Spin Foam Quantization of 2D Supergravity

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Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

David Botman

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Abstract

Loop quantum gravity, a background independent approach to unifying general relativity and quantum mechanics, has over the last 20 years been widely investigated. The aim of loop quantum gravity is to construct a background independent, non-perturbative quantum theory for the Lorentzian gravitational field in four dimensions. In this approach, the principles of quantum mechanics and general relativity are combined in a natural manner with no other additional physical assumptions. A direct consequence of this combination is that it provides a picture of quantum Riemannian geometry which is discrete at the Planck length. Loop quantum gravity predicts that we live in a space and time that is discrete at the quantum level. These quantum states of space are described in the theory by spin networks. Formally, spin networks are a directed graph with edges labelled by irreducible representations of a compact Lie group and vertices labelled by intertwiners.

In this thesis, the extension of loop quantum gravity and spin networks to include supersymmetry is presented. It is known in 4, 3 and 2 dimensions that general relativity can be formulated as a constrained BF-theory. We will show that the same is true for supergravity in 2 dimensions. After introducing the supersymmetric extension of spin networks, obtained by replacing Lie groups with super Lie groups, we present the spin foam quantization of 2d general relativity. This quantization of the constrained BF-theory formulation is obtained using the Barrett-Crane technique of imposing the classical constraint at the quantum level as a restriction on the representations summed over. We then extend this quantization procedure to the case of 2d supergravity. We find that in order to recover 2d supergravity in this framework, it is necessary to restrict the representations summed over in the spin network to the trivial representation.

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Chapter 1

Introduction

Our current understanding of gravitational phenomena is described by Einstein's Theory of General Relativity (GR). A triumph of twentieth century physics, GR has been experimentally verified and has led to relativistic astrophysics, cosmology and hopefully towards gravitational wave astronomy. Arguably more importantly, our understanding of gravity through GR has allowed the development of GPS technology that has greatly changed our lives. This incredibly successful theory however, breaks down at small scales and at high energies, which one finds for example when studying the big bang, black holes or other phenomena at the Planck scale. In order to study and understand these aspects of the universe we require a non-perturbative theory of quantum gravity. That is, a theory that is valid at all energies and scales, consistent with the principles of general relativity and formulated within the framework of quantum mechanics (QM). Quantum mechanics too, is a pillar of modern science, having opened the fields of nuclear physics, particle physics and condensed matter physics to name a few. These fields of research have led to everyday items that have drastically changed the world such as lasers, semiconductors and computers. Without the theories of general relativity and quantum mechanics our lives would be very different. So such a theory of *quantum gravity* is expected to be a major leap forward in our understanding of the universe. But there is a catch. The coherent picture of the world that was understood through pre-relativistic classical physics is destroyed by QM and GR. Each is formulated with assumptions that directly contradict each other. QM was formulated using an external time variable, the t of the Schrödinger equation. The spacetime on which QM (and quantum field theory) are defined is a fixed and nondynamical background. But this external time variable and fixed background spacetime are incompatible with GR. On the other hand, GR is formulated in terms of Riemannian geometry, assuming that the metric is a smooth and deterministic dynamical field. But QM has taught us that any dynamical field must be quantized at the quantum level. At small scales, dynamical fields are described as discrete quanta and are governed by probabilistic laws. From GR we have learned that spacetime is dynamical and we know from QM that dynamical fields are quantized and can be in a probabilistic superposition of states. This implies that at small scales there should be quanta of space and quanta of time and the superposition of spaces. But what does it mean to live in a quantum spacetime and how can we describe it? Since the development of QM in the 1930's, many physicists including Einstein, Dirac, Feynman, Weinberg, DeWitt, Wheeler, Penrose and Hawking, have attempted to unify GR and QM with varying degrees of success. Many different research directions were followed including dynamical triangulations, noncommutative geometry, Hawking's Euclidean sum over geometries, quantum Regge calculus, Penrose's twistor theory and causal sets just to name a few. Again, all these ideas had varying degrees of success in merging the conceptual ideas of GR and QM. These days the two most developed theories of quantum gravity are loop quantum gravity (LQG) and string theory. It is the former theory that we shall discuss in this thesis. We do so however, with an eye towards the latter. String theory to date, is the only theory we have currently with the potential to unify all four fundamental forces of nature. It is also the only technique we have for successfully investigating the perturbative regime. Thus if one wishes to study the potential connections between LQG and string theory, one is inevitably lead to consider supersymmetry and its implementation in the loop programme. This question, of the relationship between the two theories, has only been asked recently and as such, has not been studied in great depth. Specifically, the question of how supergravity should be included in LQG has received very little attention in the literature. As this is the case, we will examine in this thesis the more simple problem of 2d supergravity and how it is to be formulated in LQG. To do this, the 2d supergravity model first considered in [5] will be formulated as a topological field theory, known as BF-theory. In order to achieve this, it is necessary to impose a constraint on the action at the classical level. BF-theory and its connection with LQG has been well studied and there are many techniques available in this framework to allow progress on the problem of quantum gravity. Using these techniques, we wish to ask the question: how is this classical constraint on the BF-action imposed at the quantum level in the LQG theory?

The basis of LQG is the hamiltonian formulation of GR which was independently developed by Dirac [60] and Bergmann [61] in the late 1950's and it is here where this thesis begins. Shortly after, in the early 1960's, through the introduction of the ADM variables by Arnowitt, Deser and Misner [23], the algebraic complexity of the hamiltonian formulation was greatly simplified. Things were still further simplified by the self-dual connection variables developed later by Ashtekar [28, 29] and Sen [63]. The advantage to using the Ashtekar variables is that GR can be formulated as a canonical gauge theory. As such, the knowledge and techniques developed for the quantization of gauge theories can be used to tackle the problem of quantum gravity. It is not surprising that it is these Ashtekar variables that are used in the definition of LQG. Having reviewed the hamiltonian formulation of GR and introduced the Ashtekar variables, we will proceed to 2d supergravity in the superspace setting. As mentioned, the actions of supergravity can be expressed as the actions of the topological BF-theory with the addition of a constraint. Doing this, supergravity can more easily be studied in the context of LQG using the mathematical tools that have already been developed for the quantization of gauge theories. As such, the beginning of Chapter 3 is an introduction to BF-theory and GR in this context. We then proceed to the supersymmetric extension of BF-theory and write 2d supergravity as a 'constrained' super BF-action. In Chapter 4, we leave supergravity and present the foundations and principal ideas of LQG. This theory is an attempt to define a quantization of GR that is background independent and non perturbative. It is based on the idea that fixing some background metric is not appropriate when trying to describe the quantum properties of spacetime. At its foundation, the inputs to the theory are QM and GR with no additional physical requirements such as extra dimensions or supersymmetry. The basis for the Hilbert space of LQG is provided by the so called 'spin networks' which are the topic of Chapter 5. First introduced in an attempt to describe spacetime in a purely combinatorial manner, the development of spin networks was motivated more by the quantum mechanics of angular momentum than by any consideration of GR. However, after the initial formulation of LQG, it was discovered that spin networks could be used to describe the states of LQG. What is surprising, is that the spin networks at the quantum level, describe a space that is discrete. In the final chapter the spin foam quantization of 2d GR developed in [1] is presented. In order to answer the question posed above, regarding the imposing of a classical constraint at the quantum level in LQG, we will proceed to extend this spin foam quantization to include supersymmetry. Though the extension is not complete, it represents in our opinion a first step to the inclusion of supersymmetry.

Chapter 2

General relativity and supergravity

We begin this chapter with a brief overview of the hamiltonian formulation of GR, followed by the introduction of the Ashtekar variables which will be encountered again in later chapters discussing loop quantum gravity. As the focus of this thesis is 2d supergravity and its quantization in loop quantum gravity, the hamiltonian formulation will be presented. It should be pointed out that though LQG does not require supersymmetry (a symmetry relating bosonic and fermionic particles) to be a consistent and valid theory, there are compelling reasons to include it. Today string theory is the only theory that could potentially unify all four fundamental forces of nature. It is also the only successful technique for investigating the perturbative regime. But string theory *does* require supersymmetry in order to be consistent. It is a current (tentative) hypothesis that LQG, specifically the spin foam formulation, may serve as a non-perturbative and background independent framework for string theory [64, 65]. Having introduced supergravity in 2d, the supergravity model of Howe [5] is considered in detail in the context of its formulation in superspace. Chapter 2 ends with the presentation of the supergravity Lagrangians that will be written as the Lagrangians of constrained BF-theories in the following chapter.

2.1 Hamiltonian formulation of GR and the Ashtekar-Sen variables

The quantization of gravity, which at least in principle, avoids background dependence is based on the ADM approach to Dirac quantization of the hamiltonian [23]. Though the essential details will be presented in what follows, for a more thorough explanation of the details involved in the hamiltonian formulation of GR, see [24, 25, 26]. The general action for Riemannian or Lorentzian GR in $(d+1)$-dimensions and in metric variables is the Einstein-Hilbert action

$$I = \frac{1}{\kappa}\int d^{(d+1)}x\sqrt{\det(g)}R^{(d+1)}. \tag{2.1}$$

The curvature scalar associated with the metric $g_{\mu\nu}$ is $R^{(d+1)}$ and (in units where $c = 1$) $\kappa = 8\pi G$, where G is Newton's gravitational constant. The signature convention is 'mainly' plus, that is, $(-, +, \dots, +)$ or $(+, +, \dots, +)$ in the Lorentzian or Riemannian case respectively[1]. The topology of the manifold M is assumed to be of the form $\mathbb{R} \times \Sigma$, where Σ is a d-dimensional space-like hypersurface without boundary, in order to perform the $d+1$ foliation of space-time. This assumption may seem over restrictive, but a theorem of Geroch [27] states that if a space-time is globally hyperbolic (existence of Cauchy surfaces, roughly speaking manifolds that admit a smooth metric with everywhere Lorentzian signature) then it is necessarily of this type of topology. At least classically (and for the Lorentzian case) the assumption of topology seems to be no restriction. Now consider the tangential vector fields $S_i(X)$ to the hypersurface Σ_t, where $i, j, \dots$ label spatial quantities and the X are space-time coordinates labelled by lowercase Greek letters from the middle of the alphabet. The normal vector field $n^\mu(X)$ can be defined as $g_{\mu\nu}n^\mu S_i^\nu = 0$, $g_{\mu\nu}n^\mu n^\nu = -1$ and the foliation vector T can be decomposed into the basis n, S_i;

$$T = Nn + N^i S_i, \tag{2.2}$$

where N is the lapse function and $N^i S_i$ is the shift vector fields. Here we are using the standard summation convention of summing over repeated indices.

[1] See the Appendix for the general notational conventions that are used throughout this thesis.

The intrinsic metric and extrinsic curvature can now be introduced which are symmetric space-time tensors

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad K_{\mu\nu} = q_{\mu\rho} q_{\nu\sigma} \nabla^\rho n^\nu, \tag{2.3}$$

where ∇^μ is the covariant derivative. These two tensors are introduced because they are spatial, that is, they vanish when contracted with the normal vector, $q_{\mu\nu} n^\nu = 0$. The information they contain is associated with the components of the spatial fields S_i,

$$\begin{aligned} q_{ij} &= q_{\mu\nu} S_i^\mu S_j^\nu \qquad K_{ij} = \tfrac{1}{2N}[\dot{q}_{ij} - \mathcal{L}_{N^i} q_{ij}]. \\ &= g_{\mu\nu} S_i^\mu S_j^\nu, \end{aligned} \tag{2.4}$$

Here $\mathcal{L}_{N^i}$ is the Lie derivative with respect to the shift vector field N^i and the dot represents differentiation with respect to t. The metric $g_{\mu\nu}$ can now be completely expressed in terms of the induced metric q_{ij} of Σ and the functions N and N^i. Taking these variables as configuration coordinates in a phase space one performs a standard Legendre transformation of the Einstein-Hilbert action. The resulting action will depend on Σ, N, N^i and the canonically conjugate momenta π^{ij} to the induced metric. The variables (q_{ij}, π^{ij}) are known as the ADM variables. Varying the action with respect to the shift and lapse functions produces the so-called vector (diffeomorphism) constraint $V^i(q_{ij}, \pi^{ij})$ and the scalar (hamiltonian) constraint $S(q_{ij}, \pi^{ij})$ respectively [26]. With these constraints the action can be written as

$$I = \frac{1}{\kappa} \int dt \int_\Sigma d^d x [\pi^{ij} \dot{q}_{ij} - N_i V^i - NS], \tag{2.5}$$

where the hamiltonian $H = N_i V^i + NS$ is a linear combination of first class constraints and as such vanishes on solutions of the equations of motion.

2.1.1 The Ashtekar-Sen connection variables

Now we wish to make a change from the ADM variables just introduced to the connection variables first presented by Sen [63] and Ashtekar in [28, 29] (see also [30]) and extended by Barbero [31, 32] and Immirzi [33]. Using these new variables it is possible to formulate GR as a canonical gauge theory and greatly simplify the hamiltonian form of the theory. Indeed, by moving from the metric variables of the ADM formulation to these new variables allows the use of the huge number of techniques developed for the canonical quantization of gauge theories. This allowed huge steps to be made in the field of quantum gravity where before progress had virtually stopped with the ADM formulation for almost 30 years. The spatial metric q_{ij} can be expressed in terms of a set of d one-forms e_i^a defining a frame at each point of Σ (in 4d these are the triads) by

$$q_{ij} = e_i^a e_j^b \delta_{ab}, \tag{2.6}$$

where $a, b, \ldots = 1, 2, 3$ are $SU(2)$ indices (strictly speaking they are $SO(3)$ indices, however, as we wish to introduce fermions later we will consider the double cover of $SO(3)$, i.e. $SU(2)$). One may also define (the densitized triad)

$$E_a^i = \tfrac{1}{2}\epsilon^{ijk}\epsilon_{abc}e_j^b e_k^c, \tag{2.7}$$

which transforms under the vector representation of $SU(2)$. Using this definition, the inverse metric q^{ij} can be related to the densitized triad through $\det(q)q^{ij} = E_a^i E_b^j \delta^{ab}$. The $su(2)$-connection defining the covariant derivative compatible with the triad is the so-called spin connection ω_i^a, which is derived as the solution to the Cartan structure equations

$$\partial_{[i}e_{j]}^a + \epsilon^a{}_{bc}\omega_{[i}^b e_{j]}^c = 0. \tag{2.8}$$

Here the square brackets denote antisymmetrization of the indices. The new Ashtekar-Sen variables can be defined by introducing the $su(2)$-connection A_i^a as

$$A_i^a = \omega_i^a + \gamma K_i^a, \tag{2.9}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$ is the Immirzi parameter [34] and $K_i^a = K_{ij}E_b^j\delta^{ab}$ are the momenta conjugate to the densitized triad E_a^i. The new connection is nice in the sense that it too is conjugate to E_a^i. The new variables (A_i^a, E_a^i) form a conjugate pair, with the Poisson brackets of the new variables being

$$\{E_a^i, A_j^b\} = \kappa\gamma\,\delta_j^i\,\delta_b^a, \quad \{E_a^i, E_b^j\} = \{A_i^a, A_j^b\} = 0. \tag{2.10}$$

Rewriting the hamiltonian formulation of the GR action (2.5) in the new Ashtekar-Sen variables (A_i^a, E_a^i) results in the action becoming

$$I = \frac{1}{\kappa}\int dt \int d^dx \, [E_a^i \dot{A}_i^a - N^i V_i - NS - \epsilon^a G_a], \tag{2.11}$$

where the constraints are now given by

$$\begin{aligned} V_j &= E_a^i F_{ij}^a - (1+\gamma^2) K_j^a G_a, \\ S &= \frac{E_a^i E_b^j}{\sqrt{\det(\mathrm{E})}} \left(\epsilon^{ab}{}_c F_{ij}^c - 2(1+\gamma^2) K_{[i}^a K_{j]}^b \right), \\ G_a &= D_i E_a^i, \end{aligned} \tag{2.12}$$

where the F_{ij}^a appearing in the vector constraint is the curvature associated with the Ashtekar connection A_i^a. The derivative is covariant with respect to both the metric and the gauge connection, i.e. with (2.9). The third constraint is the $SU(2)$ Gauss constraint and is required to encapsulate a redundancy that occurs when expressing the components of the spatial metric q_{ij} in terms of the densitized triad E_a^i. This redundancy corresponds to the fact that one may choose different local frames e_a^i by acting on the internal indices with local $SU(2)$ rotations. The covariant derivative in the constraint is with respect to the connection A_i^a.

The argument could be made that by going from the ADM variables (q_{ij}, π^{ij}) to the new Ashtekar-Sen variables (A_i^a, E_a^i) which increases the degrees of freedom, one has made the classical theory more complicated. However the first class Gauss constraint (2.12) removes these additional degrees of freedom showing that the gauge theory phase space is indeed equivalent to the ADM phase space [26]. The advantages that come from working in this extended phase space is that canonical GR can now be formulated as a canonical gauge theory with $su(2)$-connection A_i^a and conjugate field E_a^i. Not only does this open up a possible approach to a gauge group unification with the other known forces but the problem of quantum gravity can now be attacked using the knowledge acquired and techniques that have already been developed for the canonical quantization of gauge theories. It is precisely because of this ability to formulate GR as a gauge theory that so much progress has been made in the last two decades in quantizing gravity.

2.1.2 Gauge transformations

As was alluded to before the Gauss constraint (2.12) is associated with $SU(2)$ gauge transformations and it will now be shown how gauge transformations are generated by these constraints. Starting with the Gauss constraint, defining the smeared version

$$G(\epsilon) = \int_\Sigma d^dx\, \epsilon^a G_a = \int_\Sigma d^dx\, \epsilon^a D_i E^i_a, \tag{2.13}$$

it is not difficult to show that acting on the canonical variables under the action of the Poisson brackets (2.10) one finds the familiar results

$$\delta_\epsilon A^a_i = \{A^a_i, G(\epsilon)\} = -D_i\epsilon^a, \qquad \delta_\epsilon E^i_a = \{E^i_a, G(\epsilon)\} = [E,\epsilon]^i_a. \tag{2.14}$$

Here $\epsilon = \epsilon_a\tau^a$, where τ^a are the $su(2)$ generators in the vector representation. By writing the connection and conjugate momenta as $A_i = A^a_i\tau_a$ and $E^i = E^i_a\tau^a$, the finite form of the infinitesimal transformations above are

$$A_i \rightarrow gA_ig^{-1} + g\partial g^{-1}, \qquad E^i \rightarrow gE^ig^{-1} \tag{2.15}$$

which are the standard ways in which the connection and electric field transform under gauge transformations in Yang-Mills theory. Performing the same procedure for the vector constraint (2.12) by taking the Poisson bracket of the smeared constraint

$$V(N) = \int_\Sigma d^dxN^iV_i, \tag{2.16}$$

with the canonical variables one finds,

$$\delta_N A^a_i = \{A^a_i, V(N)\} = \mathcal{L}_N A^a_i, \qquad \delta_N E^i_a = \{E^i_a, V(N)\} = \mathcal{L}_N E^i_a, \tag{2.17}$$

where $\mathcal{L}_N$ is the Lie derivative in the 'smeared' direction N^i. The vector constraint generates diffeomorphisms on the spatial surface Σ. Likewise the scalar constraint (2.12) generates coordinate time evolution up to spatial diffeomorphisms and local $SU(2)$ transformations.

The total hamiltonian of GR is a linear combination of constraints and can be written as

$$H(\epsilon, N^i, N) = G(\epsilon) + V(N^i) + S(N), \tag{2.18}$$

where

$$S(N) = \int_\Sigma d^d x \, NS \tag{2.19}$$

is the smeared scalar constraint. In a generally covariant formulation of physics, a coordinate time t has no physical meaning. One is always free to change the way space-time is coordinatized while leaving the physics invariant. This can be seen by the fact that the parameters labelling spatial diffeomorphisms N^i and coordinate time evolution N are completely arbitrary functions and the corresponding motions on the phase space must be interpreted as gauge transformations. In other words, gauge symmetry induces constraints in the canonical formulation and in turn these constraints are the generators of the gauge symmetry. The total hamiltonian generates *space-time* diffeomorphisms and as this is pure gauge the total hamiltonian is a constraint itself and vanishes on shell, $H = 0$.

2.1.3 An aside

Before moving on there are a few important things to mention. Firstly, the constraint algebra generated by the Gauss constraint does not differentiate between the Lie group $SO(3)$ and its double cover $SU(2)$ as both have the same Lie algebra. However, if one wants to include fermionic degrees of freedom, by incorporating spinors, then one is forced to use the group $SU(2)$ [35, 36]. The second thing to note is that the Ashtekar-Sen variables do not have a simple relationship with space-time fields. In particular the Ashtekar-Sen connection (2.9) cannot in general be obtained as the pull back of a space-time connection to Σ. In [37] (in 4d) it was shown though, that for the specific choice $\gamma = \pm i$ of the Immirzi parameter, the connection obtained is the pull-back of the self-dual part of a Lorentz connection ω_μ^{AB} $(A, B = 1, \ldots, 4)$. That is, A_i is the pullback of ω_μ^{+AB}, where

$$\omega_\mu^{+AB} = \tfrac{1}{2}(\omega_\mu^{AB} - \tfrac{i}{2}\epsilon^{AB}{}_{CD}\omega_\mu^{CD}). \tag{2.20}$$

The other remarkable thing about this particular choice of the Immirzi parameter is that the vector and scalar constraints (2.12) greatly simplify;

$$V_i \;=\; E_a^j F_{ij}^a,$$

$$\begin{aligned} S &= \frac{E^i_a E^j_b}{\sqrt{\det(\mathrm{E})}} \epsilon^{ab}{}_c F^c_{ij}, \\ G_a &= D_i E^i_a \end{aligned} \tag{2.21}$$

and consequently the total hamiltonian simplifies considerably. This was the original choice made by Ashtekar and loop quantum gravity was initially formulated in these variables. The reason being that in order to quantize the scalar (hamiltonian) constraint (and hence quantize gravity) it was thought to be necessary to simplify the algebraic structure of the scalar constraint. By choosing $\gamma = \pm i$, as was shown, the scalar constraint greatly simplifies and by multiplying by a factor of $\sqrt{\det(\mathrm{E})}$, the constraint becomes polynomial. There are however, a number of reasons why this choice in Immirzi parameter is no longer regarded as the preferred choice. Firstly the Ashtekar-Sen connection (2.9) is complex, i.e. $A_i \in sl(2, \mathbb{C})$, and there are technical difficulties yet to be overcome in defining the quantum theory when the connection is valued in a Lie algebra of a noncompact group. The functional analysis on spaces of such connections is still not sufficiently well-developed to construct a quantum theory. Most progress in LQG has occurred by working with connections with compact structure groups. Due to the well-known properties of the compact group $SU(2)$, such as the Haar measure and Peter-Weyl theorem, one can obtain a background independent representation of the quantum algebra and a spin network basis of the kinematic Hilbert space, which will be discussed in later chapters. In the case of $A_i \in sl(2, \mathbb{C})$ and using complex variables, the phase space of GR must be complexified and the original phase space can only be recovered by imposing the reality conditions

$$A^a_i + \bar{A}^a_i = \omega^a_i(E). \tag{2.22}$$

By making the scalar constraint polynomial, the spin connection $\omega(E)$ becomes a highly non-polynomial function. Implementing this reality condition at the quantum level by elevating it to an appropriate operator is extremely difficult to do. Thus it was accepted that γ should be real in order to remove the problems associated with needing reality conditions and dealing with noncompact Lie groups at the cost of having a non-polynomial scalar constraint. In fact, it turned out (see [26]) that the non-polynomial nature is actually required if one wishes to arrive at a well defined operator. Specifically it was shown that only scalars of density weight one could be quantized in a rigorous and background independent manner and hence the scalar constraint should not be multiplied by any power of $\sqrt{\det(E)}$.

