Calabi's extremal metrics on toric surfaces and Abreu's equation

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis is concerned with the study of the existence of extremal metrics and Abreu's equation reduced from the scalar curvature equation on toric Kähler manifolds.

Part I contains an introduction of canonical metrics in Kähler geometry. In Chapter 1, we recall the definition of Calabi's extremal metrics and the famous Yau-Tian-Donaldson conjecture which relates the existence of extremal metrics to stabilities in sense of geometric invariant theory. In Chapter 2, we review Donaldson's reduction of this problem on toric manifolds.

In Part II, we present new results in the case of toric surfaces. Based on Arrezo-Pacard-Singer's work, we prove in Chaper 3 that on every toric surface, there exists a Kähler class which admits extremal metrics. We also give examples of Kähler classes on a toric surface which admit no extremal metrics. In Chapter 4, we prove that among all toric surfaces with 5 or 6 T^2-fixed points, $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is the only one which admits Kähler classes with vanishing Futaki invariant. We also prove these Kähler classes are K-stable. Therefore by Donaldson's theorem, there exist constant scalar curvature metrics in these classes.

In Part III, we study Abreu's equation. The Bernstein theorem for Abreu's equation in dimension 2 is proved in Chapter 5. In Chaper 6, we solve a boundary value problem for Abreu's equation. Similar results for the affine maximal surface equation were proved by Trudinger and Wang. But Abreu's equation is not affine invariant, new a priori estimates are needed for these results.

Contents

Part I

Introduction

Chapter 1

Extremal metrics

The existence of canonical metrics on Kähler manifolds is one of the central problems in complex geometry. Calabi [Cal3] proposed to study the existence of extremal metrics in a given Kähler class on a Kähler manifold. In this chapter, we first recall some elementary knowledge about Kähler geometry and then introduce the notion of extremal metric.

1.1 Kähler geometry

In this section, we briefly recall some terminology and basic properties in Kähler geometry.

1.1.1 Hermitian and Kähler metrics

Let (M, J) be a compact $2n$-dimensional complex manifold, where n is the complex dimension and J is the complex structure.

Definition 1.1. A *Hermitian metric* on M is a Riemannian metric g which satisfies

$$g(JX, JY) = g(X, Y), \ \forall\ X, Y \in T_xM, \ x \in M.$$

With the Hermitian metric, we can define a 2-form ω_g on M by

$$\omega_g(X, Y) = -g(X, JY).$$

We call it the *Kähler form* of g.

Definition 1.2. A *Kähler metric* on M is a Hermitian metric g such that the associated Kähler form is closed, i.e., $d\omega_g = 0$.

Let ∇ be the Levi-Civita connection of g, then the Kähler condition, i.e., $d\omega_g = 0$ is equivalent to that the complex structure J is invariant under parallel transformation, i.e., $\nabla J = 0$.

Now, since we have an integrable complex structure, we can choose local complex coordinates $\{z^1, ..., z^n\}$ such that $z^i = x^i + \sqrt{-1}y^i$. In this coordinate system,

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}.$$

The complexified tangent bundle $TM \otimes \mathbb{C}$ is spanned by

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

We can extend the metric g $\mathbb{C}$-linearly to $TM \otimes \mathbb{C}$. One can verify that

$$g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = g(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) = 0$$

and

$$g_{i\bar{j}} = g_{\bar{j}i} = \overline{g_{\bar{i}j}}.$$

Hence, we can define a Hermitian inner product on the holomorphic tangent bundle $T^{(1,0)}M$ by

$$h = g_{i\bar{j}} dz^i \otimes d\bar{z}^j.$$

In fact, the real part of h is the Riemannian metric g and the imaginary part is the associated Kähler form. The Kähler form can be rewritten as

$$\omega_g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

in the complex coordinates. It is clear that on a Kähler manifold, a Kähler metric is uniquely determined by its Kähler form, so we usually denote a Kähler metric g by its Kähler form ω_g. Since ω_g is closed, it determines a cohomology class $[\omega_g] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$, which is called the Kähler class of ω_g. The set of Kähler forms that represent the class $[\omega_g]$ can be expressed by the set of Kähler potentials as follows,

$$\mathcal{M}([\omega_g]) = \{\phi \in C^\infty(M, \mathbb{R}) \mid \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi > 0\}.$$

Note that the volume element of ω_ϕ can be represented as $\frac{\omega_\phi^n}{n!}(= \frac{\omega_\phi \wedge \ldots \wedge \omega_\phi}{n!})$ and the volume $V(M) = \int_M \frac{\omega_\phi^n}{n!}$ depends only on the Kähler class.

1.1.2 Connection and curvatures

On a Kähler manifold, we can naturally extend the Levi-Civita connection ∇ and the Riemannian curvature tensor $\mathbb{C}$-linearly to the complexified tangent bundle $TM \otimes \mathbb{C}$. We denote by $\Gamma^{\gamma}_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ the connection and curvature coefficients in the local complex coordinates $\{z^1, ..., z^n\}$. Here indices α, β, γ, δ could be holomorphic or anti-holomorphic. When we use lower indices we indicate anti-holomorphic indices with over-bars and holomorphic indices without. For example, Γ^k_{ij} and $\Gamma^{\bar{k}}_{ij}$ are defined by

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \Gamma^k_{ij}\frac{\partial}{\partial z^k} + \Gamma^{\bar{k}}_{ij}\frac{\partial}{\partial \bar{z}^k}.$$

For a Kähler metric $g = g_{i\bar{j}}$, the Kähler condition implies that

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \ \forall\ i, j, k.$$

We have $\Gamma^{\gamma}_{\alpha\beta} = 0$ unless α, β, γ are all holomorphic or anti-holomorphic and

$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}, \quad \text{and } \Gamma^k_{ij} = g^{k\bar{l}}\frac{\partial g_{i\bar{j}}}{\partial z_l}.$$

For the curvature tensor, the only non-zero terms are $R_{i\bar{j}k\bar{l}}$. Moreover,

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}}\frac{\partial g_{p\bar{j}}}{\partial z_i}\frac{\partial g_{k\bar{q}}}{\partial \bar{z}_l}.$$

By taking trace, the Ricci tensor in local coordinates can be written as

$$R_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}} = -(\log\det(g_{k\bar{l}}))_{i\bar{j}}.$$

Hence, we can also associate it with a $(1,1)$-form called *Ricci form* as follows,

$$\begin{aligned} Ric(\omega_g) &= -\frac{\sqrt{-1}}{2}(\log\det(g_{k\bar{l}}))_{i\bar{j}}dz^i \wedge d\bar{z}^j \\ &= -\frac{\sqrt{-1}}{2}\partial\bar{\partial}(\log\det(g_{k\bar{l}})). \end{aligned}$$

It is well known that the cohomology class of the $(1,1)$-form $\frac{Ric(\omega_g)}{2\pi}$ is the first Chern class $c_1(M)$ which depends only on the complex structure of M. As we use Kähler forms to represent the Kähler metrics, we also use Ricci forms to represent Ricci curvatures.

Finally, by taking trace of Ricci tensor, we get the scalar curvature

$$S(\omega_g) = -g^{i\bar{j}}(\log\det(g_{k\bar{l}}))_{i\bar{j}}.$$

Note that when fixing a background metric ω_g and letting ω_ϕ varies in the Kähler class $[\omega_g]$, the scalar curvature $S(\omega_\phi)$ is a fourth order operator on the space of Kähler potentials.

Another fact on Kähler manifold is that when g is Kähler, for any given smooth function f, in the local coordinates the second order covariant derivative $\nabla_i \nabla_{\bar{j}} f$ equals $\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}$. Therefore,

$$\triangle_g f = g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

Here $\triangle_g$ is half of the Laplace-Beltrami operator of the metric g.

1.2 Extremal metrics

In this section we introduce Calabi's extremal metric as well as its energy functionals.

Definition 1.3 ([Cal3]). For a Kähler metric ω, *Calabi's energy* is given by

$$\mathcal{C}(\omega) = \frac{1}{V(M)} \int_M S(\omega)^2 \frac{\omega_\phi^n}{n!}. \tag{1.1}$$

Calabi [Cal3,4] proposed to study critical points of the functional in a fixed Kähler class $[\omega_g]$.

Definition 1.4 ([Cal3]). A Kähler metric in $[\omega_g]$ is called *extremal* if it is a critical point of Calabi's energy.

According to [Cal3], by computing the Euler-Lagrange equation of Calabi's energy, ω_ϕ is extremal if and only if

$$\nabla_{\bar{i}} \nabla_{\bar{j}} S(\omega_\phi) = 0, \ i, j = 1, ..., n, \tag{1.2}$$

i. e., the complex-valued gradient vector

$$X = g'^{i\bar{j}} (S(\omega_\phi))_{\bar{j}} \frac{\partial}{\partial z_i} \tag{1.3}$$

is holomorphic, where $(g'^{i\bar{j}})$ is the inverse of Hermitian matrix $(g_{i\bar{j}} + \phi_{i\bar{j}})$. In particular, if $X = 0$, the extremal metric is a constant scalar curvature (CSC) metric. Hence, if M admits no holomorphic vector fields, any extremal metric on M is a constant scalar curvature metric.

The existence of extremal metric is a rather difficult problem because (1.2) is a 6th order nolinear PDE. However, the *extremal holomorphic vector field* X given by (1.3) can be uniquely determined by the Kähler class $[\omega_g]$ no matter there exits an extremal Kähler metric on M or not. To explain it, we first define a holomorphic invariant introduced by Futaki [Fut] as an analytic invariant on a Fano manifold which arises in the study of Kähler-Einstein metrics. This invariant is formulated as a character of the Lie algebra of holomorphic vector fields and can be defined in a general Kähler class on any Kähler manifold [Cal4].

Let $\eta(M)$ be the space of all holomorphic vector fields on M. For any Kähler class $[\omega]$, we pick a Kähler metric $\omega_g \in [\omega]$. It is known that there exists a smooth function h_g on M such that,

$$S(\omega_g) - \bar{S} = \triangle_g h_g,$$

where

$$\bar{S} = \frac{1}{V(M)} \int_M S(\omega_g) \frac{\omega_g^n}{n!}$$

is the average of the scalar curvature depending only on the Kähler class. The *Futaki invariant* is defined by

$$\mathcal{F}([\omega_g], v) = \int_M v(h_g) \frac{\omega_g^n}{n!}, \quad \forall\ v \in \eta(M).$$

It was proved in [Cal4] that this invariant is independent of the choice of ω_g in $[\omega]$. For convenience, we usually write the Futaki invariant by $\mathcal{F}(\cdot)$ when the Kähler class is fixed.

Now we determine the extremal vector field following [FM]. Let $\mathrm{Aut}^0(M)$ be the identity component of the holomorphisms group of M and $\mathrm{Aut}_r(M)$ be the reductive part of $\mathrm{Aut}^0(M)$. Then $\mathrm{Aut}_r(M)$ is the complexification of a maximal compact subgroup K of $\mathrm{Aut}_r(M)$. We denote the Lie algebra of $\mathrm{Aut}_r(M)$ by $\eta_r(M)$, which induces a set of holomorphic vector fields on M. Let $v \in \eta_r(M)$ so that its imaginary part generates a one-parameter compact subgroup of K. Then if the Kähler form ω_g is K-invariant, that is, invariant under the group K, there exists a unique real-valued function θ_v (called normalized potential of v) such that

$$i_v\omega_g = \sqrt{-1}\bar{\partial}\theta_v(\omega_g), \quad \text{and} \quad \int_M \theta_v(\omega_g)\frac{\omega_g^n}{n!} = 0. \tag{1.4}$$

For simplicity, we denote the set of such potentials θ_v by Ξ_{ω_g}. Then the Futaki invariant on η_r can be written as

$$\mathcal{F}(v) = -\int_M \theta_v(\omega_g)(S(\omega_g) - \bar{S})\frac{\omega_g^n}{n!}. \tag{1.5}$$

According to [FM], an extremal vector field X in the Kähler class $[\omega_g]$ is defined by a gradient holomorphic vector field in $\eta_r(M)$,

$$g^{i\bar{j}}(\mathrm{proj}(S(\omega_g)))_{\bar{j}}\frac{\partial}{\partial z_i},$$

where $\mathrm{proj}(S(\omega_g))$ is the L^2-inner projection of the scalar curvature of ω_g to Ξ_{ω_g}. Futaki and Mabuchi proved that the definition of X is independent of the choice of K-invariant metrics in $[\omega_g]$. In fact X is uniquely determined by the Futaki invariant $\mathcal{F}(\cdot)$ as follows,

$$\mathcal{F}(v) = -\int_M \theta_v(\omega_g)\theta_X(\omega_g)\frac{\omega_g^n}{n!}, \quad \forall\, v \in \eta_r(M). \tag{1.6}$$

Since $\mathcal{F}(\cdot)$ is a character, the above relation is equivalent to

$$\mathcal{F}(v) = -\int_M \theta_v(\omega_g)\theta_X(\omega_g)\frac{\omega_g^n}{n!}, \quad \forall\, v \in \eta_c(M), \tag{1.7}$$

where $\eta_c(M)$ is the center of $\eta_r(M)$. This shows that X belongs to $\eta_c(M)$. In particular, ω_g can be replaced by a K_c-invariant Kähler form, where K_c is the Abelian compact subgroup of $\mathrm{Aut}_r(M)$ with the Lie algebra $\eta_c(M)$.

By the above discussion, one sees that a Kähler metric ω_ϕ in the Kähler class $[\omega_g]$ is extremal iff the potential ϕ satisfies a fourth-order equation with respect to the Kähler potential function ϕ,

$$S(\omega_\phi) = \bar{S} + \theta_X(\omega_\phi), \tag{1.8}$$

where $\bar{S}$ is the average of the scalar curvature of ω_g and $\theta_X(\omega_\phi) = \theta_X(\omega_g) + X(\phi)$ denotes the potential of the extremal vector field X associated to the metric ω_ϕ (cf. page 208-209 in [FM]). In particular, if we choose the Kähler class to be a multiple of the first Chern class and X vanishes, then the extremal metric, if exists, is a Kähler-Einstein metric. For example, when $[\omega_g] = 2\pi c_1(M) > 0$, and both $Ric(\omega_\phi)$ and ω_ϕ represent the first Chern class, then there exists a smooth function h_ϕ, such that

$$Ric(\omega_\phi) - \omega_\phi = \frac{\sqrt{-1}}{2}\partial\bar{\partial}h_\phi.$$

By taking trace,

$$S(\omega_\phi) - \bar{S} = \triangle h_\phi.$$

Hence (1.8) implies $\triangle h_\phi = 0$. Since the manifold is compact, h_ϕ must be a constant, which immediately implies $Ric(\omega_\phi) = \omega_\phi$. In this case, the equation can be further reduced to a second order complex Monge-Ampère equation as follows. Let h_g be the smooth function satisfying

$$Ric(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2}\partial\bar{\partial}h_g.$$

Then if ω_ϕ is Kähler-Einstein, we have

$$\begin{aligned}\omega_g + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi &= Ric(\omega_\phi)\\ &= -\frac{\sqrt{-1}}{2}\log\left(\frac{\omega_\phi^n}{\omega_g^n}\right) + Ric(\omega_g)\\ &= -\frac{\sqrt{-1}}{2}\log\left(\frac{\omega_\phi^n}{\omega_g^n}\right) + \omega_g + \frac{\sqrt{-1}}{2}\partial\bar{\partial}h_g.\end{aligned}$$

Hence, the equation is

$$(\omega_g + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi)^n = e^{h_g - \phi}\omega_g^n, \tag{1.9}$$

which is a complex Monge-Ampère equation. When $c_1(M) < 0$ or $= 0$, the Kähler-Einstein metric equation can also be formulated as complex Monge-Ampère equations with different right hand side terms.

In addition to Calabi's energy, there is another important energy functional concerning extremal metrics, called modified K-energy. The original K-energy was introduced by Mabuchi [Mab1] for CSC Kähler metrics. The modified K-energy is defined on $[\omega_g]$ by

$$\mu(\phi) = -\frac{1}{V(M)}\int_0^1\int_M \dot{\phi}_t[S(\omega_{\phi_t}) - \bar{S} - \theta_X(\phi_t)]\frac{\omega_{\phi_t}^n}{n!}\wedge dt, \tag{1.10}$$

where $\phi_t (0 \leq t \leq 1)$ is a path connecting 0 to ϕ in $\mathcal{M}([\omega_g])$. When $X = 0$, $\mu(\phi)$ reduces to Mabuchi's K-energy. It can be shown that the functional $\mu(\phi)$ is well-defined, i.e., it is independent of the choice of path ϕ_t (cf. [Gua2], [Mab1], [Sim]). Thus ϕ is a critical point of $\mu(\cdot)$ iff ϕ satisfies (1.8). Moreover by the definition of X and the relation (1.6), $\mu(\phi)$ is invariant under the group $\mathrm{Aut}_r(M)$. In [T2], Tian defined an analytic condition called properness for a functional which is equivalent to K-enegy. Then he proved that the existence of Kähler-Einstein metric in the positive first Chern class is equivalent to this properness. For the study of Kähler-Einstein metrics, different energy functionals have been introduced in [CT1, SW].

We finish this chapter by a brief review on the latest development of the study of extremal metrics concerning the uniqueness and existence.

On the uniqueness of extremal metrics, great progress has been made. The uniqueness of Kähler-Einstein metrics was pointed out by Calabi in 1950s in the case when $c_1(M) \leq 0$. In [BM], Bando and Mabuchi proved the uniqueness of the Kähler-Einstein metric in the case when $c_1(M) > 0$. For the case of CSC Kähler metrics, the answer is also positive. In [Ch], Chen proved the uniqueness of CSC Kähler metrics in any Kähler class which admits a Kähler metric with non-positive scalar curvature. In [D1], Donaldson proved the uniqueness of CSC Kähler metrics in rational Kähler classes on any projective manifold without non-trivial holomorphic vector fields. The assumption on the holomorphic vector fields was later removed by Mabuchi [Mab3]. A complete answer to general extremal metrics was given by Chen and Tian. The theorem can be stated as follows.

Theorem 1.5 ([CT2]). *Let $(M, [\omega])$ be a compact Kähler manifold with a Kähler class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. Then there is at most one extremal Kähler metric in Kähler class $[\omega]$ modulo holomorphic transformations. Namely, if ω_1 and ω_2 are two extremal Kähler metrics in the same Kähler class, then there is a holomorphic transformation σ such that $\sigma^*\omega_1 = \omega_2$.*

However, on the other hand, the existence of extremal metrics is still far from being completely understood. For the Kähler-Einstein case, this problem has been solved by Yau [Y2] when $c_1(M) = 0$, known as Calabi Conjecture, and solved by Yau, Aubin independently when $c_1(M) < 0$ [Y2, Au1]. When $c_1(M) > 0$, it is still unknown whether the manifold admits Kähler-Einstein metrics although there are some remarkable work [Siu, T1-2, WZhu]. In general, the existence of extremal metrics have been conjectured to be related to various stabilities of the underlying manifold in the sense of Geometric Invariant Theory. When the stabilities are violated, many counterexamples to the existence of canonical metrics on certain Kähler manifolds have been found [BD, R, St1]. There are also some existence results in several special cases, see, for example, [AP1-2, APS, ACGT, CLW, D1-5, Gua1]. In the next chapter, we will recall the development of extremal metrics on toric manifolds.

1.3 Relative K-stability

The relation between various notions of stabilities and the existence of Calabi's extremal metrics have been recently studied ([T1], [D1-2], [Mab3-4], etc.). The goal is to find a necessary and sufficient condition for the existence of extremal metrics in the sense of Geometric Invariant Theory ([Y3], [T2]). There is now a famous conjecture called Yau-Tian-Donaldson conjecture that will be stated in Section 1.3.1. In Section 1.3.2, we will recall Donaldson's definition of Futaki invariant for general polarized scheme. Then we state the notion of relative K-stability by [Sz], which is a main object of study in this thesis.

1.3.1 Yau-Tian-Donaldson conjecture

Around the existence of Calabi's extremal metrics, there is a well-known conjecture (cf. [Y3], [T3]):

Conjecture 1.6 ([Yau-Tian-Donaldson]). *Suppose that (M, L) is a compact complex polarized manifold. Then M admits extremal metrics in $2\pi c_1(L)$ if and only if (M, L) is stable in sense of Geometric Invariant Theory.*

Here we would like to point out that this conjecture was stated in many different ways due to different notions of stabilities. The two best known stabilities are the K-stability and the Chow-Mumford stability.

For the "only if" part of this conjecture, the first breakthrough was made by G. Tian [T2]. By introducing the concept of K-stability, he gave an answer to "only if" part for the first Chern class (if it is positive) on M (corresponding to a Kähler-Einstein manifold). Later, Donaldson extended the K-stability to general polarized varieties [D2] and made a conjecture on the relation between the K-stability and the existence of constant scalar curvature Kähler metrics. Very recently, Stoppa [St2] generalized Tian's result to a compact Kähler manifold M with a CSC Kähler metric and without any non-trivial holomorphic vector field on M. Meanwhile, a remarkable progress was made by Donaldson who showed the Chow-Mumford stability is necessary for a polarized Kähler manifold with CSC metrics when the holomorphic automorphisms group $\mathrm{Aut}(M)$ of M is finite [D1]. Donaldson's result was later generalized by T. Mabuchi to any polarized Kähler manifold M which admits an extremal metric without any assumption on $\mathrm{Aut}(M)$ [Mab4-5].

The definition of K-stability was extended by Szkelyhidi [Sz] to Kähler classes with nonzero extremal vector fields and was called relative K-stability. However it is still unknown whether it is true that the existence of general extremal metrics implies relative K-stability. Note that for Chow-Mumford stability, the answer is yes by Donaldson-Mabuchi's result. In the case of toric manifolds, we gave a positive answer [ZZ2].

The "if" part of this conjecture is more difficult because we have to solve a fourth order elliptic equation. It is a challenge in differential geometry and PDE theory. On some special manifolds, the conjecture was confirmed. On toric manifolds, Donaldson [D2] set up a strategy for this problem and he proved the conjecture for toric surface when the Kähler class admits vanishing Futaki invariant [D3-5]. We will discuss more about Donaldson's strategy on toric manifolds in the next chapter. Recently, another important progress was made on projective bundles, see [ACGT].

1.3.2 Donaldson-Futaki invariant

In this subsection, we recall the definition of Donaldson-Futaki invariant. As we said in Section 1.2, Futaki invariant was an holomorphic invariant first constructed by Futaki and Calabi on any Kähler manifold. This definition was extended to the case of Fano normal varieties in [DT]. Later, Tian defined the notion of K-stability of a Fano manifold M using this invariant and some degenerations of M. In [D2], Donaldson defined the general Futaki invariant for polarized scheme in an algebraic way. Here we state this definition as follows.

Let (M, L) be a polarized scheme, where L is an ample line bundle. Let α be a $\mathbb{C}^*$-action on (M, L). Then for any positive integer k, α induces a $\mathbb{C}^*$-action on the vector space

$$H_k = H^0(M, L^k).$$

Denote by d_k the dimension of the vector space H_k and $w_k(\alpha)$ the weight of the induced action on the highest exterior power H_k. Then d_k and w_k are given by polynomials of k as

$$d_k = a_0 k^n + a_1 k^{n-1} + \cdots,$$

$$w_k(\alpha) = b_0 k^{n+1} + b_1 k^n + \cdots.$$

Definition 1.7 ([D2]). The *Donaldon-Futaki invariant* of α on (M, L) is defined to be

$$\mathcal{F}(\alpha) = \frac{a_1 b_0 - a_0 b_1}{a_0}.$$

Donaldson also proved that when M is a smooth manifold and the $\mathbb{C}^*$-action is induced by a holomorphic vector field X, this definition coincides with Futaki's original result: let ω be a Kähler metric in $2\pi c_1(L)$, then

$$\mathcal{F}(\alpha) = -\frac{1}{4V(M)} \int_M X(g) \frac{\omega^n}{n!}, \tag{1.11}$$

where

$$g = G(S(\omega) - \bar{S}),$$

G is the Green's operator, $\bar{S}$ is the average of scalar curvature. Note that the integral in (1.11) is the original Futaki invariant in Section 1.2. Hence, when M is a manifold, Donaldson-Futaki invariant is the original Futaki invariant multiplied by a constant.

1.3.3 Notions of K-stabilities

The definition of K-stability for a polarized manifold (M, L) is related to its degenerations, called test configuration.

Definition 1.8 ([D2]). A *test configuration* for a polarized Kähler manifold (M, L) of exponent r consists of

1. a scheme $\mathcal{W}$ with a $\mathbb{C}^*$-action;

2. a $\mathbb{C}^*$-equivariant ample line bundle $\mathfrak{L}$ on $\mathcal{W}$;

3. a $\mathbb{C}^*$-equivariant flat family of schemes

$$\pi : \mathcal{W} \longrightarrow \mathbb{C},$$

where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication. We require that the fibers $(\mathcal{W}_t, \mathfrak{L}|_{\mathcal{W}_t})$ are isomorphic to (M, L^r) for any $t \neq 0$.

Note that since π is $\mathbb{C}^*$-equivariant, the $\mathbb{C}^*$-action can be restricted to the central fiber. A test configuration is called trivial if $\mathcal{W} = M \times \mathbb{C}$ is a product. Now the K-stability can be defined as follows.

Definition 1.9 ([D2]). A polarized Kähler manifold (M, L) is *K-semistable* if for any test-configuration the Futaki invariant of the induced $\mathbb{C}^*$-action on $(\mathcal{W}_t, \mathfrak{L}|_{\mathcal{W}_t})$ is nonnegative. It is called *K-stable* if in addition the equality holds if and only if the test-configuration is trivial.

It has been proved that this K-stability is a necessary condition for the existence of constant scalar curvature metrics in $2\pi c_1(L)$ on a polarized Kähler manifold (M, L) [St2].

As was pointed out, Futaki invariant is an obstruction to the existence of constant scalar curvature metric. When Futaki invariant does not vanish, i.e., the extremal vector field is nontrivial, we need a modification of the notion of K-stability. In [Sz], Szekelyhidi introduced the notion of relative K-stability based on a modified Futaki invariant as a generalization of the K-stability. Let us recall the definition of relative K-stability.

To define the modified Futaki invariant, we first need an inner product for the $\mathbb{C}^*$-actions [Sz]. Let α, β be two $\mathbb{C}^*$-actions on a polarized scheme (M, L). Suppose that A_k and B_k are the infinitesimal generators of the actions on $H^0(M, L^k)$, respectively. The inner product (α, β) is given by

$$Tr\left[\left(A_k - \frac{Tr(A_k)I}{d_k}\right)\left(B_k - \frac{Tr(B_k)I}{d_k}\right)\right] = (\alpha, \beta)k^{n+2} + O(k^{n+1}).$$

The relative K-stability is based on the following *modified Futaki invariant*,

$$\mathcal{F}_\beta(\alpha) = \mathcal{F}(\alpha) - \frac{(\alpha, \beta)}{(\beta, \beta)}\mathcal{F}(\beta), \tag{1.12}$$

where $\mathcal{F}(\alpha)$ and $\mathcal{F}(\beta)$ are Futaki invariants of α and β, respectively.

Let χ be the $\mathbb{C}^*$-action induced by the extremal vector field X. We say that a test configuration is *compatible* with χ, if there is $\mathbb{C}^*$-action $\tilde{\chi}$ on $(\mathcal{W}, \mathfrak{L})$ such that $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ is an equivariant map with trivial $\mathbb{C}^*$-action on $\mathbb{C}$ and the restriction of $\tilde{\chi}$ to $(\mathcal{W}_t, \mathfrak{L}|_{\mathcal{W}_t})$ for nonzero t coincides with that of χ on (M, L^r) under the isomorphism. Note that $\mathbb{C}^*$-action α on $\mathcal{W}$ induces $\mathbb{C}^*$-action on the central fibre $M_0 = \pi^{-1}(0)$ and the restricted line bundle $\mathfrak{L}|_{M_0}$. We denote by $\tilde{\alpha}$ and $\tilde{\chi}$ the induced $\mathbb{C}^*$-action of α and χ on $(M_0, \mathfrak{L}|_{M_0})$, respectively.

The relative K-stability is defined as follows.

Definition 1.10 ([Sz]). A polarized Kähler manifold (M, L) is *relatively K-semistable* if $\mathcal{F}_{\tilde{\chi}}(\cdot) \leq 0$ for any test-configuration compatible with χ. It is called *relatively K-stable* if in addition that the equality holds if and only if the test-configuration is trivial.

Finally, we would like to point out that since Donaldson's Futaki invariant can be defined for polarized scheme, all the above notions are also well defined for schemes. We only stated the definitions for polarized manifolds which is enough for this thesis. The main objects we study in this thesis are toric manifolds. In the next chapter, we will obtain a simplified definition for the K-stabilities on toric manifolds.

Chapter 2

Toric reduction

In [D2], Donaldson built up a program of studying the existence of constant scalar curvature metrics and stabilities on toric manifolds. He reduced the K-energy to a real functional on a polytope in $\mathbb{R}^n$, and proved that the K-stability is equivalent to the positivity of a linear functional on the polytope. Later, the reduction was extended to more general Calabi's extremal metrics [ZZ1]. In this chapter, we will describe this reduction. We also reduce the scalar curvature equation (1.8) to Abreu's equation [Ab1].

2.1 Kähler geometry on toric manifolds

In this section, we recall some background materials related to toric Kähler manifolds, for more details we refer the readers to [Ab1-2, De, D2, Gui].

A complex manifold M is called *toric*, if there is a complex torus Hamiltonian action $T^n_{\mathbb{C}}$ on M and the action has a dense free orbit, identified with $T^n_{\mathbb{C}} = (\mathbb{C}^*)^n = (S^1)^n \times \mathbb{R}^n$.

Now we assume that (M, g) is an n-dimensional toric Kähler manifold with a torus action $T \cong (\mathbb{C}^*)^n$. Then the open dense orbit of T in M induces an global coordinates $(w_1, ..., w_n) \in (\mathbb{C}^*)^n$. To do the reduction we use the affine logarithmic coordinates

$$z_i = \log w_i = \xi_i + \sqrt{-1}\eta_i.$$

Let $G_0 \cong (S^1)^n$ be a maximal compact subgroup of T. Then if g is a G_0-invariant Kähler metric, ω_g is determined by a convex function ψ_0 which depends only on $\xi_1, ..., \xi_n \in \mathbb{R}^n$ in the coordinates $(z_1, ..., z_n)$, namely

$$\omega_g = 2\sqrt{-1}\partial\bar{\partial}\psi_0, \text{ on } (\mathbb{C}^*)^n.$$

Since the torus action T is Hamiltonian, there exists a moment map

$$m : M \to \mathbb{R}^n,$$

and the image is a convex polytope in $\mathbb{R}^n$. Note that

$$\xi_i = \frac{z_i + \bar{z}_i}{2}, \qquad \eta_i = \frac{z_i - \bar{z}_i}{2i},$$

we have

$$\omega_g = \frac{\partial^2 \psi_0}{\partial \xi_i \partial \xi_j} d\xi_i \wedge d\eta_j.$$

Hence, through

$$dm_k = -i_{\frac{\partial}{\partial \eta_k}} \omega_g,$$

the moment map is given by

$$(m_1, \ldots\ldots, m_n) = \left(\frac{\partial \psi_0}{\partial \xi_1}, \ldots\ldots, \frac{\partial \psi_0}{\partial \xi_n} \right),$$

that is the gradient of ψ_0. Denote the image by

$$P = D\psi_0(\mathbb{R}^n).$$

Then P is a convex polytope. This polytope is independent of the choice of the metric g in the class $[\omega_g]$. However, P can not be an arbitrary polytope in $\mathbb{R}^n$. An interesting result [De] says that P satisfies several special conditions. *Delzant's conditions* can be stated as follows [Ab1]:

1. There are exactly n edges meeting at each vertex p.
2. The edges meeting at the vertex p are rational, i.e., each edge is of the form $p + tv_i$, $0 \le t < \infty$, $v_i \in \mathbb{Z}^n$.
3. The vectors $v_1, \cdots, v_n$ can be chosen to be a basis of $\mathbb{Z}^n$.

As a conclusion, for an n-dimensional compact toric manifold M, together with an associated Kähler class $[\omega_g]$, $(M, [\omega_g])$, there is an associated bounded convex polytope $P \subset \mathbb{R}^n$ satisfying Delzant's conditions. Conversely, from a convex polytope $P \subset \mathbb{R}^n$ satisfying Delzant's conditions, one can recover a toric manifold and the associated Kähler class $(M, [\omega_g])$. See [De] for details.

We will characterize the metric under the polytope coordinates. The polytope P can be represented by a set of inequalities of the form

$$P = \{x \in \mathbb{R}^n : \ \langle x, \ell_i \rangle \le \lambda_i, \ i = 1, 2, \cdots, d\}, \tag{2.1}$$

where ℓ_i is the normal to a face of P, λ_i is a constant, and d is the number of faces of P. Delzant's conditions can be equivalently stated as follows.

1. There are exactly n faces meeting at each vertex p.

2. The normals ℓ_i $(i = 1, 2, \cdots, d)$ are vectors in $\mathbb{Z}^n$.

3. At any given vertex p, let $\ell_{i_1}, ..., \ell_{i_n}$ be the normals to the faces at p, then $\det(\ell_{i_1}, ..., \ell_{i_n}) = \pm 1$.

