Coherence for rewriting 2-theories

General theorems with applications to presentations of Higman-Thompson groups and iterated monoidal categories.

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Declaration

The work in this thesis is my own except where otherwise stated.

[Signature]

Jonathan Asher Cohen
Dedicated to the memory of my grandmother
Shirley Esther Lipinski (1930–2006)
who always listened to my ramblings.
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Abstract

The problems of the identity of proofs, equivalences of reductions in term rewriting systems and coherence in categories all share the common goal of describing the notion of equivalence generated by a two-dimensional congruence. This thesis provides a unifying setting for studying such structures, develops general tools for determining when a congruence identifies all reasonable parallel pairs of reductions and examines specific applications of these results within combinatorial algebra. The problems investigated fall under the umbrella of “coherence” problems, which deal with the commutativity of diagrams in free categorical structures — essentially a two-dimensional word problem. It is categorical structures equipped with a congruence that collapses the free algebra into a preorder that are termed “coherent”.

The first main result links coherence problems with algebraic invariants of equational theories. It is shown that a coherent categorification of an equational theory yields a presentation of the associated structure monoid. It is subsequently shown that the higher Thompson groups $F_{n,1}$ and the Higman-Thompson groups $G_{n,1}$ arise as structure groups of equational theories, setting up the problem of obtaining coherent categorifications for these theories.

Two general approaches to obtaining coherence theorems are presented. The first applies in the case where the underlying rewriting system is confluent and terminating. A general theorem is developed, which applies to many coherence problems arising in the literature. As a specific application of the result, coherent categorifications for the theories of higher order associativity and of higher order associativity and commutativity are constructed, yielding presentations for $F_{n,1}$ and $G_{n,1}$, respectively.

The second approach does not rely on the confluence of the underlying rewriting system and requires only a weak form of termination. General results are obtained in this setting for the decidability of the two-dimensional word problem and for determining when a structure satisfying the weakened properties is coherent. A specific application of the general theorem is made to obtain a conceptually straightforward proof of the coherence theorem for iterated monoidal categories, which form a categorical model of iterated loop spaces and fail to be confluent.
# Contents

Abstract \hspace{20pt} v

Chapter 1. Introduction \hspace{20pt} 1

Chapter 2. Rewriting 2-theories \hspace{20pt} 6
  2.1. Syntax \hspace{20pt} 7
  2.2. Semantics \hspace{20pt} 11
  2.3. Relation to other systems \hspace{20pt} 14
  2.4. Coherence \hspace{20pt} 15
  2.5. Categorification \hspace{20pt} 18
  2.6. Basic properties \hspace{20pt} 20

Chapter 3. Structure monoids \hspace{20pt} 23
  3.1. Structure monoids \hspace{20pt} 23
  3.2. Structure monoids via coherence theorems \hspace{20pt} 26

Chapter 4. Coherence for complete theories \hspace{20pt} 31
  4.1. Classical lemmas \hspace{20pt} 32
  4.2. Coherence for directed theories \hspace{20pt} 35
  4.3. Coherence for invertible theories \hspace{20pt} 38

Chapter 5. Catalan categories \hspace{20pt} 41
  5.1. The groups $F_{n,1}$ and $G_{n,1}$ \hspace{20pt} 42
  5.2. $F_{n,1}$ and $G_{n,1}$ as structure groups \hspace{20pt} 44
  5.3. Catalan categories and $F_{n,1}$ \hspace{20pt} 48
  5.4. Symmetric Catalan categories and $G_{n,1}$ \hspace{20pt} 50

Chapter 6. Coherence for incomplete theories \hspace{20pt} 55
  6.1. Subdivisions \hspace{20pt} 56
  6.2. Lambek coherence \hspace{20pt} 60
  6.3. Mac Lane coherence \hspace{20pt} 61
  6.4. Finite Mac Lane coherence \hspace{20pt} 65

Chapter 7. Iterated monoidal categories \hspace{20pt} 70
  7.1. Definitions and basic properties \hspace{20pt} 70
  7.2. Proving coherence \hspace{20pt} 72

Chapter 8. Conclusion \hspace{20pt} 76

Bibliography \hspace{20pt} 79
CONTENTS
CHAPTER 1

Introduction

Coherence problems arise in category theory when one wishes to describe the free algebra generated by a particular structure. Typically, this problem boils down to solving a sequence of word problems: Which functors are equal? Which natural transformations are equal? Which modifications are equal? And so on up the dimensions. Our main interest here is in two-dimensional categorical structures. Within this context, coherence problems are related to several other problems: When are two proofs of the same theorem equivalent? When do two interpretations of the same sentence assign the same meaning? When do two programs implement the same algorithm? In order to gain some insight into the importance and meaning of coherence problems, we explore the analogy with natural and artificial languages slightly deeper.

A written language may be thought of as a collection of symbols together with rules for manipulating and combining them. A sequence of such symbols is called a sentence. A sentence is grammatical if it can be constructed via the rules of the language.

Attempting to ascribe meaning to sentences of a language is potentially fraught with difficulty. For a simple mathematical language, such as arithmetic, the meaning of a sentence is abundantly clear — it is the natural number obtained by carrying out the described calculation. For more complicated constructions, such as natural language, the problem can be significantly more difficult.

One typically wishes to assign a meaning to every possible grammatical sentence of a language. If one considers sentences to be completely independent of each other, then, for any reasonably complex language, one would need to decide on the meaning of infinitely many sentences. Such a task is unreasonable in practice. One way in which to resolve this situation is to suppose that the language is compositional. That is, that the meaning of a sentence is composed from the meaning of its subparts. It is important to note that two related claims are being made here. First, there is a collection of basic syntactic structures, which carry meaning. These can be words, such as "dog", "cat", "table", "chair" etc., or they may be more complicated phrases or sentences. The second claim is that the meaning of a sentence built from these basic pieces is a composition of the meanings of the pieces. This compositionality principle is appealing on a number of levels, not least of all because it provides a reasonable explanation for a person's ability to comprehend sentences that they hear for the first time. A more technical reason is that one can show that any recursively enumerable language can be captured by some compositional grammar [Jan96].

Within richly expressive languages, there is the potential for structural ambiguity. That is, a given sentence may have two distinct meanings even though the meanings of the individual words remains constant. For example, the sentence "The shooting of
the hunters was terrible” may mean that the hunters had terrible aim, or that it was a shame that the hunters were shot. Within the framework of compositionality, the two meanings could only have arisen from composing the words in a different manner.

When designing a computer programming language, one typically wishes to avoid the presence of any structural ambiguity. More technically, any two proofs (“compositions”) of the same typing judgement (“sentence”) must carry the same meaning [CG90, Rey91].

A language that contains no structural ambiguity whatsoever is termed “coherent”. The name stems from Mac Lane’s construction of a coherent language for a monoidal structure on a category [ML63], which is the real starting point for this thesis.

Mac Lane [ML76] attributes the motivation for his development of the theory of monoidal categories to a question of Norman Steenrod: When is there a canonical map between two specified formal combinations of modules? Steenrod was considering the category of all modules over a commutative ring and the combinations of such modules by applying the functors \( \otimes \) and \( \text{Hom} \). Monoidal categories abstract the structure of the tensor product of modules to create a bifunctor \( \otimes \) on an arbitrary category. The main result of [ML63] says, essentially, that any two \( n \)-fold products that contain the same objects in the same order are naturally isomorphic via a unique canonical natural isomorphism. Interpreting the \( n \)-fold products as parsings of sentences and natural isomorphisms as weak equivalences between parsings, this result is akin to saying that monoidal categories do not contain any structural ambiguity.

The investigation of coherence is certainly not limited to monoidal categories. Indeed, one may hope for a version of Mac Lane’s theorem for many different types of covariant structures. A covariant structure on a category \( \mathcal{C} \) consists of:

- A collection of basic functors of the form \( \mathcal{C}^n \to \mathcal{C} \).
- A collection of equations between certain pairs of formally different terms built from the basic functors.
- A collection of natural transformations between certain terms formed from the basic functors.
- A collection of equations between pairs of formally different natural transformations constructed via a sequence of compositions and substitutions of the basic natural transformations. These are typically called coherence axioms.

One of the most basic covariant structures is that of a coherently associative bifunctor \( \otimes \). This structure consists of a natural isomorphism \( \alpha : a \otimes (b \otimes c) \to (a \otimes b) \otimes c \) together with a coherence axiom stipulating that the following diagram commutes:
A special case of Mac Lane's coherence theorem for monoidal categories states that any other diagram constructed from $\alpha$, $\otimes$ and the identity natural isomorphisms commutes by virtue of the commutativity of the above diagram.

In endeavouring to construct an analogous coherence theorem for an arbitrary covariant structure carried by a category, one may ask two related questions:

1. Is a given covariant structure coherent?
2. What coherence axioms are required in order to make a given covariant structure coherent?

The main goal of this thesis is to tackle the above questions in the greatest possible generality as well as to develop applications of the resulting coherence theorems. In the following section, we give a more complete outline.

Outline

Chapter 2: The chapter starts by developing a definition of rewriting 2-theories. These form the main framework for our investigations and the chapter describes the free algebra generated by a rewriting 2-theory before showing that a rewriting 2-theory defines a Lawvere 2-theory and, hence, a covariant structure. After briefly discussing relations to other existing systems, the coherence problem is rigorously defined within the context of rewriting 2-theories. Categorifications are introduced as a method for weakening an equational variety into a categorical structure and it is shown that a coherent categorification of an equational variety defines an equivalent categorical structure to the variety. Finally, some useful general tools for working with rewriting 2-theories are introduced.

Chapter 3: Dehornoy [Deh93] introduced structure monoids as algebraic invariants of equational varieties. The main result of the chapter shows how to construct a presentation of the structure monoid of an equational variety $\mathcal{E}$ from a coherent categorification of $\mathcal{E}$. In certain situations, the structure monoid forms a group in a natural way and the result is extended to this setting.

Chapter 4: The main direction of this chapter is to generalise Mac Lane's proof of coherence for monoidal categories to rewriting 2-theories that are confluent and terminating. "Terminating" means that there are no infinite chains of non-identity morphisms, while "confluence" is the property that every span may be completed into a square, as in the following diagram:

\[
\begin{array}{ccc}
& & \\
& \swarrow & \\
\downarrow & & \\
& \searrow & \\
\end{array}
\]

Subsequently, a general coherence theorem is developed for rewriting 2-theories describing invertible covariant structures, which directly generalises the situation of monoidal categories.
Chapter 5: This chapter develops a surprising application of the results of Chapter 4. Dehornoy [Deh05] has previously shown that Thompson’s group $F$ is the structure group of the variety of semigroups and that Thompson’s group $V$ is the structure group of the variety of commutative semigroups. Dehornoy also constructed presentations of these groups using Mac Lane’s coherence axioms for the associated categorifications. In light of the results of Chapter 3, these presentations are not too surprising. Indeed, the work in Chapter 3 was directly motivated by these results. Chapter 5 begins by constructing varieties for higher-order associativity and higher-order associativity and commutativity. It is shown that the structure groups for these are the higher Thompson groups $F_{n,1}$ and the Higman-Thompson groups $G_{n,1}$, respectively. The chapter goes on to construct categorifications of these varieties and thereby to obtain new presentations of $F_{n,1}$ and $G_{n,1}$. The coherence axioms for the categorifications directly generalise Mac Lane’s axioms for the binary case, although a new class of coherence axioms is required in the higher-order case that are not present in the binary situation.

Chapter 6: It is not the case that every coherent rewriting 2-theory is terminating and confluent. This chapter develops general coherence theorems for rewriting 2-theories that are not confluent and only weakly terminating, in a precise sense. The techniques are radically different from those of Chapter 4. The driving philosophy is that a parallel pair of morphisms are equal if and only if they admit a subdivision, each face of which commutes. As such, the approach is primarily through topological graph theory, where a subdivision is defined as a certain ambient-isotopy class of planar graph embeddings whose boundary consists of the parallel pair of maps under investigation. The resulting coherence theorem is also used to construct examples of finitely presented rewriting 2-theories that cannot be made coherent via only finitely many coherence axioms, but are otherwise well behaved. The related coherence problem of when there exists a decision procedure for the commutativity of diagrams arising from a rewriting 2-theory is also briefly investigated.

Chapter 7: Iterated monoidal categories [BFSV03] arose as a categorical model of iterated loop spaces. As a rewriting 2-theory, they are particularly interesting because they possess a nontrivial equational theory on both objects and morphisms, as well as being non-confluent. A highly technical proof that iterated monoidal categories are coherent is given in [BFSV03]. After introducing iterated monoidal categories, this chapter goes on to exploit the results of Chapter 6 in order to obtain a new, conceptually straightforward proof of coherence.

The inter-dependence between chapters is indicated in the following Hasse diagram:
Throughout this thesis, we read $f \cdot g$ as "$f$ followed by $g$".
CHAPTER 2

Rewriting 2-theories

Our main goal in this chapter is to define the class of two-dimensional algebraic structures that form the basis of the following chapters. The definition that we develop uses a base set of variables. This is in contradistinction with the standard approach to two-dimensional universal algebra, which prefers a variable-free approach via categorical constructions. The reason for choosing to work with variables is rather utilitarian: it retains a strong link to first-order term rewriting theory and, therefore, preserves the strong link with various computational and linguistic constructions. A more pragmatic reason for our definition in terms of variables is that it is precisely what allows us to bring various computational and combinatorial techniques to bear on otherwise categorical constructions. The choice of working with variables has two technical implications. First, it makes the transition from a presentation of a theory to a concrete algebraic structure on an arbitrary category slightly more difficult than it otherwise might be. Second, it does not allow us to distinguish between certain different categorical structures. For instance, the map \( \iota : A \otimes A \rightarrow A \otimes A \) gives rise to two different possible semantic interpretations: a map that preserves the order of the factors and one that reverses the order of the factors. Our construction blurs the distinction between these two semantic interpretations; indeed, either choice would provide an adequate semantics for the map. It is important to note that the claim being made here is that the two maps arise purely from two different semantic interpretations and that there is, a priori, no syntactic way in which to distinguish two interpretations. No More fundamentally, the combinatorial properties of the categorical structure are unaffected by the particular semantic interpretation of the maps. Indeed, we shall see in this chapter that all of the different choices of semantic interpretations yield isomorphic structures.

Before jumping into the world of two-dimensional algebra, we seek some intuition from classical one-dimensional algebra.

When developing a classical definition of equational varieties, one starts with a graded set of function symbols \( \mathcal{F} \) and imposes a collection of equations, \( \mathcal{E} \), on the absolutely free term algebra generated by \( \mathcal{F} \) on some set of variables \( X \), which we denote by \( \mathbb{F}_{\mathcal{F}}(X) \). Quotienting out by the smallest congruence generated by \( \mathcal{E} \) on \( \mathbb{F}_{\mathcal{F}}(X) \) yields the free \( \langle \mathcal{F} | \mathcal{E} \rangle \)-algebra on \( X \), which we denote by \( \mathbb{F}_{\langle \mathcal{F} | \mathcal{E} \rangle}(X) \). It is at this point that we run into a conceptual problem: the set of variables \( X \) holds a privileged position in the construction. If we wish to obtain the free \( \langle \mathcal{F} | \mathcal{E} \rangle \)-algebra on some other set \( Y \) then we run into a problem before we even start — the very concept of an \( \langle \mathcal{F} | \mathcal{E} \rangle \)-algebra was defined with the aid of \( X \)!

The traditional way around this problem is to define an \( \langle \mathcal{F} | \mathcal{E} \rangle \)-algebra to be an algebra \( \mathbb{A} \) of type \( \mathcal{F} \) such that for any equation \((s, t) \in \mathcal{E}\) and any homomorphism \( \rho : \mathbb{F}_{\mathcal{F}}(X) \rightarrow \mathbb{A} \), we have \( \rho(s) = \rho(t) \) [BS81].
2.1. SYNTAX

The viewpoint of algebras as being induced by homomorphisms from some particular free algebra is the starting point of Lawvere theories [Law04]. Here, we consider a function symbol of arity $n$ to be a function $n \to 1$ and use the Cartesian structure of Set in order to permute, duplicate and delete variables as we please. This allows us to replace equations with commutative diagrams and yields the category $\text{Th}((F|\mathcal{E}))$. This category has finite products; indeed, its objects are just the natural numbers, where a number $n$ is considered to be the $n$-fold cartesian product of 1. The category of finite product preserving functors $\text{Th}((F|\mathcal{E})) \to \text{Set}$ forms the analogue of algebras qua homomorphisms in the classical case, allowing us to transfer the structure inherent in $\langle F|\mathcal{E} \rangle$ to an arbitrary set.

Our basic strategy in this chapter is to replicate the above arguments in the two-dimensional setting in order to provide an abstract framework for categories with algebraic structure definable in a variable-based manner. Our essential objects of study are rewriting 2-theories, which consist of a first order term rewriting system modulo a two dimensional congruence. This retains a strong link with computational structures. Indeed, syntactically, rewriting 2-theories can be seen as a generalisation of unconditional rewriting logic, which arose primarily in the study of concurrent systems [Mes92]. The syntax and algebraic semantics of rewriting 2-theories is covered in sections 2.1 and 2.2, respectively. Connections with rewriting logic and other systems are briefly outlined in Section 2.3.

The fundamental focus of this thesis is coherence for rewriting 2-theories and we introduce this concept formally in Section 2.4. Subsequently, we explore the relationship between equational varieties and coherent rewriting 2-theories in Section 2.5 before sketching some basic results in Section 2.6 that will be of frequent use.

2.1. Syntax

The purpose of this section is to introduce a general class of term rewriting systems whose semantics correspond to categories with an additional covariant structure. Concretely, we work with a term rewriting theory modulo a two-dimensional congruence. That is, a term rewriting system equipped with an equational theory on terms and an equational theory on reductions, together with an associated calculus of proof terms.

Syntactically, we shall be working with structures of the form $\langle F; T | \mathcal{E}_F; \mathcal{E}_T \rangle$, where $F$ is a set of function symbols, $T$ is a set of reduction (or transformation) rules, $\mathcal{E}_F$ is an equational theory on $F$ and $\mathcal{E}_T$ is an equational theory on $T$ containing a certain basic congruence. Our main task in this section is to describe the structure that this data generates, which forms our two-dimensional analogue of $\mathbb{F}(F|\mathcal{E})$. We begin by building the one-dimensional aspect of the structure.

**Definition 2.1.1 (Term Algebra).** Given a graded set of function symbols $F := \sum_n F_n$ and a set $X$, the absolutely free term algebra generated by $F$ on $X$ is denoted by $\mathbb{F}_F(X)$.

The next layer of structure adds an equational theory to $\mathbb{F}_F(X)$:

**Definition 2.1.2.** Given a graded set of function symbols $F$, a set $X$ and a set of equations $\mathcal{E}_F$ on $\mathbb{F}_F(X)$, we denote by $\mathbb{F}_{(F|\mathcal{E}_F)}(X)$ the quotient of $\mathbb{F}_F(X)$ by the
We can now begin to describe a two-dimensional term rewriting theory. Our first step is to define a labelled term rewriting theory:

**Definition 2.1.3 (Labelled term rewriting theory).** A labelled term rewriting theory is a structure \((\mathcal{F}; \mathcal{L}; \mathcal{T} | \mathcal{E}_\mathcal{F})_X\), where \(\mathcal{F}\) is a graded set of function symbols, \(X\) is a set of variables, \(\mathcal{E}_\mathcal{F}\) is a system of \(\mathcal{F}_X(X)\)-equations, \(\mathcal{L}\) is a set of labels and \(\mathcal{T}\) is a subset of \(\mathcal{L} \times (\mathcal{F}_X(X))^2\) satisfying the following consistency conditions:

1. If \((s_1, t_1)\) and \((s_2, t_2)\) are in \(\mathcal{T}\) then \(s_1 = s_2\) and \(t_1 = t_2\).

If \((\alpha, s, t) \in \mathcal{T}\), we write \(\alpha : s \rightarrow t\). A member of \(\mathcal{T}\) is called a labelled reduction rule.

Given a labelled term rewriting theory \((\mathcal{F}; \mathcal{L}; \mathcal{T} | \mathcal{E}_\mathcal{F})_X\), the particular choice of \(\mathcal{L}\) is irrelevant. What is important is simply that there are sufficiently many labels for the number of reduction rules. Accordingly, we shall henceforth suppress explicit mention of the labels and write \((\mathcal{F}; \mathcal{T} | \mathcal{E}_\mathcal{F})_X\) for a labelled term rewriting theory. For the remainder of this thesis, we fix an arbitrary countable infinite set \(X\) and write \((\mathcal{F}; \mathcal{T} | \mathcal{E}_\mathcal{F})\) for \((\mathcal{F}; \mathcal{T} | \mathcal{E}_\mathcal{F})_X\) when the particular choice of variable set is unimportant.

A labelled term rewriting theory embodies the basic reductions that are to generate all others. The next step is to obtain an analogue of the absolutely free term algebra for this higher dimensional layer of structure. This is achieved by the following definition, where the notation \(\bar{x}^n\) is an abbreviation for \(x_1, \ldots, x_n\) and \(\mathcal{F}([\bar{x}^n] / [\bar{x}^m])\) denotes the uniform substitution of the free variables \(\bar{x}^m\) by \(\bar{x}^n\).

**Definition 2.1.4.** Given a labelled term rewriting theory \(\mathcal{L} := (\mathcal{F}; \mathcal{T} | \mathcal{E}_\mathcal{F})_X\), the set of reductions generated by \(\mathcal{L}\) is denoted \(\mathcal{F}_\mathcal{L}(X)\) and is constructed inductively by the following rules:

\[
\begin{align*}
1_s : [s] &\rightarrow [s] & \text{(Identity)} \\
\varphi_1 : [s_1] &\rightarrow [t_1] \ldots \varphi_n : [s_n] &\rightarrow [t_n] \\
\mathcal{F}(\varphi_1, \ldots, \varphi_n) : [\mathcal{F}(s_1, \ldots, s_n)] &\rightarrow [\mathcal{F}(t_1, \ldots, t_n)] & \text{(Structure)} \\
\tau : [\mathcal{F}(\bar{x}^n)] &\rightarrow [\mathcal{G}(\bar{x}^n)] \\
\varphi_1 : [s_1] &\rightarrow [t_1] \\
\vdots & \ldots \\
\varphi_n : [s_n] &\rightarrow [t_n] & \text{(Replacement)} \\
\tau(\varphi_1, \ldots, \varphi_n) : [\mathcal{F}(\bar{x}^m / \bar{x}^n)] &\rightarrow [\mathcal{G}(\bar{t}^n / \bar{x}^m)] \\
\varphi : [s] &\rightarrow [u] \\
\psi : [u] &\rightarrow [t] & \text{(Transitivity)} \\
(\varphi \cdot \psi) : [s] &\rightarrow [t]
\end{align*}
\]

In the (Identity) rule, \([s] \in \mathcal{F}_\mathcal{L}(X)\). In the (Structure) rule, \(\mathcal{F}\) is a function symbol of rank \(n\). In the (Replacement) rule \(\tau\) is a reduction rule of rank \(n\). When the particular choice of \(X\) is irrelevant, we write \(\mathcal{F}(\mathcal{L})\) for \(\mathcal{F}_\mathcal{L}(X)\).

**Example 2.1.5.** Let \(\mathcal{L}\) be the labelled rewriting theory consisting of a single binary function symbol \(\otimes\), an empty equational theory on terms and the single reduction rule:

\[\alpha(t_1, t_2, t_3) : t_1 \otimes (t_2 \otimes t_3) \rightarrow (t_1 \otimes t_2) \otimes t_3.\]

A derivation of

\[\alpha([A \otimes (B \otimes (C \otimes D))]) \rightarrow ([A \otimes B] \otimes (C \otimes D))\]
The consistency condition in Definition 2.1.3 easily yields the following lemma, which asserts that we may equate reductions with their labels, thus providing a term calculus for the reductions.

**Lemma 2.1.6.** Let $\mathcal{L}$ be a labelled term rewriting theory. If $\alpha : s \rightarrow t$ and $\alpha : s' \rightarrow t'$ are in $\mathbb{F}(\mathcal{L})$, then $s = s'$ and $t = t'$. 

At this point, we have in hand a notion of a labelled rewriting theory, which corresponds to the usual abstract setting of rewriting modulo an equational theory on terms. We now proceed to add an equational theory on reductions to this framework. This allows us to consider problems relating to equivalences of reductions in general rewriting systems. We impose two restrictions on this structure. The first is that we may only set two reductions to be equal if they have common sources and targets since, in applications, we very rarely have a sound ontological basis for equating arbitrary reductions. The second is that we enforce the presence of certain equations that equate reductions differing only in the order of rewriting nested and/or disjoint subterms. As we shall see in the following section, this is precisely what is needed in order to ensure a sound categorical semantics. The computational effect is to equate orthogonal reductions — those that do not rewrite a critical pair. This congruence is usually dubbed the "permutation congruence" in the term rewriting literature [vOdV03]. The permutation congruence is also known as "causal equivalence" and the congruence classes that it generates correspond to the notion of Mazurkiewicz traces arising in concurrency theory. The following definition states these concepts more formally.

**Definition 2.1.7 (Rewriting 2-Theory).** A Rewriting 2-Theory is a tuple $\mathcal{R} := (\mathcal{F}; T | \mathcal{E}_T; \mathcal{E}_{T^*})$, where $(\mathcal{F}; T | \mathcal{E}_T)$ is a labelled term rewriting theory and $\mathcal{E}_T$ is a set of equations on $\mathbb{F}(\mathcal{F}; T | \mathcal{E}_T^*)$ satisfying the following consistency condition: If $\varphi_1, \varphi_2 \in \mathcal{E}_T$ and $\varphi_1 : [s_1] \rightarrow [t_1]$ and $\varphi_2 : [s_2] \rightarrow [t_2]$, then $[s_1] = [s_2]$ and $[t_1] = [t_2]$.

We further stipulate that the following equations are satisfied. We refer to these equations collectively as the standard congruence and denote them by $S(\mathcal{R})$

$$
\begin{align*}
1_s \cdot \varphi &= \varphi \\
\varphi \cdot 1_t &= \varphi \\
\varphi \cdot (\psi \cdot \rho) &= (\varphi \cdot \psi) \cdot \rho \\
F(\varphi_1, \ldots, \varphi_n) \cdot F(\psi_1, \ldots, \psi_n) &= F(\varphi_1 \cdot \psi_1, \ldots, \varphi_n \cdot \psi_n) \\
\varphi(\varphi_1, \ldots, \varphi_n) &= s(\varphi_1, \ldots, \varphi_n) \cdot \varphi(1_{t_1}, \ldots, 1_{t_n}) \\
\varphi(\varphi_1, \ldots, \varphi_n) &= \varphi(1_{s_1}, \ldots, 1_{s_n}) \cdot t(\varphi_1, \ldots, \varphi_n)
\end{align*}
$$


In the above, $F \in \mathcal{F}_n$ and $\varphi, \psi, \rho, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ are reductions in $\mathbb{F}(\mathcal{R})$ such that the above compositions are well defined.
One of the benefits of allowing additional equations on reductions beyond that provided by the standard congruence is that it allows us to study invertible reduction rules, which arise when we recast an equational theory as a rewriting system. Moreover, it provides enough flexibility for us to be able to place equations on non-invertible reduction rules, which model phenomena such as non-reversible computations.

**Definition 2.1.8 (Invertible).** Given a rewriting 2-theory \((F; T | E_T; E_T)\), a reduction rule \(\varphi : [s] \rightarrow [t]\) in \(T\) is invertible if there is a reduction rule \(\psi : [t] \rightarrow [s]\) in \(T\) and a variable substitution \(\sigma : X \rightarrow X\) such that the equations \(\varphi^\sigma : \psi^\sigma = 1_{s^\sigma}\) and \(\psi^\sigma : \varphi^\sigma = 1_{t^\sigma}\) are both in \(E_T\). A rewriting 2-theory is invertible if all of its reduction rules are invertible. We say that \(\psi\) is an inverse of \(\varphi\).

In defining particular rewriting 2-theories, we shall often just say that a reduction is invertible, without explicitly giving the data for its inverse. That is, if we say that a rewriting 2-theory contains an invertible reduction rule \(\rho\), we mean that it also contains the inverse \(\rho^{-1}\) together with the necessary equations. Before proceeding, we give an example of a rewriting 2-theory.

**Example 2.1.9.** This example gives a presentation of an invertible rewriting 2-theory involving associativity and unit reduction rules. We shall see in Section 2.2 that this example gives a presentation of the free monoidal category on a discrete category.

