Acoustic Sensor Array Signal Processing for Biomedical Applications

S. M. Akramus Salehin

BSc. Eng. (Hon 1) and Msc. Eng., University of Cape Town
Masters in ICT, Australian National University

November 2011

A THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
AT THE AUSTRALIAN NATIONAL UNIVERSITY

Applied Signal Processing Group
Research School of Engineering
College of Engineering and Computer Science
The Australian National University
Declaration

The contents of this thesis are the results of original research and have not been submitted for a higher degree to any other university or institution.

Much of the work in this thesis has been published or has been submitted for publication as journal papers or conference proceedings. In some cases, the conference papers contain overlapping material with the journal publications. The following is a list of these publications.

Patents


Journals


Conferences


The research work presented in this thesis has been performed jointly with A/Prof. Thushara D. Abhayapala. The substantial majority of this work was my own.

S. M. Akramus Salehin
College of Engineering and Computer Science,
The Australian National University,
Canberra,
ACT 0200,
Australia.
The work presented in this thesis would not have been possible without the support of a number of individuals and organizations and they are gratefully acknowledged below.

I am grateful to Assoc. Prof. Thushara D. Abhayapala for supervising me. I thank him for his patience, his encouragements and his time. He taught me how to conduct research and how to write research papers. I learned a lot from him especially in the area of acoustics signal processing.

I would like to acknowledge the helpful suggestions offered by Dr Jiang (Andrew) Zhang during the initial part of this research work. I would also like to show my gratitude to Dr Salman Durrani for his guidance in teaching practices.

I would like to acknowledge the support from and useful discussions with my fellow research students in the Applied Signal Processing Group.

I am indebted to the administrative staff at the Research School of Information Sciences and Engineering for all their help. Special mention go to Lesley Goldburg and Elspeth Davies for their assistance in arranging the conference travels undertaken during this PhD research.

I would like to thank the Australian National University and National ICT Australia for providing the scholarship without which this PhD work would not have been possible.

Lastly, I acknowledge the continuous support from my friends and my family during this difficult and stressful time. I am especially grateful to my family for their patience and moral support.
Abstract

This thesis develops array signal processing theories for selected biomedical applications involving acoustic waves. Specifically, we consider source localization in the interior of sensor arrays for lung sound localization and efficient algorithms for photoacoustic imaging. Lung sound localization provides quantitative results to the extent and location of lung disorders. Photoacoustic imaging is important for the early detection of cancer and has numerous other biomedical applications.

Previous lung sound localization methods cannot deal with multiple sources or have analytical performance measures. We propose two methods utilizing the eigen basis decomposition of the wavefield and the Minimum Variance spectrum for multiple source localization. Analytical performance measures were derived for resolution and spatial aliasing. The performance of our methods for lung sound localization together with the performance measures were proven by simulations.

We consider the photoacoustic inversion problem from a frequency invariant source localization perspective. Complete series and fast photoacoustic inversion methods have not been developed for the circular and spherical sensor geometries. A new theory is developed for photoacoustic reconstruction where the source distribution is expanded with a suitable series expansion such that separating the modes in the wavefield expansion at particular frequencies, separates the information in the source expansion. This theory is applied for photoacoustic inversion using a circular acquisition geometry. The source is expanded using a Fourier Bessel series and the coefficients are estimated by processing frequencies corresponding to the Bessel zeros. The proposed method is faster than previous approaches and the derivation is valid even for finite measurement bandwidth. This new theory is flexible enough to be applied for arbitrary sensor geometries and allows the selection of a minimum number of frequency samples for reconstruction. For previous frequency domain methods, there was no way to determine the minimum number of frequency samples required. Further, numerical experiments proved the effectiveness of our approach.
The extension of the proposed theory for photoacoustic inversion with a spherical array geometry was proposed. This new method expands the source distribution with a spherical Fourier Bessel series whose coefficients were now obtained by processing frequencies corresponding to the spherical Bessel zeros. Using computational order analysis and numerical experiments, this proposed method was shown to be faster than the backprojection and the Fourier series methods.

To enhance the reconstruction of our proposed methods, we introduced a sub-gradient based Total Variation (TV) minimization and an alternating projections post processing method. Both these methods were designed to handle the large data sets present in photoacoustic tomography. Applications of these two post processing ideas to previously proposed inversion methods are either difficult or impossible. The proposed inversion methods provide projection of the source distribution onto a set of basis functions. Therefore, these two post processing methods were developed to reconstruct a source distribution that preserves these projections and ensures that the source distribution is non-negative. Numerical experiments performed showed that reconstruction quality was improved by applying these two post processing methods. Further, the TV minimization method provided better reconstruction when compared to the alternating projections method.
### Glossary of Definitions

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{R})</td>
<td>real numbers</td>
</tr>
<tr>
<td>(\mathbb{R}^+)</td>
<td>positive real numbers</td>
</tr>
<tr>
<td>(\mathbb{N})</td>
<td>natural numbers</td>
</tr>
<tr>
<td>(i)</td>
<td>(= \sqrt{-1})</td>
</tr>
<tr>
<td>(c)</td>
<td>speed of wave propagation</td>
</tr>
<tr>
<td>(f)</td>
<td>frequency</td>
</tr>
<tr>
<td>(\omega)</td>
<td>angular frequency equal to (2\pi f)</td>
</tr>
<tr>
<td>(k)</td>
<td>wavenumber</td>
</tr>
<tr>
<td>(k_l)</td>
<td>lower frequency limit</td>
</tr>
<tr>
<td>(k_u)</td>
<td>upper frequency limit</td>
</tr>
<tr>
<td>(\sigma_n^2)</td>
<td>noise power</td>
</tr>
<tr>
<td>(A)</td>
<td>uppercase, bold letters represent matrices</td>
</tr>
<tr>
<td>(I)</td>
<td>identity matrix</td>
</tr>
<tr>
<td>(a)</td>
<td>lowercase, bold letters represent vectors</td>
</tr>
<tr>
<td>(A^T)</td>
<td>transpose of (A)</td>
</tr>
<tr>
<td>(A^*)</td>
<td>conjugate transpose of (A)</td>
</tr>
<tr>
<td>(\nabla)</td>
<td>gradient of a vector function</td>
</tr>
<tr>
<td>(\nabla^2)</td>
<td>Laplacian operator</td>
</tr>
<tr>
<td>(\text{div})</td>
<td>divergence operator for a vector field</td>
</tr>
<tr>
<td>((\cdot)^*)</td>
<td>complex conjugate operator</td>
</tr>
<tr>
<td>([\cdot])</td>
<td>integer ceiling function</td>
</tr>
<tr>
<td>(E{\cdot})</td>
<td>expectation operator</td>
</tr>
<tr>
<td>(\langle a, b \rangle)</td>
<td>inner product of vectors (a) and (b) defined as (a^*b)</td>
</tr>
<tr>
<td>(\langle \cdot, \cdot \rangle)</td>
<td>inner product defined for a specific Hilbert space</td>
</tr>
<tr>
<td>(</td>
<td>\cdot</td>
</tr>
<tr>
<td>(\text{vec}(\cdot))</td>
<td>column stacking operator for a matrix</td>
</tr>
<tr>
<td>(\text{card}(\Lambda))</td>
<td>cardinality of set (\Lambda)</td>
</tr>
<tr>
<td>(|\cdot|)</td>
<td>Euclidean or the (l_2) norm</td>
</tr>
<tr>
<td>(|\cdot|_1)</td>
<td>(l_1) norm</td>
</tr>
</tbody>
</table>
$S^2$ 2-sphere
$L^2(S^2)$ square integrable functions on the 2-sphere
$L^1(\cdot)$ Hilbert space of integrable functions
sin(\cdot) sine function
cos(\cdot) cosine function
$e^{(\cdot)}$ exponential function
log$_{10}(\cdot)$ logarithm with base 10
$\delta(\cdot)$ Dirac delta function
$\delta_{nn'}$ Kronecker delta which is 1 only when $n = n'$ otherwise 0
$J_m(\cdot)$ Bessel function of order $m$
$H^{(1)}_m(\cdot)$ Hankel function of the first kind of order $m$
$j_n(\cdot)$ spherical Bessel function of order $n$
$h^{(1)}_n(\cdot)$ spherical Hankel function of order $n$
$P_{nm}(\cdot)$ associated Legendre function of order $n$ and mode $m$
$Y_{nm}(\cdot)$ spherical harmonics of order $n$ and mode $m$
$G(\cdot)$ 3D Green’s function
$G_{2D}(\cdot)$ 2D Green’s function
$p(\mathbf{r}, t)$ pressure at vector position $\mathbf{r}$ and at time $t$
$p(\mathbf{r}, k)$ pressure at vector position $\mathbf{r}$ and at wavenumber $k$
$p_0(\mathbf{r})$ source or initial pressure distribution
$r_0$ bounding radius of source distribution
$\mathbf{r}_s$ position of sensor or position on the measurement aperture
$r_s$ radius at which sensors are placed
$r$ radial co-ordinates
$\phi, \theta$ angular co-ordinates
$\varepsilon_{\text{rel}}$ relative Mean Square Error (MSE)
2D Two Dimensions
3D Three Dimensions
a.u. arbitrary units
i.i.d. independent and identically distributed
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>DOA</td>
<td>Direction of Arrival</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
</tr>
<tr>
<td>EM</td>
<td>Electromagnetic</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>F-R</td>
<td>Frequency-Radial duality</td>
</tr>
<tr>
<td>HEELS</td>
<td>Helmholtz Equation Least Squares</td>
</tr>
<tr>
<td>MSE</td>
<td>Mean Square Error</td>
</tr>
<tr>
<td>POCS</td>
<td>Projection Onto Convex Sets</td>
</tr>
<tr>
<td>PAT</td>
<td>Photoacoustic Tomography</td>
</tr>
<tr>
<td>RF</td>
<td>Radio Frequency</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
</tr>
<tr>
<td>SOCP</td>
<td>Second Order Cone Program</td>
</tr>
<tr>
<td>STFT</td>
<td>Short Time Fourier Transform</td>
</tr>
<tr>
<td>SVD</td>
<td>Singular Value Decomposition</td>
</tr>
<tr>
<td>TAT</td>
<td>Thermoacoustic Tomography</td>
</tr>
<tr>
<td>TV</td>
<td>Total Variation</td>
</tr>
</tbody>
</table>
# Contents

Declaration i  
Acknowledgements iii  
Abstract v  
Glossary of Definitions vii  

1 Introduction 1  
1.1 Motivation .................................................. 1  
1.1.1 Lung Sound Localization ................................ 2  
1.1.2 Photoacoustic Imaging ................................... 4  
1.1.3 Problem Statement ....................................... 8  
1.2 Aims and Scope ........................................... 9  
1.3 Outline ..................................................... 10  
1.3.1 Thesis Structure ......................................... 10  
1.3.2 Contributions of Thesis ................................ 11  
References ...................................................... 15  

2 Background: Biomedical Applications and Modal Array Signal Processing 23  
2.1 Introductions ................................................. 23  
2.2 Lung Sound Localization .................................... 23  
2.2.1 Acoustic Properties of the Thorax ..................... 23  
2.2.2 Lung Sound Localization Methods ...................... 24  
2.3 Photoacoustic Tomography .................................... 26  
2.3.1 Wave Equations ......................................... 27  
2.3.2 Safety Issues ............................................ 29  
2.3.3 Acoustic Wave Propagation and Detection ............. 29  
2.3.4 Scanning Tomography .................................. 30
2.3.5 Inversion with Acoustic Lens .......................... 31
2.3.6 Inversion Methods by Computed Tomography .......... 31
2.3.7 Discussion ............................................. 42

2.4 Modal Array Signal Processing .............................. 43
  2.4.1 Circular Apertures ...................................... 45
  2.4.2 Spherical Apertures ..................................... 49
  2.4.3 Applications ........................................... 53
  2.4.4 Discussion ............................................. 56

2.5 Summary .................................................. 56

References ..................................................... 57

3 Eigen Basis Decomposition for Localizing Lung Sounds 65
  3.1 Introduction .............................................. 65
  3.2 System Model ............................................. 68
  3.3 Eigen Basis Decomposition ................................. 72
  3.4 Sound Localization ....................................... 73
    3.4.1 Orthogonality Based Algorithm ....................... 73
    3.4.2 Least Squares Based Algorithm ....................... 78
  3.5 Theoretical Performance Analysis .......................... 80
    3.5.1 Noise ................................................. 80
    3.5.2 Resolution ........................................... 82
    3.5.3 Nyquist’s criteria ................................... 86
  3.6 Simulations ............................................... 86
    3.6.1 Localizing Multiple Sources ......................... 87
    3.6.2 Performance under Different Noise Levels .......... 89
    3.6.3 Performance with Different Frequencies of Sound ... 90
  3.7 Comments and Comparison of the Proposed Algorithms ..... 92
  3.8 Summary .................................................. 93

References ..................................................... 94

4 Theory of Frequency Invariant Source Localization for Photoa-
coustic Tomography ........................................... 99
  4.1 Introduction .............................................. 99
  4.2 Photoacoustic Inverse Problem ............................ 101
  4.3 Formulation for 2D and 2.5D Circular Geometries ........ 102
  4.4 Modal Expansion of the 2D Green’s Function ............. 105
  4.5 Eigen Basis Expansion of the Source Distribution ........ 105
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5.1 Fourier Bessel Expansion</td>
<td>105</td>
</tr>
<tr>
<td>4.5.2 2D Fourier Bessel Expansion of the Source Distribution</td>
<td>106</td>
</tr>
<tr>
<td>4.6 Reconstruction using a Circular Aperture</td>
<td>107</td>
</tr>
<tr>
<td>4.6.1 Modal Filtering</td>
<td>108</td>
</tr>
<tr>
<td>4.6.2 Frequency-Radial Duality</td>
<td>108</td>
</tr>
<tr>
<td>4.6.3 Comparison with Fourier Domain Methods</td>
<td>109</td>
</tr>
<tr>
<td>4.6.4 Discrete Aperture</td>
<td>112</td>
</tr>
<tr>
<td>4.6.5 Spatial Filtering</td>
<td>113</td>
</tr>
<tr>
<td>4.6.6 Design Example: Incorporating Transducers with Different Frequency Responses</td>
<td>114</td>
</tr>
<tr>
<td>4.6.7 Numerical Experiments</td>
<td>114</td>
</tr>
<tr>
<td>4.7 Reconstruction with An Arbitrary Detection Geometry</td>
<td>116</td>
</tr>
<tr>
<td>4.7.1 Problem Statement</td>
<td>116</td>
</tr>
<tr>
<td>4.7.2 Truncation of Modal Expansion of Green’s Function</td>
<td>119</td>
</tr>
<tr>
<td>4.7.3 Proposed Frequency Domain Algorithm</td>
<td>119</td>
</tr>
<tr>
<td>4.7.4 Numerical Experiments</td>
<td>123</td>
</tr>
<tr>
<td>4.8 Reconstruction with Sparse Frequency Samples</td>
<td>127</td>
</tr>
<tr>
<td>4.8.1 Normalized 2D Fourier Bessel Expansion of Source Distribution</td>
<td>127</td>
</tr>
<tr>
<td>4.8.2 Properties of Angular Modes</td>
<td>128</td>
</tr>
<tr>
<td>4.8.3 Least Squares Estimation of Radial Basis Coefficients per Mode</td>
<td>131</td>
</tr>
<tr>
<td>4.8.4 Related Reconstruction Algorithms</td>
<td>132</td>
</tr>
<tr>
<td>4.8.5 Numerical Experiments</td>
<td>133</td>
</tr>
<tr>
<td>4.9 Summary</td>
<td>135</td>
</tr>
<tr>
<td>References</td>
<td>136</td>
</tr>
<tr>
<td>5 Frequency-Radial Duality based Photoacoustic Image Reconstruction</td>
<td>139</td>
</tr>
<tr>
<td>5.1 Introduction</td>
<td>139</td>
</tr>
<tr>
<td>5.2 Background on Photoacoustic Imaging</td>
<td>141</td>
</tr>
<tr>
<td>5.2.1 Wavefield Decomposition</td>
<td>141</td>
</tr>
<tr>
<td>5.3 Frequency-Radial Duality Based Image Reconstruction</td>
<td>142</td>
</tr>
<tr>
<td>5.3.1 Problem Statement</td>
<td>142</td>
</tr>
<tr>
<td>5.3.2 Fourier Transform on the 2-Sphere</td>
<td>143</td>
</tr>
<tr>
<td>5.3.3 Spherical Fourier Bessel Expansion of Spatial Distribution</td>
<td>144</td>
</tr>
<tr>
<td>5.3.4 Modal-Order Filtering of the Spatial Distribution</td>
<td>144</td>
</tr>
</tbody>
</table>
7.2.3 Experimental Evaluation ........................................ 213

Appendices

Appendix A Localization of Quasiperiodic, Pulsatic Signal in a Correlated Mixture 215

A.1 Introduction ...................................................... 215
A.2 Problem Statement ............................................. 216
A.3 Short-Time Fourier Transform (STFT) .......................... 218
A.4 Fundamental Frequency Estimation ............................. 219
A.5 Localization of Pulsatic Signal .................................. 220
A.6 Results ........................................................... 221
A.7 Summary .......................................................... 221
References ........................................................... 224
## List of Figures

1.1 Operation of a multisensor stethoscope for localizing lung sounds and its diagnostic ability ........................................... 3

1.2 Evolution of the stethoscope from nineteenth century monoaural devices (left) to current research into multisensor recordings to locate lung sound sources (right). ........................................... 4

1.3 Operation of photoacoustic imaging where optical or RF illumination results in acoustic waves that are detected by ultrasound transducers. ............................................................... 6

1.4 Block diagram showing the structure and relationships between the major thesis contribution areas. ........................................... 12

2.1 Photoacoustic scanning system ........................................... 30

2.2 Plane wave impinging on a circular aperture. ........................ 45

2.3 The standard spherical co-ordinate system, $r$ the radial co-ordinate, $\theta \in [0, \pi]$ elevation and $\phi \in [0, 2\pi]$ the azimuth. ............................................................... 45

2.4 Plane wave impinging on a spherical aperture. ........................ 50

3.1 System model with lung sound sources located interior of a circular sensor array placed around the chest. ......................... 68

3.2 Wavefield generated by a 2-D source at $(3, \pi/4)$. The field magnitude at the source location is infinity and the magnitude dies down very sharply. The phase information can be expected to be more different at the sensors than the magnitude. ................. 70

3.3 Transformation of the sensor data to the spatial Fourier domain. 75

3.4 Variation of the condition number $\kappa(H)$ with $kR$ and number of modes ($N$) used. ............................................................... 78

3.5 The decrease in angle, $\psi$ as sensor radius increases. ................. 85

3.6 The increase in angle, $\psi$ as frequency increases. ..................... 85

3.7 Spectrum for multiple 2-D sources with SNR = 10 dB. ............... 88
3.8 Spectrum obtained when SNR = 20 dB. .......................... 89
3.9 Spectrum showing a reduction in resolution when wavelength is increased. ................................................. 90
3.10 Spectrum showing aliasing when the wavelength is reduced. The region $\tilde{R}$ for which sources can be localized is discernible. ........ 91
3.11 Spectrum of region $\tilde{R}$ where two sound sources are present and each concentric circle represents a distance of 0.3 units. ........ 91

4.1 In the 2D problem, the source distribution is bounded within a radius of $r_0$ and the sensors are placed in the region of validity. This is known as the exterior source problem. The region of validity means that $r_s > r_0$. .................................................. 102
4.2 In the 2.5D problem, the sensor elements are long in the vertical dimension and the source distribution is enclosed within the sensor array. Using this configuration, the z-averaged source distribution is reconstructed. .................................................. 103
4.3 Input sample (a) x-y view (b) x-z view through the central axis, with arbitrary units (a.u.) for the relative absorption. ........... 115
4.4 Magnitude of Fourier Bessel coefficients at the modes $m$ and indices $\ell$ for the numerical phantom. ................................. 116
4.5 Reconstructed image using the largest 60 estimated Fourier Bessel coefficients, over both the mode $m$ and zero index $\ell$, with a SNR = 20 dB. .......................................................... 117
4.6 Reconstructed image using the largest 120 estimated Fourier Bessel coefficients, in terms of both $m$ and $\ell$, with a SNR = 20 dB. ... 117
4.7 Configuration of the source distribution and the arbitrary sensor geometry. .......................................................... 118
4.8 (a) The input spatial source distribution used in the numerical experiments. (b) Relative absorption, in arbitrary units [a.u.], through the horizontal central axis of the input spatial source distribution. .............................................. 124
4.9 Signals recorded by one of the sensors for the given input distribution. .............................................................. 124
4.10 Positions of sensors randomly placed (a) between 10 and 12 mm radius and (b) between 10 and 14 mm radius. .................... 125
4.11 Reconstructed image for a circular array of sensors, (a) mesh plot of the reconstruction and (b) profile through the horizontal central axis. Reconstructed image for sensor configuration in Fig. 4.10a, (c) mesh plot of the reconstruction and (d) profile through the horizontal central axis. Reconstructed image for sensor configuration in Fig. 4.10b, (e) mesh plot of the reconstruction and (f) profile through the horizontal central axis.

4.12 Magnitude of $j_{nm}$ for $r_0 = 3$ cm, $r_s = 20$ cm at $k = 670m^{-1}$.

4.13 Image reconstructed in the presence of aliasing for three point sources (a) proposed method using 63 frequency samples (b) frequency domain reconstruction using 400 frequency samples.

4.14 Image reconstructed with no aliasing for three point sources (a) proposed method using 27 frequency samples (b) frequency domain reconstruction using 400 frequency samples.

5.1 The problem involves estimating the spatial distribution enclosed in a spherical region of radius $r_0$ from measurements on a continuous spherical aperture at a radius $r_s$, completely enclosing the bounding region.

5.2 Plots of the spherical Bessel functions $j_n(r)$ of order $n = 0, 1$ and 2.

5.3 Numerical phantom consisting of spherical absorbers (a) cross section through the x-y plane (b) relative absorption, in arbitrary units (a.u.), through the horizontal central axis of this cross section.

5.4 Recordings by a sensor placed at azimuth $\phi = 0$ and elevation $\theta = \pi/2$ in the presence of 10 dB of noise.

5.5 Frequency spectrum of signals recorded by a sensor placed at azimuth $\phi = 0$ and elevation $\theta = \pi/2$.

5.6 Reconstruction using the proposed F-R based method with frequency range from 0 to 2 MHz and no noise (a) cross section through the x-y plane (b) relative absorption, in arbitrary units (a.u.), through the horizontal central axis of this cross section.

5.7 Reconstruction using the three different reconstruction methods with a frequency range from 100 kHz to 2 MHz and 10 dB of noise: (a) and (d) time domain backprojection method; (b) and (e) Norton-Linzer method; and (c) and (f) F-R based method.
5.8 Relative magnitudes (in a.u.) of the Fourier Bessel coefficients $\alpha_{nml}$ for the numerical phantom, note in our representation each mode $m$ has a maximum of 40 radial indices $l$. .......................... 163

5.9 Variation of run time with upper frequency limit for the different reconstruction methods. .................................................. 165

6.1 Geometric view of projection onto the fidelity constraint (A) ($l_2$ ball) in the modal domain. .................................................. 187

6.2 Iterative alternating projections onto the sets $B$ and $C$ results in $\tilde{p}_0(\alpha)$ eventually converging to a point $\tilde{p}_0 \in C \cap B$ .......................... 189

6.3 Numerical phantom composed of several circular discs, (a) cross section through the x-y plane and (b) horizontal central axis of this cross section. .................................................. 196

6.4 Reconstruction with the F-R based method in the presence of 20 dB noise for measurement bandwidth from 0 to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. Reconstruction using measurement bandwidth from 100 KHz to 1 MHz, (c) cross section through the x-y plane and (d) horizontal central axis. .................................................. 198

6.5 Resulting image after 10 iterations of the alternating projections method with measurement bandwidth from 100 KHz to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. Resulting image after 50 iterations, (c) cross section through the x-y plane and (d) horizontal central axis. Note the improvement from the reconstruction shown in Fig. 6.4c and Fig. 6.4d. ............. 199

6.6 Resulting image after 10 iterations of the alternating projections method with measurement bandwidth from 0 to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. .......................... 200

6.7 Distance of the resulting image to the positivity constraint (set B) related to the iteration index for the alternating projections method. 201

6.8 Resulting image after 10 iterations of the TV minimization method with measurement bandwidth from 100 KHz to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. Resulting image after 50 iterations, (c) cross section through the x-y plane and (d) horizontal central axis. Note the improvement from the reconstruction shown in Fig. 6.4c and Fig. 6.4d. ............. 202
6.9 Resulting image after 10 iterations of the TV minimization method with measurement bandwidth from 0 to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. 203

6.10 Relative MSE $\varepsilon_{rel}$ versus iteration index $\kappa$ for the TV minimization and the alternating projections method, for measurement bandwidth from 100 KHz to 1 MHz. 203

A.1 Test signals and results of simulations. 222

A.2 Spectrogram of the mixture using a hop size of 50 samples and a frame size of 100 samples using a Hamming window. 223
List of Tables

2.1 Comparison of the different computed tomography methods applied to photoacoustic imaging. ........................................ 43

5.1 Variation of Run Times (sec) with Frequency Upper Limit (MHz) for Different Reconstructions Methods. .................. 164
Chapter 1

Introduction

1.1 Motivation

Inverse problems are numerical problems that originate when the goal is to estimate the “hidden” or “interior” information from noisy measurements observed from “outside” [1]. This means recovering the interior composition of the body or the properties of the source responsible for the observed data by the application of a mathematical model linking the observed data to the interior environment or source. The research on inverse problems are rich and developed with many books dedicated to this subject matter [1–3]. Inverse problems arise in many areas such as medical imaging, geophysical imaging, electromagnetic scattering and in non-destructive testing.

For medical imaging, tomography is defined as looking into the body without opening it. Tomography includes X-ray imaging, Magnetic Resonance Imaging (MRI) or a number of other imaging modalities, and these have received attention for applications such as early cancer detection. Another example of an inverse problem involves the reconstruction of sharper images from blurred images with applications in astronomy and biometric identification.

Most medical imaging modalities are mathematically modeled as inverse problems and are divided into two classes: linear inverse problems and non-linear inverse problems. This thesis is concerned with linear inverse problems where the research centers around questions involving formulas that reconstruct the body’s interior from recorded data and the theoretical properties of such reconstruction formulas in terms of resolution of reconstruction, limitations on number of sensors, limitations on sensor geometry, stability of reconstruction, speed of reconstruction and reconstruction with limited view.
This thesis is only concerned with acoustic inverse problems defined as the estimation of environmental parameters or the localization of the sources responsible for the observed acoustic field. Acoustic inverse problems include acoustic holography, sonar and underwater acoustics. The literature on acoustic inverse problems is vast (see [4–8]).

For biomedical applications, ultrasound imaging is a well known acoustic inverse problem with vast biomedical applications [9] such as echocardiograms, bone sonography and fetal ultrasound. Ultrasound imaging uses reflection, from tissues, of the transmitted high frequency sound waves to create images. It became popular as it does not use harmful ionizing radiation unlike X-ray tomography. Ultrasound imaging is a type of acoustic inverse problem that recovers the parameters of the propagation environment. The second type of acoustic inverse problem recovers the source position and properties from the acoustic wavefield. For biomedical applications, lung sound localization and photoacoustic tomography belong to the second type of acoustic inverse problem and are the subject matter of this thesis.

There has been great advancement in array signal processing in recent years. In this area algorithms have been designed for robust beamforming [10], sound reproduction [11] and blind source separation [12]. The utilization of these ideas for biomedical applications has lagged behind. However, several of these ideas have been applied for ultrasound imaging [13,14] and for MEG and EEG source localization [15]. There is huge potential in developing methods for biomedical applications by incorporating the advances in array and in particular acoustic array signal processing.

1.1.1 Lung Sound Localization

Little work has been done in providing a 3D image with acoustic source locations from simultaneous multisensor recording on the chest [16–20]. The methods presented look at solving the acoustic localization for point sources in the interior of circular or cylindrical array of sensors. The geometric viewpoint of the lung sound localization problem is shown by Fig. 1.1. The existing methods cannot locate multiple sources present simultaneously, provide a measure on their performance analytically in terms of resolution and consistency of results, that is, identify when aliasing is likely to occur and in which regions of the lung the results are accurate. Furthermore, Kompis obtained different results with a different number of sensors [16]. Therefore, for localizing methods, it is impor-
1.1 Motivation

Figure 1.1: Operation of a multisensor stethoscope for localizing lung sounds and its diagnostic ability.

...tant to determine the minimum number of sensors required for consistent source localization and what factors affect this number.

Since its invention by Laennec in 1816, the stethoscope has been used qualitatively by physicians for diagnosing medical conditions of the pulmonary system (lung sounds), cardiovascular system and the gastrointestinal organs. Auscultation is used as the first mode of diagnosis due to its simplicity, non invasiveness, quick results and portability, with uses in home monitoring, emergency wards and field operations. The audible range of sound from the chest contains useful information about chest disorders since pathological processes and anatomical resonances produce sounds within this frequency range [21]. However, this information can only be processed qualitatively by physicians, and is limited by the physician’s skill and human capability to discern differences between different types of sounds. In recent years, computerized processing of lung sounds by applying signal processing and statistical methods to provide trends and detect disorders was performed by several researchers [22–25]. It must be noted that auscultation is not as reliable as chest X rays or computed tomography (CT), but is used as the first diagnostic tool and aids in deciding the diagnostic method to use next.

To get more information from lung sounds, it was suggested that (i) multisensors be used to record sounds simultaneously at different locations on the chest, (ii) determine relationship between acoustic source locations and lung disorders and (iii) use advanced spatial signal processing techniques to create an acoustic image of the chest with acoustic source locations [20]. The evolution of the stethoscope from monoaural devices and to recent research in computerized
multichannel stethoscopes is shown in Fig. 1.2. Several pulmonary diseases alter the spatial location of lung sounds, examples of this include pneumothorax and airway obstruction. By further localizing adventitious sounds within the chest, the physician can determine the severity and the location of the disorder. These applications motivate research towards a multichannel stethoscope allowing quantitative analysis of lung sounds. As a first step in the use of a multisensor device for monitoring lung sounds, several research groups have developed algorithms for mapping acoustic intensities on the thoracic surface [25–27]. In this thesis, we aim to further develop the theory of lung sound localization for using an array of acoustic sensors.

### 1.1.2 Photoacoustic Imaging

For lung sound localization, the acoustic sources are point sources whereas for photoacoustic imaging the source is distributed. We formulate the photoacoustic inverse problem as a frequency invariant source localization problem. Provided the heating pulse is sufficiently short and the source does not move significantly during the duration of the pulse, the source distribution can be separated into a product of a spatial and a temporal term. The spatial term does not depend on frequency and is frequency independent. In photoacoustic imaging, there is a linear dependence on frequency due to the temporal term which is compensated for to turn the photoacoustic reconstruction problem into a frequency invariant source localization problem. The importance of this frequency invariant formulation means that different frequency measurements convey different information about the same source distribution. The combination of all these frequency information can then be used to reconstruct the source distribution. Therefore, in
this thesis, we label photoacoustic image reconstruction as a frequency invariant source localization problem.

Photoacoustic tomography (PAT) involves reconstructing the initial pressure distribution from measured pressure waves. In PAT, the sample of interest is illuminated by a short laser pulse whereas thermoacoustic tomography (TAT) works on the same principle but the sample is irradiated by a microwave pulse. The sample absorbs part of the incident EM energy, this increases the temperature and the tissue sample expands generating an outgoing acoustic wave [29,30]. These acoustic waves are detected by ultrasound sensors surrounding the sample of interest. To reconstruct the initial source distribution (normally referred to as the initial pressure distribution), the photoacoustic inverse problem needs to be solved. The operation of the photoacoustic imaging method is shown by Fig. 1.3.

This imaging modality, just like lung sound localization, uses non ionizing radiation therefore does not pose any health risks, unlike X-rays. Further, photoacoustic imaging solves the low spatial resolution of optical imaging and the weak contrast of purely ultrasound imaging [31]. The contrast of photoacoustic tomography is not only higher than ultrasound but at multiple optical wavelengths has the potential to capture molecular information. This is the reason for the current excitement about photoacoustic tomography. In recent years, research on applying photoacoustic imaging to biomedical applications has developed quite rapidly, with uses in cancer detection, breast imaging, small animal imaging, functional imaging and tumor imaging. A comprehensive list and description of the uses of photoacoustic imaging is provided in [31].

The source reconstruction problem for photoacoustic imaging produces high resolution 3D images requiring the processing and computing of large amounts of data. Fast series solution based algorithms are available for the planar geometry [32] and for cubic sensor geometries [33]. These algorithms utilize the Fast Fourier Transform (FFT) to speed up reconstruction. For circular arrays (consisting of integrating line sensors or point sensors with 2D source distribution) and spherical arrays such fast algorithms have not been developed. The method presented in [34], for a circular array, expanded the source distribution into a complete series but the reconstruction method was based on the assumption that the source distribution provided a smooth Hankel transform of the filtered signal.

Considerations into integrating line detectors arose to solve the practical shortcomings of manufacturing small detectors with high sensitivity. [35].
Figure 1.3: Operation of photoacoustic imaging where optical or RF illumination results in acoustic waves that are detected by ultrasound transducers.

spatial resolution of the reconstructed source distribution in PAT is related to the detector size and the bandwidth of measurement [36]. Smaller detector sizes offer better spatial resolution. The use of integrating line detectors [37,38] meant that the detector size needed to be small in two dimensions only. The inversion with integrating line sensors is known as the 2.5D photoacoustic inverse problem.

A Fourier series method for the inversion for 2D photoacoustic source reconstruction was introduced by Norton [39]. This series solution was later extended to 3D, regular (planar, cylindrical and spherical) sensor geometries [28]. Norton’s solution for 2D geometries was unstable due to occurrence of divisions by zero. Solutions to this instability was solved in [34,40]. Direct implementation of the Fourier series methods of Norton [28,39] are slow, therefore approximate time domain, backprojection solutions for regular geometries in 3D were proposed in [41]. Other methods for photoacoustic inversion include the delay and sum beamforming approach [42] and the statistical approach [43]. There are numerous inversion formulas using filtration or backprojection methods [44–48]. Time reversal methods have also been applied for photoacoustic imaging in 3D [49,50]. These time reversal methods are flexible and can be extended to arbitrary geometries [49]. In 2D, Huygens principle is not valid and the time reversal methods are modified providing an approximate photoacoustic inversion [51,52].

None of the previously mentioned methods for photoacoustic inversion are as fast as the series solutions provided for cubic [33] and planar [32] acquisi-
1.1 Motivation

These series solutions provided fast inversion for photoacoustic imaging. For reconstruction on an $N \times N \times N$ discrete grid, these methods have a computational complexity of $O(N^2 \log N)$. Complete series solutions for photoacoustic inversion with the circular and spherical acquisition geometries in the frequency domain have not been proposed yet. Further, the inversion methods available in these geometries are not as fast as the series solutions for the planar and cubic sensor configurations. Moreover, the frequency domain inversion methods [28] do not provide the information content available in the frequency samples or the minimum number of frequency samples required for inversion. In this thesis, we aim to develop complete series and fast inversion methods for the circular and spherical geometries.

In photoacoustic image reconstruction, there has been limited or no application of Total Variation (TV) minimization and the Projection Onto Convex Sets (POCS) methods for improving photoacoustic inversion. The POCS method also known as the alternating projections method is the prevalent tool for image recovery in the intersection of convex sets where an initial image is projected onto the individual sets using a periodic schedule [53]. These methods have not been applied for photoacoustic reconstruction since most of the reconstruction methods do not provide projection data for the source, as is available for other imaging modalities such as MRI where TV minimization was applied in [54]. Other biomedical imaging modalities, where TV minimization improved the solution, included positron emission tomography [55] and computed tomography [56].

The idea that medical images are piecewise constant was introduced by De Lanney and Bresler [57]. The methods for photoacoustic inversion simply set all negative values in the source distribution function to zero in order to ensure non negativity of the source distribution function. For image recovery, POCS was applied in [53, 58, 59]. In photoacoustics, we need to find a source distribution that satisfies the measured data as well as the non negativity constraint, post-processing the source distribution using the alternating projections method or with the TV minimization framework would result in better images provided optical attenuation is negligible. Like other biomedical images, we expect the image to be reconstructed for photoacoustic tomography to be piecewise constant and so would benefit from a minimum TV solution. However, applying these postprocessing methods to the current reconstruction methods are difficult or impossible.
1.1.3 Problem Statement

From the above discussion we identify the following problem which has not been adequately answered previously and state it as follows:

*Design algorithms for (i) multiple point source localization with analytical performance measures for lung sound localization and (ii) frequency invariant, distributed source localization for photoacoustic tomography that provides a flexible, fast and complete series solution, all in the interior of sensor arrays.*

The complete series solution for photoacoustic tomography needs be fast for both the circular and spherical array geometries. Further, the series solution should be extendable to arbitrary sensor configurations and provide the information content present in the frequency samples.

Solutions to the point source localization method can be adapted to other areas such as speaker localization in rooms and auditorium recordings with microphone areas. The frequency invariant source localization solution has a wide area of application. Similar mathematical problems arise in sonar [60], radar [61], geophysics applications [62], ultrasound reflectivity imaging [28] and in the analysis of partial differential equations [63]. We look at these problems from a source localization perspective. Other biomedical areas where such a viewpoint was previously applied are MEG and EEG source localization [64].

The work in this thesis is motivated by the general problem of localizing sources within sensor arrays. We develop a theory of source localization in the interior of sensor arrays that can be applied to both point sources as well as distributed, frequency invariant sources. The philosophy of our approach is based on separating the information conveyed by eigen functions of the propagation environment. This basis expansion of the propagation channel also called modal expansion or wavefield decomposition. There are various applications of these modal expansions in array signal processing (see [65–67] and the references therein) and these are collectively known as modal array signal processing techniques.

The philosophy that separating the eigenfunctions separates information on the sources for point sources is not new, however, the technique presented in
this thesis specifically for lung sound localization is. For frequency invariant distributed sources, the idea that the eigenfunctions of the wavefield convey different information about the source and these can be separated by processing particular frequencies is new. We develop a general theory for frequency invariant, distributed source localization specifically for photoacoustic tomography applicable in 2D and 3D, for arbitrary enclosing arrays that allows identification of the source information conveyed by different frequency bands. This theory provides a new philosophy for solving inverse problems similar to photoacoustic tomography. Since our theory relies on modal decomposition of wavefield and uses arrays, we include our proposed methods as being part of modal array signal processing.

1.2 Aims and Scope

From the discussion presented in the motivation, we discuss the questions which will be answered in the following chapters of this thesis. Moreover, a discussion on the scope and limitations of the work done in this thesis is highlighted.

We aim to propose methods for localizing lung sounds based on the modal decomposition of the Green's function using a circular array of sensors. Use of these modal functions would allow extension to a lung sound model consisting of cylindrical layers. The localizing algorithms should be spectral based allowing lower precision sensors when compared to differences in arrival time based methods. Since lung sound localization methods provided different results with different numbers of sensors, analytical performance metrics in terms of resolution, spatial aliasing and region of localization can be calculated for the proposed methods or known beforehand. These algorithms will assume a free field wave propagation model which is valid for lung sounds within the frequency range from 100 to 1000 Hz and the lung sound velocity is constant [68]. Further, lung sound localization can be improved by first localizing the heart sound in the chest recordings. We aim to utilize the quasiperiodic (i.e., slight variations in period) property of heart sounds to localize heart sounds in these recordings.

Localizing lung sounds consist of locating point sources within a circular array. For photoacoustics, we aim to develop a new theory for source localization that are distributed and frequency invariant. From this theory, photoacoustic reconstruction methods can be designed for both 2D circular and 3D spherical acquisition geometries. Furthermore, this theory should allow determination of the source information present within the frequency bands. The source localization methods needs to be flexible enough to be used for arbitrary enclosing
arrays and should process as few frequency samples as possible over the angular basis in the source distribution expansion, rather than using all the frequency samples available as is done by the frequency domain inversion methods [28]. The methods developed should allow selecting a minimum number of frequency samples from which inversion is possible.

The general theory provided for photoacoustic reconstruction, should provide projection data for the sources allowing designing of postprocessing numerical methods (TV minimization and POCS) for enhancement of photoacoustic reconstruction for both 3D and 2D sources. This thesis should also prove the benefits of these postprocessing methods.

The general theory and methods for photoacoustic inversion will not consider the blurring effect due to the finite aperture of the sensors and will assume sensors sample a single point in space. Furthermore, the methods do not consider the partial view case when the sensors cannot fully enclose the sample under study. This scenario occurs in breast imaging using photoacoustic tomography. The sound velocity will be assumed to be spatially constant in all proposed methods for photoacoustic inversion. In photoacoustics, the variation of the speed of acoustic waves in soft tissue is neglected since this speed is approximately constant with a small variation of about 10% [69]. If this speed variation is large then this variation can be mapped using ultrasound tomography. The speed map can then be applied for photoacoustic reconstruction. However, the correction for speed variation is beyond the scope of this thesis.

Additionally, the testing of the methods proposed in this thesis using real data for both lung sound localization and photoacoustic reconstruction is beyond the scope of this thesis.

1.3 Outline

This section outlines how the thesis is organized and discusses the main contributions.

1.3.1 Thesis Structure

This thesis is organized as follows:

- The next chapter reviews the source localization methods for lung sounds, the reconstruction methods for photoacoustic imaging and gives an overview of modal array processing techniques for circular and spherical arrays.
Chapter 3 introduces our proposed methods for lung sound localization and provides analysis of the performance of these methods with respect to spatial aliasing and resolution. The effectiveness of the proposed method is shown by simulations.

Chapter 4 provides a description of our method for frequency invariant distributed source localization for PAT using a circular array. Spatial aliasing analysis of this method is also provided. This method is extended for arbitrary array geometries as well as extended to use a minimum number of data samples.

Chapter 5 extends the ideas presented in the previous chapter for 2D photoacoustic source reconstruction, to 3D with a spherical acquisition geometry. Analysis in terms of speed of reconstruction and spatial aliasing is again provided.

Chapter 6 develops a Total Variation (TV) minimization and a Projection onto Convex Sets (POCS) method in order to enhance the source reconstruction methods proposed in Chapters 4 and 5.

Chapter 7 concludes the thesis and provides a discussion on the future work areas related to the research performed in this thesis.

Appendix A provides Singular Value Decomposition (SVD) based method for localizing heart sound interferences from recordings on the chest. This is necessary to improve the lung sound localization methods presented in Chapter 3.

A block diagram showing the major areas of the thesis is shown in Fig. 1.4.

### 1.3.2 Contributions of Thesis

In this subsection, a summary of the major contributions of this thesis are provided.

**Chapter 3** proposes two new methods for source localization in the interior of sensor arrays based on cylindrical harmonic expansions of the wavefield, specifically for lung sound localization.

The first method works on the principle of separating the contribution from different modes (basis functions) of the wavefield and then removing the sensor contribution terms. This creates a unified representation for the wavefield due
Figure 1.4: Block diagram showing the structure and relationships between the major thesis contribution areas.
to the same sources but sensors placed at different positions. Then Minimum Variance (MV) spectral search is applied, providing peaks at locations of the sources.

The second method uses a least squares method to remove the sensor contributions and then uses the MV spectrum for locating sources. Analysis of the stability and the factors that affect the inversion process for the least squares and the orthogonality method is provided. The performance of both these methods for different noise levels and for different frequencies of sound is investigated by simulations.

An analytic relationship between the resolution of the proposed source localization methods and the noise level is derived. Given constraints on resolution, frequency range and noise power, we give the relationships that must be satisfied to design a sensor system for lung sound localization.

We prove that the Nyquist’s criteria for localizing sources within a sensor area is different when compared to linear arrays used for farfield Direction of Arrival (DOA) estimation. This spatial aliasing scenario results in reduction of the area of the region for which sources can be localized as the frequency increases, with the number of sensors remaining constant.

The lung sound localization algorithms can be improved by first localizing interfering heart sounds in the recordings. In Appendix A, we provide a method that uses the property that heart sounds are quasiperiodic. An SVD based method to localize heart sounds is proposed in the frequency domain. Using the Short Time Fourier Transform (STFT), the quasiperiodicity property is preserved for variations of frequency bins over the frame index. First the fundamental frequency is estimated before the heart sounds are localized in the chest recordings.

In Chapter 4, we introduce a new theory for photoacoustic inversion. For 2D circular array of sensors if the source distribution is expanded with a suitable orthogonal series, then by separating the modes in the wavefield decomposition by array weighting and summing at specifically chosen frequencies, information from the different basis functions in the source distribution can be separated.

From this idea, a new method of photoacoustic source reconstruction is introduced for a circular acquisition geometry which expands the source distribution function using a 2D Fourier Bessel series. These Fourier Bessel coefficients are estimated by processing frequencies corresponding to Bessel zeros. This method processes a lower number of frequencies for each basis function of the wavefield decomposition, hence are faster than previously proposed Fourier series methods.
Further, sparsity of the source distribution in the Fourier Bessel domain can be exploited to make reconstruction faster. Conditions for avoiding spatial aliasing are also derived. Moreover, numerical experiments are conducted to investigate the performance of the proposed method.

The idea introduced in the first part of the chapter is utilized for photoacoustic inversion concerning arbitrary sensor arrays, in the second part of this chapter. This method used a robust least squares method to estimate the Fourier Bessel coefficients at frequencies corresponding to the Bessel zeros. Numerical experiments are performed to investigate the performance of this method and quality of reconstruction is compared with reconstruction using a circular acquisition geometry.

The third part of this chapter investigates the estimation of the Fourier Bessel coefficients with a circular array using a minimum number of frequency samples. This reduces the data to be processed. Furthermore, a method on how to choose these frequency samples is also presented. Numerical experiments are conducted and compares the source reconstructed by processing a minimum number of frequency samples and that produced by the Fourier series method [70].

Chapter 5 The theory presented in Chapter 4, of using a series function expansion of the source distribution, is extended to 3D source reconstruction using a spherical acquisition geometry. The 3D source distribution is expanded with a spherical Fourier Bessel series. The coefficients of this expansion are then estimated by processing only the frequencies corresponding to the spherical Bessel zeros.