Remark 2.1.
(i) Note that if $\det(\ell_1, ..., \ell_n) = 1$ and if $\ell_i \in \mathbb{Z}^n$, the matrix $(\ell_1, \cdots, \ell_n)$ can be reduced to the unit matrix by Gauss elimination. Therefore $(\ell_1, \cdots, \ell_n)$ is a basis of $\mathbb{Z}^n$.
(ii) The constants $\lambda_1, \cdots, \lambda_d$ are not necessarily integers, and can change continuously. When they are all integers, the associated Kähler class is called integral [Gui] and from the polytope P we can recover a polarized toric manifold.
(iii) Two different polytopes may correspond to the same toric manifold $(M, [\omega_g])$. Indeed, all Delzant triangles correspond to the complex projective space $\mathbb{CP}^2$. We will discuss equivalent classes of Delzant's polytopes in Chapter 4.
(iv) We also note that the set of all Kähler classes on a toric manifold M is a finite dimensional convex cone. Moreover, a Kähler class is the first Chern class if and only if $\lambda_i = 1$ for all $i = 1, \cdots, d$ (up to translation of coordinates).

By using the Legendre transformation $\xi = (D\psi_0)^{-1}(x)$, one sees that the function (Legendre dual function) defined by

$$u_0(x) = \langle \xi, D\psi_0(\xi) \rangle - \psi_0(\xi) = \langle \xi(x), x \rangle - \psi_0(\xi(x)), \ \forall\, x \in P$$

is convex. In general, for any G_0-invariant potential ϕ in $[\omega_g]$, one gets a convex function $u_\phi(x)$ on P by using the above relation while ψ_0 is replaced by $\psi_0 + \phi$. Set

$$\mathcal{C} = \{u = u_0 + v \mid u \text{ is a convex function in } \ P, \ v \in C^\infty(\bar{P})\}.$$

It was shown in [Ab1] that there is a bijection between functions in $\mathcal{C}$ and G_0-invariant functions in $\mathcal{M}([\omega_g])$. Denote the latter by $\mathcal{M}_{G_0}([\omega_g])$. For any function u_ϕ in $\mathcal{C}$, it can be explicitly given by ([Gui, Ab2]):

$$u_\phi = \frac{1}{2}\sum_1^d (\lambda_i - \langle \ell_i, x \rangle) log(\lambda_i - \langle \ell_i, x \rangle) + f, \tag{2.2}$$

and it defines the form

$$\omega_{g_\phi} = 2\sqrt{-1}\partial\bar{\partial} m^*(\langle x, Du_\phi \rangle - u_\phi)$$

on $m^{-1}(P)$, where f is a function smooth up to boundary of P, m^* is the pull-back map of m. We usually say that a function satisfies *Guillemin's boundary condition* if it can be written in the form of (2.2).

In the coordinates $\{x_1, ..., x_n\}$, the scalar curvature of g_ϕ is given by [Ab1]

$$S(g_\phi) = -\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}, \tag{2.3}$$

where u is the Legendre function of $\psi = \psi_0 + \phi$, (u^{ij}) is the inverse matrix of $(u_{ij}) = (\frac{\partial^2 u}{\partial x_i \partial x_j})$. For simplicity in the following we will write the right hand side of (2.3) as $-u^{ij}_{ij}$. The gradient of the scalar curvature is given by

$$\nabla^{(1,0)} S = \psi^{ij} \frac{\partial S}{\partial \xi_j} \frac{\partial}{\partial z_i}.$$

Note that g_ϕ is extremal metric if and only if $\nabla^{(1,0)} S$ is holomorphic. Since $\psi^{ij} \frac{\partial S}{\partial \xi_j} = \frac{\partial S}{\partial x_i}$ is real, it is holomorphic if and only if S is an affine linear function in the x-coordinates. In fact, we can determine the potential function of the extremal vector field. As in §1.2, Let $\eta_r(M)$ be the Lie algebra of the reductive part of $Aut^0(M)$, then $\eta_c(M)$ is the Lie algebra of the torus action T on M. By (1.7) we have

Lemma 2.2 ([ZZ1]). *Let ϕ be a G_0-invariant potential in $[\omega_g]$ on M and $v \in \eta_c(M)$. Let $\theta_v = \theta_v(\omega_\phi)$ be a normalized potential of v associated to ω_ϕ as in (1.4). Then there are $2n$-numbers a_i and c_i such that*

$$\theta_v = \sum_{i=1}^{n} a_i(x_i + c_i).$$

Moreover, if v is extremal, a_i and c_i are determined uniquely by $2n$-equations,

$$\frac{vol(P)}{V(M)} \mathcal{F}(\frac{\partial}{\partial z_i}) = -\int_P [\sum_{j=1}^{n} a_j(x_j + c_j)](x_i + c_i)\, dx,\ i = 1, ..., n, \tag{2.4}$$

$$\int_P (x_i + c_i)\, dx = 0,\ i = 1, ..., n, \tag{2.5}$$

where $\mathcal{F}(\cdot)$ is the Futaki invariant.

Therefore, let θ_X be the affine linear function determined by (2.4), (2.5), the scalar curvature equation becomes a equation on the polytope,

$$-u^{ij}_{ij} = \bar{S} + \theta_X, \tag{2.6}$$

This equation is called *Abreu's equation.*

Hence, on toric manifolds, the existence of extremal metric reduces to a real PDE problem of finding smooth solutions u to equation (2.6) defined on a Delzant's polytope P such that u satisfies Guillemin's boundary condition, which is given by (2.2).

2.2 Donaldson's reduction

2.2.1 Reduction of Futaki invariant

Let $d\sigma_0$ be the Lebesgue measure on the boundary ∂P and ν be the outer normal vector field on ∂P. Then we define a measure

$$d\sigma = \frac{(\nu, x)}{\lambda_i} d\sigma_0 = \frac{1}{|\ell_i|} d\sigma_0 \tag{2.7}$$

on the face $\langle \ell_i, x \rangle = \lambda_i$ of P. Donaldson obtained the following simplification for Futaki invariant.

Lemma 2.3 ([D2]). *The Futaki invariant can be computed on the polytope by*

$$\frac{Vol(P)}{V(M)} \mathcal{F}(\frac{\partial}{\partial z_i}) = -\left(\int_{\partial P} x_i \, d\sigma - \bar{S} \int_P x_i \, dx \right). \tag{2.8}$$

For simplicity, we denote $A := \bar{S} + \theta_X$, and define a linear functional $\mathcal{L}$ by

$$\mathcal{L}(u) = \int_{\partial P} u \, d\sigma - \int_P Au \, dx. \tag{2.9}$$

Let $\mathcal{C}_1$ be the set of general convex functions with L^1 boundary value. The linear functional $\mathcal{L}$ is well defined in $\mathcal{C}_1$. By Lemma 2.2 and 2.3, A is an affine linear function in the polytope coordinates $\{x_1, ..., x_n\}$, which can be determined by the parameters in (2.1) as follows.

Proposition 2.4. *Let $A = a_0 + \sum_1^n a_i x_i$. Then $a_0, a_1, ..., a_n$ can be determined uniquely by the $n+1$-equation system*

$$\mathcal{L}(1) = 0, \;\; \mathcal{L}(x_i) = 0, \;\; i = 1, ..., n. \tag{2.10}$$

In Chaper 4, we will use this proposition to determined all the Delzant's polytopes with no more than 6 vertices and with vanishing Futaki invariant. The linear functional $\mathcal{L}$ will also play an important part in the study of K-stability later.

2.2.2 Reduction of K-energy

In this section, we transform the modified K-energy to a real functional for functions in $\mathcal{C}$. This version of K-energy was first introduced by [D2], we extended it to modified K-energy [ZZ1].

Proposition 2.5 ([D2, ZZ1]). *Let u be the Legendre function of $\psi = \psi_0 + \phi$, then there is a constant C independent of ϕ, such that*

$$\mu(\phi) = \frac{(2\pi)^n}{V(M)}(\mathcal{K}(u) + C), \tag{2.11}$$

where

$$\mathcal{K}(u) = -\int_P \log\det D^2 u\, dx + \mathcal{L}(u). \tag{2.12}$$

Proof. One can derive the formula (2.11) as in [D2]. Here we give a different proof. Denote by u_t the Legendre function of ϕ_t. By definition,

$$\phi_t(x) = x\xi_t(x) - u_t(\xi_t(x)).$$

Differentiating it and using the fact $\frac{du_t}{d\xi_{ti}} = x_i$, we have

$$\dot{\phi}_t = \sum x_i \frac{d\xi_{ti}}{dt} - \dot{u}_t - \sum \frac{du_t}{d\xi_{ti}}\frac{d\xi_{ti}}{dt} = -\dot{u}_t.$$

Changing the coordinates from ξ to x, we obtain

$$\begin{aligned}
&(\phi_t)^{ij}(\log\det D^2\phi_t)_{ij} = -[(\log\det D^2 u_t)_{ki}(u_t)^{ik} + (\log\det D^2 u_t)_k (u_t)^{ik}_i],\\
&(\phi_t)_i = x_i, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.13)\\
&\det D^2\phi_t d\xi = dx,
\end{aligned}$$

Therefore, functional (1.10) becomes

$$\begin{aligned}
\mu(\phi) &= -\frac{(2\pi)^n}{V(M)}\int_0^1\int_P \dot{u}_t[(\log\det D^2 u_t)_{ki}(u_t)^{ik} + (\log\det D^2 u_t)_k (u_t)^{ik}_i]\, dxdt\\
&\quad -\frac{(2\pi)^n}{V(M)}\int_0^1\int_P A\dot{u}_t\, dxdt.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&-\int_0^1\int_P \dot{u}_t(\log\det D^2 u_t)_{ki}(u_t)^{ik} + (\log\det D^2 u_t)_k (u_t)^{ik}_i dxdt\\
&= -\int_0^1\int_{\partial P} \dot{u}_t(\log\det D^2 u_t)_k (u_t)^{ik}\nu_i\, d\sigma_0 dt - \int_0^1\int_P (\dot{u}_t)_i(\log\det D^2 u_t)_k (u_t)^{ik}\, dxdt.
\end{aligned}$$

By Guillemin's boundary condition, it holds [D2-3]

$$(u_t)^{ik}_k \nu_i d\sigma_0 = d\sigma$$

on the boundary ∂P. Hence the first term is

$$\begin{aligned} -\int_0^1\int_{\partial P} \dot{u}_t(\log\det D^2u_t)_k(u_t)^{ik}\nu_i\, d\sigma_0 dt &= \int_0^1\int_{\partial P} \dot{u}_t(u_t)^{ik}_k\nu_i\, d\sigma_0 dt \\ &= \int_0^1\int_{\partial P} \dot{u}_t\, d\sigma dt. \end{aligned}$$

The second term is

$$\begin{aligned} \int_0^1\int_P (\dot{u}_t)_i(\log\det D^2u_t)_k(u_t)^{ik}\, dxdt &= \int_0^1\int_P (\dot{u}_t)_i(u_t)^{pq}(u_t)_{pqk}(u_t)^{ik}\, dxdt \\ &= -\int_0^1\int_P (\dot{u}_t)_q(u_t)^{pq}_p\, dxdt \\ &= -\int_0^1\int_P \frac{d(\log\det D^2u_t)}{dt}\, dt. \end{aligned}$$

Therefore we have

$$\mu(\phi) = -\frac{(2\pi)^n}{V(M)}\int_0^1 \frac{d(\int_P \log\det D^2u_t\, dx - \int_{\partial P} u_t\, d\sigma + \int_P Au_t\, dxdt)}{dt}\, dxdt.$$

The proposition follows. □

2.2.3 Reduction of K-stability

Now we consider the relative K-stability of a polarized toric manifold (M, L) which corresponds to an integral polytope P in $\mathbb{R}^n$ (i.e. when λ_i in (2.1) are integers). In [D2], Donaldson induced toric degenerations as a class of special test configuration induced by positive rational, piecewise linear functions on P. The reduction of the stability is based on these degenerations.

Recall that a *piecewise linear* (PL) function u on P is of the form

$$u = \max\{u^1, ..., u^r\},$$

where $u^\lambda = \sum a_i^\lambda x_i + c^\lambda, \lambda = 1, ..., r$, for some vectors $(a_1^\lambda, ..., a_n^\lambda) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. u is called a rational PL-function if the coefficients a_i^λ and numbers c^λ are all rational.

For a positive rational PL function u on P, we choose an integer R so that

$$Q = \{(x, t) \mid x \in P,\ 0 < t < R - u(x)\}$$

is a convex polytope in $\mathbb{R}^{n+1}$. Without loss of generality, we may assume that the coefficients a_i^λ are integers and Q is an integral polytope. Otherwise we replace u by lu and Q by lQ for some integer l, respectively. Then the $n+1$-dimensional polytope Q determines an $(n+1)$-dimensional toric variety M_Q with a holomorphic line bundle $\mathfrak{L} \to M_Q$. Note that the face $\bar{Q} \cap \{\mathbb{R}^n \times \{0\}\}$ of Q is a copy of the n-dimensional polytope P, so we have a natural embedding $i : M \to M_Q$ such that $\mathfrak{L}|_M = L$. Decomposing the torus action $T_{\mathbb{C}}^{n+1}$ on M_Q as $T_{\mathbb{C}}^n \times \mathbb{C}^*$ so that $T_{\mathbb{C}}^n \times \{\mathrm{Id}\}$ is isomorphic to the torus action on M, we get $\mathbb{C}^*$-action α by $\{\mathrm{Id}\} \times \mathbb{C}^*$. Hence, we define an equivariant map

$$\pi : M_Q \to \mathbb{CP}^1$$

satisfying $\pi^{-1}(\infty) = i(M)$. One can check that $\mathcal{W} = M_Q \backslash i(M)$ is a test configuration for the pair (M, L), called a *toric degeneration* [D2]. This test configuration is compatible to the $\mathbb{C}^*$-action χ induced by the extremal holomorphic vector field X on M. In fact, χ as a group is isomorphic to a one parameter subgroup of $T_{\mathbb{C}}^n \times \{\mathrm{Id}\}$, which acts on $\mathcal{W}$. Since the action is trivial in the direction of α, the test configuration is compatible.

The modified Futaki invariant for a toric degeneration has an explicit formula in polytope coordinates. Indeed, the following proposition relates the K-stability to the positivity of functional (2.9). It can be regarded as a generalization of Proposition 4.2.1 in [D2].

Proposition 2.6 ([D2, ZZ1]). *For a $\mathbb{C}^*$-action α on a toric degeneration on M induced by a positive rational PL-function u, we have*

$$\mathcal{F}_{\tilde{\chi}}(\tilde{\alpha}) = -\frac{1}{2Vol(P)}\mathcal{L}(u), \tag{2.14}$$

where χ is the $\mathbb{C}^*$-action induced by the extremal holomorphic vector field X, and $\mathcal{F}_{\tilde{\chi}}(\tilde{\alpha})$ is given by (1.12).

According to the above reduction, we will use the positivity of $\mathcal{L}$ as the definition of relative K-stability on toric manifolds and we usually omit the words "relative" and "for toric degenerations" for simplicity.

Definition 2.7 ([D2]). We call (M, L) is *relatively K-stable for toric degenerations* if its associated polytope satisfies $\mathcal{L}(u) \geq 0$ for all rational PL functions u on P and if $\mathcal{L}(u) = 0$ for a rational PL function u, then u must be a linear function.

Remark 2.8.
(i) In [D2] the K-stability was defined on polarized toric manifolds, that is the case when the constants λ_i in (2.1) are integers. But obviously his definition can be extended to general polytopes. (When the constants λ_i in (2.1) are not integers or rational numbers, we need to drop the word "rational" in the above definition).
(ii) The K-stability is related to Kähler class and is an intrinsic property. So if two polytopes corresponds to the same Kähler class of a toric manifold, then the K-stability of one polytope implies that of the other.

In dimension 2, that is, on toric surfaces, there is a further reduction on the positivity of $\mathcal{L}$. Following Donaldson, we say a function u is *simple PL* if there is a linear function ℓ such that $u = \max\{0, \ell\}$. If u is simple PL, the set $\mathcal{I}_u = P \cap \{\ell = 0\}$ is called the *crease* of u. We still denote by $\mathcal{C}_1$ the set of general convex functions with L^1 boundary value.

Proposition 2.9 ([D2]). *Let P be a convex polytope $P \subset \mathbb{R}^2$. Assume that A is positive and satisfies (2.10). Suppose that $\mathcal{L}(u) \geq 0$ for all convex functions $u \in \mathcal{C}_1$ but there is a nonlinear convex function $u \in \mathcal{C}_1$ such that $\mathcal{L}(u) = 0$. Then there is a simple PL function $\hat{u}$ with its crease $\mathcal{I}_{\hat{u}} \neq \emptyset$ such that $\mathcal{L}(\hat{u}) = 0$.*

We will remove the assumption $A \geq 0$ in the next chapter.

In the end of this section, we restate the conjecture of Yau-Tian-Donaldson on toric manifolds.

Conjecture 2.10 ([D2]). *A polarized toric manifold M admits extremal metrics in $2\pi c_1(L)$ if and only if (M, L) is relatively K-stable for toric degenerations.*

2.3 Recent progress on toric manifolds

In this section, we review some main results on extremal metrics on toric manifolds.

The 'only if' part of Conjecture 2.10 was proved in [ZZ2].

Theorem 2.11 ([ZZ2]). *Let (M, L) be a polarized toric manifold which admits an extremal metric in $2\pi c_1(L)$. Then (M, L) is relatively K-stable for toric degenerations.*

As we explained in last section, the relative K-stability for toric degenerations means the positivity of $\mathcal{L}$ for all PL-functions. Using Abreu's equation and Guillemin's boundary condition, one can show that

$$\mathcal{L}(f) = \int_P u^{ij} f_{ij} dx, \ f \in \mathcal{C}, \tag{2.15}$$

where u is the solution to Abreu's equation satisfying Guillemin's boundary condition. By (2.15), $\mathcal{L}$ is strictly positive for functions in $\mathcal{C}$. Hence, Theorem 2.11 implies that if M admits an extremal metric in $2\pi c_1(L)$, then for any PL-function f on P, we also have

$$\mathcal{L}(f) \geq 0.$$

Moreover the equality holds if and only if f is a linear function.

The 'if' part of the conjecture is rather difficult. Most of the recent developments occur in dimension 2. In [D2], Donalson proved that on a toric surface, if the Kähler class has vanishing Futaki invariant, then K-stability implies that the K-energy is bounded from below. Later in a series of papers [D3-5], Donaldson gave a confirmative answer to Conjecture 2.10 in this special case by a continuity method.

Theorem 2.12 ([D5]). *M admit a constant scalar curvature metric in $2\pi c_1(L)$ if and only if (M, L) is K-stable and Futaki invariant vanishes.*

In his proof, the degeneration family of the continuity method was chosen to be the perturbation of the triple $(P, A, d\sigma)$. In [D3], he obtained the interior estimates of Abreu's equation. Then under an analytical condition called M-condition, he obtained boundary estimates by blow-up arguments [D4]. These a priori estimates can be used to solve the existence of constant scalar curvature metrics. Finally, the conjecture was solved by showing that K-stability implies the a priori estimate of M-condition [D5]. In the general case with nonzero Futaki invariant, i.e., A is not a constant, the existence of solutions is still an open problem.

Another progress was made on toric Fano surfaces. A complex manifold is called Fano if its first Chern class is positive definite. It is well known that toric Fano surfaces are classified into five different types, i.e., $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the blowing-up spaces $\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}, l = 1, 2, 3$. In [CLW], X. Chen, C. Lebrun and B. Weber proved the existence of extremal metrics on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$, especially in its first Chern class. Together with the existence of Kähler-Einstein metrics on $\mathbb{CP}^2$,

$\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ [TY, Siu] and Calabi's construction of extremal metrics on $\mathbb{CP}^2\#1\overline{\mathbb{CP}^2}$ [Cal3], this implies that every toric Fano surface admits an extremal metric in its first Chern class.

In higher dimension, very little of existence result is known. In [ZZ3], under the assumption of a variational condition on the modified K-energy, the authors proved the existence of weak solutions for the extremal metrics in any dimension in the sense of general convex functions minimizing the modified K-energy. This variational condition generalizes Tian's properness condition dealing with Kähler-Einstein metrics in the positive first Chern class [T2]. The properness condition is a generalization of Moser-Trudinger inequality [T2, PSSW]. The uniqueness and regularity of this weak solution is still unknown.

In the special case for Kähler-Einstein metrics, the problem has been completely solved by Wang and Zhu.

Theorem 2.13 ([WZhu]). *A toric manifold with positive first Chern class admits a Kähler-Einstein metric if and only if the Futaki invariant vanishes in its first Chern class.*

2.4 Main results in this thesis

In Part II, we study the existence of extremal metrics and K-stability of Kähler classes on toric Kähler surfaces.

A fundamental property of toric Kähler surface is that every compact toric Kähler surface can be obtained from $\mathbb{CP}^2$ or Hirzebruch surfaces $\mathbb{F}_k$ $(k = 0, 1, 2, \cdots)$ by a succession of blow-ups at $T^2_{\mathbb{C}}$-fixed points [Ful]. In §3.1 we use this property and a result by Arrezo-Pacard-Singer [APS] to prove that on every toric surface, there exists a Kähler class which admits extremal metrics. We restate Arrezo-Pacard-Singer's theorem as Theorem 3.2 in §3.1.

In §3.2 we give examples of Kähler classes on toric surfaces which are not K-stable. By Donaldson's reduction of the K-stability, Definition 2.7, it suffices to find a polytope $P \subset \mathbb{R}^2$ satisfying Delzant's conditions and a PL function u such that the functional $\mathcal{L}(u) < 0$. Such kind of examples was found by Donaldson but we found an unstable polytope $P \subset \mathbb{R}^2$ with 9 $T^2_{\mathbb{C}}$-fixed points (the number of vertices of the polytope).

In §3.3, we remove the condition $A \geq 0$ in Proposition 2.9. Our proof uses properties of solutions to degenerate Monge-Ampère equation.

In Chapter 4, we are concerned with the existence of constant scalar curvature metrics on toric surfaces. To apply Donaldson's Theorem 2.12, we want to check when a toric surface has vanishing Futaki invariant and whether it is K-stable. It is known that a toric surface with 3 or 4 $T^2_{\mathbb{C}}$-fixed points must be $\mathbb{CP}^2$ or a Hirzebruch surface, and among these toric surfaces, $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are the only ones which have vanishing Futaki invariant and all the Kähler classes on them are K-stable. In this chapter we verify toric surfaces with 5 or 6 $T^2_{\mathbb{C}}$-fixed points.

In §4.1, we introduce a classification of polytopes in $\mathbb{R}^2$ which satisfy Delzant's conditions. That is, we regard a family of polytopes as the same class if they correspond to the same Kähler class on a toric surface. This classification is built upon the fundamental property of toric surfaces stated above.

In §4.2, we prove that among all toric surfaces with 5 or 6 T^2-fixed points, $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is the only one which admits Kahler classes with vanishing Futaki invariant. The verification of vanishing Futaki invariant involves complicated computation, as we have to check all Delzant's polytopes one by one. This was done in [WZho] but in this thesis we present a different verification for some cases of the polytopes.

In §4.3, we prove that the Kähler classes with vanishing Futaki invariant on $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ are K-stable. To verify the K-stability, by Definition 2.7 and Proposition 2.9, we need to show that the linear functional $\mathcal{L}(u) > 0$ for *all* nontrivial simple PL functions. Again the verification of K-stability is technically a difficult problem, even the number of vertices is 6.

In Part III, we study Abreu's equation. It is a fourth order partial differential equation and resembles in certain aspects to the affine maximal surface equation arising from affine geometry. We study the Bernstein theorem and the first boundary value problem for this equation.

In Chapter 5 we prove the Bernstein theorem for Abreu's equation in dimension 2. That is, we prove a smooth convex solution to

$$\sum_{i,j=1}^{2} (u^{ij})_{x_i x_j} = 0$$

in the entire space $\mathbb{R}^2$ is a quadratic function, where (u^{ij}) is the inverse matrix of (u_{ij}).

For the affine maximal surface equation this result was proved in [TW1]. Our proof is based on the a priori estimates and a rescaling argument (§5.5). This idea is similar to that for the Monge-Ampère equation [P] and the affine maximal surface equation [TW1]. But Abreu's equation is not invariant under linear transformation of coordinates $\mathbb{R}^{n+1}$. When we rotate the coordinates in $\mathbb{R}^{n+1}$ we get a more complicated 4th order pde (§5.3). We need to establish not only the a priori estimates for Abreu's equation (§5.2) but also for the new equation (§5.3).

The a priori estimates also rely on the strict convexity of solutions, which involves subtle convexity analysis, and is done in §5.4. Our convexity analysis follows in a certain way the treatment in [TW1, 5]. In §5.6 we consider a variant of the Bernstein theorem. That is we prove in dimensions 2-4 that a solution to Abreu's equation is a quadratic function if its graph is complete when equipped with Calabi's metric.

In Chapter 6, we deal with a boundary value problem for Abreu's equation, which can be formulated as a variational problem for the energy functional

$$J_0(u) = \int_\Omega \log\det D^2u\, dx - \int_\Omega fudx.$$

The Euler equation of the functional is $\sum_{i,j=1}^n (u^{ij})_{x_ix_j} = f$. We prove that in dimension 2, there exists a unique, smooth convex maximizer of J_0 in

$$S[\varphi,\Omega] = \{u \in C^2(\overline{\Omega}) \mid u \text{ is convex } u|_{\partial\Omega} = \varphi(x), Du(\Omega) \subset D\varphi(\overline{\Omega})\}, \qquad (2.16)$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and φ is a convex, smooth function defined in a neighborhood of $\overline{\Omega}$.

The proof is inspired by Trudinger-Wang's variational approach and regularity arguments in solving the affine Plateau problem. But due to the singularity of the function $\log d$ near $d = 0$ ($d = \det D^2u$), the approximation argument in [TW3, 5] does not apply directly to our problem. To avoid this difficulty we introduce in §6.2 a sequence of modified functionals J_k to approximate J_0, such that the integrand in J_k is Holder continuous at $d = 0$, and prove the existence and uniqueness of a maximizer of the functional J_k in the set $\overline{S}[\varphi,\Omega]$.

The regularity of the maximizer is our main concern. In §6.3 we establish a uniform (in k) a priori estimates for the corresponding Euler equations. In §6.4, we establish the uniform (in k) a priori estimates for the equations obtained after coordinates rotation in $\mathbb{R}^{n+1}$.

As the maximizer may not be smooth, to apply the a priori estimates we need to prove that the maximizer can be approximated by smooth solutions. We cannot prove the approximation for the functional J_0 directly as $\log d$ is singular near $d = 0$. But for maximizers of J_k, the approximation can be proved similarly as for the affine Plateau problem [TW3, TW5]. We include the proof in §6.5 and §6.6 for completeness.

The a priori estimates also relies on the strict convexity of solutions. The proof for one case is similar to that in Chapter 5 (§6.7) but for the other case the proof uses the a priori estimates, the Legendre transform and in particular a strong approximation Theorem 6.21 and is contained in §6.8.

Part II

Toric surfaces

Chapter 3

Existence and nonexistence

In this chapter focus on the two dimensional case. We will not distinguish K-stability and relative K-stability for toric degenerations. We always call K-stability. We first show that every toric surface admits an extremal metrc in Section 3.1 following the recent work [APS]. Then in Section 3.2, we present some examples of unstable Kähler class on toric surfaces. A further reduction to simple PL functions for the verification of K-stability will be given in Section 3.3. Results in this chapter are contained in [WZho].

3.1 An existence theorem

In this section, we show the following result.

Theorem 3.1. *On every toric Kähler surface M, there is a Kähler class such that M admits an extremal metric of Calabi in the class.*

Theorem 3.1 is essentially due to [APS]. Let us recall a main result in [APS]. Let (M, ω_g) be an n-dimensional compact Kähler manifold whose associated Kähler metric g is an extremal metric,

$$\Xi_S = J\nabla S + \sqrt{-1}\nabla S$$

is a holomorphic vector field on M, where S is the scalar curvature and J is the complex stucture. Suppose that G is a compact subgroup of $Isom(M, g)$ and its Lie algebra is $\mathfrak{g}$. Denote by $\mathfrak{h}$ the vector space of G-invariant hamiltonian real-holomorphic vector fields on M. Then

Theorem 3.2 ([APS, Theorem 2.1]). *Suppose (M, g) is extremal and $\mathfrak{h} \subset \mathfrak{g}$, $J\nabla s \in \mathfrak{g}$. Let G_0 be the identity component of G. Given $p_1, ..., p_m \in Fix(G_0)$*

and $a_1, ..., a_m > 0$ such that $a_{j_1} = a_{j_2}$ if p_{j_1} and p_{j_2} are in the same G-orbit, there exists $\epsilon_0 > 0$ and, for any $\epsilon \in (0, \epsilon_0)$, there exists a G-invariant extremal Kähler metric ω_ϵ on the blow-up space $\tilde{M}$ at $p_1, ..., p_m$, such that ω_ϵ lies in the class

$$\pi^*[\omega] - \epsilon^2(a_1^{\frac{1}{n-1}} PD[E_1] + ... + a_m^{\frac{1}{n-1}} PD[E_m]), \tag{3.1}$$

where $\pi : \tilde{M} \to M$ is the standard projection map, $PD[E_j]$ are the Poincare duals of the $(2n-2)$-homology class of the exceptional divisors of the blow up at p_j.

Note that when M is a toric manifold and $G = G_0$ is the compact torus action T^n, one has $\mathfrak{h} = \mathfrak{g}$ and $J\nabla s \in \mathfrak{g}$, so the conditions in the theorem hold automatically, as pointed out in corollary 2.2 in [APS].

Now we apply the above result to toric Kähler surfaces. It is known that every compact toric Kähler surface can be obtained from $\mathbb{CP}^2$ or Hirzebruch surfaces $\mathbb{F}_k$ $(k = 0, 1, 2, \cdots)$ by a succession of blow-ups at $T^2_{\mathbb{C}}$-fixed points [Ful]. More precisely, let M be a toric surface with Kähler class K corresponding to a polytope P. Then a $T^2_{\mathbb{C}}$-fixed point X of M corresponds to a vertex p of the polytope P. A blow-up of M at X is a new toric Kähler surface which corresponds to a convex polytope $\tilde{P}$ obtained by chopping off a corner of the polytope P at p. Moreover, it is known [Ful] that a Delzant polytope with m vertices $(m \geq 5)$ can be obtained by chopping off a corner from a Delzant polytope with $m-1$ vertices. Theorem 3.2 means that if there is an extremal metric on M, in the Kähler class K, then there is an extremal metric on $\tilde{M}$ in the class $\tilde{K}$ corresponding to $\tilde{P}$, provided the chopped-off corner is small and $\tilde{P}$ satisfies Delzant's conditions. Furthermore, the Kähler class $\tilde{K}$ is exactly the class given in (3.1) [Gui].

It is well known that on $\mathbb{CP}^2$, the Fubini-Study metric is a Kähler-Einstein metric in the first Chern class. For the Hirzebruch surface $\mathbb{F}_k$, the existence of extremal metrics in any Kähler class can be reduced to an ODE and can be found in [Cal3]. Therefore Theorem 3.1 follows from the above Theorem 3.2.

3.2 Unstable examples

Donaldson [D2] found Delzant polytopes with large number of vertices which are not K-stable. Here we provide different examples. We wish to find an unstable Delzant's polytope with least number of vertices.

Theorem 3.3. *For any $m > 8$, there exists a toric Kähler surface $M^{(m)}$ with unstable Kähler class, and so there is no extremal metric on $M^{(m)}$ in the class, where $M^{(m)}$ denotes a toric surface with m $T^2_{\mathbb{C}}$-fixed points.*

3.2.1 An example

Our first example is symmetric with respect to both the x_1 and the x_2 axes. So it suffices to give the vertices in the positive quarter

$$\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0\}.$$

Let α, n ($\alpha >> n > 1$) be integers to be determined later. The intersection of ∂P with the positive axes are the two points $(0, \alpha^*)$ and $(n+1, 0)$ (both points are not vertex of P), where

$$\alpha^* = \alpha + \frac{n(n+1)}{2}.$$

The vertices in the positive quarter are given by

$$\begin{aligned}
p_1 &= (1, \alpha^*), \\
p_2 &= (2, \alpha^* - 1), \\
p_3 &= (3, \alpha^* - (1+2)), \\
&\cdots\cdots \\
p_{k+1} &= (k+1, \alpha^* - (1+2+\cdots+k)), \\
&\cdots\cdots \\
p_{n+1} &= (n+1, \alpha^* - (1+2+\cdots+n)) = (n+1, \alpha).
\end{aligned}$$

The vertices in other quarters are reflections of $p_1, \cdots, p_{n+1}$ in the axes.

Let E_0 be the edge connecting the vertex $(-1, \alpha^*)$ to p_1, and E_k be the edge connecting the vertices p_k and p_{k+1}, $k = 1, 2, \cdots, n$, and E_{n+1} be the edge connecting p_{n+1} to $(n+1, -\alpha)$. Let ℓ_i be a normal at the edge E_i, given by

$$\begin{aligned}
\ell_0 &= (0, 1), \\
\ell_1 &= (1, 1), \\
\ell_2 &= (2, 1), \\
&\cdots\cdots \\
\ell_k &= (k, 1), \\
&\cdots\cdots \\
\ell_n &= (n, 1), \\
\ell_{n+1} &= (1, 0).
\end{aligned}$$

One easily verifies the polytope P satisfies Delzant's conditions.

We show that the polytope P given above is not K-stable when α, n are sufficiently large ($\alpha >> n$). As was pointed out above, it suffices to show that

$$\mathcal{L}(u) = \int_{\partial P} u\, d\sigma - \int_P Au\, dx < 0 \tag{3.2}$$

for some PL function u. Note that when computing the integral $\int_{\partial P} d\sigma$, we have

$$\begin{aligned} &\int_{E_0} d\sigma = 2, \\ &\int_{E_i} d\sigma = 1 \text{ for } i = 1, \cdots, n, \\ &\int_{E_{n+1}} d\sigma = 2\alpha. \end{aligned} \tag{3.3}$$

Note that P is symmetric with respect to the axes, the linear function A is necessarily a constant, namely $A = a_0$ (so an extremal metric must be a CSC metric if it exists). To compute the constant a_0, letting $u_0 \equiv 1$ we have

$$\begin{aligned} \mathcal{L}(u_0) &= 4(\alpha + n + 1) - a_0|P| \\ &= 4(\alpha + n + 1) - 4a_0[\alpha(n+1) + O(n^3)], \end{aligned}$$

where $|P|$ is the area of the polytope P. Hence

$$a_0 = \frac{1}{n+1} + O(\frac{1}{\alpha}). \tag{3.4}$$

We choose $\alpha >> n$ such that $O(\frac{1}{\alpha})$ is so small that can be neglected.

