Quantum Control to Generate Entanglement of Two Atoms

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A thesis submitted for the degree of Masters of Philosophy in Theoretical Physics of The Australian National University

September, 2008
Declaration

This thesis is an account of research undertaken between February 2006 and September 2008 at The Department of Physics, Faculty of Science, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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September, 2008
Acknowledgements

Thanks are due to my supervisor Joe Hope, for pointing me in the right direction, and telling me when things were too far gone. I also have to thank Andre Carvalho, who provided a great deal of guidance and on whose previous endeavours the original work on entangling two atoms in this thesis builds, and without whose help it would not have gone as well as it did. I thank Matt James for my understanding of quantum control, most of which is founded on talks he gave even when further clarification from other sources was required. Among his students Stuart Wilson also taught me a great deal about quantum control, and in particular the mechanics of applying it. I am also grateful to the other people involved in quantum control at the ANU, conversations with whom deepened my understanding.

I have also benefited greatly from the mental stability provided by my friends and family, and their support has helped me get through this intact.
Abstract

The importance of control theory when applied to systems that are essentially classical suggests that quantum technologies and detailed quantum experiments may require similar levels of control for quantum systems. This introduces several complications, one of the most prominent of which is the effect of measurement back-action on a system being controlled. Techniques including a rigorous description of the behaviour of a quantum system in the framework of control are the subject of quantum control theory.

This thesis begins with a general overview of quantum control and some of the relevant techniques used in latter chapters. Using control to cool a Bose-Einstein condensate is then considered. Classical control is insufficient to the task of resolving questions raised by feedback applied to this system. An optimised control scheme for a single atom approximation of a non-interacting BEC follows. The bulk of the thesis focuses on schemes to generate entanglement between two atoms held in an optical cavity and driven with various lasers. A theoretical model that adiabatically eliminates the cavity from the equations of motion predicts that the two atoms will reach a steady state concurrence (which is a measure of entanglement which has a value between 0 and 1 [1]) of approximately 0.11. Wang, Wiseman, and Milburn showed that by using homodyne detection to measure the light emitted from the cavity and using that to dictate the intensity of an additional control laser it is possible to raise the concurrence to $\approx 0.31$ [2]. Carvalho later showed that switching to a photodetector and using an asymmetric feedback scheme it was possible to reach values of the concurrence close to 1, with the precise value a function of the spontaneous emission in the system [3].

Here are presented results that reintroduce the cavity to the equations of motion, both confirming previous results and allowing an investigation of how the steady state concurrence is affected by the cavity parameters. Also included are a deeper investigation of how the behaviour of the system is affected by spontaneous emission than had appeared elsewhere, and examples of what the behaviour of an individual experiment might look like using stochastic numerical simulations. The numerical simulations that are the basis of this investigation indicate that the optimal cavity with which to perform this experiment is one where the cavity damping is high (i.e. the finesse is low) and the cooperativity is also high, which implies that the cavity coupling should be high and the spontaneous emission low [4]. This is unusual in a quantum optics experiment, normally attempts are made to keep the cavity damping low. The generally superior results achievable using photodetection based feedback are also confirmed.
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Feedback is acting on a system in order to accomplish some goal. Control theory is the mathematics of applying feedback, and of determining how to do so for maximum impact. Control theory provides insight into what mechanisms to use for a particular system, and also how to model those types of feedback in theoretical descriptions. It also provides the mathematical tools needed to optimise the variations of control variables to minimise costs as a system is moved towards some goal.

Whenever repeatable results are expected but conditions vary, control is essential to ensure such results, which means that good control is an essential part of many modern technologies, as well as the fabrication processes that create them. Attempts to apply quantum research to new technologies will require the same reliable results provided by conventional ones, and this will be achievable only if control can be applied correctly to these quantum systems. Scientific experiments often also require feedback control to ensure they are repeatable and to compensate for environmental factors that interfere with the system being studied. Feedback is even more important when high levels of precision are required. In an experiment where very high precision measurements are made via a laser small fluctuations in the frequency of the beam can introduce enough noise to wipe out the result, and it is through control that this can be overcome. This same level of precision is desirable in quantum systems for some experiments.

There are two general types of control. The first is open loop control, where control is applied without measuring the system. This is appropriate in situations where the initial state is predetermined and the purpose of the control is simply to move the system from the initial state to some target state, and it will also work when the initial conditions do not contribute to the final state as a consequence of the control sequence. Closed loop control measures the system and acts on the system in a way that is partially determined by the measurement result. This is the formulation that describes a driver controlling the speed of a car using the accelerator and break, or a system that modifies the cavity length of a laser to stabilise the frequency of the beam when the temperature fluctuates. The use of the measurement can vary from a proportional action on some state variable through so called PID control using the integral and derivative of the signal all the way to full fledged state estimation, where the evolution of the signal is used to calculate an estimate of the state of the system [5]. The most developed state estimation schemes evolve an estimate of the state of the system using the known dynamics and compare these predictions to the measurement results. These filter equations proposed by Kalman [6] are useful when technical noise is a factor since they provide a moderating effect on the response of the control signal to fluctuations in the measurement signal. Which choice is appropriate depends very heavily on the system being controlled.

Applying control to a quantum system introduces two main complications. The first
of these applies to both open and closed loop control, and indeed all quantum simulations, and is the size of the model. Equations of motion for a realistic quantum system have a much higher dimension than classical ones, and since they usually involve elements where a coupling with the environment has been reduced to a non-reversible term in the evolution, such as spontaneous emission, the equations often evolve not a single system but rather an ensemble average state called the density matrix. This means that even in the absence of control seemingly simple systems can involve numbers of dimensions high enough to make analytic solutions implausible and to require numerical resources that are not available. Overcoming this dimensionality problem through approximations or different numerical methods is the subject of much research in quantum theory. One means of doing this is to simulate a particular path the system may take stochastically, and then use that stochastic path to acquire some insight, or perform this process many times and take the ensemble average to recover the density matrix evolution. Two methods along these lines will be introduced in this thesis.

The second complication produced by applying control to a quantum system is that measurement produces back-action. A measurement in a quantum system is modelled by entangling the system with some measurement device that acts as a reservoir, and then taking a partial trace over the measurement reservoir. This will produce an irreversible term in the evolution. Aside from adding a new term to the evolution this means that the measurement is changing the system, and this change is usually detrimental. Careless measurement will make it impossible to achieve the goal of the control scheme. This introduces a trade-off between information gained about the system and damage done to it that is absent from classical models. Accounting for this requires a model to be developed for each measurement strategy, and the control scheme will need to be chosen to counteract the back-action as much as possible. It is possible to model systems neglecting this back-action, however the back-action can have a significant effect on the evolution. Consequently, approximating the back action away will produce results that cannot be relied on when designing an experiment.

Measurement back-action is not always a negative property of a quantum system. The back action can cause pathological states of a system that were stable and unresponsive to control in a model without back action [7] to decay [8], as is the case in a Bose-Einstein condensate presented as an example in Chapter 3. The measurement strength can also be optimised as part of the control strategy. There are a number of techniques referred to as optimal control that allow the calculation of an optimum trajectory along which to vary the control variables to accomplish the control scheme goal with a minimum cost, and the measurement strength can be used as one of these control variables. In a situation where the system is driven towards a stable final state this will end with the measurement switched off, thus ending the damage. The results of applying this to cooling a single atom held in a one dimensional trap can also be found in Chapter 3.

Quantum limits on the information that can be extracted from a system by measurement can also be a problem for some control schemes. Proportional control schemes may not be heavily effected depending on the form of the scheme, and integral schemes will generally be okay, however the quantum fluctuations in a measurement signal will have to be accounted for if the derivative forms part of the control scheme. There is a quantum formulation of filter equations first proposed by Belavkin [9] which reduces the impact of this, since the state estimation elements of a filter help overcome the effects of quantum measurement noise just as the classical filter reduces the impact of technical noise.

Quantum control theory can be used to accomplish valuable control on experiments.
For example, control methods can be used to complete high precision phase measurements with fewer photons and greater accuracy than is otherwise possible. As an example of quantum control followed from theory to an experiment, Wiseman proposed [10] and then in collaboration with Killip [11] developed a description of an experimentally realisable setup that could measure the phase of a single optical pulse significantly more accurately than the previous best methods available. This is accomplished using a homodyne detector which ordinarily can only extract roughly half of the precision possible in the phase quadrature of a pulse with unknown phase. By using low level state estimation techniques and varying the phase of the local oscillator used in the homodyne measurement so the detector would focus on the phase quadrature for most of the duration Wiseman showed it should be possible to achieve much greater accuracy. Mabuchi et al. later demonstrated this experimentally [12], achieving results significantly better using the adaptive scheme than they did using the conventional heterodyne technique in the same apparatus, although not as well as the method in theory allows due to the technical limitations of their experiment. Aside from being an interesting proof of concept, greater accuracy in phase measurements of a pulse with unknown phase could be of significant value in quantum technologies, particularly communication systems. This example is explained in more detail in Chapter 2.

In Chapters 4 and 5 the system which is the focus of my research is presented and investigated. It is shown that controlled laser pulses can be used to entangle two atoms held in an optical cavity [2]. Since entanglement is a property that can only be determined by making use of ensembles of data any attempt to predict the theoretical outcome requires the use of mathematical methods that take quantum averages into account. The model predicts the behaviour of the two atoms, and shows what level of entanglement the ensemble average achieves with differing measurement and feedback schemes. Chapter 4 describes the concurrence, which is a measure of entanglement [1], and how much concurrence can be achieved using homodyne [2] and quantum jump (or photodetection) [3] measurement schemes to drive various types of feedback. These treatments all make use of an approximation that treats the cavity as negligible called an adiabatic elimination [2]. Although this approximation is good for cavities that have a high cavity decay rate, it is not appropriate for the high finesse cavities that are common in quantum optics [13, 14, 15]. As a consequence, simulations using a non-approximate version of the equation are needed, and the results of simulations of this type that I performed are provided in Chapter 5 along with an analysis of the results. In particular it is shown that cavities with a low decay rate are in fact inefficient for this experiment, and a low finesse is necessary to achieve high values for the concurrence [4]. That chapter also includes simulations of individual trajectories in the model using the quantum trajectory method. These simulations give some idea of what might be seen on a detector in one path of an experiment. While the experiment would necessarily be performed many times to measure entanglement itself, these trajectories provide insight into the processes behind the robustness of the entanglement generation.

1.1 Thesis outline

This thesis is essentially in two parts, with the first a general introduction to quantum control theory and how classical control theory can be extended to deal with quantum systems. The second part investigates in detail the specific problem of using control to generate entanglement between two atoms held in an optical cavity, and details how this
can best be achieved.

Chapter 2 introduces some necessary control concepts, and also describes the mathematics of density matrices and master equations, which are used throughout the thesis. Quantum trajectories and continuous conditional master equations are introduced as methods that simulate individual trajectories appropriately for systems using jump and continuous measurement schemes respectively. The chapter ends with a specific example of quantum control theory, an improved method for determining the phase of an optical pulse, with both the theory and an experiment described.

Chapter 3 examines in some detail the problem of cooling a Bose-Einstein condensate using feedback. The semiclassical treatment that begins the chapter suggests the existence of states that cannot be affected by the control scheme used. A full quantum treatment of a simplified model shows that these dark states will be disrupted by the measurement back-action, and are therefore not a concern. Finally, the measurement strength in the fully quantum model is optimised, providing both an introduction to optimal control and an improvement in the temporal dependence of the scheme.

Chapter 4 begins the analysis of generating entanglement between two atoms held in an optical cavity. This chapter introduces the concurrence, which is a measure of entanglement, and a number of existing schemes to generate entanglement described in the literature. The chapter then presents an adiabatic elimination of the cavity from the equations of motion, and covers previous treatments of the feedback problem that work within that adiabatic approximation. Homodyne and jump feedback schemes are compared, the consequences of spontaneous emission are shown, the superiority of a local rather than symmetric feedback operator is demonstrated, and the effect of imperfect detection is discussed. The chapter concludes with a detailed consideration of the effect of detection and an overall comparison between the homodyne and jump schemes when the parameter spaces of detection inefficiency and spontaneous emission are scanned, and this shows that the jump scheme is superior in the region of interest.

Chapter 5 focuses on extending theoretical understanding of the jump feedback scheme. Simulations using a less approximate model of the system where the cavity is not adiabatically eliminated are analysed. The results produced by these numerical simulations show that the regime in which the adiabatic elimination is a valid approximation will produce greater entanglement than most other cavity parameters, and some additional detail of the behaviour of the system with varying parameters is also included. Finally some single path simulations and their implications are presented.
Classical control theory provides the mathematical underpinnings to design feedback schemes to ensure that a system acts as desired. Feedback can be categorised as either closed-loop control, where a system is measured and then controlled, or open-loop control, where some control scheme is imposed on a system in a predetermined fashion to accomplish some result without measuring the system being controlled. Closed-loop control can also be divided into schemes where the state of the system is in some fashion estimated, and ones where the measurement signal at a given time determines the feedback without reference to previous measurement results. These varying forms of control are all well developed for classical systems, however applying them to quantum systems introduces additional complications.

For all control schemes the choice of how best to act on a quantum system being controlled can be difficult, for if the system is going to be in a superposition of two quantum states then the control may move it both towards and away from the goal, or introduce interference effects that are counterproductive. In the case of open-loop control a solid understanding of quantum mechanics and the system being acted on may be sufficient to resolve these questions, and complications involved with measurement back-action can be avoided. In the case of measurement-based feedback control, however, this back-action adds additional complications, as it can have dramatic effects on the behaviour of the feedback scheme. In the theoretical case this requires some method to model the essentially random measurement results and how they change the system. Some mathematical tools to deal with this are described in this chapter, with an example of how control can be applied to a quantum system at the end.

2.1 Classical and semiclassical systems

Before describing the difficulties of applying control theory to a quantum system, we start with a brief description of classical control. For a classical system where knowledge of the system is absolute, adding control to equations of motion for the system is not especially difficult. If the system has a number of state variables $x_i$ and the behaviour of those variables follows a system of equations where $\dot{x}_i = f_i(x_1, \ldots, x_n, t)$, then a control corresponds to the addition of new terms to the equations of motion. The new terms for open loop control may be time varying but will not involve the system variables. Real world systems where this would be a sensible approach are varied, in the quantum world it might be used to model situations where a predetermined sequence of laser pulses will drive an atom with any initial state into a target state. If the state of an atom or molecule needs to be changed from a known initial state it can be accomplished in
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perfect conditions by simple combinations of pulses, however accounting for detuning and intensity fluctuations optimally can result in highly complicated and unintuitive pulse sequences that can nevertheless be proved to be the best means of reaching the goal [16].

Closed-loop control is modelled using control terms that include the state variables, and there are several general ways to do this. The control terms may be dependent only on the present value of one or more of the state variables. Since such a control scheme has no regard for the past behaviour of the system it is Markovian, and since it uses a control signal that is proportional to a measurement result it can be called a proportional control. An actual example of this type of control might feed a measurement result straight back into some parameter of a system, or use the measurement signal to directly trigger some action. This is the way one would model a person using the brakes and accelerator to control the speed of a car, or a physics experiment tuning a laser to keep the frequency of the beam steady when the temperature fluctuates. An alternative to simple proportional control is to take the past behaviour of the state variables into account. A basic version of this is replacing simple proportional control with proportional, integral, and differential (PID) control [5]. Integral control tracks the evolution of the system by applying a control term proportional to the integral of the measurement signal up to the present time. Although slightly more complicated to model, this form of control overcomes a possible problem with proportional control where a variable slowly deviates from a desired target. Control proportional to the derivative of a measurement signal can be used in order to track whether some variable of the system is changing rapidly, and if it is the motion of the state variable can be anticipated by the control. This does not require retaining much information about past values of the signal, however in an experiment there is a risk that noise in the detector will cause rapid shifts that make the control behave badly. Overcoming this requires a certain amount of care. A more detailed dependence on past signal behaviour is making some attempt to determine the present state of the system from the information gathered over the course of the run. State estimation is often necessary in cases where it isn’t possible to determine the initial state of the system being controlled, and frequently involves using the filter equation techniques mentioned briefly later in this chapter.

As an example, if an atom is being held in an optical trap and the control is applied by changing the intensity of the laser providing it then the control would be represented by a time varying element in the trap term in the state equations. An electrical circuit being controlled with a variable resistor would have another term to represent that resistor added to the initial model. The choice of what control to use and how to represent that action in the equations of motion is control theory, and once this is done the problem becomes a mathematical one. It is possible to model a quantum system in an approximate fashion, with measurement not affecting the system and the quantum dynamics are neglected. If this is done then the state variables being evolved behave as if they are classical, and control can be applied in a similar way. However, as will be shown for a particular example in Chapter 3, this can have the disadvantage of failing to capture important information about the behaviour of the system being modelled. In experiments where quantum behaviour not captured in such a model is explicitly sought, or when this sort of approximation fails this is not an adequate way to proceed. In order to model the quantum dynamics of a system density matrices and master equations can be used.
2.2 Density matrices and master equations

Quantum experiments frequently involve large numbers of individual experiments where the interesting property is found in the ensemble average, or classical mixtures of different states. Both of these situations can best be described using a construct called a density matrix. The benefit of the density matrix when considering control problems is greatly enhanced however by it being able also to cope with non-Hamiltonian terms in the evolution, which will be the result of coupling a system to a reservoir of a measurement device. Adding such terms results in a master equation, and this will be the result when measurements are being modelled.

2.2.1 Density matrices

A single quantum system can usually be described by use of a state vector, or ket. For example a two level atom can be described as existing in a 2 dimensional Hilbert space, with a basis vector representing each level \( \{|g\}, \{|e\} \) and a general state being some combination of the two \( |\psi\rangle = a|g\rangle + b|e\rangle \). This formulation can describe any pure state of the atom, and an equation describing the time evolution can usually be constructed. Measuring the atom will produce a result that will have some probability of showing the atom in either of the two levels, with the probability determined by the coefficients. After the measurement the atom will be in one state or the other. When a single atom is being considered, then, it seems that a control scheme could be modelled with only these elements to work with. However, in a quantum experiment there will not generally be only one instance, and the random results of measurement and other loss terms will move a group of pure states into a mixed state.

The superposition of the two basis vectors in the state \( |\psi\rangle \) above is a quantum state. If the atom is measured one state or the other will be chosen as the result. If a large number of atoms all in the state \( |\psi\rangle \) are measured then approximately \( |a|^2 \) will be found to be in state \( |g\rangle \) and \( |b|^2 \) will be found in state \( |e\rangle \). However, this would also be the result of measuring many atoms that were prepared so that that \( |a|^2 \) of them were in the state \( |g\rangle \) and the rest were in state \( |e\rangle \). This is a classical mixture, and the simple state mathematics cannot encapsulate both. The density matrix is a way to describe a quantum state that includes both superpositions and mixtures with a single mathematical construct. Given a distribution of states \( |\psi_i\rangle \) where each has some probability \( P_i \) the density matrix has the form

\[
\hat{\rho} = \sum P_i |\psi_i\rangle \langle \psi_i |. \tag{2.1}
\]

The form of the density matrix makes it clear that it is also an operator, and the evolution of the density operator and how to calculate an expectation value can be worked out from the equations that describe these processes for the state vector. The evolution in the absence of complications from non-Hamiltonian terms is

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] \tag{2.2}
\]

and the expectation value for an observable \( \hat{A} \) is

\[
\langle \hat{A} \rangle = \text{Tr}\{\hat{A} \hat{\rho} \}. \tag{2.3}
\]

When a quantum system is measured it becomes entangled with the measurement device,
which acts as a reservoir. However, the measurement device is not modelled in the dynamics. This means that the larger system-reservoir density matrix including the measurement device is reduced by taking a partial trace over the reservoir. Since the measurement entangled the device and the system this partial trace will change the state of the system, and so the evolution of the system density matrix will includes terms that model this change. These terms will not be in a Hamiltonian form, and so the evolution can not be described by Eq. 2.2. The equation of motion for the density matrix will become a master equation.

