Amenability properties and their consequences in Banach algebras

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December 2008

A thesis submitted for the degree of Doctor of Philosophy of the Australian National University
A study on the role of pragmatism in American philosophy and their
consequences in Buddhist philosophy.
Decloration

The work in this thesis is my own except where otherwise stated.

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Declaration

The undersigned hereby declares under her/his own

signature:

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Acknowledgements

Firstly, I would obviously like to thank my supervisor Dr. Rick Loy for all his help and patience over the many years. The number of times Rick has dropped everything to help with my work is unbelievable, as well as the patience he showed every time I asked him a question where I was just confusing myself. He often pointed things out that I had not considered, but which dramatically improved the sharpness of results in this work. The speed and accuracy with which he corrected drafts, even when he was not a thousand percent, was astounding. He also showed extreme trust in me, allowing me to go in my own directions with work, but on the more personal side always showed understanding to my situation whether work related or not. It has been a tremendous time, largely thanks to him.

Also, I would also like to thank very much the organisers of the Banach Algebras 2007 conference. They did a marvellous job, and Université Laval offered me generous support to attend. The author would also like to thank Professor S. Stoneway for his assistance throughout the candidature. Often taken for granted, most people reading this work understand how important \LaTeX is for typesetting mathematical work. Thanks to the people who developed \LaTeX and all its packages and extra functionality off their own back. I am sure I am not the only one who appreciates it. Your legacy is a truly good and noble one.

Finally to all the people that I care about so much (even though I don’t tell you, I hope you know), many of which I met during my time here, thankyou so much for being there always. It has meant an awful lot, during the good times, and the bad ones. After all, it is these ones that make life so special. I haven’t seen many of you in a while, hugs when I do – see you soon.
Acknowledgements

I would like to thank the following people for their support and assistance during the development of this project:

[List of acknowledgments]

This project would not have been possible without the guidance and encouragement of [Name 1], [Name 2], and [Name 3]. Their dedication and expertise have been invaluable throughout the process.

Additionally, I would like to express my gratitude to [Name 4] for their unwavering support and for providing invaluable feedback. Their contributions have been crucial to the success of this project.

Finally, I would like to thank [Name 5] for their exceptional contributions and for going above and beyond to ensure the project was completed on time. Their commitment and hard work have been truly inspiring.

I am honored to have had the opportunity to work with such a dedicated and talented team. Their support and encouragement have been invaluable throughout this project.
Abstract

The amenability of Banach algebras is at the core of studying the structure of derivations in an abstract setting. By generalizing the notion of amenability in various ways, one obtains more structural information about derivations in cases where the Banach algebra fails to be amenable. As with amenability, the usefulness of providing a general analysis comes largely from the subsequent hereditary theory developed. As well as this, one obtains much knowledge about the structure of multiplication in a given Banach algebra by examining the structure of derivations from the Banach algebra, and vice versa. We seek to continue examining these phenomena, especially in the case of approximate weak amenability and approximate identities.
Abstract

The recent rapid growth in the number of mobile devices has created an urgent demand for more efficient data management. This paper proposes a novel approach to managing large datasets by leveraging the capabilities of mobile devices. The proposed method involves the development of a distributed system that utilizes the processing power of multiple mobile devices to perform complex data operations. The system is designed to be scalable and adaptable to different types of data and applications. Experimental results demonstrate that the proposed method significantly reduces the processing time compared to traditional methods, making it a promising solution for managing large datasets in a mobile environment.
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Notation and terminology

$A$ An associative algebra, usually a Banach algebra

$B(X,Y)$ The space of bounded linear operators from a Banach space $X$ to another Banach space $Y$

$B(X) = B(X,X)$

$K(X,Y)$ The space of compact linear operators from a Banach space $X$ to another Banach space $Y$

$K(X) = K(X,X)$

$W(X,Y)$ The space of weakly compact linear operators from a Banach space $X$ to another Banach space $Y$

$A^*$ The algebraic dual of $A$

$A^\prime$ The Banach algebra dual of $A$

$A^{(n)}$ The $n$-th Banach algebra dual of $A$

$A^\check{}$ The unitization of the Banach algebra $A$

$\mathcal{M}(A)$ The multiplier algebra of $A$

$A \oplus B$ The direct sum of two algebras $A$ and $B$

$A \widehat{\otimes} B$ The projective tensor product of two Banach algebras (or spaces) $A$ and $B$

$\pi$ The linear extension of the product map $a \otimes b \mapsto ab$ to the tensor product

$Z^1(A,X)$ The collection of continuous derivations from a Banach algebra $A$ into the Banach $A$-bimodule $X$
The collection of inner derivations from a Banach algebra $A$ into the Banach $A$-bimodule $X$

$Z^1(A, X)$

$N^1(A, X)$

The unit circle group

$D$

The open unit disc in $\mathbb{C}$

$A(D)$

$\{f \in C(D) : f|_D \text{ is analytic}\}$

$H^\infty(D)$

$\{f \in \ell^\infty(D) : f \text{ is analytic}\}$

$C^{(n)}[0, 1]$ $\{f : f \text{ is } n \text{ times continuously differentiable on } [0, 1], n \geq 1\}$

$L^\varphi$

The Orlicz space associated with the Young function $\varphi$

$\ell^\varphi$

The Orlicz sequence space associated with the Young function $\varphi$

$\ell^\Phi$

The modular space associated with a collection of Young functions $\Phi = (\varphi_n)$

$G$

A locally compact group

$S(G)$

A Segal algebra on the group $G$

$\hat{G}$

The dual group of a locally compact abelian group $G$

$\Sigma$

The dual object of a locally compact group

$GL(n)$

The general linear group of invertible $n \times n$ matrices

$SU(n)$

The special unitary group of $n \times n$ matrices

$U(n)$

The unitary group of $n \times n$ matrices

$SO(n)$

The special orthogonal group of $n \times n$ matrices

$O(n)$

The orthogonal group of $n \times n$ matrices

$SL(n, \mathbb{R})$

The special linear group of $n \times n$ matrices with real coefficients

$\hat{f}$

The Fourier transform of a function $f$
Chapter 1

Overview

1.1 Approach

It is rare in mathematics that a single result leads to a dynasty of researchers and study into an area of the subject. But sometimes, that is how profound the statement of fact can be. The result of Barry Johnson ([35] Theorem 2.5), demonstrating equivalence between the amenability of a locally compact group and the vanishing of some cohomology of the $L^1$ group algebra is definitely one of the more pioneering of these monumental results. Some would argue that given the notions of amenability relevant in many categories, Johnson’s result is merely one branch of the ‘amenability tree’. But being a very early such extension to a category other than groups, and amenability of Banach algebras still being a very active area of research, this branch is indeed the trunk of the tree.

Background on the notion of amenability for groups and its extension to Banach algebras is given in [50]. However, with hindsight, we elect to take a different approach. Despite the fact that we outline the necessary background material, a basic familiarity with Banach algebras and amenability is assumed.

Derivations are a very important concept in many areas of mathematics, including algebra, differential geometry and partial differential equations. Hence, it is often desirable to know about and understand the structure of the most general forms of derivations from and into the appropriate spaces. Amenability for Banach algebras turns out to be a very important notion in achieving this.

We will start by assuming only the most minimal algebraic structure and proceed to build towards amenability for Banach algebras, hence explicating why Banach algebras are a very appropriate framework for the study of derivations. At the most elementary level, we may consider derivations from an associative...
algebra into an algebraic bimodule. For the basic definitions, consult [7] and [12]. Note that the following conditions can be phrased in terms of algebraic cohomology, see [59] for details.

Throughout this document, we shall work with algebras over the complex field \( \mathbb{C} \).

**Definition 1.1.** Let \( A \) be an algebra, and \( E \) be an \( \mathbb{A} \)-bimodule. A linear map \( D : A \to E \) is a *derivation* if for every \( a, b \in A \)

\[
D(ab) = D(a) \cdot b + a \cdot D(b).
\]

Note here that we do not yet require the module actions (or the derivations) to be continuous – \( A \) may not even be associated with a topology. When introducing the notion of amenability, Johnson included such continuity in the context of Banach algebras, but here we will omit this and introduce it later so as to ascertain how restrictive the purely algebraic notions are.

There is an immediate way of generating linear maps which are derivations: consider taking a fixed \( x \in E \) and defining

\[
\delta_x(a) = a \cdot x - x \cdot a, \quad (a \in A).
\]

Then for \( a, b \in A \),

\[
\delta_x(ab) = (ab) \cdot x - x \cdot (ab) = a \cdot (b \cdot x - x \cdot b) + (a \cdot x - x \cdot a) \cdot b = a \cdot \delta_x(b) + \delta_x(a) \cdot b,
\]

and so \( \delta_x \) is a derivation.

**Definition 1.2.** A derivation from an algebra \( A \) to an \( \mathbb{A} \)-bimodule \( E \) is said to be *inner* if it is of the form \( \delta_x \) for some \( x \in E \).

Perhaps the immediate question is, given an algebra and a bimodule, are all the derivations inner? We define this explicitly.

**Definition 1.3.** An algebra \( A \) is *contractible* if for each \( A \)-bimodule \( E \), every derivation \( D : A \to E \) is inner.

From ([7] Theorem 1.9.21) we have the following standard result:

**Theorem 1.4.** An algebra \( A \) is contractible if and only if \( A \) is finite-dimensional and semisimple.
In the language of cohomology, this is the same as saying that $H^1(A, E) = 0$ for every $A$-bimodule $E$.

The class of contractible algebras is very small; namely, they are just finite direct sums of full matrix algebras, and as such are not very interesting. Hence to obtain a larger class one might consider examining only certain types of bimodules. Amenability for Banach algebras restricts to topological dual bimodules, however since at this point we are still looking at all (not just topological) algebras, we will consider all algebraic dual bimodules:

**Definition 1.5.** An algebra $A$ is **amenable** if for each $A$-bimodule $E$, every derivation $D : A \to E^\ast$ is inner. That is, $H^1(A, E^\ast) = 0$ for every $A$-bimodule $E$.

However this turns out to be equivalent to algebraic contractibility, as can be shown from Dales' synopsis of algebraic cohomology. For suppose $H^1(A, E^\ast) = 0$, all $E$. Then by ([59] Proposition 1.9.18)

$$H^{n+1}(A, E^\ast) \simeq H^1(A, (A \otimes \cdots \otimes A \otimes E)^\ast)$$

as linear spaces, and so is 0 by hypothesis. But this means that the corresponding homology groups $H_n(A, E)$ are all trivial also. Hence by ([59] Theorem 9.2.11) we have that $A$ is finite-dimensional and semisimple.

Being the same as algebraic contractibility, this is not of much interest to us. However all is not lost, but we do have to restrict our algebras and bimodules in a more appropriate way. It turns out (via Johnson's result of course) that one way to do this is by imposing topological structure on our algebras and bimodules. The natural framework for this is Banach algebras, which are simply Banach spaces $A$ which have an associative product which is continuous, that is, there exists $K > 0$ such that $\|ab\| \leq K\|a\|\|b\|$ ($a, b \in A$).

**Henceforth in this document, unless otherwise specified, our algebras $A$ shall be Banach algebras, and our bimodules will be Banach $A$-bimodules in that they will be Banach spaces with continuous module actions. Derivations from a Banach algebra $A$ into a Banach $A$-bimodule will be taken to be continuous.**

By doing this, we obtain the **Banach algebra** concepts of contractibility and amenability. Because our algebras now have a functional analytic structure we make use of topological duals in the definition of amenability.

**Definition 1.6.** A Banach algebra $A$ is **contractible** if for each Banach $A$-bimodule $E$, every derivation $D : A \to E$ is inner.
**Definition 1.7.** A Banach algebra $A$ is *amenable* if for each Banach $A$-bimodule $E$, every derivation $D : A \to E^*$ is inner.

Even with this Banach algebra structure, it is still widely conjectured that contractibility of $A$ is equivalent to finite-dimensionality and semisimplicity; see [49] for an account, where it is shown that this is true under some fairly mild hypotheses on the Banach space structure of $A$.

Amenability for Banach algebras, on the other hand, has turned out to be an extremely fruitful and important notion in the theory of Banach algebras. Because of this, when referring to amenability and contractibility from now on, we will do so in this Banach algebra context, unless otherwise specified.

Most references introduce the notion of amenability as was done historically. In 1964 the notion of amenability for Banach algebras came about when Barry Johnson was examining the relationship between a locally compact group $G$ and its associated algebra $L^1(G)$. A locally compact group is said to be *amenable* if there is a left invariant mean on $L^\infty(G)$, see eg. [50] for details. Johnson’s landmark result is that $G$ is amenable if and only if $L^1(G)$ is amenable as a Banach algebra.

There are two reasons as to why amenability for Banach algebras is important. Firstly it is not as restrictive as contractibility; it is known that there are non-trivial Banach algebras which are amenable, for example $C(K)$, $K$ compact, $K(\ell^p), p < \infty$ and certain $L^1(G)$ algebras discussed above. Also there is a very strong hereditary theory associated to amenability, see [50], along with the knowledge that the notion of amenability imposes the existence of a bounded approximate identity. In some ways though, there is still a problem. The above-mentioned examples are essentially the only natural examples of amenable Banach algebras. Even the $\ell^p, p < \infty$ spaces have an approximate identity, are commutative, semisimple and in the Banach space context, also basic, yet are not amenable with pointwise multiplication. $B(\ell^2)$ has an identity and is semisimple, and is not amenable [44]. So, what can be done to broaden our concept of amenability to cover more situations?

There are several notions related to amenability for Banach algebras. One such is approximate amenability, where the derivations are required to be approximable by inner ones in the strong topology. There are others, and with each one, it is important to consider the tradeoff between the scope of examples satisfying the property and the strength of the hereditary theory associated with the concept.

It was implicit in the above that amenable Banach algebras are generally not
amenable as associative algebras. We may consider this more generally: are all algebras which are amenable in one class also amenable in a wider class? Pirkovskii and Selivanov ([45] Corollary 4.5) have shown that any amenable Banach algebra is also amenable when considered as a Fréchet algebra. Does this remain true for other notions of amenability (to be defined below)? For example, the authors of [37] suggest that the category of Fréchet algebras is suitable for considering approximate amenability.

It is also possible to consider classes contained within Banach algebras. When defining the appropriate (Connes) amenability for dual or von Neumann algebras, the relevant bimodules are dual normal bimodules and one considers weak*-continuous derivations, recognising the dual space structure involved. Connes amenability is certainly weaker than Banach algebra amenability for dual Banach algebras, consider $B(\ell^2)$. Note that some $B(X)$ spaces when $X$ is infinite-dimensional are Connes amenable, whereas some are not ([50] §4.4), whereas it is conjectured that none of these are amenable; for example $B(\ell^p \oplus \ell^q), p \in (2, \infty), q \in (1, 2)$ with $p^{-1} + q^{-1} = 1$ are not Connes amenable [13], whereas amenability is unknown. This is due to the dual space structure inherent in the definition of Connes amenability. Note that the cohomology of $C^*$-algebras is usually studied via their enveloping von Neumann algebras.

Amenability also fails to capture the full behaviour of operator algebras. For example, the Fourier algebra $A(SO(3))$ is not amenable. But when one defines operator amenability (by restricting to operator bimodules), this captures the amenability of the group: $G$ is amenable if and only if $A(G)$ is operator amenable ([50] Theorem 7.4.3). Is operator approximate amenability actually weaker than operator amenability? It has been recently determined ([5] §4) that $A(\mathbb{F}_2)$ fails to be operator approximately amenable, so that the solution to this question could be very difficult indeed.

As briefly alluded to above, there are essentially three major considerations with any notion of amenability for Banach algebras, namely the inclusion of wide classes of examples, the hereditary theory, and the presence of approximate identities and the associated structure of multiplication.

It was mentioned above that although amenability for Banach algebras is much ‘better’ than contractibility in that it is known to include infinite-dimensional examples, and that it is also an appropriate category to consider, there is still a lack of examples encased by the notion. Traditionally, this led to defining variants of amenability, such as weak amenability or $n$-weak amenability. We begin with weak amenability for commutative Banach algebras.
CHAPTER 1. OVERVIEW

Definition 1.8. [3] A commutative Banach algebra $A$ is weakly amenable if for each commutative $A$-bimodule $E$, any continuous derivation $D : A \to E$ is inner.

As can be seen, this involves further restricting the bimodules considered in a rather severe way. In fact, even more so than at first glance, as ([3] Theorem 1.5) shows that one needs only consider the bimodule $A^*$ in the definition. Hence for any Banach algebra one generalizes to the following:

Definition 1.9. ([33] Definition 3.3) A Banach algebra $A$ is weakly amenable if each derivation $D : A \to A^*$ is inner.

There are many examples of Banach algebras which are weakly amenable, yet not amenable. These include the $\ell^p$ spaces with pointwise product; and in fact any algebra which is spanned by its idempotents is weakly amenable ([7] Proposition 2.8.72), though may also be amenable. One may even define variants on weak amenability by considering the $n$-th dual of $A$ as the relevant bimodule ([7] Definition 2.8.75), referred to as $n$-weak amenability. However, another approach is to allow for a generalization regarding the derivations themselves.

Definition 1.10. [21] A Banach algebra $A$ is approximately amenable if, for each Banach $A$-bimodule $E$ and every continuous derivation $D : A \to E^*$, there exists a net $(x_\alpha^*) \subset E^*$ such that $D$ is given by

$$D(a) = \lim_\alpha (a \cdot x_\alpha^* - x_\alpha^* \cdot a) \quad (a \in A).$$

That is to say, $D$ is approximately inner.

([21] §6) includes examples of Banach algebras which satisfy approximate amenability but which are not amenable; these are $c_0$-sums of specific amenable algebras. Perhaps slightly less contrived is the fact that certain Banach sequence algebras including the James space are approximately amenable ([11] Corollary 3.5) but not amenable ([7] Example 4.1.45 (x)); the Fourier algebra of a locally compact group $A(G)$ is approximately amenable when $G$ is amenable and discrete ([26] Theorem 3.1) whereas [18] shows that $A(G)$ is amenable if and only if $G$ has an abelian subgroup of finite index. For any of the various notions of amenability, it is important to find examples which demonstrate that the notion is indeed different from one that is formally stronger. Ideally, each notion of amenability would include as many examples as possible, so as to fit these into the framework of the subsequent theory developed. We will seek to provide such examples in some cases. However we will now discuss that there is a fundamental tradeoff involved.
So far, we have not really discussed why one would desire to examine amenability type notions, aside from the fact that one may in turn determine the structure of given derivations. As already mentioned the typical approach is to weaken the notion of amenability considered, so as to include more examples. However the weaker the notion of amenability, the weaker one can expect the resulting theory to be. More specifically, we are referring to the hereditary theory involved. That is, given a Banach algebra which satisfies a given notion of amenability, when do, for example, its ideals and quotients also satisfy the property? After all, this is the inherent usefulness of any given notion of amenability; one may invoke the subsequently determined theory instead of verifying the property for every related algebra. In the case of amenability itself, the hereditary theory has been studied extensively ([50] §2.3). But given another notion, the second thing one must do, after finding the relevant examples, is to develop the hereditary theory. Aside from the formal definition of a notion of amenability itself, the hereditary theory is perhaps most indicative in describing the relative strength of the notion. For example, it is obvious that, formally, amenability is stronger than approximate amenability. And so is the hereditary theory. For example it is standard ([50] Theorem 2.3.7) that if $A$ is amenable, then so is any closed complemented ideal $I \subset A$. Now ([23] Corollary 4.5) and the remarks following it show that if $\alpha = (\alpha_k)$ is a collection of positive real weights, then the Feinstein algebra $A_\alpha$ is approximately amenable in the case where $\liminf_k \alpha_k < \infty$. However, $\ell^1$ is contained complementedly in $A_1$ as an ideal, where $1 = (1, 1, \ldots)$ is the constant sequence of weights, and $\ell^1$ is not approximately amenable ([11] Theorem 4.1). We will seek to examine the hereditary behaviour for various notions of amenability.

Given a Banach algebra $A$, the first question that may come to mind is as to the structure of the multiplication operation within the algebra. At one end of the scale, is $A$ even essential, by which we mean $A = \overline{A^2}$, using the notation $A^2 = \text{span}\{a_1a_2 : a_1, a_2 \in A\}$? If the underlying Banach space is not trivial and the multiplication evaluates to zero always, then this property obviously fails, and so such algebras do not have a strong structure in this sense. At the other end of the scale, $A$ might have an identity. We will see that the behaviour of the multiplication is fundamental to amenability properties. Typically, some form of approximate identity is closely linked to the amenability properties satisfied by a Banach algebra. For example, $\ell^1(\mathbb{Z})$ fails to be approximately amenable under pointwise operations ([11] Theorem 4.1) but $\ell^1(\mathbb{Z})$ is amenable under convolution via Johnson’s standard result [35]; the two underlying Banach spaces are
of course the same. Notice that under pointwise multiplication, the given algebra has no bounded approximate identity, whereas under convolution it has an identity. Thus it comes as no surprise that the existence or lack thereof of various approximate identities in a Banach algebra $A$ is central to determining the amenability behaviour of $A$. In fact any amenable Banach algebra must have a bounded approximate identity ([50] Proposition 2.2.1), immediately giving that, for example, the algebras $\ell^p, p < \infty$ are not amenable; it is much more difficult to show they fail to be approximately amenable ([11] Theorem 4.1). Some of the most fundamental open questions about approximate amenability relate to the existence of various approximate identities. It is unknown whether approximate amenability of a Banach algebra $A$ implies the existence of a bounded approximate identity (or a two-sided unbounded one) in $A$. The solution to this problem would impact on our knowledge of the hereditary properties of approximate amenability, since if $A$ is approximately amenable and possesses a bounded approximate identity, then $A \oplus A$ must also be approximately amenable ([23] §6). The seemingly innocuous question about whether $A$ being approximately amenable implies that $A \oplus A$ is approximately amenable is still unresolved. A full understanding of the existence and behaviour of approximate identities in a given Banach algebra goes a long way towards determining whether the algebra satisfies or fails various notions of amenability. We will examine the existence of various approximate identities in Orlicz spaces, which are a generalization of the $\ell^p$ spaces.

There are many standard results in Banach algebra theory which we will need. Typically we will refer to the somewhat biblical reference of Dales [7], as we have already done above. The same is the case for amenability, whence we refer to this as well as the lecture notes of Runde [50].

In our reference list, we have used the bibliographical details as supplied in the format by the source of publication (for preprints, we have adopted a consistent format), along with the plain style within Bibtex. Whilst this approach yields a reference list which is not in an entirely self-consistent format, and for which the ordering of entries is slightly askew, a list which is representative of the authentic citation of each publication is obtained.
Chapter 2

Combining approximate notions

2.1 Approach

As already mentioned, one would like to modify the original concept of amenability so as to facilitate the inclusion of further examples of Banach algebras. The approach is to weaken the definition appropriately so as to allow such examples. One approach to this is to severely restrict the choice of bimodules considered, leading to the notions of weak and $n$-weak amenability. These serve to substantially increase the number of examples encapsulated, for example the $\ell^p$ spaces are weakly amenable ([7] Proposition 2.8.72), $L^1(G)$ is weakly amenable for any locally compact group $G$ ([7] Theorem 5.6.48), $A(G)$ for any SIN group is weakly amenable if and only if the connected component of the identity $G_e$ is abelian ([17] Corollary 3.4), and $C^*$-algebras are *permantly weakly amenable*, which is to say that they are $n$-weakly amenable for all $n \in \mathbb{N}$ ([8] Theorem 2.1).

The subsequently developed alternative approach is not to restrict the bimodules, but to somewhat diminish the restriction on the structure of the derivations themselves; which leads to approximate amenability. This concept was shown to be extremely fruitful upon its introduction [21]. We seek to combine the two approaches. In the case of (standard) amenability, the innerness of derivations is known to be equivalent to the existence of various elements in $(A \hat{\otimes} A)^{**}$ and $A \hat{\otimes} A$, referred to as virtual and approximate diagonals respectively (see [50] Theorem 2.2.4). There is a natural analogue in the approximate case.

Definition 2.1. [24] A Banach algebra $A$ is said to be *pseudo-amenable* if there is a net $(x_\alpha) \subset A \hat{\otimes} A$ (not necessarily bounded) such that $a \cdot x_\alpha - x_\alpha \cdot a \to 0$ and $\pi(x_\alpha)a \to a$ for each $a \in A$. Further, $A$ is *pseudo-contractible* if the $x_\alpha$ can be chosen to be central, that is, $a \cdot x_\alpha = x_\alpha \cdot a$ for all $a \in A$ and all $\alpha$. 

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Such a net is called an approximate diagonal, which is somewhat ambiguous terminology considering that the same term is used when referring to a bounded approximate diagonal in the amenable case. We will always be explicit as to which we mean.

What, if any, is the relationship between approximate and pseudo-amenability? The $\ell^p$, $p < \infty$ algebras are pseudo-amenable ([24] Proposition 2.1) but not approximately amenable ([11] Theorem 4.1) so the notions are not equivalent; the same is true for the Fourier algebra $A(\mathbb{F}_2)$ ([26] Remark 3.4). In the presence of a central approximate identity, pseudo-amenability is weaker than approximate amenability, and in the presence of a bounded approximate identity they are equivalent ([24] Propositions 3.2 and 3.3), but in general nothing is currently known. This is yet another situation where the presence of certain approximate identities is fundamental in determining the amenability properties of a given Banach algebra. However there is no complete characterization as to how the existence of (possibly) unbounded diagonals relates to the structure of derivations from a Banach algebra $A$. As there is no clear way to combine pseudo-amenability with weak amenability (as there is no clear link with pseudo-amenability to bimodules) we restrict ourselves to combining the notions of weak and approximate amenability. The obvious way to do this is to require all derivations into the dual bimodule to be approximately inner.

**Definition 2.2.** A Banach algebra $A$ is approximately weakly amenable if for every continuous derivation $D : A \to A^*$ there is a net $(a^*_\alpha) \subset A^*$ such that

$$D(a) = \lim_{\alpha} (a \cdot a^*_\alpha - a^*_\alpha \cdot a) \quad (a \in A).$$

Given what has already been said, this is a natural combination of the two distinct methods for generalizing amenability. Approximate weak amenability has not been extensively studied, except for Segal algebras [25]. One of our main goals is to remedy this situation. It is obvious that approximately amenable algebras and weakly amenable algebras satisfy the given property, but this is of no interest. After all, it has not even been established that approximate weak amenability is totally independent of weak amenability, or of approximate amenability. The broad purpose of this chapter is to present this independence. To be specific, we seek an example of a Banach algebra, which is approximately weakly amenable, but which fails to be approximately amenable or weakly amenable.
2.2 The example

A very tentative start in this search is the fact that we know such a Banach algebra cannot possess non-trivial continuous point derivations ([21] Proposition 2.1). This immediately rules out some common Banach algebras such as $A(D)$, $H^\infty(D)$ and $C^n[0,1]$ for all $n \geq 1$. In fact, a simple argument leads to the following.

**Proposition 2.3.** Suppose that $A$ is a commutative Banach algebra which is approximately weakly amenable. Then $A$ is weakly amenable.

**Proof.** Consider a derivation $D : A \to A^*$. Then by hypothesis there is a net $(x^*_\alpha) \subseteq A^*$ such that for each $a \in A$,

$$D(a) = \lim_{\alpha} (a \cdot x^*_\alpha - x^*_\alpha \cdot a) = 0,$$

since $A^*$ is a commutative $A$-bimodule. □

That is to say, for commutative Banach algebras, approximate weak amenability and weak amenability are the same. It is known that certain Banach algebras possess no non-trivial (continuous) point derivations, but still fail to be weakly amenable.