The theory consists of a binary function symbol \(\otimes\) and a nullary function symbol \(I\). We write \(\otimes\) in infix notation. It has the following invertible reduction rules:

\[
\alpha(t_1, t_2, t_3) : t_1 \otimes (t_2 \otimes t_3) \rightarrow (t_1 \otimes t_2) \otimes t_3 \\
\lambda(t) : I \otimes t \rightarrow t \\
\rho(t) : t \otimes I \rightarrow t
\]

It has equations stating that the following diagrams commute:

<table>
<thead>
<tr>
<th>[a \otimes (b \otimes (c \otimes d))]</th>
<th>[a \otimes b]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\alpha]</td>
<td>[1 \otimes \alpha]</td>
</tr>
<tr>
<td>[(a \otimes b) \otimes (c \otimes d)]</td>
<td>[a \otimes ((b \otimes c) \otimes d)]</td>
</tr>
<tr>
<td>[\alpha]</td>
<td>[\alpha]</td>
</tr>
<tr>
<td>[\alpha \otimes 1]</td>
<td>[\alpha \otimes 1]</td>
</tr>
<tr>
<td>[(a \otimes b) \otimes c \otimes d]</td>
<td>[(a \otimes (b \otimes c)) \otimes d]</td>
</tr>
</tbody>
</table>

\[\square\]

**Definition 2.1.10.** If \(R := (F; T | E_T; E_T)\) is a rewriting 2-theory, then \([E_T + S(R)]\) denotes the smallest congruence generated by \(E_T\) and \(S(R)\) on \(\mathbb{P}(R)\). It is generated inductively by the following rules:
2.2. Semantics

In this section, we shall provide a semantics for rewriting 2-theories akin to the semantics that Lawvere theories provide for syntactically defined equational varieties. The appropriate generalisation of Lawvere theories to this setting is a special case of discrete enriched Lawvere theories — algebraic theories on categories whose hom-sets carry additional structure [Pow99, Pow05]. The presence of the standard congruence on the set of reductions is precisely what puts us in the 2-categorical setting. Had we omitted the requirement that $\mathcal{F}$ contains the standard congruence, then we would instead be in the more general setting of sesquicategories, whose relationship with term rewriting was investigated by Stell [Ste94]. As we are in the 2-categorical setting, the hom-sets are themselves categories. We shall not require any deep enriched category theory but shall make some use of the language of 2-dimensional categories, an introduction to which may be found in [KS74].

**Definition 2.2.1 (Lawvere 2-theory).** A discrete finitary Lawvere 2-theory is a small 2-category $\mathcal{L}$ with finite 2-products, together with a finite-2-product preserving identity-on-objects 2-functor $\iota : \text{Nat}^{\text{op}} \rightarrow \mathcal{L}$, where Nat is the 2-category of natural numbers and all maps between them. A map of discrete finitary Lawvere 2-theories $\mathcal{L} \rightarrow \mathcal{L}'$ is a finite-product preserving 2-functor $\Theta$ making the following diagram

\[
\begin{align*}
\varphi &= \varphi \\
\varphi_1 &= \varphi_2 \\
\varphi &= \psi \\
\psi &= \varphi \\
F(\varphi_1, \ldots, \varphi_n) &= F(\psi_1, \ldots, \psi_n) \\
\tau(\varphi_1, \ldots, \varphi_n) &= \tau(\psi_1, \ldots, \psi_n) \\
\varphi &= \psi \\
\varphi^\sigma &= \psi^\sigma \\
(\varphi_1 = \psi_1) : s \rightarrow u & \quad (\varphi_2 = \psi_2) : u \rightarrow t \\
(\varphi_1 \cdot \psi_1 = \varphi_2 \cdot \psi_2) : s \rightarrow t \\
\end{align*}
\]

(Identity) $\varphi \in \mathcal{T}$

(Inheritance) $(\varphi_1, \varphi_2) \in \mathcal{E}_T + S(\mathcal{R})$

(Symmetry)

(Structure) $F \in \mathcal{F}_n$

(Replacement) $\tau \in \mathcal{T}_n$

(Substitution) $\sigma$ a substitution.

(Transitivity)

All that remains is to quotient out by the congruence generated by an equational theory on reductions.

**Definition 2.1.11.** Given a rewriting 2-theory $\mathcal{R} := (\mathcal{F}; \mathcal{T}; \mathcal{E}_F; \mathcal{E}_T)_X$, we use $\mathcal{F}(\mathcal{F}; \mathcal{T}; \mathcal{E}_F; \mathcal{E}_T)(X)$ to denote the quotient $\mathcal{F}(\mathcal{F}; \mathcal{T}; \mathcal{E}_F)(X)/[\mathcal{E}_T + S(\mathcal{R})]$. Where explicit mention of the set $X$ is not necessary, we write $\mathcal{F}(\mathcal{R})$ for $\mathcal{F}_\mathcal{R}(X)$.

In the following section, we investigate the semantics of rewriting 2-theories and establish that a rewriting 2-theory provides a presentation of a free structure carried by a discrete category.
Since we shall not require any more sophisticated notion of Lawvere 2-theory, we use “Lawvere 2-theory” to mean “discrete finitary Lawvere 2-theory”. These are an alternative categorical presentation of strongly finitary 2-monads on $\text{Cat}$, studied in [KL93]. The way in which to visualise a Lawvere 2-theory is to think of each object as for some arbitrary category $\mathcal{D}$ (although, strictly speaking, the objects are simply natural numbers). The arrows are then maps $\mathcal{D}^n \to \mathcal{D}^m$ and the two-cells are maps between such arrows. For us, all of the arrows of $\mathcal{L}$ will be generated by basic arrows $\mathcal{D}^n \to \mathcal{D}$, corresponding to function symbols, and all of the two-cells will be generated by reduction rules.

A Lawvere 2-theory is essentially a two-dimensional analogue of a free algebra. As in the one-dimensional case, we define a category having the structure specified by $\mathcal{L}$ by product-preserving functors out of $\mathcal{L}$.

**DEFINITION 2.2.2.** A model of a Lawvere 2-theory $\mathcal{L}$ in $\text{Cat}$ is a finite-product preserving 2-functor $M : \mathcal{L} \to \text{Cat}$.

In order to relate rewriting 2-theories with Lawvere 2-theories, we need to show how to generate a Lawvere 2-theory $\text{Th}(\mathcal{R})$ from a given rewriting 2-theory $\mathcal{R}$. This would allow us to translate the purely syntactic $\mathcal{R}$ into an object that specifies an additional structure on a category.

In general, there is not a strictly unique way in which to construct $\text{Th}(\mathcal{R})$, since there may be many possible ways in which to express a given reduction rule, particularly in the case where $\mathcal{R}$ contains reduction rules such as $A \otimes A \to A$. However, as we shall see, $\text{Th}(\mathcal{R})$ is unique up to 2-isomorphism of Lawvere 2-theories, so the distinction is inessential for our purposes.

**DEFINITION 2.2.3.** Let $\mathcal{R} := \langle \mathcal{F}; \mathcal{T}; \mathcal{E}_\mathcal{F}; \mathcal{E}_\mathcal{T} \rangle$ be a rewriting 2-theory. A Lawvere 2-theory associated to $\mathcal{R}$ is a Lawvere 2-theory $\mathcal{L}$ containing precisely the following structure:

1. For every term $t \in \mathcal{F}(\mathcal{R})$ of arity $n$, there is a one-cell $|t| : n \to 1$ in $\mathcal{L}$.
2. For every reduction $\rho : [s] \to [t]$ in $\mathcal{F}(\mathcal{R})$, there is a 2-cell $|\rho| : |s| \to |t|$ in $\mathcal{L}$.
3. $s = t$ if and only if $|s| = |t|$, for terms $s, t \in \mathcal{F}(\mathcal{R})$.
4. $\sigma = \tau$ if and only if $|\sigma| = |\tau|$, for reductions $\sigma, \tau \in \mathcal{F}(\mathcal{R})$.

It is immediate from the definition that any two Lawvere 2-theories associated to a rewriting 2-theory differ only in the precise way in which the function symbols and reduction rules are represented. This immediately implies the following lemma.

**LEMMA 2.2.4.** Any two Lawvere 2-theories associated to a rewriting 2-theory $\mathcal{R}$ are 2-isomorphic.

In light of the previous lemma, the following is well-defined:
DEFINITION 2.2.5. Th(\mathcal{R}) is the Lawvere 2-theory associated to the rewriting 2-theory \mathcal{R}.

As it stands, the relationship between Th(\mathcal{R}) and \mathcal{R} is still quite vague. In the remainder of this section, we shall see how to construct Th(\mathcal{R}) from \mathcal{R} and we shall also see that no "extra" equations arise from the 2-categorical nature of Th(\mathcal{R}).

Let \mathcal{R} := \langle \mathcal{F}; \mathcal{T} | \mathcal{E}_\mathcal{F}; \mathcal{E}_\mathcal{T} \rangle be a rewriting 2-theory and let \mathcal{L} be the initial Lawvere 2-theory. That is, \mathcal{L} contains no structure other than that implied by the existence of a finite-product preserving identity-on-objects functor \iota : \text{Nat}^{op} \to \mathcal{L}. For each function symbol \( F \in \mathcal{F} \) of arity \( n \), add a one-cell \(|F| : n \to 1\) to \( \mathcal{L} \). Extend this inductively to terms by setting:

\[
|t| = \begin{cases} 
|F|(|s_1|, \ldots, |s_n|) & \text{if } t = F(s_1, \ldots, s_n) \\
|t| & \text{if } t \in \mathcal{F}
\end{cases}
\]

For each equation \( (s, t) \in \mathcal{E}_\mathcal{F} \), we enforce an equality \(|s| = |t|\) by making use of the cartesian structure of \( \mathcal{L} \). In particular, we may make use of the following operations:

- We may duplicate an object by making use of the diagonal map \( \Delta : 1 \to 2 \).
- We may delete the left hand-side of a pair of variables by making use of the first projection \( \pi_1 : 2 \to 1 \).
- We may delete the right hand-side of a pair of variables by making use of the second projection \( \pi_2 : 2 \to 1 \).
- We may commute two variables by making use of the twist map \( \tau : 2 \to 2 \).

As an example, suppose that \( \mathcal{R} \) contains the binary function symbol \( \otimes \) and the equation

\[
\otimes(a, \otimes(b, c)) = \otimes(\otimes(b, b), a).
\]

This equation can be represented by saying that the following diagram commutes:

\[
\begin{array}{c}
3 \\
\downarrow 1 \times \otimes \\
2 \\
\downarrow \tau \\
2 \\
\downarrow \Delta \times 1 \\
3 \\
\downarrow \otimes \times 1 \\
2 \\
\end{array}
\]

We may interpret the above diagram as saying that the following two deductions are equal:

\[
\frac{\otimes(a, \otimes(b, c))}{\otimes(a, \otimes(b, c))} \quad \frac{\otimes(\otimes(b, b), a)}{\otimes(\otimes(b, b), a)}
\]

\[
\frac{\otimes(a, \otimes(b, c))}{1 \times \otimes} \quad \frac{\otimes(\otimes(b, b), a)}{\otimes(\otimes(b, b), a)}
\]

Of course, there are other ways in which to represent the equation. However, any choice of diagram to represent the equation induces the same congruence on one-cells.
Next, we need to construct a two-cell $\rho$ in $\mathcal{L}$ for every reduction $\rho \in \mathbb{F}(\mathcal{R})$. We accomplish this by constructing a two-cell $|\rho| : |s| \rightarrow |t|$ for every reduction $\rho : [s] \rightarrow [t]$ in $\mathcal{T}$ and extending the construction inductively to arbitrary reductions as follows:

$$|\rho| = \begin{cases}  
|\rho'|(|\sigma_1|, \ldots, |\sigma_n|) & \text{if } \rho = \rho'(\sigma_1, \ldots, \sigma_n) \\
|\rho_1| \cdot |\rho_2| & \text{if } \rho = \rho_1 \cdot \rho_2 \\
|\rho| & \text{if } \rho \in \mathcal{T}
\end{cases}$$

As in the construction of a congruence in $\mathcal{L}$ from $\mathcal{E}_F$, there is a choice as to how to construct $|\rho|$ for a given $\rho \in \mathcal{T}$. However, in light of Lemma 2.2.4, this particular choice is inconsequential. Finally, we enforce the equation $|\sigma| = |\tau|$ for every $(\sigma, \tau) \in \mathcal{E}_T$.

From our construction of $\text{Th}(\mathcal{R})$, we have that any equation that holds in $\mathbb{F}(\mathcal{R})$ holds, after suitable translation, in $\text{Th}(\mathcal{R})$. The converse result holds but is not immediately obvious. That is, it is not clear that the fact that $\mathcal{L}$ is a 2-category does not introduce any extra equations.

Since we are only interested in models of $\mathcal{L}$ in $\textbf{Cat}$, checking that all of the 2-categorical axioms are satisfied in $\mathbb{F}(\mathcal{R})$ amounts to checking the axioms for functoriality and naturality.

Let $\tau : s \rightarrow t$ be a reduction rule of rank $n$. Naturality of $\tau$ amounts to the assertion that for all reductions $\sigma_i : s_i \rightarrow t_i$, we have:

$$\tau(1_{s_1}, \ldots, 1_{s_n}) \cdot t(\sigma_1, \ldots, \sigma_n) = s(\sigma_1, \ldots, \sigma_n) \cdot \tau(1_{t_1}, \ldots, 1_{t_n}).$$

This follows immediately from the combination of (Nat1) and (Nat2).

Suppose that $F \in \mathcal{F}_n$. Without loss of generality, we may assume that $F$ is binary. The functoriality of $F$ is established as follows:

\[
F(\phi, 1) \cdot F(1, \psi) = F(\phi \cdot 1, 1 \cdot \psi) \quad \text{by (Funct)} \\
= F(\phi, \psi) \quad \text{by (ID1) and (ID2)} \\
= F(1 \cdot \phi, \psi \cdot 1) \quad \text{by (ID1) and (ID2)} \\
= F(1, \psi) \cdot F(\phi, 1) \quad \text{by (Funct)}
\]

Our construction of $\text{Th}(\mathcal{R})$ from $\mathcal{R}$ carries the message that we may view function symbols in $\mathcal{R}$ as functors and reduction rules in $\mathcal{R}$ as natural transformations. Thus, a rewriting 2-theory can be seen as giving a syntactic specification of an additional structure carried by a category.

**Example 2.2.6.** Let $\mathcal{R}$ be the rewriting 2-theory from Example 2.1.9. Then, $\text{Th}(\mathcal{R})$ is the Lawvere 2-theory for monoidal categories.

In the following section, we discuss several systems related to rewriting 2-theories.

## 2.3. Relation to other systems

Our basic structure of a rewriting 2-theory, $\mathcal{R} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_F; \mathcal{E}_T \rangle$, simultaneously generalises several other systems, which we cover in order of increasing generality.
(1) **First order rewriting:** If both \( \mathcal{E}_F \) and \( \mathcal{E}_T \) are empty and we do not impose the standard congruence, then we are in the setting of first order term rewriting. However, our construction of \( \mathcal{F}(R) \) adds in identity reductions, which are not usually assumed to be present in term rewriting systems.

(2) **Rewriting modulo an equational theory:** If \( \mathcal{E}_T \) is empty and we do not impose the standard congruence, then we are in the setting of rewriting modulo an equational theory, with the same caveat as for standard first order term rewriting.

(3) **Calculus of Structures:** If \( \mathcal{E}_F \) is empty and we do not impose the standard congruence, then we are also in the setting of the Calculus of Structures \([Gug07, GS01]\). This is a proof theoretic framework that extends one sided Gentzen systems with the ability for inference rules to act arbitrarily deeply within a sequent.

(4) **Rewriting logic:** If \( \mathcal{E}_T \) is empty, then we are in the setting of unconditional rewriting logic \([Mes92]\). This system has its roots in concurrency theory and particularly in the notions of causal equivalence and Mazurkiewicz trace languages.

(5) **Clubs:** The notion of a fully covariant club was introduced by Kelly \([Kel72]\) as a unified framework for covariant structures carried by a category. His description of fully covariant clubs is very similar to our notion of a rewriting 2-theory, with several points of difference. First, Kelly’s calculus of proof terms is provided implicitly by the categorical setting, whereas our calculus is generated inductively. Second, Kelly works purely within the framework of two-dimensional category theory, whereas we prefer an approach via term rewriting systems, which highlights the connection with computational notions. A more substantial technical point of differentiation is that Kelly gives a variable-free presentation, which does not allow the expression of certain equations at the term level, such as the commutativity of a binary function symbol. Indeed, the only equations expressible at the term level in Kelly’s setting are the strongly regular ones — those equations \( (s, t) \) where \( \text{Var}(s) = \text{Var}(t) \), each variable appears precisely once in both \( s \) and \( t \) and the order in which the variables appear in \( s \) is the same as the order in which they appear in \( t \).

In the following section, we introduce the coherence problem, which will be our main focus throughout the thesis.

### 2.4. Coherence

There are many interrelated problems that go under the name of “coherence”. Ultimately, all of these questions relate to describing the free algebra generated by some algebraic structure on a category. The original manifestation of this problem was in Mac Lane’s investigation of monoidal categories \([ML63]\). Since all diagrams commute in the free monoidal category on a discrete category, it was this phenomenon that was originally associated with the term “coherence”. This was in keeping with work in
algebraic topology on defining algebraic operations on topological spaces together with
equations that hold only up to homotopy [Sta63].

It is not the case that all algebraically defined structures on categories enjoy the
same strong coherence property that monoidal categories do. For instance, simply re-
moving one of the coherence axioms from the definition of a monoidal category destroys
this property. This observation led to Kelly reformulating the coherence problem to ask
which diagrams commute purely as a result of the axioms [Kel72]. However, even this
question may be too strong, for we may not be able to even decide if a given diagram
commutes as a result of the axioms. The view that a coherence problem is essentially
concerned with deciding whether given diagrams commute has its roots in Lambek’s
investigation of residuated structures arising in mathematical linguistics [Lam68].

The main thrust of this thesis is the investigation of various coherence problems
for structures defined by rewriting 2-theories. While this does not cover the complete
array of possible categorical structures, it is sufficiently broad so as to encompass many
interesting and pathological examples. In this section, we set out precise definitions of
the various coherence problems.

One difficulty that arises when investigating coherence problems is that the com-
mutativity of a particular diagram may have no bearing on the question at hand. For
this reason, we need to carefully define those diagrams and reductions that are of im-
portance for us. These are the diagrams that are in “general position”. That is, they
contain the maximum number of distinct variables. Before making this precise, we
need the concept of the shape of a reduction.

**Definition 2.4.1.** Let \( \mathcal{R} \) be a rewriting 2-theory. The **shape** of a reduction \( \alpha \in \mathbb{F}(\mathcal{R}) \) is defined recursively by the following:

\[
\text{Shape}(\alpha) = \begin{cases} 
\text{Shape}(\alpha_1) \cdot \text{Shape}(\alpha_2) & \text{if } \alpha = \alpha_1 \cdot \alpha_2 \\
\tau(\text{Shape}(\alpha_1), \ldots, \text{Shape}(\alpha_n)) & \text{if } \alpha = \tau(\alpha_1, \ldots, \alpha_n) \\
F(\text{Shape}(\alpha_1), \ldots, \text{Shape}(\alpha_n)) & \text{if } \alpha = F(\alpha_1, \ldots, \alpha_n) \\
o & \text{otherwise}
\end{cases}
\]

In the system from Example 2.1.5, we have:

\[
\text{Shape}(\alpha(1_A, 1_B, 1_C)) = \text{Shape}(\alpha(1_A, 1_A, 1_A)) = \alpha(o, o, o)
\]

We now need a precise definition of the variables present in a reduction.

**Definition 2.4.2.** Given a rewriting 2-theory \( \mathcal{R} \), the **set of variables** in a reduction \( \alpha \in \mathbb{F}(\mathcal{R}) \) is defined recursively as follows:

\[
\text{Var}(\alpha) = \begin{cases} 
\text{Var}(\alpha_1) \cup \text{Var}(\alpha_2) & \text{if } \alpha = \alpha_1 \cdot \alpha_2 \\
\bigcup_{i=1}^{n} \text{Var}(\alpha_i) & \text{if } \alpha = \tau(\alpha_1, \ldots, \alpha_n) \\
\bigcup_{i=1}^{n} \text{Var}(\alpha_i) & \text{if } \alpha = F(\alpha_1, \ldots, \alpha_n) \\
o & \text{otherwise}
\end{cases}
\]

Returning to Example 2.1.5, we find that

\[
\text{Var}(\alpha(1_A, 1_B, 1_C)) = \{1_A, 1_B, 1_C\}.
\]
whereas
\[ \text{Var}(\alpha(1_A, 1_A, 1_A)) = \{1_A\}. \]

We can finally nail down what we mean when we say a reduction has the maximum possible number of variables.

**Definition 2.4.3.** Given a rewriting 2-theory \( \mathcal{R} \), a reduction \( \alpha \in F(\mathcal{R}) \) is in general position if
\[
|\text{Var}(\alpha)| = \max\{|\text{Var}(\tau)| : \tau \in F(\mathcal{R}) \text{ and } \text{Shape}(\tau) = \text{Shape}(\alpha)\}.
\]

**Example 2.4.4.** Consider the system from Example 2.1.5 augmented with the following reduction rule:
\[
\beta(x) : x \otimes x \rightarrow x
\]
Then,
\[
\alpha(1_A, 1_A, 1_B) \cdot (\beta(1_A) \otimes 1_B) : A \otimes (A \otimes B) \rightarrow A \otimes B
\]
is in general position, whereas
\[
\alpha(1_A, 1_A, 1_B) : A \otimes (A \otimes B) \rightarrow (A \otimes A) \otimes B
\]
is not in general position.

For coherence problems, we only need to focus on those diagrams whose reductions are all in general position. This allows us to define the various problems that will be our focus.

**Definition 2.4.5.** Let \( \mathcal{R} := \langle F; T | E_F; E_T \rangle \) be a rewriting 2-theory.

1. \( \mathcal{R} \) is Mac Lane coherent if any two parallel reductions in \( F(\mathcal{R}) \) are equal.
2. \( \mathcal{R} \) is Lambek coherent if there is a decision procedure for the commutativity of diagrams in \( F(\mathcal{R}) \).

Unfortunately, deciding whether a given 2-theory is coherent in either the Mac Lane or Lambek sense is often impossible.

**Theorem 2.4.6.** The decision problems for Mac Lane coherence and Lambek coherence are undecidable over the class of finitely presented rewriting 2-theories.

**Proof.** Following the work of Markov [Mar51], we know that many problems are undecidable for finitely presented monoids. Our basic strategy is to show how to encode a monoid as a rewriting 2-theory. Let \( M := \langle X | R \rangle \) be a finite presentation for a monoid. Let \( \mathcal{R}(M) \) be the rewriting 2-theory consisting of a single unary function symbol \( F \), reductions \( \tau_i : F(x) \rightarrow F(x) \) for every \( \tau_i \in X \) and relations \( (\omega_i, \omega_j) \) for every \( (\omega_i, \omega_j) \in R \). If we could solve the Mac Lane coherence problems for \( \mathcal{R}(M) \), then we could decide whether \( M \) is trivial. Similarly, if we could solve the Lambek coherence problem for \( \mathcal{R}(M) \), then we could solve the word problem for \( M \). Since both of these monoid problems are undecidable in general, so too are the associated coherence problems.

The notions of coherence that we have introduced here are focused entirely on the congruence present on reductions. Historically, this arose because equations on
terms can often be converted into coherent natural isomorphisms. We explore this phenomenon in the following section.

2.5. Categorification

The fundamental group of a topological space is usually defined as the group of homotopy-equivalence classes of based loops in the space. This definition forgets the particular relationships between any two loops lying in a given equivalence class. An alternative approach might be to define a group structure on the space of based loops together with explicit homotopies between elements. This notion of algebraic structure "up to homotopy" was introduced in [Sta63] and has been extended to handle quite general structures [Ros07]. A problem that arises with this approach is that one then needs to examine the relationships between the homotopies themselves.

Translated into our language, the above process takes a labelled rewriting theory \( \mathcal{L} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F} \rangle \) and replaces it with a rewriting 2-theory \( \mathcal{R}(\mathcal{L}) \) in which each equation in \( \mathcal{E}_\mathcal{F} \) is replaced with an invertible reduction. In order to retain the link between the \( \mathcal{L} \) and \( \mathcal{R}(\mathcal{L}) \), one needs to show that \( \text{Th}(\mathcal{L}) \simeq \text{Th}(\mathcal{R}(\mathcal{L})) \). However, this can only be the case if any two sequences of the new invertible reductions in \( \mathcal{R}(\mathcal{L}) \) having the same source and target are equal. In other words, one needs to construct \( \mathcal{R}(\mathcal{L}) \) in such a way that it is Mac Lane coherent.

**Definition 2.5.1.** A categorification of a labelled rewriting theory \( \mathcal{L} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F} \rangle \) is a rewriting 2-theory \( \mathcal{R}(\mathcal{L}) := \langle \mathcal{F}; \mathcal{T} \cup I(\mathcal{E}_\mathcal{F}) \mid \emptyset; \mathcal{E}_{\mathcal{T} \cup I(\mathcal{E}_\mathcal{F})} \rangle \), where:

1. \( I(\mathcal{E}_\mathcal{F}) \) consists of reductions \( \rho_{s,t} : s \to t \) and \( \rho_{s,t}^{-1} : t \to s \) for each \((s, t) \in \mathcal{E}_\mathcal{F}\).
2. \( \mathcal{E}_{\mathcal{T} \cup I(\mathcal{E}_\mathcal{F})} \) contains \( \mathcal{E}_t \), as well as the equations
   \[
   \rho_{s,t} \cdot \rho_{s,t}^{-1} = 1_s
   \]
   \[
   \rho_{s,t}^{-1} \cdot \rho_{s,t} = 1_t
   \]
   for each \((s, t) \in \mathcal{E}_\mathcal{F}\).

\( \mathcal{R}(\mathcal{L}) \) is a coherent categorification of \( \mathcal{L} \) if it is Mac Lane coherent.

**Example 2.5.2.** Monoidal categories, as defined in Example 2.1.9 are a coherent categorification of the theory for strict monoidal categories. This theory consists of a binary function symbol \( \otimes \), a nullary function symbol \( I \) as well as equations

\[
\begin{align*}
a \otimes (b \otimes c) &= (a \otimes b) \otimes c \\
a \otimes I &= a \\
I \otimes a &= a
\end{align*}
\]

If we take models for the theory of strict monoidal categories to be product preserving functors into \( \text{Set} \), then we recover the variety of monoids.

A categorification of an equational theory \( \mathcal{E} := \langle \mathcal{F}; \emptyset \mid \mathcal{E}_\mathcal{F} \rangle \), is a categorification of the labelled term rewriting theory \( \langle \mathcal{F}; \emptyset \mid \mathcal{E}_\mathcal{F} \rangle \). Similarly, we can define \( \text{Th}(\mathcal{L}) \) for a labelled rewriting \( \mathcal{L} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F} \rangle \) to be \( \text{Th}(\langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F}, \emptyset \rangle) \). The Lawvere 2-theory \( \text{Th}(\mathcal{E}) \) associated to an equational theory \( \mathcal{E} \) is defined analogously.
Theorem 2.5.3. Let $\mathcal{R}(\mathcal{E})$ be a categorification of the equational theory $\mathcal{E}$. There is a biequivalence of 2-categories $\text{Th}(\mathcal{R}(\mathcal{E})) \simeq \text{Th}(\mathcal{E})$ if and only if $\mathcal{R}(\mathcal{E})$ is coherent.

Proof. Let $\mathcal{E}$ be an equational theory and let $\mathcal{R}(\mathcal{E})$ be a categorification of $\mathcal{E}$.

 Suppose that $\mathcal{R}(\mathcal{E})$ is coherent. For each congruence class of 1-cells $[s] \in \text{Th}(\mathcal{E})$, pick a distinguished element $r([s])$. Define a pseudofunctor $F : \text{Th}(\mathcal{E}) \to \text{Th}(\mathcal{R}(\mathcal{E}))$ by:

- 0-Cells: Identity
- 1-Cells: $F([s]) = r([s])$
- 2-Cells: $\text{Th}(\mathcal{E})$ contains only identity 2-cells. Define $F(1_{[s]}) = 1_{r([s])}$.

