

Dissipative Solitons and Their Applications

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Declaration

All of this research is my own work except where otherwise acknowledged.

A handwritten signature in black ink, consisting of several overlapping, fluid strokes that form a cursive, somewhat abstract shape.

Wonkeun Chang

December 2009

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Abstract

Stationary solitons are well-known in media where they exist due to a balance of nonlinearity and dispersion. When more elements are considered, a richer variety of solitons can exist. In this thesis, dissipative solitons governed by the complex Ginzburg-Landau equation (CGLE) and its variants are presented. These exist in dissipative systems, *i.e.* where loss and gain occur. A detailed numerical study around the region of transition between solitons and fronts has revealed that it is rich with bifurcations. A technique called the method of moments has been applied to the CGLE with several trial functions to obtain low-dimensional systems. These reduced systems allow one to obtain approximate solutions of the CGLE, and, in particular, their region of existence.

The concept of dissipative solitons can be applied to the generation of optical pulses from a laser oscillator. One important application is the generation of high-energy pulses. The discovery of the dissipative soliton resonance phenomenon, revealed in this thesis, provides a universal theory behind recent experimental achievements in this field. An approximate expression for the location of the resonance has been obtained by constructing a reduced model using the method of moments. The existence of these high-energy pulses in the anomalous dispersion is supported by this result, and this has been confirmed by numerically solving the CGLE. The resonance turns out to be a robust phenomenon. It persists when the model is strongly perturbed with parameter management.

Dissipative soliton interactions in hybrid mode-locked lasers are studied by introducing a time-periodic term into the CGLE. The study of the effects of this active mechanism on a dissipative soliton pair has shown that the influence depends on the frequency of modulation relative to the initial pulse separation. The soliton pair may exist in different regimes of evolution, depending on the size of the modulation depth. The transition from one regime to another occurs in the form of a bifurcation. Exploiting the observations made in this study will allow for more accurate control over the generation of pulse trains in fibers.

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Chapter 1

Introduction

1.1 Nonlinear Dynamical Systems

A dynamical system is a physical reality that is modeled using mathematical equations. It provides a mathematical formalization of the rules describing the evolution of objects over the course of time or distance. Often, differential equations or difference equations are employed in describing these systems. Such an approach is useful in modeling and understanding systems in every branch of science, including physical, chemical and biological systems, and provides understanding of various phenomena in these systems.

In some cases, the system output is proportional to its input, thus making it linear. Such a system can be described by linear differential or difference equations, and their solutions can be obtained in a relatively simple manner. However, since most physical systems in nature are inherently nonlinear, the associated mathematical models are nonlinear dynamical systems expressed with nonlinear differential or difference equations. Although these may be approximated by a linear system under some specific conditions, the resulting solutions are generally insufficient for a comprehensive understanding of the system behavior.

These different types of dynamical systems can be described using a basic example from classical mechanics, presented in Fig. 1.1 [1]. This example considers the simplest case where a system has only one degree of freedom, *i.e.* the angle from the vertical θ . One could couple several identical pendula to represent a system with more than one degree of freedom.

The system is approximately linear, if the angle of oscillation θ is small, as

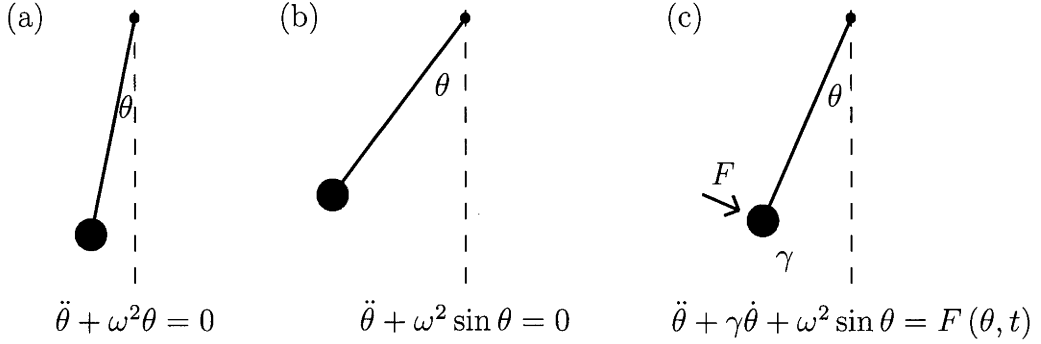


Figure 1.1: A pendulum example showing a dynamical system with one degree of freedom. θ is the angle from the vertical and the angular frequency ω is a constant. (a) The system is linear if θ is small and (b) nonlinear if θ is not small. (c) In a dissipative system, the damping coefficient γ is introduced, and an external force F is required to maintain the oscillations. [1]

shown in Fig. 1.1(a). If several of these linear oscillators are coupled together, the solution can be written as a linear superposition of the modes of a single pendulum. The system becomes nonlinear when the amplitude becomes larger. This case is depicted in Fig. 1.1(b). The linear approximation is invalid here, and the solution of a coupled set of such systems cannot be written as a linear superposition of the normal modes.

A further generalization can be made in the pendulum example by allowing energy exchange with the system's surrounding, as shown in Fig. 1.1(c). In this case, the total energy of the system is no longer conserved. The presence of friction, denoted by γ , causes the oscillation to be damped if the external energy supplied to the system is not enough. This is called a 'dissipative system'. For a coupled set of dissipative systems, the solutions, in general, cannot be found analytically, and are therefore obtained using numerical methods. Indeed, most real systems are not 'closed systems', but have energy or material interaction with their surroundings. Hence, the inclusion of dissipation in dynamical systems is essential in understanding the general behavior.

In this thesis, nonlinear dissipative systems are studied closely in the context of nonlinear optics. More specifically, the investigation focuses on soliton solutions of the so-called 'complex Ginzburg-Landau equation' (CGLE), which is a partial differential equation (PDE) with an infinite number of degrees of freedom that can

be used to model various nonlinear phenomena, including the generation of optical pulses in laser oscillators.

1.2 Solitons

‘Solitons’ refer to localized solutions of certain PDEs, which possess the remarkable property of preserving their shape during propagation. The first scientific report of soliton observation was published in 1844 by a Scottish naval engineer named John Scott Russell [2]. He observed a ‘heap of water’ in a narrow canal that propagated without change of form or diminution of speed over several kilometers. In his publication, Russell called this phenomenon the ‘wave of translation’.

The term ‘soliton’ was later coined by Zabusky and Kruskal in 1965, after a solitary wave pulse that moves with a uniform velocity proportional to its amplitude [3]. It was found by numerically solving the Korteweg de Vries (KdV) equation, which was introduced in 1895 to describe weakly nonlinear shallow water waves [4]. The KdV equation is a special case of a conserved system where the PDE is integrable. The development of the inverse scattering transformation [5] allowed one to find soliton solutions of integrable systems analytically. In 1972, Zakharov and Shabat employed this technique on a different integrable PDE, called the nonlinear Schrödinger equation (NLSE), and were also able to find its soliton solutions [6]. Owing to these historical reasons, in the past, the use of the term ‘soliton’ was limited to the localized solutions of integrable systems. However, in recent years, its use has been extended to include the localized solutions of more generalized systems.

Dissipative solitons refer to localized structures which exist for an extended period of time in dissipative systems. Their formations and structures are not generally defined by the initial conditions, but rather by the rules of the system, *i.e.* by the system parameters such as gain, loss, dispersion and nonlinearity. For a simple system that can be modeled by a single equation, these rules may be reduced to a balance between gain and loss, as well as a balance between dispersion and nonlinearity. Such analogies can be extended to more complicated systems, like the presence of animal species in nature which can be represented as elaborate forms of solitons [7]. They are localized ‘structures’, and their existence depends on the environmental conditions such as the temperature, pressure and humidity.

Optical pulses emerging out of passively mode-locked lasers are further examples of dissipative solitons. When the laser is in the stable regime of its operation, the output pulses are the result of a balance between the dispersion in the cavity and the nonlinear effects in the mode-locking device, as well as a balance between the energy supplied from the pump source and the loss in the cavity. In fact, passive mode locked laser systems can be described by using a master equation approach, which was initiated by Haus in 1975 [8]. In one form or another, such an approach leads to the CGLE [9]. One of the significant advantages of using this technique is that the properties of the Ginzburg-Landau equation and its solutions have been studied to a great extent in the physics literature [10]. Despite being approximate, the master equation describes in detail the majority of effects in the cavity, and allows one to predict all major features of pulse generation [11].

The possibility of the use of solitons in the context of nonlinear optics was first suggested by Hasegawa and Tappert in 1973 [12]. In optical communications, each bit of data is transmitted by propagating an optical pulse through a fiber. Dispersive and nonlinear effects in optical fibers cause the traveling pulses to be broadened and deformed, which may result in inter-symbol interference. In order to avoid such degradation of the communication system, pulses need to be regenerated repeatedly before they become significantly distorted. Apart from being expensive, such regeneration also becomes a bottle-neck in the system design, due to the slow response time of the electronic devices involved. Hasegawa and Tappert proposed that a balance between the dispersive and nonlinear effects in optical fibers could allow for optical solitons to be formed. As a result, they can propagate for a long distance without any significant deformation, and this in turn eliminates the need for the pulse regeneration.

The first experimental observation of optical solitons was made by Mollenauer, Stolen and Gordon in 1980 [13], when an optical source capable of generating pico-second pulses became available. In this experiment, they successfully propagated optical solitons with the width of 7 ps at a wavelength of $1.55\text{ }\mu\text{m}$ over a 700 m long single-mode fiber. Since this breakthrough, numerous experiments have been performed in order to transmit optical solitons at higher bit-rates over a longer distance [14,15]. In particular, advances in fiber amplifiers [16] allowed all-optical transmission of data with multiple wavelength channels in a single fiber using the wavelength division multiplexing, thereby substantially increasing the system band-

width. In Australia, a commercial communication system based on optical solitons was deployed between Adelaide and Perth in 2003 [17]. The use of solitons and erbium-doped fiber amplifiers enabled the transmission of nine channels each with a bit-rate of 10 Gbit/s over 5745 km of fiber without any regeneration in between.

1.3 Thesis Overview

This thesis is concerned with the study of solitons in dissipative systems. The investigation is based mainly on a study of the numerical solutions and approximate analytic solutions of these dissipative systems. Various localized soliton solutions are discussed, and their behaviors, as well as conditions for their existence, are investigated. Specifically, the study focuses on dissipative solitons in optical systems, with an emphasis on their applications in optical pulse generation from mode-locked lasers. The thesis is organized in the following way.

Chapter 2 provides background materials needed for understanding solitons in dynamical systems. Here, solitons of the NLSE are first introduced, while those in dissipative systems, and in particular of the CGLE, are revisited. A numerical technique that is employed in the study is also presented in this chapter.

Chapter 3 is devoted to the investigation of soliton bifurcations in dissipative systems. This study has found that systems modeled by the CGLE are richly filled with different types of solitons and their bifurcations near the region of transition between stable pulses and moving fronts. A bifurcation diagram is constructed by changing a system parameter. Various solutions that appear in the diagram are presented, and each bifurcation is studied in a detailed manner.

Chapter 4 focuses on studying the solutions of the CGLE using reduced models with a finite number of degrees of freedom; these are constructed by applying the method of moments. These approximate models allow one to study particular branches of dissipative soliton solutions in an analytical way. Moreover, the reduced models are useful in approximately locating the region of existence for each branch of solutions and their bifurcations in the parameter space. In particular, several models have been constructed to investigate pulsating and creeping solitons.

Chapter 5 presents the recently-discovered phenomenon of dissipative soliton resonances, where the soliton energy increases dramatically for a specific set of system parameters in the CGLE. Such high-energy soliton solutions can be employed

in the design of passively mode-locked lasers producing ultra-high energy pulses. Various aspects of the resonance phenomenon are studied and presented in this chapter, with an emphasis on its application in the generation of ultra-high energy pulses.

Chapter 6 presents the effects of the external modulation on a pair of dissipative solitons. A time-periodic term is added to the CGLE to model pulse generation in a laser which combines both passive and active mode-locking mechanisms. The time separation and the phase difference between the soliton pairs are studied under two different mechanisms of external modulations with various modulation amplitudes and frequencies. Such an investigation could become useful in generating pulse trains with more accurate control over the pulse-to-pulse separations.

The final chapter draws conclusions from this research.

Chapter 2

Solitons in Infinite-Dimensional Dynamical Systems

The concept of the soliton is one of the most fascinating ideas in nonlinear dynamics. Solitons exist in various physical systems. Examples include, solitary waves on a water surface and light pulses traveling through an optical fiber. Mathematically speaking, a soliton is a localized solution of a partial differential equation (PDE) with an infinite number of degrees of freedom. A soliton has the special feature of being undistorted over a long propagation distance. Owing to such intriguing characteristics, the soliton field has gained a lot of scientific interest, and has provided practical applications, including in fiber-optic communications.

In this chapter, solitons in different nonlinear dynamical systems are discussed. The chapter begins, in Section 2.1, with a simple case of solitons in integrable systems. This is then followed by the introduction of a more general case, *viz.* dissipative systems, focusing on the complex Ginzburg-Landau equation (CGLE) and its various soliton solutions. In Section 2.3, a numerical method employed to find the solutions of the nonlinear dynamical systems is discussed. Section 2.4 summarizes the chapter.

2.1 Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation is a universal equation that can be used to describe nonlinear conservative systems in many areas of physics, including nonlinear optics [18], heat pulses in solids [19, 20], water waves [21, 22] and various other

nonlinear phenomena. In the context of nonlinear optics, the NLSE is derived from the famous Maxwell's equations. The nonlinearity in the equation arises from the interaction of the intense electromagnetic fields with the dielectric material, which induces a nonlinear polarization response in the medium. In the derivation of the NLSE, the light is considered to propagate in one direction, and it is also assumed that the envelope of the wave packet changes slowly in comparison with the carrier frequency of the light. The later assumption is called the slowly-varying envelope approximation, and this assumption holds true for an optical pulse down to femto-second duration [23].

The standard NLSE, in its normalized form, is written as below [24]:

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi = 0, \quad (2.1.1)$$

where $\psi(t, z)$ is the slowly-varying field envelope. The physical meaning of the variables in Eq. (2.1.1) depends on the system it is describing. For example, when the NLSE is modeling the propagation of pulses in an optical fiber, t is the time frame moving with the group-velocity of the pulse, while z is the propagation distance along the fiber. A subscript denotes the derivative with respect to that variable. The fiber loss is assumed to be negligible in Eq. (2.1.1). D is the dispersion coefficient, and it takes the value $+1$ or -1 , depending on whether the group-velocity dispersion (GVD) is anomalous or normal, respectively. The nonlinear term, $|\psi|^2\psi$, represents the self-phase modulation (SPM). The above notation is used throughout this thesis. It is worth mentioning that the variables in Eq.(2.1.1) are dimensionless, and they are normalized to incorporate the magnitude of both GVD and nonlinear effects.

2.1.1 Group-Velocity Dispersion

Group-velocity dispersion (GVD) refers to a property where the light velocity depends on its wavelength. This frequency-dependent response of the medium is caused by the characteristic resonance frequencies at which the medium absorbs the interacting electromagnetic radiation through oscillations of bound electrons [18].

In the anomalous dispersion regime ($D = 1$), higher frequency components travel faster than the lower frequency components. This results in the higher frequency components being advanced towards the leading edge of the pulse while the lower

frequency components fall back towards the trailing edge of the pulse, causing pulse broadening and chirping. Fig. 2.1 shows schematically the effect of GVD on an initially unchirped pulse in the anomalous dispersion regime. On the other hand, in the normal dispersion regime ($D = -1$), frequency components are shifted to the opposite directions, and again this leads to pulse broadening and chirping.

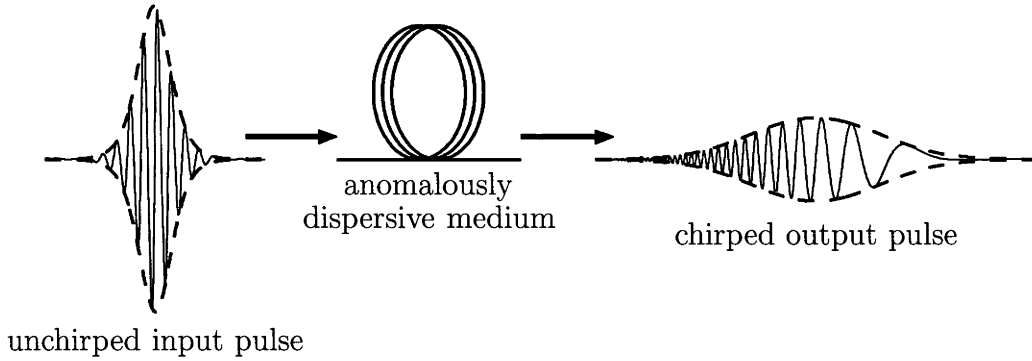


Figure 2.1: Schematic diagram of dispersive effects on an initially unchirped pulse under anomalous dispersion regime.

GVD plays a critical role in fiber optic communications, especially for ultra-high bit rate systems where short optical pulses are used in a single-mode fiber [18]. Substantial pulse broadening caused by the wide spectral width associated with the short pulses can lead to inter-symbol interference and impair the system quality significantly if the dispersion is not managed properly.

2.1.2 Self-Phase Modulation

Self-phase modulation (SPM) is one of the most common nonlinear effects in the propagation of pulses in an optical waveguide. It originates from the optical Kerr effect, a nonlinear effect arising from the third-order susceptibility $\chi^{(3)}$ of the dielectric medium, which causes a variation in the refractive index, depending on the local intensity of the light [25]. An optical pulse, under the effect of SPM, experiences a self-induced phase shift during its propagation.

The effect of SPM is that the leading edge of the pulse is down-shifted in frequency, while the trailing edge of the pulse is up-shifted in frequency, causing the pulse to be chirped. This generation of new frequency components broadens the

pulse spectrum. Fig. 2.2 shows an example of the spectral broadening caused by SPM.

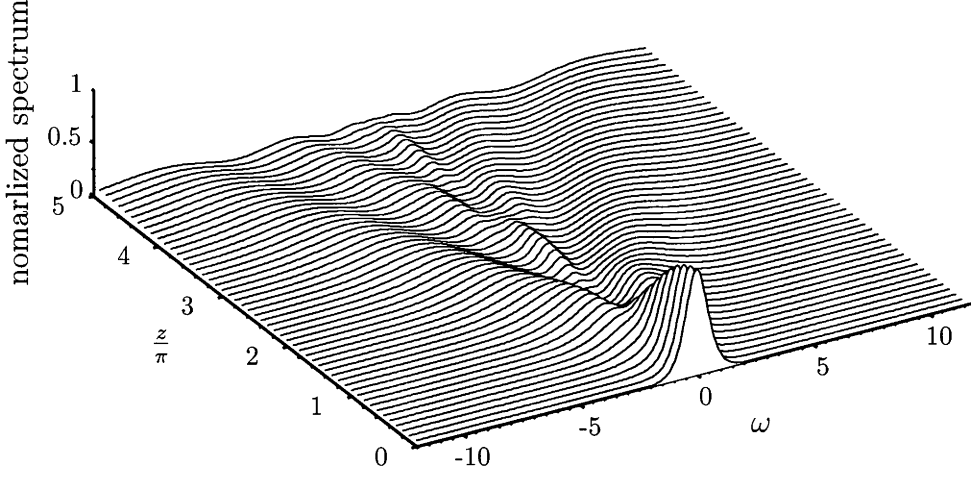


Figure 2.2: SPM-broadened spectrum in absence of GVD for an initially unchirped hyperbolic-secant pulse.

In the absence of GVD, SPM has no effect on the temporal profile or the width. The behavior of the pulse under the combined effect of GVD and SPM depends on the dispersion regime in which the pulse is traveling. Under the normal dispersion regime, lower frequencies generated on the leading edge and the higher frequencies generated on the trailing edge cause more rapid broadening of a pulse compared to a pulse under the sole effect of GVD. On the other hand, an optical soliton, which retains its profile over a long propagation distance, can be supported when the effect of SPM is combined with the anomalous dispersion regime [12].

2.1.3 Solitons of NLSE

In the anomalous dispersion regime, Eq. (2.1.1) has pulslike solutions that either do not change their shape during propagation or follow a periodic evolution pattern due to the interplay between GVD and SPM. The balance between these two independent effects is the underlying physics behind the solitons of NLSE. It turns out that a hyperbolic-secant pulse shape exactly balances out the two effects considered in Eq. (2.1.1), and hence a soliton solution of the NLSE is given by [1]

$$\psi(t, z) = \text{sech}(t) \exp\left(i\frac{z}{2}\right). \quad (2.1.2)$$

By using the symmetries of the NLSE, a single-parameter family of solutions is obtained by a simple scaling transformation and is given below:

$$\psi(t, z|q) = q \operatorname{sech}(qt) \exp\left(iq^2 \frac{z}{2}\right), \quad (2.1.3)$$

where q is a real parameter.

Generally speaking, solutions of a nonlinear PDE cannot be expressed in analytical form, except for some special cases where the dynamical system is integrable. The inverse scattering method, a technique that is a nonlinear analogue of the Fourier transform, can be applied to find the analytical solutions of such integrable systems [6]. The nonlinear dynamical system described by Eq. (2.1.1) is one of those few special cases where the system is integrable. Integrability implies that the system has as many conserved quantities as the number of degrees of freedom. Since the NLSE is a dynamical system with an infinite number of degrees of freedom, it needs an infinite number of conserved quantities for it to be integrable. It has been shown by Zakharov and Shabat that the NLSE indeed has an infinite number of conserved quantities, and applying the inverse scattering method leads to the solution for any initial condition [6].

The set of solutions given in Eq. (2.1.3) is known as the fundamental soliton family. Every localized solution of the NLSE can be expressed in terms of a nonlinear superposition of a finite number of these fundamental soliton solutions and a continuum of radiation waves [24]. Higher-order solitons of the NLSE, also known as ‘breathers’, are examples of the nonlinear superposition of these fundamental solitons. As its name suggests, a higher-order soliton changes its shape periodically along the propagation with the period $z_0 = \pi/2$. These higher-order solitons can be excited by [18]

$$\psi(t, q) = nq \operatorname{sech}(qt), \quad (2.1.4)$$

where n is a positive integer; it represents the order of the soliton, *i.e.* the number of fundamental solitons involved in the nonlinear superposition. An example of second-order soliton ($n = 2$) propagation is shown in Fig. 2.3.

In an integrable system, the solitons and the radiation waves serve as ‘nonlinear modes’ of the system. Solitons remain unchanged during interactions with the radiation waves and other solitons, apart from additional phase shifts. Soliton-soliton collisions are elastic, which means that the number of solitons and the motion

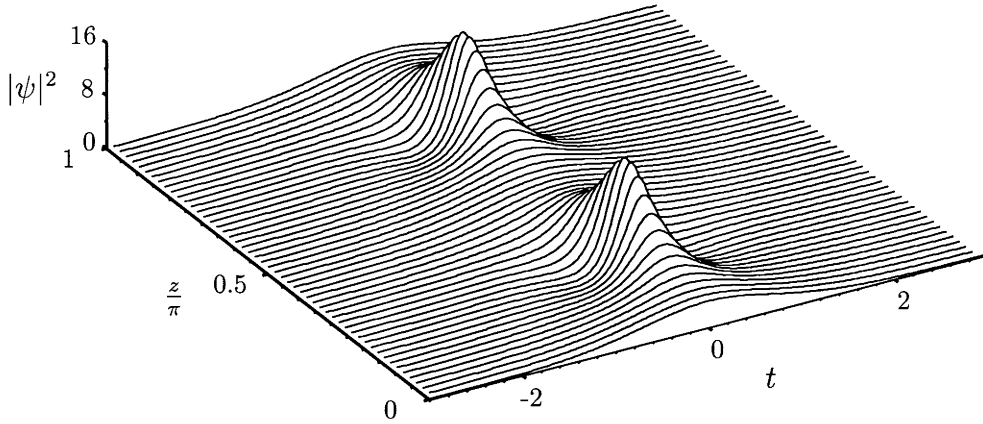


Figure 2.3: Evolution of a second-order soliton over two soliton periods, *i.e.* $z = \pi$.

of each soliton remain unchanged after a collision [24]. These interaction behaviors do not appear in non-integrable systems.

2.2 Dissipative Systems

An integrable system is considered to be an isolated system which has no interaction with its surroundings. Quantities are conserved within the system. However, this is not true in real systems. Although in some cases, approximation of a real system by an integrable system can be useful in investigating a particular phenomenon in nature to a certain degree, many complex spatio-temporal dynamics in optical [7, 24], general physical [10], chemical [26] and biological [27] systems cannot be fully understood without introducing dissipation into the dynamical system. This generalization leads to a dissipative system. In such systems, interactions between the system and its surroundings and also interactions within the system are taken into consideration.

The NLSE can be extended to model a dissipative system. This can be achieved by converting the real coefficients of Eq. (2.1.1) into complex numbers. The resulting equation is widely known as the complex Ginzburg-Landau equation (CGLE). The CGLE is a generic equation that models a variety of phenomena in systems far from thermodynamic or other equilibrium [10]. Examples include pulse genera-

tion in passively mode-locked lasers, superconductivity, superfluidity, Bose-Einstein condensation and strings in field theory. The variety of these systems is enormous. Moreover, the number of solutions to this equation is so large that the sphere of this knowledge by itself has been called ‘the world of the Ginzburg-Landau equation’ [10, 28].

The complex cubic-quintic Ginzburg-Landau equation, in dimensionless form, can be written as [24]

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{tt} + i\mu|\psi|^4\psi - \nu|\psi|^4\psi. \quad (2.2.1)$$

The left-hand-side of Eq. (2.2.1) is identical with the NLSE, and there are additional terms on the right-hand-side. δ is the linear gain-loss coefficient, the $i\beta\psi_{tt}$ term accounts for parabolic gain dispersion, ϵ and μ correspond to the cubic and quintic nonlinear gains, respectively, and the term with ν represents quintic nonlinear refractive index. The inclusion of the quintic terms is necessary for the study of stable localized structures in dissipative systems, as pulse-like solutions of cubic CGLE are known to be unstable [29].

2.2.1 Dissipative Solitons

One of the distinctive features of nonlinear dissipative systems is the formation of stable localized structures by self-organization [30–32]. Although the term ‘soliton’ was initially used for the non-varying solutions of integrable systems, its definition has been extended in recent years to include localized structures of dissipative systems. Dissipative solitons are qualitatively different from the solitons of integrable systems. Unlike those of conservative systems, solitons of dissipative systems are governed by the properties of the system rather than by the initial conditions, and, therefore, a wide range of initial conditions can evolve into a particular dissipative soliton solution [1]. Continuous external energy supply is required to keep a dissipative soliton ‘alive’. This pump of energy is essential. The soliton vanishes when this energy source is switched off, or if the system parameters are changed and they are no longer within the region of the existence of the soliton. Different parts of the soliton experience different gain and loss of energy. This means that even if the soliton appears to be stationary, there is a continuous flow of energy between its parts. Such complicated dynamics can be easily pictured by considering a pile

of sand as an example [33]. If sand is continuously added to central area of the pile while the same amount is continuously removed from the outer parts, the sand will be redistributed within the pile. This will lead to a particular shape of the sand pile. It depends on gravity and nonlinear interactions between the particles. Fig. 2.4 schematically illustrates the flow of energy in a dissipative soliton. The soliton is a result of a complicated dynamical process of energy exchange with the environment and between its own parts.

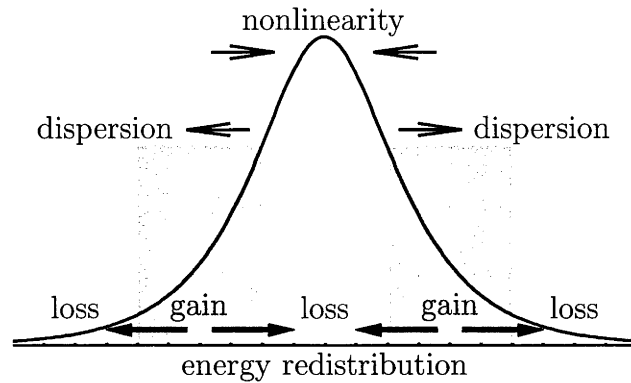


Figure 2.4: A schematic description of a dissipative soliton. The soliton has parts where energy is supplied as well as parts where energy is dissipated. The energy is redistributed within the soliton [1].

The concept of the dissipative soliton has received wide recognition in optics, physics in general, biology and medicine [7, 33]. This concept allows many diverse phenomena in nature to be understood and studied in a simple and ordered way. It represents a unifying notion in science, since it stresses the common features in a variety of complicated behaviors of self-localized formations. The most prominent feature of dissipative solitons is that they exist only when there is a continuous energy supply from an external source. This energy has to be dissipated in the medium where solitons are found. Thus, their formation requires a balance between the energy supplied and lost. This balance must be exact in order to prevent the overheating or the cooling down of the system and the subsequent disappearance of the soliton. The exact balance is provided by the highly involved dynamics in the system, which implies that dissipative solitons have to be self-organized. Their shape, amplitude, width and all other parameters are unique for given external conditions. Dissipative solitons can be natural, *i.e.* they can exist in nature as a result of a relaxation to a localized formation under the influence of solar heating

and dissipation, or they can be artificial objects created by scientists. In the latter case, they can be used in modern technology.

Examples of dissipative solitons include: ultra-short pulses from passively mode-locked lasers, soliton propagation in long-haul all-optical transmission systems, nerve pulses, localized formations in reaction-diffusion systems, vegetation clustering in arid lands, Bose-Einstein condensates in cold atoms, wave phenomena in neuron networks, spiral waves in weakly-excitabile media and traveling waves in cortical networks [7, 33]. This list is far from being complete. The variety of systems that admit dissipative solitons is enormous. The number of mathematical equations they satisfy is also very large. Nevertheless, they all have a common feature: they only ‘live’ in the presence of a continuous energy supply from an external source and with careful parameter adjustments. Thus, they can be studied using certain common rules and principles.

The region of parameter space of the CGLE in which stable stationary solitons exist has been studied extensively by Soto-Crespo, Akhmediev and Afanasjev for the anomalous dispersion case [34]. Unlike systems described by the NLSE, dissipative systems described by the CGLE admit stable pulse-like solutions even in the normal dispersion regime ($D = -1$). Analytical and numerical studies of such solutions, including their region of existence in the parameter space of the CGLE, have been conducted by Soto-Crespo *et al.* [35]. Pulse-like solutions in the case of normal dispersion have also been observed experimentally in mode-locked fiber lasers with normal average cavity dispersion [36–39] and in all-normal-dispersion fiber lasers without any anomalous dispersion in the cavity [40].

Finding sets of parameters for which the CGLE has soliton solutions is a tedious task which has been carried out in a number of previous works [24, 41]. This set is not necessarily a single region. There could be several regions of solitons, with each one having distinct soliton properties [42]. The regions can be adjacent to each other [41] or separated in the space of parameters [42]. The structure of the regions is extremely complicated. However, the solutions with the most remarkable features can be distinguished. There are various types of dissipative solitons, including families of stationary, pulsating and exploding solitons [28, 41]. Fig. 2.5 shows propagation of some of these dissipative soliton types.

The simplest type, in terms of its behavior, is a stationary soliton. A stationary soliton, as its name suggests, does not change its shape during propagation. An

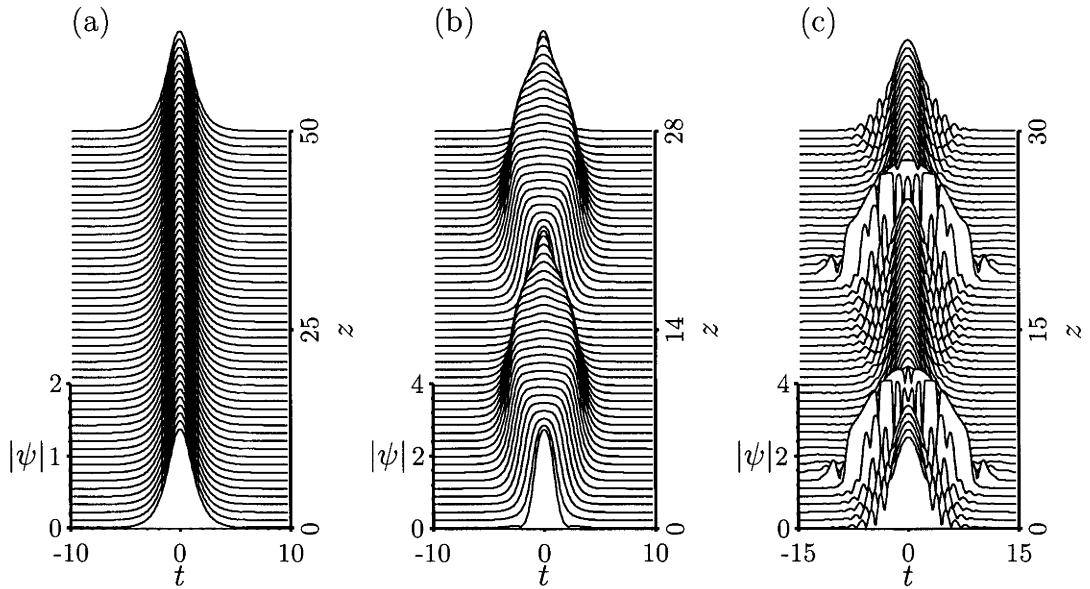


Figure 2.5: Examples of different types of dissipative solitons: (a) a stationary soliton, (b) a pulsating soliton and (c) an exploding soliton.

example of it is shown in Fig. 2.5(a). Even though it looks stationary, there is a continuous flow of energy within it, and into and out of the system, and it must not be confused with the fundamental soliton Eq. (2.1.3) of the NLSE. A stationary solution of a dissipative system does not survive in the NLSE limit of the CGLE. *i.e.* the right-hand-side of Eq. (2.2.1) does not approach zero in regions of existence of stationary solitons.

A dissipative system can also have pulsating solutions, *i.e.* solutions that oscillate periodically along the propagation direction, as in the example shown in Fig. 2.5(b). These are limit-cycles in an infinite-dimensional phase space, so the pulsating period and shape dynamics, as a function of propagation distance, are fixed. They have been found numerically [41, 43] and observed experimentally in a passively mode-locked fiber laser [44]. Again, these forms of pulsating solitons are distinct from the higher-order solitons of the NLSE mentioned in Section 2.1. They do not appear in the NLSE limit of the CGLE, and they are not the result of a nonlinear superposition of fundamental solitons.

Dissipative solitons may also exhibit exploding and self-restoring behavior. Such solitons are named ‘exploding solitons’, and they have remarkably interesting behaviors. An exploding soliton travels with almost no change in its shape for some

propagation distance. There is a build up of instability during this phase, which then leads to an explosion, *i.e.* a break up of the structure. The soliton quickly recovers its original shape after the explosion. The soliton repeats this process along its propagation as shown in Fig. 2.5(c). There is no precisely defined interval between the explosions, and every explosion is different from the others. Nevertheless, the soliton always returns to the same shape after the explosion. This type of behavior is similar to a strange attractor in the Lorentz model [45]. They were initially found by numerically solving Eq. (2.2.1) [28], and later observed experimentally in a Ti:sapphire mode-locked laser [46]. Such complicated dynamics cannot be modeled without the inclusion of the system dissipation.

Apart from the above three types of dissipative solitons that are briefly introduced here, the CGLE permits many other types of solitons which have unique behaviors and properties. In the following chapters, more examples of these different types of solitons and their transitions from one type to another are investigated in detail.

2.3 Numerical Methods

The study of solitons in a non-integrable nonlinear dynamical system, such as the CGLE, requires a numerical approach, as often the system does not generally lend itself to analytic solutions. The discovery of soliton solutions with complicated behaviors, including the ones given in the previous section, would not have been possible without the aid of a numerical analysis on a modern computer.

Out of a large number of numerical methods that can be employed to solve a PDE such as Eq. (2.2.1), a spectral method called the split-step Fourier method is one of the most commonly used techniques for the study of soliton propagation in a nonlinear medium. The main advantage of the split-step Fourier method over most other techniques is that the method makes use of the fast Fourier transform (FFT) algorithm [47] when transforming between the time domain and the frequency domain. This reduces the computational time by an order of magnitude compared with other methods that achieve the same accuracy [48].