There is another argument for a real Immirzi parameter which is based on a physical reasoning. One of the most proclaimed results so far in loop quantum gravity is the derivation of the Bekenstein-Hawking formula for the entropy of a black hole [38, 44, 42, 43, 45]. In LQG there is an 'area' operator which measures the discrete area of a 2d surface $S \subset \Sigma$ and the eigenvalues of this operator contain the Immirzi parameter. The Bekenstein-Hawking formula in this context becomes

$$S_{BH} = \frac{\gamma_0}{4\ell_P^2 \gamma} A_S, \qquad \gamma_0 = \frac{\ln 2}{\pi\sqrt{3}}. \tag{2.23}$$

For this result to match the statistical mechanical entropy given precisely by the Bekenstein-Hawking formula one must set $\gamma = \gamma_0$. Even though the arguments used to arrive at this result have been questioned and other values of the Immirzi parameter have been suggested [39, 40, 41], these values are always real.

So far, the Ashtekar-Sen variables that form the basis of loop quantum gravity were presented. It was shown how these variables allow GR to be formulated as a gauge theory. This is desirable for two reasons; first, the hamiltonian form of the theory is greatly simplified and second, the techniques developed for the quantization of gauge theories can now be used on the problem of quantizing gravity. Next we will consider 2d supergravity in the context of superspace. The method for constructing generic actions will be discussed and a number of different Lagrangians will be presented.

2.2 Supergravity

For the nongravitational forces, a unified renormalizable quantum field theory exists, the well-known standard model. The electromagnetic and weak forces are unified within the $SU(2) \times U(1)$ electroweak theory of Glashow, Salam and Weinberg. Describing the strong force, the interaction between quarks and gluons which keeps the nucleus together, there is the $SU(3)$ theory of quantum chromodynamics. It is these two theories together that make the standard model. But there is a major omission, the standard model does not contain gravity. The unification of gravity with the other known forces of nature has been one of the central problems of modern physics. Though many attempts were made, and numerous ideas were proposed, unification of the forces proved extremely difficult. The problem lies in the fact that the standard model is a quantum field theory and as such, is renormalizable. The incorporation of the principles of GR into these theories proved in the past impossible because GR describes gravity and gravity has a *dimensional* coupling constant. This dimensional coupling constant means that GR is not renormalizable. Thus what was needed was a field theory describing both gravity and the other forces that was based on the principles of quantum field theory and general relativity that, though not renormalizable, still could make predictions. Supergravity [66, 67, 68] was proposed as such a theory. It is the gauge theory of supersymmetry, the proposed symmetry between bosonic and fermionic fields. It turned out that this gauge symmetry between bosonic and fermionic fields could only be implemented in field theory if the spacetime is curved and hence only if gravity was included. The supersymmetric partner to the spin-(2) graviton is the spin-($\frac{3}{2}$) gravitino, χ_μ. Simply by coupling a gravitino to GR, namely with the action (in first order form)

$$I = \int d^4x\,\sqrt{-g}(\tfrac{1}{\kappa}R - \tfrac{i}{2}\epsilon^{\mu\nu\rho\sigma}\chi_\mu\gamma_5\gamma_\nu D_\rho\chi_\sigma), \tag{2.24}$$

one could have a theory finite even at two loops. The infinities in the S-matrix in the first and second order quantum corrections cancel due to the supersymmetry. Unfortunately in the early 1980's it became clear that above two loops supergravity was not renormalizable and the excitement and interest in supergravity dropped off. That was however, until the late 1990's during the second string revolution when supergravity in eleven dimensions made a comeback, being used to understand features of string theory and its relation with M-theory. What is more, recent developments indicate that certain supergravity theories, in certain dimensions, are actually likely to be finite.

As mentioned before, loop quantum gravity does not require supersymmetry in order to be consistent, though it can be put in by hand if desired. There are good reasons though why we should take the idea of supersymmetry in loop quantum gravity seriously. These days string theory is the only theory that could potentially unify all four fundamental forces of nature. It is also the only successful technique for investigating the perturbative regime. String theory, unlike LQG, does require supersymmetry in order to be consistent. It is a current (and tentative) hypothesis that LQG, specifically the spin foam formulation, may serve as a non-perturbative and background independent framework for string theory [65].

2.2.1 Hamiltonian formulation of 2d supergravity

Curved two-dimensional spacetime is worthy of study for at least two reasons. First, it possibly has a connection with the real world through string theory. Second, it provides the simplest possible ground for a model of quantum gravity and supergravity. There had been a barrier however that had prevented progress on the second of these lines of research. That is, the lack of a two-dimensional, non-trivial analog of the action principle for Einstein's equations. In two spacetime dimensions the only non-trivial, local analogue of Einstein's equations is

$$R - \lambda = 0. \tag{2.25}$$

There is no invariant action constructed out of the metric g_{mn} alone, that is the integral of a local Lagrangian that gives the equation of motion (2.25). Indeed if one varies the Einstein-Hilbert action $\int \sqrt{-g}(R-\lambda)$ in two dimensions, due to the curvature term being proportional to the topological Euler characteristic, one is left with the statement $\lambda = 0$ and no restriction on the metric. It is possible however, to recover the equations of motion (2.25) from a non-invariant action [46, 47]. In general, the hamiltonian equations of motion for a dynamical variable F are obtained from

$$\dot{F} = \int dx\, \eta^m(x)[F, \mathcal{H}_m(x)], \tag{2.26}$$

where in the 2d case the index $m = \perp, 1$, where $\perp$ denotes the normal direction to the spacelike hypersurface of constant time and $m = 1$ labels tangent vectors to the spacelike surface. The bracket is the normal Poisson bracket in the classical theory or a commutator divided by $i\hbar$ in the quantum case. The hamiltonian generators $\mathcal{H}_m$ are constructed from the canonical variables. The η^m are arbitrary functions of space and time and determine the surface at time $t+\delta t$ from the initial hypersurface at time t. The arbitrariness of the η^m in the hamiltonian formulation is the expression of the general covariance of the equations of motion. The action corresponding to the equations of motion (2.26) is of the form

$$I = \int dt\, dx\, (\dot{\varphi}\pi - \eta^m \mathcal{H}_m), \tag{2.27}$$

where φ, π are canonical variables. In two dimensions this action is not invariant under spacetime reparameterisations though the equations of motion are. To make contact with gravity, the hamiltonian generators are explicitly given by

$$\begin{aligned} \mathcal{H}_\perp &= \tfrac{1}{2}(k\pi^2 + k^{-1}\varphi'^2) - 2k^{-1}\varphi'' - k^{-1}\lambda \exp(\varphi), \\ \mathcal{H}_1 &= \pi\varphi' - 2\pi', \end{aligned} \tag{2.28}$$

which satisfy the necessary surface deformation algebra for any value of k, λ with central charge $z = 48\pi k^{-1}$ (see [46, 47] for full details of this algebra and the derivation of the above generators). Substituting these equations into the action (2.27), eliminating π for $\dot{\varphi}$ and varying the action with respect to φ, the equation of motion

$$\delta I/\delta\varphi = g_{mn}\delta I/\delta g_{mn} = -k^{-1}\sqrt{-g}(R - \lambda), \tag{2.29}$$

is recovered. To obtain this result, it is necessary to use the relation

$$g_{mn} = \exp(\varphi) \begin{bmatrix} (\eta^1)^2 - (\eta^\perp)^2 & \eta^1 \\ \eta^1 & 1 \end{bmatrix}, \tag{2.30}$$

between φ and the metric g_{mn}. The action (2.27) with the appropriate hamiltonian generators provides a suitable alternative to the Einstein-Hilbert action in 2d, giving the correct equations of motion with the central charge z playing the role of the gravitational constant.

The extension to 2d supergravity is rather straightforward and was (to the author's knowledge) first presented in [48]. The surface deformation algebra generated by the $\mathcal{H}^m$ admits a graded extension by including two real anticommuting supersymmetry generators $\mathcal{S}^\alpha$. These supersymmetry generators may be obtained directly from the $\mathcal{H}^m$ by taking the "square root" in a way completely similar to how supergravity in 4d is derived from gravity. This method introduces an anticommuting Majorana spinor χ, the two components of which, provide the anticommuting counterpart to the pair of canonical variables (φ, π). The graded hamiltonian generators are;

$$\begin{aligned}
\mathcal{H}_\perp &= \tfrac{1}{2}(k\pi^2 + k^{-1}\varphi'^2) - 2k^{-1}\varphi'' + 2k^{-1}m^2\exp(\varphi) \\
&\quad -i(2k)^{-1}[\chi^T\gamma_5\chi' - m\exp(\varphi/2)\bar{\chi}\chi], \\
\mathcal{H}_1 &= \pi\varphi' - 2\pi' + i(2k)^{-1}\chi^T\chi', \\
\mathcal{S} &= \gamma_5\chi\pi + 4k^{-1}\chi' - k^{-1}\varphi'\chi + 2m/k^{-1}\exp(\varphi/2)\chi \qquad (2.31)
\end{aligned}$$

and satisfy the graded extension of the surface deformation algebra for any value of the constants k, m (where m enters the equations through the spinor covariant derivative [48]) and central charge $z = 48\pi k^{-1}$. The action for 2d supergravity corresponds to the hamiltonian

$$H = \int dx\,(\eta^m\mathcal{H}_m + i\bar{\zeta}\mathcal{S}), \qquad (2.32)$$

where the spinor ζ is an arbitrary function of space and time and is the anticommuting counterpart to η. Once again, this action will neither be coordinate invariant or invariant under supersymmetry, but the equations of motion will be. Taking the action whose hamiltonian is given above and eliminating π for $\dot{\varphi}$, one obtains the following expressions for its variational derivatives;

$$\begin{aligned}
\delta I/\delta\varphi &\equiv g_{mn}\delta I/\delta g_{mn} = -k^{-1}(-g)^{1/2}\tilde{R}, \\
&\qquad \gamma_m\delta I/\delta\bar{\chi}_m = -16ik^{-1}(-g)^{1/2}\tilde{H}, \qquad (2.33)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R} &= R + 2im(-g)^{1/2}\epsilon^{mn}\bar{\chi}_m\gamma_5\chi_n + 2m^2, \\
\tilde{H} &= \tfrac{1}{2}\gamma_5(-g)^{1/2}(\epsilon^{mn}D_m\chi_n + \tfrac{1}{2}m\gamma^m\chi_m), \qquad (2.34)
\end{aligned}$$

the covariant derivative is defined as usual, $D_m = \partial_m + \frac{1}{2}\omega_m\gamma_5$ and the gamma matrices γ^m, γ^5 (refer to Appendix for details on gamma matrices) correspond to the standard Pauli matrices. The $\tilde{R}$ and $\tilde{H}$ terms (which together form a supercurvature) are related to each other by local supersymmetry transformations, as well as being covariant under coordinate reparameterisations. Later in this chapter we will examine Chamseddine's supersymmetric extension of the Jackiw-Teitelboim action of 2d supergravity (c.f. (2.80)). We would like to point out now, that if the cosmological constant is identified as $\lambda = -\frac{1}{2}\lambda'^2 = -2m^2$, then the above two terms that form the supercurvature are in exact agreement with those of Chamseddine's supersymmetric extension.

2.2.2 Superfield formulation of N=1, 2d supergravity

One of the models we wish to formulate later as a constrained BF-theory is Howe's 2d, N=1 supergravity [5]. In the following section we will proceed to give the relevant details of this paper and a brief description of this model. The reason this particular model is of interest is because the bosonic part of the relevant action is simply Einstein's gravity in two dimensions and hence is topological and proportional to the Euler number. The gravitino terms in the model ultimately make no contribution to the action. The topological nature of 2d gravity, as we will soon show, allows it to be easily formulated as a more 'simple' topological BF-theory. These BF-theories serve as a starting point in the study of background independent theories. It is due to the topological nature of this model that we will use it later on to illustrate the extension of the Barrett-Crane technique to supergravity.

As mentioned before, supergravity is the gauge theory of local supersymmetry and contains the corresponding gauge field, the gravitino which is a spinor field. Owing to the presence of spinor fields, supergravity is formulated using the tetrads (c.f. (2.6)) (Cartan formulation) of GR rather than in the metric formulation. Furthermore, we will now make a shift to working in superspace, an extension of normal spacetime as this comes with a number of advantages. Supersymmetry extends ordinary spacetime symmetries by adding N spinorial generators Q whose anticommutator produces a translation generator, $\{Q_m, Q_n\} = P_{mn}$ where the indices run from 1 to N. This symmetry can be realized on ordinary spacetime fields by transformations that mix bosons and fermions. But this 'component by component' approach is unnecessarily complicated and inconvenient. A compact approach is given by the superspace-superfield approach. Superspace is an extension of ordinary spacetime to include anticommuting coordinates in the form of Weyl spinors θ. Superfields $\Psi(x,\theta)$ are functions defined over this space. The Taylor expansion of these superfields with respect to the anticommuting coordinates is finite as the square of any anticommuting quantity vanishes. The coefficients of this expansion are the normal component fields. Also in superspace, supersymmetry is manifest being represented by translations and rotations involving both ordinary and anticommuting coordinates. A further advantage of superfields is that they automatically contain, in addition to the dynamical degrees of freedom, the unphysical auxiliary fields which are needed for the off-shell closure of the supersymmetry algebra.

2.2.3 General properties

The coordinates of 2+2 superspace are $z^M = (x^m, \theta^\mu)$, where $m, \mu = \{0, 1\}$, the x's are the standard commuting coordinates and the θ's anti-commuting. In superspace notation this can be expressed as

$$z^M z^N = (-1)^{|M||N|} z^N z^M, \tag{2.35}$$

where $|M| = 0$ (1) for bosonic (fermionic) indices. The dyads of standard GR are replaced by superdyads $E_M{}^A(x,\theta)$, which allows us to define a set of frames, that is, a basis of orthonormal one-forms

$$E^A = dz^M E_M{}^A. \tag{2.36}$$

We are using the standard summation convention of summing over repeated indices and the superspace index $A = (a, \alpha)$, where a and α are tangent space and spinor indices respectively. We can also define a covariant exterior derivative by

$$DV^A = dV^A + V^B \Omega_B^A = dz^M D_M V^A, \tag{2.37}$$

where ${\Omega_B}^A$ is the superconnection and V^A is some field transforming as a supervector under the tangent space group $GL(2,2)$. Similarly the torsion and curvature two-forms are defined by

$$\begin{aligned} T^A &= DE^A = \tfrac{1}{2}E^C \wedge E^B {T_{BC}}^A, \\ {R_A}^B &= d{\Omega_A}^B + {\Omega_A}^C \wedge {\Omega_C}^B = \tfrac{1}{2}E^D \wedge E^C {R_{CD,A}}^B, \end{aligned} \tag{2.38}$$

which satisfy the Bianchi identities

$$\begin{aligned} DT^A &= E^B \wedge {R_B}^A, \\ D{R_A}^B &= 0. \end{aligned} \tag{2.39}$$

It is well known [5] that by imposing the kinematic torsion constraints

$${T_{\beta\gamma}}^a = 2i(\gamma^a)_{\beta\gamma}, \qquad {T_{\beta\gamma}}^\alpha = {T_{bc}}^a = 0, \tag{2.40}$$

the number of independent components in the supertorsion and curvature are reduced and in fact can be expressed by one scalar superfield S, which we will now go on to show takes the form

$$S = A + \theta^\alpha \psi_\alpha + \tfrac{1}{2}\bar{\theta}\theta C, \tag{2.41}$$

where $\bar{\theta}\theta = \theta^\alpha\theta_\alpha$ and C and R are given by

$$\begin{aligned} C &= R - \tfrac{1}{2}\bar{\chi}_a\gamma^a\psi + \tfrac{i}{4}\epsilon^{ab}\bar{\chi}_a\gamma_5\chi_b A - \tfrac{1}{2}A^2, \\ R &= 2\epsilon^{mn}\partial_m\omega_n. \end{aligned} \tag{2.42}$$

(The gamma matrices here have been chosen to be real. See the Appendix for details). Explicitly the supercurvature tensor can be simplified due to the 2d Lorentz group to the form

$${R_A}^B = F{E_A}^B, \tag{2.43}$$

where $F = d\Omega$ and ${\Omega_A}^B = \Omega{E_A}^B$. From (2.40) one gets, using the Bianchi identities the following components of the super curvature;

$$\begin{aligned} R_{ab} &= \epsilon_{ab}D_\alpha D^\alpha S - \epsilon_{ab}S^2, \\ R_{\alpha\beta} &= -4(\gamma_5)_{\alpha\beta}S, \\ T_{\alpha b} &= 2i(\gamma_5\gamma_b)_\alpha{}^\beta D_\beta S. \end{aligned} \tag{2.44}$$

Clearly if S vanishes then the curvature does too and the space is flat. In this case the superdyad becomes

$$E_m{}^a = \delta_m{}^a, \quad E_M{}^\alpha = 0, \quad E_\mu{}^a = i\theta^\lambda(\gamma^a)_{\lambda\mu}, \quad E_\mu{}^\alpha = \delta_\mu{}^\alpha. \tag{2.45}$$

2.2.4 Wess-Zumino Gauge

To discuss the component formalism of supergravity we must make our way from the superfield to the component language. In order to do this it is necessary to eliminate the superfluous auxiliary fields required by the superfields to ensure off-shell closure of the supersymmetry. This is achieved by choosing the Wess-Zumino gauge for the superdyad. Under a general super-coordinate transformation the superdyad changes as

$$E_M{}^A(z) = \frac{\partial z'^N}{\partial z^M} E_N{}'^A(z'). \tag{2.46}$$

Expanding z' as a power series in θ^μ it is easy to see that a number of parameters can be used to gauge some of the components of $E_\mu{}^A$ to zero. A superdyad can always be brought to the Wess-Zumino gauge;

$$\begin{aligned} E_\mu{}^a &\sim \theta^\nu E_{\nu\mu}{}^a, \\ E_\mu{}^\alpha &\sim \delta_\mu{}^\alpha + \theta^\nu E_{\nu\mu}{}^\alpha, \end{aligned} \tag{2.47}$$

where $E_{\nu\mu}{}^a = E_{\mu\nu}{}^a$ and $E_{\nu\mu}{}^\alpha = E_{\mu\nu}{}^\alpha$, using the super coordinate transformation (2.46). Setting the first component of $E_m{}^a$ to be the dyad $e_m{}^a$ and the first component of $E_m{}^\alpha$ to be the Rarita-Schwinger field $\frac{1}{2}\chi_m{}^\alpha$ one can, using the kinematic torsion constraints (2.40) and the Bianchi identities calculate the full θ expansion of the superdyad, (see [5]). From the condition $T_{ab}{}^c = 0$ and using the definition of supertorsion (2.38) one finds that the spin connection ω_m is

$$\omega_m = -e_{am}\epsilon^{nl}\partial_n e_l{}^a + \tfrac{1}{2}\bar{\chi}_m\gamma_5\gamma^n\chi_n, \tag{2.48}$$

(see [5] for explicit details on the relation between ω_m and the superconnection $\Omega_A{}^B$). Similarly for ψ, the second component of the scalar superfield S, one finds from $T_{bc}{}^\alpha = 0$ that

$$\psi = -2i\epsilon^{mn}\gamma_5 D_m\chi_n - \tfrac{1}{2}\gamma^m\chi_m A, \tag{2.49}$$

where A is the leading component of the superfield S and the bosonic space covariant derivative is given by

$$D_m\chi_n = \partial_m\chi_n - \tfrac{1}{2}\omega_m\gamma_5\chi_n. \tag{2.50}$$

Note that due to (2.48) the bosonic space torsion is non-vanishing;

$$T_{mn}{}^a = D_m e_n{}^a - D_n e_m{}^a = \tfrac{i}{2}\bar{\chi}_m\gamma^a\chi_n. \tag{2.51}$$

Putting all this together we have the complete form for the scalar superfield S (2.41) which was given above. Note that the bosonic space curvature scalar R of the spin connection ω_m (2.48) includes the gravitino field.

2.2.5 Symmetry Transformations

In the previous section we went to the Wess-Zumino gauge in order to eliminate the auxiliary fields from the superfields to obtain the appropriate component formalism. However, in doing this, the gauge was not fixed completely and several parameters remain free. Initially there were twenty free parameters; sixteen arising from the general super-coordinate transformations (2.46) and the expansion of z' as a power series in θ and four parameters coming from local Lorentz transformations. A total of fifteen of these parameters can be used to give gauge fixing conditions. Specifically, the free parameters that remain are the zeroth component of the superspace diffeomorphisms and the zeroth component of the Lorentz transformation used to gauge away the first component of the super connection Ω_μ and the antisymmetric part of the θ coefficient. Thus our choice in gauge has reduced the original invariance from twenty parameters down to five which correspond to bosonic space reparameterisations, local Lorentz transformations and local supersymmetry. Having used both super-diffeomorphisms and Lorentz transformations to fix the gauge we see that the superdyad and superconnection obey the following infinitesimal transformation rules,

$$\begin{aligned} \delta E_A{}^M &= \xi^N\partial_N E_A{}^M + \partial_M\xi^N E_N{}^A + E_M{}^B L_B{}^A, \\ \delta\Omega_M &= \xi^N\partial_N\Omega_M + \partial_M\xi^N\Omega_N - \partial_M L, \end{aligned} \tag{2.52}$$

where the vector superfield $\xi^N(z)$ parameterizes super-diffeomorphisms and $L_A{}^B$ is the generator of Lorentz (super) transformations. There is only one generator ${L_A}^B$ in the superalgebra and it has components ${L_a}^b \simeq {\epsilon_a}^b$ and ${L_\alpha}^\beta \simeq {(\gamma_5)_\alpha}^\beta$ in the vector and spinor representations respectively. For details on the explicit calculation of the supersymmetry parameters, see [6]. Ultimately one arrives at

$$\begin{aligned}
\xi^m &= \eta^m - i\bar{\zeta}\gamma^m\theta + \tfrac{1}{4}\bar{\theta}\theta\bar{\zeta}\gamma^n\gamma^m\chi_n, \\
\xi^\mu &= \zeta^\mu - \tfrac{i}{2}\bar{\theta}\gamma^m\zeta{\chi_m}^\mu - \tfrac{1}{2}\theta^\lambda{(\gamma_5)_\lambda}^\mu\ell - \tfrac{i}{4}\bar{\theta}\theta(\gamma_5\gamma^m)^{\mu\beta}\zeta_\beta\omega_m - \\
&\quad - \tfrac{1}{8}\bar{\theta}\theta\bar{\zeta}\gamma^n\gamma^m\chi_n{\chi_m}^\mu, \\
L &= \ell - \tfrac{i}{2}\bar{\zeta}\gamma_5\theta A - i\bar{\zeta}\gamma^m\theta\omega + \tfrac{i}{4}\bar{\theta}\theta\bar{\zeta}\gamma_5\psi + \tfrac{1}{16}\bar{\theta}\theta\bar{\zeta}\gamma^m\gamma_5\chi_m A + \\
&\quad + \tfrac{1}{4}\bar{\theta}\theta\bar{\zeta}\gamma^n\gamma^m\chi_n\omega_m, \qquad (2.53)
\end{aligned}$$

where η^m, ζ^μ and ℓ correspond to bosonic space coordinate transformations, local supersymmetry and Lorentz transformations respectively. Using (2.52) and the above parameter expansions it is straightforward to calculate the local $N = 1$ supersymmetry transformations for the component fields;

$$\delta {e_m}^a = i\bar{\zeta}\gamma^a\chi_m, \qquad \delta\chi_m = 2D_m\zeta - \tfrac{1}{2}\gamma_m\zeta A, \qquad \delta A = \bar{\zeta}\psi \qquad (2.54)$$

and further more

$$\delta\omega_m = -i\bar{\zeta}\gamma_m\gamma_5\psi - \tfrac{i}{2}\bar{\zeta}\gamma_5\chi_m A, \quad \delta\psi = -i\bar{\zeta}\gamma^m(\partial_m A + \tfrac{1}{2}\chi_m A) + \zeta C. \quad (2.55)$$

Following the same procedure we find the reparameterisations, forming the group Diff(M);

$$\begin{aligned}
\delta {e_m}^a &= \eta^n\partial_n {e_m}^a + {e_n}^a\partial_m\eta^n, \\
\delta\chi_m &= \eta^n\partial_n\chi_m + \chi_n\partial_m\eta^n, \\
\delta A &= \eta^n\partial_n A, \\
\delta\omega_m &= \eta^n\partial_n\omega_m + \omega_n\partial_m\eta^n, \qquad (2.56)
\end{aligned}$$

and the local Lorentz transformations

$$\begin{aligned}
\delta {e_m}^a &= \ell{\epsilon^a}_b{e_m}^b, \\
\delta\chi_m &= \tfrac{1}{4}\ell\gamma_5\chi_m, \\
\delta A &= 0, \\
\delta\omega_m &= -\partial_m\ell. \qquad (2.57)
\end{aligned}$$

We also have the ordinary Weyl transformations, which are rescalings of the metric $g_{mn} \rightarrow \Omega^2 g_{mn}$;

$$\begin{aligned} \delta e_m{}^a &= \sigma e_m{}^a, \\ \delta \chi_m &= \tfrac{1}{2}\sigma \chi_m, \end{aligned} \tag{2.58}$$

where $\Omega = e^{\sigma}$. A space is conformally flat if there is a coordinate system such that the metric is proportional to the flat metric. Recall that in two dimensions all spaces are locally conformally flat. In [5] the notion of a superspace being superconformally flat was developed. If one naively attempts to generalize the Weyl transformations to superspace and insists on the preservation of the torsion constraints (2.40) then the scaling parameter is restricted to be a constant. To overcome this define super Weyl transformations by

$$\begin{aligned} E_M{}^a &= \Lambda E_M{}^a, \\ E_M{}^\alpha &= \Lambda^{1/2} E_M{}^\alpha + E_M{}^a \phi_a{}^\alpha, \end{aligned} \tag{2.59}$$

where $\phi_a{}^\alpha = -i(\gamma_a)^{\alpha\beta} D_\beta \Lambda^{1/2}$ is a spinor parameter. It is now possible to define a superconformally flat superspace as one for which we can choose a coordinate system such that the superdyad is related to the *flat* superdyad by the above super Weyl transformations, where in this case $E_M{}^A$ is given by (2.45). It is always possible to pick a gauge such that the above argument holds and hence any (2+2)-dimensional superspace is superconformally flat. However, if the kinematic torsion constraints are not satisfied then this does not automatically hold. It is interesting to note that in this gauge the bosonic space torsion is zero and the χ contribution to the spin-connection ω_m (2.48) drops out also. The final symmetry then is the super Weyl transformations

$$\begin{aligned} \delta e_m{}^a &= 0, \\ \delta \chi_m &= \gamma_m \lambda. \end{aligned} \tag{2.60}$$

Having discussed the properties of superspace, the details of Howe's 2d supergravity and the corresponding symmetries we will now look at some Lagrangians of various supergravity theories formulated in superspace.