### 2.2.2 Master equations

When a non-Hamiltonian term is present in the equation of motion for a density matrix the equation is called a master equation. Any measurement will introduce such a term, since measuring a quantum system will couple it to a reservoir in a non-reversible way. Experimental losses such as spontaneous emission will also have such an effect, and master equations are a common tool throughout quantum physics. These non-Hamiltonian terms can for many systems be represented using superoperators, which are essentially efficient ways of describing common types of terms in the equations. Superoperators combine operators in particular ways so that they may be acting on either side of the density matrix, which is how they are distinct from simple combined operators. Mathematically, in systems with finite dimension where the state vector has dimension \( n \) and the density operator has dimensions \( n \times n \), a superoperator can also be represented with a left multiplying matrix of dimension \( n \times n \), which can be of use when solving these equations numerically. One such superoperator is the Lindblad operator \([17]\), which has the definition

\[
\mathcal{D}[\hat{c}]\hat{\rho} = \hat{c}\rho\hat{c}^\dagger - \mathcal{A}[\hat{c}]\hat{\rho}
\]  

(2.4)

where the component superoperator \( \mathcal{A} \) which also appears independently in some equations is defined as

\[
\mathcal{A}[\hat{c}]\hat{\rho} = \frac{1}{2}(\hat{c}\rho\hat{c}^\dagger + \hat{c}^\dagger\rho\hat{c}).
\]  

(2.5)

The operator \( \hat{c} \) is replaced by the operator appropriate to the process in question. This superoperator describes a general system-reservoir interaction, and so is used for spontaneous emission, experimental losses, dephasing introduced by a measurement, or any number of other interactions. The two parts can be considered to represent the effect of jumps on the evolution through the \( \hat{c}\rho\hat{c}^\dagger \) term, and the evolution of the system interacting with the reservoir but not jumping in the terms in \( \mathcal{A} \). In the spontaneous emission case, the operator \( \hat{c} \) will be replaced by the atomic lowering operator \( |g\rangle\langle e| \) or \( \hat{\sigma} \). Adding this term to the equation of motion for the density matrix for an ensemble of \( i \) atoms which are also undergoing some Hamiltonian evolution results in the equation

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_i \gamma_i \mathcal{D}[\hat{\sigma}_i] \hat{\rho}.
\]  

(2.6)

\( \gamma_i \) is the spontaneous emission coefficient for each atom. There are many techniques for solving master equations, often involving approximations or representations using complex numbers, since problems with more than a few dimensions rapidly become unmanageable. This sort of master equation does not as yet contain any control elements, although in some cases both control and measurement can be described using terms of this form. When
measurements and control terms are added, the methods available to solve the equation will still be valid, although which is most effective may be changed by the additions.

Also of note is that the master equation models the behaviour of an ensemble average, which often is of great use. However, it may also be of interest to get a feel for a number of individual instances of an experiment, particularly if an experimentalist wants to see what sort of measurement signal an apparatus can be expected to provide. Also, if it is necessary to simulate a master equation numerically it may be the case that none of the methods that preserve the ensemble average character of the master equation can be implemented in such a way that the calculations complete in an acceptable time frame. In these cases it can be possible to simulate a large number of single paths and take the average to find an approximate version of the ensemble. The appropriate method to use this partially depend on the character of the measurements involved. When all the non Hamiltonian evolution involves quantum jumps, the evolution can be performed using quantum trajectories.

### 2.2.3 Quantum trajectories

Quantum trajectories or MonteCarlo wavefunctions are one method of dealing with a master equation that avoids simulating an entire density matrix numerically by evolving a particular wavefunction that represents one possible evolution of the initial state \[18, 19\]. This can be of significant value in cases where a system density matrix is highly complex and has a high dimension, since for an \(N\) dimensional system a quantum trajectory requires evolution of a vector of \(N\) elements whereas the density matrix is of size \(N^2\). In order to duplicate the full master equation results many paths must be evolved and then averaged, of course, which reduces the numerical benefit. Evolving many paths, however, is a task that can be distributed across many different processors, and is thus much better suited to numerical simulation on a computer cluster than the master equation itself. The trajectories method will also be beneficial for very large \(N\), since it may be possible to get acceptable results using fewer than \(N\) trajectories. The single trajectories are also of analytic interest, since they show a possible path that the system may take, thereby providing insight into the dynamics of quantum systems. The essence of the method is that at each timestep of a numerical simulation the choice is made to evolve the wavefunction using either the Schrodinger equation combined with a non-Hermitian decay term and then re-normalise, or some quantum jump appropriate to a measurement (or any other probabilistic element in the equation, e.g., spontaneous emission) where the system is placed in a state matched to the measurement outcome. For example, an atom in a cavity in the excited state will either be slowly decaying to the ground state, or will emit a photon and jump straight to the ground state.

In more detail, this method functions for equations of the form

\[
\frac{d\rho}{dt} = \frac{1}{i\hbar}[H, \rho] + \mathcal{L}_{\text{relax}}[\rho], \tag{2.7}
\]

with the relaxation operator expressible as a sum of superoperators of Lindblad form

\[
\mathcal{L}_{\text{relax}}[\rho] = \sum_{m} D[C_m]\rho, \tag{2.8}
\]

where \(C_m\) has dimensions of \(\sqrt{1/\text{time}}\). As mentioned above the Lindblad form covers a general system-reservoir interaction and this is therefore not a stringent condition on when
this method can be used. The model which is the basis for the feedback schemes that are the focus of this thesis is compatible with this technique, and use will be made of it in Chapter 5. If the Lindblad form condition is met, then the evolution may be split into the jump and non-jump parts, with the effective non-Hermitian Hamiltonian taking the form

\[ \hat{H} = H - \frac{i\hbar}{2} \sum_m C_m^\dagger C_m. \]  

(2.9)

The non-Hermitian part of this effective Hamiltonian is precisely the portion of the Lindblad operator that describes evolution due to non-jump interactions with a reservoir. After a small timestep \( \delta t \) a state \( \langle \phi(t) \rangle \) will have evolved to become

\[ \langle \phi(t + \delta t) \rangle = (1 + \frac{1}{i\hbar} \hat{H} \delta t) \langle \phi(t) \rangle \]  

(2.10)

which will have a norm squared of \( 1 - \delta p \). With a timestep small enough to make a first order calculation valid the reduction in the norm is

\[ \delta p = \sum_m \delta p_m = \delta t \sum_m \langle \phi(t) | C_m^\dagger C_m | \phi(t) \rangle. \]  

(2.11)

This value is also the probability of having some jump, which means that it is clear what the state of the system is after the timestep \( \delta t \), either there was no jump (probability \( 1 - \delta p \)), which gives the state

\[ \langle \phi(t + \delta t) \rangle = \frac{|\phi(t + \delta t)\rangle}{\sqrt{1 - \delta p}} \]  

(2.12)

or one of the possible jumps has occurred, so with probability \( \delta p_m \) the state is

\[ \langle \phi(t + \delta t) \rangle = \frac{C_m \langle \phi(t) \rangle}{\sqrt{\delta p_m / \delta p}}. \]  

(2.13)

which includes the effect of the jump portion of the Lindblad operator. By using these possible outcomes and a random number at each timestep it is possible to evolve the state. Taking the average of many such paths can be proved to recover the results of the full master equation [19, 20].

A simple way to implement a quantum trajectory method numerically is to refrain from renormalising the state when the evolution does not include a jump. This will lead to the norm decaying at a rate proportional to the decay terms in the equation, as can be seen from the description of the new norm after each timestep. A random number is chosen after each jump, and when the norm has reached that random number a jump is applied to the state. This has the advantage of not requiring a random number to be generated at each timestep, which is more numerically efficient. It also allows easy implementation of new jump types in a pre-existing simulation, since the addition of a new possible jump will change the probability of there being some jump in a way that will correspond to the new rate of decay of the norm as a consequence of introducing the new decay term. Quantum trajectories are used for simulations in Chapter 5 that evolve a single quantum system, and it is a simulation of this form that was used to generate those results.
2.2.4 Stochastic calculus and conditional master equations

The quantum trajectory method described above produces a conditional master equation that models processes that involve sharp discontinuities in the evolution of a system. Some processes that need to be modelled are continuous in character, one example being a homodyne measurement on a laser, and a different method should be used to account for them. When constructed using a white noise stochastic element a conditional master equation can include these processes. This construction also has the advantage that when formulated in an appropriate way taking the ensemble average of the conditional master equation is straightforward, and the result is the non-stochastic master equation for the system. Performing calculations that involve white noise terms introduces some additional complications, however, and to deal with them correctly stochastic calculus must be used.

Stochastic calculus

Stochastic calculus is used in classical control theory in order to model systems that include random elements. In particular, it can account for technical noise in a measurement, or factors in a classical system that cannot be properly accounted for ahead of time. It is also of great benefit when a classical control theorist wishes to apply control to a system where they cannot possess all of the information they would ideally have available to determine how to act on the system. These elements of ignorance produce systems that behave in a manner reminiscent of a quantum system, although the random measurements results follow a classical rather than a quantum probability distribution. This can easily become necessary in a real world application, since technical noise is frequently unavoidable. Stochastic calculus itself is a method of accounting for random elements by using stochastic variables, which have associated with them a probability distribution. The probability distribution is often either a Gaussian or Poisson distribution, since these represent the more common types of noise. Adding these stochastic variables to partial differential equations produces stochastic PDEs.

Unsurprisingly dealing with these stochastic PDEs is more complicated than normal partial differential equations, in particular in the limit where the noise terms are white noise they are not differentiable. There are however techniques that can be used. It is possible to generate possible trajectories for stochastic calculus problems numerically, and it is also possible to perform calculations of expectation values, where taking an ensemble average over many trajectories removes the stochastic elements from the equations. This allows an understanding of the average behaviour for the system which determines how the control variables are used. Non-Markovian control schemes will also require some means of determining what the likely state of the system is given the measurement results provided by the system, which may be difficult to calculate in real time. For Markovian feedback the introduction of noise will reduce the effectiveness of the control, and may run the risk of moving the system away from the desired state.

Stochastic calculus provides methods for dealing with differential or integral equations that include stochastic variables. There are two alternative formulations, called Ito and Stratonovich [21], with each having advantages and disadvantages in their applications. The difference between them is a consequence of the ambiguity in the definition of the integral of a random variable. A general stochastic differential equation (SDE) has the form

\[ dX_t = A(X_t)dt + B(X_t)dW_t. \]  

(2.14)
In this equation $A(X_t)$ and $B(X_t)$ are functions of the stochastic variable and $dW_t$ is the Wiener increment $dW_t = W_{t+dt} - W_t$ which is equivalent to a Gaussian white noise. The integral form of this equation is

$$X_t = X_0 + \int_0^t dt' A(X_{t'}) + \int_0^t B(X_{t'})dW_{t'}.$$ \hspace{1cm} (2.15)

The ambiguity is generated by the presence of the random variable in the second integral term, which is often the case in quantum optics applications of stochastic PDEs. If the term involving the Wiener increment did not also involve the random variable it would not be a concern. To see the ambiguity, consider a discrete time approximation for the equation

$$I = \int_{t_0}^t F_t' dW_{t'},$$ \hspace{1cm} (2.16)

with $F_t$ some stochastic quantity. The discrete time approximation divides the interval into subintervals with timestep $\Delta t$, and within these intervals the time $\tau_k = t_k + \alpha \Delta t$ is a fraction $\alpha$ of the time step from the lower bound. The discrete time version of the integral is the sum

$$I_t = \sum_{k=0}^{i-1} F_{\tau_k} (W_{t_{k+1}} - W_{t_k}),$$ \hspace{1cm} (2.17)

recalling the definition of the Wiener increment above. The stochastic quantity $F_t$ may be chosen as $W_t$, which allows the average value of the integral to be calculated, and it can be shown to be $\langle I_t \rangle = \alpha (t - t_0)$ [21]. The choice of $\alpha$ introduces the ambiguity, and it represents where in the interval the stochastic quantity $F_t$ is evaluated. Of the two common versions of quantum stochastic calculus the Ito version chooses $\alpha = 0$, and thus evaluates $F_t$ at the beginning of the interval. The Stratonovich version chooses $\alpha = 1/2$, which means the evaluation of $F_t$ is at the midpoint of the interval. The two interpretations result in different meanings for Eq. 2.15. The Stratonovich integral leads to equations that obey the conventional rules of calculus, which has the benefit of familiarity, and would seem like the superior choice to work with. However, the Ito calculus despite the new rules has the benefit of leaving $B(X_t)$ and $dW_t$ statistically independent for $t' \geq t$, which has the effect of tremendously simplifying calculations. In addition, in cases where a stochastic term representing some measurement or feedback action has been constructed phenomenologically it may not be immediately clear which of these two interpretations is correct, and using the wrong form can lead to equations producing unphysical results. It is possible to convert an equation or term of one form into the other by applying a conversion formula which will generate a correction term [22], and this conversion changes the Stratonovich SDE

$$dx = \alpha dt + \beta dW(t)$$ \hspace{1cm} (2.18)

into the Ito SDE

$$dx = [\alpha + \frac{1}{2} \beta \partial_x \beta] dt + \beta dW(t),$$ \hspace{1cm} (2.19)

with an inverse conversion also possible. Using either stochastic calculus it is possible to create a master equation that includes the particular random output of some measurement device in the evolution, and this is a conditional master equation.
Conditional Master equations

A conditional master equation is a stochastic PDE that describes the evolution of a system replacing the average decoherence and measurement terms in a master equation with stochastic variables that track measurement results and noise. Naturally, if the ensemble average of this equation is taken it must return the master equation for the system, or it isn’t correct. The Lindblad superoperator has a known stochastic form. As mentioned above, working with such an equation will seldom involve analytic solutions, since the particular measurements are stochastic, and therefore probably dependent on numerical solutions to solve. It may however be easier to describe the action of the control or measurement analytically in stochastic form and take the ensemble average to find a term for the full master equation, as in the homodyne case described below. As with the quantum trajectories method described above the continuous conditional master equation may also be used to simulate many different paths and take an average to provide numerical results that approximate the behaviour of the master equation, since this may be the only numerically tractable option, or simpler than simulating the master equation itself. Also, again as with the quantum trajectory method, individual paths may be interesting in their own right, since they provide valuable information about how an experiment might be expected to proceed. This is of particular importance when feedback is being applied, since it will give some indication of how the system responds to particular actions taken by the feedback scheme. This information may suggest ways to improve the feedback that are not immediately apparent from ensemble average results. In a case where numerous experiments that provide ensemble average results are sequential rather than concurrent, this can give the experimentalist some idea of what is happening in a particular instance given the visible output.

In a situation where the system being controlled cannot be completely determined externally, as can be the case when the measurement is noisy, it is possible to use a technique called filtering to calculate what control to use. This is of particular importance in the quantum case, but as was mentioned in Chapter 1 there are classical systems where the internal behaviour of the system or the initial conditions cannot be known or are clouded by noise, and it is classical systems of this type for which filter equations were first formulated by Kalman [6]. The filtering technique begins with a guess of the initial condition of the system under consideration and then uses the measurement signal and control to bring the state described by the filter and the state of the system together [23]. A quantum formulation of a filter equation due to Belavkin [24] can allow a control scheme where the initial state of the system is not crucial to the problem. Indeed any form of state estimation will have this effect, and there are alternatives to the full filter method for some systems. This also takes past measurements into account while applying feedback, which allows non-Markovian feedback schemes that are of benefit for some systems. A filter equation also provides a convenient mathematical framework for dealing with the inevitable uncertainty in conjugate quantum variables. Since the resulting equations are stochastic PDEs, solving them requires the same combinations of numerical techniques as any other conditional master equation. Filter equations can answer a number of different questions, and an example of a case where a filtering equation can be used is highlighted in Chapter 3.
2.2.5 Conditional master equation for homodyne detection

When the effect of some measurement in stochastic form has been determined it can be included in a conditional master equation including other terms modelling a system. Taking the ensemble average of the resulting equation will provide a master equation that includes the correct terms describing the measurement. Here is an example of this used by Wiseman and Milburn to calculate the form of the feedback and measurement terms for a homodyne detector receiving input from an optical cavity [25]. An optical cavity with damping due to coupling to an external bath through the mirrors can be represented by the master equation

$$\dot{\rho} = \mathcal{D}[a] \rho,$$  \hspace{1cm} (2.20)

where $\mathcal{D}$ is the Lindblad superoperator described in Section 2.2.2, and $a$ is the annihilation operator for the cavity mode. Photons escaping from the cavity enter a thermal bath that is not modelled, and the trace over that bath results in the Lindblad term. The conditional master equation form for a case where photons released from the cavity are detected provides some knowledge of the state of the bath, and this leads to a conditional master equation. A photodetector would introduce jumps in the state of the conditioned density matrix. A homodyne measurement produces a photocurrent, which if the $x$ quadrature is measured will take the form [25]

$$I_c(t) = \eta (a + a^\dagger) c(t) + \sqrt{\eta} \xi(t),$$  \hspace{1cm} (2.21)

with $\eta$ the quantum efficiency of the detector and $\xi(t)$ the noise introduced by the local oscillator, which is related to the previously described $dW(t)$ noise terms by $\xi(t) = dW(t)/dt$. With this photocurrent the equation for the conditioned density matrix is

$$\dot{\rho}_c = \mathcal{D}[a] \rho_c + \sqrt{\eta} \xi(t) \mathcal{H} \rho_c$$  \hspace{1cm} (2.22)

where $\mathcal{H}$ is defined as

$$\mathcal{H} \rho = a \rho + \rho a^\dagger - \text{Tr}(a \rho + \rho a^\dagger) \rho.$$  \hspace{1cm} (2.23)

This equation uses the Ito formulation of stochastic calculus, and taking the ensemble average of this equation would lead to the $\xi$ term vanishing, since the noise will have zero mean. This recovers the original equation, as would be expected.

Using an operator for feedback that is linearly proportional to the homodyne photocurrent is a logical choice, and so the feedback term for the conditional master equation will be

$$[\dot{\rho}_c]_b = \langle a + a^\dagger \rangle_c(t) + \xi(t)/\sqrt{\eta} \mathcal{K} \rho_c,$$  \hspace{1cm} (2.24)

with $\mathcal{K}$ being some superoperator that describes the feedback. However, it is not clear whether this is an Ito or Stratonovich form term. The decision as to which it is comes when the consequences of adding it to equation 2.22 are investigated, and it is possible to determine that it must be interpreted as Stratonovich to produce a physical result. Using the rules for converting between the two forms of stochastic calculus equation 2.22 can be converted to Stratonovich form and the feedback term added to produce the equation

$$\dot{\rho}_c = \mathcal{D}[a] \rho_c + \left[ -\frac{1}{2} \eta \mathcal{H}^2 + \langle a + a^\dagger \rangle_c(t) \mathcal{K} + \sqrt{\eta} \xi(t) \langle \mathcal{H} + \frac{1}{\eta} \mathcal{K}^2 \rangle \right] \rho_c.$$  \hspace{1cm} (2.25)

However, this equation isn’t as helpful for determining the ensemble average, since it is in
Stratonovich form, so it is worthwhile to convert it back into Ito form, where the ensemble average will be equivalent to the noise term $\xi(t)$ vanishing. Performing this conversion and substituting in the definition of $\mathcal{H}$ in some places to simplify results in the equation

$$\dot{\rho}_c = \mathcal{D}[\alpha] \rho_c + \mathcal{K}(\alpha \rho_c + \rho_c \alpha^\dagger) + \frac{1}{2\eta} \mathcal{K}^2 \rho_c + \sqrt{\eta} \xi(t)/(\mathcal{H} + \frac{1}{\eta}) \rho_c. \quad (2.26)$$

This result only occurs if the superoperator ordering $\mathcal{K} \mathcal{H}$ is used, which Wiseman and Milburn justify by invoking a time delay between the measurement contained in $\mathcal{H}$ and the feedback applied by $\mathcal{K}$, which is then set to 0. From the mathematical perspective this ordering is necessary to ensure that the master equation to be calculated will be trace preserving. In any case, from this Ito form equation the ensemble average and thus the non-conditional master equation can be easily calculated, and has the form

$$\dot{\rho} = \mathcal{D}[\alpha] \rho + \mathcal{K}(\alpha \rho + \rho \alpha^\dagger) + \frac{1}{2\eta} \mathcal{K}^2 \rho. \quad (2.27)$$

This equation is now linear, which is of importance since there are probability theory arguments that require this property, and it is to ensure this that the choice to treat the feedback term above as Stratonovich is made. Treating the feedback term as being of Ito form results in an eventual master equation that is not linear.

The two new terms in the equation provide the two expected effects of a quantum control scheme. The first is the feedback action that will be used to accomplish the desired goal, and the second is a diffusion term that adds the noise to the system caused by the measurement, and as expected as the quantum efficiency of the detector is reduced the damage done increases. This equation is extremely general, and indeed extra terms in the Liouville superoperator obeyed by the cavity will not change the process that derived the homodyne terms, so they can be applied for any case where a homodyne signal from a cavity is driving feedback linearly proportional to the photocurrent. This allows applications to a number of experiments in quantum atom optics.

