions (5.7)-(5.9) become

$$(-\mathbf{A_c}\mathbf{U}^{-1} + (\mathbf{B_{c_3}}C_p + \mathbf{Y}A_p)\mathbf{X}\mathbf{U}^{-1} + \mathbf{Y}B_{pu}C_c)\mathbf{\Theta_{n_c}}^T$$
$$+\mathbf{\Theta_{n_c}}(\mathbf{A_c}\mathbf{U}^{-1} - (\mathbf{B_{c_3}}C_p + \mathbf{Y}A_p)\mathbf{X}\mathbf{U}^{-1} - \mathbf{Y}B_{pu}C_c)^T$$
$$+\sum_{i=1}^{3}\mathbf{B_{c_i}}\Theta_{w_{c_i}}\mathbf{B_{c_i}}^T = 0, \quad (5.21)$$
$$\mathbf{B_{c_1}}\Theta_{w_{c_1}}D_c^T = -\mathbf{\Theta_{n_c}}C_c^T, \quad (5.22)$$
$$D_c\Theta_{w_{c_1}}D_c^T = \Theta_{w_{c_1}}. \quad (5.23)$$

If there exists a upper-bound γ and matrices $\mathbf{A_c}$, $\mathbf{B_c}$, $\mathbf{C_c}$, D_c, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{S}$, $\mathbf{N}$, $\mathbf{J}$, $\mathbf{H}$, $\mathbf{V}$, $\mathbf{U}$, Q satisfying conditions (5.15)-(5.23), then matrices A_c, B_c and C_c of a quantum controller of the form (5.5) can be obtained as

$$C_c = \mathbf{C_c}\mathbf{U}^{-1}, \quad (5.24)$$
$$B_c = \mathbf{V}^{-1}\mathbf{B_c}, \quad (5.25)$$
$$A_c = \mathbf{V}^{-1}(\mathbf{A_c} - \mathbf{Y}A_p\mathbf{X} - \mathbf{B_c}\bar{C}_p\mathbf{X})\mathbf{U}^{-1} - \mathbf{V}^{-1}\mathbf{Y}B_{pu}C_c. \quad (5.26)$$

5.3.2 Numerical optimization procedures

Our **Problem 5.1** can be formulated as minimization of the LQG cost subject to constraints including LMIs and additional nonlinear constraints which are related to rank conditions [54], [119]. In the following we present numerical algorithms based on extended LMI approach. The nonlinear constraints (5.7)-(5.9) make proposing numerical algorithms for solving **Problem 5.1** very challenging, which differs quantum LQG problem from that of conventional LQG.

Numerical Procedure 5.1: To seek a fully quantum controller, we will allow matrices Θ_{n_c} to be arbitrary skew-symmetric matrices but invertible. Let $D_c = I$ and $\Theta_{w_{c_i}} = \mathrm{diag}_{\frac{n_{w_i}}{2}}(J)$, $i = 1, 2, 3$. For simplicity, we choose $\mathbf{U} = I$ and hence $\mathbf{V} = \mathbf{S} - \mathbf{YX}$. Then, suppose $Z_{x_1} = \mathbf{A_c}, Z_{x_2} = \mathbf{B_{c_1}}, Z_{x_3} = \mathbf{B_{c_2}}, Z_{x_4} = \mathbf{B_{c_3}}, Z_{x_5} = \mathbf{C_c}, Z_{x_6} = \Theta_{n_c}, Z_{x_7} = \mathbf{X}^T, Z_{x_8} = \mathbf{Y}, Z_{x_9} = \mathbf{S}$. Introducing appropriate matrix lifting variables $Z_{v_1}, Z_{v_2}, \cdots, Z_{v_{16}}$ can linearize conditions (5.21) and (5.22). Define a symmetric matrix Z of dimension $26n \times 26n$ as $Z = VV^T$, where $V = [I \ \ Z_{x_1}^T \ \ \cdots \ \ Z_{x_9}^T \ \ Z_{v_1}^T \ \ \cdots \ \ Z_{v_{16}}^T]^T$, $Z_{v_1} = \mathbf{YX}$, $Z_{v_2} = \mathbf{V}$, $Z_{v_3} = \mathbf{V}\Theta_{n_c} = \mathbf{V_h}$, $Z_{v_4} = \mathbf{Y}A_p + \mathbf{B_{c_3}}C_p$, $Z_{v_5} = \mathbf{Y}B_{pu}$, $Z_{v_6} = \mathbf{V_h}\mathbf{C_c}^T$, $Z_{v_7} = (\mathbf{Y}A_p + \mathbf{B_{c_3}}C_p)\mathbf{X}$, $Z_{v_8} = \mathbf{A_c}\mathbf{V_h}^T$, $Z_{v_9} = \mathbf{Y}B_{pu}(\mathbf{V_h}\mathbf{C_c}^T)^T$, $Z_{v_{10}} = (\mathbf{Y}A_p + \mathbf{B_{c_3}}C_p)\mathbf{X}\mathbf{V_h}^T$, $Z_{v_{11}} = \mathbf{B_{c_1}}\Theta_{w_{c_1}}$, $Z_{v_{12}} = \mathbf{B_{c_1}}\Theta_{w_{c_1}}\mathbf{B_{c_1}}^T$, $Z_{v_{13}} = \mathbf{B_{c_2}}\Theta_{w_{c_2}}$, $Z_{v_{14}} = \mathbf{B_{c_2}}\Theta_{w_{c_2}}\mathbf{B_{c_2}}^T$, $Z_{v_{15}} = \mathbf{B_{c_3}}\Theta_{w_{c_3}}$, $Z_{v_{16}} = \mathbf{B_{c_3}}\Theta_{w_{c_3}}\mathbf{B_{c_3}}^T$. Then, we have the following set of additional constraints

$$
\begin{aligned}
& Z \geq 0; && Z_{v_8} - Z_{x_1}Z_{v_3}^T = 0; \\
& Z_{0,0} - I_{n\times n} = 0; && Z_{v_9} - Z_{v_5}Z_{v_6}^T = 0; \\
& Z_{x_6} + Z_{x_6}^T = 0; && Z_{v_{10}} - Z_{v_7}Z_{v_3}^T = 0; \\
& Z_{v_1} - Z_{x_8}Z_{x_7} = 0; && Z_{v_{11}} - Z_{x_2}\mathrm{diag}_{\frac{nw_1}{2}}(J) = 0; \\
& Z_{v_2} - Z_{x_9} + Z_{v_1} = 0; && Z_{v_{12}} - Z_{v_{11}}Z_{x_2}^T = 0; \\
& Z_{v_3} + Z_{v_2}Z_{x_6}^T = 0; && Z_{v_{13}} - Z_{x_3}\mathrm{diag}_{\frac{nw_2}{2}}(J) = 0; \qquad (5.27)\\
& Z_{v_4} - Z_{x_8}A_p - Z_{x_4}C_p = 0; && Z_{v_{14}} - Z_{v_{13}}Z_{x_3}^T = 0; \\
& Z_{v_5} - Z_{x_8}B_{pu} = 0; && Z_{v_{15}} - Z_{x_4}\mathrm{diag}_{\frac{nw_3}{2}}(J) = 0; \\
& Z_{v_6} - Z_{v_3}Z_{x_5}^T = 0; && Z_{v_{16}} - Z_{v_{15}}Z_{x_4}^T = 0; \\
& Z_{v_7} - Z_{v_4}Z_{x_7}^T = 0; && \qquad (5.28)
\end{aligned}
$$

and a rank constraint

$$\mathrm{rank}(Z) \leq n. \tag{5.29}$$

Conditions (5.21)-(5.22) for physical realizability can be expressed as

$$-Z_{v_8} + Z_{v_8}^T + Z_{v_9} + Z_{v_9}^T + Z_{v_{10}} + Z_{v_{10}}^T + Z_{v_{12}} + Z_{v_{14}} + Z_{v_{16}} = 0, \tag{5.30}$$

$$Z_{v_{11}} + Z_{v_6} = 0. \tag{5.31}$$

If we can employ a semidefinite programming to solve the feasibility problem with constraints (5.3), (5.18), (5.19) and (5.32)-(5.35) in which decision variables are Z and Q, **Problem 5.1** is solvable. Then controller matrices can be built as (5.24)-(5.26).

Numerical Procedure 5.2: To achieve better optimal control performance and seek a fully quantum controller, we will set Θ_{n_c} to be an arbitrary invertible skew-symmetric matrices and allow D_c to be an arbitrary symplectic matrix. Let $\Theta_{w_{c_i}} = \mathrm{diag}_{\frac{n_{w_i}}{2}}(J)$, $i = 1, 2, 3$. For simplicity, we choose $\mathbf{U} = I$ and hence $\mathbf{V} = \mathbf{S} - \mathbf{YX}$. Then, suppose $Z_{x_1} = \mathbf{A_c}, Z_{x_2} = \mathbf{B_{c_1}}, Z_{x_3} = \mathbf{B_{c_2}}, Z_{x_4} = \mathbf{B_{c_3}}, Z_{x_5} = \mathbf{C_c}, Z_{x_6} = \Theta_{n_c}, Z_{x_7} = \mathbf{X}^T, Z_{x_8} = \mathbf{Y}, Z_{x_9} = D_c$. Introducing appropriate matrix lifting variables $Z_{v_1}, Z_{v_2}, \cdots, Z_{v_{21}}$ can linearize conditions (5.21)-(5.23) as well as nonlinear terms $\mathbf{Y}B_{pu}D_c$ and $(\mathbf{Y}B_{pu}D_c)^T$ in inequality (5.18). Define a symmetric matrix Z of dimension $32n \times 32n$ as $Z = VV^T$, where $V = [I \quad Z_{x_1}^T \quad \cdots \quad Z_{x_{10}}^T \quad Z_{v_1}^T \quad \cdots \quad Z_{v_{21}}^T]^T$, $Z_{v_1} = \mathbf{YX}$, $Z_{v_2}=\mathbf{V}$, $Z_{v_3}=\mathbf{V}\Theta_{n_c} = \mathbf{V_h}$, $Z_{v_4} = \mathbf{Y}A_p + \mathbf{B_{c_3}}C_p$, $Z_{v_5} = \mathbf{Y}B_{pu}$, $Z_{v_6} = \mathbf{V_h}\mathbf{C_c}^T$, $Z_{v_7} = (\mathbf{Y}A_p + \mathbf{B_{c_3}}C_p)\mathbf{X}$, $Z_{v_8}=\mathbf{A_c}\mathbf{V_h}^T$, $Z_{v_9}=\mathbf{Y}B_{pu}(\mathbf{V_h}\mathbf{C_c}^T)^T$, $Z_{v_{10}}=(\mathbf{Y}A_p + \mathbf{B_{c_3}}C_p)\mathbf{X}\mathbf{V_h}^T$, $Z_{v_{11}}=\mathbf{B_{c_1}}\Theta_{w_{c_1}}$, $Z_{v_{12}} = \mathbf{B_{c_1}}\Theta_{w_{c_1}}\mathbf{B_{c_1}}^T$, $Z_{v_{13}} = \mathbf{B_{c_2}}\Theta_{w_{c_2}}$, $Z_{v_{14}} = \mathbf{B_{c_2}}\Theta_{w_{c_2}}\mathbf{B_{c_2}}^T$, $Z_{v_{15}} = \mathbf{B_{c_3}}\Theta_{w_{c_3}}$, $Z_{v_{16}} = \mathbf{B_{c_3}}\Theta_{w_{c_3}}\mathbf{B_{c_3}}^T$, $Z_{v_{17}} = D_c\Theta_{w_{c_1}}$, $Z_{v_{18}} = D_c\Theta_{w_{c_1}}D_c^T$, $Z_{v_{19}} = \mathbf{B_{c_1}}\Theta_{w_{c_1}}D_c^T$, $Z_{v_{20}} = D_c^T B_{pu}^T$, $Z_{v_{21}} = \mathbf{Y}B_{pu}D_c$. Then, we have the following set of additional constraints:

$$\begin{array}{ll}
Z \geq 0; & Z_{v_{10}} - Z_{v_7}Z_{v_3}^T = 0; \\
Z_{0,0} - I_{n\times n} = 0; & Z_{v_{11}} - Z_{x_2}\mathrm{diag}_{\frac{n_{w_1}}{2}}(J) = 0; \\
Z_{x_6} + Z_{x_6}^T = 0; & Z_{v_{12}} - Z_{v_{11}}Z_{x_2}^T = 0; \\
Z_{v_1} - Z_{x_8}Z_{x_7} = 0; & Z_{v_{13}} - Z_{x_3}\mathrm{diag}_{\frac{n_{w_2}}{2}}(J) = 0; \\
Z_{v_2} - \mathbf{S} + Z_{v_1} = 0; & Z_{v_{14}} - Z_{v_{13}}Z_{x_3}^T = 0; \\
Z_{v_3} + Z_{v_2}Z_{x_6}^T = 0; & Z_{v_{15}} - Z_{x_4}\mathrm{diag}_{\frac{n_{w_3}}{2}}(J) = 0; \\
Z_{v_4} - Z_{x_8}A_p - Z_{x_4}C_p = 0; & Z_{v_{16}} - Z_{v_{15}}Z_{x_4}^T = 0; \\
Z_{v_5} - Z_{x_8}B_{pu} = 0; & Z_{v_{17}} - Z_{x_{10}}\mathrm{diag}_{\frac{n_{w_1}}{2}}(J) = 0; \\
Z_{v_6} - Z_{v_3}Z_{x_5}^T = 0; & Z_{v_{18}} - Z_{v_{17}}Z_{x_{10}}^T = 0; \\
Z_{v_7} - Z_{v_4}Z_{x_7}^T = 0; & Z_{v_{19}} - Z_{v_{11}}Z_{x_{10}}^T = 0; \\
Z_{v_8} - Z_{x_1}Z_{v_3}^T = 0; & Z_{v_{20}} - Z_{x_{10}}^T B_{pu}^T = 0; \\
Z_{v_9} - Z_{v_5}Z_{v_6}^T = 0; & Z_{v_{21}} - Z_{x_8}Z_{v_{20}}^T = 0;
\end{array} \tag{5.32}$$

and a rank constraint

$$\mathrm{rank}(Z) \leq n. \tag{5.33}$$

Conditions (5.21)-(5.23) for physical realizability can be expressed as

$$-Z_{v_8} + Z_{v_8}^T + Z_{v_9} + Z_{v_9}^T + Z_{v_{10}} + Z_{v_{10}}^T + Z_{v_{12}} + Z_{v_{14}} + Z_{v_{16}} = 0, \tag{5.34}$$

$$Z_{v_{19}} + Z_{v_6} = 0, \tag{5.35}$$

$$Z_{v_{18}} - \mathrm{diag}_{\frac{n_{w_1}}{2}}(J) = 0. \tag{5.36}$$

Extended LMI constraints (5.3), (5.18), (5.19) can be reexpressed as

$$\begin{bmatrix} (A_pZ_{x_7}^T + B_{pu}Z_{x_5})^T & (\cdot)^T & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ A_p^T & (Z_{x_8}A_p + Z_{x_4}C_p)^T & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ -Z_{x_7} + \mathbf{N} & \mathbf{J} & -Z_{x_7} & (\cdot)^T & (\cdot)^T \\ \mathbf{J}^T & -Z_{x_8}^T + \mathbf{H} & -\mathbf{S} & -Z_{x_8}^T & (\cdot)^T \\ 0 & ([Z_{x_2}\ Z_{x_3}\ Z_{x_4}]\bar{D}_{pw})^T & 0 & ([Z_{x_2}\ Z_{x_3}\ Z_{x_4}]\bar{D}_{pw})^T & 0 \end{bmatrix} +$$

$$\begin{bmatrix} A_pZ_{x_7}^T + B_{pu}Z_{x_5} & (\cdot)^T & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ Z_{x_1} & Z_{x_8}A_p + Z_{x_4}C_p & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ A_pZ_{x_7}^T + B_{pu}Z_{x_5} & A_p - \mathbf{S}^T & -Z_{x_7}^T & (\cdot)^T & (\cdot)^T \\ Z_{x_1} - I_n & Z_{x_8}A_p + Z_{x_4}C_p & -I_n & -Z_{x_8} & (\cdot)^T \\ \bar{B}_{pw}^T & ([Z_{x_8}B_{pw}\ Z_{v_{21}}\ 0])^T & \bar{B}_{pw}^T & ([Z_{x_8}B_{pw}\ Z_{v_{21}}\ 0])^T & -I_n \end{bmatrix} < 0, \tag{5.37}$$

$$\begin{bmatrix} \mathbf{N} & (\cdot)^T & (\cdot)^T \\ \mathbf{J}^T & \mathbf{H} & (\cdot)^T \\ C_{pz}Z_{x_7}^T + D_{pz}Z_{x_5} & C_{pz} & Q \end{bmatrix} > 0, \tag{5.38}$$

$$\mathrm{Tr}(Q) < \gamma. \tag{5.39}$$

If we can employ a semidefinite programming to solve the feasibility problem with constraints (5.32)-(5.39) in which decision variables are Z, $\mathbf{N}$, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{H}$ and Q, **Problem 5.1** is solvable. Then controller matrices can be built as (5.24)-(5.26).

Our numerical procedures can be solved based on Yalmip [116], SeDuMi [117], and LMIRank [118], [119]. The LMIRank solver can only solve feasibility problems and uses a local approach to address the non-convex rank constraints, hence it is essential to find proper starting points for our algorithms [118], [119].

In our two procedures, Θ_{n_c} is allowed to be arbitrary antisymmetric matrices. However, as pointed out in [32], [81], [84], we cannot build a linear quantum stochastic controller as a suitable network of basic quantum devices if it is not in a *standard* form in the sense of Definition 3.1 proposed in Chapter 3. Thus, we need to transform the quantum controller into a *standard* form once it does not satisfy conditions in Definition 3.1.

Theorem 5.1. *Given an arbitrary real skew-symmetric matrix Θ_{n_c} ($n_c \geq 2$), there exists a real nonsingular matrices S_{n_c} such that*

$$\Theta_{n_c} = S_{n_c} \mathrm{diag}_{\frac{n_c}{2}}(J) S_{n_c}^T. \tag{5.40}$$

Then we have

$$\begin{aligned} \tilde{A}_c &= S_{n_c}^{-1} A_c S_{n_c}, & (5.41)\\ \tilde{B}_{c_i} &= S_{n_c}^{-1} B_{c_i}, & (5.42)\\ \tilde{C}_c &= C_c S_{n_c}. & (5.43)\\ \tilde{D}_c &= D_c. & (5.44) \end{aligned}$$

Furthermore, if the original closed-loop system (5.10) *is asymptotically stable, the closed-loop system of the form* (5.10) *with a new quantum controller built by $\tilde{A}_c$, $\tilde{B}_c$, $\tilde{C}_c$, $\tilde{D}_c$ is still asymptotically stable and its LQG cost is the same as original one.*

Proof. The similar proofs of relations (5.40)-(5.43) can be found in [104] and hence is omitted here. Substituting (5.40)-(5.43) into conditions (5.7)-(5.8) with some algebraic manipulations gives

$$\begin{aligned} &\tilde{A}_c \mathrm{diag}_{\frac{n_c}{2}}(J) + \mathrm{diag}_{\frac{n_c}{2}}(J)\tilde{A}_c^T + \tilde{B}_{c_1}\mathrm{diag}_{\frac{n_{w_{c1}}}{2}}(J)\tilde{B}_{c_1}^T + \\ &\quad \tilde{B}_{c_2}\mathrm{diag}_{\frac{n_{w_{c2}}}{2}}(J)\tilde{B}_{c_2}^T + \tilde{B}_{c_3}\mathrm{diag}_{\frac{n_{w_{c3}}}{2}}(J)\tilde{B}_{c_3}^T = 0, & (5.45)\\ &\tilde{B}_{c_1}\mathrm{diag}_{\frac{n_{w_{c1}}}{2}}(J)\tilde{D}_c^T = -\mathrm{diag}_{\frac{n_c}{2}}(J)\tilde{C}_c^T. & (5.46) \end{aligned}$$

By applying similarity transformation $\Gamma = \mathrm{diag}(I, S_{n_c}^{-1})$ to A, B and C, we have

$$\begin{aligned} \tilde{A} &= \Gamma \begin{bmatrix} A_p & B_{pu}C_c \\ B_{c_3}C_p & A_c \end{bmatrix} \Gamma^{-1} = \begin{bmatrix} A_p & B_{pu}\tilde{C}_c \\ \tilde{B}_{c_3}C_p & \tilde{A}_c \end{bmatrix}, & (5.47)\\ \tilde{B} &= \Gamma \begin{bmatrix} B_{pw} & B_{pu}D_c & 0 \\ B_{c_3}D_{pw} & B_{c_1} & B_{c_2} \end{bmatrix} = \begin{bmatrix} B_{pw} & B_{pu}D_c & 0 \\ \tilde{B}_{c_3}D_{pw} & \tilde{B}_{c_1} & \tilde{B}_{c_2} \end{bmatrix}, & (5.48)\\ \tilde{C} &= \begin{bmatrix} C_{pz} & D_{pz}C_c \end{bmatrix} \Gamma^{-1} = \begin{bmatrix} C_{pz} & D_{pz}\tilde{C}_c \end{bmatrix}. & (5.49) \end{aligned}$$

From (5.47), we can see that the *new* closed-loop system of the form (5.10) with A, B, C, x and z replaced by $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{x} = \Gamma x$ and $\tilde{z} = \tilde{C}\tilde{x}$ is asymptotically stable. Multiplying the left and right hand sides of each term in (5.12) with Γ and Γ^T gives

$$\begin{aligned} \Gamma A \Gamma^{-1} \Gamma S \Gamma^T + \Gamma S \Gamma^T \Gamma^{-T} A^T \Gamma^T + \Gamma B B^T \Gamma^T &= 0, \\ \tilde{A}\tilde{S} + \tilde{S}\tilde{A}^T + \tilde{B}\tilde{B}^T &= 0, \end{aligned}$$

where $\tilde{S} = \Gamma S \Gamma^T$.

The LQG cost for the *new* closed-loop system of the form (5.10) is given by

$$\begin{aligned}
\tilde{J}_\infty &= \lim_{t\to+\infty} \sup \frac{1}{t}\int_0^t \langle \tilde{z}(s)^T \tilde{z}(s)\rangle ds \\
&= \lim_{t\to+\infty} \sup \frac{1}{t}\int_0^t \mathrm{Tr}(\tilde{C}^T \tilde{C}\tilde{S}(t)) ds \\
&= \mathrm{Tr}(\tilde{C}^T \tilde{C}\tilde{S}) \\
&= \mathrm{Tr}(C^T C S) = J_\infty. \qquad (5.50)
\end{aligned}$$

This completes the proof. □

5.4 An example

The linear quantum plant below is studied in [54, Section 8].

$$\begin{aligned}
dx_p(t) &= \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 & 0 \\ 0 & -0.2 \end{bmatrix} du(t) \\
&\quad + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & -0.2 \end{bmatrix} dw_p(t), \\
dy(t) &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix} x_p(t) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} dw_p(t). \qquad (5.51)
\end{aligned}$$

For comparison with results in [54, Section 8], we work in Matlab using the same Yalmip prototyping environment and the same semi-definite program solver. Then applying our **Numerical Procedure 5.1** with $\gamma = 5.4$ proposed in the Section 5.3 to the plant (5.51), we get the following solutions:

$$A_c = \begin{bmatrix} -0.0265 & -0.2471 \\ 0.0665 & -0.1558 \end{bmatrix},$$

$$B_{c_1} = \begin{bmatrix} 0.0835 & -0.5259 \\ 0.1740 & -0.0578 \end{bmatrix},$$

$$B_{c_2} = 10^{-12}\begin{bmatrix} -0.1212 & -0.0865 \\ -0.0785 & -0.0100 \end{bmatrix},$$

$$B_{c_3} = \begin{bmatrix} 0.7786 & -0.1680 \\ 0.7468 & -0.0383 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 0.0578 & -0.5259 \\ 0.1740 & -0.0835 \end{bmatrix},$$

$$D_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let us first check if the above results solve **Problem 5.1**. The right hand sides of (5.7) and (5.8) take the numerical values $10^{-13}\begin{bmatrix} 0 & 0.6040 \\ -0.6040 & 0 \end{bmatrix}$ and $10^{-13}\times \begin{bmatrix} 0.3064 & -0.4412 \\ -0.1275 & -0.1837 \end{bmatrix}$, respectively, which indicate the quantum controller is physically realizable. The eigenvalues of the closed-loop system are $-0.0281 + 0.1030i$, $-0.0281 - 0.1030i$, $-0.0631 + 0.0901i$, $-0.0631 - 0.0901i$, so the plant-controller system is Hurwitz stable. The resulting LQG performance is 4.1651, which is a little better than the LQG cost 4.1793 in [54].