Moreover, we show by computational order analysis that our proposed method is faster than two well known methods namely the time domain backprojection method [41] and the Fourier series method proposed in [28]. The reconstruction quality and speed of our proposed method is compared to these two methods using numerical experiments.

Chapter 6 deals with convex optimization methods for enhancing photoacoustic inversion. Total Variation (TV) minimization and Projection Onto Convex Sets (POCS) methods have been used for improving the biomedical image reconstruction for other modalities. We propose a novel, subgradient based method for TV minimization using the framework that the proposed methods in Chapters 4 and 5 provides projection of the source distribution onto a set of basis functions. The proposed TV minimization method is adapted to work with these projections.

A POCS or alternating projections method is used to reconstruct a source
distribution that satisfies both the projection onto the basis functions as well as the constraint that the source distribution is a non-negative function. This chapter provides a method to allow simple computationally efficient projections onto these two sets.

The numerical experiments prove that the quality of images is improved using these two postprocessing methods (TV minimization and POCS). Furthermore, the TV minimization result removes ripple artifacts that are present after POCS postprocessing. The numerical experiments show that the TV minimization method produces better images when compared to POCS.

References


Chapter 2

Background: Biomedical Applications and Modal Array Signal Processing

2.1 Introductions

This chapter provides a literature review for this thesis. Firstly, we provide a brief discussion on the acoustic properties of the chest and the methods that have been proposed by other researchers for lung sound localization. The next section looks at the physics of the photoacoustic effect together with inversion procedures for photoacoustic tomography. The last section describes modal array signal processing techniques that have been applied to circular and spherical arrays. We only consider these two sensor geometries since these are most relevant to the work done in this thesis.

2.2 Lung Sound Localization

Firstly, we present a brief overview of sound transmission in the thorax and then move onto discussions about the methods that have been applied for lung sound localization.

2.2.1 Acoustic Properties of the Thorax

We can think of the transmission of sound in the chest as comprising of two parts: firstly, the transmission through the bronchi which is similar to transmission in
the air and secondly coupling to and transmission to the chest surface through 
the parenchyma, rib cage, fat layer and muscle layer. There have been several 
studies looking at both these transmission modes, for an in depth literature see 
Royston et al. [1].

The second part of transmission was investigated by several researchers [1-3] 
where simplified geometries were assumed and homogenized material properties 
were proven. Research by Wodicka et. al. [2] assumed a cylindrical geometry 
where the outer tissue of the thorax was treated as load on the parenchyma. 
Moreover, a similar research done by Vovk et. al. [3] included annular regions for 
the different tissue layers in the chest. In [1], the chest propagation model was 
investigated with finite element simulations. It was found in the literature, that 
for frequencies higher than 100 Hz and ignoring large bronchial segments, the 
propagation of sound originating inside the lung and traveling to the chest surface 
can be thought of as propagation through a homogeneous isotropic material.

The speed of sound for different tissues found in the chest are different. Sound 
travels through the chest wall and heart at about 1500 m/s, speed in the lung 
parenchyma is about 23 to 63 m/s and speed in the bronchi range from 222 to 
312 m/s [4]. The average speed of this propagation through the different layers 
of the chest varies between 25 to 75 m/s [5].

The frequency spectra of normal lung sounds show that lung sounds have 
frequencies from 50 to 1000 Hz [6]. Adventitious lung sounds such as crackles, 
strides and wheezes which are indicators of lung disorders or diseases have very 
high frequencies near 2000 Hz [6].

2.2.2 Lung Sound Localization Methods

Murphy proposed a lung sound localization system consisting of a backpack with 
16 stethoscopes for simultaneous recordings of lung sounds from patients which 
can later be analyzed by computer signal processing methods [7]. He improved 
his system to a stethoscope jacket which used a normal time delay method to 
locate separated adventitious sounds, producing a 3D image to aid diagnosis of 
lung diseases. His algorithm depended on methods to first separate adventitious 
sounds before applying the localizing operation. The locations were therefore 
affected by the way sounds were separated in the lung sound measurements.

A work done by Kompis used a triangulation method [4]. Kompis tested his 
method with an experimental phantom, and on four healthy subjects and one pa-
tient with pneumonia. His study showed the benefits of acoustic source localiza-
2.2 Lung Sound Localization

tion for lung sounds by detecting different patterns between healthy subjects and one patient with pneumonia. The geometry of the lung, the acoustic speed and wavelengths were considered for his model of sound propagation. However, he used the concept that the chest can be considered to be homogeneous medium for high frequencies and did not consider refraction, reflection and standing waves. Further, the data collected by the 3D array was modeled to satisfy

\[ s_j \left( t - \frac{\|x_j - y\|}{c} \right) = d^{\|x_j - y\|} \times \frac{r(y, t)}{\|x_j - y\|^2} \]  

(2.1)

where \( x_j \) is the position of the \( j^{th} \) microphone, \( s_j \) is the recorded signal and \( y \) is the position of the acoustic source. Moreover, \( c \) is the speed of sound waves, \( r(y, t) \) is the signal emitted by the source, \( d \) is the damping factor per unit length and \( \| \cdot \| \) is the Euclidean norm. If there are \( J \) microphones, we need to solve \( J \) equations for determining \( r(y, t) \) and the source location \( y \). However, triangulation methods are extremely sensitive to noise and do not work for multiple sources. Kompis’ method takes into account the time delay and the attenuation experienced at each of the microphones. Further, the experimental tests showed different results when using 8 microphones compared to that obtained for 16 microphones. Kompis concluded that 16 microphones were necessary for accurate 3D imaging of the lung but no theoretical verification of this statement was given.

Methods proposed after the results of Kompis’ work [4] showed the significance of verifying the performance of the localization methods since it is important to know when the results are valid. The two methods proposed in [8] considered the difficulties present for nearfield source localization in the interior of sensor arrays. This paper highlighted a performance measure called the \textit{beam focus} which they verified experimentally as being dependent on frequency, array radius, speed of sound and location of the source [9]. This beam shape was defined as the relative power in all areas when beamforming was applied to a specific position. The first method proposed in [8] divides the area within the sensor array into grids and the relative power for each grid position is calculated using a delay and sum beamformer. The source position is determining to be the grid with the highest relative power. The second method proposed calculates the time delay for the source by cross correlating signals from different microphones. Using the delay information, a system of equations are constructed and solved for the source location. The work [8] tested these methods on a phantom for localization of a single source. However, the experiments performed did not consider
localization of multiple sources. We observe that the performance evaluation became important for designing lung sound localization methods. Further, no theoretical ideas have developed considering design of a microphone arrays with respect to these performance measures such as resolution and spatial aliasing.

Ozer et. al. developed a matched field processing method for lung sound localization [10]. They first developed a lung sound propagation model using boundary element analysis considering the lung geometry and the different types of tissue. Their sound propagation model considered reflections, refractions and standing waves. The matched field Bartlett processor they proposed was computationally expensive and could only be used for locating a single acoustic source. Further, the performance in terms of resolution and spatial aliasing was not investigated.

2.3 Photoacoustic Tomography

The photoacoustic effect is the generation of acoustic waves caused by thermal expansion due to absorption of electromagnetic (EM) energy [11]. In photoacoustic imaging, the aim is to reconstruct the initial pressure distribution from pressure measurements taken at positions surrounding the sample under test. In Photoacoustic Tomography (PAT) the objects are excited with a laser pulse, this may also be called Optoacoustic Tomography (OAT). If however, the excitation is done with radio waves then this is referred to as Thermoacoustic Tomography (TAT).

There are many methods of tomography applied to medical imaging and these include X-rays, magnetic resonance, ultrasound, electrical impedance and many others [12]. These imaging modalities must offer high contrast and resolution while at the same time have low costs and non-harmful effects. Reducing the undesired properties while increasing the desired effects are often contradictory or impossible. Most modalities have one strength. For example, ultrasound imaging has high resolution but low contrast while electrical impedance tomography has good contrast but low resolution. In recent years, researchers looked at combining the modalities to produce hybrid tomographic systems that retain the individual strengths but alleviate the individual deficiencies. One of the most popular and successful of such a hybrid imaging modality is PAT which combines the high resolution of ultrasound imaging with the high contrast of optical imaging modalities [13].

The next part of this section provides an overview of the underlying physics
of PAT. In Section 2.3.2 we present the safety issues involved in photoacoustic imaging and in Section 2.3.3, we describe sound propagation in biological tissues and how the produced pressure signals can be measured. Sections 2.3.4 and 2.3.5 describe two ways inversion can be done. This thesis is concerned with photoacoustic inversion with computed tomography and previously derived tomographic methods are covered in Section 2.3.6. Section 2.3.7 mentions other factors that need to be considered for photoacoustic imaging.

### 2.3.1 Wave Equations

This section provides a brief overview of photoacoustic imaging together with the wave equations associated with it. Photoacoustic imaging uses pressure waves generated by the absorption of a laser pulse. It is generally assumed that the incident electromagnetic (EM) wave is absorbed instantaneously within the targeted volume of tissue. This condition can be met when the incident laser pulse duration is short such that the temporal illumination function can be approximated by a Dirac delta function $\delta(t)$. Assuming constant speed of propagation $c$ in the medium, the pressure $p(r, t)$ at a vector position $r$ and time $t$ in a lossless, linear medium is described by the inhomogeneous Helmholtz’s equation [14,15]

$$\frac{\partial^2 p(r, t)}{\partial t^2} - c^2 \cdot \nabla^2 p(r, t) = p_0(r) \frac{\partial}{\partial t} \delta(t)$$

(2.2)

where the time derivative of the impulsive pressure distribution $p_0(r)\delta(t)$ provides the driving force and $\nabla^2$ is the Laplacian operator. The initial, spatial distribution of pressure is denoted by $p_0(r)$, however, this will also be called the source or spatial distribution in this thesis.

The spatial distribution is linearly related to the distribution of the EM absorption coefficient $\mu(r)$ by

$$p_0(r) = \Gamma \Psi(r) \mu(r)$$

(2.3)

where $\Psi(r)$ is the optical fluence distribution and $\Gamma$ is the dimensionless Grüneisen coefficient which is defined by $\Gamma = \varrho c^2/C_p$ with $\varrho$ the volume expansion sensitivity and $C_p$ the isobaric specific heat capacity. By solving for $p_0(r)$, differences in the EM absorption property can be observed in the target volume. This can identify different types of tissues or cancerous regions which have different EM absorption coefficients.

The pressure recorded by an ultrasound transducer in the time domain and
at a position \( r_s \) can be described by the general solution to the inhomogeneous Helmholtz equation [16] (2.2), which is expressed as

\[
p(r_s, t) = \frac{\partial}{\partial t} \int_V p_0(r) \frac{\delta(t - \| r_s - r \|/c)}{4\pi \| r_s - r \|} \, dr
\]  

(2.4)

where \( \int_V \, dr \) represents integration over a volume of space in \( \mathbb{R}^3 \) and \( \| \cdot \| \) is a vector norm. The time domain solution is used in several reconstruction algorithms [17, 18].

Let's take Fourier transforms with respect to time \( t \) for the measured pressure data \( p(r_s, t) \), then the analysis

\[
p(r_s, k) = \int_{-\infty}^{\infty} p(r_s, t) e^{ikt} \, dt
\]  

(2.5)

and synthesis

\[
p(r_s, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(r_s, k) e^{-ikt} \, dk
\]  

(2.6)

equations for the Fourier transform are utilized where \( i = \sqrt{-1} \), the wavenumber \( k = \omega/c \) with the angular frequency \( \omega \) equal to \( 2\pi f \). \( f \) denotes the frequency and \( p(r_s, k) \) is the frequency spectrum of the time domain pressure signal \( p(r_s, t) \). Applying the Fourier transform to (2.2) yields

\[
(\nabla^2 + k^2)p(r, k) = ikcp_0(r).
\]  

(2.7)

Moreover, the Fourier transform of the measured acoustic wave (2.4) over time \( t \) can be written as

\[
p(r_s, k) = -ikc \int_V p_0(r)G(k; r_s, r) \, dr
\]  

(2.8)

where \( G(\cdot) \) is the Green's function and is defined as the fundamental solution to the following nonhomogeneous, differential equation

\[
(\nabla^2 + k^2)G(k; r_s, r) = -\delta(r_s - r).
\]  

(2.9)

In three dimensions (\( \mathbb{R}^3 \)), the Green's function is expressed as

\[
G(k; r_s, r) = \frac{e^{ik\| r_s - r \|}}{4\pi \| r_s - r \|}.
\]  

(2.10)

Photoacoustic tomography is an inverse source problem that is classified as
Diffraction tomography due to the diffractive property of acoustic waves. Techniques and methods have been borrowed from X-ray and optical tomography for PAT. Additionally, focused sensors and acoustic lenses can be used for direct photoacoustic reconstruction. Further, computed tomographic methods have been applied to PAT using unfocused sensors and requiring computer based calculations to reconstruct the source distribution $p_0(r)$.

2.3.2 Safety Issues

There are safety issues associated with PAT and TAT. Standards exist that limit the Radio Frequency (RF) or laser exposure of humans to safe levels. These standards define the maximum permissible exposure (MPE) as the level of EM radiation a person can be exposed to without harmful effects.

For TAT, the IEEE standard, Std. C95.1, 1999 edition define MPE levels for RF fields for frequencies from 3 kHz to 300 GHz [19]. This MPE can be calculated by $f/3 \times 10^{-8}$ milli Watts (mW)/cm². For PAT, the American National Standard (Z136.1-2000) defines MPE levels for laser wavelengths from 180 nm to 1 mm and for different exposure durations [20]. These standards define formulas to calculate MPE for repetitive exposures considering the wavelength of incident EM radiation, pulse duration, duration of complete exposure and pulse repetition frequency. This section highlighted the safety issues associated with PAT and what limitations this places on the intensity of EM radiation that can be used.

2.3.3 Acoustic Wave Propagation and Detection

The EM induced initial pressure causes outgoing acoustic waves that travel to the tissue surface to be detected by ultrasound transducers. Unlike in ultrasound imaging, the transducers only act as receivers therefore their emission efficiency is not important, but their sensitivity is. These ultrasound detectors are normally piezoelectric based [21]. These piezoelectric based sensors have low thermal noise, high sensitivity and can provide a wide band up to 100 MHz [22]. Other sensors based on optical detection are also available. These either exploit changes in refractive index [23] or photoacoustic based surface displacement [24], therefore are capable of non contact pressure measurements over large areas.

The propagation of acoustic waves in biological tissues can be assumed to be constant at 1540 m/s with small variations of about 10% [21,26]. For most photoacoustic inversion methods the inhomogeneity of the acoustic speed is neglected. If acoustic variation becomes significant, then first ultrasound imaging
can be applied to provide the spatial distribution of acoustic speed before applying photoacoustic inversion. Note that ultrasound wave attenuation due to scattering and absorption are dependent on frequency and temperature. The scattering normally accounts for $10-15\%$ of attenuation. Normally, attenuation increases with frequency. However, the depth limitation for photoacoustic imaging is mainly due to the penetration depth of the light or the EM pulse rather than the acoustic attenuation.

### 2.3.4 Scanning Tomography

Use of a focused transducer to scan along the surface of the tissue for photoacoustic reconstruction is similar to B-mode ultrasonography, see Fig. 2.1 showing the operation of scanning PAT. The images obtained from different transducer positions are combined sequentially thus forming a cross sectional image. For the scanning system, the imaging area is limited by the focal zone of the transducer. Further, the axial resolution is proportional to the width of the EM pulse and width of the transducer impulse response, and the lateral resolution is proportional to the focal diameter and the frequency of the measured acoustic waves. A better Signal-to-Noise Ratio (SNR) is obtained by using a higher intensity EM pulse, but this intensity must be limited to safe levels, see Section 2.3.2. Further safety constraints limit the EM pulse repetition frequency which in turn reduces the scanning speed.
2.3.5 Inversion with Acoustic Lens

Just like an optical lens, acoustic lenses can diverge or converge acoustic waves. Acoustic lenses can be employed to reconstruct the source distribution on a transparent medium in real time from which measurements can be taken without the need for scanning or computer calculations. A system for photoacoustic reconstruction using acoustic lenses was demonstrated in [27]. The image formed contained artifacts due to finite aperture effects and lens aberrations.

2.3.6 Inversion Methods by Computed Tomography

To alleviate the limitations of using focused transducers or acoustic lenses, recent researches have focused on deriving methods for photoacoustic inversion using computed tomographic methods and omnidirectional sensors. Each temporal signal received at a single sensor represents the information about the source at a particular distance relative to the sensor. The temporal and spatial measurements can provide the necessary information required to reconstruct the 3D or 2D source distribution. We also note from literature that the photoacoustic inverse problem is related to the Radon transform. This section begins by describing approximate inversion methods based on the Radon transform and then moves onto exact reconstructions with filtered backprojection methods. Next an overview of the backprojection and time reversal methods are presented, then a series solution is described. Furthermore, Fourier domain reconstruction methods together previously proposed methods for PAT that do not belong in any of the groups mentioned previously are also described.

Approximate Filtered Backprojection Methods

The solution to the partial differential equation (2.2) for PAT can also be given using the spherical mean operator

\[ p(r_s, t) = c \frac{\partial}{\partial t} (t(R p_0) (r_s, t)) \]  

(2.11)

where the spherical mean operator \( R \) also known as the spherical Radon transform is defined as

\[ (R p_0)(r_s, t) = \frac{1}{4\pi} \int_{\|y\|=1} p_0(r_s + t y) \, dS(y) \]  

(2.12)
with \( dS(y) \) denoting the surface area element on the unit sphere with respect to \( y \). This solution is derived using the Poisson-Kirchhoff formulas described in [28]. Therefore, the measured pressure \( p(r_s, t) \) for \( t \geq 0 \) contains the spherical mean of the source distribution about the sensor position \( r_s \). This integral geometric model was used in several studies of PAT. The spherical mean model is not as flexible as the wave model described in Section 2.3.1 since it is not valid when the speed is not constant.

In ideology, the spherical mean viewpoint is similar to the Radon transform used for X-ray computed tomography where averaging is done over straight lines or planes rather than spheres.

The Radon transform is normally inverted by filtered backprojection methods [29]. These methods either filter the signal in the Fourier domain or convolve the signal with a kernel prior to the backprojection step. The filtration step can also be performed after backprojection. For the spherical mean operator, the inversion process is expected to include a filtration step to adjust for attenuation due to the radial position and then sums over spheres passing through the point of interest. In the beginning, this approach did not yield any exact solutions. However, researchers developed several approximate inversion formulas with optional iterative steps for improving the reconstructed source distribution.

A method developed by Popov and Shushko [30] utilized a "straightening formula" for approximating the Radon transform with spherical means. For a plane passing through the source distribution, integrals over this plane were approximated by spherical surfaces that were tangential to this plane. For each point on the plane, a spherical surface containing this point was chosen, note that the values of the spherical means were available from the measured signals \( p(r_s, t) \) (2.11). By approximating integrals over a set of planes, the source distribution \( p_0(r) \) could be reconstructed by inverting a Radon transform. However, this approach only provided an approximation to the source distribution.

Another approach for the spherical acquisition geometry assumed the sample occupied a small volume in the center of the volume enclosed by the array. In such a case, integrals over spherical surfaces can be approximated as integrals over planes. Therefore, the spherical mean operator can be thought of as an approximation to the standard Radon transform. Numerical inversion methods for approximating the Radon transform in 2D (circular arrays) [31] and 3D (spherical arrays) [32] were conducted by Andreev et. al. Further, using this approach Kruger et. al. proposed a formula for recovering the source distribution with a
spherical observation surface \( S \) as \([33]\)

\[
p_0(r) \approx -\frac{1}{2\pi} \int_S \left[ \frac{t}{r_s^2} \frac{\partial p(r_s, t)}{\partial t} + 2p(r_s, t) \right] \frac{dS(r_s)}{t = ||r - r_s||/c} \tag{2.13}
\]

where \( r_s \) is the radius where all the sensors are placed. A similar formula was derived by Liu et. al. using what he called a “p-transform” \([17]\). The Hilbert transform was also utilized for approximating the Radon transform in \([34]\). Investigations by these researchers showed that their methods allowed good construction in the central regions with significant artifacts in other areas where the spherical means do not approximate integrals over planes.

**Exact Filtered Backprojection Methods**

For an observation surface \( S \), the spherical integrals \( g(r_s, \hat{r}) \) are known for centers on this surface (the sensors are placed on this surface, i.e., \( r_s \in S \)) where \( g(r_s, \hat{r}) \) is defined as

\[
g(r_s, \hat{r}) \triangleq \int_{||y||=1} \hat{r}^2 p_0(r_s + \hat{r} y) dS(y) = 4\pi \hat{r}^2 (R p_0)(r_s, \hat{r}). \tag{2.14}
\]

From the approximate solution, several exact filtered backprojection formulas emerged with the aim of inverting spherical integrals. These formulas differed in the order in which filtration and backprojection were performed.

Exact filtered backprojection formulas were developed for spherical surfaces \( S \) and with source distributions \( p_0(r) \) contained within this surface. These methods were local requiring only knowledge of the spherical integrals passing through points close to the point to be reconstructed. The benefits of these exact methods became clear when simple discrete implementations provided stable reconstructions.

For the spherical surface in 3D \( \mathbb{R}^3 \) a set of filtered backprojection formulas were derived in \([35]\). These formulas are as follows

\[
p_0(r) = -\frac{1}{8\pi^2} \nabla^2 \int_S \frac{g(r_s, ||r_s - r||)}{||r_s - r||} dS(r_s) \tag{2.15}
\]

and

\[
p_0(r) = -\frac{1}{8\pi^2} \int_S \left( \frac{1}{t} \frac{d^2}{dt^2} g(r_s, t) \right)_{t = ||r_s - r||} dS(r_s) \tag{2.16}
\]

where \( \nabla^2 \) is the Laplacian over the variable \( r \). Filtered backprojection methods were developed first for odd dimensions (e.g. \( \mathbb{R}^3 \)) and then for even dimensions.
(e.g. $\mathbb{R}^2$) [36]. A general solution applicable in all dimensions and stated for $\mathbb{R}^3$ is [37]

$$p_0(r) = \frac{1}{8\pi^2} \text{div} \int_S n(r_s) \left( \frac{1}{t} \frac{d}{dt} g(r_s, t) \right)|_{t=\|r_s - r\|} \, dS(r_s) \quad (2.17)$$

where div is the divergence operator for a vector field and $n(r_s)$ is the normal vector to the surface $S$ at $r_s$. The reconstructions obtained by applying (2.16) is not equivalent to applying (2.17) since these formulas have different effects on the noise present in the measurements. Further, the formulas presented provide significant artifacts when there are sources outside the surface $S$ (sensor array).

Another exact filtered backprojection method for the spherical acquisition geometry looking at the inverse of the spherical integral, first integrated the measured pressure over time

$$\Psi(r_s, t) = t \int_0^t p(r_s, t) \, dt \quad (2.18)$$

and from $\Psi(r_s, t)$ the source distribution was reconstructed by [35]

$$p_0(r) = \frac{1}{2\pi r_s} \nabla^2 \int_S \frac{\Psi(r_s, t = \|r_s - r\|)}{\|r_s - r\|} \, dS(r_s). \quad (2.19)$$

This method first backprojects the modified data (2.18) into the image space and then applies filtration by utilizing the Laplacian $\nabla^2$.

The filtered backprojection methods for 3D mentioned in this section all have $O(n^5)$ operational complexity for reconstructing a source distribution on an $n \times n \times n$ grid. Here, the number of sensors are proportional to $n^2$ and each sensor contains spherical integrals of the order $O(n)$.

For 2D ($\mathbb{R}^2$), the methods look at inverting circular integrals formulas. For performing exact filtered backprojection in 2D with $S$ the circumference of a circle are as follows [36]

$$p_0(r) = \frac{1}{4\pi r_s} \nabla^2 \int_S \int_0^{2r_s} g(r_s, t) \log |t^2 - \|r_s - r\|^2| \, dt \, dS(r_s) \quad (2.20)$$

and

$$p_0(r) = \frac{1}{4\pi r_s} \nabla^2 \int_S \int_0^{2r_s} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \frac{g(r_s, t)}{t} \right) \log |t^2 - \|r_s - r\|^2| \, dt \, dS(r_s). \quad (2.21)$$

Another formula was also derived using the Bessel $J_0(\cdot)$ and Neumann $Y_0(\cdot)$
functions of the zeroth order [37]

\[ p_0(r) = \frac{-1}{8\pi} \text{div} \int_S \mathbf{n}(r_s) h(r_s, ||r_s - r||) \, dS(r_s) \] (2.22)

where

\[ h(r_s, t) = \int_{\mathbb{R}^+} Y_0(\lambda t) \int_0^{2\pi} J_0(\lambda t') g(r_s, t') \, dt' - J_0(\lambda t) \int_0^{2\pi} Y_0(\lambda t') g(r_s, t') \, dt' \, \lambda d\lambda. \] (2.23)

The filtration step in this formula is similar to the Hilbert transform, in fact, the implementation of this method uses two Hilbert transforms. The exact filtered backprojection methods is similar to those used for inverting the 2D Radon Transform and is non local, meaning that all the values of spherical integrals \( g(r_s, t) \) needs to be known to reconstruct the source distribution at a particular point. This property is different to the local property of the 3D filtered backprojection methods. The computational order for these 2D filtered backprojection methods is \( O(n^3) \) for reconstructing a source distribution on an \( n \times n \) grid. In this case, the number of sensors are proportional to \( n \). Further each sensor recording includes \( n \) circular integrals. Note that in the filtered backprojection formulas, we have assumed that the sound velocity \( c = 1 \).

**Backprojection Methods**

We start by describing a “universal” backprojection formula derived by Xu and Wang [39]

\[ p_0(r) = \frac{2}{\Omega_0} \text{div} \int_S \mathbf{n}(r_s) \left[ \frac{p(r_s, \bar{t})}{\bar{t}} \right]_{\bar{t}=||r_s-r||} \, dS(r_s) \] (2.24)

where \( \mathbf{n}(r_s) \) is the normal vector of the observation surface \( S \) at \( r_s \) pointing towards the source, \( \bar{t} = ct \) and \( \Omega_0 \) is the solid angle for the observation surface.

This formula can be applied to all the regular observation surfaces, that is planar, cylindrical and spherical observation surfaces. For the planar geometry, the solid angle \( \Omega_0 \) is equal to \( 2\pi \) whereas for both the cylindrical and spherical surfaces the solid angle \( \Omega_0 \) is \( 4\pi \). Taking the backprojection term as

\[ b(r_s, \bar{t}) = 2p(r_s, \bar{t}) - 2\bar{t} \frac{\partial p(r_s, \bar{t})}{\partial \bar{t}} \] (2.25)
then (2.24) can be written in the backprojection form as

$$p_0(r) = \frac{1}{\Omega_0} \int_S b(r_s, \bar{t} = ||r_s - r||) \, dS(r_s)$$

(2.26)

where $dS(r_s) = dS(r_s)/||r_s - r||^2 \cdot (n(r_s) \cdot (r_s - r))/||r_s - r||$. The term $\bar{t} = ||r_s - r||$ compensates for the acoustic attenuation which is proportional to the distance the wave travels from the source to the sensor. Additionally, the factor $n(r_s) \cdot (r_s - r)/||r_s - r||$ is the angle between the normal vector and the direction vector from the source to the sensor.

This backprojection formula was derived from the Fourier series method introduced by Norton and Linzer [40]. Moreover, the exact filtered backprojection formula proposed by Finch et. al. (2.19) [35] is similar to this backprojection formula for the spherical geometry.

The “universal” formulas was simplified to provide a modified, approximate backprojection formula [18] when the distance between the source and the sensor are much greater than the wavelength ($|r_s - r| \gg 1$), $2p(r_s, \bar{t})/\bar{t}$ is negligible and $n(r_s) \cdot (r_s - r)/||r_s - r||$ can be approximated by $n(r_s)\bar{t}/||r_s||$. These assumptions are similar to those made for the approximate filtered backprojection methods and means that the source occupies the central region $||r|| \ll ||r_s||$. This modified backprojection formula is given as follows [18]

$$p_0(r) \approx -\frac{1}{2\pi} \int_S \frac{n(r_s)(-n_0(r_s))}{||r_s - r||^2} \times \left[ \frac{-\partial p(r_s, \bar{t})}{\partial \bar{t}} \right]_{\bar{t}=||r_s - r||} \, dS(r_s)$$

(2.27)

where $n_0(r_s) = r_s/||r_s||$. The backprojection method uses coherent summation over spherical surfaces with spatial weighting factors to compensate for attenuation. Also, for PAT, the derivative with respect to time of the measured pressure is used in the sums rather than just the pressure.

Other backprojection type formula were also proposed. Hoelen et. al. [41] used the time domain delay and sum beamformer applicable for a planar array for photoacoustic inversion to image blood vessels. Synthetic aperture methods were borrowed from ultrasonic imaging to provide a backprojection formula for a linear scanning, thermoacoustic system [42]. Liao et. al. also applied the synthetic aperture approach with coherent summing to solve the photoacoustic inversion problem in [43]. For the 2D inversion problem, a backprojection formula was derived in [44]. The computational complexity of the backprojection methods are similar to the exact filtered backprojection methods discussed in the previous
section.

**Time Reversal Methods**

The time reversal method provides fast reconstruction with flexibility to be adopted to any observation surface $S$ in $\mathbb{R}^3$. A description of the implementation of a time reversal method for 3D photoacoustic inversion was presented in [45]. The operation count for a time reversal method implemented numerically with the finite difference algorithm is $\mathcal{O}(n^4)$. Here, the source distribution $p_0(r)$ is reconstructed on an $n \times n \times n$ grid.

This method works with the assumption that the source occupies a region $\Omega$ completely enclosed by the measurement surface $S$ and therefore at a finite time the wave has completely left this region. For PAT with waves due to the initial pressure distribution $p_0(r)$, there will be a time $T$ when the wave has completely left the volume $\Omega$. Reconstruction of $p_0(r)$ can be done by solving the wave equation back in time with zero initial conditions at time $T$ and considering the boundary conditions. Thus, the source distribution $p_0(r)$ is the wavefield at time $t = 0$.

The time reversal method works well in odd dimensions ($\mathbb{R}^n$ with $n$ being an odd integer) where Huygen’s principle is valid. But in even dimensions ($\mathbb{R}^n$ with $n$ being an even integer), for example in 2D, Huygen’s principle is no longer valid, therefore, model based approaches for approximating time reversal methods were proposed in [46,47]. For 2D, this method requires $\mathcal{O}(n^3)$ operations.

**Series Solution**

The series solution introduced in [48] expands the source distribution with an orthonormal basis, that is

$$p_0(r) = \sum_{m=0}^{\infty} a_m u_m(r). \quad (2.28)$$

where $a_m$ are the Fourier coefficients and $u_m(r)$ are the orthonormal functions. The orthonormal functions are chosen to be the eigen functions of the Dirichlet Laplacian for the region $\Omega$ bounded by the acquisition surface (sensor arrays) and thus satisfy

$$\nabla^2 u_m(r) + \lambda_m u_m(r) = 0, \quad r \in \Omega, \quad (2.29)$$
the Dirichlet boundary condition

\[ u_m(r_s) = 0, \quad r_s \in S \quad (2.30) \]

and has unit magnitude

\[ \|u_m(r)\|^2 = \int_{\Omega} |u_m(r)|^2 \, dr = 1 \quad (2.31) \]

with \( \lambda_m \) being the corresponding eigenvalues.

Just like the filtered backprojection methods, this series solution aims to reconstruct the source distribution \( p_0(r) \) given that the spherical integrals \( g(r_s, t) \) are known. The source distribution is reconstructed by first obtaining the Fourier coefficients

\[ a_m = \int_{S} I(r_s, \lambda_m) \frac{\partial}{\partial n} u_m(r_s) \, dS(r_s) \quad (2.32) \]

where

\[ I(r_s, \lambda_m) = \int_{\mathbb{R}^+} g(r_s, t) G(\lambda_m, t) \, dt, \quad (2.33) \]

\( \partial/(\partial n) \) is the normal derivative with respect to surface \( S \) and

\[ G(\lambda_m, t) = \cos(\lambda_m t)/t \]

is the real Green’s function. This series solution is applicable to acquisition geometries whose eigenfunctions are known or can be numerically calculated.

For a cubic acquisition surface, with length \( R \), the eigen functions are

\[ u_m = \frac{8}{R^3} \sin \frac{\pi m_1 x}{R} \sin \frac{\pi m_2 y}{R} \sin \frac{\pi m_3 z}{R} \quad (2.34) \]

and eigenvalue \( \lambda_m = \pi|m|^2/R^2 \) where \( x, y \) and \( z \) represents the normal rectangular co-ordinates and \( \mathbf{m} = (m_1, m_2, m_3) \) with \( m_1, m_2, m_3 \in \mathbb{N} \). For this geometry, \( I(r_s, \lambda_m) \) can be calculated by the Fast Cosine Transform and the Fast Sine Transform can be employed to obtain \( a_m \) and then to reconstruct the source distribution \( p_0(r) \). This method is extremely fast for the cubic geometry and for reconstruction of the source distribution \( p_0(r) \) on an \( n \times n \times n \) grid requires \( \mathcal{O}(n^3 \log n) \) floating point operations. However, its speed or numerical feasibility for other geometries is not known.
Fourier Domain Methods

Norton and Linzer proposed a solution for 3D photoacoustic inversion for the planar, cylindrical and spherical observation surfaces based on the harmonic decomposition of the measured signals and the source distribution and then equating the corresponding coefficients of the Fourier series \[40\]. This method is referred to as the Fourier series method in the photoacoustic or thermoacoustic literature. They observed that the mapping of the source distribution \( p_0(\mathbf{r}) \) to the measured pressure \( p(\mathbf{r}_s, k) \) shown in (2.8) is linear. Therefore, the inversion is expected to be a linear integral with the following structure

\[
p_0(\mathbf{r}) = \int_S \int_k p(\mathbf{r}_s, k) K_k(\mathbf{r}_s, \mathbf{r}) \, dk \, dS(\mathbf{r}_s)
\]  

(2.35)

where \( k \) is the wavenumber and the integral kernel \( K_k(\mathbf{r}_s, \mathbf{r}) \) for a spherical observation surface is

\[
K_k(\mathbf{r}_s, \mathbf{r}) = \frac{1}{2\pi^2 r_s^2} \sum_{m=-\infty}^{\infty} \frac{(2m + 1) j_m(kr)}{h_m^{(1)}(kr_s)} P_m \left( \frac{\mathbf{r}_s \cdot \mathbf{r}}{\|\mathbf{r}_s\|\|\mathbf{r}\|} \right)
\]  

(2.36)

where \( P_m(\cdot) \) is the Legendre polynomial, \( j_m(\cdot) \) and \( h_m^{(1)}(\cdot) \) is the spherical Bessel function of the first kind and the spherical Hankel function of the first kind, respectively, and \( m \) represents the mode. The operation count for the Fourier series method is \( \mathcal{O}(n^6) \) for reconstructing a source distribution on an \( n \times n \times n \) grid. This is slower than both the filtered backprojection and the backprojection methods.

Norton also provided a Fourier series solution for 2D circular arrays \[49\]. We provide an in depth derivation of this solution. First the measured data and the source distribution are decomposed with the Fourier series in the angular variable as follows:

\[
p_0(\mathbf{r}) = \sum_{m=-\infty}^{\infty} p_{0,m}(r)e^{in\phi}
\]  

(2.37)

for a polar co-ordinate system \( \mathbf{r} \equiv (r, \phi) \) where \( r \) is the radial and \( \phi \) is the angular co-ordinates, points on the observation surface are \( \mathbf{r}_s \equiv (r_s, \phi_s) \), \( p_{0,m}(r) \) is the Fourier coefficients of the source distribution \( p_0(\mathbf{r}) \) and

\[
p(\mathbf{r}_s, k) = \sum_{m=-\infty}^{\infty} p_m(k)e^{im\phi_s}
\]  

(2.38)
where $p_m(k)$ is the Fourier coefficient of the pressure measured over the observation surface. We simplify the inverse problem for PAT ignoring the constants before the integral and for 2D, using the real form of the 2D Green’s function $J_0(k\|\mathbf{r}_s - \mathbf{r}\|)$, the inverse problem is formulated as

$$p(\mathbf{r}_s, k) = \int_S p_0(\mathbf{r}) J_0(k\|\mathbf{r}_s - \mathbf{r}\|) \, d\hat{S}$$  \hspace{1cm} (2.39)$$

where $\int_S \cdot d\hat{S}$ is an integration over the area occupied by the source distribution $p_0(\mathbf{r})$ within the observation surface $S$. For a circular observation surface with the sensors all placed at radius $r_s$, the inverse problem is

$$p(\mathbf{r}_s, k) = \int_0^{2\pi} \int_0^\infty p_0(\mathbf{r}) J_0(k\|\mathbf{r}_s - \mathbf{r}\|) \, r \, dr \, d\phi$$  \hspace{1cm} (2.40)$$

Substituting the addition theorem of Bessel functions

$$J_0(k\|\mathbf{r}_s - \mathbf{r}\|) = \sum_{m=-\infty}^{\infty} J_m(kr_s) J_m(kr) e^{im\phi} e^{-im\phi}$$  \hspace{1cm} (2.41)$$

into (2.40) and simplifying yields

$$p(\mathbf{r}_s, k) = \sum_{m=-\infty}^{\infty} J_m(kr_s) \mathcal{H}_m(k; p_{0,m}(r)) e^{im\phi}$$  \hspace{1cm} (2.42)$$

where the Hankel transform $\mathcal{H}_m(\cdot)$ of order $m$ is defined as

$$\mathcal{H}_m(k; f(r)) \triangleq \int_0^\infty f(r) J_m(kr) \, rdr.$$  \hspace{1cm} (2.43)$$

Therefore, from (2.38) and this definition of the Hankel transform, we get

$$\frac{p_m(k)}{J_m(kr_s)} = \mathcal{H}_m(k; p_{0,m}(r)).$$  \hspace{1cm} (2.44)$$

Since the Hankel transforms are self invertible, that is

$$f(r) = \int_0^\infty \mathcal{H}_m(k; f(r)) J_m(kr) \, kdk$$  \hspace{1cm} (2.45)$$

then the Fourier coefficients $p_{0,m}(r)$ can be obtained by the following equation

$$p_{0,m}(r) = \mathcal{H}_m \left( r; \frac{p_m(k)}{J_m(kr_s)} \right)$$  \hspace{1cm} (2.46)$$
The source distribution \( p_0(r) \) can now be reconstructed by summing the Fourier series (2.37). The practical implementation of this solution has instabilities due to the division by Bessel functions which have an infinite number of zeros. This is minimized by regularization and the fact that the presence of noise ensures exact cancelation is unlikely.

The 2D version of the 3D solution (2.35) [40] avoids this instability by utilizing the complex form of the 2D Green’ function \( H_0^{(1)}(k\|r_s - r\|) \) in photoacoustic inverse problem

\[
p(r_s,k) = \int_0^{2\pi} \int_0^\infty p_0(r)H_0^{(1)}(k\|r_s - r\|) rdr d\phi.
\]

(2.47)

Now, substituting the addition theorem for the Hankel functions

\[
H_0^{(1)}(k\|r_s - r\|) = \sum_{m=-\infty}^{\infty} J_m(kr_s)H_m^{(1)}(kr_s)e^{im(\phi_1 - \phi_2)}
\]

(2.48)

into (2.47) and solving for the Fourier coefficients of the source distribution \( p_0(r) \) yields

\[
p_{0,m}(r) = H_m \left( r; \frac{p_m(k)}{H_m^{(1)}(kr_s)} \right)
\]

(2.49)

For the spherical observation surface (3D source distribution \( p_0(r) \)), the exponentials are replaced by spherical harmonics and the Bessel and Hankel functions are replaced by the spherical Bessel and spherical Hankel functions, respectively. This leads to the solution provided by (2.35) and (2.36) [40]. The operation count for the 2D Fourier series method is \( O(n^3) \) for reconstruction on an \( n \times n \) grid.

**Other Methods**

This section describes methods that have been proposed for PAT but do not fit in the categories mentioned previously. Photoacoustic inversion can be set up as a linear matrix equation by discretizing either (2.4) or (2.8). This matrix equation is

\[
D = \hat{M}P_0
\]

(2.50)

where \( \hat{M} \) is the sensitivity matrix mapping the source distribution matrix \( P_0 \) to the matrix containing the measured data \( D \). Numerical method for linear, matrix equations can be applied to solve for \( P_0 \). An iterative least squares method
for PAT was thus proposed in [50]. Another iterative method was proposed by Xu and Wang and used the truncated conjugate gradient method to solve this matrix equation [34]. Moreover, statistical methods were applied for photoacoustic inversion. One such method transformed the measured data from the time domain to spatial co-ordinates for each sensor and then summed these with a filtration step [51]. A weighted expectation maximization algorithm was proposed by Zhang et. al. [52] and was shown to be effective in mitigating artifacts due to temporal truncation of the measured pressure signals. These statistical and numerical methods are extremely slow requiring a large amount of memory for storing large matrices.

2.3.7 Discussion

This section gave an overview of photoacoustic imaging together with background on safety issues, acoustic wave detection and propagation, and reconstruction using focused sensors and acoustic lenses. Moreover, we looked at the various inversion methods for PAT using computed tomography which are particularly important and related to the research presented in this thesis. A comparison between the different reconstruction methods for photoacoustic imaging is shown in Table 2.1.

There are other factors to consider when performing photoacoustic inversion and these are as follows: (i) the sensors may not be able to fully enclose the object under study and is referred to as the limited view problem, (ii) inhomogeneity of acoustic speed, (iii) blurring due to the finite size of sensor width and (iv) attenuation effects. The research in this thesis deals with photoacoustic inversion considering constant speed of sound, point detectors and full view acquisition. For research dealing with the other highlighted issues, the reader is referred to [13,53].

The next section provides a brief overview on modal array signal processing. Since the work in this thesis is concerned with circular and spherical arrays, we limit the modal array signal processing literature to these two array geometries.
Table 2.1: Comparison of the different computed tomography methods applied to photoacoustic imaging.

<table>
<thead>
<tr>
<th>Reconstruction Method</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximate Filtered Backprojection</td>
<td>Approximate reconstruction, providing good reconstruction only in the central region.</td>
<td></td>
</tr>
<tr>
<td>Exact Filtered Backprojection</td>
<td>Exact reconstruction. Stable reconstruction for the discrete case. Applicable for both 2D and 3D reconstruction.</td>
<td>Slow with complexity of $O(n^5)$ for 3D reconstruction.</td>
</tr>
<tr>
<td>Backprojection</td>
<td>Stable and exact reconstruction. Applicable for both 2D and 3D reconstruction.</td>
<td>Slow with complexity of $O(n^5)$ for 3D reconstruction.</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>Flexible - can be used for arbitrary sensor geometries. Fast with $O(n^4)$ complexity for 3D reconstruction.</td>
<td>Exact for only 3D reconstruction. Approximate solution for 2D reconstruction.</td>
</tr>
<tr>
<td>Series Solution</td>
<td>Exact solution for both 2D and 3D reconstruction. Extremely fast for cubic acquisition geometries.</td>
<td>Slow for geometries other than the cubic geometry.</td>
</tr>
<tr>
<td>Frequency Domain Solutions</td>
<td>Exact solution applicable for both 2D and 3D reconstruction.</td>
<td>Slow with $O(n^3)$ complexity for 2D reconstruction and $O(n^6)$ complexity for 3D reconstruction.</td>
</tr>
<tr>
<td>Other Methods (Iterative, Statistical and Linear Algebraic)</td>
<td>Flexible - can be adapted to different sensor geometries.</td>
<td>Large memory requirements. Slow.</td>
</tr>
</tbody>
</table>

2.4 Modal Array Signal Processing

The general wave equation governing the propagation of longitudinal acoustic waves in an ideal fluid\textsuperscript{1} is [54]

\[
\nabla^2 x(t, r) - \frac{1}{c^2} \frac{\partial^2}{\partial t} x(t, r) = 0
\]

\textsuperscript{1}An ideal fluid is a fluid that is incompressible and has zero viscosity
where \( r \) is spatial position co-ordinate and \( x(t, r) \) is a space-time function. The solution using separation of variables for a plane wave can be written as [54]

\[
x(t, r) = A e^{i(-\omega t + k^T r)}
\]  

(2.52)

and for a spherical wave is [54]

\[
x(t, r) = \frac{A}{4\pi \| r \|} e^{i(-\omega t + k^T r)}
\]  

(2.53)

where \((\cdot)^T\) is the transpose operator, \( A \) is the amplitude, \( \omega = 2\pi f \) is the angular frequency in radians per second and the wavenumber denoted by vector \( k \) represents the direction and speed of wave propagation and \( k = \| k \| = \omega / c \). For spherical waves the amplitude decay at the observation point is proportional to the distance from the source. In array signal processing, a nearfield source results when the impinging wavefield is spherical. If the source is significantly far from the observation point then the wave can be approximated as a plane wave and a farfield condition results.

Let us first describe what we mean by modal decomposition of wavefields. Modal or wavefield decomposition is the decomposition of wavefields on an aperture into spatially orthogonal eigenfunctions of the wave equation (2.51) with respect to a co-ordinate system best suited to the geometry of the aperture [55]. An aperture is any spatially distributed observation surface used for determining the spatial and temporal characteristics of a wavefield [55]. When the aperture is discretely sampled, then we have an array.

Circular and spherical apertures are particularly suitable for analyzing wavefields with modal decomposition techniques. Circular apertures decompose wavefields into eigen solutions of the wave equation in a cylindrical co-ordinate system and is only suitable for two dimensional (2D) wavefields. On the other hand, spherical apertures can be used for capturing and describing three dimensional (3D) wavefields by decomposing the wavefield to eigenfunctions of the wave equation in a spherical co-ordinate system. An advantage of spherical apertures over other geometries is it can provide uniform spatial selectivity for all directions in 3D. The cylindrical array provides uniform spatial selectivity for only the \( 2\pi \) space.
2.4.1 Circular Apertures

For a three dimensional, plane wave impinging on a circular aperture, illustrated by Fig. 2.2, the pressure on the aperture is

\[ P_{inc}(k, r_s) = e^{ik^T r_s} \]  

(2.54)

where \( r_s \) is the spatial position on the aperture and the time dependence \( e^{-\omega t} \) is omitted to simplify notation. For the standard spherical co-ordinate system,

![spherical coordinate system](image)

Figure 2.2: Plane wave impinging on a circular aperture.