Although in reality, the linear and nonlinear effects act together on propagation, the split-step Fourier method obtains an approximate solution by assuming that the linear and the nonlinear effects act independently in propagating the field over a

small distance dz [18]. For this approach, the CGLE can be rewritten in the form of

$$\psi_z = \left(\hat{T} + \hat{V}(\psi) \right) \psi, \quad (2.3.1)$$

where $\hat{T}$ is a differential operator that accounts for the linear effects in a medium and $\hat{V}$ is a nonlinear operator that governs the effects of the medium nonlinearities on pulse propagation. These operators are given by

$$\hat{T} = \left(i \frac{D}{2} + \beta \right) \frac{\partial^2}{\partial t^2}, \quad (2.3.2)$$

$$\hat{V}(\psi) = \delta + (\epsilon + i) |\psi|^2 + (\mu + i\nu) |\psi|^4. \quad (2.3.3)$$

$\psi(t, z_0)$ is defined to be the initial field envelope at $z = z_0$ and $\psi(t, z_0 + dz)$ to be the field envelope after a single step at $z = z_0 + dz$. An exact solution after a single step dz would be

$$\psi(t, z_0 + dz) = \exp \left(\left(\hat{T} + \hat{V} \right) dz \right) \psi(t, z_0). \quad (2.3.4)$$

This equation cannot be solved in general. In a split-step Fourier method, the approximation of the field after one segment is done in two steps. Hence Eq. (2.3.4) can be approximated to

$$\psi(t, z_0 + dz) \approx \exp \left(\hat{T} dz \right) \exp \left(\hat{V} dz \right) \psi(t, z_0). \quad (2.3.5)$$

In the first step, only the linear effects acting over the segment are calculated in the Fourier domain without the nonlinearities. This then gives an intermediate field $\phi(t)$ after a linear step, and it is given by

$$\phi(t) = \mathcal{F}^{-1} \left[\mathcal{F}[\psi(t, z_0)] \exp \left(-i \frac{D}{2} \omega^2 dz - \beta \omega^2 dz \right) \right], \quad (2.3.6)$$

where $\mathcal{F}$ represents the Fourier transform and ω is the difference in angular frequency from its carrier angular frequency ω_0 . In the numerical analysis, the discrete retarded time t has a finite number of grid points N over a fixed window which should be large enough to contain the localized solution under investigation. Then

each discrete ω_n is given by

$$\omega_n = \left(n - \frac{N}{2} \right) \frac{\pi}{t_{\max}}, \quad (2.3.7)$$

where $n = 1, 2, 3, \dots, N$ and $t_{\max}$ is a half the range of the time window.

In the next step, the nonlinear effects act alone, while the linear operator is set to be zero. Mathematically, it can be written as

$$\psi(t, z_0 + dz) \approx \phi(t) \exp(\hat{V} dz). \quad (2.3.8)$$

In order to obtain the approximate solution of Eq. (2.3.8), the fourth-order Runge-Kutta method can be employed. Then the approximation given in Eq. (2.3.8) can be written as

$$\psi(t, z_0 + dz) \approx \phi(t) + \frac{1}{6} dz (k_1 + 2k_2 + 2k_3 + k_4), \quad (2.3.9)$$

where

$$\begin{aligned} k_1 &= \hat{V}(\phi(t)) \\ k_2 &= \hat{V}\left(\phi(t) + \frac{k_1}{2}\right) \\ k_3 &= \hat{V}\left(\phi(t) + \frac{k_2}{2}\right) \\ k_4 &= \hat{V}(\phi(t) + k_3). \end{aligned} \quad (2.3.10)$$

The nonlinear operator $\hat{V}$ is calculated by substituting the intermediate solution $\phi(t)$ in place of ψ in Eq. (2.3.3).

The solution obtained after a single step is then used in approximating the solution of the next step. By repeating this procedure with a careful choice of the longitudinal step-size dz and the time window step dt , the evolution of the solution along z can be investigated. If the step-sizes chosen are too big, then the numerical output will not be an accurate approximate solution of the CGLE due to the build up of the errors. On the other hand, if the step-sizes chosen are too small, one will suffer from a long computational time without much increase in the accuracy.

The accuracy of the split-step Fourier method described above can be improved at a minimal expense of computational time by symmetrizing the operators in

Eq. (2.3.5) [49]. While Eq. (2.3.5) assumes that the nonlinear effects are lumped at the end of every segment, the symmetrized split-step Fourier method divides the segment into two, and the nonlinear effects of the whole segment are assumed to act at the mid-plane, while in the first and second halves of the segment, only the linear effects are assumed to act. Then Eq. (2.3.5) can be replaced by

$$\psi(t, z_0 + dz) \approx \psi(t, z_0) \exp\left(\hat{T} \frac{dz}{2}\right) \exp\left(\hat{V} dz\right) \exp\left(\hat{T} \frac{dz}{2}\right). \quad (2.3.11)$$

The calculation of each exponential is done in a similar manner to the non-symmetrized technique. The only difference is that the initial linear part is calculated for $dz/2$, and then the nonlinear step is performed before calculating the remaining linear part. The schematic diagram of the symmetrized split-step Fourier method is shown in Fig. 2.6.

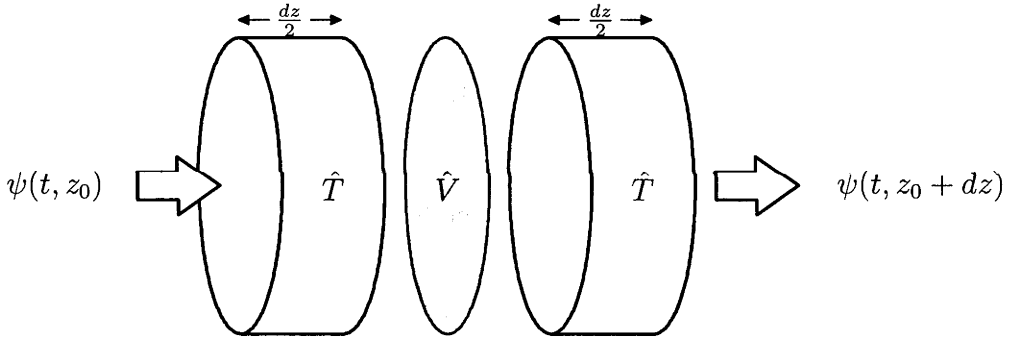


Figure 2.6: A schematic illustration of the symmetrized split-step Fourier method.

Many of the results presented in this thesis were found using the symmetrized numerical scheme presented above (and its variations to suit the model under investigation). It is important that the results obtained are good approximate solutions of the PDE involved, and not any consequence of numerical artifacts. The validity of the numerical solutions can be assured by verifying them using finer resolutions in both t and z directions.

2.4 Summary

The concept of the soliton was first initiated by studying the non-evolving localized solutions of integrable systems such as the Korteweg-de Vries (KdV) equation and the nonlinear Schrödinger equation (NLSE). These systems have no interactions with their surroundings, and the system quantities are conserved. A soliton solution of a such system is the result of the interplay between linear and nonlinear effects; it can be found analytically by using the inverse scattering technique.

On the other hand, a dissipative system, such as the complex Ginzburg-Landau equation (CGLE), describes a system that has interactions with its surroundings as well as interactions within itself. The concept of the dissipative soliton provides a new paradigm of localized structures in non-equilibrium systems. Stable soliton solutions of such systems not only require a balance between the dispersion/diffraction and the nonlinearity, but also need to satisfy the energy balance condition. Different parts of a soliton experience different gain or loss, and these often depend on its own field intensity and frequency. Even if the soliton appears to be stationary, there is a continuous flow of energy between the parts of the soliton. Dissipative solitons are formed by the mechanisms of self-organization, and their existence relies on the system parameters, rather than the initial conditions. Usually, such solutions cannot be obtained analytically, and so a numerical approach, assisted by the split-step Fourier method, can be a powerful tool for investigating solitons of dissipative systems.

There are many different types of dissipative solitons with interesting characteristics and behaviors. Each of these different types of solitons exists in a finite region in the parameter space. Varying one or more of the system parameters may lead to a transition from one type of soliton to another, and this is an interesting area of study.

Chapter 3

Bifurcations of Dissipative Solitons

In general, nonlinear dissipative systems such as the one modeled by the complex Ginzburg-Landau equation (CGLE) possess a variety of localized solutions. These include various types of soliton solutions [41], as well as sources, sinks and fronts moving at a constant velocity along the transverse direction [29]. For a given set of equation parameters, the form of the solution is usually set, *i.e.* only one solution exists for a specified situation. Occasionally, there can be a few, but not a family of them. A soliton solution can often be characterized by its energy, amplitude and width. Additionally, for a solution that evolves periodically along its propagation, parameters such as the period, shape dynamics and velocity in the transverse direction can also be monitored to characterize the solution. By changing the system parameters, *i.e.* varying the coefficients of the terms on the right-hand-side of Eq. (2.2.1), the characteristics of the solutions can be changed. As in finite dimensional nonlinear systems, the soliton parameters can change continuously, as well as *via* bifurcations. The latter refers to qualitative changes in structure and behavior of the solution, such as transforming from one type of soliton to another.

It has been shown, in previous studies [50, 51], that there is a specific range of parameter values where solitons in dissipative systems are transformed into fronts. Such transformation was first studied by Hakim, Jakobsen and Pomeau by considering the solutions of two limiting cases, real and imaginary, of the CGLE [52]. Consequently, other research showed that this is not a simple transformation [24]. Solitons in dissipative systems can, indeed, have very complicated bifurcation dia-

grams. The examples that are presented in this chapter demonstrate such complexity.

In this chapter, new numerical discoveries are presented for the region of transition between the solitons and the fronts. A bifurcation diagram is constructed for a particular region of this transition and presented in Section 3.1. The study has shown that the region is richly filled with a variety of bifurcations dividing a multiplicity of localized solutions. Findings in this region include a transformation of so-called ‘creeping solitons’ [41] into chaotic moving solitons through a sequence of period-doubling bifurcations, two symmetry-breaking bifurcations (each of different type), two different branches of solitons with multi-frequency dynamics that exist simultaneously for a given set of system parameters, and many more. Each of these bifurcations is closely investigated in the subsequent sections. This is followed by a discussion on the validity of system reductions to approximate such a highly complicated model, in Section 3.7. Finally, a chapter summary is given in Section 3.8. A careful study of these solutions and bifurcations is important for understanding the general dynamics of solitons in dissipative systems.

3.1 Bifurcation Diagram

Previous studies have shown that the CGLE admits a variety of localized stationary and pulsating solutions, as well as period-doubling and period-quadrupling bifurcations [41]. In order to investigate soliton bifurcations in a finite region bounded by a set of system parameters, Eq. (2.2.1) has been solved numerically using the split-step Fourier method outlined in Section 2.3.

Initially, the study of the creeping solitons, in particular, their regions of existence, has led to the study of this region. A creeping soliton is a special type of pulsating localized solution that changes its shape periodically and shifts a finite distance in the transverse direction after each period of oscillation. Especially, the work of Akhmediev, Soto-Crespo and Town has indicated that the CGLE admits a moving pulsating solution which has multi-frequency dynamics [41]. In this particular example, the movement of the pulsating soliton in the transverse direction is not a simple uni-directional ‘creeping’ motion, but it is rather a complicated zig-zag motion involving several different elements at different frequencies. Such interesting behavior of a solution triggered the initial investigation about its existence in the

parameter space.

A set of parameters that is known to give a creeping soliton with double period is the following: $D = 1$, $\epsilon = 0.835$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$ and $\mu = -0.11$ [41]. Eq. (2.2.1) has been solved for this set of parameters with an asymmetric initial condition. Once it converges to a creeping soliton, this solution has been used as the initial condition for finding a new solution for a different set of equation parameters in the vicinity of the first one, *viz.* ϵ is changed slightly while keeping all the other parameters fixed. By repeating this procedure, the change of the solution behaviors, as ϵ is varied, is studied. More specifically, starting from the above set of parameters, ϵ has been varied in both directions, ranging up to 0.85 and down to 0.78 with an initial step $\Delta\epsilon = 0.0002$. When an abrupt change of a soliton parameter occurs due to the jump of the solution from one type to another, the vicinity is studied carefully by changing ϵ in both directions with a smaller increment. In this way, a variety of soliton bifurcations and many co-existing different solutions in this region have been uncovered.

During the numerical computation of the propagation equation, the soliton parameter that is most often monitored is the energy, Q , carried by a certain solution. It is defined by

$$Q = \int_{-\infty}^{\infty} |\psi(t, z)|^2 dt. \quad (3.1.1)$$

For a solution that oscillates along the propagation direction z , the energy maxima and minima, denoted by $Q_{\max}$ and $Q_{\min}$ respectively, are used in constructing the bifurcation diagram. For every value of ϵ , the field is propagated a long enough distance in z before the soliton parameters are monitored, to avoid any transient behaviors. Fig. 3.1 shows the results.

Fig. 3.1 shows a region of transition from solitons to fronts. On the very left-hand-side of the Fig. 3.1, there is a pulsating soliton, indicated by a single $Q_{\max}$ value and a single $Q_{\min}$ value. On the far right-hand-side, there are two front solutions that are moving away from each other. Here, a finite value of the energy Q cannot be given since the moving front quickly fills the numerical time window with a constant amplitude, and the solutions of the CGLE are no longer localized. This is a relatively narrow region in comparison with the whole region of stationary, or even pulsating, solitons. This transition is far from being a trivial phenomenon like a simple increase of the soliton width, as it had previously been thought. As a consequence, it is a very dense region with various types of soliton solutions.

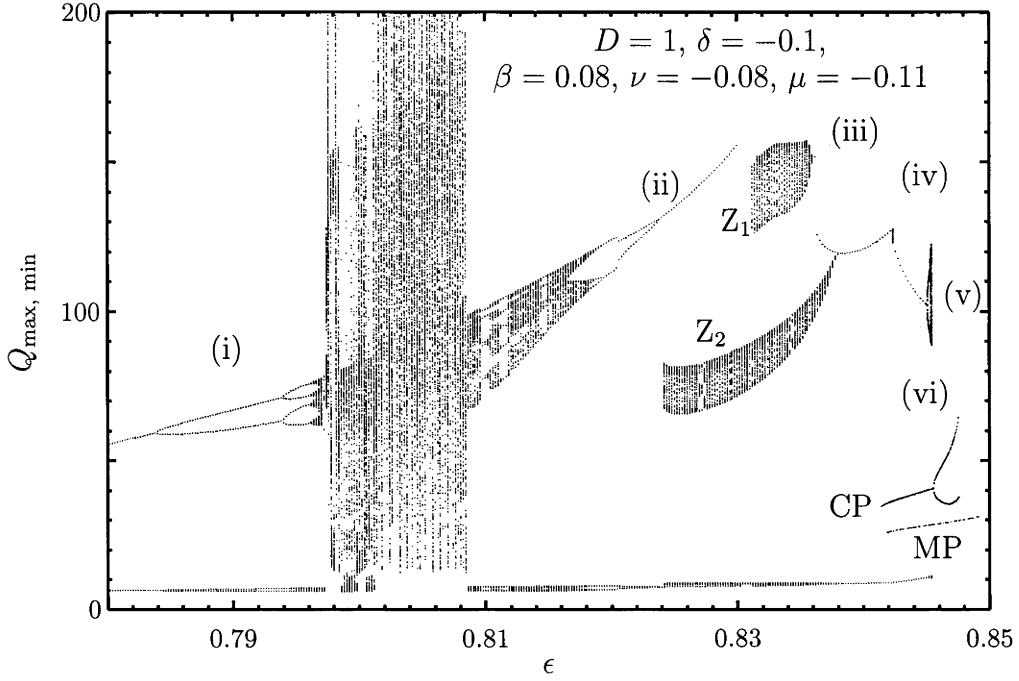


Figure 3.1: Soliton energy bifurcation diagram for varying ϵ . The rest of parameters are fixed at $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$ and $\mu = -0.11$. The shaded regions labeled with roman numerals (i) to (vi), and the branches labeled Z_1 , Z_2 , CP and MP, contain interesting solution behaviors and bifurcations.

Each of the specific regions that are shaded and labeled with roman numerals (i) to (vi), and the branches labeled Z_1 , Z_2 , CP and MP, contains interesting solution behaviors and bifurcations. Many of these regions are further resolved using smaller increments of the parameter ϵ in the numerical simulations. Varying the initial conditions allows one to find the unstable branches of solitons. In this way, all types of soliton solutions that exist in this region have been found. In the following sections, descriptions of each of these transformations are presented.

3.2 Period-Doubling Route to Chaos I

The shaded region labeled (i) shows a sequence of period-doubling bifurcations of the pulsating solitons. The pulse profile evolution at $\epsilon = 0.782$ is shown in Fig. 3.2(a). The soliton changes its shape periodically in the z direction. This motion is a limit-cycle with a closed single loop in an infinite-dimensional phase space constructed from infinitely many soliton parameters. This type of solution

can be described as a single period pulsating soliton.

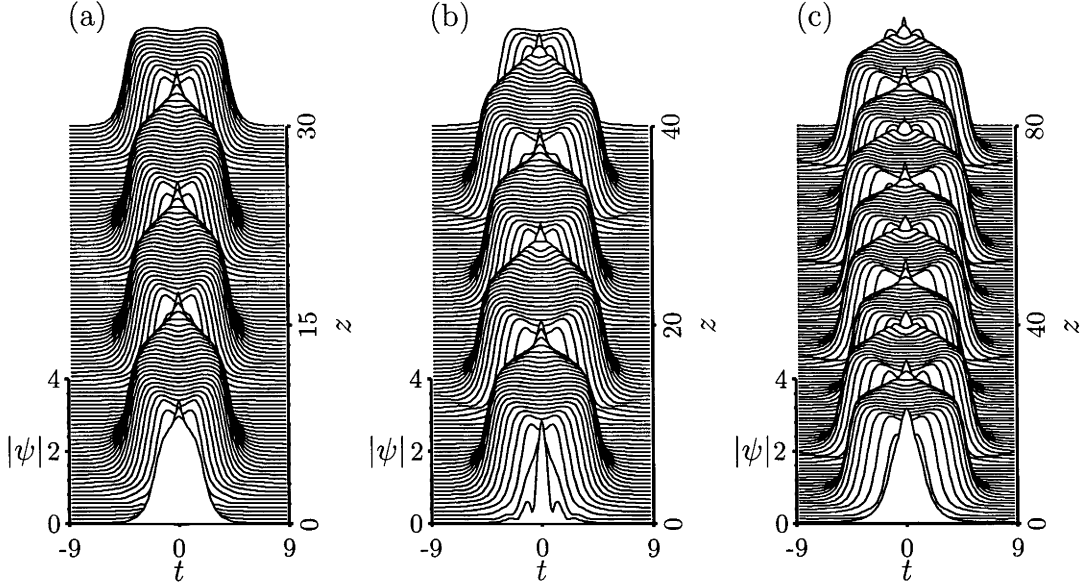


Figure 3.2: Soliton dynamics at $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and (a) $\epsilon = 0.782$ showing a single period pulsating soliton, (b) $\epsilon = 0.793$ showing period-doubling. (c) $\epsilon = 0.795$, showing period-quadrupling.

When the value of ϵ is increased sufficiently, the period of the pulsating soliton doubles. Here, this period-doubling bifurcation occurs at $\epsilon = 0.784$. A period doubling bifurcation refers to the emergence of new limit-cycle from an existing one. The trajectory in the phase space is a limit-cycle with a closed double loop, and the period becomes approximately twice that of the single-period pulsation. Consequently, in the region around $\epsilon = 0.793$ in Fig. 3.1, the energy, $Q(z)$, oscillates in a way that shows two different values of local maxima, along with two values of local minima. The two minima are very close to each other and therefore can hardly be distinguished on the scale of the Fig. 3.1. Branching of the curves corresponding to the maxima are clearly seen in Fig. 3.1. The soliton shape repeats itself every two pulsations. This is depicted in Fig. 3.2(b), which shows the profile evolution at $\epsilon = 0.793$. Period doubling bifurcations have been observed experimentally in various mode-locked laser systems including the passively mode-locked fiber lasers [44, 53–55].

A further increase in the ϵ -value, up to $\epsilon = 0.795$, shows another period-doubling bifurcation; it leads to period-quadrupling behavior with four local minima and four

local maxima for Q . The evolution of the solution profiles for this case is shown in Fig. 3.2(c). There are four pulsations in the closed loop of the limit-cycle.

This sequence of period doubling continues towards the right-hand-side edge of the shaded region (i) in Fig. 3.1. The interval $\Delta\epsilon$ for the period-doubling relative to that for period-quadrupling can be calculated, *viz.* $(0.794 - 0.784)/(0.79615 - 0.794) = 4.651$. This ratio of 4.651 is close to the generic (period-doubling) Feigenbaum ratio of 4.669, which indicates the system may exhibit a transition to chaos *via* period-doubling upon continual increase of ϵ . This is indeed the case, and the pulsating branch of solution follows a sequence of period-doubling bifurcations to chaos.

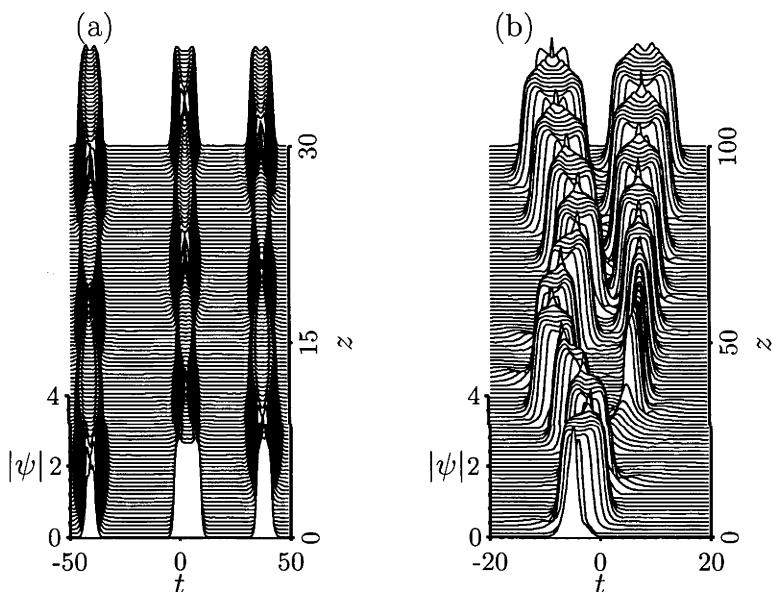


Figure 3.3: (a) An example of chaotic evolution of multiple solitons at $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.804$. (b) An example of soliton splitting in this chaotic region.

Solitons in the region around $0.798 < \epsilon < 0.808$ are unstable. They are pulsating solitons that chaotically change shape as they evolve. The chaotic motion frequently splits the soliton into two parts that are separated and in turn are transformed into chaotic solitons. The train of solitons which originates in this way fills the numerical grid, thus creating, in Fig. 3.1, the strip with a wide range of values of energy, Q . For each soliton, the range of allowed Q values is limited, but the process of their multiplication widens this range to infinity. There are no regular scenarios of soliton evolution in this range of values of ϵ . An example of chaotic

evolution of the soliton profile, where there are three solitons in the numerical window, is shown in Fig. 3.3(a). Each soliton remains localized unless it merges with another one or splits into two new solitons. An example of the latter process is shown in Fig. 3.3(b). The average distance that each soliton travels before splitting or interacting with another one depends on ϵ and the density of solitons already existing in the numerical window.

Increasing ϵ further terminates the region of chaotic solitons, and the region near $\epsilon = 0.821$, labeled (ii), corresponds to a soliton with period-two pulsations. The soliton dynamics in this region is similar to the one shown in Fig. 3.2(b), except that the energy range of Q values is now wider – soliton pulsations in region (ii) have a larger amplitude of their oscillations than those at $\epsilon = 0.793$. A soliton profile evolution in this region is shown in Fig. 3.4.

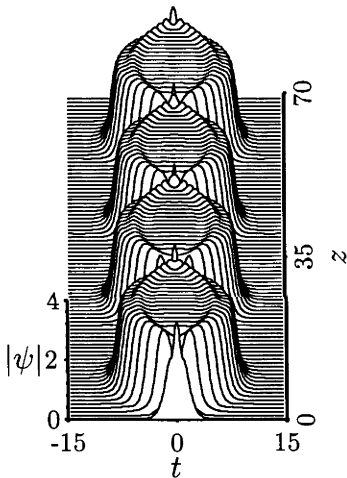


Figure 3.4: Soliton profile evolution at $\epsilon = 0.821$ showing two-period pulsation.

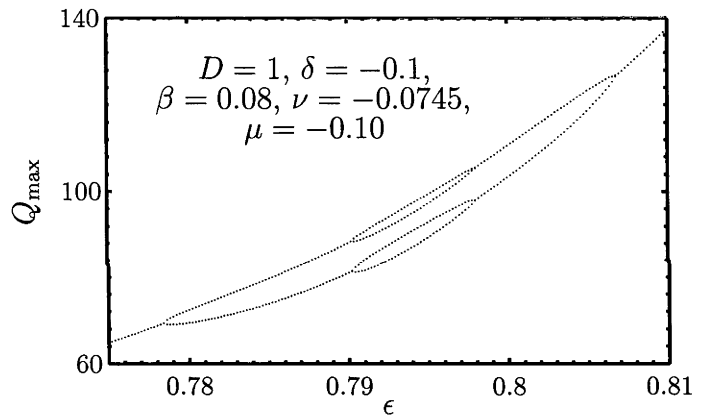


Figure 3.5: Soliton energy bifurcation diagram for varying ϵ [41]. The values of μ and ν are slightly different from those of Fig. 3.1.

Further increase in ϵ leads to a bifurcation back to single frequency pulsations. Keeping in mind that only one of the system parameters is varied here, it may happen that this region (ii) and the period-doubling regions before the chaos (i) can be connected to each other in the complete parameter space. The investigation of this requires a further numerical simulations with a slightly different set of system parameters. The study has shown that the two regions (i) and (ii) are indeed connected to each other. Fig. 3.5 is another bifurcation diagram with slightly different μ and ν , showing the evolution of the CGLE solutions as ϵ is varied [41]. This diagram

can be pictured as a different ‘layer’ of the bifurcation diagram that exists near to and parallel with Fig. 3.1 in the parameter space, where the gap between the layers corresponds to the difference in μ and ν values of these two layers.

In Fig. 3.4, the region of chaos disappears. In fact, the whole sequence of period-doubling leading to chaos disappears, and only period two and period four branches remain. There are two period-doubling bifurcations in the direction of increasing ϵ and also two period-doubling bifurcations in the direction of decreasing ϵ , and the period four branches are connected. This example also shows that the period doubling bifurcations lead to chaos, only if the system parameters are tuned correctly to reach the region of chaos.

3.3 Multi-frequency Creeping Soliton Dynamics

Two bands of solutions, labeled Z_1 and Z_2 in Fig. 3.1, are the regions where the creeping solitons with multiple frequency dynamics are found. These two wide strips correspond to two different creeping solutions, which co-exist in a certain interval of values of ϵ , *i.e.*, the behaviors of the solutions in the upper band, Z_1 , with higher-energy and the lower crescent-like strip, Z_2 , are qualitatively different. There are two or more frequencies involved in the motion of these solitons. The combination of multiple incommensurate frequencies leads to evolution where the energy, $Q(t)$, has local maxima $Q_{\max}$ and minima $Q_{\min}$ in a continuous range of values. Each frequency is related to a particular type of motion, although a complete separation, *i.e.* the attribution of each frequency to a certain type of motion, is non-trivial in a nonlinear problem. Nevertheless, the highest frequency, and hence the shortest period, is always related to the soliton pulsations. Similar to conventional uni-directional creeping solitons, each pulsation also causes the soliton to be shifted a certain amount in the transverse direction. The direction of this shift depends on the ‘global’ direction of the soliton’s movement caused by the lower frequency components at the moment of creeping. On the other hand, the lower frequency components, and hence the longer period components, are related to the velocity of the soliton’s movements in the transverse direction. Two example of the resulting dynamics, in the form of zig-zag motion found in this region, are shown in Fig. 3.6(a) and Fig. 3.6(c). It should be stressed that these two different examples presented are for exactly the same set of system parameters. The first example is for the

upper band Z_1 , and the second one is for the lower band, Z_2 , both at $\epsilon = 0.834$.

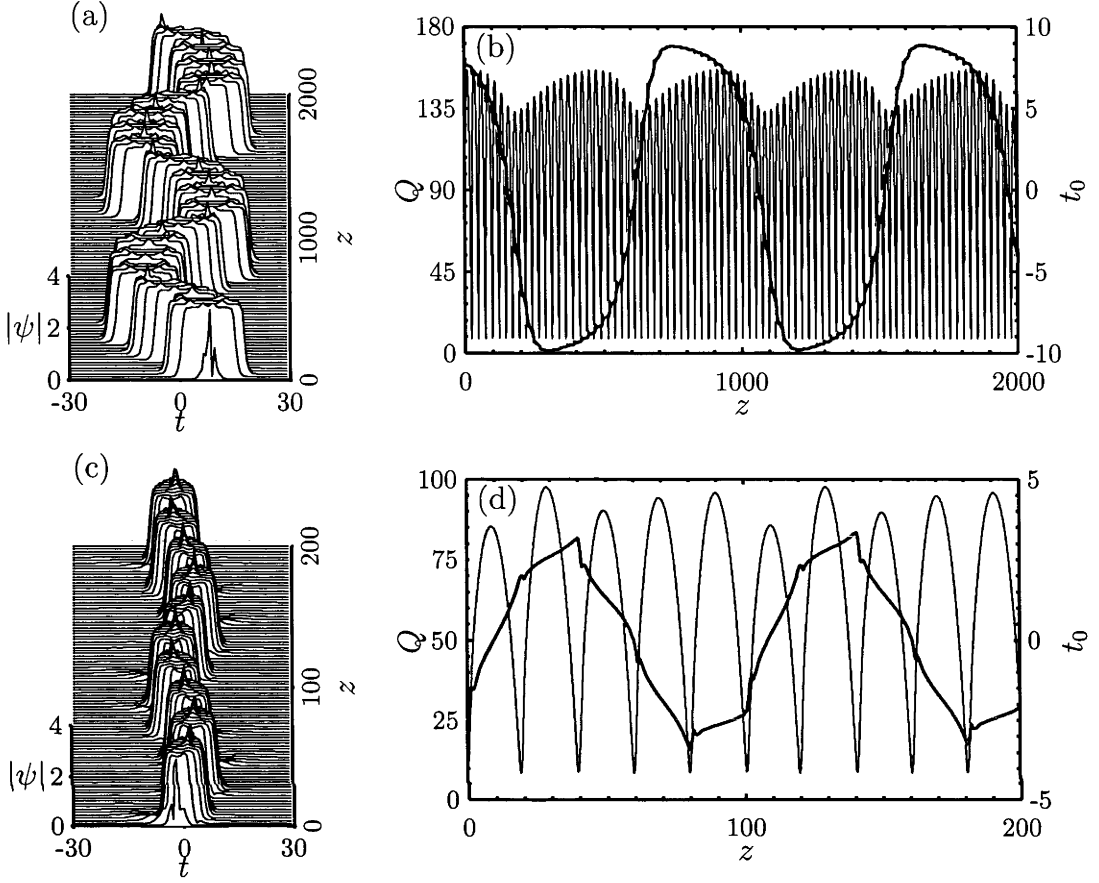


Figure 3.6: Soliton evolution profile for $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.834$ (a) in the upper band, Z_1 and (c) in the lower band, Z_2 . The energy, Q (black line) and the center of mass of the soliton t_0 (red line), as functions of propagation z , for each example, are plotted in (b) and (d), respectively.

The major difference between the two examples shown in Fig. 3.6 involves the frequencies which occur in their motions. For solitons in the upper band, Z_1 , the frequency of the transverse velocity is much lower than the frequency of the soliton width pulsations, while for the solitons in Z_2 , this frequency is comparable with the frequency of the width pulsation. The curves showing energy evolution, $Q(z)$, and the soliton's center of mass, t_0 , for these types of motion reveal the frequencies involved. These curves, for both Z_1 and Z_2 at $\epsilon = 0.834$, are shown in Fig. 3.6(b)

and Fig. 3.6(d), respectively. The center of mass t_0 is defined as

$$t_0 = \frac{\int_{-\infty}^{\infty} t |\psi(t, z)|^2 dt}{Q} \quad (3.3.1)$$

In the upper band, there are many pulsations within the ‘larger’ period change in soliton energy along the propagation direction. This clearly shows that there are two frequencies involved in this motion, despite the fact that they are incommensurate. From Fig. 3.6(a) and Fig. 3.6(b), one can see that the higher frequency component causes the soliton to pulsate and ‘creep’, while the lower frequency component causes the overall zig-zag motion.

On the other hand, in the lower band, the change in the direction of the motion occurs with a relatively short period compared with the period of the pulsating cycle. It can be seen that t_0 oscillates with a period approximately five times the period of the oscillations in Q . The center-of-mass experiences small abrupt changes every time that Q reaches a minimum value, a feature which is related to the ‘creeping’ nature of the motion of the soliton. Fig. 3.6(d) lacks the periodicity, *i.e.* the curves do not appear to repeat themselves exactly over a few periods. In fact, a further investigation has shown that this motion involves an additional frequency of significantly different magnitude, and the resulting motion is more complicated. The effect of this third frequency component becomes more evident by monitoring t_0 over a longer propagation distance z , as shown in Fig. 3.7.

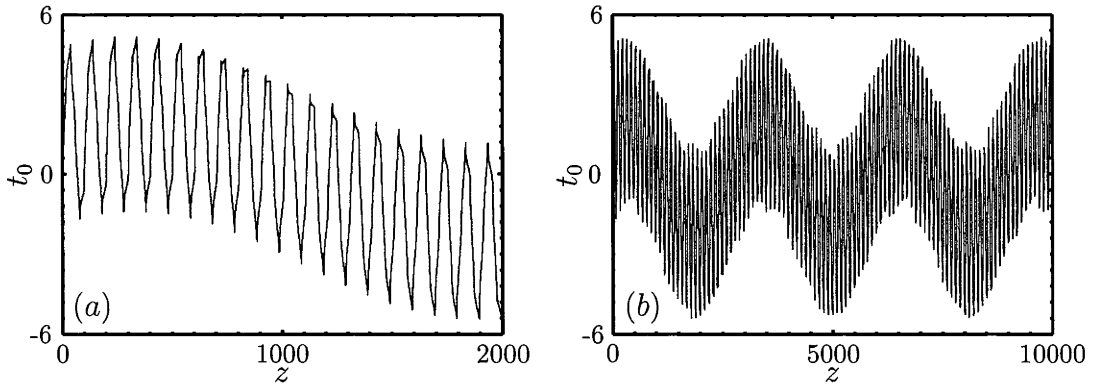


Figure 3.7: Soliton’s center-of-mass along propagation axis z for $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.834$. The same motion is shown on two horizontal scales, (a) $0 < z < 2000$ (b) and $0 < z < 10000$.

The center-of-mass does not strictly follow a single periodic motion after com-

pleting one zig-zag motion, as shown in Fig. 3.7(a). The third frequency component also causes the soliton to move in a zig-zag motion, but this time with a much longer period. The effect of this third frequency is clearly visible in Fig. 3.7(b). This third frequency component does not appear in the entire range of the lower-band Z_2 . In fact, a Fourier analysis of these frequency components has shown that the third, and hence the lowest, frequency component only appears around $\epsilon = 0.834$. In the other parts of the lower band, only two frequencies govern the dynamics of the creeping solitons.

The zig-zag motion of the pulsating solitons disappears altogether at the right-hand-side end of both the upper band Z_1 and lower band, Z_2 . There is a branch of uni-directional creeping soliton solutions that starts at the right-hand-side end of Z_1 , while the lower band Z_2 branch joins a branch of zero-velocity pulsating solitons. These two points of transformation are located in the shade region labeled (iii) in Fig. 3.1. This region of the bifurcation diagram is shown at a higher resolution in Fig. 3.8(a). The branches of pulsating and creeping solitons interact through a small region of bistability shown in Fig. 3.8(b). The nature of this bistability is explained later in Section 3.6.1.

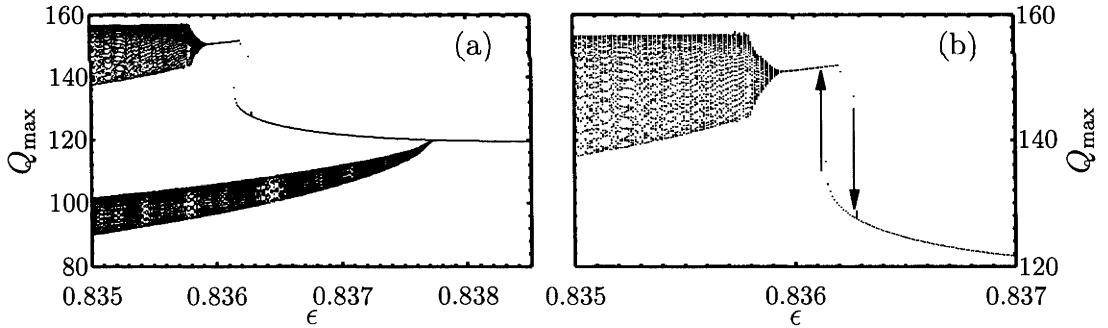


Figure 3.8: (a) Magnified region of the bifurcation diagram, shown as (iii) in Fig. 3.1. This detailed plot shows splitting of the branch of creeping solitons. (b) The branch of solitons above region (iii) that shows bistability which is not resolved on the scale of Fig. 3.1.