Next, define a 2-functor $G : \text{Th}(\mathcal{R}(\mathcal{E})) \to \text{Th}(\mathcal{E})$ by:

- 0-Cells: Identity
- 1-Cells: $G(s) = [s]$  
- 2-Cells: Since there is a 2-cell $s \to t$ in $\text{Th}(\mathcal{R}(\mathcal{E}))$ precisely when $[s] = [t]$ in $\text{Th}(\mathcal{E})$, we can define $G(\rho : s \to t) = 1_{[s]}$.

It follows from the definitions that $F \cdot G = 1_{\text{Th}(\mathcal{E})}$. Since $\mathcal{R}(\mathcal{E})$ is coherent, the two legs of the following diagram commute:

\[
\begin{array}{ccc}
G \cdot F(s) & \cong & s \\
\downarrow G \cdot F(\rho) & & \downarrow \rho \\
G \cdot F(t) & \cong & t
\end{array}
\]

It follows that $G \cdot F \cong 1$, so $\text{Th}(\mathcal{R}(\mathcal{E})) \simeq \text{Th}(\mathcal{E})$.

Conversely, suppose that $\text{Th}(\mathcal{R}(\mathcal{E})) \simeq \text{Th}(\mathcal{E})$. Then, there exist functors $F : \text{Th}(\mathcal{E}) \to \text{Th}(\mathcal{R}(\mathcal{E}))$ and $G : \text{Th}(\mathcal{R}(\mathcal{E})) \to \text{Th}(\mathcal{E})$ such that $F \cdot G \cong 1$. Suppose that $\rho_1, \rho_2 : s \to t$ are a parallel pair of 2-cells in $\text{Th}(\mathcal{R}(\mathcal{E}))$. Then, $F \cdot G(\rho_1) = F(1_{[s]}) = F \cdot G(\rho_2)$. Thus, $\mathcal{R}(\mathcal{E})$ is coherent. \(\square\)

Example 2.5.4. It follows from Theorem 2.5.3 that the theory for monoidal categories is biequivalent to the theory for strict monoidal categories.

Given two rewriting 2-theories $\mathcal{R}_1$ and $\mathcal{R}_2$, we define $\text{Th}(\mathcal{R}_1) \cup \text{Th}(\mathcal{R}_2) := \text{Th}(\mathcal{R}_1 \cup \mathcal{R}_2)$.

Corollary 2.5.5. Let $\mathcal{R} \coloneqq \langle \mathcal{F}; T \mid \mathcal{E}_F ; \mathcal{E}_T \rangle$ be a labelled rewriting theory and let $\langle \mathcal{F}; I(\mathcal{E}_F) \mid \emptyset; \mathcal{E}_I(\mathcal{F}) \rangle$ be a coherent categorification of $\langle \mathcal{F}; \mathcal{E}_F \rangle$. Then

$$\text{Th}(\mathcal{R}) \simeq \text{Th}(\langle \mathcal{F}; T \cup I(\mathcal{E}_F) \mid \emptyset; \mathcal{E}_T \cup \mathcal{E}_I(\mathcal{F}) \rangle).$$

Proof. By Theorem 2.5.3, we have

$$\text{Th}(\mathcal{R}) = \text{Th}(\langle \mathcal{F}; \emptyset \mid \mathcal{E}_F ; \emptyset \rangle \cup \langle \emptyset; T \mid \emptyset; \mathcal{E}_T \rangle) = \text{Th}(\langle \mathcal{F}; \emptyset \mid \mathcal{E}_F ; \emptyset \rangle) \cup \text{Th}(\langle \emptyset; T \mid \emptyset; \mathcal{E}_T \rangle) \simeq \text{Th}(\langle \mathcal{F}; I(\mathcal{E}_F) \mid \emptyset; \mathcal{E}_I(\mathcal{F}) \rangle) \cup \text{Th}(\langle \emptyset; T \mid \emptyset; \mathcal{E}_T \rangle) = \text{Th}(\langle \mathcal{F}; T \cup I(\mathcal{E}_F) \mid \emptyset; \mathcal{E}_T \cup \mathcal{E}_I(\mathcal{F}) \rangle).$$
The above corollary roughly states that, for a given rewriting 2-theory, we can switch between an equational theory on terms and a coherent invertible theory on terms as we please. This ability is very useful in investigating coherent structures. In the following section, we introduce some other useful concepts for investigating coherence.

2.6. Basic properties

This section is predominantly intended as a collection of basic concepts and results that will prove useful throughout the thesis.

Given a rewriting 2-theory $\mathcal{R}$, we shall frequently need to break up a reduction in $\mathbb{F}(\mathcal{R})$ into a composite of smaller reductions. Since all of the reductions in $\mathbb{F}(\mathcal{R})$ are generated by a set of reduction rules, this process must ultimately terminate. However, it is important that we have some understanding of the resulting normal forms.

**Definition 2.6.1 (Singualar).** Let $\mathcal{R} := \langle \mathcal{F}; T \mid \mathcal{E}_x; \mathcal{E}_T \rangle$ be a rewriting 2-theory. The set of singular reductions in $\mathbb{F}(\mathcal{R})$ is denoted $\text{Sing}(\mathcal{R})$ and is generated as follows:

- If $\rho \in \mathcal{T}_n$ and $[t_1], \ldots, [t_n]$ are congruence classes of terms in $\mathbb{F}(\mathcal{R})$, then $\rho(1_{t_1}, \ldots, 1_{t_n})$ is singular.
- If $F \in \mathcal{F}_n$ and $\rho$ is a singular reduction and $1 \leq i \leq n$, then
  $$F(1, \ldots, 1, \rho, 1, \ldots, 1),$$
  is singular.

**Example 2.6.2.** In the system from example 2.1.9, the reduction

$$1_a \otimes \alpha(1_b, 1_c, 1_d) : a \otimes (b \otimes (c \otimes d)) \rightarrow a \otimes ((b \otimes c) \otimes d)$$

is singular, whereas the reduction

$$\alpha(1_a, \alpha(1_b, 1_c, 1_d), 1_e) : a \otimes ((b \otimes (c \otimes d)) \otimes e) \rightarrow (a \otimes ((b \otimes c) \otimes d)) \otimes e$$

is not singular.

**Lemma 2.6.3.** Let $\mathcal{R}$ be a rewriting 2-theory. Every non-identity reduction in $\mathbb{F}(\mathcal{R})$ is equal to a composite of finitely many singular reductions.

**Proof.** Let $\mathcal{R}$ be a rewriting 2-theory and let $\rho$ be a reduction in $\mathbb{F}(\mathcal{R})$. Define the rank of $\rho$ to be

$$R(\rho) = \begin{cases} R(\rho_1) + R(\rho_2) & \text{if } \rho = \rho_1 \cdot \rho_2 \\ \sum_{i=1}^n R(\tau_i) & \text{if } \rho = F(\tau_1, \ldots, \tau_n) \\ \sum_{i=1}^n R(\tau_i) & \text{if } \rho = \sigma(\tau_1, \ldots, \tau_n) \\ 1 & \text{if } \rho \in \text{Sing}(\mathcal{R}) \end{cases}$$

We proceed by induction on $R(\rho)$ to show that $\rho$ is a composite of singular morphisms. If $R(\rho) = 1$, then $\rho$ is singular.

Suppose that $R(\rho) > 1$. Suppose that $\rho = \rho_1 \cdot \rho_2$, where neither $\rho_1$ nor $\rho_2$ is an identity reduction. Then by induction each of $\rho_1$ and $\rho_2$ is a composite of finitely many
singular reductions. Suppose that $\rho = \sigma(\tau_1, \ldots, \tau_n)$, where $\sigma: s \to t$ and $\tau_i: s_i \to t_i$ are reductions in $F(\mathcal{R})$ such that at least one $\tau_i$ is not an identity map. Then, by (Nat 1) in Definition 2.1.7, we may rewrite $\rho$ as $s(\tau_1, \ldots, \tau_n) \cdot \sigma(1_{t_1}, \ldots, 1_{t_n})$. Since $\sigma(1_{t_1}, \ldots, 1_{t_n})$ is singular by induction, we may assume that $\rho = F(\tau_1, 1_{t_2})$. Without loss of generality, suppose that $n = 2$. It follows from the functoriality of $F$ that $\rho = F(\tau_1, 1_{t_2})$. By induction, each of $\tau_1$ and $\tau_2$ is a composite of singular reductions. It follows then from the functoriality of $F$ that $\rho$ is equal to a composite of $R(\rho)$-many singular reductions.

In light of the above lemma, we know that any particular reduction is equal to a composite of only finitely many singular reductions. However, $F(\mathcal{R})$ might still contain an infinite sequence of composable reductions.

**Definition 2.6.4 (Terminating).** A rewriting 2-theory $\mathcal{R}$ is terminating if any infinite sequence of composable singular reductions in $F(\mathcal{R})$ contains cofinitely many identity reductions.

Of particular importance in many investigations of various kinds of term rewriting systems are those terms that are not the source of any non-identity reduction. Often, one would like to assign such a term to an arbitrary term.

**Definition 2.6.5 (Normal Form).** Let $\mathcal{R}$ be a rewriting 2-theory and let $[s]$ be a term in $F(\mathcal{R})$. A normal form for $[s]$ is a term $[t]$ such that there is a reduction $[s] \to [t]$ in $F(\mathcal{R})$ and there are no non-identity reductions whose source is $[t]$ in $F(\mathcal{R})$. We say that $\mathcal{R}$ has normal forms if every term in $\mathcal{R}$ has a normal form.

In an arbitrary rewriting 2-theory $\mathcal{R}$, a given term may or may not have a normal form. If $\mathcal{R}$ is terminating, then every term has at least one normal form. In the fortunate situation where every term in $\mathcal{R}$ has a unique normal form, many investigations become somewhat simpler. In order to guarantee this property, we need further restrictions on $\mathcal{R}$.

**Definition 2.6.6 (Confluent).** A rewriting 2-theory $\mathcal{R}$ is confluent if any diagram

$$
\begin{array}{c}
[t_1] \\
\rho_1 \\
\Downarrow
\end{array}
\quad \begin{array}{c}
[s] \\
\rho_2 \\
\Downarrow
\end{array}
\quad \begin{array}{c}
[t_2] \\
\gamma_2
\end{array}
$$

in $F(\mathcal{R})$ can be completed into a (not necessarily commutative) square:

$$
\begin{array}{c}
[s] \\
\rho_2
\end{array}
\quad \begin{array}{c}
[t_2] \\
\gamma_2
\end{array}
$$

**Definition 2.6.7.** A rewriting 2-theory is complete if it is terminating and confluent. Otherwise it is incomplete.

**Lemma 2.6.8.** A complete rewriting 2-theory has unique normal forms.

**Proof.** Let $\mathcal{R}$ be a complete rewriting 2-theory and let $t$ be a term in $F(\mathcal{R})$. Since $\mathcal{R}$ is terminating, $t$ has at least one normal form. Suppose that $N_1(t)$ and $N_2(t)$ are
normal forms for $t$. If $\mathcal{N}_1(t) \neq \mathcal{N}_2(t)$, then since $\mathcal{R}$ is confluent there must be a term $v$ and reductions $\mathcal{N}_i(t) \rightarrow v$ for $i \in \{1, 2\}$ in $\mathcal{F}(\mathcal{R})$, contradicting the normality of these terms.

Our investigation of coherence for rewriting 2-theories splits into two cases, corresponding to whether the theories are assumed to be complete or not with the latter case being somewhat more delicate.

In the following chapter, we establish a link between algebraic invariants and coherent categorifications of equational theories. This link is particularly useful for complete rewriting 2-theories.
In Theorem 2.5.3, we saw that a coherent categorification of an equational variety is equivalent to the original variety in the sense that it has an equivalent Lawvere 2-theory. The main purpose of this chapter is to highlight how this phenomenon arises in combinatorial algebra within the realm of structure monoids. Later, in Chapter 5, we shall exploit this connection in order to construct new presentations of some famous algebraic objects.

Structure monoids were introduced by Dehornoy [Deh93] as algebraic invariants of a certain class of equational varieties. Dehornoy subsequently showed that Higman’s groups $F$ and $V$ arise as algebraic invariants of the varieties of semigroups and of commutative semigroups, respectively [Deh05]. In particular, he showed how to construct presentations of these groups using Mac Lane’s pentagon and hexagon coherence axioms for coherently associative and commutative bifunctors.

The relations in Dehornoy’s presentations consist of two parts. First, there are the so-called geometric relations, which arise purely from the fact that a semigroup is, in the first instance, a magma. The second class of relations arise from the particular equational structure of the variety at hand. In the case of $F$, one additional class of relations are added corresponding to the Stasheff-Mac Lane pentagon [ML63] and in the case of $V$, the presentation further contains a class of relations corresponding to the Mac Lane hexagon, which encodes the essential interaction between associativity and commutativity.

The goal of this chapter is to place Dehornoy’s constructions in a more general context. More precisely, we consider coherent categorifications of equational varieties. Within this setting, Dehornoy’s geometric relations correspond to the functoriality and naturality of the associated categorical structure with the remaining relations arising from the coherence axioms.

We recall the definition of structure monoids in Section 3.1 and go on, in Section 3.2 to show that a coherent categorification of an equational variety gives rise to a presentation of the associated structure monoid. In certain favourable situations, the structure monoid can be turned into a group and we show that the construction of a presentation from a coherent categorification carries over to this setting.

3.1. Structure monoids

In this section, we recall Dehornoy’s construction of an inverse monoid associated to a balanced equational theory [Deh93].

We begin by briefly recalling and expanding upon some definitions from the previous chapter. For a graded set of function symbols $\mathcal{F}$ and a set $X$, we denote by $F_{\mathcal{F}}(X)$
the absolutely free term algebra generated by \( \mathcal{F} \) on \( X \). An equational theory is a tuple \( (\mathcal{F} | \mathcal{E}_\mathcal{F})_\mathcal{V} \), where \( \mathcal{V} \) is a set of variables, \( \mathcal{F} \) is a graded set of function symbols and \( \mathcal{E}_\mathcal{F} \) is an equational theory on \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \). A map \( \varphi : \mathcal{V} \rightarrow \mathbb{F}_\mathcal{F}(\mathcal{V}) \) is called a substitution and it extends inductively to an endomorphism \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \rightarrow \mathbb{F}_\mathcal{F}(\mathcal{V}) \). By abuse of notation, we label this latter map by \( \varphi \) as well. We use \( [\mathcal{V}, \mathbb{F}_\mathcal{F}(\mathcal{V})] \) to denote the set of all substitutions. For a term \( s \in \mathbb{F}_\mathcal{F}(\mathcal{V}) \) and a substitution \( \varphi \in [\mathcal{V}, \mathbb{F}_\mathcal{F}(\mathcal{V})] \), we use \( s^\varphi \) to denote the image of \( s \) under \( \varphi \). The support of a term \( s \) is the set of variables appearing in it. A pair of terms \((s, t)\) is balanced if they have the same support and an equational theory is balanced if every defining equation is balanced.

**Definition 3.1.1.** Given a balanced pair of terms \((s, t)\) in \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \), we use \( \rho_{s,t} \) to denote the partial function \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \rightarrow \mathbb{F}_\mathcal{F}(\mathcal{V}) \) with graph
\[
\{ (s^\varphi, t^\varphi) \mid \varphi \in [\mathcal{V}, \mathbb{F}_\mathcal{F}(\mathcal{V})] \}.
\]

For a balanced pair of terms \((s, t)\), the partial function \( \rho_{s,t} \) is functional since the support of \( t \) is a subset of the support of \( s \). The stronger restriction that the pair is balanced is required since we wish to utilise the inverse partial function \( \rho_{t,s} \) as well.

Given an equational theory \( \mathcal{E} := (\mathcal{F} | \mathcal{E}_\mathcal{F})_\mathcal{V} \), we use \([\mathcal{E}_\mathcal{F}]\) to denote the congruence generated by \( \mathcal{E}_\mathcal{F} \) on \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \) and we use \( \mathbb{F}_\mathcal{E}(\mathcal{V}) \) to denote the quotient \( \mathbb{F}_\mathcal{F}(\mathcal{V})/[\mathcal{E}_\mathcal{F}] \). Similarly, we use \([s]\) to denote the congruence class of a term \( s \) in \( \mathbb{F}_\mathcal{E}(\mathcal{V}) \). It is clear that \([u] = [\rho_{s,t}(u)]\) for any balanced equation \((s, t) \in \mathcal{E}_\mathcal{F}\) and any term \( u \in \text{dom}(\rho_{s,t}) \). However, the collection of all partial maps \( \rho_{s,t} \) for \((s, t) \in \mathcal{E}_\mathcal{F}\) is not sufficient to generate \([\mathcal{E}_\mathcal{F}]\), since equations apply to subterms as well. To this end, we introduce translated versions of the maps \( \rho_{s,t} \), that apply to arbitrary subterms.

A subterm \( s \) of a term \( t \) is naturally specified by the node where its root lies in the term tree of \( t \), which in turn is completely specified by the unique path from the root of \( t \) to the root of \( s \) in the term tree. A path in a term tree may be specified by an alternating sequence of function symbols and numbers, where the numbers indicate an argument of a function symbol. More formally, we have the following situation.

For a graded set \( \mathcal{F} := \coprod_n \mathcal{F}_n \), we set
\[
A_\mathcal{F} := \bigcup_n \bigcup_{F \in \mathcal{F}_n} \{ (F, 1), \ldots, (F, n) \}.
\]
The set of addresses associated to \( \mathcal{F} \) is denoted by \( A^*_\mathcal{F} \) and is the free monoid generated by \( A_\mathcal{F} \) under concatenation, with the unit being the empty string \( \lambda \). For a term \( t \in \mathbb{F}_\mathcal{F}(\mathcal{V}) \) and an address \( \alpha \in A^*_\mathcal{F} \), we use \( \text{sub}(t, \alpha) \) to denote the subterm of \( t \) at the address \( \alpha \). Note that \( \text{sub}(t, \alpha) \) only exists if the term tree of \( t \) contains the path \( \alpha \) and that \( \text{sub}(t, \lambda) = t \).

**Example 3.1.2.** Suppose that \( \mathcal{F} := \{F, G\} \), where \( F \) is a binary function symbol and \( G \) is a ternary function symbol. Suppose that \( \mathcal{V} \) is a set of variables. Then, the term \( t := F(w, G(x, y, z)) \) is in \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \). The term tree of \( t \) is given in Figure 1. The term \( t \) has the following subterms:
3.1. STRUCTURE MONOIDS

**Figure 1.** The term tree of $F(w, G(x, y, z))$

\[
\begin{align*}
\text{sub}(t, (F, 1)) &= w \\
\text{sub}(t, (F, 2)) &= G(x, y, z) \\
\text{sub}(t, (F, 1)(G, 1)) &= x \\
\text{sub}(t, (F, 1)(G, 2)) &= y \\
\text{sub}(t, (F, 1)(G, 3)) &= z
\end{align*}
\]

**Definition 3.1.3 (Orthogonal).** Given a graded set $\mathcal{F}$ and addresses $\alpha, \beta \in A^*_\mathcal{F}$, we say that $\alpha$ and $\beta$ are orthogonal and write $\alpha \perp \beta$ if neither $\alpha$ nor $\beta$ is a prefix of the other. Given a term $t$, and addresses $\alpha$ and $\beta$, the subterms $\text{sub}(t, \alpha)$ and $\text{sub}(t, \beta)$ are orthogonal if $\alpha \perp \beta$.

Our current addressing system is sufficient to describe translated copies of the basic operators.

**Definition 3.1.4.** Given a graded set of function symbols $\mathcal{F}$, a variable set $\mathcal{V}$, a balanced pair of terms $(s, t) \in F\mathcal{F}(\mathcal{V})$ and an address $\alpha \in A^*_\mathcal{F}$, the $\alpha$-translated copy of $\rho_{s,t}$ is denoted $\rho_{\alpha, s,t}$ and is the partial map $\mathcal{F}\mathcal{F}(\mathcal{V}) \to \mathcal{F}\mathcal{F}(\mathcal{V})$ defined as follows:

- A term $u \in \mathcal{F}\mathcal{F}(\mathcal{V})$ is in the domain of $\rho_{\alpha, s,t}$ if $\text{sub}(u, \alpha)$ is defined and is in the domain of $\rho_{s,t}$.
- For $u \in \text{dom}(\rho_{\alpha, s,t})$, the image $\rho_{\alpha, s,t}(u)$ is defined by
  \[
  \text{sub}(\rho_{\alpha, s,t}(u), \alpha) = \rho_{s,t}(\text{sub}(u, \alpha))
  \]
  and $\text{sub}(\rho_{\alpha, s,t}(u), \beta) = \text{sub}(u, \beta)$ for every address $\beta$ orthogonal to $\alpha$.

Note that $\rho_{s,t}^\alpha = \rho_{s,t}$.

We are finally in a position to introduce the structure monoid generated by an equational theory.

**Definition 3.1.5 (Structure Monoid).** Given a balanced equational theory $\mathcal{E} := \langle \mathcal{F} \mid \mathcal{E}\mathcal{F} \rangle_\mathcal{V}$, the structure monoid of $\mathcal{T}$, denoted $\text{Struct}(\mathcal{T})$, is the monoid of partial endomorphisms of $\mathcal{F}\mathcal{F}(\mathcal{V})$ generated by the following maps under composition:

\[
\{ \rho_{s,t}^\alpha \mid (s, t) \text{ or } (t, s) \in \mathcal{E}\mathcal{F} \text{ and } \alpha \in A^*_\mathcal{F} \}
\]

The structure monoid of an equational theory is readily seen to completely capture the equational theory.

**Lemma 3.1.6 (Dehornoy [Deh93]).** Let $\mathcal{E} := \langle \mathcal{F} \mid \mathcal{E}\mathcal{F} \rangle_\mathcal{V}$ be a balanced equational theory and let $t, t' \in F\mathcal{F}(\mathcal{V})$. Then $t =_\mathcal{E} t'$ if and only if there is some $\rho \in \text{Struct}(\mathcal{E})$ such that $\rho(t) = t'$.

 Given an equational theory $\mathcal{E} = \langle \mathcal{F} \mid \mathcal{E}\mathcal{F} \rangle_\mathcal{V}$ and maps $\rho_{s_1, t_1}, \rho_{s_2, t_2} \in \text{Struct}(\mathcal{T})$, the composition $\rho_{s_1, t_1} \cdot \rho_{s_2, t_2}$ may be empty. It is nonempty precisely when there exist
substitutions $\varphi, \psi \in \mathcal{V}, \mathbb{F}_T(\mathcal{V})$ such that $t_1^\varphi = s_2^\psi$. In this case, we say that the pair $(t_1, s_2)$ is unifiable and that $(\varphi, \psi)$ is a unifier of the pair. In the case where $(t_1, s_2)$ is not unifiable, the composition $\rho_{s_1, t_1} \cdot \rho_{s_2, t_2}$ results in the empty operator, which we denote by $\varepsilon$. Note that, for any operator $\rho \in \text{Struct}(T)$, we have $\rho \cdot \varepsilon = \varepsilon \cdot \rho = \varepsilon$. The existence of the empty operator makes freely computing with inverses in $\text{Struct}(T)$ impossible.

**Definition 3.1.7 (Composable).** An equational theory $\langle F \mid \mathcal{E}_F \rangle$ is composable if any pair of terms in $\bigcup_{(s, t) \in \mathcal{E}_F} \{s, t\}$ are unifiable.

Recall that an inverse monoid $M$ is one in which for each element $x \in M$, there is an element $y \in M$ such that $xyx = x$ and $yxy = y$. Dehornoy [Deh06] shows that $\text{Struct}(T)$ always forms an inverse monoid and contains the empty operator precisely when $\mathcal{E}$ is not composable. One way in which to transform $\text{Struct}(G)$ into a group is by passing to the universal group of $\text{Struct}(\mathcal{E})$, which we denote by $\text{Struct}_G(\mathcal{E})$, by collapsing all idempotents to 1. In the case where $\mathcal{E}$ is composable, the idempotent elements of $\text{Struct}(T)$ are precisely those operators that act as the identity on their domain. A particular class of composable theories is provided by a certain class of linear theories. Recall that an equation $s = t$ is linear if it is balanced and each variable appears precisely once in both $s$ and $t$. An equational theory is linear if each of its defining equations is linear.

**Lemma 3.1.8 (Dehornoy [Deh06]).** A linear equational theory containing precisely one function symbol is composable. $\square$

It follows from the above lemma that each linear equational theory containing precisely one function symbol gives rise to a structure group.

**Example 3.1.9.** The equational theories for semigroups, $S$, and for commutative semigroups, $C$, are both linear. Since these theories involve a single binary operator, Lemma 3.1.8 implies that they are composable. In this case we have that $\text{Struct}_G(S)$ is Thompson’s group $F$ and $\text{Struct}_G(C)$ is Thompson’s group $V$ [Deh05].

In the following section, we shall see how structure monoids and groups relate to coherent categorifications of equational varieties.

### 3.2. Structure monoids via coherence theorems

The main goal of this section is to show how coherent categorifications of equational theories give rise to presentations of structure monoids. We base our analysis at the level of theories, rather than of equational varieties. While this is seemingly at odds with Dehornoy’s result [Deh93] that structure monoids are independent of the particular equational presentation of a variety, differing presentations of the same variety lead to distinct categorifications and thence to distinct presentations of the structure monoid.

Dehornoy’s utilisation of the pentagon and hexagon coherence axioms in order to obtain presentations of Thompson’s groups [Deh05] is indicative of a more general relationship between structure monoids and coherent categorifications of equational
theories. The first step on the road to formalising this relationship is to construct a
monoid presentation out of a categorification of an equational theory. In light of Lemma
2.6.3, a good candidate for the generators of the monoid is provided by the singular
morphisms of the categorification. Since we shall be moving back and forth between the
structure monoid and a categorification, \( R(\mathcal{E}) \), of an equational theory \( \mathcal{E} \), there is some
danger of confusion about whether a symbol “\( \rho \)” lies in Struct(\( \mathcal{E} \)) or in \( R(\mathcal{E}) \). Thus,
in this section, we adopt the convention that an element marked as “\( \rho \)” lies in \( R(\mathcal{E}) \)
and an unmarked element “\( \rho \)” lies in Struct(\( \mathcal{E} \)). A second notational difficulty arises
due to the differing way in which elements of Struct(\( \mathcal{E} \)) and morphisms in Sing(\( R(\mathcal{E}) \))
are represented. For this reason, we give a way of rewriting singular morphisms to
more closely resemble elements of Struct(\( \mathcal{E} \)). If \( R(\mathcal{E}) = \langle F; I(\mathcal{E}_F) \mid \varnothing; E_{I(\mathcal{E}_F)} \rangle \), then a
reduction \( \hat{\rho} \in I(\mathcal{E}_F) \) with source \( s \) and target \( t \) is written as \( \hat{\rho}_{s,t} \).

**Definition 3.2.1** (Type/Address). Let \( R(\mathcal{E}) \) be a categorification of the equational
theory \( \mathcal{E} \). The type, \( T(\hat{\rho}_{s,t}) \) of a singular morphism \( \hat{\rho}_{s,t} \in \text{Sing}(R(\mathcal{E})) \) is defined inductively by:

\[
T(\hat{\rho}_{s,t}) = \begin{cases} 
T(\hat{\sigma}_{u,v}) & \text{if } \hat{\rho}_{s,t} = F(1, \ldots, 1, \hat{\sigma}_{u,v}, 1, \ldots, 1) \\
\hat{\rho}_{s,t} & \text{otherwise.}
\end{cases}
\]

The address, \( A(\hat{\rho}) \) is the word of \( A^\ast_F \) constructed as follows:

\[
A(\hat{\rho}) = \begin{cases} 
(F, i)A(\hat{\sigma}) & \text{if } \hat{\rho} = F(1, \ldots, 1, \hat{\sigma}, 1, \ldots, 1) \\
\lambda & \text{otherwise.}
\end{cases}
\]

Given a categorification \( R(\mathcal{E}) \) of an equational theory \( \mathcal{E} \), we can now construct a
monoid whose generators are the singular reductions of \( R(\mathcal{E}) \) and whose relations are
generated by the functoriality, naturality and coherence axioms.