$C_\text{abs}(\mathbb{T}) = \{ f \in L^\infty(\mathbb{T}) : \| f \| = \| f \|_\infty + \int_0^{2\pi} |f(e^{i\theta})| d\theta < \infty \}$

is an example of such a Banach algebra ([3] p2). It is clear that the example of a Banach algebra we are searching for must fail to be commutative.

As previously stated the only concerted effort to study approximate weak amenability has been carried out on Segal algebras. We recall the basic definitions involved. Given a function $f$ defined on a group $G$, we denote the left and right translations of $f$ by $x \in G$ as

$$l_x f(y) = f(xy), r_x(y) = f(yx), \quad (y \in G).$$

**Definition 2.4.** A Segal algebra on a locally compact group $G$ is a linear subspace $S(G)$ of the convolution algebra $L^1(G)$ which satisfies the following properties:

- $S(G)$ is dense in $L^1(G)$
- $S(G)$ is a Banach space under some norm $\| \cdot \|_S$ and $\| f \|_S \geq \| f \|_1$ for all $f \in S(G)$
• $S(G)$ is left-translation invariant, and $\|l_x f\|_S = \|f\|_S$ for every $x \in G$ and $f \in S(G)$, and the map $x \mapsto l_x f$ from $G$ into $S(G)$ is continuous.

Under these properties, it is standard that a Segal algebra is indeed a Banach algebra, contained as an ideal in $L^1(G)$. A Segal algebra is symmetric if the analogous statement also holds for right translations.

Note at this point that we distinguish between Segal algebras defined on a group and abstract Segal algebras ([7] Definition 4.1.8). However group Segal algebras also happen to be abstract Segal algebras ([7] p492). We also need to slightly restrict the class of groups considered.

Definition 2.5. A locally compact group $G$ is called a SIN group if there is a basis for neighbourhoods of the identity of $G$ consisting of compact sets $U_\alpha$ for which $xU_\alpha x^{-1} = U_\alpha$ for all $\alpha$ and $x \in G$.

It is standard that many commonly considered groups are of class SIN, for example all compact groups, discrete groups and abelian groups.

The main result obtained for Segal algebras is the following. We see that many of the most common Segal algebras are in fact approximately weakly amenable.

Theorem 2.6. ([25] Theorem 2.1) Suppose that $S(G)$ is a symmetric Segal algebra on a SIN group. Then $S(G)$ is approximately weakly amenable.

In particular, $L^2(G)$ is approximately weakly amenable for compact $G$. Whilst the proof of this fact is quite technical, the idea is to extend a given derivation $D : S(G) \to S(G)^*$ to a derivation $\tilde{D} : L^1(G) \to L^1(G)^*$, and use the fact that $L^1(G)$ is always weakly amenable ([50] Theorem 4.2.3). Subsequently, one uses a central approximate identity to bring the problem back into $S(G)$.

There is also a result of this type for weak amenability [20]. However the fact is that it assumes commutativity of the Segal algebra, that is, $G$ must be abelian (this result is, of course, obvious from Theorem 2.6 in light of the equivalence of weak and approximate weak amenability under the commutativity hypothesis). A partial converse to this is the following observation.

Proposition 2.7. ([20] Remark 3.2) Suppose that $G$ is a compact non-abelian group. Then $(L^2(G), \ast)$ is not weakly amenable.

This fact is established by taking the inner derivation determined by the point mass at a non-central point $x \in G$ and examining this derivation in relation to some other element $y \in G$ which does not commute with $x$. 
This indicates the possibility that there might be a compact non-abelian group $G$ such that $L^2(G)$ fulfils the required criteria. We look for a non-abelian compact group $G$ such that $L^2(G)$ is not approximately amenable. This group must be infinite.

To determine a possible approach for this, one looks to the abelian case, even though this is not what is ultimately desired. We have already chosen to investigate the case of compact $G$. Suppose then for the moment, that $G$ is abelian. Then the standard Fourier transform maps to a Fourier series, and gives a Banach algebra isomorphism of $L^2(G)$ onto $\ell^2(\hat{G})$ under pointwise multiplication. We thus need a result relating to the approximate amenability of $\ell^2$-sums.

**Lemma 2.8.** For $1 \leq p < \infty$ and any infinite set $I$, $\ell^p(I)$ fails to be approximately amenable.

This is the observation directly following Theorem 4.1 of [11]. Thus $\ell^2(\hat{G})$ fails approximate amenability, and hence $L^2(G)$ fails approximate amenability. Armed with this, we look for a generalization of the Fourier series for arbitrary compact groups. Thankfully, compact groups turn out to be exactly the groups for which such an analysis is feasible. The references [16] and [31] describe this method, which we adopt.

**Definition 2.9.** Let $G$ be a locally compact group. A unitary representation of $G$ is a homomorphism $\pi$ from $G$ into the group $U(H_\pi)$ of unitary operators on some non-trivial Hilbert space $H_\pi$ which is continuous in the strong operator topology on $U(H_\pi)$.

Thus $\pi$ is a map $\pi : G \to U(H_\pi)$ such that $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and for which $x \mapsto \pi(x)u$ is continuous from $G$ to $H_\pi$ for each $u \in H_\pi$.

From now on, when we talk about a representation $\pi$ of a group $G$, $\pi$ is assumed to be a unitary representation.

**Definition 2.10.** Let $\pi_1$ and $\pi_2$ be unitary representations of a group $G$. $\pi_1$ and $\pi_2$ are (unitarily) equivalent if there exists a unitary operator $U$ such that $\pi_2(x) = U\pi_1(x)U^{-1}$ for each $x \in G$.

We wish to consider appropriate ‘decompositions’ of representations, and in particular work with representations which cannot be decomposed in the following sense.
Definition 2.11. Let $\pi$ be a unitary representation of a group $G$ on the Hilbert space $H_\pi$. Suppose $M$ is a closed subspace of $H_\pi$. $M$ is called an invariant subspace for $\pi$ if $\pi(x)(M) \subseteq M$ for all $x \in G$. $\pi$ is reducible if $\pi$ admits a non-trivial invariant subspace, otherwise it is irreducible.

It is easy to see that given a group $G$ and a representation $\pi$, if $M$ is such a non-trivial invariant subspace, then so is $M^\perp$, and $\pi = \pi|_M \oplus \pi|_{M^\perp}$ ([16] Proposition 3.1 and Corollary 3.2). If $G$ is abelian, then every irreducible representation of $G$ is one-dimensional, consistent with the dual group $\hat{G}$ yielding the orthodox Fourier series.

Definition 2.12. For $G$ a locally compact group, the dual object of $G$, denoted by $\Sigma = \Sigma(G)$, is the collection of equivalence classes of irreducible unitary representations of $G$.

It is important to remember that we are dealing with equivalence classes. What can we say about $\Sigma(G)$ in the case that $G$ is not abelian?

Proposition 2.13. ([16] Theorem 5.2) Suppose that $G$ is compact. Then every irreducible representation of $G$ is finite-dimensional, and every unitary representation is a direct sum of irreducible representations.

This is not as readily manipulated as in the abelian group case, but is at least to some extent tractable, as we will see.

Considering that we cited the Fourier series technique as an easy way of solving the abelian case for approximate amenability, we seek to define the appropriate range space for a generalization of the Fourier transform. It will fundamentally involve the elements of $\Sigma$, as one might expect.

Definition 2.14. ([31] 28.24) Let $I$ be an index set. For each $i \in I$, let $H_i$ be a Hilbert space of dimension $d_i$. Then denote $\bigoplus_{i \in I} B(H_i)$ by $\mathcal{G}(I)$. Define a Hilbert space direct sum norm on $\mathcal{G}(I)$ by

$$
\|E\|_2 = \left( \sum_{i \in I} d_i \|E_i\|_{tr}^2 \right)^{1/2},
$$

and corresponding space $\mathcal{G}_2(I) = \{ E \in \mathcal{G}(I) : \|E\|_2 < \infty \}$; here, $\|M\|_{tr}$ denotes the trace class norm of a matrix $M$.

It is shown in ([31] Corollary 28.29) that $\mathcal{G}_2(I)$ is a Hilbert space under the inner product

$$
<E, F> = \sum_{i \in I} d_i \text{tr}(E_i F_i^*),
$$
thus generalizing the Hilbert space $\ell^2$.

One now considers the index set $\Sigma$, with $d_\sigma = \dim H_\sigma$, and uses coordinate-wise product on $\mathfrak{G}_2(\Sigma)$ to turn it into a Banach algebra. Note that if $G$ were abelian, then $\mathfrak{G}_2(\Sigma)$ would correspond to $\ell^2(\Sigma)$ both in terms of topology and multiplication. We now make use of the algebra $\mathfrak{G}_2(\Sigma)$; it puts us in a position to define an appropriate Fourier transform.

It should be noted that if one uses ([31] Definitions 28.34) to define our Fourier transform $f \mapsto \hat{f}$ for $f \in L^1(G)$, it is given in terms of the decompositions of the unitary representations of $G$. We use an equivalent definition, which is in the notation of ([16] p134), circumventing the need to specify bases for the Hilbert spaces involved.

**Definition 2.15.** The **Fourier transform** of $f \in L^1(G)$ at $\sigma \in \Sigma$ is the operator

$$\hat{f}(\sigma) = \int f(x)\sigma(x)dh(x),$$

which maps onto a dense subset of $c_0(\Sigma) = \{E \in \mathfrak{G}(I) : \|E_i\|_\text{tr} \xrightarrow{i} 0\}$. As usual, $h$ is Haar measure on the group $G$.

To make sense of this definition, one should view it in the weak sense. That is, for any $u \in H_\sigma$ we define $\hat{f}(\sigma)u$ by specifying its inner product with another arbitrary $v \in H_\sigma$, and this is given by

$$< \hat{f}(\sigma)u, v > = \int f(x) < \sigma(x)u, v > dh(x).$$

Then this Fourier transform has the desired properties.

**Theorem 2.16.** ([31] Theorems 28.40 and 28.43) Suppose that $G$ is a compact group. Then the transform $f \mapsto \hat{f}$ maps $L^2(G)$ isometrically onto $\mathfrak{G}_2(\Sigma)$. Also, this map is a $*$-isomorphism, in that $\hat{f} \ast \hat{g}(\sigma) = \hat{f}(\sigma)\hat{g}(\sigma)$ for $f, g \in L^2(G)$ and $\sigma \in \Sigma$.

Hence the Fourier transform is actually a Banach algebra isomorphism from $L^2(G)$ onto $\mathfrak{G}_2(\Sigma)$. This means that if we can establish that $\mathfrak{G}_2(\Sigma)$ fails to be approximately amenable, then $L^2(G)$ will also fail to be approximately amenable.

To aid this search, let us for the moment assume that the group has infinitely many one-dimensional representations; somewhat resembling the abelian case. Write $\Sigma_1 = \{\sigma \in \Sigma : \dim(H_\sigma) = 1\}$. Then

$$\mathfrak{G}_2(\Sigma) = \mathfrak{G}_2(\Sigma_1) \oplus \mathfrak{G}_2(\Sigma \setminus \Sigma_1),$$
with both of these summands being ideals; recall that we are using coordinatewise product. Since the product on $\mathcal{S}_2(\Sigma_1)$ coincides with the pointwise product on $\ell^2(\Sigma_1)$, and denoting $I = \mathcal{S}_2(\Sigma \setminus \Sigma_1)$, we have

$$\mathcal{S}_2(\Sigma) / I = \ell^2(\Sigma_1).$$

Note that when we write $\mathcal{S}(\Sigma)$, it is clear that we are referring to complex-valued sequences over the index set $\Sigma$. However, we will frequently encounter $\ell^p(A)$ where $A$ is a Banach algebra, which will denote $A$-valued sequences over a countable index set with summable $p$-th powers of the norms. It will always be clear from the context which we are using.

We know that $\mathcal{S}(\Sigma)$ is not approximately amenable from Lemma 2.8, and thus in this instance $\mathcal{S}_2(\Sigma)$ cannot be approximately amenable ([21] Corollary 2.1). We recall the standard fact that if two group characters are distinct, then they are non-equivalent. Hence, we have the following.

**Proposition 2.17.** To find a compact group for which $L^2(G)$ fails approximate amenability, it suffices to find such a non-abelian group with infinitely many one-dimensional representations. □

**Example 2.18.** Consider the group $G = \mathbb{T} \times SU(2)$, where $\mathbb{T}$ is the circle group and $SU(2)$ is the special unitary group of $2 \times 2$ matrices $T$ with determinant 1 such that $T^*T = I$. We know that $\mathbb{T}$ has dual group $\mathbb{Z}$ of countably many distinct one-dimensional unitary representations. We now extend these to the entire group $G$.

To this end, consider the representations as operators $\tilde{\psi} : \mathbb{T} \mapsto B(\mathbb{C})$, and extend to $\tilde{\psi} : G \mapsto B(\mathbb{C})$ by $\tilde{\psi}(t, M) = \psi(t)$. These are all clearly unitary and not equivalent for different $\psi \in \hat{\mathbb{T}}$, and so there are countably many classes of the required representations. Hence our group $G$ has infinitely many classes of irreducible one-dimensional representations, and (at last) $L^2(\mathbb{T} \times SU(2))$ with convolution cannot be approximately amenable.

It is obvious that the pattern generalizes.

**Corollary 2.19.** Suppose that $G = G_0 \times G_1$ where $G_0$ and $G_1$ are compact, and $G_0$ is infinite and abelian. Then $L^2(G)$ is not approximately amenable. □

Hence by choosing $G_1$ non-abelian, one has a range of groups for which $L^2(G)$ is approximately weakly amenable, but which is neither approximately nor weakly
amenable. Note that we may even choose $G_1$ to be finite in this context. Recall that when $G$ is non-abelian, ([20] Remark 3.2) uses any two noncommuting elements of $G$ to demonstrate the lack of weak amenability of $L^2(G)$.

We can also see that in this case (as is also known for wider classes of Banach algebras) pseudo-amenability is weaker than approximate amenability.

**Proposition 2.20.** ([24] Proposition 4.4) Suppose that a group $G$ is compact. Then $L^2(G)$ is pseudo-amenable.

This result demonstrates the apparent lack of any connection between the structure of derivations and existence of diagonals in this situation.

**Example 2.21.** The unitary group $U(2)$ of $2 \times 2$ matrices $T$ such that $T^*T = I$ has countably many one-dimensional non-equivalent unitary representations ([31] §29.48). $U(2)$ is also isomorphic (as a group) to $\frac{\mathbb{R} \times SU(2)}{\{(1,1), (1,-1)\}}$ ([16] p146), elucidating the existence of the one-dimensional representations. Thus $L^2(U(2))$ is not approximately amenable via Proposition 2.17.

### 2.3 Generalizations

#### 2.3.1 $\ell^p$-sums

Thus far, we have considered compact groups with infinitely many one-dimensional representations. A compact group is only guaranteed to have one such representation, namely the trivial one. What about compact groups with infinitely many $n$-dimensional representations? This would involve examining the space $\ell^2(M_n)$ for some fixed $n$. For the rest of this section, we will take $n$ to be some fixed natural number, and write $M(\cdot) = M_n(\cdot)$, $M = M_n(\mathbb{C})$. We obtain an appropriate result about sequences of matrices, based on a very similar result on amenability from the recent memoir [9].

**Proposition 2.22.** Let $A$ be a Banach algebra. Then $A$ is approximately amenable if and only if $M_n(A)$ is approximately amenable.

**Proof.** The ‘if’ part (which is what we really need) follows directly via the method of ([9] Theorem 2.7), replacing inner derivations with approximately inner derivations. For the converse, one invokes ([21] Proposition 2.3) and its subsequent remark, noting that since $M$ is finite-dimensional, the condition in the remark is satisfied.
This has immediately given us a way of untangling the approximate amenability of $\ell^1(M_n)$ algebras via the canonical Banach algebra isomorphism $\ell^1(M) = M(\ell^1)$. However a similar approach is also possible in the $\ell^p$ case.

**Lemma 2.23.** Let $1 \leq p \leq \infty$. Denoting an element $\Gamma = (\gamma_{ij}) \in M$, we have that

$$||\Gamma||_p^1 = \left(\sum_{i,j} |\gamma_{ij}|^p\right)^{1/p}$$

is an algebra norm on $M$.

**Proof.** The fact that $\Gamma = 0$ if and only if $||\Gamma||_p = 0$, as well as the homogeneity of $|| \cdot ||_p$, are obvious. The triangle inequality is the same as Minkowski’s inequality for $\ell^p$ spaces. Since $M$ is finite-dimensional, all norms on $M$ are equivalent, and hence $|| \cdot ||_p$ is an algebra norm. \qed

This observation immediately leads to an important identification, whose verification is elementary.

**Proposition 2.24.** For $1 \leq p \leq \infty$, there is a canonical Banach algebra isomorphism from $M(\ell^p)$ onto $\ell^p(M)$.

**Remark 2.25.** This crucially uses the fact that all norms on a fixed finite-dimensional algebra, namely $M$, are equivalent, a fact we will use again shortly to relate back to the representation space of a compact group. Unfortunately, we cannot say anything about that algebra $A_p = \ell^p(M_k)$ where $k$ is now the index of summation. We know that $A_p$ is not amenable, but whether it is approximately amenable is unknown. Note that the canonical approximate identity $e_k = \sum_{i=1}^k \delta_k$, where $\delta_k$ has an $k \times k$ identity matrix in the $k$th coordinate, is multiplier bounded, and that $A_p$ does not factor (similarly to the corresponding $\ell^p$ space), so that $A_p$ does not have a bounded approximate identity. Hence $A_p$ is not boundedly approximately amenable ([5] Theorem 3.3).

We now give the main result for this section, discussing approximate amenability for summable sequences of finite-dimensional algebras.

**Theorem 2.26.** Suppose that $F$ is a finite-dimensional normed algebra. Then for $1 \leq p < \infty$, $\ell^p(F)$ is not approximately amenable.

**Proof.** Since $F$ is finite-dimensional, it has a Wedderburn decomposition

$$F \cong \bigoplus_{i=1}^N M_{n_i} \otimes R,$$
where $R = \text{rad}(\mathcal{F})$. Hence we have the decomposition

$$
\ell^p(\mathcal{F}) \cong \bigoplus_{i=1}^{N} \ell^p(M_{n_i}) \odot \ell^p(R) \cong \bigoplus_{i=1}^{N} M_{n_i}(\ell^p) \odot \ell^p(R).
$$

If $\mathcal{F}$ is not purely radical, then Proposition 2.22, Lemma 2.8 and ([21] Corollary 2.1) show that $\ell^p(\mathcal{F})$ is not approximately amenable. On the other hand, if $\mathcal{F}$ is radical, then, being finite-dimensional, it is nilpotent, whence so is $\ell^p(\mathcal{F})$. Thus $\ell^p(\mathcal{F})$ cannot possess an approximate identity. 

**Remark 2.27.** From the isomorphism Proposition 2.24, and Proposition 2.22, it becomes clear that $\ell^\infty(\mathcal{F})$ is amenable (or approximately amenable) if and only if $\mathcal{F}$ is semisimple.

Given the obtained result for finite-dimensional algebras, there is an immediate question: for $1 \leq p < \infty$, does $\ell^p(A)$ fail approximate amenability for every Banach algebra $A$? We proceed with a technical lemma.

**Lemma 2.28.** Suppose that $A$ is a Banach algebra, and that $I$ is a closed ideal in $A$. For $1 \leq p \leq \infty$, we have an isomorphism

$$
\frac{\ell^p(A)}{\ell^p(I)} \cong \ell^p(A/I).
$$

**Proof.** Let $a = (a_n) \in \ell^p(A)$. Define $Q : \ell^p(A) \to \ell^p(A/I)$ by

$$
Q(a_n) = (a_n + I), \quad (a = (a_n) \in \ell^p).
$$

It is immediate that $Q$ is a contractive homomorphism.

Now suppose that $i = (i_n) \in \ell^p(I)$. Then $Q(i) = 0$. If $a \in \ell^p(A)$ and $Q(a) = 0$, then we have $a_n \in I$ for each $n$, and so $a \in \ell^p(I)$. Thus $\ker Q = \ell^p(I)$.

Finally, we must check that $Q$ is surjective. Given $(a_n + I) \in \ell^p(A/I)$, we know that, for $n \in \mathbb{N}$, given $\varepsilon_n > 0$, there is $i_n \in I$ such that

$$
\|a_n + i_n\| \leq \|(a_n + I)\| + \varepsilon_n.
$$

In particular, if $a_n \notin I$, then $\|a_n + I\| \neq 0$, and so we can choose $i_n \in I$ such that

$$
\|a_n + i_n\| \leq 2\|a_n + I\|. \quad (2.1)
$$

And if $a_n \in I$, then $\|a_n + I\| = 0$, so with $i_n = -a_n$, equation (2.1) still holds. So for all $n$, we may choose $i_n \in I$ so that (2.1) holds.

Thus $(a_n + i_n) \in \ell^p(A)$ by comparison, and $Q(a_n + i_n) = (a_n + I)$.

$\square$
With this, we proceed by taking a Banach algebra $A$, with $I$ a closed finite-codimensional ideal. Then $\ell^p(I)$ is a closed ideal in $\ell^p(A)$, and we have the given isomorphism. We obtain the following result.

**Corollary 2.29.** Suppose that $A$ is a Banach algebra with a closed ideal $I$ of finite codimension. Then for $1 \leq p < \infty$, $\ell^p(A)$ is not approximately amenable.

**Proof.** We immediately obtain that $\ell^p(A)/\ell^p(I)$ is not approximately amenable and by ([21] Corollary 2.1), $\ell^p(A)$ also fails approximate amenability. □

Hence, $\ell^p(A)$ fails to be approximately amenable for many examples of Banach algebras $A$, including commutative non-radical $A$ (for then $A$ admits a non-zero character). This confirms, for example, that algebras such as $\ell^1(\ell^\infty)$ are not approximately amenable.

Coming back to compact groups, the following result is formally stronger than what we had obtained previously.

**Proposition 2.30.** Let $G$ be a compact group with associated representation space $\Sigma$. Suppose that $G$ has infinitely many $n$-dimensional representations for some $n$. Then $L^2(G)$ is not approximately amenable.

**Proof.** We repeat the argument preceding Proposition 2.17, with $\Sigma_n = \{\sigma \in \Sigma : \dim(H_\sigma) = n\}$ and $I = \mathcal{G}_2(\Sigma \setminus \Sigma_n)$ to obtain

$$\mathcal{G}_2(\Sigma)/I = \ell^2(M_n) = M_n(\ell^2),$$

which is not approximately amenable. Recall that norming $M_n$ under $\| \cdot \|_{\text{tr}}$ is equivalent to norming it under $\| \cdot \|_2$. □

This enables us to examine some more groups.

**Example 2.31.** Consider the infinite product $G = \prod_{i=1}^{\infty} SU(2)$. It is known that $SU(2)$ has exactly one $n$-dimensional representation for each $n$ ([16] Theorem 5.39). Thus it is easy to verify using ([31] Theorem 27.43) that $G$ has only one 1-dimensional representation, but infinitely many $n$-dimensional representations for $n \geq 2$. We may now invoke Proposition 2.30 to see that $L^2(G)$ is not approximately amenable.
2.3. GENERALIZATIONS

2.3.2 Other directions

There are several possible generalizations to this. There is one very immediate problem with the above situation, a very definite gap in what has been established. We have a result about the approximate amenability of $L^2(G)$ for $G$ compact only when $G$ has infinitely many $n$-dimensional representations for some fixed $n$.

**Example 2.32.** Consider the compact non-abelian group $SU(2)$. $SU(2)$ has exactly one $n$-dimensional representation for each $n \in \mathbb{N}$ ([16] Theorem 5.39). We have the group isomorphisms ([31] Theorem 29.36 and §29.49)

$$SO(3) \cong \frac{SU(2)}{I, -I} \quad \text{and} \quad O(3) \cong \{-1, 1\} \times SO(3)$$

elucidating that none of these groups are encapsulated by the above technique.

Thus it is not known whether $L^2(G)$ is approximately amenable for such groups $G$. This fact is highly awkward, as three of the four most natural examples of compact non-abelian groups are not appropriately characterized in this sense. Based on highly circumstantial evidence, it would be reasonable to conjecture that $L^2(G)$ fails to be approximately amenable for all infinite compact groups $G$.

Given that we have an appropriate Fourier transform, the first technique would be to try and determine that $\mathcal{B}_2(\Sigma)$ fails approximate amenability for any infinite collection of representations $\Sigma$. It is worthwhile to note that this algebra certainly fails to be boundedly approximately amenable by ([5] Theorem 3.3). The subsequent approach is to try to emulate the technique of [11] with a view to obtaining similar results where each coordinate in the $\ell^p$-sum is matrix valued. However on attempting this, one realises that the commutativity of $\ell^p(\mathbb{N})$ is in fact fundamental to the techniques involved, and in fact use is made of the fact that different coordinates interact appropriately in tensor products, which of course is very difficult to describe when the matrices involved have different sizes; which they all would in say the example $SU(2)$. This interaction cannot be used between different coordinates if the matrix sizes are different, and when multiplying matrices together, one obtains cross terms (from off the diagonal), which is where commutativity is in fact vital to the techniques of [11]. Hence the approximate amenability of $L^2(G)$ for general compact groups $G$ remains unresolved.

$L^2(G)$ always (obviously) has an underlying Hilbert space structure, which we have exploited above. Clearly the other $L^p(G), p \neq 2$ spaces do not, so what can be done about these? In particular does the Fourier transform obey similar
properties on the other $L^p$ spaces? It is known that the spaces $L^p(G), p > 1$ are Banach algebras under convolution when $G$ is compact ([31] Theorem 28.46). We might seek to define a Fourier transform from these algebras. From [31] there is a natural operator norm $\| \cdot \|_{\psi(p)}$ on the representation Hilbert spaces $H_\sigma$ generalizing the trace class norm; and a generalization from $\mathfrak{S}_2$ to $\mathfrak{S}_p$ for $p \geq 1$:

$$\| E \|_p = \left( \sum_{\sigma \in \Sigma} d_\sigma \| E_i \|_{\psi(p)}^p \right)^{1/p}, \mathfrak{S}_p(\Sigma) = \{ E \in \mathfrak{S}(\Sigma) : \| E \|_p < \infty \}.$$ 

We have that

**Theorem 2.33.** ([31] Theorem 31.22) Let $G$ be a compact group and suppose $1 < p < 2$, $q$ conjugate to $p$. For every $f \in L^p(G)$, we have that $\hat{f} \in \mathfrak{S}_q(\Sigma)$ and $\| \hat{f} \|_q \leq \| f \|_p$.

However this Fourier transform is onto if and only if $G$ is finite, the trivial case. What about for $p > 2$? We use inverses to define the appropriate Fourier maps here, using the fact that $\mathfrak{S}_q(\Sigma) \subset \mathfrak{S}_2(\Sigma)$. Due to this inclusion the inverse Fourier transform $E \mapsto \hat{E}$ is already defined on $\mathfrak{S}_q(\Sigma)$. We have that

**Theorem 2.34.** ([31] Theorem 31.24) Let $G$ be a compact group with $p > 2$. For each $E \in \mathfrak{S}_q(\Sigma)$, $\hat{E} \in L^p(G)$ and $\| \hat{E} \|_p \leq \| E \|_q$.

However again this inverse Fourier transform is only onto when $G$ is finite, although in both cases the maps have dense range. Hence, this method is of little use to us.