![spherical coordinate system](image)

Figure 2.3: The standard spherical co-ordinate system, \( r \) the radial co-ordinate, \( \theta \in [0, \pi] \) elevation and \( \phi \in [0, 2\pi] \) the azimuth.
Background: Biomedical Applications and Modal Array Signal Processing

described by Fig. 2.3, the wavevector $\mathbf{k}$ is

$$
\mathbf{k} \equiv k \begin{pmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{pmatrix}
$$

(2.55)

and the spatial positions on the aperture $r_s$ are

$$
\mathbf{r}_s \equiv r_s \begin{pmatrix}
\sin \theta_s \cos \phi_s \\
\sin \theta_s \sin \phi_s \\
\cos \theta_s
\end{pmatrix}.
$$

(2.56)

The geometry of the circular aperture illustrated by Fig. 2.2 shows that the aperture lies in the median plane $\theta_s = \pi/2$. If we restrict the sources to a similar median plane $\theta = \pi/2$ where signals from off median planes are sufficiently attenuated then

$$
P(kr_s, \phi_s) = P_{inc}(\mathbf{k}, \mathbf{r}_s)|_{\theta_s=\pi/2, \theta=\pi/2} = e^{ikr_s \cos(\phi_s - \phi)}.
$$

(2.57)

Therefore, the circular aperture is only capable of analyzing 2D wavefields. The Jacobi-Anger expansion [56] can be utilized to decompose the incident plane waves to modes (also called circular harmonics or eigen beams)

$$
P(kr_s, \phi_s) = \sum_{n'=\infty}^{\infty} i^{n'} J_{n'}(kr_s) e^{in'\phi} e^{-in'\phi}
$$

(2.58)

where $n'$ denotes the modal index and $J_{n'}(\cdot)$ is the Bessel function. The total response of the circular aperture is the weighted spatial Fourier series expansion of the incident wavefield

$$
F_n(kr_s, \phi_s) \triangleq \frac{1}{2\pi} \int_0^{2\pi} w(kr_s, \phi_s) P(kr_s, \phi_s) e^{-in\phi} \, d\phi
$$

(2.59)

where $F_n(kr_s, \phi_s)$ are also the Fourier coefficients of the aperture and for an omnidirectional aperture (an aperture consisting of omnidirectional elements), the frequency dependent weights are $w(kr_s, \phi_s) = 1$. Substituting the modal
expansion of the wavefield (2.58) in the total response yields

\[ F_n(kr_s, \phi_s) = \frac{1}{2\pi} \sum_{n'=-\infty}^{\infty} i^{n'} J_{n'}(k r_s) \int_{0}^{2\pi} e^{-i n' \phi_s} e^{i n' \phi_s} \, d\phi_s \, e^{i n' \phi} \]

\[ = i^n J_n(k r_s) e^{-i n \phi} \quad (2.60) \]

Therefore, the application of the spatial Fourier series to the circular aperture decomposes the plane wave into modes. The modes of the circular aperture are also eigenfunctions of the acoustic wave equation (2.51) in cylindrical coordinates. In the modes, the terms dependent on the frequency and the terms dependent on the angular Direction of Arrival (DOA) of the wave are decoupled. This property is the main idea utilized for modal array processing methods and has been applied for broadband source localization [57] and beamforming [58].

By the principle of superposition, the modes for \( V \) plane waves impinging on the aperture is equivalent to the sum of modes for each incident plane wave,

\[ F_{n,t} = \sum_{\nu=1}^{V} i^n J_n(k_{\nu} r_s) e^{-i n \phi_{\nu}} \quad (2.61) \]

**Spatial Aliasing**

If microphones are used for discretely sampling the continuous circular aperture, then the theory of aliasing is extended for modal array processing. Given the wavefield is to be decomposed to a maximum of \( N \) modes (modes are counted starting from zero), applying the same reasoning for bandlimited time signals [59], the Nyquist’s criteria states that the number of equidistant microphones \( Q \) needed to avoid spatial aliasing should satisfy the following condition [60]

\[ Q > 2N. \quad (2.62) \]

But how can we determine the maximum modal index \( N \). The arguments of the Bessel functions \( kr_s \) indicate the magnitude of the modes. A rule of thumb for a first approximation gives [60]

\[ N \approx k_u r_s \quad (2.63) \]

where \( k_u \) is the largest wavenumber. This rule of thumb is based on the reasoning that the Bessel functions for modes \( n > 0 \) exceeding the argument \( k r_s \) have low magnitudes. This rule of thumb is proved formally in [63].

For the linear array, the spacing must be less than half the minimum wave-
length $\lambda_{\text{min}}$. If we combine (2.62) and (2.63), the spacing on the arc satisfies $d_s < \lambda_{\text{min}}/2$ and the sampling interval on the circle should be

$$\phi_T = \frac{2\pi}{Q} < \frac{2\pi}{2k_u r_s} = \frac{\pi}{k_u r_s}. \quad (2.64)$$

The modes for a discretely sampled aperture is

$$F_n^a(kr_s, \phi_s) = F_n(kr_s, \phi_s) + F_n^a(kr_s, \phi_s) \quad (2.65)$$

where the aliasing component $F_n^a(kr_s, \phi_s)$ is

$$F_n^a(kr_s, \phi_s) = \sum_{q=1}^{\infty} (i^g J_g(kr_s)e^{ig\phi} + i^h J_h(kr_s)e^{-ih\phi}) \quad (2.66)$$

with $g = Qq - n$ and $h = Qq + n$. These aliasing terms are residuals due to sampling which are superimposed on mode $n$. These distortions are referred to as modal aliasing. Unlike time signals which can be bandlimited by using low pass filters, no such filters are available for space time signals. However, modal aliasing can be minimized by carefully choosing the number of sensors (2.62) and the maximum spatial frequency $k_u r_s$. The upper frequency limit provides a way to find an upper bound on the mode. Modes exceeding this upper bound is attenuated and this acts as a modal filter to cut off higher modes due to the source. This modal filtering occurs due to wave propagation and by limiting the maximum frequency.

**Truncating the Modal Decomposition of 2D Wavefields**

The modal decomposition of a plane wave impinging on a circular aperture (2.58) contains an infinite number of terms. If we represent the wavefield with a finite number of modes or circular harmonics $N$, then the truncation error that results is

$$\epsilon_{\text{trunc}}(kr_s) = \sum_{n > |N|} i^n J_n(kr_s)e^{in\phi_s}e^{-in\phi}. \quad (2.67)$$

An expression for the mean, squared truncation error $|\epsilon_{\text{trunc}}(kr_s)|^2$ can be derived from Parseval’s theorem for the Fourier series expansion [61] $f(\phi_s) = \sum_{n=-\infty}^{\infty} a_n e^{in\phi_s}$ then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\phi_s)|^2 d\phi_s = \sum_{n=-\infty}^{\infty} |a_n|^2. \quad (2.68)$$
Using Parseval’s theorem, the mean squared truncation error is

\[ |\epsilon_{\text{trunc}}(k_{r_s})|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n>|N|} i^n J_n(k_{r_s}) e^{in\phi} e^{-in\phi} \right|^2 d\phi \]

\[ = 2 \sum_{n>|N|} J_n^2(k_{r_s}). \]  

(2.69)

A result from [62] simplified this truncation error to a finite number of terms

\[ |\epsilon_{\text{trunc}}(k_{r_s})|^2 = 1 - \sum_{n=0}^{N} (2 - \delta_{n0}) J_n^2(k_{r_s}) \]  

(2.70)

where the Kronecker delta \( \delta_{n0} = 1 \) only when the mode \( n = 0 \). We interpret that for small \( kr_s \), only a few modes are necessary to represent the modal decomposition of the wavefield with negligible error. Another study truncated the modal decomposition of 2D wavefield fields to \( N = [ekr_s/2] \approx k_{r_s} \) which upper bounded the normalized truncation error [63]. Practically, circular apertures are discretely sampled, therefore, there are errors due to both spatial aliasing and truncation of the modal decomposition. Design of such arrays must be done to minimize both these errors satisfying conditions (2.62) and with maximum order \( N = k_{r_s} \).

### 2.4.2 Spherical Apertures

Spherical Fourier transforms are important for modal array processing with spherical apertures. The spherical Fourier transform [64] decomposes any function \( f(\Omega) \) on the unit sphere \( S^2 \) into spherical Harmonics with analysis and synthesis equations, respectively

\[ \varphi_{nm} = \int_{\Omega \in S^2} f(\Omega) Y_{nm}^*(\Omega) d\Omega \]  

(2.71)

and

\[ f(\Omega) = \sum_{n,m} \varphi_{nm} Y_{nm}(\Omega) \]  

(2.72)

where \( \varphi_{nm} \) are the spherical Fourier coefficients for order \( n \) and mode \( m \), and the integration over the sphere \( \int_{\Omega \in S^2} (\cdot) d\Omega \triangleq \int_0^{2\pi} \int_0^\pi (\cdot) \sin \theta \, d\theta \, d\phi \).

The geometry of the spherical aperture is illustrated in Fig. 2.4. The modal
Background: Biomedical Applications and Modal Array Signal Processing

decomposition of the plane wave impinging on this spherical aperture is [55]

\[ P_{inc}(\mathbf{k}, \mathbf{r}_s) = e^{ik'r_s} = \sum_{n,m} \varphi_{nm}(kr_s, \Omega)Y_{mn}(\Omega_s) \]  

(2.73)

where \( \sum_{n,m} \triangleq \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \), \( \Omega = (\theta, \phi) \) the Direction of Arrival (DOA) of the plane wave, \( \Omega_s = (\theta_s, \phi_s) \) is the angular position on the aperture. Moreover, the spherical Fourier coefficients \( \varphi_{nm}(kr_s, \Omega) = 4\pi i^n j_n(kr_s)Y_{mn}^*(\Omega) \), \( j_n(\cdot) \) are spherical Bessel functions and \((\cdot)^*\) is the complex conjugate operator. Further, the spherical Harmonics are defined as

\[ Y_{mn}(\Omega) = \sqrt{\frac{(2n + 1)(n - m)!}{4\pi(n + m)!}} P_{mn}(\cos \theta)e^{im\phi} \]  

(2.74)

where \( P_{mn}(\cdot) \) are the associated Legendre polynomials. The term \( 4\pi i^n j_n(kr_s)Y_{mn}^*(\Omega) \) are eigen solutions of the wave equation (2.51) in the spherical co-ordinate system. Similar to the circular aperture, by working in the modal space comprising of these eigen functions, terms dependent on the frequency and those dependent on the spatial characteristics of the impinging wave are separated.

Figure 2.4: Plane wave impinging on a spherical aperture.

Spherical array processing, first multiplies the aperture response by frequency dependent weights and then integrates over the whole sphere resulting in an
aperture output

\[ p(kr_s) = \int_{\Omega_s \in S^2} P_{\text{inc}}(k, r_s) w^*(k, \Omega_s) \, d\Omega_s \]

\[ = \sum_{n,m} \varphi_{nm}(kr_s, \Omega) w^*_m(kr_s). \tag{2.75} \]

This result is obtained by using (2.73) and the spherical Fourier transform for the weights \( w^*(k, \Omega_s) = \sum_{n,m} w^*_n(kr_s) Y^*_m(\Omega_s). \) The complex conjugate weights \( w^*(k, \Omega_s) \) are used since this simplifies notation [65]. Applying weights in the spherical Fourier transform domain is called phase mode processing [66, 67]. These weights can be designed depending on the application the array is used for. For example, Meyer and Elko derived weights in the spherical Fourier space for DOA estimation in [68].

**Sampling on the Sphere**

For the spherical aperture, unlike time signal which can be bandlimited by low pass filters, there are no methods available to prevent modal aliasing. Further, sampling on the sphere is much more complicated than sampling on the circle. This section discussed sensor number and placement strategies for sampling spherical apertures when the maximum order is \( N \). This means that the modal expansion involves sums over \( (N+1)^2 \) orthogonal, basis functions.

Sampling over the sphere can be recast as the discretization of the orthonormality property of the spherical harmonics, that is

\[ \sum_{v=1}^{\Gamma} w_v Y_{mn}(\Omega_v) Y_{m'n'}(\Omega_v) = \delta_{nn'} \delta_{mm'} \tag{2.76} \]

where the Kronecker Delta \( \delta_{mn'} \) is 1 only when \( m = m' \) otherwise zero, \( \Gamma \) are the total number of sampling points and \( w_v \) is a non uniform weighting factor. Moreover, the discrete spherical Fourier Transform is

\[ \varphi_{nm} = \sum_{v=1}^{\Gamma} w_v f(\Omega_v) Y^*_{mn}(\Omega_v). \tag{2.77} \]

for function \( f(\Omega) \in S^2 \).

Spherical sampling includes many different sampling methods with different weighting factors in order to optimize different criteria. A discussion on spherical sampling can be found in [69, 70]. We discuss four different methods of sampling
Equiangle Sampling: In equiangle sampling, if the order limit is \( N \), then to modal aliasing is negligible if the number of sampling points are larger than \( 4(N + 1)^2 \). The weights \( w_i \) in this scheme compensates for the denser sampling points near the poles and are non uniform. This sampling method is simple with \( 2(N + 1) \) equiangular samples for the azimuth \( \phi \) and \( 2(N + 1) \) equiangular samples for the elevation \( \theta \). The equiangular sampling method is advantageous when samples are obtained through rotations of a sensor array. The equiangular sampling method on the sphere was studied in [64] and the values of the weights were tabulated.

Gaussian Sampling: This method reduces the number of sampling point by half when compared to the equiangular sampling scheme. Therefore, if the function \( f(\Omega) \) is order limited to \( N \) then a total of \( 2(N + 1)^2 \) samples are required. Sampling on the azimuth \( \phi \) is the same as for equiangular sampling but the samples on the elevation \( \theta \) are placed on the zeros of the Legendre polynomial \( P_n(\cdot) \) of order \( N + 1 \), that is \( P_{N+1}(\cos \theta) = 0 \) for \( \forall \nu \) [16]. The weights for this method can be taken from tables in [16].

Nearly Uniform Sampling: The problem with the equiangle and the Gaussian sampling methods are that sampling points are densely packed near the poles. Further, the weights are non uniform. The nearly uniform sampling methods solves both these problems ensuring that the sampling points are distributed nearly uniformly on the sphere. This uniform placement of sampling points results in geometries called platonic solids.

One nearly uniform sampling method introduced by Hardin and Sloane called the \( t \)-design where the sampling points \( \Omega_v \) are chosen so that [71]

\[
\int_0^{2\pi} \int_0^{\pi} f(\cos \theta, \phi) \cos \theta \, d\theta \, d\phi = \frac{1}{\Gamma} \sum_{v=1}^{\Gamma} f(\Omega_v)
\]  

holds for all order limited, functions \( f(\cos \theta, \phi) \) on the unit sphere \( S^2 \). The order limit \( N \) is taken to be less than \( t \). A table of the sampling points for different \( t \) values were provided in [71]. For this sampling method, there is no analytic method to determine the sampling points which must be determining with numerical optimization. Furthermore, the analytic expressions for modal aliasing errors and the minimum number of sampling points needed given an order limit \( N \) is not known. Both these quantities were determined from simulations in [71] where it was proved that at least \( (N + 1)^2 \) samples are needed so that modal
2.4 Modal Array Signal Processing

Aliasing is negligible. In practice, however, the number may be larger to about $1.5(N+1)^2$ when the weights are uniform.

The nearly uniform sampling scheme has the advantage of lower number of sampling points but cannot take advantage of Fast Fourier Transforms (FFTs) to speed up the spherical Fourier Transform. This is possible for both equiangle and Gaussian sampling schemes.

**Truncation of the Modal Decomposition for 3D Wavefields**

Same as for circular apertures, truncating the modal decomposition for spherical apertures leads to errors. Practical systems should be designed to minimize these errors $\epsilon_{\text{trunc}}(kr_s)$ which are quantified as

$$\epsilon_{\text{trunc}}(kr_s) = 4\pi \sum_{n=N+1}^{\infty} i^n j_n(kr_s) \sum_{m=-n}^{n} Y_{mn}^*(\Omega)Y_{mn}(\Omega).$$

Following the same approach as for circular apertures, Parseval's theorem for the spherical Fourier transform is utilized to derive the mean squared truncation error as [55]

$$|\epsilon(kr_s)|^2 = \sum_{n=N+1}^{\infty} (2n+1) j_n^2(kr_s).$$

This result can be as a sum of a finite number of terms [62]

$$|\epsilon(kr_s)|^2 = 1 - \sum_{n=0}^{N} (2n+1) j_n^2(kr_s).$$

The results obtained for the truncation are similar to those for the circular apertures mentioned in Section 2.4.1. The upper bound on the normalized absolute error can be minimized by selecting the maximum order $N = \lceil ek_{r_s}/2 \rceil \approx kr_s$ [63].

A simple intuition of this follows from the fact that the magnitudes of the spherical Bessel functions when the order $n$ exceeds the argument $kr_s$ is very small.

**2.4.3 Applications**

**Beamforming**

The concept of the modal beamformer was introduced for spherical arrays in [68,72] and for circular apertures in [57]. The processing for designing these beamformers take place in the modal domain where the modes or the harmonics
are used as building blocks for complicated beampatterns. This section describes the principle of designing a beamformer for a continuous circular aperture. The ideas in designing a beamformer for a spherical aperture are essentially the same. However, for discretely sampled apertures, spatial aliasing and modal truncation need to be considered.

To design a beamformer with a desired beampattern $F^d(kr_s, \phi)$, spatial weights are first applied to the aperture response which are then summed, i.e.

$$F^d(kr_s, \phi) = \frac{1}{2\pi} \int_0^{2\pi} w(kr_s, \phi_s) P_{inc}(k, r_s) \, d\phi_s$$  \hspace{1cm} (2.82)

where $w(kr_s, \phi_s)$ are the spatial weights with spatial Fourier series expansion defined as $w(kr_s) = \sum_{n=-\infty}^{\infty} w_n(kr_s) e^{in\phi_s}$. By substituting the modal decomposition for plane waves (2.58) into (2.82) yields

$$F^d(kr_s, \phi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} w(kr_s, \phi_s) e^{-in\phi_s} \, d\phi_s \, i^n J_n(kr_s) e^{in\phi}$$

$$= \sum_{n=-\infty}^{\infty} w_n(kr_s) i^n J_n(kr_s) e^{in\phi}. \hspace{1cm} (2.83)$$

Now defining the modal decomposition of the desired beampattern as $F^d(kr_s, \phi) = \sum_{n=-\infty}^{\infty} F_n^d(kr_s) e^{in\phi}$ with $F_n^d(kr_s)$ being the $n^{th}$ mode or circular harmonic of the desired beampattern. By examining this definition and (2.83), the $n^{th}$ mode of the beampattern $F^d_n(kr_s)$ is equivalent to $w_n(kr_s) i^n J_n(kr_s)$. Since the desired beampattern is known, the modal weights of the circular aperture can be obtained from

$$w_n(kr_s) = \frac{F_n^d(kr_s)}{i^n J_n(kr_s)}. \hspace{1cm} (2.84)$$

Note by working with the modes, we have replaced the spatial aperture weights $w(kr_s, \phi_s)$ by modal weights $w_n(kr_s)$. The advantage of working with the modes becomes apparent in designing broadband beamformers, where the frequency invariant beampattern is used $F^d(\phi) \triangleq F^d(kr_s, \phi)$ and the modal weights are calculated by $w_n(kr_s) = F_n^d / (i^n J_n(kr_s))$.

**Source Localization**

By applying modal space processing, the ESPRIT based DOA method [73] was extended to work with broadband sources for circular apertures [57]. The main
idea compensated for the frequency dependency of the modes by

$$\tilde{F}_n(\phi) = V_n(kr_s)F_n(kr_s, \phi) = e^{-in\phi}. \quad (2.85)$$

where $V_n(kr_s) = 1/(i^n J_n(kr_s))$. Other subspace based approaches such as MUSIC [74] can also be applied in the modal space for broadband source localization.

**Source Detection**

The source detection problem is important for subspace based Direction of Arrival (DOA) estimation where the number of active sources must be known beforehand. This section describes a source detection method in the modal space for circular apertures presented in [57]. When there are $I$, unit amplitude, plane waves impinging on a circular aperture, the pressure received is

$$P(kr_s, \phi_s, \phi_\nu) = \sum_{\nu=1}^{I} e^{i k r_s \cos(\phi_s - \phi_\nu)} = \sum_{\nu=1}^{I} \sum_{n=-\infty}^{\infty} i^n J_n(kr_s)e^{in\phi_s}e^{-in\phi_\nu}. \quad (2.86)$$

where $\phi_\nu$ are the DOA of the $\nu^{th}$ plane wave. Form (2.86), the modes for the circular aperture are

$$F_n(kr_s, \phi_\nu) = \sum_{\nu=1}^{I} i^n J_n(kr_s)e^{-in\phi_\nu}. \quad (2.87)$$

Lets choose a particular aperture position $\phi_0 = 0$, then assuming negligible truncation error, performing an inverse spatial Fourier series transformation for this position yields

$$P(kr_s, 0, \phi_\nu) = \sum_{\nu=1}^{I} e^{ikr_s \cos \phi_\nu} = \sum_{n=-N}^{N} F_n(kr_s, \phi_\nu). \quad (2.88)$$

The modal impulse response is obtained by transforming (2.88) to the time domain

$$p(t, \tau_\nu) = \sum_{\nu=1}^{I} \delta(t - \tau_\nu) \quad (2.89)$$

where the delays of the $\nu^{th}$ plane wave is $\tau_\nu = r_s/c \cos \phi_\nu$. The number of source are equal to the number of Dirac impulses on the modal impulse response.

Source detection for practically acoustic signals such as speech requires an adaptive method with a reference signal. Note that by working in the modal
space, an array of sensors are replaced by a single virtual sensor with different spatial characteristics called the modes. This is observed where the zeroth mode has omnidirectional spatial behavior whereas the first mode has a dipole spatial characteristic. The reference signal should be omnidirectional hence the zeroth mode of the array is used. Using this reference signal, adaptive methods such as least mean squares based or recursive least squares [75] can be applied for source detection.

2.4.4 Discussion

Extracting information from wavefields spread out both spatially and temporally requires spatially distributed, sensor arrays. Information can be extracted from wavefields by employing classical signal processing methods where operations are applied directly to the sensor outputs. Modal array signal processing methods perform signal processing in the modal space comprising of eigen solutions of the wave equation, thereby considering principles of wave propagation. We have presented the principle of modal array signal processing for circular apertures and spherical apertures. Furthermore, modal array signal processing has been applied for many acoustic applications. In [57] a broadband source localization method with circular apertures was derived and demonstrated. For the spherical array, modal array signal processing has been applied for acoustic scene analysis [76], sound recording [77] and beamforming [58,68].

2.5 Summary

This chapter provided an introduction to both lung sound localization and photoacoustic imaging together with methods that have been proposed before for solving these two inverse problems. The limitations and qualitative performance of the lung sound localization methods were discussed. For photoacoustics, we highlighted the speed of each method and their limitations. The last section provided an overview of the application of modal decomposition to array signal processing focusing on circular and spherical apertures. The methods developed in the next four chapters processes the signals in the modal space.
References


Background: Biomedical Applications and Modal Array Signal Processing


Chapter 3

Eigen Basis Decomposition for Localizing Lung Sounds

3.1 Introduction

Localization of lung sounds from multi-sensor recordings can be classified as a problem of source localization within a circular array of sensors. Ward et. al. [1] showed how to locate a single source in the interior field of the sensor array. We propose algorithms to localize multiple sources within a circular array of sensors using cylindrical harmonic functions. It is advantageous to use these functions since propagation of layered cylindrical media can be analyzed by these functions and the chest can be modeled as consisting of cylindrical layered media. Performance metrics for the algorithms accounting for noise levels, frequency of sounds, and region of localization are derived. We apply a Minimum Variance (MV) spectrum to the processed sensor recordings to obtain high resolution localization algorithms.

The stethoscope allows physicians to diagnose the pulmonary system over the auditory range. This is useful since most physiological processes and structure of the body causes sounds that resonate in the audible sound range. Research interest in localizing and analyzing lung sounds in the audible range was a result of the shortcomings of existing tools for lung diagnosis. These lung diagnosis methods involved using ultrasound techniques. However, ultrasound techniques have not been applied for lung diagnosis due to its poor performance with high frequency sounds [2]. This poor performance is caused by the high attenuation property of the lung parenchyma for high frequency sounds.

Lung sound analysis with multiple stethoscopes can be used as a first diagno-
sis tool for lung disorders. Well established methods such as computed tomography (CT) and x-rays can later be applied to confirm the results obtained from the multiple stethoscope device. Diseases or injury can cause alterations in the structure and function of the lungs resulting in changes in lung sound production and transmission. Lung consolidation, pneumothorax and airway obstructions are some of the conditions that can cause spectral and regional changes in lung sounds. If these changes are properly analyzed and localized from multi sensor recordings then the extent and location of the trauma can be acquired [3, 4].

Simultaneous recordings of breath sounds can be processed to provide a surface acoustic image of the thorax using interpolation [5]. Studies on healthy subjects and subjects with interstitial pneumothorax illustrated that thoracic surface acoustic images can provide information on the spatial extent of the disease [6].

To get more information from the lung sounds, researchers began to develop algorithms for localization of lung sounds in a 3D co-ordinate system [7-9]. One of the earliest work on this was by Kompis et. al. who presented a solution for acoustic imaging of the human chest [4]. The resolution of Kompis’s algorithm was reported from measurements to be 2cm. Moreover, he was able to show that lung sound localization can give information on lung consolidation.

Other researchers tried to either incorporate a more accurate acoustic transmission model of the lung at the expense of the localizing algorithm [9] or proposed better localizing algorithms simplifying the acoustic transmission model [8]. These methods do not work well for multiple sources and analytical formulas for performance in terms of resolution and spatial aliasing were not provided. A detailed review of lung sound localization methods is provided in Section 2.2.2.

The aim of this chapter is to propose and analyze sound localization algorithms within a circular sensor array for acoustic imaging of the chest. However, the algorithms can also be applied in sensor monitoring, hands free communication in a room or for recording sounds in an auditorium. The solution to the Helmholtz wave equation can be synthesized and analyzed for a cylindrical co-ordinate system with a set of eigen basis functions. These eigen basis functions are the cylindrical harmonics. The lung can be modeled as a layered cylindrical structure and propagation in such environments can include reflections and refractions and can be analyzed using cylindrical harmonic functions [10-12]. This chapter aims to develop localizing algorithms using these eigen basis functions and investigates their performance under different conditions. The localizing algorithms rely on this eigen basis decomposition since this will allow easy ex-
tension to a layered cylindrical model of lung sound propagation. Further, these eigen basis function were shown to be versatile for direction-of-arrival (DOA) estimation [13].

The contributions made by this chapter are discussed as follows:

- We propose two localizing algorithms using cylindrical harmonics with the aim of extending these algorithms for localizing sources within a layered cylindrical structure. The propagation through a layered cylindrical structure can be comparable to that of lung sounds propagating to the surface of the chest.

- The proposed algorithms will aim to use spectral search to locate source since spectral based methods are more accurate than triangulation. Further, methods using differences in arrival times require higher precision equipment than spectral based algorithms.

- We investigate the performance of the algorithms for different levels of noise and for different frequencies of sound.

- This chapter also derives a relationship for resolution in terms of noise level and frequency for the proposed algorithms. This relationship will be useful in designing multi sensor systems for lung sound localization. Given the resolution required, noise levels and frequency range, the number of sensors can be determined for localizing sounds within a specific region.

- We prove that the Nyquist’s criteria for localizing sources within a circular sensor array is different when compared to the Nyquist’s theorem applied to linear arrays used for farfield beamforming.

This chapter is organized as follows: Section 3.2 outlines the system model and defines the problem. Section 3.3 discusses a eigen basis decomposition method for wavefields. Section 3.4 presents two algorithms for lung sound localization that use this eigen decomposition. Section 3.5 provides theoretical analysis on the noise transformation, resolution and Nyquist’s criteria for the proposed algorithms. Section 3.6 presents and describes the simulation results obtained for localizing sound sources using the proposed algorithms. Section 3.7 discusses some useful properties of the localizing algorithms and also presents a comparison of the two algorithms. Section 3.8 concludes this chapter with a summary of major findings and discusses future work needed in acoustic imaging of the chest.
3.2 System Model

Lung sounds due to normal breathing are recorded by microphones placed on the chest as illustrated by Fig. 3.1. These microphone recordings can be processed to locate lung sounds. In this chapter, we consider localizing two dimensional lung sound sources completely surrounded by a circular array of uniformly spaced sensors (microphones) placed at a radius $R$.

![Figure 3.1: System model with lung sound sources located interior of a circular sensor array placed around the chest.](image)

The uniform spacing assumption is not a necessary condition, however, it simplifies notation and calculations. We assume that the source signals are zero mean and stationary. In most array signal processing literature, the sources are in the farfield such that the impinging wavefront is planar. The scenario presented in this chapter is different and considers the sources to be in the nearfield with cylindrical impinging wavefronts.

Let there be $Q$ sensors located at $x_q$ with $q = 1, \ldots, Q$, where $x_q \equiv (x_q, \theta_q)$, in polar coordinates, $x_q$ is the distance from the origin and $\theta_q$ is the angle to the $q^{th}$ sensor. For a circular array $x_q = R$ for all $q$. Assume that there are $V$ sources present within the region enclosed by the sensors at locations $y_v$ where $y_v \equiv (y_v, \phi_v)$.

Lung sounds are broadband and the sensor data can be separated into different frequency bins by the Discrete Fourier Transform (DFT). The data captured
by the sensors at a frequency bin with central frequency $f_0$ is

$$z(k) = \sum_{v=1}^{V} a(y_v, k)s_v(k) + n(k)$$

(3.1)

where

- $k$ is the wavenumber and $k = \frac{2\pi f_0}{c}$ with $c$ as the speed of propagation,
- $z(k)$ is the $Q \times 1$ vector of sensor recordings,
- $n(k)$ is the $Q \times 1$ vector containing additive noise,
- $s_v(k)$ is the signal emitted by the $v^{th}$ source as received at the origin, and
- $a(y_v, k)$ is the array manifold vector generated by a source located at $y_v$.

The array manifold vector is composed of elements that contain information on the attenuation and the phase change as the wave propagates from the source location to the sensors and is defined by

$$a(y_v, k) \triangleq [G_{2D}(k; x_1, y_v), \ldots, G_{2D}(k; x_Q, y_v)]^T$$

(3.2)

where

$$G_{2D}(k; x, y) = \frac{i}{4} H_0^{(1)}(k\|x - y\|).$$

(3.3)

$G_{2D}(k; x, y)$ is the Green’s function which is the fundamental solution to the 2D Helmholtz equation. The term $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind and zeroth order. The Hankel function of the first kind is used since the wavefield is radiating away from the origin. An example of a 2D wavefield for $k = 2$ (frequency of 125 Hz) is illustrated in Fig. 3.2. For multiple sources, the principle of superposition can be applied to derive the wavefield magnitude and phase at vector point $x$.

Note that for a farfield case using linear arrays, the array manifold entries for different sensors changes in phase only. However, for our case involving cylindrical wavefronts, the entries in the array manifold change in both magnitude and phase for different sensor recordings as illustrated in Fig. 3.2.

Previous work [1] for locating sources contained in the interior of a sensor array is only applicable for localizing a single source and uses an optimization:

$$a(\phi, k) \triangleq [e^{ikx_1 \sin(\phi)}, \ldots, e^{ikx_Q \sin(\phi)}]^T$$

where $\phi$ is the DOA.
Figure 3.2: Wavefield generated by a 2-D source at $(3, \pi/4)$. The field magnitude at the source location is infinity and the magnitude dies down very sharply. The phase information can be expected to be more different at the sensors than the magnitude.
transformation to cancel out the magnitude changes. This optimization transformation cannot be applied when multiple sources are involved. The algorithm developed in this chapter takes into account both the magnitude and the phase variation between sensors in order to locate multiple sources.

Localization algorithms discussed in this chapter will consider narrowband sources. This can be easily extended to a broadband scenario since broadband signals can be decomposed to a set of narrowband bins by applying a set of narrowband filters to the sensor data. The DFT decomposes the array data to different frequency bins and the proposed localization algorithms are applied independently to each of these bins. For the rest of this chapter, we will be considering data from only one frequency bin, therefore, rewriting (3.1) in matrix notation and ignoring $k$ for the narrowband case, we have

$$ z = A(Y)s + n $$

(3.4)

where

$$ A(Y) = \begin{bmatrix} a(y_1), \ldots, a(y_v) \end{bmatrix}, $$

(3.5)

$$ Y = \begin{bmatrix} y_1, \ldots, y_v \end{bmatrix}, $$

(3.6)

and

$$ s = \begin{bmatrix} s_1, \ldots, s_v \end{bmatrix}^T. $$

(3.7)

Direction of Arrival (DOA) methods aim to determine the angle only, however the localization problem, considered in this chapter, is set up similar to a DOA problem but aims to estimate $Y$ which includes both the range and angle in polar co-ordinates. In a similar fashion to DOA methods, the localization algorithms will use the correlation matrix of the received data which is

$$ R_z = E\{zz^*\}. $$

(3.8)

By substituting (3.4) into (3.8) and assuming that the noise and source signals are uncorrelated, the correlation matrix is equivalent to

$$ R_z = A(Y)E\{ss^*\}A(Y)^* + E\{nn^*\}. $$

(3.9)
3.3 Eigen Basis Decomposition

A sensor array captures information on the impinging wavefield which can be decomposed to a set of orthogonal basis functions depending on the spatial coordinates used. These basis sets are useful for synthesizing and analyzing wavefield information captured by a sensor array. For a three dimensional wavefield, spherical harmonics form the basis sets and for a two dimensional wavefield, as investigated in this chapter, cylindrical harmonics make up the basis set.

Eigen basis functions for wavefields have been applied in research pertaining to antennas. Works [14, 15] used eigen basis modes to synthesize antenna shapes. In beamforming, eigen basis modes were used for designing nearfield broadband beamformers [16, 17]. Moreover, in acoustic signal processing, these basis modes were applied to soundfield recording [18] and reproduction [19]. A more in depth review of the use of eigen decomposition of wavefields in array signal processing is presented in Section 2.4. More importantly, sound propagation through the chest can be modeled similar to a layered cylindrical media. Eigen basis modes or more specifically cylindrical harmonics have been applied to wave propagation in layered cylindrical media [10–12].

The two dimensional wavefield investigated in this chapter can be decomposed to basis functions [20, page 66]

$$B(x, y) = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(kx) J_n(ky) e^{in\theta_x} e^{-in\theta_y}$$  \hspace{1cm} (3.10)

where $J_n(\cdot)$ is the Bessel function of order $n$. This decomposition consists of an infinite number of terms and is called the addition theorem for Hankel functions, valid only when $\|x\| > \|y\|$. The decomposition can be used if the significant number of terms can be finite. The Bessel functions of finite argument approach zero as the order $n$ becomes large. Therefore, for a finite region of space bounded by a circle of radius $R$ and for the wavelength of sound being $\lambda$, the number of eigen basis functions that characterize a wavefield without incurring significant errors [21] can be limited to

$$N = \left\lfloor \frac{\pi c R}{\lambda} \right\rfloor \approx kR. \hspace{1cm} (3.11)$$

In the truncated 2D wavefield decomposition, the order $n$ spans the set $n \in [-N, \ldots, 0, \ldots, N]$ in (3.10). In the rest of this chapter, the truncated 2D Green’s function approximated by $2N+1$ basis functions is denoted by $G_{2D}(x, y)$.
where the wavenumber $k$ is dropped to make notation simpler.

### 3.4 Sound Localization

The cylindrical waves impinging on the sensor array caused by 2D sources, results in an array manifold defined by

$$A(Y) = \begin{bmatrix}
\tilde{G}_{2D}(x_1, y_1) & \cdots & \tilde{G}_{2D}(x_1, y_V) \\
\vdots & \ddots & \vdots \\
\tilde{G}_{2D}(x_Q, y_1) & \cdots & \tilde{G}_{2D}(x_Q, y_V)
\end{bmatrix}.$$  \hfill (3.12)

The elements in $A(Y)$ represent both the magnitude and phase difference of wavefields received by the sensors. Further, these elements can be represented by (3.3) or by the summation of eigen basis modes (3.10). Source and sensor location information are present in the elements of $A(Y)$. However, by considering these elements as the sum of eigen basis modes, we can separate $A(Y)$ into two independent terms containing sensor locations and source locations, respectively.

The two proposed algorithms presented in the next two subsections exploit the idea that the sensor location terms can be removed from the array manifold, $A(Y)$ and then source locations can be estimated using spectral techniques.

#### 3.4.1 Orthogonality Based Algorithm

**Fourier Series Expansion**

The angular positions of the sensors span the range $[0, 2\pi]$, allowing exploitation of the orthogonality property of exponential functions, $e^{in\theta}$. Let $z(\theta)$ be a continuous function denoting the received signal for a sensor placed at an angle $\theta$. This continuous function $z(\theta)$ is periodic on $2\pi$ and can be expressed as

$$z(\theta) = \sum_{n=-N}^{N} \alpha_n^{(R)} e^{in\theta}$$ \hfill (3.13)

where $\alpha_n^{(R)}$ is called the spatial Fourier coefficients of the sensor data for mode $n$. We can view (3.13) as a Fourier series expansion of the received signal. By multiplying both sides of (3.13) by $e^{-in\theta}$ and integrating with respect to $\theta$ over $[0, 2\pi)$, we obtain

$$\alpha_n^{(R)} = \frac{1}{2\pi} \int_{0}^{2\pi} z(\theta)e^{-in\theta} d\theta.$$ \hfill (3.14)
From (3.1)

\[ z(\theta) = \sum_{v=1}^{V} G_{2D}((R, \theta), \mathbf{y}_v) s_v + n(\theta) \]  

(3.15)

where \( n(\theta) \) is the AWGN at a sensor placed at an angle \( \theta \) on the circular array. Substituting (3.10) and (3.15) into (3.14), we get

\[ \alpha_n^{(R)} = \frac{i}{8\pi} H_n^{(1)}(kR) \sum_{v=1}^{V} J_n(ky_v)e^{-in\phi_v} + \tilde{n}_n. \]  

(3.16)

where \( \tilde{n}_n \) is the noise for the spatial Fourier coefficient at mode \( n \). Writing (3.16) in matrix notation

\[ \alpha = HJ s + \tilde{n} \]  

(3.17)

where \( \alpha = 8\pi / i \left[ \alpha_{-N}^{(R)}, \ldots, \alpha_{N}^{(R)} \right]^T \), \( H = \text{diag} \left[ H_{-N}^{(1)}(kR), \ldots, H_{N}^{(1)}(kR) \right] \) and \( J \) contains information on the source locations

\[ J = \begin{bmatrix} J_{-N}(ky_1)e^{iN\phi_1} & \ldots & J_{-N}(ky_V)e^{iN\phi_V} \\ \vdots & \ddots & \vdots \\ J_{N}(ky_1)e^{-iN\phi_1} & \ldots & J_{N}(ky_V)e^{-iN\phi_V} \end{bmatrix}. \]  

(3.18)

The spatial Fourier coefficients comprise of terms dependent on the positions of the \( V \) sources and the radius at which the sensors are placed. Contributions of the sensor angular positions are removed by transforming the array data to a spatial Fourier domain (3.14).

**Discrete Angular Samples**

In practice, we measure \( z(\theta) \) only on discrete sensor positions at \( \theta_q, q = 1 \ldots Q \). Thus, one can approximate the integral (3.14) by a summation

\[ \alpha_n^{(R)} = \frac{1}{2\pi} \sum_{q=1}^{Q} z(\theta_q)e^{-in\theta_q} \Delta\theta_q \]  

(3.19)

where \( \Delta\theta_q \) is the angular separation between the \( q^{th} \) and \( (q+1)^{th} \) sensors. If the sensors are uniformly spaced on the circle, then (3.19) can be viewed as a Discrete Fourier Transform (DFT). The operations required to transform the discrete sensor data to the spatial Fourier domain are summarized in Fig. 3.3.

\[^2\text{The discrete form of the orthogonality relationship for exponential functions applied in (3.19) is valid only if there is no aliasing. A discrete number of sensors sample the imping} \]
We write (3.19) in matrix form as

$$\mathbf{\alpha} = \frac{4}{iQ} \Xi^* \mathbf{z}$$

(3.20)

where

$$\Xi = \begin{bmatrix} e^{-iN\theta_1} & \cdots & e^{iN\theta_1} \\ \vdots & \ddots & \vdots \\ e^{-iN\theta_Q} & \cdots & e^{iN\theta_Q} \end{bmatrix}$$

(3.21)

and the columns of $\Xi$ are the $Q$ discrete samples of the orthonormal function $e^{in\theta}$.

The spatial Fourier coefficients has a component that is dependent on the radial positions of the sensor. This algorithm aims to transform the sensor data to a domain dependent only on the source locations. This would create a unified representation for data recorded by different sensor arrays for an impinging wavefield caused by sources in the same locations. However, removal of the matrix $\mathbf{H}$ from the spatial Fourier coefficients by using its inverse can cause instability of the solution when the condition number, $\kappa(\mathbf{H})$, is large.

wavefront and is analogous to sampling the function $e^{in\theta}$ at the angular positions of the sensors. For large $n$, a greater number of sensors spanning the circumference of a circle is required in order to avoid aliasing.
If the radial sensor components are not removed then the source locations can be estimated by first calculating the covariance matrix of the spatial Fourier coefficients $R_\alpha$ by

$$R_\alpha = E\{\alpha\alpha^*\}.$$  \hspace{1cm} (3.22)

In practice, the covariance matrix is not acquirable. However, $\alpha$ can be obtained for a finite number of snapshots $T$. Then a maximum likelihood approximation

$$R_\alpha \approx \frac{1}{T} \sum_{t=1}^{T} \alpha(t)\alpha(t)^*$$  \hspace{1cm} (3.23)

can be used to estimate $R_\alpha$. It is important to mention that maximum likelihood approximations improve if the value of $T$ is larger.

The Minimum Variance (MV) or the Capon's method [22] was developed to overcome poor resolution methods that were available in classical beamforming. The MV method passes the signal from the look direction and minimizes the output power from all other directions. The output power of the circular sensor array as a function of $y$ and $\phi$ is given by the MV spatial spectrum

$$\tilde{Z}(y, \phi) = \frac{1}{d(R, y, \phi)^*R_\alpha^{-1}d(R, y, \phi)}$$  \hspace{1cm} (3.24)

where

$$d(R, y, \phi) = \begin{bmatrix} H_{-N}^{(1)}(kR)J_{-N}(ky)e^{iN\phi} \\ \vdots \\ H_{N}^{(1)}(kR)J_{N}(ky)e^{-iN\phi} \end{bmatrix}.$$  \hspace{1cm} (3.25)

The spectrum is computed and plotted over the whole range of $y$ and $\phi$. It is important to mention that the spacing of the $y$ and $\phi$ in computing the spectrum must be smaller than the resolution of the MV method. From the MV spatial spectrum, the source locations are estimated by locating the peaks in the spectrum.

If the condition number, $\kappa\{H\}$ is small, we can remove the sensor radial component by

$$\tilde{\alpha} = H^{-1}\alpha$$  \hspace{1cm} (3.26)

where

$$H^{-1} = \text{diag} \left[ \frac{1}{H_{-N}^{(1)}(kR)}, \ldots, \frac{1}{H_{-N}^{(1)}(kR)}; \ldots, \frac{1}{H_{N}^{(1)}(kR)} \right].$$  \hspace{1cm} (3.27)

and $\tilde{\alpha}$ is called the modal space or the eigen space domain of the sensor array data. The covariance matrix of $\tilde{\alpha}$ can be estimated by the maximum likelihood
approximation

$$R_{\tilde{\alpha}} \approx \frac{1}{T} \sum_{t=1}^{T} \tilde{\alpha}(t)\tilde{\alpha}(t)^*$$  \hspace{1cm} (3.28)

and the source locations are the peaks in the new MV spectrum

$$Z(y, \phi) = \frac{1}{c(y, \phi)^* R_{\tilde{\alpha}}^{-1} c(y, \phi)}$$  \hspace{1cm} (3.29)

where

$$c(y, \phi) = \begin{bmatrix} J_{-N}(ky)e^{iN\phi} \\
\vdots \\
J_{N}(ky)e^{-iN\phi} \end{bmatrix}.$$  \hspace{1cm} (3.30)

Comparing (3.24) and (3.29), the MV spectrum when the radial component is removed is less computationally expensive since computing $d(R, y, \phi)$ for the entire range of $y$ and $\phi$ is more expensive than computing $c(y, \phi)$.

In normal DOA scenarios, the MV spectrum is less computationally expensive since the spectrum is obtained as a function of one variable, the DOA. Further, the MV spectrum can be computationally expensive for a large sensor array as it requires the computation of a matrix inverse. Note that high resolution subspace methods such as MUSIC [23] can be applied to the orthogonality based algorithm in place of the MV spatial spectrum.

**Condition Number of $H$**

The Hankel function is a complex function with a corkscrew like behavior with increasing argument. Since $H_{-n}^{(1)}(kR) = e^{i\pi n} H_n^{(1)}(kR)$, we have $|H_{-n}^{(1)}(kR)| = |H_n^{(1)}(kR)|$ and so the condition number for matrix $H$,

$$\kappa(H) = |H_n^{(1)}(kR)|/|H_0^{(1)}(kR)|.$$  \hspace{1cm} (3.31)

If $\kappa(H)$ is large, then removing $H$ by multiplying $\alpha$ by $H^{-1}$ can result in instability of the solution since noise is amplified and the results obtained for the source locations will be practically useless. However, in such cases regularization methods can be applied resulting in a degradation of resolution in the solutions obtained.

Further, the magnitude of the Hankel functions increase with the mode $n$ for a given argument, so the greater the number of modes used the larger the value of $\kappa(H)$, but $\kappa(H)$ decreases as values of $kR$ get larger (for cases when $kR$ is
a real number with zero imaginary component). From Fig. 3.4., we can see that as a rule of thumb if \( kR > 0.75 \times N \) then \( \kappa \{ \mathbf{H} \} \) is small (in this definition we ensure that \( \kappa \{ \mathbf{H} \} < 10^2 \)) thus ensuring a stable solution when \( \mathbf{H}^{-1} \) is multiplied to \( \alpha \).