3.4 Multiplicity of Solutions

One of the remarkable features of the region of transition between solitons and the fronts is that several solutions may exist for a same set of system parameters. This

has indeed been shown in the previous section, where two creeping solitons with different dynamics co-exist for the same set of system parameters. In fact, Fig. 3.1 exhibits bistability in most of the intervals between $\epsilon = 0.82$ and $\epsilon = 0.85$, and tristability in a smaller, but still considerable, interval of values of ϵ .

There are two branches of solutions, labeled MP (moving pulse) and CP (composite pulse), co-existing in the interval between $\epsilon \approx 0.842$ and $\epsilon \approx 0.848$. A part of this interval has tristability, having another branch of solutions at higher energy. The amplitude, and spectra of the MP and CP for the same set of system parameters, *i.e.* $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.844$, are given in Fig. 3.9.

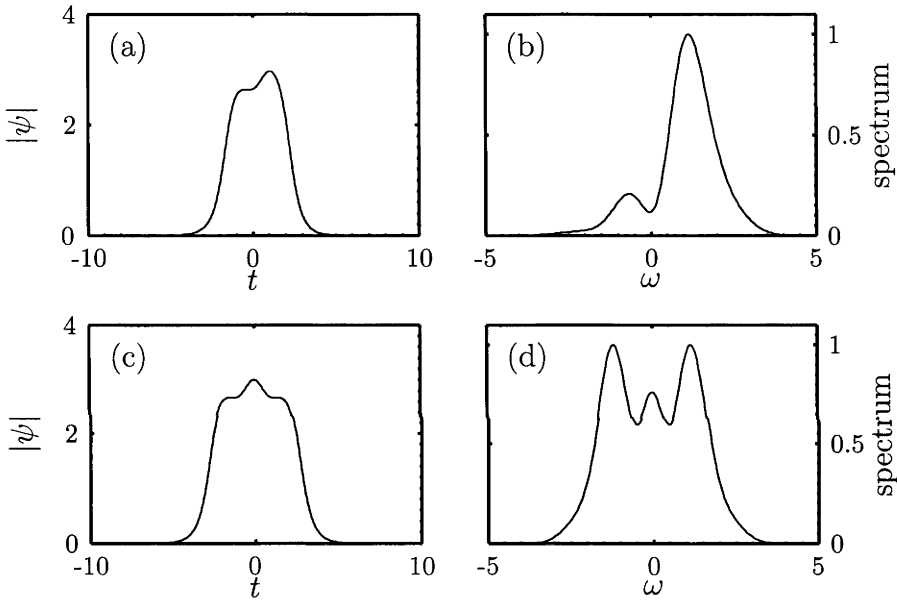


Figure 3.9: (a) Amplitude profile and (b) spectrum of a moving pulse (MP) and (c,d) those of a composite pulse (CP) for $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.844$.

The CP has a small ‘hill’ in the central part and two fronts located symmetrically on each side of the pulse, whereas the MP has a similar structure to the CP, but one of the fronts is missing. Hence, the CP can be considered as a nonlinear superposition of two fronts and a plain pulse, while the MP can be considered as a combination of a front and a plain pulse. These types of solutions have been found previously, but for a completely different set of parameters [24].

The MP, as its name suggests, has a constant velocity in the transverse direction, and has an asymmetric spectrum with two unequal peaks separated by a valley. The

inversion symmetry of the CGLE relative to the variable t allows for the existence of solitons with velocities in both positive and negative t directions. They are mirror images of each other. However, the MP always moves with the pulse leading, so that the example shown in Fig. 3.9(a) moves to the positive t direction. The front pushes the pulse from one side.

The CP, on the other hand, has central symmetry of the amplitude profile, and has zero velocity, *i.e.*, it is a stationary soliton. The novel feature of this branch found in Fig. 3.1 is a bifurcation in the middle of its range of existence. Here, a CP turns into a pulsating composite pulse. To demonstrate this new feature explicitly, the region labeled (vi) has been plotted with a higher resolution in Fig. 3.10(a).

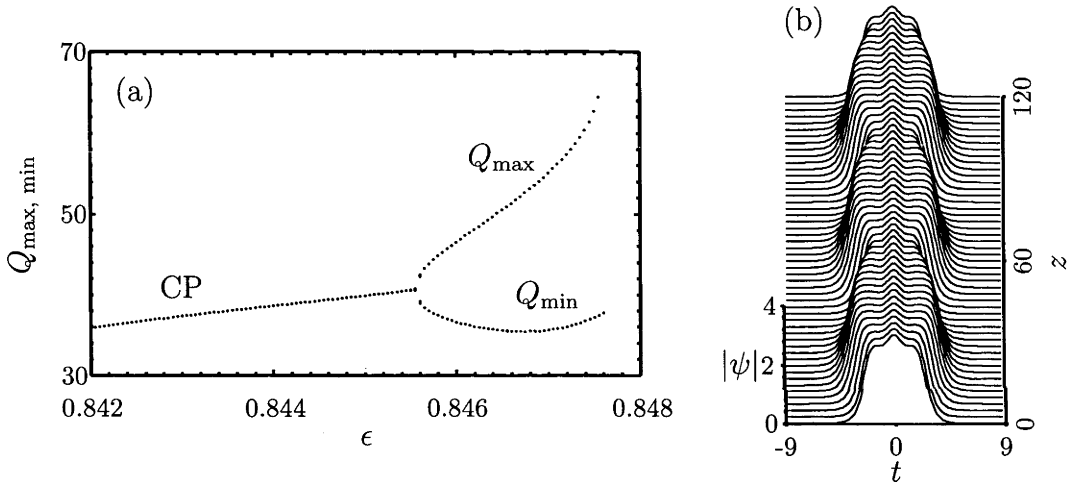


Figure 3.10: Diagram showing how a composite pulse (CP) bifurcates into a pulsating CP. This corresponds to region (vi) of Fig. 3.1. (b) Evolution of a pulsating CP for $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.847$.

In the region of stationary CP, $Q_{\max}$ and $Q_{\min}$ coincide, resulting in a single curve up to the point of bifurcation at $\epsilon \approx 0.8453$ in the direction of increasing ϵ . Above this point, the energy curve splits, thus demonstrating the pulsating nature of the solution. A pulsating CP solution is shown in Fig. 3.10(b). Such a transition from a fixed point solution to a limit-cycle is classified as an Andronov-Hopf bifurcation. It should be stressed that, in contrast to results from standard nonlinear dynamics, this bifurcation occurs in an infinite dimensional system.

These two branches of solutions, labeled MP and CP, disappear at higher ϵ near the upper boundary of Fig. 3.1. This is the boundary between solitons and fronts. Localized solutions are transformed into two fronts that move away from each other.

3.5 Period Doubling Route to Chaos II

There is one more type of soliton that exists at the same values of ϵ as MP and CP solutions. The corresponding branch is located above the branches denoted MP and CP in Fig. 3.1. It starts at the center of the shaded region labeled (iv) and ends up inside the region (v). This is a branch of uni-directional single frequency creeping solitons. Each side of the branch terminates in a bifurcation.

On the right-hand-side of the curve, this branch undergoes a series of period-doubling bifurcations. This can be seen clearly in Fig. 3.11. This figure magnifies region (v) in Fig. 3.1. The resolution of the plot given in Fig. 3.11 allows one to observe a period-quadrupling and even a period-eight solution. The first bifurcation of this series occurs at $\epsilon = 0.845125$ while the second one is at $\epsilon = 0.845325$. The interval $\Delta\epsilon$ for the period-doubling relative to that for period-quadrupling is $(0.845325 - 0.845125)/(0.845368 - 0.845325) = 4.651163$. Again, this ratio is very close to the Feigenbaum ratio of 4.669201 suggesting that it is very likely that there is a whole sequence of period-doubling bifurcations on its route to chaotic behavior.

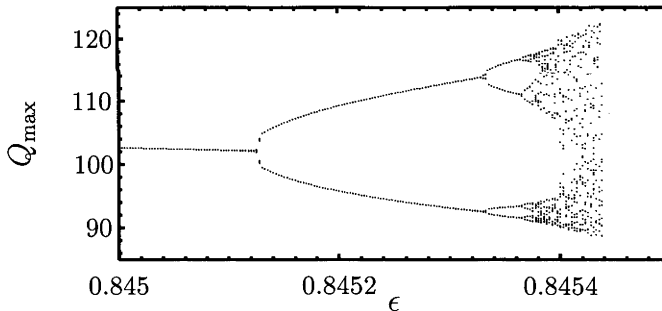


Figure 3.11: A sequence of period-doubling bifurcations of creeping solitons leading to chaos (region (v) in Fig. 3.1). Bifurcations up to period-eight can clearly be seen. A further increase of ϵ leads to transformation of solitons into two moving fronts.

It should be emphasized here that this example is the first observation of period-doubling bifurcations of creeping solitons. Earlier, there has been a report of a sequence of period-doubling bifurcations of zero-velocity pulsating solitons [41]. Due to the creeping nature of this branch, the chaotic solitons that appear at the end of this route are still moving localized solutions. The velocities of these solitons in the t direction vary chaotically, but the motion is only in one direction, and it

is determined by the initial conditions. This branch ends near $\epsilon = 0.84544$ when the period of pulsation increases indefinitely and the soliton disappears to form two moving fronts. This evolution leads to the transformation of a soliton into two fronts.

3.6 Symmetry-Breaking Bifurcations

A pitch-fork bifurcation indicates broken symmetry. The CGLE is symmetric with respect to the transformation $t \rightarrow -t$. However, solutions do not have to be symmetric for this nonlinear system. The transformation $t \rightarrow -t$ converts them into different functions which are also solutions of Eq. (2.2.1). In particular, this transformation changes the sign of the soliton velocity. Asymmetric solutions may appear as a result of a bifurcation. An example of a symmetry-breaking bifurcation in the theory of solitons was first found by Akhmediev [56]. In Hamiltonian systems, this type of bifurcation occurs in an inhomogeneous medium. Dissipative systems allow for symmetry-breaking bifurcations to occur even in homogeneous media.

There are two types of symmetry-breaking bifurcations in the theory of bifurcations, *viz.* sub-critical and super-critical. The region studied here is richly filled with bifurcations, and both of these types have been observed. Each one involves a bifurcation of a symmetric limit-cycle to two asymmetric limit-cycles. Clearly, bifurcations of limit-cycles are distinct from typical cases of symmetry-breaking involving fixed points. A sub-critical symmetry-breaking bifurcation from a stationary soliton to a moving pulse has been found by Descalzi, Brand and Cisternas for a case where the cubic Ginzburg-Landau equation is coupled to another equation [57]. This system represents two coupled nonlinear optical fibers, where one fiber also features dispersion. However, for the range of parameters presented in Fig. 3.1, the moving pulse branch always has non-zero velocity, and no bifurcations from a stationary soliton to a moving soliton are observed.

3.6.1 Sub-Critical Symmetry Breaking

A sub-critical symmetry-breaking bifurcation is observed in the region labeled (iii) in Fig. 3.1. This region is magnified and plotted in Fig. 3.8(b). The upper band of solitons, with the zig-zag motion, ends with the branch of a uni-directional creeping (asymmetric) solitons that is stable from $\epsilon \approx 0.8358$ up to $\epsilon \approx 0.8362$. The lower

branch in this plot corresponds to pulsating solitons (symmetric localized solutions). The two types of solitons co-exist over a small region of ϵ . The transition from one branch to the other depends on the direction in which the parameter ϵ is changed. It occurs at the points shown by the vertical arrows. Increasing ϵ leads to switching from the asymmetric branch to the symmetric one, as shown by the right-hand-side arrow, whereas decreasing ϵ results in the transformation shown by the left-hand-side arrow. The sign of the average velocity of the resulting creeping solitons is defined by perturbations in the system. A sub-critical bifurcation leads to a small region with hysteresis.

3.6.2 Super-Critical Symmetry Breaking

Another symmetry-breaking bifurcation is found at the end of the pulsating branch, in the region labeled (iv). The two curves to the left and to the right of this point meet with different slopes, showing that this is indeed the point of another bifurcation. Both localized solutions are pulsating on each side of the point. However, solitons on the left-hand-side of the point have zero transverse velocity, while solitons on the right-hand-side have non-zero transverse velocity. The branch of solutions corresponding to moving-pulsating solitons is degenerate, since it represents two separate creeping branches moving in opposite directions.

Detailed observations show that the curve on the left-hand-side of the bifurcation in region (iv) has a continuation in the form of a branch of unstable non-moving pulsating solitons. This continuation is indicated in Fig. 3.12(a) by the dashed-line. The points on this line have been obtained by using the same propagation technique as for the other branches. However, for the dashed-line, the solutions only remained stationary for a limited propagation distance, thus indicating their unstable nature. When symmetric initial conditions are chosen, the solutions first converge to zero-velocity pulsating solitons, but these are later transformed into non-zero velocity solutions due to the instability of solitons with zero-velocity in this region. Fig. 3.12(b) shows this process of transformation. Fig. 3.12(c) and Fig. 3.12(d) show the evolution of soliton profile before and after the transformation, respectively.

The energy, Q , pulsates for each of the above solutions. However, the amplitude of pulsations is reduced after the transformation from a non-moving pulsating soliton to a creeping one has occurred. Creeping solitons to the right of the point

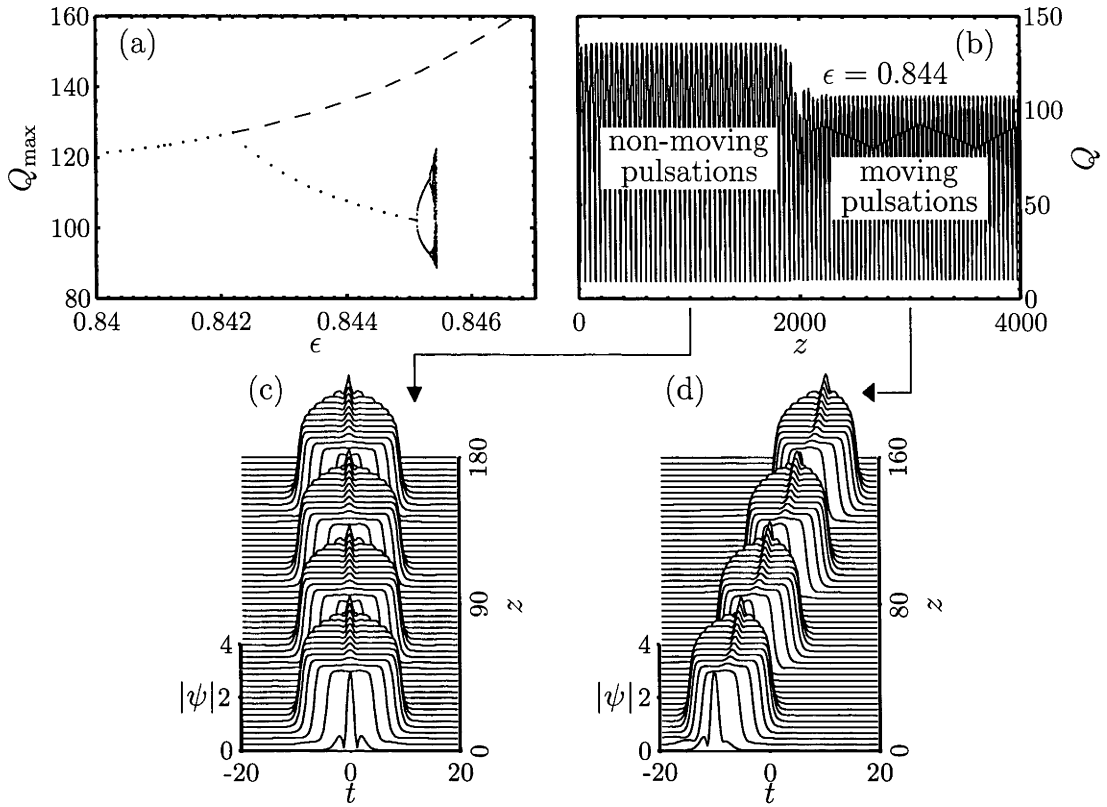


Figure 3.12: (a) Symmetry-breaking bifurcation at $\epsilon \approx 0.8425$. The dotted branch on the left of the bifurcation point corresponds to stable non-moving pulsating solitons, while the same branch to the right of the bifurcation point corresponds to creeping solitons. The branch of unstable non-moving pulsating solitons is shown with the dashed-line. This corresponds to region (iv) of Fig. 3.1. (b) Energy Q against propagation z , for $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.08$, $\mu = -0.11$ and $\epsilon = 0.844$. (c) Unstable pulsating solitons are transformed into (d) stable creeping solitons.

$\epsilon \approx 0.8425$ have a velocity that increases from zero at the point of bifurcation. These observations allow one to conclude that, according to the standard classification in the theory of bifurcations in low dimensional systems [58], the point $\epsilon \approx 0.8425$ is a super-critical symmetry-breaking bifurcation. Infinite-dimensional systems may have bifurcations with much richer structure than the simple ones appearing in the case of two-dimensional phase space. Nevertheless, the simple classification is also valid in higher-dimensional systems.

3.7 Validity of System Reductions

As the system is highly complex, finding some general simple rules governing these transitions is non-trivial. One of the possible techniques that could be used to address these issues is a reduction to a low-dimensional system similar to those performed by Tsoy, Ankiewicz and Akhmediev [59]. However, the method of moments would not be able to give an accurate answer because its predictions lack precision. The technique may give approximate boundaries of the regions, but it cannot describe the whole variety of solutions observed numerically. For example, the phenomenon of co-existence of two types of creeping solitons is beyond the reach of any technique which uses any type of reasonably simple trial function. Also, describing the zig-zag motion of the solitons by using a trial function is highly doubtful. The fine structure of bifurcations shown in Fig. 3.1 is too detailed to be approximated by any type of reduction to a low-dimensional dynamical system. Individual features of this structures may be modeled, but their relative locations in the parameter space are too intricate. Thus, numerical simulations are still the most reliable method for studying such complex dynamics in the region of transition from solitons to fronts in dissipative systems.

3.8 Summary

Dissipative systems with an infinite number of degrees of freedom have rich dynamical behaviors. Leaving aside pattern formation, and restricting one's attention to uniquely localized solutions, the number of possibilities still appears to be infinite. On increasing the cubic gain parameter, ϵ , the localized solutions can pass through regions of no solitons, stationary solitons of various types, pulsating solitons and then transformation into front solutions. In addition, each boundary contains a multiplicity of bifurcations. Moreover, several solutions can exist simultaneously.

In this chapter, it has been shown that the region of transformation from pulsating solutions to fronts is especially opulent with bifurcations and multi-stabilities. This occurs even with the simplest modeling, when all parameters except ϵ are fixed. Changing the five other parameters in Eq. (2.2.1) as well may result in a multiplicity of other transformations.

A numerical approach appears to be the most reliable way of investigating such a highly complicated system. It is highly unlikely that any type of system reduction

would model the whole variety of solutions observed in this region of transition from solitons to fronts. However, approximate boundaries of individual features may be found using such approaches.

Chapter 4

Analytical Approach: System Reduction

Dynamical systems such as those described by the complex Ginzburg-Landau equation (CGLE) have an infinite number of degrees of freedom. Although exact solutions of the CGLE do exist, they can be expressed analytically only for certain relations between the parameters of the equation. Furthermore, only stationary solutions are known in analytical form [24]. Therefore, finding various localized solutions and their bifurcation boundaries, especially in the complete multi-dimensional space of system parameters, is not simple, and usually involves massive numerical simulations with different sets of parameters and initial conditions. The complexity of the bifurcation diagram presented in Chapter 3 where only one system parameter is varied, serves as a good example of the intricacy of these problems.

One way to avoid such extensive computations is to construct a simplified model that can be used to approximate the existence range of localized solutions. By reducing the CGLE to a finite-dimensional dynamical model, one can obtain a set of ordinary differential equations (ODEs) approximating the evolution of localized solutions. This reduction can be carried out by applying the method of moments, which was originally developed by Maimistov for the perturbed nonlinear Schrödinger equation [60]. A brief outline of this method is presented in Section 4.1. Quite remarkably, by using this approach, it is possible to obtain approximate models and the region of existence in the parameter space for pulsating solitons [59], and even creeping solitons. A careful selection of the trial function is important for approximating each particular type of soliton. It should be detailed enough to describe

the soliton under investigation, yet, simple enough to lead to an approximate model with a finite set of ODEs. The derivation and the analysis of low-dimensional dynamical systems obtained for approximating both pulsating and creeping solitons are presented in the following sections. Finally, a chapter summary is given in Section 4.4.

4.1 The Method of Moments

The method of moments provides a way of dealing with a system by introducing some integral characteristics of the field under consideration [60]. Different problems described by the perturbed NLSE [60–62], as well as the CGLE [59, 63] have been successfully approximated by this approach. The method reduces a system with an infinite number of degrees of freedom to a dynamical system describing the evolution of a finite set of pulse characteristics. These characteristics may include the energy, width, center-of-mass and phase parameters. An approach that can be used to derive the evolution equations of these pulse characteristics using the method of moments is presented below.

Before beginning the derivation, the right-hand-side of Eq. (2.2.1) is defined to be $R[\psi]$ for convenience. Hence Eq. (2.2.1) becomes:

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{tt} + i\mu|\psi|^4\psi - \nu|\psi|^4\psi \equiv R[\psi], \quad (4.1.1)$$

As a starting point, a set of moments needs to be defined. In principle, there is an infinite number of equations for moments, and using the complete set would give exact results. However, in order to obtain a significant reduction in the number of degrees of freedom, only a finite number of moments are used [63]. The number of moments should correspond to the number of parameters in the trial function for describing a particular solution, *i.e.* the complexity of the solution under investigation and the intended accuracy of the approximate model determine the required number of moments. At least five pulse characteristics are required to describe pulsating and creeping solitons. Therefore, a minimum of five moments is needed. Increasing this number may improve the accuracy of the reduced model, but the complexity of the analysis then increases dramatically.

For an arbitrary localized field ψ , one can introduce two lowest-order integrals,

namely the energy Q and momentum P

$$\begin{aligned} Q &= \int_{-\infty}^{\infty} |\psi|^2 dt, \\ P &= \frac{1}{2} \int_{-\infty}^{\infty} (\psi \psi_t^* - \psi^* \psi_t) dt, \end{aligned} \quad (4.1.2)$$

which are two basic variables which evolve in z . Three higher-order generalized moments are given by the following expressions [60]:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} t |\psi|^2 dt, \\ I_2 &= \int_{-\infty}^{\infty} (t - t_0)^2 |\psi|^2 dt, \\ I_3 &= \int_{-\infty}^{\infty} (t - t_0) (\psi^* \psi_t - \psi \psi_t^*) dt, \end{aligned} \quad (4.1.3)$$

where $t_0(z)$ is center-of-mass of the field envelope.

Using the original Eq. (4.1.1), one can derive the following evolution equations for the generalized moments [60]:

$$\begin{aligned} \frac{dQ}{dz} &= i \int_{-\infty}^{\infty} (\psi R^* - \psi^* R) dt, \\ \frac{dP}{dz} &= -i \int_{-\infty}^{\infty} (\psi_t R^* + \psi_t^* R) dt, \\ \frac{dI_1}{dz} &= iDP + i \int_{-\infty}^{\infty} t (\psi R^* - \psi^* R) dt, \\ \frac{dI_2}{dz} &= -iDI_3 + i \int_{-\infty}^{\infty} (t - t_0)^2 (\psi R^* - \psi^* R) dt, \\ \frac{dI_3}{dz} &= 2P \frac{dt_0}{dz} + i \int_{-\infty}^{\infty} (2D|\psi_t|^2 - |\psi|^4) dt \\ &\quad + 2i \int_{-\infty}^{\infty} (t - t_0) (\psi_t R^* + \psi_t^* R) dt + i \int_{-\infty}^{\infty} (\psi R^* + \psi^* R) dt. \end{aligned} \quad (4.1.4)$$

The set of differential equations given in Eq. (4.1.4) is valid not only for the CGLE, but also for a large class of NLSE-type evolution equations. $R[\psi]$ can represent any additional terms put on the right-hand-side of Eq. (2.1.1).

Up to this point, the equations in Eq. (4.1.4) are exact. *i.e.* an exact solution of Eq. (4.1.1) in place of ψ would result in identities from the set of evolution equations.

However, since no stable solutions of the quintic CGLE can be expressed in an analytical form, apart from some stationary solutions, one can use a trial function with a few parameters which depend on z . The selection of the trial function is very important, and different functions are suitable for different localized solutions. Often the trial function is chosen from the general symmetries of the problem and also from results of numerical simulations. Once an appropriate trial function, with five parameters, is chosen, it is then substituted into Eq. (4.1.4), and a set of five ODEs describing the evolution of the function parameters is obtained.

4.2 Pulsating Solitons

Pulsating solitons form one set of possible localized solutions of the CGLE. The width of these solutions changes, so they contract and expand periodically along the propagation axis z . They were discovered numerically [41, 43, 44] and observed experimentally in a fiber laser [44]. They appear from stationary solitons when the latter become unstable. Hence, pulsating solitons are found on the boundary between the region of stationary solitons and fronts [41]. A pulsating soliton solution of the CGLE is a limit-cycle of an infinite-dimensional dissipative dynamical system [7]. In the pulsating region, the binding of the two interacting fronts is strong enough to form a stationary soliton, but with lower binding effect, they oscillate relative to one another.

Fig. 4.1(a) shows the evolution of a pulsating soliton profile. This particular solution is found when $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\epsilon = 0.66$, $\mu = -0.1$ and $\nu = -0.1$. In order to find the region of existence of this pulsating soliton in the parameter space, one needs to solve the CGLE numerically, as no analytic expressions for pulsating solutions and their bifurcation boundaries have been found so far. There are several parameters of the CGLE that define the system, and therefore the boundaries are surfaces in this multi-dimensional space of the system parameters. Thus finding these boundaries is not a simple task.

A more feasible solution to this problem is to vary only two of the system parameters and fix the others, and map the region on a two-dimensional plane. Although this does not give a complete region in the parameter space, it serves as a good starting point to deepen one's understanding of the solutions and their range of existence. One such plot is given in Fig. 4.1(b) [41]. The pulsating region is

enclosed by a thick line, and this region includes the pulsating solution shown in Fig. 4.1(a).

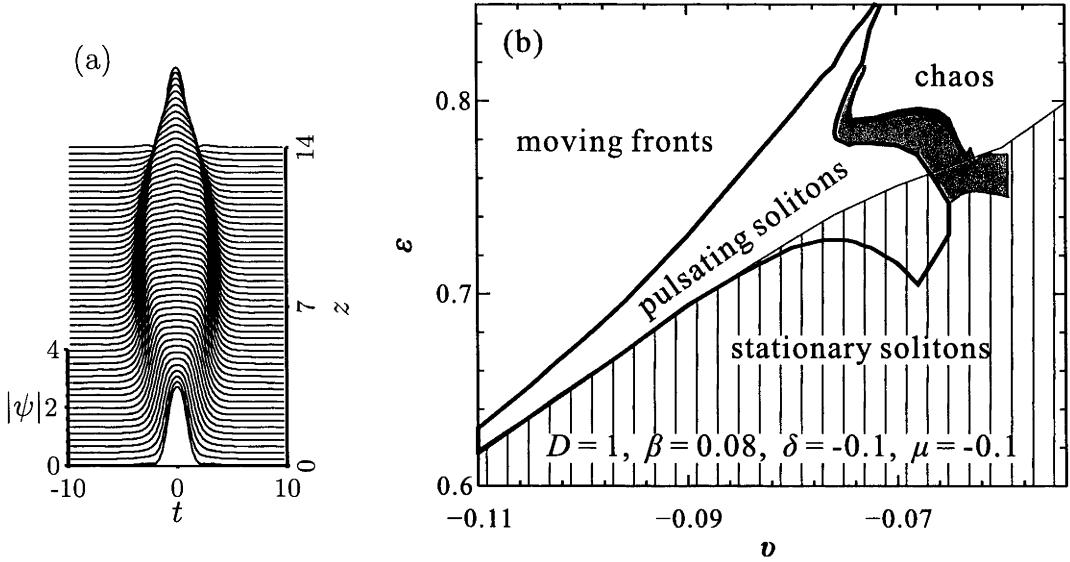


Figure 4.1: (a) An example of a pulsating soliton of the CGLE at $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\epsilon = 0.66$, $\mu = -0.1$ and $\nu = -0.1$. (b) Region of existence of pulsating solitons in (ϵ, ν) -plane of the CGLE [41].

In order to find the boundaries of a set of particular solutions, for example, the pulsating soliton shown in Fig. 4.1(a), one can start from the above set of system parameters, and, while fixing the other parameters, ϵ and ν can be varied slightly to check if the pulsating solution still exists, using earlier pulsating solution as the initial condition. This process is repeated until there is a qualitative change in the solution behavior, which then implies that there is a boundary nearby. Close to the boundary, one can improve the finesse by reducing the increment in either ϵ or ν . Once a different type of solution is known to exist on the border with the pulsating solitons, the region of existence of this solution can be also mapped using the same method. This technique even allows one to resolve areas where two or more soliton solutions exist simultaneously, as shown by the overlapping areas near the bottom right-hand-side border of the pulsating solitons in Fig. 4.1(b).

By using the method of moments with a suitable choice of the trial functions, it is possible to approximate these boundaries with a much smaller expense of the computational time.

The author would like to acknowledge that many of the results presented in this

section are the work of Tsoy, Ankiewicz and Akhmediev [59].

4.2.1 Hyperbolic-Secant Pulse

One possible trial function that can be used to model the pulsating soliton is the hyperbolic-secant function:

$$\psi(t, z) = A \operatorname{sech} \left(\frac{t - t_0}{w} \right) \exp \left(i \left(b(t - t_0) + c(t - t_0)^2 \right) \right), \quad (4.2.1)$$

where $A(z)$ is the amplitude, $w(z)$ is the width and $t_0(z)$ is position of the center-of-mass. $b(z)$ and $c(z)$ are the linear and quadratic phase coefficient, respectively. The number of parameters characterizing this trial function, five, corresponds to the number of moments used in Eq. (4.1.4), and this allows one to obtain the evolution equations of these pulse parameters.

By substituting Eq. (4.2.1) into Eq. (4.1.2) and Eq. (4.1.3), the generalized moments can be expressed in terms of the variables of the trial function. They are

$$\begin{aligned} Q &= 2A^2w, & P &= -2iA^2wb, & I_1 &= 2A^2wt_0, \\ I_2 &= (\pi^2/6) A^2w^3, & I_3 &= i(2\pi^2/3) A^2w^3c. \end{aligned} \quad (4.2.2)$$

The derivatives of Eq. (4.2.2) with respect to z are then equated to the right-hand-side of Eq. (4.1.4), and the equations for the evolution of the variables of the trial function are obtained. They are

$$\begin{aligned} Q_z &= Q \left(2\delta - 2b^2\beta + \frac{2}{3} \frac{\beta}{w^2} + \frac{2\pi^2}{3} w^2 c^2 \beta + \frac{2}{3} \frac{Q\epsilon}{w} + \frac{4}{15} \frac{Q^2\mu}{w^2} \right), \\ w_z &= w \left(2cD + \frac{8}{\pi^2} \frac{\beta}{w^2} - \frac{16\pi^2}{15} w^2 c^2 \beta - \frac{2}{\pi^2} \frac{Q\epsilon}{w} - \frac{1}{\pi^2} \frac{Q^2\mu}{w^2} \right), \\ c_z &= \frac{1}{w^2} \left(-2w^2 c^2 D - \left(\frac{4}{\pi^2} + \frac{4}{3} \right) c\beta - \frac{1}{\pi^2} \frac{Q}{w} - \frac{8}{15\pi^2} \frac{Q^2\nu}{w^2} + \frac{2}{\pi^2} \frac{D}{w^2} \right), \\ t_{0,z} &= bD - \frac{2\pi^2}{3} w^2 bc\beta, \\ b_z &= -\frac{4\pi^2}{3} w^2 bc^2\beta - \frac{4}{3} \frac{b\beta}{w^2}. \end{aligned} \quad (4.2.3)$$

To study non-moving pulsating solutions, one can make a further reduction and consider a system with only three variables, *viz.* Q , w and c . This is possible, since

for non-moving solutions, b and $t_{0,z}$ become zero for $z \rightarrow \infty$. Hence, the set of the first three ODEs in Eq. (4.2.3) can be solved to approximate the pulsating solutions and their region of existence.

4.2.2 Generalized Gaussian Pulse

Another trial function that can be considered is a combination of Gaussian and super-Gaussian types of functions, as the numerical pulsating solutions of the CGLE change their shape during one period from a bell-shaped pulse to a flat-top pulse. This can be written in the form:

$$\psi(t, z) = A \exp\left(-\frac{t^2}{w^2} - \frac{t^4}{4mw^4}\right) \exp(ict^2), \quad (4.2.4)$$

where the meanings of $A(z)$, $w(z)$ and $c(z)$ are the same as in Eq. (4.2.1). m is a constant that determines the ‘rectangularity’ of the pulse; it can be chosen arbitrarily and is independent of z . Applying the same principle as in the previous trial function, the center-of-mass t_0 and the linear phase coefficient b are deliberately omitted in Eq. (4.2.4). Hence, there are three parameters, $A(z)$, $w(z)$ and $c(z)$ that need to be solved in the three-dimensional model.

The study has shown that this choice of trial function, presented in Eq. (4.2.4) gives a better approximation of the pulsating soliton than the Gaussian function or the super-Gaussian function with a quartic term alone [59]. The value of m is not very critical, and the case $m = 0.25$ seems to agree well with the pulsating solitons of the CGLE. Indeed, other values of m can be used, but here, only the $m = 0.25$ case will be presented with arithmetic values in place of the hypergeometric function representation of the coefficients. The general case for arbitrary m is given in Appendix I.

Due to the symmetry of Eq. (4.2.4) in t , two integrals, P and I_1 become zero, while the other three moments, in terms of function variables, are given below:

$$Q = 1.051A^2w, \quad I_2 = 0.145Qw^2, \quad I_3 = 4icI_2. \quad (4.2.5)$$

Applying the same technique as outlined in the hyperbolic-secant trial function, one

can obtain a set of three equations describing the evolution of the pulse parameters:

$$\begin{aligned} Q_z &= Q \left(2\delta - \frac{3.738\beta}{w^2} - 1.158w^2c^2\beta + \frac{1.433Q\epsilon}{w} + \frac{1.146Q^2\mu}{w^2} \right), \\ w_z &= w \left(2cD + \frac{2.142\beta}{w^2} - 0.874w^2c^2\beta - \frac{0.290Q\epsilon}{w} - \frac{0.325Q^2\mu}{w^2} \right), \\ c_z &= \frac{1}{w^2} \left(-2w^2c^2D - 19.624c\beta - \frac{1.237Q}{w} - \frac{1.319Q^2\nu}{w^2} + \frac{6.453D}{w^2} \right). \end{aligned} \quad (4.2.6)$$

It should be noted that the form of the set of equations in Eq. (4.2.3) is similar to the set in Eq. (4.2.6). The only differences occur in the coefficient values for each term. This is due to the fact that the dependency of each function variables on t is the same for Eq. (4.2.1) and Eq. (4.2.4) [64].

4.2.3 Pulsating Model Analysis and Comparison

Each set of coupled ODEs, Eq. (4.2.3) and Eq. (4.2.6), is a distinct model that can be used to approximate the CGLE, Eq. (4.1.1), in the pulsating region of the system parameters. As shown in Fig. 4.2(a), the reconstructed evolution of the pulsating soliton, using the trial function, Eq. (4.2.4) and the coupled ODEs Eq. (4.2.6), shows a very good qualitative approximation of the exact pulsating soliton shown in Fig. 4.1(a). Indeed, the region in parameter space where the pulsating solutions of these two reduced models are observed is clearly in good agreement with the exact solution. In Fig. 4.2(b), the pulsating region for each model is plotted in the same slice of the (ϵ, ν) -plane as in Fig. 4.1(b), for a comparison with the CGLE model, which is the region enclosed by the thin solid line.