### 2.3 Quantum control

With the mathematical framework for dealing with quantum systems in place it is possible to consider how the application of control will work. There are two general ways to do this. The first works in the ensemble average picture represented by the master equation. Modelling this way means the first point to consider is the effect of the measurement on the evolution, or put another way, what the new terms in the master equation will be, the effect of a measurement and the measurement signal may be calculated, perhaps using a method similar to that used in the case of homodyne detection above, or it may be constructed. The control terms may be obvious results of this process, or it may be that some identifiable term in the equation is being modified over time. Obvious candidates for this include an externally applied potential, as in an atom trap, or the strength of a measurement signal. The precise forms of those operators inevitably vary, and the state of research into these variations for a particular system will be driven by how interesting it is to quantum control theorists. Detailing the broad range of possible control terms is outside the scope of this discussion, however, an example that indicates how quantum control can be applied and the results that can be achieved is provided below. This is an application of control to an experimental system to improve a phase measurement.
2.3.1 Adaptive single shot phase measurement

An example of how quantum control theory can be used is an experimental realisation of quantum control that has been implemented, and that showed improved results over what could be achieved without the control. The objective was to improve an optical phase measurement with a scheme using an adaptive homodyne detection. This was first proposed by Wiseman [10], refined in collaboration with Killip [11], and performed experimentally by Mabuchi et al. [12]. The objective is to measure the phase of a single mode optical pulse, which has applications in communications. Prior to the method proposed by Wiseman all methods available gave equivalent results, however these results are far from the theoretically achievable limit. One standard method was to use heterodyne detection, mixing the signal pulse with a far detuned local oscillator at a beam splitter and recording both outputs. This provides information about both the phase and intensity quadratures of the signal pulse, and the intensity is discarded. As Wiseman points out in the first paper proposing an alternative [10], this means the measurement is necessarily not providing as much information about the phase as could be obtained. The proposed superior method is to use a Homodyne measurement where the estimated phase of the single signal pulse based on the information collected at a given time is used to control the phase of the local oscillator. If the local oscillator is adjusted to keep the phase in quadrature for the duration of the measurement then in principle the precision of the phase measurement could reach the theoretical maximum, although doing that well is probably not possible due to experimental constraints.

The general idea proposed in Wiseman’s first paper provides numerically superior results for some cases, and is proved analytically to be superior for a special case, however the particular scheme used in the experiment is found in the paper Wiseman wrote with Killip. This paper provides a full quantum treatment of the problem, and proposes two specific schemes for how to perform such a measurement. If the phase of a system is already known a homodyne measurement can exclusively measure the phase quadrature, providing as much information as the quantum limits permit about the system. In this case, however, the phase of the system is not known, and so the idea is to estimate the phase from the total measurement outcome and to adjust the local oscillator so that the phase quadrature would be measured for a system that had the phase that is the best estimate of the pulse phase at that time. Mathematically, the local oscillator phase is adjusted so that

$$\Phi(\nu) = \hat{\phi}(\nu) + \frac{\pi}{2},$$

(2.28)

where $\hat{\phi}$ is the estimate of the system phase of the system at time $\nu$. The estimated phase is naturally some function of the homodyne photocurrent, and it is chosen as

$$\hat{\phi}(\nu) = \arg A_{\nu},$$

(2.29)

where $A_{\nu}$ is a complex number that includes the time integral of the homodyne photocurrent. $A$ is given by

$$A_{\nu} = \int_0^\nu I(\mu)e^{i\Phi(\mu)}d\mu.$$  

(2.30)

This information is sufficient to design an experiment to use an adaptive phase measurement, however what to use as the final phase of the system is also a question of note, and one that is addressed. One choice is to treat the final phase estimate of the system as simply the estimated phase at the final time, however, it is shown that if there are
more than a few photons in the pulse this is not the best way to go. A better final phase estimate is obtained using what is designated the mark II adaptive measurement, where the final phase is a combination of $\phi(\nu_{\text{final}})$ and a function of a second complex number that provides sufficient statistics for the homodyne measurement in combination with $A_\nu$. The full function gives the final phase estimate as

$$\Phi_{\text{II}} = \hat{\phi} + \arg(1 + C) \quad (2.31)$$

where $C_\nu$ is given by

$$C_\nu = e^{-2i\phi(\nu)} \int_0^\nu e^{2i\phi(\mu)} d\mu. \quad (2.32)$$

Wiseman and Killip show that this scheme provides a better final phase estimate than the heterodyne scheme, and a better estimate than the mark I version except in the one photon case, and it is this scheme that is implemented in the experiment of Mabuchi et al. Technical proof of why this is so requires more detail than is practical here, as well as a deeper understanding of the mechanics of quantum stochastic calculus than could reasonably be provided in this thesis, but it is available in the two papers cited.

In the experiment described in [12] the variances in phase estimates from the adaptive homodyne measurement are compared to those from heterodyne measurements using the same apparatus. Figure 2.1, reproduced from the paper, shows these results. In the figure it is clear that the adaptive homodyne variances are smaller than those of the heterodyne case for most of the plot, especially since the plot is logarithmic. In general for intermediate photon numbers the adaptive measurement is better than the best values achievable with the heterodyne scheme when accounting for electronic noise, and for some values it is better than the absolute best possible heterodyne result. Of course, the variances are well short of the quantum limit. It is noted that the performance of the scheme was limited by feedback bandwidth, which is one reason why the results fall short of the theoretically predicted results. The points on the plot are averages of the variance of multiple single shot experiments within ensembles of fixed phase to estimate the variance, although multiple ensembles with different phases are used for each point. The paper also includes data to show that the performance of the adaptive measurement does not depend on the phase of the signal.

Aside from an idea of how the theoretical descriptions of quantum control theory can assist with actual experiments and measurements, this is a concrete example of control theory applied to an intrinsically quantum system, and has the added benefit of including a simple state estimation element. It is in experiments of this kind that the value of quantum control theory becomes clear, since superior phase estimations on an optical pulse can be of practical value in communications, as well as in experiments dealing with subtle changes in phases of such pulses. In addition, the theoretical model from which the new method was derived depends on the quantum properties of the system, in particular the behaviour of quadrature measurement, and therefore it would be impossible to achieve this improvement using only classical control theory.

2.4 Final comments

The purpose of this chapter has been to introduce some of the mathematical tools of classical control theory, which are mostly well established, and the additional layer of complexity that is added when the problems and systems being considered are intrinsically
Figure 2.1: Plot showing experimental results from [12], variance in phase measurements against number of photons in a single signal pulse. Adaptive homodyne results are the blue crosses, heterodyne results are red circles. The thin lines are theoretical curves for heterodyne with (solid) and without (dotted) corrections for electronic noise in the detector. The agreement between the solid line and the heterodyne results verifies that the calibration of the experiment is correct. The blue dashed line is a second order fit to the adaptive data, and the thick line is the uncertainty limit given the photodetection efficiency of the experiment. As mentioned in the text, it is clear that the adaptive homodyne measurement performs significantly better than the heterodyne for intermediate photon values, although the uncertainty limit is still significantly lower.
quantum in nature. Also introduced are several methods to model quantum systems, both the ensemble average master equation and a pair of probabilistic methods to simulating individual paths, one jump based and the other continuous. All of these methods can account for the effect of measurement on a quantum system, which is the major difference between classical and quantum control problems. The significant differences in behaviour that are seen when this back-action is correctly modelled can be seen in the case of controlling a Bose-Einstein condensate, which is introduced in the next chapter.
In chapter 2 several mathematical techniques involved in control theory were presented along with some examples. In this chapter a detailed investigation is provided of a particular problem. This example is cooling a continuously pumped Bose-Einstein Condensate, or BEC. A semiclassical consideration of controlling a BEC suggests that there are problem states that could not be successfully cooled, and these treatments do not include the back-action of the measurement. Conveniently, addressing the lack of back-action also resolves the dark state problem. The chapter ends with an unpublished application of optimal control to the measurement strength in an attempt to find the best way to vary it over the course of an experiment. Numerical results provide how the measurement should vary, however it was not possible to derive an analytic solution.

3.1 BECs and atom lasers

A Bose-Einstein condensate as discussed in quantum atom optics is a dilute gas of atoms that has been cooled to temperatures low enough that the atoms in the gas adopt a collective quantum state. First experimentally realised in the mid 90’s [26, 27, 28], there are now labs around the world using them for various experiments. One potentially valuable path being pursued is that of the atom laser [29]. The atoms in a BEC must be held in a trap which is usually magnetic for atom laser experiments. Magnetic traps only trap atoms in one or two of several internal states of the atom, so it is possible to outcouple condensed atoms from the trap by changing their internal state. This can be accomplished in several ways [30, 31, 32], and this can produce a beam that has spatial and temporal coherence properties [33, 34]. In principle this atom laser beam can be used in a variety of applications, since it shares several of the properties of a laser while adding interesting new ones, including interaction within the beam and with environmental effects to which photons are not sensitive. In order to make use of these properties a high flux, low linewidth beam is desirable, as in the case of the optical laser. Achieving both these properties is possible in the same way that it is for an optical laser, by ensuring competition between a pumping mechanism and damping in the BEC that will result in gain narrowing [33]. When it becomes possible to create a continuously pumped atom laser in a lab a theoretical examination of how a pumped and outcoupled BEC might behave and the linewidth that can be achieved will be necessary.

Modelling a BEC as a quantum field requires an intractably large multimode analysis, since the BEC is in a trap with many energy levels and the atoms that form it generally possess internal degrees of freedom that are of crucial importance to the trap and laser modes. It is possible to perform a multimode analysis using semiclassical techniques such
as the Gross-Pitaevskii equation [35, 36], however. These methods can provide insight into the spatial structure of an atom laser beam and whether the beam will move towards a single mode, but they cannot provide the linewidth. This is because the linewidth of an atom laser is governed by the quantum statistics of the mode that are ignored by the semiclassical approximation, which in turn dictates that a quantum analysis is necessary [37]. Calculations of this type are usually difficult, however versions with a few modes have been published [38]. In order to generate a narrow linewidth beam it is necessary to ensure that the beam is in a single mode, and that the BEC from which it is produced is in the lowest energy state possible. It has been shown that this can be accomplished by using spatially invariant pumping of a BEC with a high interatomic interaction strength [39]. It can also be accomplished using pumping that favours the centre of the trap, since low frequency modes have greater overlap with that central region, and are therefore preferentially pumped [40]. The difficulty with these methods is that high interatomic interactions will lead to phase diffusion in the atom laser beam [41, 42], which will destroy the narrow linewidth. The motivation to explore applying control to the BEC is thus to produce a similarly low energy BEC without this difficulty, using feedback to actively cool the BEC rather than trying to pump only low energy modes.

3.2 A semiclassical treatment

Several papers provide semiclassical investigations of the results of applying control to BECs and atom lasers [39, 43], with the most complete treatment based on the Gross-Pitaevskii equation performed by Johnsson, Haine, and Hope [7], which built on those previous papers. The model is restricted to two modes, the trapped $\psi_t(x)$ and untrapped $\psi_u(x)$ modes. The trapped mode is driven by an incoherent reservoir of atoms the density of which is described by $n(x)$ with coupling to the trapped mode of $\kappa(x)$. Using only a single dimension condensate to simplify the model the dynamical equations before adding feedback take the form

$$i\hbar \frac{\partial \psi_t}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_t - i\hbar \gamma_t^{(1)} + (U_{tt} - i\hbar \gamma_t^{(2)}) |\psi_t|^2 + (U_{tu} - i\hbar \gamma_t^{(2)}) |\psi_u|^2 \right] \psi_t + \frac{i\hbar}{2} \kappa(x)n(x) \psi_t + \kappa_{out}(x) e^{ikx} \psi_u,$$

$$i\hbar \frac{\partial \psi_u}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + mgx + (U_{uu} - i\hbar \gamma_u^{(2)}) |\psi_u|^2 + (U_{tu} - i\hbar \gamma_t^{(2)}) |\psi_t|^2 \right] \psi_u + \kappa_{out}(x) e^{-ikx} \psi_t,$$

$$\frac{dn}{dt} = r - \gamma_p n(x) - \kappa(x)|\psi_t|^2 n(x) + \lambda \nabla^2 n(x).$$

The trapping potential is $V_t$, $m$ is the atomic mass, $g$ is the acceleration due to gravity with the $x$ direction defined so that $g$ is in the negative $x$ direction, and the interatomic interactions are $U_{ij} = 4\pi \hbar^2 a_{ij}/m$ with $a_{ij}$ the s-wave scattering length between the internal atomic states. $\gamma_t^{(1)}$ is the loss rate due to background gas collisions for mode $\psi_t$, $\gamma_t^{(2)}$ is the two body inelastic collision loss rate within the state and $\gamma_{tu}^{(2)}$ is that between the modes. The coupling rate between the trapped and untrapped modes is $\kappa_{out}$, the momentum kick applied by the process is $k$, $r$ is the rate of density increase of the incoherent atom cloud, $\gamma_p$ is the loss rate of that cloud, and $\lambda$ is the spatial diffusion coefficient. In order to allow spatially dependent pumping the coupling between the reservoir and the trapped
state is chosen as $\kappa(x) = \kappa_0 e^{-x^2/\sigma^2}$. The pumping terms in these equations were chosen phenomenologically.

Control is provided by assuming that it is possible to dynamically modify the trap parameters, specifically the strength and position of the minimum, and the nonlinear interaction strength of the atoms. This means that there is a manipulatable external potential $V_{f_0}^1 = a_1(t) x + a_2(t) x^2$ and an interaction strength of the form $V_{f_0}^2 = b(t) |\psi_t|^2$. The mechanics of moving the trap position and strength is fairly clear, and the nonlinear interaction strength can be modified for some atomic species by changing a magnetic field close to a Fesbach resonance [44]. The measurements used to drive the control are $<x>, <x^2>, <|\psi_t|^2>$, and in the absence of pumping and losses from the condensate an adequate choice of the functions $a_1, a_2,$ and $b$ will guarantee a reduction of energy in the condensate. Two things that this treatment does not account for are the time delay introduced between measurement and action, and the effects of the detection scheme itself. The time delay could be important, since the motion of a BEC in a trap is oscillatory, and if the delay is too great the feedback may not reduce the oscillations but rather increase them, acting as a positive feedback. However, the timescales involved suggest that this is unlikely to be a problem given realistic parameters. A greater concern is the lack of back-action from the measurement. That problem is addressed in the subsequent work discussed later in this chapter.

When feedback is applied to a condensate without pumping or loss, the feedback terms above can be shown to result in the energy either decreasing or remaining constant. In the crucial case, with pumping and outcoupling to the untrapped, or laser, state, there are additional contributions to the energy change due to the loss and pumping terms, and these are independent of the feedback. The feedback terms are the strictly nonpositive (given the choice of feedback constants) term that guaranteed energy reduction in the simple case, and also an interaction between loss and feedback that can be of varying sign depending on the trapped and untrapped wavefunctions. Several of the pumping and loss terms can also be positive, depending on the values of the pumping, outcoupling, loss, and the form of the wavefunctions. When the pumping, outcoupling, and loss are low the control will generally do well, however it can no longer be guaranteed to remove energy. Simulations show that the control can provide effective stabilisation in some regimes where the BEC would not be stable without it, and the stable lifetime can also be significantly extended. The results suggest that it should be possible to stabilise such a continuously pumped and outcoupled BEC, especially if the laser only needs to remain stable for a short period of time. There are, however, a few concerns. In general an unstable system will not be made stable, but rather will have the period of time before the instabilities reach a destructive magnitude extended, and so absolute stability is not available. In addition, it is possible for pathological modes in the BEC to become populated which will not respond to the feedback at all, given the possibly positive terms in the equation for the change in energy. The authors refer to these uncontrolled modes as dark states. There is also still no resolution to the conflict between the desire to reduce the nonlinear interactions as close to zero as possible to reduce phase diffusion in the atom laser produced. Resolving these problems in the semiclassical model would prove difficult, but fortunately, as has been discovered, the inclusion of measurement back-action actually increases the stability of the model.
3.3 A treatment including measurement back-action

The results in the previous section suggested the existence of excited dark states in a BEC that did not respond to the control. These dark states involve excitations that leave the moments of the BEC that are controlled stable, but can have widely varying behaviour of higher order moments. This would be even more of a problem without atomic interactions, since in the presence of interactions the dark states are unstable. This is partially a consequence of the absence of measurement back-action in the semiclassical treatment of the system. The dark states are affected by the back-action and become unstable, which solves the problem.

Further investigation of the dark states from the semiclassical treatment requires a quantum model of the BEC measurement and cooling system, which in general would require equations that would be very difficult to treat numerically or analytically. However, because the non-interacting, and thus linear, case would produce an atom laser with the best linewidth after leaving the condensate, if a feedback scheme can work for this case that is enough, and it is possible to consider a linear form of the equation using a quantum filter model. This also provides an example of how a filter equation can be used in quantum control. The quantum treatment of the BEC and atom laser was undertaken by Wilson et al. [8], and they showed that the dark states are disturbed by the measurement back action, and that a quantum version of the feedback can be certain of removing energy up to some limiting value.

The model for a non-interacting BEC in a harmonic potential is equivalent to that for a single atom, and with the addition of continuous position measurement results in the master equation

\[
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H_A, \rho] + \alpha D[q] \rho, \tag{3.4}
\]

where \(D\) has the usual definition described by Eq. 2.4, and the Hamiltonian is given by

\[
H_A = \frac{\rho^2}{2m} + \frac{1}{2} m \omega^2 q^2 - u(t) q. \tag{3.5}
\]

\(\alpha\) is the strength of a continuous position measurement and the control is \(u(t) q\). This master equation governs the ensemble average behaviour, however the control term only makes sense for particular trajectories because \(u(t) q\) can only be determined in a conditional equation. The stochastic master equation for a quantum filter for the model is given by

\[
d\pi_t = -\frac{i}{\hbar} [H_A, \pi_t] dt + \alpha D[q] \pi_t dt + \sqrt{\eta \alpha} [q \pi_t + \pi_t q - 2 \pi_t \text{Tr}(q \pi_t)] dW \tag{3.6}
\]

with \(dW\) a white noise related to the measurement, and \(\eta\) the detection efficiency. An analytic investigation of this model for the case where the initial state \(\pi_0\) is Gaussian and with the control chosen as \(u(t) = k \hat{p}_t\) (where \(\hat{p}_t\) is the conditional expectation for momentum) is provided in the paper. It is shown that the control detailed there will drive any Gaussian initial state to a lower energy final state. The final energy is dependent on the feedback strength, for which an optimal value can be found, and the measurement strength. Stronger measurements introduce greater disturbance, and thus increase the final energy achievable, however they can also extract energy from the condensate faster. The effects of varying these parameters can be seen in Figure 3.1, which shows the final expected energy as a function of the measurement coupling strength and the feedback strength. For the feedback strength there is a minimum, and it appears that simply
§3.3 A treatment including measurement back-action

Figure 3.1: Final expected energy as a function of the feedback parameter (left) and measurement coupling strength (right)[8]. For the plot on the left the coupling is $\alpha = 0.09$, for the varying measurement plot the feedback parameter is $k_p = -1.35$. The energy is in units of $\hbar \omega$ for constants such that $\hbar = m = \omega = 1$, and the horizontal line is the ground state energy. The cross on the plot for feedback strength represents the optimal value, whereas for the measurement strength the final energy simply increases linearly with the coupling.

having the weakest practical measurement is the best option. What these plots do not show, however, is that as the measurement strength decreases, the rate of energy removal drops off, as can be seen in Figure 3.2 which shows the evolution of the energy in the condensate for different values of the measurement strength. A way to improve on these results using optimal control will be addressed in Section 3.5.

The dark states are not Gaussian so to consider that problem Wilson et al. expand an arbitrary initial state as a sum of Gaussian states, which is also an integral over coherent states. The stochastic master equation is not linear in the state $\pi_t$, so applying it to each of the states in a decomposition and summing them will not describe the evolution of the state thus decomposed. However, the unnormalised Belavkin filter [9] is linear in the unnormalised state $\sigma_t$, and so it can be used for the elements of the decomposition. The equation for the Belavkin filter is given by

$$d\sigma_t = -\frac{i}{\hbar}[H_A, \sigma_t]dt + \alpha(q\sigma_t q - \frac{1}{2}q^2\sigma_t - \frac{1}{2}\sigma_t q^2)dt + 2\sqrt{\eta}\alpha(q\sigma_t + \sigma_t q)dq_{\text{meas}},$$

(3.7)

where $dq_{\text{meas}}$ is the measurement signal and $\sigma_0 = \pi_0$. Taking the evolution of the component states and then normalising them will provide the same result as for the stochastic master equation provided first, although in this form the results are more tractable. The decomposition gives the state $\pi_t$ as an integral of the evolution of the normalised component states. The limiting results for Gaussian initial conditions [45] dictate that the individual states are independent of the decomposition factor for large times, which then dictates that the state $\pi_t$ is equivalent to the final state arrived at using one element of the decomposition, which in turn means that any initial state will have the same final state as a Gaussian one. Consequently, once measurement back-action is accounted for the dark states present in the Haine et al. treatment do not exist. As an aside, a more common use of the filter would be to derive the unnormalised Belavkin filter first, and calculate the stochastic master equation by normalising it, which is helpful when the stochastic master equation is difficult to construct.