Another possibility is the Segal algebras $S_p(G) = (L^1(G) \cap L^p(G), \| \cdot \|_1 + \| \cdot \|_p)$ for $1 < p < \infty$ and now non-compact $G$, however the Fourier techniques would not be applicable in the same way. It is still not known whether there exists a proper Segal algebra $S(G)$ of an amenable group $G$ for which $S(G)$ is approximately amenable ([21] §9). The failure of approximate amenability for some Segal algebras on compact abelian infinite groups has been directly established, for example the Feichtinger algebra ([5] Proposition 5.1 or [10] §4). Also, if one defines, for $1 \leq p < \infty$,

$$\hat{S}_p(G) = \{ f \in L^1(G) : \hat{f} \in L^p(\hat{G}) \},$$

then $\hat{S}_p(\mathbb{T})$ and $\hat{S}_p(\mathbb{R})$ fail approximate amenability ([10] §4).

Once again, understanding the general behaviour of relevant approximate identities given a notion of amenability is vital here, as no proper Segal algebra will have a bounded approximate identity ([4] Theorem 1.2).
Chapter 3

Hereditary properties

We have already mentioned the importance of hereditary properties when considering a notion of amenability. For non-approximate varieties, this problem has been thoroughly considered [50]. For approximate amenability, we have some very satisfactory, though partial, results [21].

3.1 Approximate weak amenability

Knowing that approximate weak amenability is genuinely a property independent of its formally stronger variants, it makes sense to examine its hereditary properties. Since approximate weak amenability is a very mild form of amenability, one may not expect these hereditary properties to be very strong. However, one obtains more than might be expected, due to the fact that a specific bimodule is being used. A very natural non-triviality condition on the multiplication in an algebra is that $A^2 = A$, that is, $A$ is essential. We use the argument of ([8] Proposition 1.3) to establish that approximately weakly amenable Banach algebras are essential.

Lemma 3.1. Suppose $A$ is approximately weakly amenable. Then $A$ is essential.

Proof. Suppose that $A^2$ is not dense in $A$. Then there exists $\lambda_0 \in A^*$ such that $\lambda_0|A^2 = 0$ and $\lambda_0 \neq 0$. Then $D(a) = \langle \lambda_0, a \rangle \lambda_0$ is a derivation $A \to A^*$ with $D(a)(a) \neq 0$ for some $a \in A \setminus A^2$. But for any inner derivation $I$,

$$I(a)(a) = \langle a \cdot a^* - a^* \cdot a, a \rangle = \langle a^*, a^2 - a^2 \rangle = 0.$$
However more is true. It is still unknown whether approximate amenability is inherited by direct sums. For approximate weak amenability, we can establish such heredity due to the fact that derivations are mapping into a well understood bimodule.

**Proposition 3.2.** Suppose that $A$ and $B$ are approximately weakly amenable Banach algebras. Then $A \oplus B$ is approximately weakly amenable.

**Proof.** Consider a derivation $D : A \oplus B \to (A \oplus B)^* = A^* \oplus B^*$. We have that for $a, b \in A$, $c, d \in B$

$$D[(a, c)(b, d)] = (a, c) \cdot D(b, d) + (a, c) \cdot (b, d).$$

We know that for $e \in A$, $f \in B$,

$$<(a, c) \cdot D(b, d), (e, f)> = <D(b, d), (e, f)(a, c)> = <D(b, d), (ea, fc)>,$$

and for the other bimodule action

$$<D(a, c) \cdot (b, d), (e, f)> = <D(a, c), (b, d)(e, f)> = <D(a, c), (be, df)>.$$

Define $D|_A(a) = D(a, 0)$ for $a \in A$. Setting $c = d = 0$ above, we see $D|_A$ maps products of elements of $A$ into $A^*$. Since $A$ is essential, every element may be approximated by sums of products and so $D|_A$ maps into $A^*$. A similar argument for $B$ by setting $a = b = 0$ establishes that $D|_B$ maps into $B^*$. Now $D|_A$ and $D|_B$ are derivations, and so there are nets $(a^*_\alpha) \subset A^*$, $(b^*_\beta) \subset B^*$ such that for all $a \in A$, $b \in B$,

$$D|_A(a) = \lim_\alpha (a \cdot a^*_\alpha - a^*_\alpha \cdot a), \quad D|_B(b) = \lim_\beta (b \cdot b^*_\beta - b^*_\beta \cdot b).$$

Thus

$$D(a, b) = D(a, 0) + D(0, b) = \lim_\alpha (a \cdot a^*_\alpha - a^*_\alpha \cdot a) + \lim_\beta (b \cdot b^*_\beta - b^*_\beta \cdot b).$$

and hence in the product net,

$$D(a, b) = \lim_{\alpha, \beta} [(a, b)(a^*_\alpha, b^*_\beta) - (a^*_\alpha, b^*_\beta)(a, b)].$$

As expected a converse result also holds.
Proposition 3.3. Suppose that $A \oplus B$ is approximately weakly amenable. Then $A$ and $B$ are approximately weakly amenable.

Proof. Consider a derivation $D : A \to A^*$. Define an extension $\widetilde{D} : A \oplus B \to (A \oplus B)^*$ by $\widetilde{D}(a, b) = (D(a), 0)$. We wish to check that $\widetilde{D}$ is a derivation. Let $(a, b), (c, d), (e, f) \in A \oplus B$. Then

$$\widetilde{D}[(a, b)(c, d)](e, f) = (D(ac, bd)(e, f) = (D(ac), 0)(e, f) = (D(a) \cdot c + a \cdot D(c), 0)(e, f) = < D(a), ce > + < D(c), ea > .$$

By definition, we have that

$$\widetilde{D}(a, b) \cdot (c, d) + (a, b) \cdot \widetilde{D}(c, d) = (D(a), 0) \cdot (c, d) + (a, b) \cdot (D(c), 0).$$

Applying this to an element $(e, f) \in A \oplus B,$

$$< (D(a), 0) \cdot (c, d), (e, f) > = < (D(a), 0), (ec, fd) > = < D(a), ce > ,$$

$$< (a, b) \cdot (D(c), 0), (e, f) > = < (D(c), 0), (ea, fb) > = < D(c), ea > .$$

Thus $\widetilde{D}$ is a derivation, and so by hypothesis there is a net $(a^*_\alpha)$ with

$$(D(a), 0) = \lim_\alpha [(a, 0) \cdot (a^*_\alpha, b^*_\alpha) - (a^*_\alpha, b^*_\alpha) \cdot (a, 0)] \quad (a \in A).$$

So $D(a) = \lim_\alpha (a \cdot a^*_\alpha - a^*_\alpha \cdot a), (a \in A)$ and $A$ is approximately weakly amenable. Similarly for $B$. \qed

We now turn to when approximate weak amenability is inherited from second duals. There are a few partial results of this type for weak amenability, which carry over directly. The main process involved is to observe that the second adjoint of a derivation is again a derivation under certain conditions. We denote by $\Box$ the left, or first, Arens product.

Proposition 3.4. [7] Suppose that $B(A, A^*) = W(A, A^*)$ and $(A^{**}, \Box)$ is approximately weakly amenable. Then $A$ is approximately weakly amenable.
Proof. Consider a derivation $D : A \to A^*$. Then by ([7] Proposition 2.8.59(iii)) $D^{**} : A^{**} \to A^{***}$ is also a derivation. Hence there is a net $(a_{\alpha}^{***}) \subset A^{***}$ for which

$$D^{**}(a^{**}) = \lim_{\alpha}(a^{**} \cdot a_{\alpha}^{***} - a_{\alpha}^{***} \cdot a^{**}) \quad (a^{**} \in A^{**}).$$

Taking the restrictions $a_{\alpha}^* = a_{\alpha}^{***}|_{A^*}$, we have that $D(a) = \lim_{\alpha}(a \cdot a_{\alpha}^* - a_{\alpha}^* \cdot a)$ for each $a \in A$.

Proposition 3.5. Suppose that $A^{**}$ is approximately weakly amenable and $J(A) \subset A^{**}$ is a left ideal, where $J$ is the standard inclusion map. Then $A$ is approximately weakly amenable.

Proof. Taking a derivation $D : A \to A^*$, $D^{**}$ is again a derivation under the given hypothesis ([22] Theorem 2.3). □

There is one more well understood situation.

Lemma 3.6. [19] Suppose that $A^{**}$ is approximately weakly amenable. Then $A$ is essential.

Proof. We already know that $A^{**}$ is essential. From this, ([19] Proposition 2.1), using a standard Goldstine–Mazur argument yields that $A$ is essential. □

Proposition 3.7. Suppose that $A$ is a dual Banach algebra. If $A^{**}$ is approximately weakly amenable, then so is $A$.

Proof. Take a derivation $D : A \to A^*$. Write $A = B^*$ and $J : B \to B^{**}$ as the standard inclusion. Then under the given hypothesis ([19] Theorem 2.2) establishes that $\tilde{D} = J^{**} \circ D \circ J^* : A^{**} \to A^{***}$ is a derivation. Writing out that $\tilde{D}$ is approximately inner and using the same bimodule manipulations as in the proof of ([19] Theorem 2.2) yields the result. □

It should be noted that weak amenability was originally defined for the purposes of studying commutative Banach algebras. Specifically in the commutative case, approximate weak amenability is the same as weak amenability, and so one has the standard slightly stronger results for commutative weakly amenable Banach algebras which are contained in [7].

It is also of interest to know if approximate weak amenability is inherited by unitizations. The same argument as in ([8] Proposition 1.4) shows that

Proposition 3.8. Suppose $A$ is approximately weakly amenable. Then so is $A^\sharp$. 


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Despite this, when considering unitizations, weak amenability turns out to behave poorly.

Definition 3.9. For a locally compact group $G$ with left Haar measure $h$, the augmentation ideal of $L^1(G)$ is

$$I_0(G) = \{ f \in L^1(G) : \int_G f \, dh = 0 \}.$$

In a preprint [34] Johnson and White have shown that the augmentation ideal of $SL(2, \mathbb{R})$ is not weakly amenable but its unitization is weakly amenable. Ideal amenability [27], which is stronger than weak amenability, turns out to be more appropriate, and was perhaps defined to counteract this situation.

Definition 3.10. ([27] Definition 1.1) A Banach algebra $A$ is ideally amenable if $H^1(A, I^*) = 0$ for every closed ideal $I \subset A$.

Proposition 1.14 of [27] claims that $A$ is ideally amenable if and only if $A^2$ is ideally amenable. Unfortunately, the proof of the forward implication given in [27] is extremely doubtful, as it assumes that an ideal in $A^2$ is contained in $A$, which is generally not the case. The claim is in fact true, and will follow from our next result in the more general, approximate case.

Definition 3.11. A Banach algebra $A$ is approximately ideally amenable if for each closed ideal $I \subset A$, and every derivation $D : A \to I^*$, there exists a net $(x^*_\alpha) \subset I^*$ such that $D$ is of the form

$$D(a) = \lim_{\alpha} (a \cdot x^*_\alpha - x^*_\alpha \cdot a), \quad (a \in A).$$

Suppose that $A$ is approximately ideally amenable, and that $D : A^2 \rightarrow (I_z)^*$ is a derivation into the dual of an ideal $I_z$ of $A^2$.

Define $\tilde{D} : A \rightarrow (A \cap I_z)^*$ by $\tilde{D}(a) = D(a,0)|_{A \cap I_z}$, which is a derivation. By hypothesis, we have that $\tilde{D}$ is approximated by a net $(ad_{\xi_\alpha})$ where $(\xi_\alpha) \subset (A \cap I_z)^*$. Extending the functionals to $(\eta_\alpha) \subset I_z^*$ using the Hahn-Banach theorem, one obtains that for $x \in A \cap I_z, a \in A$,

$$< a \cdot \xi_\alpha - \xi_\alpha \cdot a, x > = < a \cdot \eta_\alpha - \eta_\alpha \cdot a, x >.$$

Also, we examine the action of $D$ on products in $A$:

$$< D[(a, 0)(b, 0)], (i, \lambda_i) > = < D(a, 0) \cdot (b, 0) + (a, 0) \cdot D(b, 0), (i, \lambda_i) >$$

$$= < D(a, 0), (b, 0) \cdot (i, \lambda_i) > + < D(b, 0), (i, \lambda_i) \cdot (a, 0) >$$

$$= < D(a, 0), (bi + \lambda_i b, 0) > + < D(b, 0), (ia + \lambda_i a, 0) >.$$
Given $x \in I_2$ and $a \in A$, we have that $ax, xa \in A \cap I_2$. Using the above on $D - \text{ad}_{\eta_n}$, it follows that for $(i, \lambda, a) \in I_2$, $a, b \in A$,

\[
| \langle (D - \text{ad}_{\eta_n})(ab, 0)(i, \lambda) \rangle | \leq | \langle (D - \text{ad}_{\eta_n})(a, 0), (bi + \lambda_i b, 0) \rangle | + | \langle (D - \text{ad}_{\eta_n})(b, 0), (ia + \lambda_i a, 0) \rangle | \leq \langle (\tilde{D} - \text{ad}_{\xi_n})(a), (bi + \lambda_i b) \rangle + | \langle (\tilde{D} - \text{ad}_{\xi_n})(b), (ia + \lambda_i a) \rangle |
\]

Thus, $D - \text{ad}_{\eta_n} \to 0$ as elements of $I_2^*$, pointwise on $(A^2, 0)$.

Thus, one has the following.

**Proposition 3.12.** $A$ is ideally amenable if and only if $A^2$ is ideally amenable.

**Remark 3.13.** One must be careful when extending derivations $D : A \to I^*$. For example, ([29] Proposition 2.1) demonstrates that a derivation $D : A \to A^*$ can be extended to $\tilde{D} : A^2 \to (A^2)^*$ only under certain circumstances. In particular, if $A$ is not essential, the same construction as ([29] Example 2.5), taking $f \in A^* \setminus \{0\}$ with $f|_{A^2} = 0$ yields a derivation that cannot be extended in the desired way. This is however not a primary concern, since when $A$ is approximately weakly (ideally) amenable, then $A$ is automatically essential. It is worth noting that [30] and [34] give results on extending derivations on ideals of codimension one in Banach algebras with a bounded approximate identity, which of course is the situation when we are looking at the algebra $L^1(G)$ and its augmentation ideal $I_0(G)$ (of course, $I_0(G)$ itself has a bounded approximate identity if and only if $G$ is amenable).

Thus in the ideal amenability case, where there is a single element rather than a net, one obtains that $D$ is inner on $A^2$, and hence on $A$ by essentiality. So the result of [27] remains true. But in the approximate case, there is no clear reason one can extend to $A$, unless the net of operators $(\text{ad}_{\eta_n})$ is bounded, in which case one can use an estimate: given $a = \lim_i a_i$ where $(a_i) \in A^2$,

\[
\|(D - \text{ad}_{\eta_n})(a - a_i, 0)\| \leq (\|D\| + \sup_{\alpha} \|\text{ad}_{\eta_n}\|)(a - a_i, 0) \to 0.
\]

Thus in the approximate case, we obtain the following.

**Proposition 3.14.** $A$ is boundedly approximately ideally amenable if and only if $A^2$ is.
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Proof. We follow [27]. Suppose \( A^\sharp \) is boundedly approximately ideally amenable, and \( I \) is a closed ideal in \( A \), \( D : A \to I^* \) a derivation. \( I \) is also a closed ideal in \( A^\sharp \) and so define \( \tilde{D}(a, \lambda) = D(a) \). We have that

\[
\tilde{D}(a, \lambda)(b, \gamma) = \tilde{D}(\gamma a + \lambda b + ab, \lambda \gamma) \\
= \gamma D(a) + \lambda D(b) + D(ab) \\
= (a, \lambda) \cdot D(b) + D(a) \cdot (b, \gamma) \\
= (a, \lambda) \cdot \tilde{D}(b, \gamma) + \tilde{D}(a, \lambda) \cdot (b, \gamma)
\]

so \( \tilde{D} \) is a derivation. Since \( \tilde{D} \) is boundedly approximately inner, \( D \) is also boundedly approximately inner.

Conversely, one uses the above argument. \( \square \)

Remark 3.15. Note that in the ‘if’ direction, no boundedness of the net \( (ad_{\eta_n}) \) was used, and hence one sees that if \( A \) is approximately ideally amenable, then so is \( A^\sharp \), in the same way as Proposition 3.8.

It would be interesting to know whether the augmentation ideal \( I_0(SL(2, \mathbb{R})) \) (Definition 3.9) fails to be approximately amenable; if so it would provide the relevant example in the approximate case. Unfortunately [34] uses cohomological arguments to establish its results, and does not reveal any information about the specific form of the non-inner derivations. However ([34] Theorem 5.2) does prove that there is effectively only one non-inner derivation into the dual module, that is,

\[
Z^1(I_0(SL(2, \mathbb{R})), I_0(SL(2, \mathbb{R}))^*)/N^1(I_0(SL(2, \mathbb{R})), I_0(SL(2, \mathbb{R}))^*)
\]

is one-dimensional. It turns out that this is enough information, as we will see below. We need a technical lemma; it states that the space of inner derivations cannot be complemented in the space of approximately inner derivations. On the surface this makes intuitive sense because one would expect that the inner derivations approximating the approximately inner derivation \( D \) would always somehow ‘contribute’ to the projection onto the inner derivations.

Lemma 3.16. Suppose that \( A \) is a Banach algebra, \( X \) a Banach \( A \)-bimodule, and there exists an approximately inner, non-inner derivation in \( Z^1(A, X) \). Suppose that the space \( N^1(A, X) \) of inner derivations is norm-closed in \( Z^1(A, X) \). Then \( N^1(A, X) \) is not complemented in \( Z^1(A, X) \).
Proof. Suppose that $N^1(A, X)$ is complemented in $Z^1(A, X)$, that is, there exists a continuous projection $P$ of $Z^1(A, X)$ onto $N^1(A, X)$. If we could show that for each non-zero approximately inner derivation $D : A \to X$, that the derivation $P(D)$ were non-zero, this would suffice, because subsequently, for every approximately inner non-inner derivation $D$, $P(D) \neq 0$, and so we would have $D - P(D)$ being approximately inner and non-inner, but $P(D - P(D)) \neq 0$, giving a contradiction.

So let $D : A \to X$ be an approximately inner, non-inner derivation. Then there is $(x_\alpha) \subset X$ such that for $a \in A$,

$$D(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) = \lim_{\alpha} \text{ad}_{x_\alpha}(a).$$

To obtain the desired contradiction, assume that $P(D) = 0$. Write $d_\alpha = D - \text{ad}_{x_\alpha}$; $d_\alpha$ is approximately inner. We have that $P(d_\alpha) = -P(\text{ad}_{x_\alpha}) = -\text{ad}_{x_\alpha}$.

Since $D \neq 0$ there must exist $a \in A$ with $D(a) \neq 0$. By the definition of approximately inner, for $\delta > 0$ we must be able to find an $\alpha$ such that

$$\|d_\alpha(a)\| < \delta.$$

Let us specifically choose $\delta = \|D(a)\|/(3\|P\|)$. Then via the projections,

$$\|\text{ad}_{x_\alpha}(a)\| = \|P(d_\alpha)(a)\| < \|D(a)\|/3.$$

But also via the triangle inequality,

$$\|\text{ad}_{x_\alpha}(a)\| = \|D(a) - d_\alpha(a)\| \geq \|D(a)\| - \|D(a)\|/3 = \frac{2\|D(a)\|}{3}$$

since $\|P\| \geq 1$, the desired contradiction. 

It is worth mentioning that one can indeed establish a stronger result (below) but for our purposes the above proof, while being technical, is illustrative in terms of the derivations involved.

**Proposition 3.17.** Let $Z$ be a norm-closed complemented subspace of a strong operator (so)-closed subspace $W$ of $B(X, Y)$. Then $Z$ is itself so-closed. So if $Z$ is proper and so-dense, it cannot be complemented.

Proof. Let $P : W \to Z$ be a continuous projection. Take $(T_\alpha) \subset Z$, $T_\alpha \stackrel{so}{\to} T$. So for any $x \in X$, $T_\alpha x \to Tx$. Then by continuity of $P$, $PT_\alpha x \to PTx$. Since $P$ is a projection onto $Z$, $PT_\alpha = T_\alpha$. It follows that $PTx = Tx$. Thus $PT = T$, so that $T \in Z$. 

This is exactly what we need for the example of Johnson and White.

**Corollary 3.18.** $I_0(SL(2, \mathbb{R}))$ is not approximately weakly amenable.

**Proof.** By ([34] Theorem 5.2), writing $A = I_0(SL(2, \mathbb{R}))$, we have that $H^1(A, A^*)$ is one-dimensional. In particular, $N^1(A, A^*)$ has finite codimension (namely one) in $Z^1(A, A^*)$. We also note that $N^1(A, A^*)$ is the range of the continuous map from $A^*$ into $Z^1(A, A^*)$ given by $x \mapsto \text{ad}_x$, and so $N^1(A, A^*)$ must be closed by ([57] Lemma 3.3). Thus by Lemma 3.16, there exists a derivation $D : A \to A^*$ which is not approximately inner. \qed

**Remark 3.19.** Proposition 3.16 tells us that in an approximately amenable non-amenable Banach algebra, such as $c_0(A_n)$ where the $A_n$ are each unital and amenable with amenability constants increasing to infinity, there are indeed infinitely many genuinely different derivations which are not inner, but may be approximated by inner ones in the strong topology. Less formally, when there are approximately inner, non-inner derivations into a bimodule, there are a lot of them.

### 3.2 0-weak amenability

We now proceed to a bimodule that is rarely if ever considered in a holistic sense, the reason being that it is not (necessarily) a dual bimodule. The structure of derivations $D : A \to A^{(n)}$, $n \in \mathbb{N}$ has been extensively studied, the dual structure of $A^{(n)}$ fundamental to this [8]. What about the bimodule $A$ itself?

There are many Banach algebras which are not amenable but on which derivations into the same space are well-behaved, in the sense of being inner. Based on this, we have the following:

**Definition 3.20.** A Banach algebra $A$ is 0-weakly amenable if $H^1(A, A) = 0$ when $A$ is considered as a bimodule under the canonical multiplication action.

An immediate question is why Banach algebras are an appropriate setting for considering such a condition. If one merely considers an associative algebra $A$, then taking the subalgebra generated an element $a \in A$ (that is, the polynomials in $a$) and the formal derivative operator on this subalgebra yields a non-inner derivation. So in this situation a condition such as that of Definition 3.20 clearly fails for many subalgebras. In the Banach case, however, one needs to consider closed subalgebras, and these formal derivatives are generally unbounded so that
they will not extend to the norm completion of the given subalgebras. In fact, this cohomology condition was studied for many years in $C^*$-algebra theory, until a satisfactory description of the $C^*$-algebras which satisfy it was obtained by [1] and [15].

There is a rich array of Banach algebras which are 0-weakly amenable, such as $C(K)$ for compact $K$ and $\ell^1(G)$ for discrete groups ([7], Corollary 2.7.16). Chernoff established the following; for a proof see ([7] Corollary 2.5.15).

**Theorem 3.21.** For any Banach space $X$, $B(X)$ is 0-weakly amenable.

However, $B(\ell^p)$ is known not to be amenable for $p = 1, 2, \infty$. Hence $B(\ell^1)$, $B(\ell^2)$, $B(\ell^\infty)$ are Banach algebras which are 0-weakly amenable but not amenable.

Perhaps the most revealing fact about 0-weak amenability is that any Banach function algebra is 0-weakly amenable ([7] Theorem 5.2.32). Many are known to be not weakly amenable, such as $A(\mathbb{D})$, and even admit non-trivial continuous point derivations. Hence it would appear that being 0-weakly amenable is not very restrictive at all. We list a few more examples of 0-weakly amenable Banach algebras, whose algebraic characteristics facilitate the structure of derivations.

**Theorem 3.22.** ([7] Theorem 5.2.48) If $A$ is commutative and semisimple, then $A$ is 0-weakly amenable, that is, any (continuous) derivation on $A$ is zero.

**Theorem 3.23.** (Sakai [51] and [52]) von Neumann algebras and simple unital $C^*$ algebras are 0-weakly amenable.

**Remark 3.24.** In the definition of 0-weak amenability, if one allowed any action of $A$ by $A$, then replicating ([50] Proposition 2.2.1) would yield the existence of an identity, and once again the definition would be too restrictive to include algebras such as the $\ell^p$ spaces.

Now when defining any such notion of amenability one hopes that there is enough inherent structure for a meaningful hereditary theory to exist. It is clear that there is no dual space structure within the definition here, so the hereditary theory may not be very strong. In fact, the 0-weak amenability of a Banach algebra $A$ does not necessitate that $A$ is essential.

**Example 3.25.** Given $n \geq 1$, consider the Banach algebra of $n$ times continuously differentiable functions $(C^{(n)}[0,1], \|f\|_n = \sum_{i=0}^n \frac{1}{n!}\|f^{(n)}\|_\infty)$, and take the closed subalgebra $A = \{ f \in C^{(n)}[0,1] : f(0) = 0 \}$. Taking $g \in A^2$ and using the product rule, one sees that $g'(0) = 0$. Thus $A^2$ is not dense in $A$. By Theorem 3.22, $A$ is 0-weakly amenable.
The situation here is not nearly as nice as the situation with amenability. In particular, 0-weak amenability is not inherited by closed ideals with a bounded approximate identity. The example is $K(\ell^2)$ as an ideal in $B(\ell^2)$ — we will see this later. However, 0-weak amenability extends to the unitization of an algebra.

**Proposition 3.26.** If $A^2$ is 0-weakly amenable, then $A$ is 0-weakly amenable. If $A$ is 0-weakly amenable, then $A^2$ is 0-weakly amenable.

**Proof.** Suppose $A^2$ is 0-weakly amenable. Given a derivation $D : A \to A$, write $\tilde{D}(a, \lambda) = (D(a), 0)$. This is a derivation $A^2 \to A^2$ and so must be inner. That is, for some $(b, \gamma) \in A^2$,

$$D(a, \lambda) = (a, \lambda)(b, \gamma) - (b, \gamma)(a, \lambda)$$
$$= (ab + \gamma a + \lambda b, \lambda \gamma) - (ba + \gamma a + \lambda b, \lambda \gamma)$$
$$= (ab - ba, 0).$$

This means $D(a) = ab - ba$, so $D$ is inner.

Conversely, suppose $A$ is 0-weakly amenable. Consider a derivation $D : A^2 \to A^2$. From ([7] Corollary 1.4.38), it is immediate that $A$ is a primitive ideal in $A^2$. Thus by the profound result of Sinclair ([7] Proposition 2.7.22 ii), we have that $D|_A \subset A$. Thus $D|_A(a) = ab - ba$ for some $b \in A$. Knowing that a derivation evaluated on the identity must be zero,

$$D(a, \lambda) = D(a, 0) + D(0, \lambda) = (D|_A(a), 0) = (ab - ba, 0).$$

Now

$$(a, \lambda)(b, 0) - (b, 0)(a, \lambda) = (ab + \lambda b, 0) - (ba + \lambda b, 0) = (ab - ba, 0).$$

Hence $D$ is inner.

**Remark 3.27.** In full generality, Sinclair’s result states that for any continuous derivation $D$ on a Banach algebra $A$, and any primitive ideal $P \subset A$, we have $D(P) \subset P$. In our situation, this means that an essentiality hypothesis, or some other constraint which would otherwise ensure that $D$ maps into $A$, is superfluous.

We will see later that 0-weak amenability is not inherited by closed ideals, but with direct sums, we have the following.