**Definition 3.2.2.** Let \( \mathcal{E} := \langle F | I(\mathcal{E}_F) \rangle \) be a balanced equational theory and let
\( R(\mathcal{E}) := \langle F; I(\mathcal{E}_F) \mid \varnothing; E_{I(\mathcal{E}_F)} \rangle \) be a categorification of \( \mathcal{E} \). The monoid \( S(R(\mathcal{E})) \) is the
monoid generated by

\[
\{ T(\hat{\rho})^{A(\hat{\rho})} \mid \hat{\rho} \in \text{Sing}(R(\mathcal{E})) \} \cup \{ \hat{\rho}^{\alpha}_{s,s} \mid (s, t) \text{ or } (t, s) \text{ in } \mathcal{E}_F \text{ and } \alpha \in A^\ast_F \}
\]

if \( R(\mathcal{E}) \) is composable and by

\[
\{ T(\hat{\rho})^{A(\hat{\rho})} \mid \hat{\rho} \in \text{Sing}(R(\mathcal{E}_F)) \} \cup \{ \hat{\rho}^{\alpha}_{s,s} \mid (s, t) \text{ or } (t, s) \text{ in } \mathcal{E}_F \text{ and } \alpha \in A^\ast_F \} \cup \{ \varepsilon \}
\]

otherwise, subject to the following relations.

- **Identity:**

\[
\hat{\rho}^{\alpha}_{s,s} \cdot \hat{\rho}^{\alpha}_{s,t} = \hat{\rho}^{\alpha}_{s,t}
\]

\[
\hat{\rho}^{\alpha}_{s,t} \cdot \hat{\rho}^{\alpha}_{s,t} = \varepsilon
\]

- **Composition:** If \( t_1 \) and \( s_2 \) are not unifiable then

\[
\hat{\rho}^{\alpha}_{s_1,t_1} \cdot \hat{\rho}^{\alpha}_{s_2,t_2} = \varepsilon
\]
• **Empty operator:**
\[
\rho_{s,t}^\alpha \cdot \varepsilon = \varepsilon
\]
\[
\varepsilon \cdot \rho_{s,t}^\alpha = \varepsilon
\]

• **Functoriality:** For \(\alpha \perp \beta\):
\[
\rho_{s,t}^\alpha \cdot \rho_{u,v}^\beta = \rho_{u,v}^\beta \cdot \rho_{s,t}^\alpha
\]

• **Naturality:** Suppose that \(\rho_{s,t} \in I(\mathcal{E}_F)\) is a generator and that some variable \(x\) appears at addresses \(\beta_1, \ldots, \beta_p\) in \(s\) and at addresses \(\gamma_1, \ldots, \gamma_q\) in \(t\). Then, for all addresses \(\alpha, \delta\) and each \(\rho_{u,v} \in I(\mathcal{E}_F)\):
\[
\rho_{s,t}^\alpha \cdot \rho_{u,v}^{\alpha_\gamma \beta \delta} = \rho_{u,v}^{\beta_\alpha \gamma \delta} \cdot \rho_{s,t}^\alpha
\]

• **Coherence:** For \((\sigma_1, \ldots, \sigma_p, \tau_1, \ldots, \tau_q) \in \mathcal{E}_I(\mathcal{E}_F)\), where each \(\sigma_i\) and \(\tau_j\) is singular, set:
\[
T(\sigma_1)^{A(\sigma_1)} \cdot \ldots \cdot T(\sigma_p)^{A(\sigma_p)} = T(\tau_1)^{A(\tau_1)} \cdot \ldots \cdot T(\tau_q)^{A(\tau_q)}
\]

The relations for functoriality and naturality in \(S(\mathcal{R}(\mathcal{E}))\) are adapted from [Deh06]. The functoriality relation is precisely the requirement that each operator \(F \in \mathcal{F}\) is a functor. The naturality condition is, in turn, precisely the requirement that each \(\rho \in I(\mathcal{E}_F)\) is a natural transformation. The rather involved addressing system in the naturality condition is due to the fact that the same variable may appear multiple times in different positions on either side of an equation. For naturality, one needs to apply a map to each of these instances of the variable simultaneously.

**Lemma 3.2.3.** Let \(\mathcal{E}\) be a balanced equational theory and let \(\mathcal{R}(\mathcal{E})\) be a categorification of \(\mathcal{E}\). Then \(S(\mathcal{R}(\mathcal{E}))\) is an inverse monoid.

**Proof.** For nonempty \(\hat{\rho} := \rho_{s,t}^{\alpha_1} \cdots \rho_{s,t}^{\alpha_k}\), set \(\hat{\rho}^{-1} := \rho_{s,t}^{\alpha_k} \cdots \rho_{s,t}^{\alpha_1}\). Since \(\rho_{s,t}\) is the inverse of \(\rho_{s,t}\), it follows that
\[
\hat{\rho} \cdot \hat{\rho}^{-1} = \rho
\]
\[
\hat{\rho}^{-1} \cdot \hat{\rho}^{-1} = \rho
\]
Since we also have that \(\varepsilon \cdot \varepsilon \cdot \varepsilon = \varepsilon\), it follows that \(S(\mathcal{R}(\mathcal{E}))\) forms an inverse monoid. \(\Box\)

We now know that both \(S(\mathcal{R}(\mathcal{E}))\) and \(\text{Struct}(\mathcal{E})\) are inverse monoids. Since there is a clear relationship between the generators of each, in order to establish that they are in fact isomorphic we need to focus on the relations. In particular, since \(\mathcal{R}(\mathcal{E})\) is an arbitrary categorification of \(\mathcal{E}\), it might contain inequivalent reductions with the same source and target. Since the elements of \(\text{Struct}(\mathcal{E})\) are partial functions completely determined by their domain and codomain, such a situation cannot occur in \(\text{Struct}(\mathcal{E})\). These considerations lead one to suspect that if we require \(\mathcal{R}(\mathcal{E})\) to be a coherent categorification of \(\mathcal{E}\), then the two monoids might in fact be isomorphic.

**Theorem 3.2.4.** Let \(\mathcal{E}\) be a balanced equational theory and let \(\mathcal{R}(\mathcal{E})\) be a categorification of \(\mathcal{E}\). The following map is an epimorphism of inverse monoids and it is an
isomorphism if and only if $\mathcal{R}(\mathcal{E})$ is coherent:

$$S(\mathcal{R}(\mathcal{E})) \xrightarrow{\Theta} \text{Struct}(\mathcal{E})$$

$$\rho_{s_1,t_1}^{\alpha_1} \cdots \rho_{s_k,t_k}^{\alpha_k} \longmapsto \rho_{s_1,t_1}^{\alpha_1} \cdots \rho_{s_k,t_k}^{\alpha_k}$$

**Proof.** By construction, $\Theta$ is a homomorphism of inverse monoids. For surjectivity, we need only show that every generator $\rho_{s,t}^{\alpha} \in \text{Struct}(\mathcal{E})$ corresponds to some singular morphism $S(\rho_{s,t}^{\alpha}) \in \text{Sing}(\mathcal{R}(\mathcal{E}))$. This singular morphism can be constructed recursively as follows:

$$S(\rho_{s,t}^{\alpha}) = \begin{cases} F(1, \ldots, 1, S(\rho_{s,t}^{\alpha}), 1, \ldots, 1) & \text{if } \alpha = (\tilde{F}, i)\beta \\ \tilde{\rho}_{s,t} & \text{if } \alpha = \lambda \end{cases}$$

It remains to show that $\Theta$ is faithful if and only if $\mathcal{R}(\mathcal{E})$ is coherent.

Suppose that $\Theta$ is faithful and let $\hat{\rho}_1, \hat{\rho}_2$ be a parallel pair of morphisms in $\mathbb{F}(\mathcal{R}(\mathcal{E}))$. Then $\Theta(\hat{\rho}_1) = \Theta(\hat{\rho}_2)$, since $\hat{\rho}_1$ and $\hat{\rho}_2$ have the same source and target. Since $\Theta$ is faithful, it follows that $\hat{\rho}_1 = \hat{\rho}_2$.

Conversely, suppose that $\mathcal{R}(\mathcal{E})$ is coherent and that $\Theta(\hat{\rho}_1) = \Theta(\hat{\rho}_2)$. Then, $\hat{\rho}_1$ and $\hat{\rho}_2$ have the same source and target. Since $\mathcal{R}(\mathcal{E})$ is coherent, it follows that $\hat{\rho}_1 = \hat{\rho}_2$. □

The above theorem is very closely linked with Theorem 2.5.3. The essential insight is that $\text{Struct}(\mathcal{E})$ is simply a monoid encoding of $\mathbb{F}(\mathcal{E})$, while $S(\mathcal{R}(\mathcal{E}))$ is a monoid encoding of $\mathbb{F}(\mathcal{R}(\mathcal{E}))$. In order to extend this correspondence to structure groups, we need to modify our presentations slightly.

**Definition 3.2.5.** Let $\mathcal{E}$ be a balanced composable equational theory and let $\mathcal{R}(\mathcal{E})$ be a categorification of $\mathcal{E}$. The group $S_G(\mathcal{R}(\mathcal{E}))$ is generated by

$$\{T(\hat{\rho})^{A(\hat{\rho})} \mid \hat{\rho} \in \text{Sing}(\mathcal{R}(\mathcal{E}))\},$$

subject to the functoriality, naturality and coherence relations from Definition 3.2.2, together with the following relation:

$$(\rho_{s,t}^{\alpha})^{-1} = \overline{\rho}_{t,s}^{\alpha}.$$ 

Following the same line of reasoning as in the proof of Theorem 3.2.4, we obtain the following relationship between $S_G(\mathcal{R}(\mathcal{E}))$ and $\text{Struct}_G(\mathcal{E})$.

**Theorem 3.2.6.** Let $\mathcal{E}$ be a balanced, composable equational theory and let $\mathcal{R}(\mathcal{E})$ be a categorification of $\mathcal{E}$. The following map is an epimorphism of groups and it is an isomorphism if and only if $\mathcal{R}(\mathcal{E})$ is coherent:

$$S_G(\mathcal{R}(\mathcal{E})) \xrightarrow{\Theta} \text{Struct}_G(\mathcal{E})$$

$$\rho_{s_1,t_1}^{\alpha_1} \cdots \rho_{s_k,t_k}^{\alpha_k} \longmapsto \rho_{s_1,t_1}^{\alpha_1} \cdots \rho_{s_k,t_k}^{\alpha_k}$$

□

**Example 3.2.7.** As we saw in Example 3.1.9, the structure group for semigroups is Thompson’s group $F$ and the structure group for commutative semigroups is Thompson’s group $V$, which is the first known finitely presented infinite simple group. It follows from Theorem 3.2.6 and Mac Lane’s coherence theorem for monoidal categories.
that we may construct a presentation for $F$ using the pentagon coherence diagram displayed in Example 2.1.9. A categorification of the theory of commutative semigroups contains an invertible reduction rule $\tau : a \otimes b \leadsto b \otimes a$. It follows from Mac Lane's results [ML63] that a coherent categorification of the theory is provided by requiring $\tau \cdot \tau = 1$, together with the pentagon axiom and the hexagon axiom, which states that the following diagram commutes:

$$
\begin{array}{ccc}
\rotatebox{90}{$\tau$} & \otimes & \alpha^{-1} \\
(a \otimes (b \otimes c)) & \rightarrow & (b \otimes c) \otimes a \\
\alpha \downarrow & & \downarrow 1 \otimes \tau \\
(a \otimes b) \otimes c & \rightarrow & (b \otimes a) \otimes c \\
\tau \otimes 1 & & \alpha^{-1} \\
(b \otimes a \otimes c) & \rightarrow & b \otimes (a \otimes c)
\end{array}
$$

This coherence theorem allows us, as a result of Theorem 3.2.6, to construct a presentation of Thompson's group $V$. These presentations for $F$ and $V$ are the same as those constructed by Dehornoy [Deh05].

Paraphrasing theorems 3.2.4 and 3.2.6, whenever we have a coherent categorification of a balanced equational theory, we automatically have a presentation of the associated structure monoid or group. As we shall see in Chapter 5, this is a reasonably powerful result, allowing us to obtain presentations of certain important infinite groups. However, before we can embark upon that investigation, we need a way of constructing coherent categorifications and proving that a given rewriting 2-theory is coherent. The following chapter solves these problems for rewriting 2-theories that are terminating and confluent, which is precisely the situation that arises in Chapter 5.
CHAPTER 4

Coherence for complete theories

In order to obtain a general coherence theorem for rewriting 2-theories, one needs to find distinguishing features of the underlying rewriting theory that make the investigation tractable. As a first port of call, one might examine Mac Lane’s proof of coherence for monoidal categories [ML63]. Looking at this proof from the angle of rewriting theory, one notices several things. First, every reduction rule in the structure, presented in Example 2.1.9, is invertible. Second, an analysis of the rewriting system consisting of only the positive maps \( \alpha, \lambda, \rho \) reveals that this subtheory is complete. Finally, one only needs to show that each term has a unique reduction to its unique normal form in order to rapidly conclude coherence. An approach along these lines is used by Johnson [Joh87] in order to develop a general coherence theorem for pasting diagrams in \( n \)-categories.

Similar considerations led Melliès to formulate the notion of “universal confluence” for a term rewriting theory within his framework of axiomatic rewriting theory [Mel02]. In order to formulate this concept within our setting, we require the notion of a commuting joining.

**DEFINITION 4.0.8 (Span).** A span, \( S \), in a rewriting 2-theory \( \mathcal{R} \) is a diagram of the form

\[
\begin{array}{c}
\varphi_1 \\
\downarrow u_1 \\
S \\
\downarrow \varphi_2 \\
u_2
\end{array}
\]

in \( \mathbb{F}(\mathcal{R}) \). A joining of \( S \) is a term \( t \) of \( \mathbb{F}(\mathcal{R}) \) together with reductions \( \psi_1 : u_1 \to t \) and \( \psi_2 : u_2 \to t \) in \( \mathbb{F}(\mathcal{R}) \). Pictorially, a joining is:

\[
\begin{array}{c}
S \\
\downarrow \varphi_2 \\
u_2 \\
\downarrow \varphi_1 \\
u_1 \\
\downarrow \psi_1 \\
t
\end{array}
\]

We call \( S \) joinable if a joining of \( S \) exists and we call \( S \) commuting-joinable if a joining exists such that the above diagram commutes.

Universal confluence is intended to capture the strong version of confluence present within monoidal categories, which coincides with the presence of pushouts in the free monoidal category on a discrete category. More specifically, it may be described as follows:

For every span \( u_1 \xleftarrow{\varphi_1} s \xrightarrow{\varphi_2} u_2 \) and for \( i \in \{1, 2\} \), there is a commuting joining \( \psi_i : u_i \to t \) such that for any other commuting joining \( \tau_i : u_i \to t \), there is a unique map \( \rho : t \to v \) making the following diagram commute:
For a general rewriting 2-theory, the map $\rho$ in the above diagram does not necessarily exist. However, whenever every reduction rule is invertible and the positive subtheory has unique normal forms, as is the case for monoidal categories, we can construct $\rho$ quite easily. Indeed, since there are maps $s \to t$ and $s \to v$, both $t$ and $v$ must have the same normal form $N(s)$. This means that there is a map $N_t : t \to N(s)$ and a map $N_v : v \to N(s)$ and we may simply take $\rho$ to be $N_t \cdot N_v^{-1}$.

When $\mathcal{R}$ contains non-invertible rules, the existence of $\rho$ is no longer guaranteed. Surprisingly though, the invertibility of the rules is not crucial for coherence. This was first demonstrated by Laplaza’s coherence theorem for categories with a directed associativity map $\alpha : a \otimes (b \otimes c) \to (a \otimes b) \otimes c$ that is not necessarily invertible [Lap72a]. Remarkably, the only coherence axiom required for this result is Mac Lane’s pentagon — precisely what is required in the invertible case.

We are now in the situation of needing to discern conditions on a confluent and terminating rewriting 2-theory that ensure Mac Lane coherence. Our approach needs to be delicate enough to handle both the invertible and the non-invertible case, since the same coherence axioms usually suffice for both. Ultimately, we shall end up with a slightly weaker and more general concept than universal confluence, essentially not requiring the existence of the map $\rho$.

Our approach requires some classical tools and lemmas from first order term rewriting theory and we briefly cover the required material in Section 4.1. In Section 4.2, we develop a practical general coherence theorem for complete rewriting 2-theories and extend this result to invertible theories in Section 4.3.

4.1. Classical lemmas

The focus of this section is on several classical lemmas that make the examination of confluence for finitely presented rewriting theories tractable. This analysis essentially reduces to enumerating over the possible ways in which two reductions can diverge within the theory. In other words, what we seek is some sort of classification of all possible spans that can arise from the theory. In light of Lemma 2.6.3, we can begin by focussing our attention on singular reductions.

**Definition 4.1.1.** A span $u \xleftarrow{\varphi_1} s \xrightarrow{\varphi_2} u_2$ is singular if both $\varphi_1$ and $\varphi_2$ are singular.

In the case where the rewriting theory is terminating, Newman’s Lemma reduces confluence to showing that every singular span is joinable.

**Lemma 4.1.2 (Newman’s Lemma [New42]).** A terminating rewriting 2-theory is confluent if every singular span is joinable. 

A singular span $u \xleftarrow{\varphi_1} s \xrightarrow{\varphi_2} u_2$ may take one of three forms:

- $\varphi_1$ and $\varphi_2$ rewrite disjoint subterms of $s$.
- $\varphi_1$ and $\varphi_2$ rewrite nested subterms of $s$.
- $\varphi_1$ and $\varphi_2$ rewrite overlapping subterms of $s$.

In practice, it is the rewriting of overlapping subterms of $s$ that can lead to non-confluence. It is, therefore, important to define precisely what we mean when we say that two reductions overlap. Before we do this, we need to identify all possible places where a reduction rule could apply.

**Definition 4.1.3 (Redex).** Let $\mathcal{L} := (\mathcal{F}; T | \mathcal{E}_\mathcal{F})_X$ be a labelled rewriting theory and let $\rho : [s] \to [t]$ be a reduction rule in $T$. For a substitution $\sigma : X \to \mathcal{F}_X(X)$ and a term $u \in [s]$, the term $u^\sigma$ is called a $\rho$-redex.

We are now in a position to define overlapping reduction rules.

**Definition 4.1.4 (Overlap).** Let $\mathcal{L} := (\mathcal{F}; T | \mathcal{E}_\mathcal{F})_X$ be a labelled term rewriting theory and let $t \in \mathcal{F}_X(X)$. Two subterms $t_1, t_2$ of $t$ overlap if they share at least one function symbol occurrence. Two reduction rules $\rho_1 : [s_1] \to [t_1]$ and $\rho_2 : [s_2] \to [t_2]$ in $T$ overlap if there is a term $t$ containing instances of a $\rho_1$-redex $r_1$ and a $\rho_2$-redex $r_2$ such that $r_1$ and $r_2$ overlap. We do not count the trivial overlap between a redex $r$ and itself unless $r$ is a redex of two different reduction rules.

**Example 4.1.5.** In the positive subtheory of the theory for monoidal categories given in Example 2.1.9, the reduction rules $\rho$ and $\lambda$ overlap on the term $a \otimes (I \otimes b)$ and the reduction rule $\alpha$ overlaps nontrivially with itself on the term $a \otimes (b \otimes (c \otimes d))$.

With Newman’s Lemma in mind, we now restrict our focus to singular reductions that rewrite overlapping terms. Unfortunately there may be infinitely many such spans, even for finitely presented theories. However, if we know that a certain span is joinable, then we automatically know that all substitution instances of it are joinable. Therefore, we can refocus our investigation on finding a minimal set of overlapping spans $S$ such that any overlapping span is a substitution-instance of a member of $S$.

**Definition 4.1.6.** Let $\mathcal{F}$ be a graded set of function symbols. Given terms $t, u \in \mathcal{F}_X(X)$, we say that $u$ is an instance of $t$ if there is a substitution $\sigma$ such that $u = t^\sigma$. A term $v \in \mathcal{F}_X(X)$ is a common instance of the terms $t, u \in \mathcal{F}_X(X)$ if it is an instance of both $t$ and $u$. The term $v$ is the most general common instance of $t$ and $u$ if any other common instance of $t$ and $u$ is also an instance of $v$.

Two terms may not have a common instance but when they do, they are guaranteed to have a most general common instance. The reader may find a proof of the following lemma in [DJ90].

**Lemma 4.1.7.** Let $\mathcal{F}$ be a graded set of function symbols. If $t, u \in \mathcal{F}_X(X)$ have at least one common instance, then they have a most general common instance.

Given two overlapping reductions, we can bootstrap the notion of most general common instance in order to obtain the “most general” way in which the two reduction
4.1. CLASSICAL LEMMAS

rules can overlap. Before we do this, however, we need to know precisely how two reduction rules overlap. This information is provided by the following lemma, a proof of which may be found in [KdV03, Lemma 2.7.7].

**Lemma 4.1.8.** Two reduction rules \( \rho_1 : s_1 \rightarrow t_1 \) and \( \rho_2 : s_2 \rightarrow t_2 \) overlap if and only if there is a non-variable subterm of \( s_1 \) that can be matched with a \( \rho_2 \)-redex or a non-variable subterm of \( s_2 \) that can be matched with a \( \rho_1 \)-redex.

In order to facilitate our definition of the “most general” overlap of two reduction rules, we need a way of specifying a distinguished subterm of a term. To this end, we use the notation \( t\{s\} \) to denote a term \( t \) with a distinguished subterm \( s \). We may apply rewrites directly to the subterm \( s \). If \( \rho : s \rightarrow s' \) is some reduction, then we may apply \( t\{\rho\} : t\{s\} \rightarrow t\{s'\} \).

**Definition 4.1.9 (Critical span).** Consider a pair of overlapping reduction rules \( \rho_1 : [\ell_1] \rightarrow [r_1] \) and \( \rho_2 : [\ell_2] \rightarrow [r_2] \). By Lemma 4.1.8, we may assume that \( \ell_1 = t\{u\} \) and that there are substitutions \( \sigma, \tau \) such that \( u^\sigma = \ell_2^\tau \). By Lemma 4.1.7, we may assume that \( u^\sigma = \ell_2^\tau \) is a most general common instance of \( u \) and \( \ell_2 \). Then, the following span arising from this overlap is called a critical span:

\[
\begin{array}{c}
[\ell_1^\sigma] \\
\xrightarrow{\rho_1^\sigma} [t^\sigma\{u^\sigma\}] \\
\xrightarrow{t\{\rho_2^\tau\}} [t\{r_2^\tau\}]
\end{array}
\]

**Example 4.1.10.** Consider the positive theory for monoidal categories given in Example 2.1.9. We then have the following reduction rules:

\[
\begin{align*}
\alpha(t_1, t_2, t_3) & : t_1 \otimes (t_2 \otimes t_3) \rightarrow (t_1 \otimes t_2) \otimes t_3 \\
\lambda(t) & : I \otimes t \rightarrow t \\
\rho(t) & : t \otimes I \rightarrow t
\end{align*}
\]

By Lemma 4.1.8, in order to find all overlaps between the reduction rules, we need only insert redexes of reduction rules as subterms of redexes of other reduction rules.

The reduction rule \( \alpha \) contains two instances of \( \otimes \). Thus, it overlaps nontrivially with itself and leads to the following critical span:

\[
\begin{array}{c}
(a \otimes b) \otimes (c \otimes d) \\
\xrightarrow{\alpha} a \otimes (b \otimes (c \otimes d)) \\
\xrightarrow{\otimes \alpha} a \otimes ((b \otimes c) \otimes d)
\end{array}
\]

Furthermore, \( \alpha \) overlaps with \( \lambda \) and \( \rho \) in three possible ways, leading to the following critical spans:

\[
\begin{align*}
(I \otimes b) \otimes c & \xleftarrow{\alpha} I \otimes (b \otimes c) \xrightarrow{\lambda} a \otimes b \\
(a \otimes I) \otimes c & \xleftarrow{\alpha} a \otimes (I \otimes c) \xrightarrow{\otimes \lambda} a \otimes b \\
(a \otimes b) \otimes I & \xleftarrow{\alpha} a \otimes (b \otimes I) \xrightarrow{\otimes \rho} a \otimes b
\end{align*}
\]
Finally, $\lambda$ and $\rho$ overlap with each other, leading to the following critical span:

\begin{equation}
4.5 \quad I \xleftarrow{\lambda} I \otimes I \xrightarrow{\rho} I
\end{equation}

This exhausts all of the critical spans arising in the theory.

As mentioned previously, the main utility of critical spans is that they drastically reduce the number of spans we need to check for joinability when investigating confluence. This result is embodied in the critical pairs lemma, so named because critical spans are usually identified with their pair of reduced terms. The reader may find a proof of the Lemma in [KdV03, Lemma 2.7.15].

**Lemma 4.1.11 (Critical Pairs Lemma).** Let $\mathcal{L}$ be a labelled rewriting theory. Every singular span in $\mathcal{L}$ is joinable if and only if every critical span in $\mathcal{L}$ is joinable. $\Box$

In the following section, we develop a general coherence theorem for rewriting 2-theories having unique normal forms. Our basic strategy is to obtain versions of Newman's Lemma and the Critical Pairs Lemma that take into account the commutativity of the diagrams involved. This leads to some additional subtleties, but the basic strategy remains close to this section.

### 4.2. Coherence for directed theories

In this section, we develop a coherence theorem for terminating and confluent rewriting 2-theories. For this, we shall need to refine our notion of confluence.

**Definition 4.2.1.** Let $\mathcal{R}$ be a rewriting 2-theory. A span $S$ in $\mathcal{F}(\mathcal{R})$ is commuting-joinable if there is a joining of $S$ that commutes in $\mathcal{F}(\mathcal{R})$. We say that $\mathcal{R}$ is commuting-confluent if every span in $\mathcal{F}(\mathcal{R})$ is commuting-joinable and we say that $\mathcal{R}$ is locally commuting-confluent if every singular span in $\mathcal{F}(\mathcal{R})$ is commuting-joinable.

We are now in a position to obtain a strong form of Newman's Lemma that includes information on the commutativity of diagrams.

**Lemma 4.2.2 (Strong Newman's Lemma).** Let $\mathcal{R}$ be a terminating, locally commuting-confluent finitely presented rewriting 2-theory. Then:

1. Every term $s \in \mathcal{F}(\mathcal{R})$ has a unique normal form $\mathcal{N}(s)$.
2. Any two reductions from $s$ to $\mathcal{N}(s)$ in $\mathcal{F}(\mathcal{R})$ are equal.

**Proof.** Part (1) follows from the classical Newman's Lemma, since $\mathcal{R}$ is terminating and confluent.

For Part (2), suppose that $s$ is a term in $\mathcal{F}(\mathcal{R})$ and let $\varphi$ and $\psi$ be two reductions from $s$ to $\mathcal{N}(s)$ in $\mathcal{F}(\mathcal{R})$. Then, it follows from Lemma 2.6.3 that $\varphi = \varphi_1 \cdot \varphi_2$, where $\varphi_1$ is singular. Similarly, $\psi = \psi_1 \cdot \psi_2$, where $\psi_1$ is singular. Suppose that $\varphi_1 : s \rightarrow u_1$ and $\psi_1 : s \rightarrow u_2$. Then, these two arrows form a singular span, which by assumption has a commuting joining $\tau_1 : u_i \rightarrow t$, where $t$ is a term in $\mathcal{F}(\mathcal{R})$ and $i \in \{1, 2\}$. Since there is a reduction $s \rightarrow t$ in $\mathcal{F}(\mathcal{R})$, it follows from Part (1) that $\mathcal{N}(t) = \mathcal{N}(s)$ and there is a reduction $\rho : t \rightarrow \mathcal{N}(s)$. For a term $a$, let $\mu(a)$ be the length of the longest
reduction from $a$ to $\mathcal{N}(a)$ in $\mathbb{F}(\mathcal{R})$ that does not contain an identity reduction. This is well defined since $\mathcal{R}$ is terminating and finitely presented. We proceed by induction on $\mu(s)$ to show that $\varphi = \psi$ by showing that the following diagram commutes in $\mathbb{F}(\mathcal{R})$:

\[
\begin{array}{c}
\varphi_1 \\
\downarrow u_1 \hspace{1cm} \downarrow \tau_1 \hspace{1cm} \downarrow \varphi_2 \\
\varphi \\
\downarrow t \\
\psi_1 \\
\downarrow u_2 \hspace{1cm} \downarrow \tau_2 \hspace{1cm} \downarrow \psi_2 \\
N(s)
\end{array}
\]

If $\mu(s) = 0$, then $s = \mathcal{N}(s)$ and $\varphi = \psi = 1_s$. Suppose that $\mu(s) > 0$. Without loss of generality, we may assume that neither $\varphi_1$ nor $\psi_1$ is $1_s$. Then, since there is a reduction from $s$ to $u_1$ and one from $s$ to $u_2$, it follows from Part (1) that $\mathcal{N}(u_1) = \mathcal{N}(u_2) = \mathcal{N}(s)$. Hence, $\mu(u_1) < \mu(s)$ and $\mu(u_2) < \mu(s)$ and it follows from induction that the subdiagrams labelled (2) and (3) in the diagram above commute. Since the diagram labelled (1) commutes by assumption, we have that $\varphi = \psi$.