Now we choose the function

$$\hat{u} = \max\{0, x_2 - \alpha\}. \tag{3.5}$$

It is a simple PL function, with crease $\mathcal{I}_{\hat{u}} = P \cap \{x_2 = \alpha\}$. By (3.3), one easily verifies that

$$\int_{\partial P} \hat{u}\, d\sigma \le 2(n+1) \sup_P \hat{u} = n(n+1)^2$$

and

$$\int_P \hat{u} dx \ge |\hat{P}| \inf_{\hat{P}} \hat{u} = \frac{n^5}{128},$$

where $\hat{P} = P \cap \{\alpha + \frac{1}{8}n^2 \le x_2 \le \alpha + \frac{1}{4}n^2\}$. Hence when n is sufficiently large and when $\alpha >> n$, by (3.4) we have

$$\mathcal{L}(\hat{u}) = \int_{\partial P} \hat{u}\, d\sigma - a_0 \int_P \hat{u}\, dx < 0. \tag{3.6}$$

Hence, the corresponding toric surface is not K-stable. Note that the polytope P above is integral, namely the corresponding constants λ_i in (2.1) are integers.

3.2.2 Unstable polytopes with less vertices

It is interesting to find polytopes with less vertices such that the corresponding toric manifolds are not K-stable. In dimension 2, if the polytope has 3 vertices, it can only be $\mathbb{CP}^2$. If it has 4 vertices, the toric manifold must be a Hirzebruch surface. Both of them admit extremal metrics in any Kähler class. Hence, in dimension 2, a polytope of an unstable toric manifold has at least 5 vertices.

Let us first consider the polytope P in Section 3.2.1. We want to find the least n such that P is not K-stable. Instead of the test function (3.5), now we choose

$$\hat{u} = \max\{0, x_2 - \alpha + k\}, \tag{3.7}$$

so that $\hat{u} = 0$ when $x_2 \le \alpha - k$ and $\hat{u}$ is linear when $x_2 \ge \alpha - k$. Let $n = 3$. We have

$$\begin{aligned} \int_{\partial P} \hat{u}\, d\sigma &= k^2 + 8k + O(1), \\ \int_P \hat{u}\, dx &= 4k^2 + 34k + O(1), \end{aligned}$$

where $O(1)$ are absolute constants (depending only on n, but here $n = 3$ is fixed), and the number 34 is the area of $P \cap \{x_2 > \alpha\}$. Hence when k is sufficiently large and $\alpha >> k$,

$$\mathcal{L}(\hat{u}) = \int_{\partial P} \hat{u}\, d\sigma - \frac{1}{4} \int_P \hat{u}\, dx = -\frac{1}{2}k + O(1) < 0. \tag{3.8}$$

Therefore when $n = 3$, the corresponding toric manifold is not K-stable. The polytope P has totally 16 vertices.

Let

$$P' = P \cap \{x_2 > -\alpha\}, \tag{3.9}$$

where P is the above polytope, with $n = 3$, so that P' has only two vertices in $\{x_2 < 0\}$ and has totally 10 vertices. It is symmetric in x_1 but not x_2. The linear function A associated with P' has now the form $A = a_0 + a_2 x_2$. By direct computation,

$$a_0 = \frac{1}{4} + O(\frac{1}{\alpha}), \quad a_2 = O(\frac{1}{\alpha^2}).$$

We can choose α large enough such that $O(\frac{1}{\alpha})$ and $O(\frac{1}{\alpha^2})$ are sufficiently small and can be ignored in the computation of $\mathcal{L}(\hat{u})$, where $\hat{u}$ is given in (3.7). Hence when k is sufficiently large and $\alpha >> k$, as above we have $\mathcal{L}(\hat{u}) < 0$. Hence the corresponding toric manifold is not K-stable.

Let P^* be the polytope with vertices given by

$$\begin{aligned}
p_0 &= (0, \alpha^*),\\
p_1 &= (1, \alpha^*),\\
p_2 &= (2, \alpha^* - 1),\\
p_3 &= (3, \alpha^* - (1+2)),\\
p_4 &= (4, \alpha^* - (1+2+3)),\\
p_5 &= (5, \alpha^* - (1+2+3+4)),\\
p_6 &= (7, \alpha^* - (1+2+3+4+10)) = (7, \alpha),\\
p_7 &= (7, -\alpha),\\
p_8 &= (0, -\alpha),
\end{aligned}$$

where $\alpha > 0$ is a large constant and $\alpha^* = \alpha + (1+2+3+4+10)$. Then

$$a_0 = \frac{2}{7} + O(\frac{1}{\alpha}), \quad a_1 = O(\frac{1}{\alpha}), \quad a_2 = O(\frac{1}{\alpha^2}).$$

Let $\hat{u}$ be the test function in (3.7). Denote $u_t = \max\{0, x_2 - \alpha + t\}$. Then when $\alpha >> t >> 1$, we have

$$\frac{d}{dt}\mathcal{L}(u_t) = \int_{\partial P^* \cap \{x_2 > \alpha\}} d\sigma - \int_{P^* \cap \{x_2 > \alpha\}} A dx + O(\frac{1}{\alpha}).$$

Therefore if

$$\int_{\partial P^* \cap \{x_2 > \alpha\}} d\sigma \leq -\delta + \int_{P^* \cap \{x_2 > \alpha\}} A dx \tag{3.10}$$

for some $\delta > 0$ independent of α, then $\mathcal{L}(\hat{u}) < 0$ when k is sufficiently large. Direct computation gives

$$\int_{\partial P^* \cap \{x_2 > \alpha\}} d\sigma = 27, \quad \int_{P^* \cap \{x_2 > \alpha\}} A dx = 27\frac{1}{7}.$$

The polytope P^* has totally 9 vertices. It is the polytope of least vertices we have found such that the corresponding toric manifold is not K-stable.

An interesting question is whether a polytope $P \subset \mathbb{R}^2$ is K-stable if it has 8 or less vertices. We believe the answer is yes for polytopes with 7 or less corners. Our computation suggests the case of 8 corners is the borderline case. The verification of K-stability is technically a difficult problem, even for the polytope with 5 vertices. In the next chapter, we consider the K-stabilty in the case A is constant.

3.3 K-stability and simple PL functions

Let P be a convex polytope $P \subset \mathbb{R}^n$, $n \geq 2$. As in [D2] we denote by $\mathcal{C}_1$ the set of convex functions f on $\overline{P}$ such that $\int_{\partial P} f \, d\sigma < \infty$. Note that for a convex function $f \in \mathcal{C}_1$, (i) f is locally uniformly Lipschitz continuous in P; (ii) when restricted to a codimension 1 face of P, f is also a convex function, (iii) f may not be continuous near the boundary, such as the function $f = 0$ in P and $f = 1$ on ∂P; (iv) f may not be uniform bounded at the vertices of P, but the value of f at vertices has no effect on the integral $\int_{\partial P} f \, d\sigma$. In this section we prove

Theorem 3.4. *Let P be a convex polytope $P \subset \mathbb{R}^2$. Suppose that $\mathcal{L}(u) \geq 0$ for all convex functions $u \in \mathcal{C}_1$ but there is a nonlinear convex function $u \in \mathcal{C}_1$ such that $\mathcal{L}(u) = 0$. Then there is a simple PL function $\hat{u}$ with its crease $\mathcal{I}_{\hat{u}} \neq \emptyset$ such that $\mathcal{L}(\hat{u}) = 0$.*

Theorem 3.4 was proved by Donaldson [D2] under the assumption $A \geq 0$ (Proposition 5.3.1, [D2]). Here we remove the condition $A \geq 0$. Theorem 3.4 is needed if one wishes to verify the K-stability of polytopes. Namely to verify the K-stability for a polytope $P \in \mathbb{R}^2$, by Theorem 3.4 it suffices to verify $\mathcal{L}(u) \geq 0$ for all simple PL functions u.

Note that in Theorem 3.4, P can be any polytope, or any bounded convex domain. From the proof of Theorem 3.4, we also have the following

Corollary 3.5. *Let P be a convex polytope $P \subset \mathbb{R}^2$. If there is a convex function $u \in \mathcal{C}_1$ such that $\mathcal{L}(u) < 0$, then there is a simple PL function $\hat{u}$ such that $\mathcal{L}(\hat{u}) < 0$.*

To prove Theorem 3.4, we first introduce some terminologies and related properties.

Extreme point. Let Ω be a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$. A boundary point $z \in \partial\Omega$ is an extreme point of Ω if there is a hyperplane L such that $z = L \cap \partial\Omega$, namely z is the unique point in $L \cap \partial\Omega$. It is known that any interior point of Ω can be expressed as a linear combination of extreme points of Ω. If Ω is a convex polytope, a boundary point $z \in \partial\Omega$ is an extreme point if and only if it is a vertex of Ω.

Supporting plane. Let u be a convex function in a domain $\Omega \subset \mathbb{R}^n$ and $z \in \Omega$ be an interior point. A hyperplane L, given by $L = \{x_{n+1} = \phi(x) \mid x \in \mathbb{R}^n\}$, is a supporting plane of u at z if $u(z) = \phi(z)$ and $u(x) \geq \phi(x)$ for any $x \in \Omega$. When

u is C^1 at z, then there is a unique supporting plane, which is the tangent plane, of u at z. For convenience we call ϕ the supporting function of u at z.

Normal mapping. Let u be a convex function in a domain $\Omega \subset \mathbb{R}^n$ and $z \in \Omega$ be an interior point. The normal mapping of u at z, $N_u(z)$, is the set of gradients of the supporting functions of u at z. For any subset $\Omega' \subset \Omega$, denote $N_u(\Omega') = \bigcup_{z \in \Omega'} N_u(z)$. If u is C^1, the normal mapping N_u is exactly the gradient mapping Du.

A degenerate Monge-Ampère equation. Let Ω be a bounded convex domain in $\mathbb{R}^n$, and u_0 be a convex function on $\overline{\Omega}$. Then

$$u(x) = \sup\{\ell(x) \mid \ell \text{ is a linear function in } \overline{\Omega} \text{ with } \ell \leq u_0 \text{ on } \partial\Omega\} \tag{3.11}$$

is the unique convex solution (generalized solution in the sense of Aleksandrov) to the Monge-Ampère equation [Gut, TW4]

$$\det D^2 u = 0 \tag{3.12}$$

in Ω, subject to the Dirichlet boundary condition $u = u_0$ on $\partial\Omega$. Here we say a (nonsmooth) convex function u is a generalized solution to the degenerate Monge-Ampère equation (3.12) if $\mu_u = 0$, where the measure μ_u is defined by

$$\mu_u(\omega) = |N_u(\omega)| \tag{3.13}$$

for any Borel set $\omega \subset \Omega$, and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$. It is well known that μ_u is a measure defined in Ω, and is called the Monge-Ampère measure [TW4]. When u is a smooth convex function,

$$\mu_u(\omega) = \int_\omega \det D^2 u \, dx.$$

A basic property of the Monge-Ampère measure is that if a sequence of convex function u_m converges to u (locally) uniformly in Ω, then $\mu_{u_m} \to \mu_u$ weakly [Gut, TW4].

Let u be a generalized solution of (3.12), given by (3.11). For any interior point $z \in \Omega$, let $L_z = \{x_{n+1} = \phi(x), x \in \mathbb{R}^n\}$ be a supporting plane of u at z. By convexity, the set $\mathcal{T} := \{x \in \Omega \mid u(x) = \phi(x)\}$ is convex. By (3.11), $\mathcal{T}$ cannot be a single point.

Lemma 3.6. *An extreme point of $\mathcal{T}$ must be a boundary point of Ω.*

Lemma 3.6 is often used in the study of Monge-Ampère equation. It can be proved as follows. If there is an interior point $y \in \Omega$ which is an extreme point of $\mathcal{T}$, by choosing proper coordinates we assume that $y = 0$, $\mathcal{T} \subset \{x_n < 0\}$, and the origin 0 is the only point of $\mathcal{T} \cap \{x_n = 0\}$. By subtracting a linear function, we assume that $L_z = \{x_{n+1} = 0\}$, namely $\phi \equiv 0$. Then for a sufficiently small $\epsilon > 0$, since the origin is an extreme point of $\mathcal{T}$, we must have $u(x) > \epsilon(x_n + 1)$ on $\partial\Omega$, which is in contradiction with (3.11).

The above results are well known to researchers in the real Monge-Ampère equation. We are now in position to prove Theorem 3.4.

Proof of Theorem 3.4. Denote $P_+ = \{x \in P \mid A(x) > 0\}$, $P_- = \{x \in P \mid A(x) < 0\}$. If u_1 and u_2 are two convex functions in $\mathcal{C}_1$ satisfying $u_1 \geq u_2$ in $\overline{P}_+$, $u_1 \leq u_2$ in $\overline{P}_-$, and $u_1 = u_2$ on ∂P, then $\mathcal{L}(u_1) \leq \mathcal{L}(u_2)$.

(i) Since for any codimension 1 face F of P, the area of F is bounded by $C\int_F d\sigma$ for some constant $C > 0$ depending only on P, there exist a small positive constant $\delta_0 > 0$, depending only on P, such that for any simple PL function $\hat{u}$ with the Lebesgue measure $|\{x \in P \mid \hat{u}(x) > 0\}| \leq \delta_0$, we have $\mathcal{L}(\hat{u}) > 0$. Therefore if $\mathcal{L}(u) > 0$ for all simple PL functions, there exists $\sigma_0 > 0$ such that $\mathcal{L}(u) > \sigma_0$ for any simple PL function $u = \max(0, \ell)$ with $|D\ell| = 1$ and $|\{x \in P \mid u(x) > 0\}| \geq \delta_0$.

(ii) Let u be a nonlinear convex function in $\mathcal{C}_1$, which is not simple PL, such that $\mathcal{L}(u) = 0$. We show that u is continuous at any codimension 1 face F (not including its codimension 2 boundary). Indeed, for any $x_0 \in F$, since u is convex, one easily verifies that $\overline{\lim}_{x\in P, x\to x_0} u(x) = \underline{\lim}_{x\in P, x\to x_0} u(x)$. Hence we can define a convex function $\tilde{u} \in \mathcal{C}_1$ by letting $\tilde{u} = u$ in P and $\tilde{u} = \lim_{x\in P, x\to x_0} u(x)$ for $x_0 \in F$ (the value of $\tilde{u}$ on the codimension 2 edges does not affect the integral $\int_{\partial P} d\sigma$). If there is a point $x_0 \in F$ at which $\tilde{u} < u$, then we have $\mathcal{L}(\tilde{u}) < \mathcal{L}(u) = 0$, a contradiction with the assumption that $\mathcal{L}(v) \geq 0$ for any $v \in \mathcal{C}_1$.

Let

$$u_+(x) = \sup\{\ell(x) \mid \ell \text{ is a linear function with } \ell \leq u \text{ in } P_- \cup \partial P\}. \tag{3.14}$$

Then $u_+ = u$ in P_- and on ∂P, and $u_+ \geq u$ in P_+. If there is a point $x \in P_+$ such that $u_+(x) > u(x)$, then $\mathcal{L}(u_+) < \mathcal{L}(u) = 0$, in contradiction with the assumption that $\mathcal{L}(v) \geq 0$ for all $v \in \mathcal{C}_1$. Hence $u_+ = u$ in P. By (3.11), u satisfies the degenerate Monge-Ampère equation (3.12) in P_+.

Next let

$$u_-(x) = \sup\{\ell(x) \mid \ell \text{ is a supporting function of } u \text{ at some point } x \in P_+\}. \tag{3.15}$$

Then $u_- = u$ in $\overline{P}_+$ and $u_- \leq u$ in $\overline{P}_-$. If there is a point $x \in P_-$ such that $u_-(x) < u(x)$, then $\mathcal{L}(u_-) < \mathcal{L}(u) = 0$, contradicting the assumption that $\mathcal{L}(v) \geq 0$ for all $v \in \mathcal{C}_1$. Hence $u_- = u$ in P.

We claim that u satisfies the degenerate Monge-Ampère equation (3.12) in the whole polytope P. Indeed, we have shown $\det D^2 u = 0$ in P^+. Hence $|N_u(P^+)| = 0$. Note that the function u_- in (3.15) can be approximated by

$$u_-^\epsilon(x) = \sup\{\ell(x) \mid \ell \text{ is a supporting function of } u \text{ at some point } x \in P_+^\epsilon\}, \tag{3.16}$$

where $P_+^\epsilon = \{x \in P \mid A(x) \geq \epsilon\} \subset P_+$. By (3.16), a supporting plane of u_-^ϵ at some point in P must also be a supporting plane of u at some point in P_+^ϵ. But since u is a generalized solution of (3.12) in P_+, we have $|N_u(P_+^\epsilon)| = 0$ by definition. Hence $|N_u(P)| = 0$ and so u_-^ϵ is a generalized solution to (3.12) in P. By the weak convergence of the Monge-Ampère measure, $u = \lim_{\epsilon\to 0} u_-^\epsilon$ is also a generalized solution to (3.12) in P.

Alternatively, the claim that u_-^ϵ is a solution to (3.12) also follows from a theorem of Aleksandrov, which states that for any convex function $w \in C(\Omega)$, the set $\{p \in \mathbb{R}^n \mid p \in N_w(x) \cap N_w(y)$ for two different points $x, y \in \Omega\}$ has measure zero, because a supporting plane of u_-^ϵ at some point in P_- is also a supporting plane of u at some point in P_+^ϵ.

(iii) Let x_0 be the mass center of P. By a translation of the coordinates we assume $x_0 = 0$ is the origin. Let $L = \{x_{n+1} = \phi(x)\}$ be a supporting plane of u at x_0. By subtracting a linear function we assume that $\phi \equiv 0$. By Lemma 3.6, the extreme points of $\mathcal{T} = \{x \in P \mid u(x) = 0\}$ are located on ∂P.

Assume $n = 2$. Then $\mathcal{T}$ is either a line segment with both endpoint on ∂P, or $\mathcal{T}$ is a polytope (which is a convex subset of P) with vertices on ∂P, by Lemma 3.6.

By a rotation of the coordinates, we assume $\mathcal{T}$ is contained in the x_1-axis in the former case, or an edge of $\mathcal{T}$ is contained in the x_1-axis and $\mathcal{T} \subset \{x_2 \leq 0\}$ in the latter case. For any point $(x_1, 0) \in P$, let

$$a(x_1) = \lim_{t \searrow 0} \frac{1}{t}(u(x_1, t) - u(x_1, 0)).$$

By convexity the limit exists and is nonnegative. Let $a_0 = \inf a(x_1)$. We must have $a_0 = 0$, otherwise denote

$$\psi(x) = \max(0, x_2), \tag{3.17}$$

and $u_1 = u\chi_-$, $u_2 = (u - a_0\psi)\chi_+$, where $\chi_- = 1$ in $\{x_2 < 0\}$ and $\chi_- = 0$ in $\{x_2 > 0\}$, and $\chi_+ = 1 - \chi_-$. Then ψ is a simple PL function,

$$u = u_1 + u_2 + a_0\psi,$$

and $u_1 + u_2$ is convex in P. Hence

$$\mathcal{L}(u) = \mathcal{L}(u_1 + u_2) + a_0\mathcal{L}(\psi) \geq a_0\sigma_0 > 0,$$

where $\sigma_0 > 0$ is the constant in (i) above. We reach a contradiction.

Since $u > 0$ in $P \cap \{x_2 > 0\}$ and the set $G_\epsilon := \{x \in P \mid u(x) < \epsilon\psi(x)\} \neq \emptyset$, we have

$$G_\epsilon \subset \{0 \leq x_2 < \delta\} \quad \text{with } \delta \to 0 \text{ as } \epsilon \to 0 \tag{3.18}$$

(otherwise by taking limit we would reach a contradiction as $\mathcal{T} \subset \{x_2 \leq 0\}$). Denote

$$\begin{aligned} u_1 &= u\chi_-, \\ u_2 &= (u - \epsilon\psi)\chi_+, \\ \tilde{u}_2 &= \max(u_2, 0). \end{aligned}$$

Then $u = u_1 + u_2 + \epsilon\psi$ and $u_1 + \tilde{u}_2$ is convex in P. Denote $\tilde{u} = u_1 + \tilde{u}_2 + \epsilon\psi$. By (ii) above we have

$$\mathcal{L}(\tilde{u}) = \mathcal{L}(u_1 + \tilde{u}_2) + \epsilon\mathcal{L}(\psi) \geq \epsilon\mathcal{L}(\psi) \geq \epsilon\sigma_0.$$

On the other hand, observing that $0 \leq \tilde{u}_2 - u_2 \leq \epsilon\delta$, we have $u \leq \tilde{u} \leq u + \epsilon\delta$. It follows that

$$\mathcal{L}(\tilde{u}) \leq \mathcal{L}(u) + C\epsilon\delta = C\epsilon\delta.$$

We obtain $\epsilon\sigma_0 \leq C\epsilon\delta$. But recall that $\delta \to 0$ as $\epsilon \to 0$. Hence when $\epsilon > 0$ is sufficiently small, we reach a contradiction. □

Remark 3.7. Part (iii) in the above argument does not apply directly to high dimensions, because (3.18) holds only when $\{x_n = 0\} \cap P \subset \mathcal{T}$, which is usually not true in high dimensions.

Chapter 4

Constant scalar curvature metrics

It is known that a toric surface with 3 or 4 $T^2_{\mathbb{C}}$-fixed points must be $\mathbb{CP}^2$ or a Hirzebruch surface, and the Futaki invariant vanishes only for $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. When a toric surface has large number of $T^2_{\mathbb{C}}$-fixed points, the verification of vanishing Futaki invariant is technically a complicated problem. In Section 4.2 we prove that among all toric surfaces of 5 or 6 $T^2_{\mathbb{C}}$-fixed points, $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is the only one which allows vanishing Futaki invariant. In Section 4.3 we will check the K-stability of $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. By Donaldson's recent work, this means the existence of constant scalar curvature metrics on the toric surface. For general toric surfaces with $m \geq 5$ $T^2_{\mathbb{C}}$-fixed points, the verification of K-stability is also a very complicated problem. We still denote by $M^{(m)}$ a toric surface with m $T^2_{\mathbb{C}}$-fixed points.

4.1 Equivalent class

Let $P^{(m)}$ be a Delzant polytope with m vertices $p_0, p_1, \cdots, p_m$ with $p_0 = p_m$. Let E_i be the edge connecting p_i and p_{i+1}, ℓ_i be the normal to the edge E_i. Also we set $\ell_m = \ell_0$, $E_\beta = E_0$.

Denote

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} : \ q_{11}q_{22} - q_{12}q_{21} = \pm 1, \ q_{ij} \in \mathbb{Z} \right\}. \tag{4.1}$$

For any Delzant polytope P, and any transform $Q \in SL(2, \mathbb{Z})$, $P' = QP$ is also a Delzant polytope. From [De], the corresponding Kähler manifold $M_{P'}$ is symplectomorphic to M_P, and $M_{P'}$ has the same complex structure and Kähler class. By this property we can introduce equivalent classes for Delzant polytopes.

Definition 4.1. We say two Delzant polytope P and P' are *equivalent* if there exists a transform $Q \in SL(2,\mathbb{Z})$ such that after proper translation and dilation, $P' = QP$. For any Delzant polytope, we denote by $[P]$ the equivalent class.

For any Delzant polytope P, and any vertex p of P, we can make a translation and transform $Q \in SL(2,\mathbb{Z})$ such that p is the origin, the edges at p lie in the coordinates axes, and P is contained in the positive quarter $\{x_1 > 0, x_2 > 0\}$. It implies that at any vertex of a Delzant polytope $M^{(m)}$, one can chop off the corner to get a new Delzant polytope $M^{(m+1)}$.

It is easy to verify that a Delzant polytope with 3 vertices must be equivalent to the polytope with vertices

$$\begin{aligned} p_0 &= (0,0), & \ell_0 &= (-1,0), \\ p_1 &= (0,1), & \ell_1 &= (1,1), \\ p_3 &= (1,0), & \ell_2 &= (0,-1). \end{aligned} \tag{4.2}$$

It is also known [Ful] that a Delzant polytope with 4 vertices must be equivalent to $P^{(4)}[k]$ for some $k = 0,1,2,\cdots$, where $P^{(4)}[k]$ is the polytope with vertices

$$\begin{aligned} p_0 &= (0,0), & \ell_0 &= (-1,0), \\ p_1 &= (0,h), & \ell_2 &= (-k,1), \\ p_2 &= (1,h+k), & \ell_3 &= (1,0), \\ p_3 &= (1,0), & \ell_3 &= (0,-1), \end{aligned} \tag{4.3}$$

where $h > 0$. Note that $P^{(4)}[k]$ has two parallel edges E_0 and E_2.

To work out the equivalent classes of Delzant polytopes with 5 or 6 vertices, we need the following lemma.

Lemma 4.2 ([Ful]). *For $m \geq 4$, every Delzant polytope $P^{(m+1)}$ can be obtained by chopping off a corner from a Delzant polytope $P^{(m)}$.*

From (4.3) and Lemma 4.2 we see that a Delzant polytope $P^{(m)}$ with $m \geq 5$ must contain a pair of parallel edges, located respectively in $\{x_1 = 0\}$ and $\{x_1 = 1\}$ after proper choice of coordinates. From (4.3) and Lemma 4.2 we also have

Lemma 4.3. *A Delzant polytope $P^{(5)}$ must belong to an equivalent class of the polytope $P^{(5)}[k]$ for some $k = 0,1,2,\cdots$,*

$$\begin{aligned} p_0 &= (0,0), & \ell_0 &= (-1,0), \\ p_1 &= (0,h), & \ell_2 &= (-k-1,1), \\ p_2 &= (t, h+(k+1)t), & \ell_3 &= (-k,1), \\ p_3 &= (1, h+(k+1)t+k(1-t)), & \ell_3 &= (1,0), \\ p_4 &= (1,0), & \ell_4 &= (0,-1), \end{aligned} \tag{4.4}$$

where t, h are positive constants and $t < 1$.

Lemma 4.3 follows from Lemma 4.2 immediately. Note that if $P^{(5)}$ is obtained by chopping off the corner p_0 or p_3, we need a translation, a dilation and a transform $Q \in SL(2,\mathbb{Z})$ to get the expression (4.4).

Similarly by (4.4) and Lemma 4.2, we have

Lemma 4.4. *A Delzant polytope $P^{(6)}$ must belong to an equivalent class of the polytope $P^{(6)}[i;k]$ for some $i = 1, 2$ or 3 and some integer $k = 0, 1, 2, \cdots$, where*
(i) $P^{(6)}[1;k]$ is given by

$$\begin{aligned}
&p_0 = (0,0), && \ell_0 = (-1,0),\\
&p_1 = (0,h), && \ell_1 = (-k-2,1),\\
&p_2 = (t, h+(k+2)t), && \ell_2 = (-k-1,1),\\
&p_3 = (s+t, h+(k+2)t+(k+1)s), && \ell_3 = (-k,1),\\
&p_4 = (1, h+(k+2)t+(k+1)s+k(1-s-t)), && \ell_4 = (1,0),\\
&p_5 = (1,0), && \ell_5 = (0,-1),
\end{aligned} \tag{4.5}$$

where h, t, s are positive constants, $t + s < 1$.
(ii) $P^{(6)}[2;k]$ is given by

$$\begin{aligned}
&p_0 = (0,0), && \ell_0 = (-1,0),\\
&p_1 = (0,h), && \ell_1 = (-k-1,1),\\
&p_2 = (t, h+(k+1)t), && \ell_2 = (-2k-1,2),\\
&p_3 = (t+s, h+(k+1)t+(k+\tfrac{1}{2})s), && \ell_3 = (-k,1),\\
&p_4 = (1, h+(k+1)t+(k+\tfrac{1}{2})s+k(1-s-t)), && \ell_4 = (1,0),\\
&p_5 = (1,0), && \ell_5 = (0,-1),
\end{aligned} \tag{4.6}$$

where h, t, s are positive constants, $t + s < 1$.
(iii) $P^{(6)}[3;k]$ is given by

$$\begin{aligned}
&p_0 = (0,0), && \ell_0 = (-1,0),\\
&p_1 = (0,h), && \ell_1 = (-k-1,1),\\
&p_2 = (s, h+(k+1)s), && \ell_2 = (-k,1),\\
&p_3 = (1, h+(k+1)s+k(1-s)), && \ell_3 = (1,0),\\
&p_4 = (1,(1-t)), && \ell_4 = (1,-1),\\
&p_5 = (t,0), && \ell_5 = (0,-1),
\end{aligned} \tag{4.7}$$

where h, t, s are positive constants such that $t, s < 1$, and $k \geq 0$ is an integer.

The verification of Lemma 4.4 is straightforward, we leave it to the reader. But we would like to point out that if the new polytope P' is obtained by chopping off the corner $(0,0)$ of $P^{(5)}[k]$ in (4.4), then P' is in the equivalent class of $P^{(6)}[3;k+1]$. We also point out that $P^{(6)}[1;0]$ and $P^{(6)}[2;0]$ are in the same equivalent class. In (i) above, we can allow $k=-1$, but $P^{(6)}[1;-1]$ is in the same class as $P^{(6)}[3;1]$.

4.2 Polytopes with vanishing Futaki invariant

4.2.1 Futaki invariant

We verify whether the Futaki invariant vanishes for the polytopes given above.

Denote

$$\begin{aligned} &b_0=\int_{\partial P} d\sigma, \quad b_1=\int_{\partial P} x_1 d\sigma, \quad b_2=\int_{\partial P} x_1 d\sigma,\\ &v_0=\int_P dx, \quad v_1=\int_P x_1 dx, \quad v_2=\int_P x_1 dx,\\ &\mathcal{F}_1=v_0b_1-v_1b_0, \quad \mathcal{F}_2=v_0b_2-v_2b_0, \quad \mathcal{F}_3=v_1b_2-v_2b_1. \end{aligned}$$

By Proposition 2.4, the linear function $A=a_0+a_1x_1+a_2x_2$ is determined by

$$\mathcal{L}(1)=0,\ \mathcal{L}(x_1)=0,\ \mathcal{L}(x_2)=0.$$

Therefore a_0, a_1, a_2 satisfy

$$\begin{aligned} &a_0v_0+a_1v_1+a_2v_2=b_0,\\ &a_0v_1+a_1\int_P x_1^2+a_2\int_P x_1x_2=b_1,\\ &a_0v_2+a_1\int_P x_1x_2+a_2\int_P x_2^2=b_2. \end{aligned}$$

Recall that the Futaki invariant vanishes if and only if A is a constant, namely $a_1=a_2=0$. Since P is contained in $\{x_1>0,x_2>0\}$, we have $v_1,v_2>0$. Hence if the Futaki invariant vanishes, the above linear equations can be reduced to

$$\frac{b_0}{v_0}=\frac{b_1}{v_1}=\frac{b_2}{v_2}. \tag{4.8}$$

Therefore we obtain

Lemma 4.5. *The Futaki invariant vanishes if and only if* $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=0$.

With Lemma 4.5, we can verify when a Delzant polytope has vanishing Futaki invariant.

To estimate $\mathcal{F}_1$, we introduce another way of expression of $P^{(m)}$, $m\geq 5$. For convenience, we use $\{x,y\}$ instead of $\{x_1,x_2\}$ to stand for the coordinates. We still assume that a pair of edges are located respectively in $\{x = 0\}$ and $\{x = 1\}$ as in the last section. Furthermore, we may assume that the origin is a vertex and the other edge passing the origin is contained in $y = 0$. It is clear that all the other edges form two piecewise linear functions f_1, f_2. Then we have

$$P^{(m)} = \{(x, y) \mid f_2(x) \le y \le f_1(x)\}.$$

For simplicity, we can use the pair (f_1, f_2) to express the polytope. Without loss of generality, we may also assume that $\{y = f_1\}$ contains no less edges than $\{y = f_2\}$.