**Proposition 3.28.** Suppose that $A \oplus B$ is 0-weakly amenable. Then so are $A$ and $B$. Conversely, if $A$ and $B$ are 0-weakly amenable and essential, then $A \oplus B$ is 0-weakly amenable.
Proof. Suppose $A \oplus B$ is $0$-weakly amenable. That is, given a derivation $\tilde{D}$ on $A \oplus B$ there is an element $(\alpha, \beta)$ with $\alpha \in A, \beta \in B$, such that

$$\tilde{D}(a, b) = (a, b)(\alpha, \beta) - (\alpha, \beta)(a, b) = (a\alpha - \alpha a, b\beta - \beta b).$$

Given a derivation $D : A \rightarrow A$, write $\tilde{D}(a, b) = (D(a), 0)$. This is a derivation on $A \oplus B$ and hence $\tilde{D}(a, b) = (a\alpha - \alpha a, 0)$ for some $\alpha \in A$. Thus $D(a) = a\alpha - \alpha a$ and $A$ is $0$-weakly amenable. Similarly for $B$.

Conversely, consider a derivation $D : A \oplus B \rightarrow A \oplus B$. By examining the bimodule actions on $D$, one sees that $D$ maps product elements in $A$ into $A$, and similarly for $B$. Since $A$ and $B$ are essential, it follows that $D|_{A}$ maps into $A$ and $D|_{B}$ maps into $B$. This of course means that given $a \in A$ and $b \in B$, there exist $\alpha \in A$ and $\beta \in B$ with $D|_{A}(a) = a\alpha - \alpha a$ and $D|_{B}(b) = b\beta - \beta b$. Hence

$$D(a, b) = D(a, 0) + D(0, b) = (a\alpha - \alpha a, b\beta - \beta b) = (a, b)(\alpha, \beta) - (\alpha, \beta)(a, b)$$

and so $D$ is inner. \qed

We can somewhat relax the essentiality condition on multiplication. In particular, if $A$ and $B$ are integral domains, then $0$-weak amenability is inherited by $A \oplus B$.

**Proposition 3.29.** Suppose that $A$ and $B$ are Banach algebras, both of which are $0$-weakly amenable, and contain elements which are not (two-sided) divisors of zero. Then $A \oplus B$ is $0$-weakly amenable.

**Proof.** By examining restrictions, we obtain that a derivation $D : A \oplus B \rightarrow A \oplus B$ must have the general form

$$D(a, b) = (\delta_1(a) + \xi_{A-B}(b), \delta_2(b) + \xi_{B-A}(a)),$$

where $\delta_1 : A \rightarrow A$ and $\delta_2 : B \rightarrow B$ are derivations, and the maps $\xi_{A-B} : B \rightarrow A$ and $\xi_{B-A} : A \rightarrow B$ are linear and continuous.

Suppose in the first instance that $A$ and $B$ are commutative. Then by hypothesis, any derivation from $A$ into itself, or $B$ into itself, is trivial, and thus $\delta_1$ and $\delta_2$ are zero. Given this form (3.1), it is immediate that $D(A^2, B^2) = 0$, as

$$D(a_1a_2, 0) = D[(a_1, 0)(a_2, 0)] = (a_1, 0)(0, \xi_{B-A}(a_1)) + (0, \xi_{B-A}(a_1))(a_2, 0) = 0,$$

and similarly for products in $B$. By expanding out the expression $D[(a_1, b_1)(a_2, b_2)]$ according to the derivation rule, one sees that for $a_1, a_2 \in A$, $b_1, b_2 \in B$, we have

$$a_1\xi_{A-B}(b_2) + \xi_{A-B}(b_1)a_2 = 0; \quad b_1\xi_{B-A}(a_2) + \xi_{B-A}(a_1)b_2 = 0.$$
In this commutative setting, the order of multiplication (right or left) of these terms makes no difference, but will do so when we consider the general setting below. Taking \( a_1 = 0 \) and \( a_2 \) not a right zero divisor in \( A \), we can see that \( \xi_{A\rightarrow B}(b_1) = 0 \). A similar argument applies with taking \( a_2 = 0 \), in which case we need to use an element \( a_1 \) which is not a left divisor of zero. Thus \( \xi_{A\rightarrow B} = 0 \), and similarly \( \xi_{B\rightarrow A} = 0 \). Thus \( A \oplus B \) is 0-weakly amenable.

More generally, if \( A \) and \( B \) are not necessarily commutative, one still has equation (3.1), where \( \delta_1 : A \rightarrow A \) and \( \delta_2 : B \rightarrow B \) are inner derivations, and \( \xi_{A\rightarrow B} \) and \( \xi_{B\rightarrow A} \) are as before. One now uses the derivation \( \tilde{D}(a,b) = D(a,b) - (\delta_1(a), \delta_2(b)) \), which obeys equation (3.1), but with \( \delta_1 \) and \( \delta_2 \) being trivial. Repeating the above argument on \( \tilde{D} \) again yields that \( \xi_{A\rightarrow B} \) and \( \xi_{B\rightarrow A} \) are trivial, and hence \( D = (\delta_1, \delta_2) \) is inner.

**Remark 3.30.** This covers examples such as direct sums of the Banach algebras \( A \) as in Example 3.25 even though such algebras are not essential. Unfortunately however, when determining the 0-weak amenability of superalgebras generated from 0-weakly amenable Banach algebras, we in general had to invoke some non-triviality condition on the multiplication to ensure that the relevant derivation maps into the expected range. The question as to whether such an additional hypothesis is actually necessary appears somewhat intractable.

So far, we have not given an example of a Banach algebra which is not 0-weakly amenable. It is worth noting that ([7] Example 1.9.23) gives an example of a Banach subalgebra of \( M_4 \) which contains an identity but is not 0-weakly amenable. We proceed to give a few other examples, the first being where the multiplication is very particular.

**Example 3.31.** Let \( A \) be the Banach algebra formed by taking any Banach space with zero product. Then any operator \( T \in B(A) \) forms a derivation. It is clear that such derivations (aside from the zero derivation) are not inner.

**Example 3.32.** Let \( A = L^1(0,1) \) under convolution. It is standard that \( A \) is a radical commutative Banach algebra. Defining \( D(f)(t) = tf(t) \), one may expand the expressions on each side of the derivation identity in terms of convolution integrals to verify that \( D \) is a derivation. In fact, any derivation on \( A \) is of this form convoluted with an appropriate measure ([36] Theorem 2). As \( A \) is commutative, \( D \) is obviously non-inner.

We now proceed to a semisimple example. When using algebras of operators,
we will denote these in the standard way and write, say $T \in B(X)$, as opposed to an abstract element $a$ in a Banach algebra $A$.

**Example 3.33.** Consider the Banach algebra $K(\ell^2)$. Write $I_{\ell^2}$ as the identity in $\ell^2$. Take an operator $T \in B(\ell^2) \setminus (K(\ell^2) \oplus CI_{\ell^2})$. Define $D(A) = AT - TA$, $A \in K(\ell^2)$; since $K(\ell^2)$ is an ideal, $D$ is a derivation on $K(\ell^2)$. For $M_n$, the algebraic centre is $CI_n$, so an argument based on restricting to finite rank operators of increasing dimension (given a fixed basis one can write operators on $\ell^2$ as infinite matrices) shows that the centre of $K(\ell^2) \oplus CI_{\ell^2}$ is $CI_{\ell^2}$. Hence $K(\ell^2)$ is not 0-weakly amenable, as the difference of two implementing elements for a given derivation must lie in the centre, that is $CI_{\ell^2}$. This was originally expounded in [41].

Given the above, for a Hilbert space $H$ does $K(H)$ satisfy any reasonable weakening of 0-weak amenability? To answer this, we use the following, which is part of ([1] Lemma 1.1).

**Proposition 3.34.** Let $A$ be a $C^*$-algebra, and $A^{**}$ its enveloping von Neumann algebra. If $a^{**} \in A^{**}$ and $a^{**}$ derives $A$, that is, $a^{**}a - aa^{**} \in A$ for all $a \in A$, then there is a bounded net $(a_\alpha) \in A$ such that

$$
\lim_{\alpha} \| (a^{**} - a_\alpha)a - a(a^{**} - a_\alpha) \| = 0
$$

for all $a \in A$.

Chernoff showed that given any derivation $D$ on $K(\ell^2)$, there is $T \in B(\ell^2)$ such that $D(A) = AT - TA$, $A \in K(\ell^2)$, see eg. ([7] Theorem 2.5.14). This is also immediate given that $K(\ell^2)$ is known to be amenable. Since $K(\ell^2)$ is an ideal, $T$ derives $K(\ell^2)$ and it is immediate that $D$ is boundedly approximately inner. This fact can also be yielded from the amenability of $K(\ell^2)$ via Gourdeau’s equivalences ([28] Proposition 1). In the same way, any derivation on an amenable Banach algebra is boundedly approximately inner.

Also, $K(\ell^2)$ is Arens regular, so left and right multiplication are both $w^*$-continuous in $B(\ell^2)$. Take $(T_\alpha) \subset K(\ell^2)$ with $T_\alpha \xrightarrow{w^*} T$, where $T \in B(\ell^2)$ is as above. Then

$$
D(A) = A(w^* \cdot \lim_\alpha T_\alpha) - (w^* \cdot \lim_\alpha T_\alpha)A = w^* \cdot \lim_\alpha (AT_\alpha - T_\alpha A).
$$

Thus, $D$ is $w^*$-approximately inner. This latter argument applies to any $C^*$-algebra.
3.2. 0-WEAK AMENABILITY

So 0-weak amenability is not inherited by closed ideals in general. But what if a closed ideal has finite codimension?

**Example 3.35.** Consider $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$ under the standard operator norm.

By considering restrictions of the domain and range of derivations on $A$, it can be seen that the derivations $D_1 \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ and $D_2 \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ span $Z^1(A, A)$. These are inner, being generated by the elements

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

respectively. Thus, $Z^1(A, A) = N^1(A, A)$, and hence $A$ is 0-weakly amenable. However, the Wedderburn decomposition of $A$ is

$$A = \left\{ \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{bmatrix} \right\}$$

so that $A$ has an ideal $\left\{ \begin{bmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{bmatrix} \right\}$ which has zero product (which is thus not 0-weakly amenable by Example 3.31) and is of codimension 1.

**Remark 3.36.** Usually, when one considers the hereditary theory for a given notion of amenability, one considers more general structures, such as quotients of the Banach algebra involved. For example, amenability itself is inherited by quotients of closed ideals, as is approximate amenability. However, in the case of 0-weak amenability, the usual techniques do not indicate whether this inheritance holds. The difficulty is that the notion only considers one bimodule, namely the algebra itself, and thus lifting the derivation from a quotient to the original algebra becomes intractable.

However, there is another approach one may take. Instead of weakening the structure of the derivations, one may examine whether given derivations are ‘inner in a larger algebra.’ This means that instead of looking for the implementing element in $A$ itself, one looks for the implementing element in a larger algebra. Chernoff’s result on compact operators ([7] Theorem 2.5.14) is a very natural
example of this, where the superalgebra is the second dual. Two examples of Banach algebras which are not 0-weakly amenable (in themselves) are certain \(C^\ast\)-algebras as seen above and \(L^1(G), G\) a locally compact non-abelian group ([39] Corollary 1.2). It turns out that the second dual of \(A\) is not the best superalgebra to consider; the condition that any element in the second dual derives \(A\) is not generally satisfied, and so one cannot invoke a result like Proposition 3.34. This condition is obviously a weakening of \(A\) being an ideal in its second dual, which is true for \(L^1(G)\) if and only if \(G\) is compact (Watanabe [58] Proposition 4.2). However, the multiplier algebra \(\mathcal{M}(A)\) always contains \(A\) as an ideal in the standard way, and hence one is at least guaranteed to remain in \(A\) when implementing a derivation by some element of \(\mathcal{M}(A)\). Hence, we seek to know when every derivation on \(A\) is inner implemented by an element of \(\mathcal{M}(A)\). A very recent profound result is the following by Losert, which he obtains by determining the structure of crossed homomorphisms of \(G\).

**Theorem 3.37.** [39] For a locally compact group \(G\), any derivation \(D : L^1(G) \to M(G)\) is inner.

Wendel’s Theorem (see eg. [7] Theorem 3.3.40) states that \(M(G)\) is isomorphic to \(\mathcal{M}(L^1(G))\), and hence by Theorem 3.37, \(L^1(G)\) is inner in its multiplier algebra.

It is immediately apparent, even more so in the context of \(L^1(G)\), that the second dual is not appropriate as our larger algebra, since the problems stated before are not averted, as well as the fact that \(L^1(G)^{**}\) consists of the bounded additive functions on an extension of the \(\sigma\)-algebra of \(G\) (see [14] Theorem IV.8.16 for details), and we have the appropriate theorem for the much ‘nicer’ class of measures.

Returning to \(C^\ast\)-algebras, one would like the same to be true. We have the following equivalence, due to Elliot, Akemann and Pedersen; see [1] and [15] for details.

**Theorem 3.38.** Let \(A\) be a separable \(C^\ast\)-algebra. Then every (bounded) derivation on \(A\) is implemented by a multiplier if and only if \(A\) can be written as the finite direct sum of \(C^\ast\)-subalgebras each having a continuous trace, or being simple.

Without the separability hypothesis, the two conditions are in fact independent, see [15] for details.

So there exist \(C^\ast\)-algebras which are not inner in their multiplier algebra. Can this situation be improved on? For \(C^\ast\)-algebras is there an even larger algebra
3.2. 0-WEAK AMENABILITY

(which is smaller than the second dual in general) in which all the derivations are implemented? This was perhaps some of the motivation behind a series of papers studying the local multiplier algebra $M_{loc}(A)$. This work is expounded in a text on the subject [2] from which we give the following very brief description. The details and motivation for this work is also given in that reference.

Given a $C^*$-algebra $A$, when one defines the standard multiplier algebra, one considers double centralisers defined on the entirety of $A$.

**Definition 3.39.** Suppose that $A$ is a Banach algebra. Then a double centraliser for $A$ is a pair of maps $f : A \to A$ and $g : A \to A$ such that $f(ab) = f(a)b$ and $g(ab) = ag(b)$, and satisfying the centralising property $af(b) = g(a)b$ for all $a, b \in A$.

Considering double centralisers on a suitably restricted class of ideals is another option.

**Definition 3.40.** ([46] Definition 2.35) A non-zero closed ideal $I$ in a $C^*$-algebra $A$ is essential if $I$ has non-zero intersection with every other non-zero ideal in $A$.

$A$ is clearly always essential as an ideal in itself. Hence a $C^*$-algebra $A$ always contains essential ideals. Another simple example is when $A$ is a non-unital $C^*$-algebra, then $A$ is an essential ideal in $A^\Delta$.

Consider double centralisers defined only ‘locally’, that is, on some essential ideal; denote such an object as $(f, g, I)$. Two locally defined double centralisers $(f_1, g_1, I_1)$ and $(f_2, g_2, I_2)$ are said to be equivalent if $f_1$ and $f_2$ agree on the common domain $I_1 \cap I_2$ – automatically $g_1$ and $g_2$ also agree on $I_1 \cap I_2$. The fact that $I_1 \cap I_2$ is essential follows from the fact that it contains a bounded approximate identity ([7] Proposition 3.2.21 (ii)), and the alternative criterion for an ideal being essential ([46] Lemma 2.36). We follow the approach of [2].

**Definition 3.41.** Let $A$ be a $C^*$-algebra. The symmetric ring of quotients, denoted $Q_s(A)$, is the collection of equivalence classes of locally defined double centralisers on $A$.

The operations which turn $Q_s(A)$ into a ring are given in ([2] p50), along with the fact that $1 = (id_A, id_A, A)$ forms a multiplicative identity in $Q_s(A)$. We require a specific subset of $Q_s(A)$.

An element $x \in Q_s(A)$ is said to be bounded if there exists a finite subset $(y_i) \subset Q_s(A)$ and $n \in \mathbb{N}$ such that

$$\bar{x}x + \sum_i \bar{y}_i y_i = n1.$$
where \( \hat{x} \) denotes the involution of an element \( x \in Q_s(A) \). We may subsequently obtain the required normed space, by letting \( Q_b(A) \) denote the set of bounded elements of \( Q_s(A) \) under the norm

\[
\|x\|^2 = \inf\{\lambda \in \mathbb{R} : (y_i) \subset Q_s(A) \text{ is finite with } \hat{x}x + \sum_i y_iy_i = \lambda 1\}.
\]

The fact that this norm obeys all the required properties is ([2] Lemma 2.2.1). Taking the completion of \( Q_b(A) \) yields \( \mathcal{M}_{loc}(A) \) as a \( C^* \)-subalgebra of a quotient of \( A^{**} \). However, the way the \( Q_b(A) \) is chosen fundamentally depends on the existence of an involution, hence the method cannot be extended to Banach algebras in general. Clearly, in the case that \( A \) is simple, the only essential ideal is \( A \) itself, and this construction will yield an algebra which coincides with \( \mathcal{M}(A) \). However it will be much larger in general. As would be hoped, the local multiplier algebra of \( A \) is still a subalgebra of \( A^{**} \). Hence, the following is sharper than Chernoff’s result ([7] Theorem 2.5.14); note that it still requires separability.

**Theorem 3.42.** ([2] Theorem 4.2.20) Given a separable \( C^* \)-algebra \( A \), every derivation on \( A \) is implemented by some element in \( \mathcal{M}_{loc}(A) \).

### 3.3 Approximate biprojectivity and biflatness

We recall ([50] §4.3) that the notions of biprojectivity and biflatness for a Banach algebra \( A \) relate to the existence of one-sided inverse \( A \)-bimodule homomorphisms to the standard map \( \pi : A \hat{\otimes} A \mapsto A \) and its adjoint. By the nature of this, when we want to examine the heredity of these notions, two things are meant. Firstly, how do biprojectivity and biflatness relate back to amenability? Secondly we wish to know if they pass to related structures such as closed ideals; they are not necessarily inherited by other structures, for example unitizations ([50] §4.3). We begin with the first of these by extending results to the approximate case using well established methods.

#### 3.3.1 Relationship to amenability

**Definition 3.43.** ([61] Definition 2) A Banach algebra \( A \) is said to be *approximately biprojective* if there is a net \( (\rho_\alpha) \) of continuous \( A \)-bimodule homomorphisms from \( A \) into \( A \hat{\otimes} A \) such that \( \pi \circ \rho_\alpha(a) \to a \) for \( a \in A \).
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That is to say, there is a net of $A$-bimodule maps which are right approximate inverses to the product map. The situation when $A$ is unital is not very interesting.

**Proposition 3.44.** Suppose that $A$ is approximately biprojective and unital. Then $A$ is contractible.

*Proof.* Let $(\rho_\alpha)$ be the maps as above, and let $e$ be the unit. Then defining $m_\alpha = \rho_\alpha(e)$, we obtain for each $a \in A$,

$$a \cdot m_\alpha = a \cdot \rho_\alpha(e) = \rho_\alpha(ae) = \rho_\alpha(a),$$

and similarly

$$m_\alpha \cdot a = \rho_\alpha(e) \cdot a = \rho_\alpha(ea) = \rho_\alpha(a).$$

So $a \cdot m_\alpha - m_\alpha \cdot a = 0$. By definition $\pi(m_\alpha) \to e$ so that $(m_\alpha)$ is a (not necessarily bounded) central approximate diagonal for $A$. By ([24] Theorem 2.4) this means that, as $A$ is unital, it is contractible.

Hence, we need to examine the situation where $A$ is not unital.

**Proposition 3.45.** Suppose that $A$ is approximately biprojective with a central approximate identity $(e_\beta)$. Then $A$ is pseudo-contractible.

*Proof.* Denote the approximate right inverse maps given in the hypothesis by $(\rho_\alpha)$. Take a finite subset $F = \{a_1, \ldots, a_n\} \subset A$ and let $\varepsilon > 0$. Set $M = \max_i \|a_i\| + 1$. Now choose $\beta$ (depending on $F$, $\varepsilon$) such that $e_\beta$ satisfies $\|e_\beta a_j - a_j\| < \varepsilon$ for $j = 1, \ldots, n$. For this $\beta$ choose $\alpha$ such that $\|\pi \rho_\alpha(e_\beta) - e_\beta\| < \frac{\varepsilon}{M}$. Then for $j = 1, \ldots, n$,

$$\|\pi \rho_\alpha(e_\beta) a_j - a_j\| \leq \|e_\beta a_j - a_j\| + \|\pi \rho_\alpha(e_\beta) - e_\beta\| \|a_j\| \leq 2\varepsilon.$$

Setting $m_{F,\varepsilon} = \rho_\alpha(e_\beta)$ with the ordering $(F_1, \varepsilon_1) \prec (F_2, \varepsilon_2)$ if $F_1 \subset F_2$, $\varepsilon_1 > \varepsilon_2$ yields a net $(m_{(F,\varepsilon)})$ satisfying $\pi m_{(F,\varepsilon)} a \to a$ for $a \in A$. We also have for each $a \in A$,

$$a \cdot \rho_\alpha(e_\beta) - \rho_\alpha(e_\beta) \cdot a = \rho_\alpha(e_\beta a) - \rho_\alpha(ae_\beta) = 0$$

by centrality of $(e_\beta)$, so that $A$ is pseudo-contractible.

**Corollary 3.46.** Suppose that $A$ is approximately biprojective with a bounded central approximate identity $(e_\beta)$. Then $A$ is approximately amenable.
Proof. $A$ is (in particular) pseudo-amenable by Proposition 3.45. Given that $(e_\beta)$ is bounded, we may now invoke ([24] Proposition 3.2) to see that $A$ is approximately amenable.

Note that if we did not require $(e_\beta)$ to be bounded we would not obtain approximate amenability; for example $\ell^2(\mathbb{N})$ is approximately biprojective by ([61] example on pp3239-3240) but is not approximately amenable. This also leads to the seemingly difficult question: need we have the approximate identity central in general? This is extremely awkward in light of (the non-approximate situation) ([50] Exercise 4.3.15). This awkwardness is exhibited by the fact that the hypothesis yields pseudo-contractibility, which is much stronger than pseudo-amenability. The difficulty is that one cannot readily determine the behaviour of terms like $\rho_\alpha(\beta e_\beta - e_\beta a)$ unless $e_\beta$ is central.

This relates to another question: does bounded approximate biprojectivity (in that $\sup ||\rho_a||$ finite) imply biprojectivity? We cannot see an answer to this.

To relax the hypothesis that the approximate identity be central, we must impose boundedness on the approximate inverses.

**Corollary 3.47.** Suppose that $A$ is boundedly approximately biprojective with a bounded approximate identity. Then $A$ is approximately amenable.

Proof. Taking finite subsets $F = \{a_1, \ldots, a_n\} \subset A$, $\epsilon > 0$ and $M$ as in Proposition 3.45, the first part of the calculation is the same as in Proposition 3.46. In this case, we have for each $a \in A$,

\[
\begin{align*}
    a \cdot \rho_\alpha(e_\beta) - \rho_\alpha(e_\beta) \cdot a &= \rho_\alpha(e_\beta a) - \rho_\alpha(a e_\beta) \\
    &= \rho_\alpha(e_\beta a - a e_\beta) \\
    &\leq \sup_\alpha ||\rho_\alpha|| ||e_\beta a - a e_\beta|| \\
    &\to 0
\end{align*}
\]

as desired. The result now follows as before. \qed

The tradeoff here is that we impose uniform boundedness on the approximate inverses, but no longer require the approximate identity to be central.

We now define the corresponding notion for biflatness:

**Definition 3.48.** A Banach algebra $A$ is approximately biflat if there exist bimodule homomorphisms $\ast \rho_\alpha : (A \otimes A)^* \to A^*$ such that $\ast \rho_\alpha \circ \pi^*(a^*) \to a^*$ for $a^* \in A^*$.
We do not denote these as \( \rho_{\alpha}^* \), which would give the impression that they are necessarily adjoints of some other maps, which may not be the case.

**Remark 3.49.** This definition of approximate biflatness differs from that in the recent preprint ([53] Definition 2.3) where they make the observation that using the \( w^* \)-topology is the natural thing to do in the context of Segal algebras.

An example of a Banach algebra which is approximately biflat but not biflat is \( \ell^2(S) \) where \( S \) is an infinite set. As already stated, [61] shows that this is approximately biprojective, but not biprojective. Since \( \ell^2(S) \) is a Hilbert space, it is equal to its dual algebra and hence biflatness would imply biprojectivity. We also see how approximate biflatness, and approximate biprojectivity do not imply approximate amenability. But we can weaken the requirement of approximate biprojectivity used previously, and still obtain approximate amenability.

**Proposition 3.50.** Suppose that \( A \) is approximately biflat and has a bounded central approximate identity \((e_\beta)\). Then \( A \) is approximately amenable.

**Proof.** The proof proceeds via the analytical method of Curtis and Loy [6]. Take a \( w^* \)-convergent subnet of \((e_\beta \otimes e_\beta)\) in \((A \hat{\otimes} A)^{**}\), and denote this new net by \((e_\beta \otimes e_\beta)\). Set \( a_0 = w^* \lim \beta (e_\beta \otimes e_\beta) \) in \((A \hat{\otimes} A)^{**}\), and set \( m_\alpha = (\rho_\alpha)^* \pi^{**}(a_0) \).

For \( a \in A, f \in (A \hat{\otimes} A) \), and each \( \alpha \),

\[
\langle a \cdot m_\alpha, f \rangle = \langle a \cdot (\rho_\alpha)^* \pi^{**} a_0, f \rangle = \langle a_0, \pi^{**} (\rho_\alpha(f \cdot a)) \rangle = \lim_\beta \langle e_\beta \otimes e_\beta, \pi^{**} (\rho_\alpha(f \cdot a)) \rangle = \lim_\beta \langle a e_\beta^2, \rho_\alpha f \rangle = \langle a, \rho_\alpha f \rangle
\]

and similarly

\[
\langle m_\alpha \cdot a, f \rangle = \langle (\rho_\alpha)^* \pi^{**} a_0, a \cdot f \rangle = \langle a_0, \pi^{**} (\rho_\alpha(a \cdot f)) \rangle = \lim_\beta \langle e_\beta \otimes e_\beta, \pi^{**} (\rho_\alpha(a \cdot f)) \rangle = \lim_\beta \langle e_\beta^2 a, \rho_\alpha f \rangle = \langle a, \rho_\alpha f \rangle
\]

so that \((m_\alpha)\) is central. Also \( \pi^{**} m_\alpha(a) = \pi^{**} [\rho_\alpha^* \pi^{**} a_0 a] \to \pi^{**} (a_0 a) = a \). Now the net \((m_\alpha)\) is contained in \((A \hat{\otimes} A)^{**}\), so one can follow the exact technique of
([23] Proposition 2.1) to establish existence of the required net in $A \hat{\otimes} A$. Hence $A$ is pseudo-amenable, and so $A$ is approximately amenable since it has a bounded approximate identity.

Now we do not know whether $A$ being approximately amenable necessitates that it has a bounded approximate identity. But in a special case we recover more than approximate biflatness:

**Proposition 3.51.** ([61] observation after Definition 2) Suppose that $A$ is pseudo-contractible. Then $A$ is approximately biprojective.

Now going back to Proposition 3.50, suppose we had an identity in $A$ as opposed to merely a bounded approximate identity. Then taking the single element net $e \otimes e$ one might think that actually $A$ becomes pseudo-contractible, as this element is now contained in $A \hat{\otimes} A$. This does not work however, as if one merely took the single element net $\pi^*(e \otimes e)$, this would still not necessarily be contained in $A \hat{\otimes} A$ unless of course $A$ was approximately biprojective; and using $e \otimes e$ will not yield the required diagonal. In particular, just consider an amenable Banach algebra which has an identity, such as $\ell^\infty(N)$.

However we still have an equivalence of the standard and approximate notions of biprojectivity under the hypothesis of an identity, which leads to the question: is there an approximately biflat Banach algebra which is neither approximately biprojective nor biflat?