Applying **Numerical Procedure 5.2** with $\gamma = 5.4$ proposed in the Section 5.3 to the plant (5.51), we get the following solutions:

$$A_c = \begin{bmatrix} -0.0265 & -0.2471 \\ 0.0665 & -0.1558 \end{bmatrix},$$

$$B_{c_1} = \begin{bmatrix} 0.0835 & -0.5259 \\ 0.1740 & -0.0578 \end{bmatrix},$$

$$B_{c_2} = 10^{-12}\begin{bmatrix} -0.1212 & -0.0865 \\ -0.0785 & -0.0100 \end{bmatrix},$$

$$B_{c_3} = \begin{bmatrix} 0.7786 & -0.1680 \\ 0.7468 & -0.0383 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 0.0578 & -0.5259 \\ 0.1740 & -0.0835 \end{bmatrix},$$

$$D_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The right hand sides of (5.7) and (5.8) take the values $10^{-13}\begin{bmatrix} 0 & 0.6040 \\ -0.6040 & 0 \end{bmatrix}$ and $10^{-13}\begin{bmatrix} 0.3064 & -0.4412 \\ -0.1275 & -0.1837 \end{bmatrix}$, respectively, which indicate the quantum controller is physically realizable. The eigenvalues of the closed-loop system are $-0.0281 + 0.1030i$, $-0.0281 - 0.1030i$, $-0.0631 + 0.0901i$, $-0.0631 - 0.0901i$, so the plant-controller system is Hurwitz stable. The resulting LQG performance is 4.1601, which is a little better than that of **Numerical Procedure 5.1**.

5.5 Concluding remarks

In this chapter, we propose two new numerical procedures based on extended LMIs approach and new physical realizability conditions, which can provide more parameters for the design of a physically realizable quantum controller of the standard form and give less conservative solutions to quantum LQG problem. For comparison, we reinvestigate the example given in [54]. It turns out that our optimization procedures proposed in this chapter may be useful in the optimal design of quantum optical networks.

Chapter 6

Optimal Controller Design for Quantum Measurement-based Feedback Control Systems with Feedback-loop Time Delay

One is often confronted with time delay mainly originated from the transition delay of signals in a quantum feedback control loop, which may cause quantum feedback control systems unstable. The effect of time delay on the control performance plays a peculiar role in quantum mechanics. It has been shown that time delay is often a source of instability of feedback control systems, which has received considerable attention in the past years [88], [90], [112]. This chapter formulates a problem of quantum feedback control of linear stochastic systems with feedback-loop time delay and then proposes a numerical procedure for optimal controller designs to solve this problem.

6.1 Introduction

Although the time delay required in quantum feedback control loops as shown in Figure 6.1 is vanishingly small and thus often neglected in previous works [29], [54], [81], [83], it can have an effect on system performance in real experiments. So ignoring the time delay may lead to design flaws and incorrect analysis conclusions. Furthermore, as shown in [32] and [84], mixed quantum-classical linear stochastic systems are, in general, represented by Linear Stochastic Differential Equations (LSDEs) which have Markov property (the memoryless property of a stochastic process), where the quantum-classical nature is captured in the matrices specifying the commutation relations of the system and signal (e.g. boson field) variables. As mentioned in Chapter 1, the problem of H^∞ control of mixed quantum-classical linear stochastic systems has been discussed in [29]; the problem of LQG control of mixed systems described by LSDEs has been investigated in [74]. However, theoretical ways to study the problem of quantum measurement-based feedback control systems with feedback-loop time delay described by LSDEs have not been addressed so far. In this chapter, we shall investigate this problem in a systematic way.

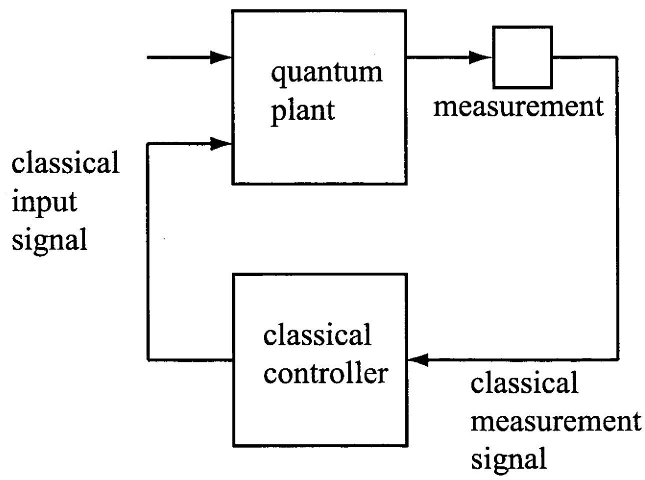

Figure 6.1: Quantum measurement-based feedback control.

The stability criteria in classical control field are often classified into two types: delay-independent criteria and delay-dependent criteria [86], [120], [121], [122], [123]. In the first case the stability property is irreverent to the size of the delay, whereas in the second one the stability property is a function of the delay size, seen as a parameter. Generally speaking, the latter ones are less conservative than the former ones, while the former ones are also useful when the effect of time delay is small. However, the developed delay-independent (or delay-dependent) stability criteria in classical control theory cannot be directly applied to quantum feedback control systems, so a quantum version of delay-independent stability criterion with an upper bound on a cost function is derived based on quantum Itō rules in this chapter. Moreover, controller designs based on the quantum version of delay-independent stability criterion suffers from severe limitations since some nonlinear and non-convex conditions, and many decision variables are involved in design procedures. We thus propose one numerical procedure for classical controller designs to overcome the limitations.

This chapter is organized as follows. Section 6.2 presents a model of closed-loop plant-controller systems with time delay. Section 6.3 develops a sufficient condition for the stability of quantum measurement-based feedback control systems subject to feedback-loop time delay, and an upper bound on a quadratic cost function is derived. Section 6.4 proposes one numerical procedure for quantum feedback controller designs. Section 6.5 presents an example to illustrate our numerical procedure. Finally, Section 6.6 gives the conclusion of this chapter.

6.2 Closed-loop plant-controller systems

This section presents our plant and controller models as well as the set-up of a closed-loop plant-controller system with time delay.

Consider a quantum plant to be controlled described by non-commutative stochastic models of the following form

$$\begin{aligned} dx_p(t) &= A_p x_p(t)dt + B_{pw}dw_p(t) + B_{pu}du(t), \\ dy_p(t) &= C_p x_p(t)dt + D_{pw}dw_p(t), \end{aligned} \tag{6.1}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_{pw} \in \mathbb{R}^{n\times n_{w_p}}$, $B_{pu} \in \mathbb{R}^{n\times n_u}$, $C_p \in \mathbb{R}^{n_{y_p}\times n}$, $D_{pw} \in \mathbb{R}^{n_{y_p}\times n_{w_p}}$ (n, n_{w_p}, n_u and n_{y_p} are even). The plant matrices should satisfy physical realizability conditions. x_p represents a vector of plant variables and w_p is a quantum noise. Suppose that $\Theta_{y_p} = \frac{F_{y_p}-F_{y_p}^T}{2i} = \mathrm{diag}_{\frac{n_{y_p}}{2}}(J)$ with $dy_p(t)dy_p(t)^T = F_{y_p}dt$. The signal $u(t)$ is a control input of the form

$$du(t) = \beta_u(t)dt + d\tilde{u}(t), \tag{6.2}$$

where $\beta_u(t)$ and $\tilde{u}$ are the signal and noise parts of $u(t)$, respectively. If the quantum output signals $y_p(t)$ are measured by homodyne detectors (HD), classical signals $y_m(t) = \mathbf{G}y_p(t)$ are produced during these processes. The matrix $\mathbf{G}$ defined as before satisfies the following condition

$$\mathbf{G}\Theta_{y_p}\mathbf{G}^T = 0 \tag{6.3}$$

with $\mathrm{rank}(\mathbf{G}) \leq \frac{n_{y_p}}{2}$, which corresponds to measurement processes.

Consider a classical controller given by

$$\begin{aligned} dx_c(t) &= A_c x_c(t)dt + B_c du_c(t), \\ y_c(t) &= C_c x_c(t), \end{aligned} \tag{6.4}$$

where $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times n_{uc}}$, $C_c \in \mathbb{R}^{n_u\times n}$. $x_c(t)$ represents a vector of classical controller variables. Assume that the quantum plant and the classical controller are initially decoupled such that $x_p(0)x_c(0)^T - (x_p(0)x_c(0)^T)^T = 0$, and w_p and w_c are independent with each other.

Since time delay often happens in a feedback loop, a closed-loop system with a classical controller as shown in Figure 6.2 can be obtained by making the identification $u(t) = y_u(t)$ and $u_c(t) = y_m(t)$, where quantum time-delay signals $y_u(t) = y_c(0) + \int_0^{t-\tau} y_c(s)ds + w_c(t)$ are produced by displacing the vectors of vacuum quantum fields w_c via modulators. Interconnecting systems (6.1) and (6.4), we have

$$dx(t) = Ax(t)dt + A_d x(t-\tau)dt + Bdw(t), \tag{6.5}$$

where $x = [x_p^T \quad x_c^T]^T, w = [w_p^T \quad w_c^T]^T$ and its dimension is $2m$, $\Theta_w = \frac{F_w - F_w^T}{2i} = \mathrm{diag}_m(J)$ with $dw(t)dw(t)^T = F_w dt$. The matrices $A = \begin{bmatrix} A_p & 0_{n\times n} \\ B_c\mathbf{G}C_p & A_c \end{bmatrix}$, $A_d = \begin{bmatrix} 0_{n\times n} & B_{pu}C_c \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}$, $B = \begin{bmatrix} B_{pw} & B_{pu} \\ B_c\mathbf{G}D_{pw} & 0_{n\times n} \end{bmatrix}$.

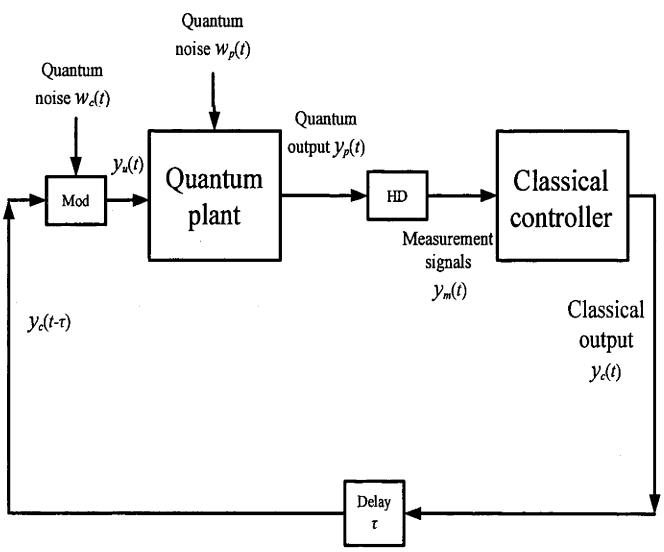

Figure 6.2: A closed-loop system with a classical controller. HD represents a homodyne detector for measurements; Mod represents a modulator.

6.3 Delay-independent stability criterion

So far we have presented a description of the plant-controller system (6.5) to be used for controller designs. In this section, we derive a delay-independent stability criterion as well as an upper bound on a cost function below for (6.5), suitably adapted to the quantum context.

The following lemma and definition will be used in the proof of Theorem 6.2.

Lemma 6.1. *If there exists a real valued function $S(t)$ of time t satisfying the differential inequality*

$$\frac{dS(t)}{dt} + cS(t) \leq \lambda, \tag{6.6}$$

where c and λ are positive real numbers, then inequality

$$S(t) \leq e^{-ct}S(0) + \frac{\lambda}{c} \tag{6.7}$$

holds, which implies that $S(t)$ is bounded for all $t \geq 0$.