We first compute $\mathcal{F}_1$ with the data of (f_1, f_2). Note that f_1 is a concave function while f_2 is convex. We denote the following parameters:

$$h = f_1(0),\ a = f_1(1) - f_1(0),\ b = f_2(1),\ k = f_1'(1),\ l = f_2'(1),$$

where k, l are integers, $l \ge 0$ and $h, a, b > 0$, $b < a + h$. For polytopes given in §4.1, it is clear that $k \le a$, $b \le l$. Let $d\sigma_1$ and $d\sigma_2$ be the boundary measures on $\partial P^{(m)}$ at the boundary parts determined by $\{y = f_1(x)\}$ and $\{y = f_2(x)\}$, respectively. As we pointed out in (2.7), $d\sigma_i$ is piecewisely a scalar multiplication of dx with the scalars determined by the outer normals of the edges. We would like to point out in particular that all the parameters h, a, b, k, l and the boundary measures $d\sigma_i$ are not chosen arbitrarily. They come from a Delzant polytope in §4.1. Hence, by computation,

$$b_0 = 2h + a - b + \int_0^1 d\sigma_1 + \int_0^1 d\sigma_2, \quad v_0 = \int_0^1 f_1(x) - f_2(x)dx,$$
$$b_1 = h + a - b + \int_0^1 xd\sigma_1 + \int_0^1 xd\sigma_2, \quad v_1 = \int_0^1 x(f_1(x) - f_2(x))dx.$$

Furthermore, write

$$f_1 = h + ax + g_1, \quad f_2 = bx + g_2.$$

Then

$$\begin{aligned}
\mathcal{F}_1 &= v_0 b_1 - v_1 b_0 \\
&= \left(h + a - b + \int_0^1 xd\sigma_1 + \int_0^1 xd\sigma_2\right) \int_0^1 h + (a-b)x + g_1 - g_2 dx \\
&\quad - \left(2h + a - b + \int_0^1 d\sigma_1 + \int_0^1 d\sigma_2\right) \int_0^1 x[h + (a-b)x + g_1 - g_2]dx \\
&= Kh + L(a-b) + \left(h + a - b + \int_0^1 xd\sigma_1 + \int_0^1 xd\sigma_2\right) \int_0^1 g_1 - g_2 dx \\
&\quad - \left(2h + a - b + \int_0^1 d\sigma_1 + \int_0^1 d\sigma_2\right) \int_0^1 x(g_1 - g_2)dx,
\end{aligned}$$

where

$$\begin{aligned} K &= \frac{a-b}{3} + \int_0^1 x d\sigma_1 - \frac{1}{2}\int_0^1 d\sigma_1 + \int_0^1 x d\sigma_2 - \frac{1}{2}\int_0^1 d\sigma_2, \\ L &= \frac{a-b}{6} + \frac{1}{2}\int_0^1 x d\sigma_1 - \frac{1}{3}\int_0^1 d\sigma_1 + \frac{1}{2}\int_0^1 x d\sigma_2 - \frac{1}{3}\int_0^1 d\sigma_2. \end{aligned}$$

Now we treat $\mathcal{F}_1$ as a functional of (g_1, g_2). We consider more general functions (g_1, g_2). Define

$$\mathcal{S}_1 = \{g \mid g \text{ is concave and } \ g(0) = g(1) = 0,\ g'(1) = k - a\},$$

$$\mathcal{S}_2 = \{g \mid g \text{ is convex and } \ g(0) = g(1) = 0,\ g'(0) = -b,\ g'(1) = l - b\}.$$

Hence, we formulate a minimizing problem

$$\inf_{g_1 \in \mathcal{S}_1, g_2 \in \mathcal{S}_2} \mathcal{F}_1. \tag{4.9}$$

For simplicity, we denote by

$$\xi = \frac{2h + a - b + \int_0^1 d\sigma_1 + \int_0^1 d\sigma_2}{h + a - b + \int_0^1 x d\sigma_1 + \int_0^1 x d\sigma_2}.$$

Then we have

$$\mathcal{F}_1 = Kh + L(a-b) + \left(h + a - b + \int_0^1 x d\sigma_1 + \int_0^1 x d\sigma_2\right)[\Phi(g_1) - \Phi(g_2)], \tag{4.10}$$

where

$$\Phi(g) = \int_0^1 (1 - \frac{x}{\xi}) g.$$

Note that $\Phi(g_1)$ and $\Phi(g_2)$ are independent when fixing h, a, b, k, l.

Lemma 4.6. *We have the following reduction.*

$$\inf_{g_1 \in \mathcal{S}_1} \Phi(g_1) = \inf_{\xi \le t < 1} (k - a)\Psi(t), \tag{4.11}$$

$$\sup_{g_2 \in \mathcal{S}_2} \Phi(g_2) = \begin{cases} \inf_{\xi \le s < 1} (l - b)\Psi(s), & \text{if } \xi \ge \frac{b}{l}, \\ \inf_{\frac{b}{l} \le s < 1} (l - b)\Psi(s), & \text{if } \xi < \frac{b}{l}, \end{cases} \tag{4.12}$$

where Ψ is a function given by

$$\Psi(x) = -\frac{x^2}{6\xi} + \frac{x}{2} + \frac{1}{6\xi} - \frac{1}{2}. \tag{4.13}$$

Proof. We first consider $\inf_{g_1 \in \mathcal{S}_1} \Phi(g_1)$. Define

$$g_{1,t} = \begin{cases} \frac{(k-a)(t-1)}{t} x, & x \in (0,t), \\ (k-a)(x-1), & x \in (t,1). \end{cases}$$

where $\xi \leq t < 1$. It is clear that $g_{1,t} \in \mathcal{S}_1$. We claim

$$\inf_{g_1 \in \mathcal{S}_1} \Phi(g_1) = \inf_{\xi \leq t < 1} \Phi(g_{1,t}). \tag{4.14}$$

Indeed, for any $g_1 \in \mathcal{S}_1$, let $\xi_0 = g_1(\xi)$, then $\xi_0 \leq g_{1,\xi}(\xi)$. There exists a $t_0 \in [\xi, 1)$ such that $\xi_0 = g_{1,t_0}(t_0)$. Since $g_1 \in \mathcal{S}_1$, by the concavity of g_1 and g_{1,t_0}, we have

$$g_1(x) \geq g_{1,t_0}(x) \text{ for } x \in [0,\xi),$$
$$g_1(x) \leq g_{1,t_0}(x) \text{ for } x \in [\xi,1].$$

It implies $\Phi(g_1) \geq \Phi(g_{1,t_0})$. Hence, it suffices to compute $\Phi(g_{1,t})$. We have

$$\begin{aligned} \Phi(g_{1,t}) &= \int_0^t (1-\frac{x}{\xi}) \frac{(k-a)(t-1)}{t} x + (k-a) \int_t^1 (1-\frac{x}{\xi})(x-1) \\ &= (k-a)\Psi(t), \end{aligned}$$

where $\Psi(t)$ is given as in (4.13).

Next, we consider $\sup_{g_2 \in \mathcal{S}_2} \Phi(g_2)$. If $\xi \geq \frac{b}{l}$, we define

$$g_{2,s} = \begin{cases} \frac{(l-b)(s-1)}{t} x, & x \in (0,s), \\ (l-b)(x-1), & x \in (s,1), \end{cases}$$

where $s \in [\xi, 1)$. If $\xi < \frac{b}{l}$, $g_{2,s}$ is defined for $s \in [\frac{b}{l}, 1)$. In either case, by a similar argument as for (4.14), one can prove

$$\sup_{g_2 \in \mathcal{S}_2} \Phi(g_2) = \sup_s \Phi(g_{2,s}). \tag{4.15}$$

Therefore the lemma follows by a computation of $\Phi(g_{2,s})$. □

The infimum of $\mathcal{F}_1$ can be estimated in the following special case.

Proposition 4.7. *Suppose that $k \geq l+1$. If the boundary measures satisfy $d\sigma_2 = dx$ and $d\sigma_1 = dx, \frac{dx}{2}$ piecewisely, then $\mathcal{F}_1 > 0$.*

Proof. The assumption $k \geq l+1$ implies $a-b > k-l \geq 1$. Note that

$$\begin{aligned}\Psi(1) &= 0,\\ \Psi(\xi) &= \frac{(2\xi-1)(\xi-1)}{6\xi} \leq 0,\\ \Psi(\frac{3\xi}{2}) &= \frac{(3\xi-2)^2}{24\xi},\end{aligned}$$

where Ψ is given by (4.13).

First, we discuss the infimum of $\mathcal{F}_1$ in two cases.

(i) When $\frac{3\xi}{2} \geq 1$, i.e., $L \geq \frac{h}{6}$, $\mathcal{F}_1$ attains its minimum when $t = s = 1$. Then

$$\begin{aligned}\inf \mathcal{F}_1 &= Kh + L(a-b)\\ &\geq \left(\frac{a-b}{2} + \int_0^1 x d\sigma_1 - \frac{1}{2}\int_0^1 d\sigma_1 + \int_0^1 x d\sigma_2 - \frac{1}{2}\int_0^1 d\sigma_2\right) h := M_1.\end{aligned}$$

(ii) When $\frac{3\xi}{2} < 1$, we have $L < \frac{h}{6}$. In this case, $\Psi(t)$ attains its maximum at $\frac{3\xi}{2}$. If $\frac{3\xi}{2} \geq \frac{b}{l}$, $\Psi(s)$ attains its maximum at $\frac{3\xi}{2}$. For $\Psi(s)$, it has two possibilities. If $\frac{3\xi}{2} < \frac{b}{l}$, $\Psi(s)$ attains its maximum at $\frac{b}{l}$. Note that $\Psi(\frac{b}{l}) \leq \Psi(\frac{3\xi}{2})$. Hence, we always have

$$\begin{aligned}&\inf \mathcal{F}_1\\ \geq\ & Kh + L(a-b) + \left(h + a - b + \int_0^1 x d\sigma_1 + \int_0^1 x d\sigma_2\right)(k+b-a-l)\Psi(\frac{3\xi}{2})\\ =\ & Kh + L(a-b) + \left(h + a - b + \int_0^1 x d\sigma_1 + \int_0^1 x d\sigma_2\right)\frac{(k+b-a-l)(3\xi-2)^2}{24\xi}\\ :=\ & M_2,\end{aligned}$$

Next, we use the boundary measure condition to estimate $\inf \mathcal{F}_1$ in the above two cases. By the assumption $d\sigma_2 = dx$,

$$\begin{aligned}K &= \frac{a-b}{3} + \int_0^1 x d\sigma_1 - \frac{1}{2}\int_0^1 d\sigma_1,\\ L &= \frac{a-b}{6} + \frac{1}{2}\int_0^1 x d\sigma_1 - \frac{1}{3}\int_0^1 d\sigma_1 - \frac{1}{12}.\end{aligned}$$

For case (i), we have

$$\inf \mathcal{F}_1 \geq M_1 = \left(\frac{a-b}{2} + \int_0^1 x d\sigma_1 - \frac{1}{2}\int_0^1 d\sigma_1\right) h > 0.$$

For case (ii), we first need to estimate L from below. Assume that the parameters h, a, b, k, l and $d\sigma_i$ are determined by the polytope (f_1, f_2). We prove that

$$L \geq \frac{a-b}{6} - \frac{1}{6}$$

under the assumption of this proposition. If $d\sigma_1 = dx$, then $L = \frac{a-b}{6} - \frac{1}{6}$. Then we assume that there is a segment $[x_0, x_0+r]$ with $d\sigma_1 = \frac{dx}{2}$. Since (f_1, f_2) defines a Delzant's polytope, we denote by E_0 the edge determined by the restriction of f_1 in the segment (x_0, x_0+r). E_0 has two adjacent edges E_1, E_2. By Delzant's condition, the directional vectors of E_1, E_0, E_2 are $(1, i+1)$, $(2, 2i+1)$, $(1, i)$ for some positive integer i, respectively. Let $l(x)$ be the linear function that $l(x) = f_1(x)$ on E_2. We construct a new polytope $(\tilde{f}_1, \tilde{f}_2)$ by

$$\tilde{f}_2 = f_2, \quad \tilde{f}_1 = \begin{cases} f_1(x), & x \in [0, x_0], \\ l(x) - \frac{r}{2}, & x \in [x_0, x_0+r], \\ f_1(x) - \frac{r}{2}, & x \in [x_0+r, 1]. \end{cases}$$

It is easy to check that the polytope $(\tilde{f}_1, \tilde{f}_2)$ still satisfies Delzant's condition. For the new polytope $(\tilde{f}_1, \tilde{f}_2)$, all the parameters are the same as (f_1, f_2) except that $\tilde{a} = a - \frac{r}{2}$. The boundary measure $d\tilde{\sigma}_1 = dx$ on $[x_0, x_0+r]$ and $d\tilde{\sigma}_1 = d\sigma_1$ on $[0,1] \setminus [x_0, x_0+r]$, while $d\tilde{\sigma}_2 = d\sigma_2$. By a simple computation,

$$\tilde{L} - L = -\frac{r}{12} + \frac{1}{2}\int_{x_0}^{x_0+r} (\frac{1}{2}x - \frac{1}{3})dx = -\frac{r}{4} + \frac{1}{4}\int_{x_0}^{x_0+r} xdx \le 0.$$

Repeat the above construction until there is no segments that $d\sigma_1 = \frac{dx}{2}$. Note that after each construction L becomes smaller. Hence,

$$L \ge \frac{a-b}{6} - \frac{1}{6} > 0.$$

It is clear that

$$K \ge \frac{a-b}{3} + \int_0^1 x\frac{dx}{2} - \frac{1}{2}\int_0^1 dx = \frac{a-b}{3} - \frac{1}{4}.$$

It follows

$$\begin{aligned} M_2 &= Kh + L(a-b) + \left(h + a - b + \int_0^1 xd\sigma_1 + \frac{1}{2}\right)(k + b - a - l)\frac{(3\xi-2)^2}{24\xi} \\ &> [(a-b)-(k-l)] \cdot \left[\frac{h}{3} + \frac{a-b}{6} - \frac{(\frac{1}{2} + h - 3\int_0^1 xd\sigma_1 + 2\int_0^1 d\sigma_1 - a + b)^2}{24(\frac{1}{2} + h + a - b + \int_0^1 xd\sigma_1)}\right]. \end{aligned}$$

Since $d\sigma_1 = dx$ or $\frac{dx}{2}$,

$$\begin{aligned} \int_0^1 xd\sigma_1 &\ge \frac{1}{4}, \\ -3\int_0^1 xd\sigma_1 + 2\int_0^1 d\sigma_1 &\le \int_0^{\frac{2}{3}} (-3x+2)dx = \frac{2}{3}. \end{aligned}$$

Here we used $a-b > k-l \geq 1$. Then

$$\inf \mathcal{F}_1 \geq M_2 > [(a-b)-(k-l)] \cdot \left[\frac{h}{3} + \frac{a-b}{6} - \frac{(\frac{7}{6}+h-a+b)^2}{24(\frac{3}{4}+h+a-b)}\right] > 0.$$

The proposition follows. □

4.2.2 $P^{(m)}$ with $m = 5, 6$

First we verify the polytope $P^{(5)}[k]$, given in (4.4).

Proposition 4.8. *The linear function A associated with $P^{(5)}[k]$ is not a constant for all $k = 0, 1, 2, \cdots$. Hence there is no toric surface $M^{(5)}$ of which the Kähler classes admit vanishing Futaki invariant.*

Proof. If $k \geq 1$, by Proposition 4.7, $\mathcal{F}_1 > 0$. If $k = 0$, we may assume that $h \geq 1 - t$. Otherwise we can make a translation, a dilation and a $SL(2, \mathbb{Z})$-transformation such that p_4 is located at the origin. By a simple computation,

$$\begin{aligned} b_0 &= 2 + 2h + t, \\ b_1 &= 1 + h + t, \\ v_0 &= h + t - \tfrac{1}{2}t^2, \\ v_1 &= \tfrac{1}{2}h + \tfrac{1}{2}t - \tfrac{1}{6}t^3. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}_1 &= \tfrac{1}{6}t^4 - \tfrac{1}{6}t^3 + (\tfrac{1}{2}t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3)h \\ &\geq \tfrac{1}{6}t^4 - \tfrac{1}{6}t^3 + (\tfrac{1}{2}t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3)(1-t) \\ &= \tfrac{t}{2}(1-t)^2 + \tfrac{1}{6}(t^3 - t^4) > 0. \end{aligned}$$

This completes the proof. □

Next we turn to $P^{(6)}$. The verification of whether $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = 0$ is elementary but for polytope with 6 vertices, the formulae are longer. Some formulas are calculated by using Maple.

Proposition 4.9. *Among the Delzant polytopes $P^{(6)}[i;k]$ in Lemma 4.4, where $i = 1, 2, 3$ and $k = 0, 1, 2, \cdots$, the polytope with vanishing Futaki invariant must be $P^{(6)}[3;0]$, and the parameter t, s, h must satisfy either*

$$s + t = 1, \tag{4.16}$$

or

$$h = 1 - t = 1 - s. \tag{4.17}$$

Proof. We check $P^{(6)}[i;k]$ for $i = 1, 2, 3$ case by case.

(i) Verification for $P^{(6)}[1;k]$. If $k \geq 1$, by Proposition 4.7, $\mathcal{F}_1 > 0$.

If $k = 0$, we need to consider both $\mathcal{F}_1$ and $\mathcal{F}_2$. By computation,

$$\begin{aligned}
b_0 &= 2 + 2h + 2t + s,\\
b_1 &= 1 + h + 2t + s,\\
v_0 &= +h + 2t + s - t^2 - ts - \tfrac{1}{2}s^2,\\
v_1 &= \tfrac{1}{2}h + t + \tfrac{1}{2}s - \tfrac{1}{6}s^3 - \tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 s - \tfrac{1}{2}ts^2,\\
b_2 &= s + 2t + t^2 + ts + (1 + s + 2t)h + h^2,\\
v_2 &= \tfrac{1}{2}s^2 - \tfrac{1}{3}s^3 + 2t^2 + 2ts - \tfrac{4}{3}t^3 - 2t^2 s\\
&\quad -\tfrac{3}{2}ts^2 + (s + 2t - ts - t^2 - \tfrac{1}{2}s^2)h + \tfrac{1}{2}h^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{F}_1 \ &= st + t^2 - 2st^2 - s^2 t - \tfrac{4}{3}t^3 - \tfrac{1}{6}s^3 + \tfrac{3}{2}s^2t^2 + \tfrac{5}{6}s^3 t + \tfrac{2}{3}t^4 + \tfrac{1}{6}s^4 + \tfrac{4}{3}st^3\\
&\quad +(t + \tfrac{1}{2}s - st - t^2 - \tfrac{1}{2}s^2 + \tfrac{1}{3}s^3 + s^2 t + st^2 + \tfrac{2}{3}t^3)h,\\
\mathcal{F}_2 \ &= -\tfrac{1}{3}s^3 - 2st^2 - ts^2 - \tfrac{4}{3}t^3 + \tfrac{10}{3}st^3 + \tfrac{7}{2}t^2s^2 + \tfrac{5}{3}ts^3 + \tfrac{5}{3}t^4 + \tfrac{1}{3}s^4\\
&\quad +(-2st - 2t^2 - \tfrac{1}{2}s^2 + 4st^2 + 3ts^2 + \tfrac{8}{3}t^3 + \tfrac{2}{3}s^3)h\\
&\quad +(-\tfrac{1}{2}s - t + t^2 + ts + \tfrac{1}{2}s^2)h^2.
\end{aligned}$$

So $\mathcal{F}_1 - \mathcal{F}_2 = U + Vh + Wh^2$, where

$$\begin{aligned}
U &= t^2 + st + \tfrac{1}{6}s^3 - \tfrac{1}{6}s^4 - t^4 - 2st^3 - 2t^2s^2 - \tfrac{5}{6}ts^3,\\
V &= t + \tfrac{1}{2}s + st + t^2 - \tfrac{1}{3}s^3 - 2ts^2 - 3st^2 - 2t^3,\\
W &= t - t^2 - ts + \tfrac{1}{2}s - \tfrac{1}{2}s^2.
\end{aligned}$$

By $t > t(t+s), s > s(t+s)$, it is clear that $W > 0$, and

$$\begin{aligned}
U &> t^3 + 2t^2 s + s^2 t - t^4 - 2st^3 - 2t^2s^2 - \tfrac{5}{6}ts^3 > 0,\\
V &> 2t^2 + \tfrac{5}{2}st + \tfrac{1}{2}s^2 - \tfrac{1}{3}s^3 - 2ts^2 - 3st^2 - 2t^3 > 0.
\end{aligned}$$

Hence $\mathcal{F}_1 - \mathcal{F}_2 > 0$.

As noted after Lemma 4.4, $P^{(6)}[3;1] = P^{(6)}[1;-1]$ (in the same class). We will check $P^{(6)}[1;-1]$ instead of $P^{(6)}[3;1]$.

We found that $\mathcal{F}_1 \neq 0$ or $\mathcal{F}_2 \neq 0$ is not true, and cannot find a combination of $\mathcal{F}_1$ and $\mathcal{F}_2$ which does not vanish for all admissible h, s, t. We have to employ $\mathcal{F}_3$.

We found the computation is simple if we choose a different coordinate system. Since $P^{(6)}[1;-1]$ is obtained by chopping off two corners on the same edge of a rectangle. Let us assume the corners of the rectangle is $(0,0),(0,h),(1,h)$, and $(1,0)$. Let $P^{(6)}[1;-1]$ be obtained by chopping off the corner $\{x_1 > 0, x_2 > 0, x_1 + x_2 < t\}$ at (0,0) and the corner $\{x_1 < 1, x_2 > 0, x_2 < x_1 + 1 - r\}$ at $(1,0)$, where $t, r > 0, t + r < 1$, and $r, t < h$. There is no loss in assuming that $r \geq t$.

By direct computation we have

$$\begin{aligned} b_1 &= 1 + h - r, \\ b_2 &= h^2 + h, \\ v_1 &= \tfrac{1}{2}h - \tfrac{1}{6}t^3 - \tfrac{1}{2}r^2 + \tfrac{1}{6}r^3, \\ v_2 &= \tfrac{1}{2}h^2 - \tfrac{1}{6}t^3 - \tfrac{1}{6}r^3. \end{aligned}$$

and

$$\mathcal{F}_3 = \tfrac{1}{2}h^2(r - r^2 + \tfrac{1}{3}r^3 - \tfrac{1}{3}t^3) - \tfrac{1}{2}h(r^2 - \tfrac{2}{3}r^3) + \tfrac{1}{6}(1-r)(r^3 + t^3).$$

Regard $\mathcal{F}_3$ is a function of h. Recall that $h \geq \max(t, r) = r$. It suffices to verify that $\mathcal{F}_3(r) > 0$, $\mathcal{F}_3'(r) > 0$ and $\mathcal{F}_3''(h) > 0\ \forall\ h > r$. By direct computation,

$$\begin{aligned} \mathcal{F}_3(r) &= \tfrac{1}{6}r^3(1-r)^2 + \tfrac{1}{6}t^3(1 - r - r^2) > 0, \\ \mathcal{F}_3'(r) &= \tfrac{2}{3}r^2(1-r)^2 + \tfrac{1}{3}r^2 - \tfrac{2}{3}rt^3 > 0, \\ \mathcal{F}_3''(h) &= r - r^2 + \tfrac{1}{3}r^3 - \tfrac{1}{3}t^3 > 0, \end{aligned}$$

where we used the conditions $1 - r > t$, $r > t$, and $t < \frac{1}{2}$.

(ii) Verification for $P^{(6)}[2;k]$. If $k \geq 1$, by Proposition 4.7, $\mathcal{F}_1 > 0$.

As noted in §4.1, $P^{(6)}[2;0]$ and $P^{(6)}[1;0]$ are in the same class, so we do not need to verify $P^{(6)}[2;0]$ here.

(iii) Verification for $P^{(6)}[3;k]$. Recall that $P^{(6)}[3;1]$ is in the same class as $P^{(6)}[1;-1]$, and the latter has been verified before. We need only to consider the cases $k \geq 2$ and $k = 0$.

If $k \geq 2$, by Proposition 4.7, $\mathcal{F}_1 > 0$.

If $k = 0$, the corresponding toric surface is $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. In this case the three pairs of edges (E_0, E_3), (E_1, E_4), and (E_2, E_5) are parallel. We may assume that

$$h \geq \max\{1 - t, 1 - s\}. \tag{4.18}$$

Otherwise by discussion in §4.1.1, we can take a $SL(2,\mathbb{Z})$-transformation and re-label the vertices such that (4.18) holds. By direct computation, we have

$$\begin{aligned} b_0 &= 1+2h+t+s, \\ b_1 &= h+t+s, \\ v_0 &= -\tfrac{1}{2}+h+t+s-\tfrac{1}{2}(t^2+s^2), \\ v_1 &= -\tfrac{1}{3}+\tfrac{1}{2}h+\tfrac{1}{2}t+\tfrac{1}{2}s-\tfrac{1}{6}(t^3+s^3). \end{aligned}$$

Hence $\mathcal{F}_1 = U+Vh$ with

$$\begin{aligned} U &= (-\tfrac{2}{3}t+\tfrac{1}{2}t^2)+(-\tfrac{2}{3}s+\tfrac{1}{2}s^2)+\tfrac{1}{6}t^4+\tfrac{1}{6}s^4+\tfrac{1}{3}+(-\tfrac{1}{3}t^3-\tfrac{1}{3}s^3) \\ &\quad +(st-\tfrac{1}{2}t^2s-\tfrac{1}{2}ts^2)+\tfrac{1}{6}(st^3+ts^3), \\ V &= -\tfrac{1}{3}+\tfrac{1}{2}t+\tfrac{1}{2}s-\tfrac{1}{2}t^2-\tfrac{1}{2}s^2+\tfrac{1}{3}t^3+\tfrac{1}{3}s^3. \end{aligned} \tag{4.19}$$

U, V can be simplified as

$$\begin{aligned} U &= \tfrac{1-t-s}{6}\Big\{(1-t)\big[(1-s)^2+t^2+t(1-s)\big]+(1-s)\big[(1-t)^2+s^2+s(1-t)\big]\Big\}, \\ V &= -\tfrac{1-t-s}{6}\Big\{\big[(1-s)^2+t^2+t(1-s)\big]+\big[(1-t)^2+s^2+s(1-t)\big]\Big\}. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{F}_1 &= \tfrac{1-t-s}{6}\Big\{(1-t-h)\big[(1-s)^2+t^2+t(1-s)\big] \\ &\quad +(1-s-h)\big[(1-t)^2+s^2+s(1-t)\big]\Big\}. \end{aligned}$$

Hence $\mathcal{F}_1 = 0$ if and only if $t+s=1$, namely (4.16), or

$$(1-t-h)\big[(1-s)^2+t^2+t(1-s)\big]+(1-s-h)\big[(1-t)^2+s^2+s(1-t)\big]=0.$$

By $h \geq \max\{1-t, 1-s\}$, the latter case is equivalent to (4.17).

In the case (4.16), each parallel pair of the edges (E_0, E_3), (E_1, E_4), and (E_2, E_5) have the same length. In the case (4.17), the edges E_0, E_2, E_4 have the same length, and edges E_1, E_3, E_5 have the same length. Under either (4.16) or (4.17), one can easily verify that $\mathcal{F}_2 = 0$.

From the above verification, we see that among all Delzant polytopes $P^{(6)}$, the polytope with vanishing Futaki invariant must be $P^{(6)}[3;0]$ and the length of the edges satisfies either (4.16) or (4.17). □

Remark 4.10.
(i) Note that the polytope satisfying (4.16) is obtained by chopping-off the same sized corner from the opposite vertices of a rectangle, and the polytope satisfying

(4.17) is obtained by chopping-off a same sized corner from each vertex of the triangle. So it is easy to see that the toric surface $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$, with a Kähler class corresponding to a polytope $P^{(6)}$ satisfying (4.16) and (4.17), has vanishing Futaki invariant. This property was also observed in [LS], Example 3.2.
(ii) The polytope satisfying (4.16) can also be obtained from the triangle by chopping off three different sized corners. Therefore in the both cases (4.16) and (4.17), the corresponding toric surface is $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$.
(iii) Let H be the hyperplane divisor of $\mathbb{CP}^2$, and D_1, D_2, D_3 be the three exceptional divisors. Then after a dilation, the Kähler class corresponding to (4.16) is

$$3H - aD_1 - bD_2 - (3-a-b)D_3 \tag{4.20}$$

and the Kähler class corresponding to (4.17) is

$$3H - c(D_1 + D_2 + D_3), \tag{4.21}$$

where a, b, c are positive constants, $a+b<3$, and $c<\frac{3}{2}$.

In conclusion, we have

Theorem 4.11. *If a toric surface $M^{(m)}$, $m \leq 6$, admits a Kähler class with vanishing Futaki invariant, then it must be one of the following manifolds:*

$$\mathbb{CP}^2, \quad \mathbb{CP}^1 \times \mathbb{CP}^1, \quad \textit{or} \;\; \mathbb{CP}^2\#3\overline{\mathbb{CP}^2}.$$

In [WZho] we proved Propositions 4.8, 4.9 by verifying the Futaki invariant for all the polytopes $P^{(6)}[i;k]$ case by case directly. Here we verified the Futaki invariant for some polytopes by a different way, using Proposition 4.7 in §4.2.1.

4.3 Verification of K-stabiity

In a series of papers [D2-D5], Donaldson made a great progress on Conjecture 2.10 on toric surfacs.

Theorem 4.12 ([D5]). *M admits CSC(constant scalar curvature) metrics in $2\pi c_1(L)$ if and only (M, L) is K-stable and Futaki invariant vanishes.*

As in previous section, we have worked out all the toric surfaces $M^{(m)}$, where $m \leq 6$ with vanishing Futaki invariant. As we mentioned in Section 4.1, the existence is well-known when $m=3$ or 4. When $m=6$, the only toric surface allowing vanishing Futaki invariant is $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. All such Kähler classes have been characterized in Section 4.2. We will check the K-stability of these classes. Combing with Donaldson's theorem, we have

Theorem 4.13. *The toric manifold $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is K-stable, and admits a CSC metric, in any Kähler class with vanishing Futaki invariant.*

As shown in last section, the polytope corresponding to $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is $P^{(6)}[3;0]$, whose vertices are given in (4.7) with $k=0$. If $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ has vanishing Futaki invariant in a Kähler class, the parameters h, s, t must satisfy either (4.16) or (4.17).

Proposition 4.14. *Let $P := P^{(6)}[3;0]$ be the polytope given in (4.7) with $k=0$ and h,s,t satisfying either (4.16) or (4.17). Then for any nontrivial simple PL function u with its crease $\mathcal{I}_u \neq \emptyset$, we have $\mathcal{L}(u) > 0$.*

Proof. Let u be a simple PL function. We can write it as

$$u(x) = u_{\theta,r}(x) = \max\{x_1\cos\theta + x_2\sin\theta - r, 0\} \tag{4.22}$$

where $(\cos\theta, \sin\theta)$ is the normal to the crease $\mathcal{I}_u$, $r\in\mathbb{R}^1$ is the distance from the crease $\mathcal{I}_u$ to the origin (when $\theta\in(0,\frac{\pi}{2})$).

Denote

$$\begin{aligned} \mathcal{I}_r &= \{x\in\bar{P} \mid u_{\theta,r}(x) = 0\}, \\ P_r &= \{x\in P \mid u_{\theta,r}(x) > 0\}, \end{aligned}$$

where $\mathcal{I}_r = I_{u_r}$ is the crease of u_r. In our proof below, we will fix θ and let the parameter r change. So in the following we will drop the parameter θ. Define

$$F(r) = \mathcal{L}(u_r), \qquad f(r) = F''(r).$$

Assume $\mathcal{I}_r \neq \emptyset$ for $r\in(\underline{r},\bar{r})$ and $\mathcal{I}_r=\emptyset$ when $r<\underline{r}$ or $r>\bar{r}$. Obviously $F(\underline{r}) = F(\bar{r}) = 0$. We want to prove $F(r)>0$ for all $r\in(\underline{r},\bar{r})$.

A simple computation shows that $F\in C^1(\underline{r},\bar{r})$, and

$$F'(r) = \int_{\partial P\bigcap P_r} d\sigma - A|P_r|. \tag{4.23}$$

Let y_r, y_r' be the two end points of the crease $\mathcal{I}_r$, lying respectively on the edges E and E'. If none of y_r or y_r' is a vertex of P, F is twice differentiable in r and

$$f(r) = \frac{\sigma(y_r)}{\sin\alpha} + \frac{\sigma(y_r')}{\sin\alpha'} - A\,l_r, \tag{4.24}$$

where l_r is the length of the crease $\mathcal{I}_r$, α, α' are respectively the angles between the crease $\mathcal{I}_r$ and E, E', and $\sigma(y_r)$, $\sigma(y'_r)$ are the values of the density function of the measure $d\sigma$ on the edges E and E'. Since P has 6 vertices only, f has at most 4 discontinuous points.

Proposition 4.14 now follows from the following observations.

(i) By our choice of $\underline{r}$ and $\bar{r}$, obviously $F(\underline{r}) = F(\bar{r}) = 0$. By (4.24) we also have $F'(r) > 0$ for $r > \underline{r}$, near $\bar{r}$; and $F'(r) < 0$ for $r < \bar{r}$, near $\bar{r}$. Hence $F(r) > 0$ for $r > \underline{r}$, near $\underline{r}$; and for $r < \bar{r}$, near $\bar{r}$.

(ii) Note that at $r = \underline{r}$ or $r = \bar{r}$, the crease $\mathcal{I}_r$ must contain a vertex of P. By making a transform $Q \in SL(2,\mathbb{Z})$, where $SL(2,\mathbb{Z})$ is given in (4.1), we may assume that $p_0 = (0,0)$ is the vertex contained in $\mathcal{I}_r$ at $r = \underline{r}$. By the expression (4.23), we have $\underline{r} = 0$. Then p_3 is the vertex contained in $\mathcal{I}_r$ with $r = \bar{r}$, and r is the distance from 0 to the crease $\mathcal{I}_r$.

(iii) When we increase the value of r from $\underline{r}$ to $\bar{r}$, the crease $\mathcal{I}_r$ will pass across the other 4 vertices at $r = r_1, r_2, r_3, r_4$ with $\underline{r} \le r_1 \le r_2 \le r_3 \le r_4 \le \bar{r}$. The crease may contain at most two vertices. Choose $\hat{r} \in [r_2, r_3]$ such that there are three vertices on each side of the crease $\mathcal{I}_{\hat{r}}$, or there are two vertices on each side of $\mathcal{I}_{\hat{r}}$ and two on the line $\mathcal{I}_{\hat{r}}$. The positivity of F in the latter case is immediate when the former case is proved. There are three sub-cases in the former case,

(a) $p_0, p_1, p_5 \mid p_2, p_3, p_4$; (b) $p_0, p_4, p_5 \mid p_1, p_2, p_3$; (c) $p_0, p_1, p_2 \mid p_3, p_4, p_5$.

(iv) The length l_r is monotone increasing for $r \in (\underline{r}, \hat{r})$. At any given $r \in (\underline{r}, \hat{r})$, if $\mathcal{I}_r$ contains no vertex of P, the quantity $\frac{\sigma(y_r)}{\sin\alpha} + \frac{\sigma(y'_r)}{\sin\alpha'}$ is locally a constant, and so $f(r)$ is monotone decreasing near r.