If the sensor radial component is to be removed, there are two conditions that need to be considered when choosing the number of modes to use, given both the wavenumber, \( k \) and the radius of the region of interest, \( R \). These two conditions include the stability condition and (3.11) resulting in \( N \) being

\[
kR < N < \frac{4}{3} kR.
\]

(3.32)

### 3.4.2 Least Squares Based Algorithm

Similar to the orthogonality based algorithm, the least squares based algorithm removes the sensor contributions to transform the sensor array data to a eigen basis domain, \( \tilde{\alpha} \). The source locations can be estimated by peaks in the MV spectrum (3.29) after an estimate of the covariance matrix in the eigen space domain is calculated (3.28).

From (3.10), each element in the array manifold matrix, \( \mathbf{A}(\mathbf{Y}) \) consists of summation of orthogonal basis functions of 2D wavefields. Therefore, the matrix
\( A(Y) \) can be separated into two matrices as

\[
A(Y) = \frac{i}{4} \Gamma \Upsilon 
\]

(3.33)

where

\[
\Gamma = \begin{bmatrix}
H^{(1)}_{-N}(kR)e^{-iN\theta_1} & \ldots & H^{(1)}_{-N}(kR)e^{iN\theta_1} \\
\vdots & \ddots & \vdots \\
H^{(1)}_{-N}(kR)e^{-iN\theta_Q} & \ldots & H^{(1)}_{-N}(kR)e^{iN\theta_Q}
\end{bmatrix} 
\]

(3.34)

and

\[
\Upsilon = \begin{bmatrix}
J_{-N}(ky_1)e^{iN\phi_1} & \ldots & J_{-N}(ky_V)e^{iN\phi_V} \\
\vdots & \ddots & \vdots \\
J_{-N}(ky_1)e^{-iN\phi_1} & \ldots & J_{-N}(ky_V)e^{-iN\phi_V}
\end{bmatrix} 
\]

(3.35)

One of these matrices, \( \Gamma \) contains data on the sensor locations. The other matrix, \( \Upsilon \) contains the data on the source locations.

From the array manifold, we need to remove the contribution of sensor locations. Given that the sensor locations are known, we can construct the matrix, \( \Gamma \). The contribution of the sensor locations from the sensor recording vector, \( z \) can be removed by using the Moore-Penrose pseudo-inverse of \( \Gamma \), denoted by \( \Gamma^\dagger \). This pseudo-inverse is

\[
\Gamma^\dagger = [\Gamma^*\Gamma]^{-1}\Gamma^* 
\]

(3.36)

Multiplying the sensor recording vector by the pseudo inverse (3.37), transforms the data to the modal space, \( \tilde{\alpha} \) containing source location matrix, source signal received at origin and the modified noise \( \hat{n} \). The operation shown in (3.37) is equivalent to a least squares approximation.

\[
\tilde{\alpha} = \frac{4}{i} \Gamma^\dagger z 
\]

(3.37)

\[
= \Upsilon s + \hat{n} 
\]

(3.38)

**Pseudo-Inverse of \( \Gamma \)**

The calculation of the Moore-Penrose pseudo-inverse for \( \Gamma \) is equivalent to (3.36) only when \( \Gamma \) is not close to singular. From trials constructing \( \Gamma \) for several different arrangement in a circular array, it was observed that in most cases \( \Gamma \) was close to singular. The reason for \( \Gamma \) being close to singular in certain situations is because elements in \( \Gamma \) can only be approximated to a certain number of decimal places and the magnitude of these elements can be very small. In such a case,
\( \Gamma^\dagger \) can be calculated by using Singular Value Decomposition (SVD). The SVD of \( \Gamma \in \mathbb{C}^{Q \times (2N+1)} \) is
\[
\Gamma = UDF^* \tag{3.39}
\]
where \( U \in \mathbb{C}^{Q \times Q} \) and \( F \in \mathbb{C}^{(2N+1) \times (2N+1)} \) are orthogonal matrices, and \( D \) is a \( Q \times (2N+1) \) diagonal matrix
\[
D = \begin{bmatrix}
\xi_1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \xi_2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \xi_3 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \xi_p & 0 & \ldots & 0
\end{bmatrix} \tag{3.40}
\]
where \( \xi_i \) are the singular values of \( \Gamma \) with \( p = \min\{Q, 2N+1\} \) and \( \xi_1 > \xi_2 > \ldots > \xi_p > 0 \).

From (3.40), \( \Gamma^\dagger \) can be obtained by
\[
\Gamma^\dagger = FD^\dagger U^* \tag{3.41}
\]
where
\[
D^\dagger = \begin{bmatrix}
\frac{1}{\xi_1} & 0 & 0 & \ldots & 0 \\
0 & \frac{1}{\xi_2} & 0 & \ldots & 0 \\
0 & 0 & \frac{1}{\xi_3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \frac{1}{\xi_p} \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix} \tag{3.42}
\]

For more information on the Moore-Penrose pseudo-inverse, the reader is referred to the article [24].

### 3.5 Theoretical Performance Analysis

#### 3.5.1 Noise

In this section, we analyze how the measurement noise at each sensor affects the performance of the proposed localization algorithms. We assume that the additive noise at each sensor is zero mean white, complex Gaussian and the noise
3.5 Theoretical Performance Analysis

at different sensors are uncorrelated. The covariance of the noise matrix is

\[ R_n = E\{nn^*\} = \sigma_n^2 I \]  

(3.43)

where \( \sigma_n^2 \) is the noise power at each sensor and \( I \) represents an identity matrix.

In literature, DOA algorithms applied to scenarios with correlated noise such as sonar applications led to biasing of the estimate and a degradation in resolution [25]. In this subsection, we analyze what effect the two proposed algorithms have on the noise covariance matrix and if these transformations result in the noise being correlated.

Firstly, for the orthogonality based algorithm, the noise covariance matrix after transformation into the eigen basis domain becomes

\[ R_n = \left( \frac{4}{Q} \right)^2 \sigma_n^2 \mathbf{H}^{-1} \mathbf{\Xi}^* \mathbf{\Xi} (\mathbf{H}^{-1})^* . \]  

(3.44)

Since the series \( \{e^{in\theta}\}_{n=-N}^N \) is orthogonal, \( \mathbf{\Xi}^* \mathbf{\Xi} = Q \mathbf{I} \). Further,

\[ \mathbf{H}^{-1} (\mathbf{H}^{-1})^* = (\mathbf{H} \mathbf{H}^*)^{-1} = (\mathbf{H}^* \mathbf{H})^{-1} \]

\[ = \text{diag} \left[ \frac{1}{|H_{N}^{(1)}(k_R)|^2}, \ldots, \frac{1}{|H_{0}^{(1)}(k_R)|^2}, \ldots, \frac{1}{|H_{N}^{(1)}(k_R)|^2} \right] . \]  

(3.45)

and the modified covariance matrix can be simplified to

\[ R_n = \frac{16}{Q} \sigma_n^2 (\mathbf{H}^* \mathbf{H})^{-1} . \]  

(3.46)

From (3.46), the structure of the noise covariance matrix remains diagonal meaning that the noise at the different modes are uncorrelated after the transformation to the new basis domain. The transformation of sensor recordings in the eigen basis domain creates an analogy between the noise at the sensors and the noise at the different modes. Further, the noise distribution remains Gaussian but is scaled differently at the different modes.

Secondly, for the least squares based algorithm, the noise covariance matrix after transforming the received data into the eigen basis domain becomes

\[ R_n = 16 \sigma_n^2 \Gamma^\dagger \Gamma^* \]

\[ = 16 \sigma_n^2 [\Gamma^* \Gamma]^{-1} \Gamma^* \Gamma [\Gamma^* \Gamma]^{-1} . \]  

(3.47)

We can decompose \( \Gamma \) to

\[ \Gamma = \mathbf{\Xi} \mathbf{H} . \]  

(3.48)
and simplifying (3.47), we get

$$R_n = \frac{16}{Q} \sigma_n^2 (H^* H)^{-1}. \quad (3.49)$$

Similar to the orthogonality based algorithm, the modified noise covariance matrix maintains its diagonal structure.

Both the proposed algorithms transform the sensor data to a modal domain. This transformation preserves the uncorrelated nature of the noise between different sensors with a similar uncorrelated behavior between the different modes. However, the noise is scaled by a factor which differs between the different modes. This noise scaling is symmetrical about the zeroth order, meaning that mode $n$ and $-n$ undergo the same level of noise scaling. The higher the modulus of the order of a mode $|n|$ the lower the noise scaling. Form the modifications that occur to the noise covariance matrix, we can conclude that both algorithms perform the same transformation on the sensor data.

### 3.5.2 Resolution

The angular resolution of DOA estimation methods depends on the number of sensors and the SNR. These DOA estimation methods relied on correlation of the phase vector of the source signal and the assumed phase if the signal was present at a test direction. If the test direction and the source direction were equal, a maximum value in the spectrum was obtained. In a similar manner, the two proposed algorithms localize sound sources based on the correlation of the received signal to the vector $c(y, \phi)$. This subsection aims to present a theoretical analysis to the factors affecting resolution in the two proposed algorithms.

We start our analysis by assuming that the source signals are zero mean, Gaussian with variance, $\sigma_s^2$. The data covariance matrix in the eigen basis domain is

$$R = \sigma_s^2 c(y, \Phi) c^*(y_0, \phi_0) + \frac{16}{Q} \sigma_n^2 (H^* H)^{-1}. \quad (3.50)$$

To simplify the analysis, we assume there is only one source located at $(y_0, \phi_0)$. Using the identity for simplifying the inverse of a sum of matrices defined in [26, page 490], the inverse of the covariance matrix is

$$R^{-1} = \frac{1}{a\sigma_n^2} (H^* H)^{-1} \left( I - \frac{c(y_0, \phi_0) c^*(y_0, \phi_0) 1/(a\sigma_n^2)(H^* H)^{-1}}{\sigma_s^{-2} + c^*(y_0, \phi_0) 1/(a\sigma_n^2)(H^* H)^{-1} c(y_0, \phi_0)} \right). \quad (3.51)$$

where $a = 16/Q$. Using (3.51) and (3.24), the output from the MV spectrum
can be derived as
\[
Z^{-1}(y, \phi) = \frac{1}{a \sigma_n^2} \left( \langle c^*(y, \phi) H^* H c(y, \phi) \rangle - \frac{|c^*(y, \phi) H^* H c(y_0, \phi_0)|^2}{a \sigma_n^2 / \sigma_s^2 + c^*(y_0, \phi_0) H^* H c(y_0, \phi_0)} \right).
\]

(3.52)

We define the inner product between two vectors, \(a\) and \(b\) as a scalar equivalent to \(a^* b\) and is denoted by \(\langle a, b \rangle\). The output of the MV spectrum (3.52) can be written in terms of the inner products of vectors as
\[
Z^{-1}(y, \phi) = \frac{1}{a \sigma_n^2} \left( \langle H c(y, \phi), H c(y, \phi) \rangle - \frac{|\langle H c(y, \phi), H c(y_0, \phi_0) \rangle|^2}{a \sigma_n^2 / \sigma_s^2 + \langle H c(y_0, \phi_0), H c(y_0, \phi_0) \rangle} \right).
\]

(3.53)

The modulus is introduced in (3.53) since \(Z(y, \phi)\) is a real number. The maximum value of \(Z(y, \phi)\) occurs at \(y = y_0\) and \(\phi = \phi_0\). It is important to mention that the MV spectrum obtained is the same whether we remove \(H\) or not. However, removal of \(H\) upper bounds the number of modes, \(M\) that we can use for a specified region of interest.

The MV spectrum has a peak at the source location which decreases gradually to a minimum, hence the 3dB point can be used to measure the sharpness of this decrease and give a good measure of the resolution of the proposed algorithms. The 3dB point occurs at \((y, \phi)\) satisfying
\[
\frac{Z^{-1}(y_0, \phi_0)}{Z^{-1}(y, \phi)} = \frac{1}{2}.
\]

(3.54)

For a large SNR, the 3dB point is close to the source location, therefore
\[
\langle H c(y, \phi), H c(y, \phi) \rangle \approx \langle H c(y_0, \phi_0), H c(y_0, \phi_0) \rangle = |b|^2.
\]

(3.55)

The peak of the MV spectrum occurs at \(y = y_0\) and \(\phi = \phi_0\), and using (3.55), \(Z^{-1}(y_0, \phi_0)\) is
\[
Z^{-1}(y_0, \phi_0) = \frac{1}{a \sigma_n^2} \left( \frac{|b|^2}{a \sigma_n^2 / \sigma_s^2 + |b|^2} \right) = \frac{1}{a^2 \sigma_n^4 / \sigma_s^2 + a \sigma_n^2 |b|^2} \left( \frac{a \sigma_n^2 |b|^2}{\sigma_s^2} \right)
\]

(3.56)

and \(Z^{-1}(y, \phi)\) is
\[
Z^{-1}(y, \phi) = \frac{1}{a \sigma_n^2 / \sigma_s^2 + a \sigma_n^2 |b|^2} \left( \frac{a \sigma_n^2}{\sigma_s^2} |b|^2 + |b|^4 - \frac{1}{\langle H c(y, \phi), H c(y_0, \phi_0) \rangle^2} \right).
\]

(3.57)
Substituting (3.56) and (3.57) into (3.54), we get

\[
\frac{a\sigma_n^2/|b|^2}{a\sigma_n^2/(\sigma_s^2|b|^2) + 1 - |\langle Hc(y, \phi), Hc(y_0, \phi_0) \rangle|/|b|^4} = \frac{1}{2}.
\] (3.58)

The expression \( \langle Hc(y, \phi), Hc(y_0, \phi_0) \rangle /|b|^2 \) is the cosine of the angle between the vectors \( Hc(y, \phi) \) and \( Hc(y_0, \phi_0) \), we will denote this angle as \( \psi \). In (3.58), the modulus value of the inner product bounds the values of \( \cos(\psi) \) between 1 and 0. When the angle between the two vectors is small, the inner product is close to 1 and when the inner product is close to 0 the angle is large. The angle between these two vectors at the 3dB increases as the SNR \( (\sigma_n^2/\sigma_s^2) \) decreases and is more noticeable if we simplify (3.58) to

\[
\left( \frac{|\langle Hc(y, \phi), Hc(y_0, \phi_0) \rangle|}{|b|^2} \right)^2 = 1 - \frac{16}{Q|b|^2} \frac{\sigma_n^2}{\sigma_s^2}. \] (3.59)

The angle between \( Hc(y, \phi) \) and \( Hc(y_0, \phi_0) \) at the 3 dB point as a function of SNR is

\[
\psi_{3dB} = \cos^{-1} \left( \sqrt{1 - \frac{16}{Q|b|^2} \frac{\sigma_n^2}{\sigma_s^2}} \right)
\] (3.60)

and is valid only when \( \sigma_s^2 > \sigma_n^2 \). From (3.60), the resolution increases as \( \psi_{3dB} \) decreases i.e., as the noise power decreases. In summary, the resolution is inversely proportional to \( \psi_{3dB} \).

From the solution obtained the following remarks can be made

- The resolution is affected by the noise power which distorts the correlation point, \( c(y_0, \phi_0) \) in the sensor data and pushes this correlation point to overlap nearby correlation points \( c(y_0 + \delta, \phi_0 + \varsigma) \) where \( \delta \) and \( \varsigma \) are small in magnitude. Therefore the higher the noise power the lower the resolution and the larger the angle between the source location and the 3 dB vector.

- Increasing the sensors creates more noise averaging thus reducing the effective noise power. This results in a higher resolution and a similar effect can be observed in DOA algorithms.

- The radius at which the sensors are placed (radius of the chest) has an effect on the resolution, shown by the presence of \( H \) in (3.59)). This factor of sensor radius is not present in DOA algorithms. From Fig. 3.5., we can observe that for two points within the sensor radius, the angle \( \psi \) decreases as the sensor radius is increased, hence the \( \psi_{3dB} \) occurs for points...
that are further apart. Therefore, the resolution decreases as sensor radius increases given that the radius of the region of interest is constant and thus the number of modes used remains the same.

- The matrices $H$, $c(y, \phi)$ and $c(y_0, \phi_0)$ are dependent on the wavenumber, $k$ and so dependent on the frequency. Testing for two points in close proximity, we can observe from Fig. 3.6 that as the frequency increases the angle between these two points (placed at different radii), $\psi$ increases. A similar result was obtained for points at different angles. Therefore, an increase in frequency increases both the radial and the angular resolution of the proposed algorithms.

![Figure 3.5: The decrease in angle, $\psi$ as sensor radius increases.](image)

![Figure 3.6: The increase in angle, $\psi$ as frequency increases.](image)
3.5.3 Nyquist’s criteria

Consider (3.19) as a sampling scenario. Here, coefficients of a signal with angular frequency $n$ needs to be recovered given the sampling frequency is $2\pi/Q$. According to Nyquist’s theorem, the sampling frequency must be greater than twice the highest frequency of the signal. Since we have $Q$ sensors over $2\pi$ radians, we require

$$N < \frac{1}{2} Q.$$  \hspace{2cm} (3.61)

Substituting (3.11) in (3.61) for an arbitrary radius, $\tilde{R}$, Nyquist’s criteria is satisfied if

$$\tilde{R} < \frac{\lambda Q}{2\pi e}.$$  \hspace{2cm} (3.62)

Works [27, 28] discussed spatial aliasing effects for the case of linear arrays. Spatial aliasing in linear arrays prevented localization of all sources. However, for localizing sources within a circular array, aliasing can be removed by reducing the radius of region $\tilde{R}$ where sources need to be located. As the frequency of sources increase, this radius reduces. This scenario contains an aliased region where $\tilde{R} > \lambda Q/(2\pi e)$ and a non-aliased region where $\tilde{R} < \lambda Q/(2\pi e)$.

The result from the Nyquist’s criteria gives an important interpretation towards sensor position in localizing sources. Assuming that we want to localize all sources within a radius $\tilde{R}$ then from the Nyquist’s criteria (3.62), the minimum number of sensors $\tilde{Q}$ required can be calculated. Further, these sensors can be placed at any radius greater that $\tilde{R}$. Although, placing the sensors at a large radius can diminish their sensitivity to low power sources. In the sensor recordings noise is present, therefore increasing the number of sensors from $\tilde{Q}$ results in a better resolution since the maximum likelihood estimations (3.23) and (3.28) become more accurate.

3.6 Simulations

The simulations investigate the performance of the two proposed algorithms in localizing sound sources for different noise levels and for different frequencies. A circular array consisting of 40 uniformly spaced sensors on the circumference of a circle is used to record sounds from the sources. The radius of this circle is set to 8 units. The average chest diameter varies according to gender. The approximate average male and female chest diameters are 30 cm and 26 cm respectively [29].
To correspond to a male chest, 1 unit needs to represent 1.875 cm and for a female chest, 1 unit needs to represent 1.625 cm. We have used units since this allows the simulations to be scaled for a wide range of dimensions.

The source signals and the noise are modeled as stationary zero-mean white Gaussian processes. Further, the noise at each sensor is independent of the noise at any other sensor. The noise power received by the sensors is defined from the total signal power at the origin. For $V$ sources, the SNR at each of the sensors is defined to be

$$SNR = 10 \log_{10} \left[ \frac{\sum_{v=1}^{V} P_{v,0}}{\sigma^2_N} \right]$$

where $P_{v,0}$ is the power of the $v^{th}$ source at the origin and $\sigma^2_N$ is the noise power.

The simulations are performed with narrowband sources and for each trial 100 snapshots are taken. The impact of aliasing on the localization methods are investigated in Section 3.6.3. The recorded signals are then discrete Fourier transformed within the desired frequency band. Operations described in Section 3.4 are performed on the data set to obtain MV spectral estimates using the two proposed algorithms. The MV spectral estimate shows peaks at locations where sound sources are present. This chapter will not investigate the effect of increasing the number of sensors or the number of snapshots. These factors were previously investigated in works [30,31] for linear arrays.

The two proposed algorithms provide similar MV spectra and so for brevity, one set of results are illustrated in the following subsections.

### 3.6.1 Localizing Multiple Sources

The environment consists of eight uncorrelated sound sources placed at different radii. The marks "X"s in Fig. 3.7a. shows the actual locations of the eight sources. The SNR is set to 10 dB and the wavelength of the sources is 4 units. Scaling for an average male chest gives wavelength of the sound sources to be 7.5 cm. The speed of sound in lung parenchyma varies between 25 - 75 m/s [32]. Taking the lower speed, the frequency of the sound sources is 333 Hz. Gavriely et al reported spectral characteristics of normal lung sounds to lie approximately between 50 and 1000 Hz [33,34]. A more detailed description of acoustic propagation in the chest was presented in Section 2.2.1. For lung sound localization, the performance of the algorithms are considered only within this frequency range.
X shows the actual position of the sound sources

(a) X-Y view of the spectrum. This is a polar plot with angle versus radius. Successive concentric circles represent an increase of one unit of distance from the center

(b) 3-D plot of the spectrum

Figure 3.7: Spectrum for multiple 2-D sources with SNR = 10 dB.

The use of units for the radius and wavelength can be considered to be a
powerful representation and allows the spectrum obtained to be flexible. The dimensions of the chest varies from one person to another. Suppose a lung sound localization device providing a spectral estimate for different people represents the radius in units which can be scaled for application to the specific chest diameter (measured beforehand with a tape measure). Further, the speed of sound in the lung varies. Therefore, wavelength represented in units can be scaled and represented to frequencies for different speeds using the relationship \( c = f \lambda \).

Peaks in the MV spectral estimate as illustrated in Fig. 3.7b. correspond accurately to actual source locations. In Fig. 3.7a., peaks are represented by the light colored regions. The peaks decrease in height as the radius is increased. Further, the source lying on equal radius as the sensors (radius = 8 units) cannot be detected by both proposed algorithms. Concerning resolution of the algorithms at 10 dB noise, the sources at radius of 5 and 6 are detected as one source since the resolution is not high enough to give two peaks.

3.6.2 Performance under Different Noise Levels

![Figure 3.8: Spectrum obtained when SNR = 20 dB.](image)

In the second scenario, the power of sound emitted by the eight sources is kept constant, but noise power is reduced by half to set the SNR to 20 dB. Even at a reduced noise power, the source at radius of 8 units cannot be detected.
Compared to the previous trial with SNR equal to 10 dB, the peaks corresponding to the source locations are higher and narrower. According to (3.60), this scenario has a resolution that is approximately two times larger than the previous scenario with SNR equal to 10 dB. This is illustrated by Fig. 3.8, where two distinct peaks are obtained for sources at radius of 5 and 6 units.

### 3.6.3 Performance with Different Frequencies of Sound

![Figure 3.9: Spectrum showing a reduction in resolution when wavelength is increased.](image)

In this scenario, the wavelength is increased to 7 units. Using dimensions of an average male chest and speed of sound in the chest as 25 m/s, this wavelength corresponds to a frequency of 190.5 Hz. Comparing Fig. 3.9 and Fig. 3.7, we can see that the resolution is reduced when the wavelength is increased or the frequency is reduced, agreeing with results obtained for the theoretical resolution analysis. Given that by experimental verification under a known wavelength and noise power the resolution can be determined, then (3.60) can use this initial resolution to give resolution of both algorithms for different wavelengths and noise power. Thus, the result obtained using (3.60) is important in the performance analysis of the proposed localizing algorithms.

In this trial, wavelength is reduced to 1 unit. As before, converting to an average male chest dimension, the frequency is increased to 1333 Hz. For lung sounds, this high frequency does not have a high intensity. However, for the purpose of demonstrating aliasing for the proposed algorithms, we will use this
Figure 3.10: Spectrum showing aliasing when the wavelength is reduced. The region $\tilde{R}$ for which sources can be localized is discernible.

Figure 3.11: Spectrum of region $\tilde{R}$ where two sound sources are present and each concentric circle represents a distance of 0.3 units.

frequency to show that simulation results obtained Fig. 3.10 and Fig. 3.11 are in agreement with the Nyquist’s criteria described in subsection 3.5.3. The aliasing that occurs in localizing sources within a circular area is different from the aliasing that occurs for a linear array. This difference in aliasing was discussed in subsection 3.5.3. The simulation results illustrated by Fig. 3.10 and Fig. 3.11 prove that high frequencies cause aliasing, the region for which sources can be localized is reduced to $\tilde{R}$. From simulation results, $\tilde{R}$ is approximately equal to
2.4 units and agrees with the result obtained by applying (3.62) to this scenario. Further, within the region of radius $R$, the resolution is higher than when the wavelength was equal to 4 units. In summary, simulations verify that increasing the frequency results in a higher resolution but reduces the radius of the region in which sources can be localized.

From simulation results presented in this subsection, several deductions to the performance of the algorithms can be made. These deductions are as follows:

- For a set up where the number of sensors and noise power is known, experimental determination of resolution at a certain frequency can be used to calculate the resolutions at other frequencies and noise powers using (3.60) for the proposed algorithms.

- For localizing sources within a frequency range, (3.62) can be applied to determine the radius of the region where sources can be located without aliasing.

- Given the radius of the region, minimum acceptable resolution and frequency range, the number of sensors can be determined using both (3.60) and (3.62).

### 3.7 Comments and Comparison of the Proposed Algorithms

Lung sound measurements at multiple locations can give information on the lung sounds both spectrally i.e. in the frequency domain and the regional distribution of the sounds. Alterations from the normal lung sounds can occur due lung injury or disease such as pneumothorax, lung consolidation, asthma and airway obstruction. These alterations involve a change in frequency content, quantifying and locating sounds for different frequency bins can be used to detect lung abnormalities [3, 35]. The two proposed algorithms can be modified to be used over the entire range of lung sounds by separating this frequency range into a set of narrowband frequency bins and then applying either one of the two proposed algorithms iteratively.

From the simulations presented in the previous section, it was shown that both the proposed algorithms have similar performance. The orthogonality based algorithm is less computationally expensive since it requires calculation of one matrix inverse whereas the least squares based algorithm involves calculating the
inverse of two different matrices. Both algorithms perform the same transformation to the sensor data and the MV spectrum remains the same whether the contributions due the sensor radius is removed or not removed. However, not removing the sensor radius component increases the computational expense of the MV spectrum.

Further, the following comments pertaining to both proposed localizing algorithms can be made

- Both the localizing algorithms can work without previous estimates of source locations.
- Since both the algorithms calculate covariance matrices for the modified sensor recordings, other spatial spectral methods such as MUSIC or its variants can be applied instead of MV spectrum.
- For large sensor arrays and considering the frequency of sound, the dimension can be reduced to $2N + 1$ by converting to the eigen basis sets of a 2D wavefield. This reduces the computation expense of the proposed algorithms.

3.8 Summary

We have proposed two algorithms for localizing sound sources within a circular array of sensors by decomposing the wavefield to a set of eigen basis functions. These two algorithms can be applied for the purpose of acoustic imaging of the chest. However, in this thesis, we have assumed that the velocity of sound in the chest is isotropic. Future work will look at extending these algorithms for a layered cylindrical media that is characteristic of the chest and include reflections, refractions and standing waves.

The resolution of both algorithms increase with a decrease in noise power and increase with an increase in the frequency of the sound. We derived a theoretical relationship for resolution in terms of the noise level and frequency. Further, increase in frequency results in a reduction in the radius of the region for which sound sources are localized provided the number of sensors remain the same. This reduction in radius was a result of Nyquist’s criteria applied to this scenario. The Nyquist’s criteria and results from the resolution analysis can be applied in designing a localization system for the lung sounds given resolution, frequency range and noise power.
The lung sound localization method introduced in this chapter would provide a better performance if the interfering heart sounds can be located in the sensor recordings. Appendix A describes a method to do this based on the quasiperiodic nature of the heart sounds.

References


Chapter 4

Theory of Frequency Invariant Source Localization for Photoacoustic Tomography

4.1 Introduction

In the previous chapter, we derived methods for point source localization within a circular array for lung sound localization. In this chapter, we derive a solution for localizing a frequency invariant, distributed source in the interior of a circular aperture for the purpose of photoacoustic imaging. The flexibility of the proposed solution is shown by its extension to arbitrary sensor geometries. Furthermore, the proposed method can be modified to use a minimum number of samples and a method to determine these samples are also presented.

In this chapter, we propose a novel method for the 2D and 2.5D circular geometries (circular array with integrating line sensors) that expands the source distribution function (initial pressure distribution) in a 2D Fourier Bessel domain. Previous reconstruction methods in these geometries were presented in [1–3]. To reconstruct the source distribution, we estimate the Fourier Bessel coefficients from frequency samples corresponding to the Bessel zeros. This method provides an exact inversion since the exact source distribution is reconstructed given ideal conditions of infinite measurement bandwidth, a continuous measurement aperture and absence of noise. Further, the proposed method provides a complete series solution.

The proposed method does not require infinite bandwidth and conditions for reconstruction for the finite bandwidth case is provided. Further, this method
was extended for discrete apertures and a rule was derived to avoid spatial aliasing. The proposed method is faster than previously proposed methods (see Section 2.3.6) since it only uses a subset of frequency samples and can exploit sparsity of the source distribution in the Fourier Bessel domain.

Previous methods proposed for photoacoustic image reconstruction include the delay and sum beamforming approach [4] and the statistical approach [5]. Time domain methods were proposed for simple sensor geometries such as cylindrical, spherical and planar geometries by Xu et. al. [6]. For these simple geometries frequency domain methods were also discussed in [7].

In practice and due to physical limitations, the sensor geometry may differ from the simple geometries. It is advantageous to place the sensors as close as possible to the sample under study to minimize the acoustic attenuation. Previous approaches for photoacoustic image reconstruction for arbitrary sensor geometries uses a time reversal approach [8, 9] and can only be applied for 3D sensor geometries. In practice, time reversal methods can be applied to 2D since the errors due to truncating the signals quickly become smaller than errors due to noise. Furthermore, the time reversal methods require the sensor geometry to be sufficiently smooth.

We extend the proposed method for regular, circular geometries to arbitrary sensor geometries for photoacoustic imaging using the frequency spectra of the recorded signals. Arbitrary acquisition geometries occur in animal imaging, blood vessel imaging and other in vivo applications of photoacoustic imaging. Our method provides no restriction on the smoothness of the sensor manifold, but requires the sensors to be placed outside the enclosing circular region containing the spatial source distribution. Frequency domain methods in photoacoustic tomography has the advantage of being able to easily compensate for the sensor frequency response. The source distribution is expanded using a 2D Fourier Bessel expansion, whereby the Fourier Bessel coefficients of the source distribution expansion is then obtained using a robust least squares solution at the frequencies corresponding to the Bessel zeros.

We extend our method of estimating the Fourier Bessel coefficients of the source distribution to use a minimum number of samples. Furthermore, we propose a novel method that allows the determination of the optimum number and set of frequency samples required for reconstruction. Previous frequency domain algorithms [1, 7, 10] cannot provide information to the optimum number of frequency samples to use, but assumes a larger number of samples will provide better reconstruction.
This chapter is organized as follows. The next section briefly restates the photoacoustic wave equations. Section 4.3 derives the mathematical formulation for the 2D and 2.5D photoacoustic inverse problem. Section 4.4 describes the modal expansion of the 2D Green’s function and Section 4.5 introduces the 2D Fourier Bessel series expansion of the source distribution. The next section describes the photoacoustic reconstruction method for regular, circular geometries from a frequency invariant source localization perspective together with numerical experiments performed to show images reconstructed with the proposed method. Section 4.7 extends the method proposed in Section 4.6 to arbitrary sensor geometries by utilizing a robust least squares solution. Numerical experiments to validate this method are also described. Moreover, Section 4.8 extends the method in Section 4.6 to use a minimum number of frequency samples. Numerical experiments are conducted to show the effectiveness of the method using a sparse set of frequency samples. The last section summarizes the main ideas presented in this chapter.

### 4.2 Photoacoustic Inverse Problem

From Section 2.3.1, the photoacoustic wave equation is [11]

\[
\frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} - c^2 \nabla^2 p(\mathbf{r}, t) = p_0(\mathbf{r}) \frac{\partial}{\partial t} \delta(t)
\]  

(4.1)

which is an inhomogeneous Helmholtz equation provided the speed of the acoustic waves \( c \) is constant, \( p(\mathbf{r}, t) \) is the pressure at a vector position \( \mathbf{r} \) and at time \( t \), \( \delta(\cdot) \) is a Dirac delta function and \( p_0(\mathbf{r}) \) is the distribution of the initial pressure.

In this chapter, we refer to the initial pressure distribution \( p_0(\mathbf{r}) \) as the source distribution. In photoacoustic tomography, the inverse problem of estimating the source distribution from pressure measurements for \( \mathbf{r} \in \mathbb{R}^3 \) in the frequency domain is

\[
p(\mathbf{r}_s, k) = -ikc \int_V p_0(\mathbf{r})G(k; \mathbf{r}_s, \mathbf{r}) \, d\mathbf{r}
\]  

(4.2)

where \( \int_V(\cdot) \, d\mathbf{r} \) is the integration over a volume of space \( V \subset \mathbb{R}^3 \), the Fourier transform is taken over time \( t \), \( \mathbf{r}_s \) is the sensor position and \( G(\cdot) \) is the three dimensional Green’s function. By analyzing (4.2), we can deduce that the source distribution is frequency invariant, i.e. the source distribution function is the same for all frequencies. Hence, in this thesis, the problem of estimating \( p_0(\mathbf{r}) \) is labeled as a frequency invariant, source localization problem.
Figure 4.1: In the 2D problem, the source distribution is bounded within a radius of $r_0$ and the sensors are placed in the region of validity. This is known as the exterior source problem. The region of validity means that $r_s > r_0$.

### 4.3 Formulation for 2D and 2.5D Circular Geometries

Firstly, we formulate the 2D inverse problem for photoacoustic tomography using a circular array of ultrasound sensors. Let's define a cylindrical co-ordinate system with vector point identified by $(r, z)$ where $\mathbf{r} = (r, \phi)$ is a 2D polar co-ordinate vector where $r$ is the distance from the origin and $\phi$ is the angular position, and $z$ is the vertical co-ordinate. Further, the sensor positions are denoted by $(r_s, z_s)$. A circular sensor geometry is considered where all the sensors are placed on the circumference of a circle of radius $r_s$. We describe the 2D formulation as the estimation of the source distribution $p_0(r)$, $r \in \mathbb{R}^2$ that depends on two co-ordinates and is independent of the vertical direction $z$ along the axis of the cylinder. The reconstruction of the 2D $p_0(r)$ now requires a circular array of point sensors and the geometry of the problem is illustrated by Fig. 4.1. Further, the source distribution is considered to be zero outside a bounding radius $r_0$.

The sensors are all placed at a constant distance from the origin $r_s$ and at a single vertical co-ordinate $z_s$, therefore, the sensor positions vary over only the angular position $\phi_s$. The received pressure at a wavenumber $k$ for a sensor placed at an angular position $\phi_s$ is

$$p(\phi_s, k) = -ikc \int_S p_0(\mathbf{r}) G_{2D}(k; \mathbf{r_s}, \mathbf{r}) \, d\mathbf{r}$$  \hspace{1cm} (4.3)
4.3 Formulation for 2D and 2.5D Circular Geometries

Figure 4.2: In the 2.5D problem, the sensor elements are long in the vertical dimension and the source distribution is enclosed within the sensor array. Using this configuration, the z-averaged source distribution is reconstructed.

where \( \int_S (\cdot) \, d\mathbf{r} \) is the integration over a surface \( S \subseteq \mathbb{R}^2 \) and \( G_{2D}(\cdot) \) is the 2D Green's function. The 2D Green's function can be derived from the 3D Green's function \( G(k; (\mathbf{r}, z), (\mathbf{r}', z')) \) by [12]

\[
G_{2D}(k; \mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} G(k; (\mathbf{r}, z), (\mathbf{r}, z_s)) \, dz_s.
\] (4.4)

We observe from (4.4) that integration over one vertical co-ordinate \( z_s \) removes dependence on the other vertical co-ordinate \( z \). Moreover, equation (4.3) represents the 2D photoacoustic inverse problem. To reconstruct the 2D source distribution, measurements over two degrees of freedom are required that is the angular position of the sensors \( \phi_s \) and the frequency \( k \).

The 2.5 Dimensional problem allows the source distribution \( p_0(\mathbf{r}, z) \) to vary over all three dimensions, however, now integrating line sensors are placed in a circular geometry. The geometry of this problem is illustrated in Fig. 4.2. Similar to the 2D problem, the source distribution is confined within a bounding radius of \( r_0 \) and the sensors are placed at a constant radius \( r_s \). The integration over the vertical direction is performed physically by the integrating line sensors and the received pressure is

\[
p(\phi_s, k) = \int_{-\infty}^{\infty} p((\phi_s, z_s); k) \, dz_s.
\] (4.5)
Substituting (4.5) into (4.2) yields

\[
p(\phi, k) = -ikc \int_S \int_{-\infty}^{\infty} p_0(r, z) \int_{-\infty}^{\infty} G(k; (r_s, z_s), (r, z)) \, dz_s \, dz \, dr
\]  
(4.6)

and now using definition for the 2D Green’s function (4.4), the inner integral can be simplified to give

\[
p(\phi, k) = -ikc \int_S \int_{-\infty}^{\infty} p_0(r, z) \, dz G_{2D}(k; r_s, r) \, dr
\]  
(4.7)

By defining the z-averaged source distribution \( \overline{p_0}(r) \) as

\[
\overline{p_0}(r) \triangleq \int_{-\infty}^{\infty} p_0(r, z) \, dz
\]  
(4.8)

and substituting into (4.7) yields

\[
p(\phi, k) = -ikc \int_S \overline{p_0}(r) G_{2D}(k; r_s, r) \, dr
\]  
(4.9)

A more detailed proof for the 2.5D inverse formulation is provided in [1,2].

In the literature, for source reconstruction in a cylindrical geometry for both ultrasonic reflectivity imaging and photoacoustic imaging the range of integration in the radial direction for (4.3) and (4.9) was taken from 0 to infinity [1, 7, 10]. In our formulation, we know that the source is bounded within the sensor radius, further, we assume that the source distribution is zero outside a region with bounding radius \( r_0 \). Therefore, we use a definite integration for the radial parameter from 0 to \( r_0 \) in our formulation.

To summarize, both the 2D and the 2.5D inverse problem equates to solving the same mathematical problem as can be observed by comparing (4.3) and (4.9). For the 2D problem the source distribution is assumed a priori to be two dimensional whereas for the 2.5D problem the z-averaged source distribution is reconstructed.
4.4 Modal Expansion of the 2D Green’s Function

The Green’s function in 2D for the exterior case where all the sources are enclosed by the sensors is

\[ G_{2D}(k; r_s, r) = \frac{i}{4} H_0^{(1)}(k||r_s - r||) \]  

(4.10)

where \( H_0^{(1)}(\cdot) \) is the Hankel function of the first kind and zeroth order, and \( || \cdot || \) is the Euclidean or \( l_2 \) norm. The addition theorem can be used to expand the 2D Green’s function [13, p.66] as

\[
H_0^{(1)}(k||r_s - r||) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr_s) J_n(kr) e^{-im\phi} e^{im\phi_s} 
\]  

(4.11)

which is valid when \( r_s > r \) and \( J_n(\cdot) \) represents a Bessel function of order \( n \). The region of validity of the addition theorem (4.11) for a source distribution \( p_0(r) \) with bounding radius \( r_0 \) is the region where \( r_s > r_0 \) and is shown in Fig. 4.1. This expansion is called eigen basis or modal expansion of the Green’s function and was used for ultrasound reflectivity imaging [7] and for photoacoustic reconstruction in [6]. Moreover, this modal expansion was applied for source localization within circular sensor arrays in [14,15] and in Chapter 3. The advantage of using this expansion is that the radius and frequency, and angular position for both sensor and source are represented by separable basis functions.

4.5 Eigen Basis Expansion of the Source Distribution

4.5.1 Fourier Bessel Expansion

The Fourier Bessel expansion is a more efficient expansion for finite duration signals and was used for speech and acoustic signals [16]. The \( m^{th} \) order Fourier Bessel expansion of a finite duration signal \( f(t) \) in the interval \( 0 < t < t_u \) is [17]

\[
f(t) = \sum_{\ell=0}^{\infty} \zeta_{\ell} J_m \left( \frac{z_{m\ell}}{t_u} t \right) 
\]  

(4.12)

where the modes \( m \) must be integers and \( z_{m\ell} \) is the \( \ell^{th} \) root of the Bessel function of order \( m \) i.e., \( J_m(z_{m\ell}) = 0 \). The orthogonality property of the Bessel function
considering different zero indices $\ell$ is \cite{18}

$$
\int_0^{t_u} J_m\left(\frac{z_{m\ell}}{t_u} t\right) J_m\left(\frac{z_{m\ell'}}{t_u} t\right) t \, dt
\begin{cases}
\frac{t_u^2}{2} J_{m+1}(z_{m\ell})^2 & \text{if } \ell = \ell' \\
0 & \text{otherwise.}
\end{cases}
$$

(4.13)

Using the above orthogonality property, the analysis equation for the Fourier Bessel series can be written as

$$
\zeta_\ell = \frac{2}{t_u^2 J_{m+1}(z_{m\ell})^2} \int_0^{t_u} f(t) J_m\left(\frac{z_{m\ell}}{t_u} t\right) t \, dt.
$$

(4.14)

4.5.2 2D Fourier Bessel Expansion of the Source Distribution

From the Fourier Bessel series, we define the 2D Fourier Bessel series expansion for the source distribution as

$$
p_0(r) = \sum_{m=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \beta_{m\ell} J_m\left(\frac{z_{m\ell}}{r_0} r\right) e^{im\phi}
$$

(4.15)

where $m$ is called the mode, $\ell$ is the zero index since $z_{m\ell}$ is the $\ell^{th}$ root of $J_m(\cdot)$ and $\beta_{m\ell}$ is a complex Fourier Bessel coefficient. The radial component is expanded using the normal Fourier Bessel series. This series expansion can represent any complex 2D function in polar co-ordinates for a finite interval between $0 \leq r < r_0$. The synthesis equation for the 2D Fourier Bessel expansion considering the orthogonality relationship of the Bessel functions (4.13) and the exponential functions is

$$
\beta_{m\ell} = \frac{1}{\pi r_0^2 J_{m+1}(z_{m\ell})^2} \int_0^{2\pi} \int_0^{r_0} p_0(r) J_m\left(\frac{z_{m\ell}}{r_0} r\right) e^{-im\phi} r \, dr \, d\phi.
$$

(4.16)

From this 2D Fourier Bessel expansion, we can reconstruct the spatial source distribution if the Fourier Bessel coefficients $\beta_{m\ell}$ can be recovered from the measured signals. The following comments can be made with regard to this Fourier Bessel expansion of the source distribution $p_0(r)$:

- The expansion represents the source distribution’s dependence on the radial and angular variables using separable, orthogonal basis functions.
• The basis functions dependent on the radial position is the same as that in the modal expansion of the 2D Green’s function (4.11) and the same can be said for the basis function dependent on the angular position.

• The modes of the Bessel functions and the exponential functions are coupled together because the same coupling is present in the modal expansion of the wavefield (4.11).

• The source distribution \( p_0(r) \) can be represented by its sample values, however, by expanding the source distribution in a different domain a more compact representation is possible requiring estimation of fewer terms in order to obtain \( p_0(r) \).

### 4.6 Reconstruction using a Circular Aperture

Given that the surface \( S \) is a circular region with bounding radius \( r_0 \), the signal received by a sensor at angular position \( \phi_s \) and wavenumber \( k \) can be specified as

\[
p(\phi_s, k) = -ikc \int_0^{r_0} \int_0^{2\pi} p_0(r)G_{2D}(k; r_s, r) \, d\phi \, rdr.
\]  

(4.17)

Substituting (4.15) and (4.11) into (4.17) yields

\[
p(\phi_s, k) = \frac{k}{4} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\ell=1}^{\infty} H_n^{(1)}(kr_s) e^{im\phi_s} \\
\times \left[ \frac{1}{r} \right] \beta_{n\ell} J_m \left( \frac{z_{n\ell}}{r_0} \right) J_n(kr) \, rdr \\
\times \int_0^{2\pi} e^{im\phi} e^{-in\phi} \, d\phi.
\]  

(4.18)

Furthermore, we use the orthogonality property of exponential functions

\[
\int_0^{2\pi} e^{im\phi} e^{-in\phi} \, d\phi = \begin{cases} 2\pi & \text{if } n = m, \\ 0 & \text{otherwise} \end{cases}
\]  

(4.19)

to simplify (4.18) to

\[
p(\phi_s, k) = \sum_{m=-\infty}^{\infty} a_m(k) e^{im\phi_s}
\]  

(4.20)
where

\[ a_m(k) = \frac{\pi kc}{2} H_m^{(1)}(kr) \sum_{\ell=1}^{\infty} \beta_{\ell m} \int_0^{r_0} J_m(kr) J_m \left( \frac{z_{\ell m} r}{r_0} \right) r \, dr. \] (4.21)

Note that (4.20) is the spatial Fourier series expansion of the received signal on a continuous aperture as a function of the aperture angle \( \phi_a \).

### 4.6.1 Modal Filtering

We can estimate the \( a_m(k) \) from the sensor recordings \( p(\phi_a, k) \) by using the Fourier series analysis equation

\[ a_m(k) = \frac{1}{2\pi} \int_0^{2\pi} p(\phi_a, k) e^{-im\phi_a} \, d\phi_a. \] (4.22)

We refer to \( a_m(k) \) as modal coefficients and these outline the angular distribution of the source function. Looking back at the modal array signal processing methods discussed in Section 2.4, our method processes the measured signals in the modal space by processing the modal coefficients of the circular aperture \( a_m(k) \). The processing of signals in the modal space was useful in several other applications (see Section 2.4).

From the recordings of an ideal continuous aperture, it is possible to separate the different modes \( m \) in the source distribution expansion (4.21). This concept of separating the modes will be termed modal filtering. These modes \( m \) then needs to be separated into different zero indices \( \ell \) in order to estimate the 2D Fourier Bessel coefficients \( \beta_{\ell m} \).

### 4.6.2 Frequency-Radial Duality

The following theorem shows how to estimate the source distribution coefficients \( \beta_{\ell m} \) from modal coefficients \( a_m(k) \) at a specific set of frequencies.