Proof. Integrating the both sides of (6.6), we obtain

$$S(t+h) - S(t) + c \int_{t}^{t+h} S(s)ds \leq \lambda h, \quad (h \geq 0).$$

From the above inequality, we can infer that

$$\frac{dS(t)}{dt} \leq -cS(t) + \lambda,$$

which implies that $S(t)$ is bounded for all $t > 0$. Suppose that $\frac{dS(t)}{dt} + cS(t) = r$ $(r \leq \lambda)$. Solving the above equation, we have $S(t) = e^{-ct}S(0) + \frac{r}{c} \leq e^{-ct}S(0) + \frac{\lambda}{c}$. □

Definition 6.1. A linear system is said to be bounded stable if there exists a real valued function $S(t) = \langle V(t) \rangle$ satisfying inequality (6.6), where $V(t)$ represents an abstract internal energy for the system at time t.

Assume that system (6.5) is bounded stable in the sense of Definition 6.1. Then we can associate a infinite-horizon quadratic cost function

$$J_\infty = \lim_{t_f \to +\infty} \frac{1}{t_f} \int_{t_0}^{t_f} \left\langle [x(s)^T \quad x(s-\tau)^T] R [x(s)^T \quad x(s-\tau)^T]^T \right\rangle ds \tag{6.8}$$

with $R > 0$ as a performance measure for (6.5). Later we will find a minimum upper bound for the cost function (6.8).

The following theorem relates the stability of the system (6.5) to certain linear matrix inequalities.

Theorem 6.2. *The system* (6.5) *is bounded stable in the sense of Definition 6.1 with*

$$S(t) = \langle V(t)\rangle = \left\langle e^{-ct}x(t)^T Px(t) + e^{-ct}\int_{t-\tau}^{t} x(s)^T Qx(s)ds \right\rangle \tag{6.9}$$

if there exist real matrices $P, Q > 0$, *satisfying the following linear matrix inequality*

$$\begin{bmatrix} A^T P + PA + Q & PA_d \\ A_d^T P & -Q \end{bmatrix} \leq 0. \tag{6.10}$$

Furthermore, suppose all quantum noises are canonical (hence $F_w = I_{2m} + i\mathrm{diag}_m(J)$ *and then the cost function* (6.8) *should satisfy*

$$J_\infty \leq \mathrm{Tr}(B^T PB). \tag{6.11}$$

Proof. We construct a Lyapunov-Krasovskii functional defined as in (6.9). Applying quantum Itō rule to (6.9), we have

$$\begin{aligned}
& d\langle V(t)\rangle \\
=& e^{-ct}\left\langle dx(t)^T Px(t) + x(t)^T Pdx(t) + dx(t)^T Pdx(t) - cx(t)^T Px(t)dt\right\rangle + \\
& e^{-ct}\left\langle x(t)^T Qx(t) - x(t-\tau)^T Qx(t-\tau) - c\int_{t-\tau}^{t} x(s)^T Qx(s)ds \right\rangle dt \\
=& e^{-ct}\langle x(t)^T[A^T P + PA]x(t) + 2x(t)^T PA_d x(t-\tau) + x(t)^T Qx(t) - cx(t)^T Px(t)\rangle\, dt - \\
& e^{-ct}\left\langle c\int_{t-\tau}^{t} x(s)^T Qx(s)ds - x(t-\tau)^T Qx(t-\tau)dt + \mathrm{Tr}\left(dw(t)^T B^T PBdw(t)\right)\right\rangle dt \\
=& e^{-ct}\langle x(t)^T[A^T P + PA]x(t) + 2x(t)^T PA_d x(t-\tau) + x(t)^T Qx(t) - cx(t)^T Px(t)\rangle\, dt + \\
& e^{-ct}\left\langle \sum_{j,k}^{n} dw_j(t)[B^T PB]_{jk}dw_k(t) - x(t-\tau)^T Qx(t-\tau) + c\int_{t-\tau}^{t} x(s)^T Qx(s)ds \right\rangle dt \\
=& e^{-ct}\langle x(t)^T[A^T P + PA]x(t) + 2x(t)^T PA_d x(t-\tau) - cx(t)^T Px(t) + \mathrm{Tr}(B^T PBF_w)\rangle dt + \\
& e^{-ct}\left\langle x(t)^T Qx(t) - c\int_{t-\tau}^{t} x(s)^T Qx(s)ds - x(t-\tau)^T Qx(t-\tau)\right\rangle dt,
\end{aligned} \tag{6.12}$$

where $\langle dw(t)\rangle = 0$ and $\left\langle dw(t)dw(t)^T\right\rangle = F_w dt.$

From (6.12), we get that

$$\begin{aligned}
& \frac{d\langle V(t)\rangle}{dt} + c\langle V(t)\rangle \\
=& \left\langle e^{-ct}\left(x(t)^T[A^T P + PA]x(t) + 2x(t)^T PA_d x(t-\tau) + x(t)^T Qx(t)\right)\right\rangle + \\
& \left\langle e^{-ct}\left(\mathrm{Tr}(B^T PBF_w) - cx(t)^T Px(t) + cx^T(t)Px(t) - c\int_{t-\tau}^{t} x(s)^T Qx(s)ds\right)\right\rangle + \\
& \left\langle e^{-ct}c\int_{t-\tau}^{t} x(s)^T Qx(s)ds - x(t-\tau)^T Qx(t-\tau)\right\rangle
\end{aligned}$$

$$=\langle e^{-ct}(x(t)^T[A^TP+PA+Q]x(t)+2x(t)^TPA_dx(t-\tau)-x(t-\tau)^TQx(t-\tau)+\lambda)\rangle +$$

$$=\left\langle e^{-ct}\left([x(t)^T \;\; x(t-\tau)^T]\begin{bmatrix} A^TP+PA+Q & PA_d \\ A_d^TP & -Q \end{bmatrix}[x(t)^T \;\; x(t-\tau)^T]^T+\lambda\right)\right\rangle, \quad (6.13)$$

where $\lambda = \mathrm{Tr}(B^TPBF_w)$.

Now suppose that (6.10) holds. Then, we get

$$\left\langle e^{-ct}[x(t)^T \;\; x(t-\tau)^T]\begin{bmatrix} A^TP+PA+Q & PA_d \\ A_d^TP & -Q \end{bmatrix}[x(t)^T \;\; x(t-\tau)^T]^T + e^{-ct}\lambda\right\rangle$$
$$\leq e^{-ct}\lambda \leq \lambda, \quad (6.14)$$

where $c > 0$ and $0 < e^{-ct} \leq 1$. Therefore, the system (6.5) is bounded stable in the sense of Definition 6.1.

From (6.13), we can infer that there exists a positive symmetric $R=\begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$ defined as in (6.8) such that

$$\begin{aligned}
\frac{d}{dt}\langle V(t)\rangle &\leq e^{-ct}[x(t)^T, x(t-\tau)^T]\begin{bmatrix} A^TP+PA+Q & PA_d \\ A_d^TP & -Q \end{bmatrix}[x(t)^T \;\; x(t-\tau)^T]^T \\
&\quad +e^{-ct}\lambda \\
&\leq [x(t)^T \;\; x(t-\tau)^T]\begin{bmatrix} A^TP+PA+Q & PA_d \\ A_d^TP & -Q \end{bmatrix}[x(t)^T \;\; x(t-\tau)^T]^T+\lambda \\
&\leq \left\langle [x(t)^T \;\; x(t-\tau)^T]\begin{bmatrix} A^TP+PA+Q & PA_d \\ A_d^TP & -Q \end{bmatrix}[x(t)^T \;\; x(t-\tau)^T]^T+\lambda\right\rangle \\
&\leq \left\langle [x(t)^T \;\; x(t-\tau)^T]\begin{bmatrix} -R_1 & -R_2 \\ -R_2^T & -R_3 \end{bmatrix}[x(t)^T \;\; x(t-\tau)^T]^T+\lambda\right\rangle. \quad (6.15)
\end{aligned}$$

Integrating both sides of the inequality (6.15) from t_0 to t_f, we obtain

$$\langle V(t_f) - V(t_0)\rangle \leq \int_{t_0}^{t_f} \langle \lambda - [x(s)^T \;\; x(s-\tau)^T]R[x(s)^T \;\; x(s-\tau)^T]^T\rangle\, ds. \quad (6.16)$$

Combining the above proof with Lemma 6.1, we know that $\langle V(t)\rangle$ is bounded $\forall\, t > 0$. So, We can conclude that $\frac{\langle V(t_f)\rangle}{t_f} = 0$ and $\frac{\langle V(0)\rangle}{t_f} = 0$ as t_f goes to $+\infty$. From (6.8) and (6.16), we thus have the following relation by dividing the both sides of (6.16) by t_f and then taking the limit as $t_f \to \infty$

$$\begin{aligned}
J_\infty \leq \lambda &= \mathrm{Tr}(B^TPB(I_{2m} + i\mathrm{diag}_m(J))) \\
&= \mathrm{Tr}(B^TPB) + i\mathrm{Tr}(B^TPB\mathrm{diag}_m(J)) \\
&= \mathrm{Tr}(B^TPB),
\end{aligned}$$

where the last equality follows from the fact that $\text{diag}_m(J)$ is a antisymmetric matrix with $[\text{diag}_m(J)]_{j,k} = -1, 1, 0$ and the matrix $B^T PB$ is symmetric, which implies that $\text{Tr}(B^T PB\text{diag}_m(J)) = 0$. Therefore, the corresponding guaranteed cost controller in the form of (6.4) is an optimal guaranteed cost controller in the sense that under this controller the upper bound on the closed-loop cost function (6.8) is minimized. □

In subsequent sections, we will focus our attention to solve the problem below.

Problem 6.1: Given a cost bound parameter $\lambda > 0$, find a classical controller of the form (6.4) with controller matrices A_c, B_c, C_c such that the following conditions hold.

1. Condition (6.3) should be satisfied.

2. There exist symmetric matrices $P > 0$, $Q > 0$ satisfying (6.10).

3. An upper bound condition (6.11) should be satisfied.

Problem 6.1 can be transformed into a rank constrained LMI problem [54], [119], which can be solved based on Yalmip [116], SeDuMi [117], and LMIRank [118]. The details of rank constrained LMI problems for classical controller designs will be solved in Section 6.4.

6.4 Controller designs

In this section, we will present a numerical procedure for classical controller designs to solve **Problem 6.1**.

It should be noted that matrices A and A_d contain plant and controller matrices. So the terms PA and PA_d as well as their corresponding symmetric matrices in inequality (6.10) make plant and controller matrices mixed together, which causes difficulties in controller designs. In order to separate the former from the later, we now extend the method proposed in [75] by introducing auxiliary variables N, M, X_1, Y_1, Y_2, where $MN^T + X_1 Y_1 = I_n$, $N^T X_1 + Y_2 M^T = 0$; N and M are invertible and X_1, Y_1, Y_2 are symmetric. Let $P = \begin{bmatrix} Y_1 & N \\ N^T & Y_2 \end{bmatrix}$ and

$$P\Pi = \begin{bmatrix} I_n & Y_1 \\ 0_{n\times n} & N^T \end{bmatrix} \text{ with } \Pi = \begin{bmatrix} X_1 & I_n \\ M^T & 0_{n\times n} \end{bmatrix}.$$