(v) Since the edge E_0 and E_5 at p_0 are located respectively in the x_2-axis and x_1-axis, one easily verifies that when the crease $\mathcal{I}_r$ passes through the vertices p_1 or p_5 (as r increases), the quantities $\frac{\sigma(y_r)}{\sin\alpha} + \frac{\sigma(y'_r)}{\sin\alpha'}$ has a jump-down.

(vi) Therefore in case (a), $f(r)$ is monotone decreasing for $r \in (\underline{r}, \hat{r})$, and increasing for $r \in (\hat{r}, \bar{r})$. The monotonicity of f implies that F has only one local maximum in $(\underline{r}, \bar{r})$ and so by (i) above, F must be positive in $(\underline{r}, \bar{r})$.

(vii) In case (b), then the crease $\mathcal{I}_r$ passes through p_5 at r_1 and passes through p_4 at r_2, and $\underline{r} \le r_1 \le r_2 \le \hat{r} \le r_3$. When $r \in (r_2, r_3)$, the end points y_r, y'_r of the crease $\mathcal{I}_r$ are on the parallel edges E_0 and E_3. From (vi) above, $f(r)$

is monotone decreasing in $(\underline{r}, r_2)$ (and increasing in $(r_3, \bar{r})$). Assume for a moment that f is a negative constant for $r \in (r_2, r_3)$. Then as in (vi) we infer again that F has only one local maximum point in $(\underline{r}, \bar{r})$, and so F must be positive.

When $r \in (r_2, r_3)$, the end points y_r, y'_r of the crease $\mathcal{I}_r$ are on the parallel edges E_0 and E_3. Hence we have $\alpha' = \alpha$ and $f = \frac{2}{\sin\alpha} - \frac{A}{\sin\alpha}$. In the case (4.16), we have $A = \frac{2h+2}{h-(1-t)^2} > 2$ and so $f < 0$. In the case (4.17), we have $A = \frac{6}{1+2ht} > 2$ and also $f < 0$.

(viii) In case (c), note that by a dilation, a translation, and a transform $Q \in SL(2, \mathbb{Z})$ of the coordinates, we can take any vertex of P at the origin. Hence a similar proof as (vii) implies that $F(r) > 0$ for $r \in (\underline{r}, \bar{r})$. $\square$

Remark 4.15. The existence of CSC metrics for case (4.17) was also obtained in [CH] by Calabi flow method. When $t = s = h = \frac{1}{2}$, the Kähler class on $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is half of the first Chern class, in this case the CSC metric is a Kähler-Einstein metric, and was obtained in [Siu], [TY].

Part III

Abreu's equation

Chapter 5

Bernstein theorem

5.1 Introduction

In this chapter, we study convex solutions to Abreu's equation

$$\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0 \tag{5.1}$$

on $\mathbb{R}^n$, where (u^{ij}) is the converse matrix of (u_{ij}). For simplicity in the following we will write the left hand side as u^{ij}_{ij}. It is locally the Euler-Lagrange equation of the functional

$$A_0(u) = \int_\Omega \log \det D^2 u \, dx. \tag{5.2}$$

Abreu's equation can also be written in the following form

$$U^{ij} w_{ij} = 0, \;\; w = [\det D^2 u]^{-1}, \tag{5.3}$$

where (U^{ij}) is the cofactor matrix of (u_{ij}). It is known that (U^{ij}) is divergence free, namely

$$\sum_i \frac{\partial U^{ij}}{\partial x_i} = 0 \;\; \forall \, j.$$

Abreu's equation is similar to the *affine maximal surface equation*

$$U^{ij} w_{ij} = 0, \;\; w = [\det D^2 u]^{-(1-\theta)}, \tag{5.4}$$

whose energy functional is the *affine area functional* [Cal2]

$$A_\theta(u) = \int_\Omega [\det D^2 u]^\theta dx, \tag{5.5}$$

where $\theta = \frac{1}{n+2}$.

In [TW1], the authors proved the Bernstein theorem for (5.4) in the two dimensional case, which was conjectured by S.S. Chern. Namely a smooth convex solution to (5.4) in the entire space $\mathbb{R}^2$ is a quadratic function. The main result of this chaper is the following Bernstein theorem for Abreu's equation.

Theorem 5.1. *Let $n = 2$ and u be an entire convex solution to (5.1). Then u is a quadratic polynomial.*

Remark 5.2. The Bernstein theorem has an interesting geometric explanation. Let us define the metric

$$g = u_{ij}dx^i dx^j + u^{ij}dy_i dy_j$$

under the coordinates (x^1, x^2, y_1, y_2) on $\mathbb{R}^4$, or

$$g' = u_{ij}dx^i dx^j + u^{ij}d\theta_i d\theta_j$$

under the coordinates $(x^1, x^2, \theta_1, \theta_2)$ on $\mathbb{R}^2 \times (S^1)^2$. These metrics are Kählerian and their scalar curvatures are $-u^{ij}_{ij}$. Therefore the above result implies that if $(\mathbb{R}^4, g)$ or $(\mathbb{R}^2 \times (S^1)^2, g')$ is scalar flat, it must be flat.

5.2 Interior estimates

In this section, we establish a priori estimates for Abreu's equation. We consider the more general equation with non-homogeneous right hand term

$$U^{ij}w_{ij} = f, \quad w = [\det D^2u]^{-1} \quad \text{in } \Omega, \tag{5.6}$$

where Ω is a bounded convex domain and $f \in L^\infty(\Omega)$. The functional $A_0(u)$ is replaced by

$$J_0(u) = A_0(u) - \int_\Omega fu\, dx. \tag{5.7}$$

First we recall an upper bound of the determinant obtained by Donaldson [D2].

Lemma 5.3. *Let u be a convex smooth solution to Abreu's equation in Ω. Suppose that u satisfies $u < 0$ in Ω and $u = 0$ on $\partial\Omega$. Then there is a constant C depending only on $\sup_\Omega |\nabla u|$, $\sup_\Omega |u|$, $\sup |f|$ such that*

$$\det D^2u \leqslant \frac{C}{(-u)^n}. \tag{5.8}$$

Proof. Let

$$z = -\log w + \log(-u)^\beta + |Du|^2,$$

where β is a positive number to be determined later. Then z attains its maximum at a point $p \in \Omega$. Hence, at p, it holds

$$z_i = 0, \; u^{ij} z_{ij} \le 0.$$

By computation,

$$z_i = -\frac{w_i}{w} + \frac{\beta u_i}{u} + 2u_{ki}u_k, \tag{5.9}$$

$$z_{ij} = -\frac{w_{ij}}{w} + \frac{w_i w_j}{w^2} + \frac{\beta u_{ij}}{u} - \frac{\beta u_i u_j}{u^2} + 2u_{kij}u_k + 2u_{ki}u_{kj}. \tag{5.10}$$

On the other hand, since $\det D^2 u = w^{-1}$,

$$u^{ab}u_{abi} = (-\log w)_i = -\frac{w_i}{w}.$$

Therefore we have

$$u^{ij}z_{ij} = -u^{ij}\frac{w_{ij}}{w} + u^{ij}\frac{w_i w_j}{w^2} + \frac{\beta n}{u} - \beta\frac{u^{ij}u_i u_j}{u^2} - 2\frac{w_k}{w}u_k + 2\Delta u.$$

By (5.9),

$$\begin{aligned} u^{ij}\frac{w_i w_j}{w^2} &= \beta^2 u^{ij}\frac{u_i u_j}{u^2} + \frac{4\beta|Du|^2}{u} + 4u_{ij}u_i u_j, \\ \frac{w_k}{w}u_k &= \frac{\beta|Du|^2}{u} + 2u_{ij}u_i u_j. \end{aligned}$$

By (5.10) and equation (5.6),

$$u^{ij}z_{ij} = -f + \frac{\beta n}{u} + 2\triangle u + \frac{2\beta|Du|^2}{u} + (\beta^2 - \beta)u^{ij}\frac{u_i u_j}{u^2} \le 0.$$

Choosing $\beta = n$, we have

$$-f + \frac{n^2}{u} + 2\triangle u + \frac{2n|Du|^2}{u} \leqslant 0.$$

Hence,

$$(-u)[\det D^2 u]^{\frac{1}{n}} \le (-u)\triangle u \le C$$

at p. The lemma follows. $\square$

Remark 5.4. In the 2 dimensional case, we point out that the assumption $u = 0$ on $\partial\Omega$ can be removed by using the cut off function $\eta = \log(R^2 - |x|^2)$ and the relation

$$u^{11} + u^{22} = \frac{\Delta u}{\det D^2 u}.$$

See [TW3].

To estimate the determinant from below, we consider the Legendre function of u. If u is smooth, the Legendre function of u is defined on the domain $\Omega^* = Du(\Omega)$, given by

$$u^*(y) = x \cdot y - u(x), \tag{5.11}$$

where x is the point determined by $y = Du(x)$. By differentiating the formula $y = Du(x)$, we have

$$\frac{dy}{dx} = D^2u(x).$$

Hence we have

$$\det D^2u(x) = [\det D^2u^*(y)]^{-1}.$$

The Legendre transform of u^* is u, and

$$u(x) = x \cdot y - u^*(y)$$

at $x = Du^*(y)$. The dual functional with respect to the Legendre function is given by

$$J_0^*(u^*) = A_0^*(u^*) - \int_{\Omega^*} f(Du^*)(yDu^* - u^*) \det D^2u^* dy, \tag{5.12}$$

where

$$A_0^*(u^*) = -\int_{\Omega^*} [\det D^2u^*] \log\det D^2u^* \, dy. \tag{5.13}$$

If u is a solution to Abreu's equation (5.6) in Ω, it is a local maximizer of the functional J_0. Hence u^* is a critical point of J_0^* under local perturbation, so it satisfies the Euler equation of the dual functional J_0^*, namely in Ω^*

$$U^{*ij}w^*_{ij} = f(Du^*) \det D^2u^*, \quad w^* = \log\det D^2u^*, \tag{5.14}$$

where (U^{*ij}) is the cofactor matrix of (u^*_{ij}).

Lemma 5.5. *Let u^* be a smooth convex solution to (5.14) in Ω^* in dimension 2. Assume that $u^* < 0$ in Ω^*, $u^* = 0$ on $\partial\Omega^*$. Then there is a constant C depending only on $\sup_{\Omega^*} |\nabla u^*|$, $\sup_{\Omega^*} |u^*|$, $\sup |f|$ such that*

$$\det D^2u^* \le \frac{C}{(-u^*)^2}.$$

Proof. Consider

$$z = w^* + \log(-u^*)^\alpha + \beta|\nabla u^*|^2,$$

where the constants α and β will be given below. By assumption, z tends to $-\infty$ near $\partial\Omega$, so it attain its maximum at some point $p \in \Omega$. At p we have

$$z_i = 0,\ u^{*ij} z_{ij} \leqslant 0,$$

where

$$\begin{aligned} z_i &= w_i^* + \alpha \frac{u_i^*}{u^*} + 2\beta u_{ki}^* u_k^*, \\ z_{ij} &= w_{ij}^* + \alpha \frac{u_{ij}^* u^* - u_i^* u_j^*}{u^{*2}} + 2\beta u_{kij}^* u_k^* + 2\beta u_{ki}^* u_{kj}^*. \end{aligned}$$

Note that $u^{*ij} u_{kij}^* = w_k^*$. Hence,

$$\begin{aligned} 0 \geq u^{*ij} z_{ij} &= u^{*ij} w_{ij}^* + \alpha \frac{nu^* - u^{*ij} u_i^* u_j^*}{u^{*2}} + 2\beta w_k^* u_k^* + 2\beta \triangle u^* \\ &= f + \alpha \frac{nu^* - u^{*ij} u_i^* u_j^*}{u^{*2}} - 2\beta(\alpha \frac{u_k^*}{u^*} + 2\beta u_{kl}^* u_l^*) u_k^* + 2\beta \triangle u^* \\ &\geq f - \alpha \frac{\sum u^{*ii} |\nabla u^*|^2}{u^{*2}} + 2\alpha\beta \frac{|\nabla u^*|^2}{u^*} + 2\beta(1 - 2\beta|\nabla u^*|^2) \triangle u^* + \frac{\alpha n}{u^*}. \end{aligned}$$

Choosing $\beta = \frac{1}{4 \sup |Du^*|^2}$ and using $u^{*11} + u^{*22} = \frac{\Delta u^*}{\det D^2 u^*}$ in dimension 2, we obtain

$$f - \alpha \frac{|\nabla u^*|^2}{u^{*2}} \frac{\Delta u^*}{\det D^2 u^*} + 2\alpha\beta \frac{|\nabla u^*|^2}{u^*} + \beta \triangle u^* + \frac{\alpha n}{u^*} \leq 0.$$

If

$$\frac{\beta}{2} \triangle u^* - \alpha \frac{|\nabla u^*|^2}{u^{*2}} \frac{\Delta u^*}{\det D^2 u^*} \leq 0,$$

we obtain

$$(-u^*)^2 \det D^2 u^* \leqslant C$$

at p. Otherwise, we have

$$f + 2\alpha\beta \frac{|\nabla u^*|^2}{u^*} + \frac{\beta}{2} \triangle u^* + \frac{\alpha n}{u^*} \leq 0.$$

Therefore, we also obtain

$$(-u^*)^2 \det D^2 u^* \leq (-u^*)^2 (\Delta u^*)^2 \leq C$$

at p. The lemma follows by choosing $\alpha = n = 2$. □

To apply the above determinant estimates, we first introduce the *modulus of convexity* for convex functions. The modulus of convexity of u at x is defined by

$$h_{u,x}(r) = \sup\{h \geq 0 \mid S_{h,u}(x) \subset B_r(x)\},\ r > 0 \tag{5.15}$$

and the modulus of convexity of u on Ω is defined by

$$h_{u,\Omega}(r) = \inf_{x\in\Omega} h_{u,x}(r), \tag{5.16}$$

where

$$S_{h,u}(x) = \{y \in \Omega \mid u(y) < h + a_x(y)\}$$

and a_x is a tangent plane of u at x. When no confusions arise, we will also write $S_{h,u}(x)$ as $S_{h,u}$ or S_h, for brevity.

For a convex domain $\Omega \subset \mathbb{R}^n$, it is known [TW4] that there is a unique ellipsoid $E \subset \mathbb{R}^n$ containing Ω, which attains the minimal volume among all ellipsoids containing Ω, such that

$$\frac{1}{n}E \subset \Omega \subset E,$$

where $\frac{1}{n}E = \{\frac{1}{n}(x - x_0) \mid x \in E\}$ and x_0 is the centre of E. When E is a unit ball, we say Ω is *normalized.*

Proposition 5.6. *Let u be a solution to equation (5.7). Assume $f \in C^\alpha(\Omega)$, $u = 0$ on $\partial\Omega$, and $\inf_\Omega u = -1$. Assume also that Ω is normalized. Then for any $\Omega' \subset \Omega$, there is a constant C depending only on α, $dist(\Omega', \partial\Omega)$, f and the modulus of convexity $h_{u,\Omega}$ such that*

$$|u|_{C^{4,\alpha}(\Omega')} \leq C.$$

Proof. For any $x \in \Omega$, by Lemma 5.4, we have

$$\det D^2u(x) \leq C$$

where C is a constant depending only on f and $\delta = dist(x, \partial\Omega)$. Let $y = Du(x) \in \Omega^*$. By (5.15), (5.16), we have

$$S_{\delta^*,u^*}(y) \subset \Omega^*,$$

where $\delta^* = h_{u,\Omega}(\frac{\delta}{2})$. Furthermore, since $|Du^*| \leq diam(\Omega)$, we also have

$$dist(y, \partial\Omega^*) \geq \frac{\delta^*}{2diam(\Omega)}.$$

Hence, by Lemma 5.5,

$$\det D^2u(x) = [\det D^2u^*(y)]^{-1} \geq C',$$

where C' is a constant depending only on f, δ and $h_{u,\Omega}$.

Once the determinant $\det D^2u$ is bounded, we also have the Hölder continuity of $\det D^2u$ by Caffarelli-Gutierrez's Hölder continuity for linearized Monge-Ampère equation [CG]; and the $C^{2,\alpha}$ regularity for u by Caffarelli's $C^{2,\alpha}$ estimates for Monge-Ampère equation [Caf1, JW]. Higher regularity then follows from the standard elliptic regularity theory. □

We will estimate in §5.4 the modulus of convexity for the solution u in dimension 2. In §5.3 we consider the change of Abreu's equation under a coordinate transformation and establish the a priori estimates for the equation after the transformation.

5.3 Equations under transformations in $\mathbb{R}^{n+1}$

From this section, we always assume that $f = 0$. Abreu's equation is invariant under transformations of the x-coordinates in $\mathbb{R}^n$, but it changes when taking transformations in $\mathbb{R}^{n+1}$. We note that the affine maximal surface equation is invariant under uni-modular transformations in $\mathbb{R}^{n+1}$, which plays an important part [TW1]. In this section we will derive the new equation under a rotation in $\mathbb{R}^{n+1}$ and establish the a priori estimates for it.

For our purpose it suffices to consider the rotation $z = Tx$, given by

$$z_1 = -x_{n+1}, \tag{5.17}$$

$$z_2 = x_2, \ \ldots, \ z_n = x_n, \tag{5.18}$$

$$z_{n+1} = x_1, \tag{5.19}$$

which fixes $x_2, \ldots, x_n$ axes.

Assume that the graph of u, $\mathcal{M} = \{(x, u(x)) \in \mathbb{R}^{n+1} \mid x \in \Omega\}$ can be represented by a convex function $z_{n+1} = v(z_1, \ldots, z_n)$ in z-coordinates, in a domain $\hat{\Omega}$. To derive the equation for v, we compute the change of the functional A_0.

$$\begin{aligned} A_0(u) &= \int_\Omega \log\det D^2u\, dx \\ &= \int_\Omega \left[\log \frac{\det D^2u}{(1+|Du|^2)^{\frac{n+2}{2}}} + \log(1+|Du|^2)^{\frac{n+2}{2}}\right] dx \\ &= \int_{\mathcal{M}} \left[\log K + \log(1+|Du|^2)^{\frac{n+2}{2}}\right](1+|Du|^2)^{-\frac{1}{2}} d\Sigma, \end{aligned} \tag{5.20}$$

where K is the Gaussian curvature of $\mathcal{M}$ and $d\Sigma$ the volume element of the hypersurface. It is easy to verify that

$$u_1 = -\frac{1}{v_1},\ u_2 = \frac{v_2}{v_1},\ \ldots,\ u_n = \frac{v_n}{v_1}, \tag{5.21}$$

where $v_i = \frac{\partial v}{\partial z_i}$. So we have

$$1 + |Du|^2 = \frac{1 + |Dv|^2}{v_1^2}.$$

Lemma 5.7. *Let u be a solution of (5.1). Let T and v be as above. Then v satisfies the equation*

$$V^{ij}(d^{-1})_{ij} = g, \tag{5.22}$$

where (V^{ij}) is the cofactor matrix of (v_{ij}), $d = \det D^2 v$ and

$$g = 2v^{kl}v_{kl1}\frac{1}{v_1} - (n+2)\frac{v_{11}}{v_1^2}. \tag{5.23}$$

Proof. By (5.20) we have

$$\begin{aligned} A_0(u) &= \int_\Sigma \left[\log K + \log(1 + |Du|^2)^{\frac{n+2}{2}}\right](1 + |Du|^2)^{-\frac{1}{2}} d\Sigma \\ &= \int_{\hat\Omega} \left[\log \frac{\det D^2 v}{(1 + |Dv|^2)^{\frac{n+2}{2}}} + \log(\frac{1 + |Dv|^2}{v_1^2})^{\frac{n+2}{2}}\right](v_1^2)^{\frac{1}{2}} dz \\ &= \int_{\hat\Omega} [\log \det D^2 v - \frac{n+2}{2}\log(v_1^2)](v_1^2)^{\frac{1}{2}} dz := \hat{A}_0(v). \end{aligned}$$

One can now verify directly that (5.22) is the Euler-Lagrange equation of $\hat{A}_0(v)$. □

Next we prove a determinant estimate for equation (5.22). Assume v satisfies

$$\begin{aligned} &v \geq 0,\ v \geq z_1,\ v_1 \geq 0, \ \text{ and} \\ &v(0) \ \text{ is as small as we want.} \end{aligned} \tag{5.24}$$

Let

$$\hat{v} = v - \epsilon z_1 - c \quad \text{and} \quad \hat\Omega_{\epsilon,c} = \{z \mid \hat{v}(z) < 0\},$$

where ϵ and c are positive constants in $(0, \frac{1}{2})$. Then $\hat{v}$ satisfies

$$\hat{V}^{ij}(\hat{d}^{-1})_{ij} = 2\hat{v}^{kl}\hat{v}_{kl1}\frac{1}{\hat{v}_1 + \epsilon} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1 + \epsilon)^2} := \hat{g}.$$

For simplicity, we omit the hat on v and write the above equation as

$$V^{ij}(d^{-1})_{ij} = g, \tag{5.25}$$

where

$$g = 2v^{kl}v_{kl1}\frac{1}{v_1+\epsilon} - (n+2)\frac{v_{11}}{(v_1+\epsilon)^2}. \tag{5.26}$$

Lemma 5.8. *Let v $(= \hat{v})$ be as above. Then there exists a positive constant C depending only on $\sup_{\hat{\Omega}_{\epsilon,c}}|v|$ and $\sup_{\hat{\Omega}_{\epsilon,c}}|Dv|$, such that*

$$\det D^2 v \le \frac{C}{(-v)^n}.$$

Proof. Consider

$$z = \log w - \beta \log(-v) - A|Dv|^2,$$

where $w = d^{-1}$ and β, A are positive numbers to be determined below. Observe that $v < 0$ in $\hat{\Omega}_{\epsilon,c}$ and $v = 0$ on $\partial\hat{\Omega}_{\epsilon,c}$. Then z attains its minimum at a point p in $\hat{\Omega}_{\epsilon,c}$. Hence, at p, it holds

$$z_i = 0,\ v^{ij}z_{ij} \ge 0.$$

By computation,

$$z_i = \frac{w_i}{w} - \frac{\beta v_i}{v} - 2Av_{ki}v_k, \tag{5.27}$$

$$z_{ij} = \frac{w_{ij}}{w} - \frac{w_iw_j}{w^2} - \frac{\beta v_{ij}}{v} + \frac{\beta v_iv_j}{v^2} - 2Av_{kij}v_k - 2Av_{ki}v_{kj}, \tag{5.28}$$

$$\frac{w_k}{w} = -v^{ij}v_{ijk}. \tag{5.29}$$

Hence,

$$g = -2\frac{w_1}{w}\frac{1}{v_1+\epsilon} - (n+2)\frac{v_{11}}{(v_1+\epsilon)^2}.$$

Therefore, by (5.28) and equation (5.25),

$$\begin{aligned} v^{ij}z_{ij} &= -2\frac{w_1}{w}\frac{1}{v_1+\epsilon} - \frac{(n+2)v_{11}}{(v_1+\epsilon)^2} - \frac{v^{ij}w_iw_j}{w^2} - \frac{\beta n}{v} + \frac{\beta v^{ij}v_iv_j}{v^2} \\ &\quad +2A\frac{w_k}{w}v_k - 2A\triangle v. \end{aligned}$$

By (5.27),

$$\begin{aligned} \frac{v^{ij}w_iw_j}{w^2} &= \beta^2 v^{ij}\frac{v_iv_j}{v^2} + 4A\beta\frac{|Dv|^2}{v} + 4A^2v_{ij}v_iv_j, \\ \frac{w_1}{w} &= \frac{\beta v_1}{v} + 2Av_{k1}v_k, \\ \frac{w_k}{w}v_k &= \frac{\beta|\nabla v|^2}{v} + 2Av_{ij}v_iv_j. \end{aligned}$$

It follows

$$\begin{aligned} v^{ij}z_{ij} &= -(n+2)\frac{v_{11}}{(v_1+\epsilon)^2} - 4A\frac{v_{11}v_1}{v_1+\epsilon} - 4A\sum_{k=2}^{n}\frac{v_{1k}v_k}{v_1+\epsilon} - 2\beta\frac{v_1}{(v_1+\epsilon)v} \\ &\quad -\frac{\beta n}{v} - 2A\triangle v - 2A\beta\frac{|Dv|^2}{v} - (\beta^2-\beta)v^{ij}\frac{v_iv_j}{v^2}. \end{aligned}$$

We choose $\beta > 1$ such that $\beta^2 - \beta > 0$. By the positive definiteness of v_{ij}, it holds $v_{1k}^2 \leq v_{11}v_{kk}$ for any $k = 2, ..., n$, so there is C' depending on $|Dv|$, such that

$$\sum_{k=2}^{n}\frac{|v_{1k}v_k|}{v_1+\epsilon} \leq \sum_{k=2}^{n}\frac{1}{4}v_{kk} + C'\frac{v_{11}}{(v_1+\epsilon)^2} \leq \frac{1}{4}\triangle v + C'\frac{v_{11}}{(v_1+\epsilon)^2}. \tag{5.30}$$

Then we have

$$-(2+n-2AC')\frac{v_{11}}{(v_1+\epsilon)^2} - 4A\frac{v_{11}v_1}{v_1+\epsilon} - \frac{2\beta v_1}{(v_1+\epsilon)v} - \frac{\beta n}{v} - A\triangle v \geq 0. \tag{5.31}$$

Choose A small enough such that

$$2+n-2AC' > 0.$$

By a Schwarz inequality, there is a $C_0 > 0$ depending on $|v_1|$ such that

$$-(2+n-2AC')\frac{v_{11}}{(v_1+\epsilon)^2} - 4A\frac{v_{11}v_1}{v_1+\epsilon} \leq C_0A^2v_{11}. \tag{5.32}$$

Now by (5.31), (5.32), we have

$$\begin{aligned} 0 &\leq -\frac{2\beta v_1}{(v_1+\epsilon)v} - \frac{\beta n}{v} - (A - C_0A^2)\triangle v \\ &= \frac{2\beta\epsilon}{(v_1+\epsilon)v} - \frac{\beta(n+2)}{v} - (A-C_0A^2)\triangle v \\ &\leq -\frac{\beta(n+2)}{v} - (A-C_0A^2)\triangle v. \end{aligned}$$

Finally, we choose A small enough such that $A - C_0A^2 \geq \frac{A}{2}$. It follows

$$(-v)\triangle v(p) \leq C$$

at p. Hence, choosing $\beta = n$, the lemma follows by

$$e^{z(x)} \geq e^{z(p)} = d^{-1}(-v)^{-n}e^{-A|Dv|^2} \geq [\frac{1}{n}\triangle v(-v)]^{-n}e^{-A|Dv|^2} \geq C.$$

$\square$

5.4 Modulus of convexity estimate

In this section, we establish the modulus of convexity estimate for solutions to Abreu's equation in dimension 2 with vanishing boundary condition.

Let $u^{(k)}$ be a sequence of solutions to Abreu's equation in normalized domains Ω^k with vanishing boundary condition, and let $\mathcal{M}^k$ be their graphs. By taking a subsequence, we suppose that Ω^k converges to a convex domain Ω in $\mathbb{R}^2$, and $u^{(k)}$ converges to a convex function u defined on Ω.

As in [TW1], we consider the domains

$$D^k = \{(x, x_3) \in \mathbb{R}^3 \mid u^{(k)}(x) < x_3 < 0\}, \tag{5.33}$$

$$D = \{(x, x_3) \in \mathbb{R}^3 \mid u(x) < x_3 < 0\}. \tag{5.34}$$

Then D^k converges to D. The graph $\mathcal{M}$ of u is understood as $\partial D - \{x_3 = 0\}$, so $\mathcal{M}^k$ converges in Hausdorff distance to $\mathcal{M}$. We extend the definition of u to $\partial\Omega$ such that for any boundary point x_0,

$$u(x_0) = \liminf_{x \to x_0,\ x \in \Omega} u(x).$$

For any given interior point x_0', let $l(x)$ be a supporting function of u. Denote the contact set by

$$\mathcal{C} = \{x \mid u(x) = l(x)\}.$$

We first prove a key lemma.

Lemma 5.9. *If $x_0 \in \partial\Omega \cap \partial\mathcal{C}$, then $u(x_0) = 0$.*

Proof. We prove this lemma by contradiction as in [TW1]. If $u(x_0) < 0$, by the convexity of u, the segment connecting $(x_0, u(x_0))$ and $(x_0', u(x_0'))$ lies on $\mathcal{M}$. Denote $p_0 = (x_0, u(x_0)) \in \mathcal{M}$. Since the equation is invariant under linear transformation of the x variables, we may suppose without loss of generality that

$$x_0 = (-1, 0),\ x_0' = (0, 0),$$

and

$$\Omega \subset \{x_1 \geq -1\}.$$

Then the segment

$$\{(0, 0, t) \mid u(-1, 0) < t < 0\}$$

lies in $\mathcal{M}$. Adding a linear function to $u^{(k)}$ and u, we can further suppose that

$$u(-1, 0) = -1,\ l = -x_1 - 2,$$

which is a supporting function of u at x_0'. In addition, we consider the line

$$L = \{(-1, t, -1) \mid t \in \mathbb{R}\}.$$

It is clear that $\mathcal{M} \cap L$ must be a single point (Case I) or a segment (Case II). In Case II, we may suppose that p_0 is an end point of the segment which is

$$\{(-1, t, -1) \mid -1 < t < 0\}.$$

Later, we will discuss the two cases separately.

Now we can first translate the origin to p_0 and then make the rotation (5.17)-(5.19) such that $\mathcal{M}$ can be represented by a convex function v near p_0. Therefore, we have the change of coordinates

$$z_1 = -x_3 - 1, \quad z_2 = x_2, \quad z_3 = x_1 + 1.$$

By convexity, $\mathcal{M}^k$ can also be represented by $z_3 = v^{(k)}(z_1, z_2)$ near p_0, respectively. By Lemma 5.7, $v^{(k)}$ is a solution of the equation (5.22) near the origin. As we know that $\mathcal{M}^k$ converges in Hausdorff distance to $\mathcal{M}$, in new coordinates, $v^{(k)}$ converges locally uniformly to v. It is clear that

$$\begin{aligned} &v(0) = 0, \quad v \geq 0, \text{ when } -1 \leq z_1 \leq 0 \text{ and} \\ &v \geq z_1, \text{ when } 0 \leq z_1 \leq 1 \end{aligned}$$

and the two line segments

$$\{(t, 0, 0) \mid -1 \leq t \leq 0\}, \quad \{(t, 0, t) \mid 0 \leq t \leq 1\}$$

lie on the graph of v.

As in (5.25), let $\hat{v}^{(k)} = v^{(k)} - \frac{1}{2}z_1$ and $\hat{v} = v - \frac{1}{2}z_1$. Also as in (5.25), in the following computation we omit the hat for simplicity. Then

$$v \geq \frac{1}{2}|z_1| \quad \text{and} \quad v(z_1, 0) = \frac{1}{2}|z_1|. \tag{5.35}$$

Let

$$\tilde{\mathcal{C}} = \{z \mid v(z) = 0\}.$$

Observe that

$$\mathcal{M} \cap L = \{(z_1, z_2, 0) \mid (z_1, z_2) \in \tilde{\mathcal{C}}\}$$

in z-coordinates.

Case I. In this case, v is strictly convex at $(0,0)$. The strict convexity implies that Dv is bounded on $S_{h,v}(0)$ for small $h > 0$. Hence, by locally uniform convergence, $Dv^{(k)}$ are uniformly bounded on $S_{\frac{h}{2},v^{(k)}}(0)$. By Lemma 5.8, we have the determinant estimate

$$\det D^2 v^{(k)} \leq C \tag{5.36}$$

near the origin.

For $\delta \leq \frac{h}{2}$, by (5.35), $S_{\delta,v}(0) \subset \{-\frac{1}{2}\delta \leq z_1 \leq \frac{1}{2}\delta\}$ and $(\pm\frac{\delta}{2}, 0) \in \partial S_{\delta,v}(0)$. In the z_2 direction, we define

$$\kappa_\delta = \sup\{|z_2| \mid (z_1, z_2) \in S_{\delta,v}(0)\}.$$

By comparing the images of $S_{\delta,v}(0)$ under normal mapping of v and the cone with bottom at $\partial S_{\delta,v}(0)$ and top at the origin,

$$|N_v(S_{\delta,v}(0))| \geq C\frac{\delta}{\kappa_\delta}. \tag{5.37}$$

The definition of normal mapping N_v has been given in §3.3.

On the other hand, by the lower semi-continuity of normal mapping,

$$N_v(S_{\delta,v}(0)) \subseteq \liminf_{k\to\infty} N_{v^k}(S_{\delta,v}(0)).$$

By (5.36),

$$\begin{aligned} |N_v(S_{\delta,v}(0))| &\leq \liminf_{k\to\infty} |N_{v^{(k)}}(S_{\delta,v}(0))| \\ &= \liminf_{k\to\infty} \int_{S_{\delta,v}(0)} \det D^2 v^{(k)} \\ &\leq C|S_{\delta,v}(0)| \\ &\leq C\delta\kappa_\delta. \end{aligned} \tag{5.38}$$

Hence, (5.37), (5.38) imply $\kappa_\delta \geq C > 0$, where C is independent of δ. Again by the strict convexity, $\kappa_\delta \to 0$ as $\delta \to 0$. The contradiction follows.