**Theorem 4.6.1 (Frequency-Radial duality)** The source distribution \( p_0(r) \) variation over the radial parameter \( r \) for each mode \( m \) are contained in the Bessel series summed over the zero index \( \ell \). Information from a single zero index \( \ell \) can be separated in each mode \( m \) at frequencies corresponding to the Bessel zeros i.e., \( k = z_{\ell m}/r_0 \). Therefore, the 2D Fourier Bessel coefficient \( \beta_{\ell m} \) for a particular mode \( m \) and zero index \( \ell \) can be obtained by multiplying a suitable weight to
the modal coefficient $a_m(k)$ at $k = \frac{z_{mt}}{r_0}$,

$$\beta_{mt} = h_{mt} a_m \left( \frac{z_{mt}}{r_0} \right)$$  \hspace{1cm} (4.23)

where the weight $h_{mt}$ is

$$h_{mt} = \frac{4}{\pi c r_0 H_m^{(1)}(r_s \frac{z_{mt}}{r_0}) z_{mt} [J_{m+1}(z_{mt})]^2}.$$  \hspace{1cm} (4.24)

**Proof**

We use the orthogonality relationship for Bessel functions [18], (4.13) in (4.21) at $k = \frac{z}{r_0}$ to obtain

$$a_m(k)|_{k=z_{mt}/r_0} = \frac{\pi z_{mt} c}{2r_0} H_m^{(1)} \left( \frac{z_{mt}}{r_0} r_s \right) \frac{r_0^2}{2} [J_{m+1}(z_{mt})]^2 \beta_{mt}$$  \hspace{1cm} (4.25)

and by applying the definition of $h_{mt}$ provided by (4.24) in (4.25) we obtain (4.23).

By calculating the Fourier Bessel coefficients $\beta_{mt}$ from the measured signals, we can reconstruct the source distribution $p_0(\mathbf{r})$ using the Fourier Bessel series synthesis equation (4.15).

In the photoacoustic literature, there are exact and approximate inversion methods. More information on these exact and approximate inversion methods was provided in Section 2.3.6. Exact inversion methods are defined as inversion methods that can recover the source distribution $p_0(\mathbf{r})$ exactly given an infinite bandwidth of measured signals in the absence of noise and a continuous observation surface. The method described in this section is an exact inversion method, since an infinite number of Fourier Bessel coefficients $\beta_{mt}$ for the source distribution expansion can be calculated if the bandwidth in infinite and a continuous circular aperture is available. Therefore, the source distribution $p_0(\mathbf{r})$ can be reconstructed exactly using the synthesis equation (4.15).

### 4.6.3 Comparison with Fourier Domain Methods

This section begins with a brief overview of the Fourier Domain algorithms, first proposed for ultrasound imaging [7] and then modified for photoacoustic imaging [6,19]. A different description of this method with Hankel transforms was presented in Section 2.3.6. This algorithm described in 2D expands the
theory of frequency invariant source localization for photoacoustic tomography

source distribution function as

\[ p_0(\mathbf{r}) = \sum_{m=0}^{\infty} \int_0^{\infty} \alpha_m(k) k J_m(kr) \, dk \, e^{im\phi} \quad (4.26) \]

and expands the measured signals as

\[ p(\mathbf{r}, k) = \sum_{m=0}^{\infty} \hat{\alpha}_m(k) e^{im\phi_s}. \quad (4.27) \]

We can calculate \( \hat{\alpha}_m(k) \) by the inverse of (4.27) (a transform similar to (4.22)) over all the sensors which are placed in a circle. From \( \hat{\alpha}_m(k) \), we get \( \alpha_m(k) \) by

\[ \alpha_m(k) = \frac{\hat{\alpha}_m(k)}{c(k) H_m^{(1)}(kr_s)} \quad (4.28) \]

where \( c(k) \) is a constant equal to \( \pi k c/2 \). To compute \( p_0(\mathbf{r}) \) at a particular point, we take the Hankel transform over \( k \) of \( \alpha_m(k) \) and then sum over the modes (4.26). Note that \( \alpha_m(k) \) is equivalent to \( \int_0^{\pi} \int_0^{2\pi} p_0(\mathbf{r}) J_m(kr) e^{-im\phi} \, dr \, d\phi \) and the orthogonality of the Bessel functions over an infinite interval \([18]\)

\[ \int_0^{\infty} k J_m(kr) J_m(kr') \, dk = \frac{1}{r} \delta(r - r') \quad (4.29) \]

is used to recover the source distribution at a particular vector position \( \mathbf{r} \), shown in (4.26). One of the drawbacks of the Fourier Domain methods are that they are computationally expensive requiring the sum of a large number of terms at every point (see (4.26)). Further, Fourier Domain methods require an infinite bandwidth otherwise the relationship shown in (4.29) is no longer valid. The Fourier Domain method was modified in \([6,19]\) to recover the source distribution in the time domain reducing the computational complexity.

Rather than integrating over a frequency range, the proposed algorithm considers the natural integration that occurs as a wave propagates through a region of space and is therefore not affected by the spatial sampling issues due to a discrete sensor. The concept of frequency-radial duality introduced in this chapter is novel and has not been utilized previously for photoacoustic imaging or ultrasound imaging. Since we know that the source distribution is bounded in a radial region \( r_0 \), we need to only consider frequencies corresponding to the Bessel zeros \( k = m \pi / r_0 \) and these frequencies are only resolved to a single mode. In addition, the source distribution can be sparse in the Fourier Bessel domain and
summation can be done over only the largest modes and indices. These lead to a large reduction in computational complexity compared to the Fourier Domain methods where all frequencies are used and resolved to all modes. It is important to mention that a lower bound $r_0$ only means that less frequency samples are used and has no effect on the resolution of the reconstructed image.

For the circular geometry, a solution proposed by Norton [3] was unstable due to possibility of divisions by zero. Our method does not have such a shortcoming since the values of Bessel functions $J_{m+1}(z_{mt})$ will never be zero. The computational order of both the method in [3] and the Fourier domain method for reconstruction on an $N \times N$ grid requires $\mathcal{O}(N^3)$ flops. Further, the time reversal methods have an operation count of $\mathcal{O}(N^3)$. A review of the various photoacoustic inversion methods and their computational order can be found in Section 2.3.6. The method proposed in [20] used a similar source expansion, however their method requires that the source provides a smooth Hankel transform.

The proposed method avoids numerical complications in calculating the Hankel transforms for high modes where the Bessel functions are highly oscillatory [21]. The spatial Fourier series expansion of the aperture for $N$ modes with $N_k$ frequency samples requires $\mathcal{O}(N_k N \log N)$ floating point operations with the Fast Fourier Transform (FFT). Given that there is maximum of $N_R$ zero indices for any mode $m$, then calculation of all the Fourier Bessel coefficients requires $\mathcal{O}(N N_R)$ operations. Now from the Fourier Bessel coefficients, the source distribution $p_0(\mathbf{r})$ can be reconstructed using the synthesis equation (4.15). The sums over the zero indices $\ell$ for a single mode $m$ is first done with a fast Bessel series sum [22] and has operation count $\mathcal{O}(N_R \log N_R)$. From there the FFT is used giving the total operation count for the synthesis step as $\mathcal{O}(N^2 \log N)$ assuming $N_R = N$ and $N_k = N$. Therefore, the proposed method has an operation count of $\mathcal{O}(N^2 \log N)$ and comparable to the fast inversion methods for the planar [23] or cubic acquisition geometries [24]. A new method that optimizes the frequency domain method was derived in [25] and has a similar operation count to our method but does not provide a complete series solution or reduce the number of frequency samples to process for each mode $m$.

One pertinent question in photoacoustic imaging is how to recover the image with a discrete aperture and avoid distortions due to spatial aliasing. Both the Fourier-domain and the time-domain methods require infinite bandwidth and a continuous aperture. Theoretical validation to their extension to the discrete and finite bandwidth case has not been provided. The next two sections highlight the advantages of our approach in considering a discrete aperture and spatial
4.6.4 Discrete Aperture

In the previous sections, estimation of the Fourier-Bessel coefficients was done assuming ideal conditions with infinite bandwidth and a continuous aperture. Exact reconstruction of the source distribution \( p_0(r) \) is possible under these ideal conditions. In this section, we provide the conditions under which the source distribution can be reconstructed with a discrete aperture and a bandwidth limited by the frequency response of the ultrasound transducer.

Provided there are \( Q \) uniformly placed sensors at a radius \( r_s \), then the discrete aperture response at a wavenumber \( k \), as a vector is

\[
p(\phi_s, k) = [p(\phi_s^{(1)}, k), \ldots, p(\phi_s^{(Q)}, k)]^T
\]

where \( \phi_s^{(q)} \) is the angular position of the \( q^{th} \) sensor. The modal coefficients \( a_m(k) \) can be calculated with a discrete version of the spatial Fourier transform (4.22)

\[
a_m(k) = \frac{1}{2\pi} e(m)^T p(\phi_s, k) \Delta\phi_s
\]

where

\[
e(m) = [e^{-im\phi_s^{(1)}}, \ldots, e^{-im\phi_s^{(Q)}}]^T.
\]

and \( \Delta\phi_s \) is the angular spacing between the sensors.

For the discrete aperture, both temporal and spatial aliasing can occur. Temporal aliasing can be avoided by using a sampling frequency that is greater than twice the frequency response upper limit of the transducer. Given that we need to decompose the wavefield to a finite number of modes \( m \in [-M, \ldots, M] \), a minimum number of sensors are required which must satisfy

\[
Q > 2M.
\]

Additionally, the contributions of the modes higher than \( M \) should be negligible at this wavenumber in order to avoid spatial aliasing. The spatial aliasing that can occur due to overlapping of the higher modes is referred to as modal aliasing, more details on this is provided in [26] and in Section 2.4.1. Therefore, given a discrete number of sensors, to avoid modal aliasing the Fourier Bessel expansion of the source distribution \( p_0(r) \) should be bandlimited i.e., coefficients for modes greater than \( M \) should be negligible for the transducer frequency response. Fur-
4.6 Reconstruction using a Circular Aperture

ther, reconstruction of the source distribution is only possible if this expansion has significant terms for modes $m \in [-M, \ldots, M]$.

4.6.5 Spatial Filtering

It is important to avoid spatial aliasing since this can cause blurring and distortion in the reconstructed image. The bandlimit restriction of the source distribution $p_0(\mathbf{r})$ to avoid aliasing limits the use of this method for practical scenarios. In the literature, there is no method prescribed to perform spatial filtering for photoacoustic image reconstruction. However, for the method described in this chapter spatial filtering is possible.

The orthogonality relationship for the Bessel functions (4.13) can be evaluated for a continuous range of frequencies as [27]

$$\int_0^{r_0} J_m(kr) J_m\left(\frac{z_m \ell}{r_0} r\right) rdr$$

$$= \begin{cases} \frac{r_0^2}{2} [J_{m+1}(z_m \ell)]^2 & \text{if } k = \frac{z_m \ell}{r_0}, \\ \frac{z_m \ell J_{m-1}(z_m \ell)}{k^2 - (z_m \ell/r_0)^2} J_m(kr_0) & \text{otherwise.} \end{cases} \tag{4.34}$$

The Bessel function $J_m(kr_0)$ in (4.34) higher than the zeroth mode are close to zero for values of $kr_0$ lower than a particular upper limit. This upper limit increases with the mode $m$. Using this property, a rule of thumb used in array signal processing and in source localization [26] to avoid spatial aliasing is stated as follows: the maximum number of modes $M$ present for a particular frequency response upper limit $k_u$ is

$$\text{minimize } M \text{ such that } M > k_u r_0. \tag{4.35}$$

By limiting the frequency upper limit, spatial filtering is achieved i.e., the number of modes $m$ of the source distribution $p_0(\mathbf{r})$ in the measured signals are limited. This criterion also governs the number of sensors needed to avoid spatial aliasing with respect to the frequency upper limit. Since if the number of modes is $M$, we require more than $2M$ sensors.
4.6.6 Design Example: Incorporating Transducers with Different Frequency Responses

This section proposes a method to combine the measurements from two different types of ultrasound sensors with different frequency responses. Practically, ultrasound transducers have a limited frequency response, therefore, two types of sensors with different frequency responses can be combined to increase the measurement bandwidth in photoacoustic imaging. This would ultimately lead to better quality reconstruction since more information about the initial pressure distribution is now available.

We assume that the first type of sensor has a frequency response lower limit $k_l^{(1)}$ and upper limit $k_u^{(1)}$ and the second sensor has a frequency response lower limit $k_l^{(2)}$ and upper limit $k_u^{(2)}$. From (4.35), the impinging wavefield on the first transducer contains $M^{(1)}$ significant modes and the second transducer has $M^{(2)}$ significant modes. The number of transducers should satisfy (4.33) and can be placed at different angular positions on the circumference of a circle with radius $r_s$. If there is no overlap in the frequency responses at $k$ then the modal coefficient $a_m(k)$ can be estimated by (4.31) using the measurements from one type of sensor, otherwise measurements from both types of sensors can be used resulting in a more accurate estimate of the Fourier Bessel coefficient $\beta_{ml}$. By transforming to the modal space, we get a set of modal coefficients that are invariant with respect to the angular placement of the different types of sensors.

Furthermore, optimizations in reducing the number of sensors are possible if there is an overlap in the frequency responses. If the highest frequency response of the first type of sensor minus the overlap is $k_u^{(1)}$, then we need to find the maximum number of modes $M$ using this new frequency upper limit, hence, the required number of sensors of the first type can be reduced. Note that the measurements from both sensors will be used in estimating the Fourier Bessel terms within the frequency band overlap.

4.6.7 Numerical Experiments

In this section we describe the numerical experiments performed to validate our proposed algorithm. The numerical phantom used in the simulations is shown in Fig. 4.3, with $r_0$ as 15 mm, speed of propagation $c$ is 1.5 mm/µs (speed of sound in soft tissue) and the sensors are placed in a circle at a radius of 50 mm. The bandwidth of measurement is from 0 to 3 MHz, this means that modes up
4.6 Reconstruction using a Circular Aperture

Figure 4.3: Input sample (a) x-y view (b) x-z view through the central axis, with arbitrary units (a.u.) for the relative absorption.

to 180 need to be resolved (4.35), therefore, 380 sensors are placed uniformly around the source to avoid spatial aliasing. Also, for the zeroth mode, only 60 zero indices can be recovered. In the numerical experiments, we approximated the signal received at the sensor for each of the required frequencies \( k = z_{mf}/r_0 \) using a quadrature approximation to the double integral shown in (4.3).

In this work, we are interested in estimating the source distribution \( p_0(\mathbf{r}) \) using snapshots differing in frequencies rather than snapshots differing in time. Therefore, the noise to be introduced in the simulations has to be defined differently. Assuming that we use frequencies in the range from \( k_\ell \) to \( k_u \) then the power of the signal between this range is

\[
P_{\text{signal}} \triangleq \frac{1}{k_u - k_\ell} \int_{k_\ell}^{k_u} |f(\gamma)|^2 \, d\gamma. \tag{4.36}
\]

Since we are working with discrete samples, (4.36) is modified to

\[
P_{\text{signal}} = \frac{1}{k_u - k_\ell} \sum_{\gamma=2}^{\Upsilon} |f(\gamma)|^2 (\tau(\gamma) - \tau(\gamma - 1)). \tag{4.37}
\]

In (4.37), there are \( \Upsilon \) non-uniform, discrete samples in the frequency range of interest, \( f(\gamma) \) is the signal recorded at the \( \gamma^{th} \) frequency sample and \( \tau(\gamma) \) is the frequency at sample \( \gamma \), arranged in ascending order. The SNR in dB can then be defined in the normal way as \( 10 \log_{10}(P_{\text{signal}}/\sigma_n^2) \) where \( \sigma_n^2 \) is the noise power.
Further, the noise is AWGN. For the simulations a SNR of 20 dB was used, with 20 measurements available at each required frequency to average out the noise.

We applied our proposed method to the frequency samples in order to estimate $\beta_{mt}$. The values of the Fourier Bessel coefficients $\beta_{mt}$ of the input source distribution is shown in Fig. 4.4. We observe that the magnitude of most coefficients are negligible. A reconstruction using only the largest estimated 60 Fourier Bessel coefficients, over both the mode $m$ and zero index $\ell$, is illustrated by Fig. 4.5 and using 120 Fourier Bessel coefficients is illustrated by Fig. 4.6. It is observed that a better reconstruction with improved resolution results if more coefficients $\beta_{mt}$ are used. However, this increases the computational expense.

4.7 Reconstruction with An Arbitrary Detection Geometry

4.7.1 Problem Statement

Frequency domain algorithms for photoacoustic tomography for planar, cylindrical and spherical sensor geometries were presented by [7] and are well known. The least squares solution [28] by discretizing the integral operation in (4.3) or the partial differential equation (4.1) can be used to provide a solution for photoacoustic inversion with an arbitrary sensor geometry. However, the least squares method requires storage of large matrices (see Section 2.3.6). In this section, we provide a better least squares solution in the frequency domain which does not
4.7 Reconstruction with An Arbitrary Detection Geometry

Figure 4.5: Reconstructed image using the largest 60 estimated Fourier Bessel coefficients, over both the mode $m$ and zero index $\ell$, with a SNR = 20 dB.

Figure 4.6: Reconstructed image using the largest 120 estimated Fourier Bessel coefficients, in terms of both $m$ and $\ell$, with a SNR = 20 dB.
require discretization of any operator and involves solving a set of smaller least squares problems to estimate the Fourier Bessel coefficients $\beta_{ml}$ of the source distribution $p_0(r)$. Thus, solving the large memory requirements.

The sensor geometry may differ from the simple geometries in [7] due to physical limitations and by placing the sensors as close as possible to the object to be imaged minimizes the attenuation of the acoustic waves.

We look at the problem of reconstructing the spatial distribution $p_0(r)$ in the frequency domain given that the sensors are placed in an arbitrary geometry completely enclosing the spatial source distribution, the configuration is illustrated by Fig. 4.7. It is not a requirement that the sensor manifold to be sufficiently smooth but the source distribution $p_0(r)$ is enclosed in a circular region of radius $r_0$ i.e., the source distribution is zero at all time outside of this region. Moreover, the sensors are placed outside of this region with the radial and angular position of the $j^{th}$ sensor denoted by $r_j$ and $\phi_j$ respectively and the total number of sensors are denoted by $J$. We concentrate on solving this problem in $\mathbb{R}^2$ and neglect the additive white Gaussian noise (AWGN) present in the sensor recordings while presenting our method in order to simplify notation.

![Sensor Manifold](image)

Figure 4.7: Configuration of the source distribution and the arbitrary sensor geometry.

Rather than start with a continuous aperture, we solve the problem assuming discrete sensors. The area of detection for each sensor is not considered and the sensors are regarded as point detectors. In photoacoustic tomography, the
4.7 Reconstruction with An Arbitrary Detection Geometry

The detection area of the sensors causes blurring in the reconstruction of the spatial distribution \( p_0(\mathbf{r}) \). The deconvolution methods required to deblur the images is beyond the scope of this thesis. In addition, the transducers have a finite bandwidth with lower frequency limit \( k_l \) and upper frequency limit \( k_u \). The proposed reconstruction method does not assume infinite bandwidth and takes into account this bandwidth limitation.

### 4.7.2 Truncation of Modal Expansion of Green’s Function

We rewrite the modal expansion of the 2D Green’s function presented in Section 4.4 as

\[
H_0^{(1)}(k||\mathbf{r}_s - \mathbf{r}||) = \sum_{n=-\infty}^{\infty} \alpha_n(k; \mathbf{r}) H_n^{(1)}(k \mathbf{r}_s)e^{in\phi_s} 
\]  

(4.38)

where

\[
\alpha_n(k; \mathbf{r}) = J_n(kr)e^{-in\phi}. 
\]  

This explicitly separates the contribution due to sensor positions, and the terms dependent on the source positions are grouped into a new parameter \( \alpha_n(k; \mathbf{r}) \). We note that the drawback to this expansion is that it involves an infinite number of terms. However, truncation to \( N \) terms can be done for a particular wavenumber \( k \) with negligible error if \( N = \lfloor kr_0 \rfloor \) by using the result presented in Section 4.6.5. This truncation can be applied since the Bessel functions \( J_m(kr) \) of orders higher than \( \lfloor kr_0 \rfloor \) are close to zero for arguments \( kr \) smaller than \( kr_0 \).

### 4.7.3 Proposed Frequency Domain Algorithm

We have introduced the modal expansion for the wavefield (4.38) and the 2D Fourier Bessel expansion of the source distribution \( p_0(\mathbf{r}) \) in Section 4.5.2, we will use these to propose a method to obtain the Fourier Bessel coefficients from the frequency spectra of the sensor recordings. Equation (4.3) for the sensor readings in the frequency domain can be simplified by substituting the wavefield decomposition (4.38) and the Fourier Bessel expansion of the source distribution (4.15). As a result the received signal at the \( j^{th} \) sensor can be written as

\[
p(r_j, \phi_j, k) = \frac{k\rho}{4} \sum_{n=-\lfloor kr_0 \rfloor}^{\lfloor kr_0 \rfloor} \hat{\alpha}_n(k; \mathbf{r}) H_n^{(1)}(kr_j)e^{in\phi_j} 
\]  

(4.40)
where

\[ \hat{\alpha}_n(k; r) = \int \alpha_n(k; r)p_0(r) \, dr \]
\[ = \sum_{m=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \beta_{m\ell} \int_0^{r_0} J_m \left( \frac{z_{m\ell} r}{r_0} \right) J_n(kr) r \, dr \]
\[ \times \int_0^{2\pi} e^{im\phi} e^{-in\phi} \, d\phi. \]  

(4.41)

Equation (4.40) is similar to the expansion characterizing the outgoing wave from a source field. Moreover, the truncation of the modal expansion is applied in (4.40) to avoid summation of infinite terms. The advantage of using separable basis in both the modal expansion and the Fourier Bessel expansion can be observed from (4.41), thus allowing separation of the two integrals. Further simplification is possible since the orthogonality property of the exponential functions means that the second integral is zero when \( m \neq n \) and equal to \( 2\pi \) when \( m = n \). Therefore, \( \hat{\alpha}_n(k; r) \) is simplified to

\[ \hat{\alpha}_n(k; r) = 2\pi \sum_{\ell=1}^{\infty} \beta_{n\ell} \int_0^{r_0} J_n \left( \frac{z_{n\ell} r}{r_0} \right) J_n(kr) r \, dr. \]  

(4.42)

Truncation of the wavefield decomposition to a finite number of modes \( n \) results in a filtering of the modes \( m \) in the source distribution expansion. The source distribution is also truncated to the same modal indices, as observed by the simplification of (4.41) to (4.42) by utilizing the orthogonality of the exponential functions. Another conclusion we can draw from (4.42) is that if we separate the modes \( n \) in the wavefield decomposition this in turn separates the modes \( m \) in the source distribution expansion.

**Least Squares Formulation**

Let's assume that the modes have been separated, we still need a way to separate the parts dependent on the zero indices \( \ell \). For this, we can use the orthogonality property of the Bessel functions (4.13) and use frequencies corresponding to the Bessel zeros \( k = z_{n\ell'}/r_0 \). Hence, at \( k = z_{n\ell'}/r_0 \), the coefficient \( \hat{\alpha}_{n'}(z_{n\ell'}/r_0; r) \) becomes

\[ \hat{\alpha}_{n'} \left( \frac{z_{n\ell'} \ell}{r_0}; r \right) = \beta_{n\ell'} \pi r_0^2 \left[ J_{n'+1}(z_{n\ell'}) \right]^2, \]  

(4.43)

the contributions of the other zero indices when \( \ell \neq \ell' \) becomes zero, see (4.13). To make notations simpler, we define a new variable \( \gamma_{n\ell'} = \beta_{n\ell'} \pi r_0^2 J_{n'+1}(z_{n\ell'}) \),
so that
\[ \hat{\alpha}_{n'} \left( \frac{z_{n'\ell'}}{r_0}; \mathbf{r} \right) = \gamma_{n'\ell'} J_{n'+1}(z_{n'\ell'}). \] (4.44)

Note that the roots of the Bessel functions of the negative and positive modes \( n \) are equal i.e., \( z_{n'\ell'} = z_{(-n')\ell'} \) since \(( -1 )^n J_n(kr) = J_{-n}(kr)\). Therefore,
\[ \hat{\alpha}_{-n'} \left( \frac{z_{n'\ell'}}{r_0}; \mathbf{r} \right) = \gamma_{(-n')\ell'} J_{-n'-1}(z_{n'\ell'}). \] (4.45)

At \( k = \frac{z_{n'\ell'}}{r_0} \) the value of \( \hat{\alpha}_n \left( \frac{z_{n'\ell'}}{r_0}; \mathbf{r} \right) \) for the other modes \( (n \neq n' \text{ and } n \neq -n') \) is
\[ \hat{\alpha}_n \left( \frac{z_{n'\ell'}}{r_0}; \mathbf{r} \right) = p_n \left( \frac{z_{n'\ell'}}{r_0} \right) J_n(z_{n'\ell'}). \] (4.46)

where
\[ p_n \left( \frac{z_{n'\ell'}}{r_0} \right) = \sum_{\ell=1}^{\infty} \beta_{n\ell} \frac{2\pi r_0 z_{n\ell} J_{n-1}(z_{n\ell})}{(z_{n'\ell'})^2 - (z_{n\ell})^2}. \] (4.47)

The previous result is obtained by applying the following identity of the Bessel functions [27]
\[ \int_0^{r_0} J_n(ar)J_n(br)r \, dr = \frac{br_0 J_n(ar_0)J_{n-1}(br_0) - ar_0 J_n(ar_0)J_n(br_0)}{a^2 - b^2}. \] (4.48)

Let's define two sets
\[ A = \{ n | n \in \mathbb{Z}, [-kr_0] \leq n \leq [kr_0], n \neq n', n \neq -n' \} \] (4.49)

and
\[ B = \{ n | n \in \mathbb{Z}, n = n' \cup n = -n' \}. \] (4.50)

Using these two definitions and substituting (4.44), (4.45) and (4.46) into (4.40) at \( k = \frac{z_{n'\ell'}}{r_0} \), we get
\[ p \left( r_j, \phi_j, \frac{z_{n'\ell'}}{r_0} \right) = \frac{z_{n'\ell'}}{4r_0} \left[ \sum_{n \in A} p_n \left( \frac{z_{n'\ell'}}{r_0} \right) J_n(z_{n'\ell'}) H_n^{(1)} \left( \frac{z_{n'\ell'}}{r_0} r_j \right) e^{in \phi_j} \right] \]
\[ + \sum_{n \in B} \gamma_{n'\ell'} J_{n+1}(z_{n'\ell'}) H_n^{(1)} \left( \frac{z_{n'\ell'}}{r_0} r_j \right) e^{in \phi_j} \left. \right]. \] (4.51)
Using all the $J$ sensors, we can turn (4.51) into a matrix equation

$$
P\left(\frac{z_{n't'}}{r_0}\right) = \frac{z_{n't'}}{4r_0} S\left(\frac{z_{n't'}}{r_0}\right) \gamma\left(\frac{z_{n't'}}{r_0}\right)
$$

(4.52)

where

$$
P\left(\frac{z_{n't'}}{r_0}\right) = \left[ p\left( r_1, \phi_1, \frac{z_{n't'}}{r_0}\right), \ldots, p\left( r_J, \phi_J, \frac{z_{n't'}}{r_0}\right) \right]^T,
$$

(4.53)

$(-)^T$ being the transpose operator,

$$
\gamma\left(\frac{z_{n't'}}{r_0}\right) = \left[ p_{-N}\left(\frac{z_{n't'}}{r_0}\right), \ldots, \gamma(-n')e', \ldots, \gamma_{n'}e', \ldots, p_N\left(\frac{z_{n't'}}{r_0}\right) \right]^T,
$$

(4.54)

$$
S\left(\frac{z_{n't'}}{r_0}\right) = \left[ s_{-N}\left(\frac{z_{n't'}}{r_0}\right), \ldots, s_N\left(\frac{z_{n't'}}{r_0}\right) \right],
$$

(4.55)

for $n \in A$

$$
s_n\left(\frac{z_{n't'}}{r_0}\right) = \left[ J_n(z_{n't'})H_n^{(1)}\left(\frac{z_{n't'}}{r_0}\right) e^{i\phi_1}, \ldots, \right.
$$

$$
J_n(z_{n't'})H_n^{(1)}\left(\frac{z_{n't'}}{r_0}\right) e^{i\phi_J}\right]^T,
$$

(4.56)

and for $n \in B$

$$
s_n\left(\frac{z_{n't'}}{r_0}\right) = \left[ J_{n+1}(z_{n't'})H_n^{(1)}\left(\frac{z_{n't'}}{r_0}\right) e^{i\phi_1}, \ldots, \right.
$$

$$
J_{n+1}(z_{n't'})H_n^{(1)}\left(\frac{z_{n't'}}{r_0}\right) e^{i\phi_J}\right]^T.
$$

(4.57)

From the matrix equation (4.52), we devise a least squares solution to get

$$
\gamma(z_{n't'}/r_0) = p(z_{n't'}/r_0) \text{ (the sensor reading which we have)}
$$

using the pseudo-inverse

$$
\hat{S}\left(\frac{z_{n't'}}{r_0}\right) = \left[ S\left(\frac{z_{n't'}}{r_0}\right)^H S\left(\frac{z_{n't'}}{r_0}\right) \right]^{-1} S\left(\frac{z_{n't'}}{r_0}\right)^H
$$

(4.58)

where $(-)^H$ is the conjugate transpose operator i.e.,

$$
\gamma\left(\frac{z_{n't'}}{r_0}\right) = \frac{4r_0}{z_{n't'}e'} \hat{S}\left(\frac{z_{n't'}}{r_0}\right) p\left(\frac{z_{n't'}}{r_0}\right).
$$

(4.59)

From $\gamma(z_{n't'}/r_0)$, we only need the values corresponding to $\gamma(-n')e$ and $\gamma_{n'}e$ and
the rest can be discarded. With these two values, we can get \( \beta_{n'\ell'} \) and \( \beta_{(-n')\ell'} \) by

\[
\beta_{n'\ell'} = \frac{\gamma_{n'\ell'}}{\pi r_0^2 J_{n'+1}(z_{n'\ell'})}
\]

(4.60)

This least squares procedure is applied iteratively to frequencies corresponding to Bessel zeros for the other modes \( n \) and zero indices \( \ell \), to calculate the set of Fourier Bessel coefficients present within the sensor bandwidth. Subsequently, using the Fourier Bessel series synthesis equation (4.15), the source distribution \( p_0(r) \) can be reconstructed.

**Discussion on the Proposed Method**

Since the sensors have a finite bandwidth, we are only able to recover a subset of the Fourier Bessel coefficients. These coefficients have mode \( m \) satisfying \( \{m|m \in \mathbb{Z}, -[k_u r_0] \leq m \leq [k_u r_0]\} \) and for these modes the zero indices \( \ell \) belong to the set \( \{\ell|\ell \in \mathbb{Z}, k_{l\ell} r_0 \leq z_{m\ell} \leq k_u r_0\} \). Further, aliasing needs to be considered. The number of sensors \( J > 2N \) i.e., \( J > 2k_u r_0 \) to prevent aliasing (see Section 4.6.5). Additionally, this ensures that the least squares solution always exists since the size of \( \gamma(z_{n'\ell'}/r_0) \) will be less than the number of sensors.

The Helmholtz Equation Least Squares (HELS) method [29] is quite popular in acoustic holography and can be applied for this case to map the acoustic signals to a circular manifold, from which the method described in Section 4.3 can be applied. However, the HELS method is not very robust requiring regularization and the number of modes required is determined by trial and error. The ill-conditioning of HELS method occurs because the magnitude of the Hankel function increase with the absolute value of the mode. A more detailed discussion on the ill-condition of inverting a matrix containing Hankel functions is provided in Section 3.4.1. However, in our proposed method the Hankel function is coupled to the Bessel function for which the magnitude decreases with the absolute value of the mode \( m \) and so is not affected by the ill-conditioning problem in the HELS method.

**4.7.4 Numerical Experiments**

This section describes the numerical experiments conducted and the results obtained to prove the effectiveness of the proposed algorithm. The input source distribution is shown in Fig. 4.8. The signals generated in the time domain by the infinite cylinders are calculated using the formulas provided in [11] and these
have been proven by experimental data.

Figure 4.8: (a) The input spatial source distribution used in the numerical experiments. (b) Relative absorption, in arbitrary units [a.u.], through the horizontal central axis of the input spatial source distribution.

Figure 4.9: Signals recorded by one of the sensors for the given input distribution.

The signals are generated for a period $T_s$ of 50 $\mu$s and is sampled at 4 MHz. An example of a recorded signal for one of the sensors for the given input distribution is shown in Fig. 4.9. The sensor bandwidth is set between 0 and 1 MHz. The time domain signals are filtered to take account of the sensor bandwidth. Moreover, for each sensor, we calculate the power of the received signal and add white Gaussian noise ensuring an SNR of 20 dB. The SNR at the $j^{th}$ sensor is defined according to the following equation: $SNR = 10 \log_{10}(\frac{\int_0^{T_s} |p(r_j, \phi_j, t)|^2 \, dt}{(T_s \sigma_n^2)})$ where $\sigma_n^2$
is the noise power. The signals are padded with zeros ensuring that the frequency spectra has 200 samples in the sensor bandwidth. The Fast Fourier Transform (FFT) is applied to the recorded signals and a simple, linear interpolation is applied to get values for frequencies corresponding to the Bessel zeros.

The radius \( r_0 \) is set at 10 mm and the reconstruction using the method proposed in section 4.3 for a circular sensor manifold at a radius of 10 mm is illustrated by Fig. 4.11a and Fig. 4.11b. The reconstruction using our proposed method for sensors placed randomly between radius 10 mm and 12 mm (note that the angular positions are uniformly distributed between 0 and \( 2\pi \)) is illustrated by Fig. 4.11c and Fig. 4.11d. The sensor positions in this configuration is shown by Fig. 4.10a. This reconstructed source distribution is similar to that for the circular geometry. The sensor placement can be varied further, with sensors placed between radius of 10 mm and 14 mm, see Fig. 4.10b, the reconstruction results are illustrated by Fig. 4.11e and Fig. 4.11f. From Fig. 4.11, we can observe that the variations in the sensor placement has negligible effect on the reconstructed image. We have used sensors placed at random positions since this tests our proposed method under the worst possible scenario in terms of sensor geometry. Our proposed method is capable of reconstruction under this condition, hence will work for practical acquisition geometries which are much smoother than the sensor geometries considered in the simulations. In the numerical experiments, 84 sensors were used and modes from \(-41\) to \(41\) were considered.

![Figure 4.10: Positions of sensors randomly placed (a) between 10 and 12 mm radius and (b) between 10 and 14 mm radius.](image-url)
Figure 4.11: Reconstructed image for a circular array of sensors, (a) mesh plot of the reconstruction and (b) profile through the horizontal central axis. Reconstructed image for sensor configuration in Fig. 4.10a, (c) mesh plot of the reconstruction and (d) profile through the horizontal central axis. Reconstructed image for sensor configuration in Fig. 4.10b, (e) mesh plot of the reconstruction and (f) profile through the horizontal central axis.
4.8 Reconstruction with Sparse Frequency Samples

The method proposed in Section 4.3, uses different frequency samples for the different modes $m$. In this section, we present a method that uses a minimum number of frequency samples for photoacoustic reconstruction for the 2D and 2.5D inverse problem and the same frequency samples for each mode $m$. Additionally, a method for selecting these frequencies is introduced. Again to simplify notation, we rewrite the 2D Green's function modal expansion separating the terms dependent on radial positions and those dependent on the angular positions (4.11) as

$$G_{2D}(k; r, r_s) = \frac{i}{4} \sum_{n=-\infty}^{\infty} R_n(k, r_s, r)e^{-in\phi}e^{in\phi_s}$$

where $R_n(k, r_s, r) \equiv H_n^{(1)}(kr_s)J_n(kr)$. Section 4.7 rewrote this modal expansion by separating the terms dependent on the source and sensor locations.

4.8.1 Normalized 2D Fourier Bessel Expansion of Source Distribution

The source distribution $p_0(r)$ can be expanded as a sum of infinitely many orthonormal basis functions with the normalized 2D Fourier Bessel series expansion. The synthesis and analysis expansions of this expansion are, respectively,

$$p_0(r, \theta) = \sum_{m, \ell} \beta_{m\ell} N_{m\ell} J_m \left( \frac{z_{m\ell}}{r_0} r \right) e^{im\phi}$$

and

$$\beta_{m\ell} = \int_0^{r_0} \int_0^{2\pi} p_0(r, \phi) N_{m\ell} J_m \left( \frac{z_{m\ell}}{r_0} r \right) e^{-im\phi} d\phi \, dr$$

where $\beta_{m\ell}$ are complex Fourier Bessel coefficients for the normalized expansion, $\sum_{m, \ell} \triangleq \sum_{m=-\infty}^{\infty} \sum_{\ell=0}^{\infty}$, and

$$N_{m\ell} = \sqrt{2}/(r_0J_{m+1}(z_{m\ell}))1/\sqrt{2\pi}$$
are the normalizing terms. A similar expansion without the normalizing terms was previously applied for 2D photoacoustic tomography in Sections 4.3 and 4.7. Since the source distribution \( p_0(r, \theta) \) is real, estimating the coefficients for positive modes \( m \) is sufficient for reconstruction. The following theorem proves this concept.

**Theorem 4.8.1 (Relationship between coefficients of the positive and negative modes)** The Fourier Bessel coefficients \( \tilde{\beta}_{m\ell} \) for the positive and the negative modes are related by the formula

\[
\tilde{\beta}_{(-m)\ell} = \tilde{\beta}_{m\ell}^* \tag{4.65}
\]

where \((\cdot)^*\) denotes the complex conjugate operator and this relationship is valid for real source distributions \( p_0(r, \phi) \).

**Proof**

Since \( p_0(r, \phi) \) is real, then \( p_0(r, \phi) = p_0(r, \phi)^* \) and so

\[
p_0(r, \phi) = \sum_{m, \ell} \tilde{\beta}_{m\ell}^* N_{m\ell} J_m \left( \frac{z_{m\ell}}{r_0} \right) e^{-im\phi}. \tag{4.66}
\]

The roots of \( J_m(\cdot) \) and \( J_{-m}(\cdot) \) are equivalent since \( J_m(r) = (-1)^m J_{-m}(r) \). Further we can substitute this relationship between Bessel functions and

\[
J_{m+1}(z_{m\ell}) = -J_{m-1}(z_{m\ell}) = -(1)^{m-1} J_{-m+1}(z_{m\ell}) = (-1)^m J_{-m+1}(z_{m\ell}) \tag{4.67}
\]

into (4.66) to yield

\[
p_0(r, \phi) = \sum_{m, \ell} \tilde{\beta}_{m\ell}^* \frac{N_{(-m)\ell}}{(-1)^m} (-1)^m J_{-m} \left( \frac{z_{m\ell}}{r_0} \right) e^{-im\phi} \tag{4.68}
\]

where the basis function is equivalent to that for mode \(-m\) and the coefficient is that for mode of \(m\), therefore coefficients from positive and negative modes are related by (4.65).

**4.8.2 Properties of Angular Modes**

Firstly, we expand the received signals with respect to the exponential functions (we call this the spatial Fourier transform as the exponentials vary in terms of
the angular positions of the sensors)

\[ p(\phi_s, k) = \sum_{n=-\infty}^{\infty} a_n(k)e^{in\phi_s}. \quad (4.69) \]

These spatial modal coefficients \( a_n(k) \) can be calculated by

\[ a_n(k) = \frac{1}{2\pi} \int_{0}^{2\pi} p(\phi_s, k)e^{-in\phi_s} \, d\phi_s. \quad (4.70) \]

Now, we substitute the Fourier Bessel expansion of the spatial distribution (4.62) and (4.61) into (4.3) (\( S \) is a circular region with bounding radius of \( r_0 \)) to get

\[
p(\phi_s, k) = \frac{k_c}{4} \sum_{m, \ell} \sum_{n} \beta_{m\ell} N_{ml} e^{in\phi_s} \\
\times \int_{0}^{r_0} R_m(k, r_s, r) J_m \left( \frac{z_{m\ell}}{r_0} r \right) \, rdr \int_{0}^{2\pi} e^{i\phi_s} e^{-in\phi_s} \, d\phi_s. \quad (4.71)
\]

The orthogonality of the exponential functions simplifies (4.71) to

\[
p(\phi_s, k) = \sum_{m} \left[ \sum_{\ell} \frac{\pi k_c}{2} \beta_{m\ell} N_{ml} \right] \\
\times \int_{0}^{r_0} R_m(k, r_s, r) J_m \left( \frac{z_{m\ell}}{r_0} r \right) \, rdr \] e^{im\phi_s}. \quad (4.72)
\]

Studying the equations (4.72) and (4.69), we can conclude that \( a_m(k) \) is equivalent to the term in square brackets in (4.72).

Subsequently, by expanding and separating terms in \( a_m(k) \) we get

\[ a_m(k) = \frac{c_n}{2} \frac{1}{\sqrt{2\pi}} \sum_{\ell} j_{m\ell}(k, r, r_s) \beta_{m\ell} \quad (4.73) \]

where

\[ j_{m\ell}(k, r, r_s) = k H_m^{(1)}(kr_s) \frac{\sqrt{2}}{r_0 J_{m+1}(z_{m\ell})} \times \int_{0}^{r_0} J_m(kr) J_m \left( \frac{z_{m\ell}}{r_0} r \right) \, rdr. \quad (4.74) \]

Considering (4.73), the angular modes consist of a weighted sum of Fourier Bessel coefficients \( \beta_{m\ell} \) for a particular mode \( m \) over the zero indices \( \ell \).

**Theorem 4.8.2 (Evaluating \( j_{m\ell}(k, r, r_s) \))** The weights for the angular modes...
$a_m(k)$ can be evaluated without approximating an integral with

$$j_{ml}(k, r, r_s) = \frac{\sqrt{2}}{z_{ml}/(kr_0) - kr_0/z_{ml}} J_m(kr_0) H_m^{(1)}(kr_s)$$  \hspace{1cm} \text{(4.75)}$$

if $k$ does not equal any of the $z_{ml}/r_0$ over all zero indices $\ell$ for that particular mode $m$, and

$$j_{ml}(k, r, r_s) = \frac{kr_0}{\sqrt{2}} H_m^{(1)}(kr_s) J_m+1(z_{ml})$$  \hspace{1cm} \text{(4.76)}$$

if $k = z_{ml}/r_0$.

**Proof**

An identity exists for evaluating the definite integral of the product of two Bessel functions [27] and which can be stated as follows

$$\int_0^{r_0} J_m(\mu r)J_m(\nu r) \, r \, dr = \frac{\nu r_0 J_m(\mu r_0)J_{m-1}(\nu r_0) - \mu r_0 J_{m-1}(\mu r_0)J_m(\nu r_0)}{\mu^2 - \nu^2}.$$  \hspace{1cm} \text{(4.77)}$$

The stated identity is valid provided that both $\mu$ and $\nu$ are not equal to $z_{ml}/r_0$, otherwise, the integral can be evaluated using the orthogonality relationship for the Bessel functions (4.13). This identity is used to remove the integral operator in (4.74)

$$j_{ml}(k, r, r_s) = H_m^{(1)}(kr_s) \frac{\sqrt{2}}{r_0 J_{m+1}(z_{ml})} \frac{k z_{ml} J_m(kr_0) J_{m-1}(z_{ml})}{k^2 - (z_{ml}/r_0)^2},$$  \hspace{1cm} \text{(4.78)}$$

further, using $J_{m-1}(z_{ml}) = -J_{m+1}(z_{ml})$ (4.67), we get

$$j_{ml}(k, r, r_s) = \frac{\sqrt{2}}{r_0} \frac{k z_{ml}}{(z_{ml}/r_0)^2 - k^2} H_m^{(1)}(kr_s) J_m(kr_0),$$  \hspace{1cm} \text{(4.79)}$$

and subsequently, with some algebraic manipulations, we can derive (4.75). If $k = z_{ml}/r_0$, we apply the orthogonality relationship of Bessel functions (4.13) to (4.74) to yield

$$j_{ml}(k, r, r_s) = k H_m^{(1)}(kr_s) \frac{\sqrt{2}}{r_0 J_{m+1}(z_{ml})} \frac{r_0^2}{2} J_{m+1}(z_{ml})^2$$  \hspace{1cm} \text{(4.80)}$$

and with some algebraic manipulations, we can derive (4.76). □

So far we have described what information about the Fourier Bessel coefficients $\tilde{\beta}_{ml}$ the angular modes $a_m(k)$ contain. In the next subsection, we will
4.8.3 Least Squares Estimation of Radial Basis Coefficients per Mode

Numerically evaluating the magnitude of $j_{ml}(k, r, r_s)$ at different frequencies, a pattern emerged. Observing the magnitude of $j_{ml}(k, r, r_s)$ at a particular frequency in Fig. 4.12, we observe that for some modes, $|j_{ml}|$ peaks at only one zero index $l$ for a mode $m$ and the value of this index decreases with mode. Further, this has the effect of filtering certain zero indices $l$ at a particular frequency. We note that these peaks occur for a set of frequencies close to $z_{ml}/r_0$. By carefully picking frequency values where different zero indices $l$ peak, we can ensure that the matrix of $j_{ml}(k, r, r_s)$ at a particular mode $m$ will be full rank and invertible. Provided that for a certain mode $m$, there are $q$ zero indices that peak within the frequency response of the transducer, then we can set up the
following linear equation (ignoring noise)

$$A_m = J_m \gamma_m$$

(4.81)

where $A_m = [a_m(k_1), \ldots, a_m(k_q)]^T$, $\gamma_m = [\beta_m \ell_1, \ldots, \beta_m \ell_q]^T$ and the $q \times q$ square matrix

$$J_m = \begin{bmatrix} j_{m \ell_1}(k_1, r, r_s) & \cdots & j_{m \ell_q}(k_1, r, r_s) \\ \vdots & \ddots & \vdots \\ j_{m \ell_1}(k_q, r, r_s) & \cdots & j_{m \ell_q}(k_q, r, r_s) \end{bmatrix}.$$  

(4.82)

The solution to this linear equation can be done by $\gamma_m = J_m^{-1} A_m$ and provides the Fourier Bessel coefficients $\tilde{\beta}_m \ell$ of the source distribution $p_0(r)$. Using these coefficients, the source distribution can be reconstructed by applying (4.62).