This model makes use of the noninteracting condensate and yet can stabilise it, which
Figure 3.2: Results from numerical simulations of the control strategy described in Section 3.3 when the measurement strength is varied. The feedback strength is $k_p = -1.3$, and as in Figure 3.1 the energy is in units of $\hbar \omega$ with $\hbar = m = \omega = 1$. The various lines correspond to $\alpha$ having the values: 0.1 red, small dashed; 0.2 blue, solid; 0.4 black, dotted; 0.8 green, large dashed. It is clear in the plot that the dotted line for $\alpha = 0.4$ is below the other lines while the energy is rapidly reduced, and the line for $\alpha = 0.2$ is appreciably lower than the line for $\alpha = 0.1$ well after the other lines have reached a steady state. $\alpha = 0.1$ does however provide the best steady state energy despite it reaching this value long after the others have.
suggests that the trade-off between condensate stability and beam phase diffusion that was a feature of the semiclassical treatments will not be as much of an issue in actual experiments. This is a model without pumping or outcoupling so the subject is not yet closed, however the quantum control treatment has provided an indication of how the problems raised by the semiclassical model can be solved, as well as indicating that the measurement back-action does not render control ineffective. One additional question raised by this investigation has to do with the final energy achieved by the control. It has been seen that the measurement strength used influences both the time at which the system reaches its eventual steady state energy and the value for that energy steady state. With the measurement strong the time is shorter and the final energy is higher. Is seems logical that the optimal strategy would therefore involve modifying the measurement strength so that it is initially high but is gradually turned off to reach the best possible final state. In order to describe precisely how the measurement strength should be varied it is possible to turn to the methods of optimal control, which addresses exactly this sort of question.

3.4 Optimal control

In general the goal of optimal control is to minimise an integral of some cost function, chosen based on the system and the objective. Naturally, the choice of cost function will have a significant effect on the outcome of the process and how difficult it is to solve the problem.

A general cost function can be expressed as

$$V(u) = \int_0^{t_f} f_0(t, x(t), u(t))dt,$$

with $x$ the vector of state variables, $u$ the vector of control variables, and the optimal control $u^*$ is such that $V(u^*) < V(u)$ for all $u$. Depending on the problem the endpoint of the time integral may be open, or may be a particular time at which the final state of the system is of interest.

Choosing the cost function is nontrivial, and different choices of cost function can strongly influence how well the system performs. As an example, the objective in the BEC case above is to reduce the energy of the condensate. If the cost function is chosen as the energy with a variable measurement strength being optimised then the solution has an infinite control. In order to ensure a physical control trajectory the control variable can be added to the cost function. This obviously rules out an infinite control, and promotes efficient use of the control variable. In a simple case $u^2$ is a good choice, although other forms are used for more elaborate forms of control. With this addition a single atom approximation of the BEC equations which is developed below produces a control with two measurement pulses at different times, one brief interval of very strong measurement and after a time delay a weaker one. An example of this sort of solution is shown in Figure 3.3. This sort of solution may be difficult to implement, and a continuous alternative with less precise timing requirements is preferable. In this case it is possible to achieve that by making a different choice of cost function. Instead of using the energy directly it is possible to use the covariance to achieve an energy reduction. The covariance matrix is not directly tied to the energy of the system, however they will both be reduced by the control. The behaviour of the solution when this choice is made is significantly different, as will be discussed below, and can be seen in Figure 3.3. The measurement strength now
follows a continuous trajectory, and the energy is in fact reduced further for the same initial conditions. There are too many ways to tweak the cost function to cover all the options, however these are the considerations that led to the choices made in the next section.

Once the choice of cost function has been established there are several means of solving the actual problem. Some cases can be solved by making use of the Pontryagin maximum principle [46]. This converts the problem from a minimum of an integral equation to a system of partial differential equations, one of which involves taking a maximum. There are some systems for which this technique cannot determine the optimal control, but if it can it can make the optimal control problem solvable, either analytically or numerically. Another way to convert the integral into something more easily tractable is called dynamic programming. With this method the minimum of an integral is converted into the minimum of a PDE, and this is the method used below to answer the question about an optimal way of varying the measurement of a BEC to cool it.

Dynamic programming is based on what is called the principle of optimality [46]. This states that if you take an optimal control \( u^*(t) \) and break it into two parts at some time \( t \), then the second part will itself be an optimal strategy to take the system from the value at time \( t \) to the target final state. This principle allows a mathematical derivation of the dynamic programming solution to a problem of the form in Eq. 3.8. For a problem where the goal is to minimise

\[
V(x_1, t_1) = \int_{t_0}^{T} f_0(x(t), u(t)) dt,
\]

subject to the conditions

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = (x)_0, \quad t_0 \leq t \leq T,
\]

then the optimum cost up to a time \( t \), \( V_t \) can be shown to be [46]

\[
V_t(x, t) = -\min_{u \in \Omega} \{ \nabla_x V(x, t) \cdot f(x, u) + f_0(x, u) \}.
\]

where \( \Omega \) is the set of all possible controls \( u \), and the optimal control \( u^* \) is simply the control for which this is accomplished. Equation 3.11 is called the Hamilton-Jacobi-Bellman (HJB) equation, and solving it provides the optimal cost, which in turn provides the optimal control. The minimum over an integral equation has now become a minimum of a partial differential equation, which is as previously mentioned easier to solve both analytically and numerically. There are cases where the minimum can be evaluated analytically even if the resulting PDE cannot, which again greatly simplifies the process of solving the equation.

As an aside, the overall technique of optimal control theory doesn’t deal with any of the quantum aspects of the problem, which means that this overview of dynamic programming could be describing its use in a classical or a quantum case. The distinction between them comes primarily in the equations being used to evolve the system. There are additional differences though. For example, in the BEC case the varying parameter that will be optimised is the measurement strength, which is unlikely to have any optimising effects on a classical system as there is no measurement back-action to cause a trade-off between information gained and destruction of the state.
3.5 Optimal control of an atom in a well

As previously mentioned the form of the control for a non interacting BEC suggests that there will be an optimal control strategy. Dynamic programming is used for this first attempt at determining it. In order to apply optimal control to the scheme for cooling a non-interacting BEC, the equations of motion need to be reduced to something easier to use, and a choice of cost function has to be made. Because the equations of motion for the non-interacting case can be approximated by those for a single atom held in a trap, and so these provide the more numerically tractable equations that are the basis of this analysis. The results can be applied to the BEC easily enough, although in the interacting case they will no longer be strictly valid, and the approximation will become invalid as the interaction strength and the extent to which it affects the behaviour of the condensate under this feedback scheme increase. The reason an optimal control scheme is needed is that although a measurement is necessary to determine how to change the trapping potential holding the condensate, it will involve interacting with the condensate in a way that will add heat, as can be seen from the varying effects of different measurement strengths in Figure 3.2. Strong measurements result in a higher steady state energy, but low measurement strengths extract energy at a much lower rate. The intuitive answer is that the better strategy is to measure hard at the beginning and reduce the measurement strength over time, and dynamic programming provides an optimal strategy to accomplish this.

As in the rest of this chapter the equations are restricted to only one dimension, which means there are in effect 6 state variables that must be modelled to completely described the motion of the atom. These variables are the position $\langle q \rangle^2$, the momentum $\langle p \rangle^2$ the product of the momentum and position $\langle qp \rangle$, and the variances and cross variance $V_q$, $V_p$, and $V_{qp}$. The state of the system is also assumed to be Gaussian when the equations of motion are derived, which both simplifies the calculations and means that a direct transfer of the results to an experiment would have to be undertaken with care. For the case described in Section 3.3, with the units transformed so they match, these variables obey the following equations of motion:

\[
\begin{align*}
\frac{\partial \langle q \rangle^2}{\partial t} &= \omega \langle p \rangle + u^2 \frac{4\eta}{3m\omega^2} V_q^2 \quad (3.12) \\
\frac{\partial \langle p \rangle^2}{\partial t} &= -\omega \langle q \rangle^2 + 2k_p \langle p \rangle + u^2 \frac{4\eta}{2m\omega^2} V_{qp}^2 \quad (3.13) \\
\frac{\partial \langle qp \rangle}{\partial t} &= 2\omega \langle p \rangle^2 - \langle q \rangle^2 + k_p \langle q \rangle + u^2 \frac{4\eta}{3m\omega^2} V_q V_{qp} \quad (3.14) \\
\frac{\partial V_q}{\partial t} &= \omega V_{qp} - u^2 \frac{4\eta}{3m\omega^2} V_q^2 \quad (3.15) \\
\frac{\partial V_p}{\partial t} &= -\omega V_{qp} + u^2 \left( \frac{\hbar^2}{2m} - \frac{4\eta}{2m\omega^2} V_{qp}^2 \right) \quad (3.16) \\
\frac{\partial V_{qp}}{\partial t} &= 2\omega (V_p - V_q) - u^2 \frac{4\eta}{3m\omega^2} V_q V_{qp} \quad (3.17)
\end{align*}
\]

The variables $m$ and $\omega$ are the mass of the atom and the trapping frequency as normal, and $\eta$ is the detector efficiency. The variation in control is provided by the time varying measurement strength $u^2$, and the question is whether it is possible to determine an optimal choice of $u$ over time to reduce the atom to the lowest possible energy. As mentioned
above intuitively this will involve the measurement strength decreasing over time, and trial and error along those lines might produce a good approximation to the best possible final state. However, using the method of dynamic programming described above it is possible to generate an equation that will precisely describe that control trajectory. The cost function for this case will be

\[ f_0(x, u) = \tilde{f}_0(x) + k_\alpha u^a \]  

where the function \( \tilde{f}_0 \) is chosen to determine what precisely is being minimised, and \( x \) indicates the set of all the state variables. There are two obvious choices for the cost function. The first is the energy of the atom which would be \((\langle q \rangle^2 + \langle p \rangle^2 + v_q + v_p)/4\). Obviously, if the purpose of the exercise is to reduce the energy of the atom then using the energy as the cost function should work. However, it is also possible to minimise the variances, which is \( v_q \cdot v_p - v_{qp}^2 \). This is because the control will reduce the variances along with the energy, and so they can also be used to determine the best trajectory for the control. As it happens in this case the variance cost provides a better behaved output, as will be seen below. The term proportional to the measurement strength serves to ensure that an initially infinite strength measurement is not offered as a solution, and to include the cost of applying the control itself.

The optimal control problem is to minimise

\[ V(x, t) = \min_{u \in \Omega} \left\{ \int_t^T f_0(x(t, u), u(t)) dt \right\} \]  

where \( \Omega \) is the set of possible trajectories for the measurement strength. This integral can be split into the parts before and after some slight elapsed time \( \delta t \), which allows the first part to be resolved using a Taylor series expansion, and the second part is just the optimal cost from the new time by the principle of optimality\[46\]. The result is

\[ V = \min \{ \delta t f_0(x(t, u), u) + O(\delta t^2) + V(x(t + \delta t, u), t + \delta t) \}. \]

A second round of Taylor series expansion and taking the derivative of both sides allows the conversion to the Hamilton Jacobi Bellman equation

\[ \frac{\partial}{\partial t} V(x, t) = -\min \{ f_0(x(t, u), u) + \nabla_x V(x, t) \cdot f(x, u) \}. \]

Substituting the evolution and cost function equations into this equation allows the value of the total cost derivative to be simplified, since several terms are independent of the control variable. We split the terms in the equations of motion for the state variables into those parts which do not include the control, designated \( \tilde{f}(x) \) and those which do, designated \( \tilde{g}(x) \), and remove the terms without the control from the minimum. The resulting equation

\[ \frac{dV}{dt} = -\tilde{f}_0(x) - \nabla_x V(x, t) \cdot \tilde{f}(x) - \min_{u \in \Omega} \left\{ k_\alpha u^a + \nabla_x V(x, t) \cdot \tilde{g}(x) u^2 \right\} \]

has a minimum that can be calculated by taking the derivative with respect to the control and setting it equal to zero. This provides the value of the optimal control \( u^* \), which in turn allows the resolution of the minimum, and thus the derivation of a partial differential
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3.5 Optimal control of an atom in a well

Equation for $V$. Taking the derivative gives the value for $u^*$ as

$$u^*(x, t) = \left( \frac{-2 \nabla_x V(x, t) \cdot \hat{g}(x)}{a k_0} \right)^{1/2}$$  \hspace{1cm} (3.23)

when $\nabla_x V(x, t) \cdot \hat{g}(x) < 0$, or when $\nabla_x V(x, t) \cdot \hat{g}(x) \geq 0$ the optimal control is 0. Substituting the non-trivial case into equation 3.22 above results in an equation for the total cost

$$\frac{dV}{dt} = -f_0(x) - \nabla_x V(x, t) \cdot \hat{f}(x) - k_0 u^a - \nabla_x V(x, t) \cdot \hat{g}(x) u^a \cdot$$  \hspace{1cm} (3.24)

To solve the equation the variable power of the control in the cost function, $a$ is chosen to be 2. Since this is only a partial differential equation, or rather a system of them, it can be solved numerically. It is at this point that choice of the function $f_0$ makes a difference, since the energy cost requires numerically simulating all 6 system variables, as well as the total cost $V$. By contrast, since the equations for the evolution of the variances do not depend on the actual state variables when a variance area cost is applied and neither does the equation for the total cost the state variables do not need to be evolved to solve the equations. This means that solving the variance area equations numerically involves half as many variables, and is thus less intensive. Solving the equations with a numerical package can give trajectories for the optimal control $u^*$. Particular choices of initial values will change these trajectories, and the code that produces these particular paths includes a specified time interval over which the cost is optimised, which can produce unusual results for short times. An example of the trajectory for each of the two different cost functions is shown in Figure 3.3. The variance goal dictates a smooth reduction of the measurement strength, which is what was expected at the outset. The two pulse control of the energy goal extracts significant information about the system initially and then checks the system again at a later time, which also makes sense. Either solution is reasonable, although in practice the variance goal would likely be easier to implement. Changing the parameters of the simulation changes the shape of the energy cost function trajectory greatly, however in the variance area cost function case the general shape of the trajectory is consistent unless the time available is too short.

The consistent form of the solution in the variance area case, as well as the relatively simple form of the equation for $V(x, t)$ suggests that it might be possible to find an analytic solution for the total cost $V$ in that case. However, an ansatz must be made for the entire PDE for $V$. It can be solved analytically in a series expansion for the time variable. Unfortunately I have extended this series expansion out beyond tenth order in $t$ and there is no indication that this series will terminate or represent a series expansion of some combination of known functions. The principle problem in the expansion is that although increasing orders of $t$ for short times might suggest that the terms are shrinking the higher order terms also include extremely high orders of the variances which cannot be assumed to be small. With the values of the terms not decaying and with nothing to suggest that higher order terms will cancel there would only be an analytic solution if the series is the expansion of some combination of functions, however as the order in time increases the terms become increasingly complicated rather than conforming to any simple pattern. With no pattern readily apparent the series expansion was abandoned.

While this example was a standard application of optimal control theory, we have applied it to a feature of the system that would not require optimisation for a classical system, the strength of the measurement that controls the feedback. In this chapter the treatment is one dimensional and applies a Gaussian approximation, which means that for
Figure 3.3: Results from numerical simulations of the optimal control strategies described in the text. The plots show examples of the variable measurement strength (left) and the behaviour of the energy (right) for an energy (top) or variance (bottom) choice of cost function. The shape of the variance goal plot is fairly consistent across choice of parameters and total time for the control, with variations of what appears to be a similar curve. The energy goal plot will maintain the two action shape in many cases, however this shape offers little hope of analytic tractability. The values on the axes of the plots provide little information in this case, since the parameters used in these simulations are partially arbitrary, rather than connected to any particular atom or experiment.
an actual experiment it would be of limited utility. The benefit of these limitations is that it is possible to get a solution, which may not be true in the 3D case. There also isn’t any pumping or outcoupling in the model. However, the fact that this solution was achievable suggests that if the quantum treatment in Section 3.3 were extended to include these absent elements and the same general behaviour was present then dynamic programming may be able to provide an optimal control scheme there as well, although probably only for the less intensive variance cost version. If an experiment were being performed which used an approximately one dimensional condensate then the variance goal solution could be used, although approximations necessary to perform these calculations would mean that precisely following the path dictated would no longer be optimal. It should be close though, and the smooth shape means it would not be hard to implement. By contrast the energy cost solutions make use of fairly precise timing, and would therefore be difficult to use effectively without simulations with more detail.

3.6 Commentary

Modeling a BEC using semiclassical methods led to the expectation that control could provide a method to reduce the energy of the BEC. Since there are experiments working on producing a continuous atom laser that require a means of reducing this energy, this is of interest. These treatments were incomplete, however, in that they neglected to include the effect of the measurement on the BEC. Without including this back-action the semiclassical treatments cannot be relied on as the basis for an actual experiment, and so a more complete scheme is necessary. A version including back action for a non-interacting condensate without outcoupling showed that with back-action included control is still possible, as well as resolving a question about dark states present in the previous solutions. It also provided the basis for an investigation of optimising the measurement strength, which is of course meaningless when there is no back action to force a trade-off. There are still unanswered questions in this model, however further extension thereof is outside the scope of this thesis. In the next chapter the discussion of the use of feedback control to generate entanglement between two atoms that is the focus of this thesis begins.
Cooling a BEC
Chapter 4

Concurrence of Two Atoms in a Cavity

In this chapter the model which is the basis for the rest of this thesis is presented. Two atoms are held in an optical cavity, and they are driven by an additional laser tuned to a resonance that converts the atoms into two level systems. In the absence of other factors the behaviour is simple, the atoms oscillate between the ground and excited states, and some photons are emitted into the cavity rather than the driving field. The atomic states in this system are entangled. To describe how entangled, I use a measure of entanglement called the concurrence [1], which ranges from 0 – 1. Without feedback atoms in the symmetric subspace will reach a concurrence of 0.11 [2]. Previous work used an approximate model where the cavity was adiabatically eliminated. Using a homodyne based measurement scheme and a symmetric control operator Wang Wiseman and Milburn raised the concurrence to 0.31 [2]. Carvalho later showed that a photocounting scheme using the same operator that achieves a concurrence of 0.45 [3]. Including spontaneous emission in this model significantly reduces the achievable concurrence, however it has been shown that using a local control operator greatly reduces this effect. This also significantly improves the concurrence when there is no spontaneous emission. The jump measurement scheme was shown to achieve maximum entanglement without spontaneous emission, and reaches concurrences above 0.9 for reasonable spontaneous emission rates [3]. Previous work performed by Carvalho considering the effects of imperfect detection [3] feeds into my own research comparing the homodyne and jump schemes when scanning the detection efficiency and spontaneous emission. This analysis confirms that the jump scheme is superior when the concurrence is high. In Chapter 5 the adiabatic approximation that underlies all of what follows is relaxed. This chapter also includes descriptions of other methods of achieving entanglement between two atoms, and some discussion of the relative strengths and weaknesses of these methods as compared to each other and method that is the main focus of this thesis.

4.1 Entanglement and concurrence

Entanglement is a property of quantum systems that provides one of the starkest differences from classical mechanics. A pure state is described as entangled if it cannot be factorised, and a mixed state is entangled if it cannot be expressed as a combination of factorisable pure states. This statement of what entanglement is does not however provide any clear way to determine the entanglement of a state. Since entanglement is a crucial part of a number of experiments and areas of research, quantum computing in particu-
lar, some idea of how entangled a state is is crucial, and indeed several ways to quantify this have been investigated. A consideration of the behaviour of a non separable state in quantum mechanics suggests that it is sensible to speak of an ordering of states as being more or less entangled than each other. A measure of entanglement should agree with this apparent behaviour, rating the least separable states as most entangled, and then decreasing monotonically.

The maximally entangled states for two coupled two level systems can be constructed. The basis states for the Hilbert space are of the form \{\langle gg\rangle, \langle eg\rangle, \langle ge\rangle, \langle ee\rangle\}, where what precise labels are used depends on what particular two level systems are involved. Later in this chapter a pair of two level atoms will be considered, and so that will be the language used to present the formula to calculate the entanglement of a mixed state. In such a system there are four maximally entangled states, the antisymmetric

\[ \frac{1}{\sqrt{2}}(\langle eg\rangle - \langle ge\rangle) \]  

(4.1)

and symmetric

\[ \frac{1}{\sqrt{2}}(\langle eg\rangle + \langle ge\rangle) \quad \frac{1}{\sqrt{2}}(\langle gg\rangle + \langle ee\rangle) \quad \frac{1}{\sqrt{2}}(\langle gg\rangle - \langle ee\rangle) \]  

(4.2)

Bell States.