2.2.6 Supergravity Lagrangians

Before considering various models of supergravity and their corresponding Lagrangians, let's first examine briefly the general construction of actions in superspace. For a more in depth discussion of this topic see [70]. To construct a locally supersymmetric action one must first covariantize all derivatives and the measure. For integrals over all of superspace, analogous with ordinary space, we will use the generalized determinant E of the superdyad as a density to define a covariant measure. Explicitly, this measure takes the form [69]

$$E = e(1 + \tfrac{1}{2}i\theta^{\alpha}\gamma^{m}{}_{\alpha}{}^{\beta}\chi_{m\beta} + \bar{\theta}\theta[\tfrac{1}{4}iA + \tfrac{1}{8}\epsilon^{mn}\chi_{m}{}^{\alpha}\gamma^{5}{}_{\alpha}{}^{\beta}\chi_{n\beta}]). \tag{2.61}$$

Using this we can write actions of the form

$$I = \int d^2xd^2\theta\, EL, \tag{2.62}$$

where L is a general real scalar superfield constructed out of covariant matter fields and derivatives. By construction, the transformations of L and E are such that the action is left invariant [70]. The first Lagrangian to consider is $L = 1$. In this case we have the action

$$\begin{aligned} I &= \int d^2xd^2\theta\, E \\ &= \int d^2x\, [\tfrac{i}{4}A + \tfrac{1}{8}\epsilon^{mn}\chi_m\gamma^5\chi_n], \end{aligned} \tag{2.63}$$

which would correspond to the cosmological constant in bosonic gravity. Here A is an as-yet uneliminated auxiliary field, so this action is on-shell. By itself this action only has trivial solutions but if added to other Lagrangians may yield nontrivial contributions. We will illustrate later how a cosmological constant can be added to a supergravity model. The next simplest case to consider is when $L = S$ (2.41), which corresponds to the supergravity of Howe [5]. For this choice we have the action

$$\begin{aligned} I &= \int d^2xd^2\theta\, ES \\ &= \int d^2x\, e(\tfrac{1}{2}C + \tfrac{i}{2}\chi_m\gamma^m\psi + [\tfrac{i}{4}A^2 + \tfrac{1}{8}\epsilon^{mn}\chi_m\gamma^5\chi_nA]) \\ &= \int d^2x\, \epsilon^{mn}\partial_m\omega_n. \end{aligned} \tag{2.64}$$

This is (at least locally) a total derivative and is well known to be a topological invariant, the Euler number $\chi(M)$. This topological invariant measures the inability to construct globally flat coordinates. In 2d, due to the symmetry properties of the Riemann tensor, it has only one independent component which we take as R_{1212}. With the following relations for the Ricci tensor and scalar;

$$\det(g)R_{mn} = g_{mn}R_{1212}, \quad \det(g)R = 2R_{1212}, \tag{2.65}$$

it is easy to see that the Einstein equations,

$$R_{mn} - \tfrac{1}{2}g_{mn}R \equiv 0, \tag{2.66}$$

are trivial, the Einstein tensor vanishes identically in 2d. *Any* metric is a solution to the vacuum Einstein equations. That the Einstein-Hilbert action in 2d is locally a total derivative follows [2];

$$\begin{aligned} \mathcal{L} = \sqrt{\det(g)}R &= \det(e)\, e^{ma}e_a{}^n R^p{}_{mpn} \\ &= \det(e)\, e^{ma}e^{nd}\epsilon_{ad}\, \partial_{[m}\omega_{n]} \\ &= \tfrac{1}{2}\det(e)\det(e^{-1})\, \epsilon^{mn}\, \partial_{[m}\omega_{n]} = \epsilon^{mn}\partial_m\omega_n, \end{aligned} \tag{2.67}$$

where $R^{ab}{}_{mn} = \epsilon^{ab}\partial_{[m}\omega_{n]}$, assuming that $D_m e_n{}^a = 0$. The spin-connection ω_m is defined by the Cartan structure equations (2.8) and in what follows we will be dealing with an $SO(2)$ connection. As the Einstein-Hilbert action (2.1) in 2d is a total derivative we would naively expect from Stokes' theorem that the action is just equal to zero on manifolds without boundary. As is well known, this is not the case. For compact, orientable 2d manifolds we have the Gauss-Bonnet theorem (see [3] for details),

$$\frac{1}{2\pi}\int K d\sigma = \chi(M), \tag{2.68}$$

where $K = \frac{-R_{1212}}{\det(g)}$ is the total curvature and $d\sigma = \sqrt{\det(g)}dx^1 \wedge dx^2$ is the oriented area element of the manifold. The Euler characteristic $\chi(M)$ only depends on the genus of the manifold and is given by $\chi(M) = 2 - 2h$, where h is the genus. Using the Gauss-Bonnet theorem (2.68) and (2.65) we easily see that

$$\int_M d^2x\sqrt{\det(g)}R = 4\pi\chi(M). \tag{2.69}$$

From this we can see that the above action (2.64) is proportional to the Euler number;

$$\begin{aligned} I &= \int d^2x\, \epsilon^{mn}\partial_m\omega_n \\ &= \int d^2x\, eR \sim \chi(M) \end{aligned} \tag{2.70}$$

and that Howe's 2d, $N = 1$ supergravity is trivial and reduces to 2d Einstein gravity. One should note that the spin-connection in the above Lagrangian is that of (2.48) and therefore contains gravitino terms. To see that these gravitino terms make no contribution to the Euler number it is convenient to decompose the Rarita-Schwinger field in the following Lorentz covariant way

$$\chi_m = \gamma_m\phi + \lambda_m, \tag{2.71}$$

where λ_m satisfies the condition, $\gamma \cdot \lambda = 0$. Furthermore, due to the two dimensional identity

$$\gamma^m\gamma^n\gamma_m = 0, \tag{2.72}$$

we can write λ_m as

$$\lambda_m = \gamma^n\gamma_m D_n\alpha, \tag{2.73}$$

for some spinor α (see [11] for details). Using the supergauge transformation (2.54), we can set α to zero. In this gauge the bosonic space torsion is zero and the gravitino terms in (2.48) drop out giving no contribution in the above action. It is this very gauge that is used in proving that all (2+2)-dimensional superspace are superconformally flat. A proof that the 'super Euler number' reduces to the ordinary one;

$$\chi(M) = \frac{i}{2\pi}\int d^2x d^2\theta\, ES = \frac{1}{2\pi}\int d^2x d^2\theta\, eR \tag{2.74}$$

regardless of which gauge is picked, can be found in [7].

Further supergravity models can be constructed along these lines by considering higher order terms in the supercurvature S [69]. Though these models often have the nice property of being non-trivial, they do not lend themselves to being formulated as constrained BF-theories. As such we will not discuss them here.

The final action we wish to discuss with an eye on formulating it as a BF-theory is the supersymmetric extension of the Jackiw-Teitelboim model [9, 46, 47, 10] studied by Chamseddine [8]. The Jackiw-Teitelboim model is obtained by dimensional reduction of the (2+1)-dimensional Einstein-Hilbert action leading to the two dimensional action

$$I = \frac{1}{4\pi}\int d^2x\, e\phi(R-\lambda), \tag{2.75}$$

where the scalar curvature R is equated to the cosmological constant λ through a Lagrange multiplier, the dilaton field ϕ. In two dimensions it is not possible to decouple the dilaton field. It turns out, using the superspace formalism of Howe, that the supersymmetric extension of the Jackiw-Teitelboim model is quite easy to obtain as one can simply replace the fields with the corresponding superfields. The supergravity action presented by Chamseddine is therefore

$$I = \frac{-i}{8\pi}\int d^2x d^2\theta\, E\Phi(S-\lambda'), \tag{2.76}$$

where the dilaton superfield Φ has the theta-expansion

$$\Phi = \phi + \theta^\alpha \Lambda_\alpha + i\bar{\theta}\theta F. \tag{2.77}$$

The role of Φ is to impose the constraint

$$S - \lambda' = 0, \tag{2.78}$$

which in component form is

$$\begin{array}{rcl} A & = & \lambda', \qquad \psi = 0, \\ C = 0 \quad \Rightarrow \quad R & = & \frac{i}{4}\epsilon^{mn}\bar{\chi}_m\gamma_5\chi_n A - \frac{1}{2}A^2. \end{array} \tag{2.79}$$

Integrating over θ and eliminating the auxiliary fields by their equations of motion, we recast the action in component fields,

$$I = \frac{1}{4\pi}\int d^2xe\Big\{\phi\Big(R+\tfrac{1}{2}\lambda'^2-\tfrac{i}{4}\lambda'\epsilon^{mn}\bar{\chi}_m\gamma^5\chi_n\Big)-2i\Lambda\Big(\epsilon^{mn}\gamma^5D_m\chi_n+\tfrac{1}{4}\lambda'\gamma^m\chi_m\Big)\Big\} \tag{2.80}$$

This is $N = 1$, 2d superfield supergravity in component form, in Wess-Zumino gauge, with $D_m\chi_n$ (2.50) and R (2.79) imposed, with residual local supersymmetries and Weyl invariance, derived from Chamseddine's supergravity action (2.76). If the Rarita-Schwinger fields are set to zero we recover the bosonic Jackiw-Teitelboim model with cosmological constant $\lambda = -\frac{1}{2}\lambda'^2$. In comparison, we see that Howe's form of supergravity (2.64) reduces to the Euler characteristic (2.70). Later we will show how these actions can be formulated as constrained BF-theories with $OSp(1|2)$ gauge group.

This first chapter began with a brief introduction to the hamiltonian formulation of general relativity in the ADM variables. In this formulation, the solutions to the hamiltonian constraints are incredibly difficult to solve. In fact, very limited progress could be made in the field of quantum gravity using these variables. But with the introduction of the Ashtekar-Sen connection variables, GR could be formulated as a canonical gauge theory. Though it seems that changing to these new variables increases the degrees of freedom, complicating the classical theory, the first class Gauss constraint removes them. Thus the phase space of the gauge theory and the ADM phase space are equivalent. What's more, by using the new variables and formulating GR as a gauge theory, one could use the large number of techniques that had been developed for the quantization of gauge theories to attempt to quantize GR. It is no surprise then, as we shall see in a later chapter, that loop quantum gravity is formulated using the Ashtekar-Sen variables. The next section introduced supersymmetry and more specifically supergravity. As has been mentioned a number of times before, supersymmetry is not a necessary requirement for loop quantum gravity to be consistent. Indeed, it must be put in by hand. Though the importance that supergravity plays in string theory and the possibility of a connection between string theory and loop quantum gravity suggest that the idea of supersymmetry in loop quantum gravity is one worth pursuing. Having presented for completeness the hamiltonian formulation of 2d supergravity, it was then formulated in superspace coordinates. Two models were presented, the first being the $N = 1$, 2d supergravity of Howe [5] and then second was Chamseddine's supersymmetric extension [8] of the bosonic Jackiw-Teitelboim model of 2d GR. In the next chapter we will discuss a topological field theory known as BF-theory and the corresponding super BF-theory and show how these two theories of supergravity can be expressed as BF-theories with certain constraints. Ultimately we wish to see how these classical constraints may be imposed at the quantum level in the context of LQG.

Chapter 3

BF-theory

Topological field theories offer an exciting possibility to combine ideas from physics and mathematics. They are quantum field theories with no physical degrees of freedom and their properties are fully determined by the global structure of the manifold they are defined on. A remarkable feature is that for many topological theories, like Donaldson theory and Chern-Simons (CS) theory, the expectation values of the observables are topological invariants. In the case of Chern-Simons theory, this provides a three dimensional interpretation of the theory of knots. The Wilson loops in the theory are closely related to the Jones Polynomials of knot theory [71]. Another important application of CS theory is in three dimensional gravity. With the Poincaré group as the gauge group, the CS action is the Einstein-Hilbert action [21], giving a gauge theory formulation of gravity in three dimensions. However, Chern-Simons theory is defined only in three dimensions. The generalization to arbitrary dimensions gives rise to BF-theory. These BF-theories, like CS theory, describe the moduli space of flat connections.

This chapter begins with a discussion of the general properties of BF-theory, which is a Schwarz type topological gauge theory. Defined on spacetimes of any dimension, these theories are topological and hence background independent, needing no pre-existing metric or any other such geometric structure on spacetime in order to formulate them. Having presented the basics of BF-theory we will show how 2d GR can be formulated as a BF-theory with the addition of some constraints. These constraints have an important role to play and will be examined in more depth later in the chapter. Following this we will proceed to introduce the supersymmetric extension of BF-theory [72, 73, 74], which is a Witten type topological theory. Deviating from the usual procedure in the literature, the 'super' BF-theory will be formulated with supergroups. Two different 2d supergravity models will be presented as super BF-theories with the addition of constraints on the action.

3.1 Schwarz type topological gauge theories

Topological field theories (for an excellent review see [75]) can be classified into two types, both of which we will discuss in this chapter including some well-known examples. The first type that we shall discuss are the Schwarz type [76]. These theories are defined by their quantum action being given by (in general)

$$S_q(\Phi, g) = S_c(\Phi) + \{\mathcal{Q}, V(\Phi, g)\}, \tag{3.1}$$

where $S_c(\Phi)$ is a metric independent classical action which is not a total derivative and depends on the field content Φ which contains gauge fields, ghosts and lagrange multipliers. $\mathcal{Q}$ is the BRST operator corresponding to the local gauge symmetry of the theory and $V(\Phi, g)$ is some field and metric dependent functional. As the classical action is metric independent the classical energy-momentum tensor is zero. Defining the complete energy-momentum tensor $T_{\alpha\beta}$ by the change in the action under infinitesimal deformations of the metric

$$\frac{\delta S_q}{\delta g_{\alpha\beta}} = \frac{1}{2}\int \sqrt{g}\, T_{\alpha\beta}; \tag{3.2}$$

if (3.1) holds then

$$T_{\alpha\beta} = \left\{\mathcal{Q}, \tfrac{2}{\sqrt{g}}\tfrac{\delta V}{\delta g^{\alpha\beta}}\right\}. \tag{3.3}$$

It follows from this that the partition function is metric independent. Examples of theories that satisfy the above properties include Chern-Simons theory in 3d, Abelian BF theory and 2 and 3 dimensional non-Abelian BF models. In the case of non-Abelian BF models, when the dimension $n > 3$, even starting with a metric independent classical action, properties (3.1) and (3.3) do not hold and further work is required to show that the resulting quantum theory is indeed topological.

3.1.1 BF-theory basics

In general, BF-theory can be defined on space-times of any dimension. As such, in this discussion on the generic properties of BF-theory, we take space-time to be any n-dimensional oriented, smooth manifold M. The gauge group G can be any Lie group whose Lie algebra $\mathfrak{g}$ is equipped with an invariant, non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. We choose a principal G-bundle P with base space M and the fields of the theory are a connection A on P and an ad(P)-valued $(n-2)$-form B on M. The curvature of A is the ad(P)-valued two-form F. Here, ad(P) is the vector bundle associated with P through the adjoint action of G on its Lie algebra. Choosing a local trivialization we can think of A, B and F as a $\mathfrak{g}$-valued one-form, $(n-2)$-form and two-form respectively. The Lagrangian density formed from these fields is

$$L = \langle B, F \rangle, \tag{3.4}$$

which is an n-form, by taking the wedge product of the differential parts of B and F and the Lie algebra parts are paired using the bilinear form. When the gauge group G is semisimple the bilinear form can be taken as the Killing form $\langle x, y \rangle = tr(xy)$, where the trace is taken in the adjoint representation. The equations of motion derived from this Lagrangian by varying with respect to the connection A and the B-field are respectively

$$\begin{aligned} dB(x) &= 0, \\ F(x) &= 0. \end{aligned} \tag{3.5}$$

These equations of motion are solved by any constant B-field and flat connection and locally, all solutions are equivalent. This is clearly true for the connection as all flat connections locally, are equivalent up to gauge transformations. To see that the B-field solutions to (3.5) are equivalent it is necessary to observe that the BF-action is invariant under another symmetry. This additional symmetry is

$$A \to A, \qquad B \to B + d_A \xi, \tag{3.6}$$

for some $(n-3)$-form ξ and where d_A is the exterior covariant derivative. These symmetries are gauge symmetries in the sense that any two solutions differing by this symmetry should be considered as physically equivalent. Due to the fact that locally all closed forms are exact, it follows that if A is flat, any B-field which satisfies the constraint (3.5) can be written locally as $d_A\xi$. The solutions to the equations of motion are equivalent locally and as the classical theory has no local degrees of freedom it is topological and of the form (3.1) with $V = 0$.

3.1.2 2d GR as a constrained BF-theory

With the basics of BF-theory presented previously, we wish to consider in this section the points of contact and difference between 2d GR and 2d BF-theory. It will be shown how GR can be formulated as a BF-theory with a constraint applied to the B field. As was mentioned, these constraints play a vital role. Recall that BF-theories are topological and have zero physical degrees of freedom, yet GR does (at least in dimensions greater than two). These constraints 'free up' degrees of freedom as we will explain later.
BF-theory in 2d is defined by the action

$$I = \frac{-k}{2} \int tr(B \wedge F), \tag{3.7}$$

where B is a Lie algebra valued scalar field and F is the Lie algebra valued two-form curvature of the spin connection ω. In the particular case of 2d GR we consider an Abelian $SO(2)$ connection on the frame bundle. Using the matrix representation of the $so(2)$ generator

$$\tau^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we can write the curvature as $F^{\alpha}{}_{\beta} = f\tau^{\alpha}{}_{\beta}$, where $f = d_A\omega$ is a real valued two-form. Letting $B = b\tau$, the action (3.7) becomes

$$S_{BF} = k \int bf, \tag{3.8}$$

where we have used $tr[\tau\tau] = -2$. The equations of motion are the standard BF equations; the B-field is covariantly constant and the spin-connection is required to be flat. Like 2d BF-theory, 2d GR has no local degrees of freedom and is topological. The curvature term in the action is a total derivative and the action is proportional to the Euler characteristic $\chi(M)$ [77] (c.f. 2.70). However it differs to BF-theory in that the equations of motion of 2d GR are solved by any spin-connection ω, not necessarily one that is flat. We can follow the construction of 4d GR as a constrained BF-theory [51] to get the analogous result in the 2d case. By using a two-form Lagrange multiplier ϕ, we add a constraint to the original BF-theory (3.7)

$$\begin{aligned} I &= \frac{-k}{2}\int tr(B\wedge F - \phi(B^2+\mathbf{1})) \\ &= k\int (bf - \phi(b^2-1)). \end{aligned} \tag{3.9}$$

Now the equations of motion obtained by varying ω, B and ϕ, respectively, are

$$\begin{aligned} d_A b &= 0, \\ f\tau &= 2\phi(x) b\tau, \\ -\tfrac{1}{2}tr[b\tau b\tau] &= 1. \end{aligned} \tag{3.10}$$

As $\phi(x)$ is arbitrary and free to take on any value we see that the connection ω is no longer required to be flat. In fact it can be any value as is the case for 2d GR. Equation (3.10) is the 2d analogue of the Plebanski constraint which we will discuss in detail in Chapter 5. Substituting this constraint back into the action (3.7) and picking $k = \frac{1}{8\pi G}$, we obtain

$$S_{GR} = \frac{sgn(e)}{8\pi G}\int f, \tag{3.11}$$

which is just the 2d Einstein-Hilbert action,

$$I = \frac{1}{16\pi G}\int d^2x\,\sqrt{detg}R, \tag{3.12}$$

in the 'dyad' formalism, where $sgn(e)$ is the determinant of the dyad. Note that the two signs of $\sqrt{\det g}$ correspond to the two solutions, $B = \pm\tau$, of the Plebanski constraint. In 4d the addition of the Plebanski constraint enhances the number of degrees of freedom. This is due to the constraint constraining a constrainer, the Lagrange multiplier B. Here though, the situation is slightly different. The constraint enlarges the space of classical

solutions of the equations of motion. This can easily be seen as the curvature (3.10) is no longer zero but arbitrary. But this does not increase the number of degrees of freedom because of gauge equivalence. In the 4d case, the temporal components of the B field are Lagrange multipliers while in 2d the B field is simply the conjugate variable to the spatial component of the connection. So the constraint does not constrain a multiplier and no additional degrees of freedom arise.
As is expected the 2d case is much simpler than in the 4d case, which will be presented later. Now we will extend BF-theory to include supersymmetry by using superfields. After introducing the basics, two supergravity models will be formulated as super BF-theories with constraints.