This sort of formulation reduces computational complexity since the size of $J_m$ depends on how many zero indices $\ell$ peak within the bandwidth of the measured signals. The noise averaging capacity of this approach by using more frequency points is left for future work. In the presence of noise, the least squares solution,

$$\min_{\gamma_m} \| A_m - J_m \gamma_m \|_2$$

can be used. Moreover, a sparse set of frequencies are used since a frequency will amplify one, unique zero index $\ell$ for all the modes $m$.

### 4.8.4 Related Reconstruction Algorithms

A novel algorithm for 2D photoacoustic reconstruction using a circular array of sensors was proposed in section 4.3. This algorithm required frequencies corresponding to the Bessel zeros for a particular mode. In comparison, the algorithm presented in this section is more flexible in terms of the frequencies that it can use. The FFT has a computational expense of $O(N \log N)$, moreover, this increase in cost of the FFT is multiplied over the number of sensors. Another advantage of the method presented in this chapter is that it is able to discern the maximum amount of information over the frequency range, since for the algorithm described in Section 4.3, if the frequency corresponding to the zero of a Bessel function is just outside the bandwidth of the sensors the value of this Fourier Bessel coefficient $\beta_{m \ell}$ is lost, although it may produce a peak in a lower frequency within the sensor bandwidth.

Other algorithms that can be applied to the 2D and 2.5D circular geometries were described in [1, 10], and in Sections 2.3.6 and 4.6.3. These methods are de-
rived with the assumption of infinite bandwidth i.e., applies the Hankel transform using Bessel functions over an infinite frequency range. Further, the number of modes that are needed in the summation formulas is not explicitly mentioned. These algorithms require the application of Hankel transforms and so requires zero padding to increase the number of frequency samples. The evaluation of the Hankel transform for the higher modes \( m \) has numerical instabilities [21]. Further, the optimization that less modes are present for a lower frequency is not applied.

### 4.8.5 Numerical Experiments

In these numerical experiments, we compare the performance of the proposed method using an optimized, minimum number of frequency samples with the frequency domain method described in [1]. In the simulation set-up, the speed of sound was 1500 m/s (acoustic speed in tissue) and point sources were used. The measured signals for a point source at \((r, \phi)\) is

\[
p(\phi_s, k) = \frac{k}{4} H_0^{(1)}(k[r^2 + r_s^2 - 2rr_s \cos(\phi_s - \phi)])^{1/2}. \tag{4.83}
\]

The signals were recorded for a frequency range from 10 kHz to 500 kHz with zero noise.

In the first set of simulations, the sensors were placed at a radius \( r_s = 100 \) mm, the bounding radius of the spatial distribution \( r_0 = r_s \) and three point sources were placed at a radius of 50 mm at angular position of 0, \( 2\pi/3 \) and \( 4\pi/3 \). There were 32 uniformly placed sensors and the results obtained for both methods are illustrated by Fig. 4.13. It is observed that the frequency domain method has less artifacts due to aliasing compared to the proposed algorithm.

In the second set of simulations the number of sensors were increased to 170 and these were now placed at 40 mm. Three point sources were placed at a radius of 15 mm and at angular positions of 0, \( 2\pi/3 \) and \( 4\pi/3 \). In this case no aliasing is present and the quality of reconstructed images produced by both methods are similar as shown by Fig. 4.14. The simulations were performed with MATLAB ver 7.00(R14) on a personal computer with an AMD Athlon 2.21 GHz cpu and 1.50 GB of RAM. The computation time for the second set of simulations using the proposed algorithm with a minimum set of frequency samples is 778.2656 s and no information was precomputed. The proposed method mentioned in Section 4.6 working with frequencies corresponding to the Fourier Bessel zeros
Figure 4.13: Image reconstructed in the presence of aliasing for three point sources (a) proposed method using 63 frequency samples (b) frequency domain reconstruction using 400 frequency samples.

Figure 4.14: Image reconstructed with no aliasing for three point sources (a) proposed method using 27 frequency samples (b) frequency domain reconstruction using 400 frequency samples.

took 307.6987 s. The previously proposed frequency domain [1] method took 2871.4 s, clearly both the proposed methods are much faster. The method using the sparse frequency samples is slower than that proposed in Section 4.6 since the least squares calculation has a high computational cost. In the simulations, we did not take into account the cost of FFT as sensor, measured data was
4.9 Summary

The main ideas presented in this chapter are summarized as follows:

- In this chapter, we have proposed a novel, exact method for photoacoustic image reconstruction for the 2D and 2.5D circular geometry. This method provides a complete series solution. Further, this proposed method is faster than previously proposed methods and does not assume infinite bandwidth. This method can be easily discretized and a relationship between the number of sensors and upper frequency limit was provided in order to avoid spatial aliasing. In this method, we expand the source distribution in the 2D Fourier Bessel domain. The source is reconstructed by estimating the Fourier Bessel coefficients from frequency samples corresponding to the Bessel zeros.

- This chapter has introduced a method for photoacoustic reconstruction in the frequency domain when the sensor manifold has an arbitrary geometry. There has been no method proposed to do this before in the frequency domain, however, only time reversal methods were proposed previously to deal with arbitrary sensor geometries in 3D. These time reversal based methods require that the sensor geometry be sufficiently smooth. The proposed method enforces no such restriction on the sensor geometry or dimension of the problem, and uses a robust least squares solution to estimate the Fourier Bessel coefficients of the source distribution from frequency samples corresponding to the Bessel zeros. However, the proposed method has its drawbacks in that all the sensors must be placed outside a circular region of space containing the source distribution.

- We introduced a novel method of photoacoustic image reconstruction using a minimum number of frequency samples. Provided that there is no aliasing, the images reconstructed are similar to previous methods using a large number of frequency samples. However, in the case of aliasing there are more aliasing artifacts when a lower number of frequency samples are used. Further, a method to calculate the optimum, minimum number of frequency samples was proposed together with how to select these frequency samples.
• Using computational complexity analysis, we have shown the lower computational order of our method compared to previous reconstruction methods. Since all the 2D photoacoustic inversion methods have a computational complexity of $\mathcal{O}(N^3)$ for reconstruction on an $N \times N$ discrete grid, we compared the run times for the proposed method with frequencies corresponding to the Bessel zeros and that using a minimum number of frequency samples to the frequency domain method (see Section 2.3.6). The results proved that both our methods were faster than the frequency domain method.

References


Chapter 5

Frequency-Radial Duality based Photoacoustic Image Reconstruction

5.1 Introduction

In Chapter 4, complete series solutions for photoacoustic tomography was developed for the 2D and 2.5D circular geometry. This method consisting of processing the measured signals in the modal space. In this chapter, we aim to extend the concepts described in the last chapter for 3D photoacoustic inversion using a spherical aperture.

Photoacoustic image reconstruction is an inverse problem requiring the estimation of the initial spatial distribution of pressure by processing a set of recorded signals. However, existing reconstruction methods [1-3] deal with a large amount of data due to the high measurement bandwidths. A novel frequency domain method is proposed for the spherical sensor configuration where the spatial pressure distribution is expanded using a spherical Fourier Bessel series. By processing only the frequencies corresponding to the Bessel zeros, the Fourier Bessel coefficients can be estimated and the spatial distribution recovered. This reduction in data processing provides a faster solution than previous reconstruction methods. Further, new insights relating the amount of information processing required with the spatial characteristics are provided.

This chapter considers the case where the sensors are placed in a spherical configuration surrounding the sample under study and the spatial distribution is bounded within a known radius. Other configurations are possible and recon-
struction algorithms have been provided for these different set ups [3-8]. The spherical configuration can be useful for imaging external organs, small animals or testing blood samples for cancer. A Fourier series [1] and a time domain backprojection method [2] was proposed for image reconstruction for the spherical geometry. Speed of image reconstruction is usually slow using these algorithms since a large number of samples need to be processed. More efficient and faster reconstruction algorithms are still needed for photoacoustic imaging.

The main contribution of this chapter are as follows. A new method for photoacoustic image reconstruction is proposed which expands the spatial pressure distribution using a spherical Fourier Bessel series. A natural basis function for the radial component is introduced which has not been previously mentioned in the photoacoustic literature. The proposed method processes only the frequencies corresponding to the Bessel zeros to estimate the spherical Fourier coefficients and subsequently, recover the spatial distribution. A new concept of frequency-radial duality is introduced which shows that sampling at the Bessel zero frequencies separates the information from the different radial basis functions.

Frequency-radial duality provides new insights into the information content given a finite bandwidth and volume of space occupied by the initial pressure distribution. The proposed method is proved to be faster than both the backprojection [2] and the Fourier series (Norton-Linzer) [1] methods using order analysis and numerical experiments. Furthermore, the computational order analysis provides conditions under which the Norton-Linzer method is faster than the backprojection algorithm. Moreover, spatial aliasing ideas from spherical array signal processing are extended for photoacoustic imaging.

This chapter is organized as follows. The next section provides some theoretical background to the underlying wave equation associated with photoacoustic imaging. Section 5.3 outlines the problem statement, provides a description of the spherical Fourier Bessel expansion of the initial spatial distribution of pressure and introduces the proposed frequency-radial duality (F-R) based algorithm. Section 5.4 extends the proposed method considering aliasing and discrete spatial sampling. Section 5.5 compares the F-R based method with the Norton-Linzer and the backprojection methods. Further, a computational order analysis is provided for these three methods. A description of the numerical experiments performed to validate and compare the F-R based method with previous methods are provided in Section 5.6. The last section concludes the chapter by providing a summary of the main ideas.
5.2 Background on Photoacoustic Imaging

A review of the physics of photoacoustic imaging can be found in Section 2.3.1. The photoacoustic inverse problem in the frequency domain is

\[ p(r_s, k) = -ikc \int_V p_0(r)G(k; r_s, r) \, dr. \quad (5.1) \]

where \( i \equiv \sqrt{-1}, \) the wavenumber \( k = \frac{2\pi f}{c}, \) \( f \) denotes the frequency and \( p(r_s, k) \) is the measured pressure at sensor position \( r_s. \) Moreover, the source distribution (also known as the initial pressure distribution) is \( p_0(r), \) \( r \) is a vector co-ordinate in \( \mathbb{R}^3 \) and \( G(\cdot) \) is the Green’s function. This Fourier transform was used in photoacoustic image reconstruction methods described in [3, 6]. From (5.1), the spatial distribution \( p_0(r) \) is frequency invariant, hence the estimation of such a distribution is classified as a frequency invariant, distributed source localization problem.

5.2.1 Wavefield Decomposition

In the previous section, the Green’s function was mentioned and this function in \( \mathbb{R}^3 \) using the standard spherical co-ordinate system [9] is expressed as

\[ G(k; r_s, r) = \frac{e^{ik|r_s-r|}}{4\pi ||r_s - r||} \quad (5.2) \]

with sensor position \( r_s \equiv (r_s, \Omega_s), \) source position \( r \equiv (r, \Omega), \) \( \Omega_s \equiv (\theta_s, \phi_s) \) and \( \Omega \equiv (\theta, \phi). \) This Green’s function is defined as the fundamental solution to the 3D Helmholtz equation. Next, we introduce the wavefield or modal decomposition of the Green’s function as a sum of orthogonal basis functions,

\[ G(k; r_s, r) = \sum_{n,m} R_n(k; r_s, r)Y_{nm}(\Omega_s)Y_{nm}^*(\Omega) \quad (5.3) \]

where

\[ R_n(k; r_s, r) = ik h_{nm}^{(1)}(kr_s)j_n(kr) \quad (5.4) \]

valid for \( k > 0 \) and \( r_s > r; \) with \( \sum_{n,m} \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^{n}, \) \( n \) as the order, \( m \) as the mode, \( h_{nm}^{(1)}(\cdot) \) and \( j_n(\cdot) \) are the spherical Hankel function of the first kind and the spherical Bessel function, respectively, and \( (\cdot)^* \) denotes the complex conjugate.
The spherical harmonics are denoted by $Y_{nm}(\cdot)$ and are defined as

$$Y_{nm}(\Omega) \triangleq \sqrt{\frac{(2n + 1)(n - m)!}{4\pi(n + m)!}} P_{nm}(\cos \theta)e^{im\phi}$$

where the polynomial function $P_{nm}(\cdot)$ is the associated Legendre function.

This decomposition introduced here will be used for deriving our proposed algorithm and was used previously in ultrasound reflectivity imaging [1]. This modal decomposition has the advantage of separating components dependent on the source position form those dependent on the sensor position and thus found application in direction of arrival (DOA) estimation [10] and in biomedical acoustic source localization within circular sensor arrays [11, 12].

### 5.3 Frequency-Radial Duality Based Image Reconstruction

This section introduces our novel algorithm for photoacoustic imaging under certain ideal conditions. Extension for practical application is considered in the subsequent section. In this section we elaborate the problem statement, the novel spherical Fourier Bessel expansion of the spatial distribution, and highlight the two components of our proposed method for photoacoustic tomography.

#### 5.3.1 Problem Statement

We develop our algorithm by considering a hypothetical continuous aperture on the 2-sphere $S^2$ at a radius of $r_s$. Further, the problem of photoacoustic image reconstruction is an inverse problem requiring the estimation of the spatial distribution $p_0(r, \Omega)$ (specifying $p_0(r)$ in expression (5.1) in a standard spherical co-ordinate system) from measurements $p(\Omega_s, k)$ (specifying the measured pressure $p(r_s, k)$ due to acoustic waves on a spherical manifold at $r_s$). This chapter considers the case where the sensor aperture is specified by samples on the sphere $S^2$ completely enclosing the spatial distribution and Fig. 5.1 illustrates the geometry of the problem formulation, with the spatial distribution being zero at all spatial points with radii larger then $r_0$. 

Figure 5.1: The problem involves estimating the spatial distribution enclosed in a spherical region of radius $r_0$ from measurements on a continuous spherical aperture at a radius $r_s$, completely enclosing the bounding region.

### 5.3.2 Fourier Transform on the 2-Sphere

This section provides a more formal description of the spherical Fourier transform which was discussed previously for modal array signal processing for spherical apertures (see Section 2.4.2). The spherical harmonic functions form a complete orthonormal basis function on the 2-sphere $S^2$ with respect to the natural inner product [13]. Provided that the aperture response is square integrable on the sphere i.e., $p(\cdot, k) \in L^2(S^2)$, the spherical harmonics decomposition [13], or the spherical Fourier transform [14] analysis and synthesis equations of the aperture response function at wavenumber $k$ is, respectively

\begin{equation}
\varphi_{nm}(k) = \int_{\Omega_s \in S^2} p(\Omega_s, k) Y_{nm}^*(\Omega_s) \, d\Omega_s
\end{equation}

and

\begin{equation}
p(\Omega_s, k) = \sum_{n,m} \varphi_{nm}(k) Y_{nm}(\Omega_s)
\end{equation}

where $\int_{\Omega_s \in S^2} d\Omega_s \triangleq \int_0^{2\pi} \int_0^\pi \sin \theta_s d\theta_s d\phi_s$ and represents integration over the unit sphere. The spherical Fourier coefficients $\varphi_{nm}$ are complex terms and equality in expression (5.7) means convergence in the mean, however, if the aperture function is sufficiently smooth, point-wise convergence can be assumed. The spherical Fourier transform introduced in this section will be applied in later
parts of this chapter.

5.3.3 Spherical Fourier Bessel Expansion of Spatial Distribution

The spatial distribution $p_0(r)$ can be expanded as a sum of orthogonal basis functions (synthesis equation)

$$p_0(r) = \sum_{n,m} \sum_{\ell=1}^{\infty} \alpha_{nm\ell} j_n \left( \frac{z_{n\ell}}{r_0} r \right) Y_{nm}(\Omega)$$  \hspace{1cm} (5.8)

where $z_{n\ell}$ is the $\ell^{th}$ root of $j_n(\cdot)$, therefore $\ell$ denotes the Bessel zero index and $\alpha_{nm\ell}$ are complex, spherical Fourier Bessel coefficients. Note that $\ell$ also acts as an index for the radial basis function and will also be called the radial index. This expansion will be referred to as the spherical Fourier Bessel expansion. The corresponding analysis equation to calculate $\alpha_{nm\ell}$ can be written as

$$\alpha_{nm\ell} = \frac{2}{r_0^3 J_{n+1}(z_{n\ell})} \int_0^{r_0} \int_{\Omega \in S^2} p_0(r) Y_{nm}^* (\Omega) j_n \left( \frac{z_{n\ell}}{r_0} r \right) d\Omega \ r^2 dr.$$  \hspace{1cm} (5.9)

The analysis equation can be explained by the spherical Fourier transform described in the previous subsection and by the spherical Fourier Bessel series, which is used for the radial basis function. This spherical Fourier Bessel series is derived from the well known Fourier Bessel series which found application in expanding acoustic wavefields [15].

If we can estimate all the spherical Fourier Bessel coefficients $\alpha_{nm\ell}$ then we can reconstruct the spatial distribution $p_0(r)$ using the synthesis equation (5.8). Such an orthogonal expansion (5.8) has not been proposed before in either photoacoustic or ultrasonic reflectivity imaging. However, the authors have previously introduced a 2D Fourier Bessel expansion for the spatial distribution which was applied for 2D photoacoustic imaging in a circular geometry [16] and for 2.5D photoacoustic reconstruction for a cylindrical geometry [17] (see also Chapter 4).

5.3.4 Modal-Order Filtering of the Spatial Distribution

So far we have introduced the wave equations related to photoacoustic imaging, the wavefield modal expansion and the spherical Fourier Bessel expansion. This section describes a methodology that can separate the information from the different modes and orders in the spatial distribution expansion (5.8).
Firstly, we define a new function
\[
\beta_{nm}(r) = \sum_{\ell=1}^{\infty} \alpha_{n\ell m} j_{\ell} \left( \frac{z_{n\ell}}{r_0} \right)
\] (5.10)
thus, the spatial distribution expansion can be shortened to
\[
p_0(r) = \sum_{n,m} \beta_{nm}(r) Y_{nm}(\Omega).
\] (5.11)
This definition allows a simpler notation in describing the following Theorem.

**Theorem 5.3.1 (Modal-Order Filtering)** Taking the spherical Fourier transform of the aperture response \(p(\Omega, k)\) separates the information from the different orders \(n\) and modes \(m\) in the spherical Fourier Bessel expansion of the spatial distribution. Hence, the spherical Fourier coefficient \(\phi_{n,m}(k)\) of order \(n\) and mode \(m\) is only dependent on the coefficients of the same order and mode in the spherical Fourier Bessel expansion of the spatial distribution \(p_0(r)\) i.e.,
\[
\phi_{nm}(k) = -ikc \int_{0}^{\infty} \beta_{nm}(r) R_n(k; r_s, r) r^2 dr.
\] (5.12)

**Proof**

By substituting the Fourier Bessel expansion (5.11) and the modal expansion of the Green's function (5.3) into (5.1), the received signal by a sensor at angular position \(\Omega_s\) for wavenumber \(k\) is
\[
p(\Omega_s, k) = -ikc \int_{V} \sum_{n', m'} \beta_{n', m'}(r) Y_{n'm'}(\Omega) \times \sum_{n,m} R_n(k; r_s, r) Y_{nm}^*(\Omega) Y_{nm}(\Omega_s) \ dV
\]
\[
= -ikc \sum_{n', m'} \sum_{n,m} \int_{\Omega \in S^2} Y_{n'm'}(\Omega) Y_{nm}^*(\Omega) \ d\Omega
\]
\[
\times \int_{0}^{\infty} \beta_{n'm'}(r) R_n(k; r_s, r) r^2 dr Y_{nm}(\Omega_s).
\] (5.13)
This can be simplified further by applying the orthogonality property of the spherical harmonic functions in the first integral, over the angular domain to
yield

\[ p(\Omega_s, k) = \sum_{n,m} \left( -ikc \int_0^{r_0} \beta_{nm}(r) R_n(k; r_s, r) r^2 \, dr \right) Y_{nm}(\Omega_s). \] (5.14)

By comparing (5.14) to (5.7), the spherical Fourier transform of the aperture response for order \( n \) and mode \( m \) is (5.12).

The advantage of choosing separable basis expansions for the radial and angular components in both the wavefield modal expansion and the spatial distribution expansion allows the integration over a spherical volume \( V \) to be separated to two independent integrals (5.13). One of these integrals is over the radial parameter while the other is over the angular parameter. Further, the separation of integrals allows the exploitation of the orthogonality property of the spherical harmonic functions reducing summation over four parameters to summation over two (5.14).

So far, we can separate out the information from the different orders and modes, however, to estimate the spherical Fourier Bessel coefficients in (5.10) a further separation of the radial indices is required.

### 5.3.5 Frequency-Radial Duality

The following theorem describes a method to separate the information from the different radial basis functions.

**Theorem 5.3.2 (Frequency-Radial Duality)** The spherical Fourier coefficient \( \varphi_{nm}(k) \) taken at frequencies corresponding to the Bessel zeros, more specifically at \( k = z_{n\ell}/r_0 \), is dependent only on the spherical Fourier Bessel coefficient corresponding to the Bessel zero index \( \ell \) and given by

\[ \varphi_{nm}(k)|_{k=z_{n\ell}/r_0} = \frac{(z_{n\ell})^2}{2} \frac{1}{c r_0} \int_{z_{n\ell}}^{r_0} (z_{n\ell}) h_n^{(1)} \left( \frac{z_{n\ell}}{r_0} \right) \alpha_{n\ell} \] (5.15)
Proof

Substituting (5.4) and (5.10) into (5.12), we get the spherical Fourier coefficient

\[ \varphi_{nm}(k) = k^2c \sum_{\ell' = 1}^{\infty} h_n^{(1)}(kr_s) \]

\[ \times \int_0^{r_0} \alpha_{nm\ell'} j_n \left( \frac{zn_{\ell'}}{r_0} \right) j_n(kr)r^2 \, dr \]  

(5.16)

and for \( k = \frac{z_{\ell'}}{r_0} \) this becomes

\[ \varphi_{nm} \left( \frac{zn_{\ell'}}{r_0} \right) = \left( \frac{zn_{\ell'}}{r_0} \right)^2 c \sum_{\ell' = 1}^{\infty} \alpha_{nm\ell'} h_n^{(1)} \left( \frac{zn_{\ell'}}{r_0} \right) \]

\[ \times \int_0^{r_0} j_n \left( \frac{zn_{\ell'}}{r_0} \right) j_n \left( \frac{zn_{\ell'}}{r_0} \right) r^2 \, dr . \]  

(5.17)

The integral in (5.17) can be simplified by using the orthogonality property of the spherical Bessel function [15]

\[ \int_0^{r_0} j_n \left( \frac{zn_{\ell'}}{r_0} \right) j_n \left( \frac{zn_{\ell'}}{r_0} \right) r^2 \, dr \]

\[ = \begin{cases} \frac{r_0^3}{2} J_{n+1}(zn_{\ell'}) & \text{if } \ell = \ell', \\
0 & \text{otherwise}. \end{cases} \]  

(5.18)

Now, the spherical Fourier coefficient \( \varphi_{nm}(\frac{zn_{\ell'}}{r_0}) \) in (5.17) can be written as

\[ \varphi_{nm} \left( \frac{zn_{\ell'}}{r_0} \right) = \left( \frac{zn_{\ell'}}{r_0} \right)^2 c \frac{r_0^3}{2} J_{n+1}(zn_{\ell'}) \frac{J_n(zn_{\ell'})}{J_{n+1}(zn_{\ell'})} h_n^{(1)} \left( \frac{zn_{\ell'}}{r_0} \right) \alpha_{nm\ell} \]  

(5.19)

and with further algebraic simplification yields (5.15).

\[ \square \]

From the aperture response \( p(\Omega_s, k) \), we can get \( \varphi_{nm}(\frac{zn_{\ell'}}{r_0}) \) by applying the spherical Fourier transform at \( k = \frac{z_{\ell'}}{r_0} \). Subsequently, using the result obtained in (5.15), we can estimate the spherical Fourier Bessel coefficient \( \alpha_{nm\ell} \) of the spatial distribution \( p_0(r) \) by

\[ \alpha_{nm\ell} = \frac{2\varphi_{nm}(\frac{zn_{\ell'}}{r_0})}{(zn_{\ell'})^2cr_0 J_{n+1}(zn_{\ell'})h_n^{(1)}([zn_{\ell'}/r_0]r_s)} . \]  

(5.20)

Measurements are available over different angular positions, however, the different frequency samples provides the extra dimension required to derive the radial information of the spatial distribution \( p_0(r) \). This is shown by Theorem
5.3.2.

In practice the sensor has a finite frequency response, let $k_l$ denote the lower limit and $k_u$ the upper limit of this frequency response. The coefficients $\alpha_{n\ell \ell}$ that can be estimated must have radial indices $\ell$ and order $n$ such that $k_l \leq z_{n\ell}/r_0 \leq k_u$. Therefore, only a finite number of coefficients $\alpha_{n\ell \ell}$ can be obtained due to the finite bandwidth of measurement and so an estimate of $p_0(r)$ is reconstructed. The number of orders $n$ that needs to be accounted depends only on the upper frequency limit $k_u$. This is due to the fact that the value of the first root $z_{n,1}$ increases with order $n$ for the spherical Bessel functions as illustrated by Fig. 5.2. Thus, the largest order $n$ satisfies

$$\text{maximize } n$$

$$\text{such that } \frac{z_{n,1}}{r_0} \leq k_u. \quad (5.21)$$

It is intuitive that to estimate $p_0(r)$ which occupies a larger area of space would require more information or more samples. This idea is proved by our proposed method where $r_0$ specifies a bounding radius and affects the location of frequency samples. If $r_0$ is large then the spacing between adjacent frequency samples used for the same order decreases, that is

$$k_{i+1} - k_i = \frac{z_{n,i+1}}{r_0} - \frac{z_{n,i}}{r_0} = \frac{z_{n,i+1} - z_{n,i}}{r_0}. \quad (5.22)$$
Therefore more frequency samples are used for a given measurement bandwidth and that a larger number of coefficients $\alpha_{nm\ell}$ can be obtained from the same measurement bandwidth. The effect of the bounding radius on the information required has not been previously mentioned in the photoacoustic literature and cannot be accounted for by the previous proposed algorithms. Our method shows that the smaller the area occupied by $p_0(r)$ within the sensor manifold, the less the information that is required for reconstruction (lower number of frequency samples need to be processed) and so the reconstruction is faster.

5.3.6 Algorithm Simplification

To estimate $p_0(r)$, we only need to estimate half the spherical Fourier Bessel coefficients $\alpha_{nm\ell}$ due to the fact that the spatial $p_0(r)$ is a real function. Therefore $p_0(r) = p_0^*(r)$, and so the complex conjugate of the spherical Fourier Bessel expansion is

$$p_0^*(r) = \sum_{n,m} \sum_{\ell=1}^{\infty} \alpha_{nm\ell}^* j_n \left( \frac{z_{n\ell}}{r_0} r \right) Y_{nm}^*(\Omega). \quad (5.23)$$

Note that, unlike the other terms, the spherical Bessel functions are always real. Further, if we exploit the following relationship $Y_{nm}^*(\Omega) = (-1)^m Y_{n(-m)}(\Omega)$ [13] then

$$p_0(r) = \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} j_n \left( \frac{z_{n\ell}}{r_0} r \right) \sum_{m=-n}^{n} \alpha_{nm\ell}^* (-1)^m Y_{n(-m)}(\Omega). \quad (5.24)$$

From the previous equation, we can infer that

$$\alpha_{n(-m)\ell} = (-1)^m \alpha_{nm\ell}^* \quad (5.25)$$

and so if we can estimate $\alpha_{nm\ell}$, we can also obtain $\alpha_{n(-m)\ell}$. In practice we can choose to estimate $\alpha_{nm\ell}$ for only the positive modes ($m \geq 0$).

5.4 Extension to Discrete Spatial and Temporal Samples

In this section, we discuss the impact of discrete time and discrete spatial samples. For the case of discrete time samples, the sampling frequency must be more than twice the upper limit of the frequency response of the sensor. This condition ensures that "temporal aliasing" is avoided.

In this chapter, we do not consider the blurring effect caused by the finite size
of the sensor but assume that sensors sample a single point in the spatial domain. In planar array geometries, spatial aliasing can be avoided by ensuring that the distance between the sensors is smaller than half the wavelength associated with the upper frequency limit of the sensor response. If the spherical Fourier transform of the aperture response is order limited i.e. \( \phi_{nm}(k) = 0 \) for \( n > N \), to avoid spatial aliasing using an equiangular sampling method [14] both azimuth \( \phi \) and elevation \( \theta \) are each sampled by \( 2N + 1 \) equiangular spaced samples (total of \( 4(N + 1)^2 \) samples). A method based on Gaussian quadrature only requires \( N + 1 \) samples for the elevation [9]. It is possible to space the sensors uniformly on the sphere, reducing the total number of sensors to \( (N + 1)^2 \), however, there is no analytical expression for the sensor positions [18]. A discussion of sampling schemes for the 2-sphere can be found in Section 2.4.2.

The order limiting value \( N \) can be determined by borrowing a concept from spherical array signal processing. This concept states that aliasing is negligible provided that the product of the upper wavenumber limit \( k_u \) of the sensor response and the radius of the sensors placement \( r_s \) is smaller than \( N \) [10, 19],

\[
k_u r_s < N. \tag{5.26}
\]

In our case, we can assume that the order limit is provided from the spatial distribution expansion \( p_0(r) \). Therefore, we assume that the Fourier Bessel coefficients \( a_{nmf} \) is negligible for \( n > N \). For the rest of the chapter, we employ this assumption and will only consider equiangular sensor placement.

For a discrete array of point sensors, a discrete version of the spherical Fourier transform is used applying a simple rectangular quadrature,

\[
\tilde{\phi}_{nm}(k) = \frac{1}{4(N + 1)^2} \sum_{u=1}^{2(N+1)} \sum_{q=1}^{2(N+1)} a_{uq} p(\Omega_{s,uq}, k) \\
\times Y_{nm}^*(\Omega_{s,uq}) \sin(\theta_q) \tag{5.27}
\]

where the azimuth positions are indexed by \( u \), the elevation by \( q \) and the spatial weighting \( a_{uq} = 1 \) for all \( u \) and \( q \). Better quadrature methods such as the trapezoidal method can be applied, however, the spatial weighting \( a_{uq} \) needs to be evaluated accordingly.

Another important point to consider is that if we use the fast Fourier transform (FFT) on the time domain data, then the frequency samples are uniformly spaced. The required frequency values corresponding to the Bessel zeros
$k = z_{nt}/r_0$ may not be available. Provided that with zero padding, there are sufficient frequency samples, we can apply a simple linear interpolation method to calculate the frequency spectra corresponding to the Bessel zeros,

$$f \left( \Omega_s, \frac{z_{nt}}{r_0} \right) = f(\Omega_s, k_u) + \frac{z_{nt}/r_0 - k_u}{k_{u+1} - k_u} \left( f(\Omega_s, k_{u+1}) - f(\Omega_s, k_u) \right)$$

(5.28)

where $f(\Omega_s, k)$ denotes the value of the frequency sample for sensor position $\Omega_s$ and at frequency $k$, $k_u$ is the frequency available directly smaller than $k = z_{nt}/r_0$ ($k = z_{nt}/r_0$ lies between $k_u$ and $k_{u+1}$). The non-uniform Fourier transform can also solve this problem without using any interpolation. Moreover, for a better estimate other interpolation methods such as polynomial interpolation can be applied. However, this may increase the computational complexity.

### 5.5 Comparison with Previous Reconstruction Methods

We compare our proposed method of photoacoustic tomography with another frequency domain, Fourier series method proposed by Norton and Linzer [1] for ultrasound reflectivity imaging which was later applied to photoacoustic imaging [2]. We refer to this Fourier series method as the “Norton-Linzer” method. Subsequently, we evaluate theoretically, the computational complexity of our proposed method with two other photoacoustic imaging methods applicable to spherical geometries.

#### 5.5.1 Frequency Domain Algorithm

The Norton-Linzer method [1] for photoacoustic image reconstruction using the spherical geometry also specifies a bounding radius $r_0$ for the spatial distribution. This solution is briefly described in this subsection. This method utilizes modal-order filtering without specifying a basis expansion for the radial parameter. Taking the spherical Fourier transform over the aperture results $p(\Omega_s, k)$ in (5.14) yields

$$\varphi_{nm}(k) = k^2 c \int_0^{r_0} \beta_{nm}(r) j_n(kr) r^2 \, dr \, h_n^{(1)}(kr).$$

(5.29)
This series method utilizes a different orthogonality property of the spherical Bessel functions and over the wavenumber rather than the radius,

\[ \int_0^\infty k^2 j_n(kr) j_n(kr') \, dk = \frac{\pi}{2r^2} \delta(r - r'). \]  
(5.30)

This orthogonality property is utilized to calculate \( \beta_{nm}(r) \) for all \( r \) by integrating the spherical Fourier coefficients multiplied a factor over all the frequency samples. This operation is formally described by

\[ \int_0^\infty \frac{\varphi_{nm}(k)}{c h_n^{(1)}(kr)} j_n(kr') \, dk \]
\[ = \int_0^\infty \beta_{nm}(r) \int_0^\infty k^2 j_n(kr) j_n(kr') \, dk \, r^2 dr \]
\[ = \int_0^\infty \beta_{nm}(r) \frac{\pi}{2r^2} \delta(r - r') \, r^2 dr \]
\[ = \frac{\pi}{2} \beta_{nm}(r'), \]  
(5.31)

by changing the subject of the resulting equation, the radial dependent coefficients are

\[ \beta_{nm}(r') = \frac{2}{\pi} \int_0^\infty \frac{\varphi_{nm}(k)}{c h_n^{(1)}(kr)} j_n(kr') \, dk. \]  
(5.32)

Subsequently, the spatial distribution can be reconstructed by evaluating the following equation for all reconstruction points

\[ p_0(\mathbf{r}) = \sum_{n,m} \beta_{nm}(r) Y_{nm}(\Omega). \]  
(5.33)

The proposed reconstruction methodology and the Norton-Linzer solution presented here produce exact reconstruction if the measurement bandwidth is infinite. However, ultrasound transducers used in photoacoustic imaging have a finite measurement bandwidth, hence, the orthogonality property of the spherical Bessel functions in (5.30) utilized in the Norton-Linzer solution may not be valid. Further, the Norton-Linzer solution requires an extra operation of integrating over all frequency samples which is avoided in our methodology by sampling at frequencies corresponding to the Bessel zeros and allowing the propagation channel and the spherical configuration to perform a natural integration over the radius. Utilizing integration over the radius has the advantage of avoiding errors due to discretization of a continuous integral. Furthermore, the integration over the frequency increases the computation complexity of the series solution and
all spherical Fourier coefficients needs to be calculated at all frequencies. Our methodology is more efficient requiring the spherical Fourier coefficients for a single order and only for frequencies corresponding to the Bessel zeros.

The direct calculation of the spherical Fourier Bessel coefficients can provide an efficient means of storage of the image if it is sparse in this domain rather than directly storing the samples at discrete points of the spatial distribution. Further, extra calculation to convert to the spherical Fourier Bessel domain is avoided if the frequency-radial duality (F-R) based approach is used.

The proposed methodology provides a new framework for information content for reconstruction in the spherical geometry. It is possible to optimize the reconstruction based on prior information on the bounding radius $r_0$. This approach proves that a smaller $r_0$ requires the processing of different and a fewer number of frequency samples for the same bandwidth. The Norton-Linzer solution cannot provide such an insight.

With the proposed frequency-radial duality method, we can tell which basis functions in the spatial distributions expansion is recovered for a given measurement bandwidth. This formalizes the observation that for values of lower frequency response $k_i$ greater than zero, large structures cannot be reconstructed but their edges are visible. These large structures contain information from the lower zero indices whose Bessel zero frequencies are lower than $k_i$. The Norton-Linzer method cannot be used to provide such a formal reason for this lack of reconstruction of large structures.

### 5.5.2 Computational Complexity Analysis

Firstly, we analyze the computational complexity of the time domain backprojection method presented in [2] which calculates the spatial distribution by the following operation

$$p(r_0) = -\frac{r_0^2}{2\pi c^2} \int_{\Omega_s \in S^2} \frac{1}{t} \frac{\partial p(\Omega_s, t)}{\partial t} \bigg|_{t = \|r_s-r\|/c} d\Omega_s. \quad (5.34)$$

This time domain method was proposed to overcome the high computational complexity of the Norton-Linzer method [2]. The operation count for this algorithm was calculated to be $O(N_R N^4)$. This operation count is a result of analyzing the discretization of any backprojection method [2, 20] where $N_R$ is the number of radial samples in the reconstructed image and $N$ represents the usual order limit. Note that the number of sensors is proportional to $N^2$. In the
literature, it is common to assume that $N_R = N$ and so the order is simplified to $O(N^5)$. Other filtration or backprojection methods for the 3D reconstruction problems mentioned in [20, 21] have a similar computational order of $O(N^5)$. However, the time reversal methods [22] have a complexity of $O(N^3)$. In order to apply the time domain solution, the sensors must be placed in the “farfield” i.e. $|k|r_s \gg 1$. Both the Norton-Linzer and the proposed F-R based methods allow the sensors to be placed as close as possible to the spatial distribution resulting in a higher Signal to Noise Ratio (SNR). Further, this restriction means that the lower frequency measurements cannot be processed using the backprojection method. The backprojection algorithm is an approximate reconstruction method whereas both the Norton-Linzer and the F-R based methods are exact reconstruction methods.

A direct implementation of the Norton-Linzer method results in a computational order of $O(N^6)$. Both the F-R based method and the Norton Linzer method uses the spherical Fourier transform which can be optimized to speed up image reconstruction. By changing the order of summation, the inverse spherical Fourier transform for a function $f(\theta, \phi)$ defined on a unit sphere can be written as

$$f(\theta, \phi) = \sum_{m=-N}^{N} e^{im\phi} \sum_{n=-|m|}^{N} \phi_{nm} P_{n|m|}(|\cos \theta|)$$

where $P_{n|m|}$ is the normalized, associated Legendre function. The inner sum over the orders $n$ for a single value of $\theta$ includes a maximum of $N$ terms and is evaluated for $2N$ values of $\theta$ and $2N$ number of modes with a operation cost of $O(N^3)$. All the sums over the orders are evaluated and then stored. Next, the sums over the modes $m$ are calculated, for a single angular position $(\theta, \phi)$, involves $2N$ sums. Since the total number of angular positions is proportional to $N^2$, the computational complexity of evaluating the sums over the modes for all the angular positions is $O(N^3)$. The overall computational order of the optimized, inverse spherical Fourier transform is $O(N^3) + O(N^3) = O(N^3)$. Using a similar approach, the computational complexity of the spherical Fourier transform can be reduced to $O(N^3)$. Moreover, a fast implementation of the inverse and the forward spherical Fourier transform was introduced in [23] with an order of $O(N^2 \log(N))$.

We divide the Norton-Linzer method into three steps: The first step calculates the spherical Fourier transform at each of the $N_k$ frequency samples and has order of $O(N^3 N_k)$ with the optimized spherical Fourier transform. The second step cal-
calculates the radial dependent coefficient $\beta_{nm}(r)$ for each of the $N_R$ radial samples. For a single radial position, $N^2$ coefficients are calculated, each requiring integration over $N_k$ frequency samples, and so has an operation order of $O(N^2 N_k)$. The operation count to calculate the radial coefficients for all $N_R$ radial samples is $O(N^2 N_k N_R)$. The last step uses the optimized spherical Fourier transform to reconstruct the spatial distribution from the radial coefficients (5.33) and has an order of $O(N^3 N_R)$ since there are $N^2$ radial coefficients and $N^2$ angular samples for each of the $N_R$ radial samples. The overall order for this optimized, Norton-Linzer method is $O(N^3 N_k) + O(N^2 N_k N_R) + O(N^3 N_R) = O(N^3 N_k)$, assuming $N = N_R$.

In practice, the Norton-Linzer method needs a large number of frequency samples and so the time domain signals are zero padded before applying the Fourier transform. Whether, the Norton-Linzer method is faster than the back-projection method depends on the relationship between $N_k$ and $N^2$. If $N_k > N^2$, then the Norton-Linzer method is slower than the time domain method. Furthermore, there is no method available which gives the minimum number of frequency samples $N_k$ required for image reconstruction using the Norton-Linzer method.

Using the fast spherical Fourier transform, the first and the last steps now have complexity of $O(N^2 \log(N) N_k)$ and $O(N^2 \log(N) N_R)$, respectively. Therefore, applying the fast spherical Fourier transform does not change the overall complexity of the Norton-Linzer method.

Lastly, we analyze the computational complexity of our proposed F-R based approach. To utilize the advantage of the optimized spherical Fourier transform, the spherical Fourier coefficients $\varphi_{nm}(k)$ are calculated for frequency samples on either side of frequency values corresponding to Bessel zeroes, and then interpolated using the linear interpolation method mentioned in (5.28) for each order $n$ to calculate the coefficients at frequencies corresponding to Bessel zeroes. The computational order of this operation is $O(N_{Z_k} N^3)$ where $N_{Z_k}$ are the reduced number of frequency samples processed and with the fast spherical Fourier transform this reduces to $O(N_{Z_k} N^2 \log N)$. Lets denote the maximum number of radial indices $\ell$ for any order by $Z_{\text{max}}$. Calculating the spherical Fourier Bessel coefficients $\alpha_{nm\ell}$ from all the $p_{nm}(z_{n\ell}/r_0)$ (total number of spherical Fourier coefficients proportional to $N^2$) is of order $O(N^2 Z_{\text{max}})$. The synthesis step (5.8) can be optimized by first calculating $\beta_{nm}(r)$ from the Fourier Bessel coefficients $\alpha_{nm\ell}$ using (5.11). The sum is over a maximum of $Z_{\text{max}}$ terms and is evaluated for $N_R$ radial samples and $N^2$ Fourier Bessel coefficients resulting in an operation count of $O(N^2 N_R Z_{\text{max}})$. Subsequently, an optimized
or the fast spherical Fourier transform can be applied for all radial samples to give an operation count of \( O(N^3 N_R) \) or \( O(N^2 \log(N) N_R) \), respectively. The overall order of this method using the optimized spherical Fourier transform is \( O(N^3 N_{zk}) + O(N^2 Z_{\text{max}}) + O(N^2 N_R Z_{\text{max}}) + O(N^3 N_R) = O(N^4) \), provided that \( N_{zk} = N \), \( Z_{\text{max}} \propto N_R \), and \( N_R = N \).

If we apply the fast spherical Fourier transform and fast transforms for the Bessel series sum \([24]\) (to calculate \( \beta_{nm}(r) \) from the Fourier Bessel coefficients \( \alpha_{nm\ell} \)), we reduce the overall complexity of the F-R based method to \( O(N^3 \log N) \). Since our proposed method only uses the synthesis step of the Bessel series numerical complications in calculating the inverse transform is avoided. We conclude that the F-R based method is faster than both the optimized Norton-Linzer methods, the time reversal methods and the backprojection methods. The advantage of the F-R based method not apparent from the computational order analysis are the lower number of frequency samples used for each order which can speed up the reconstruction. The method to optimize the Norton-Linzer method presented in this section have not been mentioned previously in the photoacoustic literature. Depending on the number of frequency samples used, this analysis also proves that the Norton-Linzer method can be optimized to be as fast as the time reversal methods and faster than the backprojection method.

### 5.6 Numerical Simulations

In this section, we describe some numerical experiments conducted to verify the validity of the proposed F-R based reconstruction method. Further, we compare the performance and the quality of the images reconstructed with the Norton-Linzer and the backprojection methods.

The numerical phantom used in the simulations is shown in Fig. 5.3 and consists of six spherical absorbers with centers, \( r_\ell \) at the origin, \( (r = 3\text{mm}, \text{azimuth } \phi = 0, \text{elevation } \theta = \pi/2), (r = 8\text{mm}, \phi = 0, \theta = \pi/2), \) \( (r = 6\text{mm}, \phi = \pi, \theta = \pi/2), (r = 8\text{mm}, \phi = 2/3\pi, \theta = \pi/2) \) and \( (r = 6\text{mm}, \phi = 5/3\pi, \theta = \pi/2) \). These spherical absorbers are 12mm, 1mm, 2mm, 3mm, 2.5mm and 1.5mm in radius, respectively. In addition, they have absorption intensities \( \mu_0 \) (in a.u.) of 1, 7, 5, 10, 2, and 8, respectively. In mathematical notation, the uniform spherical absorbers are normally written as \( p_0(r) = \mu_0 U(r_a - \|r - r_\ell\|) \) where \( r_a \) is the radius of the sphere and the \( U(\xi) \) is a unit step function which is zero when \( \xi < 0 \) and one when \( \xi \geq 0 \). We assume that the speed of propagation is constant and is set at \( c = 1500 \text{ m/s}, \) which is the typical speed of sound in biological
tissue, and the optical pulse duration is short. With these two assumptions the photoacoustic signal generated by a uniform, spherical absorber is [25, 26]

\[ p(r_s, t) = \mu_0 \Gamma U(r_a - |d_r - ct|) \frac{d_r - ct}{2d_r} \]  

(5.36)

where the distance from the center of the absorber \( r_s \) to the sensor position \( r_s \) is denoted by \( d_r = \|r_s - r_c\| \) and the Grüneisen coefficient \( \Gamma \) is assumed to be equal to \( c^2 \). Moreover, this formula was validated using experimental data [25].

The following parameter values were chosen for the numerical experiments: the bounding radius \( r_0 \) was set to 15mm, the sensor radius \( r_s \) was set to 50mm and the sampling frequency was chosen to be 20 MHz. The input distribution was assumed to be order limited with a limit \( N \) of 30. Therefore, to avoid spatial aliasing, \( 2N + 1 = 62 \) point sensors sample both the azimuth and the elevation with a total of 3844 sensors placed in an equiangular fashion on a sphere centered at the origin. In practice, the recorded signal can be taken by a single sensor which is rotated for several irradiation of the sample to cover the required spatial samples. Moreover, this can be an array of sensors that is rotated after the sample is radiated with a laser pulse.