Performing congruence transformations on inequality (6.10) with transformation matrix diag(Π, I_{2n}), we have

$$\begin{aligned}
&\begin{bmatrix} \Pi^T & 0_{2n\times 2n} \\ 0_{2n\times 2n} & I_{2n} \end{bmatrix} \begin{bmatrix} A^TP+PA+Q & PA_d \\ A_d^TP & 0_{2n\times 2n} \end{bmatrix} \begin{bmatrix} \Pi & 0_{2n\times 2n} \\ 0_{2n\times 2n} & I_{2n} \end{bmatrix} \\
=& \begin{bmatrix} \Pi^T(A^TP+PA)\Pi + F & \Pi^TPA_d \\ A_d^TP\Pi & -Q \end{bmatrix} < 0 \\
=& \begin{bmatrix} A_pX_1 + X_1A_p^T & A_p & 0 & 0 \\ A_p^T & Y_1A_p + A_p^TY_1 & 0 & 0 \\ 0 & 0 & 0 & -Q_2 \\ (B_{pu}C_c)^T & (Y_1B_{pu}C_c)^T & -Q_2^T & 0 \end{bmatrix} + \\
& \begin{bmatrix} F_1 & \mathbf{A_c}^T + F_2 & 0 & B_{pu}C_c \\ \mathbf{A_c} + F_2^T & \mathbf{B_c}C_p + (\mathbf{B_c}C_p)^T + Q_1 & 0 & Y_1B_{pu}C_c \\ 0 & 0 & -Q_1 & 0 \\ 0 & 0 & 0 & -Q_3 \end{bmatrix} < 0, \qquad (6.17)
\end{aligned}$$

where $\mathbf{A}_c = Y_1A_pX_1 + NB_c\mathbf{G}C_pX_1 + NA_cM^T$, $\mathbf{B}_c = NB_c\mathbf{G}$, $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}$, $F = \Pi^TQ\Pi = \begin{bmatrix} F_1 & F_2 \\ F_2^T & Q_1 \end{bmatrix}$ with $F_1 = X_1Q_1X_1 + MQ_2^TX_1 + X_1Q_2M^T + MQ_3M^T$, $F_2 = X_1Q_1 + MQ_2^T$. The upper bound condition (6.11) can be rewritten as

$$\begin{aligned}
&\mathrm{Tr}\left(B_{pw}B_{pw}^TY_1 + B_{pu}B_{pu}^TY_1 + B_c\mathbf{G}D_{pw}(B_c\mathbf{G}D_{pw})^TY_2\right) + \\
&\mathrm{Tr}\left(B_{pw}(NB_c\mathbf{G}D_{pw})^T + B_c\mathbf{G}D_{pw}B_{pw}^TN\right) \le \gamma. \qquad (6.18)
\end{aligned}$$

For simplicity, we choose $M = I_n$ and hence $N = I_n - Y_1X_1$, $X_1N + Y_2 = 0$ and $\mathbf{A}_c = Y_1A_pX_1 + NB_c\mathbf{G}C_pX_1 + NA_c$. Introducing appropriate matrix lifting variables and the associated equality constraints can linearize nonlinear condition (6.3) and nonlinear entries in inequality (6.17). Introducing appropriate matrix lifting variables and the associated equality constraints can linearize nonlinear condition (6.3) and nonlinear entries in (6.17)-(6.18).

Let $Z_{x_1} = B_c$, $Z_{x_2} = C_c^T$, $Z_{x_3} = X_1$, $Z_{x_4} = Y_1$, $Z_{x_5} = Y_2$, $Z_{x_6} = N$, $Z_{x_7} = N^T$, $Z_{x_8} = \mathbf{G}$, $Z_{x_9} = \mathbf{G}^T$, $Z_{x_{10}} = Q_1$, $Z_{x_{11}} = Q_2^T$. Define a symmetric matrix Z of dimension $25n \times 25n$ as $Z = \mathbf{V}\mathbf{V}^T$, where $\mathbf{V} = [I_n \ Z_{x_1}^T \ \cdots \ Z_{x_{11}}^T \ Z_{v_1}^T \ \cdots \ Z_{v_{13}}^T]^T$, $Z_{v_1} = Y_1X_1$, $Z_{v_2} = Y_1B_{pu}$, $Z_{v_3} = Y_1B_{pu}C_c$, $Z_{v_4} = \mathbf{G}\Theta_{y_p}$, $Z_{v_5} = \mathbf{G}\Theta_{y_p}\mathbf{G}^T$, $Z_{v_6} = \mathbf{G}^TB_c^T$, $Z_{v_7} = \mathbf{G}^TB_c^TY_2$, $Z_{v_8} = Z_{v_6}Z_{v_7}^T$, $Z_{v_9} = NB_c\mathbf{G}$, $Z_{v_{10}} = X_1Q_2$, $Z_{v_{11}} = X_1N$, $Z_{v_{12}} = X_1Q_1$, $Z_{v_{13}} = X_1Q_1X_1$. The symmetric matrix Z should satisfy the following conditions:

$$
\begin{array}{ll}
Z \geq 0; & Z_{0,0} - I_{n\times n} = 0; \\
Z_{x_3} - Z_{x_3}^T = 0; & Z_{x_4} - Z_{x_4}^T = 0; \\
Z_{x_5} - Z_{x_5}^T = 0; & Z_{x6} - Z_{x7}^T = 0; \\
Z_{x10} - Z_{x10}^T = 0; & Z_{v_1} - Z_{x_4} Z_{x_5}^T = 0 \\
Z_{v_1} - I + Z_{x6} = 0; & Z_{v_2} - Z_{x_5} B_{pu} = 0; \\
Z_{v_3} - Z_{v_2} Z_{x_3}^T = 0; & Z_{v_4} - Z_{x_9} \Theta_{y_p} = 0; \\
Z_{v_5} - Z_{v_4} Z_{x_9}^T = 0; & Z_{v_5} = 0; \\
Z_{v_6} - Z_{x_9} Z_{x_1}^T = 0; & Z_{v_7} - Z_{v_6} Z_{x_5}^T = 0; \\
Z_{v_8} - Z_{v_7} Z_{v_6}^T = 0; & Z_{v_9} - Z_{x6} Z_{v6}^T = 0; \\
Z_{v_{10}} - Z_{x_3} Z_{x_{11}}^T = 0; & Z_{v_{11}} - Z_{x_3} Z_{x_7}^T = 0; \\
Z_{v_{12}} - Z_{x_3} Z_{x10}^T = 0; & Z_{v_{13}} - Z_{v_{12}} Z_{x_3}^T = 0; \\
Z_{v_{13}} + Z_{x_6} = 0; & Z_{v_{11}} + Z_{x_5} = 0; \\
Z_{x_8} - Z_{x_9}^T = 0; &
\end{array}
\tag{6.19}
$$

and a rank constraint

$$
\operatorname{rank}(Z) \leq n. \tag{6.20}
$$

Stability conditions are given as

$$
\begin{bmatrix}
A_p Z_{x_4} + Z_{x_4} A_p^T + Z_{v_{13}} & A_p + \mathbf{A}_c^T & 0_{n\times n} & B_{pu} Z_{x_2}^T \\
A_p^T + \mathbf{A}_c & Z_{x_4} A_p + A_p^T Z_{x_4} + Z_{v_{10}} & 0 & Z_{v_3} \\
0 & 0 & 0 & -Z_{x_{11}}^T \\
(B_{pu} Z_{x_2}^T)^T & Z_{v_3}^T & -Z_{x_{11}} & 0
\end{bmatrix} +
$$
$$
\begin{bmatrix}
Z_{v_{10}} + Z_{v_{10}}^T + Q_3 & Z_{v_{12}} + Z_{x_{11}} & 0 & 0 \\
(Z_{v_{12}} + Z_{x_{11}})^T & Z_{v_9} C_p + (Z_{v_9} C_p)^T & 0 & 0 \\
0 & 0 & -Z_{x10} & 0 \\
0 & 0 & 0 & -Q_3
\end{bmatrix} < 0, \tag{6.21}
$$

$$
\Pi^T P \Pi = \begin{bmatrix} Z_{x_4} & I_n \\ I_n & Z_{x_5} \end{bmatrix} > 0, \tag{6.22}
$$

$$
Q = \begin{bmatrix} Z_{x10} & Z_{x_{11}} \\ Z_{x_{11}}^T & Q_3 \end{bmatrix} > 0, \tag{6.23}
$$

where Q_3 is a symmetric matrix.

An upper bound condition is given as

$$\mathrm{Tr}\left(B_{pw}B_{pw}^T Z_{x_5} + B_{pu}B_{pu}^T Z_{x_5} + Z_{v_8} + 2B_{pw}D_{pw}^T Z_{v_9}^T\right) \leq \gamma. \tag{6.24}$$

If we can employ a semi-definite programming to solve the feasibility problem with constraints (6.19)-(6.24) in which decision variables are Z, $\mathbf{A}_c$ and Q_3, we have

$$\begin{aligned} A_c &= Z_{x_7}^{-1}(\mathbf{A}_c - Z_{x_4}A_pZ_{x_3} - Z_{v_9}C_pZ_{x_3}), \\ B_c &= Z_{x_1}, \\ C_c &= Z_{x_2}^T. \end{aligned}$$

Remark 6.3. It should be noted that the above procedure with some γ may give ill-conditioned solutions. Thus γ has to be chosen carefully to generate a meaningful controller. Furthermore, the LMIRank solver can only solve feasibility problems and uses a local approach to address the non-convex rank constraints, so it is essential to find proper starting points for our numerical procedures [119]. Otherwise improper ones may also return ill-conditioned solutions. Sometimes imposing some additional conditions on the decision variables which lead to ill-conditioned solutions can help us quickly find proper starting points. When reasonable results are obtained by applying our procedure with new constraints, we remove the additional conditions, set the results as our starting points $\mathbf{V}_0$ and then use the procedure again with the original conditions to get final results. This fact will be illustrated by an example given in Section 6.5.

6.5 An example

In this section, we present an example to test our numerical procedure developed in Section 6.5.

Consider a quantum plant to be controlled

$$\begin{aligned} dx_p &= \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} x_p(t)dt + \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} dw_p + \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} du(t), \\ dy_p &= \begin{bmatrix} -3 & 1 \\ 4 & -2 \end{bmatrix} x_p(t)dt + dw_p, \end{aligned} \tag{6.25}$$

where the quantum plant matrices satisfy physical realizability conditions (3.30). Applying our numerical procedure to the quantum plant (6.25) with $\gamma = 2$, we get the following solutions:

$$A_c = \begin{bmatrix} -12.7262 & -1.8521 \\ 3.7122 & -0.1574 \end{bmatrix}, \quad B_c = \begin{bmatrix} -5.7952 & 3.8640 \\ -1.3860 & 0.9239 \end{bmatrix},$$
$$C_c = \begin{bmatrix} -0.0691 & -0.0210 \\ -0.1115 & -0.0469 \end{bmatrix}, \quad \mathbf{G} = 10^{-14}\begin{bmatrix} 0.0708 & -0.5597 \\ 0.1626 & -0.0860 \end{bmatrix}.$$

Clearly, the matrix $\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix}$ is ill-conditioned (not physically meaningful). So an extra condition should be added to the original conditions such that $Z_{x_9}(1,1) = \mathbf{G}_{11} > 1$. Then we employ a semi-definite programming to solve the feasibility problem with constraints (6.19)-(6.24) as well as the additional condition $\mathbf{G}_{11} > 1$ and obtain reasonable solutions, which are set as starting points V_0. Applying our procedure again with constraints (6.19)-(6.24) and V_0, we can get the following solutions:

$$A_c = \begin{bmatrix} -545.8124 & -552.1781 \\ 130.8006 & 113.9618 \end{bmatrix}, \quad B_c = \begin{bmatrix} -1.2601 & 3.0471 \\ -5.3781 & 0.9453 \end{bmatrix},$$
$$C_c = \begin{bmatrix} -1.2055 & -0.0643 \\ -2.4110 & -0.1286 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -0.7362 & 0.4323 \\ -2.8301 & 1.6618 \end{bmatrix},$$

$$\Pi^T P \Pi = \begin{bmatrix} 13.9188 & 11.7573 & 1.0000 & 0 \\ 11.7573 & 52.8529 & 0 & 1.0000 \\ 1.0000 & 0 & 0.0886 & -0.0197 \\ 0 & 1.0000 & -0.0197 & 0.0235 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.3315 & -0.0927 & -0.5801 & -1.5521 \\ -0.0927 & 0.0261 & 0.1623 & 0.4344 \\ -0.5801 & 0.1623 & 1.0155 & 2.7169 \\ -1.5521 & 0.4344 & 2.7169 & 7.2697 \end{bmatrix}.$$

Now we check that if the resulting solutions satisfy the constraints listed in **Problem 6.1**. The eigenvalues of (6.21) are -16.5055, -8.6424, -0.3882, -0.0001, -0.0001, -0.0001, -0.0001, -0.0001 and thus condition (6.21) is negative. It is easily checked that P and Q are positive symmetric matrices, and $\mathbf{G}\mathrm{diag}_{\frac{n_{w_{k_1}}}{2}} \mathbf{G}^T = 10^{-12}\begin{bmatrix} 0 & 0.7521 \\ -0.7521 & 0 \end{bmatrix}$. Furthermore, the upper bound is $1.9760 < 2$. Therefore, the resulting solutions are reasonable for the classical controller designs.