Case II. In this case,

$$\tilde{C} = \{(0, z_2) \mid -1 < z_2 < 0\}.$$

We define the following linear function:

$$l_\epsilon(z) = \delta_\epsilon z_2 + \epsilon$$

and $\omega_\epsilon = \{z \mid v(z) < l_\epsilon\}$, where δ_ϵ is chosen such that

$$v(0, \frac{\epsilon}{\delta_\epsilon}) = l(0, \frac{\epsilon}{\delta_\epsilon}) = 2\epsilon,\ v(0, -\frac{\epsilon}{\delta_\epsilon}) = l(0, -\frac{\epsilon}{\delta_\epsilon}) = 0.$$

We can suppose that ϵ is small enough such that ω_ϵ is contained in a small ball near the origin. Hence, $Dv^{(k)}$ is uniformly bounded. By comparing the image of ω_ϵ under normal mapping of v and the cone with bottom at $\partial\omega_\epsilon$ and top at the origin, we have

$$|N_v(\omega_\epsilon)| \geq C\delta_\epsilon. \tag{5.39}$$

On the other hand, $\omega_\epsilon \subset \{-\epsilon \leq z_1 \leq \epsilon\}$ since $v \geq \frac{1}{2}|z_1|$. By the convexity and the assumption above, $\omega_\epsilon \subset \{-\frac{\epsilon}{\delta_\epsilon} \leq z_2 \leq \frac{\epsilon}{\delta_\epsilon}\}$. Therefore,

$$|\omega_\epsilon| \leq C\frac{\epsilon^2}{\delta_\epsilon}.$$

$v^{(k)} - l_\epsilon$ still satisfies equation (5.22). Applying the determinant estimate in Lemma 5.8 to $v^{(k)} - l_\epsilon$ and by a similar argument as in (5.38),

$$|N_v(\omega_\epsilon)| \leq C\frac{\epsilon^2}{\delta_\epsilon}. \tag{5.40}$$

Combining (5.39) and (5.40), we have

$$\frac{\epsilon^2}{\delta_\epsilon^2} \geq C.$$

However, according to our construction, $\frac{\epsilon}{\delta_\epsilon}$ goes to 0 as ϵ goes to 0, which induces a contradiction. □

Remark 5.10. The following property has been used in the above proof, and will also be used in the next chapter. Assume that u is a 2-dimensional convex function satisfying

$$u(0) = 0, \quad u(x) > 0 \text{ for } x \neq 0 \text{ and } u(x_1, 0) \geq C|x_1|. \tag{5.41}$$

Then

$$\frac{|N_u(S_{h,u}(0))|}{|S_{h,u}(0)|} \to \infty \text{ as } h \to 0.$$

In other words, if

$$\det D^2 u \leq C$$

and u vanishes on boundary, then u is C^1 in Ω. This property can be extended to high dimension if

$$u(0) = 0, \quad u(x', x_n) \geq C|x_n| \text{ and } u(x', x_n) \geq C|x'|^2, \tag{5.42}$$

where $x' = (x_1, ..., x_{n-1})$.

It is also known that a generalized solution to

$$\det D^2 u \geq C$$

in a domain in $\mathbb{R}^2$ must be strictly convex. This result was first proved by Aleksandrov but a simple proof can be found in [TW4].

Now we can prove the estimate for the modulus of convexity.

Proposition 5.11. *Let $u^{(k)}$ be a solution to Abreu's equation in a normalized domain Ω^k with vanishing boundary value and $\inf_{\Omega^k} u = -1$. Then there is a positive function $h(r)$ $(r > 0)$ such that the function*

$$h_{u^{(k)},\Omega^k}(r) \geq h(r),$$

where $h_{u^{(k)},\Omega^k}$ is defined in (5.16).

Proof. We also prove this proposition by contradiction. Let $\Omega^k \to \Omega$ and $u^{(k)} \to u$ as before. If there exists $r_0 > 0$ such that $h_{u^{(k)},\Omega^k}(r_0) \to 0$, then the limit u is not strictly convex. So we may assume the contact set

$$\mathcal{C} = \{x \mid u(x) = l(x)\}$$

is not a single point set, where $l(x)$ is a supporting function of u at the origin.

By Lemma 5.9, there must be an point $p \in \partial\mathcal{C}$ lying in the interior of Ω. By making a linear transformation of x and adding a linear function to u, there is no loss of generality in assuming that $p = (0,0)$, $l(x) = 0$, and the segment

$$\{(x_1, 0) \mid -1 \leq x_1 \leq 0\}$$

lies in $\mathcal{C}$. Here the transformation does not change Abreu's equation.

To reduce this case to the model as in Lemma 5.9, we make the following construction. For any $\epsilon > 0$, we consider a linear function

$$l_\epsilon = \epsilon x_1 + \epsilon$$

and a subdomain $\Omega_\epsilon = \{u < l_\epsilon\}$. Let T_ϵ be the coordinates transformation that normalizes Ω_ϵ and

$$u_\epsilon(y) = \frac{1}{\epsilon} u(x), \quad y \in \tilde{\Omega}_\epsilon$$

where $y = T_\epsilon x$ and $\tilde{\Omega}_\epsilon = T_\epsilon(\Omega_\epsilon)$. Note that Abreu's equation is invariant under the above transformation. By choosing a subsequence, $\tilde{\Omega}_\epsilon$ converges to a normalized domain $\tilde{\Omega}$, u_ϵ converges to a convex function $\tilde{u}$ on $\tilde{\Omega}$. Denote by $\mathcal{M}_\epsilon$ the graph of u_ϵ. We also have $\mathcal{M}_\epsilon$ sub-converges in Hausdorff distance to a convex surface $\tilde{\mathcal{M}} \in \mathbb{R}^3$. It is clear that $(0,0)$ is a boundary point of $\tilde{\Omega}$ and $\tilde{\mathcal{M}}$ contains the two segments

$$\{(t,0,0) \mid -1 \le t \le 0\}, \quad \{(0,0,t) \mid 0 \le t \le 1\}$$

By Lemma 5.9 we reach the contradiction. □

5.5 Proof of Theorem 5.1

Let u be an entire solution to Abreu's equation on $\mathbb{R}^2$. By adding a linear function, we assume u attains its minimum 0 at the origin. We claim that there are constants $C > 0$ such that

$$0 < C \le \det D^2 u \le C^{-1}. \tag{5.43}$$

To prove it, we first recall a lemma.

Lemma 5.12 ([Caf2]). *Let u be a locally uniformly convex function in $\mathbb{R}^n$. Then for any $y \in \mathbb{R}^n$ and $h > 0$, there is a point x such that y is the centre of mass of the level set $S_{h,u}(x)$.*

By Lemma 5.12, for any $h > 0$, there is $x_h \in \mathbb{R}^n$ such that 0 is the centre of mass of $S_{h,u}(x_h)$. Let T_h be the linear transformation which normalizes $S_{h,u}(x_h)$ and

$$u_h(y) = \frac{u(x) - u(x_h) - Du(x_h)(x - x_h)}{h}, \quad y \in \Omega_h, \tag{5.44}$$

where $y = T_h(x)$ and $\Omega_h = T_h(S_{h,u}(x_h))$. Noting that 0 is the centre of mass of Ω_h, we have $u_h = 1$ on $\partial\Omega_h$ and $\inf_{\Omega_h} u_h = u_h(T_h(x_h)) = 0$. By Proposition 5.11, u_h is strictly convex, uniformly in h. By Lemmas 5.3 and 5.5, we have the estimate

$$C_1 \le \det D^2 u_h \le C_2 \ \text{ in } B_{1/2n}(0). \tag{5.45}$$

Hence by Proposition 5.6, $\|u_h\|_{C^4(B_{1/4n}(0))} \le C$ for some C independent of h. We obtain

$$C_1|y|^2 \le u_h(y) - Du_h(0)y - u_h(0) \le C_2|y|^2. \tag{5.46}$$

Note that by (5.44),

$$\frac{u(x)}{h} = u_h(y) - Du_h(0)y - u_h(0).$$

It follows

$$C_1|T_h x|^2 \le h^{-1}u(x) \le C_2|T_h x|^2.$$

Let Λ_h, λ_h be the largest and the least eigenvalues of the transformation T_h. Then

$$\Lambda_h \le C_3 h^{-\frac{1}{2}},\ \lambda_h \ge C_4 h^{-\frac{1}{2}}, \tag{5.47}$$

where C_3 depends on C_1, C_2 and $\sup_{\partial B_1} u$ and C_4 depends on C_1, C_2 and $\inf_{\partial B_1} u$. Letting $h \to \infty$, we obtain (5.43).

Finally, the Bernstein theorem for Abreu's equation in dimension 2 follows from the following result for two dimensional elliptic equations.

Proposition 5.13 ([B, Ho, Mi]). *Suppose u is a solution to the elliptic equation*

$$\sum_{i,j=1}^{2} a_{ij}u_{ij} = 0 \text{ in } \mathbb{R}^2 \tag{5.48}$$

such that

$$|u(x)| = o(|x|) \text{ as } |x| \to \infty. \tag{5.49}$$

Then u is a constant.

The use of Proposition 5.13 is inspired by [Tr]. Instead of using the above proposition, we can also use Proposition 5.6 and a rescaling argument, as in [TW1, 2].

5.6 A variant of Bernstein theorem

In this section, we will prove a Bernstein property for Abreu's equation under the assumption of completeness in Calabi's metric. The argument originates from Yau's gradient estimate for harmonic functions on complete Riemannian manifolds [Y].

Let Ω be a domain in $\mathbb{R}^n$ and u be a strictly smooth convex function on Ω. Denote by $\mathcal{M}_u$ the graph of u. Calabi introduced an Riemannian metric on $\mathcal{M}_u$, given by [Cal1]

$$g = u_{ij}dx^i dx^j. \tag{5.50}$$

In [JL], [Mc], the authors proved the Bernstein property for affine maximal surfaces in dimensions 2 and 3 under the assumption of completeness with respect to Calabi's metric. In the case of Abreu's equation, we have a similar result.

Theorem 5.14. *Let u be a solution to (5.1). Assume $2 \leq n \leq 4$ and the graph $\mathcal{M}_u$ is complete with respect to g. Then u is a paraboloid.*

Denote by ∇ and $\triangle$ the gradient and Laplace operators in the Calabi metric. Let f be a smooth function on Ω. Then

$$|\nabla f|^2 = u^{ij} f_i f_j, \tag{5.51}$$
$$\triangle f = u^{ij} f_{ij} + \frac{1}{2} u_i^{ij} f_j, \tag{5.52}$$

where the subscripts i, j means the usual partial derivatives in x_i and x_j. Note that Abreu's equation can be written as

$$u^{ij} L_{ij} - u^{ij} L_i L_j = 0, \tag{5.53}$$

where

$$L = \log \det D^2 u.$$

Hence, Abreu's equation can also be rewritten as a second order equation in Calabi's metric,

$$\triangle L = \frac{1}{2} |\nabla L|^2. \tag{5.54}$$

By direct computation, Ricci curvature of Calabi's metric is given by [Cal1, JL]

$$R_{ij} = \frac{1}{4} (u^{pq} u^{rs} u_{pri} u_{qsj} - u^{pq} u^{rs} u_{pqr} u_{sij}).$$

Lemma 5.15. *The following formula holds in all dimensions,*

$$R_{ij} \geq -\frac{1}{16} |\nabla L|^2 u_{ij}. \tag{5.55}$$

Proof. For any point $p \in \Omega$, we can make a linear transformation to the coordinates such that at p, $u_{ij} = \delta_{ij}$ and R_{ij} is diagonal. Then

$$\begin{aligned} R_{ii} &= \frac{1}{4} (u_{pri}^2 - u_{ppr} u_{rii}) \\ &= \frac{1}{4} (u_{pri}^2 - L_r u_{rii}) \geq -\frac{1}{16} |\nabla L|^2. \end{aligned}$$

□

Lemma 5.16. *We also have*

$$\triangle |\nabla L|^2 \geq \frac{5-n}{8n} |\nabla L|^4 + \langle \nabla L, \nabla |\nabla L|^2 \rangle. \tag{5.56}$$

Proof. For any point $p \in \Omega$, as above we make a linear transformation to the coordinates such that at p, $u_{ij} = \delta_{ij}$ and $L_k = 0$ for $k > 1$. Then

$$\begin{aligned} Ric(\nabla L, \nabla L) &= \frac{1}{4}\sum_{p,r}(u_{pr1}^2 - u_{ppr}u_{r11})L_1^2 \\ &= \frac{1}{4}\sum_{p,r}(u_{pr1}^2 - L_r u_{r11})L_1^2 \\ &\geq \frac{1}{4}\sum_{p}(u_{pp1}^2 - L_1 u_{111})L_1^2 \\ &= \frac{1}{4}(u_{111}^2 - L_1 u_{111} + \sum_{p\geq 2} u_{pp1}^2)L_1^2 \end{aligned}$$

Using an elementary inequality, we have

$$\begin{aligned} Ric(\nabla L, \nabla L) &\geq \frac{1}{4}\left[u_{111}^2 - L_1 u_{111} + \frac{1}{n-1}(\sum_{p\geq 2} u_{pp1})^2\right] L_1^2 \\ &= \frac{1}{4}\left[u_{111}^2 - L_1 u_{111} + \frac{1}{n-1}(L_1 - u_{111})^2\right] L_1^2 \\ &= \frac{1}{4}\left[\frac{n}{n-1}u_{111}^2 - \frac{n+1}{n-1}L_1 u_{111} + \frac{1}{n-1}L_1^2\right] L_1^2 \\ &\geq -\frac{1}{4}\frac{1}{n-1}\left[\frac{(n+1)^2}{4n} - 1\right] L_1^4 = -\frac{n-1}{16n}|\nabla L|^4. \qquad (5.57) \end{aligned}$$

Applying (5.54), (5.57) to the Bochner formula,

$$\begin{aligned} \triangle|\nabla L|^2 &= 2|\nabla^2 L|^2 + 2Ric(\nabla L, \nabla L) + 2\langle \nabla L, \nabla \triangle L\rangle \\ &\geq \frac{2}{n}|\triangle L|^2 - \frac{n-1}{8n}|\nabla L|^4 + \langle \nabla L, \nabla |\nabla L|^2\rangle \\ &= \frac{5-n}{8n}|\nabla L|^4 + \langle \nabla L, \nabla |\nabla L|^2\rangle. \qquad (5.58) \end{aligned}$$

□

Proof of Theorem 5.14. Fix a point p in $\mathbb{R}^n$. Denote the geodesic ball centered at p with radius R by $B_R(p)$. Let r be the distance function

$$r(\cdot) = d_g(p, \cdot).$$

Assume

$$z = 2\log(R^2 - r^2) + \log|\nabla L|^2.$$

Since $z = -\infty$ on the boundary of the ball, it attains its maximum at an interior point q. We may suppose that q lies outside the cut-locus of p, that is, r^2 is smooth at q. Otherwise, we can use the approximation argument as in [SY] or [JL, Mc].

Choosing the normal coordinates at the maximum point, we have

$$z_i = 0, \; z_{ii} \leq 0.$$

By computation,

$$z_i = -2\frac{(r^2)_i}{R^2 - r^2} + \frac{(|\nabla L|^2)_i}{|\nabla L|^2}, \tag{5.59}$$

$$z_{ii} = -2\frac{(r^2)_i^2}{(R^2 - r^2)^2} - 2\frac{(r^2)_{ii}}{R^2 - r^2} + \frac{(|\nabla L|^2)_{ii}}{|\nabla L|^2} - \frac{(|\nabla L|^2)_i^2}{|\nabla L|^4}. \tag{5.60}$$

Using the fact $|\nabla r| = 1$, we have

$$\triangle r^2 = 2r \triangle r + 2. \tag{5.61}$$

Substituting (5.59) and (5.61) into $\triangle z \leq 0$, we obtain

$$-24\frac{r^2}{(R^2 - r^2)^2} - 2\frac{2r \triangle r + 2}{R^2 - r^2} + \frac{\triangle |\nabla L|^2}{|\nabla L|^2} \leq 0. \tag{5.62}$$

By Lemma 5.16,

$$\begin{aligned} \frac{\triangle |\nabla L|^2}{|\nabla L|^2} &\geq \frac{5-n}{8n}|\nabla L|^2 + 8\frac{\langle \nabla L, r\nabla r\rangle}{R^2 - r^2} \\ &\geq (\frac{5-n}{8n} - \epsilon)|\nabla L|^2 - C_\epsilon \frac{r^2}{(R^2 - r^2)^2}, \end{aligned} \tag{5.63}$$

where C_ϵ is a positive constant depending on the small $\epsilon > 0$.

Next, we deal with the term $r \triangle r$. Denote $R^* = d_g(p, q)$. Assume $R^* > 0$. When $2 \leq n \leq 4$, by Lemma 5.16,

$$\triangle |\nabla L|^2 \geq \langle \nabla L, \nabla |\nabla L|^2 \rangle.$$

Hence, by the maximum principle,

$$\max_{B_{R^*}(p)} |\nabla L|^2 = \max_{\partial B_{R^*}(p)} |\nabla L|^2.$$

However, $z = (R^2 - r^2)|\nabla L|^2$ attains its maximum at q. So

$$\max_{B_{R^*}(p)} |\nabla L|^2 = \max_{\partial B_{R^*}(p)} |\nabla L|^2 = |\nabla L|^2(q).$$

Therefore from Lemma 5.15 we have the Ricci curvature bound,

$$R_{ij}(x) \geq -\frac{1}{16}|\nabla L|^2(x) u_{ij}(x) \geq -\frac{1}{16}|\nabla L|^2(q) u_{ij}(x)$$

for any x in $B_{R^*}(p)$. By the Laplace comparison theorem,

$$r\triangle r \leq (n-1)[1+|\nabla L|(q)r] \tag{5.64}$$

in $B_{R^*}(p)$. This inequality also holds at q when $R^*=0$. Indeed, when $R^*=0$, it means $q=p$. For any $\delta>0$, we have

$$R_{ij}(x) \geq -\left(\frac{1}{16}|\nabla L|^2(q)+\delta\right)u_{ij}(x)$$

in $B_s(q)$ provided s is sufficiently small. Then the inequality follows by taking $\delta \to 0$.

Substitute (5.63), (5.64) into (5.62) to get

$$\begin{aligned}
0 \;&\geq\; -24\frac{r^2}{(R^2-r^2)^2} - 4\frac{(n-1)(1+|\nabla L|(q)r)+1}{R^2-r^2} \\
&\quad +(\frac{5-n}{8n}-\epsilon)|\nabla L|^2(q) - C_\epsilon\frac{r^2}{(R^2-r^2)^2} \\
&\geq\; (\frac{5-n}{8n}-\epsilon-\epsilon')|\nabla L|^2(q) - (24+C_\epsilon+C_{\epsilon'})\frac{r^2}{(R^2-r^2)^2} - \frac{4n}{R^2-r^2}
\end{aligned}$$

When $2 \leq n \leq 4$, choosing ϵ and ϵ' small enough, we obtain

$$e^{z(q)} \leq C_1R^2 + C_2.$$

Here C_1, C_2 are positive constants independent of R. Thus by the definition of z,

$$(R^2-r^2)^2|\nabla L|^2(x) = e^{z(x)} \leq e^{z(q)} \leq CR^2 + C.$$

This implies $|\nabla L|(p)=0$ by letting $R \to 0$. Since p can be any point, so L must be a constant. Namely $\det D^2u$ =const.. Hence u defines an affine complete parabolic affine hypersphere. By [Cal1], u is a paraboloid.

Chapter 6

Boundary value problems

We continue to investigate Abreu's equation. In this chapter, we study a boundary value problem for Abreu's equation, which can be formulated as an variational problem for the energy functional.

6.1 Introduction

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$ and φ a smooth, uniformly convex function defined in a neighborhood of $\overline{\Omega}$. Define

$$S[\varphi,\Omega] = \{u \in C^2(\Omega) \mid u \text{ is convex } u|_{\partial\Omega} = \varphi(x), Du(\Omega) \subset D\varphi(\overline{\Omega})\}. \tag{6.1}$$

In [TW3, 5], N. Trudinger and X.J. Wang studied the *affine Plateau problem.* A special case of the affine Plateau problem is to maximize the affine area functional

$$A_\theta(u) = \int_\Omega [\det D^2u]^\theta dx, \tag{6.2}$$

in $S[\varphi,\Omega]$, where $\theta = \frac{1}{n+2}$. We formulate an analogous variational problem for the energy functional of Abreu's equation as follows. As we pointed out in Chapter 5, the energy functional is given by

$$A_0(u) = \int_\Omega \log\det D^2u\, dx. \tag{6.3}$$

As in [TW3], we can consider the more general functional,

$$J_0(u) = A_0(u) - \int_\Omega fu\, dx, \tag{6.4}$$

where $f \in L^\infty(\Omega)$. So we have a similar variational problem for Abreu's equation, that is to find a function u in $S[\varphi, \Omega]$ such that

$$J_0(u) = \sup\{J_0(v) \mid v \in S[\varphi, \Omega]\}. \tag{6.5}$$

The Euler equation of the functional (6.4) is Abreu's equation

$$U^{ij} w_{ij} = f, \tag{6.6}$$

where

$$w = [\det D^2 u]^{-1}. \tag{6.7}$$

In [TW3], Trudinger and Wang proved the existence of smooth maximizers of J_θ in $S[\varphi, \Omega]$. The main result in this chapter is as follows.

Theorem 6.1. *Suppose the domain Ω is bounded and smooth. Assume $f \in C^\infty(\Omega)$. If $n = 2$, there exists a unique, smooth, locally uniformly convex maximizer u of the variational problem (6.5).*

The variational problem (6.5) extends the first boundary value problem for equation (6.6),

$$\begin{aligned} u &= \varphi \quad \text{on } \partial\Omega, & (6.8)\\ Du &= D\varphi \ \text{ on } \partial\Omega. & (6.9) \end{aligned}$$

Indeed, if we have a classical, locally uniformly convex solution $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ to (6.6), u will also solve (6.5) uniquely. The uniqueness follows from the concavity of the functional A_0.

The proof of Theorem 6.1 is inspired by Trudinger-Wang's variational approach and regularity arguments in solving the affine Plateau problem. But due to the singularity of the function $\log d$ near $d = 0$, the approximation argument in [TW3, 5] does not apply directly to our problem. To avoid this difficulty we introduce a sequence of modified functionals J_k to approximate J_0, and prove that the limit of the maximizers of J_k is a maximizer of (6.5). For the regularity, we need to establish uniform a priori estimates for maximizers of J_k.

6.2 A modified functional

Since the set $S[\varphi, \Omega]$ is not closed, we introduce

$$\overline{S}[\varphi, \Omega] = \{u \in C^0(\overline{\Omega}) \mid u \text{ is convex } u|_{\partial\Omega} = \varphi(x), N_u(\Omega) \subset D\varphi(\overline{\Omega})\}, \tag{6.10}$$

where N_u is the normal mapping of u, introduced in §3.3. Note that $\overline{S}[\varphi, \Omega]$ is closed under the locally uniform convergence of convex functions. In [ZZ3], we proved the upper semi-continuity of the functional A_0, which implies the existence of a maximizer of J_0 in $\overline{S}[\varphi, \Omega]$. To apply the a priori estimates to the maximizer, we need a sequence of smooth solutions to Abreu's equation to approximate the maximizer. Since the penalty method in [TW3] does not apply to J_0, we consider a functional of the form

$$J(u) = A(u) - \int_\Omega f u\, dx, \tag{6.11}$$

where

$$A(u) = \int_\Omega G(\det D^2 u) dx. \tag{6.12}$$

Here $G(d) = G_\delta(d)$ is a smooth concave function on $[0, \infty)$ which depends on a constant $\delta \in (0,1)$ and satisfies the following conditions.

(a) $G(d) = \log d$ when $d \geq \delta$.

(b) $G'(d) > 0$ and there exist constants $C_1, C_2 > 0$ independent of δ such that for any $d > 0$

$$\begin{aligned} G''(d) &\geq -C_1 d^{-2}, \\ \left| \frac{dG'''(d)}{G''(d)} \right| &\leq C_2. \end{aligned}$$

(c) The function $F(t) = G(d)$, where $t = d^{\frac{1}{n}}$, is smooth in $(0, +\infty)$ and satisfies

$$\begin{aligned} &F(0) > -\infty, \quad F''(t) < 0, \\ &\lim_{t\to 0} F'(t) = \infty, \quad \lim_{t \to 0} tF'(t) \leq C_3, \end{aligned}$$

where C_3 is a positive constant.

Remark 6.2.
(i) The condition $F''(t) < 0$ in (c) implies that the functional A is concave.
(ii) The concavity of F, $F''(t) < 0$, is equivalent to $dG''(d) + \frac{n-1}{n} G'(d) < 0$; and $\lim_{t\to 0} F'(t) = \infty$ is equivalent to $d^{\frac{n-1}{n}} G'(d) \to \infty$ as $x \to 0$.
(iii) A function G satisfying properties (a)-(c) will be given in (6.18) below.

The Euler equation of the functional J is

$$U^{ij} w_{ij} = f, \tag{6.13}$$

where

$$w = G'(\det D^2 u) \tag{6.14}$$

and (U^{ij}) is the cofactor matrix of $D^2 u$.

Remark 6.3. Equation (6.13), (6.14) is invariant under unimodular linear transformation. If we make a general non-degenerate linear transformation $T: \ y = Tx$ and let $\tilde{u}(y) = u(x)$, then $\tilde{u}(y)$ is a solution of

$$\tilde{U}^{ij}\tilde{w}_{ij} = f, \ \tilde{w} = \tilde{G}'(\det D^2\tilde{u}),$$

where $\tilde{G}(\tilde{d}) = G(|T|^2\tilde{d})$, $\tilde{d} = \det D^2\tilde{u}$. Here $\tilde{G}$ is a smooth concave function satisfying (a), (b), (c) with $\tilde{\delta} = |T|^{-2}\delta$, $\tilde{C}_1 = C_1$, $\tilde{C}_2 = C_2$, $\tilde{C}_3 = C_3$.

Now we study the existence and uniqueness of maximizers to the functional $J(u)$. The treatment here is similar to that in [TW3] and [ZZ3].

First, we extend the functional J to $\overline{S}[\varphi, \Omega]$. It is clear that the linear part in J is naturally well-defined. It suffices to extend $A(u)$ to $\overline{S}[\varphi, \Omega]$. Since u is convex, u is almost everywhere twice-differentiable, i.e., the Hessian matrix (D^2u) exists almost everywhere. Denote the Hessian matrix by $(\partial^2 u)$ at those twice-differentiable points in Ω. Recall that a convex function on Ω induces a Monge-Ampère measure $\mu[u]$ through its normal mapping. This measure is a Radon measure and can be decomposed into a regular part and a singular part as follows,

$$\mu[u] = \mu_r[u] + \mu_s[u].$$

It was proved in [TW3] that the regular part $\mu_r[u]$ can be given explicitly by

$$\mu_r[u] = \det \partial^2 u \, dx$$

and $\det \partial^2 u$ is a locally integrable function. Therefore for any $u \in \overline{S}[\varphi, \Omega]$, we can define

$$A(u) = \int_\Omega G(\det \partial^2 u) dx. \tag{6.15}$$

Next, we state an important property of $A(u)$. For any Lebesgue measurable set E, by the concavity of G and Jensen's inequality,

$$\begin{aligned} \int_E G(\det \partial^2 u) dx &\leq |E| G\left(\frac{\int_E \det \partial^2 u \, dx}{|E|}\right) \\ &\leq |E| G(|E|^{-1}\mu[u](E)). \end{aligned} \tag{6.16}$$

By the assumption (a), $d^{-1}G(d) \to 0$ as $d \to \infty$. Note that G is bounded from below. So the above integral goes to 0 as $|E| \to 0$. With this property, we have an approximation result for the functional $A(u)$. For $u \in \overline{S}[\varphi, \Omega]$, let

$$u_h(x) = h^{-n} \int_{B_1(x)} \rho(\frac{x-y}{h}) u(y) dy,$$

where $h > 0$ is a small constant and $\rho \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \rho = 1$. Suppose that u is defined in a neighborhood of Ω such that u_h is well-defined for any $x \in \Omega$. A fundamental result is that that $(D^2 u_h) \to (\partial^2 u)$ almost everywhere in Ω [Z]. We have therefore obtained as in [TW1],

Lemma 6.4. *Let $u \in \overline{S}[\varphi, \Omega]$, we have*

$$\int_\Omega G(\det \partial^2 u) dx = \lim_{h \to 0} \int_\Omega G(\det \partial^2 u_h) dx.$$

Finally, the existence of maximizers of J in $\overline{S}[\varphi, \Omega]$ follows from the following upper semi-continuity of the functional $A(u)$ with respect to uniform convergence.

Lemma 6.5. *Suppose that $u_n \in \overline{S}[\varphi, \Omega]$ converge locally uniformly to u. Then*

$$\limsup_{n \to \infty} \int_\Omega G(\det \partial^2 u_n) dx \leq \int_\Omega G(\det \partial^2 u) dx.$$

Proof. The proof is also inspired by [TW1], see also [ZZ3]. Subtracting G by the constant $G(0)$, we may suppose that $G(0) = 0$. By Lemma 6.4, it suffices to prove it for $u_n \in C^2(\overline{\Omega})$ and we may assume that u_n converges uniformly to u in $\overline{\Omega}$.

Denote by S the supporting set of $\mu_s[u]$, whose Lebesgue measure is zero. By the upper semi-continuity of the Monge-Ampère measure, for any closed subset $F \subset \Omega \setminus S$,

$$\limsup_{n \to \infty} \int_F \det D^2 u_n dx \leq \int_F \det \partial^2 u \, dx. \tag{6.17}$$

For given $\epsilon, \epsilon' > 0$, let

$$\Omega_k = \{x \in \Omega \setminus S \mid (k-1)\epsilon \leq \det \partial^2 u < k\epsilon\}, \ k = 0, 1, 2, ...,$$

and $\omega_k \subset \Omega_k$ be a closed set such that

$$|\Omega_k \backslash \omega_k| < \frac{\epsilon'}{2^{|k|}}.$$

For each ω_k, by concavity of G and (6.17), we have

$$\begin{aligned}
\limsup_{n \to \infty} \frac{1}{|\omega_k|} \int_{\omega_k} G(\det D^2 u_n) dx &\leq \limsup_{n \to \infty} G\left(\frac{\int_{\omega_k} \det D^2 u_n dx}{|\omega_k|}\right) \\
&\leq G\left(\frac{\int_{\omega_k} \det \partial^2 u \, dx}{|\omega_k|}\right) \\
&\leq G(k\epsilon).
\end{aligned}$$

It follows

$$\begin{aligned}\limsup_{n\to\infty}\int_{\omega_k} G(\det D^2u_n)dx &\le G(k\epsilon)|\omega_k| \\ &\le G((k-1)\epsilon)|\omega_k| + G(\epsilon)|\omega_k| \\ &\le \int_{\Omega_k} G(\det\partial^2 u)dx + G(\epsilon)|\Omega_k|.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\int_{\bigcup\omega_k} G(\det D^2u_n)dx \le \int_{\Omega} G(\det\partial^2 u)dx + G(\epsilon)|\Omega|.$$

By (6.16), letting ϵ go to 0, we can replace the domain of the left hand side integral by Ω. The lemma is proved. □

For the uniqueness of maximizers, one can check that Lemma 2.3 in [TW3] also holds for $J(u)$. That is

Lemma 6.6. *For any maximizer u of $J(\cdot)$, the Monge-Ampère measure $\mu[u]$ has no singular part.*

In conclusion, we have obtained the existence and uniqueness of maximizers of J in $\overline{S}[\varphi,\Omega]$.

Theorem 6.7. *Let Ω be a bounded, Lipschitz domain in $\mathbb{R}^n$. Suppose φ is a convex Lipschitz function defined in a neighborhood of $\overline{\Omega}$ and $f \in L^\infty(\Omega)$. There exists a unique function in $\overline{S}[\varphi,\Omega]$ maximizing J.*

Proof. The existence follows from the upper semi-continuity of $A(u)$. For the uniqueness, note that by the concavity of the functional, if there exist two maximizers u and v, then $\partial^2 u = \partial^2 v$ almost everywhere. Hence by Lemma 6.6 we have $\mu[u] = \mu[v]$. By the uniqueness of generalized solutions to the Dirichlet problem of the Monge-Ampère equation, we conclude that $u = v$. □

In Theorem 6.7, we only need the Lipschitz condition on Ω and φ. But later for the regularity, we must assume the smoothness as stated in Theorem 6.1. We also point out that the above argument applies to the functional J_0, and the existence and uniqueness of maximizers also hold for J_0.

At the end of this section we also point out the existence of functions G satisfying properties (a)-(c) above. A function in our mind satisfies

$$G(d) = \begin{cases} \frac{\delta^{-\theta}}{\theta(1-\theta)}d^\theta - \frac{\theta\delta^{-1}}{1-\theta}d + \log\delta - \frac{1+\theta}{\theta}, & d < \delta, \\ \log d, & d \ge \delta, \end{cases} \tag{6.18}$$

where $\theta = \frac{1}{n+2}$. One can check that $G \in C^2(0,\infty)$ and C^3 except at $d = \delta$. It is easy to see that G satisfies (a) and (c). We can also check that G satisfies (b) except at $d = \delta$. Hence, we can always mollify G to have a sequence of smooth functions satisfying the properties (a)-(c) to approximate it.

For our purpose of studying J_0, we choose a sequence of functions $G_k = G_{\delta_k}$ satisfying (a)-(c) with $\delta_k \to 0$ as $k \to \infty$, and consider the functionals

$$J_k(u) = A_k(u) - \int_\Omega fu\,dx, \tag{6.19}$$

where

$$A_k(u) = \int_\Omega G_k(\det D^2u)dx. \tag{6.20}$$

By Theorem 6.7, there exists $u^{(k)} \in \overline{S}[\varphi,\Omega]$ maximizing the functional J_k in $\overline{S}[\varphi,\Omega]$. It is clear that $u^{(k)}$ converges to a convex function u_0 in $\overline{S}[\varphi,\Omega]$. In the rest of this chaper, we will prove that in dimension 2, u_0 solves the problem (6.4). It suffices to prove that u_0 is smooth in Ω and satisfies Abreu's equation.

6.3 Interior estimates

In this section, we establish the interior estimates for the equation (6.13).