The signals are recorded for \( T_0 = (r_s + r_0)/c = 44/\mu s \). The noise associated with photoacoustic imaging can be considered to be additive white Gaussian noise (AWGN) and is independent for the different sensor recordings. We defined the signal to noise ratio (SNR) in our experiments as

\[ SNR = 10 \log_{10} \left( \frac{\int_0^{T_s} |p(r_{ref}, t)|^2 \, dt}{T_s \sigma_n^2} \right) \]  

(5.37)

where \( \sigma_n^2 \) is the noise power and the reference sensor \( r_{ref} \) is the sensor at angular position \( (\phi = 0, \theta = \pi/2) \). We used an SNR of 10dB and the signals recorded at the reference sensor is illustrated by Fig. 5.4. The frequency spectrum of this signal is shown in Fig. 5.5. We can observe that most of the energy is concentrated within the 0 to 2MHz bandwidth.

The numerical experiments were conducted in MATLAB on a personal computer with a Intel Core 2 Duo, 2.4 GHz processor with 2 GB of RAM. We implemented a time domain backprojection, the optimized Norton-Linzer and the optimized F-R based method. Reconstruction was done for a plane with \( \theta = \pi/2 \) and 120 radial samples \( N_R \) and also, 120 angular samples \( N_\Omega \). Before applying the reconstruction algorithms, the time domain signal is filtered with
Figure 5.3: Numerical phantom consisting of spherical absorbers (a) cross section through the x-y plane (b) relative absorption, in arbitrary units (a.u.), through the horizontal central axis of this cross section.
Figure 5.4: Recordings by a sensor placed at azimuth $\phi = 0$ and elevation $\theta = \pi/2$ in the presence of 10 dB of noise.

Figure 5.5: Frequency spectrum of signals recorded by a sensor placed at azimuth $\phi = 0$ and elevation $\theta = \pi/2$. 
a pass band filter equivalent to the sensor frequency response. This reduces the high frequency noise, however, better denoising algorithms can be applied if better noise suppression is required.

In the first set of simulations, the sensor frequency response is set from 0 to 2 MHz and the reconstruction from applying the F-R based method is shown by Fig. 5.6. No noise was added in this numerical experiment. We can observe that each absorbing sphere is discernible and all the salient features of the numerical phantom is visible from the reconstructed image. Since there is no noise, the difference between the reconstructed and the numerical phantom is due to spatial sampling. If more sensors are used then the reconstruction is a much better approximation to the numerical phantom.

The second set of numerical experiments, adds 10 dB of noise to the sensor recordings. Further, the sensor frequency response is changed from 100 KHz to 2 MHz, so that the restriction of \( |k|r_s \gg 1 \) is satisfied for the approximate backprojection method. The reconstructed images from using the three reconstruction methods are illustrated in Fig. 5.7. The Norton-Linzer method has the best quality of reconstructed image with a larger portion of the largest spherical disk being visible. The quality of the F-R based reconstruction is comparable to the Norton-Linzer method. However, the contrast in the time domain, back-projection reconstruction is lower where the absorber with absorption intensity of 2 is not visible and the reconstructed image is not as smooth as the other reconstructed images. Since the low frequency components are not used, the large spherical absorber of 12mm is not fully reconstructed with only its edges being visible in all the reconstructions.

The Fourier Bessel coefficients \( \alpha_{nm} \) for the numerical phantom is shown in Fig. 5.8. The numerical phantom is compressible in the Fourier Bessel domain, this compressibility can allow for efficient storage as well as speeding up reconstruction by using only the significant coefficients in the synthesis step of the proposed method. For a bandwidth from 100 KHz to 2 MHz, the radial indices of 1 and 2 for order 0 and the radial index of 1 for orders 1 and 2 cannot be recovered. Therefore, the large spherical absorber cannot be fully reconstructed (Fig. 5.7) since the low frequency Fourier Bessel coefficients cannot be recovered. However, when these coefficients can be estimated as in the first simulation with bandwidth from 0 to 2 MHz, then this large structure is fully reconstructed (see Fig. 5.6).

In the next set of simulations, we compared the speed of reconstruction for the different methods. The variation of the speed of reconstruction with the
Figure 5.6: Reconstruction using the proposed F-R based method with frequency range from 0 to 2 MHz and no noise (a) cross section through the x-y plane (b) relative absorption, in arbitrary units (a.u.), through the horizontal central axis of this cross section.
Figure 5.7: Reconstruction using the three different reconstruction methods with a frequency range from 100 kHz to 2 MHz and 10 dB of noise: (a) and (d) time domain backprojection method; (b) and (e) Norton-Linzer method; and (c) and (f) F-R based method.
Figure 5.8: Relative magnitudes (in a.u.) of the Fourier Bessel coefficients $a_{nm\ell}$ for the numerical phantom, note in our representation each mode $m$ has a maximum of 40 radial indices $\ell$. 
Table 5.1: Variation of Run Times (sec) with Frequency Upper Limit (MHz) for Different Reconstructions Methods.

<table>
<thead>
<tr>
<th>Freq. Upper Limit (MHz)</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backprojection</td>
<td>2724</td>
<td>2922</td>
<td>2879</td>
<td>2867</td>
<td>2887</td>
<td>2865</td>
</tr>
<tr>
<td>Fourier Series</td>
<td>125</td>
<td>153</td>
<td>180</td>
<td>217</td>
<td>267</td>
<td>319</td>
</tr>
<tr>
<td>F-R Based</td>
<td>25</td>
<td>28</td>
<td>32</td>
<td>37</td>
<td>44</td>
<td>53</td>
</tr>
</tbody>
</table>

upper frequency limit is considered. The lower frequency limit was kept constant at 100 KHz. Zero padding was applied before the FFT was applied to the recorded data. The number of frequency samples that the Norton-Linzer method processes for an upper frequency limit of 2 MHz was 305, this is about three times lower than $N^2 = 900$. In addition, the F-R based method processes a maximum of 40 frequency samples for each order, since the zero order has 40 radial indices within this bandwidth. The runtimes are depicted by Table 5.1 and Fig. 5.9. The F-R based method has the fastest reconstruction times, followed by the Norton-Linzer method. The rate of increase of reconstruction times with increasing frequency upper limit is slower with the F-R based method when compared to the Norton-Linzer method. The backprojection method is unaffected by the frequency upper limit. This is true provided that the samples in the reconstructed image and the number of sensors remain constant. Otherwise, the increase in frequency upper limit would require a larger number of sensors and more samples in the reconstructed image because a higher frequency bandwidth means a higher resolution of image can be reconstructed. In this experiment, we only consider the time taken to process more frequency samples between the two frequency domain methods as the bandwidth increases. From the results, the backprojection method is more than 10 times slower than the other two methods and the F-R based method is roughly about 5 times faster than the Norton-Linzer method. These results corroborate the computational order analysis performed in the previous section.

5.7 Summary

In this work, we introduced a novel algorithm for photoacoustic image reconstruction using a spherical sensor geometry. The proposed method is an exact and complete series solution for photoacoustic inversion. In photoacoustics, the speed of reconstruction and the computational complexity are big factors. The proposed F-R based method, using theoretical analysis and simulations, was
Figure 5.9: Variation of run time with upper frequency limit for the different reconstruction methods.
shown to be both less computationally complex and faster than the time domain, backprojection [2] and the Norton-Linzer/Fourier series methods. The reconstructed images were of similar quality as the Norton-Linzer method and was much better than the approximate backprojection method. Moreover, unlike the backprojection method, there is no restriction on sensor placement to ensure farfield conditions. The computational order analysis disapproves assumptions made in previous research that the Norton-Linzer method is slower than the backprojection method. By simple optimizations of the spherical Fourier transform, the Norton-Linzer method can be faster than the backprojection method when the number of frequency samples to be used is less than the square of the order limit.

The source distribution was expanded to a spherical Fourier Bessel domain, and the F-R based method estimates the Fourier Bessel coefficients in order to recover the initial spatial distribution. The F-R based method applies both the new concept of frequency-radial duality and modal-order filtering and is an exact reconstruction method if the bandwidth is infinite. The concept of frequency-radial duality also proves that when the source distribution occupies a smaller region of space, the number of frequency samples that need to be used for each order is reduced. This provides a new information theoretic criteria relating volume with the number of samples that need to be processed.

The ideas presented in Chapter 4 of extending the method to arbitrary sensor geometries and using the same frequency samples for all modes in the wavefield and source expansion can also be applied to this method for 3D photoacoustic inversion.

References


Chapter 6

Convex Optimization Methods for Enhancing Photoacoustic Tomography

6.1 Introduction

This chapter proposes an alternating projection (also known as the Projection onto Convex Sets (POCS)) and a Total Variation (TV) minimization method\(^1\) for photoacoustic tomography in the Fourier Bessel domain. Both these post processing methods for the Frequency-Radial duality (F-R) based reconstruction [3, 4] result in improved image quality and the recovery of some lost information. Constrained TV formulation is possible since working in the Fourier Bessel domain, the projection onto the fidelity constraint can done by a projection on an \(l_2\) ball.

Rudin, Osher and Fatemi first introduced the TV objective function in [5]. This objective function found uses in several image restoration problems such as denoising, inpainting and deblurring (see [2, 6–9]). Methods based on penalization of the TV minimization become popular for solving inverse problems [10]. Several numerical methods such as the subgradient method [11], second-order cone programming [12] and level set methods [13] have been proposed to solve the TV minimization problem.

The TV functional reconstructs images that have a spatially sparse gradient consisting of large constant regions and sharp edges. This property is desir-

---

\(^1\)The reader is referred to [1] for a description on alternating projection method and to [2] for a review on TV minimization methods.
able for medical imaging where the image consists of material properties which change in different regions. Therefore, the Total Variation objective allows to separate different objects clearly which is particularly important for tomographic imaging. The idea that medical images are often piecewise constant has been recognized before and incorporated into a TV minimization problem by Delaney and Bresler [14]. The TV objective function has been successfully applied to computed tomography [15], in magnetic resonance imaging (MRI) [16] and in positron emission tomography (PET) [17].

The contributions made by this chapter are as follows:

- The Frequency-Radial (F-R) duality based methods developed in Chapters 4 and 5 obtains the projections of the source distribution onto a set of basis functions (Fourier Bessel series). Since in practice, the measurement bandwidth is finite, a set of source distributions can satisfy these projections. Moreover, the source distribution in photoacoustic tomography is always non-negative. This chapter derives an alternating projections (or POCS) method that finds a source distribution in the intersection of these two sets i.e., the source distribution is non-negative and satisfies the calculated projections on the Fourier Bessel series.

- We develop a constrained TV formulation in the Fourier Bessel domain. This formulation has the advantage of converting the projection onto the fidelity constraint to a simple projection onto an $l_2$ ball as well as reducing the dimension and magnitude of this fidelity constraint. Moreover, the constrained formulation is not affected by the choice of a Lagrangian parameter unlike the unconstrained TV minimization formulation.

- A projected subgradient method is designed to solve the TV minimization problem. Other optimization methods such as Second Order Cone Programs (SOCP) or interior point methods can be applied, but these are not as scalable as subgradient methods which is capable of handling the large data sets present in photoacoustic imaging.

- This chapter proposes a photoacoustic, TV formulation and its numerical solution with a modified subgradient projection method for the circular and the spherical geometry. Proof of convergence of this subgradient method is presented. Further, the improvements achieved using either the proposed alternating projections method or the TV minimizations are shown by numerical experiments. Applying the TV minimization or the alternating
projections method has the capability of recovering some lost information particularly in the case when the measurement bandwidth lower frequency limit is higher than zero.

This chapter is organized as follows. The next section provides an introduction to Total variation (TV) minimization and the subgradient method. Section 6.3 introduces the constrained TV formulation for photoacoustic imaging. Section 6.4 describes the Frequency-Radial duality (F-R) based method and how projections onto the constraints can be done in the Fourier Bessel domain. Section 6.5 describes the proposed alternating projections method and Section 6.6 extends the constrained TV formulation to the 3D spherical geometry. The numerical algorithm for TV minimization is described in Section 6.7. Description and results of the numerical experiments are presented in Section 6.8 and the last section summarizes the main contributions of the chapter.

6.2 Total Variation (TV) Minimization for Inverse Problems

In inverse image restoration problems, an image $x$ in a real Hilbert space $(\mathcal{H}, \|\cdot\|)$ must be obtained form observed data,

$$y = Lx + n \quad (6.1)$$

where $n \in \mathcal{H}$ is the additive noise component and $L$ is the linear operator representing the channel. Numerous approaches to obtain $x$ can be found in [18] and the references therein. The image recovery problem can be cast as a convex optimization problem

$$\min \mathcal{J}(x) \text{ such that } x \in C = \bigcap_{i=1}^{M} C_i \quad (6.2)$$

where $\mathcal{J} : \mathcal{H} \rightarrow \mathbb{R}$ is a convex, objective and $C_i, i \in 1, ..., M$ are convex constraints from a priori information about the original image and the model (6.1).

For the objective function a Total Variation (TV) semi-norm can be used and was made popular by Osher and Rudin [19].

**Definition 6.2.1 (Total Variation (TV) Function)** If $x$ is an image in a bounded, convex region $\tilde{\Omega}$ and $x \in L^1(\tilde{\Omega})$ (Hilbert space of integrable functions),
we define the TV function as

\[ TV(x) \triangleq \int_{\Omega} \| \nabla x \| \, dx \]  \quad (6.3)

where \( \nabla x \) is the vector of the weak, partial derivatives of \( x \) and \( \| \cdot \| \) is the Euclidean (\( l_2 \)) norm.

The TV objective is useful in recovering functions that are piecewise smooth since it preserves sharp discontinuities. By reducing the TV norm, we restore images that belong to the bounded variation (BV) image model. With the TV seminorm as the objective function, the restoration problem can now be formulated as a constrained optimization problem

\[
\text{minimize } TV(x) \text{ subject to } \| Lx - y \|^2 \leq \delta \quad (6.4)
\]

with an inequality constraint where \( \delta \) is the fidelity bound. This constrained optimization problem is convex and nonlinear since both the TV objective function and the ellipsoidal fidelity constraint are convex and nonlinear. The inequality constraint ensures that the resulting image is within a given Euclidean distance \( \delta \) of the actual measured data.

The unconstrained problem with a Lagrangian parameter \( \lambda \) is

\[
\text{minimize } TV(x) + \lambda \| Lx - y \|^2, \lambda \geq 0. \quad (6.5)
\]

This is easier to implement in practice and is equivalent to the constrained formulation (6.4) under certain assumptions and applying Langrangian theory [7]. However, it is computationally intensive to find the exact value of the Langrangian parameter \( \lambda \). The value of this parameter in many TV minimization methods is set in an adhoc manner e.g. [6,19] which has a significant impact on the solution [10].

The TV minimization formulation has two significant drawbacks

1. **Staircase Effect:** This effect produces images with a lower TV than that of the original image.

2. **Knowledge of the Noise Environment:** The bound \( \delta \) requires specific information about the noise. Further, this bound can incorporate other sources of data inconsistencies e.g. in [15], this bound included cone beam artifacts for computed tomographic reconstruction.
Practically, TV minimization algorithms are implemented in a discrete setting. Therefore, if we are dealing with 2D images, they are discretized on a $P \times Q$ sampling grid. The discrete version of the TV function $TV(\mathbf{X})$ for $\mathbf{X} = [X_{p,q}] \in \mathbb{R}^{P \times Q}$ used in this chapter is

$$TV(\mathbf{X}) = \sum_{p=1}^{P-1} \sum_{q=1}^{Q-1} \left( |X_{p,q} - X_{p+1,q}| + |X_{p,q} - X_{p,q+1}| \right)$$

$$+ \sum_{p=1}^{P-1} |X_{p,Q} - X_{p+1,Q}| + \sum_{q=1}^{Q-1} |X_{P,q} - X_{P,q+1}|.$$  \hspace{1cm} (6.6)

where $| \cdot |$ is the complex modulus operator (for real arguments reduces to the modulus operator). This is known as the $l_1$ based anisotropic TV, other TV formulations such as the isotropic formulation [2] can also be used. The discrete image $\mathbf{X}$ can be dealt as a vector in the Euclidean space $\mathcal{H} = \mathbb{R}^{PQ}$ by applying the column stacking operator $\text{vec}(\cdot)$ i.e. $\mathbf{x} = \text{vec}(\mathbf{X})$. Further, by introducing suitable difference matrices $[\Gamma_{p,q}]_{1 \leq p \leq P-1, 1 \leq q \leq Q-1} \in \mathbb{R}^{2 \times PQ}$, $[\Gamma_{p,q}]_{1 \leq p \leq P-1} \in \mathbb{R}^{1 \times PQ}$ and $[\Gamma_{p,q}]_{1 \leq q \leq Q-1} \in \mathbb{R}^{P \times 2}$, the TV objective can be expressed as

$$TV(\mathbf{x}) = \sum_{p=1}^{P-1} \sum_{q=1}^{Q-1} \|\Gamma_{p,q}\mathbf{x}\|_1 + \sum_{p=1}^{P-1} |\Gamma_{p,Q}\mathbf{x}| + \sum_{q=1}^{Q-1} |\Gamma_{P,q}\mathbf{x}|.$$  \hspace{1cm} (6.7)

where $\| \cdot \|_1$ is the $l_1$ norm.

This discretization method can be easily extended to 3D images where $\mathbf{X} \in \mathbb{R}^{P \times Q \times R}$. The TV semi-norm $TV : \mathcal{H} \to \mathbb{R}$ is a convex function where the Hilbert space $\mathcal{H} = \mathbb{R}^T$; for the 2D problem $T = PQ$. In the rest of this chapter, the TV function will be assumed to be operated on a column stacked image matrix.

### 6.2.1 Preliminaries on the Subgradient Method

There are several different algorithms in the image processing literature in order to perform Total Variation (TV) minimization. In [13] and [20], level set methods were proposed for the minimizing the TV and in [21] a dual formulation was applied. Several other approaches are outlined in [2,8,9]. In this chapter, we use an adaptation of the standard subgradient descent algorithm for TV minimization. This approach of using subgradients for TV minimization was previously applied in [22] and in [23]. Subgradients are used since the TV objective function is not smooth and is non-differentiable.

Here, we provide a brief introduction to the subgradient method, for a more
in depth description of subgradient methods the reader is referred to [24]. Given a convex function $J: \mathbb{R}^T \to \mathbb{R}$, we can minimize this function using the subgradient function by using the iteration

$$x^{(k+1)} = x^{(k)} - \gamma_{k}g^{(k)} \quad (6.8)$$

where $x^{(k)}$ solution at the $k^{th}$ iterate, $\gamma_{k}$ is the $k^{th}$ step size and $g^{(k)}$ is any subgradient of $J$ at $x^{(k)}$. The subgradient method takes a step in the negative direction of the subgradient at each iterate. Moreover, the subgradient of $J$ at $x$ is defined as any vector that satisfies the following inequality condition $J(y) \geq J(x) + \langle g, y - x \rangle$ for all $y$. The subdifferential of the function $J$ at $x$ is the set of all subgradients of $J$ at $x$ and is normally denoted by $\partial J(x)$. If the function $J$ is differentiable at $x$ then $\nabla J(x)$ is its unique subgradient i.e. $\partial J(x) = \{\nabla J(x)\}$ and the subgradient method reduces to the normal gradient descent method. Furthermore, $x^*$ is a global minimizer of $J$ if and only if $0 \in \partial J(x^*)$.

To solve the constrained convex optimization problem,

$$\text{minimize } J(x) \text{ subject to } x \in \mathcal{C} \quad (6.9)$$

where $\mathcal{C}$ is a closed convex set, an extension of the subgradient method known as the projected subgradient method is applied. This method is given by

$$x^{k+1} = \mathcal{P}_C(x^{(k)} - \gamma_{k}g^{(k)}) \quad (6.10)$$

where $\mathcal{P}_C$ is the projection onto $\mathcal{C}$. The projection operator is defined for all $x \in \mathbb{R}^T$ as the unique point $\mathcal{P}_C(x) \in \mathcal{C}$ such that $\|x - \mathcal{P}_C(x)\| = d(x, \mathcal{C}) = \inf_{y \in \mathcal{C}} \|x - y\|$ with $d(\cdot, \mathcal{C})$ defined as the distance function.

### 6.3 Constrained TV Model for Photoacoustic Imaging

The photoacoustic inverse problem requires reconstructing the source distribution (normally called the initial pressure distribution) from measured acoustic signals taken all around the sample under study. These acoustic signals are a result of the thermal expansion of the sample due to an incident laser pulse. The wave equations associated with photoacoustic tomography (PAT) were discussed
in Section 2.3.1. The integral equations for the photoacoustic inversion problem in the time domain is

\[ p(r_s, t) = \frac{\partial}{\partial t} \int_V p_0(r) \frac{\delta(t - \|r_s - r\|/c)}{4\pi \|r_s - r\|} \, dr \]  

(6.11)

where \( \int_V (\cdot) \, dr \) represents integration over a volume of space in \( \mathbb{R}^3 \), \( p(r_s, t) \) are the measured data at \( r_s \) and \( p_0(r) \) is the source distribution at \( r \). The integral equation in the frequency domain is

\[ p(r_s, k) = -ikc \int_V p_0(r) G(k; r_s, r) \, dr \]  

(6.12)

where \( k \) is the wavenumber, and \( G(k; r_s, r) \) is the Green’s function and in three dimensions (\( \mathbb{R}^3 \)) is

\[ G(k; r_s, r) = \frac{e^{ik\|r_s - r\|}}{4\pi \|r_s - r\|}. \]  

(6.13)

Section 4.3 described the 2D photoacoustic inverse problem which has the following integral equation

\[ p(r_s, k) = -ikc \int_S p_0(r) G_{2D}(k; r_s, r) \, dr \]  

(6.14)

where \( \int_S (\cdot) \, dr \) is the integration over the area occupied by \( p_0(r) \) and the two dimensional (\( \mathbb{R}^2 \)) Green’s function is

\[ G(k; r_s, r) = \frac{i}{4} H_0^{(1)}(k \|r_s - r\|) \]  

(6.15)

where \( H_0^{(1)}(\cdot) \) is the Hankel function of the first kind and order zero.

We discretize the photoacoustic inverse problem with the matrix \( P_0 \) being the discretely sampled source distribution \( p_0(r) \), matrix \( P \) containing the measured, acoustic signals and \( \bar{M} \) is called the sensitivity matrix that maps the source distribution \( P_0 \) to the measured data \( P \). The inverse problem can be written as a matrix equation

\[ P = \bar{M} P_0. \]  

(6.16)

This matrix equation can be discretized by either using the time domain inverse formulation (6.11) or the frequency domain inverse formulation (6.12). Moreover, for 2D, this matrix equation should be a discrete form of (6.14). By using the
column stacking operator vec(·), the matrix equation can be rewritten as follows

\[ p = Mp_0 + n \]  \hspace{1cm} (6.17)

where \( p_0 = \text{vec}(P_0) \) is the vector denoting the source distribution, \( p = \text{vec}(P) \) is the vector containing the measured acoustic signals and the new sensitivity matrix \( M \) links \( p_0 \) to \( p \). The noise vector \( n \) is included in the formulation and is zero mean, additive, white Gaussian noise with variance of \( \sigma_n^2 \). The noise is independent with respect to sensor position and frequency.

The constrained TV formulation for photoacoustic tomography should yield an image \( p_0^* \) that is

\[ p_0^* = \arg\min \ TV(p_0) \]  \hspace{1cm} (6.18)

subject to the following two convex constraints, the data fidelity constraint (the source distribution \( p_0(r) \) satisfying this constraint are denoted by set \( A \)):

\[ \|p - Mp_0\| \leq U\sigma_n^2 \]  \hspace{1cm} (6.19)

where the vectors \( p \in \mathbb{R}^U, p_0 \in \mathbb{R}^T \) and the non negativity constraint (the set of source distributions \( p_0(r) \) satisfying this constraint are denoted by set \( B \)):

\[ p_0 \geq 0. \]  \hspace{1cm} (6.20)

For photoacoustic tomography, the operator \( M \) remains linear since it is a discrete version of the linear mapping between the measured pressure and the source distribution \( p_0(r) \) (6.11), (6.12) or (6.14). The unconstrained formulation of TV minimization (6.5) is normally used since the projection on the inequality constraint is usually expensive and requires solving a second order cone program (SOCP).

The non negativity constraint is included since the source distribution \( p_0(r) \) has values greater than or equal to zero. The positivity constraint is also found in the TV minimization for cone beam tomography [15]. The constrained TV minimization formulation differs from the standard TV minimization problems [7,9,21] in that it includes two convex constraints.
6.3.1 Difficulties in Applying TV Minimization to Photoacoustic Tomography

The backprojection [25] and the Norton-Linzer [26] method described in Sections 2.3.6 and 2.3.6 cannot be formulated to the TV minimization problem presented in the previous section. These two methods give a unique, single image and not a set of images over which a TV minimization can be applied. Moreover, these methods do not work with projected data on some basis function as in the image recovery problems described in [1,18] which can easily be formulated to a TV minimization problem. Other photoacoustic inversion problems described in Section 2.3.6 have the same restriction.

However, the method of solving a linear equation for photoacoustic tomography (see Section 2.3.6) can be formulated easily to a TV minimization problem. The standard techniques for solving linear equation can be used for photoacoustic tomography with the discrete, linear equation (6.16). An iterative method based on solving this linear equation was presented in [27]. The problem with using the TV minimization formulation, from this linear solution, is that due to the large data sets involved with PAT, the sensitivity matrix $M$ can be very large. Furthermore, the projection onto the ellipsoidal, fidelity constraint is itself an SOCP and can be expensive, requiring the computation of the inverse of $M$ which is usually ill conditioned and large.

The constrained, TV minimization is made more complicated due to two convex constraints. This can be solved easily by solving the unconstrained problem (6.5) [15]. However, as mentioned previously, the solution is affected by the choice of the Lagrangian parameter $\lambda$.

We refer to the photoacoustic inversion methods presented in Chapters 4 and 5 as the Frequency-Radial duality (F-R) based reconstruction method. This method works with projection data and its formulation into a TV minimization problem will be presented in the next section.

6.4 TV Minimization Formulation in the Fourier Bessel Domain

We consider the problem of reconstructing the source distribution $p_0(\mathbf{r})$ with a circular or cylindrical array geometry with infinitely long sensors in the $z$ direction (2.5 dimensional formulation). The solution for the circular geometry
and for the 2.5 dimensional cylindrical geometry was presented in Chapter 4. In both these cases the source distribution \( p_0(\mathbf{r}) \) was contained in a ball \( B^2(r_0) \) with a bounding radius of \( r_0 \). We consider this geometry since we do not have to consider errors due to partial view. Extension of applying this methodology for TV minimization in 3D with spherical arrays will be discussed in Section 6.6.

Before providing a brief description of the F-R based reconstruction, we introduce the following two definitions.

**Definition 6.4.1 (Square integrable function space)** We define the Hilbert space of square integrable functions in a circular region \( L^2(B^2(r_0)) \) with a bounding radius \( r_0 \) and equipped with the inner product in polar co-ordinates

\[
\langle p_0(\mathbf{r}), f(\mathbf{r}) \rangle \triangleq \int_0^{2\pi} \int_0^{r_0} p_0(\mathbf{r}) f(\mathbf{r})^* r \, dr \, d\phi
\]

where \( r \) is the radial, \( \phi \) is the angular co-ordinate and \((\cdot)^*\) is the complex conjugate operator. This inner product then induces the norm \( \|p_0(\mathbf{r})\| \triangleq \|p_0(r, \phi)\| \triangleq \langle p_0(\mathbf{r}), p_0(\mathbf{r}) \rangle^{1/2} \).

**Definition 6.4.2 (Complex sequence space \( l^2 \))** For a countable index set \( \Lambda \), we represent the space of square summable, complex sequences \( l^2(\Lambda) \) as the space of sequences \( u = \{u_\nu\}_{\nu \in \Lambda} \) with norm

\[
\|u\|_{l^2(\Lambda)} \triangleq \left( \sum_{\nu \in \Lambda} |u_\nu|^2 \right)^{1/2}
\]

where \(|(\cdot)|\) denotes the complex modulus.

The normalized Fourier Bessel series \( \{\Psi_{m\ell}\}_\Lambda \), the set \( \Lambda = \{m, \ell| -\infty < m < \infty, 1 \leq \ell < \infty, m \in \mathbb{Z}, \ell \in \mathbb{Z}\} \) is a complete, orthonormal basis of \( L^2(B^2(r_0)) \) where

\[
\Psi_{m\ell} \triangleq \frac{1}{\sqrt{\pi}} \frac{1}{r_0 J_{m+1}(z_{m\ell})} J_m(r \, z_{m\ell}/r_0) e^{im\phi},
\]

\( J_m(\cdot) \) are the Bessel functions with mode \( m \) and \( z_{m\ell} \) is the \( \ell^{th} \) root of \( J_m(\cdot) \) and \( \ell \) is the zero or radial index. The Fourier Bessel series expansion of the source distribution \( p_0(\mathbf{r}) \in L^2(B^2(r_0)) \) is

\[
p_0(\mathbf{r}) = \sum_{(m, \ell) \in \Lambda} \beta_{m\ell} \Psi_{m\ell}
\]

where \( \beta_{m\ell} \in l^2(\Lambda) \) are complex Fourier Bessel coefficients. The analysis equation
for this expansion is $\beta_{ml} = \langle p_0(\mathbf{r}), \psi_{ml} \rangle$, since we will be working with the discrete version of $p_0(\mathbf{r})$, a method of calculating the Fourier Bessel coefficients for the discrete case involves using numerical integration methods such as trapezoidal methods to replace integrals with sums in evaluating the inner product.

### 6.4.1 Frequency-Radial Duality (F-R) Based Reconstruction

This section describes the F-R based inversion method introduced in Chapter 4 using operator notation which is suitable for formulating a TV minimization or a Projection Onto Convex Sets (POCS) method.

The photoacoustic inverse problem can be written with operator notation as follows. The measured pressure $p(\mathbf{r}_s, k)$ at sensor position $\mathbf{r}_s$ and frequency $k$ is

$$p(\mathbf{r}_s, k) = \langle \mathcal{M} p_0 \rangle(\mathbf{r}_s, k) + n(\mathbf{r}_s, k)$$

(6.25)

where $\mathcal{M}$ is an operator transforming the source distribution according to (6.14) and $n(\mathbf{r}_s, k)$ is zero mean, additive, white Gaussian noise with variance of $\sigma_n^2$.

The F-R based inversion method calculates the Fourier Bessel coefficients $\beta_{ml}$ by processing the frequencies corresponding to the Bessel zeroes. Aperture functions at a frequency, $k$ and for sensors placed in a circle of constant radius $\mathcal{R}$, $p(\phi_s, k)$ constitute square, integrable function on the unit circle $\mathcal{S}$ i.e. $L^2(\mathcal{S})$. The functions $\{1/\sqrt{2\pi}e^{im\phi_s}\}$ forms a complete, orthonormal basis in this space and the expansion is referred as the modal expansion where the modal coefficients $a_m(k)$ at a frequency $k$ can be calculated from

$$a_m(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} p(\phi_s, k)e^{-im\phi_s} d\phi_s.$$

(6.26)

These modal coefficients $a_m(k)$ at the wavenumbers corresponding to Bessel zeroes, i.e. $k = z_{ml}/r_0$ are equal to weighted Fourier Bessel coefficients $\beta_{ml}$ for mode $m$ and zero or radial index $\ell$

$$\beta_{ml} = h_{ml}a_{ml}$$

(6.27)

where

$$a_{ml} \triangleq a_m(k)|_{k=z_{ml}/r_0}$$

(6.28)
and the weighting factor

\[ h_{ml} = \frac{2\sqrt{2}}{\pi c H_m^{(1)}(z_{ml}r_s/r_0) z_{ml} J_{m+1}(z_{ml})}. \]  

(6.29)

The F-R based method uses (6.27) to calculate \( \beta_{ml} \) and then the source distribution is reconstructed using the Fourier Bessel series synthesis equation (6.24).

### 6.4.2 Noise in the Modal Space

Let's denote the operation to calculate the modal coefficient \( a_m(k) \) (6.26) by \((A_m)_{\mathbf{p}}(k)\). Applying this operator to (6.25) at \( k = z_{ml}/r_0 \) results in

\[ (A_m)_{\mathbf{p}}(z_{ml}/r_0) = (A_m (A_p_0))(z_{ml}/r_0) + (A_m n)(z_{ml}/r_0). \]  

(6.30)

If we are to formulate the TV minimization problem in the modal space, we need to determine how the fidelity bound will change. The following definition proves that the noise power or variance \( \sigma_n^2 \) is unchanged.

**Theorem 6.4.1** [White Gaussian Noise in \( L^2 \)] Given a zero mean white Gaussian noise with variance \( \sigma_n^2 \) in \( L^2(\Omega) \) represented by a random variable \( n(r) \) where \( r \in \Omega \subseteq \mathbb{R}^U \), such that for any function \( \Psi_i(r) \in L^2(\Omega) \) the complex scalar \( n_i \)

\[ n_i \triangleq \int_\Omega n(r)\Psi_i(r)^* \, dr = \langle n(r), \Psi_i(r) \rangle \]

is also a zero mean Gaussian random variable with variance [28, eqn 8.1.35]

\[ E(|n_i|^2) = \sigma_n^2 \int_\Omega |\Psi_i(r)|^2 \, dr = \sigma_n^2 (\|\Psi_i(r)\|_{L^2})^2. \]

When the function \( \Psi_i(r) \) are orthonormal, \( \mathbf{n} = \{n_i\}_{i=1}^{\infty} \) are a vector of \( i.i.d. \), zero mean Gaussian random variable with variance \( \sigma_n^2 \). From Theorem 6.4.1, the orthonormal, modal expansion of the noise at mode \( m \), \((A_m n)(z_{ml}/r_0)\) remains a zero mean Gaussian random variable with variance of \( \sigma_n^2 \).

### 6.4.3 Reduced Dimension Fidelity Constraint

An exact source distribution \( p_0(r) \) can be reconstructed in the ideal case when the measurement bandwidth is infinite since only then can an infinite number of Fourier Bessel coefficients \( \beta_{ml} \) be estimated. In practice, the measurement
bandwidth is finite with the lower frequency limit of \( k_l \) and an upper frequency limit of \( k_u \), then the set of the coefficients that can be recovered is \( \hat{\Lambda} = \{ m, \ell | k_l \leq z_{m\ell}/r_0 \leq k_u, m \in \mathbb{Z}, \ell \geq 1, l \in \mathbb{Z} \} \). The index limited source distribution \( p_0(r) \) can be interpreted as the result of applying an index limiting operator on the original source distribution.

**Definition 6.4.3 (Index Limiting Operator)** From the Fourier Bessel expansion, we define an index limiting operator on the Hilbert space \( L^2(B^2(r_0)) \), as a truncated Fourier Bessel transform

\[
(\mathcal{B}_\Lambda p_0)(r) = \sum_{\{m,\ell\} \in \Lambda} \langle p_0(r), \Psi_{m\ell} \rangle \Psi_{m\ell}
\]

and can be thought of as a filter on \( B^2(r_0) \). This operator is compact \([29]\) and is also a projection. The index limiting operator is idempotent and self adjoint. The complement of this index limiting operator is

\[
(\mathcal{B}_{\Lambda^c} p_0)(r) = p_0(r) - (\mathcal{B}_\Lambda p_0)(r)
\]

where \( \Lambda^c \) is the complement of set \( \Lambda \).

In the modal space, the fidelity constraint (A), using (6.30), can be expressed as

\[
\sum_{\{m,\ell\} \in \Lambda} |(A_m p)(z_{m\ell}/r_0) - (A_m(M p_0))(z_{m\ell}/r_0)|^2 \leq \text{card}(\hat{\Lambda}) \sigma_n^2
\]

where \( \text{card}(\hat{\Lambda}) \) for a set \( \hat{\Lambda} \) is the cardinality operator returning the number of elements in \( \hat{\Lambda} \). Let \( \tilde{\beta}_{m\ell} \) be the calculated Fourier Bessel coefficients from the measured pressure, i.e.

\[
\tilde{\beta}_{m\ell} = h_{m\ell}(A_m p)(z_{m\ell}/r_0)
\]

and \( \beta_{m\ell} \) are the actual Fourier Bessel coefficients of the source distribution \( p_0(r) \), i.e.

\[
\beta_{m\ell} = h_{m\ell}(A_m(M p_0))(z_{m\ell}/r_0).
\]

Then the fidelity constraint (6.33) is simplified to

\[
\sum_{\{m,\ell\} \in \Lambda} |a_{m\ell} - \tilde{a}_{m\ell}|^2 \leq \text{card}(\hat{\Lambda}) \sigma_n^2
\]
where $\tilde{a}_{m\ell} = \hat{\beta}_{m\ell}/h_{m\ell}$ and $a_{m\ell} = \beta_{m\ell}/h_{m\ell}$. In matrix form, the fidelity constraint (A) is

$$\|a - \tilde{a}\|_{\beta(\Lambda)}^2 \leq \text{card}(\Lambda)\sigma_n^2$$

(6.37)

where $\tilde{a}$ is a vector of calculated modal coefficients $\tilde{a}_{m\ell}$ and $a$ is a vector of the actual modal coefficients $a_{m\ell}$ for $\{m, \ell\} \in \Lambda$. Moreover, the fidelity constraint is now a spheroidal rather than an ellipsoidal constraint.

Working in the Fourier Bessel domain has the following advantages:

- The fidelity constraint becomes a spheroidal constraint from an ellipsoidal constraint. Projecting onto ellipsoidal constraints are computationally expensive and requires solving an SOCP. However, projections onto spheroidal constraints have a closed form analytical formula (projection onto a $l_2$ ball).

- The dimension of the fidelity constraint in the Fourier Bessel domain is lower, this reduces the data resulting in lower computation costs. Further, with a F-R based reconstruction viewpoint the number of frequency samples for each mode is reduced to only those corresponding to the Bessel zeros.

- The fidelity bound becomes tighter.

Further reductions in dimension and magnitude of the fidelity bound is possible by utilizing the property that the source distribution $p_0(r)$ is always real.

**Theorem 6.4.2** Since the source distribution $p_0(r)$ is always real, the Fourier Bessel coefficients for the positive and the negative modes have the following relationship

$$\beta_{(-m)\ell} = \beta^*_{m\ell}.$$  

(6.38)

For proof of this theorem see Section 4.8.1. From the previous theorem, we can derive the following relationship between the modal coefficients.

**Corollary 6.4.1** From Theorem 6.4.2, the modal coefficients $a_{m\ell}$ for the negative and the positive modes are related by the following formula

$$a_{(-m)\ell} = \frac{h^*_{m\ell}}{h_{m\ell}} a^*_{m\ell}$$

(6.39)

**Proof**

Substituting (6.27) into (6.38), we get

$$h_{(-m)\ell}a_{(-m)\ell} = h^*_{m\ell} a^*_{m\ell}$$

(6.40)
Subsequently, using the following identity of Hankel functions $H_{-m}(\cdot) = (-1)^m H_m(\cdot)$ [30] and the following relationship $J_{-m+1}(z_{m\ell}) = (-1)^m J_{m+1}(z_{m\ell})$ [4] in (6.29) yields

$$h(-m)e = h_{m\ell}.$$  
(6.41)

Now, substituting (6.41) into (6.40) and simplifying gives (6.39).

From Corollary 6.4.1, we can derive the following relationship

$$|a_{(-m\ell)} - \tilde{a}_{(-m\ell)}| = \left| \frac{h_{m\ell}^*}{h_{m\ell}} a_{m\ell}^* - \frac{h_{m\ell}^*}{h_{m\ell}} \tilde{a}_{m\ell}^* \right|$$

$$= \left| \frac{h_{m\ell}^*}{h_{m\ell}} \right| |a_{(m\ell)} - \tilde{a}_{(m\ell)}|$$

$$= |a_{(m\ell)} - \tilde{a}_{(m\ell)}|$$  
(6.42)

since $|h_{m\ell}^*/h_{m\ell}| = 1$.

The result derived in (6.42) can be used to further reduce the dimension of the fidelity constraint (A), which becomes:

$$\|a - \tilde{a}\|_{\tilde{\Lambda}}^2 \leq \text{card}(\tilde{\Lambda}) \sigma_n^2$$  
(6.43)

where the set $\tilde{\Lambda} = \{m, \ell | \{m, \ell\} \in \hat{\Lambda}, m \geq 0\}$ and contains only the zeroth and positive modes. The cardinality of the new set $\tilde{\Lambda}$ is half the previous set $\hat{\Lambda}$ if we do not consider the zeroth modes.

### 6.4.4 Projection Operators for the Constraint Sets

If we neglect noise, then the constrained TV formulation in the Fourier Bessel domain is

$$\begin{align*}
\text{minimize} & \quad \text{TV}(p_0) \\
\text{subject to} & \quad \langle p_0(r), \Psi_{m\ell} \rangle = \tilde{\beta}_{m\ell} \text{ for } \forall \{m, \ell\} \in \hat{\Lambda} \\
& \quad p_0 \geq 0
\end{align*}$$  
(6.44)

This simplifies the fidelity constraint and projections onto this set is simple (ensuring that the required Fourier Bessel coefficients are a specific value for the reconstructed source distribution) and computationally less expensive than an spheroidal constraint in the presence of noise.

The non negativity constraint (B) remains the same for ideal, noiseless case and in the presence of noise. The operator $\mathcal{P}_B$ for projection onto the positivity
constraint is simply

\[
(\mathcal{P}_B p_0) = \begin{cases} 
0 & \text{if } p_{0,q} < 0 \\
p_{0,q} & \text{otherwise}
\end{cases}
\quad \text{for } \forall q.
\]

(6.45)

This operator can be interpreted as setting all negative values to zero in the source distribution.

Projection onto the fidelity constraint before transforming to the Fourier Bessel is itself an SOCP problem and requires the computation of the inverse of the sensitivity matrix \( M \) which is large and can be ill conditioned. However, approximate methods such as subgradient projection method [13] or the set theoretic Projection onto Convex Sets (POCS) [1] can be used to project onto the fidelity constraint. Transforming into the Fourier Bessel domain makes the fidelity constraint \((\mathcal{A})\) (6.43) into a spheroidal constraint. The projection onto this constraint (projection onto an \( l^2 \) ball) can be done by the following projection operator

\[
(\mathcal{P}_\Lambda a) = \begin{cases} 
a & \text{if } \|a - \tilde{a}\|^2_{l^2(\Lambda)} \leq \text{card}(\Lambda) \sigma_n^2 \\
\tilde{a} + v & \text{otherwise}
\end{cases}
\quad \text{6.46}
\]

where the vector \( v \) is defined as

\[
v = \frac{a - \tilde{a}}{\|a - \tilde{a}\|_{l^2(\Lambda)}} \sqrt{\text{card}(\Lambda) \sigma_n}
\quad \text{6.47}
\]

This projection operator is simple to implement and a geometric interpretation of the projection is shown in Fig. 6.1. This projection operator affects the modal coefficients, working with the Fourier Bessel coefficients \( \beta_{ml} \) would change the spheroidal constraint into an ellipsoidal constraint since the magnitude of the weighting term \( h_{ml} \) is different for different modes \( m \) and zero indices \( \ell \). Moreover, projection onto an ellipsoidal set is computationally expensive and requires solving an SOCP.

The Fourier Bessel coefficients \( \beta_{ml} \) for the set \( \Lambda \) can be calculated by the inner product \( \langle p_0(r), \Psi_{ml} \rangle \). From these Fourier Bessel coefficients we can calculate all the modal coefficients \( a_{ml} \) for the set \( \Lambda \) by using (6.27) and then form the vector \( a \). A new set of modal coefficients \( a' = (\mathcal{P}_\Lambda a) \) are obtained from the projection onto the fidelity constraint \((\mathcal{A})\). Further, a new set of Fourier Bessel coefficients \( \beta'_{ml} \) can be calculated using (6.27) with \( a'_{ml} \).

For the negative modes, the new Fourier Bessel coefficients are obtained by
applying Theorem 6.4.2 and so the new Fourier Bessel coefficients are available for the set $\Lambda$. Now, the projection operator onto the fidelity constraint (A) for the source distribution $p_0(r)$ is

$$\left( P_A p_0 \right)(r) = \left( B_{\Lambda C} p_0 \right)(r) + \sum_{\{m,\ell\} \in \Lambda} \beta_{m\ell} \Psi_{m\ell}. \quad (6.48)$$

The F-R based reconstruction method calculates the projection of the source distribution $p_0(r)$ on the Fourier Bessel series. Therefore, the formulation of the TV minimization from a F-R based reconstruction viewpoint provides important advantages that were discussed in this section. Further, this projection concept of the F-R based reconstruction allows application of alternating projections (also called POCS) method for post processing the reconstructed photoacoustic image. The alternation projections method for F-R based reconstruction will be presented in the next section.

## 6.5 Alternating Projections for Photoacoustic Reconstruction

The alternating projections method is a simple but slow algorithm for finding a point in the intersection of two convex sets by applying a sequence of projections.
Like gradient and subgradient based methods alternating projections can be slow but is useful when there are analytical ways of doing the projections on the convex sets. Cheney and Goldstein [31] proved that when the intersection of the two sets is nonempty then the sequence of points generated will converge to a point in the intersection of these sets. However, when the sets do not intersect, the alternating projection method provides two points that have minimum distance with one point in the first set and the other belonging to the second set.

For the frequency radial duality based reconstruction, we need to find a real source distribution

$$\text{find } p_0(\mathbf{r}) \text{ such that } p_0(\mathbf{r}) \in C \cap B$$

(6.49)

where set $B$ is the non negativity constraint $p_0(\mathbf{r}) \geq 0$ and the constraint $C$ preserves the calculated Fourier Bessel coefficients $\tilde{\beta}_{m}\ell$ and is expressed as

$$\langle p_0(\mathbf{r}), \Psi_m \rangle = \tilde{\beta}_{m}\ell \text{ for } \forall \{m, \ell\} \in \hat{A}.$$  

(6.50)

The projection on the positivity constraint or set $B$ was defined in (6.45). The projection operator onto the set $C$ is described by the next theorem.