6.6 Concluding remarks

In this chapter, we investigate a quantum measurement-based feedback control system subject to feedback-loop time delay. A delay-independent stability criterion and an upper bound on a cost function are derived for such quantum feedback control systems with time delay. One numerical procedure is proposed for classical controller designs. An example is presented to test our procedure. These results are expected to give useful guidelines for the future quantum feedback control experiments with time delay.

Chapter 7

H^{∞} Control of Quantum Feedback Control Systems with Feedback-loop Time Delay

In Chapter 6, we have proposed a numerical procedure to design a linear feedback controller that not only makes a quantum feedback control system stable but also guarantees an upper bound for the performance functional. In modern control theory, H^{∞} control technique can also solve the problem of linear classical time-delay systems. This chapter deals with the H^{∞} controller synthesis problem for a quantum measurement-based feedback control system with time delay in a feedback control loop.

7.1 Introduction

The last decade has witnessed the emergence of a fully developed theory for robust control in the form of H^{∞} optimal control [124], [125], [126], [127]. The backbone of H^{∞} synthesis is the small gain theorem [128], [129], [130]. The most promising feature of the H^{∞} controller is the guaranteed stability margin that it provides in the face of a norm bounded perturbation [124], [125]. The problem of H^{∞} optimal control of classical time-delay systems is studied in [131], [132]. As mentioned in Chapter 1, the H^{∞} technique generalized from modern control theory has already been successfully applied to the area of quantum feedback control systems without time delay in recent years. An H^{∞} synthesis problem for a class of linear quantum stochastic systems has been formulated and solved in [29]. The paper [56] presents an experimental realization of a coherent quantum feedback control system using H^∞ control theory. In [50], a coherent H^∞ control problem has been considered for a class of linear quantum systems described by complex quantum stochastic differential equations in terms of annihilation operators only. However, the problem of the H^∞ control for quantum feedback control systems subject to feedback-loop time delay has not been solved so far, where the plant is quantum and the designed controller is classical.

Therefore, this chapter is concerned with applying the H^∞ control technique to quantum measurement-based feedback control systems with feedback-loop time delay. The main contributions of this chapter are as follows. Firstly, we develop a linear model for a quantum measurement-based feedback control system with time delay. This model is a mixed quantum-classical system, which contains time-delay system variables. Secondly, the properties of our model such as stability, dissipation and gain are characterized in algebraic terms, which then lead to a new quantum version of the Bounded Real Lemma (BRL). Finally, we present a numerical procedure for H^∞ controller designs based on the new version of the Strict BRL developed in this chapter.

This chapter is organized as follows. Section 7.2 presents the set-up of closed-loop plant-controller systems with time delay. Section 7.3 investigates basic performance characteristics such as dissipativity, gain, stability, etc for the model developed in Section 7.2. Section 7.4 proposes a numerical procedure to build a quantum feedback controller, which is illustrated with an example. Finally, Section 7.5 gives the conclusion of this chapter.

7.2 Closed-loop systems

In this section, we develop a model for a quantum measurement-based feedback control system with time delay, which will be used in the following sections.

Consider a quantum plant described by non-commutative stochastic models of the following form

$$\begin{aligned} dx_p(t) &= A_p x_p(t)dt + B_w dw(t) + B_{v_p} dv_p(t) + B_u du(t), \\ dz(t) &= C_z x_p(t)dt + D_u du(t) + D_w dw(t), \\ dy_p(t) &= C_p x_p(t)dt + D_{v_p} dv_p(t), \end{aligned} \tag{7.1}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_w \in \mathbb{R}^{n\times n_w}$, $B_{v_p} \in \mathbb{R}^{n\times n_{v_p}}$, $B_u \in \mathbb{R}^{n\times n_u}$, $C_z \in \mathbb{R}^{n_z\times n}$, $D_u \in \mathbb{R}^{n_z\times n_u}$, $D_w \in \mathbb{R}^{n_{y_p}\times n_w}$, $C_p \in \mathbb{R}^{n_{y_p}\times n}$, $D_{v_p} \in \mathbb{R}^{n_{y_p}\times n_{v_p}}$ (n, n_w, n_u, n_{y_p} and n_{v_p} are even). x_p represents a vector of plant variables. $w(t)$ represents a disturbance signal of the form

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t), \tag{7.2}$$

where $\beta_w(t)$ and $\tilde{w}$ are the signal and noise parts of $w(t)$, respectively; $\Theta_{\tilde{w}} = \frac{F_{\tilde{w}} - F_{\tilde{w}}^T}{2i} = J_{\frac{n_w}{2}}$ with $d\tilde{w}(t)d\tilde{w}(t)^T = F_{\tilde{w}}dt$. v_p represents additional quantum noises and $\Theta_v = \frac{F_v - F_v^T}{2i} = J_{\frac{n_{v_p}}{2}}$ with $dv_p(t)dv_p(t)^T = F_v dt$. The signal $u(t)$ represents a control input of the form

$$du(t) = \beta_u(t)dt + d\tilde{u}(t), \tag{7.3}$$

where $\beta_u(t)$ and $\tilde{u}$ are the signal and noise parts of $u(t)$, respectively. $z(t)$ represents a performance output. Classical signals $y_m(t) = \mathbf{G}y_p(t)$ are produced by the quantum output signals $y_p(t)$ being measured via homodyne detectors (HD) where the matrix G has been defined in Chapter 6, which satisfies the following condition

$$\mathbf{G}\Theta_{y_p}\mathbf{G}^T = 0 \tag{7.4}$$

with $\text{rank}(\mathbf{G}) \leq \frac{n_{y_p}}{2}$.

Consider a classical controller given by

$$\begin{aligned} dx_c(t) &= A_c x_c(t)dt + B_c du_c(t), \\ y_c(t) &= C_c x_c(t), \end{aligned} \tag{7.5}$$

where $A_c \in \mathbb{R}^{n \times n}$, $B_c \in \mathbb{R}^{n \times n_{u_c}}$, $C_c \in \mathbb{R}^{n_u \times n}$. $x_c(t)$ represents a vector of classical controller variables. Assume that the quantum plant and the classical controller are initially decoupled such that $x_p(0)x_c(0)^T - (x_p(0)x_c(0)^T)^T = 0$.

The closed-loop system with a classical controller as shown in Figure 7.1 is obtained by making the identification $u(t) = y_u(t)$ and $u_c(t) = y_m(t)$, where quantum time-delay signals $y_u(t) = y_c(0) + \int_0^{t-\tau} y_c(s)ds + w_c(t)$ are produced by displacing the vectors of vacuum quantum fields w_c via modulators. Interconnecting systems (7.1) and (7.5), we have

$$\begin{aligned} dx(t) &= Ax(t)dt + A_d x(t-\tau)dt + Bdw(t) + B_v dv(t), \\ dz(t) &= Cx(t)dt + C_d x(t-\tau)dt + Ddw(t) + D_v dv(t), \end{aligned} \tag{7.6}$$

where $x = [x_p^T \quad x_c^T]^T$ represents a vector of closed-loop system variables; w is defined as in (7.2); $v(t) = [v_p(t)^T \quad w_c(t)^T]^T$ represents additional noise sources. Assume that w, v_p, w_c are independent with each other. We define $\beta_z(t) = Cx(t) +$

$C_d x(t-\tau) + D\beta_w(t)$. The closed-loop system matrices $A = \begin{bmatrix} A_p & 0_{n\times n} \\ B_c\mathbf{G}C_p & A_c \end{bmatrix}$, $A_d = \begin{bmatrix} 0 & B_uC_c \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_w \\ 0 \end{bmatrix}$, $B_v = \begin{bmatrix} B_{v_p} & B_u \\ B_c\mathbf{G}D_{v_p} & 0 \end{bmatrix}$, $C = [C_z \quad 0]$, $C_d = [0 \quad D_u\mathbf{G}C_c]$, $D = D_w$, $D_v = [0 \quad D_u\mathbf{G}]$.

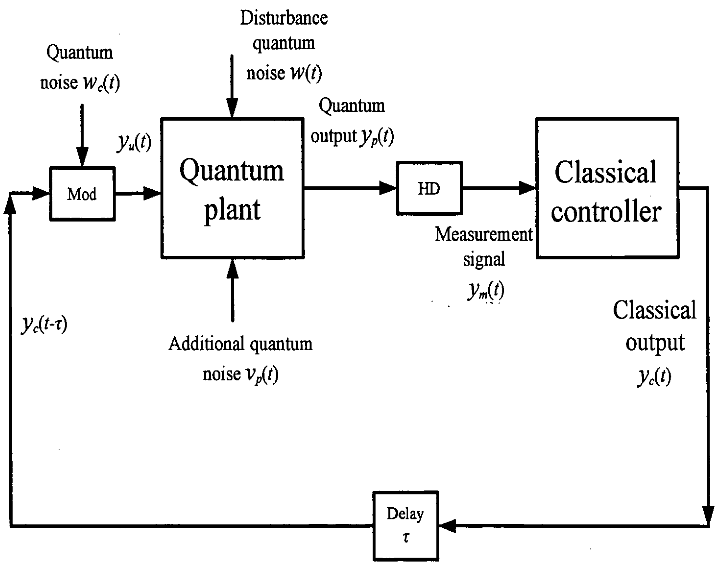

Figure 7.1: The closed-loop system with a classical controller. HD represents a homodyne detector for measurements; Mod represents a modulator.

7.3 Performance characteristics

The purpose of this section is to discuss basic performance characteristics such as dissipativity, gain, stability, etc., for a quantum measurement-based feedback control system with feedback-loop time delay of the form (7.6) developed in Section 7.2.

7.3.1 Dissipativity of time-delay closed-loop systems

Since a supply rate is a function of input and output and the output $z(t)$ of the system (7.6) contains time-delay system variables, the delay terms should be considered as an argument of supply rate. In order to define dissipation for the system (7.6), we use the quadratic supply rate below

$$r(\beta_z(t), \beta_w(t)) = [x(t)^T \ x(t-\tau)^T \ \beta_w(t)^T] W \begin{bmatrix} x(t) \\ x(t-\tau) \\ \beta_w(t) \end{bmatrix}, \tag{7.7}$$

where W is a symmetric matrix of the form

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix}. \tag{7.8}$$

For the system (7.6), we define a candidate storage function as

$$V(t) = x(t)^T P x(t), \tag{7.9}$$

where symmetric matrices P is positive definite.

Definition 7.1. (Dissipation) The quantum measurement-based feedback control system with feedback-loop time delay is said to be dissipative with respect to the supply rate (7.7) if there exists a nonnegative-definite functional $V(t)$ defined as in (7.9) such that

$$\langle V(t)) \rangle - \langle V(0) \rangle + \int_0^t \langle r(\beta_z(t), \beta_w(t)) \rangle ds \leq \lambda t, \tag{7.10}$$

for all $t > 0$. Also, the system (7.6) is said to be strictly dissipative if there exists a matrix $\hat{W} = \begin{bmatrix} \hat{W}_{11} & 0 & 0 \\ 0 & \hat{W}_{22} & 0 \\ 0 & 0 & \hat{W}_{33} \end{bmatrix}$ with $\hat{W}_{11} > 0$, $\hat{W}_{22}$, $\hat{W}_{33} \geq 0$, such that inequality (7.10) holds with the matrix W replaced by matrix $W + \hat{W}$.

Theorem 7.1. *The quantum measurement-based feedback control system with feedback-loop time delay is dissipative with respect to the supply rate* (7.7) *if and only if there exists a real positive definite matrix P such that the following matrix inequality is satisfied*

$$\begin{bmatrix} A^T P + PA + W_{11} & PA_d + W_{12} & PB + W_{13} \\ A_d^T P + W_{12}^T & W_{22} & W_{23} \\ B^T P + W_{13}^T & W_{23}^T & W_{33} \end{bmatrix} \leq 0. \tag{7.11}$$

Furthermore, the system (7.6) *is strictly dissipative with respect to the supply rate* (7.7) *if and only if there exists a real positive definite symmetric matrix P such* *that the following matrix inequality is satisfied*

$$\begin{bmatrix} A^TP+PA+W_{11}+\hat{W}_{11} & PA_d+W_{12} & PB+W_{13} \\ A_d^TP+W_{12}^T & W_{22}+\hat{W}_{22} & W_{23} \\ B^TP+W_{13}^T & W_{23}^T & W_{33}+\hat{W}_{33} \end{bmatrix} \le 0. \tag{7.12}$$

Proof. Applying quantum Itō rule to (7.9), we have

$$\begin{aligned} d\langle V(t)\rangle &= \langle dx(t)^TPx(t)+x(t)^TPdx(t)+dx(t)^TPdx(t)\rangle \\ &= \langle x(t)^T[A^TP+PA]x(t)+x(t)^TPA_dx(t-\tau)+x(t)^TPB\beta_w(t)\rangle\, dt + \\ &\quad \langle x(t-\tau)^TA_d^TPx(t)+\beta_w(t)B^TPx(t)+\lambda\rangle dt, \end{aligned} \tag{7.13}$$

where $\lambda = \mathrm{Tr}(B^TPBF_{\tilde{w}})+\mathrm{Tr}(B_v^TPB_vF_v) > 0$.