3.2 Super BF-theory

As we have seen, BF-theory is a topological theory of Schwarz type. This is not the case for the supersymmetric extension. The super BF-theory is a Witten type topological field theory [75]. This is a quantum field theory described by a BRST exact quantum action

$$S_q(\Phi, g) = \{Q, V(\Phi, g)\}, \tag{3.13}$$

where Φ represents the fields in the theory and g is the metric of the underlying manifold M. Q is the BRST operator corresponding to the local gauge symmetries and V is some field and metric dependent functional. The relevant BRST transformations are determined from the symmetries one wishes to study. Witten type field theories are in a sense a model that enables one to study the moduli space $\mathcal{M}$ of the theory under consideration. The moduli space is defined as the space of fields which are solutions to certain equations, modulo the symmetries one wishes to study. Roughly we have

$$\mathcal{M} = \frac{\{\Phi \in S | D\Phi = 0\}}{\{\text{symmetries}\}}, \tag{3.14}$$

where S denotes the space of fields and D is some appropriately defined operator acting on the fields. Generally, $\mathcal{M}$ will be finite and the theory has a finite number of degrees of freedom. Now we will proceed to discuss the details of super BF-theory formulated using supergroups.

3.2.1 2d supergravity as a constrained BF-Theory

One can extend the formulation of 2d gravity, described in the previous section, to the supersymmetric case of supergravity in a super BF formulation. This is achieved by extending the symmetry group from $SO(2)$ to the supergroup $OSp(1,2)$. One can then introduce the familiar B field of BF-theory as a scalar field taking values in the corresponding Lie superalgebra. Similarly, from the connection A the supercurvature F, a Lie superalgebra valued two-form field strength can be calculated. With these components, the action can be constructed and takes the form

$$I = \int_M Str(B \wedge F), \tag{3.15}$$

where one should note that Str stands for the $OSp(1,2)$ invariant bilinear form which is unique up to an overall constant factor (see Appendix for details) and the manifold M has the standard structure $M^2 \simeq \mathbb{R} \times \Sigma^1$. The equations of motion mimic the standard case;

$$\begin{aligned} F &= 0, \\ d_A B &= 0, \end{aligned} \tag{3.16}$$

with the curvature being equal to zero and the super Gauss constraint imposing the usual $SU(2)$ gauge invariance and the additional supersymmetric constraint. We will now illustrate this procedure in detail by presenting the supersymmetric extension of the Jackiw-Teitelboim model of 2d gravity and the 2d model of Howe.

3.2.2 The supersymmetric Jackiw-Teitelboim model

It has been known for some time that the (1+1)-dimensional gravity model of Jackiw and Teitelboim has a topological and gauge invariant formulation [13]. It is also known [12] that the supersymmetric extension of the Jackiw-Teitelboim model (2.76) can be formulated in the framework of a topological super BF-theory. This formulation is based on the two dimensional graded de Sitter group, the supergroup $OSp(1,2)$[1];

[1]See Appendix for details.

$$
\begin{aligned}
&[P_a, P_b] = -\tfrac{1}{4}\lambda'^2\epsilon_{ab}J, \qquad [P_a, J] = \epsilon_a{}^b P_b,\\
&[Q_\alpha, J] = \tfrac{1}{2}(\gamma^5)_\alpha{}^\beta Q_\beta, \qquad [P_a, Q_\alpha] = \tfrac{1}{4}\lambda'(\gamma_a)_\alpha{}^\beta Q_\beta,\\
&\{Q_\alpha, Q_\beta\} = -2i(\gamma^a)_{\alpha\beta}P_a + i\lambda'(\gamma^5)_{\alpha\beta}J,
\end{aligned} \tag{3.17}
$$

with the parameter λ'^2 taking on positive, vanishing or negative values (note that the cosmological constant is $\lambda = \frac{1}{2}\lambda'^2$, not λ'). The graded invariant quadratic form consistent with a non-degenerate Casimir operator (except for $\lambda' = 0$ corresponding to the supersymmetric extension of $ISO(2)$. For details on this case see [75]) is;

$$
\langle P_a, P_b\rangle = \eta_{ab}, \quad \langle J, J\rangle = 4/\lambda'^2, \quad \langle Q_\alpha, Q_\beta\rangle = -(8i/\lambda')\epsilon_{\alpha\beta}, \tag{3.18}
$$

with all other relations equal to zero. Using this we can evaluate the BF action

$$
I = \frac{1}{4\pi}\int Str(B\wedge F), \tag{3.19}
$$

where $F = dA + A\wedge A$ is the Lie superalgebra (3.17) valued, two-form field strength associated with the gauge field

$$
A_m = e_m{}^a P_a - \omega_m J + \tfrac{1}{2}\chi_m{}^\alpha(\gamma^5)_\alpha{}^\beta Q_\beta \tag{3.20}
$$

and

$$
B = b^a P_a + bJ + b^\alpha Q_\alpha \tag{3.21}
$$

is a scalar field, again taking values in the above Lie superalgebra. Calculating the curvature with respect to the connection A_m (3.20), we have

$$
\begin{aligned}
F_{mn} &= F^a_{mn}P_a + f_{mn}J + F^\alpha_{mn}Q_\alpha \qquad (3.22)\\
&= (\partial_m e_n{}^a - \epsilon^a{}_b\omega_m e_n{}^b - \tfrac{i}{4}\chi_m{}^\alpha(\gamma^a)_\alpha{}^\beta\chi_{n\beta})P_a\\
&+ (-\partial_m\omega_n - \tfrac{1}{8}\lambda'^2\epsilon_{ab}e_m{}^a e_n{}^b - \tfrac{i}{8}\lambda'\chi_m{}^\alpha(\gamma^5)_\alpha{}^\beta\chi_{n\beta})J\\
&+ (D_m\chi_n{}^\beta(\gamma^5)_\beta{}^\alpha - \tfrac{1}{8}\lambda' e_n{}^b\chi_m{}^\beta(\gamma^5)_\beta{}^\gamma(\gamma_b)_\gamma{}^\alpha)Q_\alpha
\end{aligned}
$$

and using the scalar superfield B and doing a little rearranging we find the component form of the action (3.19) is,

$$
I = \frac{1}{4\pi}\int \; d^2x \;\; \Big\{b_a\epsilon^{mn}\Big(\partial_m e_n{}^a - \epsilon^a{}_b\omega_m e_n{}^b - \tfrac{i}{4}\chi_m{}^\alpha(\gamma^a)_\alpha{}^\beta\chi_{n\beta}\Big)
$$

$$
\begin{aligned}
- & \tfrac{4}{\lambda'^2} b \epsilon^{mn}\Big(\partial_m \omega_n + \tfrac{1}{8}\lambda'^2 \epsilon_{ab} e_m{}^a e_n{}^b + \tfrac{i}{8}\lambda' \chi_m{}^\alpha (\gamma^5)_\alpha{}^\beta \chi_{n\beta}\Big) \\
+ & \tfrac{4i}{\lambda'} b_\alpha \Big(\epsilon^{mn} D_m \chi_n{}^\beta (\gamma^5)_\beta{}^\alpha + \tfrac{1}{4} e \lambda' \chi_m{}^\beta (\gamma^m)_\beta{}^\alpha\Big)\Big\}. \qquad (3.23)
\end{aligned}
$$

The role of the scalar superfield B, as in any BF-theory, is to impose the constraint $F_{mn} = 0$. If we integrate out the B-field we should recognize that the condition imposed by b_a, that is $F_{mn}{}^a = 0$, is nothing but the standard torsion condition of supergravity, $T_{bc}{}^a = 0$ (2.40). Solving this constraint we can write the spin-connection ω_m in terms of $e_m{}^a$ and $\chi_m{}^\alpha$ and recover the result (2.48)

$$
\omega_m = -e_{am}\epsilon^{nl}\partial_n e_l{}^a + \tfrac{1}{2}\bar{\chi}_m \gamma_5 \gamma^n \chi_n. \qquad (3.24)
$$

The other conditions, $f_{mn} = 0$ and $F_{mn}{}^\alpha = 0$, imposed by integrating out b and b_α, are exactly those for constant supercurvature. Specifically,

$$
\begin{aligned}
F_{mn}{}^\alpha = 0 \quad &\Rightarrow \quad \psi = 0, \\
f_{mn} = 0 \quad &\Rightarrow \quad C = 0 \qquad (3.25)
\end{aligned}
$$

and we have the supercurvature (2.41) being a constant

$$
S = A, \qquad (3.26)
$$

where we have identified A with λ' through its equation of motion. This restriction to constant supercurvature is the same as that imposed by the dilaton superfield (2.77) in the supersymmetric Jackiw-Teitelboim model (c.f. (2.78)). To make the connection between the above BF-theory formulation using the supergroup $OSp(1,2)$ and the supersymmetric extension of the Jackiw-Teitelboim action (2.76) complete, note that if we identify the component fields of the superfield B as $b = \lambda'^2\phi/2$ and $b^\alpha = -\lambda'\Lambda^\alpha/2$ (where b^α is Grassmann valued to avoid the vanishing of $b^\alpha b^\beta \epsilon_{\alpha\beta}$) and substitute the relation for the spin-connection (2.48) back into the action above we recover the action of Chamseddine (2.80).

The final thing we wish to mention about this gauge formulation of the supersymmetric Jackiw-Teitelboim model is the relation between supergauge transformations and local Lorentz, diffeomorphism and $N = 1$ supersymmetry transformations. If we look at the supergauge transformation

$$
\delta_\epsilon A_m = D_m \epsilon = \partial_m \epsilon + [A_m, \epsilon], \qquad (3.27)
$$

with gauge parameter $\epsilon = \eta^a P_a + \ell J + \zeta^\alpha Q_\alpha$, we eventually get

$$\begin{aligned}
\delta e_m{}^a &= \partial_m \eta^a + e_m{}^b \epsilon^a{}_b \ell + i\bar{\zeta}\gamma^a \chi_m, \\
\delta \omega_m &= \partial_m \ell - \tfrac{1}{4}\lambda'^2 \epsilon_{ab} e_m{}^a \eta^b - \tfrac{i}{2}\lambda' \bar{\zeta}\gamma^5 \chi_m, \\
\delta \chi_m &= 2D_m \zeta - \tfrac{1}{2}\lambda' \gamma_m \zeta + \tfrac{1}{4}\ell \gamma_5 \chi_m - \tfrac{1}{8}\lambda' \chi_m \gamma^5 \gamma_a \eta^a.
\end{aligned} \tag{3.28}$$

If we identify the transformations involving ζ and ℓ with $N = 1$ supersymmetry and local Lorentz transformations respectively we clearly see that they correspond to the transformations discovered by Howe [5] (c.f. (2.54), (2.55) and (2.57)), where we identify λ' with the scalar auxiliary field A. The transformations with parameter η^a can be identified with diffeomorphisms up to local Lorentz and supersymmetry transformations in the constant supercurvature geometry.

3.2.3 Howe's supergravity

Starting from the $OSp(1,2)$ BF-theory action (3.19) we saw, once written out in components, that we could recover Chamseddine's supersymmetric extension of the Jackiw-Teitelboim model (2.80) after redefining the B-field and substituting in the solution for the spin-connection. Is it possible to recover Howe's action (2.64) through the imposing of constraints and field redefinitions? Starting from the BF-theory action we can reduce it down to

$$I = \frac{-1}{4\pi}\int d^2x \, \tfrac{4}{\lambda'^2} b \epsilon^{mn} \Big(\partial_m \omega_n + \tfrac{1}{8}\lambda'^2 \epsilon_{ab} e_m{}^a e_n{}^b + \tfrac{i}{8}\lambda' \chi_m{}^\alpha (\gamma^5)_\alpha{}^\beta \chi_{n\beta}\Big) \tag{3.29}$$

by substituting in the solutions from the constraints imposed by b_a and b_α, that is, (2.48) and $\psi = 0$. If we now put $b = -\lambda'^2/4$, we get

$$I = \frac{1}{4\pi}\int d^2x \, \epsilon^{mn} \Big(\partial_m \omega_n + \tfrac{1}{8}\lambda'^2 \epsilon_{ab} e_m{}^a e_n{}^b + \tfrac{i}{8}\lambda' \chi_m{}^\alpha (\gamma^5)_\alpha{}^\beta \chi_{n\beta}\Big), \tag{3.30}$$

which in superfield notation is

$$I = \frac{1}{4\pi}\int d^2x d^2\theta \, E(S - \lambda'). \tag{3.31}$$

Therefore in order to recover Howe's supergravity action it is necessary to take the limit $\lambda' \to 0$, which is the super analogue of the standard conformal-Poincaré contraction. To briefly summarize, the superfield action (3.31) is equivalent to the $OSp(1,2)$ super BF-theory action (3.19), only in the special case of $b = \frac{1}{4}\lambda'^2$, which only agrees with Howe when $\lambda' \to 0$. However, starting with the super BF action (3.19), one may also recover Chamseddine's supergravity action (2.80) with the identification $b = \lambda'^2\phi/2$ and $b^\alpha = -\lambda'\Lambda^\alpha/2$ and the elimination of the auxiliary field A. Thus, both Howe's and Chamseddine's supergravity, though not equivalent, can both be derived from super BF-theory under the appropriate conditions.

In taking the limit $\lambda' \to 0$, it is important to note that the supersymmetry transformations generated by supergauge transformations no longer close off-shell. Also we are still restricted to the case of constant super curvature, i.e $\psi = C = 0$ and $S = A = \lambda'$. Howe's supergravity is not restricted to this situation and allows arbitrary supercurvature. To achieve this, rather than simply putting $b = -\lambda'^2/4$, we will add an algebraic constraint to the BF-theory action such that this particular value of b is a solution. A possibility is to add the constraint through a Lagrange multiplier two-form Σ,

$$\Sigma(B^2 - \tilde{\Lambda}). \tag{3.32}$$

Adding this constraint to the BF-action we now have the constrained action

$$I = \int_M Str(BF + \Sigma(B^2 - \tilde{\Lambda})). \tag{3.33}$$

Mimicking the 2d bosonic case, varying the action with respect to the Lagrange multiplier gives the result

$$B^2 = \tilde{\Lambda}. \tag{3.34}$$

It is indeed possible to choose $\tilde{\Lambda}$ in such a way that the solutions to this constraint, in component form, are

$$\begin{aligned} b_a &= 0, \\ b &= \pm\frac{\lambda'^2}{4}, \\ b_\alpha &= 0. \end{aligned} \tag{3.35}$$

To see this more explicitly, note that the fundamental representation of $OSp(1,2)$ is three dimensional. The Lie superalgebra is then generated by five 3×3 matrices which are given explicitly in the Appendix. Expressing the scalar field B and $\tilde{\Lambda}$ as 3×3 graded matrices one can see that the entries can be chosen in such a way that the above constraints hold. So, from the super BF-action with the added constraint, we have the following equations of motion by varying with respect to the connection ω_m, scalar field B and Lagrange multiplier $\tilde{\Lambda}$ respectively;

$$\begin{aligned} dB &= 0, \\ F &= \Sigma B, \\ B^2 &= \tilde{\Lambda}. \end{aligned} \tag{3.36}$$

One can clearly see that as the Lagrange multiplier Σ is arbitrary, the field strength no longer has to be zero and the supercurvature can be nonzero which is the case in the Howe model. Thus starting with the constrained BF-action, imposing the constraint $B^2 = \tilde{\Lambda}$ on the B field, with $\tilde{\Lambda}$ chosen in such a way that the values (3.35) of the B field hold, then once again the action (3.30) can be recovered without the restriction that the supercurvature be zero. This however is not yet the action of Howe. As was pointed out before, the limit $\lambda' \rightarrow 0$ must be taken. But in doing this the Lie superalgebra reduces to the supersymmetric extension of $ISO(2)$ which has no non-degenerate invariant bilinear form. Despite this, it is still possible to construct an invariant super BF-action for this group. This is achieved by rescaling the components of the connection and scalar field B as follows [78],

$$\begin{aligned} e^a_m &\rightarrow ke^a_m, \quad \omega_m \rightarrow k\omega_m, \quad \chi_m \rightarrow \sqrt{k}\chi_m, \\ b_a &\rightarrow \frac{1}{k}b_a, \quad b \rightarrow \frac{1}{k}b, \quad b_\alpha \rightarrow \frac{1}{\sqrt{k}}b_\alpha \end{aligned} \tag{3.37}$$

and taking the limit $k \rightarrow 0$.

The idea of a quantum field theory with no physical degrees of freedom and whose properties are fully determined by the global structure of the manifold they are defined on may seem boring. Despite that these topological fields often have the feature of describing topological invariants which is worthy of study in itself, we saw in this chapter that these theories make direct contact with physics. A particular example of this kind of theory is Chern-Simons theory, which provides a description of 3d GR, and its generalization to arbitrary dimensions in the form of BF-theories. After discussing the properties and construction of the Schwarz type BF-theories we demonstrated how 2d GR can be formulated as a BF-theory with a constraint term added to the action. This constraint had the effect of placing a restriction on the scalar field in the theory and substituting this constraint back into the BF-action we arrived at the 2d GR action. This procedure directly follows the 4d case which will be discussed in Chapter 5. Extending this method to include supersymmetry by replacing the gauge group with a supergroup led us to consider the super BF-theories. Explicitly, using the gauge supergroup $OSp(1,2)$ two supergravity models were formulated as super BF-theories. For the second of these models, that of Howe, it was shown that by copying the procedure for 2d gravity by adding a constraint term to the action, certain restrictions could be imposed on the scalar field B. When these constraints were put back into the BF-action, the recovery of Howe's supergravity action was possible only in the limit of λ' going to zero. Though this resulted in the reduction of the gauge group to the supersymmetric extension of $ISO(2)$, which does not have a non-degenerate bilinear form, it was shown that a super BF-action could still be constructed with the appropriate rescaling of the component fields of the theory.

Chapter 4

Loop Quantum Gravity

Standard quantum field theory provides an excellent unification of the principles of quantum mechanics and special relativity. The standard model and its ability to describe the known particles and their interactions is a perfect example of a quantum field theory on the fixed background geometry of Minkowski spacetime. As such, the standard model can only be considered as an approximation of the description of fundamental interactions when gravity is negligible, as in the lab. Using the techniques of quantum field theory on curved spacetime would extend the domain of applicability of the standard model to situations where a nontrivial, but weak gravitational field is present. Such situations would occur when the spacetime curvature is small compared with the Plank scale. In the case of strong gravitational fields, such as at singularities, it appears that a full theory of quantum gravity is required. The principles of GR and quantum mechanics must be unified into one consistent theory. Considering the gravitational interaction, one can see that it is fundamentally different from the other known forces. The degrees of freedom of the gravitational field are encoded in the spacetime geometry. The spacetime geometry is fully dynamical and the notion that space is absolute and a 'stage' on which things happen no longer makes sense. The gravitational field defines the geometry on which its own degrees of freedom and those of matter fields propagate. Looking at the lessons of GR, implementing the principles of general covariance completely cuts out the concept of absolute space. According to Einstein the world is relational. There is no absolute space and it only makes sense to describe physical entities in relation to other physical entities. Unfortunately the consequences of this view point in quantum physics are yet to be understood. In fact, when dealing with quantum gravity, the standard procedure to date has been *perturbative* quantum gravity. In standard perturbative approaches one attempts to describe gravitational interactions using the same techniques used in the standard model. These techniques require a nondynamical background which arbitrarily separates the degrees of freedom of the gravitational field into two terms. The first term represents a background geometry $\eta^{(bg)}_{\mu\nu}$ which is fixed and the second term represents dynamical metric fluctuations $h_{\mu\nu}$. The spacetime metric is given as

$$g_{\mu\nu} = \eta^{(bg)}_{\mu\nu} + h_{\mu\nu}. \tag{4.1}$$

However this strategy leads to a number of significant difficulties. Firstly, conventional perturbative QFT of GR based on the above split metric leads to a non-renormalizable theory. To get rid of these ultraviolet divergences one must resort to string theory. Furthermore, as was discussed previously LQG shows that at the Planck scale the structure of spacetime is discrete. Physical spacetime possibly has no short-distance structure at all. The assumption of a smooth, fixed background $\eta^{(bg)}_{\mu\nu}$ implicit in (4.1) may be precisely the cause of the ultraviolet divergences. Finally the idea of fixing part of the metric directly contradicts the physical lessons of GR. If we are to take seriously the ideas of GR and use them to guide us towards a theory of quantum gravity, the relevant spacetime geometry is the one determined by the full gravitational field $g_{\mu\nu}$. Loop quantum gravity is an attempt to define a quantization of gravity that is background independent and therefore also non perturbative. LQG is based on the idea that (4.1) is not appropriate for describing the quantum properties of spacetime. The inputs to the theory are just QM and GR. No other additional physical assumptions, such as extra dimensions or supersymmetry, are required. The theory is based on the Hamiltonian quantization of GR in the Ashtekar-Sen variables which was presented in Chapter 2. In terms of these variables GR is formulated as a background independent $SU(2)$ gauge theory.

In this chapter we will present a standard strategy for the canonical quantization of GR, ultimately leading to the description of a Hilbert space of states and the kinematics of the theory. In order to implement any quantization procedure a choice has to be made, which quantities to promote to quantum operators. In QFT, this quantization leads to the creation and annihilation operators a and $a^\dagger$. What characterizes LQG is the choice of a different algebra, based on the holonomies of the gravitational connection. The holonomy, which is the matrix of the parallel transport along a curve [1], will be presented in this chapter. It is from these holonomies that state functions can be defined in LQG and the kinematic Hilbert space can be described as will be demonstrated.

4.1 Canonical quantization of GR

As was shown in the previous chapters, GR can be formulated as a constrained hamiltonian system. The quantization of generally covariant systems was first considered by Dirac [49] and consists of the following generalized steps [26, 50]:

(i) Define the classical phase space $(\mathsf{M}, \{.,.\})$, including a set of first class constraints and (possibly) a hamiltonian. In order to quantize the phase space, one must choose a submanifold $\mathcal{C}$ of M called the configuration space, the coordinates of which have vanishing Poisson brackets amongst themselves. We will assume that M is a cotangent bundle $\mathsf{M} = T^*\mathcal{Q}$ and it is natural to take $\mathcal{C} = \mathcal{Q}$.

(ii) Find a representation of the phase space variables as linear operators on some kinematical Hilbert space $\mathcal{H}_{kin}$ satisfying the commutation relations $\{\ ,\ \}_{PB} \to \frac{-i}{\hbar}[\ ,\ ]$. Specifically, consider the spaces $C^\infty(\mathcal{C})$ and $V^\infty(\mathcal{C})$ of smooth functions and vectors fields on $\mathcal{C}$ respectively. The pair $C^\infty(\mathcal{C}) \times V^\infty(\mathcal{C})$ form a Poisson-Lie algebra defined by the relation $[(f,v),(f',v')] = (v[f'] - v'[f], [v,v'])$, where $v[f]$ denotes the action of a vector field on a function. The space of fibre coordinates of M, called momentum space, generates preferred elements of $V^\infty(\mathcal{C})$ through $(v_p[f])(q) := (\{p,f\})(q)$, where q,p are configuration and momentum coordinates respectively. The elements (f, v_p)

[1] In LQG, the holonomy is defined on a curve, not necessarily closed, which is the usual definition.

form a subalgebra $\mathcal{B}$ of the Lie algebra $C^\infty(\mathcal{C}) \times V^\infty(\mathcal{C})$. The subalgebra $\mathcal{B}$ is closed in the sense that for every element $(f'', v_{p''}) \in \mathcal{B}$ the relation $[(f, v_p), (f', v_{p'})] = (f'', v_{p''})$ holds. One then wishes to find all irreducible representations $\pi : \mathcal{B} \longrightarrow \mathcal{L}(\mathcal{H}_{kin})$ of $\mathcal{B}$ as linear operators on $\mathcal{H}_{kin}$, such that the commutation relations

$$\begin{aligned} \pi([a,b]) &= \tfrac{-i}{\hbar}[\pi(a), \pi(b)], \\ \pi(a^*) &= \pi(a)^\dagger \end{aligned} \tag{4.2}$$

are implemented for all $a, b \in \mathcal{B}$ and where the $*$ and $\dagger$-relations are complex conjugation and the adjoint operation respectively.