**Theorem 6.5.1 (Projection operator to preserve $\tilde{\beta}_{m}\ell$)** The projection operator to preserve the calculated Fourier Bessel coefficients $\tilde{\beta}_{m}\ell$ of the source distribution $p_0(\mathbf{r})$ is given by

$$(P_C p_0)(\mathbf{r}) = p_0(\mathbf{r}) + \sum_{\{m, \ell\} \in \hat{A}} (\tilde{\beta}_{m}\ell - \langle p_0(\mathbf{r}), \Psi_m \rangle)\Psi_m$$

(6.51)

and is interpreted as first removing the contributions of the Fourier Bessel basis function for mode $m$ and zero index $\ell$ which can be calculated from the measured pressure and then adding the contributions corresponding to the calculated Fourier Bessel coefficients.

**Proof**

The projection operator adds contributions corresponding to the calculated Fourier Bessel coefficients to the complement of the index limited source distribution $p_0(\mathbf{r})$, that is

$$(P_C p_0)(\mathbf{r}) = (\mathcal{B}_{\hat{A}^C} p_0)(\mathbf{r}) + \sum_{\{m, \ell\} \in \hat{A}} \tilde{\beta}_{m}\ell\Psi_m$$

(6.52)

where $\hat{A}^C$ is the complement of set $\hat{A}$. Substituting (6.32) into (6.52) and simplifying yields (6.51).
The alternating projections method for photoacoustic tomography, starts with an initial source distribution $\tilde{p}_0^{(0)}(r) = \sum_{\{m,\ell\} \in \Lambda} \beta_{m\ell} \Psi_{m\ell} \in C$, and then alternately project onto $B$ and $C$:

$$\tilde{p}_0^{(k)}(r) = (\mathcal{P}_B \tilde{p}_0^{(k)})(r)$$

$$\tilde{p}_0^{(k+1)}(r) = (\mathcal{P}_C \tilde{p}_0^{(k)})(r)$$

$$k = 0, 1, 2, \ldots$$

(6.53)

This generates a sequence of source distributions which eventually converge to a point $\tilde{p}_0^* \in C \cap B$ as shown in Fig. 6.2. The number of iterations are usually set in advance.

### 6.6 Extension to 3D Spherical Geometry

The F-R based methodology for the 3D spherical geometry is described in detail in Chapter 5. In this section, we provide a brief description and outline how the proposed alternating projections and the TV minimization methods can be applied to the spherical case. The source distribution $p_0(r)$ where $r \in \mathbb{R}^3$ is
a point in the standard spherical co-ordinate system and is contained within a
3D ball $B^3(r_0) \subseteq \mathbb{R}^3$ with a bounding radius of $r_0$. The Hilbert space of square
integrable functions $L^2(B^3(r_0))$ contained within this ball is equipped with the
following inner product

$$\langle p_0(\mathbf{r}), f(\mathbf{r}) \rangle = \int_{\Omega \in S^2} \int_0^{r_0} p_0(\mathbf{r}) f^*(\mathbf{r}) \, r \, \mathrm{d}r \, \mathrm{d}\Omega \quad (6.54)$$

where $\mathbf{r} \triangleq (r, \Omega)$, the angular component $\Omega \triangleq (\theta, \phi)$ and integration over the
2-sphere $S^2$  is $\int_{\Omega \in S^2} \mathrm{d}\Omega \triangleq \int_0^{2\pi} \int_0^{\pi} \sin \theta \, \mathrm{d}\theta \, \mathrm{d}\phi$.

The source distribution $p_0(\mathbf{r})$ in $L^2(B^3(r_0))$ can be expanded using the or-
thonormal, spherical Fourier Bessel series

$$p_0(\mathbf{r}) = \sum_{\{n,m,\ell\} \in \Lambda} \alpha_{nml} \Phi_{nml} \quad (6.55)$$

where $\alpha_{nml}$ are complex spherical Fourier Bessel coefficients for order $n$, mode
$m$ and zero index $\ell$, and the set $\Lambda = \{n,m,\ell\; | \; -\infty \leq n,m \leq \infty; 1 \leq \ell \leq \infty; n,m,\ell \in \mathbb{Z}\}$. The basis functions are defined as follows

$$\Phi_{nml} = \frac{\sqrt{2}}{r_0^{3/2} j_{n+1}(z_{nl})} j_n \left( \frac{z_{nl}}{r_0} \right) Y_{nm}(\Omega) \quad (6.56)$$

where $Y_{nm}(\cdot)$ are the normalized spherical harmonics and $z_{nl}$ is the $\ell^{th}$ root of
the spherical Bessel functions $j_n(\cdot)$ of order $n$.

The F-R based method calculates the spherical Fourier Bessel coefficients for
the source distribution $p_0(\mathbf{r})$ by first calculating the spherical Fourier coefficients
of the aperture response $p(\mathbf{r}_s, k)$ ($\mathbf{r}_s$ are the sensor positions) at frequencies
corresponding to the Bessel zeros (more specific at frequencies $k = z_{nl}/r_0$),

$$\varphi_{nml} \triangleq \varphi_{nm}(k)|_{k=z_{nl}/r_0} = \int_{\Omega_s \in S^2} p(\Omega_s, k) Y_{nm}^*(\Omega_s) \, \mathrm{d}\Omega_s \quad (6.57)$$

where $\varphi_{nm}(k)$ are the spherical Fourier coefficients at frequency equal to $k$. The
aperture response $p(\Omega_s, k)$ varies in terms of the sensor angular position $\Omega_s$ since
the sensors are placed on a spherical manifold of constant radius $r_s$. From the
spherical Fourier coefficients the spherical Bessel coefficients of the source distri-
bution $p_0(\mathbf{r})$ can be obtained by

$$\alpha_{nml} = \hat{h}_{nl} \varphi_{nml} \quad (6.58)$$
where the weighting term is

\[ h_{nt} = \frac{\sqrt{2r_0}}{(z_{nt})^2 c j_{n+1}(z_{nt}) h_{n}^{(1)}([z_{nt}/r_0]r_s)} \] (6.59)

and \( h_{n}^{(1)}(\cdot) \) is a spherical Hankel function of the first kind. The source distribution \( p_0(\mathbf{r}) \) is then reconstructed by applying the spherical Fourier Bessel series synthesis equation (6.55). As before due to the finite measurement bandwidth, a bandlimited source distribution is obtained where \( \hat{\Lambda} = \{n, m, \ell | -n \leq m \leq n, k_\ell \leq z_{nt}/r_0 \leq k_u \} \).

Since the source distribution is positive, the spherical Fourier Bessel coefficients for the negative and the positive modes are related according to \( \alpha_{n(-m)\ell} = (-1)^m \alpha_{nmt}^* \) (see Section 5.3.6). This means that the spherical Fourier coefficients at the Bessel zeros \( \varphi_{nmt} \) for the positive and negative modes are related as follows:

\[ \varphi_{n(-m)\ell} = (-1)^m (h_{nt}/h_{nt}) \varphi_{nmt}^* \]

And so by the same algebraic manipulation as in (6.42), we get

\[ |\varphi_{n(-m)\ell} - \tilde{\varphi}_{n(-m)\ell}| = |\varphi_{nmt} - \tilde{\varphi}_{nmt}| \]

where \( \varphi_{nmt} \) would be the actual spherical Fourier coefficients without noise and \( \tilde{\varphi}_{nmt} \) are the calculated spherical Fourier coefficients from the measured data with noise. Now the TV formulation for the spherical geometry is

\[
\begin{align*}
\text{minimize} & \quad \text{TV}(\mathbf{p}_0) \\
\text{subject to} & \quad \|\mathbf{\varphi} - \tilde{\mathbf{\varphi}}\|_{\ell_2(\hat{\Lambda})}^2 \leq \text{card}(\bar{\Lambda}) \sigma_n^2 \\
& \quad \mathbf{p}_0 \geq 0
\end{align*}
\]

where \( \mathbf{\varphi} \) and \( \tilde{\mathbf{\varphi}} \) are the vectors containing the actual spherical Fourier coefficients \( \varphi_{nmt} \) and the calculated spherical Fourier coefficients \( \tilde{\varphi}_{nmt} \), respectively. The set \( \bar{\Lambda} \subset \hat{\Lambda} \) with \( m \geq 0 \). As before we have reduced the dimension of the fidelity constraint by working in the spherical Fourier domain and which now requires a similar projection as that for an \( l^2 \) ball (6.46). The spherical Fourier transform does not change the variance of the noise due to Theorem 6.4.1 and the fact the spherical harmonics form a complete orthonormal basis on the 2-sphere \( S^2 \).

The TV minimization formulation for the spherical case has the same form as that for the circular or for the 2.5D cylindrical case and so can be solved with the same method. A subgradient projection method for solving this TV minimization is presented in the next section.
For the TV minimization problem, we modify the optimization problem in (6.9) to

\[
\text{minimize } TV(p_0) \text{ subject to } p_0 \in \mathcal{C}
\]

and denote \( p_0^* \) as solution to this problem which can be solved by the projected gradient method (6.10) with \( g_{TV}^{(k)} \in \partial TV(p_0^{(k)}) \) being a subgradient of \( TV(p_0^{(k)}) \) (6.7) at the \( k^{th} \) step. For our definition of TV(\( p_0 \)), this subgradient \( g_{TV}^{(k)} \) is calculated by

\[
g_{TV}^{(k)} = \sum_{p=1}^{P-1} \sum_{q=1}^{Q-1} \text{sgn}(\Gamma_{p,q}^T \Gamma_{p,q} p_0^{(k)}) + \sum_{p=1}^{P-1} \text{sgn}(\Gamma_{p,q}^T \Gamma_{p,q} p_0^{(k)})
\]

where \( \text{sgn}(p_0) \) for a vector \( p_0 \) is a vector of equal dimensions whose co-ordinates are the signs of the corresponding co-ordinates in \( p_0 \) or zero if the corresponding co-ordinate is zero.

Next, the step size is chosen according to the following theorem.

**Theorem 6.7.1 (Square summable but not summable step sizes)** Step sizes can be chosen to satisfy the following conditions

\[
\sum_{k=1}^{\infty} \gamma_k^2 < \infty, \sum_{k=1}^{\infty} \gamma_k = \infty
\]

therefore, \( \lim_{k \to \infty} p_0^{(k)} = p_0^* \), these conditions ensure convergence of the subgradient method. The proof of this convergence result can be found in [32].

We choose step sizes

\[
\gamma_k = \frac{D}{k + 1}
\]

where \( D \) is a constant set a priori by the user. The step sizes therefore satisfies the conditions set in Theorem 6.7.1. The subgradient method is a slow algorithm and several ways can be used to speed it up by choosing optimum step sizes using level sets [13] and using localization or bundle methods. However, the scope of this chapter is to propose a computationally simple TV minimization method capable of handling the large data sets in photoacoustic tomography. Moreover, the practitioner can stop the TV minimizing algorithm after a few iterations.
The projected subgradient method can be easily extended for other TV formulations such as the isotropic TV formulation and is not only restricted to differentiable TV functions.

The main problem in applying the constrained TV minimization method to photoacoustic tomography is the presence of two convex constraints: the fidelity and the non-negativity constraint. The next section presents a method to deal with this in a manner so as to ensure convergence.

### 6.7.1 Projection for Multiple Convex Constraints

Projection operators such as the projection onto the positivity constraint $P_B$ and the projection onto the fidelity constraint $P_A$ are non-expansive mappings. Non-expansive mappings are defined as follows.

**Definition 6.7.1 (Non-expansive Mapping)** A mapping or operator $T: \mathcal{H} \rightarrow \mathcal{H}$ on an Hilbert space $\mathcal{H}$ is called non-expansive if it satisfies the following property

$$\|T(x) - T(y)\| \leq \|x - y\| \quad (6.65)$$

for $\forall x, y \in \mathcal{H}$.

Now to project onto the intersection of both these convex sets $C = A \cap B$ requires several alternative projections where the computational complexity of projection operator $P_B$ is dependent on the computational complexity of the Fourier Bessel series synthesis (6.24) and analysis steps (calculation of $\beta_{ml}$ from $p_0(r)$). Using the Fast Fourier Transform (FFT) and the fast Bessel series transforms [33], the computational complexity of the Fourier Bessel synthesis and analysis steps are $O(Q^2 \log Q)$ where discrete version of $p_0$, $P_0$ is in $\mathbb{R}^{Q \times Q}$.

Let's define a new operator $(P_C x) \triangleq (P_B \circ P_A p_0)$. Given $y \in C$, then since $y \in A$, we have $\|(P_A p_0) - y\| \leq \|p_0 - y\|$ and since $y \in B$ also, we have $\|(P_B \circ P_A p_0) - y\| \leq \|(P_A p_0) - y\|$. Combining both these inequalities, the new operator has the following property

$$\|(P_C p_0) - y\| \leq \|p_0 - y\| \quad (6.66)$$

and only requires one projection for each set with a computational order of $O(2Q^2 \log Q)$ where the constant 2 is required since there is one synthesis and one analysis step for $P_A$. The new operator $P_C$ is a non-expansive mapping.
Now the new projected subgradient method is given by

\[ p_0^{k+1} = \mathcal{P}_C(p_0^{(k)} - \gamma g_{TV}) \]  

(6.67)

where the starting value \( p_0^{(0)} \) is the reconstructed image from the F-R based method with a projection onto the non negativity constraint.

To prove the convergence of this new projected subgradient method, we start by defining \( v^{(k+1)} = p_0^{(k)} - \gamma g_{TV} \) i.e. the subgradient update before applying the operator \( \mathcal{P}_C \). The distance to the solution is

\[ ||v^{(k+1)} - p^*_0||^2 = ||p_0^{(k)} - \gamma g_{TV} - p^*_0||^2 \]

\[ = ||p_0^{(k)} - p_0^*||^2 - 2\gamma g_{TV}^T(p_0^{(k)} - p_0^*) + \gamma^2||g_{TV}||^2 \]

\[ \leq ||p_0^{(k)} - p_0^*||^2 - 2\gamma\|TV(p_0^*) - TV(p_0^*)\| + \gamma^2||g_{TV}||^2. \]  

(6.68)

Further from the property of the operator \( \mathcal{P}_C \) (6.66), we note that

\[ ||p_0^{(k+1)} - p_0^*|| = ||(\mathcal{P}_C v^{(k+1)}) - p_0^*|| \]

\[ \leq ||v^{(k+1)} - p_0^*||. \]  

(6.69)

Combing the two inequalities (6.68) and (6.69) yields

\[ ||p_0^{(k+1)} - p_0^*||^2 \]

\[ \leq ||p_0^{(k)} - p_0^*||^2 - 2\gamma\|TV(p_0^*) - TV(p_0^*)\| + \gamma^2||g_{TV}||^2 \]  

(6.70)

and so the proof proceeds in the same way as the proof of convergence for the ordinary subgradient method [32].

The image produced after a set number of iterations may not be in the feasible set \( C \). The previous convergence result shows that the resulting image after each iteration gets closer and closer to this feasible set in terms of the \( l_2 \) norm.

### 6.7.2 Summary of Algorithm

The steps in the constrained TV minimization for photoacoustic tomography, in the Fourier Bessel domain, can be summarized as follows:

1. Compute \( p_0^{(0)} \) as the reconstructed image from the F-R based method with a projection onto the non negativity constraint.

2. Choose a maximum number of iterations \( K \).
3. Set the step size according to (6.64).

4. Calculate the subgradient \( g_{TV}^{(\kappa)} \) at \( p_0^{(\kappa)} \) using (6.62).

5. Compute \( v^{(\kappa+1)} = p_0^{(\kappa)} - \gamma_\kappa g_{TV}^{(\kappa)} \).

6. Compute the approximate projection \( \mathcal{P}_C v^{(\kappa+1)} \) which applies the projection onto the fidelity constraint \( \mathcal{P}_A \) (6.48) and then the projection onto the non negativity constraint \( \mathcal{P}_B \) described in (6.45). Then set \( p_0^{(\kappa+1)} = (\mathcal{P}_C v^{(\kappa+1)}) \).

7. Increment the iteration counter \( \kappa \) and loop to step 4 while \( \kappa < K \).

8. End.

The computational complexity of the TV minimization is determined by the number of times the projection onto the fidelity constraint is performed (most costly step). For \( K \) iterations, we have \( K \) projections onto the fidelity constraint \( A \) and so the computational complexity of the TV minimization step is \( O(2K Q^2 \log Q) \). Note that the computational complexity of the alternating projections method with the same number of iterations \( K \) is the same since the projection to preserve the calculated Fourier Bessel coefficients \( \mathcal{P}_C \) requires one analysis step and one synthesis step of the Fourier Bessel series.

The F-R based reconstruction for the circular or the 2.5D cylindrical geometry with the FFT and the fast Bessel series transforms have a computational order of \( O(Q^2 \log Q) \). The use of either the alternating projections or the TV minimization methods proposed in this chapter will not change the computational complexity of the F-R based method provided the number of iterations is small i.e. \( K \ll Q \). However, these methods do add extra post processing computations to the F-R based reconstruction methodology.

### 6.8 Numerical Experiments

Numerical experiments were performed to investigate the improvements in the image quality after applying the proposed alternating and the TV minimization methods to the reconstructed images from an F-R based reconstruction method. These numerical experiments were done for the 2D case. However, it is simple to extend the numerical experiments to the 3D spherical array geometry where we expect similar results.
Figure 6.3: Numerical phantom composed of several circular discs, (a) cross section through the x-y plane and (b) horizontal central axis of this cross section.

The numerical phantom used in the simulations is shown in Fig. 6.3. The signals generated by the circular discs were produced according to the formula described in [34]. Further this formula was verified by experimental measurements. The speed of sound $c$ was set at 1500 m/s which is the speed of sound in biological tissue. Both the bounding radius $r_0$ and the sensor radius $r_s$ was set to 10 mm. The recorded pressure signals were low pass filtered at a frequency of 1 MHz and so 84 sensors were used to avoid aliasing, i.e., number of sensors is greater than $2k_wr_s$.

Noise was added to the generated pressure signals according to

$$\text{SNR} = 10 \log_{10} \left( \frac{\int_0^{T_s} |p(r_{ref} \cdot t)|^2 dt}{T_s \sigma_n^2} \right)$$

where the reference sensor is the sensor placed at a angular position $\phi = 0$ and $T_s$ is total amount of time for which the signals are recorded. This time $T_s$ was set to $(r_s + r_0)/c$ which is the total time for all signals generated within the ball bounded by $r_0$ to reach the sensors. In the numerical experiments, 20 dB of noise was added to the generated signals.

The optimum source distribution $p_0$ is the input phantom shown in Fig. 6.3
6.8 Numerical Experiments

and the relative Mean Square Error (MSE) is defined as

\[ \varepsilon_{rel} = \frac{\| \mathbf{p}_0^5 - \mathbf{p}_0^\circ \|}{\| \mathbf{p}_0^\circ \|}. \]  

(6.72)

The reconstructed images by applying the F-R based reconstruction method to the generated data for different measurement bandwidths are shown in Fig. 6.4. We observe that large structures are not fully reconstructed when the lower frequency limit is larger than zero. However, the boundaries of these large structures are present in these reconstructions. This is because the low frequency components of the numerical phantom cannot be obtained from these measurements. Further, both the reconstructed images contain ripply artifacts.

The application of the alternating projections method to the reconstructed images for a measurement bandwidth from 100 KHz to 1 MHz after 10 and 50 iterations are shown in Fig. 6.5. The alternating projections method produces images that are significantly better than those produced by the F-R based reconstruction alone. This is observed from Fig. 6.10, where the relative MSE \( \varepsilon_{rel} \) of the F-R based reconstruction is the relative MSE at \( \kappa = 0 \). Moreover, some of the lost, low frequency information can be recovered by applying alternating projections. The resulting image for a measurement bandwidth from 0 to 1 Hz after 10 iterations of the alternating projections method is shown in Fig. 6.6. The improvements in this case are not as significant as before when the lower frequency limit was higher than zero. Therefore, the alternating projections methods provides better improvements for images reconstructed when the lower limit of the measurement bandwidth is larger than zero. In practice, the ultrasound sensors used to record the pressure signals have a measurement bandwidth lower limit that is higher than zero, hence the alternating projections method is suitable for this scenario.

The alternating projections method after an infinite number of iterations produces an image that is in the intersection of both sets. After each iteration, the resulting image gets closer to the optimum image and to each of the constraint sets. A distance measure to one of these sets is an indication of how close the resulting image is to the optimum value. A plot of the distance to the positivity constraint (set B) \( d(\tilde{p}_0^{(\kappa)}(r), B) \) versus the iteration index is shown in Fig. 6.7. The plot shows the result of performing alternative projections on the F-R based reconstructed image with a measurement bandwidth from 100 KHz to 1 MHz. This shows that significant improvements occur up to about 10 iterations, also observed in Fig. 6.10. Therefore, after a few iterations of the
Figure 6.4: Reconstruction with the F-R based method in the presence of 20 dB noise for measurement bandwidth from 0 to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. Reconstruction using measurement bandwidth from 100 KHz to 1 MHz, (c) cross section through the x-y plane and (d) horizontal central axis.
6.8 Numerical Experiments

Figure 6.5: Resulting image after 10 iterations of the alternating projections method with measurement bandwidth from 100 KHz to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis. Resulting image after 50 iterations, (c) cross section through the x-y plane and (d) horizontal central axis. Note the improvement from the reconstruction shown in Fig. 6.4c and Fig. 6.4d.
alternative projections no significant improvement in image quality is obtained. Moreover, this can be observed by comparing the resulting images shown in Fig. 6.5 where the improvement after 10 iterations from the original reconstructed image in Fig. 6.4 is much greater than that from the 10^{th} iteration to the 50^{th} iteration.

The TV minimization method performs alternative projections as well as minimizing the Total Variation of the image for each iteration. The step size constant $D$ was chosen to be 0.1 in the numerical experiments. The resulting images for a measurement bandwidth from 100 KHz to 1 MHz for 10 and 50 iterations are shown in Fig. 6.8. Same as the alternating projections method, much improvement occurs in the first few iterations as shown in Fig. 6.10. The TV minimization method is also capable of recovering the lost low frequency information and has the added advantage of removing the ripply artifacts and denoising the resulting image. The resulting image after 10 iterations of the TV minimization method for a measurement bandwidth from 0 to 1 MHz is shown in Fig. 6.9. As before, the TV minimization method provides more improvement when the lower frequency limit of measurement is higher than zero. The TV minimization method produces better contrast, smoother images when compared to the alternating projections method allowing better identification of the different discs in the numerical phantom. As shown in Fig. 6.10, the rate of

Figure 6.6: Resulting image after 10 iterations of the alternating projections method with measurement bandwidth from 0 to 1 MHz. (a) cross section through the x-y plane and (b) horizontal central axis.
6.9 Summary

We have proposed a subgradient based TV minimization for photoacoustic tomography capable of dealing with the large data sets. Further, by working in the Fourier Bessel domain a constrained TV formulation is possible since the projection onto the fidelity constraint can be performed by a simple projection onto the $l_2$ ball. The constrained TV formulation has the advantage that the image reconstructed is not dependent on a Lagrangian parameter. The F-R based method recontracts images from projections onto a subset of the Fourier Bessel series expansion of the source distribution. Therefore, from this viewpoint and that the source distribution is always non negative, an alternative projections method was proposed to satisfy both the non negativity and the F-R based projection constraints. From the numerical experiments, it was proven that both the proposed TV minimization and the alternating projections method can recover some lost low frequency information and produce better quality with a few number of iterations than only using the F-R based reconstruction. The TV minimization method having the same computational complexity as the alternative

Figure 6.7: Distance of the resulting image to the positivity constraint (set B) related to the iteration index for the alternating projections method.

reduction of relative MSE $\varepsilon_{\text{rel}}$ is higher for the TV minimization method when compared to the alternative projections method.
Figure 6.8: Resulting image after 10 iterations of the TV minimization method with measurement bandwidth from 100 KHz to 1 MHz. (a) cross section through the x-y plane and (b) horizontal central axis. Resulting image after 50 iterations, (c) cross section through the x-y plane and (d) horizontal central axis. Note the improvement from the reconstruction shown in Fig. 6.4c and Fig. 6.4d.
Figure 6.9: Resulting image after 10 iterations of the TV minimization method with measurement bandwidth from 0 to 1 MHz, (a) cross section through the x-y plane and (b) horizontal central axis.

Figure 6.10: Relative MSE $\varepsilon_{rel}$ versus iteration index $\kappa$ for the TV minimization and the alternating projections method, for measurement bandwidth from 100 KHz to 1 MHz.
projections method has the added advantage of denoising the image, removing ripply artifacts and improving the contrast.

Future investigations can look at applying the ideas of TV minimization and alternative projections onto convex sets for photoacoustic tomography when the sound speed is not constant and for the partial view problem. However, to apply these methods to these cases, there is a need to quantify the fidelity bound to these scenarios as there are additional sources of data inconsistencies together with noise.

References


Chapter 7

Conclusions and Future Work

In this chapter, we draw conclusions from the work done in the thesis considering the motivation outlined in Section 1.1, and the aims and scope outlined in Section 1.2. Moreover, possible directions for future work are also suggested.

7.1 Conclusions

We separate the conclusions from the major chapters.

- **Chapter 3** solved the lung sound localization problem using a wavefield decomposition consisting of cylindrical harmonics. Moreover, we derived two performance metrics relating the resolution to the frequency and noise power, and the radius of region for which localization is possible. Numerical experiments showed that our proposed methods were effective in localizing multiple sound sources in the interior of sensor arrays and that the theoretical performance measures agreed with the results.

  These metrics should be known for a lung sound localization method since for lung sound localization, it is important to know the accuracy and the validity of the results obtained. Moreover, the spatial aliasing and resolution formula derived can be used in designing sensor systems for lung sounds given a required resolution, frequency range and noise power. These analysis will determine the number of sensors required to match the requirements.

- **Chapter 4** developed a theory for photoacoustic inversion in order to reconstruct a distributed source distribution (initial pressure distribution). We developed a theory for photoacoustic inversion from a frequency invari-
Conclusions and Future Work

This theory was applied in deriving a method for photoacoustic inversion using circular arrays. The information from each of the modes (basis functions in the wavefield decomposition) was separated by weighting and summing the aperture response with suitable weights. The theory proved that separation of information from each of the basis functions of the source expansion can be done by separating modes at frequencies corresponding to the Bessel zeros. By applying this theory, the Fourier Bessel coefficients in the source expansion were estimated and the source distribution reconstructed.

This method is valid for finite measurement bandwidths unlike previously proposed methods and is faster than previously proposed frequency domain methods. Numerical experiment was performed validating the effectiveness of our method for source reconstruction. Further, computational order analysis proved that our method is faster than previous inversion methods for the circular acquisition geometry.

This theory was used to design a method for photoacoustic inversion using an arbitrary sensor geometry without the shortcoming of time reversal methods which are applicable only in 3D. Our proposed method utilized a robust least squares method to separate wavefield modes at frequencies corresponding to the Bessel zeros. This method proved to be stable. Numerical experiments conducted showed that the reconstruction was of similar quality as those produced with a circular acquisition geometry.

In the third part of this chapter, we utilized our proposed theory to find the minimum number of frequency samples that are required for photoacoustic source (image) reconstruction. Further, a method was devised on how to select these frequencies. Previous, frequency domain reconstruction methods cannot provide a way to determine the minimum number of frequency samples to use. Our first method was extended to provide reconstruction with these minimum number of frequency samples. Numerical experiments proved the effectiveness of this method as well as showed that the methods developed in this part and the first part of the chapter were faster than the previously proposed frequency domain method.

• Chapter 5 devised a method to solve the 3D photoacoustic inversion problem using a spherical acquisition geometry. The source was expanded using a spherical Fourier Bessel series and the coefficients were now estimated by processing frequency samples corresponding to the zeros of spherical Bessel
functions. This method did not process all the frequencies for each mode of
the wavefield expansion, but a reduced number of frequencies correspond-
ing to the Bessel function zeros at that mode was necessary. Computa-
tional order analysis and numerical experiments proved that the proposed
method is faster than a backprojection and a Fourier series method. The
reconstructions using the proposed method was of similar quality as those
produced by the Fourier series method and better than those produced by
the backprojection method.

- **Chapter 6** used the framework that the methods developed in Chapters
  4 and 5 provide projections onto the basis functions in the source expan-
sion. With this idea, an alternating projections method or POCS (projec-
tion onto convex sets) was devised to provide a source that satisfies these
projections as well ensures that the source distribution functions is non
negative. Further, a subgradient based Total Variation (TV) minimization
method was developed that satisfies both the projection and the non neg-
ativity constraint while at the same time minimizing the TV of the source
distribution.

Application of these post processing methods with other reconstruction
methods are difficult or impossible. The numerical experiments performed
proved that considerable improvement in discerning different structures as
well as removal of artifacts is possible from a few iterations of these post-
processing (POCS or TV minimization) methods. These methods were
designed to be applicable to large data sets present in photoacoustic to-
mography and does not increase the computational order of the proposed
reconstruction methods mentioned in Chapters 4 and 5. Moreover, TV
minimization was shown to provide superior results when compared to the
POCS method by removing more of the ripple artifacts present in photoa-
coustic inversion with the same number of iterations.

### 7.2 Future Research Directions

Based on the work done in this thesis, we propose three main research direction
out of several possible future work areas, which could result in more efficient
inversion methods, more robust source localization methods and the design of
array signal processing methods using modal decomposition to provide solutions
to related problems.
7.2.1 Remove Restrictions

For lung sound localization, we assumed that the speed of sound in the chest was known in advance. However, in real scenarios, this speed varies from person to person. Further, previously proposed methods gave different results when the assumed speed did not match the actual speed. Therefore, the lung sound localization method needs to be modified to be robust with respect to mismatch in sound velocity.

The photoacoustic reconstruction methods proposed did not consider the finite sensor width which causes blurring of the source distribution. Further investigations as to how the sensor shape affects the estimation of the Fourier Bessel coefficients of the source distribution can be done. Future work can look at designing algorithms to reduce this effect when estimating these coefficients.

The partial view problem is also an important problem in photoacoustic imaging where the sensors cannot completely enclose the sample under study. This occurs in breast tumor detection where only a hemispherical array is possible. Future extensions needs to devise a method to estimate the Fourier Bessel coefficients given the limited view and investigate what limitations this imposes on the reconstructed image. Further, investigations using postprocessing methods such as TV minimization or POCS to improve the reconstruction in the limited view case can be performed.

This thesis assumed that the velocity of sound waves remained constant. However, in biological tissues, this can vary spatially. Given that the speed is a function of spatial parameters and this function is known, how can we estimate the Fourier Bessel coefficients of the source distribution. Again, research can be conducted to determine if the postprocessing methods (POCS and TV minimization) can be useful in this scenario.

There are many different acquisition geometries that occur in photoacoustic tomography. This thesis did not consider reconstruction methods for planar and cylindrical geometries. Future work can look into applying the proposed theory to develop reconstruction algorithms for these two geometries.

7.2.2 Other Areas of Application

The photoacoustic inverse problems is similar to several other problems. We are mainly interested in applying the theory developed in this thesis to develop reconstruction methods for Synthetic Aperture Radar (SAR) and sonar imaging, where the mathematical model is similar or close to the photoacoustic inversion
7.2 Future Research Directions

problem.

7.2.3 Experimental Evaluation

The algorithms proposed were not verified with real or measured data. For lung sound localization for the frequencies considered, the freefield acoustic model has been proven by experimental measurements, and the proposed method has worked well in the numerical simulations conducted. Verifications with experimental would further enhance the applicability of this method.

For photoacoustic tomography, the methods were tested with simulated data generated by models that have been extensively verified by experiments. Given that the blurring due to sensor width is small and speed variations of acoustic waves are small, our proposed methods would work in practice. Nevertheless, we suggest that the proposed photoacoustic reconstruction algorithms be tested with experimental data to validate their effectiveness.
1.3.1 Reaction Kinetics

The reaction kinetics of the process are the study of how the rate of a reaction changes with time. This involves understanding the factors that influence the reaction rate, such as temperature, concentration, and catalyst effectiveness. By analyzing these variables, one can optimize the reaction conditions to achieve the desired product yield and reaction rate. Kinetic studies are crucial in the development of new chemical processes, as they provide insights into the mechanisms of reaction and the conditions under which they operate. This information is essential for scale-up to industrial processes and for understanding the behavior of reactions under various conditions.
Appendix A

Localization of Quasiperiodic, Pulsatic Signal in a Correlated Mixture

A.1 Introduction

This appendix proposes a novel method to track the fundamental frequency of a quasiperiodic, pulsatic signal mixed with a correlated signal. Time domain based SVD, independent and principal component analysis fails for the correlated case. For a quasiperiodic signal, the signal obtained by keeping the frequency bin constant and varying the frame index is also quasiperiodic for all frequency bins. Since the second signal and the quasiperiodic signal overlap in the time-frequency domain, this property is preserved only for a subset of the frequency bins. We estimate the fundamental frequency by applying an SVD analysis with different frame lengths to these frequency bin signals. From this SVD analysis, we determine the number of frames for which the sum of the ratio of the first to the second singular value is largest over all the frequency bins. From this number of frames, we can calculate the fundamental frequency. By discarding the frequency bins where the quasiperiodic behavior is destroyed and reconstructing the time domain signal, the pulsatic signal can be localized in the mixture. This method can be applied for analyzing physiological, quasiperiodic signals such as heart sounds, and ECG and EEG signals.

In signal processing literature, there are several methods for the extraction and estimation of the fundamental frequency of periodic or cyclostationary signals in an uncorrelated mixture [1,2]. However, there has been a recent focus on
quasiperiodic signals since these occur in biological rhythmic signals and geophysical data such as climate variations. Even speech can be modeled as a quasiperiodic signal [3]. Fundamental frequency estimation in the time-frequency domain using surface reconstruction method applied to speech was proposed in [3]. Moreover, extraction of quasiperiodic signals in white, Gaussian noise was studied recently in [4]. Works on estimating the fundamental frequency of a quasiperiodic signal mixed with a correlated signal has not been studied before. Further, no time frequency analysis has considered the localization of a pulsatile signal. A measurement based heuristic for localizing pulsatile, quasiperiodic heart sounds mixed with lung sounds was introduced in [5]. In this appendix, we propose an algorithm for estimating the fundamental frequency and localizes a quasiperiodic signal in a correlated mixture.

This appendix is organized as follows: Section 2 introduces the problem, provides the definitions and the assumptions used. Section 3 gives a brief outline of the short-time Fourier transform (STFT). Section 4 discusses the proposed fundamental frequency estimation algorithm in the time-frequency domain and Section 5 presents a solution to localize the pulsatile signal within the correlated mixture. The results obtained on theoretical and real mixtures using these algorithms is presented in Section 6. Section 7 provides a summary of the contributions and the important ideas.

A.2 Problem Statement

We consider the problem of estimating the fundamental frequency (related to the period) of a quasiperiodic, pulsatile signal mixed with another signal. We will denote the quasiperiodic signal as $p(t)$ and the second signal as $s(t)$ in the time domain and begin this section by introducing definitions for a quasiperiodic signal and a pulsatile signal.

A quasiperiodic signal has the property that $p(t) = p(t + T) + \epsilon$ where $\epsilon$ is a small error value and $T$ is the period of the signal during a small time interval. This property is similar to a purely, periodic signal where the error value would be zero. The fundamental frequency and therefore the period of a quasiperiodic signal varies with time. We will assume that the rate of variation of the period is very small. A quasiperiodic signal is defined in terms of harmonic functions as

$$p(t) = \sum_{n=0}^{\infty} a_n \cos(n[2\pi f_0(t)]t + \theta_n(t)).$$  \hspace{1cm} (A.1)
where $\theta_n(t)$ is the phase of the $n^{th}$ harmonic at time $t$, $a_n$ is the coefficient of the $n^{th}$ harmonic, and $f_0$ is the fundamental frequency. It is significant to mention that the fundamental frequency is a function of time. In a periodic signal, $f_0$ is constant for all $t$ and the Fourier transform of a periodic signal consists of components that are integer multiples of the fundamental frequency. We assume that the fundamental frequency varies slowly with time. Further, we discretize this variation over a time segment equivalent to the period at that instant as

$$
p(t) = \begin{cases} 
a_1 \sum_{n=0}^{\infty} a_n g_n(f_1, t) & \text{for } 0 < t < T_1 \\
a_2 \sum_{n=0}^{\infty} a_n g_n(f_2, t) & \text{for } T_1 < t < T_1 + T_2 \\
\vdots & 
\end{cases}
$$

where $a_1$ and $a_2$ are scaling factors, $g_n(f, t) = \cos(n[2\pi f]t + \theta_n(t))$, $f_1$ the fundamental frequency during the first segment with period and duration $T_1 = 1/f_1$ and $f_2$ is the fundamental frequency of the second time segment with period and duration $T_2 = 1/f_2$. Note that $f_1 = f_2 + \delta_2$ where $\delta_2$ is small compared to the values of the fundamental frequencies.

Next, we define what we mean by a pulsatic signal in this paper. The signal $p(t)$ is also pulsatic, hence

$$
p(t)|_{0 < t < T} = \begin{cases} 
p_j(t) & \text{for } t_j < t < t_j + \Gamma_j, \ j = 1, 2, 3, \ldots \\
0 & \text{otherwise within a period } T
\end{cases}
$$

such that

$$
\frac{T - \sum_j \Gamma_j}{\sum_j \Gamma_j} \geq \frac{1}{3}.
$$

The condition in (A.4) ensures that a significant portion of the signal $p(t)$ is continuously at zero and the threshold value of 1/3 is arbitrarily chosen.

The pulsatic, quasiperiodic signal $p(t)$ is mixed with a second signal $s(t)$ together with additive white, Gaussian noise, $n(t)$ given by

$$
x(t) = p(t) + s(t) + n(t).
$$

In this problem, there is no reference signal or a second mixture with a different
linear combination of the two signals. This means that blind source separation (BSS) techniques cannot be applied in separating one signal from the other. We need to estimate the fundamental frequency of \( p(t) \) during a single time segment and localize the regions in times when this signal is not continuously zero i.e. locations of \( p_j(t) \) for all \( j \).

However, a further complication arises in that the signals \( p(t) \) and \( s(t) \) are not independent, but are correlated. The signal \( s(t) \) can be quasiperiodic but the range of its fundamental frequency must not overlap with that of \( p(t) \). The signal \( s(t) \) is a continuous signal, more specifically, with no continuous regions equal to zero. Therefore, the time-frequency regions of the signal \( p(t) \) and \( s(t) \) can overlap but there will be regions where the repeating pattern of \( p(t) \) is much stronger than in other regions.

We will use the short-time Fourier transform (STFT) to convert the signal to a time-frequency domain. This transform is introduced briefly in the next section.

### A.3 Short-Time Fourier Transform (STFT)

The discrete STFT of a sequence \( i(t) \) is defined as

\[
S(l, k) = \sum_{n=0}^{N-1} j(n + lH)w(n)e^{-in2\pi k/N}
\]  

(A.6)

for \( k = 0, 1, \ldots, N - 1 \). In (A.6), \( l \) is the time frame index, \( k \) is the frequency bin index, \( H \) is the hop size and \( w(n) \) is a window function and \( i \triangleq \sqrt{-1} \). The STFT is the Fourier transform of successive windowed signals. In this appendix, we use the Hamming window with \( N = 100 \) and \( H = 50 \). For hamming windows, a hop size equal to \( N/2 \) ensures that successive frames overlapping in time are weighted equally. The overlap-add method gives the synthesis equation to recover the original time signal. First, the windowed time frame is covered

\[
\hat{j}(l, n) = \frac{1}{N} \sum_{k=0}^{N-1} S(l, k)e^{in2\pi k/N}
\]  

(A.7)

and then these windowed time frames are overlapped and summed

\[
j(t) = \sum_{l=0}^{L-1} \hat{j}(l, n - lH).
\]  

(A.8)
Normally, the spectrogram is the log magnitude of the STFT, $20 \log_{10} |S(l, k)|$.

### A.4 Fundamental Frequency Estimation

When the signals $p(t)$ and $s(t)$ are independent, an SVD (singular value decomposition) based method in the time domain is enough to detect the fundamental frequency of $p(t)$. This method checks the ratio of the first singular value and the second singular value for different lengths of time samples. The period is estimated as the length of sample window for which this ratio is a maximum. This method was applied for EEG signals in [2].

However, when the two signals are correlated then this time domain SVD method fails. The STFT of $p(t)$ gives $N$ signals $S(:, k)$, showing how frequency components $k$ varies with time frames. The quasiperiodic and pulsatic property of $p(t)$ is inherited by $S(:, k)$ for frequencies where $p(t)$ has a non zero component at any time frame. The spectra of $s(t)$ and $p(t)$ can overlap, but our proposed method can only be applied successfully if there are a set of frequency indices where the overlap of the spectra of $s(t)$ on the spectra of $p(t)$ is minimal or the quasiperiodic property of $p(t)$ is dominant. Our solution sets up matrices from the spectra waveforms $S(:, k)$, thereby increasing the dimension/signals over which we can estimate the fundamental frequency. Firstly, we set up a matrix

$$S_L(:, k) = \begin{bmatrix} S(1, k) & S(2, k) & \ldots & S(L, k) \\ S(L + 1, k) & S(L + 2, k) & \ldots & S(2L, k) \end{bmatrix}$$ (A.9)

and then obtain its SVD, $S_L(:, k) = U \Lambda V^T$ where $U$ and $V$ are orthonormal vectors; $\Lambda$ is $\text{diag}\{\sigma_1, \sigma_2 : 0\}$ with $\sigma_1 \geq \sigma_2 \geq 0$. If $s(t)$ is not added to the mixture, then $S_L(:, k)$ is rank 1 with the ratio $\sigma_1/\sigma_2$ equivalent to infinity provided $H \times L$ corresponds to $1/f_1$.

Our method is iterative, estimating the number of frames that corresponds to the period of a signal. The number of frames $L$ can be converted to the number of time samples by $H \times L$. We iterate over a specified range of test frame numbers $L_t \leq L \leq L_u$ calculating the ratio $\sigma_1(L, k)/\sigma_2(L, k)$, the frame corresponding to the fundamental frequency is

$$\arg \max_L \sum_{k=0}^{N-1} \frac{\sigma_1(L, k)}{\sigma_2(L, k)}. \quad (A.10)$$

Those frequency bins where the overlap of $s(t)$ and $p(t)$ are significant contributes
less to this sum since the matrix $S_L(:, k)$ is not rank 1. But for frequency bins where this matrix is close to rank 1 provides a much larger contribution.

This method can deal with small shifts in the fundamental frequency, since only two singular values are required, therefore only two repeating segments are needed. Secondly, since the fundamental frequency is determined from the number of frames then as long as frequency shift $\delta$ equivalent to shift in period $1/\delta < H/2$, the next repeating pattern will still occupy the same number of frames due to the picket-fence effect of the Fourier transform [6]. This effect occurs since the Discrete Fourier Transform (DFT) represents the signal with integer multiples of a frequency that is the inverse of the sample length, for all components lying in the middle of these integer frequencies.

### A.5 Localization of Pulsatic Signal

The idea for localizing the pulsatic signal in the mixture is simple. From the estimate of the fundamental frequency, we keep the repeating pattern in the spectral waveform for frequency bins where the ratio of the first and second singular values is high and set the value of the spectra where this ratio is low to zero. From there, we reconstruct the resulting time domain signal which indicates the locations of the pulses in $p(t)$.

We will denote the frame number corresponding to the fundamental frequency as $L_f$. If we arrange the ratio of the singular values at $L_f$ in a descending order i.e. $\sigma_1^{-1}(L, k_{j-1})/\sigma_2^{-1}(L, k_{j-1}) > \sigma_1^{-1}(L, k_j)/\sigma_2^{-1}(L, k_j)$ and find the index that provides the maximum difference between the neighboring ratios

$$\arg \max_j \left( \frac{\sigma_1^{-1}(L_f, k_{j-1})}{\sigma_2^{-1}(L_f, k_{j-1})} - \frac{\sigma_1^{-1}(L_f, k_j)}{\sigma_2^{-1}(L_f, k_j)} \right).$$

This frequency bins corresponding to indices lower than or equal to this index represent the set of frequencies where the overlap of $s(t)$ and $p(t)$ are significant. We expect that there will be a sharp drop in the ratio as the test matrix changes from being singular to being rank 2. These frequency sets are set to zero in the STFT domain. For the other frequency ranges, we only require the portion that is repeated which can be obtained by

$$S(l, k) = u_1 \sigma_1(L_f, k)v_1^T$$

where $u_1$ is the first column of $U$ and $v_1$ is the first column of $V$. 

A.6 Results

In this section, we discuss the results obtained by using the proposed algorithm on mixtures where the time domain SVD was not effective. We used a known pulsatic signal as shown in Fig. A.1a and a continuous, correlated signal $s(t)$ (see Fig. A.1b). The signals were sampled at 2756.3 Hz. A spectrogram plot of the mixture is illustrated by Fig. A.2. Our method obtained the correct $L_f$ as 45 which gives the period as 2250 samples. The test input included the period of the pulsatic signals varying between 2100 to 2300 time samples. The algorithm could track period shifts only in increments of 50 samples. If a higher sensitivity is required then the hop size needs to be reduced. The resulting time domain signal from the output of the localization algorithm is illustrated by Fig. A.1d, we can observe that the location of the pulses in this signal corresponds to the location of $p(t)$ in the mixture.

We used auscultation recordings for two different individuals with recordings from five different places on the chest [7]. The task was to track the change in the heart rate and localize the heart sound pulses from the lung sounds. The algorithms provided the same performance as with the test data, verified by human inspection using the spectrogram of these signals.

A.7 Summary

In this appendix, we have proposed a novel algorithm in the time-frequency domain in order to estimate the fundamental frequency and localize a pulsatic, quasiperiodic signal in the presence of another correlated signal. However, the quasiperiodicity of a signal is present in the frequency variations of the signal with respect to the frame index. Presence of the correlated signal destroys this property for some frequency bins, but we can detect the frequency bins for which this quasiperiodicity is dominant by applying the proposed SVD based method. Using only these dominant frequency bins, we can determine the fundamental frequency and localize the pulsatic signal.
Localization of Quasiperiodic, Pulsatic Signal in a Correlated Mixture

Figure A.1: Test signals and results of simulations.

(a) $p(t)$

(b) $s(t)$

(c) Mixture of $s(t)$ and $p(t)$ with 20 dB of AWGN.

(d) output from the localizing algorithm
Figure A.2: Spectrogram of the mixture using a hop size of 50 samples and a frame size of 100 samples using a Hamming window.
References