The two lower boundary lines are the boundary between stationary solutions and pulsating solutions for each reduced model, and hence there is an Andronov-Hopf bifurcation. However, there is no region where multiple solutions co-exist in the reduced models, which suggests that the reduced models cannot resolve such fine details of the soliton transitions. In the reduced model, the transition from the pulsating to stationary solutions and vice versa occurs on the same surface in the parameter space, and there is no formation of a hysteresis loop.

On the other hand, on crossing the two upper boundary lines, there is a transition from pulsating solitons to moving fronts. The period of the limit-cycle increases indefinitely upon approaching the upper boundary. At any fixed ν , the period of

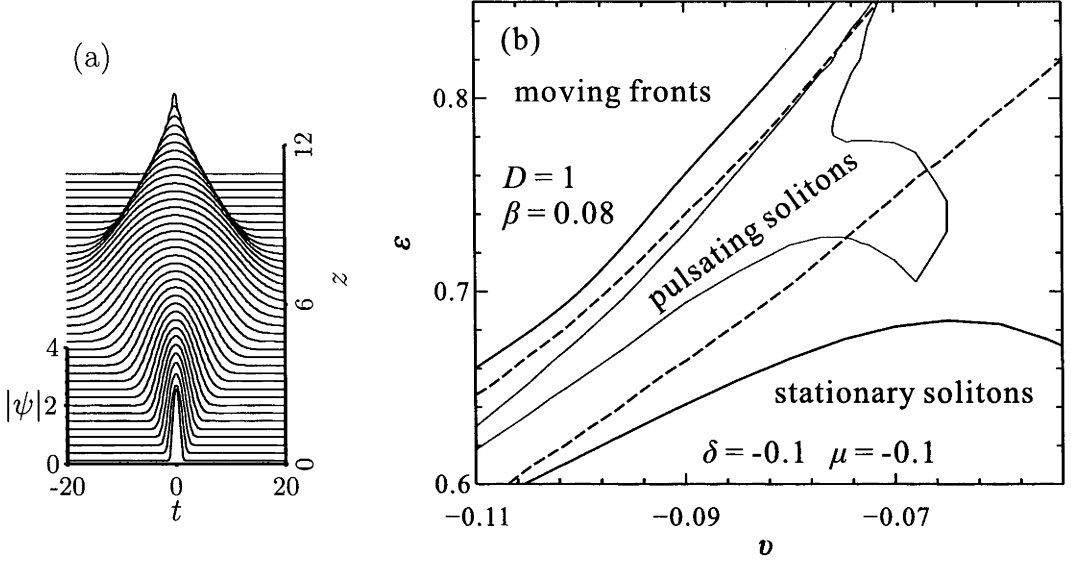


Figure 4.2: (a) Reconstructed soliton from the trial function Eq. (4.2.4) and the solution of Eq. (4.2.6) at $D = 1$, $\epsilon = 0.66$, $\delta = -0.1$, $\beta = 0.08$, $\mu = -0.1$ and $\nu = -0.1$. (b) Region of existence of pulsating solitons in (ϵ, ν) -plane of both reduced models. The regions between the two solid lines and two dashed lines correspond to the pulsating region in Eq. (4.2.3) and Eq. (4.2.6), respectively. The pulsating region of the exact model is also plotted (thin solid line) for a comparison [59].

oscillation increases with ϵ . Once the parameters are above this line, the energy Q and the width w increase monotonically with z . The ratio Q/w remains more or less constant, as the amplitude A remains mostly unchanged. This closely represents two fronts moving away from each other, as observed in numerical solutions of the CGLE.

Fig. 4.3 compares the dependence of the oscillation period of the limit-cycle on ϵ , while the other parameters are fixed at $D = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.09$ and $\mu = -0.1$ for three different models, namely, Eq. (4.1.1), Eq. (4.2.3) and Eq. (4.2.6). Although there are clear differences between the exact model and the reduced models, as well as between the reduced models themselves, the qualitative features of all three curves are the same. They all start from a finite value near the lower boundary of the pulsating region, and increase to infinity as ϵ reaches the upper boundary. Fig. 4.3 also indicates that the reduced model constructed from the generalized Gaussian function, Eq. (4.2.4) gives a more accurate approximation for the pulsating solutions of the CGLE than the one using the hyperbolic-secant

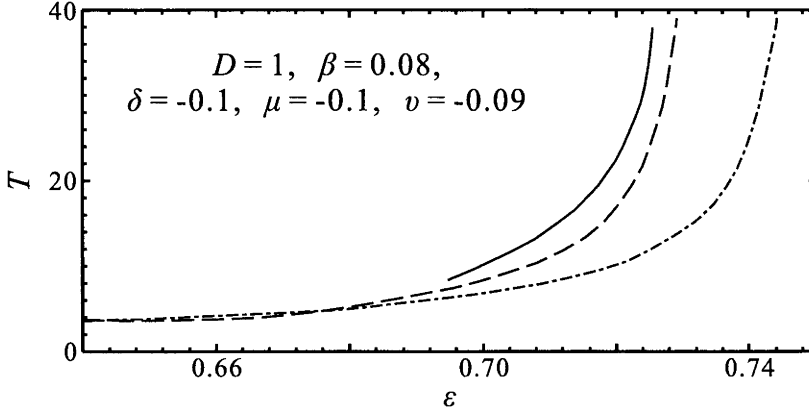


Figure 4.3: Period of pulsations T as function of ϵ . The solid line is the result of numerical simulations of the CGLE, Eq. (4.1.1). The dot-dashed line correspond to the model Eq. (4.2.3) and the dashed line to Eq. (4.2.6) [59].

trial function. This is due to the profile of Eq. (4.2.4) being more rectangular in its shape, and this shape is closer to the shape observed for the pulsating solutions of the CGLE.

4.3 Creeping Solitons

A creeping soliton is a special type of pulsating localized solution that changes its shape periodically and shifts a finite distance in the transverse direction after each period of oscillation. The value of the shift is constant for each period, and hence the soliton has a finite average velocity, although the motion occurs as a step-by-step translation in one direction. In most cases, solitons that move in this way have wide flat-top profiles bounded by a front on each side. The two fronts move asymmetrically in time, thus creating creeping movements of the whole ‘worm-like’ formation.

Creeping solitons were first observed in numerical simulations [28]. Their existence, for various dissipative systems, was confirmed in later publications [41, 65]. An example of a creeping soliton is shown in Fig. 4.4(a). Creeping solitons exist in a certain range of the equation parameters. This particular solution is found at $D = 1$, $\delta = -0.1$, $\beta = 0.101$, $\epsilon = 1.3$, $\mu = -0.3$ and $\nu = -0.101$. Starting from this set of initial parameters, bifurcation boundaries between various types of dissipative soliton solutions in a finite region of the (ϵ, ν) -plane were found using a

similar approach to that outlined in Section 4.2. Fig. 4.4(b) shows these boundaries around the region of existence of the creeping solitons.

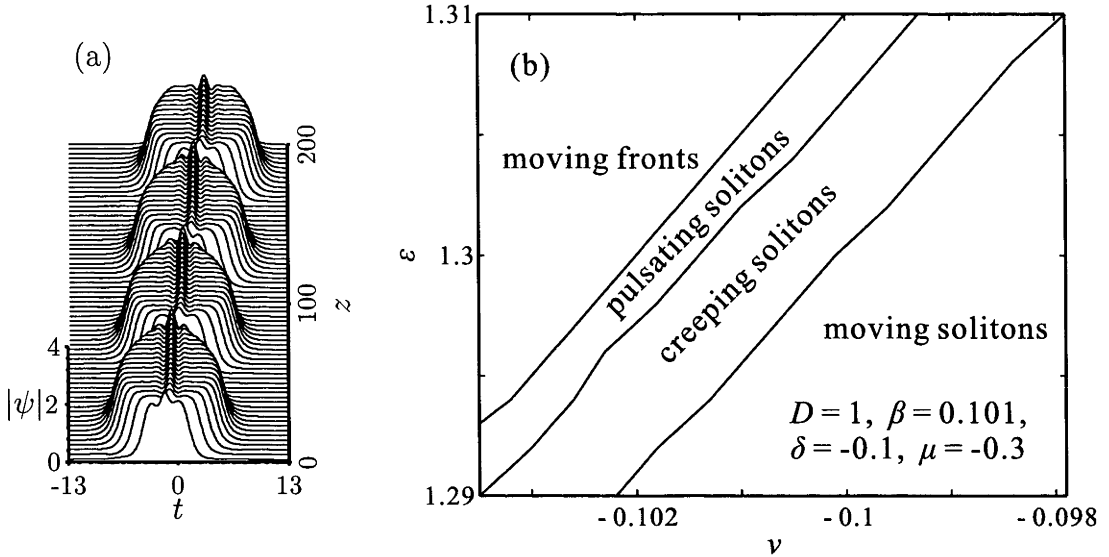


Figure 4.4: (a) An example of a creeping soliton of the CGLE at $D = 1$, $\delta = -0.1$, $\beta = 0.101$, $\epsilon = 1.3$, $\mu = -0.3$ and $\nu = -0.101$. (b) Region of existence of creeping solitons in (ϵ, ν) -plane of the CGLE.

Creeping solitons exist in a central diagonal strip in the (ϵ, ν) -plane. Below this strip, pulsations disappear and pulses are transformed into fixed shape solitons moving with constant velocity. These constant profile pulses are moving along the t -axis. Above the diagonal strip, the solitons are still pulsating, but their average velocity is zero. Thus, the creeping feature disappears on the middle solid curve in Fig. 4.4(b).

Inside the strip, the soliton energy oscillates due to the oscillations of the soliton width. These oscillations are shown in Fig. 4.5(a). The period of the oscillations varies in the range 50 – 60 across the strip. In each oscillation, the center-of-mass of the soliton moves by a finite increment. This step-wise motion of the center-of-mass is shown in Fig. 4.5(b). Due to the symmetry of the CGLE with respect to inversion of the t -axis, a soliton can equally well move to the left as to the right, depending on the initial condition.

At the upper boundary of the central diagonal strip in Fig. 4.4(b), the soliton creeping behavior ceases. Translations after each period disappear and the soliton simply pulsates with zero transverse velocity. These pulsating solitons exist in a narrower strip above the region for creeping solitons. The period of pulsations

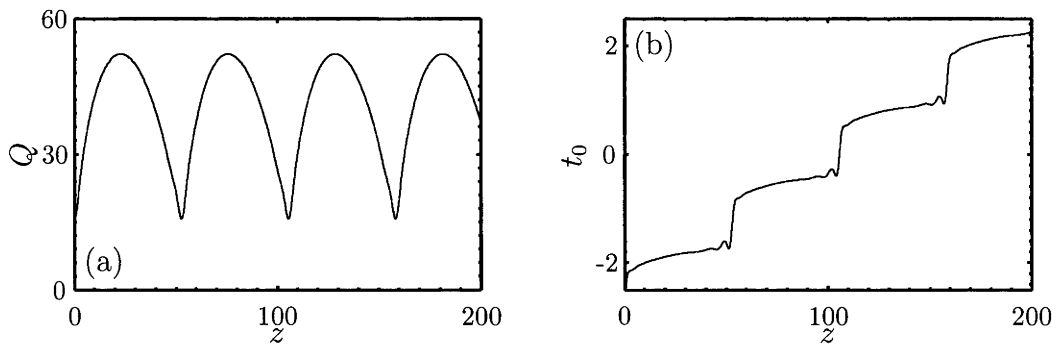


Figure 4.5: (a) Energy Q and (b) the center-of-mass t_0 against z for the creeping soliton shown in Fig. 4.4(a). Due to the symmetry in t , the soliton could also move in the opposite direction if the initial condition is changed.

increases and becomes infinite at the upper boundary of this second strip. Above the second strip, any localized solution increases its width and transforms itself into a pair of fronts moving away from each other. The narrow width of the strip indicates that the creeping solitons involve quite a subtle balance.

At the lower boundary of the central strip, the pulsations stop and creeping behavior disappears. The soliton profile, as well as its energy, Q , become constant. The soliton moves with a constant velocity to the right or to the left. Each of the boundaries in Fig. 4.4(b) represents a bifurcation in the dynamics of the dissipative soliton.

There is no technique available for describing pulsating solutions using exact solutions of the governing equation. Consequently, there is little hope of finding analytical solutions for creeping solitons. However, the method of moments can be applied to the CGLE to model creeping solitons as was done for the pulsating solitons in Section 4.2. The model needs to be extended to include the translational motion of the localized solutions. This requires an additional variable for the velocity of the soliton, and this increases the dimensionality of the finite-dimensional dynamical system. The velocity is also a periodic function of propagation distance z , and it thus describes the creeping feature of the soliton.

4.3.1 Asymmetric Generalized Gaussian Pulse

The numerical example presented in Fig. 4.4(a) shows that the creeping soliton changes its shape, during one period, from a bell-shaped pulse to a flat-top pulse.

Its velocity is generally constant, with an irregular ripple superimposed on it. Therefore, a variant of a ‘super-Gaussian’ function is considered as the trial function to reduce the system dimensionality. This is given by

$$\psi(t, z) = a(m) \sqrt{\frac{Q(z)}{f(z)}} \exp \left(-\frac{\sqrt{m}y}{w(z)} - \frac{y^4}{32w(z)^4} + i(b(z)y + c(z)y^2) \right), \quad (4.3.1)$$

where $y = t - t_0(z)$, and the meanings of $w(z)$, $b(z)$ and $c(z)$ are the same as in the previous trial functions. The constant m can be chosen arbitrarily, but it is independent of z , while a is a constant that depends on m . It is chosen in such a way that the total energy, $\int_{-\infty}^{\infty} |\psi|^2 dt$, equals $Q(z)$. For example, if $m = 0.008$, a should be 0.519548. The trial function given in Eq. (4.3.1) is deliberately chosen to be non-symmetric in y , otherwise the velocity and linear chirp would approach zero after a short transient. This would mean that only three ODEs would remain at the end of the derivation, and the creeping effect could be lost. The above trial function is detailed enough to show the creeping effects, yet the required integrals can still be obtained analytically with the help of hyper-geometric functions.

The generalized moments in Eq. (4.1.2) and Eq. (4.1.3), for any m , are given by

$$\begin{aligned} P &= -iQ(z) \left(b(z) + 4\sqrt{m} \frac{k_t}{k_a} c(z)w(z) \right), \\ I_1 &= Q(z) \left(2\sqrt{m} \frac{k_t}{k_a} w(z) + t_0(z) \right), \\ I_2 &= 4 \frac{k_b}{k_a} Q(z) w^2(z), \\ I_3 &= \frac{16i}{3k_a} w(z) Q(z) (3k_b c(z)w(z) - \sqrt{m} k_c b(z)), \end{aligned} \quad (4.3.2)$$

where the constants k_i are given by the following expressions:

$$\begin{aligned} k_a &= \Gamma \left(\frac{1}{4} \right) H \left(\frac{1}{2}, \frac{3}{4}; m \right) + 8m \Gamma \left(\frac{3}{4} \right) H \left(\frac{5}{4}, \frac{3}{2}; m \right), \\ k_t &= \Gamma \left(-\frac{1}{4} \right) H \left(\frac{1}{2}, \frac{5}{4}; m \right) + 2m \Gamma \left(-\frac{3}{4} \right) H \left(\frac{3}{2}, \frac{7}{4}; m \right), \\ k_b &= \Gamma \left(\frac{3}{4} \right) H \left(\frac{1}{4}, \frac{1}{2}; m \right) + 8m \Gamma \left(\frac{5}{4} \right) H \left(\frac{3}{4}, \frac{3}{2}; m \right), \\ k_c &= 3 \Gamma \left(\frac{3}{4} \right) H \left(\frac{1}{2}, \frac{5}{4}; m \right) + 8m \Gamma \left(\frac{5}{4} \right) H \left(\frac{3}{2}, \frac{7}{4}; m \right). \end{aligned} \quad (4.3.3)$$

In these terms, the function $H(r, s; m)$ represents the generalized hypergeometric function ${}_0F_2(\{\}; r, s; m^2)$. It should be noted that for $m \rightarrow 0$, Eq. (4.3.1) becomes symmetric, and hence the expected results for a symmetric function are obtained.

Now, the derivatives of Eq. (4.3.2) with respect to z are taken, and they are equated with the right-hand-sides of Eq. (4.1.4). After rearrangement, one can obtain five coupled ODEs for the z -dependent soliton parameters. Eq. (4.3.4) presents five equations for the evolution of the soliton parameters for the $m = 0.0025$ case.

$$\begin{aligned}
Q_z &= Q \left(2\delta - 2b^2\beta + 1.080wbc\beta - \frac{0.511\beta}{w^2} - 10.902w^2c^2\beta + \frac{0.466Q\epsilon}{w} + \frac{0.117Q^2\mu}{w^2} \right), \\
w_z &= w \left(2cD + 0.634wbc\beta - \frac{0.099bD}{w} + \frac{0.065\beta}{w^2} - 6.448w^2c^2\beta - \frac{0.066Q\epsilon}{w} - \frac{0.024Q^2\mu}{w^2} \right), \\
t_{0,z} &= 0.859w^3c^2\beta + 0.987bD - 10.671w^2bc\beta - 0.035Q\epsilon - \frac{0.015Q^2\mu}{w} + \frac{0.090\beta}{w}, \\
b_z &= 1.764w^3c^3\beta - 21.920w^2bc^2\beta + 0.027bcD - 0.071Qc\epsilon - \frac{0.030Q^2c\mu}{w} - \frac{0.019c\beta}{w} \\
&\quad - \frac{1.006b\beta}{w^2} - \frac{0.012Q}{w^2} - \frac{0.004Q^2\nu}{w^3} + \frac{0.026D}{w^3}, \\
c_z &= 0.172w^2c^3\beta - 2c^2D - 2.142wbc^2\beta + \frac{0.198bcD}{w} - \frac{0.007Qc\epsilon}{w} - \frac{0.003Q^2c\mu}{w^2} - \frac{2.957c\beta}{w^2} \\
&\quad + \frac{0.060b\beta}{w^3} - \frac{0.043Q}{w^3} - \frac{0.015Q^2\nu}{w^4} + \frac{0.095D}{w^4}. \tag{4.3.4}
\end{aligned}$$

The set of ODEs for general m can be presented instead of the arithmetic values given in Eq. (4.3.4), but these are extremely lengthy and cumbersome, involving numerous hyper-geometric functions, and hence are not presented in this thesis. The case $m = 0.0025$ gives a reasonably good agreement with the numerical solutions of the CGLE. The small degree of asymmetry mimics the real soliton and produces a small average velocity as the soliton creeps. This case has been studied extensively, and the results are presented below. Although the dependence of each trial function parameter on t is same as for the previous two trial functions used for the study of the pulsating solutions, Eq. (4.3.4) contains a lot more terms, as there are more coefficients that do not cancel to zero, due to the asymmetry involved in Eq. (4.3.1). However, as expected, all the terms shown in the previous two sets of ODEs, Eq. (4.2.3) and Eq. (4.2.6), are present in Eq. (4.3.4) and differ only in the numerical values of their coefficients.

4.3.2 Creeping Model Analysis

By solving the set of coupled ODEs in Eq. (4.3.4), pulsating solitons with non-zero transverse velocity can be found in a certain range of the original equation parameters. Fig. 4.6(a) shows a reconstructed soliton profile evolution from the trial function Eq. (4.3.1) and solution of Eq. (4.3.4) at $D = 1$, $\epsilon = 1.45$, $\delta = -0.1$, $\beta = 0.101$, $\mu = -0.3$ and $\nu = -0.101$, while $m = 0.0025$. Although fine details of the CGLE solutions are not present in the reduced model, the qualitative features of the creeping dynamics are reasonably well-approximated. The results for the region

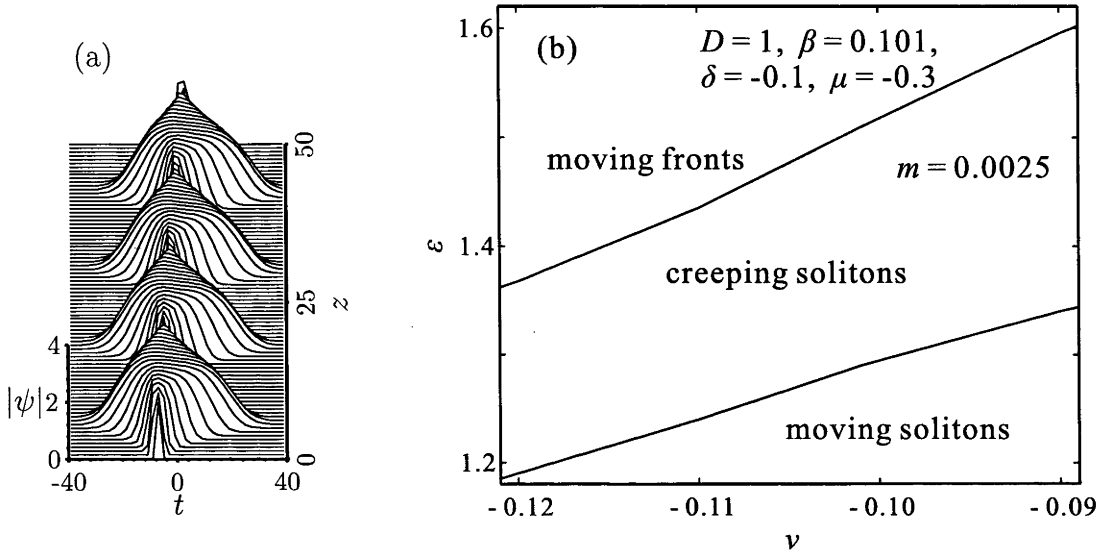


Figure 4.6: (a) Reconstructed soliton from the trial function Eq. (4.3.1) and solution of Eq. (4.3.4) at $D = 1$, $\epsilon = 1.45$, $\delta = -0.1$, $\beta = 0.101$, $\mu = -0.3$ and $\nu = -0.101$, while $m = 0.0025$. (b) Region of existence of creeping solitons in (ϵ, ν) -plane of the reduced model.

of existence of creeping solitons in this reduced model are shown in Fig. 4.6(b). As with the CGLE model shown in Fig. 4.4(b), creeping solitons occur in a central diagonal strip in Fig. 4.6(b). For lower values of ϵ , the limit cycle contracts to a fixed point with all parameters, including the velocity, being constant. This transition is an Andronov-Hopf bifurcation. Thus below the diagonal strip, solitons with constant shape and constant velocity exist in the same way as in the numerical simulations of the CGLE.

All the soliton parameters in the region of creeping solitons are oscillating. This solution is a limit-cycle in the five-dimensional phase space. Two projections of this limit-cycle onto a two-dimensional space of parameters are shown in Fig. 4.7. For

lower values of ϵ , the parametric plot tends to a round shape, and hence the velocity of the center-of-mass and the energy Q are approximately sinusoidal functions of z . Indeed, with $\epsilon = 1.29$, the velocity appears to be almost sinusoidal. For larger values of ϵ , the parametric plot becomes more elongated. This kind of irregularity is more evident as ϵ reaches the upper boundary of the creeping range. The closed curve parametric plots become larger in area, since both $t_{0,z}$ and Q have a larger range of oscillations when ϵ increases.

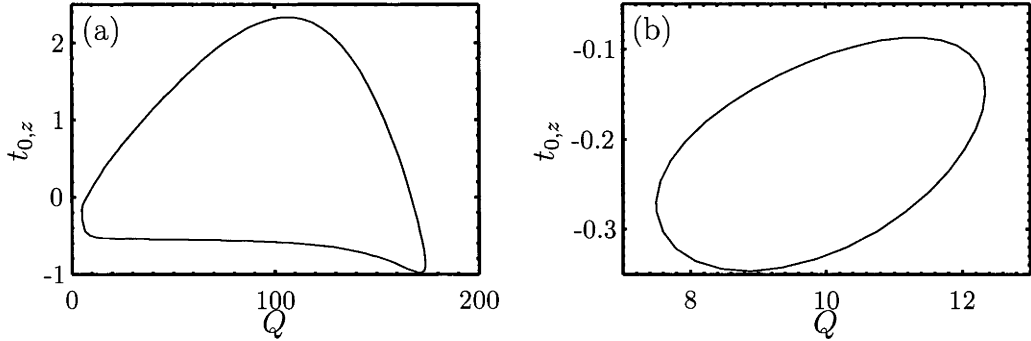


Figure 4.7: (a) Examples of limit-cycles in $(Q, t_{0,z})$ -plane for the reduced model at $D = 1$, $\delta = -0.1$, $\beta = 0.101$, $\mu = -0.3$, $\nu = -0.101$, (a) $\epsilon = 1.45$ and (b) $\epsilon = 1.29$, while $m = 0.0025$.

Periodic evolution of the energy, Q , and creeping motion of the center-of-mass of the soliton t_0 along z , are shown in Fig. 4.8. Qualitatively, these curves are similar to those shown in Fig. 4.5. Clearly, one cannot expect a better fit because of the dramatic reduction of the number of degrees of freedom when using the trial function. However, the possibility of modeling of the creeping solitons using a simple trial function is remarkable in itself.

As ϵ increases, the period of all five parameters increases. At the upper limit of existence of creeping solitons, the soliton becomes wider while retaining a constant amplitude, so the energy Q becomes approximately proportional to the width. At the top of the ‘creeping’ range, the period becomes infinite and there is a bifurcation to solitons with constantly increasing width. This behavior corresponds to the splitting of a soliton into a pair of fronts moving away from each other in numerical simulations of the CGLE. However, the velocity of the center-of-mass for this soliton is not zero, as was the case in the numerical solutions. Thus, in the reduced model, there is only a single boundary separating creeping solitons from moving fronts, as shown in Fig. 4.6. Again, this is hardly surprising if one takes into account

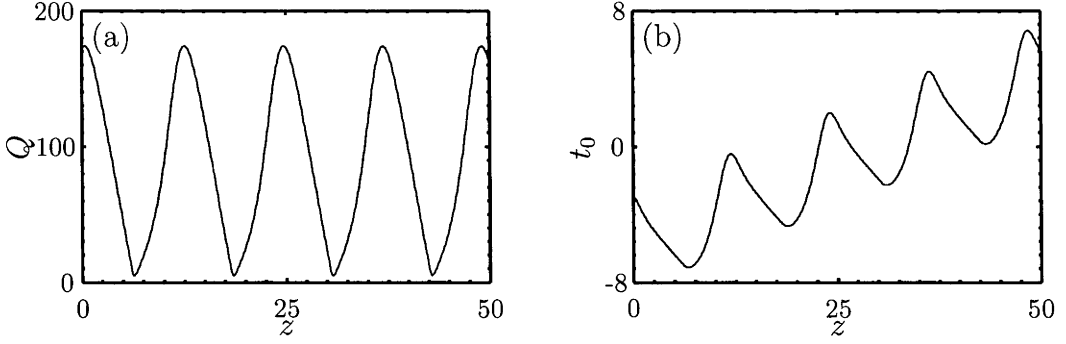


Figure 4.8: (a) Energy Q and (b) the center-of-mass t_0 against z for the reduced model at $D = 1$, $\epsilon = 1.45$, $\delta = -0.1$, $\beta = 0.101$, $\mu = -0.3$ and $\nu = -0.101$, while $m = 0.0025$.

the drastic reduction in the number of degrees of freedom in the simplified model. Despite these discrepancies, the qualitative location of the region of existence of creeping solitons and the boundary slopes are remarkably similar to those obtained in the CGLE. For example, in each case, the lower boundary slope is $\Delta\epsilon/\Delta\nu \approx 4.9$.

For $m = 0.0025$, $\nu = -0.101$, the transition from fixed point to creeping soliton occurs at $\epsilon \approx 1.29$, while the bifurcation from creeping to moving fronts occurs at $\epsilon = \epsilon_0 \approx 1.5092$. For fixed ν , the parameters Q , w , $t_{0,z}$, b and c all oscillate with the same period, T . Upon approaching the upper bifurcation boundary, pulse energy, average velocity and period all increase. Numerical analysis shows that the period T varies as

$$T \approx \frac{0.6912}{\epsilon_0 - \epsilon}, \quad (4.3.5)$$

and hence it becomes infinite at the upper bifurcation point.

4.4 Summary

Studying different types of the complex Ginzburg-Landau equation (CGLE) solutions and finding the region of their existence in the parameter space is a tedious work involving extensive numerical investigations, as there are no analytical expressions known for them, except for some specific stationary solutions. In this chapter, a new approach based on the method of moments has been introduced to reduce the dimensionality of the system. The technique reduces a partial differential equation, such as the CGLE to a set of ordinary differential equations (ODEs)

that approximately describes the evolution of the soliton parameters. Pulsating soliton solutions and creeping soliton solutions of the CGLE have been studied using this approach. Although it may not reproduce the solution details, especially for creeping solitons, which have very intricate behaviors, the qualitative features are well-described by the reduced model. Particularly, the region of existence of the pulsating and creeping solitons agrees reasonably well with the region plotted from the numerical simulations of the CGLE. With a careful selection of the trial function, this technique allows one to approximate the region of a particular family of soliton solutions under investigation with much less computational time.

Chapter 5

Dissipative Soliton Resonances

So far in this thesis, the concept of the dissipative solitons has been discussed in the context of the theoretical and fundamental basis of nonlinear dynamical systems. However, the dissipative soliton is not just of pure scientific interest, as this concept can find applications in the generic modeling of phenomena that exist in nature and in various modern technologies. One of the most important applications is the generation of ultra-short optical pulses with record-high energy [66]. Femto-second optical pulses with extremely high energy provide modern experimental science and technology with new tools. Applications include the generation of exceptionally wideband, or supercontinuum, light sources [67] that find numerous uses in biomedical imaging [68]. They can offer promising alternatives to the acceleration of charged particles [69]. They can be used for molecular finger printing [70], fabrication of optical couplers [71] and nanostructures in transparent dielectrics such as silica glass [72], high-density optical three-dimensional data storage [73] and high harmonic generation of atto-second pulses in the X-ray range [74]. Chemical reactions which occur on femto-second time scales can be studied by using ultra-short optical pulses [75]. Zewail was awarded the Nobel Prize in Chemistry in 1999 for the development of this technique and opening up the field of femto-chemistry. Medical applications include accurate and precise laser surgery with no destructive effects on surrounding tissues [76], non-invasive manipulation of the structural machinery of cells and tissues [77], dissection of biological samples [78], accurate non-destructive dentistry [79] and delicate eye surgery [80]. The list of possible applications in all areas of human activity seems to be unlimited [81]. Thus, producing shorter and more powerful pulses is vital for further progress in science and technology.

Until recently, techniques for generating ultra-short high-energy optical pulses were mostly based on the concept of chirped-pulse amplification [82]. This required a cascade of delicate amplifiers at the output of a mode-locked oscillator [83–86], and, as a result, the required equipment increased dramatically in size and price. Recently, however, femto-second laser oscillators with pulse energies in excess of 100 nJ and peak powers that exceed several megawatts have been demonstrated [87,88]. They have the potential to replace more complex and expensive ultra-short pulse amplifier aggregates. Solid-state lasers [89] and fiber lasers have also been studied vigorously in this regard [90–92].

While progress in generating high-energy pulses out of these lasers has been remarkable, most of the results have only been obtained experimentally by using ‘trial and error’ techniques for each specific case. There is no well-defined general theory behind these achievements. The main problem is that each particular laser design requires specific modeling and numerical simulations [11,93]. There is no ‘inverse technique’ to uncover a system that will produce a specific required output pulse, due to the large variety of laser designs [94].

However, there is a general approach which allows one to approximate the main properties of pulses generated by these devices. It is based on a single master equation derived using certain approximations [8]. The resulting equation is the complex Ginzburg-Landau equation or a modification of it. The active nonlinearity in the equation must be at least quintic in order to model the stable generation of pulses. Thus, the simplest model that can be used is the complex cubic-quintic Ginzburg-Landau equation (CGLE). This technique can be used for modeling stable pulse train generation by both solid-state [8,95] and fiber ring lasers [11]. Moreover, Renninger, Chong and Wise recently designed a so-called ‘dissipative soliton laser’ that generates pulses which can be directly compared with pulse solutions of the CGLE [96]. Depending upon the device and the phenomena that needs to be described, the model can be complemented with additional terms in the equation. These could be higher-order dispersion terms, step-wise change of parameters, etc. The additional terms may modify the output pulse and the conditions for its generation, but in general, they do not destroy the solution completely. If a certain effect is robust, it will persist despite all the modifications.

In the past, most scientific attention has been focused on cavity dispersion issues. Although soliton propagation in the anomalous dispersion regime can provide neat

hyperbolic-secant-profile pulses, the search for higher-energy and wider-bandwidth laser outputs has led to the design of cavities with strong dispersion management, featuring an average dispersion shifted into the normal regime [90, 91, 97]. Pulses in the normal dispersion regime are as stable as those in the case of anomalous dispersion. Observations also demonstrated that, when the magnitude of the normal dispersion coefficient grows larger, a useful increase of pulse energy results [37, 39]. The principle of dispersion management eventually evolved into the concepts of parabolic-pulse and all-normal-dispersion lasers [40, 98]. The resulting stationary output pulses are usually strongly chirped, making them wider in the temporal domain. However, further compression outside the cavity can be used to shorten the pulses down to the femto-second range.

One more theoretical step forward has been made by Akhmediev, Soto-Crespo and Grelu [66]. It has been shown numerically that the energy of dissipative solitons of the CGLE increases dramatically when their parameters are located on a special surface in the space of parameters. As this process resembles the resonance phenomenon in the theory of oscillators, it has been named ‘dissipative soliton resonance’ (DSR). Here, the definition of ‘resonance’ is extended to include an internal resonance between the system parameters, which, when related in a certain way, enable the support of high energy pulses. The investigation has shown that the effect occurs over a significantly large region of the system parameters in the normal dispersion regime. Basically, there is a co-dimension-one space of the complete parameter space which admits infinitely large pulse energies. Thus, the DSR effect has predictive features and, in principle, can be used to design laser systems for the generation of high energy pulses.

Despite its apparent simplicity, finding the location of the DSR in the parameter space is highly complicated. Even in the simplest model, the number of parameters in the CGLE is beyond easy handling, and therefore, the model requires a significant number of numerical simulations to describe the pulse shape and its parameters. This can take a long time, even with the use of modern computer equipment. Moreover, it is non-trivial to make a correspondence between the real experimental parameters and those used in the CGLE model, and hence this requires a special derivation. While the net cavity dispersion can, in a straightforward manner and after a certain normalization, be represented by the dispersion parameter in the equation, the values of the rest of the coefficients are complicated

functions of specific laser parameters which can be adjusted experimentally. This is particularly true for the nonlinear ones. However, their exact representations are not significantly important in understanding the laser operation at an intuitive level.

The work presented in this chapter is not an attempt to describe any particular laser model. Rather, the DSR effect is considered at a phenomenological level. The chapter begins with the numerical discovery of this phenomenon in Section 5.1. In the next section, one of the reduced models obtained using the method of moments in Chapter 4 is employed to model the DSR, and an approximate analytical expression for the location of the resonance surface in the parameter space is presented. This is then followed by a study of the DSR in the anomalous dispersion regime, and the effect of the quintic nonlinearity on the resonance. Section 5.5 presents a study on the robustness of the DSR phenomenon, when parameter management is introduced in the laser model. In Section 5.6, the compression of the highly-chirped pulses out of such lasers is briefly discussed, and finally, the chapter summary follows in Section 5.7.

5.1 Ultra-High Energy Pulses of the CGLE

In order to study the generation of ultra-high energy pulses from a passively mode-locked laser, a master equation approach [8], based on the CGLE is employed [99]. The master equation approach avoids the problem of separate models for specific laser designs, but it still gives a good approximation when the changes due to pulse evolution are relatively small during any individual round-trip of the cavity. It allows one to find the critical parameters of the system that generate the pulse with the highest possible energy [66].

The master equation is given by [100]

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi + \nu|\psi|^4\psi = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{tt} + i\mu|\psi|^4\psi. \quad (5.1.1)$$

This equation is identical to Eq. (2.2.1), except that D is no longer normalized and is now a variable that can be any real number. Hence, Eq. (5.1.1) has total of six system parameters instead of five, as used in the earlier chapters. Despite this increase in the number of parameters, such de-normalization allows one to directly

compare the solutions of Eq. (5.1.1) with the experimental results, as the dispersion is most often the main control parameter in experiments involving generation of ultra-high energy pulses. Nevertheless, the definition for the sign of D is still the same, and hence D is positive in the anomalous dispersion regime and negative in the normal dispersion regime.