Lemma 6.8. *Let u be the convex smooth solution to (6.13) in Ω. Assume that $u < 0$ in Ω and $u = 0$ on $\partial\Omega$. Then there is a positive constant C depending only on* $\sup|\nabla u|$, $\sup|u|$, $\sup|f|$ *and independent of δ, such that*

$$\det D^2u \le \frac{C}{(-u)^n}.$$

Proof. Let

$$z = -\log d - \log(-u)^\beta - |\nabla u|^2,$$

where β is a positive number to be determined later. Then z attains its minimum at a point p in Ω. We may assume that $d(p) > \delta$ so that $w = d^{-1}$ in a small neighborhood of p. Otherwise, the estimate follows directly. Hence, at p, it holds

$$z_i = 0,\ u^{ij}z_{ij} \ge 0.$$

We can rewrite z as

$$z = \log w - \log(-u)^\beta - |\nabla u|^2$$

near p. By computation,

$$z_i = \frac{w_i}{w} - \frac{\beta u_i}{u} - 2u_{ki}u_k, \tag{6.21}$$

$$z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta u_{ij}}{u} + \frac{\beta u_i u_j}{u^2} - 2u_{kij}u_k - 2u_{ki}u_{kj}. \tag{6.22}$$

On the other hand, since $\det D^2 u = w^{-1}$,

$$u^{ab}u_{abi} = (-\log w)_i = -\frac{w_i}{w}.$$

Therefore we have

$$u^{ij}z_{ij} = u^{ij}\frac{w_{ij}}{w} - u^{ij}\frac{w_i w_j}{w^2} - \frac{\beta n}{u} + \beta\frac{u^{ij}u_i u_j}{u^2} + 2\frac{w_k}{w}u_k - 2\triangle u.$$

By (6.21),

$$\begin{aligned} u^{ij}\frac{w_i w_j}{w^2} &= \beta^2 u^{ij}\frac{u_i u_j}{u^2} + \frac{4\beta|Du|^2}{u} + 4u_{ij}u_i u_j, \\ \frac{w_k}{w}u_k &= \frac{\beta|Du|^2}{u} + 2u_{ij}u_i u_j. \end{aligned}$$

It follows

$$u^{ij}z_{ij} = f - \frac{\beta n}{u} - 2\triangle u - \frac{2\beta|Du|^2}{u} - (\beta^2 - \beta)u^{ij}\frac{u_i u_j}{u^2} \geq 0.$$

Choosing $\beta = n$, we have

$$(-u)[\det D^2 u]^{\frac{1}{n}} \leq (-u)\triangle u \leq C$$

at p. The lemma follows. □

For the lower bound estimate of the determinant, we consider the Legendre function u^* of u. As we mentioned in §5.2, if u is smooth, u^* is defined on $\Omega^* = Du(\Omega)$, given by

$$u^*(y) = x \cdot y - u(x),$$

where x is the point determined by $y = Du(x)$. Differentiating $y = Du(x)$, we have

$$\det D^2 u(x) = [\det D^2 u^*(y)]^{-1}.$$

The dual functional with respect to the Legendre function is given by

$$J^*(u^*) = A^*(u^*) - \int_{\Omega^*} f(Du^*)(yDu^* - u^*)\det D^2 u^* dy,$$

where

$$A^*(u^*) = \int_{\Omega^*} G([\det D^2u^*]^{-1}) \det D^2u^* dy.$$

If u is a solution to equation (6.13) in Ω, it is a local maximizer of the functional J. Hence u^* is a critical point of J^* under local perturbation, so it satisfies the Euler equation of the dual functional J^*, namely in Ω^*

$$u^{*ij}w^*_{ij} = -f(Du^*), \tag{6.23}$$

where

$$w^* = G(d^{*-1}) - d^{*-1}G'(d^{*-1}),\ d^* = \det D^2u^*. \tag{6.24}$$

Note that on the left hand side of (6.23), it is u^{*ij}, the inverse of (u^*_{ij}).

Lemma 6.9. *Let u^* be a smooth convex solution to (6.23) in Ω^* in dimension 2. Assume that $u^* < 0$ in Ω^* and $u^* = 0$ on $\partial\Omega^*$. Then there is a positive constant C depending only on* $\sup|\nabla u^*|$, $\sup|u^*|$, $\inf f$ *and independent of δ such that*

$$\det D^2u^* \le \frac{C}{(-u^*)^2}.$$

Proof. We consider

$$z = -\log d^* - \log(-u^*)^\beta - \alpha|\nabla u^*|^2,$$

where α, β are positive numbers to be determined below. Since z tends to ∞ on $\partial\Omega^*$, it must attain its minimum at some point $p \in \Omega^*$. At p we have

$$z_i = 0,\ u^{*ij}z_{ij} \geqslant 0.$$

By (6.24), we compute

$$\begin{aligned} w^*_i &= G''(d^{*-1})d^{*-3}d^*_i, & (6.25)\\ w^*_{ij} &= -G'''(d^{*-1})d^{*-5}d^*_id^*_j - 3G''(d^{*-1})d^{*-4}d^*_id^*_j + G''(d^{*-1})d^{*-3}d^*_{ij}. & (6.26)\end{aligned}$$

On the other hand, by computation,

$$\begin{aligned} z_i &= -\frac{d^*_i}{d^*} - \beta\frac{u^*_i}{u^*} - 2\alpha u^*_{ki}u^*_k, & (6.27)\\ z_{ij} &= -\frac{d^*_{ij}}{d^*} + \frac{d^*_id^*_j}{d^{*2}} - \beta\frac{u^*_{ij}}{u^*} + \beta\frac{u^*_iu^*_j}{u^{*2}} - 2\alpha u^*_{kij}u^*_k - 2\alpha u^*_{ki}u^*_{kj}. & (6.28)\end{aligned}$$

It follows

$$u^{*ij}z_{ij} = -\frac{u^{*ij}d^*_{ij}}{d^*} + \frac{u^{*ij}d^*_id^*_j}{d^{*2}} - \beta\frac{n}{u^*} + \beta\frac{u^{*ij}u^*_iu^*_j}{u^{*2}} - 2\alpha\frac{d^*_k}{d^*}u^*_k - 2\alpha \triangle u^*.$$

By (6.26) and equation (6.23), we have

$$\frac{u^{*ij}d^*_{ij}}{d^*} = -\frac{d^{*2}}{G''(d^{*-1})}f + \frac{d^{*-1}G'''(d^{*-1})}{G''(d^{*-1})}\frac{u^{*ij}d^*_i d^*_j}{d^{*2}} + 3\frac{u^{*ij}d^*_i d^*_j}{d^{*2}}.$$

We may assume that $f(p) < 0$. By the condition (b) for G,

$$\frac{d^{*2}}{G''(d^{*-1})} \le -C_1^{-1}, \quad \left|\frac{d^{*-1}G'''(d^{*-1})}{G''(d^{*-1})}\right| \le C_2.$$

Hence,

$$\frac{u^{*ij}d^*_{ij}}{d^*} \ge C_1^{-1}\inf f + (3 - C_2)\frac{u^{*ij}d^*_i d^*_j}{d^{*2}}.$$

So we have

$$u^{*ij}z_{ij} \ge -C_1^{-1}\inf f + (C_2 - 2)\frac{u^{*ij}d^*_i d^*_j}{d^{*2}} - \frac{\beta n}{u^*} + \beta\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} - 2\alpha\frac{d^*_k}{d^*}u^*_k - 2\alpha \triangle u^*.$$

By (6.27),

$$\begin{aligned}
\frac{u^{*ij}d^*_i d^*_j}{d^{*2}} &= \beta^2\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} + 4\alpha\beta\frac{|\nabla u^*|^2}{u^*} + 4\alpha^2 u^*_{lk}u^*_l u^*_k, \\
\frac{d^*_k}{d^*}u^*_k &= -\beta\frac{|\nabla u^*|^2}{u^*} - 2\alpha u^*_{lk}u^*_l u^*_k.
\end{aligned}$$

Therefore

$$\begin{aligned}
&-C_1^{-1}\inf f + [\beta + (C_2 - 2)\beta^2]\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} - \frac{n\beta}{u^*} \\
&+[4(C_2 - 2) + 2]\alpha\beta\frac{|\nabla u^*|^2}{u^*} + [4(C_2 - 2) + 4]\alpha^2 u^*_{lk}u^*_l u^*_k - 2\alpha \triangle u^* \ge 0.
\end{aligned}$$

Choose α small enough such that

$$[4(C_2 - 2) + 4]\alpha^2 u^*_{lk}u^*_l u^*_k \le \alpha \triangle u^*.$$

Using the fact $u^{*11} + u^{*22} = \frac{\Delta u^*}{\det D^2 u^*}$ in dimension 2, we have

$$\frac{u^{*ij}u^*_i u^*_j}{u^{*2}} \le \frac{|\nabla u^*|^2}{u^{*2}}\frac{\Delta u^*}{\det D^2 u^*}.$$

It follows

$$-C_1^{-1}\inf f + C'\frac{|\nabla u^*|^2}{u^{*2}}\frac{\Delta u^*}{\det D^2 u^*} - \frac{\beta n}{u^*} + C''\frac{|\nabla u^*|^2}{u^*} - \alpha \triangle u^* \ge 0,$$

where C', C'' are constants depending only on α, β, C_1 and C_2. If

$$\frac{\alpha}{2} \triangle u^* - C'\frac{|\nabla u^*|^2}{u^{*2}}\frac{\Delta u^*}{\det D^2 u^*} \le 0,$$

we obtain

$$(-u^*)^2 \det D^2 u^* \leqslant C$$

at p. Otherwise, we have

$$-C_1^{-1} \inf f - \frac{\beta n}{u^*} + C'' \frac{|\nabla u^*|^2}{u^*} - \frac{\alpha}{2} \triangle u^* \geq 0.$$

Hence, we also obtain

$$(-u^*)^2 \det D^2 u^* \leq (\triangle u^*)^2 (-u^*)^2 \leqslant C$$

at p. The lemma follows by choosing $\beta = n = 2$. □

We would like to point out that

(i) the determinant estimates above is independent of δ. This leads us to use the approximation $\{G_k\}$;

(ii) The estimate depends only on $\inf f$. This is crucial in §6.8;

(iii) In Lemma 6.9, the estimate only holds in dimension 2. Since we do not have the relation $u^{*11} + u^{*22} = \frac{\triangle u^*}{\det D^2 u^*}$, we can not deal with the term $\frac{u^{*ij} u_i^* u_j^*}{u^{*2}}$ in the proof. This is why we can not extend Theorem 6.1 to higher dimensions.

By Caffarelli-Gutierrez's Hölder continuity for linearized Monge-Ampère equation [CG] and Caffarelli's $C^{2,\alpha}$ estimates for Monge-Ampère equation [Caf1, JW], we have the following a priori estimates.

Lemma 6.10. *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of (6.13) in dimension 2.*

(i) Assume $f \in L^\infty(\Omega)$. Then

$$\|u\|_{W^{4,p}(\Omega')} \leq C$$

for any $p > 1$ and $\Omega' \subset\subset \Omega$, where C depends on n, p, $\sup|f|$, $dist(\Omega', \partial\Omega)$ and the modulus of convexity of u.

(ii) Assume $f \in C^\alpha(\Omega)$. Then

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C$$

for any $\alpha \in (0,1)$ and $\Omega' \subset\subset \Omega$, where C depends on n, α, $\sup|f|$, $dist(\Omega', \partial\Omega)$ and the modulus of convexity of u.

Note that the modulus of convexity has been explained in §5.2.

6.4 Equations after rotations in $\mathbb{R}^{n+1}$

In order to establish the estimate of the modulus of convexity, we also need to consider the equation under rotations in $\mathbb{R}^{n+1}$.

As in §5.3, we consider the rotation $z = Tx$, given by

$$z_1 = -x_{n+1}, \tag{6.29}$$
$$z_2 = x_2, \ ..., \ z_n = x_n, \tag{6.30}$$
$$z_{n+1} = x_1, \tag{6.31}$$

which fixes $x_2, ..., x_n$ axes. Assume that the graph of u, $\mathcal{M} = \{(x, u(x)) \in \mathbb{R}^{n+1} \mid x \in \Omega\}$ can be represented by a convex function $z_{n+1} = v(z_1, ..., z_n)$ in z-coordinates, in a domain $\hat{\Omega}$. To derive the equation for v, we compute the change of the functional A_0.

$$\begin{aligned} A(u) &= \int_\Omega G(\det D^2 u) dx \\ &= \int_\Omega G\left(\frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+2}{2}}}(1+|Du|^2)^{\frac{n+2}{2}}\right) dx \\ &= \int_{\mathcal{M}} G\left(K(1+|Du|^2)^{\frac{n+2}{2}}\right)(1+|Du|^2)^{-\frac{1}{2}} d\Sigma, \end{aligned} \tag{6.32}$$

where K is the Gaussian curvature of $\mathcal{M}$ and $d\Sigma$ the volume element of the hypersurface. Following the computation in §5.3, we have

$$1 + |Du|^2 = \frac{1 + |Dv|^2}{v_1^2}.$$

Hence we obtain

$$A(u) = \int_{\hat{\Omega}} G(v_1^{-(n+2)} \det D^2 v)(v_1^2)^{\frac{1}{2}} dz := \hat{A}(v). \tag{6.33}$$

In addition,

$$\begin{aligned} \int_\Omega f(x) u(x) dx &= \int_{\mathcal{M}} f u (1+|Du|^2)^{-\frac{1}{2}} d\Sigma \\ &= \int_{\hat{\Omega}} f(v, z_2, ..., z_n)(-z_1)(v_1^2)^{\frac{1}{2}} dz. \end{aligned}$$

Let

$$\hat{J}(v) = \hat{A}(v) - \int_{\hat{\Omega}} f(v, z_2, ..., z_n)(-z_1)(v_1^2)^{\frac{1}{2}} dz.$$

After computing the Euler equation for the functional $\hat{J}(v)$, we have

Lemma 6.11. *Let u be a solution of (6.13). Let T and v be as above. Then v satisfies the equation*

$$V^{ij}(d^{-1})_{ij} = g - f_1 z_1 v_1 + f_1 z_1 + f \tag{6.34}$$

in the set $\{z \mid v_1^{-(n+2)} d > \delta\}$, where (V^{ij}) is the cofactor matrix of (v_{ij}), $d = \det D^2 v$ and

$$\begin{aligned} g &= 2v^{kl} v_{kl1} \frac{1}{v_1} - (n+2)\frac{v_{11}}{v_1^2}, \\ f &= f(v, z_2, ..., z_n), \\ f_1 &= \frac{\partial f}{\partial x_1}(v, z_2, ..., z_n). \end{aligned}$$

Remark 6.12. In the proof of strict convexity in §6.7, we will use the upper bound estimate for $\det D^2 v$ given below. Since the lower bound for $\det D^2 v$ is not needed, we do not give the explicit form of the equation for v outside the set $\{z \mid v_1^{-(n+2)} d > \delta\}$ in this lemma.

Next we prove a determinant estimate for v. Assume v satisfies

$$\begin{aligned} &v \geq 0, \ v \geq z_1, \ v_1 \geq 0, \ \text{ and} \\ &v(0) \ \text{ is as small as we want.} \end{aligned} \tag{6.35}$$

Let

$$\hat{v} = v - \epsilon z_1 - c \text{ and } \hat{\Omega}_{\epsilon,c} = \{z \mid \hat{v}(z) < 0\},$$

where ϵ and c are positive constants in $(0, \frac{1}{2})$. Then $\hat{v}$ satisfies

$$\hat{V}^{ij}(\hat{d}^{-1})_{ij} = \hat{g} - \hat{f}_1 z_1(\hat{v}_1 + \epsilon) + \hat{f}_1 z_1 + \hat{f} \tag{6.36}$$

in the set $\{z \mid (\hat{v}_1 + \epsilon)^{-(n+2)} d > \delta\} \cap \hat{\Omega}_{\epsilon,c}$, where $\hat{d} = \det D^2 \hat{v}$, and

$$\hat{g} = 2\hat{v}^{kl}\hat{v}_{kl1}\frac{1}{\hat{v}_1 + \epsilon} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2}, \tag{6.37}$$

$$\hat{f} = f(\hat{v} + \epsilon z_1 + c, z_2, ..., z_n), \tag{6.38}$$

$$\hat{f}_1 = \frac{\partial f}{\partial x_1}(\hat{v} + \epsilon z_1 + c, z_2, ..., z_n). \tag{6.39}$$

Lemma 6.13. *Let $\hat{v}$ be as above. Then there exists $C > 0$ depending only on $\sup|\hat{f}|$, $\sup|\nabla \hat{f}|$, $\sup_{\hat{\Omega}_{\epsilon,c}} |\hat{v}|$ and $\sup_{\hat{\Omega}_{\epsilon,c}} |D\hat{v}|$, but independent of δ, such that*

$$(-\hat{v})^n \det D^2 \hat{v} \leq C.$$

Proof. Consider

$$z = \log w - \beta \log(-\hat{v}) - A|D\hat{v}|^2,$$

where $w = \hat{d}^{-1}$, and β, A are positive numbers to be determined below. Then z attains its minimum at a point p $\hat{\Omega}_{\epsilon,c}$. Hence, at p, it holds

$$\eta_i = 0, \ \hat{v}^{ij}\eta_{ij} \geq 0.$$

We can suppose that $p \in \{z \mid (\hat{v}_1 + \epsilon)^{-(n+2)}\hat{d} > \delta\}$. Otherwise, we have

$$(\hat{v}_1 + \epsilon)^{-(n+2)}\hat{d} \leq \delta$$

and then the estimate follows. By computation,

$$\eta_i = \frac{w_i}{w} - \frac{\beta \hat{v}_i}{\hat{v}} - 2A\hat{v}_{ki}\hat{v}_k, \tag{6.40}$$

$$\eta_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta \hat{v}_{ij}}{\hat{v}} + \frac{\beta \hat{v}_i \hat{v}_j}{\hat{v}^2} - 2A\hat{v}_{kij}\hat{v}_k - 2A\hat{v}_{ki}\hat{v}_{kj}, \tag{6.41}$$

$$\frac{w_k}{w} = -\hat{v}^{ij}\hat{v}_{ijk}. \tag{6.42}$$

By (6.37),

$$\hat{g} = -2\frac{w_1}{w}\frac{1}{\hat{v}_1 + \epsilon} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1 + \epsilon)^2}.$$

Therefore we have

$$\begin{aligned} \hat{v}^{ij}\eta_{ij} = & -\frac{\hat{v}^{ij}w_i w_j}{w^2} - \frac{w_1}{w}\frac{2}{\hat{v}_1 + \epsilon} - \frac{\beta n}{\hat{v}} - (n+2)\frac{\hat{v}_{11}}{(\hat{v}_1 + \epsilon)^2} + \frac{\beta \hat{v}^{ij}\hat{v}_i\hat{v}_j}{\hat{v}^2} + 2A\frac{w_k}{w}\hat{v}_k \\ & -2A\triangle\hat{v} - \hat{f}_1 z_1(\hat{v}_1 + \epsilon) + \hat{f}_1 z_1 + \hat{f}. \end{aligned}$$

By (6.40),

$$\begin{aligned} \frac{\hat{v}^{ij}w_i w_j}{w^2} &= \beta^2 \hat{v}^{ij}\frac{\hat{v}_i\hat{v}_j}{\hat{v}^2} + 4A^2\hat{v}_{ij}\hat{v}_i\hat{v}_j + +4A\beta\frac{|D\hat{v}|^2}{\hat{v}}, \\ \frac{w_1}{w}\frac{2}{\hat{v}_1 + \epsilon} &= \frac{2\beta\hat{v}_1}{(\hat{v}_1 + \epsilon)\hat{v}} + 4A\frac{\hat{v}_{1k}\hat{v}_k}{\hat{v}_1 + \epsilon}, \\ \frac{w_k}{w}\hat{v}_k &= \beta\frac{|D\hat{v}|^2}{\hat{v}} + 2A\hat{v}_{ij}\hat{v}_i\hat{v}_j. \end{aligned}$$

Hence, we have

$$\begin{aligned} \hat{v}^{ij}\eta_{ij} = & -(n+2)\frac{\hat{v}_{11}}{(\hat{v}_1 + \epsilon)^2} - 4A\left(\frac{\hat{v}_{11}\hat{v}_1}{\hat{v}_1 + \epsilon} + \sum_{k=2}^{n}\frac{\hat{v}_{1k}\hat{v}_k}{\hat{v}_1 + \epsilon}\right) - \frac{2\beta\hat{v}_1}{(\hat{v}_1 + \epsilon)\hat{v}} - 2A\triangle\hat{v} \\ & -\frac{\beta n}{\hat{v}} - \frac{2A\beta|D\hat{v}|^2}{\hat{v}} - (\beta^2 - \beta)\frac{\hat{v}^{ij}\hat{v}_i\hat{v}_j}{\hat{v}^2} - \hat{f}_1 z_1(\hat{v}_1 + \epsilon) + \hat{f}_1 z_1 + \hat{f}. \end{aligned} \tag{6.43}$$

We choose $\beta > 1$ such that $\beta^2 - \beta > 0$. By the positive definiteness of $\hat{v}_{ij}$, it holds $\hat{v}_{1k}^2 \leq \hat{v}_{11}\hat{v}_{kk}$ for any $k = 2, ..., n$, so there is C' depending on n and $|D\hat{v}|$, such that

$$\sum_{k=2}^{n} \frac{|\hat{v}_{1k}\hat{v}_k|}{\hat{v}_1 + \epsilon} \leq \frac{1}{4}\sum_{k=2}^{n} \hat{v}_{kk} + C'\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} \leq \frac{1}{4}\Delta\hat{v} + C'\frac{\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2}. \tag{6.44}$$

It follows

$$\begin{aligned} &-\frac{(2n+4-4AC')\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} - 4A\frac{\hat{v}_{11}\hat{v}_1}{\hat{v}_1+\epsilon} - \frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} - A\triangle\hat{v} - \frac{\beta n}{\hat{v}} \\ &\quad -2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f} \geq 0. \end{aligned} \tag{6.45}$$

Choosing A small enough such that $2n + 4 - 4AC' > 0$. Then by a Schwarz inequality, there exists a $C_0 > 0$ depending only on $|D\hat{v}|$ such that

$$-\frac{(2n+4-4AC')\hat{v}_{11}}{(\hat{v}_1+\epsilon)^2} - 4A\frac{\hat{v}_{11}\hat{v}_1}{\hat{v}_1+\epsilon} \leq C_0A^2\hat{v}_{11}. \tag{6.46}$$

By (6.45), (6.46), we have

$$0 \leq C_0A^2\hat{v}_{11} - \frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} - \frac{\beta n}{\hat{v}} - A\triangle\hat{v} - 2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f}.$$

Choosing A small enough furthermore such that $C_0A^2 \leq \frac{A}{2}$, and observing that

$$\frac{2\beta\hat{v}_1}{(\hat{v}_1+\epsilon)\hat{v}} = \frac{2\beta}{\hat{v}} - \frac{2\beta\epsilon}{(\hat{v}_1+\epsilon)\hat{v}} \geq \frac{2\beta}{\hat{v}},$$

we have

$$-\frac{\beta(n+2)}{\hat{v}} - \frac{A}{2}\triangle\hat{v} - 2A\beta\frac{|D\hat{v}|^2}{\hat{v}} - \hat{f}_1 z_1(\hat{v}_1+\epsilon) + \hat{f}_1 z_1 + \hat{f} \geq 0,$$

which implies

$$(-\hat{v})\Delta\hat{v} \leq C$$

at p. Hence, choosing $\beta = n$, the lemma follows by

$$e^{\eta(x)} \geq e^{\eta(p)} = \hat{d}^{-1}(-\hat{v})^{-n}e^{-A|D\hat{v}|^2} \geq [\frac{(-\hat{v})\Delta\hat{v}}{n}]^{-n}e^{-A|D\hat{v}|^2} \geq C.$$

□

6.5 Second boundary value problem

In order to construct approximation solutions to the maximizer of $J(u)$, we study the second boundary value problem for equation (6.13). This section is just a modification of the second boundary problem in [TW3]. We include it here for completeness. Throughout this section, we will denote by d the determinant $\det D^2u$ for simplicity.

We study the existence of smooth solutions to the following problem.

$$U^{ij}w_{ij} = f(x,u), \text{ in } \Omega, \tag{6.47}$$

$$w = G'(d), \text{ in } \Omega, \tag{6.48}$$

$$w = \psi, \text{ on } \partial\Omega, \tag{6.49}$$

$$u = \varphi, \text{ on } \partial\Omega, \tag{6.50}$$

where Ω is a smooth, uniformly convex domain in $\mathbb{R}^n$, φ, ψ are smooth functions on $\partial\Omega$ with

$$0 < C_0^{-1} \le \psi \le C_0.$$

$f \in L^\infty(\Omega \times R)$ is nondecreasing in u and there is $t_0 \le 0$ such that

$$f(x,t) \le 0, \ t \le t_0.$$

We note that this condition is not needed if u is bounded from below.

By Inverse Function Theorem, $w = G'(d)$ has an inverse function $d = g(w)$. g is an decreasing function which goes to 0 as $w \to \infty$ and goes to ∞ as $w \to 0$. To solve the problem (6.47)-(6.50), we first consider the approximating problem

$$U^{ij}w_{ij} = f, \text{ in } \Omega, \tag{6.51}$$

$$\det D^2u = \eta_k g(w) + (1-\eta_k), \text{ in } \Omega, \tag{6.52}$$

where φ and ψ satisfy (6.49), (6.50) and $\eta_k \in C_0^\infty(\Omega)$ is the cut-off function satisfying $\eta_k = 1$ in $\Omega_k = \{x \in \Omega \mid dist(x, \partial\Omega) > \frac{1}{k}\}$.

Proposition 6.14. *Suppose that $f \in L^\infty$ satisfies the condition above. If (u,w) is the C^2 solution of (6.51), (6.52), there is a constant depending only on $diam(\Omega)$, f, φ, ψ and independent of k, such that*

$$C^{-1} \le w \le C, \ \textit{in } \Omega, \tag{6.53}$$

$$|w(x) - w(x_0)| \le C|x - x_0|, \ \textit{for any } x \in \Omega, x_0 \in \partial\Omega. \tag{6.54}$$

Proof. The proof of the upper bound for w is totally the same as that for affine maximal surface equation in [TW3] by considering the auxiliary function

$$z = \log w + A|x|^2$$

and using the condition $F'(0) = \infty$ in (c). By $w \le C$, we have $\det D^2u \ge C$. Suppose that v is a smooth, uniformly convex function such that $D^2v \ge K > 0$ and $v = \psi$ on $\partial\Omega$. Then, if K is large,

$$U^{ij}v_{ij} \ge KU^{ii} \ge K[\det D^2v]^{\frac{n-1}{n}} \ge CK \ge f,$$

which implies $U^{ij}(v-w)_{ij} \ge 0$. By maximum principle, $v - w \le 0$. We thus obtain

$$w(x) - w(x_0) \ge -C|x - x_0|, \text{ for any } x \in \Omega, x_0 \in \partial\Omega. \tag{6.55}$$

To prove the lower bound of w, let

$$z = \log w + w - \alpha h(u),$$

where $\alpha > 0$ is a constant to be determined later and h is a convex, monotone increasing function such that,

$$h(t) = t, \text{ when } t \ge -t_0 \text{ and } h \ge -t_0 - 1, \text{ when } t \le -t_0.$$

Assume that z attains its minimum at x_0. If x_0 is near $\partial\Omega$, by (6.55), $z(x_0) \ge -C$. Otherwise, x_0 is away from the boundary. Hence, we have, at x_0,

$$\begin{aligned} 0 = z_i &= \frac{w_i}{w} + w_i - \alpha h'(u)u_i, \\ 0 \le z_{ij} &= \frac{w_{ij}}{w} - \frac{w_iw_j}{w^2} + w_{ij} - \alpha h''(u)u_iu_j - \alpha h'(u)u_{ij}. \end{aligned}$$

By maximum principle,

$$\begin{aligned} 0 \le u^{ij}z_{ij} &= \frac{f}{dw} - \frac{u^{ij}w_iw_j}{w^2} + \frac{f}{d} - \alpha h''(u)u^{ij}u_iu_j - \alpha h'(u)n \\ &\le \frac{f}{dw} + \frac{f}{d} - \alpha h'(u)n. \end{aligned}$$

If $u(x_0) \le t_0$, $f \le 0$, which immediately induces a contradiction. Hence, $u(x_0) \ge t_0$, and $h'(u(x_0)) \ge h'(t_0)$. Then choosing α large enough, we obtain $d \le C$ at x_0 by the assumption (a). Using the relation between w and d, we have $w(x_0) \ge C$. By definition,

$$z = \log w + w - \alpha h(u) \ge z(x_0) \ge -C.$$

This implies $w \geq C$.

Similarly, with the upper bound of the determinant, we can construct a barrier function v from above for w and prove

$$w(x) - w(x_0) \leq C|x - x_0|.$$

In conclusion, the proposition has been proved. $\square$

With the above estimates, we have the higher order and global estimates of (6.51), (6.52) by using Caffarelli-Gutierrez's Hölder continuity for linearized Monge-Ampère equation [CG] and Caffarelli's $C^{2,\alpha}$ estimates for Monge-Ampère equation [Caf1, JW] repeatly. By an application of the degree theory (see [TW3] for details), there exists a solution u_k to the approximating problem (6.51), (6.52). Finally, taking $k \to \infty$, we obtain

Theorem 6.15. *The second boundary problem (6.47)-(6.50) admits a solution* $u \in W^{2,p}_{loc} \cap C^{0,1}(\overline{\Omega})(p > 1)$ *with* $\det D^2 u \in C^0(\overline{\Omega})$. *Moreover, if* $f \in C^\alpha(\overline{\Omega} \times \mathbb{R})$ $(0 < \alpha < 1)$, *then* $u \in C^{4,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$.

6.6 Approximation

We will use a penalty method and solutions to the second boundary value problem to construct a sequence of smooth convex solutions to (6.13) to approximate the maximizer of $J(u)$. This section is similar to §6 in [TW3].

First, we consider a second boundary value problem with special non-homogenous term f. Let $B = B_R(0)$ be a ball with $\Omega \subset\subset B$ and $\varphi \in C^2(\overline{B})$ be a uniformly convex function in B vanishing on ∂B. Suppose H is a nonnegative smooth function defined in the interval $(-1, 1)$ such that

$$H(t) = \begin{cases} (1-t)^{-2n}, & t \in (\frac{1}{2}, 1), \\ (1+t)^{-2n}, & t \in (-1, \frac{1}{2}). \end{cases} \tag{6.56}$$

Extend the function f to B such that

$$f(x, u) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ h(u - \varphi(x)) & \text{if } x \in B \setminus \Omega, \end{cases}$$

where $h(t) = H'(t)$.

Lemma 6.16. *Let $f(x,u)$ be as above. Suppose $\partial\Omega$ is Lipschitz continuous. Then there exists a locally uniformly convex solution to the second boundary problem*

$$\begin{aligned} U^{ij}w_{ij} &= f(x,u) \ \ in\ B, \\ w &= G'(d),\ in\ B, \\ u &= \varphi \ \ on\ \partial B, \\ w &= 1 \ \ on\ \partial B \end{aligned} \tag{6.57}$$

with $u \in W^{4,p}_{loc}(B) \cap C^{0,1}(\overline{B})$, for all $p < \infty$, and $w \in C^0(\overline{\Omega})$.

Proof. By the discussion of the second boundary problem in last section, it suffices to prove that for any solution u to (6.57), $|f(x,u)| \le C$ for some constant C independent of u. Note that by our choice of H, a solution to (6.57) is bounded from below.

First, we prove an estimate of the determinant near the boundary ∂B. By the definition of H and the convexity of u, f is bounded from above near ∂B. For any boundary point $x_0 \in \partial B$, we suppose by a rotation of axes that $x_0 = (R, 0, ..., 0)$. There exists $\delta_0 > 0$ independent of x_0 such that f is bounded from above in $B \cap \{x_1 > R - \delta_0\}$. Choose a linear function $l = ax_1 + b$ such that $l(x_0) < u(x_0) = 0$ and $l > u$ on $x_1 = R - \delta_0$. Let

$$z = w + \log w - \beta \log(u - l),$$

where $\beta > 0$ is to be determined below. If z attains its minimum at a boundary point on ∂B, by the boundary condition $w = 1$, $z \ge -C$ near ∂B. If z attains its minimum at a interior point $y_0 \in \{u > l\}$, we have, at y_0,

$$0 = z_i = w_i + \frac{w_i}{w} - \beta\frac{(u-l)_i}{u-l}, \tag{6.58}$$

$$z_{ij} = w_{ij} + \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \beta\frac{(u-l)_{ij}}{u-l} + \beta\frac{(u-l)_i(u-l)_j}{(u-l)^2}. \tag{6.59}$$

By (6.58),

$$\frac{w_i}{w} = \frac{\beta}{1+w}\frac{(u-l)_i}{u-l}.$$

It follows by (6.59) and equation (6.57)

$$0 \le u^{ij}z_{ij} = \frac{f}{d} + \frac{f}{dw} - \frac{\beta n}{u-l} + \left[\beta - \frac{\beta^2}{(1+w)^2}\right]\frac{u^{ij}(u-l)_i(u-l)_j}{(u-l)^2}.$$

We may suppose that $w \le 1$. Choose β large enough such that

$$\beta - \frac{\beta^2}{(1+w)^2} \le 0.$$

So we have $w(y_0) \geq C$. Therefore, $\det D^2u \leq C$ near ∂B.