Determining the entanglement of other states requires a definition of a measure of entanglement. One such measure is the entanglement of formation \[47\], which for a pair of coupled two level systems can be determined for a density matrix \(\rho\) of systems \(A\) and \(B\) by taking all possible pure state decompositions of \(\rho\) into pure states \(\psi_i\) with probabilities \(p_i\), such that

\[ \rho = \sum_i p_i \psi_i \langle \psi_i |. \]  

(4.3)

For each of these pure states the entanglement \(\mathcal{E}\) is the entropy of either subsystem \[48\],

\[ E_f(\psi) = -\text{Tr}(\rho_A \log_2 \rho_A) = -\text{Tr}(\rho_B \log_2 \rho_B) \]  

(4.4)

where \(\rho_A\) is a partial trace of the state over subsystem \(B\), and \(\rho_B\) is similarly defined. The entanglement of formation of the initial mixed state \(\rho\) is the average entanglement of the pure states in the decomposition, minimised over all possible decompositions \[1\].

\[ \mathcal{E}(\rho) = \min \sum_i p_i E_f(\psi_i). \]  

(4.5)

This decomposition is not unique, and in principle it can be difficult to ensure that the decomposition that in fact minimises this quantity is chosen, a problem which is magnified when using more than a pair of two level systems. It is also difficult to code this formula in such a way that a computer can reliably calculate it, which is necessary for bulk numerical simulations.

Wootters derived an explicit formula in \(\rho\) to calculate the entanglement of formation, which is arrived at in the following way \[1\]. First, an operation is defined which is a bit flip, having the form

\[ |\tilde{\psi}\rangle = \sigma_y |\psi^*\rangle, \]  

(4.6)

where \(|\psi^*\rangle\) is the complex conjugate of the state \(|\psi\rangle\) in a two level basis, and in the same
basis $\sigma_y$ is the matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The result of applying this operator to a single qubit will be a bit flip. For a pair of two level systems the spin flipped state denoted by the tilde is

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y), \quad (4.7)$$

with the complex conjugate taken in the standard basis as before. With this bit-flip operator it is possible to calculate the entanglement not only of the mixed states that are of primary interest, but also of pure states. The entanglement of a pure state, equivalent to Eq. 4.5 can be calculated using

$$E(\psi) = E(C(\psi)), \quad (4.8)$$

where $C$, which is named the "concurrence" is defined as

$$C(\psi) = |\langle \psi | \psi^* \rangle|, \quad (4.9)$$

and the function $E$ is given by

$$E(C) = h \left( \frac{1 + \sqrt{1 - C^2}}{2} \right) \quad (4.10)$$

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x).$$

The function $E$ is monotonically increasing, and it ranges from 0 to 1 when the concurrence $C$ ranges from 0 to 1. This $E$ is equivalent to the entanglement of formation for a pure state calculated via the method given previously, and since the range is the same as the range of the concurrence the concurrence itself is also a valid measure of entanglement, and although the specific values may differ from the entanglement of formation the ordering of states does not. Inspecting the extreme states to ensure the behaviour is as expected, the spin flip operation does not change either of the Bell states mentioned above except for adding a minus sign, and therefore the concurrence of either of those states is 1, and calculation will show that the function $E$ gives the value 1 when the concurrence is 1. An unentangled pure state, for example $|\ge\rangle$ will when spin-flipped be mapped to an orthogonal state, and will thus have a concurrence of 0, which leads to a value from $E(C)$ of 0 as well.

For a mixed state the formula is slightly different, the entanglement of a mixed state of two qubits is given by

$$E(\rho) = E(C(\rho)), \quad (4.11)$$

with the concurrence for a density matrix defined as

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \quad (4.12)$$

The $\lambda_i$s in this equation are the eigenvalues in decreasing order of size of the Hermitian matrix $R \equiv \sqrt{\tilde{\rho}} \tilde{\rho} \sqrt{\tilde{\rho}}$, or the square roots of the eigenvalues of the non-Hermitian matrix $\tilde{\rho}$. The $\lambda_i$s will be non-negative real numbers, and the concurrence will have a value between 0 and 1 as it did for the pure state definition. This equation will give the same result as the pure state version in those cases where the density matrix represents a pure state.

With an explicit formula to calculate the entanglement of formation for mixed and pure
states it would be possible to treat a numerical or analytic investigation of a system where entanglement is being created in these terms. However, it is in fact more straightforward to discuss such a system using the concurrence, which has been shown to be a valid measure of entanglement as well. This will be the choice made in both the rest of this chapter and Chapter 5. All of the values given can naturally be converted into the values for entanglement of formation using the formula given.

4.2 Entangling a pair of atoms

One system in which entanglement can be generated is two coupled atoms trapped in an optical cavity. There are a number of ways to do this, some of which shall be discussed before the model that forms the basis for the rest of the thesis is introduced. As the description of these methods is an overview only, the calculations behind them are not included, with only the models, mechanisms and results discussed.

4.2.1 Cavity-loss-induced entanglement

An early model for generating entanglement between two atoms in a cavity was proposed by Plenio et al [49]. Their system placed two two level ions in a linear trap surrounded by an optical cavity with mirrors with low reflectivity, with the cavity mode resonant with the transition between the ground and excited states of the atoms, and a photon detector at the cavity output. Beginning from an initial condition where the atoms were in the ground state and the cavity was empty, one atom is moved to the excited state using a laser. The spontaneous emission rate from excited atoms is assumed to be much lower than the cavity loss. Plenio et al calculated the dynamics of the system and showed that if a photon did not leave the cavity the state would move towards an entangled state of the two atoms. Since the detector would indicate if a photon escaped the cavity it would be possible to determine that the entangled state had been created once the required time had passed. Using a form of the quantum trajectory technique described in section 2.2.3 Plenio et al derived the conditional time evolution of the system, and from this determined that the probability of a photon escaping the cavity, and therefore of the scheme failing to entangle the two atoms, was $\frac{1}{2}$. Where the detector is perfect the fidelity between the atomic state and the maximally entangled state is close to one until the elapsed time approaches the inverse of the spontaneous emission rate. However, where the detector is imperfect the fidelity of the two atom state drops significantly, as does the relative entanglement of the state.

The 50% failure rate of this method is a problem, as are the consequences of spontaneous emission and imperfect detectors, both of which are experimental realities that must be grappled with. The use of feedback can improve the scheme, as demonstrated by Hong and Lee [50]. They take the model proposed by Plenio et al and amend it by substituting three level atoms for the two level atoms. The atoms in the scheme proposed by Hong and Lee have a single excited state and two degenerate grounds states called $|L\rangle$ and $|R\rangle$, which are coupled in the cavity to the excited state by left and right circularly polarised light respectively. The atoms are prepared in the state $|L\rangle$ and a single photon is injected as before, but is now left polarised. The detector is changed to detect the polarisation of an escaping photon, and if a right polarised photon is detected then an atom must have decayed from the excited state into the state $|R\rangle$. Because which atom did this cannot be determined, the atoms must be in an entangled state combining the two ground states.
The failure rate of the system can be dramatically reduced by redirecting any photon that is not right polarised into the cavity again, thereby allowing an experimentalist to simply wait until a right polarised photon is detected. If the probability of creating the entangled state is close to the 50% achieved by Plenio et al it should only require a few cavity decay times before the probability of creating the entangled state is high. An entangled state combining two hyperfine ground states is also not susceptible to spontaneous emission. Spontaneous emission from the excited state during the creation of the entangled state is still a potential problem which would be minimised through the choice of atomic species and cavity decay rate. An additional benefit derives from the use of the photon detector. In the Plenio et al scheme the triggering of the photodetector signaled failure, whereas the detector in the Hong and Lee scheme triggers when it has succeeded, so imperfect detectors do not reduce the fidelity of the entangled states as they do for Plenio et al. The experimental geometry and the controllable source of left circularly polarised photons are, however, more challenging to create experimentally than the simpler scheme.

Both of these schemes rely on cavity loss to include entanglement, and thus have the disadvantage of requiring cavities with particular properties. They also depend heavily on the introduction of single photons. Entanglement schemes that use different mechanisms can reduce overcome these issues.

4.2.2 Measurement induced entanglement

Sørensen and Mølmer have proposed two related schemes to generate entanglement using measurement rather than cavity loss. Both of the schemes make use of three level atoms, with two hyperfine ground states and a single excited state. In the first scheme the atoms are held in a leaky cavity resonant with the transition from the excited state to one of the ground states, called $|1\rangle$ [51]. The cavity is set up so if both atoms are in the other ground state, $|0\rangle$, the light should shine straight through the cavity. A laser is used to introduce light at one of the mirrors and a detector is set up at the same end of the cavity, so if a photon is reflected by the atoms it will be detected. If this occurs both atoms cannot be in the state $|0\rangle$. Sørensen and Mølmer use this to generate the entanglement. The two atoms are prepared in a specific ground state, creating a combined state of

$$\cos^2(\phi)|00\rangle + \sqrt{2}\cos(\phi)\sin(\phi)\frac{|01\rangle + |10\rangle}{\sqrt{2}} + \sin^2(\phi)|11\rangle. \quad (4.13)$$

If resonant light is reflected from within the cavity then the $|00\rangle$ state is projected out, and if the angle $\phi$ is small then the atoms will be in close to the maximally entangled state. Better results can be achieved by using a method with two detections, rotating both atoms after the first detection so the $|11\rangle$ state becomes the $|00\rangle$ state and can be projected out by the second measurement. In theory the result is a perfect maximally entangled state.

This scheme can produce high entanglement states with relative certainty, and does not require the cavity to be as carefully constructed as the cavity loss methods. However, it relies on detecting photons reflected from the atoms in the cavity. Sørensen and Mølmer calculated the probability of success for the single detection method for varying levels of fidelity to the maximally entangled state, and for the two detection method, and their probability of success is low. In ideal conditions with perfect detectors and single photons shining on the cavity the two detection method will work about half the time. The more
Concurrence of Two Atoms in a Cavity

Figure 4.1: Probability ($P_s$) of successfully generating an entangled pair with a fidelity to the maximally entangled state of greater than 0.9 using the Sørensen and Mølmer scheme with coherent light as the cavity input. Full lines assume a single detection, and dashed lines are for double detection. $g^2/\kappa\gamma$ is the ratio of the cavity coupling $g$, decay $\kappa$ and spontaneous emission $\gamma$ parameters. The upper (lower) curves are for a detector efficiency of 1 (0.5) [51].

realistic case with the light introduced in a coherent state and thus possibly including multiple photons is shown in figure 4.1, and the probability of success is lower.

In a subsequent paper Sørensen and Mølmer indicate that this scheme is very sensitive to imperfections in the cavity and to imperfect mode matching, and so propose an alternative [52]. The new method still uses a pair of three level atoms in a cavity, but the resonant mode of the cavity is now detuned from the atomic transitions and each atom is prepared in an equal superposition of the two ground states. The measurement that induces the entanglement in the new scheme is performed by passing light through the cavity, and then performing a homodyne measurement on the light emerging from the cavity. The presence of atoms in state $|1\rangle$ in the cavity will alter the phase of the light, and using a homodyne measurement allows the number of atoms in state $|1\rangle$ to be established. If the result is 1 then the desired entangled state will have been created. Sørensen and Mølmer are able to calculate the expected fidelity, or alignment with the maximally entangled state, for a variety of parameters, as seen in figure 4.2.

This scheme allows the production of high fidelity states given high quality cavities. However, creating the highest fidelity states requires countless repetitions of the process, and unlike the previous scheme presented by Sørensen and Mølmer it is possible due to the nature of the homodyne measurement to incorrectly believe an entangled state has been created. Both these schemes overcome the reliance on single photon sources that is a weakness of the other methods thus far discussed, although the first scheme works better using single photons rather than coherent pulses.

4.2.3 Controllable degrees of entanglement and entangled-state cycles

The methods described in the previous subsections focus on methods for creating maximally entangled states. There have also been methods proposed for creating states with controllable amounts of entanglement. Clark and Parkins describe one such system [53]. They use a pair of five level atoms, with two hyperfine ground states, two excited states and an additional state. These atoms are coupled to a cavity and a combination of lasers creating two raman transitions with different coupling strengths between the ground states,
one for each excited state. The purpose of this arrangement is to produce effectively two level atoms where the coupling from state $|0\rangle$ to state $|1\rangle$ is not equal to the coupling of the inverse transition, a description which is accurate given the right choices of parameters and energy levels. Their calculations show that this is equivalent to a coupling between the atoms and a reservoir that is squeezed in the quantum optics sense. The dynamics of this system allows the creation of states with any desired level of entanglement, within the limits of experimental precision, through combinations of laser pulses. There is however a problem created by the existence of a state that is decoupled from the transitions, and into which it is possible for the atoms to decay from the excited states. The degree of this decay limits the maximum concurrence achievable by the scheme. This model is interesting as an example of a means for generating entanglement, however it focuses on creating a controllable amount of entanglement, rather than the maximum possible, and is therefore difficult to compare to the others discussed here. It also is more technically complicated than the schemes intended to create the greatest degree of entanglement.

Chia and Parkins extend the analysis of this system by considering the case where the two coupling strengths are equal, and also generalise to more than two atoms [54]. With equal coupling strengths their master equation for the evolution of the atoms has no steady state solution. Instead, Chia and Parkins use a quantum trajectory method to show that with all the atoms in state $|0\rangle$ initially the atoms will either achieve an entangled state with some probability or enter into one of a number of cycles between states. These behaviours are characterised through the number of photons that leave the cavity system, which would then be detected, allowing an experimentalist to also make the distinction. Perhaps the most interesting from the perspective of entanglement is the case where there are no photons detected, in which case the calculations show for an even number of atoms a maximally entangled state is produced after time sufficient for other contributions to the initial state to have decayed away. This distinguishes this method for generating entanglement, since few entanglement schemes can be readily extended to the multipartite case. Figure 4.3 shows the probability that there will be no photon transition for a given number of atoms. For large numbers of atoms the probability can be approximated by $\sqrt{2/\pi N}$, where $N$ is the number of atoms. This setup does share the difficulties of some other schemes that the signal of success is the absence of signal.
Figure 4.3: Probability $P_N(m = 0)$ after sufficient time for the other components of the state to decay away for the Chia and Parkins setup [54]. $N$ is the even number of atoms and $m$ is the $J_x$ component of their collective angular momentum. The $m = 0$ state is a maximally entangled multipartite state. The crosses represent calculations, and the curve is the approximate match for large $N$ of $\sqrt{2}/\pi N$.

which will be problematic in an experiment using non ideal detectors, and if the desired outcome is not achieved the whole system must be reset to a specific initial state before the experiment can begin again.

In the case where a photon is emitted or when an odd number of atoms is used the system will necessarily continue to emit photons, resulting in behaviour that Chia and Parkins label entangled-state cycles. If the measurement performed on the stream of photons is a simple photon count, the atoms will enter into a cycle between two state characterised by the $J_x$ component of their collective angular momentum. The maximally entangled state has $J_x = 0$, with the other states having values ranging between $-\frac{N}{2}$ and $\frac{N}{2}$, with the value for a specific state designated $m$. Because the photodetection measurement provides information about $m^2$ the atoms will evolve towards a cycle between the states for $+m$ and $-m$, with some probability for each $m^2$, although once the first transition occurs it is not possible to evolve to $m = 0$. Which set of states the atoms are rotating between is distinguishable from the rate of photon emission from the cavity. The states between which the atoms rotate are also entangled to some degree. In the two atom case the only cycle (for $m^2 = (N/2)^2 = 1$) is between states which in the normal $J_z$ basis are $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which are also maximally entangled. For more than 2 atoms the states have varying levels of entanglement but are not maximally entangled, which means they are of less value. As with the Clark and Parkins method described previously, the range of different entangled states that can be created is interesting, but because the value is with creating specific entangled state rather than the maximum achievable entanglement these spin cycles cannot be compared to the other schemes here discussed.

Of all the methods described in this section the Chia and Parkins method seems most interesting, as it is capable in theory of producing maximally entangled states, and is the only one that has been extended to more than two atoms. However, there is no consideration of the impact of imperfect experiments on the scheme, so it is not clear whether it would work in the presence of dephasing factors, or how effectively a real
4.3 Entanglement through quantum control

The methods used to generate maximum entanglement described in the previous section make use of various properties of atoms and cavities to generate the entanglement, and all of them may require multiple iterations of an experiment to get an entangled pair of atoms. They also have limited feedback and measurement schemes. An alternative method making use of direct feedback on the atoms may prove more effective, and it is one such scheme on which the rest of this thesis is focused. Using direct feedback on the atoms also permits some experimental imperfections to be overcome in a way not open to the methods in the previous section.

This model uses two level atoms, which means that the descriptions of concurrence above can be applied directly. Experiments with this setup exist already [13, 14, 15], so in principle predictions of entanglement can be verified, provided the approximations made in order to perform calculations are appropriate for present experiments.

4.3.1 The model

The system is a pair of two level atoms coupled to a mode in a cavity in which they are trapped. The atoms are driven by an additional laser external to the cavity, which will cause them to rotate between the ground and excited states. These atoms will emit photons into the cavity, that may then escape. Escaping photons will be measured and the measurement results will control feedback provided by an additional laser. In addition, the atoms may spontaneously emit photons. The two atoms in this system can be treated as qubits, which means entangling them may be of value in the ongoing research into quantum computing, however the primary goal is the entanglement itself. The first step to determine the master equation obeyed by this system is to determine what the terms are that govern a single atom in the cavity.

Wang, Wiseman, and Milburn (WWM) developed a model for a system where the laser
driving the cavity also drives the atoms \[2\], which initially produces an equation which is slightly more complicated than that for the model used here. They transform their density matrix to render the average number of photons in the cavity mode 0, and after this transformation they arrive at Eq. 4.15 by a different path. For the simpler derivation where the driving laser is not in the cavity mode and the average number of photons in the cavity is already 0 the evolution of a single atom in the cavity will obey the equation

\[
\rho = -i\Omega \left[ (\sigma^+ + \sigma), \rho \right] + \kappa D[a]\rho + \gamma D[\sigma]\rho + \frac{-ig}{2} \left[ (\sigma^+ a + \sigma a^\dagger), \rho \right].
\]  

(4.14)

The atom operators \(\sigma^+\) and \(\sigma\) are the raising and lowering operators for the atoms, and have the definition \(\sigma = |g\rangle\langle e|\) and \(\sigma^+ = |e\rangle\langle g|\). The annihilation operator \(a\) is for the cavity mode to which the atom is coupled, which is of great interest since it is expected in general to contain few photons, and also controls the signal which reaches the detector. The driving laser has effective Rabi frequency \(\Omega\), and unlike the cavity it is assumed to be of intensity sufficient that the atoms have no effect on it, and is therefore treated as a classical field. The \(\Omega\) term which is the interaction with the driving laser and the term with the cavity coupling \(g\) which is the interaction with the cavity mode are unsurprisingly similar. The cavity damping term \(\kappa D[a]\rho\) with \(D\) having the definition \(D[\sigma] \rho \equiv \epsilon\rho\epsilon^\dagger - \frac{1}{2} (\epsilon^\dagger\epsilon\rho + \rho\epsilon\epsilon^\dagger)\) as in Chapter 2 represents the loss of photons from the cavity that will reach whatever detector is applied. The term with a \(\gamma\) coefficient is the spontaneous emission term. Additional damping terms could be included, however they will only complicate the model at this stage. Adding the second atom primarily changes the terms so that the single atom operators are replaced with sums of the operators for each of two atoms, so the new equation is

\[
\dot{\rho} = -i\Omega \left[ (J_+ + J_-), \rho \right] + \kappa D[a]\rho + \gamma_1 D[\sigma_1] \rho + \gamma_2 D[\sigma_2] \rho + \frac{-ig}{2} \left[ (J_+ a + J_- a^\dagger), \rho \right].
\]  

(4.15)

with \(J_- = \sigma_1 + \sigma_2\) and \(J_+ = \sigma_1^+ + \sigma_2^+\). The subscripts indicate to which atom the operator applies, and also distinguishes the two spontaneous emission terms, and in principle the spontaneous emission coefficients need not be equal. Figure 4.4 also shows a measurement device and an additional feedback laser that will be added to the model in the next section.