(iii) Constraints are promoted to self-adjoint operators in $\mathcal{H}_{kin}$. For GR these are the vector, scalar and Gauss constraints (2.12). The number of representations which support the constraints and the hamiltonian as operators is usually limited if at all possible. Typically the constraints are not in $\mathcal{B}$ and corresponding operators will involve ordering ambiguities. However it is usually possible to find a domain $\mathcal{D}_{kin} \subset \mathcal{H}_{kin}$ on which all the operators and their adjoints are defined and which they leave invariant.

(iv) Find the space of solutions to the quantum constraints. Generally, solutions to the constraints do not lie in $\mathcal{H}_{kin}$, but are in the space of linear functionals on $\mathcal{D}_{kin}$. One must then define the appropriate inner product which defines the notion of physical probability and leads to the Hilbert space $\mathcal{H}_{phys}$ of physical states.

(v) Finally one must find a complete set of operators, which along with their adjoints, commute with all the constraints. These represent the gauge invariant observables.

4.2 The Loop program

In this section a more formal definition of LQG will be presented following the above steps of Dirac's method for the quantization of generally covariant theories. For convenience we will deal with the quantization of $3+1$ dimensional GR, invariant under internal $SU(2)$ gauge transformations and $3d$ diffeomorphisms as this is the most physically relevant theory and the one most commonly found in the literature.

4.2.1 Classical phase space and Poisson algebra $\mathcal{B}$

The first step in quantizing GR was completed previously in Chapter 2. The phase space M is coordinatized by the new Ashtekar-Sen variables (A_i^a, E_a^i), where A_i^a is the $SU(2)$ connection (2.9) over the $3d$ hypersurface Σ and E_a^i is the $su(2)$-valued densitized triad (2.7). There Poisson brackets are given by

$$\{E_a^i(x), A_j^b(y)\} = \delta_j^i \delta_a^b \delta(x,y), \tag{4.3}$$

with all others being zero. Clearly, it is natural to take the space of smooth $SU(2)$ connections over Σ as the configuration space $\mathcal{C} = \mathcal{A}$. Next we need to define $\mathcal{C}^\infty(\mathcal{A})$ the space of smooth functions on the connection. In order to ensure that the notion of differentiability is well defined for elements of $\mathcal{C}^\infty(\mathcal{A})$ and that they transform nicely under $SU(2)$ gauge transformations we will introduce the holonomy of a connection [50, 51], which has these properties.

Given a one dimensional orientated curve $e : [0,1] \longrightarrow \Sigma$ mapping the parameter $s \in [0,1] \longrightarrow x^m(s)$, the holonomy (or parallel propagator) $h_e(A) \in G$ (in this case specifically $SU(2)$) is denoted by

$$h_e(A) = \mathcal{P}\exp \int_0^1 ds\ \dot{e}^m(s)\, A_m^a(e(s))\, \tau_a \equiv \mathcal{P}\exp \int_e A, \tag{4.4}$$

where $\dot{e}^m(s) = dx^m(s)/ds$ is the tangent to the curve, τ_a is a basis of the Lie algebra of the group G and $\mathcal{P}$ stands for a path ordered exponential. More precisely, the holonomy as given above is the unique solution to the ordinary differential equation

$$\tfrac{d}{ds} h_e[A,s] + \dot{e}^m(s)\, A_m\, h_e[A,s] = 0, \tag{4.5}$$

with the boundary condition $h_e[A,0] = 1$ and

$$h_e(A) = h_e[A,1]. \tag{4.6}$$

The connection A is a rule which defines the meaning of parallel-transporting a vector in a representation R of G from on point to another nearby one. Along the curve e, a vector v is parallel-transported to the vector $R(h_e(A))v$. There are some important properties of the holonomy worthy of mention. Firstly, the definition of the holonomy $h_e(A)$ is independent of the parametrization of the path e. Secondly, the holonomy is a representation of the group of oriented paths. That is, the holonomy of a single point is the identity and given two paths e_1 and e_2 such that e_2 begins where e_1 ends so that we can form the path $e = e_2 e_1$, then

$$h_e(A) = h_{e_2}(A)\, h_{e_1}(A), \tag{4.7}$$

where the operation on the right is group multiplication. Also the property $h_{e^{-1}}(A) = h_e^{-1}(A)$ holds, where the inverse holonomy is simply the holonomy of the same connection along the same curve but with opposite orientation. As mentioned above, the holonomy transforms nicely under gauge transformations. Under a gauge transformation generated by the Gauss constraint (2.12c), that is $SU(2)$ gauge transformations, the holonomy transforms as

$$h'_e(A) = g(x(1))\, h_e(A)\, g^{-1}(x(0)). \tag{4.8}$$

Under the action of diffeomorphisms, which are generated by the vector constraint (2.12) the holonomy transforms as

$$h_e(\phi^* A) = h_{\phi^{-1}(e)}(A), \tag{4.9}$$

where $\phi \in \mathrm{Diff}(\Sigma)$ and $\phi^* A$ denotes the action of ϕ on the connection.
As the behaviour of the connection under gauge transformations (generated by the six constraints (2.12) for the case of GR) is quite straight forward, it makes it easy to construct gauge invariant functions such as the Wilson loop $\mathrm{Tr}(h_\gamma(A))$, where γ is a closed loop. This is in fact where the name loop quantum gravity originated from. Geometrically, the holonomy is a functional of the connection that defines the notion of parallel-transport for, in this case, $SU(2)$ spinors along the path e. As a functional of a closed path e, it encapsulates all the information of the field A_i^a [52]. For these reasons it is the holonomy that is taken as the basic functional of the connection. There

is one more important thing to mention about the holonomy. The aim of LQG is to quantize gravity in a background independent way. A connection is a one-form and hence is smeared (naturally integrated) over one dimensional submanifolds of Σ, where natural in this sense means without using a background metric. With the holonomy being the path-ordered exponential of $\int_e A$, this is in accordance with the desire to construct a background independent quantum field theory.

Now taking the holonomy as the fundamental variable rather than the connection we run into a problem. The holonomy smears the connection A in one direction so in order to obtain a well-defined Poisson algebra it is necessary to also smear the densitized triad E. From considering the relation (4.3) it is clear that E must be smeared in at least two dimensions. Taking ϵ_{ijk} as the totally antisymmetric, (background independent), tensor density of weight -1, then $E_a = \epsilon_{ijk} E^i_a dx^j \wedge dx^k$ is a two-form of weight 0 and E is naturally smeared in two dimensions. Again consistent with the desire for background independence. One is led to consider the fluxes

$$E_a(S) = \int_S E_a, \tag{4.10}$$

where S is a two-dimensional, open surface embedded in Σ. Under gauge transformations and spatial diffeomorphisms respectively, $E(S) = E_a(S)\tau^a$ transforms as

$$\begin{aligned} E^g(S) &= \int_S \mathrm{Ad}_g E, \\ E^\phi(S) &= E(\phi^{-1}(S)). \end{aligned} \tag{4.11}$$

Though the transformation under spatial diffeomorphisms is nice, the one under gauge transformations is not. The idea however, is to use $E_a(S)$ as the building blocks of more complicated functions of E which will be gauge invariant and for which the corresponding quantum operators will be well-defined. Even though the functions $h(A)$ and $E_a(S)$ are appropriately smeared to give a well-defined Poisson bracket, this smearing also complicates the calculation considerably. Technically, the functions are regularized, the Poisson bracket of the regularized functions is calculated and finally the regulator is removed to arrive at a well-defined symplectic structure. These calculations can be

found in detail in [25]. It is possible though to be slightly more general at this stage by introducing the *cylindrical* functions.
A graph Γ embedded in Σ is an ordered collection of orientated paths $e \subset \Sigma$ meeting (at most) at their end points. Let $l = 1, \ldots, N$ be the number of edges in the graph. Given a smooth function $f : SU(2)^N \to \mathbb{C}$ and a graph Γ, the couple (Γ, f) defines a functional of A,

$$\Psi_{\Gamma,f}(A) = f(h_{e_1}(A), \ldots, h_{e_N}(A)). \tag{4.12}$$

The space Cyl is defined as the linear space of all functionals $\Psi_{\Gamma,f}(A)$, for all Γ and f. These functionals are called cylindrical functions. Clearly the cylindrical functions depend on the ordering of the graph Γ. Changing the ordering or the orientation of a graph is equivalent to changing the ordering of the arguments of f, or replacing arguments with their inverse. Also, the cylindrical functions f are not necessarily gauge invariant.
Equipped with the appropriate topology (see [26]), Cyl is dense in the space of all continuous functionals of A [2]. The Poisson algebra $\mathcal{B}$, the appropriate representation of which will define our kinematical Hilbert space $\mathcal{H}_{kin}$ will be based on the the space of functions Cyl. In calculating the Poisson bracket between the flux (4.10) and a cylindrical function it is important to take into account how the path e intersects the surface S, i.e if e is tangential to S or intersects it at one point if at all. These details are important to consider but are easily dealt with and the end result is

$$\{E_a(S), \Psi\} := v_a^S[\Psi], \tag{4.13}$$

where v_a^S [25, 26] are vector fields on the space of cylindrical functions Cyl. The Poisson algebra $\mathcal{B}$ is generated by the vector fields v_a^S and the functionals $\Psi \in Cyl$, through the relation $[(\Psi, v), (\Psi', v')] = (v[\Psi'] - v'[\Psi], [v, v'])$.

4.2.2 The kinematical Hilbert space $\mathcal{H}_{kin}$

The representation theory of $\mathcal{B}$ was studied recently in [53] and the analysis is not yet fully complete. However, by requiring the irreducible representation

[2]Technically one considers a distributional extension $\bar{\mathcal{A}}$ of $\mathcal{A}$ such that for an element $A \in \bar{\mathcal{A}}$, $h_e(A) = 1$ may vary discontinuously as e is varied continuously.

to admit flux operators which are well-defined and self-adjoint and the representation to be spatially diffeomorphism invariant, a unique representation exists. It is the cylindrical functions presented above that are the candidates for states in $\mathcal{H}_{kin}$. To define the Hilbert space $\mathcal{H}_{kin}$, an inner (scalar) product needs to be defined on Cyl. Thus a measure on the space of holonomies is needed to obtain a definition of this inner product. Given a cylindrical function $\Psi_{\Gamma,f}(A) \in Cyl$, a positive, normalized state μ_{AL} is defined as

$$\mu_{AL}(\Psi_{\Gamma,f}) = \int \left(\prod_{e \subset \Gamma} dh_e \right) f(h_{e_1}, \ldots, h_{e_N}), \tag{4.14}$$

where $h_e \in SU(2)$ and dh is the Haar measure on $SU(2)$. The state μ_{AL} is called the Ashtekar-Lewandowski measure [54] and using it the inner product is defined

$$\begin{aligned} \langle \Psi_{\Gamma',f} | \Psi_{\Gamma'',g} \rangle &\equiv \mu(\overline{\Psi_{\Gamma',f}} \Psi_{\Gamma'',g}) \\ &= \int \left(\prod_{e \subset \Gamma} dh_e \right) \overline{f(h_{e_1}, \ldots, h_{e_N})}\, g(h_{e_1}, \ldots, h_{e_N}), \end{aligned} \tag{4.15}$$

where Dirac notation has been used, with the cylindrical functions becoming wavefunctions of the connection and the graph $\Gamma = \Gamma' \bigcup \Gamma''$. These wavefunctions correspond to the kinematical states $\Psi_{\Gamma,f}(A) = \langle A | \Psi_{\Gamma,f} \rangle = f(h_{e_1}, \ldots, h_{e_N})$. It is important to note that the two couples (Γ, f) and (Γ', f') may define the same functional. For example, let Γ be the union of N' edges in Γ' and M'' other edges in Γ'' and let $f(h_{e_1}, \ldots, h_{e_{N'}}, h_{e_{N'+1}}, \ldots, h_{e_{N'+M''}}) = f'(h_{e_1}, \ldots, h_{e_{N'}})$ depend trivially on the graph Γ''. Then clearly $\Psi_{\Gamma,f} = \Psi_{\Gamma',f'}$. Because of this, any two functionals $\Psi_{\Gamma',f'}$ and $\Psi_{\Gamma'',g''}$ can be rewritten as the functionals $\Psi_{\Gamma,f}$ and $\Psi_{\Gamma,g}$ having the same graph Γ which is the union of Γ' and Γ'' with $N' + M''$ edges. Therefore the inner product defined above is valid for any two functionals in Cyl;

$$\langle \Psi_{\Gamma',f'} | \Psi_{\Gamma'',g''} \rangle \equiv \langle \Psi_{\Gamma,f} | \Psi_{\Gamma,g} \rangle. \tag{4.16}$$

The kinematical Hilbert space of quantum gravity is the completion of Cyl in the norm defined by the inner product (4.15) [3].

[3]The space of the Cauchy sequences Ψ_n, where $||\Psi_m - \Psi_n||$ converges to zero.

Before moving on to define a basis for $\mathcal{H}_{kin}$, there is grounds for objecting to the above definition of $\mathcal{H}_{kin}$; it is nonseparable. Normally this would be disastrous in the context of flat-space quantum field theory, but because of diffeomorphism invariance in the general-relativistic context, this turns out to be harmless. As will be shown, the 'excessive size' of $\mathcal{H}_{kin}$ turns out to be just gauge and once the diffeomorphism gauge is factored away, the physical Hilbert space $\mathcal{H}_{phys}$ is separable.

4.2.3 An orthonormal basis of $\mathcal{H}_{kin}$

An orthonormal basis of $\mathcal{H}_{kin}$ is constructed using the Peter-Weyl theorem. It states that a basis for the Hilbert space $L^2[SU(2)]$ is given by the matrix elements of the irreducible representations of the group. Given a function $f \in L^2[SU(2)]$, it can be written as the sum over unitary irreducible representations of $SU(2)$,

$$f(g) = \sum_j \sqrt{2j+1}\, f_j^{\alpha\beta}\, R^j_{\alpha\beta}(g), \tag{4.17}$$

where

$$f_j^{\alpha\beta} = \sqrt{2j+1}\int_G dg\, R^j_{\alpha\beta}(g^{-1}) f(g) \tag{4.18}$$

and dg is the Haar measure of $SU(2)$. The normalized representation matrices [55] are $\Pi^j_{\alpha\beta} := \sqrt{2j+1}\, R^j_{\alpha\beta}$, which satisfy the orthogonality condition for unitary irreducible representations of $SU(2)$,

$$\int_{SU(2)} dg\, \Pi^j_{\alpha\beta}(g)\, \Pi^{j'}_{\gamma\delta}(g^{-1}) = \delta^{jj'}\delta_{\alpha\gamma}\delta_{\beta\delta} \tag{4.19}$$

Now given an arbitrary cylindrical function $\Psi_{\Gamma,f}(A) \in Cyl$ and using the Peter-Weyl theorem, it can be expressed as

$$\begin{aligned}\Psi_{\Gamma,f}(A) &= f(h_{e_1}(A), \ldots, h_{e_N}(A)) \qquad (4.20)\\ &= \sum_{j_1,\cdots,j_N} f^{\alpha_1,\cdots,\alpha_N,\beta_1,\cdots,\beta_N}_{j_1,\cdots,j_N} \Pi^{j_1}_{\alpha_1\beta_1}(h_{e_1}(A)), \cdots, \Pi^{j_N}_{\alpha_N\beta_N}(h_{e_N}(A)),\end{aligned}$$

where $f^{\alpha_1,\cdots,\alpha_N,\beta_1,\cdots,\beta_N}_{j_1,\cdots,j_N}$ is given by the inner product(4.15) of the cylindrical function and the tensor product of irreducible representations,

$$f^{\alpha_1,\cdots,\alpha_N,\beta_1,\cdots,\beta_N}_{j_1,\cdots,j_N} = \langle \Pi^{j_1}_{\alpha_1\beta_1}, \cdots, \Pi^{j_N}_{\alpha_N\beta_N} | \Psi_{\Gamma,f}\rangle. \tag{4.21}$$

The product of components of (normalized) irreducible representations $\prod_{i=1}^{N} \Pi^{j_i}_{\alpha_i\beta_i}(h_{e_i})$ associated with the N edges $e \subset \Gamma$ is a complete basis of $\mathcal{H}_{kin}$ provided a simple redundancy is removed. To see this redundancy consider the finite dimensional subspace $\widetilde{\mathcal{H}}_\Gamma$ of $\mathcal{H}_{kin}$ formed by the cylindrical functions with support on a particular graph Γ. Then the set of vectors $\prod_{i=1}^{N} \Pi^{j_i}_{\alpha_i\beta_i}(h_{e_i})$ is not a basis because the same vector appears in $\widetilde{\mathcal{H}}_\Gamma$ and $\widetilde{\mathcal{H}}_{\Gamma'}$ if $\Gamma \subset \Gamma'$. But because all $\widetilde{\mathcal{H}}_\Gamma$ vectors belong to the trivial representation of the paths that are in Γ' and not in Γ, the redundancy can easily be removed. A basis for $\mathcal{H}_{kin}$ is given by the vectors $\prod_{i=1}^{N} \Pi^{j_i}_{\alpha_i\beta_i}(h_{e_i})$, where $j_i = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ never takes the value zero. Defining for a graph Γ the *proper* graph subspace $\mathcal{H}_\Gamma$ as the subset of $\widetilde{\mathcal{H}}_\Gamma$ spanned by the basis states with $j_i > 0$, then it can be shown that all proper subspaces $\mathcal{H}_\Gamma$ are orthogonal and span $\mathcal{H}_{kin}$

$$\mathcal{H}_{kin} \sim \bigoplus_{\Gamma} \mathcal{H}_\Gamma \tag{4.22}$$

Loop quantum gravity is an attempt to quantize the gravitational field in a background independent and non perturbative way. Its fundamental assumptions are that GR and QM are correct and that one should consider the idea of QM formulated to be compatible with general covariance seriously. At high energy the Einstein equations may be modified but the general-relativistic notions of spacetime are assumed to be correct so LQG deliberately avoids the practice of splitting the metric (4.1). In this chapter, the canonical quantization of GR, formulated as a $SU(2)$ gauge theory in the Ashtekar-Sen variables, was presented. From the connection the holonomy was constructed and likewise from the densitized triad a smeared, two-form flux was defined. It was these two entities that, under the Poisson bracket, formed the algebra of basic field functions. Using the holonomy, we introduced the cylindrical functions, the space of which was defined as the linear space of all functionals of the connection over all graphs. The cylindrical functions, together with the fluxes formed a Poisson algebra. We showed that the appropriate representation of this algebra produced the kinematical Hilbert space $\mathcal{H}_{kin}$ of states and using the Peter-Weyl theorem a basis was constructed out of unitary irreducible representations of $SU(2)$. This kinematic Hilbert space is the space of arbitrary wave functionals of the connection. Next we will consider the states of $\mathcal{H}_{kin}$ that are invariant under local gauge transformations. These states are known as spin networks and will be the subject of the next chapter.

Chapter 5

Spin Networks and Spin Foams

In the previous chapter it was shown that an element of the Hilbert space of cylindrical functions of the holonomy can be expressed as a sum of unitary irreducible representations of the gauge group $SU(2)$. Furthermore, using the Peter-Weyl theorem, a complete and orthonormal basis of this kinematic Hilbert space can be found. In this chapter we will discuss the states of this Hilbert space which are invariant under local $SU(2)$ gauge transformations. These states are known as spin networks. They were first introduced by Penrose [83] in an attempt to find a purely combinatorial description of spacetime. Initially, the development of spin networks was motivated more by the quantum mechanics of angular momentum than by considerations of GR. However, after the initial formulation of LQG, Rovelli and Smolin [84] in 1995 discovered that the spin networks of Penrose in fact can be used to describe the states of LQG. The spin networks describe a discrete space at the quantum level. We begin this chapter by considering the most basic example of a state in the kinematic Hilbert space of LQG that is locally gauge invariant, the well known Wilson loop. This state is then generalized to the spin network states, which form a complete basis for the Hilbert space of gauge invariant states.

In order to address supergravity in the context of LQG we extend the idea of spin networks to include supersymmetry. This requires the replacement of gauge groups with supergroups, something that has received very little attention in the literature. Having introduced the super spin networks we go on to discuss the most widely studied spin network model, the Barrett-Crane model. This model attempts to quantize GR by expressing the path integral of a BF theory action as a spin foam (a sum over histories of spin networks) and then implementing the constraint imposed classically on the BF action at the quantum level. This is done by restricting the representations that are summed over in the spin foam. Later we will show how this procedure, of imposing a classical constraint at the quantum level by a restriction on representations, can be used in the case of 2d supergravity.

5.1 Gauge invariant states of $\mathcal{H}_{kin}$

The kinematical space $\mathcal{H}_{kin}$ is the space of arbitrary wave functionals $\Psi(A)$ (4.20) of the connection. We are interested in the states that are invariant under local $SU(2)$ gauge transformations. These states are solutions to the quantum Gauss constraint

$$D_i \frac{\delta}{\delta A_i^a(\vec{\tau})} \Psi(A) = 0 \tag{5.1}$$

and define the Hilbert space $\mathcal{H}_{kin}^G$. (Note that previously the *classical* Gauss constraint (2.12c) acted through the Poisson brackets (4.3). In the quantum theory, the Gauss constraint appears as a functional derivative). The simplest example of a state in $\mathcal{H}_{kin}^G$ is the well-known Wilson loop state. The Wilson loop is defined as the trace of the holonomy around a closed curve e,

$$W_e(A) := tr[h_e(A)]. \tag{5.2}$$

The $SU(2)$ invariance of $W_e(A)$ is implied by the behavior of the holonomy under a gauge transformation (4.8) generated by the Gauss constraint (2.12) and the invariance of the trace. Notice that using (4.7), (i.e. $h_e = h_{e_1} h_{e_2}$), the Wilson loop could also be defined as

$$W_e(A) = tr[h_{e_1} h_{e_2}]. \tag{5.3}$$

Furthermore, one could define the Wilson loop on the graph γ'' (see Fig. 5.1) with trivial dependence on the third argument h_{e_3}.

It is important to note that there is no physical distinction between these different graphs when evaluating the Wilson loop.

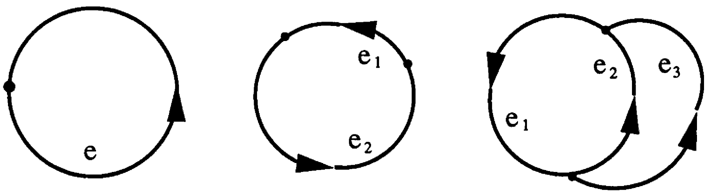

Figure 5.1: The three graphs γ, γ' and γ'' on which the Wilson loop function above is defined. As the Wilson loop has trivial dependence on the argument h_{e3}, there is no physical distinction between these graphs.

This simple gauge invariant function can be generalized by considering an arbitrary matrix representation M of $SU(2)$. Then clearly the function

$$W_\gamma^M(A) = tr[M(h_e(A))] \tag{5.4}$$

is also a gauge invariant function. If M is a unitary irreducible representation of spin j, denoted by Π^j_{mn}, then the function

$$W_\gamma^j(A) = tr[\Pi^j(h_e(A))] \tag{5.5}$$

is the simplest example of a spin network [51, 83, 84, 85]. This function is represented on the left of Fig. 5.2.