By the preceding lemma, we know that each term in a terminating, locally commuting-confluent and finitely presented rewriting 2-theory has a unique reduction to a unique normal form. In order to pass from this fact to a general coherence theorem, we need a way of extending this result to arbitrary parallel pairs of reductions. The property that turns out to be most useful for achieving this is for every reduction to be monic.

**Definition 4.2.3.** A rewriting 2-theory $(\mathcal{F}; \mathcal{T} | \mathcal{E}_\mathcal{F}; \mathcal{E}_\mathcal{T})$ is monic if whenever $(\varphi_1 \cdot \psi, \varphi_2 \cdot \psi) \in \mathcal{E}_\mathcal{T}$ modulo the basic congruence, we have $(\varphi_1, \varphi_2) \in \mathcal{E}_\mathcal{T}$.

Recall that an arrow $g : b \to c$ in a category $\mathcal{C}$ is called “monic” if for every pair of arrows $f_1, f_2 : a \to b$ in $\mathcal{C}$, if $f_1 \cdot g = f_2 \cdot g$ then $f_1 = f_2$. The following lemma follows immediately from the construction of $\mathbb{F}(\mathcal{R})$ for a rewriting 2-theory $\mathcal{R}$.

**Lemma 4.2.4.** If $\mathcal{R}$ is a monic rewriting 2-theory, then every arrow in $\mathbb{F}(\mathcal{R})$ is monic.

We now have all the necessary ingredients for a general coherence theorem.

**Theorem 4.2.5 (Coherence).** A finitely presented rewriting 2-theory is Mac Lane coherent if it is monic, terminating and locally commuting-confluent.

**Proof.** Let $\mathcal{R}$ be a rewriting 2-theory satisfying the hypotheses. Suppose that $\tau_1$ and $\tau_2$ are two reductions $s \to t$ in general position in $\mathbb{F}(\mathcal{R})$. By Lemma 4.2.2, there is a unique arrow $\rho : t \to \mathcal{N}(t)$ and $\mathcal{N}(s) = \mathcal{N}(t)$. So, $\tau_1 \cdot \rho$ and $\tau_2 \cdot \rho$ are two arrows $s \to \mathcal{N}(s)$ in $\mathbb{F}(\mathcal{R})$. Lemma 4.2.2 implies that $\varphi_1 \cdot \rho = \varphi_2 \cdot \rho$. Since $\mathcal{R}$ is monic, it follows from Lemma 4.2.4 that $\varphi_1 = \varphi_2$.

Theorem 4.2.5 effectively reduces the problem of showing that a rewriting 2-theory is coherent to showing that the underlying term rewriting system is terminating and confluent. In order to make effective use of this coherence theorem, we establish a strong form of the Critical Pairs Lemma.
LEMMA 4.2.6 (Strong Critical Pairs Lemma). A rewriting 2-theory is locally commuting-confluent if and only if every critical span is commuting-joinable.

PROOF. By definition, every critical span in a locally commuting-confluent rewriting 2-theory is commuting-joinable. For the converse direction, let $\mathcal{R}$ be a rewriting 2-theory in which every critical span is commuting-joinable. Let $S := u \xrightarrow{\varphi} s \xrightarrow{\psi} v$ be a singular span in $F(\mathcal{R})$. We can distinguish three possibilities for this span:

1. $\varphi$ and $\psi$ rewrite disjoint subterms of $s$. Without loss of generality, we may assume that $s = F(t_1, t_2)$, that $\varphi = F(\varphi', 1_{t_2})$ and that $\psi = F(1_{t_1}, \psi')$; where $\varphi' : t_1 \rightarrow t'_1$ and $\psi' : t_2 \rightarrow t'_2$. Then, we have
   $$F(\varphi', \psi') : F(t_1, t_2) \rightarrow F(t'_1, t'_2).$$
   By the functoriality of $F$, we have
   $$F(\varphi', 1) : F(1, \psi') = F(\varphi' \cdot 1, 1 \cdot \psi') = F(1 \cdot \varphi', \psi' \cdot 1) = F(1, \psi') \cdot F(\varphi', 1).$$
   So, $S$ is commuting-joinable to $F(t'_1, t'_2)$.

2. $\varphi$ and $\psi$ rewrite nested subterms of $s$. Without loss of generality, we may assume that $s = p(q)$, that $\varphi = \varphi'\{1_q\}$ and that $\psi = 1_p\{\psi'\}$; where $\varphi' : p \rightarrow p'$ and $\psi : q \rightarrow q'$. Then we have
   $$\varphi'\{\psi'\} : p(q) \rightarrow p'(q').$$
   If $\varphi'$ is an instance of a left-linear reduction rule, then using (Nat 1) and (Nat 2) we get that $S$ is commuting-joinable to $p'\{q'\}$ via the following chain of equalities:
   $$\varphi'\{1_q\} \cdot p'(\psi) = \varphi'\{\psi'\} = p(\psi') \cdot \varphi\{1_q\}.$$  
   A similar argument works when $\varphi'$ is an instance of a non left-linear rule, using step (1) to rewrite the residuals of $q$ in parallel.

3. $\varphi$ and $\psi$ rewrite overlapping subterms of $s$. Without loss of generality, we may assume that $\varphi : s \rightarrow s_1$ and $\varphi : s \rightarrow s_2$. By the definition of a critical span, $S$ is then a substitution instance of a critical span, which is commuting-joinable by assumption.

By the construction of $F(\mathcal{R})$, it follows that $S$ is commuting-joinable.

EXAMPLE 4.2.7. In this example we utilise Theorem 4.2.5 to obtain a straightforward proof of the coherence theorem for categories with a directed associativity map. This coherence theorem is the main result of [Lap72a].

Let $\mathcal{R}$ be the rewriting 2-theory consisting of a single binary function symbol $\otimes$, the reduction rule
$$\alpha(x, y, z) : x \otimes (y \otimes z) \rightarrow (x \otimes y) \otimes z$$
and the left-hand diagram from Example 2.1.9 as an equation on reductions. Then, $\mathcal{R}$
is terminating by induction with the ranking function

$$\rho(t) = \begin{cases} 
\rho(a) + 2\rho(b) - 1 & \text{if } t = a \otimes b \\
1 & \text{otherwise.}
\end{cases}$$

The only critical span in this system arises as

$$\alpha(1_a, 1_b, 1_a \otimes 1_b) \quad a \otimes (b \otimes (c \otimes d)) \quad (a \otimes b) \otimes (c \otimes d) \quad a \otimes ((b \otimes c) \otimes d)$$

By the equation we placed on reductions, this critical pair is commuting-joinable to

$$(a \otimes b) \otimes (c \otimes d).$$

By Lemma 4.2.6, $\mathcal{R}$ is locally commuting-confluent. By Lemma 4.2.4, $\mathcal{R}$ is also monic. So, we may apply Theorem 4.2.5 and conclude that $\mathcal{R}$ is coherent.

Considering the above results, one may be tempted to massage an incomplete
rewriting 2-theory into a complete one and thus apply the coherence theorems. Indeed, the famous Knuth-Bendix completion algorithm [KB70] achieves precisely this. Unfortunately, such a procedure typically adds additional reduction rules to the rewriting theory and this is certainly the case with the Knuth-Bendix algorithm. Rather than simplifying the coherence problem, this additional structure results in a new rewriting 2-theory with a completely independent coherence problem whose solution sheds very little light on the coherence problem for the original theory.

In the following section, we tackle the problem of coherence for invertible rewriting 2-theories.

4.3. Coherence for invertible theories

It is not immediately obvious whether Theorem 4.2.5 can be extended in any meaningful way to invertible rewriting 2-theories. The reason for this is that such systems are necessarily non-terminating. We can, however, sidestep this problem by restricting our attention to an orientation of a rewriting 2-theory.

**Definition 4.3.1 (Orientation).** Let $\mathcal{R} := (\mathcal{F}; \mathcal{T} | \mathcal{E}_{\mathcal{F}}; \mathcal{E}_{\mathcal{T}})$ be a rewriting 2-theory. An orientation of $\mathcal{R}$ is a function $\mathcal{O} : \mathcal{T} \to \{1, -1\}$ such that:

1. $\mathcal{O}(\alpha) = 1$ for any non-invertible rule $\alpha$.
2. For an invertible pair of rules $(\alpha, \beta)$, either:
   - $\mathcal{O}(\alpha) = 1$ and $\mathcal{O}(\beta) = -1$, or
   - $\mathcal{O}(\alpha) = -1$ and $\mathcal{O}(\beta) = 1$.

Given an orientation on a rewriting 2-theory $\mathcal{R}$, we can restrict our attention to a directed subtheory of $\mathcal{R}$.

**Definition 4.3.2.** Given a rewriting 2-theory $\mathcal{R} := (\mathcal{F}; \mathcal{T} | \mathcal{E}_{\mathcal{F}}; \mathcal{E}_{\mathcal{T}})$ with orientation $\mathcal{O}$, a reduction rule $\alpha \in \mathcal{T}$ is positive if $\mathcal{O}(\alpha) = 1$ and negative otherwise. The
positive subtheory of $R$ relative to $O$ arises from $R$ by discarding all negative reduction rules from $T$ and discarding all equations from $E_T$ that contain an instance of a negative reduction rule.

Working relative to an orientation, we can now extend Theorem 4.2.5 to invertible theories.

**Theorem 4.3.3 (Coherence).** A finitely presented invertible rewriting 2-theory is Mac Lane coherent if it has an orientation whose positive subtheory is terminating and locally commuting-confluent.

**Proof.** Let $R$ be an oriented rewriting 2-theory satisfying the hypotheses and let $R^+$ be its positive subtheory. For a reduction $\psi \in F(R)$, we write $\psi^{-1}$ for its inverse. Suppose that $\varphi : A \to B$ is a reduction in $F(R)$. By Lemma 2.6.3,

$$\varphi = A \xrightarrow{\varphi_1} s_1 \xrightarrow{\varphi_2} s_2 \to \cdots \xrightarrow{\varphi_{n-1}} s_{n-1} \xrightarrow{\varphi_n} B$$

where each $\varphi_i$ is singular. Say that $\varphi_i$ is positive if it contains an instance of a positive reduction rule and negative otherwise. Since $R^+$ is terminating and locally commuting-confluent, Lemma 4.2.2 implies that each term $t \in F(R)$ has a unique positive map $N_t : t \to N(t)$ to a unique normal form $N(t)$. We claim that each rectangle in the following diagram commutes:

\[
\begin{array}{cccccc}
A & \xrightarrow{\varphi_1} & s_1 & \xrightarrow{\varphi_2} & s_2 & \cdots \xrightarrow{\varphi_{n-1}} & s_{n-1} & \xrightarrow{\varphi_n} & B \\
N_A & \downarrow & N_{s_1} & \downarrow & N_{s_2} & \cdots & \downarrow & N_{s_{n-1}} & \downarrow & N_B \\
N(A) & \xrightarrow{N(\varphi_1)} & N(s_1) & \xrightarrow{N(\varphi_2)} & N(s_2) & \cdots & \xrightarrow{N(\varphi_{n-1})} & N(s_{n-1}) & \xrightarrow{N(\varphi_n)} & N(B) \\
\end{array}
\]

If $\varphi_i$ is positive, then it follows immediately from Lemma 4.2.2 that $\varphi_i \cdot N_{s_i} = N_{s_{i-1}}$. If $\varphi_i$ is negative, then Lemma 4.2.2 implies that $\varphi_i^{-1} \cdot N_{s_{i-1}} = N_{s_i}$, which implies that $\varphi_i \cdot N_{s_i} = N_{s_{i-1}}$. Since each rectangle commutes, we have $\varphi \cdot N_B = N_A$, which implies that $\varphi = N_A \cdot N_B^{-1}$. Since $N_A$ and $N_B$ are unique and we did not rely on a particular choice of $\varphi$, we conclude that $R$ is coherent.

**Example 4.3.4.** In this example, we sketch a proof of Mac Lane coherence for monoidal categories. From Example 4.1.10, we know a set of critical spans for a certain positive subtheory of monoidal categories. This subtheory is terminating, as is readily verified by the following ranking function:

$$\rho(t) = \begin{cases} 
\rho(a) + 2\rho(b) - 1 & \text{if } t = a \otimes b \\
1 & \text{otherwise.}
\end{cases}$$

In order to conclude coherence, we need only show that every critical span is commuting-confluent. From the definition of monoidal categories, we know that critical spans (4.1) and (4.3) are commuting-joinable. In his original definition of monoidal categories [ML63], Mac Lane included additional axioms providing commuting joinings.
for the remaining critical spans. However, Kelly later showed [Kel64] that these critical spans are commuting-joinable as a consequence of the pentagon and triangle axioms for monoidal categories. It follows from Lemma 4.2.6 that the positive subtheory for monoidal categories is locally commuting-confluent. We may then apply Theorem 4.3.3 and conclude that the theory for monoidal categories is Mac Lane coherent.

The basic approach to Mac Lane coherence outlined in this chapter of proving termination and then analysing the critical spans can be successfully used to obtain coherence theorems for various other structures arising in the literature, such as distributive categories [Lap72b] and weakly distributive categories [CS97]. In light of the results of Chapter 3, this approach may potentially be used to construct presentations of structure monoids and groups. The following chapter details a successful application of this strategy to constructing presentations of the Higman-Thompson groups.
CHAPTER 5

Catalan categories

Thompson’s groups \( F \) and \( V \) [Tho80] are important objects arising within combinatorial group theory. The group \( F \) was originally introduced by Thompson in his investigation of word problems in finitely generated simple groups. This group was later rediscovered by homotopy theorists as the automorphism group of a free homotopy-idempotent [Dyd77b, Dyd77a, FH93]. The group \( F \) has several interesting properties. For instance, it is finitely presentable, has a simple commutator subgroup, has only abelian quotients, does not contain a nonabelian free group, is totally orderable and has exponential growth [CFP96].

In unpublished notes, Thompson showed that the group \( V \) is a finitely presented infinite simple group — the first known group of this type. McKenzie and Thompson [MT73] later described \( F \) as a group generated by the variety of semigroups. As we saw in Chapter 3, the relation of \( F \) with associativity is again reflected by the fact that it is the structure group for the variety of semigroups. We also saw that the group \( V \) is the structure group for commutative semigroups and sketched how Dehornoy’s presentations for these groups [Deh05] arise from the coherence theorems for coherently associative and commutative bifunctors.

As shown by Higman [Hig74], \( V \) is in fact a member of the infinite family of groups \( G_{n,r} \); where \( n > 1 \) and \( r > 0 \) are integers. In particular, \( V \cong G_{2,1} \). These groups share many of the properties of \( V \). For instance, they are infinite, finitely presentable and are either simple or have a simple subgroup of index 2. Brown [Bro87] subsequently showed that Thompson’s group \( F \) fits into a similar infinite family \( F_{n,r} \), where \( F \cong F_{2,1} \).

In Section 5.1, we recall Brown’s definitions of \( F_{n,1} \) and \( G_{n,1} \). These groups are defined in a very similar way to \( F \) and \( V \), which may lead one to wonder whether they too are structure groups of certain equational varieties.

In Section 5.2 we introduce \( n \)-catalan algebras, which encode a notion of associativity for an \( n \)-ary function symbol and prove that \( F_{n,1} \) is the structure group of the variety of \( n \)-catalan algebras, directly generalising the relation between \( F \) and associativity. We follow this with a definition of symmetric \( n \)-catalan algebras, which encode a notion of associativity and commutativity for an \( n \)-ary function symbol and we show that \( G_{n,1} \) is the structure group of the variety of symmetric \( n \)-catalan algebras.

We know from Chapter 3 that a coherent categorification of a balanced composable equational variety yields a presentation for the associated structure group. In Section 5.3, we construct a coherent categorification of the variety of \( n \)-catalan algebras, thus obtaining a presentation for \( F_{n,1} \). Finally, in Section 5.4, we construct a coherent
categorification of the variety of symmetric n-catalan categories, thus obtaining a presentation for $G_{n,1}$. These presentations are closely linked to Dehornoy’s presentations for $F$ and $V$, which we sketched in Example 3.2.7.

The rewriting 2-theories that we construct in sections 5.3 and 5.4 are also interesting from a purely categorical point of view as they directly generalise the Mac Lane pentagon and hexagon coherence axioms for commutative and associative bifunctors. For functors of arity greater than 2, new coherence phenomena appear, which are not present in the classical binary case.

5.1. The groups $F_{n,1}$ and $G_{n,1}$

In Chapter 3, we saw that Thompson’s groups $F$ and $V$ arise as structure groups of certain balanced equational theories and we subsequently obtained presentations for these groups via coherent presentations of their associated categorical theories. In this section, we introduce generalisations of these groups due to Brown [Bro87] and Higman [Hig74], which we call $F_{n,1}$ and $G_{n,1}$, respectively. In the following sections, we shall see how the aforementioned process of constructing presentations for $F$ and $V$ generalises to this broader class of groups.

There are several paths to defining the groups $F_{n,1}$ and $G_{n,1}$, all of which relate to the fact that each of these groups arises as a subgroup of the automorphism group of a Cantor set. Of the myriad of definitions available, we choose to follow the description of Brown [Bro87], which utilises certain equivalence classes of pairs of finite rooted trees.

**Definition 5.1.1 (Tree).** The set of n-ary trees is defined inductively as follows:

- The graph consisting solely of a single vertex is an n-ary tree.
- If $T_1, \ldots, T_n$ are n-ary trees then the following is also an n-ary tree:

```
          .
         /\    \\/
        T_1  T_2  \ldots  T_n
```

The root of an n-ary tree is the unique vertex of valence 0 or $n - 1$. The leaves of a rooted tree $T$ are the vertices of valence 0 or 1 and we denote the set of leaves by $\ell(T)$.

**Definition 5.1.2 (Expansion).** A simple expansion of an n-ary tree $T$ is the tree obtained by replacing a leaf $v$ of $T$ with the following:

```
    v
   / \  \\/
  \alpha_1(v)  \alpha_2(v)  \ldots  \alpha_n(v)
```

In the above diagram, each $\alpha_i$ is simply a label for the relevant leaf. An expansion of an n-ary tree is a tree obtained by making finitely many successive simple expansions.

Given two trees $T_1$ and $T_2$ having a common expansion $S$, we say that $S$ is a minimal common expansion if any other expansion $S'$ of $T_1$ and $T_2$ is an expansion of $S$. 
5.1. THE GROUPS $F_{n,1}$ AND $G_{n,1}$

**Lemma 5.1.3** (Higman [Hig74]). Any two finite n-ary trees have a minimal common expansion. 

The underlying sets of the groups $F_{n,1}$ and $G_{n,1}$ consist of certain formal expressions called tree diagrams.

**Definition 5.1.4** (Tree diagram). An n-ary tree diagram is a triple $(T_1, T_2, \sigma)$, where $T_1$ and $T_2$ are n-ary trees having the same number of leaves and $\sigma$ is a bijection $\ell(T_1) \to \ell(T_2)$.

As in the case of trees, we may talk about expansions of tree diagrams.

**Definition 5.1.5.** A simple expansion of an n-ary tree diagram $(T_1, T_2, \sigma)$ is an n-ary tree diagram $(T'_1, T'_2, \sigma')$ obtained by the following procedure:

- $T'_1$ is the simple expansion of $T_1$ along the leaf $l$.
- $T'_2$ is the simple expansion of $T_2$ along the leaf $\sigma(l)$.
- $\sigma'$ is the bijection $\ell(T'_1) \to \ell(T'_2)$ defined by setting $\sigma'(k) = \sigma(k)$ for $k \in \ell(T_1) \setminus \{l\}$ and $\sigma'(\alpha_i(l)) = \alpha_i(\sigma(l))$.

An expansion of an n-ary tree diagram $(T_1, T_2, \sigma)$ is any n-ary tree diagram obtained by making finitely many successive simple expansions of $(T_1, T_2, \sigma)$.

Let $\sim$ be the equivalence relation on the set of n-ary tree diagrams obtained by setting $(T_1, T_2, \sigma) \sim (T'_1, T'_2, \sigma')$ whenever $(T_1, T_2, \sigma)$ and $(T'_1, T'_2, \sigma')$ possess a common expansion. Let $[(T_1, T_2, \sigma)]$ denote the equivalence class of $(T_1, T_2, \sigma)$ modulo $\sim$. We call $[(T_1, T_2, \sigma)]$ an n-ary tree symbol.

**Definition 5.1.6.** For $n \geq 2$, we set $G_{n,1}$ to be the group whose underlying set is the collection of n-ary tree symbols, together with the following group structure:

- Given two n-ary tree symbols $[(T_1, T, \sigma)]$ and $[(T'_1, T_2, \sigma')]$, it follows from Lemma 5.1.3 that we may assume that $T = T'$. We define their product to be $[(T_1, T, \sigma)][(T, T_2, \sigma')] = [(T_1, T_2, \sigma \cdot \sigma')]$.
- The inverse of $[(T_1, T_2, \sigma)]$ is $[(T_2, T_1, \sigma^{-1})]$.
- The unit element is $[(T, T, \text{id})]$.

It follows from the definitions that any n-ary tree is an expansion of the tree consisting solely of a single vertex. Thus, the leaves of an n-ary tree may be seen as a subset of the free monoid on \{1, \ldots, n\}. Therefore, we may order the leaves of the tree lexicographically, which is equivalent to ordering the leaves left-to-right when drawn on a page. We say that an n-ary tree symbol $[(T_1, T_2, \sigma)]$ is order-preserving if $\sigma$ is an isomorphism of ordered sets; that is, if $\sigma$ preserves this ordering.

**Definition 5.1.7.** For $n \geq 2$, we set $F_{n,1}$ to be the subgroup of $G_{n,1}$ consisting of the order-preserving n-ary tree symbols.

The groups $F_{n,1}$ and $G_{n,1}$ generalise Thompson’s original groups $F$ and $V$, since we have $F_{2,1} \cong F$ and $G_{2,1} \cong V$. They also share several of the interesting properties of $F$ and $V$ as surveyed in [Sco92]. In the following section, we shall realise $F_{n,1}$ as the
structure group of higher-order associativity and $G_{n,1}$ as the structure group of higher
order associativity and commutativity.

5.2. $F_{n,1}$ and $G_{n,1}$ as structure groups

Our goal in this section is to realise $F_{n,1}$ and $G_{n,1}$ as structure groups. Since both
of these groups are built using maps between n-ary trees, we take our set of function
symbols to be $F := \{ \otimes \}$, where $\otimes$ is an n-ary function symbol. For a set of variables
$V$, there is an obvious bijection between $F_x(V)$ and the set of n-ary trees whose leaves
are labelled by members of $V$. We denote the absolutely free term algebra generated
by $\{ \otimes \}$ on the set $V$ by $F_\otimes(V)$ and we denote the free monoid generated by $V$ under
concatenation by $V^*$. 

Our basic strategy is to first realise $F_{n,1}$ as a structure group by constructing an
equational theory $E$ such that $[E]$ equates any two terms $t_1, t_2 \in F_x(V)$ that contain
precisely the same variables in the same order and such that no variable appears more
than once in either $t_1$ or $t_2$. In the binary case, this is achieved by imposing associativity.
So, $E$ ought to be an analogue of associativity for $n > 2$. Once we have this realisation
of $F_{n,1}$ we need only add the ability to arbitrarily permute variables in order to obtain
a realisation of $G_{n,1}$ as a structure group.

5.2.1. Catalan Algebras and $F_{n,1}$

Associativity of a binary function symbol is
sufficient to establish that any two bracketings of the same string are equal. The way
in which one establishes this fact is to show that any bracketing of a string is equal to
the left-most bracketing. So, for an n-ary function symbol to be associative, we need
equations which imply that any bracketing of a term is equivalent to the left-most one.
In order to simplify notation, for integers $i \leq j$, we use the symbol $x_i^j$ to denote the
list $x_i, x_{i+1}, \ldots, x_j$. If $i > j$, then $x_i^j$ is the empty list.

DEFINITION 5.2.1 (n-Catalan algebras). For $n \geq 2$, the theory of n-Catalan algebras
consists of an n-ary function symbol $\otimes$ together with the following equations, where
$0 < i < n$:

$$\otimes(x_1^i, \otimes(x_{i+1}^{i+n}, x_{i+n+1}^{2n-1})) = \otimes(x_1^{i-1}, \otimes(x_i^{i+n-1}, x_{i+n}^{2n-1}))$$

We denote the theory of n-Catalan algebras by $C_n$.

The reason for the name of n-catalan algebras is that the set of all terms having
$k$ occurrences of the symbol $\otimes$ and containing precisely one variable is in bijective
correspondence with the set of n-ary trees having $k$ internal nodes, which has cardinality
equal to the generalised Catalan number $\frac{1}{(n-1)k+1} \binom{n}{k}$, [Sta99]. The rather opaque
equational theory of n-Catalan algebras is rendered somewhat more understandable by
viewing the induced equations on the term trees, which for $n = 3$, yields the following:
In order to apply the strategy from the binary case to the $n > 2$ case, we need to define what we mean by the left-most bracketing of a term $t$. Intuitively, this is the term having the same variables as $t$ in the same order, with all instances of \( \otimes \) appearing at the left.

**Definition 5.2.2 (Underlying list).** Let $t \in \mathbb{F}_\otimes(\mathcal{V})$. The underlying list of $t$ is the word of $\mathcal{V}^*$ defined inductively by

\[
U(t) = \begin{cases} 
U(t_1) \cdot \ldots \cdot U(t_n) & \text{if } t = \otimes(t_1, \ldots, t_n) \\
\ v & \text{otherwise}
\end{cases}
\]

Given the underlying list of a term, we can define the left-most bracketing by recursively adding all instances of \( \otimes \).

**Definition 5.2.3 (Left-most bracketing).** Let $t \in \mathbb{F}_\otimes(\mathcal{V})$. If $U(t) = t_1 \cdot \ldots \cdot t_{n+k(n-1)}$, then the left-most bracketing of $t$ is defined recursively by

\[
\text{Imb}(t^n_{1+k(n-1)}) = \text{Imb}(\otimes(t_1^n, t_{n+1}^{n+k(n-1)})).
\]

**Example 5.2.4.** In the table below, the right-hand term is the left-most bracketing of the left-hand term.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t$</th>
<th>$\text{Imb}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\otimes(a, \otimes(b, c), d))$</td>
<td>$\otimes(\otimes(a, b), c), d)$</td>
</tr>
<tr>
<td>3</td>
<td>$\otimes(a, \otimes(b, c, d), \otimes(e, f, g))$</td>
<td>$\otimes(\otimes(a, b, c), d, e), f, g)$</td>
</tr>
<tr>
<td>4</td>
<td>$\otimes(a, b, c, \otimes(d, e, \otimes(f, g, h), i), j))$</td>
<td>$\otimes(\otimes(a, b, c, d), e, f, g, h, i, j)$</td>
</tr>
</tbody>
</table>

We wish to establish that any term $\mathbb{F}_\otimes(\mathcal{V})$ is equal, in $\mathbb{F}_{C_n}(\mathcal{V})$, to its left-most bracketing. To this end, we define a rewriting theory based on $C_n$.

**Definition 5.2.5.** $C_n$ is the labelled rewriting theory consisting of an $n$-ary function symbol \( \otimes \), together with the following reductions, where $0 < i < n$:

\[
\alpha_i : \otimes(x_1^i, \otimes(x_{i+1}^{i+n}, x_{i+n+1}^{2n-1})) \rightarrow \otimes(x_1^{i-1}, \otimes(x_i^{i+n-1}, x_{i+n}^{2n-1}))
\]

The reduction rules of $C_n$ always move a term “closer” to its left-most bracketing. This observation is formalised in the following lemma.

**Proposition 5.2.5.** $C_n$ is terminating and confluent. Given a term $t \in \mathbb{F}_\otimes(\mathcal{V})$, its unique normal form in $\mathbb{F}_{C_n}^-(\mathcal{V})$ is given by $\text{Imb}(t)$.