The rest proof of this theorem is similar to that in ([29, Theorem 4.2]), so it is omitted. □

7.3.2 Stability and Bounded Real Lemma

In this subsection, we study stability and the Bounded Real Lemma for quantum measurement-based feedback control systems with feedback-loop time delay of the form (7.6).

Using Definition 6.1 presented in Chapter 6, we obtain the following theorem.

Theorem 7.2. *The quantum measurement-based feedback control system with feedback-loop time delay is bounded stable in the sense of Definition 6.1 with* $S(t) = \langle V(t)\rangle = \langle x(t)^TPx(t)\rangle$ *if the system* (7.6) *is dissipative with respect to the supply rate* (7.7) *with matrices* $W = \mathrm{diag}(X,0,0)$ *and* $X \ge cP > 0$ $(c > 0)$.

Proof. From (7.13), we get

$$\begin{aligned} &\frac{d\langle V(t)\rangle}{dt} + c\langle V(t)\rangle \\ =&\langle x(t)^T[A^TP+PA]x(t)+2x(t)^TPA_dx(t-\tau)+2x(t)^TPB\beta_w(t)\rangle + \\ &\langle \lambda + cx(t)^TPx(t)\rangle \\ \le& \left\langle [x(t)^T\ x(t-\tau)^T\ \beta_w(t)^T] \begin{bmatrix} A^TP+PA+X & PA_d & PB \\ A_d^TP & 0 & 0 \\ B^TP & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ \beta_w(t) \end{bmatrix} \right\rangle + \\ &\lambda. \end{aligned} \tag{7.14}$$

If (7.11) with the matrix $W = \mathrm{diag}(X,0,0)$ holds, then we can infer from (7.14) that the inequality $\frac{d\langle V(t)\rangle}{dt} + c\langle V(t)\rangle \le \lambda$ holds, which implies that the system (7.6) is bounded stable in the sense of Definition 6.1. This completes the proof. □

Definition 7.2. The quantum measurement-based feedback control system with feedback-loop time delay is said to be bounded real with disturbance attenuation $g > 0$ if the system (7.6) is dissipative in the sense of Definition 7.1 with the supply rate

$$\begin{aligned} & r(\beta_z(t), \beta_w(t)) \\ &= \beta_z(t)^T \beta_z(t) - g^2 \beta_w(t)^T \beta_w(t) \\ &= [x(t)^T \; x(t-\tau)^T \; \beta_w(t)^T] \begin{bmatrix} C^T C & C^T C_d & C^T D \\ C_d^T C & C_d^T C_d & C_d^T D \\ D^T C & D^T C_d & D^T D - g^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ \beta_w(t) \end{bmatrix}. \quad (7.15) \end{aligned}$$

Furthermore, the system (7.6) is said to be strictly bounded real with disturbance attenuation $g > 0$ if the system (7.6) is strictly dissipative in the sense of Definition 7.1 with the supply rate (7.15).

Now combining Theorem 7.1 with Definition 7.2, we obtain the following Theorem. (e.g., see also [29], [130] for the similar proof.)

Theorem 7.3. *(Bounded Real Lemma) The quantum measurement-based feedback control system with feedback-loop time delay is bounded real with finite L^2 gain from β_w to β_z less than $g > 0$ with respect to supply the rate* (7.15) *if and only if there exists a non-negative symmetric matrix P such that*

$$\begin{bmatrix} A^T P + PA + C^T C & PA_d + C^T C_d & PB + C^T D \\ A_d^T P + C_d^T C & C_d^T C_d & C_d^T D \\ B^T P + D^T C & D^T C_d & D^T D - g^2 I \end{bmatrix} \leq 0 \quad (7.16)$$

or, equivalently

$$\begin{bmatrix} A^T P + PA & PA_d & PB & C^T \\ A_d^T P & 0 & 0 & C_d^T \\ B^T P & 0 & -gI & D^T \\ C & C_d & D & -gI \end{bmatrix} \leq 0. \quad (7.17)$$

Furthermore, the system (7.6) *is strictly bounded real with finite L^2 gain from β_w to β_z less than $g > 0$ with respect to the supply rate* (7.15) *if and only if there exists non-negative symmetric matrices P, $\hat{W} = \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $Q \geq cP > 0$ $(c > 0)$, such that*

$$\begin{bmatrix} A^T P + PA + C^T C + Q & PA_d + C^T C_d & PB + C^T D \\ A_d^T P + C_d^T C & C_d^T C_d & C_d^T D \\ B^T P + D^T C & D^T C_d & D^T D - g^2 I \end{bmatrix} \leq 0 \quad (7.18)$$

or, equivalently

$$\begin{bmatrix} A^TP+PA+Q & PA_d & PB & C^T \\ A_d^TP & 0 & 0 & C_d^T \\ B^TP & 0 & -gI & D^T \\ C & C_d & D & -gI \end{bmatrix} \leq 0. \tag{7.19}$$

We are now in a position to present our main result concerning H^{∞} controller synthesis for the model (7.6).

Theorem 7.4. *The quantum measurement-based feedback control system with feedback-loop time delay is bounded stable in the sense of Definition 6.1 with* $S(t) = \langle V(t)\rangle = \langle x(t)^TPx(t)\rangle$, *and also satisfies the following relation*

$$\int_0^t \langle \beta_z(s)^T\beta_z(s) + x(s)^TQx(s)\rangle\, ds \leq g^2 \int_0^t \langle \beta_w(s)^T\beta_w(s)\rangle ds + \mu_1 + \mu_2 t \tag{7.20}$$

with t, μ_1, μ_2, $c > 0$ *and* $Q > cP$, *if the system* (7.6) *is strictly bounded real with disturbance attenuation* $g > 0$.

Proof. Consider the following index:

$$\begin{aligned} &J_{zw}(t) \\ =&\beta_z(t)^T\beta_z(t) - g^2\beta_w(t)^T\beta_w(t) + x(t)^TQx(t) \\ =&[x(t)^T\ x(t-\tau)^T\ \beta_w(t)^T]\begin{bmatrix} C^TC+Q & C^TC_d & C^TD \\ C_d^TC & C_d^TC_d & C_d^TD \\ D^TC & D^TC_d & D^TD-g^2I \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau) \\ \beta_w(t) \end{bmatrix}. \end{aligned} \tag{7.21}$$

From (7.13) and (7.21), we have

$$\begin{aligned} &\int_0^t \langle J_{zw}(s)\rangle\, ds \\ \leq &\int_0^t \left(\frac{d\langle V(s)\rangle}{ds} + \langle J_{zw}(s\rangle\right) ds + \langle V(0)\rangle \\ = &\int_0^t \left\langle [x(s)^T\ x(s-\tau)^T\ \beta_w(s)^T]\Xi \begin{bmatrix} x(s) \\ x(s-\tau) \\ \beta_w(s) \end{bmatrix} \right\rangle ds + \lambda t + \langle V(0)\rangle, \end{aligned} \tag{7.22}$$

where $\Xi = \begin{bmatrix} A^TP+PA+C^TC+Q & PA_d+C^TC_d & PB+C^TD \\ A_d^TP+C_d^TC & C_d^TC_d & C_d^TD \\ B^TP+D^TC & D^TC_d & D^TD-g^2I \end{bmatrix}$.

By Theorem 7.4, we know that the system (7.6) is strictly bounded real with disturbance attenuation $g > 0$ if and only if $\Xi \le 0$. From (7.21) and (7.22), we can get (7.20) with $\mu_1 = \langle V(0)\rangle$ and $\mu_2 = \lambda$.

From $\Xi \le 0$, we can get the following relation by taking the derivative of both sides of (7.22) with respect of t.

$$\begin{aligned}&\langle J_{zw}(t)\rangle + \frac{d\langle V(t)\rangle}{dt}\\ &= \left\langle \beta_z(t)^T\beta_z(t) - g^2\beta_w(t)^T\beta_w(t) + x(t)^TQx(t)\right\rangle + \frac{d\langle V(t)\rangle}{dt} \le \lambda,\end{aligned}$$

for all $t > 0$, which implies $\frac{d\langle V(t)\rangle}{dt} + c\langle V(t)\rangle \le \frac{d\langle V(t)\rangle}{dt} + \left\langle x(t)^TQx(t)\right\rangle < \lambda$ when $\beta_w = 0$ and $Q > cP$. So, the system (7.6) is bounded stable in the sense of Definition 6.1. This completes the proof. □

7.4 H^∞ controller synthesis

In this section, we consider the problem of H^∞ controller design for quantum measurement-based feedback control systems with time delay. The problem of H^∞ controller design is first formulated in Subsection 7.4.1 and then in Subsection 7.4.2 we propose a numerical procedure to solve the problem using LMIs technique. An example is given to illustrate our procedure in Subsection 7.4.3.

7.4.1 H^∞ controller synthesis objective

Now we formulate our H^∞ controller synthesis objective as follows:

Problem 7.1: (H^∞ controller synthesis) Given a disturbance attenuation parameter $g > 0$ and a parameter $c > 0$, the aim of H^∞ controller design for The quantum measurement-based feedback control system with feedback-loop time delay is to find a classical controller of the form (7.5), such that the following conditions hold:

1. The nonlinear condition (7.4) should be satisfied.

2. The closed-loop plant-controller system (7.6) is bounded stable in the sense of Definition 6.1.

3. The closed-loop plant-controller system (7.6) satisfies the relation (7.20).

7.4.2 H^∞ controller designs

According to Theorem 7.4, **Problem 7.1** can be solved if (7.19) holds. It can be seen that plant and controller matrices are mixed together in terms PA, PA_d, PB and PB_d as well as their corresponding symmetric matrices in inequality (7.19), which complexes controller designs. In order to separate the former from the later, we now extend the method proposed in [75] by introducing auxiliary variables $\mathbf{N}$, $\mathbf{M}$, $\mathbf{X}$, $\mathbf{Y}$, where $\mathbf{M}\mathbf{N}^T + \mathbf{X}\mathbf{Y} = I_n$; $\mathbf{N}$ and $\mathbf{M}$ are invertible and $\mathbf{X}$, $\mathbf{Y}$ are symmetric. Let $P = \begin{bmatrix} \mathbf{Y} & \mathbf{N} \\ \mathbf{N}^T & * \end{bmatrix}$ and $P\Pi = \begin{bmatrix} I_n & \mathbf{Y} \\ 0 & \mathbf{N}^T \end{bmatrix}$ with $\Pi = \begin{bmatrix} \mathbf{X} & I_n \\ \mathbf{M}^T & 0 \end{bmatrix}$.

Performing congruence transformations on inequality (7.19) with transformation matrix $\Gamma = \mathrm{diag}(\Pi, I_{2n}, I_{2n}, I_{2n})$, we have

$$\Gamma^T \begin{bmatrix} A^TP + PA + cP & PA_d & PB & C^T \\ A_d^TP & 0 & 0 & C_d^T \\ B^TP & 0 & -gI & D^T \\ C & C_d & D & -gI \end{bmatrix} \Gamma$$

$$= \begin{bmatrix} \Pi^T(A^TP + PA + cP)\Pi & \Pi^TPA_d & \Pi^TPB & \Pi^TC^T \\ A_d^TP\Pi & 0 & 0 & C_d^T \\ B^TP\Pi & 0 & -gI & D^T \\ C\Pi & C_d & D & -gI \end{bmatrix}$$

$$= \begin{bmatrix} A_p\mathbf{X}+\mathbf{X}A_p^T & A_p & 0 & B_uC_c & B_w & \mathbf{X}C_z^T \\ A_p^T & \mathbf{Y}A_p+\mathbf{B_c}C_p & 0 & \mathbf{Y}B_uC_c & \mathbf{Y}B_w & C_z^T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_c^TB_u^T & (\mathbf{Y}B_uC_c)^T & 0 & 0 & 0 & C_c^TD_u^T \\ B_w^T & B_w^T\mathbf{Y} & 0 & 0 & 0 & D^T \\ C_z\mathbf{X} & C_z & 0 & D_uC_k & D & 0 \end{bmatrix} +$$

$$\begin{bmatrix} R_1 & \mathbf{A_c}^T + R_2 & 0 & 0 & 0 & 0 \\ \mathbf{A_c}+R_2^T & A_p^T\mathbf{Y}+C_p^T\mathbf{B_c}^T+R_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -gI & 0 \\ 0 & 0 & 0 & 0 & 0 & -gI \end{bmatrix} \leq 0, \tag{7.23}$$

where $\mathbf{A_c} = \mathbf{N}A_c\mathbf{M}^T + \mathbf{Y}A_p\mathbf{X}\mathbf{M}^T + NB_c\mathbf{G}C_p\mathbf{X}$, $\mathbf{B_c} = \mathbf{N}B_c\mathbf{G}$, $R = \Pi^TQ\Pi = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$.