Given the graph γ, depicted in the center of Fig 5.2 for which an orientation

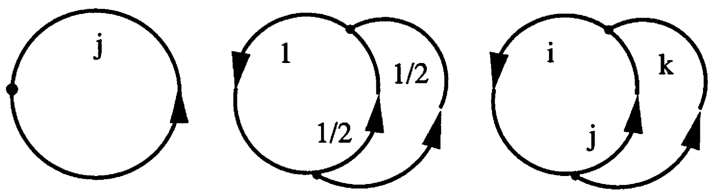

Figure 5.2: Examples of increasingly more generalized spin networks.

and ordering has been fixed, one can see that an irreducible representation j_e (not the trivial representation) has been assigned to each edge $e \in \gamma$. To each node an intertwiner n_i must also be assigned. Before continuing with this discussion on spin networks, let's briefly mention a few details about intertwiners.

5.1.1 Aside: Intertwiners

Take N irreducible representations $j_1, \ldots, j_N$ and consider the tensor product of their Hilbert spaces

$$\mathcal{H}_{j_1,\ldots,j_N} = \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}. \tag{5.6}$$

The space $\mathcal{H}_{j_1,\ldots,j_N}$ can be decomposed into a sum of irreducible spaces. The subspace $\mathcal{H}^0_{j_1,\ldots,j_N}$ which transforms in the trivial representation, that is, the subspace formed by invariant vectors, is k-dimensional where k is the multiplicity of the trivial representations appearing in the decomposition. It is a Hilbert space and as such one can choose an orthonormal basis. The elements ι of this basis are called intertwiners between the representations $j_1, \ldots, j_N$. To illustrate this explicitly, the elements $\iota^{\alpha_1,\ldots,\alpha_N} \in \mathcal{H}_{j_1,\ldots,j_N}$ are tensors that are invariant under the action of $SU(2)$ on all their indices

$$\Pi^{j_1\alpha_1}_{\beta_1}(h(A)) \cdots \Pi^{j_N\alpha_N}_{\beta_N}(h(A))\iota^{\beta_1,\ldots,\beta_N} = \iota^{\alpha_1,\ldots,\alpha_N}. \tag{5.7}$$

Furthermore, the space $\mathcal{H}_j$ of representation j has its dual space $\mathcal{H}_{j^*}$ of dual representation j^*. An intertwiner ι between n dual representations $j_1^*, \ldots, j_n^*$ and m representations $j_1, \ldots, j_m$, is a covariant map

$$\iota : \bigotimes_{i=1,\ldots,n} \mathcal{H}_{j_i^*} \to \bigotimes_{k=1,\ldots,m} \mathcal{H}_{j_k}. \tag{5.8}$$

The intertwiner ι_n associated with a node is in the tensor product of the representations associated with the edges adjacent to the node.

5.1.2 Spin networks

The more complicated spin network associated with the middle graph of Fig. 5.2 can be defined by taking the different representation matrices of spins $1, 1/2$ and $1/2$ and evaluating them on the holonomy along the respective edges. The spin network function is explicitly given by

$$S_\gamma(A) = \Pi^1(h_{e_1}(A))^{ij}\Pi^{\frac{1}{2}}(h_{e_2}(A))_{\alpha\beta}\Pi^{\frac{1}{2}}(h_{e_3}(A))_{\gamma\delta}\, \sigma_i^{\alpha\gamma}\sigma_j^{\beta\delta}, \tag{5.9}$$

where $i, j = 1, 2, 3$ are vector indices and $\alpha, \beta = 1, 2$ are spinor indices. At the two nodes one has the tensor product of two fundamental representations and one adjoint representation of $SU(2)$. The decomposition of this tensor product contains one copy of the trivial representation and therefore there is only one intertwiner; the Pauli matrices $\sigma_i^{\alpha\beta}$. The spin network function S is gauge invariant due to the fact that the Pauli matrices are invariant tensors in the tensor product of representations $1 \otimes \frac{1}{2} \otimes \frac{1}{2}$ which is where gauge transformations act. By considering the graph on the right of Fig. 5.2, it is easy to see how to generalize the idea of a spin network function to arbitrary representations. Given an invariant tensor (intertwiner) $\iota \in j \otimes k \otimes l$, the spin network is expressed as

$$S_\gamma(A) = \Pi^j(h_{e_1}(A))_{\alpha_1\beta_1}\Pi^k(h_{e_2}(A))_{\alpha_2\beta_2}\Pi^l(h_{e_3}(A))_{\alpha_3\beta_3}\,\iota^{\alpha_1\alpha_2\alpha_3}\iota^{\beta_1\beta_2\beta_3}. \quad (5.10)$$

The generalization of this construction to arbitrary graphs is straightforward. To each edge e one associates a representation j_e, and to each node n and intertwiner ι_n in the tensor product of the representations associated with the edges adjacent to the node. A spin network in Σ is defined as the triple $S = (\gamma, j_e, \iota_n)$, with coloring of the edges and nodes j_e, ι_n respectively. Each spin network S defines a state $|S\rangle$ by

$$\begin{aligned} \langle A|S\rangle &= S_\gamma(A) \\ &= \bigotimes_{n\subset\gamma}\iota_n \bigotimes_{e\subset\gamma}\Pi^{j_e}(h_e(A)). \end{aligned} \quad (5.11)$$

Intertwiners in the tensor product of an arbitrary number of irreducible representations can be expressed in terms of basic intertwiners between three irreducible representations. What this means is that any node joining an arbitrary number of edges can be decomposed into a number of nodes, each of which only joins three edges. In the case of $SU(2)$, these basic intertwiners are uniquely defined up to normalization, and are related by the Clebsh-Gordon coefficients. Following the previous chapter we see that the spin network states S form a complete and orthonormal basis in the gauge invariant kinematical Hilbert space $\mathcal{H}^G_{kin}$.

Before continuing, we should briefly comment on the other constraints associated with GR; the vector constraint (2.12a), which generates diffeomorphisms, and the scalar constraint (2.12b), which generates coordinate time evolution. Though the vector constraint can easily be imposed at the quantum level [50], having a natural (unitary) action on the states of $\mathcal{H}_{kin}$, the scalar constraint is a different story. The precise form of the quantum scalar constraint is not yet settled and there are a number of mathematically consistent definitions arising from ambiguities in the quantization procedure of the constraint [51]. As such, the space of solutions of the quantum scalar constraint remains an open issue in LQG.

5.1.3 Including supersymmetry

Similar to the case of bosonic gravity, it is possible to construct a loop quantization of supergravity. This provides the framework for the super spin networks and the corresponding models of supergravity. Following what was shown for the case of gravity with $SU(2)$ gauge group in the previous chapter, the kinematical states of loop quantum supergravity will be $OSp(1|2)$ spin networks[1]. Before moving on to discuss the states in the Hilbert space $\mathcal{H}_{kin}$ invariant under local gauge transformations, the super spin networks, we would like to consider how the loop quantization procedure presented previously extends to the supergravity case.

Supergravity in terms of the Ashtekar-Sen variables was first studied in [16]. In this formulation, for the $N = 1$ case, the canonical variables are the Lie algebra valued spin connection A_i^a and its super-partner, the gravitino field ψ_i^α. These two variables, as was shown in [56], together form a graded super Lie algebra valued connection

$$A_i = A_i^a J_a + \psi_i^\alpha Q_\alpha, \tag{5.12}$$

where in the case of the spin connection being $su(1,1)$ valued, the super Lie algebra formed is $osp(1,2)$, (c.f. (3.20), where we have made the association $(P_a, J) \rightarrow J_a$ and the J_a now refer to the generators of $sp(2) \cong su(1,1)$ with dimension $D = 1+1$). Similarly, if E_a^i and π_α^i are conjugate momenta of A_i^a and ψ_i^α respectively, then the graded momenta are defined as

$$E^i = E_a^i J^a + \pi_\alpha^i Q^\alpha. \tag{5.13}$$

[1]In this thesis we have not discussed the dynamics of LQG which leads to spin foams. One way to view spin networks is as the boundary states of some spin foam, which is the path integral for LQG.There are still many unanswered questions with regards to the dynamics of LQG and how to recover it in the spin foam formulation. See [86] for more details on these problems.

In the fundamental representation of $OSp(1,2)$ the super-Lie algebra is generated by the five 3×3 matrices U_A $(A=1,\ldots,5)$ (see Appendix and [56]), where A labels the five generators of $OSp(1,2)$. Using these we can define

$$\begin{aligned} A_i^A &= (A_i^a, \psi_i^\alpha), \\ E_A^i &= (E_a^i, \pi_\alpha^i). \end{aligned} \tag{5.14}$$

The canonical analysis of $D=1+1$ supergravity on a manifold $M=\mathbb{R}\times\Sigma$ begins with the super BF action (3.19). The canonical variables are the spatial components of the superconnection A and their corresponding conjugate momenta, the spatial components of the superdyad E. The total hamiltonian consists of two constraints: (1) the super Gauss constraint $DE=0$ imposes a vanishing supertorsion and generates local $OSp(1,2)$ gauge transformations, specifically local supersymmetry and $SU(1,1)$ gauge transformations (c.f. (3.28)) for the $2d$ supergravity case of Howe [5], (2) $F=0$ which imposes flatness on the superconnection and generates 'topological' gauge transformations on the momenta E. This constraint is composed of spatial diffeomorphisms on Σ and the constraint generating time evolution. It is now possible to 'loop quantize' the theory following the same procedure presented above for bosonic gravity. The algebra of kinematic observables is given by considering cylindrical functions of the superconnection A (c.f. (4.12)), which depends only on the superholonomies of A along the edges of some graph $\gamma\subset\Sigma$,

$$\Psi_{\gamma,F}(A) = f(H_{e_i}(A), e_i \in \gamma). \tag{5.15}$$

Note in this expression we have used the superholonomy $H_{e_i}(A)$, which is an element of $Osp(1,2)$. It is constructed in the same was as the usual holonomy (c.f. (4.4)), only that now the connection has been replaced with the superconnection [62]. By taking the supertrace of the superholonomy in the fundamental 3 dimensional representation of $OSp(1,2)$, we once again have the Wilson loop, (see Appendix for definition of supertrace)

$$\begin{aligned} W_e(A) &= Str\mathcal{P}\exp\int_0^1 ds\; \dot{e}^m(s)\, A_m^a(e(s))\,\tau_a \\ &= Str\mathcal{P}\exp\int_e A \\ &= Str[H_e], \end{aligned} \tag{5.16}$$

where again $e : [0,1] \longrightarrow \Sigma$ is a closed, one dimensional orientated curve parameterized by $s \in [0,1] \rightarrow x^m(s)$. The term $\dot{e}^m(s) = dx^m(s)/ds$ is the tangent to the curve and τ_a is a basis element of the Lie algebra of $OSp(1,2)$. It was with these Wilson loops that the loop representation of supergravity in the chiral representation was first constructed [56]. Even though the Wilson loop is gauge invariant, the superholonomies are not. However, to construct $OSp(1,2)$ spin networks it is necessary to impose gauge invariance, i.e. invariance under $Osp(1,2)$ at the nodes (vertices) n of the graph γ. Following the procedure for the pure gravity case, this invariance of the cylindrical functions would read as

$$f(g^{-1}_{(s_1)}H_e(A)g_{(s_2)}) = f(H_e(A)), \tag{5.17}$$

where s_1 , s_2 respectively denote the beginning and ending nodes of the edge $e \in \gamma$. In fact, cylindrical functions of the superholonomies $H(A)$ are actually defined on the matrix elements $M[H(A)]$ of the supergroup elements. As these matrices do not commute, one must be careful and choose a full ordering of the edges e. The precise definition of gauge invariance, and also of the super spin networks, will depend on whatever convention of ordering is chosen. This is similar to the problem found when dealing with quantum groups. Using the (super)Haar measure $d\mu$ on $OSp(1,2)$ which was investigated in [57], one can introduce the measure product of $d\mu$ on all edges $e \in \gamma$ which defines a natural inner product on the space of square integrable cylindrical functions and a basis of the resulting gauge invariant kinematical Hilbert space $\mathcal{H}^G_{kin}$ can be found. This basis is the $OSp(1,2)$ super spin networks [58, 59].

5.1.4 The Hilbert space

To show that the $OSp(1,2)$ spin networks do indeed form a basis of the gauge invariant kinematical Hilbert space of cylindrical functions, we need to address two key problems. One is the definition of the inner product of spin network states, which allows us to show that any two different spin networks are orthogonal and linearly independent. The other problem to address is the completeness of the Hilbert space, that is can any state in the Hilbert space be expressed as a sum of super spin networks? In the case of $SU(2)$, these two problems are solved successfully as we saw in the previous chapter, through the Haar measure and the Peter-Weyl theorem respectively. The end result is that the $SU(2)$ spin networks form a linearly independent basis of the Hilbert space of cylindrical functions in LQG. When the construction of spin networks is extended to the supersymmetric case, we need to construct them in a way and find the rules that are consistent with the representation theory of supergroups. At first sight, the extension to supersymmetric spin networks seems pretty straightforward. Every edge of the spin network is labelled with a representation of the Lie group of the theory. So one must relabel all the edges with the corresponding representations of the supergroup. Similarly with the vertices, these too must be relabelled with the appropriate intertwiners. One immediate question that can be raised here is what is the case at a trivalent vertex? Is it possible to decompose the three adjacent representations (a tensor product of irreducible representations) into a direct sum of irreducible representations. This is certainly true in the $su(2)$ case, but not for super Lie algebras. One example of this is in the construction of $OSp(2,2)$ super spin networks. Fortunately in the specific case that we are interested, namely $OSp(1,2)$ we do not have to worry about this. Its reducible representations are fully reducible and furthermore, any finite dimensional representation can be obtained from the direct product of fundamental representations.

5.1.5 Super spin networks

Recall that a spin network state of quantum gravity, denoted by $|S\rangle$ consists of a closed graph γ with edges labelled by the representations of $SU(2)$ and vertices by intertwining operators. The *super* spin networks are defined in the same way, simply by replacing $SU(2)$ with the Lie supergroup $OSp(1,2)$. As the steps showing that super spin networks form a basis of the Hilbert space of (super)cylindrical functions follows very closely the case of spin networks presented earlier in this chapter, we will give a more general description. For a more detailed explanation of this process we refer the reader to [59]. Starting with a manifold M we define a super-Lie algebra ($osp(1,2)$) valued 1-form connection A (for example (3.20)). The components of this connection are smooth functions over M so we denote the space of smooth connections on the manifold as $\mathcal{A}$. Let $\mathcal{C}^{\infty}(\mathcal{A})$ be the space of continuous functionals on $\mathcal{A}$ as defined previously in section 4.2.1. Now by introducing an inner product between the states of $\mathcal{C}^{\infty}(\mathcal{A})$ and completing in the norm, we wish to define the Hilbert space $\mathcal{L}^2$. Recall that the (super)holonomy of a connection is simply an element of some (super)-Lie group. Therefore the (super)cylindrical functions, defined as functionals of holonomies of connections, are also functions of some group manifold. Examining the steps presented earlier, this inner product is defined by integration over the group manifold (4.15). In order to do this it was necessary to introduce a unique, left and right invariant measure, the Haar measure. To extend this construction to the supersymmetric case one must generalize the Haar measure to allow integration for Lie supergroups. We refer the reader to [57] for further details on the super Haar measure and Haar integral. In the particular case of $OSp(1,2)$, it is possible to define a super Haar measure on the space of functionals of the super holonomies which is both left and right invariant. This is not always the case for other supergroups. With the integration theory on the space of super connections A established, we can define the inner product of cylindrical functions. If we consider some graph γ, recall that the role of the holonomy, in some irreducible representation j_{e_i}, is to assign an element $H_{e_i}(A)$ of the group G to the edge $e_i \in \gamma$. In this context, the measure for the space of smooth connections can be expressed as

$$\mathcal{D}\mathcal{A} = \bigotimes_{e_i} dH_{e_i}. \tag{5.18}$$

This measure is called the generalized Ashtekar-Lewandowski measure and is entirely analogous to the $SU(2)$ case (c.f. (4.14)). Now consider a graph $\Gamma(e_i, n_j)$, where e_i, n_j denote the *ith*-edge and *jth*-node respectively and the cylindrical functions $\Psi_{\Gamma,f}$ defined over this graph. Using the super Haar measure on G^n, we can define the inner product between two super cylindrical functions as

$$\langle \Psi_{\Gamma,f} | \Psi_{\Gamma,g} \rangle := \int_{G^n} \prod_{e \subset \Gamma} dH_{e_i}\, \overline{f(H_{e_1}, \ldots, H_{e_N})}\, g(H_{e_1}, \ldots, H_{e_N}). \tag{5.19}$$

(Note that in the case of super spin networks, the inner product defined here is essentially of the same form as the inner product of standard $SU(2)$ spin networks previously defined (4.15)). Entirely analogous to the bosonic case, if $\mathcal{H}_{j_e}$ represents the Hilbert space on which the irreducible representation j_e is defined, then the total Hilbert space associated to super spin networks can be defined as the tensor product of these spaces,

$$\mathcal{H} = \bigotimes_{v_j} \mathcal{H}^{v_j} = \bigotimes_{v_j} \Big(\bigotimes_{e_i} \mathcal{H}_{e_i} \Big)^{v_j}, \tag{5.20}$$

where the edges e_i meet at the same vertex v_j. Likewise, in the case of $OSp(1,2)$, since any product of finite irreducible representations is completely reducible, the tensor product of the Hilbert spaces can be decomposed into the direct sum of Hilbert spaces on which the irreducible representations of $OSp(1,2)$ are defined,

$$\mathcal{H}_{e_1} \bigotimes \cdots \bigotimes \mathcal{H}_{e_n} = \bigoplus_{j_m} \mathcal{H}_{j_m}. \tag{5.21}$$

Now, using a generalization of the Peter-Weyl theorem to include supergroups presented in [59], one can show that the Hilbert space of functions of the superconnection can be expressed as a direct sum over irreducible representations of $OSp(1,2)$. That is, using the generalization of (4.17) and the orthogonality condition (4.19) to include supergroups, one can show that the super spin network states are indeed orthogonal and form a complete basis for the gauge invariant Hilbert space $\mathcal{L}^2[OSp(1,2)]$ of functions of the superconnection.

5.2 The Barrett-Crane model

In four dimensions the description of the dynamics of gravity, known as spin foam representations, was motivated by lattice discretizations of the path integral of gravity in the covariant formulation [79, 80, 81]. The scope of this thesis does not include the representation of the dynamics of gravity using spin foams as 2d GR is topological and hence has no dynamics. However, we wish to finish this chapter with a discussion of one of the most widely studied spin foam models in the literature, the Barrett-Crane (BC) model [82]. The reason for considering this model is that in the final chapter it will be shown how the features of this model have been used to quantize gravity in two dimensions and calculate the corresponding partition function. We wish to know, can these features be extended to the case of supergravity? To illustrate these features, we will focus on Riemannian GR. This is because the starting point of the BC model is the BF action in four dimensions and BF theory is only well understood in the case of compact gauge groups. Working with compact gauge groups limits one to Riemannian quantum gravity. Starting with the $SO(4)$ BF-action

$$I = \int B_{AB} \wedge F^{AB}, \tag{5.22}$$

where both B and F are two-forms and $A, B = 1, \ldots, 4$ are Lie algebra indices. Note that this treatment of BF-theory differs from before, where in this case the B and F fields take value in the adjoint of $SO(4)$, which is represented as the antisymmetric part of the vector representation and hence the antisymmetric indices AB. Now if the B-field is replaced with

$$B^{AB} = \epsilon^{AB}{}_{CD}\, e^C \wedge e^D, \tag{5.23}$$

one gets precisely the action of GR. The B-field of BF theory can be identified with the gravitational field $e \wedge e$. The constraint on the B-field is called the Plebanski constraint and transforms BF theory into GR. Adding this constraint to the BF action gives the Plebanski action of $SO(4)$ GR,

$$I = \int B^{AB} \wedge F_{AB} + \Lambda_{ABCD} B^{AB} \wedge B^{CD} + \mu\, \epsilon^{ABCD} \Lambda_{ABCD}, \tag{5.24}$$

where μ is a four-form and Λ an antisymmetric tensor. Varying with respect to μ gives the constraint $\epsilon^{ABCD}\Lambda_{ABCD} = 0$. This reduces the number of independent components of Λ to 20. Now variation with respect to Λ imposes 20 algebraic equations on the 36 components of the B-field. The (non-degenerate) solutions to the equations obtained by varying the multipliers are

$$\begin{aligned} B^{AB} &= \pm\epsilon^{AB}{}_{CD} e^C \wedge e^D, \\ B^{AB} &= \pm e^A \wedge e^B, \end{aligned} \tag{5.25}$$

expressed in terms of the remaining 16 components of the tetrad field. As stated before, substituting the first of these solutions, which is the Plebanski constraint, into the original action gives Palatini's formulation of GR[2]. The second term is topological and has no effect on the classical equations of motion. How is this Plebanski constraint imposed directly at the quantum level in the BC model? The key idea is that the path integral for BF theory in four dimensions

$$I = \int DB\, DA \exp[i \int B^{AB} \wedge F_{AB}], \tag{5.26}$$

can be defined as a sum over spin foams (see [51, 50] for details). The BC model provides a definition of the path integral of gravity from the formal expression

$$I = \int DB\, DA\; \delta[B \to \epsilon^{ABCD} e_C \wedge e_D] \exp[i \int B^{AB} \wedge F_{AB}]. \tag{5.27}$$

Here the term $\delta(B \to \epsilon^{ABCD} e_C \wedge e_D)$ expresses the Plebanski constraint (5.23), an immediate consequence of which is

$$\epsilon_{ABCD}\, B^{AB} B^{CD} = 0. \tag{5.28}$$

Indeed, it is this restriction that can be imposed in a systematic way directly on the spin foams that define (5.26), that is, at the quantum level. To see this, note that the Lie algebra of $SO(4)$ is $su(2) \oplus su(2)$. The irreducible representations of $SO(4)$ are labelled by pairs of representations of $SU(2)$, namely by two spins $j = (j_+, j_-)$. The B-field is a two-form with values in the adjoint representation of $SO(4)$, and the generators of the two $SU(2)$ groups are $B^a_\pm = P^a_{\pm AB} B^{AB}$, where $P^a_{\pm AB}$ is a projector onto the selfdual components of $SO(4)$. Two invariants of $SO(4)$ can be constructed from B^{AB}; the scalar invariant

$$C = B_{AB} B^{AB} = |B|^2 \tag{5.29}$$

[2]If one substitutes the first constraint (5.25a) on the B-field into the BF-action (5.22) and set the variation of the action to zero, the field equations $e \wedge F = 0$ and $d_A(e \wedge e) = 0$ are obtained. To see that these equations are an extension of the vacuum Einstein equations, note that $d_A(e \wedge e) = 0$ is equivalent to $e \wedge d_A e = 0$ and when e is one-to-one this implies $d_A e = 0$. If e is used to pull back the connection A to a metric preserving connection Γ, the equation $d_A e = 0$ says that Γ is torsion-free, so Γ is the Levi-Civita connection of the metric. This allows one to write the term $e \wedge F$ in terms of the Riemann tensor. Furthermore $e \wedge F$ is in fact proportional to the Einstein tensor, so $e \wedge F$ is equivalent to the vacuum Einstein equation [85].

and the pseudo-scalar invariant

$$\tilde{C} = \epsilon_{ABCD} B^{AB} B^{CD}. \tag{5.30}$$

These invariants are not the $SO(4)$ Casimirs but are indeed related. Immediately we can see that the pseudo-scalar invariant is constrained to zero because of (5.28). The $SO(4)$ representations in which the pseudo-scalar invariant vanishes are called 'simple' representations. It is straightforward to calculate the value of the pseudo-scalar invariant in the (j_+, j_-) representation

$$\begin{aligned} \epsilon_{ABCD} B^{AB} B^{CD} &= B_+^a B_{+a} - B_-^a B_{-a}, \\ &= j_+(j_+ + 1) - j_-(j_- + 1). \end{aligned} \tag{5.31}$$

From this and (5.28) one obtains the result $j_+ = j_-$. The representations that satisfy this constraint, those such that $(j_+, j_-) = (j, j)$, are the simple representations and are labelled by the single spin j. This is the quantum version of the constraint (5.23). Though the details have not been presented here, when one calculates the partition function from the path integral (5.26) for BF theory, in the spin foam formulation, the resulting expression includes a term which is summed over representations. This quantum constraint suggests that quantum GR can be obtained by restricting the sum over representations in the partition function to just the simple representations. This procedure defines the class of BC models. It should be noted that due to the different ways in which the constraints can be discretized there are various versions of the BC model in the literature. It is still unclear to date, which version is indeed discretized GR. Having now seen how a classical constraint added to a BF action was first imposed at the quantum level, we will proceed to illustrate in the next chapter this procedure in more detail for 2d GR and calculate the partition function using spin networks. We will then go on to show how this procedure can also accommodate the case of supergravity.