**Proof.** We construct a ranking function on $\mathbb{F}_\otimes(\mathcal{V})$, which establishes that $\mathbb{F}_{C_n}^-(\mathcal{V})$ is terminating, that for every term $t \in \mathbb{F}_\otimes(\mathcal{V})$ there is a reduction $t \rightarrow \text{Imb}(t)$ and that $\text{Imb}(t)$ is a normal form for $t$. We begin by defining the length of $t$.

\[
L(t) = \begin{cases} 
\sum_{i=1}^n L(t_i) & \text{if } t = \otimes(t_1^n) \\
1 & \text{otherwise}.
\end{cases}
\]

Define the rank, $R(t)$, of $t$ inductively by setting $R(t) = 0$ if $t \in \mathcal{V}$ and

\[
R(\otimes(t_1^n)) = \sum_{i=1}^n R(t_i) + \sum_{i=2}^n (i - 1)L(t_i) - \frac{n(n-1)}{2}.
\]
We proceed by double induction on \( R(t) \) and \( L(t) \). If \( L(t) = 1 \) then the statement is trivial. If \( t = \otimes(t^n_i) \) and \( t = \text{imb}(t) \), then \( R(t) = R(t_1) + \sum_{i=2}^{n}(i-1) - \frac{n(n-1)}{2} = R(t_1) \) and it follows inductively that \( R(t) = 0 \). Conversely, if \( R(t) = 0 \), then \( t = \text{imb}(t) \), since otherwise we would have \( L(t_i) > 0 \) for some \( 2 \leq i \leq n \), from which it would follow that \( R(t) > 0 \).

Suppose that \( L(t) > 1 \) and \( R(t) > 0 \), so that \( t = \otimes(t^n_i) \). Let \( i \) be the greatest integer with the property that \( t_i \notin \mathcal{V} \). If \( i = 1 \), then \( t = \text{imb}(t) \) by induction on \( L(t) \). If \( i > 1 \), then \( t_i = \otimes(u^n_i) \) and \( \alpha_i : t \to t' \), where

\[
t' = \otimes(t_{i-2}^n, \otimes(t_{i-1}, u_{1}^{n-1}), u_n, t_{i+1}^n).
\]

We then have:

\[
R(t) - R(t') = R(t_{i-1}) + R(\otimes(u^n_i)) + (i-2)L(t_{i-1}) + (i-1)L(\otimes(u^n_1)) - R(\otimes(t_{i-1}, u_{1}^{n-1})) - R(u_n) - (i-2)L(\otimes(t_{i-1}, u_{1}^{n-1})) - (i-1)L(u_n) = \sum_{j=2}^{n}(j-1)L(u_j) + \sum_{j=1}^{n-1}L(u_j) - \sum_{j=2}^{n}(j-1)L(u_{j-1}) = (n-1)L(u_n).
\]

Since \( L(u_n) \geq 1 \), we have \( R(t') < R(t) \) and the proposition follows by induction on \( R(t) \).

Since each reduction rule in \( C_n^- \) is a directed version of an equation in \( C_n \), we immediately have the following corollary.

**Corollary 5.2.7.** For any \( t \in F_n(\mathcal{V}) \), we have \( t = c_n \text{imb}(t) \). 

In order to manipulate elements of \( \text{Struct}(C_n) \) effectively, we introduce the notion of a seed.

**Definition 5.2.8 (Seed).** Let \( \mathcal{F} \) be a graded set of function symbols on some set \( \mathcal{V} \) and let \( \rho \) be a partial function \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \to \mathbb{F}_\mathcal{F}(\mathcal{V}) \). A seed for \( \rho \) is a pair of terms \( s, t \in \mathbb{F}_\mathcal{F}(\mathcal{V}) \) such that the graph of \( \rho \) is equal to \( \{(s^\varphi, t^\varphi) \mid \varphi \in [\mathcal{V}, \mathbb{F}_\mathcal{F}(\mathcal{V})]\} \).

In particularly nice cases, we can construct seeds for any operator in a structure monoid.

**Lemma 5.2.9 (Dehornoy [Deh00]).** Let \( T \) be a balanced equational theory that contains precisely one function symbol. Then, each operator \( \rho \in \text{Struct}(T) \) admits a seed.

It follows from Lemma 5.1.3 that \( C_n \) is composable and we may, therefore, form the group \( \text{Struct}_G(C_n) \). In order to facilitate the passage from members of \( \text{Struct}_G(C_n) \), to members of \( F_{n,1} \), we introduce the tree generated by a term.

**Definition 5.2.10.** For a term \( t \in \mathbb{F}_\otimes(\mathcal{V}) \), let \( T(t) \) denote the \( n \)-ary tree obtained via the following construction:
5.2. F

n,1 AND G

n,1 AS STRUCTURE GROUPS

If \( t = \otimes(t_1, \ldots, t_n) \), then \( T(t) \) is equal to:

\[
T(t_1) \cdot T(t_2) \cdots T(t_n)
\]

Otherwise, \( T(t) \) is the single vertex.

We now have all the tools required to show that \( F_{n,1} \) is the structure group of \( n \)-catalan algebras.

**Theorem 5.2.11.** \( \text{Struct}_G(C_n) \cong F_{n,1} \).

**Proof.** We denote the seed of \( \rho \in \text{Struct}_G(C_n) \), which exists by Lemma 5.2.9, by \( (s_\rho, t_\rho) \). We claim that the following map is an isomorphism:

\[
\text{Struct}_G(C_n) \xrightarrow{\Theta} F_{n,1} \\
\rho \mapsto [(T(s_\rho), T(t_\rho), id)]
\]

It is routine to see that \( \Theta \) is a homomorphism. Suppose that \( \rho, \rho' \in \text{Struct}_G(C_n) \) and that \( \Theta(\rho) = \Theta(\rho') \). It follows that \( \rho \) and \( \rho' \) have the same seed, so \( \rho = \rho' \) and \( \Theta \) is faithful.

By Lemma 3.1.6, in order to establish that \( \Theta \) is surjective, we need only show that \( t_1 =_{C_n} t_2 \) whenever \( t_1, t_2 \in \mathbb{F}_\otimes(V) \) and \( U(t_1) = U(t_2) \). By Corollary 5.2.7, we have \( t_1 =_{C_n} \text{Imb}(t_1) =_{C_n} \text{Imb}(t_2) =_{C_n} t_2 \), so \( \Theta \) is surjective and, hence, an isomorphism. \( \square \)

5.2.2. Symmetric Catalan Algebras and \( G_{n,1} \). We saw in Section 5.1 that the leaves of a tree may be ordered by the lexicographic ordering on their addresses. An \( n \)-ary tree symbol \( [\{T_1, T_2, \sigma\}] \) may thereby be viewed as a pair of tree diagrams, together with a permutation of the leaves of \( T_1 \). Thus, in order to obtain an equational theory whose structure group is \( G_{n,1} \) we need to add the ability to arbitrarily permute variables in Catalan algebras. Recalling that the symmetric group is generated by transpositions of adjacent elements, we are led to the following definition.

**Definition 5.2.12 (Symmetric \( n \)-Catalan Algebras).** The theory of symmetric \( n \)-catalan algebras extends that of \( n \)-catalan algebras with the following equations, where \( 1 \leq i < n \):

\[
\otimes(x_1^{i-1}, x_i, x_{i+1}, x_{i+2}^n) = \otimes(x_1^{i-1}, x_{i+1}, x_i, x_{i+2}^n).
\]

We denote the theory of symmetric \( n \)-catalan algebras by \( SC_n \).

Symmetric \( n \)-catalan algebras essentially add an action of the symmetric group on the indices of \( \otimes \). In general, this is sufficient to induce an action of a symmetric group on the variables of any term in \( \mathbb{F}_\otimes(V) \). In the binary case, we recover the definition of commutative semigroups.

**Theorem 5.2.13.** \( \text{Struct}_G(SC_n) \cong G_{n,1} \).

**Proof.** For \( \rho \in \text{Struct}_G(SC_n) \), let \( (s_\rho, t_\rho) \) represent its seed, which exists by Lemma 5.2.9. Since \( SC_n \) is linear, \( s_\rho \) and \( t_\rho \) are linear and \( \text{supp}(s_\rho) = \text{supp}(t_\rho) \).
Let $\pi(\rho)$ be the permutation of $\text{supp}(s_{\rho})$ induced by the permutation $U(s_{\rho}) \rightarrow U(t_{\rho})$. A similar argument to the proof of Theorem 5.2.11 establishes that the following map is an isomorphism:

\[
\text{Struct}_G(C_n) \xrightarrow{\theta} G_{n,1} \\
\rho \mapsto [(T(s_{\rho}), T(t_{\rho}), \pi(\rho))]
\]

We now know that $F_{n,1}$ and $G_{n,1}$ are the structure groups of Catalan algebras and of symmetric Catalan algebras, respectively. We also know that if we can construct coherent categorifications of these algebras, then we can apply Theorem 3.2.6 to obtain presentations of these groups. In the following section, we set about the task of constructing a coherent categorification of Catalan algebras.

### 5.3. Catalan categories and $F_{n,1}$

In order to obtain a presentation for $\text{Struct}_G(C_n)$ and, hence, for $F_{n,1}$ along the lines of that provided by Dehornoy for $F$ [Deh05], we need to obtain a coherent categorification of $C_n$. The immediate problem is discerning a set of diagrams whose commutativity imply the commutativity of all diagrams generated by the categorification. As we shall see in this section, the following definition suffices for this purpose. While the coherence axioms that we have chosen may seem slightly cryptic, the reason for their choice will become apparent in the proof that the resulting categorification is coherent. We shall make frequent use of the following useful shorthand: For $1 \leq i \leq n$ and a morphism $\rho : t_i \rightarrow t'_i$, we set

\[
\otimes^i(\rho) = \otimes(1_{t_1}, \ldots, 1_{t_{i-1}}, \rho, 1_{t_{i+1}}, \ldots, 1_{t_n}).
\]

**Definition 5.3.1.** The rewriting 2-theory for $n$-Catalan categories is denoted $C_n$ and consists of:

- An $n$-ary function symbol $\otimes$.
- For $1 \leq i < n$, an invertible reduction rule $\alpha_i$ of the following form:
  \[
  \alpha_i(x_{1}^{2n-1}) : \otimes(x_{i}^{1}, \otimes(x_{i+1}^{1+n}), x_{i+n+1}^{2n-1}) \xrightarrow{\otimes(1_{i}^{i-1}, \otimes(x_{i+1}^{1+n-1}), x_{i+n}^{2n-1})}
  \]

**Pentagon axiom:** For $1 \leq i \leq n-1$, the following diagram commutes, where $X = x_{1}^{i-1}$ and $Z = x_{1}^{n-i-1}$:
Adjacent associativity axiom: For $1 \leq i \leq n-2$, the following diagram commutes, where $X = x_{i-1}^i$ and $Z = z_{i-1}^{n-i-2}$:

\[
\begin{align*}
\otimes(X, y_1, \otimes(y_2^{n+1}), \otimes(y_{n+2}^{2n+1}), Z) \\
\otimes(X, \otimes(y_1^n), y_{n+1}, \otimes(y_{n+2}^{2n+1}), Z) \\
\otimes(X, \otimes(y_1^n), \otimes(y_{n+1}^{2n-1}), y_{2n}^{2n+1}, Z) \\
\otimes(X, \otimes(y_1^n), \otimes(y_{n+1}^{2n-1}), y_{2n+1}^{2n-1}, Z)
\end{align*}
\]

In the case where $n = 2$, the pentagon axiom reduces to Mac Lane's pentagon axiom for monoidal categories from Example 2.1.9 and the adjacent associativity axiom is empty, so we recover the usual definition of a coherently associative bifunctor.

In the special case where $n = 3$, the adjacent associativity axiom leads to a single coherence axiom, illustrated by the following diagram. The other axioms given in this chapter may be unpacked in this special case in a similar manner.

We wish to apply Theorem 4.3.3 to $\mathbb{C}_n$ in order to show that it is a coherent categorification of $\mathcal{C}$. In order to do this, we need to find a positive orientation of $\mathbb{C}_n$ that is terminating and locally commuting-confluent. Let $\mathbb{C}_n^+$ be the positive subtheory of $\mathbb{C}_n$ that contains $\alpha_i$ for $0 < i < n$. This is equivalent, as a rewriting theory, to the
5.4. SYMMETRIC CATALAN CATEGORIES AND $G_{n,1}$

system $C_n^-$ introduced in the last section. Therefore, we know from Proposition 5.2.6 that $C_n^-$ is terminating. So, it remains to show that $C_n^-$ is locally commuting-confluent.

**Lemma 5.3.2.** $C_n^-$ is locally commuting-confluent.

**Proof.** By Lemma 4.2.6, we only need to show that every critical span in $C_n^-$ is commuting-joinable. Suppose that $\alpha_i$ and $\alpha_j$ overlap, where

$$\alpha_i : \otimes(t_i^i, \otimes(t_{i+1}^{i+n}, t_{i+n+1}^{2n-1}) \rightarrow \otimes(t_i^{i-1}, \otimes(t_i^{i+n-1}, t_{i+n+1}^{2n-1})).$$

Suppose first that $\otimes(t_i^i, \otimes(t_{i+1}^{i+n}, t_{i+n+1}^{2n-1})$ is an $\alpha_j$-reduct. For the overlap to be nontrivial, we must have $j \neq i$. If $j \neq i \pm 1$, then the critical span arising from the overlap is commuting-joinable by naturality. If $j = i \pm 1$, then the critical span arising from the overlap is commuting-joinable by the adjacent associativity axiom. If the overlap arises because $\otimes(t_{i+1}^{i+n})$ is an $\alpha_j$-reduct, then there are two possibilities. If $j \neq n$, then the critical span arising from the overlap is commuting-joinable by naturality. Otherwise, the critical span arising from the overlap is commuting-joinable by the pentagon axiom.

The only other possible overlap arises when some $t_k$ is an $\alpha_j$-reduct. In this case, the critical span arising from the overlap is commuting-joinable by naturality. □

Since $C_n^-$ is terminating and locally commuting-confluent, we may apply Theorem 4.3.3.

**Theorem 5.3.3.** $C_n$ is a coherent categorification of $C_n$.

With Theorem 5.3.3 in hand, we can obtain a presentation for $F_{n,1}$, which generalises the presentation for $F$ given in [Deh05].

**Corollary 5.3.4.** $S_G(C_n) \cong F_{n,1}$

**Proof.** By Theorem 5.3.3 and Theorem 3.2.6, we have $S_G(C_n) \cong \text{Struct}_G(C_n)$. It follows then from Theorem 5.2.11 that $S_G(C_n) \cong F_{n,1}$. □

In the following section, we shall obtain a coherent categorification of $SC_n$ and, thereby, a presentation of $G_{n,1}$.

5.4. Symmetric Catalan categories and $G_{n,1}$

Our goal in this section is to construct a coherent categorification of symmetric Catalan algebras. The coherence theorem for Catalan categories, Theorem 5.3.3, reduces this problem to ensuring that any two sequences of transpositions of the objects appearing in a term realise the same permutation. In other words, our categorification needs to somehow encode a presentation of the symmetric group whose generators correspond to transpositions of adjacent variables. Such a presentation is well known, having been constructed by Moore [Moo96]. This presentation has generators $T_1, \ldots, T_{n-1}$ and the following relations:

$$T_i^2 = 1 \quad \text{for } 1 \leq i \leq n - 1$$

$$(T_i T_{i+1})^3 = 1 \quad \text{for } 1 \leq i \leq n - 2$$

$$(T_i T_k)^2 = 1 \quad \text{for } 1 \leq i \leq k - 2$$
With this presentation in mind, we may now construct a reasonable categorification of $SC_n$. Recall our shorthand that for $1 \leq i \leq n$ and a morphism $\rho : t_i \to t'_i$, we have
\[ \otimes^i(\rho) = \otimes(1_{t_1}, \ldots, 1_{t_{i-1}}, \rho, 1_{t_{i+1}}, \ldots, 1_{t_n}). \]

**Definition 5.4.1.** For $n \geq 2$, the rewriting 2-theory for symmetric $n$-catalan categories, denoted $SC_n$, is the extension of the theory for $n$-catalan categories with an invertible reduction rule $\tau_i$ for $1 \leq i \leq n - 1$ such that
\[ \tau_i(t^n_i) : \otimes(t^{i-1}_1, t_i, t_{i+1}, t^n_{i+2}) \xrightarrow{\sim} \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) , \]
satisfying the following axioms:

**Involution axiom:** For $1 \leq i \leq n - 1$, the following diagram commutes:

\[
\begin{array}{ccc}
\otimes(t^n_i) & \xrightarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) \\
1 & \downarrow & \tau_i \\
\otimes(t^n_i) & \xleftarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2})
\end{array}
\]

**Compatibility axiom:** For $2 \leq i \leq n$ and $1 \leq j \leq n - 2$, the following diagram commutes, where $W = w_1^i$ and $Z = z_1^{n-1}$:

\[
\begin{array}{ccc}
\otimes(W, \otimes(x, y^{n-1}_1), y_n, Z) & \xrightarrow{\otimes^i(\tau_j)} & \otimes(W, \otimes(y^{i-1}_1, y_{j+1}, y_j, y^{n-1}_{j+2}), Z) \\
\alpha_{i-1} & \downarrow & \otimes^i(\alpha_{j+1}) \\
\otimes(W, \otimes(x, y^{i-1}_1, y_{j+1}, y_j, y^{n-1}_{j+2}), y_n, Z) & \xleftarrow{\otimes^i(\tau_{i+1})} & \otimes(W, \otimes(x, y^{n-1}_1), y_n, Z)
\end{array}
\]

**3-cycle axiom:** For $1 \leq i \leq n - 2$, the following diagram commutes:

\[
\begin{array}{ccc}
\otimes(t^n_i) & \xrightarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) \\
\tau_{i+1} & \downarrow & \tau_i \\
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xleftarrow{\tau_{i+1}} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2})
\end{array}
\]

\[
\begin{array}{ccc}
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xleftarrow{\tau_{i+1}} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) \\
\tau_i & \downarrow & \tau_{i+1} \\
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xrightarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2})
\end{array}
\]

\[
\begin{array}{ccc}
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xrightarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) \\
\tau_{i+1} & \downarrow & \tau_i \\
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xleftarrow{\tau_{i+1}} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2})
\end{array}
\]

\[
\begin{array}{ccc}
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xrightarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) \\
\tau_{i+1} & \downarrow & \tau_i \\
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xleftarrow{\tau_{i+1}} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2})
\end{array}
\]

\[
\begin{array}{ccc}
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xrightarrow{\tau_i} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) \\
\tau_{i+1} & \downarrow & \tau_i \\
\otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2}) & \xleftarrow{\tau_{i+1}} & \otimes(t^{i-1}_1, t_{i+1}, t_i, t^n_{i+2})
\end{array}
\]
**Hexagon axiom:** For $1 \leq i \leq n - 1$, the following diagram commutes, where $W = w_i^{-1}$ and $Z = z_1^{n-i-1}$:

\[
\begin{array}{c}
\otimes(W, \otimes(x^n_i), y, Z) \\
\tau_i \\
\otimes(W, y, \otimes(x^n_i), Z) \\
\alpha_i^{-1} \\
\otimes(W, x_1, \otimes(x^n_2), y, Z) \\
\otimes^i(\tau_1) \\
\otimes(W, y, x_1^{n-1}, x_n, Z) \\
\otimes(W, x_1, \otimes(y, x_2^n), Z) \\
\otimes^i(\tau_{n-1} \cdots \tau_1) \\
\otimes(W, \otimes(x_1, y, x_2^{n-1}), x_n, Z) \\
\end{array}
\]

The hexagon axiom ensures that we may replace a transposition of the form $\tau_i(t_i^{n-1}, \otimes(u^n_1), t_i^{n-1})$ with a sequence of transpositions involving only the terms $t_i^{n-1}$ and $u^n_1$. One might posit the commutativity of a diagram that serves the same purpose for a morphism of the form $\tau_i(t_i^{n-1}, \otimes(u^n_1), t_i^{n-1})$. Doing so leads to the *dual hexagon diagram*, which has the following form, for $2 \leq i \leq n$ and $W = w_i^{i-2}$ and $Z = z_1^{n-i}$:

\[
\begin{array}{c}
\otimes(W, x, \otimes(y^n_i), Z) \\
\tau_i \\
\otimes(W, \otimes(y^n_i), x, Z) \\
\alpha_i \\
\otimes(W, \otimes(y^n_i, y^n_1), y_n, Z) \\
\otimes^i(\tau_1 \cdots \tau_n) \\
\otimes(W, \otimes(y^n_2, x), Z) \\
\otimes(W, \otimes(y^n_1, x), y_n, Z) \\
\otimes^i(\tau_{n-1} \cdots \tau_1) \\
\otimes(W, y_1, \otimes(y_2^{n-1}, x, y_n), Z) \\
\end{array}
\]

**Lemma 5.4.2.** The dual hexagon diagram commutes in $\mathbb{F}(\mathcal{S}C_n)$.

**Proof.** Tracing around the dual hexagon diagram, we obtain the following morphism:

\[(5.1) \quad \alpha_i^{-1} \cdot \otimes^i(\tau_{n-1}) \cdot \alpha_i \cdot \otimes^i(\tau_1 \cdots \tau_{n-1})^{-1} \cdot \alpha_i^{-1}.
\]

In order to show that the dual hexagon diagram commutes, we need to show that

\[(5.1) = \tau_i.
\]

By functoriality, we have:

\[\otimes^i(\tau_1 \cdots \tau_{n-1})^{-1} = \otimes^i(\tau_1)^{-1} \cdots \otimes^i(\tau_{n-1})^{-1}.
\]

By functoriality and the involution axiom, we have $\otimes^i(\tau_j)^{-1} = \otimes^i(\tau_j)$. From the compatibility axiom, we also know that $\otimes^i(\tau_j) = \alpha_i^{-1} \cdot \otimes^i(\tau_{j-1}) \cdot \alpha_i$. It follows from
these observations that:

\[(5.2) \quad (5.1) = (5.3) = (5.4) = a^i - a^{-i}\]

It follows from the hexagon axiom that \((5.4) = \tau_i\). By the involution axiom, we then have:

\[\tau_i \cdot a^{-i} \cdot \otimes^{i+1}(\tau_{n-1}) \cdot \alpha_i \cdot \otimes^i(\tau_1) \cdot \alpha_i^{-1}\]

Therefore, the dual hexagon diagram commutes in \(F(\mathcal{C}_n)\).

In the \(n = 2\) case, the axiomatisation of \(\mathcal{C}_n\) reduces to the theory of a coherently associative and commutative bifunctor given in Example 3.2.7. The main result of this section establishes that \(\mathcal{C}_n\) is a suitable generalisation of this case.

**THEOREM 5.4.3.** \(\mathcal{C}_n\) is a coherent categorification of \(\mathcal{C}_n\).

**PROOF.** By Theorem 5.3.3 and Corollary 2.5.5, we may assume that all of the associativity maps are strict equalities. Thus, an object of \(F(\mathcal{C}_n)\) may be represented as \(\otimes(t^n_m)\), where each \(t_i\) is a variable and \(m = n + k(n - 1)\), for some \(k \geq 0\). Lemma 5.4.2 and the hexagon axiom imply that it suffices to consider transpositions of adjacent variables. So, for a given object \(t := \otimes(t^n_m)\), we need only consider the \(m - 1\) induced transposition natural isomorphisms

\[T_i(t^m) : \otimes(t_1^{i-1}, t_i, t_{i+1}, t_{i+2}, t^n_{i+2}) \rightarrow \otimes(t_1^{i-1}, t_{i+1}, t_i, t^n_{i+2})\]

In order to establish coherence, we have to show that every permutation of \(t^n_m\) is unique. That is, we have to show that the induced transposition maps satisfy the defining relations for the symmetric group of order \(m\).

The compatibility axiom implies that each \(T_i\) is unique. By the naturality of the maps \(T_i\), we have \(T_i \cdot T_k = T_k \cdot T_i\) for all \(1 \leq i \leq k - 2\). The involution axiom implies that \(T_i^2 = 1\). Thus, it only remains to establish that \((T_i \cdot T_{i+1})^3 = 1\). For \(n = 2\), we may use the proof from Mac Lane [ML63]. Suppose that \(n \geq 3\). Since the associativity maps are taken to be strict equalities, we may assume that \(t\) has the form \(\otimes(R, \otimes(S, t_i, t_{i+1}, t_{i+2}, U), V)\), where \(R, S, U\) and \(V\) are sequences of variables. The result then follows from the 3-cycle axiom.

We can now construct a presentation of \(\text{Struct}_G(\mathcal{C}_n)\) and, therefore, of \(G_{n,1}\), which generalises the presentation for \(V\) given in [Deh05].

**COROLLARY 5.4.4.** \(S_G(\mathcal{C}_n) \cong G_{n,1}\)

**PROOF.** By Theorem 5.4.3 and Theorem 3.2.6, we have \(S_G(\mathcal{C}_n) \cong \text{Struct}_G(\mathcal{C}_n)\). It follows then from Theorem 5.2.13 that \(S_G(\mathcal{C}_n) \cong G_{n,1}\).
applicable. In the following chapter, we develop more general coherence theorems that relax those assumptions somewhat.
CHAPTER 6

Coherence for incomplete theories

In Chapter 4, we developed a general Mac Lane coherence theorem for terminating and confluent rewriting 2-theories. This result has wide applicability, including the investigation of Catalan categories presented in Chapter 5.

Unfortunately, it is simply not the case that every coherent rewriting 2-theory has unique normal forms. For instance, the theory consisting of a unary function symbol \( F \) and the single reduction rule \( F(x) \rightarrow F(F(x)) \) is non-terminating, but easily seen to be coherent. A stronger counterexample to the hope that coherent structures have unique normal forms is provided by the theory of iterated monoidal categories [BFSV03] whose coherence problem we investigate in the following chapter. These structures arise as a categorical model of iterated loop spaces and fail to be confluent, so the tools of Chapter 4 do not apply.

We are thus faced with the problem of determining sufficient conditions for coherence in terms of the underlying rewriting system of a 2-theory that do not rely on either termination or confluence. This leads to the related problem of determining whether, for any finitely presented labelled rewriting theory, there is always a finite set of diagrams whose commutativity implies the commutativity of all diagrams built from the theory.

This chapter sets out to solve several related coherence questions by vigourously pursuing the idea that two morphisms with the same source and target in a free covariant structure on a discrete category commute precisely when they admit a planar subdivision such that each face is an instance of naturality, or of functoriality or of one of the coherence axioms. The guiding intuition behind this approach is that a span that cannot be completed into a square can never appear in such a subdivision.

Section 6.1 lays the foundations for this chapter by providing precise definitions of the various concepts related to subdivisions of parallel pairs of arrows and determining conditions that ensure that each parallel pair of arrows has only finitely many subdivisions. This quickly leads, in Section 6.2, to a general Lambek coherence theorem. Section 6.3 provides a more refined analysis of the possible subdivisions of a parallel pair of reductions in a finitely presented labelled rewriting theory and exploits this analysis to obtain a general Mac Lane coherence theorem. Finally, Section 6.4 constructs examples of labelled rewriting theories that cannot be made coherent via only finitely many coherence axioms.
6.1. Subdivisions

When one is working with rewriting 2-theories or categorical algebraic structures more generally, one typically draws diagrams representing morphisms in the free structure. The purpose of this section is to formalise these diagrams as ambient isotopy classes of planar directed graphs. This provides a mathematical setting for the manner in which one typically shows that a parallel pair of morphisms is equal: by finding a subdivision of the pair whose faces commute by virtue of functoriality, naturality and the coherence axioms. Within this setting, we examine properties that the underlying rewriting theory of a 2-theory must satisfy in order to ensure that each parallel pair of morphisms admits only finitely many such subdivisions. This forms the basis for the coherence theorems developed in the remainder of the chapter.

A subdivision of a parallel pair of reductions is, in the first instance, a collection of reductions having the same source and target. This collection forms a graphical structure.

**Definition 6.1.1.** An st-graph is a labelled directed graph $G$ (possibly with loops and multiple edges) together with two distinguished vertices $u$ and $v$, called the source and target of $G$ respectively, such that for any other vertex $w \in G$, there exist paths $u \to w$ and $w \to v$ in $G$.

By Lemma 2.6.3, we know that every reduction generated by a rewriting 2-theory is a composite of singular reductions. Before we introduce the graph associated to a labelled rewriting theory, we need to deal with a subtlety that arises due to the presence of an equational theory on terms. Let $\mathcal{R} := (\mathcal{F}; T | \mathcal{E}_F; \mathcal{E}_T)$ be a labelled rewriting theory. By the functoriality of the function symbols $F \in \mathcal{F}$, every equation in $\mathcal{E}_T$ induces an equation on reductions. Thus, we may form the quotient $\text{Sing}(\mathcal{R})/\mathcal{E}_T$. We call a member of $\text{Sing}(\mathcal{R})/\mathcal{E}_T$ an absolutely singular reduction.