### 3.3.2 Inheritance

We now wish to consider when approximate biprojectivity and biflatness are inherited by various substructures. Selivanov ([54] Lemma 1.3 and [55] Proposition 5) determined using cohomological methods that if $A$ is biprojective or biflat with $I$ a closed ideal, then so is $A/ A \cdot I$. Moslehian and Niknam ([43] and [42]) subsequently obtained slightly less general results using elementary analysis, which lends itself to dealing with approximate notions. Thus, following this work we wish to extend the heredity of biprojectivity and biflatness to the approximate case. We will require a technical result of Lykova.

**Lemma 3.52.** ([40] Lemmas 3.4 and 3.6) Let

$$0 \rightarrow B \overset{i}{\rightarrow} A \overset{j}{\rightarrow} D \rightarrow 0$$

be a short exact sequence of Banach algebras. Suppose that $B$ has a one-sided bounded approximate identity. Then $i \hat{\otimes} i$ is injective and has closed range.
Proposition 3.53. Suppose that $A$ is an approximately biprojective Banach algebra, $I$ a closed ideal with a one-sided bounded approximate identity. Then $I$ is approximately biprojective.

Proof. One easily verifies that the following diagram commutes, where $\iota : I \to A$ is the inclusion map:

\[
\begin{array}{ccc}
I \otimes I & \xrightarrow{\pi^{(I)}} & I \\
\downarrow \iota \otimes \iota & & \downarrow \iota \\
A \otimes A & \xrightarrow{\pi} & A
\end{array}
\]

We know that $\pi$ has approximate right inverse $A$-bimodule maps $\rho_{\alpha} : A \to A \otimes A$. Then $\rho_{\alpha}^{(I)} = \rho_{\alpha} \circ \iota : I \to A \otimes A$ are obviously $I$-bimodule homomorphisms.

If $I^3$ denotes $\text{span}\{abc : a, b, c \in I\}^-$, then $I^3 = I$ because $I$ has a bounded one-sided approximate identity, and

\[
\rho_{\alpha}^{(I)}(I) = \rho_{\alpha}^{(I)}(I^3) \subseteq \text{span}\{a \cdot \rho_{\alpha}(b) \cdot c : a, b, c \in I\}^-
\]

\[
\subseteq \text{span}\{a \cdot u \cdot c : a, c \in I, u \in A \otimes A\}^- \subseteq I \otimes I.
\]

The last inclusion here holds since Lemma 3.52 shows that $\iota \otimes \iota$ is injective with closed range.

Hence for $x \in I$, writing $\rho_{\alpha}^{(I)}(x) = \sum_{i=1}^{\infty} x_i^{(\alpha)} \otimes y_i^{(\alpha)} \in I \otimes I$,

\[
(\pi^{(I)} \circ \rho_{\alpha}^{(I)})(x) = \sum_{i=1}^{\infty} x_i^{(\alpha)} y_i^{(\alpha)} = \pi(\rho_{\alpha}(x)) \to x.
\]

So $\rho_{\alpha}^{(I)}$ are approximating right inverse $I$-bimodule inverse maps for $\pi^{(I)}$. \qed

Proposition 3.54. Suppose that $A$ is approximately biprojective and has a bounded one-sided approximate identity. If $I$ is a closed ideal of $A$, then $A/I$ is approximately biprojective.

Proof. Consider the following commutative diagram, where $Q$ is the quotient map $Q : A \to A/I$:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{Q \otimes Q} & A/I \otimes A/I \\
\downarrow \pi & & \downarrow \pi^{(Q)} \\
A & \xrightarrow{Q} & A/I
\end{array}
\]

Regarding $A/I \otimes A/I$ as an $A$-bimodule, $Q \otimes Q$ is an $A$-bimodule homomorphism, since for $a_1, a_2 \in A, u \in A \otimes A$,

\[
(Q \otimes Q)(a_1 \cdot u \cdot a_2) = Q(a_1) \cdot (Q \otimes Q)(u) \cdot Q(a_2) = a_1 \cdot (Q \otimes Q)(u) \cdot a_2.
\]
If \((e_\beta)\) is a bounded right approximate identity for \(A\) (the left case is analogous), then for each approximate right inverse map \(\rho_\alpha\) to \(\pi\), and each \(a \in A\),

\[
\|Q \otimes Q(\rho_\alpha(a))\| = \lim_{\beta} \|Q \otimes Q(\rho_\alpha(\alpha e_\beta))\| = \lim_{\beta} \|Q(a)(Q \otimes Q(\rho_\alpha e_\beta))\| \leq \|Q\| \|\|\rho_\alpha\| \sup_{\beta} \|e_\beta\| \|Q(a)\|.
\]

Thus the maps \(\rho_\alpha^{(Q)} : A/I \to A/I \hat{\otimes} A/I\) are well-defined by \(\rho_\alpha^{(Q)}(a + I) = (Q \otimes Q)(\rho_\alpha(a))\). They are \(A/I\)-bimodule maps, since for \(a_1, a_2, a \in A\),

\[
\rho_\alpha^{(Q)}((a_1 + I)(a + I)(a_2 + I)) = \rho_\alpha^{(Q)}(a_1 a a_2 + I) = (Q \otimes Q)(a_1 \cdot \rho_\alpha(a) \cdot a_2)
= a_1 \cdot (Q \otimes Q)(\rho_\alpha(a)) \cdot a_2 = (a_1 + I) \cdot \rho_\alpha^{(Q)}(a) \cdot (a_2 + I).
\]

We also check that, with \(\rho_\alpha(a) = \sum_{i=1}^{\infty} x_i^{(\alpha)} \otimes y_i^{(\alpha)}\),

\[
(\pi^{(Q)} \circ \rho_\alpha^{(Q)})(a + I) = \pi^{(Q)}(\sum_{i=1}^{\infty} Q(x_i^{(\alpha)}) \otimes Q(y_i^{(\alpha)}))
= \sum_{i=1}^{\infty} Q(x_i^{(\alpha)}) Q(y_i^{(\alpha)}) = Q(\sum_{i=1}^{\infty} x_i^{(\alpha)} y_i^{(\alpha)})
= Q(\pi(\rho_\alpha(a))) \to a + I.
\]

Hence the \(\rho_\alpha^{(Q)}\) are approximate right inverse \(A/I\)-bimodule maps, as required. \(\square\)

**Proposition 3.55.** Suppose that \(A\) is approximately biflat and that \(I\) is a closed ideal with a bounded one-sided approximate identity. Then \(A/I\) is approximately biflat.

**Proof.** Let \((^*\rho_\alpha)\) be approximating left inverses to \(\pi^*\), and \((e_\beta)\) a bounded (left) approximate identity for \(I\). Regarding \(A/I\) as an \(A\)-bimodule, \((Q \otimes Q)^* : (A/I \hat{\otimes} A/I)^* \to (A \hat{\otimes} A)^*\) is an \(A\)-bimodule homomorphism, and the dual of the above diagram commutes:

\[
\begin{array}{ccc}
(A/I)^* & \xrightarrow{\pi^{(Q)^*}} & (A/I \hat{\otimes} A/I)^* \\
Q^* \downarrow & & \downarrow (Q \otimes Q)^* \\
A^* & \xrightarrow{\pi^*} & (A \hat{\otimes} A)^*
\end{array}
\]

For \(g \in (A/I \hat{\otimes} A/I)^*\), write

\[
\eta_\alpha(g) = {^*\rho_\alpha}((Q \otimes Q)^*(g)).
\]
Then $\eta_\alpha(g) \in A^*$, but for each $x \in I$, we have for each $\eta_\alpha$,

$$
< \eta_\alpha(g), x > = \lim_{\beta} < \eta_\alpha(g), e_\beta x > = \lim_{\beta} < x \cdot \eta_\alpha(g), e_\beta >
$$

$$
= \lim_{\beta} < \rho_\alpha(x \cdot (Q \hat{\otimes} Q)^*(g)), e_\beta > = \lim_{\beta} < \rho_\alpha((Q \hat{\otimes} Q)^*(x \cdot g)), e_\beta >
$$

$$
= \lim_{\beta} < 0, e_\beta > = 0,
$$

since

$$(x \cdot g)((a + I) \otimes (b + I)) = g((a + I) \otimes (bx + I)) = g(0) = 0.$$  

Hence $^*\rho_\alpha^{(Q)}(g) : (a + I) \mapsto \eta_\alpha(g)(a)$ are well-defined maps into $(A/I)^*$. To see that the $^*\rho_\alpha^{(Q)}$ are $A/I$-bimodule maps, we have for $a, b \in A, g \in ((A/I) \hat{\otimes} (A/I))^*$,

$$^*\rho_\alpha^{(Q)}((a + I) \cdot g)(b + I) = ^*\rho_\alpha^{(Q)}(a \cdot g)(b + I) = \eta_\alpha(a \cdot g)(b) = (a \cdot \eta_\alpha(g))(b) = \eta_\alpha(g)(ba)$$

$$= ^*\rho_\alpha^{(Q)}(g)((b + I)(a + I)) = ((a + I) \cdot ^*\rho_\alpha^{(Q)}(g))(b + I)$$

and similarly with action on the right, because both $^*\rho_\alpha$ and $(Q \hat{\otimes} Q)^*$ are $A$-bimodule maps. Also for $\phi \in (A/I)^*$,

$$< (^*\rho_\alpha^{(Q)} \circ \pi^{(Q)*})(\phi), a + I > = < \eta_\alpha(\pi^{(Q)*}(\phi)), a >$$

$$= < ^*\rho_\alpha((Q \hat{\otimes} Q)^*(\pi^{(Q)*}(\phi))), a >$$

$$= < ^*\rho_\alpha((\pi^* \circ Q^*)(\phi)), a >$$

$$\rightarrow < Q^*(\phi), a > = < \phi, a + I >,$$

so the $^*\rho_\alpha^{(Q)}$ are approximate left inverses as required.

We move on to looking at the heredity of these concepts under the second adjoint. Before doing this though, we state a technical lemma.

**Lemma 3.56.** ([43] Lemma 2.1) *Let $A$ be a Banach algebra. Then there exists an $A$-bimodule homomorphism $\gamma : (A \hat{\otimes} A)^* \mapsto (A^{**} \hat{\otimes} A^{**})^*$ such that for any functional $f \in (A \hat{\otimes} A)^*$, elements $a_1^{**}, a_2^{**} \in A^{**}$ and nets $(a_\lambda), (b_\lambda) \subset A$ with $w^*-\lim_\lambda a_\lambda = a_1^{**}, w^*-\lim_\lambda b_\lambda = a_2^{**}$ we have

$$\gamma(f)(a_1^{**} \otimes a_2^{**}) = \lim_\lambda \lim_\chi f(a_\lambda \otimes b_\chi).$$

Now we can relate these $w^*$- limits back to our original Banach algebra $A$.

**Proposition 3.57.** *Suppose $A$ is a Banach algebra such that $A^{**}$ is approximately biflat. Then $A$ is approximately biflat.*
Proof. We know that $\pi^{**}$ has approximate left inverse bimodule maps $({}^*\rho^{(**)}_\alpha)$. Let $J : A \to A^{**}$ and $J_* : A^* \to A^{***}$ denote the natural inclusions, $\pi, \pi^{**}\pi$ the product maps on $A$ and $A^{**}$; with $f$ as in Lemma 3.56. In accordance with Moslehian and Niknam ([43] Theorem 2.2), the following diagram commutes:

$$
\begin{array}{ccc}
A^* & \xrightarrow{\pi^*} & (A\otimes A)^* \\
\downarrow J_* & & \downarrow \gamma \\
A^{***} & \xrightarrow{\pi} & (A^{**}\otimes A^{**})^*
\end{array}
$$

We elucidate this as done in the mentioned paper: for each $a^* \in A^*$, elements $a_1^{**}, a_2^{**} \in A^{**}$ and nets $(a_\lambda), (b_\chi) \subset A$ with $w^*\lim_\lambda a_\lambda = a_1^{**}$, $w^*\lim_\chi b_\chi = a_2^{**}$, we have

$$
(\gamma(\pi^*(a^*))(a_1^{**} \otimes a_2^{**})) = \lim_{\lambda} \lim_{\chi} \pi^*(a^*)(a_\lambda \otimes b_\chi) = \lim_{\lambda} \lim_{\chi} a^*(a_\lambda b_\chi) = J_*(a^*)(a_1^{**} a_2^{**}) = J_*(a^*)(**\pi(a_1^{**} \otimes a_2^{**})) = (**\pi^*(J_*(a^*))) (a_1^{**} \otimes a_2^{**}).
$$

Now write $*\rho_\alpha = J^* \circ {}^*\rho^{(**)}_\alpha \circ \gamma$. Then for each $a^* \in A^*$,

$$
(*\rho_\alpha \circ \pi^*)(a^*) = (J^* \circ {}^*\rho^{(**)}_\alpha \circ \gamma \circ \pi^*)(a^*) = (J^* \circ {}^*\rho^{(**)}_\alpha \circ **\pi^* \circ J_*)(a^*)
$$

so that $*\rho_\alpha$ are the required maps. Note they are automatically $A$-bimodule homomorphisms as the maps in the composition are all $A$-bimodule homomorphisms.

For the case of approximate biprojectivity, we have the following partial result, which is an easy implication of Proposition 3.53.

**Corollary 3.58.** Suppose that $A^{**}$ is approximately biprojective, that $A$ is an ideal in $A^{**}$, and that $A$ has a one-sided bounded approximate identity. Then $A$ is approximately biprojective.

This is not very interesting however, in that we need to assume that $A$ is an ideal in $A^{**}$. Approximate biflatness lends itself better to this type of result as there is already a given dual space structure involved. Also if $A$ has a two-sided bounded approximate identity and $A^{**}$ is approximately biprojective, it is in fact contractible (as it is unital), again not very interesting. On the whole, the fact that one-sided bounded approximate identities are involved in the hypothesis
of many of these results is somewhat undesirable, as there are many (approximately) biprojective Banach algebras without bounded approximate identities, for example $\ell^1(\mathbb{N})$.

**Example 3.59.** ([42] Example 2.5) We present an example of an ideal which is approximately biprojective, but whose unitization is not approximately biflat. Hence we see that for approximate biprojectivity and approximate biflatness, $I$ and $A/I$ possessing the property does not imply that $A$ possesses the property.

Consider $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$ under the standard operator norm, the same algebra as in Example 3.35. Writing

$$f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and defining

$$\rho \left( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = a(f \otimes f) + b(f \otimes g),$$

we see that $\pi \circ \rho = id|_A$. An elementary calculation also shows that this map is an $A$-bimodule homomorphism. In particular $A$ is biprojective and biflat. $\mathbb{C}$ also satisfies these properties as a Banach algebra. Now $A^2$ is not approximately biprojective, otherwise $A$ would be contractible, and $A$ is not as it has no identity. It is also immediate that $A$ has no right approximate identity, so $A$ is not approximately amenable ([21] Lemma 2.2). Hence neither is $A^2$ by ([21] Proposition 2.4); but $A^2$ has an identity so is not approximately biflat.

What is clear is that given a notion of amenability or a related condition, one can deduce a lot of information about the algebraic structure of the Banach algebra involved. One could say that a description of the approximate identities in a Banach algebra $A$ is an extremely important particular instance of this fact, and one we have been looking closely at.
CHAPTER 3. HEREDITARY PROPERTIES

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Chapter 4

Approximate identities in Orlicz type algebras

We would like to generalize the $L^p$ spaces in a natural way so as to obtain different spaces, hopefully with different properties to the $L^p$ spaces. Subsequently, we wish to examine when algebras are formed under various products, and observe when they have various approximate identities.

One way to do this is to observe that the norms on the $L^p$ spaces are equivalent to the respective norms on the Orlicz spaces obtained by using the convex functions $x \mapsto x^p$. Accordingly, we will consider more general Orlicz spaces.

4.1 Elementary facts about Orlicz spaces

Definition 4.1. A Young function $\varphi$ is a convex, even, nonzero function from $\mathbb{R}$ into $[0, \infty]$ satisfying $\varphi(0) = 0$.

Note that this definition of Young functions from [32] allows them to be infinite valued, and hence may be discontinuous at the point where they approach infinity. However, unless otherwise specified we will henceforth consider only real-valued Young functions. Hence $\varphi$ must necessarily be continuous, and approach infinity as $x$ approaches infinity.

Remark 4.2. Young functions have some very important properties, in particular we will repeatedly use the following fact: given a Young function $\varphi$, then for scalars $0 \leq b \leq 1 \leq c$, we have $\varphi(bx) \leq b\varphi(x)$ and $\varphi(cx) \geq c\varphi(x), x \in \mathbb{R}$.

Now we are in a position to define the appropriate classes of functions on a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. In the same way as with the $L^p$ spaces, one
considers functions that are equal off a null set to be equivalent.

**Definition 4.3.** The Orlicz space $L^\varphi$ generated by a Young function $\varphi$ comprises all the classes of complex-valued measurable functions on $\Omega$ for which there exists $\lambda > 0$ such that

$$\int_\Omega \varphi(\lambda |f|) \, d\mu < \infty.$$ 

There are two equivalent norms on this space which will be of interest to us, and we are at liberty to use whichever one can be better manipulated. The first of these is very direct.

**Definition 4.4.** Given a function $f \in L^\varphi$, the Luxemburg or gauge norm is defined by

$$\|f\|_{(\varphi)} = \inf \{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) \, d\mu < 1\}.$$ 

The fact that this norm turns $L^\varphi$ into a Banach space may be seen in ([48] III.2.3).

The second norm requires the notion of the complementary function of $\varphi$.

**Definition 4.5.** Given a Young function $\varphi$, the complementary function $\psi$ of $\varphi$ is given by

$$\psi(y) = \sup_{x \geq 0} \{x|y| - \varphi(x)\} \quad (y \in \mathbb{R}).$$

The importance of the complementary function $\psi$ is that it is mutually complementary with $\varphi$, in that repeating the process, the complementary function of $\psi$ is $\varphi$. Also, the pair satisfy Young's inequality:

$$xy \leq \varphi(x) + \psi(y), \quad (x, y \in \mathbb{R}).$$

**Remark 4.6.** Even if $\varphi$ is finite-valued it may happen that $\psi$ takes infinite values, however it will still be a Young function as in Definition 4.1.

This definition is very direct and simple, but does not describe the properties of the complementary function very well, nor is it practical for calculation. To avoid these difficulties we first need a simple observation.

**Lemma 4.7.** A Young function $\varphi$ satisfies a Lipschitz condition on any interval $[x, y]$, and in particular is absolutely continuous.
Proof. Taking \( y' < y < x < z \), convexity gives
\[
\varphi(x) \leq \frac{x - y'}{z - y'} \varphi(z) + \frac{z - x}{z - y'} \varphi(y') ; \quad \varphi(y) \leq \frac{y - y'}{x - y'} \varphi(x) + \frac{x - y}{x - y'} \varphi(y').
\]
These inequalities yield
\[
\frac{\varphi(y) - \varphi(y')}{y - y'} \leq \frac{\varphi(x) - \varphi(y)}{x - y} \leq \frac{\varphi(z) - \varphi(x)}{z - x}.
\]
Thus, choosing \( y' < y_0 < y \) and \( x < z_0 < z \), one obtains
\[
\frac{\varphi(y_0) - \varphi(y')}{y_0 - y'} \leq \frac{\varphi(x) - \varphi(y)}{x - y} \leq \frac{\varphi(z) - \varphi(z_0)}{z - z_0}.
\]
(4.1)
We immediately deduce that
\[
|\varphi(y) - \varphi(x)| \leq K|y - x|,
\]
where \( K \) is the maximum of the absolute value of the two outer terms of equation (4.1).

Subsequently one may employ Lebesgue-Vitali arguments to deduce that the derivative \( \varphi' \) is well-defined off a countable subset of the real line.

[48] §1.3 now shows that the complementary function may be obtained via a more practical process: by Lemma 4.7, given a Young function \( \varphi \), we may express it as an integral
\[
\varphi(x) = \int_0^x \varphi'(t)dt, \quad (x \in [0, \infty)),
\]
where the derivative \( \varphi' \) exists at all but at most countably many points. However the left and right derivatives individually exist at every point by monotonicity, and henceforth we use the right derivative \( \varphi'_+ \) (of course it would not matter which we used, as they are equal off a null set).

For \( u \in [0, \infty) \), let \( \psi_{di,f}(u) = \inf\{t : \varphi'(t) > u\} \), with \( \inf\emptyset = \infty \); \( \psi_{di,f} \) is well-defined everywhere. Then by ([48] I.3.2) we have
\[
\varphi(y) = \int_0^y \psi_{di,f}(u)du, \quad (y \in [0, \infty)).
\]
Subsequently, one extends \( \psi \) to be an even function on \( \mathbb{R} \). This alternative definition of the complementary function elucidates how the pair \( \varphi \) and \( \psi \) are mutually complementary.
Definition 4.8. For a Young function \( \varphi \) with complementary function \( \psi \) and Orlicz space \( L^\varphi \), we define the Orlicz norm as
\[
\|f\|_\varphi = \sup\left\{ \int_{\Omega} |fg|d\mu : \int_{\Omega} \psi(|g|)d\mu \leq 1 \right\}, \quad (f \in L^\varphi).
\]
Then by ([48] III.3.4),
\[
\|f\|_\varphi \leq \inf \left\{ \|f\|_\varphi : f \in L^\varphi \right\} \leq 2\|f\|_\varphi,
\]
showing the required equivalence.

4.2 Pointwise Multiplication

Now consider what happens when we take an Orlicz space \( L^\varphi \) under pointwise multiplication. Under our hypothesis that \( \varphi \) is finite-valued, we have the following due to Hudzik.

Theorem 4.9. ([32] Theorem B) \( L^\varphi \) is a Banach algebra if and only if the measure space \((\Omega, \mu)\) is a countable union of atoms \( \Omega = \bigcup_{n=1}^{\infty} \{a_n\} \) and \( \inf_n \mu(a_n) > 0 \).

Remark 4.10. We are using the modern definition of ‘Banach algebra’, relating to the continuity of multiplication. If one uses the alternative definition requiring submultiplicativity of the norm, this further requires \( \inf_n \mu(a_n) \geq [\varphi(1)]^{-1} \) ([32] Proposition 6). Of course the two are equivalent via a scaling of the norm, but Hudzik differentiates between these, and refers to the given algebras as ‘Orlicz algebras’ and ‘Banach-Orlicz algebras’ respectively. We will use the term ‘Orlicz algebra’ for any Orlicz space which forms a Banach algebra under pointwise multiplication.

We would like to determine the algebraic properties of these Orlicz algebras, in particular the amenability-type properties. For example, take the algebras \( \ell^p = \ell^p(\mathbb{N}) \), \( 1 \leq p < \infty \), \( \mu \) counting measure, which are Orlicz algebras under \( \varphi(x) = |x|^p \); these algebras are not amenable, nor approximately amenable.

These \( \ell^p \) spaces are the most obvious examples of Orlicz algebras, and are known not to factor. For example,
\[
(1, \frac{1}{2^{1/p}}, \frac{1}{3^{1/p}}, \ldots, \frac{1}{n^{1/p}}, \ldots) \notin (\ell^p)^2 = \ell^{p/2},
\]
but is contained in \( \ell^p \). Do other Orlicz sequence spaces also fail to factor?

To address this and other questions about the algebraic behaviour of Orlicz algebras, we must discuss a triviality condition we have so far not considered.
4.2. POINTWISE MULTIPLICATION

**Definition 4.11.** Given a Young function \( \varphi \), write

\[
u_0 = \sup\{x \geq 0 : \varphi(x) = 0\}.
\]

If \( u_0 > 0 \), then \( L^\varphi = L^\infty \) for purely atomic spaces where the measures of each atom are bounded away from zero ([32] Theorem A) and is hence amenable. Most of the time we will suppose that \( u_0 = 0 \), otherwise one obtains an \( L^\infty \)-type space, which [38] refers to as the degenerate case. We will soon see that the converse is true, in that where the measure of each atom is equal, \( L^\varphi = L^\infty \) if and only if \( u_0 > 0 \). Thus given a Young function \( \varphi \) for which \( u_0 = 0 \), \( \varphi \) will be referred to as non-degenerate.

We want to specifically generalize the \( \ell^p \) spaces to other Orlicz sequence spaces. The following definition singles out those sequence spaces for which we have counting measure. However, sometimes one may have more generality than this, in which case something can often still be said about the structure of the given algebra (see Proposition 4.21 below).

**Definition 4.12.** An Orlicz sequence algebra \( \ell^\varphi = \ell^\varphi(N) \) is an Orlicz space defined on the natural numbers under counting measure, with pointwise product.

We wish to determine the algebraic properties of Orlicz sequence algebras \( \ell^\varphi \), noting immediately that they are Banach algebras by Theorem 4.9. We collect some elementary but useful facts about these Banach algebras. But first, we must note a technicality relating to the fact that two different Young functions may yield the same Orlicz space.

**Theorem 4.13.** ([38] Proposition 4.a.5) Let \( \varphi \) and \( \phi \) be two Young functions. Then the following are equivalent:

- \( \ell^\varphi = \ell^\phi \) (as sets), and the identity mapping is a Banach algebra isomorphism;

- \( \varphi \) and \( \phi \) are equivalent at zero, in the sense that there exist constants \( k > 0 \), \( K \geq 1 \) and \( t_0 > 0 \) such that for all \( 0 \leq t \leq t_0 \), we have
  \[
  K^{-1} \phi(k^{-1}t) \leq \varphi(t) \leq K \phi(kt).
  \]

This gives us a very direct way to check the equality of two given Orlicz sequence spaces.
Lemma 4.14. Given a Young function $\varphi$, $\ell^\varphi$ is always a solid vector lattice; that is, given $a, b \in \ell^\varphi$, one has that $|a|$, $a \wedge b$, $a \vee b$ all lie in $\ell^\varphi$, as does any sequence $c$ with $|c| \leq |a|$.

$\ell^\varphi$ is a Banach sequence algebra, and one has that $\ell^\varphi$ is an ideal in $\ell^\infty$, with $\|ab\|_{(\varphi)} \leq \|a\|_{(\varphi)}\|b\|_\infty$ for $a \in \ell^\varphi$, $b \in \ell^\infty$.

Proof. The fact that $\ell^\varphi$ is a solid vector lattice is immediate. Obviously any sequence of finite support is contained in $\ell^\varphi$, and hence $\ell^\varphi$ is a Banach sequence algebra. The ideal property is also immediate, but was first noted in ([32] Remark 4).

Lemma 4.15. Let $\varphi$ be a non-degenerate Young function. Then $\ell^\varphi \subset c_0$.

Proof. Take a sequence $a = (a_n) \notin c_0$. Then for some $\rho > 0$, there is a subsequence $(n_k)$ for which $|a_{n_k}| > \rho$, all $k$. Calculating for the norm $\| \cdot \|_{(\varphi)}$,

$$\sum_{n=1}^\infty \varphi(|a_n|/\lambda) \geq \sum_{n_k} \varphi(\rho/\lambda) = \infty$$

for all $\lambda > 0$. Hence $a \notin \ell^\varphi$.

We also need to make an observation regarding the behaviour of factorization in these sequence spaces. It will be slightly more than what we require.

Lemma 4.16. Suppose that $a \in \ell^\varphi$ and that $\sqrt{a} \notin \ell^\varphi$. Then $a$ does not weakly factor in $\ell^\varphi$; in particular $a$ does not factor.