This equation can be numerically simulated directly even when feedback is included, and it will be in Chapter 5. Initially, however, it helps to consider a simpler form of the equation. In particular, if the cavity damping \(\kappa\) is greater than the other parameters of the system it is possible to adiabatically eliminate the cavity \[55, 56\]. This assumes a cavity where photons swiftly escape, without significant opportunity to influence or interact with the atoms once they have been emitted into it. By contrast in a low \(\kappa\) cavity a photon that was emitted into the cavity mode would have a non-negligible probability of being reabsorbed by one of the atoms, rather than escaping. The effects of a low \(\kappa\) will be important when detection and feedback are added to the system, since a photon that does not escape is not going to carry information to the control scheme, and this effect will become visible when the restriction to the adiabatic regime is relaxed in Chapter 5.

The process of adiabatic elimination begins with a partial expansion of the density matrix \(\rho\) in the photon numbers of the cavity mode. The result of this is

\[
\rho = \rho_0 |0\rangle\langle 0| + (\rho_1 |1\rangle\langle 0| + \text{H.C}) + \rho_2 |1\rangle\langle 1| + (\rho_2' |2\rangle\langle 0| + \text{H.C}) + O(\lambda^3),
\]  

(4.16)

with \(\lambda = g/\kappa\), which is small. Substituting this into the master equation, neglecting terms
The terms involving only the atoms are encapsulated in the superoperator $\mathcal{L}$ for brevity. These equations can be solved by assuming that $\hat{p}_1$ and $\hat{p}_2'$ are 0, which means that it can be shown that

$$\rho_1 = \frac{-i\tilde{g}}{\kappa}[J-\rho_0 - J-\rho_2],$$

$$\rho_2' = \frac{i\tilde{g}}{\sqrt{2}\kappa}J-\rho_1.$$  

(4.18)

These results can then be substituted into the two remaining equations to give

$$\dot{\rho}_0 = \mathcal{L}\rho_0 - \frac{g^2}{2\kappa}[J_+J_0 + \rho_0J_+J_- - J_+J_2 - J_2J_-] + \kappa\rho_2,$$

$$\dot{\rho}_2 = \mathcal{L}\rho_2 + \frac{g^2}{\kappa}[J_-\rho_0J_+ - J_-\rho_2J_+] - \kappa\rho_2.$$  

(4.19)

These two equations can be summed, and terms involving $\rho_2$ can be neglected since $\rho_2$ is of order $g^2/\kappa$, which gives the final adiabatically eliminated equation for the system as

$$\dot{\rho} = -i\Omega [(J_+ + J_-), \rho] + \Gamma\mathcal{D}[J_-]\rho + \gamma_1\mathcal{D}[\sigma_1]\rho + \gamma_2\mathcal{D}[\sigma_2]\rho,$$  

(4.20)

with the new coefficient $\Gamma$ defined as $g^2/\kappa$, the collective decay rate of the atoms and cavity. This equation is much easier to deal with. In addition, if the decay rate of the cavity is significantly greater than the spontaneous emission rates they can be neglected, which reduces the equation for the system to the Dicke model [57]

$$\dot{\rho} = -i\Omega [(J_+ + J_-), \rho] + \Gamma\mathcal{D}[J_-]\rho.$$  

(4.21)

Incidentally, a quantum optics concept referred to as the cooperativity is commonly defined as $g^2/\kappa\gamma_{se}$ [58], which means that the approximation that spontaneous emission is small relative to $\Gamma$ might be considered to be the statement that the cooperativity is high. This is not entirely true, however, since the adiabatic approximation that led to this equation is only valid when few photons remain in the cavity, which is similar to requiring a large $\kappa$, and high cooperativity cavities are usually designed to operate within what is known as the strong coupling regime when $g \gg \kappa$ [13, 14, 15]. It is possible for the cooperativity $\Gamma$ to be large when $\kappa \gg g$ if $g$ is sufficiently larger than $\gamma$, however, this is not the normal design of cavities used in quantum optics. Although this approximation results in a simpler equation to deal with, it will be discussed in Section 4.3.3 that even small values of $\gamma/\Gamma$ can have a drastic effect on the steady state concurrence achievable, so this would be a bad approximation for an actual experiment. Also, as mentioned above many experimental systems have $g \gg \kappa$, and for any such experiment the adiabatic elimination of the cavity is of dubious applicability. The extent to which it fails will be addressed in Chapter 5.
The Dicke model does have the advantage of producing analytic results, although it must be born in mind that they can not be relied on when the approximations to create this model fail.

An analytic investigation of the behaviour of the Dicke model is possible [55]. The symmetry with respect to interchange of atoms makes the two qubit states \{\ket{gg}, \ket{ge}, \ket{eg}, \ket{ee}\} a poor choice of basis for this, and the angular momentum states an obvious one. The angular momentum states can be divided into symmetric \((j = 1)\)

\[
\ket{1} = \ket{gg}, \quad \ket{2} = \frac{\ket{ge} + \ket{eg}}{\sqrt{2}}, \quad \ket{3} = \ket{ee},
\]  \quad (4.22)

and anti-symmetric \((j = 0)\)

\[
\ket{4} = \frac{\ket{ge} - \ket{eg}}{\sqrt{2}}.
\]  \quad (4.23)

subspaces, and with only the terms in the Dicke model, it is impossible for the system to move between these two subspaces.

The state \(\ket{4}\) is a stationary state of the equation, and it is in fact the antisymmetric Bell state mentioned in Section 4.1. Thus, in the absence of measurement and spontaneous emission a state prepared in the antisymmetric bell state will remain maximally entangled. Of course, it is likely that some form of control scheme would be needed to get to this state in the first place, and it is unlikely that a real experiment could be constructed in such a way that the spontaneous emission would not be a factor. Nevertheless, this does mean that if a control scheme can push the system into this state it could then be switched off if there is no need to contend with spontaneous emission.

In the symmetric subspace none of the three basis states is a stationary solution, which means that any initial condition will evolve into a combination of the three. Since state \(\ket{2}\) is one of the symmetric Bell states from Section 4.1, this means that any initial condition will dynamically generate a state with some entanglement. However, it has been shown that the concurrence generated in this way is only about .11 [55], which is much less than the value of 1 for the maximally entangled Bell state.

### 4.3.2 Adding feedback

In order to do better than a concurrence of 0.11 it is necessary to change the system directly. Applying feedback implies that some form of measurement of the system is necessary. Of the several possible measurement schemes that could be used the choice made here is to use the photons leaking from the cavity to perform photodetection or homodyne measurements, both of which have well developed theoretical underpinnings. The form of the feedback considered here is an additional laser field applied to the atoms, either as a pulse when a photodetector is used for the measurement, or continuously with a homodyne scheme. Both feedback schemes are Markovian, which means they do not depend on previous states of the system. This means there is no need for a filter equation or any sort of state analysis, and thus avoids the need to perform state estimation using the signal before acting. This is obviously a significantly easier experiment to perform, and is also easier to model.

The terms for the homodyne case were derived by Wiseman and Milburn [25, 59] for a control Hamiltonian proportional to the measurement signal, and applied to both atoms equally. In this case the signal from the homodyne detector is continuous, and the control Hamiltonian varies with time, but is applied always. Adding this measurement scheme
\[ \dot{\rho} = -\frac{i}{\hbar} \Omega [(J_+ + J_-), \rho] + \Gamma \mathcal{D}[J_-] \rho + \frac{1}{\Gamma} \mathcal{D}[F] \rho - \frac{i}{\hbar} [F, -iJ_- \rho + i\rho J_+] . \] (4.24)

The feedback is a product between the photocurrent signal of the homodyne detection and some operator \( F \). The choice of control operator will determine how effective the feedback is at generating entanglement. This equation was used by Wang, Wiseman and Milburn [2] to show that using a symmetric feedback operator this scheme allows an increase in the concurrence to .31 of maximum when the atoms are in a symmetric initial state. A symmetric feedback scheme also does not mix symmetric and antisymmetric subspaces, which means that there is still no mechanism whereby a system which begins in one can move to the other. The effect of different choices of feedback within this equation has received some investigation [60], both for this symmetric scheme and for local control schemes similar to those described in Section 4.3.3.

The second choice for detection method is using a simple photodetector, which detects photons escaping from the cavity or quantum jumps. In this case the detection does not add anything to the equation, since the atom and cavity collective decay term in the Dicke model, \( \Gamma \mathcal{D}[J_-] \rho \), already encapsulates the dynamics of a photon escaping from the cavity, and now it is simply being detected. Given the discrete nature of photocounting a continuous feedback term as in the homodyne case makes less sense, and instead some action taken whenever a photon is detected is the obvious choice. The equation for jump feedback was also calculated by Wiseman [59], and was applied to this model by Carvalho and Hope [3]. The result is

\[ \dot{\rho} = -\frac{i}{\hbar} \Omega [(J_+ + J_-), \rho] + \Gamma \mathcal{D}[U J_-] \rho . \] (4.25)

The choice of unitary feedback operator \( U \) is again free but likely to affect the success of the scheme, and it will be applied after each jump. The reason why the feedback can simply be directly included in the cavity decay term can be seen when the superoperator \( \mathcal{D}[U J_-] \rho \) is expanded. The term with the feedback operator included expands into

\[ \mathcal{D}[U J_-] \rho = UJ_- \rho J_+ U^\dagger - \frac{1}{2} (J_+ U J_- \rho + \rho J_+ U^\dagger U J_-) , \] (4.26)

and the use of a unitary operator ensures that the feedback will vanish in the two decay terms. This means that only the term that accounts for jumps will include the feedback, which as was mentioned when this superoperator was introduced in Section 2.2.2 is also the term that encapsulates actual escape of photons. Carvalho and Hope [3] investigated the jump scheme used with the feedback operator \( U_{fb} = \exp(-i\lambda J_x) \), which is the same control used by Wang, Wiseman and Milburn in the homodyne case described above. They showed that this increases the steady state concurrence for symmetric initial conditions to .45 of the maximum.

Simulations using an adaptive timestep Runge-Kutta algorithm evolved for sufficient time to establish a steady state density matrix execute swiftly for either version of the system. It is also possible to solve for the steady state solution to the jump equations analytically with the mathematics application Mathematica, and the results are in close agreement with numerical simulations. In the homodyne case it was not possible to duplicate this feat. Using numerical methods Carvalho and Hope generated Figure 4.5 [3], which shows the dependence of these peak concurrence values on the combination of the
driving strength $\Omega$ and the parameter in the control operator $\lambda$. The peak concurrence values are found on narrow peaks in complicated structures, and the values of these two parameters would have to be very tightly constrained in an actual experiment. On the other hand, this complicated structure would allow for a detailed comparison between theory and experiment. It is noteworthy that the jump measurement scheme will have no effect on a state in the antisymmetric space regardless of the choice of control, since the state $|4\rangle$ is still a steady state solution to the equation. This is because the action of the superoperator in Eq. 4.25 on state $|4\rangle$ is 0, and so the feedback operator $U$ will always be acting on 0. The homodyne scheme is not necessarily similarly kind to the otherwise maximally entangled antisymmetric state. Since there is a term with the feedback operator acting directly on the density matrix $\rho$ in Eq. 4.24, preservation of the antisymmetric steady state depends on the precise form of the feedback. Using $J_x$ control does preserve the independence of the subspaces, so with this choice of control the homodyne scheme also does not disturb the antisymmetric state. These results show an improvement in the generated entanglement using these control schemes, however the analysis can be taken further. In particular, it is not clear what the best choice of feedback operator would be for either scheme. Using a symmetric feedback is an obvious first choice in WWM’s scheme, since they use the cavity laser that affects both atoms to provide feedback. A symmetric scheme is not the best choice, however, and one reason why is the effect of spontaneous emission.

4.3.3 Spontaneous emission and local control

Thus far the spontaneous emission has been assumed to have a negligible effect on the behaviour of the system. When the spontaneous emission is turned on the effect on the two symmetric feedback schemes thus far discussed is dramatic, with the steady state concurrence greatly reduced even for relatively small values of $\gamma_1$ and $\gamma_2$ [3]. For simplicity $\gamma_1$ and $\gamma_2$ are presumed to be the same in what follows, although that restriction could easily be relaxed if there was some reason to do so, and numerically either choice is easy.
Entanglement through quantum control

Figure 4.6: This figure shows the effect of spontaneous emission on the steady state concurrence as a function of $\Omega$ and $\lambda$. The symmetric control $\exp(-i\lambda J_z)$ is used for each case, and the spontaneous emission is $\gamma/\Gamma = 0.01$. Even for this relatively small value of the spontaneous emission both the homodyne [4] (left) and quantum jump [3] (right) cases have their peak concurrence reduced to the point of being negligible.

The analytic calculation of the density matrix steady state for the jump feedback case is still possible with spontaneous emission included. The significant effect of spontaneous emission in the system is not totally surprising, however, since spontaneous emission introduces a mechanism by which the symmetry of the model is broken. This means with spontaneous emission it is now possible to transition between the symmetric and antisymmetric subspaces, and the steady state of the system will now be unique for a given set of parameters, and is in general very different to the steady state without emission. The antisymmetric and symmetric subspaces can and will interfere, and the steady state concurrence for both control schemes is drastically reduced. Figure 4.6 shows how stark the drop in steady state concurrence is for both the homodyne and jump measurements schemes, and although the jump scheme is less affected both schemes provide concurrence less than 0.1 of the maximum even for a very low value of the spontaneous emission $\gamma$.

The nature of the problem suggests a solution, however. The effect of spontaneous emission is so stark precisely because it is the only part of the master equation that is not symmetric with respect to atom interchange. However, the choice of symmetric feedback was an arbitrary one, and changing to a local feedback operator would mean that the spontaneous emission is not the only symmetry breaking term, in both the homodyne and jump equations. Carvalho and Hope tried a non-symmetric feedback operator with both the quantum jump [3] and homodyne [4] measurement schemes. They chose a feedback operator of the form $U = U_1 \otimes I$, which is a local control. The specific scheme used is $U = \exp(-i\lambda \sigma_x)$, with $\sigma_x = \sigma_1 + \sigma_1^\dagger$, which in a jump feedback experiment would be the feedback laser applying a pulse to only one of the atoms. This local feedback operator works better than the symmetric one even when the spontaneous emission is not included in the homodyne scheme, giving a higher maximum steady state concurrence, as shown in Figure 4.7, and the effectiveness is not affected as dramatically when spontaneous emission is included, simply reducing the height of the peak, as shown in Figure 4.8. It has been shown that in the absence of feedback a local control can raise the steady state concurrence...
Concurrence of Two Atoms in a Cavity

Figure 4.7: This figure shows the same comparison as Figure 4.5, except the feedback is now the local $\exp(-i\lambda\sigma_x)$ control. The peak in the plot of steady state concurrence as a function of $\Omega$ and $\lambda$ for the homodyne [4] (left) measurement is higher than it was for the symmetric case, but the quantum jump [3] (right) scheme has a much starker difference, where the steady state concurrence has its' maximum value except when one or other laser is off.

in the homodyne case to as much as 80% of the value of the Bell state [60].

It is however for the jump feedback that the effect of the change to local control is most stark. In the absence of spontaneous emission the local control produces a maximally entangled steady state for nearly all values of the control strength parameter $\lambda$ and the atom driving laser effective Rabi frequency $\Omega$, as shown in Figure 4.7. We can prove that the system has been moved entirely into the state $|4\rangle$. In fact, the areas where the concurrence is not at this peak value are those regions where one or other of the lasers has been turned off, and the zero concurrence result is expected. The drastic improvement in final concurrence is carried over to the case where the spontaneous emission is present, shown in Figure 4.8. Although the peak value is no longer 1, the maximum concurrence, which is determined by the ratio $\gamma/\Gamma$, is still high, with spontaneous emission of $\gamma = 0.01\Gamma$ resulting in a peak concurrence $\approx 0.95$. This peak value can also still be reached with a broad range of values for the two laser parameters. This is of tremendous interest to anyone intending to perform an experiment, since it means that the values of these two parameters do not need to be locked down as they would for the homodyne or symmetric jump cases. It is also of note that this result will occur for any feedback operator of the form $U = U_1 \otimes I$, not just the specific one mentioned above. These results were initially arrived at numerically, however for the jump case as has been mentioned it is also possible to solve the equations analytically [4], and the analytic solutions confirm the numerical results.

The reason for this drastic improvement in the jump case is clear from the analytic solutions. Except where one laser or the other is off the steady state of the system is the angular momentum state $|4\rangle$, and so the concurrence is 1. When spontaneous emission is a factor the analytic solutions can again be used to determine the steady state, which
Figure 4.8: As with Figure 4.6 this figure shows the effect of spontaneous emission on the steady state concurrence as a function of $\Omega$ and $\lambda$. Here the local control $\exp(-i\lambda\sigma_z)$ is used for each case, and the spontaneous emission is $\gamma/\Gamma = 0.01$. Unlike the symmetric feedback case both the homodyne [4] (left) and quantum jump [3] (right) cases have their peak concurrence reduced only moderately, and preserve the bulk of their structure. It is noteworthy the the jump feedback still produces a nearly maximally entangled state, even with the level of emission that destroyed both symmetric feedback schemes.

when expanded in powers of $\gamma/\Gamma$ reduces to [4]

$$
\rho_{44} \approx 1 - O(\gamma/\Gamma),
\rho_{ij} \approx O(\gamma/\Gamma), \ i, j \neq 4.
$$

(4.27)

The variation in $\Omega$ and $\lambda$ seen in Figure 4.8 is a consequence of the now introduced competition between the feedback and the emission, however the form of the steady state makes it clear that the concurrence will continue to be high.

The strong effect that the local control has on the system when spontaneous emission is included is also shown when the evolution of the concurrence for specific values of the driving and feedback strength is considered. Figure 4.9 shows the difference in the evolution of the concurrence when the system is prepared in the antisymmetric Bell state $|4\rangle$ for various values of the spontaneous emission, both with and without the control. The spontaneous emission, working as it does to move both atoms into the ground state, destroys the entangled antisymmetric state. Since there are no dynamics in the uncontrolled equation to reverse this process the concurrence drops with a decay rate being proportional to the emission strength, as would be expected. When the control is activated it competes with the emission, and the balance between the two factors determines the steady state concurrence. The third pair of lines on that figure also show that the control will work to oppose other dephasing terms, with that particular simulation including an additional dephasing term of the form $\gamma_{\text{deph}}D[p_i]\sigma_1$, with the spontaneous emission and dephasing coefficients $\gamma = \gamma_{\text{deph}} = 0.01\Gamma$. With the control active the concurrence reaches a steady state at a value lower than the emission alone would produce, but still high. The additional term could capture any negative effect on the system, which means that this feedback scheme will counteract anything damaging the entanglement. Provided the ratio $\gamma/\Gamma \ll 1$ the jump scheme will result in a highly entangled state. The ratio of the
spontaneous emission coefficient to the atom and cavity collective decay $\Gamma$ used to produce Figure 4.9 is consistent with some recent experimental cavities [61].

Also, as Figure 4.8 showed the variation in final concurrence changes slowly from the optimal value of $\lambda$, although large variations produce a greater dip, and it changes very little with fluctuations in $\Omega$, which as previously mentioned would be of great benefit to an experiment. What these figures do not show is the effect of forms of control other than that mentioned above. In the absence of emission a different local control for jump feedback cannot change the steady state, and thus the final concurrence, but it can change the rate at which a system beginning in a state other than $|4\rangle$ will approach it. In the presence of emission this decrease in rate can have an effect on the steady state concurrence, since this value is the result of a competition between the two rates acting on the system. However, the variations are not large, and are small for perturbations around the $\sigma_x$ choice made for the simulations shown[3].

4.3.4 Detection inefficiencies

An additional consideration that must be included in order to bring the model in line with experiment is the effect of detection inefficiencies on the feedback. Since applying the control at the appropriate times is a key part of any feedback scheme it is to be expected that poor detection will reduce the effectiveness of the scheme. An initial analysis of this extension is provided by Carvalho and Hope [3] for the jump case, and it indicates inefficient detection has a surprisingly small effect on the eventual concurrence. Adding imperfect detection to the jump model is made simple by the nature of the equation. A failed detection is a jump without feedback, so we simply include a detection efficiency parameter $\eta$, and modify the master equation so it reads

$$\dot{\rho} = -\frac{i}{\hbar} [\sigma_i, \rho] + \gamma_i \mathcal{D}[\sigma_i] \rho + \Gamma \eta \mathcal{D}[J_+] \rho + \Gamma (1 - \eta) \mathcal{D}[J_-] \rho.$$  \hspace{1cm} (4.28)
This equation has the desired properties when $\eta$ is 0 or 1, since it reduces to the case with no feedback or the case with perfect feedback respectively. This transformation does not change the earlier reasoning that ensured that the antisymmetric state $|4\rangle$ was the only steady state of the system [3], and thus with $\gamma_i = 0$ imperfect detection should only reduce the rate at which a given initial state is moved towards the state $|4\rangle$, and thus maximum concurrence, just as for an inferior choice of local control operator. This is shown in Figure 4.10, which also shows the behaviour when spontaneous emission is present. In this case, again as with an inferior control operator, the steady state concurrence achieved is reduced, although even with a detection efficiency $\eta = 0.5$, the reduction is not substantial in the limit where $\Gamma/\gamma \gg 1$, which is what has been considered here. The results in the paper reproduced here rely on numerical simulations, however they are validated by the analytic steady state calculations previously mentioned, that can be adapted to include the detector efficiency in addition to the other terms. However, in this case the analytic results do not provide one important piece of information, which is the increase in time it takes a system to reach the steady state. Since this is the more prominent effects of imperfect detection, it is clear that relying exclusively on steady state calculations would have been misleading in this case.