By the above determinant estimate near ∂B, it follows that $|Du|$ is bounded near ∂B. By the convexity of u,

$$\sup_B |Du| \leq C.$$

Next, we prove that f is bounded from below. We note that by the Lipschitz continuity of $\partial\Omega$, there exists positive constants r, κ such that for any $p \in B \setminus \Omega$, there is a unit vector γ such that the round cone $\mathcal{C}_{p,\gamma,r,\kappa} \subset B \setminus \Omega$, where

$$\mathcal{C}_{p,\gamma,r,\kappa} := \{x \in \mathbb{R}^n \mid |x-p| < r, \ \langle x-p, \gamma\rangle > \cos\kappa\}.$$

Assume that $M = -\inf_B f$ is attained at $x_0 \in B$. If $x_0 \in \Omega$, then $M = \|f\|_{L^\infty(\Omega)}$. If $x_0 \in B \setminus \Omega$, we have

$$M = 2n[1 + u(x_0) - \varphi(x_0)]^{-2n-1},$$

that is,

$$u(x_0) - \varphi(x_0) = \left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}} - 1.$$

Let l_0 be the tangent plane of φ at x_0. Since we have the gradient estimate of u, there exists a uniform δ_0 such that

$$0 \leq 1 + u(x) - \varphi(x) \leq 2\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}}$$

and

$$0 \leq 1 + u(x) - l_0(x) \leq 2\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}}$$

in the cone $\mathcal{C}_{x_0,\gamma,\delta_0\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}},\kappa}$. Let $\omega_0 = \{x \mid u(x) < l_0(x)\}$. It is clear that when M is sufficiently large,

$$\mathcal{C}_{x_0,\gamma,\delta_0\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}},\kappa} \subset \omega_0.$$

Integrating by parts, we have

$$\begin{aligned}
\int_{\omega_0} U^{ij} w_{ij}(u - l_0)dx &= -\int_{\omega_0} U^{ij} w_j (u - l_0)_i dx \\
&= -\int_{\partial\omega_0} wU^{ij}(u - l_0)_i \gamma_j dS + \int_{\omega_0} w \det D^2u\, dx,
\end{aligned}$$

where dS is the volume element of $\partial\omega_0$. $u - l_0$ vanishes on the boundary, so $U^{ij}(u - l_0)_i\gamma_j \geq 0$. The first integral on the right-hand side is negative. Hence, we obtain

$$\int_{\omega_0} f(x,u)(u - l_0)dx \leq \int_{\omega_0} w \det D^2u\, dx \leq C. \tag{6.60}$$

Note that the last inequality follows by the condition $\lim_{t\to 0} tF'(t) \le C_3$ in the assumption (c) on G. Estimating the integral in the cone, we have

$$\int_{\omega_0} f(x,u)(u-l_0)dx \ge 2^{-2n-1}M\cdot\left[1-2\left(\frac{M}{2n}\right)^{-\frac{1}{2n+1}}\right]\cdot C\cdot\left(\frac{M}{2n}\right)^{-\frac{n}{2n+1}}. \quad (6.61)$$

Therefore $M\le C$ follows from (6.60), (6.61).

Finally, we prove that f is bounded from above. For any $\delta>0$, let

$$\Omega_\delta=\{u<-\delta\}\subset B$$

and γ be the unit outward normal on $\partial\Omega_\delta$. We have

$$\begin{aligned}\int_{\Omega_\delta} U^{ij}w_{ij}(u+\delta)dx &= -\int_{\Omega_\delta} U^{ij}w_j u_i dx\\ &= -\int_{\partial\Omega_\delta} wU^{ij}u_i\gamma_j dS+\int_{\Omega_\delta} w\det D^2u\,dx\\ &\ge -\int_{\partial\Omega_\delta} wU^{ij}u_i\gamma_j dS\\ &= -\int_{\partial\Omega_\delta} wU^{\gamma\gamma}u_\gamma dS\\ &= -\int_{\partial\Omega_\delta} wu_\gamma^n K_s dS\\ &\ge -C\sup_{\partial\Omega_\delta} w\sup_B |Du|^n,\end{aligned}$$

where dS is the volume element of $\partial\Omega_\delta$ and K_s is the Gaussian curvature of $\partial\Omega_\delta$. Letting $\delta\to 0$, by $w=1$ on ∂B and the gradient estimate,

$$\int_B f(x,u)u\,dx\ge -C.$$

By a similar argument as in the proof of lower bound, if $u-\varphi$ is sufficiently close to 1 at some point $x\in B\setminus\Omega$, $u-\varphi$ is sufficiently close to 1 nearby in $B\setminus\Omega$. This implies the integral can be arbitrary large, which is a contradiction. Hence, f is bounded and the lemma follows. □

Now we prove that the maximizer of $J(u)$ can be approximated by smooth solutions to (6.13). This approximation was proved for the affine Plateau problem in [TW3] by a penalty method. We will also use this method.

Theorem 6.17. *Let Ω and φ be as in Theorem 6.7. Suppose $\partial\Omega$ is Lipschitz continuous. Then there exist a sequence of smooth solutions to equation (6.13) converging locally uniformly to the maximizer u.*

Proof. The proof for this approximation in [TW3] is very complicated, so we use a simplified proof in [TW5].

Let $B = B_R(0)$ be a large ball such that $\Omega \subset B_R$. By assumption, φ is defined in a neighborhood of Ω, so we can extend u to B such that φ is convex in B, $\varphi \in C^{0,1}(\overline{B})$ and φ is constant on ∂B. Adding $(|x| - R + \frac{1}{2})_+^2$ to φ, where

$$(|x| - R + \frac{1}{2})_+ = \max\{|x| - R + \frac{1}{2}, 0\},$$

we assume that φ is uniformly convex in $\{x \in \mathbb{R}^n \mid R - \frac{1}{2} < |x| < R\}$. Consider the second boundary value problem (6.57) with

$$f_j(x, u) \quad = \quad \begin{cases} f & \text{in } \Omega, \\ H_j'(u - \varphi) & \text{in } B_R \setminus \Omega, \end{cases}$$

where $H_j(t) = H(4^j t)$ and H is defined by (6.56). By Lemma 6.16, there is a solution u_j satisfying

$$|u_j - \varphi| \le 4^{-j}, \ x \in B_R \setminus \Omega. \tag{6.62}$$

By the convexity, u_j sub-converges to a convex function $\bar{u}$ in B_R as $j \to \infty$. Note that $\bar{u} = \varphi$ in $B_R \setminus \Omega$. Hence, $\bar{u} \in \overline{S}[\varphi, \Omega]$ when restricted in Ω. We claim that $\bar{u}$ is the maximizer.

Let v_j be an extension of u, given by

$$v_j = \sup\{l \mid l \in \Phi_j\},$$

where Φ_j is the set of linear functions in B_R satisfying

$$\begin{aligned} l(x) \le \varphi(x) & \quad \text{when } |x| = R \text{ or } |x| \le R - \frac{1}{j}, \text{ and} \\ l(x) \le u_j(x) & \quad \text{when } R - \frac{1}{j} < |x| < R. \end{aligned}$$

By our assumption, φ is uniformly convex in $B_R \setminus B_{\frac{R}{2}}$. By (6.62), $|u_j - \varphi| \le 4^{-j} = o(j^{-2})$, $x \in B_R \setminus \Omega$. So we have

$$v_j = u_j \qquad \text{in } B_R \setminus B_{R-\frac{1}{2j}}, \tag{6.63}$$

$$v_j = \varphi \qquad \text{in } B_{R-\frac{2}{j}} \setminus \Omega, \tag{6.64}$$

$$|v_j - \varphi| \le |u_j - \varphi| \qquad \text{in } B_{R-\frac{1}{2j}} \setminus B_{R-\frac{2}{j}} := D_j. \tag{6.65}$$

Now we consider the functional

$$J_j(v) = \int_{B_R} G(\det \partial^2 v)dx - \int_\Omega fvdx - \int_{B_R \setminus \Omega} H_j(v - \varphi)dx.$$

Subtracting G by the constant $G(0)$, we may assume that $G(0) = 0$. Note that u_j is the maximizer of J_j in $\overline{S}[u_j, B_R]$ and $v_j \in \overline{S}[u_j, B_R]$. So we have

$$J_j(v_j) \le J_j(u_j).$$

In the following, we denote by $J_j(v, E)$ the functional J_j over the domain E. By (6.63), we have

$$J_j(v_j, B_{R-\frac{1}{2j}}) \le J_j(u_j, B_{R-\frac{1}{2j}}). \tag{6.66}$$

By (6.64), (6.65), we obtain

$$-\int_{B_{R-\frac{1}{2j}}\setminus\Omega} H_j(u_j - \varphi)dx \le -\int_{B_{R-\frac{1}{2j}}\setminus\Omega} H_j(v_j - \varphi)dx. \tag{6.67}$$

For any $\epsilon > 0$, by the upper semi-continuity of the functional $A(u)$,

$$\begin{aligned} \int_{B_{R-\frac{2}{j}}\setminus\Omega} G(\det \partial^2 u_j)dx &\le \int_{B_{R-\frac{2}{j}}\setminus\Omega} G(\det \partial^2 \varphi)dx + \epsilon \\ &= \int_{B_{R-\frac{2}{j}}\setminus\Omega} G(\det \partial^2 v_j)dx + \epsilon \end{aligned} \tag{6.68}$$

provided j is large enough. In addition, by (6.16),

$$0 \le \int_{D_j} G(\det \partial^2 v)dx \le |D_j| G(|D_j|^{-1}\mu[v](D_j)) \to 0 \tag{6.69}$$

as $j \to \infty$, where $v = u_j$ or v_j.

Hence, by (6.66)-(6.69) and the upper semi-continuity of the functional $A(u)$,

$$J(u) = J(v_j) \le J(u_j) + \epsilon \le J(\bar{u}) + 2\epsilon.$$

provided j is large enough. By taking $\epsilon \to 0$, this implies $\bar{u}$ is the maximizer. By the uniqueness of maximizers in Theorem 6.7, we obtain $\bar{u} = u$. □

Remark 6.18. We remark that the above approximation does not holds for the maximizer of the functional J_0. The reason is that since $\log d$ is not bounded from below, we do not have the property

$$\left| \int_E \log\det \partial^2 u \, dx \right| \longrightarrow 0,$$

as $|E| \to 0$. This is why we introduce the function G and consider the modified functional $J(u)$.

By Theorem 6.17, for each k, there exists a smooth solutions $u_j^{(k)}$ to

$$U^{ij}w_{ij} = f, \tag{6.70}$$

where

$$w = G_k'(\det D^2 u), \tag{6.71}$$

which converges locally uniformly to the maximizer $u^{(k)}$ of (6.19). Then we have

$$u_j^{(k)} \longrightarrow u_0, \ j, k \to \infty. \tag{6.72}$$

As we explained in §6.3, if u_0 is strictly convex, the interior a priori estimates of $u_j^{(k)}$ will be independent of k and j. Hence, by taking limit, we have the interior regularity of u_0 in Ω. Moreover, by the construction of G_k, u_0 will be a solution to Abreu's equation (6.6). Therefore we have

Theorem 6.19. *Let u_0 be as above. Assume that $f \in C^\infty(\Omega)$. Then if u_0 is a strictly convex function, $u_0 \in C^\infty(\Omega)$ and solves (6.5).*

In the last two sections, we will show the strict convexity of u_0.

6.7 Strict convexity I

We prove the strict convexity of u_0 in dimension 2. Let $\mathcal{M}_0$ be the graph of u_0. If u_0 is not strictly convex, $\mathcal{M}_0$ contains a line segment. Let $l(x)$ be a tangent function of u_0 at the segment and denote by

$$\mathcal{C} = \{x \in \Omega \mid u_0(x) = l(x)\}$$

the contact set. According to the distribution of extreme points of $\mathcal{C}$, we consider two cases as follows. For the definition of extreme point, see §3.3.

Case (a) $\mathcal{C}$ has an extreme point x_0 which is an interior point of Ω.

Case (b) All extreme points of $\mathcal{C}$ lie on $\partial\Omega$.

In this section, we exclude Case (a).

Proposition 6.20. *$\mathcal{C}$ contains no extreme points in the interior of Ω.*

Proof. Since x_0 is an interior point in Ω, there is a linear function a such that $a(x_0) > u_0(x_0)$ and $a < u$ on $\partial\Omega$.

By (6.72), we can choose a sequence of smooth functions $u_k = u_{j_k}^{(k)}$ converging to u_0 such that u_k is the solution to (6.70). Let $\mathcal{M}_k$ be the graph of u_k. Then $\mathcal{M}_k$

converges in Hausdorff distance to $\mathcal{M}_0$. There is no loss of generality in assuming that $l(x) = 0$, x_0 is the origin and the segment $\{(x_1, 0) \mid 0 \leq x_1 \leq 1\} \subset \mathcal{C}$.

For any $\epsilon > 0$, we consider a linear function

$$l_\epsilon = -\epsilon x_1 + \epsilon$$

and a subdomain $\Omega_\epsilon = \{u < l_\epsilon\}$. Let T_ϵ be the coordinates transformation that normalizes Ω_ϵ. Define

$$u_\epsilon(y) = \frac{1}{\epsilon} u(x), \ u_{k,\epsilon} = \frac{1}{\epsilon} u_k(x), \quad y \in \tilde{\Omega}_\epsilon \tag{6.73}$$

where $y = T_\epsilon x$ and $\tilde{\Omega}_\epsilon = T_\epsilon(\Omega_\epsilon)$. After this transformation, we have the following observations:

(i) By Remark 6.3, $u_{k,\epsilon}$ satisfies the equation (6.13) with

$$G = G_{k,\epsilon}(d) = G_k(\epsilon |T_\epsilon|^2 d), \quad \delta = \delta_{k,\epsilon} = \frac{\delta_k}{\epsilon |T_\epsilon|^2}$$

and the right hand term ϵf. Note that $|T_\epsilon| \geq C\epsilon^{-1}$, so $\delta_{k,\epsilon} \leq C\delta_k \to 0$ for a constant C independent of ϵ.

(ii) Denote by $\mathcal{M}_\epsilon$, $\mathcal{M}_{k,\epsilon}$ the graphs of u_ϵ, $u_{k,\epsilon}$, respectively. Taking $k \to \infty$, it is clear that $u_{k,\epsilon} \to u_\epsilon$ and $\mathcal{M}_{k,\epsilon}$ converges in Hausdorff distance to $\mathcal{M}_\epsilon$. Then taking $\epsilon \to 0$, we have that the domains $\tilde{\Omega}_\epsilon$ sub-converges to a normalized domain $\tilde{\Omega}$ and u_ϵ sub-converges to a convex function $\tilde{u}$ defined in $\tilde{\Omega}$. We also have $\mathcal{M}_\epsilon$ sub-converges in Hausdorff distance to a convex surface $\tilde{\mathcal{M}}_0 \in \mathbb{R}^3$.

(iii) The convex surface $\tilde{\mathcal{M}}_0$ satisfies

$$\tilde{\mathcal{M}}_0 \subset \{y_1 \geq 0\} \cap \{y_3 \geq 0\} \tag{6.74}$$

and $\tilde{\mathcal{M}}_0$ contains two segments

$$\{(0, 0, y_3) \mid 0 \leq y_3 \leq 1\}, \ \{(y_1, 0, 0) \mid 0 \leq y_1 \leq 1\}. \tag{6.75}$$

Hence, by (i), (ii), (iii), we can suppose that there is a solution $\tilde{u}_k$ to

$$U^{ij} w_{ij} = \epsilon_k f \ \text{ in } \tilde{\Omega}_k, \tag{6.76}$$

where

$$w = \tilde{G}'_{\tilde{\delta}_k}(\det D^2 u), \tag{6.77}$$

and $\tilde{\delta}_k, \epsilon_k \to 0$, such that the normalized domain $\tilde{\Omega}_k$ converges to $\tilde{\Omega}$, $\tilde{u}_k$ converges to $\tilde{u}$ and the graph of $\tilde{u}_k$, denoted by $\tilde{\mathcal{M}}_k$ converges in Hausdorff distance to $\tilde{\mathcal{M}}_0$.

It is clear that in y-coordinates, $\tilde{\mathcal{M}}_0$ is not a graph of a function near the origin, so we need to rotate the $\mathbb{R}^3$ coordinates. Since the equation (6.13) is invariant under unimodular transformation, we may suppose

$$\tilde{\Omega} \subset \{y_1 \geq 0\}.$$

Adding a linear function to $\tilde{u}$, $\tilde{u}_k$, we replace (6.74), (6.75) by

$$\tilde{\mathcal{M}}_0 \subset \{y_1 \geq 0\} \cap \{y_3 \geq -y_1\} \tag{6.78}$$

and $\tilde{\mathcal{M}}_0$ contains two segments

$$\{(0,0,t) \mid 0 \leq t \leq 1\}, \ \{(t,0,-t) \mid 0 \leq t \leq 1\}. \tag{6.79}$$

Let

$$L = \{(y_1, y_2, y_3) \in \tilde{\mathcal{M}}_0 \mid y_1 = y_3 = 0\}.$$

L must be a single point (Case I) or a segment (Case II). In Case II, we may also suppose that 0 is an end point of the segment which is

$$\{(0,t,0) \mid -1 < t < 0\}.$$

Later, we will discuss the two cases separately.

Now we make the rotation

$$z_1 = -y_3, \quad z_2 = y_2, \quad z_3 = y_1$$

such that $\tilde{\mathcal{M}}_0$ can be represented by a convex v near the origin. By convexity, $\tilde{\mathcal{M}}_k$ can also be represented by $z_3 = v^{(k)}(z_1, z_2)$ near p_0, respectively. $v^{(k)}$ is a solution of the equation given in Lemma 6.11 near the origin. As we know that $\tilde{\mathcal{M}}_k$ converges in Hausdorff distance to $\tilde{\mathcal{M}}_0$, in new coordinates, $v^{(k)}$ converges locally uniformly to v. It is clear that

$$\begin{aligned} &v(0) = 0, \quad v \geq 0, \text{ when } -1 \leq z_1 \leq 0 \text{ and} \\ &v \geq z_1, \text{ when } 0 \leq z_1 \leq 1 \end{aligned}$$

and the two line segments

$$\{(t,0,0) \mid -1 \leq t \leq 0\}, \quad \{(t,0,t) \mid 0 \leq t \leq 1\}$$

lie on the graph of v.

As in (6.36), let $\hat{v}^{(k)} = v^{(k)} - \frac{1}{2}z_1$ and $\hat{v} = v - \frac{1}{2}z_1$. In the following computation we omit the hat for simplicity. Then

$$v \geq \frac{1}{2}|z_1| \quad \text{and} \quad v(z_1, 0) = \frac{1}{2}|z_1|. \tag{6.80}$$

Let

$$\tilde{C} = \{z \mid v(z) = 0\}.$$

Observe that

$$L = \{(z_1, z_2, 0) \mid (z_1, z_2) \in \tilde{C}\}$$

in z-coordinates. The contradiction arguments for the Cases I, II are very similar to those in Lemma 5.9 in Chapter 5. For completeness, we include the details here.

Case I. In this case, v is strictly convex at $(0,0)$. The strict convexity implies that Dv is bounded on $S_{h,v}(0)$ for small $h > 0$. Hence, by locally uniform convergence, $Dv^{(k)}$ are uniformly bounded on $S_{\frac{h}{2},v^{(k)}}(0)$. By Lemma 6.13, we have the determinant estimate

$$\det D^2 v^{(k)} \leq C \tag{6.81}$$

near the origin.

For $\delta \leq \frac{h}{2}$, by (6.80), $S_{\delta,v}(0) \subset \{-\frac{\delta}{2} \leq y_1 \leq \frac{\delta}{2}\}$ and $(\pm\frac{\delta}{2}, 0) \in \partial S_{\delta,v}(0)$. In the z_2 direction, we define

$$\kappa_\delta = \sup\{|z_2| \mid (z_1, z_2) \in S_{\delta,v}(0)\}.$$

By comparing the images of $S_{\delta,v}(0)$ under normal mapping of v and the cone with bottom at $\partial S_{\delta,v}(0)$ and top at the origin,

$$|N_v(S_{\delta,v}(0)\})| \geq C\frac{\delta}{\kappa_\delta}.$$

By the lower semi-continuity of normal mapping,

$$N_v(S_{\delta,v}(0)) \subseteq \liminf_{k\to\infty} N_{v^k}(S_{\delta,v}(0)),$$

then

$$N_v(S_{\delta,v}(0)) = N_v(S_{\delta,v}(0)) \subseteq \liminf_{k\to\infty} N_{v^{(k)}}(S_{\delta,v}(0)).$$

By (6.81),

$$\begin{aligned} |N_v(S(\delta))| &\leq \liminf_{k\to\infty} |N_{v^{(k)}}(S_{\delta,v}(0))| \\ &= \liminf_{k\to\infty} \int_{S_{\delta,v}(0)(\delta)} \det D^2 v^{(k)} \\ &\leq C|S_{\delta,v}(0)| \\ &\leq C\delta\kappa_\delta. \end{aligned} \tag{6.82}$$

Hence, $\kappa_\delta \geq C > 0$, where C is independent of δ. Again by the strict convexity, $\kappa_\delta \to 0$ as $\delta \to 0$. The contradiction follows.

Case II. In this case,

$$\tilde{\mathcal{C}} = \{(0, z_2) \mid -1 < z_2 < 0\}.$$

We define the following linear function:

$$l_\epsilon(z) = \delta_\epsilon z_2 + \epsilon$$

and $\omega_\epsilon = \{z \mid v(z) \leq l_\epsilon\}$, where δ_ϵ is chosen such that

$$v(0, \frac{\epsilon}{\delta_\epsilon}) = l(0, \frac{\epsilon}{\delta_\epsilon}) = 2\epsilon,\ v(0, -\frac{\epsilon}{\delta_\epsilon}) = l(0, -\frac{\epsilon}{\delta_\epsilon}) = 0.$$

We can suppose that ϵ is small enough such that ω_ϵ is contained in a small ball near the origin. Hence, $Dv^{(k)}$ is uniformly bounded. By comparing the image of ω_ϵ under normal mapping of v and the cone with bottom at $\partial\omega_\epsilon$ and top at the origin,

$$|N_v(\omega_\epsilon)| \geq C\delta_\epsilon. \tag{6.83}$$

On the other hand, $\omega_\epsilon \subset \{-\epsilon \leq z_1 \leq \epsilon\}$ since $v \geq |z_1|$. By the convexity and the assumption above, $\omega_\epsilon \subset \{-\frac{\epsilon}{\delta_\epsilon} \leq z_2 \leq \frac{\epsilon}{\delta_\epsilon}\}$. Therefore,

$$|\omega_\epsilon| \leq C\frac{\epsilon^2}{\delta_\epsilon}.$$

Furthermore, subtracting all $v^{(k)}$ by l_ϵ, they still satisfy the same equation. By the determinant estimate in Lemma 6.13 and a similar argument as in (6.82),

$$|N_v(\omega_\epsilon)| \leq C\frac{\epsilon^2}{\delta_\epsilon}. \tag{6.84}$$

Combining (6.83) and (6.84),

$$\frac{\epsilon^2}{\delta_\epsilon^2} \geq C.$$

However, according to our construction, $\frac{\epsilon}{\delta_\epsilon}$ goes to 0 as ϵ goes to 0. The contradiction follows.

□

6.8 Strict convexity II

In this section, we rule out the Case (b) that all extreme points of $\mathcal{C}$ lie on the boundary $\partial\Omega$.

First, we need a stronger approximation. In the case of the affine Plateau problem, this approximation was obtained by [TW5]. Here, we extend it to our functional $J(u)$.

Theorem 6.21. *Let φ, Ω be as in Theorem 6.7 and u be the maximizer of the functional J in $\overline{S}[\varphi,\Omega]$. Assume that $\partial\Omega$ is lipschitz continuous. Then there exist a sequence of smooth solutions $u_m \in W^{4,p}(\Omega)$ to*

$$U^{ij}w_{ij} = f_m = f + \beta_m\chi_{D_m} \ \ in\ \Omega \tag{6.85}$$

such that

$$u_m \longrightarrow u \quad uniformly\ in\ \Omega, \tag{6.86}$$

where $D_m = \{x \in \Omega \mid dist(x,\partial\Omega) < 2^{-m}\}$, χ is the characteristic function, and β_m is a constant. Furthermore, we can choose β_m sufficient large ($\beta_m \to \infty$ as $m \to \infty$) such that for any compact subset $K \subset N_\varphi(\Omega)$,

$$K \subset N_{u_m}(\Omega) \tag{6.87}$$

provided m is sufficient large.

Proof. We assume that $G(0) = 0$ and $G \geq 0$ by subtracting the constant $G(0)$. The proof are divided into four steps.

(i) Let $B = B_R(0)$ be a large ball such that $\Omega \subset B_R$. By assumption, φ is defined in a neighborhood of Ω, so we can extend u to B such that φ is convex in B, $\varphi \in C^{0,1}(\overline{B})$ and ϕ is constant on ∂B. Consider the second boundary value problem with

$$f_{m,j} \;=\; \begin{cases} f + \beta_m\chi_{D_m} & \text{in } \Omega, \\ H_j'(u-\varphi) & \text{in } B_R \setminus \Omega, \end{cases}$$

where $H_j(t) = H(4^j t)$ is given by (6.56). By Lemma 6.16, there is a solution $u_{m,j}$ satisfying

$$|u_{m,j} - \varphi| \leq 4^{-j}, \ x \in B_R \setminus \Omega. \tag{6.88}$$

(ii) By the convexity, $u_{m,j}$ sub-converges to a convex function u_m as $j \to \infty$ and $u_m = \varphi$ in $B_R\backslash\Omega$. Note that $u_m \in \overline{S}[\varphi,\Omega]$ when restricted in Ω. By Theorem

6.17, u_m is the maximizer of the functional

$$J_m(v) = \int_\Omega G(\det \partial^2 v)dx - \int_\Omega (f + \beta_m \chi_{D_m})vdx \tag{6.89}$$

in $\overline{S}[\varphi, \Omega]$.

(iii) Since $u_m \in \overline{S}[\varphi, \Omega]$, u_m converges to a convex function u_∞ in $\overline{S}[\varphi, \Omega]$ as $m \to \infty$. We claim that u_∞ is the maximizer u. The proof is as follows.

Define

$$\varphi_* = \sup\{l(x) \mid l \text{ is a tangent plane of } \varphi \text{ at some point in } B_R \setminus \overline{\Omega}\}.$$

Then $\varphi_* \in \overline{S}[\varphi, \Omega]$ and $v \geq \varphi_*$ for any $v \in \overline{S}[\varphi, \Omega]$. We consider the maximizer u. Let

$$\tilde{u}_m = \sup\{l(x) \mid l \text{ is linear, } l \leq u \text{ in } \Omega \text{ and } l \leq \varphi_* \text{ in } D_m\}.$$

Then $\tilde{u}_m \in \overline{S}[\varphi, \Omega]$ and $\tilde{u}_m = \varphi_*$ in D_m. Since u is convex, it is twice differentiable almost everywhere. By the definition of $\tilde{u}_m$, $\tilde{u}_m = u$ at any point where $D^2 u > 0$ when m is sufficiently large. Therefore, we have $\det \partial^2 \tilde{u}_m \to \det \partial^2 u$ a.e.. By the upper semi-continuity of the functional $A(u)$ and Fatou lemma,

$$\lim_{m\to\infty} \int_\Omega G(\det \partial^2 \tilde{u}_m)dx = \int_\Omega G(\det \partial^2 u)dx.$$

It follows that for a sufficiently small $\epsilon_0 > 0$,

$$J(u) \leq J(\tilde{u}_m) + \epsilon_0 \tag{6.90}$$

provided m is sufficiently large.

On the other hand, we consider the functional J_m. By (ii), u_m is the maximizer of J_m in $\overline{S}[\varphi, \Omega]$, so we have

$$J_m(\tilde{u}_m) \leq J_m(u_m). \tag{6.91}$$

Note that $u_m \geq \varphi_* = \tilde{u}_m$ in D_m. Hence, we obtain

$$\int_{D_m} \beta_m u_m dx \geq \int_{D_m} \beta_m \tilde{u}_m dx.$$

By the definition of J_m, it follows

$$J(\tilde{u}_m) \leq J(u_m) + \epsilon_0. \tag{6.92}$$

for sufficiently large m.

Finally, by (6.90), (6.92) and the upper semi-continuity of $A(u)$,

$$\begin{aligned} J(u) &\leq J(\tilde{u}_m) + \epsilon_0 \\ &\leq J(u_m) + \epsilon_0 \\ &\leq J(u_\infty) + 2\epsilon_0. \end{aligned}$$

By taking $\epsilon_0 \to 0$, this implies that u_∞ is the maximizer. By the uniqueness of maximizers, $u_\infty = u$.

(iv) It remains to prove (6.87). We claim that for any fixed m,

$$\lim_{\beta_m \to \infty} u_m(x) \leq \varphi_*(x). \tag{6.93}$$

We prove it by contradiction. Suppose that there is $x_0 \in D_m$ such that $u_m(x_0) \geq \varphi_*(x_0) + \epsilon_0$ for some $\epsilon_0 > 0$. Since u_m and φ_* are uniformly Lipschitz continuous, $u_m(x) \geq \varphi_*(x) + \frac{\epsilon_0}{2}$ in a ball $B_{C\epsilon_0}(x_0)$ for some constant C. Let

$$u_{m*} = \sup\{l(x) \mid l \text{ is linear, } l \leq u_m \text{ in } \Omega \text{ and } l \leq \varphi_* \text{ in } D_m\}.$$

Then $u_{m*} \in \overline{S}[\varphi, \Omega]$, and satisfies

$$u_{m*} \leq u_m \text{ in } \Omega, \; u_{m*} = \varphi_* \text{ in } B_{C\epsilon_0}(x_0).$$

Hence,

$$J_m(u_m) - J_m(u_{m*}) = J(u_m) - J(u_{m*}) - \beta_m \int_{D_m} u_m - u_{m*} dx$$

becomes negative when β_m is sufficiently large. This is a contradiction to that u_m is a maximizer of J_m. □

Remark 6.22. If $\varphi \in C^1$, we can restate (6.87) in the theorem as

$$|D(u_m - \varphi)| \to 0 \text{ uniformly on } \partial\Omega. \tag{6.94}$$

Now we deal with Case (b). By Theorem 6.22, there exists a solution $u_m^{(k)}$ to

$$U^{ij} w_{ij} = f_m, \tag{6.95}$$

where

$$w = G_k'(\det D^2 u), \tag{6.96}$$

such that

$$u_m^{(k)} \longrightarrow u^{(k)}, \; m \to \infty.$$

and for any compact set $K \subset D\varphi(\Omega)$,

$$K \subset Du_m^{(k)}(\Omega) \tag{6.97}$$

for large m. Hence, we can choose a sequence $m_k \to \infty$ such that

$$u_k := u_{m_k}^{(k)} \longrightarrow u_0. \tag{6.98}$$

Lemma 6.23. *Assume that Ω and φ are smooth. Then $\mathcal{M}_0$ contains no line segments with both endpoints on $\partial\mathcal{M}_0$.*

Proof. Suppose that L is a line segment in $\mathcal{M}_0$ with both end points on $\partial\mathcal{M}_0$. By subtracting a linear function, we suppose that $u_0 \geq 0$ and l lies in $\{x_3 = 0\}$. By a translation and a dilation of the coordinates, we may further assume that

$$L = \{(0, x_2, 0) \mid -1 \leq x_2 \leq 1\} \tag{6.99}$$

with $(0, \pm 1) \in \partial\Omega$. Note that by Remark 6.3, these transformations do not change the essential properties of equation (6.13).

Since φ is a uniformly convex function in a neighborhood of Ω and $\varphi = u_0$ at $(0, \pm 1)$, L must be transversal to $\partial\Omega$ at $(0, \pm 1)$. Hence, by $u_0 = \varphi$ on $\partial\Omega$ and the smoothness of φ and $\partial\Omega$, we have

$$u_0(x) = \varphi(x) \leq \frac{C}{2} x_1^2, \; x \in \partial\Omega.$$

By the convexity of u_0,

$$u_0(x) \leq \frac{C}{2} x_1^2, \; x \in \Omega. \tag{6.100}$$

Now we consider the Legendre function u_0^* of u_0 in $\Omega^* = D\varphi(\Omega)$, given by

$$u_0^*(y) = \sup\{x \cdot y - u_0(x), x \in \Omega\}, \; y \in \Omega^*.$$

Note that $(0, \pm 1) \in \partial\Omega$. By the uniformly convexity of φ, $0 \notin D\varphi(\partial\Omega)$. Hence, $0 \in \Omega^*$. By (6.99), (6.100) and the smoothness of φ, we have

$$\begin{aligned} u^*(0, y_2) &\geq |y_2|, && (6.101) \\ u^*(y) &\geq \frac{1}{2C} y_1^2. && (6.102) \end{aligned}$$

On the other hand, by the approximation (6.98), the Legendre function of u_k, denoted by u_k^* satisfying the equation

$$u^{*ij} w^*{}_{ij} = -f_{m_k}(Du^*), \tag{6.103}$$

where

$$w^* = G_k(d^{*-1}) - d^{*-1}G'_k(d^{*-1}). \tag{6.104}$$

By (6.97), u_k^* is smooth in $\Omega_{\epsilon_k}^*$ with $\epsilon_k \to 0$ as $k \to \infty$, where

$$\Omega_{\epsilon_k}^* = \{y \in \Omega^* \mid dist(y, \partial\Omega^*) > \epsilon_k\}.$$

By (6.101), (6.102), u_0^* is strictly convex at 0. Then $\{y \mid u_0^* < h\} \subset \Omega_{\epsilon_k}^*$ providing m is sufficiently large. Note that u_k^* converges to u_0^*. By Lemma 6.9, we have the estimate

$$\det D^2 u_k^* \le C$$

near the origin in Ω^*. Note that in Lemma 6.9, C depends on $\inf f$ but not on $\sup f$. In other words, the large constant β_{m_k} in (6.85) does not affect the bound C. Therefore sending $k \to \infty$, we obtain

$$\det D^2 u_0^* \le C$$

in the sense that the Monge-Ampère measure of u_0^* is an L^∞ function. This is a contradiction with (6.101), (6.102) according to Remark 5.10. $\square$

In conclusion, we have proved that u_0 is strictly convex in Ω in dimension 2. Theorem 6.1 follows from Theorem 6.19.

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