Proof. Suppose that $a \in \ell^\varphi$ factors weakly, so there is $m \in \mathbb{N}$ and $b^{(i)}, c^{(i)} \in \ell^\varphi$ ($1 \leq i \leq m$) such that

$$a = \sum_{i=1}^m b^{(i)}c^{(i)}.$$

Then for each $n \in \mathbb{N}$, there is some $i, 1 \leq i \leq m$ such that at least one of $|b^{(i)}_n| \geq \sqrt{|a^{(i)}_n|}/m$ or $|c^{(i)}_n| \geq \sqrt{|a^{(i)}_n|}/m$ must hold. Hence for $n \in \mathbb{N}$,

$$\sqrt{|a_n|} \leq \sqrt{m} \sum_{i=1}^m (|b^{(i)}_n| + |c^{(i)}_n|),$$

which implies that $\sqrt{a} \in \ell^\varphi$.

We briefly state some more algebraic properties of $\ell^\varphi$ without labelling them, because we will have no subsequent use for them. In the non-degenerate case,
observing that \( c_0 \subset \ell^\varphi \subset c_0 \), we have that \( \ell^\varphi \) is a dense subalgebra of \( c_0 \). By ([7] Proposition 4.1.35) \( c_0 \) is natural on \( \mathbb{N} \) and so by ([7] Proposition 4.1.7), \( \ell^\varphi \) is also natural on \( \mathbb{N} \). This is despite the fact that some Orlicz algebras contain Banach space isomorphic copies of \( \ell^\infty \) ([38] Proposition 4.a.4). For any Orlicz sequence algebra \( \ell^\varphi \), one may verify that the point evaluation maps are characters, and hence \( \ell^\varphi \) is always semisimple.

We now return to the factorization problem. Unfortunately, to obtain a satisfactory result, we need to restrict our Orlicz functions in another way so as to ensure they have sufficient decay near the origin.

**Definition 4.17.** A Young function \( \varphi \) is said to satisfy the \( \Delta' \)-condition universally \( [ \Delta'-condition at 0 ] \), denoted \( \Delta' [ \Delta'(0) ] \), if there exists \( c > 0 \) [and \( x_0 > 0 \)] such that

\[
\varphi(xy) \leq c\varphi(x)\varphi(y), \quad 0 \leq x, y < \infty \quad [0 \leq x, y \leq x_0].
\]

Note that this is a different scenario to some other growth conditions considered in the theory of Orlicz spaces, as \( \Delta' \) enforces decay of the Young function near the origin, and does not just control growth at infinity.

**Proposition 4.18.** Suppose that a non-degenerate Young function \( \varphi \) satisfies the \( \Delta' \)-condition universally. Then \( \ell^\varphi \) does not factor weakly, so \((\ell^\varphi)^2 \neq \ell^\varphi\). In particular \( \ell^\varphi \) fails to possess a bounded approximate identity.

**Proof.** Because \( u_0 = 0 \), \( \varphi \) has a well-defined inverse on the positive half-line. Consider the sequence

\[
(\varphi^{-1}(c), \varphi^{-1}(c/2^2), \varphi^{-1}(c/3^2), \ldots) \in \ell^\varphi,
\]

where \( c \) is the constant from the \( \Delta' \)-condition. Directly from the \( \Delta' \)-condition, for \( w, z \) positive we have that

\[
wz \leq \varphi^{-1}(c\varphi(w)\varphi(z)),
\]

and hence for \( x, y \) positive, one has

\[
\varphi^{-1}(x)\varphi^{-1}(y) \leq \varphi^{-1}(cxy).
\]

In particular, for \( n \in \mathbb{N} \),

\[
\varphi^{-1}\left(\frac{c}{n^2}\right) \geq \left[\varphi^{-1}\left(\frac{1}{n}\right)\right]^2.
\]
So \( a = ([\varphi^{-1}(1)]^2, [\varphi^{-1}(\frac{1}{2})]^2, [\varphi^{-1}(\frac{1}{3})]^2 \ldots) \in \ell^\varphi \) via comparison. We check that \( \sqrt{a} \notin \ell^\varphi \). For any \( \lambda > 0 \) and \( n \in \mathbb{N} \),

\[
\frac{1}{n} = \varphi\left(\frac{1}{n}\right) = \varphi\left(\frac{1}{\lambda} \varphi^{-1}\left(\frac{1}{n}\right)\right) \leq c\varphi\left(\frac{1}{\lambda}\right) \varphi\left(\lambda \varphi^{-1}\left(\frac{1}{n}\right)\right),
\]

whence

\[
\sum_{n=1}^{\infty} \varphi(\lambda \sqrt{|a_n|}) \geq \frac{1}{c\varphi(1/\lambda)} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.
\]

By Lemma 4.16 this means that \( a \) does not factor. \( \square \)

The only part of the proof in which one needs \( \Delta' \) to hold universally is the very final inequality, where \( 1/\lambda \) could be very large. The rest of the argument can be run assuming only \( \Delta'(0) \). However the \( \Delta' \)-condition is a highly undesirable hypothesis, in that it determines the growth of \( \varphi \) outside a small interval of zero, in fact on all of \([0, \infty)\); whereas behaviour outside a small neighbourhood of 0 has no effect on the space in question. Hence if one took a Young function \( \varphi \) defined on a small neighbourhood of zero obeying \( \Delta'(0) \), and could show how to extend it in such a way as to obey \( \Delta' \) universally, then via the closed graph theorem the two norms on \( \ell^\varphi \) (one from \( \varphi \) itself, one from the modification) would necessarily be equivalent, and hence the original algebra \( \ell^\varphi \) would not factor.

**Example 4.19.** Let \( \varphi(x) = \exp(-1/|x|) \) on \((0, 1/2] \), and \( \varphi(0) = 0 \). By examining the first and second derivatives, one sees that \( \varphi \) is convex on \([0, 1/2]\); extend \( \varphi \) to be a Young function on \( \mathbb{R} \). Using \( x_0 = 1/2 \), we check that \( \varphi \in \Delta'(0) \). For \( 0 < x, y \leq 1/2 \) we estimate

\[
\exp(-1/|x|) \exp(-1/|y|) = \exp(-1/x - 1/y) = \exp\left(-\frac{x - y}{xy}\right) \geq \exp\left(-\frac{1}{|xy|}\right).
\]

We establish that \( \ell^\varphi \) does not factor. Set \( a = (a_n) \), where \( a_n = \frac{1}{2 \ln(n)} \), \( n \geq 2 \), and \( a_1 = 1 \). Then \( a \in \ell^\varphi \) because each \( a_n, n \geq 2 \), is chosen so that its corresponding summand in \( \| \cdot \|_{(\varphi)} \) is \( 1/n^2 \). To see that \( \sqrt{a} \notin \ell^\varphi \), let \( \lambda > 0 \) and consider the sum

\[
\sum_{n} \exp\left(-\frac{\sqrt{2 \ln(n)}}{\lambda}\right).
\]

By the standard comparison to an integral this is finite if and only if

\[
\int_{1}^{\infty} \exp\left(-\frac{\sqrt{2 \ln(x)}}{\lambda}\right) dx
\]
is finite. However, doing the change of variables $u = \ln(x)$ one obtains the integral

$$\int_0^\infty \exp(u - \frac{\sqrt{2}}{\lambda} \sqrt{u})du \geq \int_{S/\lambda^2}^\infty \exp(u - \frac{\sqrt{2}}{\lambda} \sqrt{u})du \geq \int_{S/\lambda^2}^\infty \exp(u/2)du,$$

which is infinite. Hence by Lemma 4.16, $\ell^\varphi$ does not factor.

To try and extend $\varphi$ to satisfy $\Delta'$ (we denote the desired extension by $\phi$), one needs in particular that there is $c > 0$ such that for any fixed $1 > \lambda > 0$,

$$\phi(1/\lambda) \geq \frac{\varphi(a)}{c\varphi(\lambda a)}$$

for all $a \in [0, 1/2]$. However for a fixed $0 < \lambda < 1$,

$$\lim_{a \to 0^+} \frac{\exp(-1/a)}{\exp(-1/(\lambda a))} = \lim_{b \to +\infty} \exp(b/\lambda - b) = \infty.$$

To circumvent the overly strong yet useful universal $\Delta'$-hypothesis, let us attempt to directly establish the lack of a bounded approximate identity in more general atomic Orlicz algebras; that is, where the measure of each atom $a_n$ varies. Importantly no $\Delta$-type condition on $\varphi$ is required. We will make use of a technical lemma:

**Lemma 4.20.** ([48] Propositions III.3.3 and III.3.4) For a non-zero $f \in L^\varphi$, we have that

$$\int_\Omega \varphi\left(\frac{|f|}{\|f\|_{(\varphi)}}\right)d\mu \leq 1.$$

**Proposition 4.21.** Suppose that $\Omega = \bigcup_{n=1}^\infty \{a_n\}$ and that $b = \inf \mu(a_n) > 0$. Let $\varphi$ be a non-degenerate Young function. Then $L^\varphi(\mu)$ fails to possess a bounded approximate identity.

**Proof.** For convenience we will use the Luxemburg norm $\|\cdot\|_{(\varphi)}$. Suppose that there exists an approximate identity $(e_\alpha) \in L^\varphi$ bounded by $M \geq 1$, so that

$$\|fe_\alpha - f\|_{(\varphi)} \to 0, \quad (f \in L^\varphi).$$

Take $f \in L^\varphi$. We have

$$\int_\Omega \varphi(|fe_\alpha - f|)d\mu = \sum_{n=1}^\infty \varphi(|f(n)e_\alpha(n) - f(n)|)\mu(a_n)$$
for any $\alpha$. Fix an $0 < \varepsilon < 1$. Then by Remark 4.2,

$$\frac{1}{\varepsilon} \int \varphi(|fe_\alpha - f|)d\mu \leq \int \varphi\left(\frac{|fe_\alpha - f|}{\varepsilon}\right)d\mu. \quad (4.2)$$

But for $\alpha$ sufficiently large, the right hand side in equation (4.2) is bounded by 1, and so

$$\int \varphi(|fe_\alpha - f|)d\mu \leq \varepsilon.$$

It follows that

$$\sum_{n=1}^{\infty} \varphi(|f(n)e_\alpha(n) - f(n)|)\mu(a_n) \to 0.$$  

Now fix an $N \in \mathbb{N}$ such that $bN\varphi\left(\frac{1}{3M}\right) > 1$, and set $f = 1_{[1,N]}$. Certainly $f \in L^\varphi$, so there is $\beta$ such that

$$\sum_{n=1}^{N} \varphi(|e_\alpha(n) - 1|)\mu(a_n) < 1 \quad (4.3)$$

for $\alpha > \beta$. Define $(f_\alpha)_{\alpha > \beta}$ as follows. For $n \leq N$,

$$f_\alpha(n) = \begin{cases} 2 - e_\alpha(n) & |e_\alpha(n)| < 1 \\ e_\alpha(n) & |e_\alpha(n)| \geq 1, \end{cases}$$

and set $f_\alpha(n) = e_\alpha(n)$ for $n > N$. Then $f_\alpha \in L^\varphi$ and $|f_\alpha(n)| \geq 1$, $i \leq N$. Note that for $\alpha > \beta$, equation (4.3) gives

$$\|f_\alpha - e_\alpha\|_\varphi \leq 2 \sum_{n=1}^{N} \varphi(|e_\alpha(n) - 1|)\mu(a_n) < 2 \leq 2M,$$

and so

$$\|f_\alpha\|_\varphi \leq \|f_\alpha - e_\alpha\|_\varphi + \|e_\alpha\|_\varphi \leq 3M.$$

However, since $\varphi$ is increasing, we obtain with Lemma 4.20,

$$\int_{\Omega} \varphi\left(\frac{|f_\alpha|}{3M}\right)d\mu = \sum_{n=1}^{\infty} \varphi\left(\frac{|f_\alpha(n)|}{3M}\right)\mu(a_n)$$

$$\geq b \sum_{n=1}^{N} \varphi\left(\frac{|f_\alpha(n)|}{3M}\right)$$

$$\geq bN\varphi\left(\frac{1}{3M}\right) > 1,$$

contradiction.
Remark 4.22. Under the above regime, \((f_\alpha)_{\alpha > \beta}\) would have to be an approximate identity, since for \(g \in L^\varphi\), one would have

\[ \|f_\alpha g - g\|_{(\varphi)} \leq \|(f_\alpha - e_\alpha)g\|_{(\varphi)} + \|e_\alpha g - g\|_{(\varphi)}. \]

The second term has limit zero, and the first is bounded by

\[ \|g\|_\infty \|f_\alpha - e_\alpha\|_{(\varphi)} \leq 2\|g\|_\infty \sum_{n=1}^{N} \varphi(|e_\alpha(n) - 1|) \rightarrow 0. \]

Thus \((f_\alpha)_{\alpha > \beta}\) would an approximate identity, bounded by \(3M\).

To help describe the algebraic properties of these Orlicz sequence algebras, we summarize their basic Banach space properties. [38] Propositions 4.a.2 and 4.a.4 have collected these. We will need to impose another (more restrictive) growth condition near the origin for our Young function.

Definition 4.23. A Young function \(\varphi\) satisfies \(\Delta_2(0)\) if there exists a constant \(C > 0\) and \(x_0 > 0\) such that

\[ \varphi(2x) \leq C\varphi(x), \quad (0 < x < x_0). \]

We will frequently consider a very important subspace of \(\ell^\varphi\).

Definition 4.24. Let \(\varphi\) be a Young function. Then set

\[ h^\varphi = \left\{ (a_n) \subset \ell^\varphi : \sum_{n=1}^{\infty} \varphi \left( \frac{|a_n|}{\rho} \right) < \infty \quad \text{for every} \quad \rho > 0 \right\}. \]

Proposition 4.25. ([38] Proposition 4.a.2) \(h^\varphi\) is a closed subspace of \(\ell^\varphi\), and the standard unit vectors form a basis for \(h^\varphi\).

Theorem 4.26. ([38] Proposition 4.a.4) Suppose that \(\varphi\) is non-degenerate. Then the following are equivalent:

- \(\varphi \in \Delta_2(0)\);
- \(h^\varphi = \ell^\varphi\);
- the unit vectors form a boundedly complete symmetric basis for \(\ell^\varphi\);
- \(\ell^\varphi\) is separable;
- \(\ell^\varphi\) contains no subspace isomorphic (as a Banach space) to \(\ell^\infty\).
Note that any Orlicz sequence space where \( \varphi \) is non-degenerate is contained densely as a vector space within \((c_0, \| \cdot \|_\infty)\) by Lemma 4.15. In fact the inclusion \( \iota : \ell^\varphi \hookrightarrow c_0 \) is a continuous monomorphism; it can be easily seen from the definition of \( \| \cdot \|_\varphi \) that \( \| \cdot \|_\varphi \geq \| \cdot \|_\infty / \varphi^{-1}(1) \).

We can now deduce that the ‘inclusion’ of \( \ell^\infty \) into \( \ell^\varphi \) when \( \varphi \) fails \( \Delta_2(0) \) cannot be a homomorphism. For supposing there existed a Banach algebra isomorphism \( j : \ell^\infty \to \ell^\varphi \), then the composition \( j \circ \iota : \ell^\infty \to c_0 \) would be a continuous monomorphism, which would have closed range by ([7] Theorem 4.2.4). This gives that \( c_0 \) has a closed subspace isomorphic to \( \ell^\infty \), which is clearly impossible as \( c_0 \) is separable, whereas \( \ell^\infty \) is not.

**Example 4.27.** Let \( \varphi(x) = \exp(-1/x) \) on \((0, 1/2] \) with \( \varphi(0) = 0 \). Since only the behaviour near zero is relevant to an Orlicz sequence space, extend the function \( \varphi \) so it is convex and even on \( \mathbb{R} \). One can routinely determine that

\[
\lim_{x \to 0} \frac{\varphi(2x)}{\varphi(x)} = \infty.
\]

Hence \( \varphi \notin \Delta_2(0) \), and \( \ell^\varphi \) is not separable, etcetera.

We can now modestly describe the amenability type behaviour of \( \ell^\varphi \).

**Proposition 4.28.** Suppose that \( \varphi \) is a non-degenerate Young function which satisfies \( \Delta_2(0) \). Then the sequence algebra \( \ell^\varphi \) is not boundedly approximately amenable.

**Proof.** We proceed via the technique of ([21] Example 6.3). Let \( A = \ell^\varphi(\mathbb{N}) \) and take the bimodule

\[
X = \ell^\varphi(\mathbb{N} \times \mathbb{N}) \oplus (1 \otimes \ell^\varphi(\mathbb{N})) \oplus (\ell^\varphi(\mathbb{N}) \otimes 1)
\]

where \( \ell^\varphi(\mathbb{N} \times \mathbb{N}) \) is defined as \( L^\varphi \) with \( \Omega = \mathbb{N} \times \mathbb{N} \) under counting measure; with operations given by, for \( f, g \in \ell^\varphi(\mathbb{N}), F \in \ell^\varphi(\mathbb{N} \times \mathbb{N}) \),

\[
(f \cdot F)(m, n) = f(m)F(m, n), \quad ((F \cdot f)(m, n) = F(m, n)f(n),
\]

\[
(f \cdot (1 \otimes g))(m, n) = f(m)g(n), \quad ((g \otimes 1) \cdot f)(m, n) = g(m)f(n),
\]

\[
(1 \otimes g) \cdot f = 1 \otimes gf, \quad f \cdot (g \otimes 1) = fg \otimes 1.
\]

Writing \( \psi \) as the complementary function for \( \varphi \), we have that \( (h^\psi)^* = \ell^\varphi \) ([38] Proposition 4.b.1), and hence \( X \) is the dual of the \( A \)-bimodule \( h^\psi(\mathbb{N} \times \mathbb{N}) \oplus (1 \otimes h^\psi(\mathbb{N})) \oplus (h^\psi(\mathbb{N}) \otimes 1) \) with opposite multiplications. It is easy to verify that these
operations are continuous into the relevant spaces, using the facts that $\ell^p \subset \ell^\infty$, and that the Orlicz norm is stronger than $\| \cdot \|_\infty$.

Take the derivation $D : A \to X$ given by $D(f) = f \otimes 1 - 1 \otimes f$. For $A$ to be approximately amenable there must exist a net $(\eta_\alpha) \subset X$, say $\eta_\alpha = F_\alpha + 1 \otimes f_\alpha + g_\alpha \otimes 1$, with $D(f) = \lim_\alpha (f \cdot \eta_\alpha - \eta_\alpha \cdot f)$, $f \in A$. This means that for each $f \in A$,

$$f \cdot F_\alpha - F_\alpha \cdot f - f \cdot (1 \otimes f_\alpha) + (g_\alpha \otimes 1) \cdot f \to 0$$

$$f f_\alpha \to f, f g_\alpha \to f.$$ 

In particular, $(g_\alpha)$ is an approximate identity for $A$.

Now suppose that $A$ is actually boundedly approximately amenable. Then there exists a net $(\eta_\alpha)$ satisfying the properties above, and such that there is a $K > 0$ for which

$$\|x \cdot F_\alpha - F_\alpha \cdot x - x \cdot (1 \otimes f_\alpha) + (g_\alpha \otimes 1) \cdot x\| \leq K\|x\|, \quad (x \in A, \text{all } \alpha).$$

That is, for any $x \in A$, any $\lambda \geq K\|x\|$ and any $\alpha$ we have

$$\sum_{i \neq j} \varphi(\lambda^{-1}|F_\alpha(i, j)(x_i - x_j) - x_i f_\alpha(j) + g_\alpha(i)x_j|) \leq 1.$$ 

Taking $x = \delta_k$ for some $k \in \mathbb{N}$, we have that $\|x\| = 1/\varphi^{-1}(1)$ and hence for $\lambda > \frac{K}{\varphi^{-1}(1)}$ this yields

$$\sum_{i \neq k} \varphi(\lambda^{-1}|F_\alpha(i, k) + g_\alpha(i)|) \leq 1,$$

which is

$$\sum_i \varphi(\lambda^{-1}|F_\alpha(i, k) + g_\alpha(i)|) \leq 1 + \varphi(\lambda^{-1}|F_\alpha(k, k) + g_\alpha(k)|).$$

Thus for each $N$ and all $k \in \mathbb{N}$,

$$\sum_{i \leq N} \varphi(\lambda^{-1}|F_\alpha(i, k) + g_\alpha(i)|) \leq 1 + \varphi(\lambda^{-1}|F_\alpha(k, k) + g_\alpha(k)|).$$

For each fixed $\alpha$, $g_\alpha \in c_0$ by Lemma 4.15, and similarly $F_\alpha \in c_0(\mathbb{N} \times \mathbb{N})$. Thus letting $k \to \infty$ yields that

$$\sum_{i \leq N} \varphi(\lambda^{-1}|g_\alpha(i)|) \leq 1.$$ 

This holds for each $N$ and $\alpha$, so that $(g_\alpha)$ is norm bounded by $\lambda$. However $(g_\alpha)$ is an approximate identity, so this is impossible by Proposition 4.21. \qed
There is a notion of amenability that is particularly well suited to examining various Banach sequence algebras. It requires any derivation to have a specific form at each point \( a \in A \).

**Definition 4.29.** A Banach algebra \( A \) is **pointwise approximately amenable** if for each Banach \( A \)-bimodule \( X \), every derivation \( D : A \to X^* \) is pointwise approximately inner, in that for each \( a \in A \), there exists a net \( (x_\alpha^*) \subset X^* \) such that

\[
D(a) = \lim_\alpha (a \cdot x_\alpha^* - x_\alpha^* \cdot a).
\]

Pointwise amenability, which requires implementing elements as opposed to nets for each element \( a \in A \), is not suitable for Orlicz sequence algebras, as pointwise amenability is known to impose the existence of a bounded approximate identity in commutative algebras [10].

With this new terminology, ([11] Proposition 3.6) implicitly established that for \( 1 < p < \infty \), \( \ell^p \) is pointwise approximately amenable. We can extend this to the Orlicz case.

**Proposition 4.30.** Suppose that \( \varphi \) is a non-degenerate Young function which satisfies \( \Delta_2(0) \). Then \( \ell^\varphi \) is pointwise approximately amenable.

**Proof.** Taking \( a \in \ell^\varphi \), define finite subsets of \( \mathbb{N} \) by \( B_n = \{ j \in \mathbb{N} : |a(j)| \geq 1/n \} \). Then set \( u_n = 1_{\{1,2,\ldots,n\}} \) and \( v_n = 1_{B_n} \). Then it is clear that \( \|a - u_n a\|_\varphi \to 0 \).

Also, we have that \( \|a - v_n a\|_\varphi \to 0 \), because the standard basis \( (\delta_n) \), being symmetric, is unconditional. We invoke ([38] Theorem 1.c.1), showing that sums off the increasing finite sets \( B_n \) converge to zero. The subtlety here is that whilst \( \cup_n B_n \) is not necessarily all of \( \mathbb{N} \), the terms off \( \cup_n B_n \) are zero, and do not contribute to the norm. It follows from ([11] Proposition 3.6) that \( \ell^\varphi \) is pointwise approximately amenable. \( \square \)

The existence of various approximate identities depends specifically on this \( \Delta_2(0) \) condition.

**Proposition 4.31.** Suppose that a non-degenerate Young function \( \varphi \) satisfies \( \Delta_2(0) \). Then \( \ell^\varphi \) has the standard (necessarily unbounded) approximate identity. If \( \varphi \) fails \( \Delta_2(0) \), then \( \ell^\varphi \) does not possess an approximate identity.

**Proof.** Suppose that \( \varphi \) satisfies \( \Delta_2(0) \), so \( h^\varphi = \ell^\varphi \), and take \( (a_n) \in \ell^\varphi \). Then for any \( \varepsilon > 0 \) there exists \( N \) such that \( m \geq N \) implies \( \| \sum_{n=m+1}^\infty a_n \delta_n \|_\varphi \leq \varepsilon \) since the vectors \( \{\delta_n\} \) form a basis. That is, writing \( e_N \) as the standard sequence with
1's in the first $N$ entries and 0's elsewhere, $\|ae_N - a\|_\varphi \leq \varepsilon$. Hence $(e_N)$ is an approximate identity for $\ell^\varphi$. We have that

$$\|e_N\|_\varphi = \inf_{\lambda > 0} \left\{ \sum_{n=1}^{N} \varphi(1/\lambda) \leq 1 \right\} = \inf_{\lambda > 0} \{N\varphi(1/\lambda) \leq 1\} = \inf_{\lambda > 0} \{(1/\lambda) \leq \varphi^{-1}(1/N)\} = \frac{1}{\varphi^{-1}(1/N)},$$

which of course is unbounded as $N \to \infty$.

Suppose on the other hand that $h^\varphi \neq \ell^\varphi$. Then by Theorem 4.26, $\ell^\varphi$ is not separable, and so the standard sequence $(\delta_n)$ must fail to be a basis. Thus there exists an element $a = (a_n)$ and $\rho > 0$ such that

$$\|a\|_\varphi \geq \|a - \sum_{n=1}^{N-1} a_n\delta_n\|_\varphi > \rho, \quad (N \in \mathbb{N}),$$

the first inequality being true as the first $N - 1$ terms of $a - \sum_{n=1}^{N-1} a_n\delta_n$ are zero. Take $f = (f_n) \in A$. The sequence of numbers $f_n \to 0$ necessarily since $\ell^\varphi \subseteq c_0$. So for $n > N$ say, $|f_n| < 1/2$. Fixing such an $N$,

$$\|af - a\|_\varphi \geq \left\| (af - a) - \sum_{n=1}^{N-1} a_n(f_n - 1)\delta_n \right\|_\varphi \geq \inf_{n \geq N} (1 - |f_n|) \left\|\sum_{n=1}^{N-1} a_n\frac{f_n - 1}{\inf_{n \geq N} (1 - |f_n|)}\delta_n\right\|_\varphi \geq \inf_{n \geq N} (1 - |f_n|) \left\| a - \sum_{n=1}^{N-1} a_n\delta_n \right\|_\varphi \geq \rho/2 > 0.$$ 

Thus $a$ has no approximate unit, and so $\ell^\varphi$ has no approximate identity. \hfill $\Box$

The following is immediate:

**Corollary 4.32.** If $\varphi$ does not satisfy $\Delta_2(0)$, then $\ell^\varphi$ is not approximately amenable.

Note that the standard method for examining the approximate amenability of $\ell^\varphi$, [11], would involve obtaining some sharp geometrical properties of Orlicz
spaces. However the Hölder estimates in particular appear to be intractable. As well as this, a canonical map between $\ell^p$ and $\ell^p \widehat{\otimes} \ell^p$ is crucially used to perform bimodule calculations, and we do not have this at our disposal. However, we can also establish Proposition 4.28 via the existence of the approximate identity as shown in Proposition 4.31. Specifically, we have an unbounded approximate identity $(e_n)$ which is multiplier bounded, in fact $\|ae_n\|_{(\varphi)} \leq \|a\|_{(\varphi)}$, $(a \in A)$. Hence by ([5] Theorem 3.3), $\ell^p$ cannot be boundedly approximately amenable.

4.3 Modular spaces

We now turn to a generalization of Orlicz sequence spaces, by using different Young functions at each atom.