By numerically solving Eq. (5.1.1) using the technique outlined in Section 2.3, Akhmediev, Soto-Crespo and Grelu were able to find a region in the parameter space where stable dissipative solitons have energies much higher than those reported before [66]. Since the focus is on the generation of high-energy pulses, it is useful to define the total energy Q of the output pulse. This is given by

$$Q = \int_{-\infty}^{\infty} |\psi(t, z)|^2 dt. \quad (5.1.2)$$

For a stationary soliton solution, the energy Q converges to a finite value. Fig. 5.1 shows the energy Q against D for three different values of ϵ , while the other parameters are fixed. One of the most notable features of these curves is that the energy

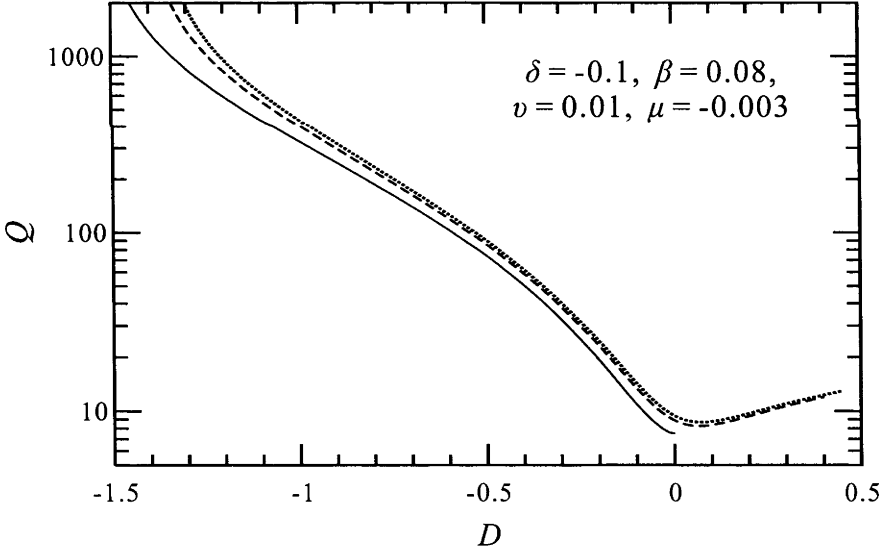


Figure 5.1: Soliton energy Q against dispersion parameter D for $\epsilon = 1$ (solid line), $\epsilon = 2$ (dashed line) and $\epsilon = 3$ (dotted line), while $\delta = -0.1$, $\beta = 0.08$, $\nu = 0.01$ and $\mu = -0.003$ [66].

Q grows indefinitely when the dispersion is pushed further into the normal dispersion regime, which agrees well with the previous experimental observations [37, 39]. In fact, the energy becomes almost infinite when the dispersion reaches the edge

of the stability range on the left-hand-side. This suggests that the only practical limits in increasing the energy of pulses out of passively mode-locked lasers are the availability of the pump power and the stability of the pulses against higher-order terms that are not taken into account in Eq. (5.1.1).

The region of existence of this special class of soliton solutions is shown in Fig. 5.2 in the grey shaded region. Only the two-dimensional projection in the (ϵ, D) -plane is shown, while the other parameters are fixed at $\delta = -0.1$, $\beta = 0.08$, $\nu = 0.01$ and $\mu = -0.003$. The left-hand-side boundary of this region corresponds

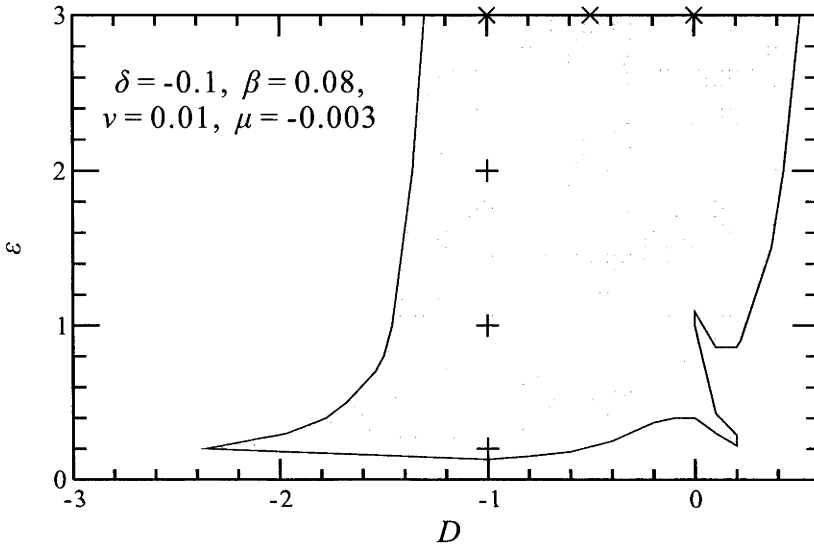


Figure 5.2: Region of existence of solitons in the (ϵ, D) -plane. The left-hand-side boundary corresponds to the ultra-high energy pulses. Soliton profiles and spectra of the pulses located at the points marked 'x' and '+' are presented in Fig. 5.3 and Fig. 5.4, respectively [66].

to the ultra-high energy ends of the curves in Fig. 5.1. Hence, this boundary is termed the 'resonance curve', *i.e.* the energy increases indefinitely as D approaches the boundary on the left-hand-side. Further to the left of the boundary, two fronts move away from each other and no localized solutions are found. Stable solitons are observed in both normal and anomalous dispersion regimes, although the resonance curve, and hence the ultra-high energy pulses, only exist in the normal dispersion regime for the two-dimensional slice presented in Fig. 5.2.

The pulse shapes and the spectra at the points marked 'x' and '+' in Fig. 5.2 are presented in Fig. 5.3 and Fig. 5.4, respectively. The evolution of the pulse shape and its spectrum, with changes in the dispersion D while other parameters

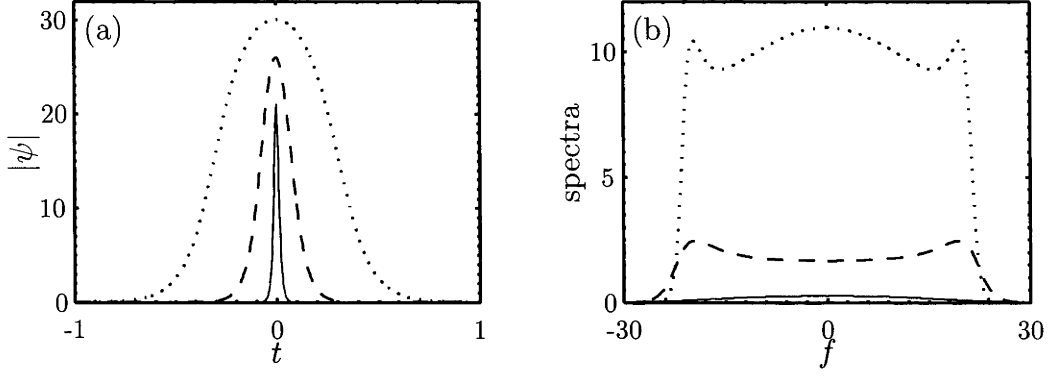


Figure 5.3: (a) Profiles and (b) spectra for $D = 0$ (solid line), $D = -0.5$ (dashed line) and $D = -1$ (dotted line), while $\delta = -0.1$, $\beta = 0.08$, $\epsilon = 3$, $\nu = 0.01$ and $\mu = -0.003$. The curves correspond to the points marked ‘ $\times$ ’ along the upper horizontal line in Fig. 5.2 [66].

are fixed, are shown in Fig. 5.3. The plots demonstrate that, as D approaches the resonance curve, the pulse width increases dramatically, while the amplitude stays relatively constant. The spectrum becomes almost rectangular in shape with two or three peaks. The spectral intensity depends strongly on the dispersion, while the width of the spectrum stays almost unchanged. Hence, the dramatic increase in the pulse energy near the resonance is mainly the result of the increase in temporal pulse width, rather than any increase in the pulse peak power. On the other hand, in the spectral domain, the increase in the spectral intensity rather than the spectral width is the main contributor of the energy gain.

It should be noted that, when carrying out numerical simulations close to the resonance, the pulse shapes change quickly with a change of parameters. In moving from point to point in this range, the step-sizes, numerical windows and the number of points in the numerical grid need to be changed, making these relatively simple procedures quite time consuming. Hence, Fig. 5.1 has an arbitrary upper limit on Q and further calculations were not carried out. This does not mean that solitons with even higher energy cannot exist.

Fig. 5.4 shows pulse shape and spectrum evolution when the cubic gain ϵ is changed, while other parameters, including D , are fixed. Unlike the spectra shown in Fig. 5.3(a), the spectral shape stays more or less the same while its width becomes wider as ϵ increases. The spectral shape is almost rectangular and has three maxima for all three values of ϵ . Also, the pulse’s temporal width narrows and the amplitude

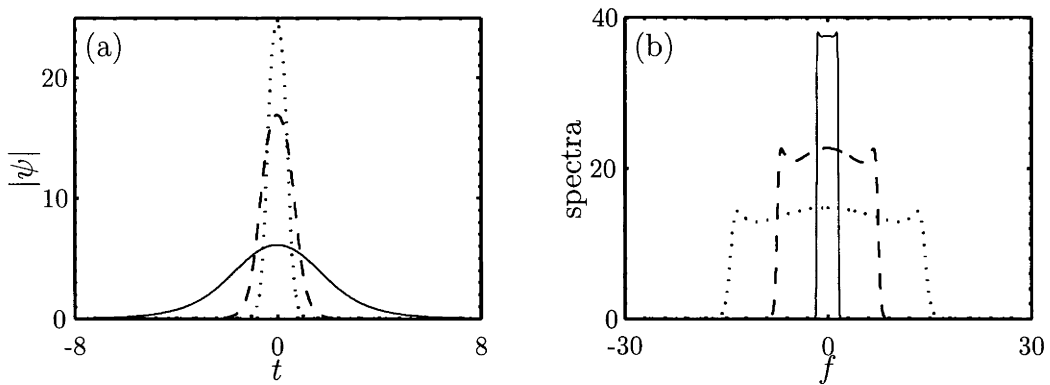


Figure 5.4: (a) Profiles and (b) spectra for $\epsilon = 0.2$ (solid line), $\epsilon = 1$ (dashed line) and $\epsilon = 2$ (dotted line), while $D = -1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = 0.01$ and $\mu = -0.003$. The curves correspond to the points marked ‘+’ along the vertical line $D = -1$ [66].

increases.

Most of the experiments dealing with the generation of high-energy pulses show three-peak spectra, which resemble the pulse spectrum near the resonance curve [89,90,95,101,102]. This demonstrates that there is a strong connection between the generation of high energy pulses in a laser oscillator and the high energy solutions of the CGLE presented in this study.

The pulses shown in Fig. 5.3 and Fig. 5.4 are far from being transform-limited, *i.e.* they are all strongly chirped. The chirp is almost linear around the central high intensity part of the pulse. Again, this is in agreement with previous experimental studies [95,102]. This strong chirp can be used beneficially to shorten the pulse width using a dispersive medium. Such a technique is further discussed in Section 5.6.

5.2 Analytical Approach

There are no direct analytical techniques using a straightforward procedure that allow one to find the location of the resonance in the parameter space. This is related to the fact that analytic solutions of the CGLE are only known for very specific values of the parameters [100]. However, there is a way to approximate the location of the resonance by simplifying the model described by Eq. (5.1.1). One can obtain a reduced system by applying the method of moments [60], as has

been done in Chapter 4, and solve the finite-dimensional system defined by a finite set of ordinary differential equations (ODEs). In fact, it has been found that the generalized Gaussian trial function, Eq. (4.2.4), which has been used to approximate the pulsating solitons, is also useful in modeling high energy pulses of the CGLE. The trial function is given by

$$\psi(t, z) = A \exp\left(-\frac{t^2}{w^2} - \frac{t^4}{4mw^4}\right) \exp(ict^2), \quad (5.2.1)$$

and the evolution equations for the function parameters become

$$\begin{aligned} Q_z &= Q \left(2\delta - \frac{3.738\beta}{w^2} - 1.158w^2c^2\beta + \frac{1.433Q\epsilon}{w} + \frac{1.146Q^2\mu}{w^2} \right), \\ w_z &= w \left(2cD + \frac{2.142\beta}{w^2} - 0.874w^2c^2\beta - \frac{0.290Q\epsilon}{w} - \frac{0.325Q^2\mu}{w^2} \right), \\ c_z &= \frac{1}{w^2} \left(-2w^2c^2D - 19.624c\beta - \frac{1.237Q}{w} - \frac{1.319Q^2\nu}{w^2} + \frac{6.453D}{w^2} \right), \end{aligned} \quad (5.2.2)$$

for $m = 0.25$. Again, setting $m = 0.25$ gives good results over a wide range of the system parameters. Setting $m < 0.25$ gives pulses that are more rectangular in shape, which slightly improves the results near the resonance. However, the differences in the final results are very small over the whole interval where $m < 0.25$. Thus, only the case when $m = 0.25$ is presented here. The equations for arbitrary positive real m are presented in the Appendix I. Another possible trial function is a quartic trial function. Its shape is also more rectangular than Eq. (5.2.1), and hence gives a better approximation near the resonance. The expression for this trial function and its complete set of evolution equations are presented in Appendix IV.

The dynamical system Eq. (5.2.2) has a relatively simple structure. The right-hand-sides of the three equations are rational expressions of Q , w and c . Using standard techniques, one can find the system fixed points and study their stability. The results for fixed values of δ , ν , β and μ are shown in Fig. 5.5. The grey region consists of stable fixed points. The black curve in the middle is the resonance curve where energy Q increases to infinitely large values.

The shape and position of the resonance curve are in reasonably good agreement with those of the exact curve found numerically in Fig. 5.2. The shape of the region on the right-hand-side of the resonance curve is also in good agreement with the numerical results. Quite remarkably, even a fine feature like the ‘hamster tail’,

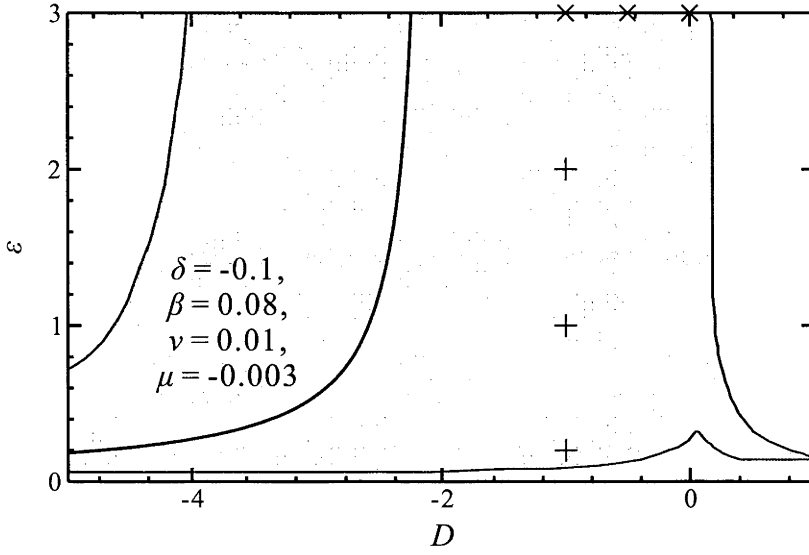


Figure 5.5: Region of existence of stable fixed points for the dynamical system given by Eq. (5.2.2) in the (ϵ, D) -plane. The energy Q and the width w increase to infinity at the resonance curve shown by a solid line in the middle of the stable region. Reconstructed pulse profiles and spectra at the points marked ‘ $\times$ ’ and ‘+’ are presented in Fig. 5.6 and Fig. 5.7, respectively [66].

a small protrusion in the right bottom corner of the region in Fig. 5.2, is well reproduced by the trial function approximation. However, the region to the left of the resonance curve appears to be an artifact of the method. The exact model described by Eq. (5.1.1) does not have this region of stable solitons to the left of the resonance curve. This could be related to the fact that more possibilities for instability appear in the infinite-dimensional dynamical system. These details may however be lost in a simple three-dimensional approximation such as the one given by Eq. (5.2.2).

Using the soliton parameters obtained as fixed points of the dynamical system given in Eq. (5.2.2) and the trial function Eq. (5.2.1), one can reconstruct the soliton profiles and their spectra. Such reconstructions have been done for the points marked ‘ $\times$ ’ and ‘+’ in Fig. 5.5, which are presented in Fig. 5.6 and Fig. 5.7, respectively. As D changes, the qualitative changes in both the profiles and the spectra of the reduced model agree well with those of the CGLE soliton solutions. Starting from the value $D = 0$ and approaching the resonance curve, the soliton width and amplitude increase, as shown in Fig. 5.6(a). However, the amplitude increases slowly and its maximum value is restricted due to the saturation related

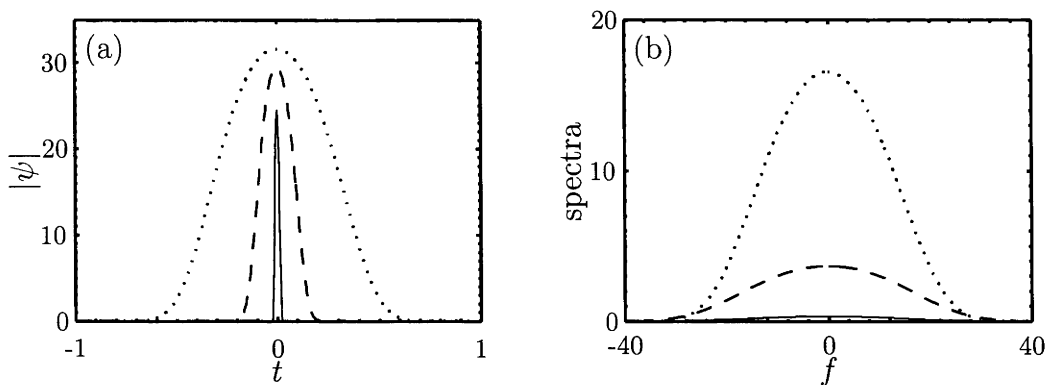


Figure 5.6: (a) Profiles and (b) spectra of the reconstructed pulse using the trial function Eq. (5.2.1) and the solutions of the ODEs in Eq. (5.2.2) for $D = 0$ (solid line), $D = -0.5$ (dashed line) and $D = -1$ (dotted line), while $\delta = -0.1$, $\beta = 0.08$, $\epsilon = 3$, $\nu = 0.01$, $\mu = -0.003$ and $m = 0.25$. The curves correspond to the points marked ‘ $\times$ ’ along the upper horizontal line in Fig. 5.5.

to negative μ . Thus, the growth in energy, Q , close to the resonance is mainly due to the increase in the pulse width. In order to obtain high-energy ultra-short pulses, a pulse compression technique has to be used outside the cavity.

The spectrum also increases in strength, as can be seen from Fig. 5.6(b). The exact shape of the spectrum is different from the exact one, as the simple trial function Eq. (5.2.1) cannot reproduce the additional side peaks. The real chirp is more complicated than the quadratic function of t used in the trial function. However, the observations from the reduced model are in qualitatively good agreement with the solutions of the CGLE shown in Fig. 5.3.

The reconstructed pulse profile and spectra for various values of ϵ are presented in Fig. 5.7. Both pulse profiles and spectra are qualitatively comparable with the numerical results of the CGLE shown in Fig. 5.4, except for the additional side maxima in the spectra. These are again related to the differences in chirps between the exact numerical results and the simple approximation Eq. (5.2.1).

To demonstrate the existence of the resonance phenomenon, plots of energy Q as a function of D are presented in Fig. 5.8(a) for three different values of ϵ . The curves clearly show that there is an indefinite growth of energy as D approaches the point of resonance from either side. Especially, the curves on the right-hand-side of the resonance are in excellent qualitative agreement with the curves obtained by solving the CGLE, as shown in Fig. 5.1. Moreover, the resonance in the approximate model

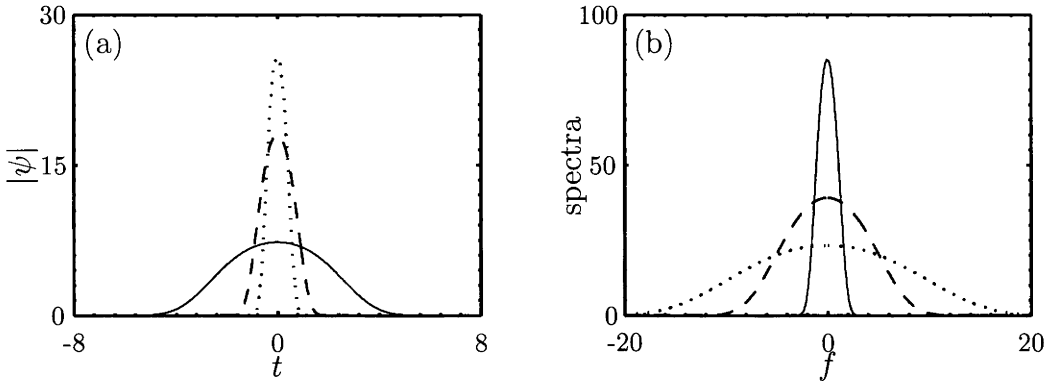


Figure 5.7: (a) Profiles and (b) spectra of the reconstructed pulse using the trial function Eq. (5.2.1) and the solutions of the ODEs in Eq. (5.2.2) for $\epsilon = 0.2$ (solid line), $\epsilon = 1$ (dashed line) and $\epsilon = 2$ (dotted line), while $D = -1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = 0.01$, $\mu = -0.003$ and $m = 0.25$. The curves correspond to the points marked ‘+’ the vertical line $D = -1$ in Fig. 5.5.

is more pronounced, since these curves can be plotted to very high values of Q . In the present plots, the upper limit extends up to the value of $Q \approx 10^6$, which is hard to achieve in the numerical simulations of the CGLE due to sampling difficulties. The chirp parameter c is also plotted against D in Fig. 5.8(b). The curves indicate that at the point of the resonance, the chirp parameter becomes zero, and the sign of c changes as it crosses the resonance point. Comparing Fig. 5.8 with Fig. 5.1, one can see that the curves on the left-hand-side of the resonance are absent from the CGLE model. These solutions may represent unstable solitons of the CGLE, and therefore would not be observed in experiments.

The fact that the pulse width increases while its spectral width does not decrease appears to contradict Fourier transform rules. However, the pulses here are not transform-limited. They are chirped, which allows for the observed growth of the width without any corresponding compression of the pulse spectra. On approaching the resonance, the chirp, c , approaches zero, but the Fourier integral is over an ever-widening range. Simple analytic calculations, presented in Appendix III, show that these factors compensate, and, as a result, the spectral width approaches a constant.

5.2.1 Resonance Curve

One of the main advantages of the low-dimensional approximation is that the resonance curve can be found analytically. Near the resonance curve, the soliton

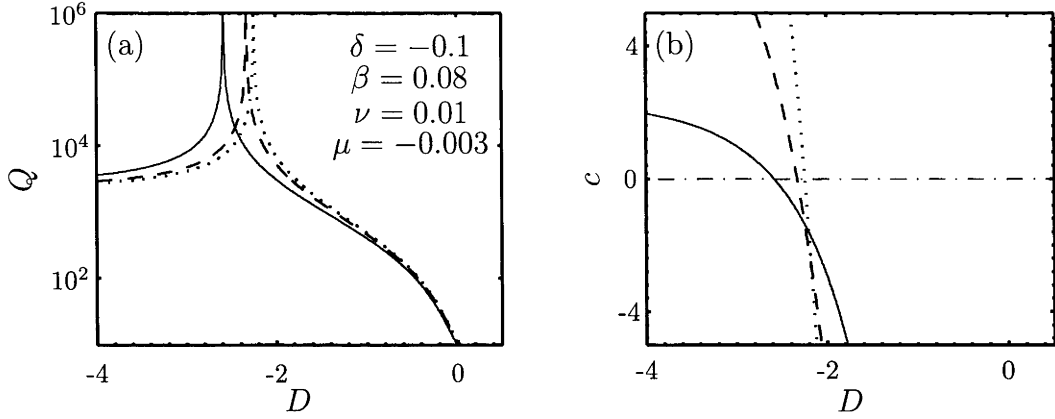


Figure 5.8: (a) Energy Q and (b) chirp parameter c against dispersion parameter D of the reduced model Eq. (5.2.2) for $\epsilon = 1$ (solid curves), $\epsilon = 2$ (dashed curves) and $\epsilon = 3$ (dotted curves). Other parameters are fixed at $\delta = -0.1$, $\beta = 0.08$, $\nu = 0.01$, $\mu = -0.003$ and $m = 0.25$.

parameters have the following asymptotic behavior: $Q \rightarrow \infty$, $w \rightarrow \infty$ and $c \rightarrow 0$. In this case, the values Q/w and cw approach constants, with chirp c being negative. Using these observations, one can derive an analytic expression for the resonance curve in terms of the system parameters. Detailed derivations for a general m value are presented in the Appendix II. For the generalized Gaussian trial function of Eq. (5.2.1) with $m = 0.25$, one can obtain:

$$\begin{aligned} \frac{Q}{w} &= -\frac{0.363k}{\mu}, \\ cw &= -\sqrt{\frac{0.472\delta\mu - 0.035k\epsilon}{\beta\mu}}, \end{aligned} \quad (5.2.3)$$

where

$$k = 1.588\epsilon + \sqrt{2.522\epsilon^2 - 9.632\delta\mu}. \quad (5.2.4)$$

Then, the equation for the resonance curve takes the form

$$D = \frac{k\beta(0.330k\nu - 0.853\mu)}{\mu(0.135k\epsilon - 1.793\delta\mu)}. \quad (5.2.5)$$

In any two-dimensional projection of the complete parameter space, Eq. (5.2.5) can be represented as a curve.

For the set of system parameters used in Fig. 5.5, the first term, 2δ , of the first equation in Eq. (5.2.2) is very small in comparison with the other terms. Thus, this term can be neglected, and much simpler expressions for Q/w and cw can be obtained. These are

$$\begin{aligned}\frac{Q}{w} &= -\frac{1.152\epsilon}{\mu} \quad (> 0), \\ cw &= -0.336\frac{\epsilon}{\sqrt{-\beta\mu}} \quad (< 0).\end{aligned}\tag{5.2.6}$$

Then the equation for the resonance curve becomes

$$D = \left(\frac{7.778\nu}{\mu} - \frac{6.333}{\epsilon} \right) \beta.\tag{5.2.7}$$

Near the resonance, the soliton increases its width indefinitely, while keeping its amplitude constant. Indeed, the amplitude of this wide soliton is $A \approx \sqrt{-\epsilon/\mu}$. It remains constant as the width, and hence energy, approach high values.

The resonance curve in Fig. 5.5 is very well-approximated by Eq. (5.2.7). Qualitatively, it has the same features as the resonance curve obtained from numerically solving the CGLE, shown in Fig. 5.2, apart from a horizontal shift to the left along the D axis. Thus, one can select coefficients in Eq. (5.2.7) and shift the curve to describe the resonance curve of the CGLE. Adjusting one of the coefficients in Eq. (5.2.7) gives

$$D = \left(\frac{3.8\nu}{\mu} - \frac{6.333}{\epsilon} \right) \beta.\tag{5.2.8}$$

This allows one to obtain virtually the exact curve presented in Fig. 5.2. Eq. (5.2.8) is simply a lateral translation of the curve given by Eq. (5.2.7) to more positive values of D .

One of the important physical conclusions from the above calculations is that the resonance phenomenon occurs continuously along a surface in the six-dimensional space of equation parameters, and is not restricted to a particular line. The resonance curve that has been presented in Fig. 5.5 is merely a section of this surface in the plane of parameters D and ϵ . This means that the resonance can be found over a wide range of parameters of the system. When dealing with passively mode-locked lasers of virtually any design, one can find the resonance by choosing the cavity parameters from a broad range of values.

In many experimental set-ups generating ultra-high energy pulses [89,90,95], the cavity dispersion is chosen to be normal ($D < 0$). From a physical point of view, this choice is dictated by the evolution of knowledge in chirped pulse oscillators [95]. The results obtained with the reduced model show that, overall, this choice is not a restriction for high-energy soliton generation by a single optical oscillator. The point of resonance can be moved along the surface by changing the parameters of the system. In particular, the resonance curve can be moved towards the anomalous dispersion regime. For example, this can occur when ν and μ are both negative. From Eq. (5.2.7), this then leads to the condition

$$\frac{\nu}{\mu} > \frac{0.814}{\epsilon}, \quad (5.2.9)$$

for D to be positive. This simple consideration shows that it is not necessary to have a cavity with normal dispersion in order to build single oscillators that generate high-energy pulses.

5.3 DSR in the Anomalous Dispersion Regime

The DSR example presented in Section 5.1 occurs in the normal dispersion regime. This is a good representation of recent experimental observations of high energy pulses from passively mode-locked lasers for both fiber [90] and solid-state [87] types. Indeed, the majority of these observations are in the normal dispersion regime of operation. However, the simple reduced model that has been derived in Section 5.2 and a few experimental pieces of evidence [87, 103] demonstrate that high energy pulses can also be generated in the anomalous dispersion regime.

As a guideline for discovering the effect of the DSR in the anomalous dispersion regime, the equations that have been obtained from the reduced model in the previous section can be used. Firstly, one must satisfy the condition given in Eq. (5.2.9). Here, a negative ν should be chosen, since μ must be negative and ϵ must be positive for stable pulses to exist [34]. Using this relationship and the approximate location of the resonance curve in the parameter space calculated using Eq. (5.2.7), one can reveal the region of high-energy pulses in the anomalous dispersion regime by solving Eq. (5.1.1) numerically.

An example of such a region is shown in Fig. 5.9. It is a contour plot of a

region in the (ϵ, D) -plane showing lines of equal energy, Q . The other system parameters are fixed at $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.001$ and $\mu = -0.002$. The

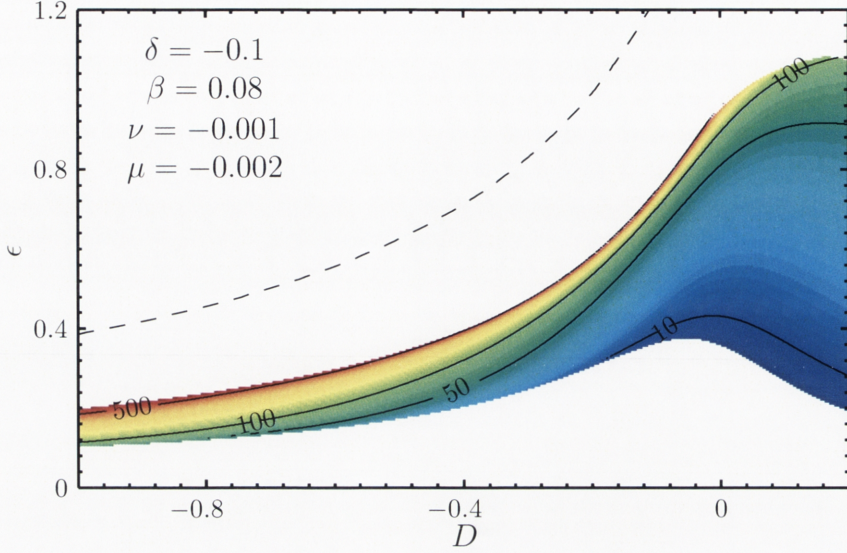


Figure 5.9: Region of existence of stable solitons in (ϵ, D) -plane. Color indicates the soliton energy, and this is highest at the upper boundary of the region (red). The contour lines drawn inside this region correspond to fixed values of the pulse energy with the highest value, $Q = 500$, and the lowest value, $Q = 10$. The other system parameters are fixed at $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.001$ and $\mu = -0.002$. The dashed line plotted above the region is the approximate location of the resonance curve found using Eq. (5.2.7).

upper boundary of the colored region corresponds to the ultra-high energy pulses of the CGLE. This upper edge extends to the region with zero and even into normal dispersion, in contrast to the case found for positive ν in Fig. 5.2. Solitons have the smallest energy, Q , at the lower edge of the region. No stable stationary solitons exist outside the colored region.

The approximate resonance curve for the above set of parameters is also plotted in Fig. 5.9. As the trial function Eq. (5.2.1) is not the exact solution of Eq. (5.1.1), the curve is located at an appreciable distance from the ‘real’ resonance curve found numerically by solving the CGLE. Moreover, in the anomalous dispersion regime, this difference becomes more marked. It is likely that the trial function, Eq. (5.2.1) could be revised in this region to have a better fit. However, despite the simplification, the approximate resonance curve is very much parallel with the upper boundary of the region, thus showing a good qualitative agreement with the

CGLE model.

In Fig. 5.9, it appears that the resonance effect is more pronounced in the normal dispersion regime. When moving to the anomalous dispersion regime, the values of soliton energy at the upper boundary become smaller. In fact, this is not true, and such appearance is only due to the limitation of the numerical grid in Fig. 5.9. More study has demonstrated that, when near the resonance, the increase in the pulse energy becomes much more dramatic in the anomalous dispersion regime than in the normal dispersion. Hence, solitons in the anomalous dispersion experience a marked increase in energy near the resonance, reaching a level of energy similar to that in the normal dispersion.

To verify this point, the curve for soliton energy, Q against D , in the vicinity of the resonance in the anomalous dispersion, is plotted in Fig. 5.10 with a much finer resolution in D . This curve represents the horizontal slice of Fig. 5.9 at $\epsilon = 1$, which is near the upper limit of the region of existence. The curves for other values of ϵ near $\epsilon = 1.0$ are similar to the curve in Fig. 5.10. These calculations

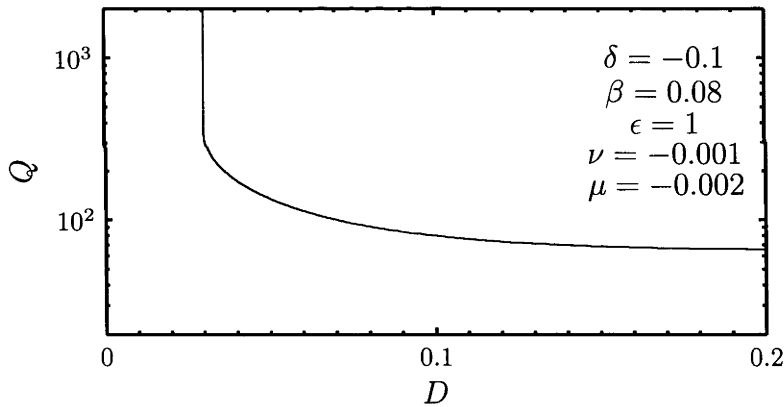


Figure 5.10: DSR effect in the anomalous dispersion regime. The energy of the soliton increases quickly near $D \approx 0.03$, which is the resonance point for $\delta = -0.1$, $\beta = 0.08$, $\epsilon = 1$, $\nu = -0.001$ and $\mu = -0.002$.

have demonstrated that the energy growth near the resonance in the anomalous dispersion regime exhibits a similar character to the normal dispersion case shown in Fig. 5.1.

Fig. 5.11 presents the pulse profiles and the spectra for three different locations on the curve in Fig. 5.10. Namely, these are at $D = 0.1$ (solid line), $D = 0.05$ (dashed line) and $D = 0.0305$ (dotted line). When approaching the resonance, the

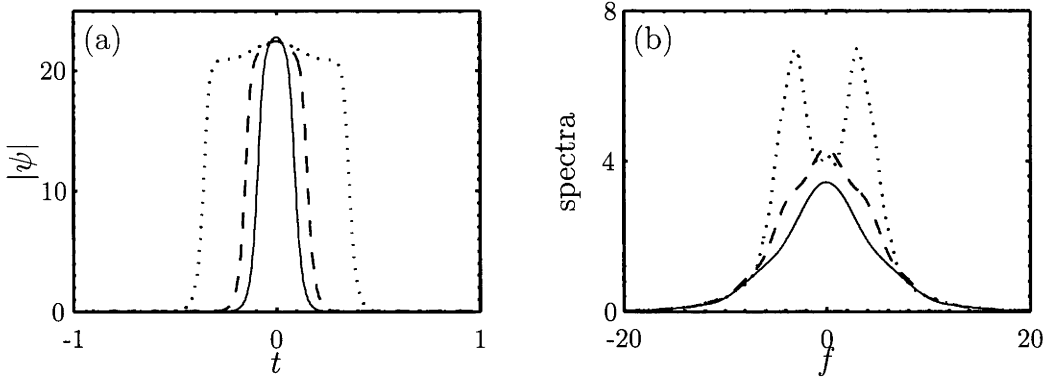


Figure 5.11: (a) Soliton profiles and (b) their spectra when the solutions approach the resonance. These are for $D = 0.1$ (solid line), $D = 0.05$ (dashed line) and $D = 0.0305$ (dotted line). The other system parameters are $\epsilon = 1$, $\delta = -0.1$, $\beta = 0.08$, $\nu = -0.001$ and $\mu = -0.002$.

pulse profile widens and becomes nearly rectangular in shape, without any apparent increase in amplitude. The spectrum, on the other hand, increases its amplitude and reveals two side maxima. The main features of these solitons are similar to those in the normal dispersion case. These observations are in line with experimental results [87, 103] of high energy solitons in the case of net anomalous dispersion in the cavity.