**Definition 6.1.2 (Reduction graph).** Let $\mathcal{L} = (\mathcal{F}; T | \mathcal{E}_F) \subseteq \mathcal{R}$ be a labelled rewriting theory. The expression $\text{Red}(\mathcal{L})$ denotes the reduction graph of $\mathcal{L}$. This graph has

- **Vertices:** The set $\mathcal{F}(\mathcal{F}; \mathcal{E}_F)(X)$.
- **Edges:** Absolutely singular reductions in $\mathcal{F}(\mathcal{F}; \mathcal{E}_F)(X)$.

The reduction graph of a rewriting 2-theory is the reduction graph of its underlying labelled rewriting theory.

A subdivision corresponds to a particular way of embedding an st-graph in the oriented plane. Given a graph $G$, we use $|G|$ to denote its geometric realisation. We write $\mathbb{R}^2$ for the plane with the clockwise orientation. We use $G(s, t)$ to denote the set of paths from $s$ to $t$ in $G$.

**Definition 6.1.3.** Let $G$ be a graph and $\alpha, \beta \in G(s, t)$. A pre-subdivision of $\langle \alpha, \beta \rangle$ is a pair $(S, \varphi)$ such that:

1. $S$ is an st-graph with source $s$ and target $t$.
2. $\{\alpha, \beta\} \subseteq S \subseteq G$.
3. $\varphi : |S| \to \mathbb{R}^2$ is a planar embedding.
(4) For every edge $\gamma \in S$, the image $\varphi(|\gamma|)$ is contained in the region of $\mathbb{R}^2$ bounded by $\varphi(|\alpha|)$ and $\varphi(|\beta|)$.

We use $\text{PSub}_G(\alpha, \beta)$ to denote the set of all pre-subdivisions of $\langle \alpha, \beta \rangle$ in $G$.

The definition of pre-subdivisions admits too many different embeddings of the same graph. To this end, we define a useful equivalence relation on pre-subdivisions. In the present context, we say that two embeddings $f, g : G \hookrightarrow \mathbb{R}^2$ are *ambiently isotopic* if there is an isotopy $h$ of the identity map of $\mathbb{R}^2$ such that $h|_f = g$. In other words, $f$ and $g$ are ambiently isotopic if they differ only by a continuous deformation of $\mathbb{R}^2$. Intuitively, $f$ and $g$ are ambiently isotopic when they differ only by the size and shape of their faces.

Given a graph $G$ and $\alpha, \beta \in G(s, t)$, let $\langle S_1, \varphi \rangle$ and $\langle S_2, \psi \rangle$ be pre-subdivisions of $\langle \alpha, \beta \rangle$. Define $\sim$ to be the equivalence relation on $\text{PSub}_G(\alpha, \beta)$ generated by setting $\langle S_1, \varphi \rangle \sim \langle S_2, \psi \rangle$ if:

1. $S_1 = S_2$.
2. $\varphi$ and $\psi$ are ambiently isotopic.

Ambient isotopy is still not quite enough to identify all subdivisions representing the same categorical diagram. The reason for this is that reflecting the plane about some axis maps a subdivision to an equivalent categorical diagram. Let $E(2)$ be the Euclidean group of the plane — the group of all rotations, translations and reflections of the plane.

We write $\text{Sub}_G(s, t)$ for the quotient $(\text{PSub}_G(s, t)/\sim)/E(2)$.

**Definition 6.1.4.** For a directed graph $G$ and $\alpha, \beta \in G(s, t)$, a subdivision of $\langle \alpha, \beta \rangle$ is a member of $\text{Sub}_G(s, t)$. For a labelled rewriting theory $\mathcal{L}$, a subdivision of a parallel pair of reductions $\alpha, \beta \in \mathcal{F}(\mathcal{L})$ is a subdivision of $\langle \alpha, \beta \rangle$ in $\text{Red}(\mathcal{L})$. The set of all such subdivisions is denoted $\text{Sub}_G(\alpha, \beta)$.

Recall that a directed graph $G$ is *locally finite* if $G(s, t)$ is finite for all vertices $s, t \in G$. The following sequence of lemmas establishes a correspondence between local finiteness and finitely many subdivisions.

**Lemma 6.1.5.** For a directed graph $G$ and a finite planar st-subgraph $S \leq G(s, t)$ with source $s$ and target $t$, there are only finitely many subdivisions of $\alpha, \beta \in G(s, t)$ having graph $S$.

**Proof.** Since we only consider embeddings of $S$ up to ambient isotopy and Euclidean group action, a subdivision with graph $S$ is completely determined by the set of edges mapped to the region bounded by $\varphi(|\gamma_1|)$ and $\varphi(|\gamma_2|)$ for every parallel pair of paths $\gamma_1, \gamma_2 \in S$. Since $S$ is finite, there are only finitely many possibilities for this.

**Lemma 6.1.6.** An st-graph with source $s$ and target $t$ is finite if and only if it has finitely many planar st-subgraphs with source $s$ and target $t$.

**Proof.** ($\Rightarrow$) A finite graph has finitely many subgraphs, so it certainly has finitely many planar subgraphs.
Suppose that $G$ is an infinite st-graph with source $s$ and target $t$. Each path from $s$ to $t$ in $G$ determines a planar subgraph of $G$, hence $G$ has infinitely many planar subgraphs with source $s$ and target $t$.

Combining Lemma 6.1.5 and Lemma 6.1.6, we obtain the desired correspondence.

**Lemma 6.1.7.** If $G$ is a directed graph containing vertices $s$ and $t$, then $G(s,t)$ is finite if and only if $\text{Sub}_G(\alpha,\beta)$ is finite for all $\alpha, \beta \in G(s,t)$

### 6.1.1. Ensuring local finiteness.

By Lemma 6.1.7, in order to ensure that every parallel pair of paths in a directed graph has finitely many subdivisions, we need only establish that the graph is locally finite. To this end, we make the following definition.

**Definition 6.1.8.** Let $G$ be a directed graph. A quasicycle in $G$ is a pair $(T,t)$ such that:

1. $T$ is an infinite chain $t_0 \rightarrow t_1 \rightarrow \ldots$ in $G$
2. $t$ is a vertex in $G$.
3. $G$ contains a path $t_i \rightarrow t$ for all $i \in \mathbb{N}$.

![A quasicycle](figure1.png)

**Figure 1. A quasicycle**

Quasicycles earn their name by being a slightly weaker notion than a cycle. Figure 1 gives an example of a quasicycle that is not a cycle. On the other hand, we have the following easy result.

**Lemma 6.1.9.** Let $C$ be a directed cycle and $c$ be a vertex in $C$. Then, $(C,c)$ is a quasicycle.

For a directed graph $G$ and a vertex $s \in G$, we use $\text{Out}_G(s)$ to denote the set $\{t \in V(G) : G \text{ contains an edge } s \rightarrow t\}$. We say that $G$ is finitely branching if $\text{Out}_G(s)$ is finite for all vertices $s \in G$. One of our main technical tools is the following graphical version of König's Tree Lemma.

**Lemma 6.1.10.** A finitely branching directed graph is locally finite if and only if it contains no quasicycles.

**Proof.** Let $G$ be a labelled finitely branching directed graph.

$(\Rightarrow)$ Suppose that $G$ contains a quasicycle $(T,t)$, where $T = t_0 \xrightarrow{\alpha_0} t_1 \xrightarrow{\alpha_1} \ldots$. If $t_i = t$ for some $i \in \mathbb{N}$ then $G(t_i,t_j)$ is infinite for all $j > i$. So, suppose that $t_i \neq t$ for all $i \in \mathbb{N}$. Since $t_i \rightarrow t$ for all $i \in \mathbb{N}$, there must be infinitely many pairs $(i,\beta_i)$, where $i \in \mathbb{N}$ and $\beta_i : t_i \rightarrow t$ is a path that does not factor through $t_j$ for any $j > i$. So, $G(t_0,t)$ is infinite.
Suppose that $G(s,t)$ is infinite. Since $\text{Out}_{G}(s)$ is finite, it follows from the pigeon hole principle that there must exist some $s_0 \in \text{Out}_{G}(s)$ and an edge $\alpha_0 : s \to s_0$ such that $G(s_0,t)$ is infinite. Continuing recursively, we obtain an infinite chain $s \xrightarrow{\alpha_0} s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \ldots$ such that $G$ contains a path $s_i \to t$ for all $i \in \mathbb{N}$. So, $G$ contains a quasicycle.

By making use of the reduction graph of a labelled rewriting theory, we can shift our terminology for directed graphs to labelled rewriting theories.

**Definition 6.1.11.** A labelled rewriting theory $\mathcal{L}$ is quasicycle-free if every quasicycle in $\text{Red}(\mathcal{L})$ contains cofinitely many identity reductions. It is locally finite if $\text{Red}(\mathcal{L})$ is locally finite and it is finitely branching if $\text{Red}(\mathcal{L})$ is finitely branching.

Recall that an equation $s = t$ is called balanced if $s$ and $t$ contain precisely the same variables and it is called linear if it is balanced and each variable appears precisely once in each of $s$ and $t$.

**Definition 6.1.12.** A labelled rewriting theory $\mathcal{L} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F} \rangle$ is term-linear if $\mathcal{E}_\mathcal{F}$ contains only linear equations.

A reduction rule $\alpha : [s] \to [t]$ is called non-increasing if $\text{Var}(t) \subseteq \text{Var}(s)$. A labelled rewriting theory $\mathcal{L}$ is non-increasing if every reduction rule in $\mathcal{L}$ is non-increasing.

**Proposition 6.1.13.** A finitely presented labelled rewriting theory is finitely branching if it is term-linear and non-increasing.

**Proof.** Let $\mathcal{L} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F} \rangle$ be a finitely presented non-increasing term-linear labelled rewriting theory. Without loss of generality, we may assume that $\mathcal{T} = \{\rho\}$. Suppose that the vertex $[s]$ in $\text{Red}(\mathcal{L})$ is infinitely branching. Since $\rho$ is non-increasing, there must be infinitely many terms $s_1, s_2, \ldots \in [s]$ containing the same number of unary and binary function symbols as $\rho$, such that each $s_i$ contains a $\rho$-redex as a subterm. But this is impossible, since $\mathcal{E}_\mathcal{F}$ is linear.

A labelled rewriting theory that is not term-linear may be infinitely branching, even if it is finitely presented and non-increasing.

**Example 6.1.14.** Let $\mathcal{L}$ be the labelled rewriting theory consisting of the binary function symbol $\mathcal{F}$, the equation $s = \mathcal{F}(s,s)$ and the reduction rule $\rho : t \to t'$. Then, in $\text{F}(\mathcal{L})$, we have:

$$t = \mathcal{F}(t,t) = \mathcal{F}(\mathcal{F}(t,t),t) = \mathcal{F}(\mathcal{F}(\mathcal{F}(t,t),t),t) = \ldots$$

The reduction rule $\rho$ induces maps from $[t]$ to:

$$(6.1) \quad t', \mathcal{F}(t',t), \mathcal{F}(\mathcal{F}(t',t),t), \mathcal{F}(\mathcal{F}(\mathcal{F}(t',t),t),t), \ldots$$

Since the terms in (6.1) are pairwise unequal, $\mathcal{L}$ is infinitely branching.

Lemmas 6.1.7 and 6.1.10 imply that a finitely branching quasicycle-free labelled rewriting theory has only finitely many subdivisions for every parallel pair of reductions. A ready supply of such theories is provided by the following observation, which follows immediately from the definitions.
LEMMA 6.1.15. A terminating labelled rewriting theory is quasicycle-free.

By Lemma 6.1.9, a quasicycle-free directed graph is acyclic. The following theorem establishes that every face of a subdivision in an acyclic graph is itself a parallel pair of paths. It was originally discovered by Power [Pow90] in his investigation of pasting diagrams in 2-categories.

THEOREM 6.1.16 (Power [Pow90]). A planar st-graph is acyclic if and only if every face has a unique source and target.

Theorem 6.1.16 readily leads to the following result by induction over the number of faces in a subdivision.

PROPOSITION 6.1.17. Let \( \mathcal{R} := (\mathcal{F}; T \mid \mathcal{E}_\mathcal{F}; \mathcal{E}_T) \) be an acyclic rewriting 2-theory and let \( \alpha, \beta \in \text{Red}(\mathcal{R})(s, t) \). Then, the following statements are equivalent:

1. \( \alpha = \beta \) in \( \mathbb{F}(\mathcal{R}) \).
2. There is a subdivision of \( \langle \alpha, \beta \rangle \) in \( \text{Red}(\mathcal{R})(s, t) \) such that each face commutes in \( \mathbb{F}(\mathcal{R}) \).
3. There is a subdivision of \( \langle \alpha, \beta \rangle \) in \( \text{Red}(\mathcal{R})(s, t) \) such that each face is either an instance of functoriality, or an instance of naturality or an instance of one of the equations in \( \mathcal{E}_T \).

In the following section, we use the tools developed so far to tackle the Lambek coherence problem.

6.2. Lambek coherence

With Proposition 6.1.17 and Lemma 6.1.7, one may be inclined to think that a Lambek coherence theorem should be immediately forthcoming, since we know that every quasicycle-free finitely branching rewriting 2-theory has only finitely many subdivisions for each parallel pair of reductions and we can just check every face to see whether it is an instance of functoriality, naturality or a coherence axiom. There is, however, one catch — we may not be able to decide whether a given face is an instance of an axiom.

DEFINITION 6.2.1 (Unification). Let \( \mathcal{F} \) be a ranked set of function symbols on a set \( X \) and \( \mathcal{E}_\mathcal{F} \) be an equational theory on \( \mathbb{F}_\mathcal{F}(X) \). An \( \mathcal{E}_\mathcal{F} \)-unification problem is a finite set:

\[ \Gamma = \{(s_1, t_1), \ldots, (s_n, t_n)\}, \]

where for \( 1 \leq i \leq n \), we have that \( s_i \) and \( t_i \) are in \( \mathbb{F}_\mathcal{F}(X) \). A unifier for \( \Gamma \) is a homomorphism \( \sigma : X \rightarrow \mathbb{F}_\mathcal{F}(X) \) such that \( \sigma(s_i) = \mathcal{E}_\mathcal{F}\sigma(t_i) \) for all \( 1 \leq i \leq n \). The set \( \Gamma \) is unifiable if it admits at least one unifier.

Unification theory is an important technical component of automated reasoning and logic programming, as it provides a means for testing whether two sequences of terms are syntactic variants of each other. A good survey of the field is provided by [BS94]. In the case where the theory \( \mathcal{E}_\mathcal{F} \) is empty, the unification problem is readily shown to be decidable (see [BS94] for details). Unfortunately, the equational unification problem is in general undecidable.
DEFINITION 6.2.2. A labelled rewriting theory \( \langle F; T \mid \mathcal{E}_F; \mathcal{E}_T \rangle \) has decidable term unification if \( \langle F \mid \mathcal{E}_F \rangle \) has a decidable unification problem.

We can finally establish a general Lambek coherence theorem.

**Theorem 6.2.3** (Lambek Coherence). A finitely branching quasicycle-free rewriting 2-theory with decidable term unification is Lambek Coherent.

**Proof.** Let \( \mathcal{R} \) be a rewriting 2-theory satisfying the hypotheses and let \( \alpha, \beta \in \text{Red}(\mathcal{R})(s,t) \). By Lemma 6.1.7, we can enumerate the subdivisions of \( \langle \alpha, \beta \rangle \). Since each subdivision has only finitely many faces and \( \mathcal{R} \) has decidable term unification, we may apply Proposition 6.1.17 to determine whether every face of a subdivision commutes in \( \mathbb{F}(\mathcal{R}) \).

Unfortunately, we may not be able to determine whether a labelled rewriting theory is quasicycle-free.

**Corollary 6.2.4.** It is undecidable whether a finitely branching rewriting 2-theory with decidable term unification is quasicycle-free.

**Proof.** The rewriting 2-theory constructed in the proof of Theorem 2.4.6 has an empty equational theory on terms and so has decidable term unification. It follows from Theorem 6.2.3 that, were we able to determine whether the theory is quasicycle-free, then we would be able to decide whether a finite monoid presentation has a decidable word problem.

As a particular application of Theorem 6.2.3, any terminating rewriting 2-theory with an empty equational theory on terms is Lambek coherent. This includes, amongst others, categories with a directed associativity [Lap72a]. The unification problem for an associative binary symbol \( \otimes \) together with an identity \( I \) for \( \otimes \) is decidable [BS94]. It follows then, from Theorem 6.2.3 that the following rewriting 2-theories are Lambek coherent (in each case we need only check that the 2-theory is terminating):

- Distributive categories with strict associativities and strict units [Lap72b].
- Weakly distributive categories with strict associativity and strict units [CS97].

An example of a non-terminating theory that is Lambek-coherent is provided by the system \( F(x) \to F(F(x)) \), since this is easily seen to be quasicycle-free.

In the following section, we continue our investigation of quasicycle free theories and derive sufficient conditions for such a system to be Mac Lane coherent.

6.3. Mac Lane coherence

The last section was concerned with deciding whether a given parallel pair of morphisms is equal or, equivalently, whether a given diagram in general position commutes. Our rough goal in this section is to find a minimal set of diagrams in general position whose commutativity implies the commutativity of all other such diagrams in \( \mathbb{F}(\mathcal{R}) \) for some rewriting 2-theory \( \mathcal{R} \). To this end, we define what it means for one subdivision to be finer than another. The driving idea is that we only wish to consider those subdivisions that do not embed into a finer subdivision.
DEFINITION 6.3.1. Let $G$ be a directed graph and $\alpha, \beta \in G(s, t)$ and $(S_1, \varphi), (S_2, \psi) \in \text{Sub}_G(\alpha, \beta)$. We say that $(S_1, \varphi)$ is coarser than $(S_2, \psi)$ if there is a graph embedding $\Lambda : S_1 \to S_2$ making the following diagram commute. In this case, we also say that $(S_2, \psi)$ is finer than $(S_1, \varphi)$ and we write $(S_1, \varphi) \preceq (S_2, \psi)$.

![Diagram]

We define the refinement order to be the antisymmetric closure of $\preceq$.

It is immediate from the definitions that the set of subdivisions of a parallel pair of morphisms forms a poset under refinement. We shall abuse notation slightly in the following definition and write $\preceq$ for the refinement order.

DEFINITION 6.3.2. Let $G$ be a directed graph and $\alpha, \beta \in G(s, t)$. A maximal subdivision of $(\alpha, \beta)$ is a maximal element of $(\text{Sub}_G(\alpha, \beta), \preceq)$.

The idea behind the definition of a maximal subdivision is that these are precisely the ones which cannot be further subdivided. This leads to the following lemma.

LEMMA 6.3.3. A finitely branching quasicycle-free rewriting 2-theory is Mac Lane coherent if and only if every parallel pair of reductions in general position admits a maximal subdivision, each face of which commutes.

PROOF. The direction $(\Leftarrow)$ follows from induction over the number of faces. For the other direction, let $\mathcal{R}$ be a finitely branching quasicycle-free rewriting 2-theory. Let $\alpha, \beta \in \text{Red}_\mathcal{R}(s, t)$. Since $\mathcal{R}$ is quasicycle-free and finitely branching, it follows from Lemma 6.1.7 that $\text{Sub}_\mathcal{R}(\alpha, \beta)$ is finite. Therefore, $(\alpha, \beta)$ admits a maximal subdivision $(S, \varphi)$. By Theorem 6.1.16, every face of $(S, \varphi)$ has a unique source and target. Since $\mathcal{R}$ is Mac Lane coherent, each of these faces commutes. \qed

In order to make Lemma 6.3.3 effective, we need to characterise those parallel pairs of morphisms that can occur as faces of a maximal subdivision.

DEFINITION 6.3.4 (Zig-zag subdivision). Let $G$ be a directed graph and $\alpha, \beta \in G(s, t)$. Suppose that

$$\alpha = s \overset{a_0}{\longrightarrow} a_0 \overset{a_1}{\longrightarrow} \cdots \overset{a_{n-1}}{\longrightarrow} a_{n-1} \overset{a_n}{\longrightarrow} t$$

$$\beta = s \overset{b_0}{\longrightarrow} b_0 \overset{b_1}{\longrightarrow} \cdots \overset{b_{m-1}}{\longrightarrow} b_{m-1} \overset{b_m}{\longrightarrow} t$$

and that each $\alpha_i$ and $\beta_i$ is singular. Let $U$ be the forgetful functor from directed graphs to graphs that forgets the direction of edges. A zig-zag subdivision of $(\alpha, \beta)$ is a subdivision $(S, \varphi)$ of $(\alpha, \beta)$ such that $U(S)$ contains a path from $U(a_i)$ to $U(b_j)$ for some pair $(i, j)$, with $0 \leq i \leq n - 1$ and $0 \leq j \leq m - 1$. We call the preimage of this path the zig-zag of $S$. 
DEFINITION 6.3.5 (Diamond). Let $G$ be a directed graph. A pair $\alpha, \beta \in G(s, t)$ is called a diamond if it does not admit a zig-zag subdivision.

The idea behind the definition of a diamond is that any subdivision containing a face that admits a zig-zag subdivision cannot be a maximal subdivision. This is made precise in the following proposition.

**Proposition 6.3.6.** Let $G$ be an acyclic directed graph and $\alpha, \beta \in G(s, t)$. Every face of a maximal subdivision of $\langle \alpha, \beta \rangle$ is a diamond.

**Proof.** Let $G$ be an acyclic directed graph and let $(S, \varphi)$ be a maximal subdivision of $\alpha, \beta \in G(s, t)$. By Theorem 6.1.16, every face of $S$ has a unique source and target. That is, every face consists of a parallel pair of reductions $\eta, \psi : u \rightarrow v$. Suppose that $\langle \eta, \psi \rangle$ is a face of $S$ that is not a diamond. That is, it admits a zig-zag subdivision. So, we have

$$
\eta = u \xrightarrow{\eta_1} w \xrightarrow{\eta_2} v \quad \text{and} \quad \psi = u \xrightarrow{\psi_1} x \xrightarrow{\psi_2} v,
$$

and a zig-zag $\gamma$ between $w$ and $x$ that is a part of a subdivision of $\langle \eta, \psi \rangle$. By maximality, $\gamma$ must be contained in $S$. Since $\langle \eta, \psi \rangle$ is a face, $\varphi(|\gamma|)$ cannot lie in the region bounded by $\varphi(|\eta|)$ and $\varphi(|\psi|)$. So, we are in one of the situations depicted in Figure 3.

![Figure 3. Possible embeddings of $\gamma$.](image)

Suppose that we are in the situation depicted in the left hand diagram of Figure 3. Since $\gamma$ is contained in $S$ and since $S$ is an st-graph, there is a path $s \xrightarrow{\delta} u$. By planarity, $\delta$ must factor through a vertex in $\gamma$ or $\eta_2$ or $\psi_2$. If $\delta$ factors through a vertex in $\eta_2$ or $\psi_2$ then it is clear that $G$ contains a cycle, contradicting the fact that $G$ is acyclic. So, we must have $s \xrightarrow{\delta_1} z \xrightarrow{\delta_2} u$ for some vertex $z$ in $\gamma$. However, since $\gamma$ appears...
in a subdivision of \((\eta, \psi)\), there is a path \(u \xrightarrow{\zeta} z\) in \(G\). Then, \(\delta_2 \cdot \zeta\) forms a cycle in \(G\), contradicting the fact that \(G\) is acyclic. So, \(\gamma\) cannot be embedded as in the left hand picture of Figure 3. Dually, it cannot be embedded as in the right hand picture of Figure 3.

Therefore, the zig-zag \(\gamma\) must be embedded within the face bounded by \((\eta, \psi)\), contradicting the maximality of \((S, \varphi)\). So, \((\eta, \psi)\) must be a diamond. □

Combining Lemma 6.3.3 and Proposition 6.3.6, we obtain our general version of the Mac Lane Coherence theorem.

**THEOREM 6.3.7 (Coherence).** A finitely branching quasicycle-free rewriting 2-theory \(\mathcal{T}\) is Mac Lane coherent if and only if every diamond in \(\text{Red}(\mathcal{R})\) commutes in \(\mathbb{F}(\mathcal{R})\). □

Theorem 6.3.7 says that in order to show that a finitely branching rewriting 2-theory is Mac Lane coherent, we need to do two things:

(1) Show that \(\mathbb{F}(\mathcal{R})\) is quasicycle-free.
(2) Show that every diamond commutes.

At the outset, showing that every diamond commutes can be a daunting task. We can guide our investigations by exploiting the properties of critical spans.

**DEFINITION 6.3.8.** Let \(\mathcal{R}\) be a rewriting 2-theory and let \(\varphi_1\) and \(\varphi_2\) be singular morphisms in \(\mathbb{F}(\mathcal{R})\). We call \((\varphi_1, \psi_1)\) the source span in a diagram of the following form:

\[
\begin{array}{ccc}
\varphi_1 & \rightarrow \\
\psi_1 \downarrow & & \downarrow \psi_2 \\
\end{array}
\]

If \(\varphi_1\) and \(\psi_1\) are singular, then there are three possibilities for a diamond with source span \((\varphi_1, \psi_1)\):

(1) \(\varphi_1\) and \(\psi_1\) rewrite disjoint subterms.
(2) \(\varphi_1\) and \(\psi_1\) rewrite nested subterms.
(3) \(\varphi_1\) and \(\psi_1\) rewrite overlapping subterms. Without loss of generality, we may assume that \((\varphi_1, \psi_1)\) forms a critical span.

By analogy with Lemma 4.2.6, one may hope to reduce the problem to only examining diamonds whose source span is critical. Unfortunately, as the following two examples show, there may be more than one diamond whose source span performs a given pair of nested or disjoint rewrites.

**EXAMPLE 6.3.9.** In this example we construct a terminating rewriting 2-theory that has more than one diamond with the same source span performing a nested pair of rewrites. Let \(\mathcal{R}\) be the 2-theory consisting of unary functor symbols \(I, J\) and \(H\), together with the following reduction rules:
Then, $F(R)$ contains the following diagram:

\[
\begin{array}{ccc}
I(I(a)) & \\ & \searrow^{J(I(a))} & \nearrow^{J(I(a))} \\
I(a) & & H(a) \\
& \swarrow^{J(I(a))} & \nwarrow^{J(I(a))} \\
& I(I(a)) & \\
\end{array}
\]

Since there is no reduction $J(J(a)) \rightarrow H(a)$, both parallel reductions form diamonds.

**Example 6.3.10.** In this example we construct a terminating rewriting 2-theory that has more than one diamond with the same source span performing a disjoint pair of rewrites. Let $R$ be the rewriting 2-theory consisting of unary functor symbols $I$ and $J$, the binary functor symbol $\otimes$ and the following reduction rules:

\[
\begin{align*}
I(x) &\rightarrow J(x) \\
J(x) \otimes I(x) &\rightarrow H(x) \\
I(x) \otimes J(x) &\rightarrow H(x)
\end{align*}
\]

Then, $F(R)$ contains the following diagram:

\[
\begin{array}{ccc}
I(A) \otimes J(A) & \\ & \searrow^{I(A) \otimes J(A)} & \nearrow^{J(A) \otimes J(A)} \\
I(A) \otimes I(A) & & H(A) \\
& \swarrow^{I(A) \otimes J(A)} & \nwarrow^{J(A) \otimes I(A)} \\
& J(A) \otimes I(A) & \\
\end{array}
\]

Since there is no reduction $J(A) \otimes J(A) \rightarrow H(A)$, both parallel reductions form diamonds.

Examples 6.3.9 and 6.3.10 serve to warn us that the collection of diamonds behaves a lot more subtly than the collection of spans, which are the typical objects of study in traditional term rewriting theory. In the next section, we look at when a labelled rewriting theory cannot be made into a Mac Lane coherent rewriting 2-theory by only finitely many coherence axioms.

### 6.4. Finite Mac Lane coherence

In light of Theorem 6.3.7, we have a reasonable strategy for determining whether a given rewriting 2-theory is Mac Lane coherent. However, we are still left with the problem of determining whether a given finitely presented labelled rewriting theory can be extended to a finitely presented Mac Lane coherent rewriting theory. Bearing
in mind the results of the previous section, we have two reasonable candidates for ensuring this property: quasicycle freeness and termination. In this section, we show that neither of these conditions suffice in general.

Given a rewriting 2-theory $\mathcal{R} := \langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F}; \mathcal{E}_\mathcal{T} \rangle$, we say that the labelled rewriting theory $\langle \mathcal{F}; \mathcal{T} \mid \mathcal{E}_\mathcal{F} \rangle$ is the reduct of $\mathcal{R}$.