The next consideration is what effect inefficient detection has on the homodyne scheme, and whether the combination of imperfect detection and spontaneous emission might result in there being a region where the performance of the homodyne scheme is superior. Determining this analytically isn’t possible given the intractability of the homodyne scheme, so to obtain some certainty in this result it is necessary to test a large parameter space numerically. The comparison must be between the peak steady state concurrence in $Q$ and $\lambda$ for each scheme for a given combination of $\eta$ and $\gamma$.

For the jump feedback the plot in $\eta$ and $\gamma$ is shown in Figure 4.11. This particular version of the plot was generated using a large number of iterations of the program used to generate the numerical results presented earlier, since at the time it was first created the analytic solution for the jump steady state had not been devised. The analytic calculations produce an equivalent plot. Producing this plot numerically for very low detection...
Figure 4.11: This figure shows the highest steady state concurrence on a plot of $\Omega$ against $\lambda$ as a function of $\eta$ and $\gamma$ for a photodetection scheme and a local control operator $U_\text{fb} = \exp(-i\lambda J_z)$. The figure shows that the concurrence is still 1 for very low $\eta$ when there is no spontaneous emission as predicted by analytic calculations, and the contours of roughly constant peak concurrence run diagonally, indicating the competition between feedback and spontaneous emission. Even with spontaneous emission the steady state concurrence is high across much of the plot.

Efficiency indicates the increase in time required for the low detection efficiency systems to reach the steady state. The increase in time required becomes a significant factor in the simulations for $\eta \approx 0.2$ and the time required increases as $\eta$ is reduced, with the point for $\eta = 0.01$ requiring 6 times as long as the majority of points on the plot to get there. The $\gamma$ axis is cut off at $\gamma/\Gamma = 0.2$, however simulations using a lower density of points indicated that the general shape of the plot extends into the higher spontaneous emission region. The equivalent plot for the homodyne measurement requires the use of a version of the homodyne master equation including imperfect detection, which looks like this

$$
\dot{\rho} = -i\Omega[J_+ + J_-, \rho] + \Gamma D[J_+]\rho + \gamma_1 D[J_-]\rho + \gamma_2 D[J_-]\rho + \frac{1}{\eta} D[F]\rho - i\sqrt{\Gamma}[F, J_-\rho + \rho J_+].
$$

(4.29)

Using this equation Figure 4.12 is produced. Unlike the quantum jump version there are no analytic results to compare this plot to, however the duration of the simulations and individual points considered in detail provide confidence that these points are close to the best that can be achieved for each value. The simulations do take longer to execute for the homodyne equation, and so there are fewer points on this plot. At first inspection the values appear to be uniformly lower, with a similar behaviour to the equivalent jump plot. Of course, in the homodyne case the peculiarity introduced by the perfect values at very low spontaneous emission is absent. There are a couple of differences between this figure and Figure 4.11, for example the decay in concurrence appears to follow a slope with a slightly different character, and there is also a sudden dip close to $\eta = 0.05$. It also appears that there is a region where the peak concurrence is almost flat for high emission and low detection efficiency, with a value of the concurrence of $\approx 0.1$. Subtracting the homodyne and jump plots reveals that there is a small region where the homodyne feedback scheme provides a superior result to the jump, however the higher values fall within the sector bounded by detection efficiency $\eta \approx 0.04 - 0.3$, and spontaneous emission $\gamma/\Gamma \approx 0.0133 - 0.14$, in which region the overall values of concurrence are lower than the original...


Figure 4.12: This figure shows the highest concurrence on a plot of $\Omega$ against $\lambda$ for each value of $\eta$ and $\gamma$ for a homodyne measurement scheme and a local control operator. It is clear from this plot as compared to Figure 4.11 that the homodyne scheme is worse than the photodetection scheme across most of this range. The lower point density as compared to the jump plot is a consequence of the greater time necessary to simulate the homodyne equation for a given set of parameters, and the absence of an analytic check requiring that the simulations be evolved over a longer time to ensure they reach the steady state.

uncontrolled case. The maximum increase is $\approx 0.052$, which isn’t a big enough increase to make this region workable, and so although the homodyne provides a better result, both feedback schemes are resulting in concurrences that show minimal improvement due to feedback. Determining the precise reason for this behaviour would require further research that is unlikely to be of much value, and does not seem like it would provide any particularly useful insight. Regardless, it is clear that the jump scheme is better in the regime where any experiment is going to take place. This conclusion justifies neglecting the homodyne feedback scheme while investigating additional aspects of the jump scheme in Chapter 5. The comparison does indicate, however, that if for experimental reasons homodyne detection is possible with a higher quantum efficiency than the best available individual photodetector the homodyne scheme could provide better results and should be considered.

The results from Carvalho and Hope [3] strongly argue that an experiment to create entanglement between two atoms along these lines is feasible, and also that the entanglement that can be produced using photodetection and some form of local control are the best way to perform such an experiment. The theoretical calculations indicate a very high level of fidelity to the maximally entangled state as compared to other methods for generating entanglement described in section 4.2. Other calculations not presented here also indicate that unlike many of those schemes no specific steady state is required to begin the experiment, simplifying the process. However, there are some areas where further detail would be useful. One obvious consideration is the adiabatic elimination itself. As mentioned previously, cavities intended for QED experiments are usually designed to be in the strong coupling limit, that is, where the cavity coupling $g \gg \kappa, \gamma$ [13, 14, 15]. The adiabatic elimination, conversely, required that $\kappa \gg g, \gamma$. It is possible that the results from the adiabatically eliminated version of the equation are not applicable to a cavity far outside this regime. It is also not clear precisely where the adiabatic elimination breaks
Another area in need of further investigation is the behaviour of an experiment over the course of a single run. Since all of the above work is performed using density matrices and master equations, the results speak only about the ensemble average, the results obtained when the outcome of many experiments is combined. It would be of use to investigate what an experimentalist might expect to see over the course of a single run. These questions are considered in the next chapter.
In the preceding chapter schemes to create entanglement between two atoms held in an optical cavity were discussed. When the cavity has a decay constant $\kappa$ greater than the other parameters of the system a version of the equation of motion with the cavity adiabatically eliminated is be a good approximation. With this equation it was possible to achieve extremely high concurrence and thus entanglement using a photodetector to detect quantum jumps, and a local control where a laser applied a pulse to an atom whenever a detection occurred. This scheme performed better than a homodyne measurement and continuous feedback modulated by the photocurrent, which had previously been proposed. It was also shown that this high ensemble average concurrence was robust to spontaneous emission, and to imperfect detection, and that in the parameter regimes where control scheme produced high values of concurrence the jump scheme was more effective than the homodyne one. What was not considered is the behaviour of the system when the cavity is not suitable for an adiabatic elimination. This work was performed in conjunction with Carvalho and Hope, and has been published [4].

5.1 Outside the adiabatic regime

As was mentioned at the end of Chapter 4, QED experiments usually run in what is called the strong coupling regime, where $g \gg \kappa$ [13, 14, 15], and when this is true there is reason to doubt that the adiabatic elimination performed in Section 4.3.1 is valid. Consequently, although it is not easily susceptible to analytic investigation, the full master equation for the two atom system should be considered. It was also not clear ahead of time whether the adiabatic regime is the best one in which to perform such an experiment, and this question is answered by performing simulations in other regimes.

5.1.1 The full master equation with imperfect detection and spontaneous emission

The master equation for the model without feedback is Eq 4.15. In Chapter 4 the adiabatic elimination was performed before feedback and imperfect detection were added, but in the jump case it is easy to see how to make the necessary changes even without the adiabatic elimination. Since the photons leaving the cavity are being detected directly, the feedback operator should be added into the cavity decay term $\kappa D[a]$. Similarly, the change to include imperfect detection is made to the cavity decay term in the same way it was made to the collective atom and cavity term when it was first introduced. The resulting master
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The equation is

\[ \dot{\rho} = -i\Omega [(J_+ + J_-), \rho] + \eta \kappa \mathcal{D}[U a] \rho + (1 - \eta) \kappa \mathcal{D}[a] \rho + \gamma_1 \mathcal{D}[\sigma_1] \rho + \gamma_2 \mathcal{D}[\sigma_2] \rho - \frac{iq}{2} \left[(J_+ a + J_- a^\dagger), \rho\right]. \] (5.1)

This equation does not allow the analytic calculation of the steady state that was possible for the adiabatically eliminated version of the jump equation. Numerical simulation of this full equation, in contrast, is not substantially different from what is required for the adiabatically eliminated version. The inclusion of the cavity mode does drastically change the dimension of the density matrix, however. All the simulations discussed in the rest of this chapter are performed by treating the cavity mode using a truncated Fock basis.

This means that the density matrix dimension is increased by the number of levels of the cavity mode included, with the Hilbert space expanding from \(2^n\) to \((2N)\) dimensions, with \(N\) the number of Fock basis states included. This will lead to vastly increased execution time for the simulations. Attempts were made to optimise the calculations, however fundamentally the crucial point remains choosing where to truncate the Fock basis. It is relatively simple, for a given set of cavity parameters, to determine how big the cavity basis needs to be to ensure that the highest level simulated has an occupation of approximately 0. In general basis states reaching 4–10 photons proved sufficient, however some of the low \(\kappa\) simulations required significantly higher dimensions, which introduced some numerical inaccuracy. As an idea of the numerical scale, increasing the cavity basis size into high values can be the difference between simulations for particular values of each parameter taking hours or a day when only seconds or minutes are required when the basis size is in the 4–10 range. The degree to which the execution time is increased is determined by the other system parameters, so it is impossible to give a general description of the increase. There are also numerical problems that make the results inaccurate when the cavity basis size approaches 50 which impose a practical limit on the cavities that can be simulated. However, as will be seen, there is no reason to expect that cavities requiring such large basis states would be of interest for generating entanglement. It should also be noted that with the additional complications of a low \(\kappa\) cavity included the simple structures that were the result of the adiabatic quantum jump simulations vanish. The plateau structures in the adiabatic simulations are replaced by significant changes in \(C\) as \(\Omega\) and \(\lambda\) vary, especially in cases where the peak concurrence achieved is low.

Before launching into the analysis of cavities not covered by the adiabatic elimination the results of the full equation simulated in the adiabatic regime should be compared to the result from the adiabatically eliminated equation. Figure 5.1 shows the results of these simulations side by side with a reproduction of the equivalent pictures from Chapter 4, and it can be seen that the two pictures do indeed match.

5.1.2 Concurrence from the full model

As a first guess as to what would be a worthwhile region to consider using this full model, a generic high cooperativity system in the strong coupling region might be considered, since the ratio \(\Gamma/\gamma\), which is the cooperativity, was the important figure of merit in the adiabatic case. Picking a cavity with what would be considered good parameters can however lead to results very different from the adiabatic case, and also to low peak steady state concurrences. An example of this can be seen in Figure 5.2, which has a cooperativity of \(C = 180\) and yet has a relatively low peak concurrence of only \(\approx 0.61\). It is possible to
§5.1 Outside the adiabatic regime

Figure 5.1: This figure shows a comparison between the results of the adiabatic and full simulations in the adiabatic regime. The plots on the left duplicate the plots for $\gamma/\Gamma = 0$ (a) and $\gamma/\Gamma = 0.01$ (c) from previous work [3]. The plots on the right use the full master equation simulation with $g = 10$, $\kappa = 100$, which means that $\Gamma = 1$. This is well into the adiabatic range, and the agreement of the plots b) and d) with the equivalent adiabatic simulation results is close enough that there is no visible difference.

compare the peak concurrence of an equivalent cooperativity adiabatic system by looking at the matching value on Figure 4.11, which is $\approx 0.95$. In addition to the inferior peak concurrence, the simple plateau structure has also been lost, with peaks and oscillations in the surface. From this plot and others like it that are not included common features of the behaviour of the concurrence can be investigated, however. The first point to note are the symmetries of the structure in the plot. After considering a number of plots it becomes apparent that the peaks are periodic in $\tilde{\lambda}$, and symmetric through the origin, so that changing the sign of both $\Omega$ and $\tilde{\lambda}$ does not change the value. Careful inspection of Figure 4.8b shows that this is true for the adiabatic simulations as well, but with the reduced values of the concurrence that are the result of the strong coupling cavities it is more prominent, and thus easier to notice.

Another aspect absent from the adiabatic simulations is the drop in concurrence when $\Omega$ is large. The value at which this occurs changes, but for all the non adiabatic systems there is a driving which eventually becomes so great that the entanglement is destroyed. Although difficult to check numerically it is possible that this is caused by very few photons being emitted into the cavity by the atoms. If the first term of Eq. 5.1 becomes completely dominant the feedback will have no opportunity to act. The adiabatic simulations were performed with low $\Omega$ because the adiabatic elimination required $\kappa$ larger than every other parameter, and so the approximation is expected to fail when $\Omega$ of a similar order to $\kappa$. This failure is confirmed by Figure 5.3, which shows how high values of $\Omega$ cause the concurrence to drop even when the other parameters of the system are smaller than $\kappa$. The values for $\Omega$ beyond which the steady state concurrence is too low to be useful can be manipulated, since as would be expected changing the parameters $\kappa$, $g$, and $\eta$ changes the $\Omega$ scale of the peak, and multiplying all those parameters by the same amount only changes the scale.

An experiment using the jump control scheme could be most swiftly performed using a pre-existing experimental cavity, however the parameters used for cavity QED result in poor steady state concurrence, as can be seen in Figure 5.4. The two parts of
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Figure 5.2: Steady state concurrence as a function of feedback and driving strengths for a cavity with parameters \((g = 3, \kappa = 1, \gamma = 0.05, \text{ and } \eta = 0.7)\) MHz. Despite having a fair cooperativity \(C = 180\), the peak concurrence is much lower than an adiabatic cavity would manage with the same \(\gamma\) and \(\eta\), only reaching a value of \(\approx 0.61\).

Figure 5.3: In this figure we can see that for high \(\Omega\) the concurrence drops off for a cavity with parameters \((g = 5, \kappa = 25, \gamma = 0.05)\) MHz, which is within the domain of the adiabatic approximation when \(\kappa \gg \Omega\). It is clear that when this condition is not met the steady state concurrence drops. The figure shows only one quadrant since symmetries relate them.
Outside the adiabatic regime

Figure 5.4: In this figure the steady state concurrence as a function of $\tilde{\lambda}$ and $\Omega$ is shown for two experimental cavities, figure a) is a cavity at the Georgia Institute of Technology with $(g, \kappa, \gamma/2) = 2\pi(17, 7, 3)\text{MHz}$ [62], and figure b) is one quadrant of the plot for a cavity at the University of Bonn, with parameters $(g, \kappa, \gamma) = 2\pi(18.0, 0.43, 2.61)\text{MHz}$ [63]. In both cases the detection efficiency $\eta$ is set to 0.5. The cooperativities are very different but both cavities are evidently very poor for this experiment, with the Georgia cavity not even reaching $1 \times 10^{-5}$. In addition, the highest values are reached in a regime where the control is inactive. The Bonn cavity has concurrence values that are even smaller, and again doing so in the margins where $\tilde{\lambda}$ is tiny or 0, so the control is off.

this plot show particular experimental cavities. The first uses the cavity parameters for the cavity described by Fortier et al. at the Georgia Institute of Technology [62], with $(g, \kappa, \gamma/2) = 2\pi(17, 7, 3)\text{MHz}$, which means a cooperativity of $C \approx 7$, which is quite low. The simulation uses a detection efficiency $\eta = 0.5$, which is higher than the reported value. The detection efficiency could be increased without changing the QED parameters of the experiment, so the simulation used a better $\eta$ to determine whether this improvement alone would allow high concurrences. For these parameters the concurrence achievable by the feedback is not even 0.001, and for most parameters the feedback is so unhelpful that better results are achieved when the feedback isn’t actually applied. For small $\Omega$ the beginnings of the familiar shape of the concurrence plot is visible, but as $\Omega$ grows the feedback becomes actively harmful, and even when it is helping the values of concurrence are much smaller than the uncontrolled case. The second plot in the figure uses the parameters for the cavity described by Dotsenko at the University of Bonn [63], with $(g, \kappa, \gamma) = 2\pi(18.0, 0.43, 2.61)\text{MHz}$ and a cooperativity of $C \approx 290$. The much higher cooperativity is no help, with the plot over a single quadrant showing concurrences even lower than the other experimental cavity when the result is anything other than 0, and most of the spikes are for precise values of the parameters which would be difficult to lock on to. Clearly neither of these cavities could be used in an experiment to create and measure concurrence.

Considering the adiabatic case it is clear that the only objective for a higher concurrence was to have a higher cooperativity. Outside the adiabatic regime this is no longer true, and it is therefore worth considering what the important parameter or ratio of parameters might be. In order to determine this, a series of plots where the cooperativity is held constant but the ratios between the parameters changes can be considered. Inspecting such a series, as shown in Figure 5.5, suggests a solution to the problem. It appears that the plots with a higher $\kappa$ and a lower spontaneous emission have a higher
peak concurrence.

5.1.3 Optimal parameter regimes for maximising concurrence

It actually isn’t surprising that a higher $\kappa$ might be a prerequisite for a high steady state concurrence. The investigation at the end of Chapter 4 showed that the steady state concurrence was determined by the competition between the feedback and the detrimental factors, such as the spontaneous emission. In the adiabatic case the effectiveness of the feedback was determined by the parameter $\Gamma$ and the measurement efficiency $\eta$, since $\Gamma$ controlled the rate at which photons escaped from the system. In the full simulation, however, the rate at which photons leave the system is the decay rate from the cavity, $\kappa$. Considering a single trajectory a low $\kappa$ means that photons will be held in the cavity longer, which means that the probability that a photon will be reabsorbed by an atom rather than emitted is increased. This will affect the ensemble average by reducing the contribution of the feedback term, the coefficient of which includes $\kappa$. Therefore, the reason why the good QED cavities have such poor performance for the concurrence is because they have sought to minimise the cavity decay, and in order to have a good cavity to generate entanglement via our feedback scheme it is necessary to ensure that $\kappa \gg \gamma$.

This effect can be seen in Figure 5.6, which has parameters $g, \kappa, \gamma = 2\pi(5, 3, 0.01)\text{MHz}$ and a cooperativity of $C = 833$. Interestingly, although with $g \approx \kappa$ the adiabatic elimination would be of dubious accuracy, the plot in $\Omega$ and $\lambda$ strongly resembles the plots generated in the adiabatic regime. The greater importance of the ratio $\kappa/\gamma$ can be seen by the comparison with Figure 5.7, which is only changed by lowering $\kappa$, thus increasing the cooperativity but generating a lower peak steady state concurrence. From this and other plots not shown the importance of $\kappa$ is clear, however, how the system behaves as $g$ changes is not.

As mentioned previously, increasing $g$ would be expected to increase the $\Omega$ range of the plot. Also, $g$ represents the coupling between the cavity and the atoms so a higher $g$ could be better, since it will increase the probability of a photon being placed in the cavity. The real picture is more complicated, as is shown in the series of plots in Figure 5.8. As expected when $g$ is too low the concurrence suffers, and an increase leads to the structure similar to the adiabatic case. However, past some value of $g$ in this case the peak concurrence no longer increases, and the structure merely stretches in $\Omega$, and appears to be deteriorating, although the plateau structure makes it hard to tell if the peak value has been reached in this range. In addition, increasing $g$ leads to the simulations requiring more time to reach the steady state and higher values of $\Omega$ to be included in the parameter range to see the peaks, as inspection of the axes will show. This consequence of the changing scale of the structure means that it takes increasingly large amounts of time to perform the numerical simulations, which makes it impractical to extend the range in $\Omega$ indefinitely to determine what the peak value of the concurrence is. It is also worth noting that the inferior plot in that figure is in fact the one where the adiabatic approximation is most appropriate, which suggests that although the high peak concurrence plot looks a lot like an adiabatic plot the adiabatic approximation itself is not in fact valid for parameters in this regime. Interestingly the concurrence in the narrow region where Figure 5.8a) has its’ peak value is close to the result from the adiabatic simulation, although what this might mean is unclear.