5.4 Effect of the Quintic Nonlinearity

The correct choice of system parameters, and ν in particular, is a necessary condition for the appearance of DSR. The role of the higher-order reactive nonlinearity term, ν in the model, requires special attention. For a fiber laser, the cubic reactive nonlinearity comes mainly from the nonlinear susceptibility of the fiber, and can be considered positive, albeit very small. However, the quintic part of reactive nonlinearity results from the combined effects of the fiber, mode-locker, etc., and its value is unpredictable. In the case of Kerr-lens mode-locking, the net cavity nonlinearity can be an even more complicated function that depends on the cavity design. Nonetheless, it is certain that, for high intensity pulses, the quintic nonlinearity cannot be ignored.

To study the dependence of the soliton energy on ν , three curves of Q against ϵ are plotted in Fig. 5.12, each with a different value of ν . The curves end at those

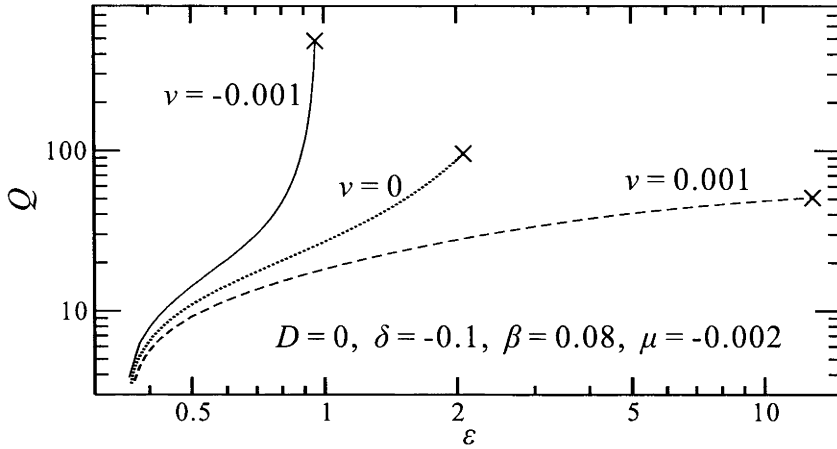


Figure 5.12: Energy Q of the pulse against ϵ for positive (solid line), zero (dotted line) and negative (dashed line) values of ν . The common system parameters are $D = 0$, $\delta = -0.1$, $\beta = 0.08$ and $\mu = -0.002$. The curve for negative ν shows the resonant increase of the energy Q .

values of ϵ where no stable solution is found. In this study, the net dispersion, D , is taken to be zero, as a transitional value between normal and anomalous dispersion. In fact, if the resonance phenomenon is found for $D = 0$, it is very likely that the DSR will also exist in the anomalous dispersion region. It has been found that changing ν from a positive to a negative value increases the maximum reachable soliton energy and transforms the $Q(\epsilon)$ dependence into a resonance-like curve. This can be clearly seen in Fig. 5.12. The curve for $\nu = -0.001$ has the fastest energy increase in a small region of ϵ , while the curve for $\nu = 0.001$ shows a very slow increase. To show this difference clearly, a logarithmic scale for ϵ is used. The curves for more negative values of ν show an even faster growth of the energy against ϵ .

The pulse profiles and their spectra at the points on these curves with the highest energy, indicated by three 'x's, are shown in Fig. 5.13. The pulses for positive ν are narrower and higher than those for negative ν . In the case of positive ν , as the cubic gain ϵ increases, the pulse becomes narrower and of higher intensity. The opposite happens for negative ν , where the increase of energy as ϵ increases is mainly due to the increase of the pulse width. This is particularly true as one gets closer to the resonance.

These observations suggest that the region of existence of high-energy pulses should be constructed for the case of negative ν . Although the value, and even the

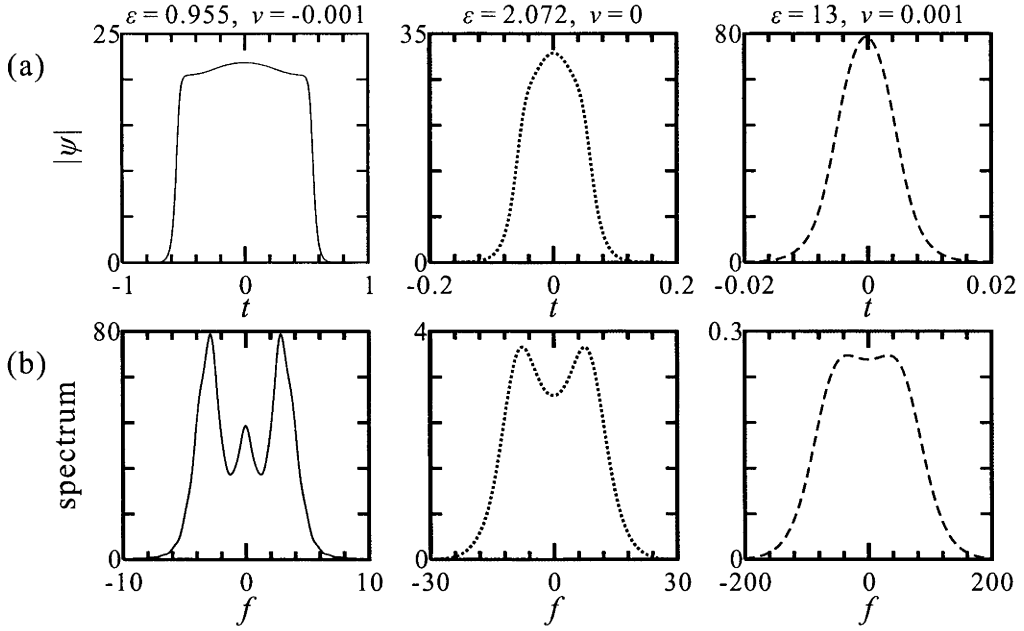


Figure 5.13: (a) Pulse profiles and (b) spectra of solutions near resonance. The values of ϵ and ν for each case are written above the plots. These three cases correspond to the three ‘ $\times$ ’s marked at the end of each curve in Fig. 5.12.

sign, of ν for laser systems are usually unknown, a comparison of pulse shapes and energies at a qualitative level can give an indication of it.

5.5 DSR in Laser Models

The CGLE is a good model for describing optical oscillators with passive mode-locking [8]. However, a better approximation is required in specific cases. Then a question arises as to whether the resonance is a robust phenomenon relative to perturbations of the model itself: will the phenomenon still exist if the model is made closer to a real laser system? One way to find out the answer for this question is to introduce parameter management into the system. Thus, one can allow for changes in the medium in which the pulse propagates, and this means that the parameters of the governing equation also change. In a typical fiber laser [11, 93], the pulse periodically passes through an erbium-doped fiber, mode-locking device and a standard single-mode fiber. Passing through the amplifier and mode-locking element in the cavity can be described by the nonlinear non-conservative terms of the master equation, while propagation through the single-mode fiber is determined

simply by second-order dispersion. In solid-state lasers, such as Ti:sapphire lasers with Kerr-lens mode-locking [104], prisms and mirrors provide pure chromatic dispersion, while the crystal itself serves simultaneously as an amplifier and nonlinear transmission element. Thus, a periodic change in the parameters of the CGLE will describe the propagation of the pulse in laser cavities more accurately [105].

If the pulse variations are modest, then the averaged model [106] with constant parameters is good enough to describe the pulse generation process. On the other hand, modifications are certainly needed if the pulse experiences relatively large changes during each round trip around the cavity. Hence, in order to study whether the periodicity in the cavity will result in the disappearance of the DSR, a model which allows for relatively large changes in the pulse evolution is introduced.

A sketch of the laser model considered in this section is shown in Fig. 5.14(a). It consists of three main elements. (1) A single-mode fiber, of normalized length L_2 , where only second-order chromatic dispersion takes place. (2) A piece of fiber with normalized length L_1 where amplification, nonlinearity and mode-locking occur. (3) A 80/20 coupler located in the middle of the element of length L_1 . Another example that could be described qualitatively by this model is a solid state laser with Kerr-lens mode-locking [104, 107]. In the latter case, the mode-locking and amplification are concentrated inside the laser crystal and the rest of the cavity can be described as a dispersive line. The model is very simple and serves only to demonstrate the principle. If the DSR exists in this model, it is expected that it will also appear in more complicated systems.

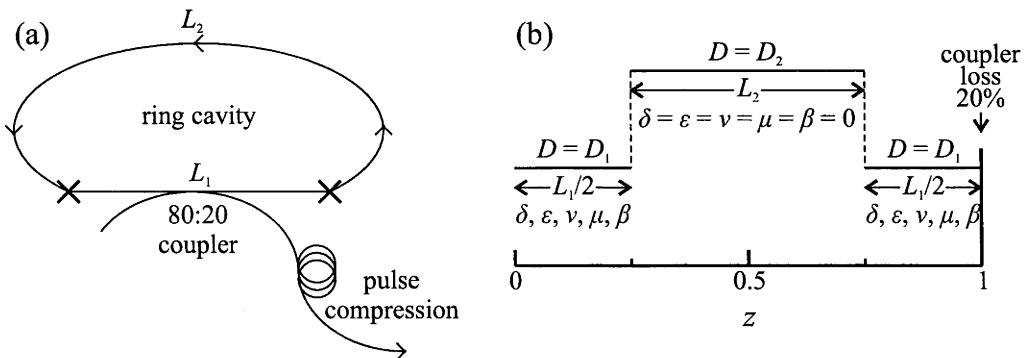


Figure 5.14: (a) Schematic diagram of the model and (b) map of the parameters of the CGLE for a single round-trip of the pulse in the cavity. Pulses travel counter-clockwise around the nonlinear (L_1) and linear (L_2) sections of the loop. At the coupler, 20% of the light is tapped off.

The propagation of a pulse through a laser cavity is a periodic process [105] that can be modeled by Eq. (5.1.1), but here the parameters D , δ , β , ϵ , μ and ν are periodic functions of z , where the propagation distance z is normalized to the cavity round-trip length. The period is defined by the cavity round-trip, and the parameters are step-wise functions of z . The parameter map used in this work is presented in Fig. 5.14(b). Clearly, there is a great variety of laser designs [94, 108–111] and this particular map cannot cover all of them. However, as has been mentioned earlier, the model serves as an example which shows that the resonance phenomenon persists in the perturbed model.

In this model, amplification and nonlinear effects are concentrated in a section of the cavity of length L_1 , while a section of length L_2 serves as the dispersive line. In the section of length L_1 , Eq. (5.1.1) is solved numerically with its dispersion parameter D_1 in place of D , while in the section of length L_2 , the dispersion D_2 is the only effect that acts on the pulse. Therefore, in L_2 , Eq. (5.1.1) can be reduced to a simpler version:

$$i\psi_z + \frac{D_2}{2}\psi_{tt} = 0. \quad (5.5.1)$$

A stationary regime of pulse evolution in the cavity is shown in Fig. 5.15. It is reached after a few round-trips, provided that the system parameters are chosen in the range of stable soliton solutions. For the case shown in Fig. 5.15, the parameters are $L_1 = 0.1$, $L_2 D_2 = -0.02$, $D_1 = -1$, $\epsilon = 1.0$, $\delta = 1.1$, $\beta = 0.08$, $\nu = 0.01$ and $\mu = -0.003$. The pulse evolution is periodic, with the width increasing in the dispersive part of the cavity and the amplitude increasing in the part of the cavity with amplification. The pulse energy drops by 20% each time the pulse passes the coupler. The pulse energy Q , is monitored at the end of each round-trip, *i.e.* at $z = 1$ in Fig. 5.14(b). The pulses outside the cavity have an energy which is 20% of the value of Q before the coupler inside the cavity. The positive value of δ which occurs over length L_1 on each loop does not cause the background to become unstable, as the 20% coupler loss experienced on each round-trip is more than the growth factor, $\delta L_1 < 1$.

Fig. 5.16 shows the pulse energy Q at $z = 1$, *i.e.* before the pulse passes the coupler, against the dispersion parameter, D_1 . The three distinct curves in this plot are calculated for three values of the linear dispersion, $L_2 D_2$. The case $L_2 D_2 = 0$ corresponds to the case without the purely dispersive piece. Thus, the pulse propagation is described by a single CGLE with constant coefficients and is disturbed

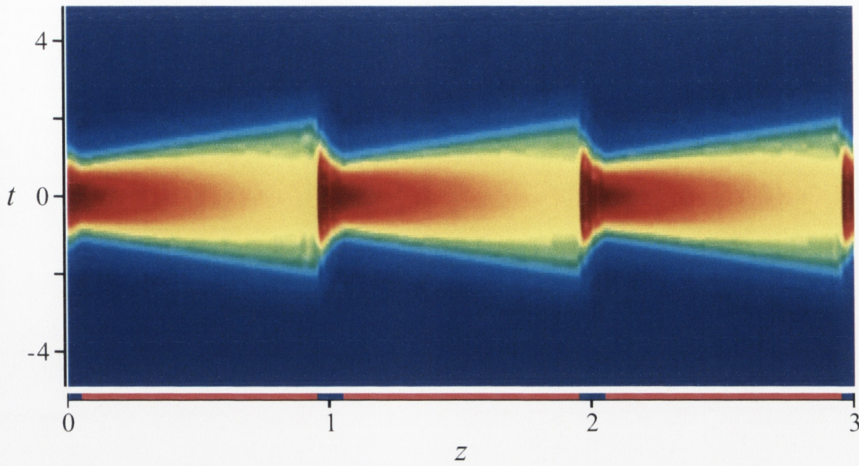


Figure 5.15: Evolution of the soliton within the dispersion-managed cavity for three consecutive round-trips. The pulse shape is repeated at the end of each round-trip, hence generating fixed shape solitons at the output of the oscillator. On the z axis, the blue line denotes section L_1 of the cavity, while the red line corresponds to section L_2 . The system parameters are $L_1 = 0.1$, $L_2 D_2 = -0.02$, $D_1 = -1$, $\epsilon = 1.0$, $\delta = 1.1$, $\beta = 0.08$, $\nu = 0.01$ and $\mu = -0.003$.

only by the coupler. This case is the closest to the model studied in Section 5.1.

The major feature of the plot in Fig. 5.16 is that Q increases to infinitely large values at the left-hand-side of each curve. This happens for every value of the linear dispersion, $L_2 D_2$. This feature is identical to those observed in the continuous model which was presented in Section 5.1. As shown here, the phenomenon does not disappear. Thus, the DSR is a robust phenomenon that reveals itself even when the model is perturbed.

The pulse profiles, the corresponding spectra and the phase are shown in Fig. 5.17, for $D_1 = -1.1$ (solid line), $D_1 = -1.2$ (dashed line) and $D_1 = -1.3$ (dotted line), where $L_2 D_2 = -0.02$ and the remaining parameters have the same values as the ones in Fig. 5.16. Although the spectral shapes are not exactly the same as in the continuous model, the qualitative changes when approaching the resonance follow the same scenario.

When $L_2 D_2$ is changed to more negative values, the energy Q also increases, as can be seen from Fig. 5.16. Fig. 5.18 shows the change in the pulse profiles and their spectra in this case. Specifically, three curves are given for $L_2 D_2 = 0$ (solid line), $L_2 D_2 = -0.02$ (dashed line) and $L_2 D_2 = 0$ (dotted line), while $D_1 = -1.1$ and other parameters are the same as in Fig. 5.16.

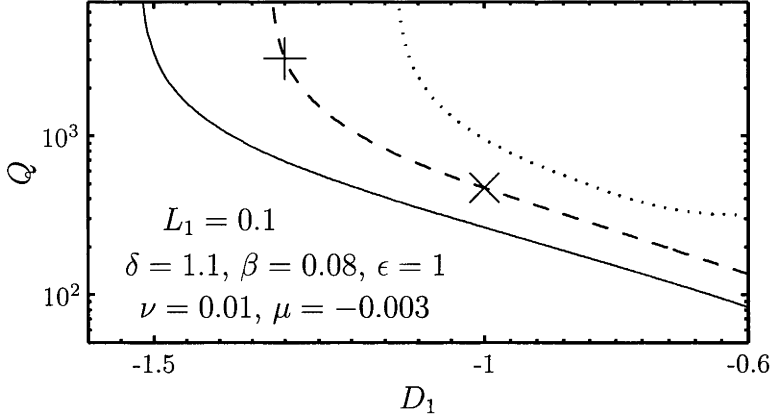


Figure 5.16: Soliton energy Q in the L_1 section of the oscillator against the dispersion parameter, D_1 , for $L_2D_2 = 0$ (solid line), $L_2D_2 = -0.02$ (dashed line) and $L_2D_2 = -0.04$ (dotted line). The other system parameters are $L_1 = 0.1$, $\delta = 1.1$, $\beta = 0.08$, $\epsilon = 1$, $\nu = 0.01$ and $\mu = -0.003$. The oscillator output pulses at the positions marked ‘ $\times$ ’ and ‘+’ are compressed and the results are presented in Section 5.6.

A comparison with the data in Fig. 5.17 shows that changing L_2D_2 has the same qualitative effect as changing the dispersion D_1 in the nonlinear section of the cavity. Moreover, the curves in Fig. 5.16 have similar shapes, which suggests that changes in D_1 and D_2 may have similar effects. In fact, the three curves in Fig. 5.16 can be approximated by the following empirical expression:

$$\log_{10}(Q) = \frac{2.8}{\langle D \rangle + 2.2}, \quad (5.5.2)$$

where the average dispersion $\langle D \rangle$ is defined by

$$\langle D \rangle = D_1 + 10L_2D_2. \quad (5.5.3)$$

Therefore, the results in Fig. 5.16 can be represented by using the $L_2D_2 = 0$ curve only, if D_1 is replaced by $\langle D \rangle$ on the horizontal axis. Thus, the system functions like a single element with an effective value of dispersion. However, this is only an approximation.

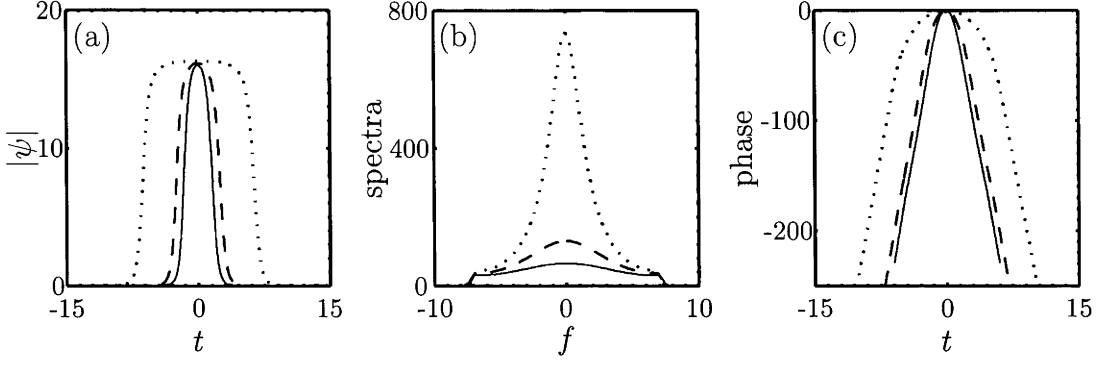


Figure 5.17: (a) Pulse profiles, (b) the spectra and (c) the phase for $D_1 = -1.1$ (solid line), $D_1 = -1.2$ (dashed line) and $D_1 = -1.3$ (dotted line), while $L_2D_2 = -0.02$. The rest of the parameters are same as the ones in Fig. 5.16.

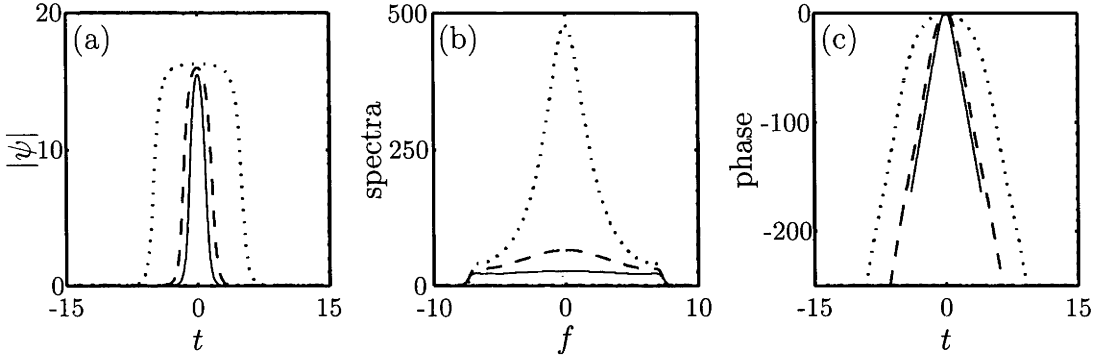


Figure 5.18: (a) Pulse profiles, (b) the spectra and (c) the phase for $L_2D_2 = 0$ (solid line), $L_2D_2 = -0.02$ (dashed line) and $L_2D_2 = 0$ (dotted line). The rest of the parameters are same as the ones in Fig. 5.16.

5.6 Pulse Compression

Although the energy of the pulse output generated from the oscillator has a high value near the resonance, it has a long pulse width. The peak amplitude is usually limited by the parameters of the mode-locking device, *i.e.* by the saturation of the nonlinear transmission, *i.e.* basically by ϵ and μ in our model. The increase in the energy is achieved mainly by the increase in the pulse width. Nevertheless, the output pulse from the oscillator is highly chirped, and this feature can be used to compress the pulse [112] and hence obtain high-energy ultra-short pulses.

The phase profiles of the output pulses, for the model discussed in the previous section, are shown in Fig. 5.17(c) and Fig. 5.18(c). The phase is approximately parabolic in the central part of the pulse and linear in the pulse wings. Each

pulse is very far from being transform-limited. The presence of the chirp makes it possible to carry out pulse compression using linear devices [112]. One of the simplest methods of compressing such pulses is to use a linearly-dispersive medium, such as a pair of gratings [113] or a dispersive fiber, to compensate for the linear component of the chirp. A compression scheme is known to work well for a linearly-chirped pulse [18]. If the initial pulse is Gaussian, then its shape is retained after the compression. In the case shown above, the pulse is not Gaussian and the chirp is not completely linear across the whole pulse. Nonetheless, a good compression can still be achieved.

Mathematically speaking, compression is achieved with the use of Eq. (5.5.1) where D_2 replaced by the dispersion characteristic of the compressor, chosen to be positive. This particular value influences only the length of propagation distance needed for achieving the optimal compression. Fig. 5.19 shows the results for the numerical pulse compression of two different pulses taken at the output of the oscillator.

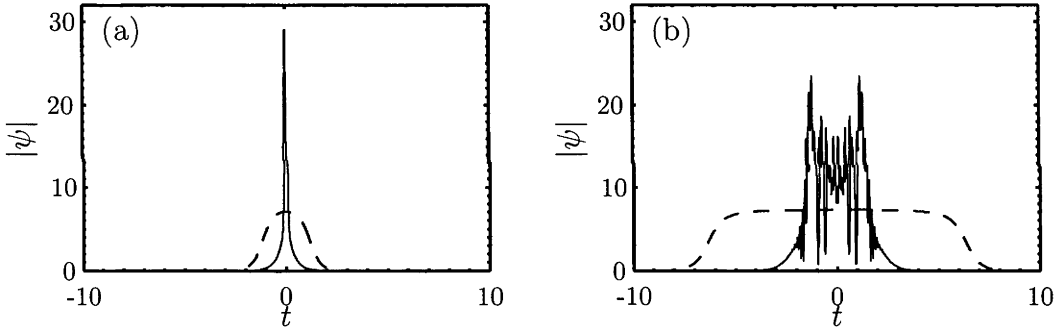


Figure 5.19: Pulse profiles before (dashed lines) and after (solid lines) the linear compression of the pulse. The oscillator output pulses are chosen for the set of parameters shown in Fig. 5.16, where $L_2 D_2 = -0.02$, (a) $D_1 = -1.0$ (' $\times$ ' in Fig. 5.16) and (b) $D_1 = -1.3$ ('+' in Fig. 5.16).

When the parameters of the oscillator are not too close to the resonance point, the pulse can be compressed effectively to a fraction of its initial width. The resulting pulse is close to being bell-shaped with a single maximum. The compression result for the pulse output at the point marked ' $\times$ ' in Fig. 5.16 is shown in Fig. 5.19(a). The dashed line in this figure shows the pulse shape before compression while the solid curve corresponds to the pulse shape after the compression.

When the parameters are chosen closer to the resonance point, a significant

part of the chirp deviates far from being linear. Thus, linear compression does not completely eliminate the chirp. The pulse is compressed, but not up to the transform-limit. Fig. 5.19(b) shows the compression result for the pulse output at the point marked ‘+’ in Fig. 5.16. The dashed line shows the pulse shape out of the oscillator, while the solid curve corresponds to the compressed pulse. The remaining chirp leads to a pulse with very narrow width, but it is multi-peaked. Pulses of this shape still can be used in practice, but, in some applications, transform-limited pulses are more desirable. For these latter cases, such pulses can be achieved with the help of more sophisticated dispersive lines [114–116] at the output of the oscillator, operating near the resonance. A choice of parameters which is different from the set shown in Fig. 5.16 can also improve the pulse chirp and subsequently the results for pulse compression. More detailed work is needed in this direction.

5.7 Summary

Dissipative soliton resonance (DSR) is a phenomenon where the energy of a soliton in a dissipative system becomes infinitely high at certain values of the system parameters. Specifically, it has been found that systems that can be modeled using the complex cubic-quintic Ginzburg-Landau equation (CGLE) admit a region of parameters with stable solitons, whose energy goes to infinity at the boundary of that region. The edge of this region in the system parameter space is a surface in the multi-dimensional system parameter space. Thus, the energy of the dissipative solitons remains infinite even when the parameters are changed in a continuous way along that hyper-surface. This phenomenon can be useful in designing optical oscillators generating pulses with exceptionally high energies.

In order to find the DSR using an analytical procedure, the method of moments can be used with a suitable trial function. The technique allows one to obtain an equation which approximately describes the position of the resonant surface in the multi-dimensional parameter space. This analytical technique has revealed that the DSR phenomenon can also exist in the anomalous dispersion regime. This requires the quintic nonlinear coefficient ν to be negative. Further studies on the role of ν in the DSR have suggested that the resonance is more significant in negative ν cases.

Also, the robustness of the DSR phenomenon has been tested for a laser model

with periodic parameter management. The study has indeed shown that the phenomenon persists when the model is strongly perturbed in such a way. The output pulses resulting from the DSR are usually highly chirped. This allows for the pulse duration to be significantly reduced using a linear dispersive line at the output of the laser oscillator.

When designing a laser generating pulses with ultra-high energy, preliminary modeling is essential to provide estimates for the range of parameters that would lead to the generation of pulses with the highest possible energy. These new solutions presented in this chapter should be considered when designing such systems.

Chapter 6

Effect of External Modulation

In order to generate a train of pulses rather than continuous wave (CW) light out of a laser oscillator, one needs to induce a fixed phase relationship between the modes oscillating in the cavity. Passive mode-locking is one of the ways to achieve this [117–120]. This technique relies on the transmission characteristics of the mode-locking device and their dependence on the intensity of the optical radiation. In this way, the instability of the CW regime of operation drives the laser into pulsations. There is a variety of ways to obtain nonlinear transmission: each method has its own advantages and disadvantages [121–124]. The main advantage of passive mode-locking is that pulses are generated in a self-organized way, taking the form of dissipative solitons [7, 41].

Applying external periodic modulation is another efficient way of generating pulses in laser cavities [108, 125–131]. This technique is well-known as ‘active mode-locking’. There are two different techniques for active mode-locking. In the first one, the optical length of the cavity is modulated by using a change in the refractive index of a crystal inside the cavity, and so the phase is being modulated. In the second case, the gain or loss in the cavity is modulated. Application of external periodic modulation with a well-defined temporal period allows for the generation of a sequence of pulses with the given separation.

An interesting, and relatively less explored, possibility for generating pulses is a combination of both passive and active mode-locking mechanisms in a sort of hybrid mode-locking. The cavity-synchronous phase or amplitude modulation technique [131, 132] transforms passively mode-locked optical oscillators into actively mode-locked lasers. Combining passive and active mode-locking in the same device results

in a new class of optical oscillators capable of generating short pulses. The pulses or pulse pairs may already appear in the cavity due to the passive mode-locking mechanism, while the active mode-locking defines their exact location inside the cavity round-trip. Hence, passive mode-locking provides favorable conditions for pulse emission, whereas active mode-locking modifies their shape or time location. In this way, pulse generation processes may be controlled more efficiently, thus providing one with versatile optical tools.

The pulses emitted by such lasers are expected to exhibit some peculiar properties, depending on the adjustment of the many relevant parameters. In particular, in combined passively and actively mode-locked lasers, the interactions among neighboring pulses should be different with respect to the case of purely passive or active lasers. Although there are various experimental works demonstrating this technique [130–132], a full understanding of the effect of the modulating process in determining the properties of the generated pulses is far from being reached.

The description of such interactions naturally raises the question of how to take the presence of external modulation into account when modeling the system using the master equation approach [8]. The natural solution would be that of adding a time-periodic term to the complex cubic-quintic Ginzburg-Landau equation (CGLE) [128, 129]. In this chapter, stable pulse solutions of the CGLE, which has been modified in such a way, are studied. In particular, the effect of an external time-periodic modulation on already existing CGLE pulse pairs is considered. The presence of the additional modulation may be expected to improve the stability of pulse-to-pulse time separation, for example. Experimental results on this subject were recently reported in [130, 133]. Here, a comprehensive numerical investigation on this problem is presented.

The chapter begins in Section 6.1, by introducing a model which incorporates both the passive and active mode-locking. In the following sections, the effect of a periodic external modulation on soliton pairs is presented. In particular, the influence of the phase modulation is examined in Section 6.2, while in Section 6.3, the effect of gain-loss modulations on soliton pairs is investigated. There are various ranges of modulation periods that can be applied to an existing pulse pair. Depending on the value relative to the initial soliton pair separation, they can be classified into three categories: intermediate, short and long periods of modulation. All these three cases are considered in both Section 6.2 and Section 6.3. Finally, a

summary is given in Section 6.4.

6.1 Model

To model the laser system, an additional time-periodic term, corresponding to active mode-locking, is added to the usual passive mode-locking terms in the CGLE. Thus, the modified CGLE is given by

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi + \nu|\psi|^4\psi = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{tt} + i\mu|\psi|^4\psi + \alpha \cos(\Omega t)\psi, \quad (6.1.1)$$

The parameters and notation in Eq. (6.1.1) are the same as in Eq. (2.2.1). The additional term, $\alpha \cos(\Omega t)\psi$, corresponds to the active mechanism. This term produces two more parameters, namely the modulation depth α and its frequency Ω . When α is real, the active mode-locking is achieved by sinusoidally changing the refractive index inside the cavity, while, when α is imaginary, cavity gain-loss is modulated for achieving mode-locking. Hence, in this chapter, the case when α is real is considered in Section 6.2, while the results for imaginary α are presented in Section 6.3. It should be noted that the sign of α only shifts the periodic modulation by $\pi/2$ radians. Hence the sign of α which is most suitable for each specific case is chosen in the examples in this chapter.

Soliton pairs in dissipative systems have been studied extensively, both theoretically [134–137] and experimentally [138, 139]. They can exist even without the presence of external modulation [140], and their formation requires a special adjustment of the laser system parameters [141]. In this work, it is assumed that this case has been reached, and the external time-periodic modulation is applied to the properly adjusted passive laser system.

Fig. 6.1(a) shows the region of existence of the stable soliton pairs (dark grey region labeled ‘SP’) in the absence of the modulation term [141]. This region is within a region of existence of single stable solitons (light grey region labeled ‘S’), and occupies much a narrower range of the system parameters than the single solitons. It is assumed that the soliton pair solutions in the middle of the region are likely to be the most stable ones, and hence a pair solution from the point marked ‘×’, *i.e.* $D = 1$, $\delta = -0.2$, $\beta = 0.5$, $\epsilon = 1.4$, $\nu = -0.075$ and $\mu = -0.5$, is chosen as the initial condition for studying the influence of the external modulation. The profile

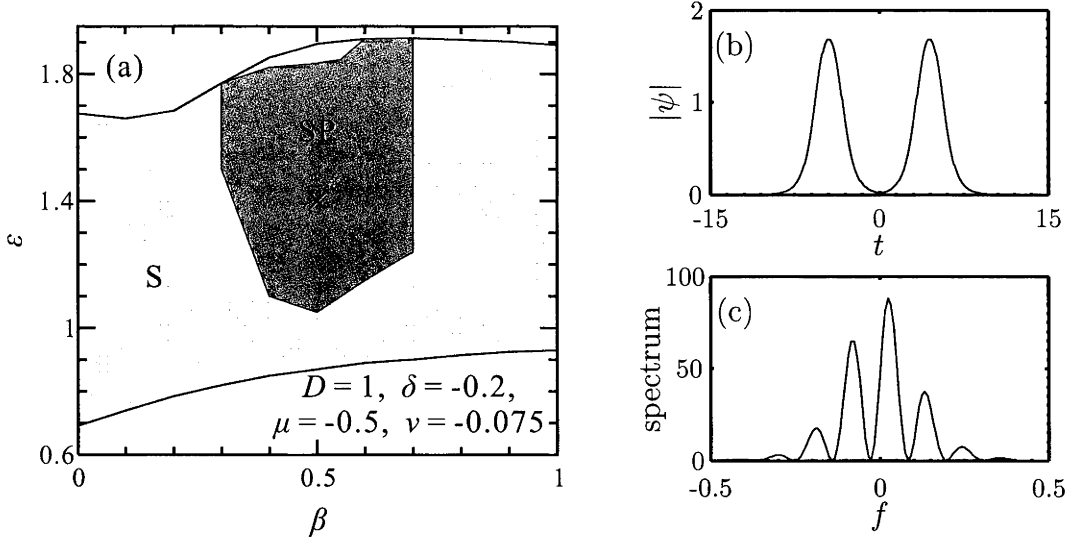


Figure 6.1: (a) Region of existence of stable stationary soliton pairs denoted, ‘SP’. This region is entirely located within the region of stable single solitons ‘S’ [141]. The set of parameters of the CGLE in the middle of the region ‘SP’, *i.e.* $D = 1$, $\delta = -0.2$, $\beta = 0.5$, $\epsilon = 1.4$, $\nu = -0.075$ and $\mu = -0.5$, is used throughout this chapter. The pair profile and its spectrum for the point marked ‘x’ in absence of the external modulation are presented in (b) and (c), respectively.

and the spectrum of this pulse pair are presented in Fig. 6.1(b). This particular pair has separation $\rho_i = 8.9230$. The spectrum is asymmetric, and it corresponds to a phase difference between the pair of $\phi_i = \pi/2$. The characteristics for this pair depend entirely upon the parameters of the passive laser system. Indeed, this pair is not the only possible choice, but it is used as the initial condition throughout this work, since scanning the entire parameter space looking for stable pairs is a tedious task which deserves more specialized efforts [134–140]. The focus here is on the effect of the external modulation. Therefore only two parameters, α and Ω , which are directly responsible for the external modulation, are varied. The other parameters, which are responsible for passive mode-locking, are fixed at those values presented above. In this way, new knowledge on how the modulation influences the cavity soliton pairs can be extracted.

At this stage, it is useful to define a reference value for the modulation frequency. A natural choice would be to define it using the initial pair separation, ρ_i , of the pulse pair shown in Fig. 6.1(b). It can be defined in such a way that the period of the periodic potential that it creates exactly corresponds to the soliton separation.

Thus, the reference modulation frequency Ω_0 can be defined as

$$\Omega_0 = \frac{2\pi}{\rho_i}. \quad (6.1.2)$$

The modulation frequency Ω can take a wide range of values, and the physical effects associated with it are significantly diverse; they depend on the value of the frequency relative to Ω_0 . Hence, it is appropriate to categorize the investigation into three parts, depending on the frequency of the modulation. Specifically, in this work, the modulation period is said to be intermediate when $\Omega \approx \Omega_0$, short when $\Omega \gg \Omega_0$ and long when $\Omega \ll \Omega_0$.