Chapter 6

Spin foam 2D quantum gravity

Spin foam models provide a non-perturbative approach to quantum gravity [51, 85, 89]. There are two ways in which they can be motivated, as a rigorous method of performing a covariant path integral quantization, or as emerging histories in (canonical) LQG. To date, these approaches to quantum gravity have had little to do with the main perturbative approach, string theory (see, however, [65, 91]). One of the main obstacles to linking these approaches to string theory is that the latter requires supersymmetry. This can be accommodated in the spin foam formulation by the promotion of gauge groups to supergroups. Unfortunately this possibility is something that has barely been investigated. Here the extension of the spin foam approach to the quantization of supergravity is considered. In this chapter we review the spin foam quantization of 2d GR developed in [1], beginning with a simple review of 2d GR on compact manifolds without boundary. This procedure involves the discretization of the 2d BF-theory via a triangulation of the base manifold which will allow the calculation of the corresponding partition function. By implementing the Plebanski constraint at the quantum level by restricting the representations summed over in the spin foam, the partition function of 2d GR is calculated. This procedure will then be extended to the 2d supergravity case, focusing on the model of Howe [5].

6.1 2d Riemannian GR on compact manifolds

The action for Riemannian GR in 2d is the Einstein-Hilbert action (2.1)

$$I = \frac{1}{16\pi G}\int d^2x\sqrt{\det(g)}R, \tag{6.1}$$

where $\det(g)$ is the determinant of the Riemannian 2d metric g_{mn}, $m,n = 1,2$, G is Newton's constant and R is the Ricci scalar. (Compare this with the Einstein equation discussed previously (2.25), in which the cosmological constant has been included to allow nontrivial equations of motion). As was shown previously, this action is topological and proportional to the Euler characteristic

$$I_{GR} = \frac{\chi(M)}{4G}. \tag{6.2}$$

The first step towards covariantly quantizing 2d GR is to calculate its partition function. This can formally be given by

$$Z = \int \mathcal{D}g\, e^{\frac{i}{\hbar}I_{GR}}. \tag{6.3}$$

It is not necessary to know what the measure $\mathcal{D}g$ on the infinite dimensional space of metrics modulo diffeomorphisms is, provided we assume it is normalized with $\int \mathcal{D}g = 1$. From (6.2) we clearly see that the action does not depend on the metric, but only on the topology of the manifold via the Euler characteristic. Therefore the partition function can be expressed as

$$Z = e^{\frac{i\chi(M)}{4\hbar G}}, \tag{6.4}$$

where the condition on the measure being normalized has been used. In background independent theories, both orientations of the manifold M are summed over in the Feynman integral. In doing this the partition function and the transition amplitudes are real. By summing over both orientations of the manifold in the Feynman path integral, the final result for the 2d GR partition function is

$$Z = \int \mathcal{D}g\Big(e^{\frac{i}{\hbar}I_{GR}} + e^{-\frac{i}{\hbar}I_{GR}}\Big) = 2\cos\left(\frac{\chi(M)}{4\hbar G}\right). \tag{6.5}$$

In the following sections it will be shown how this result is recovered from a spin foam quantization procedure.

6.1.1 The frame bundle and its topology

Considering 2d compact manifolds one would expect from Stokes' theorem

$$\int_M d\omega = \oint_{\partial M} \omega, \tag{6.6}$$

that (3.11) is zero due to $f = d\omega$ (3.8) and $\partial M = 0$. However for arbitrary topology it is not possible to globally define the dyad e^m, with the exception of the torus. This is equivalent to saying that it is not possible to define a continuous, non-vanishing vector field on 2d manifolds with $\chi(M) \neq 0$. Thus the dyads, and consequently the spin-connection, must be defined separately on each local open region U_I. The fields e_I and e_J, where the regions U_I and U_J overlap, are related to each other by an $SO(2)$ rotation $R_{IJ}(x) = e^{\psi_{IJ}(x)\tau}$,

$$e_I = R_{IJ} e_J, \tag{6.7}$$

where ψ_{IJ} is the transition function from region U_I to U_J and τ is the generator of the $so(2)$ algebra. The spin connection can only be defined locally (with the exception of the torus), and transforms under gauge transformations as

$$\omega_J = \omega_I + d\psi_{IJ}. \tag{6.8}$$

As Stokes' theorem (6.6) is a global argument it only applies when we can define the dyad and the connection globally. But as mentioned before, this is only possible for the torus for which $\chi(M) = 0$. In this particular case of GR on a torus, Stokes' theorem is not in contradiction with (2.69). Considering the principal bundle $P(M, SO(2))$ with 2d base manifold M, a nontrivial result from Stokes' theorem can be computed. First however, it is necessary to partition M, into regions in which w_I is a locally defined one-form, by a triangulation of the manifold M. This triangulation Δ consists of triangles meeting at edges which meet at points. It has an orientation inherited from M and the curvature can be regarded as being at the points of the triangulation, (those familiar with simplicial Regge geometry [87] will recognize this construction). For reasons that will soon become apparent, it is more convenient to work with a dual triangulation Δ^* with its faces dual to the original points, edges dual to edges and dual points correspond to the original faces of the triangulation. Along the boundary e_{IJ} between faces f_I and

f_J we assign an orientation and have

$$\begin{aligned}\int_M f &= \sum_{f_I} \int_{f_I} f = \sum_{f_I} \oint_{\partial f_I} \omega_I \\ &= \sum_{IJ} \omega_I = \sum_{(IJ)} \int_{e_{IJ}} (\omega_I - \omega_J) \\ &= \sum_{(IJ)} \int_{e_{IJ}} d\psi_{IJ}. \end{aligned} \tag{6.9}$$

Here (IJ) means unordered pairs and the transformation equation (6.8) has been used. From this we see that the curvature of the manifold is independent of the connection and only depends on the transition functions ψ_{IJ} which encode the topology of the bundle. There is a topological invariant of an $SO(2)$ principal bundle with a 2d base manifold which completely characterizes the topology called the Euler number $e(P)$. This is defined as the integral over the first Euler class of the manifold which in this case is just f

$$e(P) = \frac{1}{2\pi} \int f. \tag{6.10}$$

Thus from (2.69), (3.11) and the formula for the Euler number (6.10), we have

$$e(P) = \chi(M), \tag{6.11}$$

which is a well-known theorem by Gauss-Bonnet-Chern-Avez [4]. This result can be taken one step further by noting the following. In GR, the spin-connection ω is a connection on the principal bundle $P(M, SO(2))$. But because the connection satisfies the Cartan equations (2.8) it can be defined via the dyad fields e^m. These fields provide an isomorphism between the vector bundle on which ω acts and the tangent bundle. Thus ω acts on a bundle isomorphic to the tangent bundle and hence is in the frame bundle, the principal bundle of local rotations of the tangent bundle. It follows from (6.11) that in order to view 2d GR as a constrained BF-theory it must be ensured that the connection of the BF-theory is on a bundle with the same topology as the principal frame bundle. That is,

$$e(P_{fr}, SO(2)) = \chi(M). \tag{6.12}$$

When it comes to quantizing the constrained BF-theory, the non-triviality of the bundle must be considered and transition functions ψ_{IJ} picked that satisfy (6.12).

6.1.2 Discrete bundles

The idea of a discrete bundle will be introduced in this section. That is, a bundle with a discretized (triangulated) manifold. Consider a 2d base manifold with dual triangulated decomposition Δ^* over M. As discussed previously this cellular decomposition consists of 2d faces (labelled $f_I, f_J, f_K, \ldots$) dual to the 0-simplices of the original triangulization and edges e_{IJ} between faces which meet at *trivalent* vertices v_{IJK} dual to the original faces in Δ. Assuming that each face f_I is contained in some open subset $U_I \subset M$ the transition functions t_{IJ} [1] on the overlap $U_I \cap U_J$ (which also contains the edge e_{IJ}) define the bundle. Parameterizing the edge e_{IJ} with $s \in [0,1]$ we restrict the transition function to this edge, given by $t_{IJ}(s)$. The transition functions are now maps from the edges to G. They satisfy the following conditions for all points of s on which they are defined;

$$
\begin{aligned}
t_{II}(s) &= 1 && (6.13)\\
t_{IJ}(s)t_{JI}(s) &= 1 && (6.14)\\
t_{IJ}(s)t_{JI}(s)t_{KI}(s) &= 1. && (6.15)
\end{aligned}
$$

Note that the third condition holds only on the vertices as this is where all three functions are defined. Up to gauge transformations (which are now maps from the perimeter of a face f_I to G) the transition functions above define a principal bundle $P(\Delta^*, G)$ over the dual cellular complex. Associated to each edge is the variation of the transition function, which for the case of $G = SO(2)$, is given by

$$n_{IJ} = \frac{1}{2\pi} \int_{e_{IJ}} d\psi_{IJ}(s). \tag{6.16}$$

Following the gauge fixing arguments of [1], the bundle is characterized by the set of integers $n_{IJ} \in \pi_1(G)$, the first homotopy group of G [1]. Following

[1]To see this more explicitly, note that a gauge transformation in U_I adds a real number λ_{IJ}, the variation of the gauge transformation along the edge e_{IJ}, to each n_{IJ}. Writing

from the expression for the Euler number (6.10) and the fact that $n_{IJ} = n_{JI}$, one gets

$$e(P) = \frac{1}{2}\sum_{IJ} n_{IJ}. \tag{6.17}$$

From the above construction a 'discrete bundle' $P(\Delta^*, G)$ is defined as a principal G-bundle over a 2d cellular complex Δ^* with to each edge e_{IJ} the assignment of an element of $\pi_1(G)$ (up to gauge transformations).

6.2 Spin foam quantization of BF-theory

In this section, the quantization of BF-theory using spin foams will be discussed. Formally the partition function of BF-theory can be written as,

$$Z = \int dB\, d\omega\, e^{i\frac{k}{\hbar}\int_M tr[BF]}. \tag{6.18}$$

In order to explicitly calculate this partition function, a triangulation of the base manifold must be made. As can be seen from Fig. 6.1 the vertices of the dual triangulation, Δ^*, are trivalent. The continuous fields ω and B can now be replaced with discrete variables on Δ^*. The connection is replaced by the $SO(2)$ group element

$$g_{IJ} = e^{\frac{k}{\hbar}\int_{e_{IJ}}\omega}, \tag{6.19}$$

associated with the edge e_{IJ}. Since the edge e_{IJ} has the opposite orientation to e_{JI} we have the condition

$$g_{IJ}g_{JI} = 1. \tag{6.20}$$

Parameterizing $SO(2)$ as $g_{IJ} \equiv e^{\phi_{IJ}\tau}$, where $\phi_{IJ} \in [0, 2\pi]$, this is equivalent to

$$\phi_{IJ} + \phi_{JI} = 0. \tag{6.21}$$

$\lambda_{IJ} = \lambda_I^i - \lambda_I^j$, where i,j label the vertices of the edge e_{IJ}, the gauge transformation can be written as $n_{IJ} \to n_{IJ} + \lambda_I^i - \lambda_I^j + \lambda_J^j - \lambda_J^i$. Now it is always possible to choose a gauge transformation such that the transition functions satisfy $t_{IJ}(0) = t_{IJ}(1) = 1$. In this 'edge' gauge, the transition functions are a map, $t_{IJ} : S^1 \to G$. The gauge transformations deform this map smoothly and so it is the homotopy class that is relevant. In the case of $SO(2)$, n_{IJ} is just the number of times the transition function of an edge wraps around the group. It is an integer and in $\pi_1(SO(2))$.

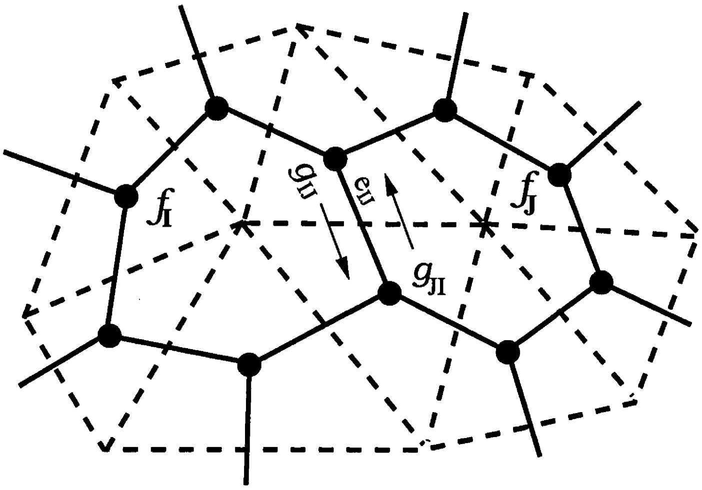

Figure 6.1: The original triangulation Δ is shown by the dashed lines and the dual triangulation Δ^* by solid lines. The group elements g_{IJ} and g_{JI} are on the edge e_{IJ}, which joins the two faces f_I and f_J.

On each face f_I the B-field is replaced with $B_a = b_a\tau \in so(2)^2$. Recalling that the curvature of M is located at the vertices of the initial triangulation Δ which correspond to the faces of Δ^*, a group element g_I related to the curvature on a face f_I can be defined. For a face f_I enclosed by faces $f_{J_1}, \ldots, f_{J_m}$

$$g_I = e^{\phi_I\tau} \equiv g_{IJ_1}, \ldots, g_{IJ_m} = e^{(\phi_{IJ_1}+\cdots+\phi_{IJ_m})\tau}, \tag{6.22}$$

where $\phi_I \in [0, 2\pi)$. For $g_I = 1$ on f_I, the discretized connection is flat. To simplify latter calculations, it is convenient to use the square root[3] of g_I, given by ${g_I}^{1/2} = e^{\frac{\phi_I}{2}\tau}$. Discretizing the BF action (3.7) by replacing the continuous variables with the new discrete variables gives

$$I_{BF} = -\frac{\hbar}{k}\sum_I tr[B_I g_I^{\frac{1}{2}}]. \tag{6.23}$$

[2]Where we take $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[3]The reason for taking the square root is that it allows one to avoid the doubling of the degrees of freedom introduced by the discretization, (see [1] for details).

In the limit of fine triangulations this action approximates the continuous BF-action (3.7)[4]. We are now in a position to perform the integration of the partition function (6.18).

$$Z = \int \prod_{IJ} dg_{IJ} \prod_{I} dB_I \, e^{\frac{ik}{\hbar} I_{BF}} = \int \prod_{IJ} dg_{IJ} \prod_{I} dB_I \, e^{-i \sum_I tr[B_I g_I^{\frac{1}{2}}]}, \quad (6.24)$$

where dg_{IJ} is the Haar measure of $SO(2)$. For dB_I we take the Lebesgue measure on $so(2)$, $dB_I = db_I$. Using

$$-tr[Bg] = -tr\left[b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi, \end{pmatrix}\right] = 2b \sin\phi, \quad (6.25)$$

one has

$$\begin{aligned} \int dB_I \, e^{-i\, tr[B_I g_I^{1/2}]} &= \int db_I \; e^{2ib_I \sin(\phi_I/2)} \\ &= 2\pi\, \delta(2\sin(\phi_I/2)) \\ &= 2\pi\, \delta(\phi_I) \\ &= \delta(g_I), \end{aligned} \quad (6.26)$$

where $\delta(g)$ is the delta function on the group. (Had g_I been used in the action rather than the square root, the result $\delta(g_I) + \delta(-g_I)$ would have been obtained). This results in the expression

$$Z_{BF} = \int \prod_{IJ} dg_{IJ} \prod_{I} \delta(g_I) \quad (6.27)$$

for the standard partition function for BF-theories. Here it is important to note that the δ-function on the group can be expanded in terms of irreducible representations using the Plancherel expansion. For the case of $SO(2)$, the irreducible representations are labelled by an integer n and have character $\chi^{(n)}(g) = e^{in\phi}$. Thus the delta function can be expanded as

$$\delta(\phi) = \frac{1}{2\pi} \sum_n e^{in\phi}, \quad (6.28)$$

[4]In the limit in which the triangulation is fine and the face f_I has area d^2x, one has $g_I^{\frac{1}{2}} = 1 + \frac{1}{2} f\tau d^2x$, where f in this expression is the curvature on the face. Thus to first order in the area, we have $tr[B_I g_I^{\frac{1}{2}}] \to tr[B(1+F)] = tr[BF]$.

which is simply the Fourier expansion of the δ-function. Substituting this into (6.27), giving the Haar measure as the normalized measure of the angle and replacing the sum over representations with a sum over faces f_I gives

$$Z = \sum_{n_I} \int \prod_{IJ} \frac{d\phi_{IJ}}{2\pi} e^{i\sum_I n_I \phi_I}, \tag{6.29}$$

where the first sum is over all possible assignments of a representation n_I to each face f_I. Using the group element (6.22) associated with the curvature of a face and the fact that $\phi_{IJ} + \phi_{JI} = 0$,

$$Z = \sum_{n_I} \int \prod_{IJ} \frac{d\phi_{IJ}}{2\pi} e^{i\sum_{IJ}(n_I - n_J)\phi_{IJ}}. \tag{6.30}$$

Using the formula for the orthogonality of characters, the integrals can be evaluated. Doing this one arrives at the partition function for unconstrained BF-theory,

$$Z_{BF} = \sum_n 1. \tag{6.31}$$

Just like the 4d partition function, (6.31) diverges. This partition function is an example of an (extremely simple) spin foam model where the faces are labelled by representations of the gauge group. In this simple case of 2d $SO(2)$ spin foams, the 'Clebsch-Gordon conditions' force the representations to be equal over all faces.

6.2.1 Quantum GR on trivial bundles

Recall that we are trying to investigate the validity of the Barrett-Crane technique for obtaining quantum GR. That is, view 2d Riemannian GR as a BF-theory with an imposed Plebanski constraint (3.10) and finding the corresponding constraint on the representations summed over in the spin foam model. The connection ω has been identified with the group element (6.19) associated with the edge e_{IJ}. The quantity which represents the B-field in the spin foam must also be found. Noting that the Fourier expansion of the B-field can be expressed in terms of the δ-function on the compact angle $\phi_I \in [0, 2\pi]$ and that the δ-function can be expanded as a sum over the representations n_I

$$\int db_I\, e^{ib_I\phi_I} = 2\pi\delta(\phi_I) = \sum_{n_I} e^{in_I\phi_I}, \tag{6.32}$$

one finds that the discrete variable n_I is the quantized version of the continuous variable B_I. The Plebanski constraint (3.10) on the B-field can now be interpreted as a constraint on the representations summed over in the spin foam model. Similar to the way angular momentum is associated with the generators of the rotation group, the continuous field $B \in so(2)$ can be identified with the $so(2)$ generator J (see [88] for a detail discussion of this identification of the B-field with the generators of the gauge group in the analogous 4d case). In this case, the generator J in the representation n, is simply $n\tau$. The Plebanski constraint (3.10) is now;

$$\begin{aligned} -\frac{1}{2}tr[BB] &= -\frac{1}{2}tr[JJ] = C \\ &= -\frac{1}{2}tr[n\tau n\tau] = n^2, \\ &\Rightarrow\; n^2 = 1. \end{aligned} \tag{6.33}$$

Note that $-\frac{1}{2}tr[JJ]$ is in fact the Casimir of $SO(2)$. The Plebanski constraint is imposed on the spin foam sum (6.29), by restricting the representations summed over to those satisfying the above condition. The divergent BF partition function (6.31) becomes

$$Z_{GR} = \sum_n \delta_{n^2,1} = 2, \tag{6.34}$$

which is now clearly finite. Recall that the partition function of BF-theory diverges. This is because the B-field has a symmetry under translations. The Plebanski constraint fixes this gauge which results in a finite partition function. As mentioned before, when dealing with GR it is necessary to consider bundles isomorphic to the frame bundle. But the only case in 2d where the frame bundle is trivial is for the torus for which $e(P_{fr}) = \chi(T_2) = 0$. Substituting this into the partition function (6.5) gives an answer of 2, which agrees exactly with the result from the spin foam formulation.

6.3 Quantum GR

Previously to each open subset U_I of a nontrivial bundle was associated a Lie algebra-valued one-form ω_I, the connection. In the case of a triangulated manifold, a connection ω_I is associated to each *face*, f_I. When dealing with the trivial bundle it was sufficient to consider the group element $g_{IJ} = e^{\frac{k}{\hbar}\int_{e_{IJ}}\omega}$, which satisfied the condition $g_{IJ}g_{JI} = 1$. However when working with nontrivial bundles it is necessary to consider *two* group elements assigned to the edge e_{IJ}. One coming from the connection ω_I on the face f_I and the other from ω_J associated with the face f_J. Due to the orientation on the edge, these two group elements g_{IJ} and g_{JI} are respectively denoted;

$$g_{IJ} = e^{\frac{k}{\hbar}\int_{e_{IJ}}\omega_I}, \quad g_{JI} = e^{\frac{k}{\hbar}\int_{e_{JI}}\omega_J}. \tag{6.35}$$

In the case of $G = SO(2)$, the connections are related by $\omega_I = \omega_J + d\psi_{IJ}$. From this and the expression (6.16) for $n_{IJ} \in \pi_1(SO(2))$,

$$g_{IJ}g_{JI} = e^{\frac{k}{\hbar}2\pi n_{IJ}\tau}, \tag{6.36}$$

which is equal to unity in the trivial case. Following the same procedure for the quantization as demonstrated on the torus, one gets (6.29) for the partition function. We can replace the sum over faces f_I to a sum over (ordered) pairs IJ by replacing ϕ_I in (6.29) with ϕ_{IJ} given by the curvature relation (6.22), where now $\phi_{IJ} + \phi_{JI} = \frac{k}{\hbar}2\pi n_{IJ}$ due to (6.36). Using this new relation for ϕ_{IJ},

$$Z = \sum_{n_I}\int\prod_{IJ}\frac{d\phi_{IJ}}{2\pi}e^{i\sum_{IJ}\left((n_I-n_J)\phi_{IJ}+n_J\frac{k}{\hbar}2\pi n_{IJ}\right)}, \tag{6.37}$$

(c.f. (6.31)). After integrating and using the expression (6.17) for the Euler number in terms of elements of $\pi_1(SO(2))$,

$$Z = \sum_n e^{in\frac{k}{\hbar}2\pi\sum_{IJ}n_{IJ}} = \sum_n e^{in\frac{2\pi k}{\hbar}e(P)}. \tag{6.38}$$

At this point it is interesting to observe how this result corresponds to the classical BF-theory. Recall that the solutions to the equations of motion of 2d BF-theory are *flat* connections, which only exist on trivial principal bundles, i.e. $e(P) = 0$. Performing the sum above for BF-theory we get the partition function

$$Z = \delta(k/\hbar e(P)). \tag{6.39}$$

Thus the partition function of BF-theory is zero unless the theory is defined on a trivial bundle. BF-theory only exists on trivial bundles. In order to get the right result for GR three conditions must be met;

$$\begin{aligned} &1) \quad \text{set } k = 1/8\pi G \\ &2) \quad \text{fix the bundle to be isomorphic to the} \\ &\quad\quad \text{frame bundle, that is require } e(P) = \chi(M) \\ &3) \quad \text{impose the Plebanski constraint } n^2 = 1. \end{aligned} \tag{6.40}$$

Imposing these three conditions on (6.38), the final result for the partition function is achieved,

$$Z = \sum_n \delta_{n^2,1} e^{in\frac{1}{4\hbar G}\chi(M)} = 2\cos\left(\frac{\chi(M)}{4\hbar G}\right). \tag{6.41}$$

This is the 2d Riemannian quantum GR partition function (6.5) that was obtained earlier using formal arguments and the Gauss-Bonnet theorem.