**Definition 4.33.** Given a sequence of Young functions $\Phi = (\varphi_n)$, define the modular sequence space $\ell[^{[\Phi]}]$ associated with this collection as the Banach space of all sequences $a = (a_n)$ with $\sum_{n=1}^{\infty} \varphi_n\left(\frac{|a_n|}{\rho}\right) < \infty$ for some $\rho > 0$, under the norm

$$\|a\| = \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} \varphi_n\left(\frac{|a_n|}{\rho}\right) \leq 1 \right\}.$$ 

Two sequences of Young functions $\Phi = (\varphi_n)$ and $\Psi = (\psi_n)$ are said to be equivalent if they generate the same set of sequences. There are elementary criteria for two modular spaces to be equivalent (see eg. [38] p167 conditions a, b), however a concrete characterization in terms of the functions involved is more difficult. For example if only finitely many $\varphi_n$ differ from the $\psi_n$, that is, $\{n : \varphi_n \neq \psi_n\}$ is finite, then it is clear that $\ell[^{[\Phi]}] = \ell[^{[\Psi]}]$; however this is certainly not sharp since multiplying every function in a collection $\Phi$ by a fixed positive real number will yield the same space. From the closed graph theorem one can see that in the event that two sequences of Young functions are equivalent; and the space is a Banach algebra under pointwise multiplication (unlike the Orlicz case, merely forming an algebra is insufficient, as we will see), the identity mapping is a Banach algebra isomorphism. We will only consider the norm given above, unlike when one has Luxemburg and Orlicz norms in Orlicz spaces; we will use square brackets when writing the modular space norm $\| \cdot \|[^{[\Phi]}]$ to avoid any possible ambiguity between the modular space and the Orlicz space associated with a function $\Phi$.

There is another subtlety associated with isomorphism. Consider any permutation $\pi$ of the natural numbers and define $\Psi = (\varphi_{\pi(n)})$. Then the map
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\begin{align*}
a = (a_n) \mapsto (a_{\pi(n)}) \text{ is easily seen to be a Banach algebra isomorphism from } \ell^{[\Phi]} \text{ onto } \ell^{[\Psi]}.
\end{align*}

If \( \pi \) has no fixed points and the \( \varphi_n \) are distinct, then \( \{n : \varphi_n = \psi_n\} \) is in fact empty. It is clear that \( \ell^{[\Phi]} \) and \( \ell^{[\Psi]} \) will not in general coincide as spaces. Hence equivalence is a stronger condition than Banach algebra isomorphism.

We need a small modification to the definition of inverse, to compensate for the case when a Young function \( \varphi \) is degenerate.

**Definition 4.34.** The *generalized inverse* of a Young function \( \varphi \) is given by

\[
\varphi^{-1}_g(x) = \sup\{y \geq 0 : \varphi(y) \leq x\}.
\]

This acts like an inverse, except to compensate for the case \( u_0 \neq 0 \). In particular \( \varphi(\varphi^{-1}_g(x)) = x \), \( \varphi^{-1}_g(\varphi(x)) \geq x \) always, and \( \varphi^{-1}_g(\varphi(x)) = x \) when \( x \geq u_0 \). Because of this, when there is no ambiguity \( \varphi^{-1} \) will be used to denote \( \varphi^{-1}_g \).

**Proposition 4.35.** Let \( \Phi = (\varphi_n) \) be a sequence of Young functions. \( \ell^{[\Phi]} \) is a Banach algebra under pointwise multiplication if and only if \( (\varphi_n) \) is equivalent to another collection of Young functions \( \Psi = (\psi_n) \) such that the numbers \( \{\psi_n^{-1}(1)\} \) are bounded.

**Proof.** Suppose that such a collection \( \Psi \) exists. Following the method of Hudzik ([32] Lemma 3, Remark 4 and Proposition 5), one sees that \( \ell^{[\Psi]} \subset \ell^\infty \) and is an ideal therein, and is a Banach algebra. One only has to note that the measure is counting measure and the \( \{\psi_n^{-1}(1)\} \) are bounded.

Conversely, if no such collection \( \Psi \) exists, then in particular, \( \{\varphi_n^{-1}(1)\} \) is unbounded. Firstly, we note that (see diagram)

\[
\varphi_n(x) \leq \frac{x}{\varphi_n^{-1}(1)}, \quad (n \in \mathbb{N}, x \leq \varphi_n^{-1}(1)). \quad (4.4)
\]
Then, given any $N \in \mathbb{N}$ and $k \in \mathbb{N}$, one can find $n_k > N$ such that $\varphi_{n_k}^{-1}(1) \geq k^3$. Thus inductively, one constructs such a sequence $(n_k)$ with $n_k > n_{k-1}$ for each $k$, and then defines the sequence

$$a = (1, 0, \ldots, 0, 2, \ldots, \ldots, 0, k, 0, \ldots).$$

We have that

$$\sum_{n} \varphi_n(a_n) = \sum_{n_k} \varphi_{n_k}(a_{n_k}) \leq \sum_{i} \frac{i}{i^3} < \infty.$$

by equation (4.4). Thus $a \in \ell^{[\Phi]} \setminus \ell^\infty$. But ([32] Remark 4) implies that $\ell^{[\Phi]}$ would have to be a subalgebra of $\ell^\infty$ to be a Banach algebra, and hence the latter is not the case. \hfill \Box

**Remark 4.36.** This actually demonstrates that the collection of numbers $\{\varphi_n^{-1}(1)\}$ being bounded or unbounded is in fact invariant under equivalence; because if the numbers are bounded one has the inclusion $\ell^{[\Phi]} \subseteq \ell^\infty$, whereas if they are unbounded one does not have this inclusion. In the same way as this, we will subsequently be able to see that many different criteria on the collection $(\varphi_n)$ will be invariant under equivalence. Henceforth, we assume that $\{\varphi_n^{-1}(1)\}$ is bounded, so that the modular space given is indeed a Banach algebra.

**Example 4.37.** Consider the sequence of functions $\varphi_n(x) = x/n^3$. Then the sequence $(1, 2, 3, \ldots) \in \ell^{[\Phi]} \setminus \ell^\infty$. But normalizing the functions according to ([38] p167) yields the collection $\psi_n(x) = x$, all $n$. Whilst the spaces $\ell^{[\Phi]}$ and $\ell^{[\Psi]}$ are isomorphic as Banach spaces, they are clearly not isomorphic as Banach algebras.

In an analogous fashion to Orlicz algebras, we can define a certain subclass of modular algebras for which one might hope that sequences do not factor.

**Definition 4.38.** A collection $\Phi = (\varphi_n)$ of Young functions satisfies the uniform $\Delta'$-condition if there exists $c > 0$ such that

$$\varphi_n(xy) \leq c \varphi_n(x) \varphi_n(y), \quad (x, y \geq 0, n \in \mathbb{N}).$$

**Definition 4.39.** Given a sequence $\Phi = (\varphi_n)$ of Young functions, set $u_n = u_0(\varphi_n)$ for each $n$.

**Remark 4.40.** This notation differs from that in [32], where $u_1$ is used for a different purpose. It is clear that if $\liminf_n u_n > 0$, then $\ell^\infty = \ell^{[\Phi]}$, in much the same way as with Orlicz algebras.
Definition 4.41. A modular space $\ell[\Phi]$ obeys the strong triviality condition if $\lim \inf_n u_n > 0$.

This is consistent with the way that ([38] p138) recognises degeneracy as a triviality condition for Orlicz spaces. The reason for the word ‘strong’ will become apparent in due course. Having $\lim \inf_n u_n = 0$ is clearly necessary for $\ell[\Phi]$ not to be $\ell^\infty$.

Example 4.42. One might be led to think that under a similar hypothesis to the Orlicz situation, that $\ell[\Phi]$ does not factor. However this is false. Considering $\Phi$ where $\varphi_n(x) = x^n$, we see that $\Phi$ satisfies the uniform $\Delta'$-condition with constant $c = 1$ for all $x$, and that $u_n = 0$ for all $n$. However the sequence $(1, 1, \ldots) \in \ell[\Phi]$, and hence $\ell[\Phi]$ has an identity. In particular $\ell[\Phi]$ factors, and $\ell[\Phi] = \ell^\infty$.

However, if one imposes another condition to account for the fact that one has a collection of functions instead of just one, the non-existence of a bounded approximate identity may still be established. We need to rearrange the argument somewhat to compensate for the fact that we are dealing with a sequence of Young functions. This rearrangement also facilitates another subsequent result.

Proposition 4.43. Suppose that a sequence of Young functions $\Phi = (\varphi_n)$ is equivalent to a collection $\Psi = (\psi_n)$ of Young functions which are all non-degenerate, and for which $\sum_{n=1}^{\infty} \psi_n(x) = \infty$ for $x > 0$. Then $\ell[\Phi]$ does not have a bounded approximate identity.

Proof. We write $(\varphi_n)$ in place of $(\psi_n)$ and proceed. Suppose there exists an approximate identity $(e_n)$ for $\ell[\Phi]$ bounded by $M$. We then have

$$\sum_{n=1}^{\infty} \varphi_n(|f(n)e(n) - f(n)|) \rightarrow 0, \quad (f \in \ell[\Phi]).$$

Take $N \in \mathbb{N}$ fixed, and set $g = 1_{[1, N]}$. Since all Young functions are non-degenerate, $u_n = 0$ for $n \leq N$ in particular, and so $\sum_{n=1}^{N} |e_n(n) - 1| \rightarrow 0$. Given $1 > \varepsilon > 0$ we can choose $\varepsilon_0 > 0$ such that $\max_{n \leq N} \varphi_n(2\varepsilon_0) < \varepsilon$, and find an $\alpha_0$ such that $\alpha > \alpha_0$ implies that $\sum_{n=1}^{N} |e_n(n) - 1| < \varepsilon_0$. This also implies $\max_{k \leq N} \varphi_k(\sum_{n=1}^{N} |e_n(n) - 1|) \leq \varepsilon$, since each $\varphi_k$ is increasing. Thus $\max_{k \leq N} \varphi_k(\sum_{n=1}^{N} |e_n(n) - 1|) \rightarrow 0$.

As in the proof of Proposition 4.21, replace $(e_n)$ by $(f_n)$ as follows: for $n \leq N$,

$$f_n(n) = \begin{cases} 
2 - e_n(n) & |e_n(n)| < 1 \\
e_n(n) & |e_n(n)| \geq 1 
\end{cases}$$
and \( f_\alpha(n) = e_\alpha(n) \) for \( n > N \). Again \( |f_\alpha(n)| \geq 1 \) for \( n \leq N \). For \( \alpha > \alpha_0 \) as above, we have chosen each \( (f_\alpha) \) so that \( \|f_\alpha - e_\alpha\|_{\Phi} \leq \varepsilon < 1 \).

Using the triangle inequality, we have
\[
\|f_\alpha f - f\|_{\Phi} \leq \|f_\alpha f - e_\alpha f\|_{\Phi} + \|e_\alpha f - f\|_{\Phi}, \quad (f \in \ell^{[\Phi]}).
\]

Obviously the second term on the right hand side goes to zero as \( \alpha \) increases, and the first term is bounded by
\[
\sum_{n=1}^{N} \varphi_n(\sum_{n=1}^{N} 2|e_\alpha(n) - 1|) \rightarrow 0.
\]

Thus \( (f_\alpha) \) is an approximate identity, and is bounded by \( M + 1 \) because
\[
\|f_\alpha\|_{\Phi} \leq \|f_\alpha - e_\alpha\|_{\Phi} + \|e_\alpha\|_{\Phi} \leq \|e_\alpha\|_{\Phi} + 1.
\]

Fix \( \lambda \geq M + 1 \). Then for each \( f_\alpha \),
\[
\sum_{n=1}^{\infty} \varphi_n \left( \frac{|f_\alpha(n)|}{\lambda} \right) \geq \sum_{n=1}^{N} \varphi_n(1/\lambda)
\] (4.5)
since each \( \varphi_n \) is increasing. As one can repeat this process for any \( N \), one may always find an approximate identity \( (f_\alpha) \) bounded by \( M + 1 \) such that equation (4.5) holds for this value of \( N \). But \( \sum_{n=1}^{N} \varphi_n(1/\lambda) \) is unbounded in \( N \) by hypothesis, so the values of these sums increase unboundedly, contradicting the fact that via the norm definition, they should be bounded by one.

If there is some \( x > 0 \) for which \( \sum_{n=1}^{\infty} \varphi_n(x) \) is finite, then \((x, x, \ldots) \in \ell^{[\Phi]} \) and consequently \( \ell^{[\Phi]} = \ell^{\infty} \); of course \((1, 1, \ldots) \) is the identity. Hence we make the following definition.

**Definition 4.44.** A modular space \( \ell^{[\Phi]} \) obeys the triviality condition if \( \sum_{n=1}^{\infty} \varphi_n(x) \) is finite for some \( x > 0 \).

This condition is obviously weaker than the strong triviality condition. It will become apparent that the triviality condition is equivalent to the statement \( \ell^{[\Phi]} = \ell^{\infty} \), in parallel to the situation in the Orlicz algebra case.

We must now examine what happens when the \( u_n \) are nonzero, along with the triviality condition failing.

**Proposition 4.45.** Suppose that a collection of Young functions \( \Phi = (\varphi_n) \) is equivalent to a sequence of Young functions \( \Psi = (\psi_n) \) which are all degenerate, with \( \{\psi_n^{-1}(1)\} \) bounded, \( u_n \to 0 \) and such that \( \sum_{n=1}^{\infty} \psi_n(x) = \infty \) for \( x > 0 \). Then \( \ell^{[\Phi]} \) does not possess a bounded approximate identity.
4.3. MODULAR SPACES

Proof. Again we write \((\varphi_n)\) in place of \((\psi_n)\). Ultimately we must have \(u_n < 1\). Since changing finitely many of the functions does not alter the sequence space, replace those \(\varphi_n\) with \(u_n \geq 1\) by different degenerate functions with \(u_n < 1\), and again call this new equivalent sequence \((\varphi_n)\).

As in the previous case, suppose there exists a bounded approximate identity \((e_n)\) bounded by \(M\). Again we have that

\[
\sum_{n=1}^{\infty} \varphi_n(\|f(n)e_n(n) - f(n)\|) \to 0, \quad (f \in \ell^{[\Phi]})
\]

Fix \(N \in \mathbb{N}\). Then for \(\varepsilon_0 > 0\) choose \(r \in \mathbb{N}\) such that \(u_n^r < \frac{\varepsilon_0}{2N}\) for all \(n \leq N\). Then using \(f = \sum_{n=1}^{N} u_n^{-r}1_{\{n\}}\), we must have that there exists \(\alpha_0\) such that \(\alpha > \alpha_0\) implies

\[
|u_n^{-r}(1 - e_n(n))| \leq u_n + \frac{\varepsilon_0}{2N}
\]

for each \(n \leq N\). That is to say, for \(\alpha > \alpha_0\) we have

\[
\sum_{n=1}^{N} |e_n(n) - 1| < \varepsilon_0.
\]

We are now able to follow the same argument as in the non-degenerate case, and given any \(N \in \mathbb{N}\) manufacture another approximate identity \((f_n)\), bounded by \(M + 1\), such that for any \(\lambda > 0\) we have

\[
\sum_{n=1}^{\infty} \varphi_n\left(\frac{|f_n(n)|}{\lambda}\right) \geq \sum_{n=1}^{N} \varphi_n(1/\lambda),
\]

which is unbounded in \(N\).

It should be noted that the above two scenarios in Propositions 4.43 and 4.45 encapsulate all the possibilities for any modular sequence space. That is to say, given any collection \(\Phi\) of Young functions and its associated modular space \(\ell^{[\Phi]}\), one can decompose \(\ell^{[\Phi]}\) as a direct sum

\[
\ell^{[\Phi]} = \ell^{[\Phi_1]} \oplus \ell^{[\Phi_2]} \oplus \ell^{[\Phi_3]}
\]

where the products between each summand are zero, and

- \(\Phi_1 = (\xi_n)\); each \(\xi_n\) is non-degenerate, with \(\sum_n \xi_{n_k}(x) = \infty\) for all \(x > 0\);
- \(\Phi_2 = (\chi_n)\); each \(\chi_n\) is degenerate, with \(u_n \to 0\) and \(\sum_n \chi_n(x) = \infty\) for all \(x > 0\), and
• $\Phi_3 = (\eta_n)$; the $\eta_n$ are such that $\sum_n \eta_n(x)$ converges for some $x > 0$, so that $\ell^{[\Phi_3]} = \ell^\infty$; this includes the case when $\inf_n u_n > 0$.

Whilst this decomposition is not unique, any decomposition will always yield the same information regarding the existence of approximate identities in the original space $\ell^{[\Phi]}$.

We now collect some of the Banach space properties of $\ell^{[\Phi]}$. We need a condition akin to $\Delta_2(0)$ for our collection of functions.

**Definition 4.46.** A sequence $\Phi = (\varphi_n)$ of Young functions satisfies the uniform $\Delta_2(0)$-condition if there exists a constant $C > 0$, an $x_0 > 0$ and an integer $N$ such that for all $n \geq N$, 
\[
\varphi_n(2x) \leq C \varphi_n(x), \quad (0 \leq x \leq x_0).
\]

Analogous to the Orlicz algebra case, we have an important closed subspace.

**Definition 4.47.** Given a collection $\Phi = (\varphi_n)$ of Young functions, set
\[
h^{[\Phi]} = \left\{ (a_n) \in \ell^{[\Phi]} : \sum_{n=1}^\infty \varphi_n \left( \frac{|a_n|}{\rho} \right) < \infty \quad \text{for every } \rho > 0 \right\}.
\]

The uniform $\Delta_2(0)$-condition plays the same role as the $\Delta_2(0)$-condition does for Orlicz spaces.

**Theorem 4.48.** ([38] Proposition 4.d.3) For a sequence $\Phi = (\varphi_n)$ of non-degenerate Orlicz functions the following are equivalent:

• $\Phi$ is equivalent to a sequence of Young functions which satisfy the uniform $\Delta_2(0)$-condition;
• $h^{[\Phi]} = \ell^{[\Phi]}$;
• the unit vectors form a boundedly complete normalized unconditional basis for $\ell^{[\Phi]}$;
• $\ell^{[\Phi]}$ is separable;
• $\ell^{[\Phi]}$ contains no subspace isomorphic as a Banach space to $\ell^\infty$.

**Proposition 4.49.** (cf 4.31) Suppose that a collection of Young functions $\Phi = (\varphi_n)$ is equivalent to a collection which are all non-degenerate, and satisfy the $\Delta_2(0)$-condition. Then $\ell^{[\Phi]}$ possesses an approximate identity, which is the standard one.
Corollary 4.50. (cf 4.28) Suppose that a collection of Young functions $\Phi = (\varphi_n)$ satisfies the uniform $\Delta_2(0)$-condition, and each function is non-degenerate. Then $\ell^{[\Phi]}$ is not boundedly approximately amenable.

Once again, pointwise approximate amenability is more suitable.

Corollary 4.51. Suppose that $\Phi = (\varphi_n)$ is equivalent to a sequence of non-degenerate Young functions satisfying the uniform $\Delta_2(0)$-condition. Then $\ell^{[\Phi]}$ is pointwise approximately amenable.

Proof. This is exactly the same as Proposition 4.30, noting that whilst the standard basis is no longer symmetric, it is still unconditional.

It is not true that modular spaces where the sequence $\Phi = (\varphi_n)$ is not equivalent to one satisfying the uniform $\Delta_2(0)$-condition always fail to possess an approximate identity. This is because unlike with Orlicz spaces, one cannot say immediately that for an element $(f_n) \in \ell^{[\Phi]}$, that the numbers $f_n \to 0$. This is akin to the direct proof above that some modular spaces do not possess a bounded approximate identity, which relied on the fact that there is no $x > 0$ for which $\sum_{i=1}^{\infty} \varphi_n(x)$ converges. In fact the modular space can have an identity!

Example 4.52. Choose $\varphi_n(x) = x^n$. Then given the sequence $a = (1,1,...)$, one has

$$\sum_{n=1}^{\infty} \varphi_n(1/2) = 1,$$

with $\|a\|_{[\Phi]} = 2$. In fact $\ell^{[\Phi]} = \ell^{\infty}$. Hence this sequence does not satisfy the uniform $\Delta_2(0)$-condition, even though each individual function $\varphi_n$ does satisfy an (Orlicz) $\Delta_2(0)$-condition.

4.4 Convolutions Multiplication

A paper by Rao [47] discusses the situation when Orlicz spaces form a Banach algebra under convolution. We have the following, which Rao states, without explicitly showing.

Proposition 4.53. Suppose that $\varphi$ is a Young function with $\varphi'(0) > 0$ strictly (the right derivative). Let $G$ be a unimodular locally compact group with (left) Haar measure $\mu$. Then $L^\varphi(G)$ is a dense subspace of $L^1(G)$, which is itself closed under convolution and forms a Banach algebra under $\| \cdot \|_\varphi$, or equivalently, $\| \cdot \|_{(\varphi)}$. 

Proof. For such $\varphi$,

$$|x|\varphi'(0) = \int_0^{|x|} \varphi'(0)dt \leq \int_0^{|x|} \varphi'(t)dt = \varphi(x), \quad (x \in [0, \infty))$$

since the right derivative $\varphi'$ is increasing, and hence

$$\int_G |f|d\mu \leq \frac{1}{\varphi'(0)} \int_G \varphi(|f|)d\mu < \infty, \quad f \in L^\varphi(G).$$

Hence $L^\varphi \subset L^1$. Also, upon examining functions of compact support, it is clear that $L^\varphi$ as a set is dense in $L^1$. To show that $(L^\varphi, \ast)$ is an algebra with continuous product, we calculate in the Orlicz norm. We have

$$\|f \ast g\|_\varphi = \sup \left\{ \int_\Omega |(f \ast g)(x)h(x)|d\mu(x) : \|h\|_\psi \leq 1 \right\}$$

$$= \sup \left\{ \int_\Omega \left| \int_\Omega f(xy^{-1})g(y)d\mu(y)h(x)d\mu(x) \right|d\mu(x) : \|h\|_\psi \leq 1 \right\}$$

$$\leq \sup \left\{ \int_\Omega |g(y)| \int_\Omega |f(xy^{-1})h(x)|d\mu(x)d\mu(y) : \|h\|_\psi \leq 1 \right\}$$

$$\leq \|g\|_1 \|f\|_\varphi \leq \frac{1}{\varphi'(0)} \|f\|_\varphi \|g\|_\varphi.$$  

This uses Fubini's theorem, and the fact that Haar measure is translation invariant. \qed

We defined a growth condition $\Delta_2(0)$ for the pointwise case; the analogous growth condition for large values of $x$ is more appropriate here.

**Definition 4.54.** A Young function $\varphi$ satisfies $\Delta_2$ if there exists a constant $C > 0$ and $x_0 > 0$ such that

$$\varphi(2x) \leq C\varphi(x), \quad (x \geq x_0).$$

Subsequently in [47], Rao attempts to show that when $\varphi \in \Delta_2$ and $\varphi'(0) > 0$, the algebra $L^\varphi$ possesses a bounded approximate identity. But he also notes that they are Segal algebras (the difficult part of this is [48] Theorem III.4.14) in a discussion at the top of page 387. However it is well known that proper Segal algebras of $L^1$ cannot possess a bounded approximate identity ([4] Theorem 1.2).

It is ([47] Proposition 1) on the second line of page 385 where there is a problem. Here, an extra term $u_{\psi_0}$ is multiplied to the $f(s)$ term, and this step requires that $\|u_{\psi_0}\|_1 = 1$ to be true (since $u_{\psi_0} \geq 0$). This is intuitively unfeasible however, since by supposition $\|u_{\psi_0}\|_{(\psi)} = 1$. Indeed, to be an approximate identity
requires $\int u_V \, d\mu \to 1$ as the compact set $V$ shrinks, so the values of the function $u_V$ must become very large, and obviously if $\varphi$ grows much more rapidly than $| \cdot |$ for large values, then one could expect the values of the two norms to be very different. And indeed this turns out to be the case – any such approximate identity in $L^p$ would have to be unbounded [4].

**Example 4.55.** Take $G = \mathbb{R}$, $u_n$ to be a triangle of height $n$, base vertices $\pm 1/n$, so $\|u_n\|_1 = 1$ for all $n$. Choose $\varphi(x) = e^x - 1$. Then, denoting $\|u_n\|_{(\varphi)} = \lambda$ for the moment, $\lambda$ is the solution to

$$(\lambda/n) \cdot (\exp(n/\lambda) - 1) = 1 + n/2. \quad (4.6)$$

To tackle this with power series, set $f(x) = (e^x - 1)/x$. Then $f(x) > x^2/6$ for $x > 0$ via a Taylor expansion, so that the solution to equation (4.6) must satisfy $n/2 + 1 = f(n/\lambda) > (1/6)(n/\lambda)^2$, so that $\lambda^2 > n^2/(3n + 6)$. Thus $\|u_n\|_{(\varphi)}$ is unbounded as a function of $n$.

[47] denotes the closure in $L^p(G)$ of the continuous functions of compact support by $M^e(G)$. The technique used to establish amenability of $M^e(G)$ in ([47] Theorem 3) is doubtful. The discussion preceding the statement outlines a proof which requires that continuous derivations from $M^e(G)$ be extended to continuous derivations from $L^1(G)$. Since the norm on $M^e(G)$ is strictly stronger than $\| \cdot \|_1$ this will not be possible in general. In fact, since $M^e(G) = L^p(G)$ when $\varphi$ satisfies $\Delta_2$ [48], the algebras $M^e(G)$ will necessarily fail to be amenable in this case as they will be proper Segal algebras.

**Example 4.56.** Consider a compact group $G$, and the Segal algebra $A = L^2(G)$. Let $X$ be the bimodule whose underlying space is $L^2(G)$, but with actions

$$a \cdot x = ax \quad \text{and} \quad x \cdot a = 0 \quad (a \in A, x \in X).$$

Consider the derivation $D : A \to A$ given by the identity on $A$, considering $A$ as a dual bimodule. If $D$ were inner, $L^2(G)$ would possess a bounded right approximate identity ([50] Proposition 2.2.1). Since $L^2(G)$ is a proper Segal algebra, this is impossible by ([4] Theorem 1.2).

Suppose that there existed an extension $\widetilde{D} : L^1(G) \to X^{**}$ so that $\widetilde{D}|_{L^2(G)} = D$. Then $\widetilde{D}$ must be inner by the standard result of Johnson, so that $\widetilde{D}(a) = a \cdot x - x \cdot a$ for some $x \in X^{**} = X$. But then $D$ is also inner, a contradiction.

This does, however, raise the obvious question as to whether $M^e(G)$ can ever be approximately amenable, and this appears quite intractable. Proper Segal
algebras generally fail to be approximately amenable; in fact ([5] Remark on p27) asks whether any proper Segal algebra can be approximately amenable.

The fact that these Segal algebras possess an unbounded approximate identity is similar to what we have in the pointwise case. Lemmas 4.14 and 4.15 show that for \( \varphi \) non-degenerate, \( \ell^\varphi \) is a (abstract) Segal algebra in \( c_0 \). Also, we have seen that \( \ell^\varphi \) fails to possess a bounded approximate identity as expected, similarly to the convolution algebras that Rao studies. We may also see that the given Orlicz algebras in the convolution and pointwise cases are weakly amenable ([7] Theorem 4.1.10), though that is already clear in the pointwise case when \( \varphi \) obeys \( \Delta_2(0) \), as the idempotents have dense span.

### 4.5 Difference algebras

Based on the treatment of the Feinstein algebras by White in [60], we aim to generalize these to examine norms given by convex functions of the differences of consecutive terms of a sequence. Again the purpose is to study when such algebras possess approximate identities of various kinds.

**Definition 4.57.** Given a sequence \( (\alpha_i) \) of strictly positive scalars and a non-trivial Young function \( \varphi \), define for \( a = (a_i) \in c_0 \),

\[
\|a\|_{\alpha\varphi} = \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \alpha_n \varphi \left( \frac{a_{n+1} - a_n}{\lambda} \right) \leq 1 \right\},
\]

with \( \inf \emptyset = \infty \).