**Definition 6.4.1** (Finitely Mac Lane coherent). A finitely presented labelled rewriting theory is finitely Mac Lane coherent if it is the reduct of a finitely presented Mac Lane coherent rewriting 2-theory.

It is not a priori obvious whether there exist theories that are not finitely Mac Lane coherent. We can simplify our investigation of this point somewhat by focusing on the most basic diamonds.

**Definition 6.4.2** (Basic diamond). A diamond $\Delta_1$ appearing in the reduction graph of a labelled rewriting theory $\mathcal{L}$ is basic if it satisfies the following properties:

1. For every substitution $\sigma$ and diamond $\Delta_2$, if $\Delta_1 = \Delta_2^\sigma$, then $\sigma$ is a variable renaming.
2. For every unary functor $F \in \mathcal{F}(\mathcal{L})$ and every diamond $\Delta_2$, if $F(\Delta_2) = \Delta_1$, then $F = 1$.

The following theorem is immediate from Theorem 6.3.7.

**Theorem 6.4.3.** A finitely presented, finitely branching labelled rewriting theory $\mathcal{L}$ is finitely Mac Lane coherent if and only if $\text{Red}(\mathcal{L})$ contains finitely many basic diamonds, up to variable renaming.

In the following example, we construct an example of a quasicycle-free labelled rewriting theory that is not finitely Mac Lane coherent.

**Example 6.4.4.** Let $\mathcal{L}$ be the labelled rewriting theory containing unary function symbols $F, G, I$ and $H$, together with the following reduction rules:

- $I(x) \rightarrow G(I(x))$
- $I(x) \rightarrow F(I(x))$
- $F(x) \rightarrow F(F(x))$
- $G(x) \rightarrow G(G(x))$
- $F(x) \rightarrow H(x)$
- $G(x) \rightarrow H(x)$

In order to show that $\mathcal{L}$ is quasicycle-free, it suffices to show that there is no term $t \in \mathcal{L}(\mathcal{L})$ such that there are infinitely many reductions with target $t$ in $\mathcal{L}(\mathcal{L})$. Let $\mathcal{L}^{-1}$ be the labelled rewriting theory with the same function symbols as $\mathcal{L}$ and a reduction rule $t \rightarrow s$ for every reduction rule $s \rightarrow t$ in $\mathcal{L}$. By Proposition 6.1.13, $\mathcal{L}^{-1}$ is finitely branching, so $\mathcal{L}$ is quasicycle-free.
However, $\mathbb{F}(\mathcal{L})$ contains the following diagram:

\[
\begin{array}{c}
G(I(a)) \longrightarrow G^2(I(a)) \longrightarrow G^3(I(a)) \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
H(I(a)) \quad H^2(I(a)) \quad H^3(I(a)) \\
\downarrow \quad \downarrow \quad \downarrow \\
F(I(a)) \longrightarrow F^2(I(a)) \longrightarrow F^3(I(a)) \longrightarrow \cdots
\end{array}
\]

Since there are no reductions $H^i(a) \to H^j(a)$ for $i \neq j$, no finite collection of diamonds with source $I(a)$ implies the commutativity of all others. So, $\mathcal{L}$ contains infinitely many substitution-reduced diamonds and it follows from Theorem 6.4.3 that it is not finitely Mac Lane coherent.

Example 6.4.4 works by constructing infinitely many substitution-reduced diamonds sharing a common source span. Terminating rewriting theories are far better behaved.

**Lemma 6.4.5.** Let $\mathcal{L}$ be a finitely branching terminating labelled rewriting theory. Then, for every term $t \in \mathbb{F}(\mathcal{L})$, the set

\[\text{Out}(t) = \{t' \in \mathbb{F}(\mathcal{L}) : \exists \text{ a reduction } t \to t' \text{ in } \mathcal{L}\}\]

is finite.

**Proof.** Let $\mathcal{L}$ be a finitely branching terminating labelled rewriting theory and let $t \in \mathbb{F}(\mathcal{L})$. Suppose that $\text{Out}(t)$ is infinite. Let $\text{Out}(t)_n$ be the set of terms $t' \in \text{Out}(t)$ such that a path of minimal length $t \to t'$ in $\mathbb{F}(\mathcal{L})$ contains $n$ edges. Since $\mathcal{L}$ is finitely branching and $\text{Out}(t)$ is infinite, $\text{Out}(t)_n$ is finite and nonempty for all $n \in \mathbb{N}$. This implies that $\mathcal{L}$ is not terminating, contradicting our assumptions. Thus, $\text{Out}(t)$ is finite. \qed

It follows from the above lemma that there are only finitely many substitution-reduced diamonds with a given source span in a finitely branching, terminating labelled rewriting theory. One may be led by this observation to posit that such a theory is necessarily finitely Mac Lane coherent. However, there is still the possibility that there are infinitely many distinct substitution-reduced diamonds, since the diamonds may possess different source spans. This problem proves to be insurmountable, as demonstrated in the following example.

**Example 6.4.6.** In this example, we construct a finitely branching, terminating labelled rewriting theory that is not finitely Mac Lane coherent. Let $\mathcal{L}$ be the labelled rewriting theory consisting of the following function symbols:

- Nullary: $W$
- Unary: $S, S', T, T'$
- Binary: $F$
together with the following reduction rules:

\[
\begin{align*}
F(F(a, b), c) & \xrightarrow{\pi} F(a, b) \\
F(S(a), b) & \xrightarrow{\alpha} W \\
F(a, T(b)) & \xrightarrow{\beta} W \\
S(a) & \xrightarrow{\sigma} S'(a) \\
T(a) & \xrightarrow{\tau} T'(a)
\end{align*}
\]

Then, \(\mathcal{F}(\mathcal{L})\) contains the following infinite sequence of diamonds:

\[
\begin{align*}
F(S'(a), T(a)) \\
\xrightarrow{\beta} F(S(a), T(a)) & \xrightarrow{\alpha} W \\
& \xrightarrow{\alpha} F(S(a), T'(a)) \\
& \xrightarrow{\beta} F(F(S'(a), b), T(c)) \\
& \xrightarrow{\beta} F(F(S(a), b), T(c)) \\
& \xrightarrow{\pi} F(F(S(a), b), T'(c)) \\
& \xrightarrow{\pi} F(S(a), b) \\
& \xrightarrow{\pi} F(F(S(a), b), T(d)) \\
& \xrightarrow{\beta} F(F(F(S'(a), b), c), T(d)) \\
& \xrightarrow{\beta} F(F(F(S(a), b), c), T(d)) \\
& \xrightarrow{\pi} F(F(F(S(a), b), c), T'(d)) \\
& \xrightarrow{\pi} F(S(a), b) \\
& \xrightarrow{\pi} F(F(S(a), b), c)
\end{align*}
\]

Since no diamond in the above sequence is a substitution-instance of another, it follows from Theorem 6.4.3 that \(\mathcal{L}\) is not finitely Mac Lane coherent.
In this chapter, we have developed very general tools for investigating coherence problems in non-confluent and non-terminating rewriting 2-theories. In the following chapter, we apply these tools to a concrete theory arising in algebraic topology.
Iterated monoidal categories

The coherence theorems developed in Chapter 6 are primarily useful for investigating non-confluent and/or non-terminating categorical structures. As we have seen previously in Chapter 4 a vast array of categorical structures suffer from neither of these deficiencies. This might lead one to suspect that any “natural” categorical structure is both confluent and terminating. Unfortunately this is not the case. In this chapter, we investigate the theory of iterated monoidal categories [BFSV03], which arise naturally as a categorical model of iterated loop spaces. This theory possesses two features making its coherence problem difficult: it has a non-trivial equational theory at the term level and it is non-confluent. A coherence theorem is developed in [BFSV03], which says that there is a unique map in an n-fold monoidal category between two terms without repeated variables. The proof proceeds via an intricate double induction on the number of variables and the dimension of the outermost tensor product in the target of a morphism. In this chapter, we exploit Theorem 6.3.7 to provide a more conceptually straightforward proof of this theorem.

7.1. Definitions and basic properties

An n-fold monoidal category contains n monoidal structures linked via “interchange” maps. The presentation given in [BFSV03] endows each tensor product with strict associativity and unit constraints. One of the interesting features of this structure is that the n tensor products all have the same unit. This fact, coupled with the equational theory on terms, allows for some unexpected interplay between the interchange maps. For instance, it is not immediately obvious that the structure is non-confluent. This section introduces n-fold monoidal categories and explores some of the subtleties that arise.

**Definition 7.1.1.** The rewriting 2-theory for n-fold monoidal categories is denoted \( \mathcal{M}_n \) and consists of the following.

1. \( n \) binary functor symbols: \( \otimes_1, \ldots, \otimes_n \)
2. A nullary functor symbol \( I \)
3. For \( 1 \leq i \leq n \):
   \[
   a \otimes_i (b \otimes_i c) = (a \otimes_i b) \otimes_i c
   
   a \otimes_i I = a
   
   I \otimes_i a = a
   \]
(4) For each pair \((i, j)\) such that \(1 \leq i < j \leq n\), there is a reduction rule, called interchange:
\[
\eta_{a,b,c,d}^{ij} : (a \otimes_j b) \otimes_i (c \otimes_j d) \rightarrow (a \otimes_i c) \otimes_j (b \otimes_i d)
\]

The interchange rules are subject to the following conditions:

(1) Internal unit condition: \(\eta_{a,b,I,I}^{ij} = \eta_{I,I,a,b}^{ij} = id_{a} \otimes_{j} b\)

(2) External unit condition: \(\eta_{a,I,I,b}^{ij} = \eta_{I,a,I,b}^{ij} = id_{a} \otimes_{i} b\)

(3) Internal associativity condition: The following diagram commutes:

\[
\begin{array}{c}
(a \otimes_j b) \otimes_i (c \otimes_j d) \otimes_i (e \otimes_j f) \xrightarrow{\eta_{a,b,c,d}^{ij} \otimes_i id_c \otimes_i f} ((a \otimes_i c) \otimes_j (b \otimes_i d)) \otimes_i (e \otimes_j f) \\
\downarrow id_a \otimes_j b \otimes_i \eta_{c,d,e,f}^{ij} \\
(a \otimes_j b) \otimes_i ((c \otimes_i e) \otimes_j (d \otimes_i f)) \xrightarrow{\eta_{a,b,c}^{ij} \otimes_i id_d \otimes_i f} (a \otimes_i c \otimes_i e) \otimes_j (b \otimes_i d \otimes_i f) \\
\end{array}
\]

(4) External associativity condition: The following diagram commutes:

\[
\begin{array}{c}
(a \otimes_j b \otimes_j c) \otimes_i (d \otimes_j e \otimes_j f) \xrightarrow{\eta_{a,b,c,d,e}^{ij}} ((a \otimes_j b) \otimes_i (d \otimes_j c)) \otimes_j (c \otimes_i f) \\
\downarrow \eta_{a,b,c,d}^{ij} \otimes_j id_c \otimes_j f \\
(a \otimes_i d) \otimes_j ((b \otimes_j c) \otimes_i (e \otimes_j f)) \xrightarrow{id_a \otimes_i d \otimes_j \eta_{b,c,e,f}^{ij}} (a \otimes_i d) \otimes_j (b \otimes_i e) \otimes_j (c \otimes_i f) \\
\end{array}
\]

(5) Giant hexagon condition: The following diagram commutes:

\[
\begin{array}{c}
((a \otimes_k b) \otimes_j (c \otimes_k d)) \otimes_i ((c \otimes_k f) \otimes_j (g \otimes_k h)) \\
\eta^{jk} \otimes_k \eta^{jk} \\
((a \otimes_j c) \otimes_k (b \otimes_j d)) \otimes_i ((c \otimes_j g) \otimes_k (f \otimes_j h)) \\
\eta^{ij} \\
((a \otimes_j c) \otimes_k (b \otimes_j d)) \otimes_i ((c \otimes_j g) \otimes_k (f \otimes_j h)) \\
\eta^{jk} \otimes_j \eta^{jk} \\
\end{array}
\]

In the giant hexagon, \((i, j, k)\) is such that \(1 \leq i < j < k \leq n\) and the labels have the evident components.

Since the terms appearing in each reduction rule of \(\mathcal{M}_n\) are linear, we immediately obtain the following lemma.
Lemma 7.1.2. A reduction \([s] \rightarrow [t]\) in \(\mathbb{F}(\mathcal{M}_n)\) is in general position if and only if \(s\) and \(t\) contain no repeated variables.

Because of the fact that an \(n\)-fold monoidal category is strictly associative and has a strict unit, we can derive various maps via Eckmann-Hilton style arguments. Two of these maps will be of particular use to us. In the following, we assume that \((i, j)\) is such that \(1 \leq i < j \leq n\). The derived maps are as follows:

1. Dimension raising: \(a \otimes_i b \rightarrow a \otimes_j b\). This represents the following composition:

\[
(a \otimes_i I) \otimes_i (I \otimes_j b) \rightarrow (a \otimes_i I) \otimes_j (I \otimes_i b) \rightarrow a \otimes_j b
\]

2. Twisted dimension raising: \(a \otimes_i b \rightarrow b \otimes_j a\). This represents the following composition:

\[
(I \otimes_j a) \otimes_i (I \otimes_j b) \rightarrow (I \otimes_i b) \otimes_j (a \otimes_i I) \rightarrow b \otimes_j a
\]

With the above maps, it is easy to see that iterated monoidal categories do not have unique normal forms.

Lemma 7.1.3. If \(n \geq 2\), then \(\mathbb{F}(\mathcal{M}_n)\) is not confluent.

Proof. The following span is not joinable:

\[
a \otimes_i b \rightarrow^{\iota_{a,b}^{i,n}} a \otimes_n b
\]

\[
b \otimes_n a
\]

In the following section, we tackle the coherence problem for \(\mathcal{M}_n\).

7.2. Proving coherence

Our first step in investigating the coherence problem for iterated monoidal categories is to bring them into the realm of applicability of Theorem 6.3.7.

Proposition 7.2.1. \(\mathcal{M}_n\) is quasicycle-free.

Proof. Let \(\mathcal{M}_n^{-1}\) be the rewriting 2-theory that arises by replacing the reduction rules \(\eta^{ij}\) in \(\mathcal{M}_n\) with the following reduction rules, where \(1 \leq i < j \leq n\):

\[\xi_{a,b,c,d}^{ij} : (a \otimes_i c) \otimes_j (b \otimes_i d) \rightarrow (a \otimes_j b) \otimes_i (c \otimes_j d).\]

Given an object \([s] \in \mathbb{F}(\mathcal{M}_n^{-1})\), we may assume that \(s\) contains no instances of \(I\). It follows that if there is a reduction \([s] \rightarrow [t]\) in \(\mathbb{F}(\mathcal{M}_n^{-1})\), then \(t\) contains the same variables as \(s\), as well as the same number of function symbols. Since there are only finitely many such possibilities, \(\mathcal{M}_n\) is quasicycle-free.
Let $t$ be a term in $\mathbb{F}(\mathcal{M}_n)$. For a set $X \subseteq \text{Var}(t)$, we write $t_X$ to denote the term resulting from substituting $I$ for each variable in $X$. For instance, $(a \otimes_i b) \otimes_j (c \otimes_i d) - \{b,d\} = a \otimes_j c$. We say that a term $u$ is in a term $t$ and write $u \in t$ if there is some $X \subseteq \text{Var}(t)$ such that $t_X = u$. Of crucial importance to us is the following result of [BFSV03].

**Theorem 7.2.2 ([BFSV03]).** Let $t$ and $u$ be terms in $\mathbb{F}(\mathcal{M}_n)$. A necessary and sufficient condition for the existence of a reduction $t \rightarrow u$ in $\mathbb{F}(\mathcal{M}_n)$ is that, for each $a, b \in \text{Var}(t)$, if $a \otimes_i b \in t$, then one of the following holds:

- There is some $j > i$ such that $a \otimes_j b \in u$
- There is some $j > i$ such that $b \otimes_j a \in u$

Theorem 7.2.2 gives us the technical tool that we need in order to show that various parallel pairs of maps are not diamonds. We begin our analysis of the collection of diamonds of $\mathbb{F}(\mathcal{M}_n)$ with diamonds whose source span rewrites disjoint subterms.

**Lemma 7.2.3.** Let $a \otimes_i b \in \mathbb{F}(\mathcal{M}_n)$ and suppose that there are maps $\varphi : a \rightarrow a'$ and $\psi : b \rightarrow b'$. Then, in the following diagram, the square labelled $(d)$ is a commutative diamond and there is a map $a' \otimes_i b' \rightarrow c$:

![Diagram](image_url)

**Proof.** The square labelled $(d)$ commutes by functoriality and it is easy to see that it does not admit a zig-zag subdivision, so it is a diamond. The tricky part is showing the existence of a map $a' \otimes_i b' \rightarrow c$.

Let $x, y \in \text{Var}(a' \otimes_i b')$ and suppose that $x \otimes_k y \in a' \otimes_i b'$. There are a few cases to consider.

- If $x, y \in a'$, then $\alpha$ implies that there is some $m \geq k$ such that $x \otimes_m y \in c$ or there is some $m > k$ such that $y \otimes_m x \in c$.
- If $x, y \in b'$, then $\beta$ implies that there is some $m \geq k$ such that $x \otimes_m y \in c$ or there is some $m > k$ such that $y \otimes_m x \in c$.
- If $x \in a'$ and $y \in b'$, then $x \otimes_i y \in a' \otimes b$. So, by $\alpha$, there is some $m \geq i$ such that $x \otimes_m y \in c$ or there is some $m > i$ such that $y \otimes_m x \in c$.

Putting all of the above facts together, it follows from Theorem 7.2.2 that there is a map $a' \otimes_i b' \rightarrow c$.

Next, we investigate diamonds whose initial span rewrites nested subterms. For a term $a$ and a subterm $b \leq a$, we write $a\{b\}$ to represent this nested term.

**Lemma 7.2.4.** Let $a\{b\} \in \mathbb{F}(\mathcal{M}_n)$ and suppose that there are maps $\varphi : a\{b\} \rightarrow a'\{b\}$ and $\psi : b \rightarrow b'$. Then, in the following diagram, the square labelled $(d)$ is a commutative
diamond and there is a map \( a'\{b'\} \rightarrow c \):

\[
\begin{array}{c}
\diamondsuit \quad a\{b'\} \\
\quad \downarrow \quad \phi \\
\diamondsuit \quad a\{b\} \\
\quad \downarrow \quad \varphi \\
\diamondsuit \quad a'\{b'\} \\
\quad \downarrow \quad \alpha \\
\diamondsuit \quad c \\
\end{array}
\]

PROOF. The square labelled (d) commutes by naturality. The rest of the proof is similar to that of Lemma 7.2.3. \( \Box \)

We now know that source spans of the only remaining diamonds in \( F(\mathcal{M}_n) \) rewrite overlapping terms.

**7.2.1. Interchange + associativity.** Let \( j > i \). The first way in which interchange and associativity can interact is in the term \( X \otimes_i (c \otimes_j d) \otimes_i (e \otimes_j f) \). Without loss of generality, we may assume that \( X = a \otimes_j b \), because we could always take \( X = X \otimes_j I \). The resulting span then gets completed into the internal associativity axiom. One may then apply Theorem 7.2.2 to show that there is no other diamond with the same initial span.

The second way in which interchange can interact with associativity is in the term \( (a \otimes_j b) \otimes_i (c \otimes_j d \otimes_j e) \). In this case, we get the following square, where the labels have the evident components.

\[
\begin{array}{c}
(a \otimes j b) \otimes_i (c \otimes j d \otimes j e) \\
\downarrow \eta \\
(a \otimes \eta_i (c \otimes j d) \otimes j (b \otimes \eta_i e)) \\
\downarrow \delta \otimes j 1 \\
(a \otimes \eta_i (c \otimes j d \otimes j e)) \\
\end{array}
\]

The above square commutes by substituting \( (a \otimes_j I \otimes_j b) \otimes_i (c \otimes_j d \otimes_j e) \) for the source and using the external associativity axiom. Theorem 7.2.2 easily yields that there can be no other diamonds with the same initial span.

Similarly, a critical span arises at \( (a \otimes_j b \otimes_j c) \otimes_i (d \otimes_j e) \). The analysis is similar to the previous case by inserting a unit to obtain \( (a \otimes_j b \otimes_j c) \otimes_i (d \otimes_j I \otimes_j e) \).

**7.2.2. Interchange + interchange.** Let \( i < j < k \). An overlap between interchange rules occurs at \( (a \otimes_j b) \otimes_i ((c \otimes_k d) \otimes_j (e \otimes_j f)) \). Since we have strict units, we may assume that \( a = a_1 \otimes_k a_t \) and \( b = b_1 \otimes_k b_2 \). We then obtain the initial span of the giant hexagon axiom. The hexagon forms a diamond and it follows from Theorem 7.2.2 that there are no other diamonds with this initial span.

**7.2.3. Interchange + units.** The critical spans arising from the interaction of interchange with units yield the various Eckmann-Hilton maps. As we have seen, these are not always joinable. When they are, they commute by the following lemma.

**Lemma 7.2.5.** The following diagrams commute in \( F(\mathcal{M}_n) \), where \( 1 \leq i < j < k \leq n \):
PROOF. This follows from [BFSV03, Lemma 4.22]. More explicitly it follows from the giant hexagon axiom by making the following substitutions:

1. \( a \otimes_i b = ((a \otimes_k I) \otimes_j (I \otimes_k I)) \otimes_i ((I \otimes_k I) \otimes_j (I \otimes_k b)) \)
2. \( a \otimes_i b = ((I \otimes_k I) \otimes_j (a \otimes_k I)) \otimes_i ((I \otimes_k b) \otimes_j (I \otimes_k I)) \)
3. \( a \otimes_i b = ((I \otimes_k a) \otimes_j (I \otimes_k I)) \otimes_i ((I \otimes_k I) \otimes_j (b \otimes_k I)) \)
4. \( a \otimes_i b = ((I \otimes_k I) \otimes_j (I \otimes_k a)) \otimes_i ((b \otimes_k I) \otimes_j (I \otimes_k I)) \)

\[ \square \]

7.2.4. Putting it all together. We have seen that \( \mathcal{F}(\mathcal{M}_n) \) is quasicycle-free and that every diamond in \( \mathcal{F}(\mathcal{M}_n) \) commutes. We can therefore apply Theorem 6.3.7 to obtain the coherence theorem for iterated monoidal categories.

**Theorem 7.2.6.** If \( a \) and \( b \) are terms of \( \mathcal{F}(\mathcal{M}_n) \) having no repeated variables, then there is at most one reduction \( a \rightarrow b \) in \( \mathcal{F}(\mathcal{M}_n) \).

\[ \square \]

Theorem 6.3.7 provided a valuable strategy for proving the above coherence theorem for iterated monoidal categories. Although some careful combinatorial investigations were still required, the overall proof is conceptually straightforward. This demonstrates that even reasonably complicated coherence problems for quasicycle-free rewriting 2-theories may be comparatively easily attacked with the tools from Chapter 6.
CHAPTER 8

Conclusion

We have developed rewriting 2-theories as an abstract framework for studying coherence problems for covariant categorical structures. While general coherence theorems have been developed previously for certain classes of covariant structures [Pow89, Lac02], the work has typically been at an abstract categorical level and so does not yield any techniques for constructing specific coherence diagrams. More recent work on this problem has yielded an approach to obtaining coherence axioms for invertible theories [FHK]. However, the coherence axioms chosen in [FHK] are all of the diagrams in general position. Certainly, this vastly over-axiomatises most theories and the authors in [FHK] note:

“It is not clear what general scheme would select coherence diagrams correctly” in accordance with what one expects for specific examples of algebraic structures known.”

The work in Chapter 4 on complete rewriting 2-theories and in Chapter 6 on quasicycle-free rewriting 2-theories does, however, provide a general scheme for selecting coherence diagrams in accordance with what one would expect for particular algebraic structures. For complete rewriting 2-theories, one only needs to select a joining of each critical span in order to obtain a complete set of coherence axioms. For quasicycle-free rewriting 2-theories, a complete set of coherence axioms is provided by the basic diamonds. It is, however, generally more difficult to construct coherence axioms in the quasicycle-free case. This is demonstrated by the intricate investigation required in Chapter 7 for iterated monoidal categories as opposed to the relatively straightforward investigation of Catalan categories in Chapter 5.

Our combinatorial approach to coherence has the additional benefit of retaining a close link to classical one-dimensional universal algebra. This has allowed us, in Chapter 3, to use coherent categorifications of balanced equational theories to build presentations of the associated structure monoids and groups. Combined with our general techniques in chapters 4 and 6 for constructing coherence axioms, this provides a powerful toolkit for developing presentations of groups and monoids. This combination came to the fore in Chapter 5, where we constructed new presentations for the higher Thompson groups $F_{n,1}$ and the Higman-Thompson groups $G_{n,1}$.

In the following section, we outline some additional questions raised by the thesis.

Further questions

Both of our general coherence theorems for incomplete theories, Theorem 6.2.3 and Theorem 6.3.7, rely on the underlying structure being quasicycle-free. One may well call this condition into question and wonder whether we can get away with a weaker
condition. For Lambek coherence, quasicycle-freeness does not capture all covariant structures known to be Lambek coherent. For example, braided monoidal categories are certainly not quasicycle free and yet their Lambek coherence problem is solvable via the Reidemeister moves [JS93]. However, employing the Reidemeister moves adds an additional rewrite system to the reductions, thus expanding the amount of information available.

**Question 1.** Are there general properties that ensure Lambek coherence for non quasicycle-free rewriting 2-theories?

The reliance on quasicycle-freeness for Mac Lane coherence seems more fundamental. However, two crucial ingredients of our theory rely predominantly on acyclicity: Theorem 6.1.16 establishes that the faces of a subdivision are themselves st-graphs, while Proposition 6.3.6 shows that the faces of a maximal subdivision are diamonds.

**Question 2.** What conditions on an acyclic rewriting 2-theory ensure Mac Lane coherence?

It seems likely that acyclic rewriting 2-theories in which every diamond commutes are Mac Lane coherent. The major obstruction to showing this is that maximal subdivisions are no longer guaranteed to exist.

As noted in Chapter 2, rewriting 2-theories with an empty set of coherence axioms correspond to the unconditional fragment of rewriting logic. Meseguer has shown a strong connection between rewriting logic and models of concurrency [Mes92]: the congruence classes of terms correspond to states of the system, while reductions correspond to processes. A parallel pair of reductions that are equal correspond in this framework to a truly concurrent pair of processes. That is, they correspond to a pair of processes that may be safely run in parallel. In this way, one may view coherence axioms as specifications that certain parallel pairs of processes that seemingly interact with one another may in fact be safely run in parallel. In this way, the Lambek coherence problem asks about the existence of a decision procedure for determining which processes may be safely run in parallel. This computational interpretation of the Lambek coherence problem motivates a more refined investigation of decision procedures for the commutativity of diagrams arising from rewriting 2-theories.

**Question 3.** What is the computational complexity of deciding whether a diagram commutes in the structure generated by a Lambek coherent rewriting 2-theory?

As we saw in Chapter 2, determining whether a finitely presented rewriting 2-theory is Mac Lane coherent is in general undecidable. However, this does not rule out the possibility of developing algorithms for tackling Mac Lane coherence. Indeed, there exist many successful algorithms for determining whether a term rewriting theory is terminating, even though this problem is also undecidable in general — a powerful such algorithm is provided by the dependency pairs method [Art00, HM05].

**Project 4.** Develop algorithms for constructing coherent categorifications of labelled rewriting theories and for determining whether a given rewriting 2-theory is Mac Lane coherent.
A finite presentation of a coherent categorification of an equational theory leads to an infinite set of singular morphisms. Thus, the presentations constructed in Chapter 3 yield an infinite presentation of the associated structure monoid or group. This presentation has the nice property of imbuing the orbit graph of the resulting monoid or group with the geometry of the categorical structure. However, many structure groups are in fact finitely presentable. In particular, this is the case for the groups $F_{n,1}$ and $G_{n,1}$.

**Question 5.** *Are there properties of a coherent categorification of an equational theory that imply the finite presentability of the associated structure monoid or group?*

Our investigations have been wide-ranging, touching on topics from category theory, computer science, universal algebra and group theory. This has allowed us to use computational insights to prove theorems about mathematical objects, to construct general coherence theorems that yield information about the actual coherence diagrams and to build presentations of groups using very general techniques. Hopefully future work will continue to exploit techniques across traditional subject boundaries, so as to illustrate connections and to foster dialogue.
Bibliography