6.2 Phase Modulation

The influence of periodic phase modulation on soliton pairs and their stability was first studied by Wabnitz [142]. Phase modulation is evidently an important mechanism for active mode-locking of lasers [129]. In Eq. (6.1.1), the phase modulation can be introduced by having a non-zero, real value of modulation depth, α . Physically, this corresponds to introducing a time-periodic cavity length modulation.

6.2.1 Intermediate Periods of Modulation

First, the case of phase modulation periods being comparable with the initial pair separation is considered. Examples of such modulations are depicted in Fig. 6.2. There are only three different modulation frequency examples shown in this figure,

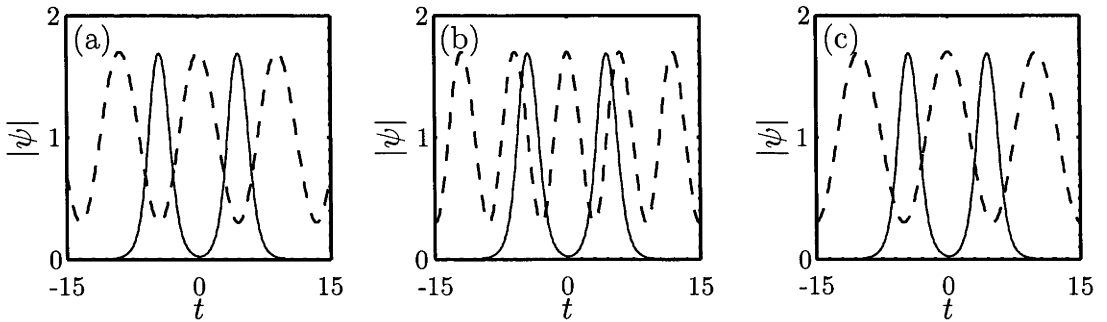


Figure 6.2: External modulation (dashed line) applied to the soliton pair (solid line), where (a) $\Omega = \Omega_0$, (b) $\Omega = (3/2)\Omega_0$ and (c) $\Omega = (8/9)\Omega_0$.

i.e. $\Omega = \Omega_0$ in Fig. 6.2(a), $\Omega = (3/2)\Omega_0$ in Fig. 6.2(b) and $\Omega = (8/9)\Omega_0$ in

Fig. 6.2(c). However, comprehending other modulation frequencies on the pulse pair should be straightforward from these plots. For $\Omega = \Omega_0$, the troughs of the modulation coincide with the peaks of the pulses, as shown in Fig. 6.2(a). For other values of Ω , the troughs are located either between the peaks of the initial pair, or outside both of them, depending on whether $\Omega > \Omega_0$ or $\Omega < \Omega_0$, respectively.

The phase modulation amplitude parameter α is adjusted for a number of different values of Ω . When $\alpha = 0$, the periodic potential vanishes, and the pair separation corresponds to the one which is given by passive mode-locking, *i.e.* by the rest of the parameters of the CGLE. When α increases, the stable pulse separation also changes. The resulting changes depend on the particular value of Ω , as is shown in Fig. 6.3. Specifically, Fig. 6.3(a) shows the variation of ρ when Ω

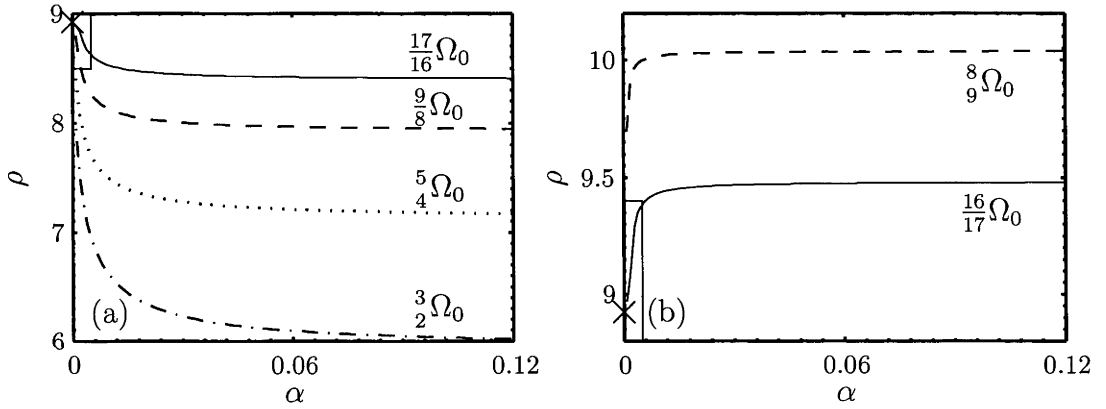


Figure 6.3: The pair separation ρ versus the modulation depth α is shown for (a) $\Omega > \Omega_0$ and (b) $\Omega < \Omega_0$.

is greater than Ω_0 , while Fig. 6.3(b) shows the variation of ρ when Ω is smaller than Ω_0 . All curves start from the same point, which corresponds to the initial time separation ρ_i , and converge to a new separation that is defined by the value of Ω . More precisely, $\rho \rightarrow 2\pi/\Omega$ as α is increased. This convergence is faster for the curves in Fig. 6.3(b), since in this case the external modulation pushes solitons further away from each other, and the CGLE interactions between the pair become weaker as the pair separation is increased.

Interestingly, the presence of a periodic phase modulation not only influences the time separation but also the phase difference between the solitons. The phase difference ϕ between the pair becomes either 0 or $\pm\pi$, depending on the value of Ω relative to Ω_0 , when α is increased from 0. When Ω is above the critical value,

i.e. $\Omega > \Omega_0$, the phase difference ϕ converges to 0, while it converges to $\pm\pi$ when $\Omega < \Omega_0$. The new resulting types of soliton pairs are shown in Fig. 6.4. The profile,

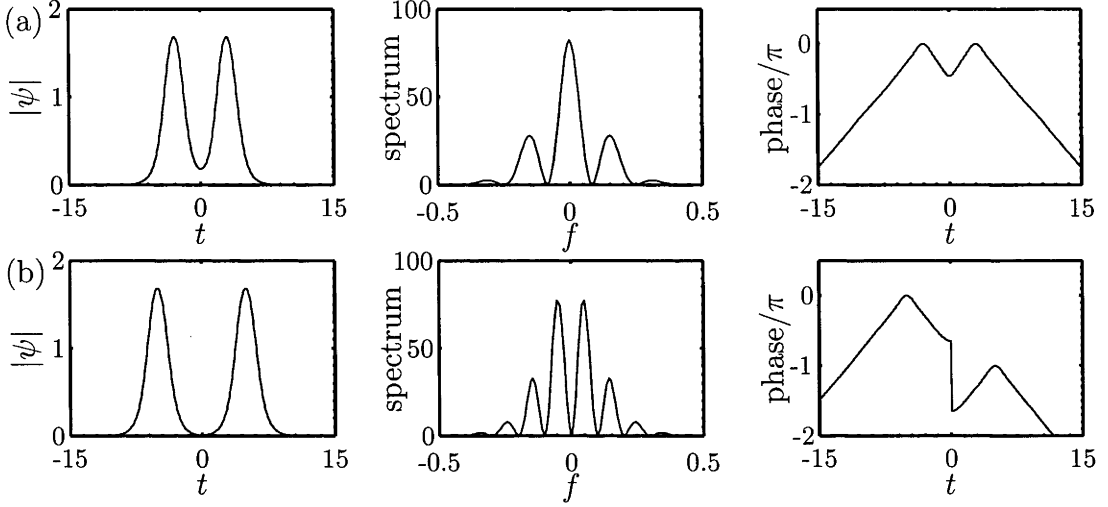


Figure 6.4: Soliton pair profile, spectrum and phase when $\alpha = 0.12$ for (a) $\Omega = (3/2)\Omega_0$ and (b) $\Omega = (8/9)\Omega_0$. Note that the phase difference ϕ is zero when $\Omega > \Omega_0$ and $\pm\pi$ when $\Omega < \Omega_0$.

phase and the spectrum are presented for two pairs, (a) for $\Omega = (3/2)\Omega_0$ and (b) for $\Omega = (8/9)\Omega_0$, for $\alpha = 0.12$. In the $\Omega > \Omega_0$ case, the periodic potential compresses the two pulses, while in the opposite case it decompresses the pair. The presence of such an external force pushes the soliton pair into the next stationary position.

One of the most interesting observations here is that the parameters of the soliton pair under the influence of the periodic modulation are not unique. Such unexpected behavior is easily observed for the case where the period of the potential coincides with the soliton separation ρ_i , *i.e.* $\Omega = \Omega_0$. The values of the pair separation, ρ , and the phase difference, ϕ , versus the modulation depth, α , in this case are shown in Fig. 6.5. One can see that the curves are not single-valued. The path of the fixed point of the dynamical system depends on whether α is increased or decreased. This feature of the fixed points is indicated by the arrows in the diagram Fig. 6.5. Thus the curves that describe the pulse pair separation and phase difference exhibit hysteresis, and two types of soliton pairs may exist simultaneously. It can be also seen from Fig. 6.5 that the phase difference ϕ approaches the value π when α reaches some particular value. When this limit is reached, the phase difference stays constant upon further increase of α . However, the separation ρ continues to change. On the other hand, when α is decreased, the phase difference

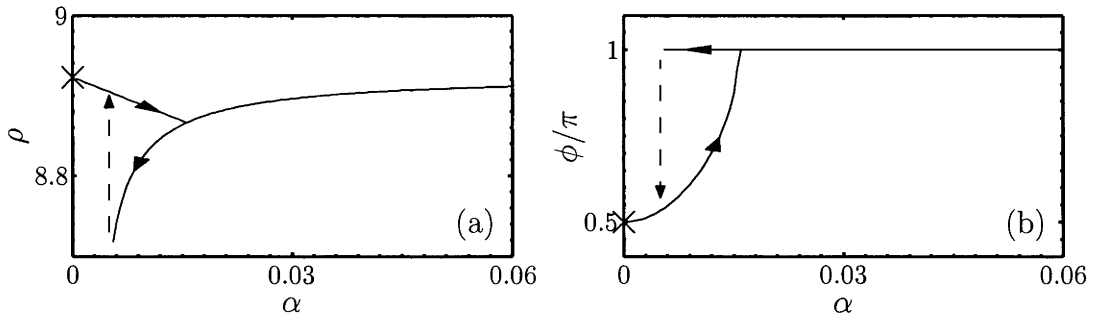


Figure 6.5: (a) Pair separation ρ and (b) the phase difference ϕ versus the modulation depth α for $\Omega = \Omega_0$. Bistability is observed in the interval $0.005 < \alpha < 0.015$. The solution follows the branch designated by an arrow when α is varied in the positive or in the negative directions.

and separation jump to their second values at a lower threshold. This transition is shown in Fig. 6.5 by the vertical dashed lines with arrows.

A careful study has revealed that similar transitions occur for the parameters of the soliton pairs at other values of Ω . They are not shown in Fig. 6.3, as the transitions occur on a much smaller scale in α . Magnified portions of the curves near small modulation amplitudes are shown in Fig. 6.6. These plots correspond to the rectangular boxes in Fig. 6.3 surrounded by the red lines. The curves of Fig. 6.6(a) and Fig. 6.6(b) refer to the cases when Ω is either slightly above or slightly below the value Ω_0 , respectively. In Fig. 6.6(a), the time separation and the phase difference decrease when α increases, whereas in Fig. 6.6(b), each of these parameters increases. At small α , the soliton separation ρ stays almost constant, while the phase difference between the solitons, ϕ , moves gradually to either 0 or π , depending on Ω . When one of these two values for the phase difference is reached, a bifurcation occurs. Above the bifurcation point, the phase difference stays constant, while the soliton separation changes until it adjusts to the value fixed by the periodic potential. The difference between the two values of α that correspond to the bifurcations, *i.e.* the interval of bistability, is greatest when Ω is equal to the critical value Ω_0 . This is easily understood, because the influence of the potential on the soliton pair is minimal when $\Omega = \Omega_0$.

Another interesting feature of the soliton pair dynamics appears at values of α that are even small in the scale of Fig. 6.6. One can see that in each of Fig. 6.6(a) and Fig. 6.6(b), there is a jump in the separation ρ between the initial separation ρ_i , denoted with the cross '×' on the vertical axis, and the actual branch of solutions.

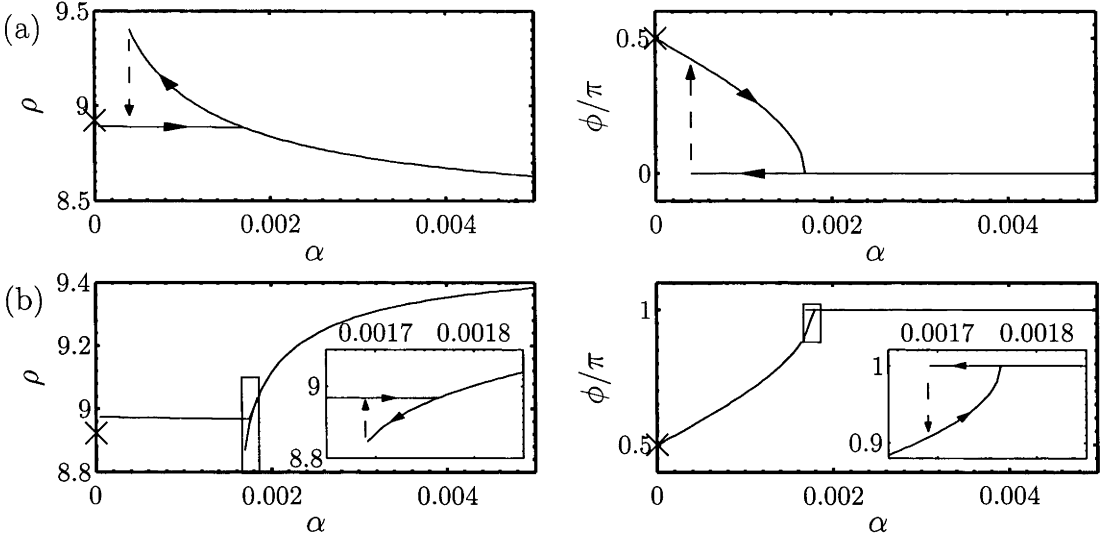


Figure 6.6: Pair separation ρ and the phase difference ϕ versus the modulation depth α for (a) $\Omega = (17/16)\Omega_0$ and (b) $\Omega = (16/17)\Omega_0$. The two plots in the left-hand-side panels are magnifications of the parts of the curves in Fig. 6.3 within the rectangles bounded by red lines. Bistability is observed when (a) $0.0004 < \alpha < 0.0017$ and (b) $0.00169 < \alpha < 0.00176$. The arrows show the branches of the curves that are followed when α is varied in the positive or negative directions.

This jump is due to the initial phase difference between the solitons, $\phi_i = \pi/2$, at zero α . This phase difference results in asymmetric soliton pairs that possess non-zero velocity [136, 139]. The direction of motion of the pair is defined by whether the above phase difference is positive or negative. However, when a periodic potential with a sufficiently high amplitude is applied, it prevents the soliton pair from moving. Thus, when the parameter α grows from zero, the soliton pair has to re-adjust its phase difference to the new conditions. The readjustment occurs at very small values of α . Indeed, small α may not prevent the pair from moving across the periodic potential. However, the velocity of such motion becomes a function of α . There is a threshold value of α above which the soliton pair motion stops completely.

As a matter of fact, there are two regimes of evolution for the soliton pairs in the periodic potential. At small values of the modulation depth, the walls of the periodic potential cannot capture the soliton pair. Solitons can move across the periodic potential and converge to an oscillating moving state, being disturbed periodically. The phase difference between the two solitons in the pair remains close

to $\pi/2$. This means that the pair is formed mainly by the passive mode-locking mechanism of the CGLE. The phase trajectory of the solution on the interaction plane in this case is shown in Fig. 6.7(a). The definition of interaction plane is given in the work of Akhmediev, Ankiewicz and Soto-Crespo [140] and should be clear from the labels on the axes in Fig. 6.7.

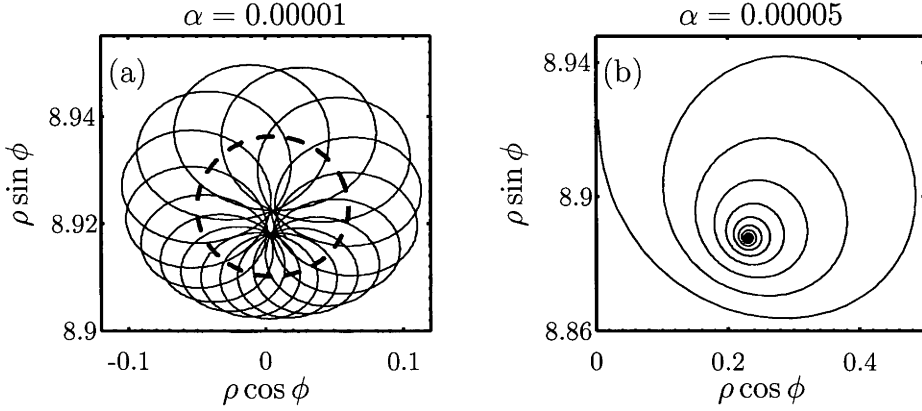


Figure 6.7: Trajectories on the interaction plane [140] showing convergence of the soliton pair to a stable state when $\Omega = (17/16)\Omega_0$. (a) At small values of α , the pair jumps over the periodic potential causing a double spiral on convergence. This converges to the limit-cycle shown by the red dashed line. (b) At larger values of α , the pair converges directly to a fixed ρ , without jumping between the periods. Thus, convergence to a fixed point occurs as a single spiral.

Due to the motion of the pair across the periodic potential, the trajectory on the interaction plane has a double spiraling feature. This means that the motion contains two frequencies. One of the frequencies is defined by the internal oscillations of the ‘soliton molecule’. Thus, the two solitons bound together oscillate relative to their common center-of-mass. The second, lower, frequency is defined by the motion of the pair across the periodic potential. It is defined by the period of the potential and the velocity of the pair. One complete rotation around the point located at $(0.0046, 8.9233)$ in Fig. 6.7(a) corresponds to this motion across one period of the potential, while smaller loops of the trajectory correspond to the internal oscillations.

These internal oscillations eventually decay, and the trajectory converges to a limit-cycle around the same point $(0.0046, 8.9233)$. The convergence occurs over a scale of more than a hundred periods. Hence the double spiral represents only a small part of the trajectory. The resulting limit-cycle is marked in Fig. 6.7(a) as a

red dashed line. This complicated motion of the soliton pair can be seen in Fig. 6.8. The stripes that correspond to each soliton show acceleration and deceleration of the soliton pair that are synchronized with the motion in the periodic potential. Furthermore, oscillations still exist and are more frequent, showing internal pulsation of the pair.

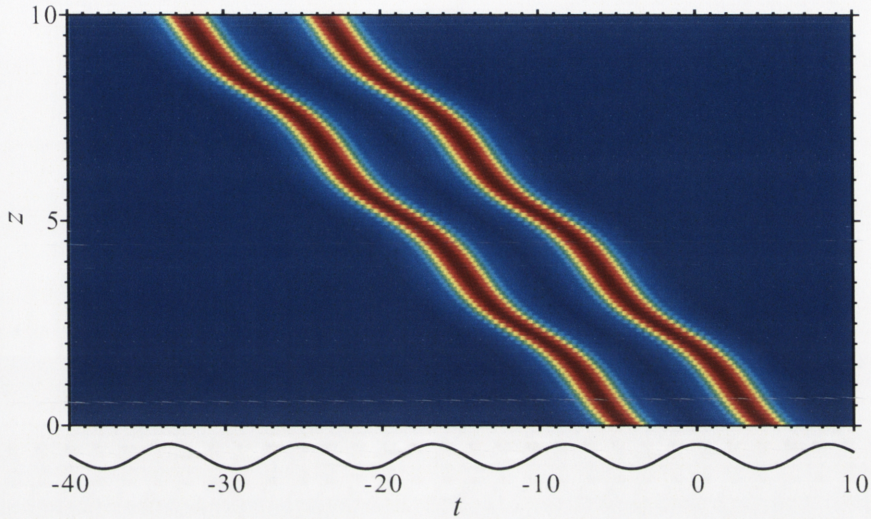


Figure 6.8: Motion of the dissipative soliton pair across the periodic potential. This case corresponds to the trajectory on the interaction plane shown in Fig. 6.7(a). The sinusoid below the plot shows the periodic potential. The soliton pair experiences internal decaying oscillations, as well as periodic acceleration and deceleration, when moving across the potential.

When α grows larger than a certain threshold, the periodic potential confines the pair completely, and, as a result, the soliton pair cannot move across it. The phase trajectory in this case is shown in Fig. 6.7(b). The periodic motion of the pair across the potential is stopped, and the trajectory converges into a stable fixed point along a simple single spiral. The fixed point here corresponds to a pair where the phase difference between solitons is not $\pi/2$, 0 or $\pm\pi$. The phase difference shifts to 0 or π when α increases. The plots presented in Fig. 6.6 are related to these fixed points.

6.2.2 Short Periods of Modulation

So far, a detailed investigation has been made on the dynamics of soliton pair separation in the situation of having a phase modulation period which is close to

the pair separation, as defined by the properties of the passive mode-locked cavity. The modulation frequency may also take values in quite a different range. Fig. 6.9(a) and Fig. 6.9(b) show two cases where $\Omega = 2\Omega_0$ and $\Omega = 3\Omega_0$, respectively. Hence, the modulation period is now a fraction of the pair separation.

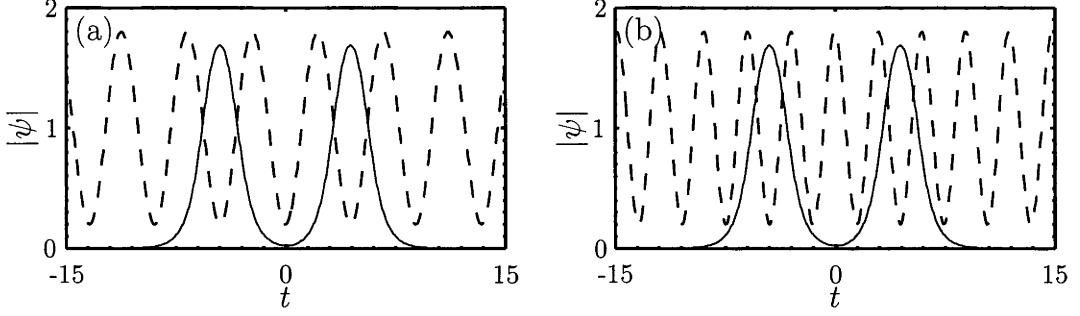


Figure 6.9: External phase modulation (dashed line) with two different frequencies (a) $\Omega = 2\Omega_0$ and (b) $\Omega = 3\Omega_0$ applied to the initial dissipative soliton pair (solid line).

Following the results presented above for medium range periods of modulation, both cases presented in Fig. 6.9 would be expected to exhibit a synchronization between the external frequency and soliton separation. Moreover, the soliton separation ρ and the phase difference between the pair ϕ are predicted to be influenced by the strength of modulation, α . The results for these two characteristics versus α are shown in Fig. 6.10.

For each case, a hysteresis loop of the parameters is observed at small modulation amplitudes, with the loops being very similar to each other. Fig. 6.10 shows that two different soliton pairs exist simultaneously over a small range of α values. The particular form of the solution depends on the direction in which the modulation amplitude α is varied.

The two modulation frequencies considered in Fig. 6.10 are frequency harmonics of the case $\Omega = \Omega_0$ which was shown in Fig. 6.5. By comparing the curves in Fig. 6.5 and Fig. 6.10, one can see that they are similar except for $\Omega = 2\Omega_0$. In Fig. 6.5 and Fig. 6.10(b), the branch moving in the direction of decreasing modulation amplitude jumps to the second branch, while in Fig. 6.10(a), this does not happen. Instead, the pair merges and forms a single soliton solution, which is another stable solution in this region, at the end of the branch. Such a difference in the solution behavior arises from the fact that the periodic potential has a minimum between the solitons

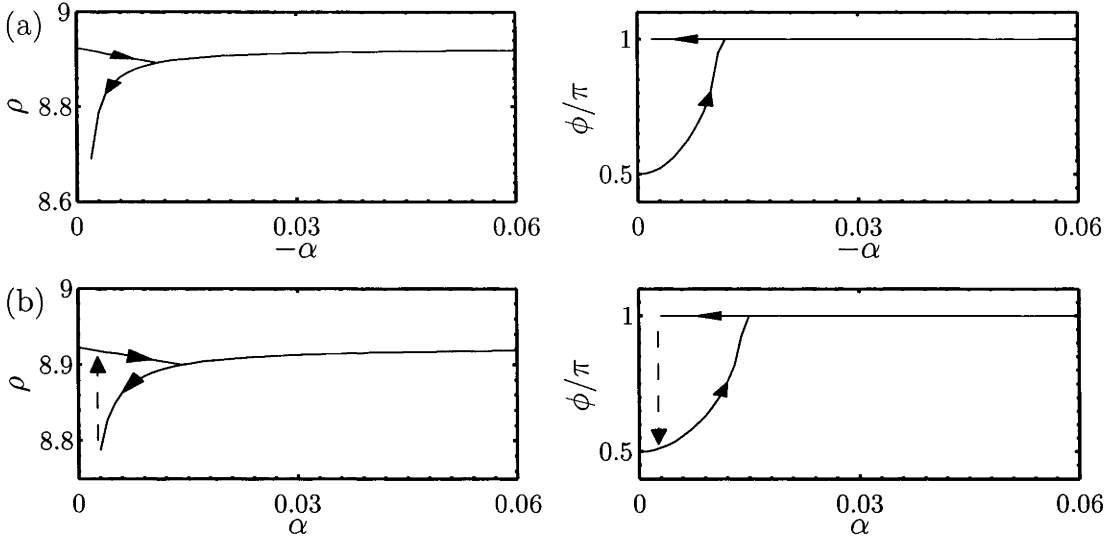


Figure 6.10: Soliton pair separation ρ and phase difference between the pair ϕ versus modulation depth α for (a) $\Omega = 2\Omega_0$ and (b) $\Omega = 3\Omega_0$, applied as shown in Fig. 6.9.

in the case $\Omega = 2\Omega_0$, while it has a maximum in each of the two other cases. When a maximum exists, it is more likely that the two solitons will stay apart.

6.2.3 Long Periods of Modulation

There are a few experimental papers which have recently appeared in the literature [130, 131, 133] on the effect of synchronous phase modulation on pulse pairs from mode-locked fiber lasers. In all of these works, the period of the imposed phase modulation was much higher than the observed soliton pair separation. Therefore it appears that, in those experiments, a different regime was studied with respect to the situations that have been presented above. Fig. 6.11(a) shows the location of the pulses in the pair relative to three different external modulations with long periods of modulation that are similar to the ones considered in those experiments.

As the two pulses are located within a single potential well, the two pulses experience external pressure towards each other. As a result, the pulse separation reduces when the amplitude of the modulation increases. The numerical results for the soliton separation in this case are shown in Fig. 6.11(b). The separation reduces from the original value of 8.923 down to approximately 5.178 for each of the three

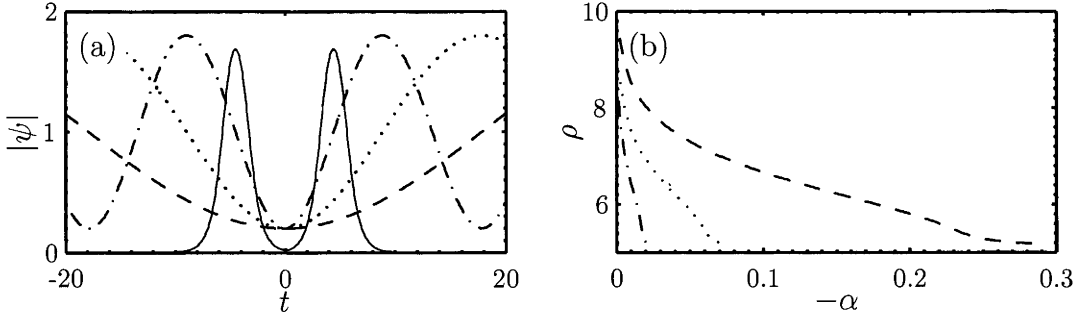


Figure 6.11: (a) External phase modulation applied to the soliton pair solution (solid line) for three modulation frequencies, namely, $\Omega = (1/8)\Omega_0$ (dashed line), $\Omega = (1/4)\Omega_0$ (dotted line) and $\Omega = (1/2)\Omega_0$ (dash-dotted line). (b) Soliton pair separation ρ versus the modulation depth α for three modulation frequencies presented in (a).

modulation frequencies, namely, $\Omega = (1/8)\Omega_0$ (dashed line), $\Omega = (1/4)\Omega_0$ (dotted line) and $\Omega = (1/2)\Omega_0$ (dash-dotted line). The pair separation, $\rho = 5.278$, is the minimal distance at which the two solitons can exist as separate units. Further compression of the pulse separation results in the merging of the two solitons into a single pulse. Such merging occurs when α reaches the values -0.283 , -0.073 and -0.020 for modulation frequencies of $(1/8)\Omega_0$, $(1/4)\Omega_0$ and $(1/2)\Omega_0$, respectively. The relative phase between the solitons quickly converges down from the original value of $\pi/2$ at $\alpha = 0$ to zero at $\alpha \approx -0.01$ for all three cases. These observations are in line with the experimental work of Hsiang *et al.* [131]. Thus, relatively low modulation frequencies, *i.e.* long time periods, allow one to efficiently control the time separation of soliton pairs inside the cavity.

6.3 Gain-Loss Modulation

An imaginary amplitude of the modulation corresponds to modulation of the cavity linear gain-loss parameters. Such modulation may involve various physical processes. Despite this difference, the action of gain-loss modulation can be qualitatively similar to that of phase modulation and can allow one to achieve active mode-locking with good control over the pulse separation.

6.3.1 Intermediate Periods of Modulation

When Ω is comparable with Ω_0 , synchronization of the pulse separation with the modulation period would be expected. Fig. 6.12(a) shows that, indeed, the soliton separation reduces to the same value of modulation period as the modulation depth is increased. The soliton separation reduces from its original value, 8.923, as the amplitude is increased, until it approaches 7.19. The phase difference between the solitons, being $\pi/2$ at zero amplitude, quickly approaches zero at very small amplitudes, and remains equal to zero upon further increases of the modulation amplitudes.

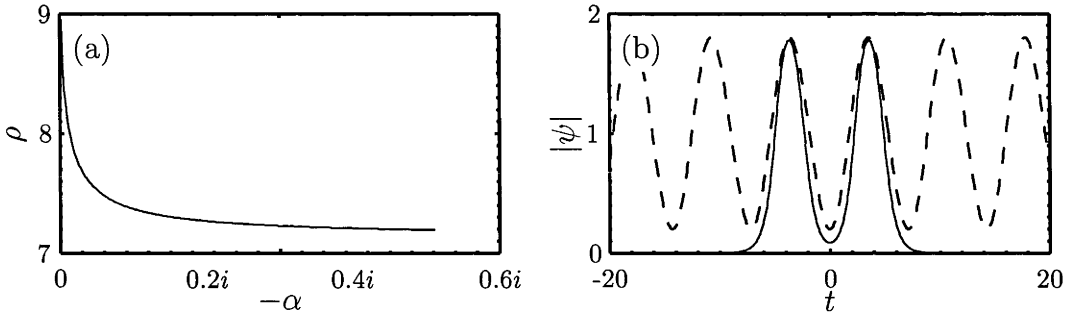


Figure 6.12: (a) Soliton separation ρ versus α for $\Omega = (5/4)\Omega_0$. Phase difference is zero in the whole interval except for a region of small α which cannot be resolved in the scale of this figure. (b) Location of two solitons (solid line) in the pair relative to periodic gain-loss modulation (dashed line), shown for $\alpha = -0.4i$.

The synchronization itself can be seen in Fig. 6.12(b). Two solitons are compressed and their maxima coincide with the maxima of the periodic modulation. On the other hand, further increases in the modulation depth result in the generation of a pulse train in the crests of the periodic gain-loss modulation; this occurs at around $\alpha = -0.51i$ for $\Omega = (5/4)\Omega_0$.

In Fig. 6.13, the results for the case where the modulation frequency is smaller than Ω_0 are presented. A very small modulation amplitude shifts ρ to a new value of $2\pi/\Omega$ and ϕ to $\pm\pi$. Interestingly, another change appears at comparatively high amplitudes of modulation. For $\Omega = (4/5)\Omega_0$, α has to be around $-0.25i$ in order for the single branch of soliton pairs to split into two branches. One of the split branches does not experience much change after the bifurcation. However, the second branch appears with both a smaller pulse separation and smaller phase difference after the bifurcation. For both branches, the soliton pair shape and

location within the periodic modulation are shown in Fig. 6.13(b) and Fig. 6.13(c). At small amplitudes, the position of the pair coincides with the peaks of the periodic gain-loss modulation, whereas, when the amplitude is larger the soliton pair stays on the ‘shoulders’ of the periodic modulation, after the pair goes through a complicated region of transition.

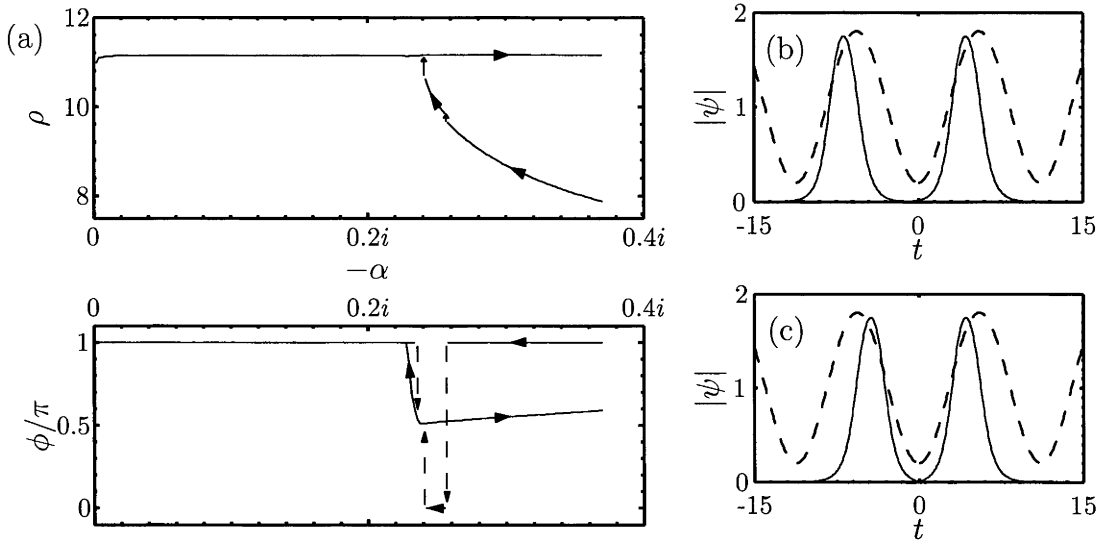


Figure 6.13: (a) Soliton separation ρ and phase difference between the solitons ϕ versus α for $\Omega = (4/5)\Omega_0$. The splitting into two types of solitons occurs at $\alpha \approx -0.24i$. Above this threshold, soliton pairs have branches of 0, $\pi/2$ and π phase difference between the pulses. The arrows show the sequence of changes when increasing or decreasing α . The location of two solitons (solid line) in the pair relative to this periodic modulation (dashed line) is shown for the solution representing (b) the upper branch and (c) the lower branch when $\alpha = 0.3i$. Solitons ‘prefer’ certain ‘equilibrium’ positions in the pattern, thus leading to two types of bound states.

Fig. 6.13(b) and Fig. 6.13(c) show that an equilibrium position for a single soliton appears on the slopes of the periodic modulation. Thus, the soliton pair can be formed from two solitons either on the same slope or on two different slopes. Each of these solutions may exist independently, and they can also exist simultaneously, thus creating a bifurcating structure. Again, at large modulation depths, the instability of the zero amplitude background causes a train of pulses to be generated.

For both modulation frequency cases shown in Fig. 6.12 and Fig. 6.13, bistability that is similar to the ones present in Fig. 6.6 is observed at very small amplitudes. The bifurcation locations and their ranges are too small to be seen in Fig. 6.12 and

Fig. 6.13, but can be pictured without difficulty from the ones shown in Fig. 6.6.