The behaviour of a changing $g$ may be more obvious with a series of plots where the best case scenario is not as good as in Figure 5.8, and this is shown in Figure 5.9.
§5.1 Outside the adiabatic regime

Figure 5.5: This figure shows the steady state concurrence as a function of $\Omega$ and $\lambda$ for 4 different cavities, which all have the same cooperativity. For all of these plots the detection efficiency $\eta = 0.5$, and the cooperativity $C = 100$. For plot a) $(g = 5, \kappa = 1, \gamma = 0.25)$MHz, for plot b) $(g = 2.5, \kappa = 2.5, \gamma = 0.25)$MHz, for plot c) $(g = 2.5, \kappa = 1, \gamma = 0.0625)$MHz, and for plot d) $(g = 5, \kappa = 2, \gamma = 0.125)$MHz. Note the starkly differing values of concurrence as evidenced by the changing scales. Comparison of the plots suggests that a high $\kappa$ produces better results, and the comparison between plot c) and plot d) shows that when all three cavity parameters are doubled the only change is that the $\Omega$ scale is doubled as well, which is expected. Also, for plot b) the feedback is so weak compared to the spontaneous emission that the steady state concurrence is only non-zero for cases where the feedback itself is turned off.
Beyond the Dicke Model

Figure 5.6: Steady state concurrence as a function of feedback and driving strengths for a cavity with parameters \((g = 5, \kappa = 3, \gamma = 0.01, \eta = 0.7)\)MHz. This cavity is in accordance with the criteria for a high peak concurrence, and the shape of the plot is similar to that seen in the adiabatic case. The cooperativity \(C = 833\) is high, however decreasing \(\kappa\) to increase \(C\) would result in a lower peak concurrence.

Figure 5.7: The high concurrence cavity in Figure 5.6 with \(\kappa = 0.75\)MHz, which means the cooperativity is now a huge \(C = 3332\), but both the peak concurrence and the general plot have deteriorated.
Figure 5.8: This figure is a comparison between 4 different plots where only the cavity coupling $g$ is changed. The starting point was the simulation resulting in the high concurrence plot in Figure 5.6, which is reproduced as figure b) here. The constant parameters have values ($\eta = 0.7, \kappa = 3, \gamma = 0.01$)MHz. The values of $g$ are a) $g = 1$MHz, b) $g = 5$MHz, c) $g = 20$MHz, and d) $g = 50$MHz. Although increasing $g$ from plot a) to plot b) has resulted in an improvement, the further increase to plot c) has if anything led to deterioration, and the shift to even higher $g$ for plot d) seems to have decreased the peak value further. In addition, increasing $g$ leads to the simulations requiring more time to reach the steady state and higher values of $\Omega$ to be included in the parameter range to see the peaks, as inspection of the axes will show.
Figure 5.9: This figure is another comparison between 4 different plots where only the cavity coupling $g$ is changed. In this case the starting point was the simulation resulting in Figure 5.5 c). The constant parameters have values $(\eta = 0.5, \kappa = 1, \gamma = 0.0625)\text{MHz}$. The values of $g$ are a) $g = 0.5\text{MHz}$, b) $g = 2.5\text{MHz}$, and c) $g = 10\text{MHz}$. As before an increase in $g$ up to a point somewhere near the order of magnitude of $\kappa$ is valuable, and after this point the principle effect of the increase is to stretch the $\Omega$ scale of the plot, with a possible slight deterioration of the peak concurrence value.

In this figure the same increase of peak concurrence to a point is seen, and again the peak concurrence stops increasing when $g$ reaches a value similar to $\kappa$. Once again, the increase in $g$ appears primarily to be increasing the range in $\Omega$ while doing little else. The peak value is not decreasing significantly, which suggests that in an experiment where fine enough control of $\Omega$ and $\lambda$ is available the parameter $g$ need not be all that tightly controlled. However, the behaviour in $g$ is probably complicated enough that it would take a more thorough investigation to determine precisely how the plots change as $g$ does. Also, an attempt to increase the value of $g$ even further in these plots ran into the numerical problem previously mentioned when very high numbers of basis states for the cavity are required. This is a consequence of the significantly increased average photon number in the cavity when the cavity coupling is high.

The summary of what has been determined from the full master equation is thus that in a physical experiment the most important point is to keep $\kappa \gg \gamma$, and that a value of $g$ of a similar order of magnitude or greater than $\kappa$ will lead to the best results. Since the
spontaneous emission is generally controlled by the atomic species chosen, achieving this 
combination will require a cavity with high coupling and high decay. When the experiment 
is very good it will not be especially important what values of $\Omega$ and $\lambda$ are used, since 
the variation in those parameters will be small, as with the adiabatic results. Values 
of concurrence that are higher than the uncontrolled case can be achieved with a cavity 
that doesn’t fit these criteria. In this case tighter control of the lasers and probably a 
simulation similar to these ahead of time will be required. Since the plots where the peak 
concurrence is highest are so similar to the adiabatic plots, it also seems that the answer 
to the question asked at the beginning of this section is that the best region in which to 
perform the experiment is either the adiabatic regime, or this second sweet spot where $g$ 
is of order $\kappa$. The adiabatic region is likely an easier combination of cavity parameters to 
work with.

5.2 Single shots

When performing a quantum computing experiment, or indeed in any situation where the 
objective is to prepare a pair of entangled atoms and then use them, it will be necessary 
to know how a particular pair of atoms might behave in cases where the ensemble average 
has a high or low concurrence, and whether there is some signal that the two atoms are 
entangled that would be available to an experimentalist. This cannot be investigated with 
the master equations presented thus far, since they describe the behaviour of the ensemble 
average.

Considering the behaviour of single experiments requires a simulation that tracks the 
stochastic behaviour of a single instance. Such a simulation can be created using the 
method of quantum trajectories [18, 19] as described in Chapter 2, which evolves a sys-
tem under an effective Hamiltonian and decay terms, and applies stochastic jumps when 
appropriate. Simulations using this method can describe correctly the behaviour of the 
quantum jump measurement used in our control scheme. Of the six terms in Eq. 5.1 
the first and last, which cover the driving of the atoms and the cavity coupling, will be 
part of the effective Hamiltonian. The four decoherence terms which take the form of the 
superoperator $D$ acting using various operators of the system will provide the four possible 
jumps in the equation and also the non-hermitian elements of the evolution. The possible 
jumps are a photon being detected and feedback applied, a photon being released but not 
detected, and spontaneous emission from either atom. The probability of each jump is 
dynamically calculated by the simulation, and which is more prevalent is a function of the 
system parameters, so the behaviour of the simulation will vary widely when the system 
parameters are changed.

It was mentioned during the discussion of local control with jump detection in Section 
4.3.3 that in the adiabatic case when the two atoms are in the antisymmetric state $|4\rangle$ 
and the spontaneous emission is 0 there will be no more quantum jumps, and thus no 
more detections of photons and feedback pulses. The antisymmetric state coupled to an 
empty cavity is still a dark state in the full equation, and this means that when the two 
atoms are in the maximally entangled state there will be no photons in the cavity, and 
thus nothing to escape and provide a signal at the detector. The single shot simulations 
agree with this, and will only move from the maximally entangled state if a spontaneous 
emission event occurs. The fact that the cavity is empty when the system is entangled 
can be seen in Figure 5.10, which is a plot from a point in Figure 5.6 on the plateau 
where the ensemble average concurrence is very close to 1. In this case a small burst
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Figure 5.10: This figure shows a single trajectory for the parameters \((\Omega = 1.2, \lambda = 1.5, g = 5, \kappa = 3, \eta = 0.7, \gamma = 0.01)\)MHz. The upper plot shows the concurrence, and the lower plot the number of photons in the cavity. For parameters that result in an ensemble average concurrence very close to 1 it is not surprising that the atoms are placed in the antisymmetric state swiftly and remain there for the duration of the simulation. The absence of photons in the cavity once the antisymmetric bell state is reached is in line with the expected behaviour.

of photons early on has provided enough control to place the atoms in the maximally entangled state, where they remain for an extended period, and if a spontaneous emission event had occurred they would have been placed in the ground state and then returned to the fully entangled state swiftly. Simulating a path in a system with larger spontaneous emission and a lower value for \(\Omega\) provides an inferior but also more interesting plot which is shown in Figure 5.11. In this case, comparison between the photon and concurrence plots shows a number of drops in photon number without changes to the concurrence, which are photons that left the cavity but were not detected. This explains the width of the windows where the concurrence is not at the peak value, with the undetected photons delaying progress towards the peak state. The atoms will reach the peak state and remain there until a spontaneous emission occurs, and when such an event occurs the control drives the atoms back to the maximally entangled state in a varying time interval similar to the time it took them to reach it initially. Figure 5.12 shows that even when the ensemble average concurrence is a somewhat dismal 0.26 on a single run of the experiment the atoms can spend some time in the maximally entangled state, although the interval before spontaneous emission knocks them back down may be very short.

What these quantum trajectory simulations make plain is that in experiments where producing high entanglement states reliably and with a long lifetime is important the high concurrence cavities are clearly the way to go. Since concurrence cannot be measured in an experiment without many instances, the appearance of any particular trajectory for a given set of system parameters doesn’t provide much information about the steady state concurrence that can be expected. Therefore it isn’t possible to infer from a few such trajectories whether a cavity is a good one for this experiment. A large number of these trajectories taken together, however, should provide an approximate density matrix, and if the concurrence is calculated from this average the results from earlier in the chapter
Figure 5.11: This figure shows a single trajectory for the parameters $(\Omega = 1, \lambda = 1.5, g = 5, \kappa = 3, \eta = .7, \gamma = .1)\text{MHz}$. With more spontaneous emission than Figure 5.10 the atoms do not remain in state $|4\rangle$ once they reach it, but are returned to it by the control after each emission event. The intermittent dark periods in the cavity matching the periods of maximal entanglement support the statement that when the atoms are in the maximally entangled state there are no cavity photons, and thus no signal.

Figure 5.12: This figure shows a single trajectory for the parameters $(\Omega = .825, \lambda = 1.5, g = 5, \kappa = 2, \eta = .5, \gamma = .125)\text{MHz}$. This is the cavity used for Figure 5.5d), and the ensemble average steady state concurrence for these values of $\Omega$ and $\lambda$ is 0.26. It is noteworthy that even though the average is low, the atoms can still reach the maximally entangled dark state and remain there for a brief period.
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will be reproduced if enough trajectories have been used to create it.

The questions posed at the end of Chapter 4 have now been addressed. It is apparently better to work with a cavity either in the regime where the adiabatic elimination is valid, or when \( g \) is close to \( \kappa \) and \( \kappa \gg \gamma \), and the strong coupling limit combines parameters that make it getting a high steady state concurrence improbable. Having a high value for \( \kappa \) and a low spontaneous emission is the key to getting a good result. The single trajectory simulations indicate that an experimentalist would have a signal to indicate when the system is maximally entangled. Therefore, the information necessary to attempt an experimental realisation of the jump feedback scheme is now available. Since the feedback is Markovian adding the control to an already existing experiment should be possible, however the requirement for a cavity otherwise unusual in the QED world probably requires a new cavity. Although cavities exist that can trap two atoms and individually address them [64], and cavities with a high cooperativity are common, combining this with a cavity where \( \kappa \) is large may be technically difficult. Once a system with \( \kappa \gg \gamma \), and \( g \) either small or of order \( \kappa \) is constructed it would not be difficult to use the methods in this chapter to determine how best to choose the feedback and driving strengths.

5.3 Future directions

The more detailed theoretical consideration of the experiment indicate that it continues to enjoy advantages over the other means of generating entanglement described in section 4.2. If the cavity chosen for the experiment is a good one then the peak concurrence, which is a proxy for the fidelity to the maximally entangled state used as a criterion by the other methods, is high. The individual run simulations also indicate that when the concurrence is not high the maximally entangled state is being created for brief periods, which may be sufficient for the aims of an experiment. The method also has no need for single photon sources, does not require carefully prepared initial states, and includes a strong signal of success. Because of these properties there is good reason to consider an experiment using this scheme instead of other methods.

There are still unanswered questions about the interrelation of the various parameters, which are primarily a function of the large parameter space. Optimal parameters for achieving high concurrence have been found, however, so there is no need to answer them. Of more interest would be extending the scheme to increase the numbers of atoms that can be entangled using feedback. The method to calculate the concurrence may be extended for more than two atoms [65, 66] and has been implemented numerically for up to 8. An adiabatically eliminated equation similar to Eq. 4.28 described in Section 4.3.3 could be used to determine an effective feedback scheme for more than two atoms, and if this can be generalised then many atom experiments might be feasible. The rapidly increasing dimensionality of the master equation when more than two atoms are used will make this difficult, however.
Chapter 6

Conclusions

6.1 Summary

The purpose of this thesis has been to provide an idea of how quantum control can be used, demonstrate to an extent why quantum control is necessary, and to cover in detail a particular problem relating to generating entanglement between two atoms. In the early chapters the first two objectives were the focus, with an introduction to several of the methods used in quantum control theory. This was supplemented by sketches of several cases in which those methods have been used, either to answer questions that can only be addressed with a full quantum treatment, or to provide superior results to those accessible classically. The extended example of cooling a Bose-Einstein condensate showed both an approximate semiclassical treatment of the problem and a quantum model that included the measurement back-action. The research indicating the existence of control immune dark states in the semiclassical model was detailed, followed by the quantum treatment that indicated that the measurement back-action that the classical theory did not account for would break these states. In order to use the control scheme in an experiment, a model that includes the outcoupling and pumping from the semiclassical treatment and the back action from the quantum one is necessary. This was followed by an application of optimal control theory and specifically dynamic programming to optimise control of an atom in a well, the equations for which are also an approximation of the equations for a non-interacting BEC.

After these overview chapters the focus moved to the investigation of a model of two atoms held in an optical cavity and driven through feedback to create and stabilise entanglement between the atoms. This is the area in which my original work was focused and a quantum treatment of the problem is essential to a full understanding of the system.

In the absence of spontaneous emission the two atoms will become partially entangled. The degree of this can be determined using a measure of entanglement called the concurrence which ranges from $0 - 1$ [1], and it has been shown by Schneider and Milburn that the atoms reach a concurrence of approximately $0.11$ [55]. Wang, Wiseman, and Milburn showed that using a homodyne detection scheme and a photocurrent proportional feedback could raise the concurrence to $0.31$ [2]. This is a significant improvement, however their model failed to account for the spontaneous emission of the trapped atoms.

In this thesis I described the alternative feedback scheme considered in [3], that used a simple photodetector and applied a control laser pulse whenever a photon was detected. This scheme achieved a steady state concurrence of $0.45$, however it also did not account for spontaneous emission. When Carvalho included spontaneous emission, both this and the homodyne schemes were unable to create significant entanglement, because there was no mechanism where the symmetric and antisymmetric subspaces of the two atom sys-
tem where connected other than the spontaneous emission, and they interfered. Carvalho
overcame this problem by changing to using a non-symmetric feedback operator, and dis-
covered that this led to much better results for the homodyne scheme and exceptional
results of almost perfect entanglement for the quantum jump scheme, depending on the
strength of the spontaneous emission. It was also confirmed by a combination of Carvalho’s
analytic calculations and simulations that I performed that for combinations of the detec-
tion efficiency and spontaneous emission where the achievable entanglement is significantly
greater than that reached by an uncontrolled system the quantum jump scheme produces
better results.

However, all of this work was performed using equations that had undergone an adia-
batic elimination of the cavity in which the atoms where being held. This approximation
indicated that any photons emitted by the atoms into the cavity would be released before
they could further influence the system. For a poor cavity this is appropriate, however
many quantum optics or quantum information experiments are performed with high fi-
nesse cavities for which this approximation is invalid. Because of this shortfall, and the
possibility that a cavity of this sort might achieve still better results a simulation of a
master equation including the cavity was necessary.

I performed these simulations, modifying pre-existing code to account for the cavity
by including the cavity levels up to a cutoff chosen as appropriate for each simulation, and
was able to show that cavities in the strong coupling limit, with high cavity coupling and
low cavity decay rates, are in fact a poor choice for this experiment. The reason for this is
that the photons emitted from the cavity are the source of information for the feedback,
which means that if they are held in the cavity for a significant period of time the feedback
rate decreases. This means that the adiabatic regime is in fact the area where the best
results occur. I have also investigated carefully the behaviour of the system when the
parameters are varied individually, which allowed me to determine that it is also possible
to do well in a regime where the parameters $g$ and $\kappa$ are approximately equal, with the
ratio $\kappa/\gamma$ being the most important factor determining the effectiveness of the feedback.

Also provided were simulations using a single path which would simulate possible single
outcomes of this experiment. This provided more detailed ideas of the sort of behaviour
that an experimentalist might see, in particular the signal at the detector that indicates
that the atoms are for the moment entangled, and what an ensemble average concurrence
might translate to in terms of such an individual experiment.

Before this research there were theoretical questions that needed to be addressed before
an experimental implementation of this system could be seriously considered. Whether
the jump scheme is everywhere better than the homodyne, whether strong entanglement
is achievable with present cavities, what system parameters would provide high values of
entanglement, and how a particular instance of the experiment would behave. These ques-
tions have here been addressed, and therefore, as was mentioned at the end of Chapter 5,
there should be enough information to actually perform the experiment. An experimental
setup with two atoms equally coupled to a cavity mode with individual atom addressing
exists [64], and the feedback scheme itself is Markovian and so in principle straightforward
to create. There is a difficulty in that the optimal conditions established by the full
model equations are for a cavity with strong coupling, low spontaneous emission, and a
high cavity decay, which may require different geometry than the cavities conventionally
used by quantum optics experiments. Since cavities commonly in use for cavity QED
experiments are usually designed to have very low cavity decay it will be necessary to
create a new cavity for this purpose. It also may be difficult to achieve low spontaneous
emission and high coupling in a low finesse, and thus high decay cavity. It would also be necessary to implement a scheme to detect whether the atoms are in fact entangled, which may be complicated by the geometry needed to create the sort of cavity required by the experiment.

6.2 Future directions

The two-atom model is sufficiently characterised in this work, and is ready for experimental implementation. There are extensions of the model that could be considered, however. It has been suggested that a low spontaneous emission regime could be reached by making use of raman transitions between ground atomic hyperfine states, as do the methods described in section 4.2. This would make the two levels in the atom the hyperfine ground states, thus drastically reducing the spontaneous emission rate. If this worked there would be a wider range of cavity parameters usable to generate high peak concurrence. In principle this approach could work, and at first glance it seems that the equation of motion for such a system would reduce to the equation modeled throughout this thesis. If this were confirmed by analytic calculations then it would present an alternate means of performing this experiment. Whether using raman transitions with a cavity with different parameters would be easier or better would depend on the specifics of a preexisting experimental setup.

One of the motivations for investigating methods for generating atom-atom entanglement suggests another possible direction for this model. This is the prospect of quantum computing, for which vastly larger numbers of atoms would need to be entangled in this manner. A system of several entangled atoms would also be interesting to create in its own right. Consequently an investigation of a version of the system with multiple atoms could be fruitful. There is a formulation of the concurrence for more than two atoms [65, 66], and it has been implemented numerically for up to 8 atoms. This allows a numerical analysis of the full master equation similar to what has been presented here. It should also be possible to adiabatically eliminate the cavity in the same way as for the two atom case, which would allow a rapid evaluation of the possible value of the quantum jump feedback scheme, perhaps using the same sort of non-symmetric feedback operator. The added complexity of a three or more atom system would likely render an analytic investigation impossible, which would in turn mean that if the feedback schemes from the two atom case are of little value it may be difficult to deduce what to try to get a better result. On the other hand, such an investigation may find that the three atom case can be controlled in a similar vein to the two atom one, which might lead to a generalised system that can be used for an arbitrary number of atoms. If such a scheme could be constructed it would be of significant value in quantum information.

It is worth noting that the numerical methods necessary to simulate the multiple atom model would likely not be different from those used in the two atom case. These methods may however be rendered numerically intractable due to the significant increase in processing power required to deal with the rapidly increasing dimensionality. This is especially true of the master equation without the adiabatic elimination, since it already involves more variables due to simulation of the cavity levels. For many atoms it may become necessary to simulate many trajectories and calculate the average density matrix, rather than evolving it directly with a master equation.
Bibliography