6.4 Quantum supergravity

Not to belabor the point, but LQG does not require supersymmetry in order to be consistent. As has been shown it is a background independent, non-perturbative theory of quantum gravity relying solely on the principals of GR and QM. Though if one wishes to make contact with string theory, currently the best attempt at unifying the laws of physics, one must at least consider supersymmetry and how it is to be implemented in LQG. With this thought in mind, we will now lay out how the ideas presented in this chapter may be extended to include supersymmetry. The starting point of this procedure is the partition function. As was explained in detail previously, the action of Howe's 2d, N=1 supergravity in superfield notation reduces to a total derivative and, like 2d GR, is topological. The gravitino terms in the connection have no effect and the action (2.70) is proportional to the Euler number $\chi(M)$. As such, we would expect the partition function to be the same as that of bosonic GR, namely (6.5). The partition function for Chamseddine's supersymmetric extension of the Jackiw-Teitelboim model was explicitly given in [8]. As we shown, both these models can be formulated as super BF-theories. To illustrate the procedure for extending the method [1] introduced in this chapter to supergravity, we shall take as our starting point the partition function of the super BF-action.

6.4.1 Discretization of the super BF-action

Starting with the action for super BF-theory (3.15), the partition function can formally be written as,

$$\begin{aligned} Z &= \int dA\, dB\, e^{i\frac{k}{\hbar}I} = \int dA\, dB\, e^{i\frac{k}{\hbar}\int Str[BF]} \\ &= \int dA\, \delta(F). \end{aligned} \tag{6.42}$$

The first step towards evaluating the partition function involves formally integrating out the B-field from the path integral, which is possible as it appears linearly in the exponential. As expected, what remains is an integral over flat connections. In order to make sense of the delta function on the curvature in the remaining path integral, it is convenient to triangulate the manifold. As before, the base manifold is decomposed into a 2d dimensional triangulation Δ. Again, it is the dual triangulation Δ^* that will actually be used. Discretizing the connection A, as in lattice gauge theory, an element of $OSp(1,2)$, the parallel transport $g_e(A)$, is associated with each edge e. The curvature F is measured through the superholonomies $h_I = (g_{e_1}, \dots, g_{e_k})$ i.e. the products of parallel transports, around each face f_I, bounded by k edges e,

$$\prod_{e \in \partial f} g_e \equiv e^{iF}. \tag{6.43}$$

Consequently the curvature F can be thought to be located at the center of the dual faces, which corresponds to the nodes of the original triangulation. The condition of zero curvature becomes the condition that the superholonomies vanish. That is, the product of group elements around each face has to be the unit element. The procedure for discretizing the B-field is entirely analogous to the bosonic case presented before. The continuous B-field in the super BF-theory is replaced by an element $B_I = b^a P_a + bJ + b^\alpha Q_\alpha \in OSp(1,2)$ on a face f_I in the dual triangulation. Replacing the continuous variables in the classical super BF-action with the discrete variables $[B_I, h_I]$ gives the discrete action,

$$I = -\frac{\hbar}{k}\sum_I Str[B_I h_I], \tag{6.44}$$

where the sum is over all faces f_I. Substituting this discrete action into the formal expression for the partition function (6.42) and replacing the measure dA with the super Haar measure dh_e gives

$$Z = \int \prod_e dh_e \prod_I dB_I \, e^{-i\sum_I Str[B_I h_I]}, \tag{6.45}$$

where the first product is over all edges and the second over all faces of the dual triangulation. Also note that dh_e represents the super Haar measure on edge e, whereas h_I represents the superholonomy around the face f_I. Now integrating out the discrete B-field, the discretized version of the super BF-action has the following partition function

$$Z = \int \prod_e dh_e \prod_I \delta(g_{e_1}, \dots, g_{e_k}), \tag{6.46}$$

where $(g_{e_1}, \dots, g_{e_k})$ is the product of group elements associated with the edges e bounding the face f_I. The delta function in the above equation vanishes unless this product of group elements is the unit element. At this point in the procedure for the $SO(2)$ case presented before, the delta function was expressed as a sum over irreducible representations of $SO(2)$ using the Plancherel expansion (6.28). More generally, the delta function can be expressed as a sum of characters of irreducible representations

$$\delta(g) = \sum_\Lambda \dim(\Lambda)\, \chi_\Lambda(g). \tag{6.47}$$

The extension of this expression for the delta function to the case of supergroups, follows from a generalization of the Peter-Weyl theorem, (see [59] and references therein). Before continuing, we should briefly discuss the representation theory of $OSp(1,2)$ [92]. Each irreducible representation of $OSp(1,2)$ contains two adjacent $SU(2)$ representations, labelled by the spins n and $n-\frac{1}{2}$, with $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The spin n is taken as the label of the $OSp(1,2)$ representation;

$$R^n = V^n \oplus V^{n-\frac{1}{2}}, \qquad n \geq \frac{1}{2}. \tag{6.48}$$

In the case of $n = 0$ one has the trivial one dimensional representation. Looking at Fig. 6.2, one sees that the $su(2)$ generators act as usual on

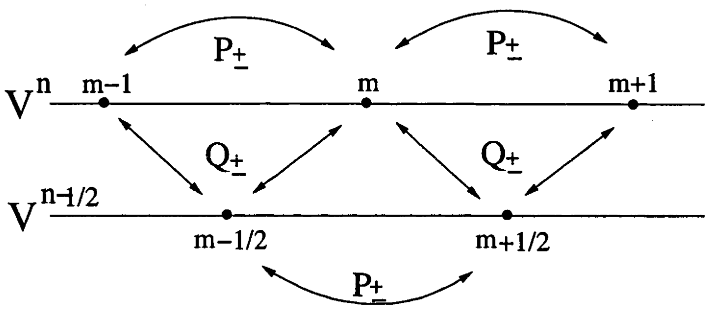

Figure 6.2: A graphical illustration of the structure of the representation R^n of $OSp(1,2)$. The identification $P_a \rightarrow P_\pm$ and $Q_\alpha \rightarrow Q_\pm$ has been made to make clear the action of the generators of the Lie superalgebra on the representation.

each level. The fermionic generators allow one to go from one level to another. Clearly the dimension of a representation R^n is $4n+1$. Formally, the quadratic Casimir operator is [58],

$$C = P^a P_a + JJ + \epsilon^{\alpha\beta} Q_\alpha Q_\beta \tag{6.49}$$

and the eigenvalues of the Casimir are;

$$C_n = n(n+\tfrac{1}{2}). \tag{6.50}$$

Returning to the question of expanding the delta function of a group element in terms of characters of irreducible representations, when dealing with supergroups, it is necessary to replace the dimension of the representation with the super-dimension. For $OSp(1,2)$ the dimension is $(4n+1)$ and the super-dimension is 1 for all finite dimensional irreducible representations. Thus, expressing the delta function over the group elements in the partition function in terms of characters of irreducible representations results in the partition function being

$$Z = \int \prod_e dh_e \prod_I \sum_n \chi_n(g_{e_1}, \ldots, g_{e_k}). \tag{6.51}$$

This is the partition function for super BF-theory with gauge group $OSp(1,2)$. In the $SO(2)$ case it was relatively simple to evaluate the value of the corresponding bosonic partition function because the characters of irreducible representations of $SO(2)$ are known and are very simple. For the spin foam quantization of 4d GR with $SO(4)$ gauge group, it is possible to use the character decomposition formula which decomposes the character of a given representation of a product of group elements into a product of Wigner D-functions in that representation [89]. Unfortunately, for the supergroup $OSp(1,2)$ the character formulas are quite complex [90] and make the explicit evaluation of the super BF partition function incredibly difficult. To the author's knowledge, the method presented in [20] is the only attempt in the literature to solve this partition function (actually in the 3d case). This method avoids explicitly using the characters of $OSp(1,2)$, instead representing the partition function diagrammatically using *circuit diagrams*. As a result of this method, the partition function is finally expressed in terms of the Wigner $6j$-symbols of $su(2)$. Unfortunately the evaluation of the partition function in this form is also difficult and it remains on open problem to date.

Though we were unable to explicitly evaluate the partition function of super BF-theory in 2d with $OSp(1,2)$ gauge group, we know that in order to make contact with the supergravity model of Howe, it is necessary to restrict the representations summed over in the partition function. It is this restriction that we will now discuss.

6.4.2 The quantum constraint

To formulate 2d supergravity from super BF-theory, it was shown previously (c.f. 3.33) to be necessary to add a constraint to the action. This is also the case for bosonic GR in two, three and four dimensions. The reason being that only *flat* connections are solutions to the equations of motion of BF-theory, whereas *any* connection is a solution of 2d GR. By adding a constraint to the BF-action, the number of degrees of freedom is increased. The added constraint effectively constrains a constrainer. Specifically for the case of Howe's 2d supergravity, it was shown that by adding the constraint $(B \wedge B - \tilde{\Lambda})$ to the action via a Lagrange multiplier two-form would lead to 2d supergravity provided a particular value for the variable $\tilde{\Lambda}$ was chosen. Integrating out the Lagrange multiplier from the action (3.33) gave the Plebanski constraint

$$Str(B \wedge B) = Str(\tilde{\Lambda}) \tag{6.52}$$

The question was, how should this constraint be imposed at the *quantum* level? The answer to this question was illustrated in the example of the Barrett-Crane model of 4d gravity. Adding a constraint to the classical super BF-action results at the quantum level in a restriction of the representations summed over in the spin foam model. Likewise, for 2d GR, it was shown that by adding a constraint to the BF-action at the classical level, the representations summed over were restricted and the partition function of the BF-theory became finite and agreed with that of GR. Even though it was not possible to explicitly calculate the partition function of 2d $OSp(1,2)$ supergravity, it appears that this should also be the case. Following the example of the Barret-Crane model, if the B-field of the original super BF-action is identified as a generator of $OSp(1,2)$, then the Plebanski constraint gives

$$\begin{aligned} Str(\tilde{\Lambda}) &= Str(B \wedge B) \\ &= Str(b_a b_b P^a P^b + b^2 J^2 + b_\alpha b_\beta Q^\alpha Q^\beta). \end{aligned} \tag{6.53}$$

One can see that this expression is related to the $OSp(1,2)$ Casimir[5] (6.49). Recalling that the Casimir has eigenvalues $C_n = n(n+\frac{1}{2})$ on an irreducible representation n, and setting $\tilde{\Lambda}$ to the values (3.35), calculated previously to recover Howe's supergravity, we have the result;

$$C_n = Str(\tilde{\Lambda})$$

[5]This expression is actually the supertrace version of the $osp(1,2)$ Killing form.

$$
\begin{aligned}
&= \frac{\pm\lambda'^2}{4} \\
&= n(n+\tfrac{1}{2}).
\end{aligned} \tag{6.54}
$$

So, to recover Howe's supergravity from the spin foam formulation of super BF-theory, it is necessary to restrict the representations summed over in the spin foam to those representations n that satisfy the above condition. However, there is a problem. Recall that the final step in writing 2d supergravity as a constrained super BF-theory was to take the limit of $\lambda' \to 0$, corresponding to the supersymmetric extension of $ISO(2)$. When this limit is taken the representations, due to the above condition, are restricted to the value $n = 0$. Therefore *all* the representations on the edges of the spin foam are simply the trivial one-dimensional representation. Thus it appears from this general argument, that it is not possible to formulate Howe's 2d supergravity as a spin foam model and recover the correct partition function (6.5). It does appear however, that the procedure described in this chapter would be suitable for formulating 2d supergravity models with nonzero cosmological constant as spin foam models. It is the cosmological constant in these models that ultimately restricts the representations summed over in the spin foam models. Thus, in order to have a nontrivial spin foam, it is necessary to have a nonzero cosmological constant.

Chapter 7

Conclusion

To date, LQG represents one of the most promising approaches to the open problem of constructing a quantum theory of the gravitational field. What characterizes this attempt from others is the assumption that the lessons learned from QM and GR should be considered seriously, without any further assumptions; mathematical or physical. Making no assumptions on how physics should behave at the Planck scale, combining QM and GR and trying to push them to their extreme consequences leads to some amazing consequences. Most notably the description of the quantum properties of space at the smallest scales, where the classical continuous space of our everyday experience takes on a discrete, granular nature. LQG provides a diffeomorphism invariant, background independent, non perturbative quantum theory of geometry and is therefore an appealing candidate for a theory of quantum gravity. That is not to say that LQG is complete. There remain many open questions, including its connection to string theory and inclusion of supergravity. By studying more simple and tractable models, it is possible to make progress and not only answer certain questions but learn what questions are good to ask and will lead one in the right direction. It is with this idea in mind that we analyzed the simple model of 2d supergravity in LQG, with the hope of finding answers that indicate the properties we would expect to find in the full and complete theory of quantum gravity. Having discussed the basics of GR and 2d supergravity in the hamiltonian and superspace setting respectively, in Chapter 3 we presented the topological BF-theory as the starting point into our investigation of supergravity in LQG. The reason was twofold; the methods for formulating BF-theory in the context of LQG is already well known, as well is the idea of expressing supergravity actions as BF-actions with a classical constraint imposed. However, to the author's knowledge, this thesis represents the first time that the 2d supergravity model in [5] has been expressed in this manner, imposing a constraint on the classical B-field through an extension of the cosmological constant. After presenting the foundations of LQG in Chapter 4, we introduce the quantum states of the Hilbert space of LQG, the spin networks, in Chapter 5. These are the eigenstates of the quantum operators corresponding to classical geometrical quantities. In other words, they are discrete quantum excitations of space itself. They are one dimensional objects label by the representations of the gauge group under consideration and invariant under spatial diffeomorphisms. Something quite different from the usual Fock states of standard quantum field theory. The question of how to extend these spin networks to include supersymmetry is an interesting one that has received little attention in the literature. In the case of this thesis the topic of interest, that is $OSp(1,2)$ spin networks, could be found from the bosonic networks by simply replacing the gauge group with the supergroup $OSp(1,2)$ and relabelling all the edges accordingly. Having formulated the notion of a supersymmetric spin network, in the final chapter we presented the spin foam quantization procedure developed in [1], beginning with a brief discussion on the topological nature of 2d bosonic gravity including the calculation of the partition function. As the action of Howe's 2d supergravity is also topological in nature and equivalent to the Euler character, it was expected to share the same partition function and hence would be a good candidate for attempting to extend the spin foam quantization procedure to supergravity. It was shown that in order to quantize a constrained BF-theory using spin foams that it is necessary to ensure that the topology of the bundle is the same as the frame bundle. It is also necessary to discretize the continuous variables of the theory and this was accomplished by using a triangulation, or more accurately the dual triangulation of the base manifold. With the action of the BF-theory in a discrete form it is possible to explicitly calculate the partition function of 2d $SO(2)$ BF-theory, which diverges. However, by adding a Plebanski constraint at the classical level to the BF-action, corresponding at the quantum level to a restriction of the representations summed over, it was possible to achieve a finite partition function. Not only this, but by satisfying a number of other conditions, the partition function of the constrained BF-theory was shown to be equivalent to that of 2d GR. The final step was to extend this procedure to supergravity and answer the question of how to impose the classical constraint on the super BF-action at the quantum level in the case of including supersymmetry. Initially the procedure follows the bosonic case; the action is discretized by a triangulation of the base manifold and the partition function is calculated. Unfortunately due to the complicated nature of $OSp(1,2)$ characters, we have not succeeded in explicitly calculating the partition function for 2d supergravity. The final consideration was how the classical constraint on the super BF-action should be implemented at the quantum level. It was found that the value of the cosmological constant at the classical level, directly determined which representations of the spin foam should be summed over at the quantum level. As Howe's model of supergravity has as its gauge group the supersymmetric extension of $ISO(2)$ and a zero cosmological constant, the only possible spin foam model is the trivial one, i.e all representations on the edges of the spin foam are the one-dimensional trivial representation. The procedure though, of formulating 2d supergravity as a spin foam model with a restriction on the representations coming from the classical action appears to be applicable for models with a nonzero cosmological constant. However, it remains an open problem, how one could explicitly calculate the partition functions of such models.

Appendix A

Conventions

Throughout this thesis the following index notation has been used unless otherwise stated. Greek letters from the middle of the alphabet $\mu, \nu, \ldots$ are used as 4d spacetime indices and as the indices of the anticommuting spinor coordinates, θ, of superspace. The indices $m, n, \ldots$ are 2d spacetime indices and lower case Latin letters $i, j, \ldots$ are 3d spatial indices, (and occasionally $SU(2)$ vector indices). Lower case Latin letters $a, b, \ldots$ from the beginning of the alphabet are tangent space indices in 2, 3 and 4 dimensions. Spinor indices are given by the Greek letters from the beginning of the alphabet $\alpha, \beta, \ldots$ Upper case indices $M, N, \ldots$ are superspace coordinate indices and $A, B, \ldots$ the corresponding tangent space indices. The upper case indices $I, J, \ldots$, when discussing the triangulation of a manifold, label the faces (and also the edges and vertices).

A.1 Two dimensional superspace

The coordinates of (2+2)-superspace are $z^M = (x^m, \theta^\mu)$, where the x^m (θ^μ) are even (odd) elements of a Grassmann algebra. As elements of a Grassmann algebra, the following relation holds;

$$z^M z^N = (-1)^{|M||N|} z^N z^M, \tag{A.1}$$

where $|M| = 0$ (1) for bosonic (fermionic) indices M.
The bosonic metric and antisymmetric tensor are;

$$\eta_{ab} = \mathrm{diag}(-1, +1), \quad \epsilon_{ab} = -\epsilon_{ba}, \quad \epsilon_{01} = 1,$$

$$\epsilon^{ab} = -\epsilon_{ab}, \qquad \epsilon_{ab}\epsilon^{bc} = \delta_a{}^c. \tag{A.2}$$

The fermionic metric is

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}, \qquad \epsilon_{12} = 1 = -\epsilon_{21}, \qquad \epsilon_{11} = \epsilon_{22} = 0 \tag{A.3}$$

and the contractions are given by

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta_\alpha{}^\gamma, \qquad \epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = 2. \tag{A.4}$$

The gamma matrices are chosen to be real and satisfy

$$\gamma^a\gamma^b = \eta^{ab} - \epsilon^{ab}\gamma^5. \tag{A.5}$$

With $\gamma^5 = \gamma^0\gamma^1$, the following relations can be deduced;

$$[\gamma^a, \gamma^b] = -2\epsilon^{ab}\gamma^5, \quad [\gamma^a, \gamma^5] = 2\epsilon^{ab}\gamma_b, \quad \gamma^a\gamma^5 = \epsilon^{ab}\gamma_b. \tag{A.6}$$

An explicit representation is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

The gamma matrices are related to the standard Pauli matrices σ, by

$$\gamma^0 = -\gamma_0 = i\sigma_2, \qquad \gamma^1 = \gamma_1 = \sigma_1, \qquad \gamma^5 = \sigma_3. \tag{A.8}$$

A.2 Supermatrices

Supergroups can be conveniently represented by matrices acting on a superspace (supermatrices). Supermatrices act linearly on the coordinates of superspace leaving invariant the partition among even and odd coordinates. The coordinates of superspace form an $(m+n)\times 1$ column vector with the first m entries (last n entries) being even (odd) elements of the Grassmann algebra. The $(m+n)\times(m+n)$ supermatrix is written in partitioned block form

$$M = \begin{pmatrix} A_{m\times m} & B_{m\times n} \\ C_{n\times m} & D_{n\times n} \end{pmatrix}, \tag{A.9}$$

where the components $A_{ij}, D_{\alpha\beta}$ $(B_{i\alpha}, C_{\alpha i})$ are even (odd) elements of the Grassmann algebra.
The addition and multiplication of supermatrices according to the rules;

$$\begin{aligned}(M_1+M_2)_{MN} &= (M_1)_{MN}+(M_2)_{MN},\\ (M_1M_2)_{MN} &= \sum_P (M_1)_{MP}(M_2)_{PN},\end{aligned} \tag{A.10}$$

is such that it produces another supermatrix.

A.3 The supertrace

The basic invariant of a supermatrix is the supertrace,

$$Str(M) = tr(A) - tr(D) = \sum_{P=1}^{m+n} (-1)^{|P|} M_{PP}, \tag{A.11}$$

which is defined so that the cyclic property

$$Str(M_1M_2) = (-1)^{|M_1||M_2|} Str(M_2M_1) \tag{A.12}$$

holds for arbitrary supermatrices.

A.4 The superalgebra of OSp(1,2)

The superalgebra $osp(1,2)$ is the simplest of the orthosymplectic groups and can be viewed as the supersymmetric version of $su(2)$. It contains three bosonic generators P_a, J which form the Lie algebra $su(2)$ and two fermionic generators Q_α. The non-vanishing commutation relations are,

$$\begin{aligned}&[P_a, P_b] = 2\lambda'^2 \epsilon_{ab} J, \qquad [J, P_a] = \epsilon_a{}^b P_b,\\ &[J, Q_\alpha] = \tfrac{1}{2}(\gamma^5)_\alpha{}^\beta Q_\beta, \quad [P_a, Q_\alpha] = \tfrac{\lambda'}{\sqrt{2}}(\gamma_a)_\alpha{}^\beta Q_\beta,\\ &\{Q_\alpha, Q_\beta\} = \sqrt{2}(\gamma^a)_{\alpha\beta} P_a - \lambda'(\gamma^5)_{\alpha\beta} J,\end{aligned} \tag{A.13}$$

with the parameter λ'^2 taking on positive, vanishing or negative values[1]. The graded invariant quadratic form consistent with a non-degenerate Casimir

[1]The sign of λ'^2 dictates whether J is a compact ($so(2)$) or noncompact ($so(1,1)$) generator of $sp(2)$. The author would like to thank the examiners of this thesis for pointing this out.

operator (except for the supersymmetric extension of $iso(2)$) is;

$$\langle P_a, P_b\rangle = \eta_{ab}, \quad \langle J, J\rangle = 2\lambda'^2, \quad \langle Q_\alpha, Q_\beta\rangle = \tfrac{\lambda'}{2}\epsilon_{\alpha\beta}, \tag{A.14}$$

with all other relations equal to zero.

The Casimir operator is $C = \eta_{ab}J^aJ^b + J^2 - \lambda'\epsilon_{\alpha\beta}Q^\alpha Q^\beta$ and has the eigenvalues $n(n+\frac{1}{2})$ on the representation labelled by the (half)-integer n. The dimension of an irreducible representation of $osp(1,2)$ is $(4n+1)$.

The fundamental representation of $osp(1,2)$, which is 3 dimensional, is generated by five 3×3 matrices. They are given explicitly as,

$$\begin{aligned} P_0 &= \frac{\lambda'}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, P_1 = \frac{\lambda'}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Q_1 &= \sqrt{\frac{\lambda'}{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Q_2 = \sqrt{\frac{\lambda'}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{A.15}$$

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