However, the case when $\Omega = \Omega_0$ does not have such a bistability when gain-loss is modulated instead of the phase. As shown in Fig. 6.14, the curve is single valued. Upon varying α , the pair separation ρ and the phase difference ϕ change only slightly from their initial values, ρ_i and ϕ_i , and the influence of the periodic modulation is minimal.

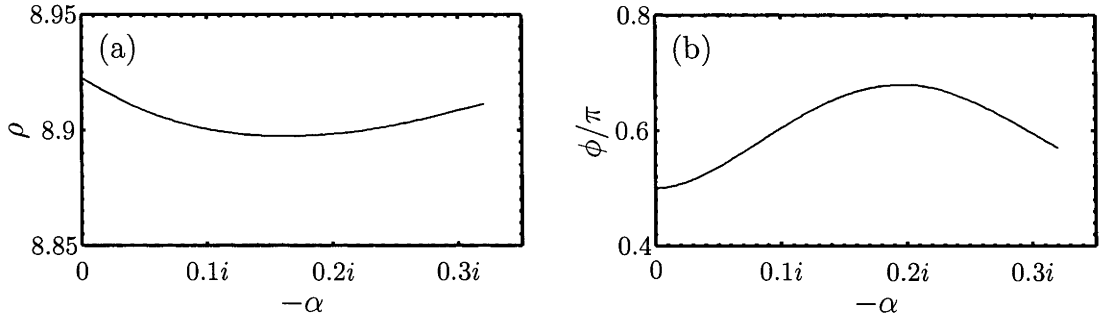


Figure 6.14: (a) Soliton pair separation ρ and (b) the phase difference ϕ versus the gain-loss modulation depth α for period of modulation $\Omega = \Omega_0$.

6.3.2 Short Periods of Modulation

When the period of modulation is much smaller than the soliton separation, the action of linear gain-loss parameters tend to be averaged across the soliton pair. In that case, the periodic modulation can be approximated by constant parameters, and the equation is reduced to a pure CGLE, where the averaged periodic gain-loss modulation is incorporated into the linear gain-loss term δ . Here, two examples, where such averaging does not apply completely, are considered. These are $\Omega = 2\Omega_0$ and $\Omega = 3\Omega_0$. For these two cases, both the soliton separation and the phase difference are plotted against the modulation amplitude, as shown in Fig. 6.15.

When increasing the amplitude of the modulation, bifurcation and hysteresis phenomena, which are very similar to those of Fig. 6.9, are observed. Such similarity occurs in each case, except for the specific values of α where the bifurcation occurs. Changes in soliton separation are relatively small – much less than one percent of the total separation. At the same time, the phase difference between the solitons changes from $\pi/2$ to π . Thus, the phase difference should be the main parameter to monitor in experiments when observing this type of bifurcation. By comparing

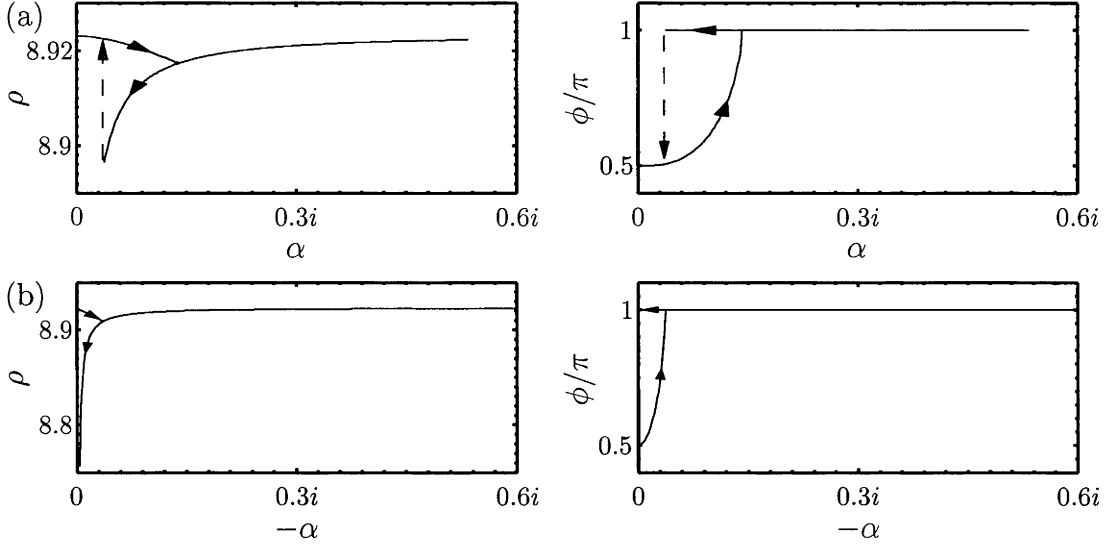


Figure 6.15: Soliton pair separation ρ and the phase difference between the pair ϕ versus the modulation depth α for (a) $\Omega = 2\Omega_0$ and (b) $\Omega = 3\Omega_0$.

Fig. 6.15 and Fig. 6.9, one can conclude that soliton pairs are more robust against changes in the gain-loss parameters than in the case of phase modulation.

6.3.3 Long Periods of Modulation

An imaginary amplitude of modulation, α , means that the gain-loss balance is modulated along the t axis, so it represents an addition to the linear loss term $i\delta\psi$ and can shift an area that originally had loss into gain. Thus, with the applied periodic term, the conditions for generating solitons change with t . Solitons may become unstable at certain points in time across the modulation period, and some preferable positions for solitons along the t axis may be created. These are called ‘equilibrium positions’. They are defined either by the overall balance between gain and loss or by the gradient of gain-loss across the period.

The equilibrium positions can be identified in numerical simulations simply by propagating the solitons. Whatever positions they had initially, single solitons or pairs of them will eventually move to an equilibrium position. For the chosen values of the system parameters, the equilibrium positions of solitons found in that way are shown in Fig. 6.16. It can be seen that these equilibrium positions vary with the amplitude of modulation α . The cases of $\alpha = 0.1i$, $\alpha = 0.2i$ and $\alpha = 0.3i$ are presented for $\Omega = (1/8)\Omega_0$ in Fig. 6.16(a), Fig. 6.16(b) and Fig. 6.16(c), respectively.

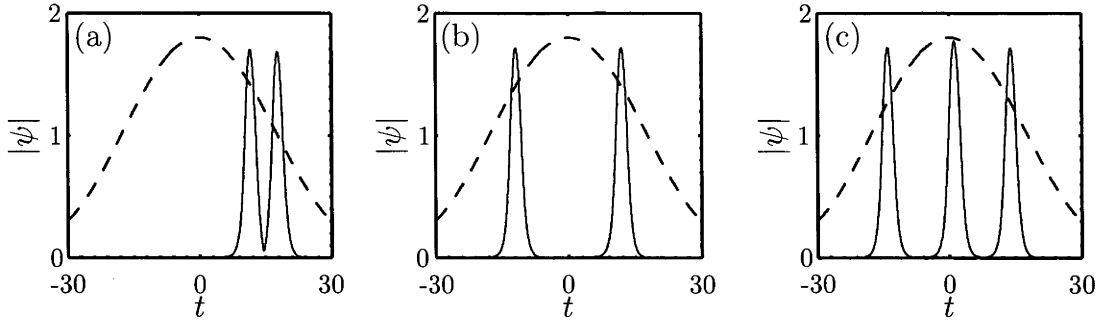


Figure 6.16: Equilibrium positions of the soliton pairs and single solitons shown by solid lines when (a) $\alpha = 0.1i$, (b) $\alpha = 0.2i$ and (c) $\alpha = 0.3i$. Dashed curve shows the periodic linear gain-loss function with $\Omega = (1/8)\Omega_0$. In (c), an additional soliton appears in the middle due to the instability of the background. Its position is not fixed and may vary along t during evolution.

When the amplitude of the modulation is $\alpha = 0.1i$, which is smaller than the linear loss term $\delta = 0.2$, the modulation does not destroy the soliton pair. Starting from any point within the period of the modulation, the pair moves to one of the equilibrium positions shown in Fig. 6.16(a). It may be located either on the negative or positive slope of the potential. Only one of them is shown on the plot.

An example of soliton pair evolution in this case is shown in Fig. 6.17(a). In this example, the pair is initially located at $t = 0$, where the linear loss is minimal. The pair moves to the right because of the specified positive sign of the phase difference $\pi/2$ between the solitons in the pair. If the sign is negative, the pair moves to the left. In a short distance, the pair arrives at an equilibrium position and stays there forever. In this simulation, the pair continues to oscillate around the equilibrium state, approaching the limit-cycle. For other sets of parameters, the limit-cycle may be transformed into a fixed point.

For a higher value of the amplitude $\alpha = 0.2i$, the gain is exactly balanced by the loss at $t = 0$. At other values of t , the linear loss prevails. The bound state does not exist and the pair splits into two separate solitons. Two equilibrium states of solitons for this case are shown in Fig. 6.16(b). They are located symmetrically on the positive and negative slopes of the gain-loss curve. As the background for the dissipative terms stays negative, noise does not grow, and single solitons are stable across the whole periodic potential. Equilibrium positions are shifted closer to the center $t = 0$ in comparison to the case with $\alpha = 0.1i$. The soliton pair becomes

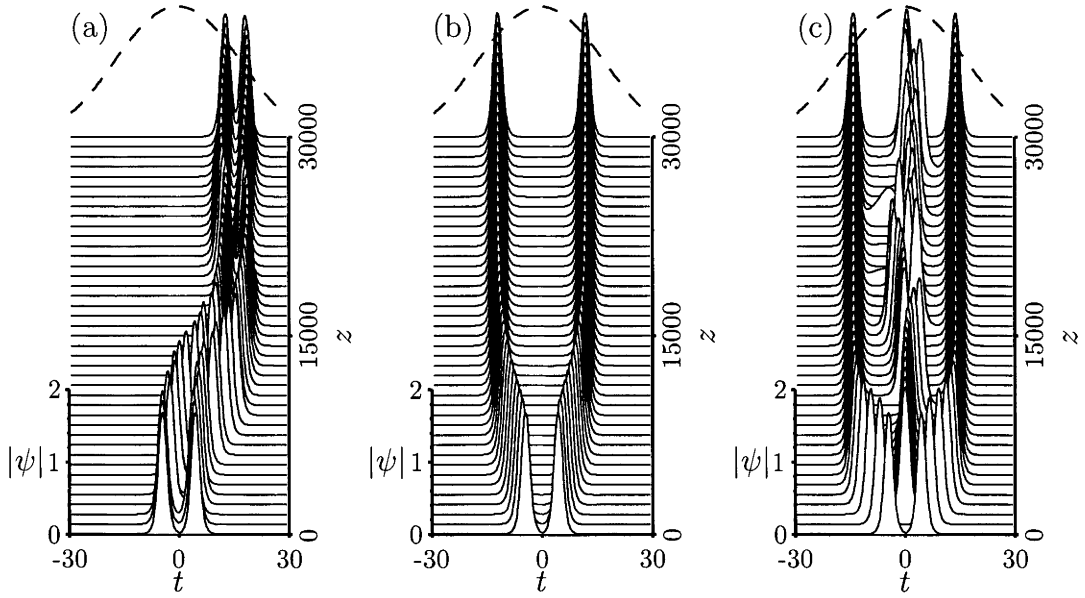


Figure 6.17: Soliton pair evolution under the effect of the gain-loss modulation with $\Omega = (1/8)\Omega_0$ (dashed line). (a) $\alpha = 0.1i$; the pair remains bound and occupies the equilibrium position as a bound state. (b) $\alpha = 0.2i$; the pair splits and each soliton occupies the equilibrium position. (c) $\alpha = 0.3i$; the pair splits and each soliton occupies the equilibrium position in the periodic potential well. The background between the two solitons is unstable and it creates additional chaotically moving solitons.

unstable and splits into two independent solitons right from the beginning of its evolution. The simulation showing this behavior is displayed in Fig. 6.17(b). After splitting, each soliton moves to an equilibrium position which remains for the rest of the simulation.

An interesting case occurs when the modulation amplitude $\alpha = 0.3i$. As this value is higher than the linear loss term $\delta = 0.2$, there are two points at the modulation curve that cancel gain-loss in the linear approximation. These are approximately the equilibrium states of the solitons. They are shown in Fig. 6.16(c). The interval between the equilibrium positions is unstable, since the linear gain is positive here. Thus, additional solitons may be created due to the spontaneous amplification of noise. One of them is shown in Fig. 6.16(c). The simulation starts from a soliton pair and shows it splitting and then the shift of each soliton to its equilibrium position. A third soliton is created between them. It moves chaotically, due to the instability of the background. For some z , two solitons may exist in the

region of instability simultaneously.

6.4 Summary

In this chapter, the influence of an external periodic potential on a pair of dissipative solitons has been studied. The investigations were carried out for two different methods of modulation, *i.e.* phase modulation and gain-loss modulation.

In the case of phase modulation, with the modulation periods being comparable with the initial pair separation, the pair separation approaches $2\pi/\Omega$ and the phase difference between the pair becomes either 0 or $\pm\pi$, depending upon the modulation frequency as the modulation depth is increased. These transitions from one type of soliton pair to another comprise hysteresis loops. For a very small value of the modulation depth, the periodic potential is not strong enough to confine the moving pairs. This causes periodic acceleration and deceleration of the soliton pairs when moving across the potential. The hysteresis loops are also present for higher modulation frequency values, where the periods are a fraction of the initial separation. For long periods of modulation, the pairs are compressed and are forced closer to each other when the modulation depth is increased.

When the gain-loss is modulated, solitons split and end up finding a position at one of the two equilibrium states of the periodic potential. The complicated dynamics of this process has been illustrated by a few numerical examples.

These results may become useful in experimental studies with lasers that have simultaneously passive and active mode-locking mechanisms. One of the possibilities found in this work has been confirmed in the experimental results of Hsiang *et al.* [131]. Other cases considered here are also worthy of experimental testing.

Chapter 7

Conclusions

In this thesis, a detailed study on dissipative solitons of the complex Ginzburg-Landau equation (CGLE) has been presented. The existence and characteristics of dissipative solitons in the context of nonlinear dynamics have been discussed and presented in the first three chapters, and their applications to optical pulse generation have been considered in the later chapters. From these investigations, various new aspects of dissipative solitons have been discovered and some important results have been obtained. The important properties and features of dissipative solitons found from this study are listed below.

- Dissipative solitons are localized structures that exist for an extended space or period of time, due to the mechanism of self-organization, in a system that has interactions with its surroundings as well as within itself, such as the CGLE [7]. A soliton solution of the CGLE is the result of a balance between gain and the loss, and also a balance between the dispersion and the nonlinearity. A system modeled by the CGLE admits various types of solitons with interesting behaviors; these cannot be observed in conservative systems. Solitons of the CGLE include stationary, pulsating, creeping and exploding solitons [41]. Their existence, however, requires a careful selection of the parameters which define the system. Each branch of soliton solutions exists only over a certain range of these parameters.
- There are specific ranges of system parameter values where solitons in dissipative systems are transformed into fronts. This type of transformation in the CGLE has been studied numerically by varying a single system parameter,

viz. the nonlinear gain coefficient ϵ , around the region of the transition. The study has revealed that this transformation is not a simple transition, and indeed it possesses a very rich bifurcation structure. In this region, there is a sequence of period-doubling bifurcations for pulsating solitons, as well as another sequence of period-doubling bifurcations for creeping solitons. Both of these branches of solutions reach chaos at the end of the sequence. There are also two symmetry-breaking bifurcations. In one of them, a creeping soliton is transformed into a pulsating soliton in a sub-critical symmetry breaking, whereas in the other one, a super-critical symmetry-breaking bifurcation is observed when a pulsating soliton is transformed into a creeping soliton. Also, creeping solitons with multiple frequency dynamics have been observed in this region. It has been found that there are two branches of these multi-frequency creeping solitons with zig-zag motions co-existing in a certain range of the system parameters. The study has shown that a significant part of this region of transition has bistability and even tristability.

- There are no analytical approaches that can exactly describe the bifurcations and the region of existence of all the solitons found above. However, approximate solutions of these are found by constructing simplified models from the CGLE. The models are constructed using the method of moments and a suitable trial function for the particular type of soliton solution that is being modeled. This approach reduces the infinite dimensional partial differential equation to a set of ordinary differential equations, and allows one to obtain the evolution of a few soliton characteristics over time. Such a technique has been applied to both pulsating solitons and creeping solitons. Some details of the solutions of the reduced models and the exact model do not match, due to the dramatic simplification of the system. However, the solutions of the approximate models show good qualitative agreement with the exact models, and the region of existence for each soliton is well-approximated in the reduced models.
- A region of the CGLE parameter space where ultra-high energy solitons exist has been found numerically [66]. These pulses were initially found in the normal dispersion regime, which in fact agrees with many experimental observations made during the generation of high-energy pulses. The energy increases

to approach infinity at a particular set of system parameters, and the term ‘dissipative soliton resonance’ (DSR) has been coined for this phenomenon. By constructing a reduced model using the method of moments, an expression for the approximate location of the resonance in the complete parameter space has been found. The expression suggested that the high-energy pulses could also be obtained in the anomalous dispersion regime. The existence of the resonance in the anomalous dispersion has been confirmed numerically; it requires the quintic refractive index ν to be negative. A study on the effect of the sign of ν on the resonance has shown that the phenomenon is more pronounced when ν is negative. The robustness of DSR against model perturbations has been investigated, and it has been shown that the phenomenon still exists when dispersion management is introduced into the cavity design. These results can be useful in constructing passively mode-locked laser systems that generate record-high energy pulses out of single oscillators, without the need for additional amplifiers. The high-energy pulses generated near the resonance are very wide and have almost linear chirp across the central part of the pulse. For the generation of high-energy ultra-short pulses, a dispersive medium can be used at the output of the laser oscillator operating near the resonance, in order to compensate the linear chirp and effectively compress the pulses.

- The influence of a periodic external modulation on a pair of dissipative solitons has been carried out. Both phase modulation and gain-loss modulation of the cavity, as well as various ranges of modulation frequencies have been considered in this study. The study of the effect of phase modulation showed that, as the modulation depth is increased, the pair tends to stay at two adjacent minima of the periodic modulation, and the phase difference between the pair becomes either 0 or $\pm\pi$, depending on the modulation frequency relative to the initial pair separation. At a lower modulation depth, there is a transition from one type of soliton pair to another, and the bifurcation consists of a hysteresis loop. For an even smaller value of the modulation depth, the periodic potential is not strong enough to confine the moving pair. When the period of the external modulation is large compared with the initial pair separation, the pairs are compressed and move closer to each other as the modulation amplitude is increased. This agrees with the experimental

work of Hsiang *et al.* [131]. In the case of gain-loss modulation, the pair splits and ends up finding a new position at one of two equilibrium states of the periodic potential. Numerical examples have shown that, during this transition, the system exhibits complicated dynamics with hysteresis loops. Exploiting the mechanism of such soliton interactions in hybrid mode-locked lasers, *i.e.* those where the pulses are generated by applying both active and passive mode-locking simultaneously, would allow for more accurate control of the frequency and stability of the generated pulse trains.

Finding qualitatively new solutions of the CGLE significantly broadens the ‘world of the Ginzburg-Landau equation’ [10]. Indeed, the new solutions can be useful in all areas of science where this universal equation may be applied.

Appendices

I Equations for Generalized Gaussian Pulse

In this appendix, the general equations for the three-dimensional dynamical system generated by the symmetric generalized Gaussian trial function given by Eq. (4.2.4) with arbitrary real constant m is provided. First, the moments are defined by

$$Q = \sqrt{m}e^m K_{\frac{1}{4}}(m)A^2w, \quad I_2 = m \left(\frac{K_{\frac{3}{4}}(m)}{K_{\frac{1}{4}}(m)} - 1 \right) Qw^2, \quad I_3 = 4icI_2, \quad (\text{I-1})$$

where $K_p(r)$ is the modified Bessel function of order p and argument r .

The dynamical system for the evolution of the soliton energy, Q , width w and quadratic chirp coefficient c are given by

$$\begin{aligned} Q_z &= Q \left(2\delta + s_1 \frac{\beta}{w^2} + s_2 c^2 w^2 \beta + s_3 \frac{Q\epsilon}{w} + s_4 \frac{Q^2 \mu}{w^2} \right), \\ w_z &= w \left(2cD + r_1 \frac{\beta}{w^2} + r_2 c^2 w^2 \beta + r_3 \frac{Q\epsilon}{w} + r_4 \frac{Q^2 \mu}{w^2} \right), \\ c_z &= \frac{1}{w^2} \left(-2c^2 w^2 D - v_1 \frac{D}{w^2} - v_2 \frac{Q}{w} - v_3 \frac{Q^2 \nu}{w^2} + v_4 c\beta \right). \end{aligned} \quad (\text{I-2})$$

The coefficients in the above equations are given in analytical form:

$$\begin{aligned} s_1 &= 1 - \frac{3K_{\frac{3}{4}}(m)}{K_{\frac{1}{4}}(m)}, & s_2 &= 8m - \frac{8mK_{\frac{3}{4}}(m)}{K_{\frac{1}{4}}(m)}, \\ s_3 &= \frac{2K_{\frac{1}{4}}(2m)}{\sqrt{m}K_{\frac{1}{4}}^2(m)}, & s_4 &= \frac{2K_{\frac{1}{4}}(3m)}{mK_{\frac{1}{4}}^3(m)}, \end{aligned} \quad (\text{I-3})$$

and

$$\begin{aligned}
r_1 &= \frac{(-4(3+m)(-1+2m)K_{\frac{1}{4}}^2(m) + mK_{\frac{1}{4}}(m)r_{1b} + mK_{\frac{3}{4}}(m)r_{1c})}{4mK_{\frac{1}{4}}(m)(K_{\frac{1}{4}}(m) - K_{\frac{3}{4}}(m)) + 4mK_{\frac{1}{4}}(m)(K_{\frac{1}{4}}(m) - K_{\frac{3}{4}}(m))}, \\
r_2 &= \frac{2 \cdot 2^{\frac{3}{4}} e^{-m} m^{\frac{1}{4}} K_{\frac{3}{4}}(m)r_{2b} + K_{\frac{1}{4}}(m)r_{2c}}{K_{\frac{1}{4}}^2(m) - K_{\frac{1}{4}}(m)K_{\frac{3}{4}}(m)}, \\
r_3 &= \frac{K_{\frac{1}{4}}(2m)K_{\frac{3}{4}}(m) - K_{\frac{1}{4}}(m)K_{\frac{3}{4}}(2m)}{\sqrt{m}K_{\frac{1}{4}}^2(m)(K_{\frac{1}{4}}(m) - K_{\frac{3}{4}}(m))}, \\
r_4 &= \frac{K_{\frac{1}{4}}(3m)K_{\frac{3}{4}}(m) - K_{\frac{1}{4}}(m)K_{\frac{3}{4}}(3m)}{mK_{\frac{1}{4}}^4(m) - mK_{\frac{1}{4}}^3(m)K_{\frac{3}{4}}(m)}, \tag{I-4}
\end{aligned}$$

where

$$\begin{aligned}
r_{1b} &= 2(5 - 28m)K_{\frac{3}{4}}(m) + 2(26m - 11)K_{\frac{5}{4}}(m) + 8mK_{\frac{7}{4}}(m), \\
r_{1c} &= 4mK_{\frac{7}{4}}(m) - 12K_{\frac{3}{4}}(m), \\
r_{2b} &= 2\sqrt{2}m\Gamma\left(\frac{5}{4}\right)H\left(\frac{5}{4}, \frac{3}{2}, 2m\right) - \Gamma\left(\frac{3}{4}\right)H\left(\frac{3}{4}, \frac{1}{2}, 2m\right), \\
r_{2c} &= 4(1 + m)K_{\frac{1}{4}}(m) - 4mK_{\frac{5}{4}}(m). \tag{I-5}
\end{aligned}$$

The function $H(i, j, k)$ in Eq. (I-5) is the hypergeometric function $({}_1F_1)$ with arguments (i, j, k) while $\Gamma(r)$ is the Euler gamma function.

Finally, the coefficients v_i are defined by

$$\begin{aligned}
v_1 &= -\frac{K_{\frac{1}{4}}(m) - 3K_{\frac{3}{4}}(m)}{4mK_{\frac{1}{4}}(m) - 4mK_{\frac{3}{4}}(m)}, \\
v_2 &= -\frac{K_{\frac{1}{4}}(2m)}{m^{\frac{3}{2}}\left(4K_{\frac{1}{4}}^2(m) - 4K_{\frac{1}{4}}(m)K_{\frac{3}{4}}(m)\right)}, \\
v_3 &= \frac{K_{\frac{1}{4}}(3m)}{3m^2K_{\frac{1}{4}}^2(m)\left(-K_{\frac{1}{4}}(m) + K_{\frac{3}{4}}(m)\right)}, \\
v_4 &= \frac{(-1 + 8m)K_{\frac{1}{4}}(m)}{m\left(K_{\frac{1}{4}}(m) - K_{\frac{3}{4}}(m)\right)} + \frac{2m\left(-9K_{\frac{3}{4}}(m) + 5K_{\frac{5}{4}}(m)\right)}{m\left(K_{\frac{1}{4}}(m) - K_{\frac{3}{4}}(m)\right)}. \tag{I-6}
\end{aligned}$$

All equations above are valid for arbitrary real positive constant m . For any m ,

the coefficients can be calculated and their arithmetic values can be found. This has been done for $m = 0.25$ in Eq. (4.2.5) and Eq. (4.2.6).

II Resonance Curve

Near the resonance curve, $Q \rightarrow \infty$, $w \rightarrow \infty$ and $c \rightarrow 0$. The terms of order c and w^{-2} in Eq. (I-2) are close to zero and can be neglected. The dynamical system is then simplified and takes the form:

$$\begin{aligned} Q_z &= Q \left(2\delta + s_2 c^2 w^2 \beta + s_3 \frac{Q\epsilon}{w} + s_4 \frac{Q^2 \mu}{w^2} \right), \\ w_z &= w \left(r_2 c^2 w^2 \beta + r_3 \frac{Q\epsilon}{w} + r_4 \frac{Q^2 \mu}{w^2} \right), \\ c_z &= \frac{1}{w^2} \left(-2c^2 w^2 D - v_2 \frac{Q}{w} - v_3 \frac{Q^2 \nu}{w^2} \right), \end{aligned} \quad (\text{II-1})$$

where the coefficients s_i , r_i and v_i are given by Eq. (I-3), Eq. (I-4) and Eq. (I-6) respectively.

The fixed points near the resonance curve are defined by the zeros of the right-hand-side of Eq. (II-1):

$$\begin{aligned} \left(2\delta + s_2 c^2 w^2 \beta + s_3 \frac{Q\epsilon}{w} + s_4 \frac{Q^2 \mu}{w^2} \right) &= 0, \\ \left(r_2 c^2 w^2 \beta + r_3 \frac{Q\epsilon}{w} + r_4 \frac{Q^2 \mu}{w^2} \right) &= 0, \\ \left(2c^2 w^2 D + v_2 \frac{Q}{w} + v_3 \frac{Q^2 \nu}{w^2} \right) &= 0. \end{aligned} \quad (\text{II-2})$$

Near the resonance curve, the values Q/w and cw approach constants. Solving the first two equations in Eq. (II-2) for Q/w and cw , one can obtain:

$$\frac{Q}{w} = \frac{k}{2(r_2 s_4 - r_4 s_2)\mu}, \quad cw = -\sqrt{\frac{k(r_4 s_3 - r_3 s_4)\epsilon + 4r_4(r_2 s_4 - r_4 s_2)\delta\mu}{2(r_4 s_2 - r_2 s_4)^2 \beta \mu}}, \quad (\text{II-3})$$

where

$$k = \epsilon(r_3 s_2 - r_2 s_3) + \sqrt{(r_3 s_2 - r_2 s_3)^2 \epsilon^2 + 8r_2(r_4 s_2 - r_2 s_4)\delta\mu}. \quad (\text{II-4})$$

Now, by substituting Eq. (II-3) into the third equation in Eq. (II-2), one can obtain the expression for the resonance curve in terms of the system parameters:

$$D = \frac{k\beta (2(r_2s_4 - r_4s_2)v_2\mu + kv_3\nu)}{4\mu(k(r_3s_4 - r_4s_3)\epsilon + 4r_4(r_4s_2 - r_2s_4)\delta\mu)}. \quad (\text{II-5})$$

The term containing 2δ in Eq. (II-2) is much smaller than the other terms for the set of parameters that we use in numerical simulations. A much simpler equation for the resonance curve is obtained by neglecting this term. Following the same procedures as above, we find

$$\frac{Q}{w} = \frac{(r_2s_3 - r_3s_2)\epsilon}{(r_4s_2 - r_2s_4)\mu}, \quad cw = -\sqrt{\frac{(r_3s_2 - r_2s_3)(r_4s_3 - r_3s_4)\epsilon^2}{(r_4s_2 - r_2s_4)^2\beta\mu}}. \quad (\text{II-6})$$

Then the expression for the resonance curve becomes:

$$D = \frac{(r_4s_2 - r_2s_4)v_2\beta\mu + (r_2s_3 - r_3s_2)v_3\beta\epsilon\nu}{2(r_4s_3 - r_3s_4)\epsilon\mu}. \quad (\text{II-7})$$

III Finite Spectral Width Near the Resonance

For a pulse with constant amplitude A , and width w , one can readily estimate the width of the spectrum for the basic model upon approaching the high energy line where the width becomes infinite. The Fourier transform is defined as

$$F(\omega) = \frac{A}{\sqrt{2\pi}} \int_{-w/2}^{w/2} \exp(ict^2 + i\omega t) dt, \quad (\text{III-1})$$

with chirp $c = -r/w$ and constant r , as explained earlier. Hence

$$|F(\omega)| = \frac{A}{2} \sqrt{\frac{w}{2r}} |\text{erf}(g_-) + \text{erf}(g_+)|, \quad (\text{III-2})$$

where the arguments of the error functions erf , are given by:

$$g_{\pm} = \frac{1}{2}(-1)^{\frac{1}{4}} \sqrt{\frac{w}{r}} (r \pm \omega). \quad (\text{III-3})$$

Therefore, in taking the limit as $w \rightarrow \infty$ leads to

$$\begin{aligned} |F(\omega)| &= A \sqrt{\frac{w}{2r}} & \text{if } \omega^2 < r^2 & \text{ and} \\ |F(\omega)| &= 0 & \text{if } \omega^2 > r^2. \end{aligned} \quad (\text{III-4})$$

The total energy ($\approx \int_{-r}^r |F(\omega)|^2 d\omega$), is then clearly proportional to the width, w , while the spectral width is fixed at $2r$. Thus, the presence of chirp means that the spectral width near the high energy is not very narrow, whereas one would expect a very narrow spectrum for a transform limited pulse. Even for cases with gaussian amplitude, which are thus not close to the resonance, the spectrum remains approximately of constant width as the pulse width increases.

IV Quartic Pulse

There is another trial function that could be considered for modeling the dissipative soliton resonance phenomena. It is related to the one in the main text, but is not a limiting case of it. This trial function involves only the quartic term:

$$\psi(t, z) = A \exp\left(-\frac{t^4}{w^4}\right) \exp(ict^2). \quad (\text{IV-1})$$

Eq. (IV-1) gives a slightly better approximation of the soliton characteristics when the soliton energy reaches extremely high values. However, the differences are not very dramatic. For the sake of completeness, the main equations are given in this appendix.

The moments for the quartic trial function, *i.e.* Eq. (IV-1), are given by the following expressions:

$$Q = 2^{\frac{3}{4}} \Gamma\left(\frac{5}{4}\right) A^2 w, \quad I_2 = \frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{2} \Gamma\left(\frac{1}{4}\right)} Q w^2, \quad I_3 = 4icI_2. \quad (\text{IV-2})$$

The dynamical system is described by the same three coupled ordinary differential equations of Eq. (I-2) except for the coefficients that take the following

particular forms:

$$s_1 = -\frac{12\pi}{\Gamma^2\left(\frac{1}{4}\right)}, \quad s_2 = -\frac{8\pi}{\Gamma^2\left(\frac{1}{4}\right)}, \quad s_3 = \frac{4}{\Gamma\left(\frac{1}{4}\right)}, \quad s_4 = \frac{8\sqrt{2}}{3^{\frac{1}{4}}\Gamma^2\left(\frac{1}{4}\right)}, \quad (\text{IV-3})$$

while the coefficients r_i are given by:

$$\begin{aligned} r_1 &= \frac{6\pi}{\Gamma^2\left(\frac{1}{4}\right)} - \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\pi}, & r_2 &= \frac{4\pi}{\Gamma^2\left(\frac{1}{4}\right)} - \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi}, \\ r_3 &= \frac{-2 + \sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}, & r_4 &= -\frac{8\sqrt{-3 + 2\sqrt{3}}}{3\Gamma^2\left(\frac{1}{4}\right)}, \end{aligned} \quad (\text{IV-4})$$

and for v_i :

$$v_1 = -3, \quad v_2 = \frac{\Gamma\left(\frac{1}{4}\right)}{2\pi}, \quad v_3 = \frac{4\sqrt{2}}{3 \cdot 3^{\frac{1}{4}}\pi}, \quad v_4 = -\frac{4\Gamma^2\left(\frac{1}{4}\right)}{\pi}. \quad (\text{IV-5})$$

The expression for the resonance curve is the same as that in the previous section with the coefficients given by Eq. (IV-3), Eq. (IV-4) and Eq. (IV-5).

On neglecting 2δ term in the dynamical system, one can find, for the two constants:

$$\frac{Q}{w} = -\frac{1.155\epsilon}{\mu}(> 0), \quad cw = -0.269\frac{\epsilon}{\sqrt{-\beta\mu}}(< 0). \quad (\text{IV-6})$$

In this case, the resonance curve takes the form

$$D = \left(\frac{7.611\nu}{\mu} - \frac{6.208}{\epsilon} \right) \beta, \quad (\text{IV-7})$$

which is slightly closer to the exact curve than Eq. (5.2.7). This is because Eq. (IV-1) provides fairly rectangular pulses which more precisely match the actual pulses near the resonance.

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List of Publications

Journal Articles

1. W. Chang, A. Ankiewicz and N. Akhmediev, “Creeping solitons of the complex Ginzburg-Landau equation with a low-dimensional dynamical system model,” *Physics Letters A* **362**, 31 – 36 (2007).
2. W. Chang, A. Ankiewicz, J. M. Soto-Crespo and N. Akhmediev, “Creeping solitons in dissipative systems and their bifurcations,” *Physical Review E* **76**, 016607 (2007).
3. W. Chang, A. Ankiewicz, J. M. Soto-Crespo and N. Akhmediev, “Dissipative soliton resonances,” *Physical Review A* **78**, 023830 (2008).
4. W. Chang, A. Ankiewicz, J. M. Soto-Crespo and N. Akhmediev, “Dissipative soliton resonances in laser models with parameter management,” *Journal of Optical Society of America B* **25**, 1972–1977 (2008).
5. W. Chang, J. M. Soto-Crespo, A. Ankiewicz and N. Akhmediev, “Dissipative soliton resonances in anomalous dispersion regime,” *Physical Review A* **79**, 033840 (2009).
6. W. Chang, N. Akhmediev and S. Wabnitz, “Effect of external periodic potential on pairs of dissipative solitons,” *Physical Review A* **80**, 013815 (2009).
7. W. Chang, N. Akhmediev, S. Wabnitz and M. Taki, “Influence of external phase and gain-loss modulation on bound solitons in laser systems,” *Journal of Optical Society of America B* **26**, 2204–2210 (2009).

Conference Contributions

8. W. Chang, A. Ankiewicz and N. Akhmediev, "Creeping solitons in dissipative systems," Australian Conference on Optical Fibre Technology, Melbourne, Australia, 10–13 July 2006.
9. W. Chang, A. Ankiewicz, N. Akhmediev and J. M. Soto-Crespo, "Self-propelled solitons in dissipative systems," Nonlinear Photonics, Quebec, Canada, 2–6 September 2007.
10. W. Chang, J. M. Soto-Crespo, A. Ankiewicz and N. Akhmediev, "Multiplicity of soliton transformations in the vicinity of the boundaries of their existence," SPIE Microelectronics, MEMS and Nanotechnology, Canberra, Australia, 4–7 December 2007.
11. W. Chang, A. Ankiewicz, J. M. Soto-Crespo and N. Akhmediev, "Dissipative soliton resonances in laser models," Australian Institute of Physics Congress, Adelaide, Australia, 30 November – 5 December 2008.
12. W. Chang, N. Akhmediev and S. Wabnitz, "Dissipative soliton bound states under the influence of external phase modulation in passively mode-locked lasers," International Workshop on Dissipative Solitons, Adelaide, Australia, 29 November – 3 December 2009.