Nonlinear conditions (7.4) and nonlinear entries in inequality (7.23) can be linearized by introducing appropriate matrix lifting variables and associated equality constraints. For simplicity, we choose $\mathbf{M} = I_n$, so $\mathbf{A_c} = \mathbf{N}A_c + \mathbf{Y}A_p\mathbf{X} + NB_c\mathbf{G}C_p\mathbf{X}$ and $N = I_n - \mathbf{YX}$. Let $Z_{x_1} = \mathbf{Y}$, $Z_{x_2} = \mathbf{X}$, $Z_{x3} = C_c^T$, $Z_{x_4} = B_c^T$, $Z_{x_5} = \mathbf{N}$, $Z_{x_6} = \mathbf{G}$, $Z_{x_7} = \mathbf{G}^T$. Define a symmetric matrix Z of dimension $15n \times 15n$ as $Z = \mathbf{VV}^T$, where $\mathbf{V} = [I_n \quad Z_{x_1}^T \quad \cdots \quad Z_{x_7}^T \quad Z_{v_1}^T \quad \cdots \quad Z_{v_7}^T]^T$, $Z_{v_1} = \mathbf{Y}B_u$, $Z_{v_2} = \mathbf{Y}B_uC_c$, $Z_{v_3} = \mathbf{YX}$, $Z_{v_4} = \mathbf{N}B_c$, $Z_{v_5} = \mathbf{N}B_c\mathbf{G}$, $Z_{v_6} = \mathbf{G}\Theta_{y_p}$, $Z_{v_7} = \mathbf{G}\Theta_{y_p}\mathbf{G}^T$. Then, we have the following set of additional constraints

$$
\begin{array}{ll}
Z \geq 0; & Z_{0,0} - I_{n\times n} = 0; \\
Z_{x_1} - Z_{x_1}^T = 0; & Z_{x_2} - Z_{x_2}^T = 0; \\
Z_{v_1} - Z_{x_1}B_u = 0; & Z_{v_2} - Z_{v_1}Z_{x_3}^T = 0 \\
Z_{v_3} - Z_{x_1}Z_{x_2}^T = 0; & Z_{v_3} - I + Z_{x_5} = 0; \\
Z_{v_4} - Z_{x_5}Z_{x_4}^T = 0; & Z_{v_5} - Z_{v_4}Z_{x_7} = 0; \\
Z_{v_6} - Z_{x_6}\Theta_{y_p} = 0; & Z_{v_7} - Z_{v_6}Z_{x_6}^T = 0; \\
Z_{v_7} = 0; &
\end{array} \tag{7.24}
$$

and a rank constraint

$$
\operatorname{rank}(Z) \leq n. \tag{7.25}
$$

Stability conditions are given as

$$
\begin{bmatrix}
A_pZ_{x_2} + Z_{x_2}A_p^T & A_p & 0 & B_uZ_{x_3}^T & B_w & Z_{x_2}C_z^T \\
A_p^T & Z_{x_1}A_p + Z_{v_5}C_p & 0 & Z_{v_2} & Z_{x_1}B_w & C_z^T \\
0 & 0 & 0 & 0 & 0 & 0 \\
Z_{x_3}B_u^T & Z_{v_2}^T & 0 & 0 & 0 & Z_{x_3}D_u^T \\
B_w^T & B_w^TZ_{x_1} & 0 & 0 & 0 & D^T \\
C_zZ_{x_2} & C_z & 0 & D_uZ_{x_3}^T & D & 0
\end{bmatrix} +
$$

$$
\begin{bmatrix}
R_1 & \mathbf{A_c}^T + R_2 & 0 & 0 & 0 & 0 \\
\mathbf{A_c} + R_2^T & A_p^TZ_{x_1} + (Z_{v_5}C_p)^T + R_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -gI & 0 \\
0 & 0 & 0 & 0 & 0 & -gI
\end{bmatrix} \leq 0, \tag{7.26}
$$

$$
\Pi^TP\Pi = \begin{bmatrix} Z_{x_1} & I_n \\ I_n & Z_{x_2} \end{bmatrix} > 0, \tag{7.27}
$$

$$
\begin{bmatrix} R_1 - cZ_{x_1} & R_2 - cI_n \\ R_2^T - cI_n & R_3 - cZ_{x_2} \end{bmatrix} > 0. \tag{7.28}
$$

For a given disturbance attenuation parameter $g > 0$ and a parameter $c > 0$, if we can employ a semi-definite programming to solve the feasibility problem with constraints (7.24)-(7.28) in which decision variables are Z, $\mathbf{A_c}$, and R, **Problem 7.1** is solvable, see [54], [116], [118], [119]. Then the designed controller matrices can be built as

$$\begin{aligned} C_c &= Z_{x3}^T, \\ B_c &= Z_{x4}^T, \\ A_c &= Z_{x5}^{-1}(\mathbf{A_c} - Z_{x_1}A_pZ_{x2} - Z_{v_5}C_pZ_{x_2}). \end{aligned}$$

7.4.3 An example

In this subsection, we present an example to test our numerical procedure developed in subsection 7.4.2.

Consider a quantum plant to be controlled

$$\begin{aligned} dx(t) &= \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} x(t)dt + \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} dw_p + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} dv_p(t) + \\ & \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} du(t), \\ dy(t) &= \begin{bmatrix} -3 & 1 \\ 4 & -2 \end{bmatrix} x(t)dt + dw_p(t), \\ dz(t) &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x(t)dt + du(t), \end{aligned} \tag{7.29}$$

where system matrices satisfy physical realizability conditions.

Applying our method proposed in Subsection 7.4.2 to the quantum plant (7.29) with $g = 7, c = 0.01$, we get the following solutions:

$$A_c = \begin{bmatrix} -0.7079 & 0.5772 \\ 1.0946 & -0.0970 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.0059 & 0.0507 \\ -0.3360 & -0.3194 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 3.194 & 0.507 \\ -3.360 & -0.59 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1.0209 & 0.2065 \\ -0.0035 & -0.0007 \end{bmatrix},$$

$$\Pi^T P \Pi = \begin{bmatrix} 0.8205 & 1.7504 & 1.0000 & 0 \\ 1.7504 & 17.0245 & 0 & 1.0000 \\ 1.0000 & 0 & 6.9585 & -2.6406 \\ 0 & 1.0000 & -2.6406 & 1.8490 \end{bmatrix},$$

$$R = \begin{bmatrix} 1.2919 & -4.2307 & 0.0100 & 0.0000 \\ -4.2307 & 14.9839 & -0.0000 & 0.0100 \\ 0.0100 & -0.0000 & 15.2019 & -0.0264 \\ 0.0000 & 0.0100 & -0.0264 & 15.1508 \end{bmatrix}.$$

Now we check that if the resulting solutions satisfy the constraints listed in **Problem 7.1**. The eigenvalues of (7.26) are -16.7443, -8.1605, -7.6262, -7.0943, -7.0001, -6.0220, -2.4694, -0.0835, -0.0313, -0.0001, -0.0377, 0, 0, 0, which indicate condition (7.26) is non-positive. It is easily checked that $R > c\Pi^T P \Pi > 0$ holds. Therefore, the resulting solutions satisfy all the conditions, which are reasonable for the classical controller designs.

7.5 Concluding remarks

In this chapter, we present a linear model for the H^∞ control of quantum measurement-based feedback control system subject to feedback-loop time delay, which is the main concern of this chapter. Stability and dissipation theory is developed for this model and from this a new quantum version of the Bounded Real Lemma is derived for our model. The H^∞ controller synthesis problem is also investigated in this chapter. An example is given to present our controller design procedure.

Chapter 8

Conclusions and Future Research

8.1 Conclusions

The work of this thesis can be grouped into two parts. The first part including Chapter 3 and 4 is devoted to mixed quantum-classical feedback network synthesis and a quantum realization of a linear classical stochastic system. The second part consisting of Chapter 5-7 focuses on controller designs for quantum feedback control systems with or without time delay. The main contributions of this work can be summarized as follows.

For the first part, Chapter 3 has developed a network theory for synthesizing linear dynamical mixed quantum-classical stochastic systems of the *standard* form in a systematic way based on three new physical realization constraints. Then one feedback architecture is proposed for this realization. Chapter 4 has shown that under certain technical assumptions, a class of classical linear stochastic systems in a certain form can be realized by quantum linear stochastic systems. It is anticipated that the main results of the work will aid in facilitating the implementation of classical linear systems with fast quantum optical devices (eg. measurement-based feedback control), especially in miniature platforms such as nanophotonic circuits.

For the second part, Chapter 5 has presented two numerical procedures based on extended LMIs approach to solve a quantum LQG problem, which can provide more parameters for the design of a physically realizable quantum controller of the standard form and give less conservative solutions to quantum LQG problem. For comparison, we reinvestigate the example given in [54]. It turns out that our optimization procedure proposed in this chapter can be used to improve overall the closed-loop control performance. Chapter 6 and Chapter 7 have concerned the influences of time delay in quantum feedback control systems. Chapter 6 has investigated classical controller designs for quantum feedback control systems with feedback-loop time delay. A delay-independent stability criterion as well as an upper bound on a cost function has been derived for a quantum measurement-based feedback control system, and from this we proposed one numerical procedure for classical controller designs. Chapter 7 has investigated H^∞ controller synthesis problems for quantum feedback control of linear stochastic systems with feedback-loop time delays. The dissipation property is characterized in linear matrix inequality forms, which can lead to a quantum version of the Bounded Real Lemma for quantum measurement-based feedback control systems subject to feedback-loop time delay. A numerical procedure is proposed for a quantum feedback control system with time delay based on this version of the Strict Bounded Real Lemma.

8.2 Future research

Here we indicate the area of future research that follows naturally from this thesis.

It is well-known in the control community that there are intrinsic conflicts between achievable performance and system robustness. A well thought controller design is to make some suitable tradeoffs between performance and system robustness. It is therefore desirable to develop design techniques that can optimally and systematically perform such performance and robustness tradeoffs. In Chapter 5 we have considered an extended LMI approach to coherent quantum LQG control design, which only addresses the issue of performance in the presence of quantum noise processes using LMI technique to calculate the dynamic controller parameters. H^∞ control is to minimize the disturbance effect on the output of the plant, which also plays an important role in the quantum controller design. The mixed multi-objective LQG and H^∞ optimal control problem for classical systems has been widely studied in previous works [133], [134], etc. The designed controller not only can guarantee the resulting closed-loop system satisfying a pre-specified H^∞ disturbance attenuation level for all admissible parameter uncertainties, but also provides an upper bound for the LQG cost function, which is minimized using a strict LMI convex optimization approach. However, the problem of multi-objective LQG and H^∞ optimal quantum controller design has not been solved so far. In our future work, we will apply the multi-objective optimization control technique to the quantum linear stochastic systems setting.

As mentioned in Chapter 6, the stability criteria can be classified into two categories, namely delay-independent and delay-dependent. In Chapter 6 and 7, we have derived delay-independent stability conditions for quantum measurement-based feedback control systems subject to feedback-loop time delays. In the future work, we will plan to theoretically develop delay-dependent stability conditions for quantum measurement-based feedback control systems described by linear stochastic differential equations. We also plan to extend multi-objective control technique to the quantum measurement-based feedback control systems with feedback-loop time delay.

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