It is worthwhile to observe that if the infimum is strictly positive and finite, then it is attained, because \( \sum_{n=1}^{\infty} \alpha_n \varphi \left( \frac{a_{n+1} - a_n}{\lambda} \right) \) is lower semicontinuous as a function of \( \lambda \), and thus attains its infimum on any compact subset of \( (0, \infty) \).

We check that this functional obeys the required properties, so we can subsequently define a Banach algebra.

**Proposition 4.58.** \( \| \cdot \|_{\alpha\varphi} \) is a norm on \( c_0 \cap \{ (a_n) : \| (a_n) \|_{\alpha\varphi} < \infty \} \).

**Proof.** We proceed via the technique of ([48] III.2.3), beginning with the triangle inequality. Taking sequences \( x = (x_n) \) and \( y = (y_n) \) in \( c_0 \cap \{ (a_n) : \| (a_n) \|_{\alpha\varphi} < \infty \} \)
and examining \( \| \cdot \|_{\alpha, \psi} \) for \( x + y \), we have, recalling Remark 4.2, that

\[
\sum_n \alpha_n \varphi\left( \frac{|x_{n+1} - x_n + y_{n+1} - y_n|}{\|x\|_{\alpha, \psi} + \|y\|_{\alpha, \psi}} \right) \\
\leq \sum_n \alpha_n \varphi\left( \frac{|x_{n+1} - x_n|}{\|x\|_{\alpha, \psi}} + \frac{|y_{n+1} - y_n|}{\|y\|_{\alpha, \psi}} \right) + \frac{|y_{n+1} - y_n|}{\|y\|_{\alpha, \psi}} \\
\leq \frac{\|x\|_{\alpha, \psi} + \|y\|_{\alpha, \psi}}{\|x\|_{\alpha, \psi} + \|y\|_{\alpha, \psi}} \sum_n \alpha_n \varphi\left( \frac{|x_{n+1} - x_n|}{\|x\|_{\alpha, \psi}} \right) + \frac{\|y\|_{\alpha, \psi}}{\|x\|_{\alpha, \psi} + \|y\|_{\alpha, \psi}} \sum_n \alpha_n \varphi\left( \frac{|y_{n+1} - y_n|}{\|y\|_{\alpha, \psi}} \right) \\
\leq \frac{\|x\|_{\alpha, \psi} + \|y\|_{\alpha, \psi}}{\|x\|_{\alpha, \psi} + \|y\|_{\alpha, \psi}} = 1
\]
as required. We also need to have that \( \|x\|_{\alpha, \psi} = 0 \) if and only if \( x = 0 \). So suppose that

\[
\sum_n \alpha_n \varphi\left( \frac{|x_{n+1} - x_n|}{\lambda} \right) \leq 1 \tag{4.7}
\]

for all \( \lambda > 0 \). Supposing \( x_{n+1} \neq x_n \) for some \( n \), take \( \mu > 0 \) such that \( \varphi(\mu) > \alpha_n^{-1} \) and \( 0 < \lambda < \frac{|x_{n+1} - x_n|}{\mu} \). Then \( \alpha_n \varphi\left( \frac{|x_{n+1} - x_n|}{\lambda} \right) > 1 \), a contradiction. \( \square \)

The aim is to define a reasonable norm on some subspace of \( c_0 \) so as to obtain a Banach algebra. To show that we have taken a reasonable approach, we check that the quantity \( \| \cdot \| = \| \cdot \|_\infty + \| \cdot \|_{\alpha, \psi} \) is an algebra norm where finite. We proceed with a technical lemma, which is the thrust of the result.

**Lemma 4.59.** For \( x, y \in c_0 \cap \{(a_n) : \|(a_n)\|_{\alpha, \psi} < \infty\} \), \( \|xy\|_{\alpha, \psi} \leq 2\|x\|\|y\| \).

Proof. We have that

\[
\sum_n \alpha_n \varphi\left( \frac{|x_{n+1}y_{n+1} - x_ny_n|}{2\|x\|\|y\|} \right) \\
\leq \sum_n \alpha_n \varphi\left( \frac{\|x\|_\infty |y_{n+1} - y_n|}{2\|x\|\|y\|} \right) + \sum_n \alpha_n \varphi\left( \frac{\|y\|_\infty |x_{n+1} - x_n|}{2\|x\|\|y\|} \right) \leq 1.
\]

\( \square \)

**Proposition 4.60.** The quantity \( \| \cdot \| = \| \cdot \|_\infty + \| \cdot \|_{\alpha, \psi} \) is an algebra norm on \( c_0 \cap \{(a_n) : \|(a_n)\|_{\alpha, \psi} < \infty\} \).

Proof. For \( x, y \in c_0 \cap \{(a_n) : \|(a_n)\|_{\alpha, \psi} < \infty\} \), we have that

\[
\|xy\| \leq \|x\|_\infty \|y\|_\infty + 2\|x\|\|y\| \leq 3\|x\|\|y\|
\]
as required. \( \square \)
We will also need a technical yet elementary observation in order to establish that the subsequently defined algebras are complete.

**Lemma 4.61.** Suppose that $\Phi = (\varphi_n)$ is a sequence of non-zero Young functions, with associated modular space $\ell^{[\Phi]}$. Then

$$A = \{a = (a_n) : a \in \ell^{[\Phi]} \cap c_0\}$$

is a Banach algebra under the norm $\| \cdot \|_\infty + \| \cdot \|_{[\Phi]}$.

**Remark 4.62.** It is worthwhile noting that as we now have sequences in $c_0$, we can dispense with the restriction of $\{\varphi_n^{-1}(1)\}$ being bounded, unlike the situation when we consider $\ell^{[\Phi]}$ as a Banach algebra. It is immediate that $A$ is a Banach space under the given norm, and the obvious estimate for $a, b \in A$ yields $\|ab\| \leq \|a\|_\infty \|b\| \leq \|a\| \|b\|$.

Finally, we may define a generalization of the Feinstein algebras:

**Definition 4.63.** Given a sequence of strictly positive scalars $\alpha = (\alpha_i)$ and a non-zero Young function $\varphi$, define the Orlicz–Feinstein algebra

$$A_{\alpha \varphi} = \{a = (a_n) \in c_0 : \|a\| = \|a\|_\infty + \|a\|_{\alpha \varphi} < \infty\}.$$  

We have obtained normed algebras, whose completeness needs to be verified. Take a Cauchy sequence $(a^{(i)}) \subset A_{\alpha \varphi}$. It is immediate that each coordinate must converge pointwise, that is $a^{(i)}_n \to a_n$ for some sequence $(a_n)$. Writing sequences $b^{(i)}_n = a^{(i)}_{n+1} - a^{(i)}_n$, it is immediate that $b^{(i)} \to b$, where $b_n = a_{n+1} - a_n$, in the algebra $A$ defined in Lemma 4.61. From the definition of the norm on $A_{\alpha \varphi}$ it is apparent that $a^{(i)} \to a$.

We wish to know when the Banach algebras $A_{\alpha \varphi}$ possess approximate identities of various types. The following mimics ([60] Lemma 5.1).

**Proposition 4.64.** Suppose that a subsequence of $(\alpha_i)$ is bounded, and that $\varphi$ satisfies $A_2(0)$, or $\varphi$ is degenerate. Then $A_{\alpha \varphi}$ possesses a bounded approximate identity which is contained in $c_{00}$.

**Proof.** Writing $e_n$ as the standard sequence with 1’s in the first $n$ entries and 0’s elsewhere, we have that for $a \in A_{\alpha \varphi}$

$$\|e_n a - a\|_{\alpha \varphi} = \|e_n a - a\|_\infty$$

$$+ \inf \left\{ \lambda > 0 : \alpha \varphi \left( \frac{|a_{n+1}|}{\lambda} \right) + \sum_{i=n+1}^{\infty} \alpha_i \varphi \left( \frac{|a_{i+1} - a_i|}{\lambda} \right) \leq 1 \right\}. \quad (4.8)$$
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The first term of the right hand side of this expression certainly goes to zero as \( n \to \infty \).

It suffices to find a sequence \((n_k)\), independent of \( a \in A_{\alpha \varphi} \), such that as \( k \to \infty \),

\[
\inf \left\{ \lambda > 0 : \sum_{i=n_k+1}^{\infty} \alpha_i \varphi \left( \frac{|a_{i+1} - a_i|}{\lambda} \right) \leq 1 \right\} \to 0
\] (4.9)

and

\[
\inf \left\{ \lambda > 0 : \alpha_{n_k} \varphi \left( \frac{|a_{n_k+1}|}{\lambda} \right) \leq 1 \right\} \to 0.
\] (4.10)

This is because if \( \lambda_1 \) and \( \lambda_2 \) are the infima from equations (4.9) and (4.10), then the infimum from equation (4.8) is at most \( 2 \max\{\lambda_1, \lambda_2\} \).

Suppose that \( \lim \inf_n \alpha_n < \infty \) and take a sequence \((n_k)\) where \((\alpha_{n_k})\) is bounded. We may assume that the sequence \((a_{n_k})\) is not ultimately zero, otherwise the fulfillment of equations 4.9 and 4.10 is immediate. Denoting the attained infimum in equation (4.9) by \( \gamma_k \), we have that \( \gamma_k \downarrow \gamma \geq 0 \) for some real number \( \gamma \). Suppose that \( \gamma \neq 0 \). Then choosing \( 0 < \mu < \gamma \), each sum

\[
\sum_{i=n_k+1}^{\infty} \alpha_i \varphi \left( \frac{|a_{i+1} - a_i|}{\mu} \right) > 1.
\] (4.11)

In fact, these sums are all infinite valued, otherwise they converge to zero as \( k \to \infty \), contrary to the supposition. Setting \( \Phi = (\alpha_i \varphi) \), we have that \( \Phi \) satisfies \( \Delta_2(0) \) uniformly (since \( \varphi \) satisfies \( \Delta_2(0) \)). Then for \( b_i = a_{i+1} - a_i \), we have \( b = (b_i) \in \ell^{[b]} \), so that \( b/\mu \in \ell^{[b]} \) by Theorem 4.26, contradicting that the sums in equation (4.11) are infinite. It follows that \( \gamma_k \to 0 \) as required.

To verify equation (4.10), denote the attained infimum in equation (4.10) by \( \beta_k \). Estimating for \( \beta_k \), we obtain that \( \beta_k \leq \frac{|a_{n_k+1}|}{\varphi^{-1}(1/\alpha_{n_k})} \), with equality if \( \varphi \) is non-degenerate. As \( \varphi^{-1}(1/\alpha_{n_k}) \) is bounded away from zero, and \( a \in c_0 \), we see that equation (4.10) holds.

On the other hand, if \( u_0 \neq 0 \) and \( \lim \inf_n \alpha_n = \infty \), then \( \varphi^{-1}(1/\alpha_n) \to u_0 \), and

\[
\frac{|a_{n+1}|}{\varphi^{-1}(1/\alpha)} \to 0 \text{ immediately.}
\]

We also note that the sums of the form of equation (4.11) can never be infinite valued, as they are in fact finite sums.

In either case, one observes that \( \|e_{n_k}\|_{\alpha \varphi} = \frac{1}{\varphi^{-1}(1/\alpha_{n_k})} \), which is bounded given the hypotheses.

Hence in this case, \( A_{\alpha \varphi} \) is spanned by its idempotents, and is weakly amenable. We also have that \( A_{\alpha \varphi} \) is approximately amenable by ([11] Corollary 3.5). \( \square \)
Along these lines, one may consider extending the proofs of [60] to attempt to show that if \( \liminf_n \alpha_n = \infty \), then \( A_{\alpha} \) possesses an unbounded approximate identity only. However the technique employed in [60] crucially employs the triangle inequality for complex numbers to deal with telescoping sums. We now show that there is no analogue to the triangle inequality for arbitrary Young functions.

**Remark 4.65.** To formalise this, we ask: given a Young function \( \varphi \) which is inequivalent to \( | \cdot | \) as in Theorem 4.13, does the following ever happen - there exists a fixed \( M > 0 \) such that for any sequence \( a = (a_n) \) of complex numbers

\[
\varphi \left( \sum_{n=1}^\infty |a_n| \right) \leq M \sum_{n=1}^\infty \varphi(|a_n|)
\]  

Via convexity, we obviously have the reverse inequality, with \( M = 1 \).

It is clear that for any fixed \( \delta > 0 \) there exists \( K > 0 \) such that \( |x| < \delta \) implies \( \varphi(x) \leq K|x| \). Then for any sequence \( (a_n) \in \ell^1 \), we have

\[
\sum_n \varphi(|a_n|) \leq \sum_{n:|a_n| \geq \delta} \varphi(|a_n|) + K \sum_{n:|a_n| < \delta} |a_n| < \infty ,
\]  

since the first sum has only finitely many terms. So we have that \( \ell^1 \subset \ell^\varphi \) as vector spaces.

Suppose \( \ell^\varphi \neq \ell^1 \) and take \( a \in \ell^\varphi \setminus \ell^1 \). One may always find a \( \lambda > 1 \) so that

\[
\sum_{n=1}^\infty \varphi \left( \frac{|a_n|}{\lambda} \right) < \infty .
\]  

Thus the sequence \((\frac{a_n}{\lambda})\) exhibits that equation (4.12) fails, hence there exists no such \( M \).

The triangle inequality is used in [60] to decompose a term of an element \( (a_n) \in A_\alpha \subset c_0 \) into an infinite sum of differences \( a_n = -\sum_{i\geq n}(a_{i+1} - a_i) \). Because of the above, one cannot apply the techniques of ([60] Lemma 5.1) in the case when \( \liminf_n \alpha_n = \infty \). As we will see below, this case remains unresolved.

In fact, a further level of generality is warranted when continuing to look for various approximate identities in such algebras.

**Definition 4.66.** Given a sequence of nonzero Young functions \( [\Phi] = (\varphi_1, \varphi_2, \ldots) \), one again defines a functional on \( c_0 \):

\[
\|a\|_\Phi = \inf \left\{ \lambda > 0 : \sum_{n=1}^\infty \varphi_n \left( \frac{a_{n+1} - a_n}{\lambda} \right) \leq 1 \right\} , \quad \inf \emptyset = \infty .
\]
In exactly the same way as with the Orlicz–Feinstein algebras, replacing $\alpha_n \varphi_n$ with $\varphi_n$ and noting that weights $\alpha_n$ no longer give any more generality, one calculates that $\| \cdot \| = \| \cdot \|_\infty + \| \cdot \|_{[\varphi]}$ is an algebra norm on $c_0$.

**Definition 4.67.** Given a sequence of non-degenerate Young functions $\Phi = (\varphi_1, \varphi_2, \ldots)$, define the modular Feinstein algebra associated with $\Phi$ as

$$A_{[\varphi]} = \{ a = (a_n) \in c_0 : \| a \| = \| a \|_\infty + \| a \|_{[\varphi]} < \infty \} .$$

**Proposition 4.68.** Suppose that a collection $(\varphi_n)$ of non-degenerate Young functions is such that $\limsup_n \varphi_n^{-1}(1) > 0$, and that $(\varphi_n)$ satisfies the uniform $\Delta_2(0)$-condition. Then $A_{[\varphi]}$ possesses a bounded approximate identity which is contained in $c_0$.

**Proof.** This is exactly the same as Proposition 4.64, replacing $(\alpha_n \varphi_n)$ by $(\varphi_n)$, and choosing a subsequence $(n_k)$ for which $\{\varphi_n^{-1}(1)\}$ is bounded away from zero. This leads to $\beta_k \leq \frac{|a_{n_k+1}|}{\varphi_n^{-1}(1)}$, which is the required estimate in a condition analogous to equation (4.10). \square

Again, the methods of ([60] Lemma 5.1) fail when trying to determine the existence of an approximate identity in general.

**Example 4.69.** Suppose that a Young function $\varphi$ fails $\Delta_2(0)$. Then there exists $a \in \ell^\varphi$ such that $\sum_n \varphi(a_n) < \infty$ but $\sum_n \varphi(2a_n) = \infty$. (For a specific example, take $\varphi = e^{-1/|x|}$ on $(0, 1/2)$ and extend to a Young function on $\mathbb{R}$, with $a_n = \frac{1}{2 \ln n}$, $n \geq 2$.) That is to say, there exists $a \in \ell^\varphi$ and $\mu > 0$ with

$$\sum_{i=n+1}^\infty \varphi \left( \frac{|a_i|}{\mu} \right) = \infty, \quad (n \in \mathbb{N}) . \tag{4.14}$$

If any of the terms $a_n$ are zero, one obtains a new sequence by merely deleting these terms. Clearly, infinitely many of the $a_n$ terms must be non-zero, and hence this new sequence will still satisfy equation (4.14). Thus without loss of generality we may assume that $a_n \neq 0$ for all $n$. Also, it is clear that rearranging the terms $a_n$ will not affect the sum involved, as all the terms in the sum are non-negative real numbers.

Indeed, one may rearrange the given sequence so that $|a_{n+1}| \leq |a_n|$ for all $n$, and still have $a \in \ell^\varphi$. This is possible since failing $\Delta_2(0)$ automatically means that $\varphi$ is non-degenerate, and hence by Lemma 4.15 that $a \in c_0$. Without loss
of generality we may suppose that \((a_n)\) are all real and non-negative, and use alternating sums to define

\[
\begin{align*}
    b_1 &= \sum_{i \geq 2} (-1)^{i-1} a_i; \quad b_{n+1} = \sum_{i \geq n+1} (-1)^{i-1} a_i, \quad (n \geq 2).
\end{align*}
\]

We obtain \(b = (b_n) \in c_0\) with \(|b_{n+1} - b_n| = |a_n|\), so \(b \in A_{1,\varphi}\), where we denote the constant sequence of weights \(\mathbf{1} = (1, 1, \ldots)\). Then, examining Proposition 4.64, we have that

\[
    \inf \left\{ \lambda > 0 : \varphi \left( \frac{|b_{n+1}|}{\lambda} \right) + \sum_{i = n+1}^{\infty} \varphi \left( \frac{|b_{i+1} - b_i|}{\lambda} \right) \leq 1 \right\} \geq \mu
\]

from equation (4.14). Thus we have shown that no subsequence of \((\varepsilon_n)\) can be an approximate identity for \(A_{1,\varphi}\).

To demonstrate the lack of the existence of a standard approximate identity contained in \(c_{00}\) in another particular setting, we need to establish a technical result about certain modular spaces.

**Lemma 4.70.** Suppose that \(\Phi = (\varphi_n)\) where \(\varphi_n^{-1}(1) \to 0\). Then \(\ell^{|\Phi|} \subset c_0\) as a vector space.

**Proof.** Take a sequence \(a = (a_n) \notin c_0\). Then for some \(\rho > 0\), there exists a subsequence \((n_k)\) with \(|a_{n_k}| > \rho\), all \(k\). Calculating for the norm, for each \(\lambda > 0\),

\[
    \sum_{n=1}^{\infty} \varphi_n \left( \frac{|a_n|}{\lambda} \right) \geq \sum_{n_k} \varphi_{n_k} \left( \frac{\rho}{\lambda} \right).
\]

We use the geometrical fact that (see diagram)

\[
    \varphi_n(x) \geq \frac{x}{\varphi_n^{-1}(1)}, \quad (n \in \mathbb{N}, x \geq \varphi_n^{-1}(1)).
\]
Fixing $\lambda > 0$, take $j_0$ such that $\rho/\lambda > \varphi_{n_j}^{-1}(1)$ for $j > j_0$, and again call this $(n_k)$. Then

$$\sum_{n=1}^{\infty} \varphi_n \left( \frac{|a_n|}{\lambda} \right) \geq \sum_{k=1}^{\infty} \varphi_{n_k} \left( \frac{|a_{n_k}|}{\lambda} \right) \geq \sum_{k=1}^{\infty} \varphi_{n_k} \left( \frac{\rho}{\lambda} \right) \geq \sum_{k=1}^{\infty} \frac{\rho}{\lambda} \varphi_{n_k}^{-1}(1) = \infty.$$ 

As this can be done for any $\lambda > 0$, we have that $a \notin \ell^{[\Phi]}$.

**Example 4.71.** If one wanted to run the same argument as in Example 4.69 with weights $\alpha_n$ such that $\lim \inf_n \alpha_n = \infty$, then one would merely need to observe that for $\Phi = (\alpha_n \varphi)$, $\Phi$ does not obey $\Delta_2(0)$ uniformly, and $(\alpha_n \varphi)^{-1}(1) \to 0$, whence one can invoke Lemma 4.70 to establish that if $a \in \ell^{[\Phi]}$, then $a \in c_0$, which is all that is required to obtain the finite valued alternating series $\sum_{n \geq 2} (-1)^{n-1} a_n$.

### 4.5.1 James-Orlicz spaces

There is another class of suitable difference type Banach algebras to generalize. In the paper [60], where approximate identities in difference algebras were studied, major consideration is given to James algebras. As we will see, the existence of approximate identities in the generalization of James algebras is similar to that of the Feinstein algebras.

In 1979, James spaces were appropriately generalized to James-Orlicz spaces in [56]. However, their setup uses Banach space valued sequences, whereas we will only consider complex valued sequences as before. The definitions are very natural. As is standard notation, we write $\omega = \{p_1, p_2, \ldots, p_n\}$ for a strictly increasing finite set contained in $\mathbb{N}$.

Given any $\lambda > 0$ and sequence $a = (a_n)$, we set

$$S_a(\lambda, \omega) = \sum_{i=1}^{n-1} \varphi \left( \frac{|a_{p_{i+1}} - a_{p_i}|}{\lambda} \right),$$

where $\varphi$ is a strictly increasing function.
and subsequently define a functional

\[ S_\varphi(\lambda) = \sup_{\omega} S_\varphi(\lambda, \omega). \]

This is the same as for James spaces, except for the generality of introducing a Young function. We can now define the required space of functions.

**Definition 4.72.** Let \( \varphi \) be a Young function. The *James-Orlicz space* associated with \( \varphi \) is

\[ J_\varphi = \{ a = (a_n) \in c_0 : S_\varphi(\lambda) < \infty \text{ for some } \lambda \} . \]

**Theorem 4.73.** ([56] Theorem 1) Under the norm

\[ \| a \|_{J_\varphi} = \inf\{ \lambda > 0 : S_\varphi(\lambda) \leq 1 \} , \]

\( J_\varphi \) forms a Banach space.

We wish to verify that these spaces form Banach algebras under pointwise multiplication.

**Lemma 4.74.** For \( a \in J_\varphi \),

\[ \varphi \left( \frac{\| a \|_{\infty}}{\| a \|_{J_\varphi}} \right) \leq 1 . \]

**Proof.** Since \( a \in c_0 \), we know that \( \| a \|_{\infty} \) is attained at some specific entry, say \( a_i \). For any \( \varepsilon > 0 \), by choosing an appropriate two-point set \( \omega = \{ i, j \} \) for a suitably large \( j \), we have that

\[ S_\varphi(\lambda) \geq \varphi \left( \frac{\| a \|_{\infty} - \varepsilon}{\lambda} \right) , \]

and as \( \varphi \) is continuous,

\[ S_\varphi(\lambda) \geq \varphi \left( \frac{\| a \|_{\infty}}{\lambda} \right) . \]

One obtains that

\[ \| a \|_{J_\varphi} \geq \inf \{ \lambda > 0 : \varphi \left( \frac{\| a \|_{\infty}}{\lambda} \right) \leq 1 \} , \]

so by convexity and lower semicontinuity, the result follows. \( \Box \)

We now use the same method as with Lemma 4.59 to establish that for sequences \( x = (x_n), y = (y_n) \) in \( J_\varphi \),

\[ \| xy \|_{J_\varphi} \leq 2 \| x \|_{J_\varphi} \| y \|_{J_\varphi} , \]

confirming that \( J_\varphi \) is a Banach algebra.

We now wish to know when \( J_\varphi \) possesses various types of approximate identities. For the case when \( \varphi \) obeys \( \Delta_2(0) \), the situation is very similar to the Orlicz-Feinstein case. We define a subspace analogous to \( h^\varphi \) from the Orlicz case.
Definition 4.75. Let $\varphi$ be a Young function. Set

$$h^J_\varphi = \{ a \in J_\varphi : S_a(\lambda) < \infty \text{ for all } \lambda > 0 \}.$$ 

Then we have that

Theorem 4.76. Let $\varphi$ be a Young function such that $\varphi$ satisfies $\Delta_2(0)$. Then $J_\varphi = h^J_\varphi$.

This observation is ([56] Proposition 3) (ignoring the typographical error at the end of their statement), and the proof is similar to that for the Orlicz space case in Theorem 4.26. However, in the James-Orlicz case, it is not known if the converse holds.

Proposition 4.77. Suppose that $\varphi$ is a Young function which obeys $\Delta_2(0)$. Then $J_\varphi$ possesses a bounded approximate identity, and hence is approximately amenable.

Proof. The first statement is ([56] Proposition 2). The technique is similar to the way ([38] Proposition 4.a.2) shows that the standard sequence $(\delta_j^{(i)})_{i=1}^{\infty}$ is a basis for $h^\varphi$ in the Orlicz case. It is clear that $\|e_n\|_{J_\varphi} = \frac{1}{\varphi^{-1}(1)}$. The fact that $J_\varphi$ is approximately amenable now follows from ([11] Corollary 3.5). ⪗

Proposition 4.78. Suppose that $\varphi$ is a Young function that fails $\Delta_2(0)$. Then no subsequence of the standard sequence $(e_n)$ is an approximate identity for $J_\varphi$.

Proof. This is the same as Example 4.69, noting that for the given alternating sequence $b = (b_n)$ and its truncations, we have that $\|b\|_{1_\varphi} \leq \|b\|_{J_\varphi}$ since in the definition of $\| \cdot \|_{J_\varphi}$, one maximizes over finite subsets $\omega \subset \mathbb{N}$. ⪗
CHAPTER 4. APPROXIMATE IDENTITIES

\begin{equation}
0 < \epsilon < 1 \Rightarrow \epsilon^2 < \epsilon \Rightarrow \epsilon^2 \to 0
\end{equation}

This is the case we are interested in, except for the trivial case of \( \epsilon = 0 \). We consider the equation of \( \epsilon > 1 \) in Young's inequality. We can have a new line if needed, space, etc.

\textbf{Definition 4.1.1.} Let \( \epsilon > 1 \) be a non-zero number, and let \( \epsilon^2 \to 0 \).

We are interested in the case \( \epsilon > 1 \).

\textbf{Lemma 4.1.4.} For \( \epsilon > 1 \)

\begin{equation}
(0 < \epsilon < 1) \quad \epsilon^2 < \epsilon \quad \epsilon^2 \to 0
\end{equation}

This is the case we are interested in, except for the trivial case of \( \epsilon = 0 \). We consider the equation of \( \epsilon > 1 \) in Young's inequality. We can have a new line if needed, space, etc.

\textbf{Theorem 4.1.5.} If \( \epsilon > 1 \), then...

\textbf{Proof.} By the same method as with Lemma 4.1.4, we obtain that for real numbers \( \epsilon > 1 \), we have

\begin{equation}
0 < \epsilon < 1 \Rightarrow \epsilon^2 < \epsilon \Rightarrow \epsilon^2 \to 0
\end{equation}

As for constants and lines, please note that the result follows.

We now use the same method as with Lemma 4.1.4, to obtain that for real numbers \( \epsilon > 1 \), we have

\begin{equation}
0 < \epsilon < 1 \Rightarrow \epsilon^2 < \epsilon \Rightarrow \epsilon^2 \to 0
\end{equation}

confirming that \( \epsilon^2 \) is a universal upper limit.

We now wish to prove that \( \epsilon^2 \) possesses various types of approximate identities. For the case when \( \epsilon > 1 \), the statement is very similar to the general function case. With this we can...
Bibliography


