Essays in Population, R&D and Economic Growth

by

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Abstract

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This thesis examines the dynamic interplay between fertility and economic growth and both the general conditions for and characteristics of long run R&D-based economic growth in the absence of population growth in a developed economy.

This thesis comprises five chapters. Chapter 1 motivates the topic, reviews the literature and provides an overview of the thesis. Chapters 2 - 5 contain four essays, each developing a theoretical model to:

1. explain the baby boom-bust and the possibility of a baby bounce-back;
2. establish the general conditions for positive long run economic growth under the alternative assumptions of a growing and stagnant population;
3. show that the notions of non-linear knowledge accumulation and non-scale growth are logically independent;

4. explain how individuals may become relatively ignorant over time in the absence of population growth.

Dynamics of Fertility and Growth: Baby Boom, Bust and Bounce-Back extends Galor & Weil (1996) by introducing a Constant Elasticity of Substitution (CES) production function for child rearing. The existence of goods and services, as an alternative input to maternal time, generates a baby boom-bust cycle: fertility rises in the first phase where women are at home, raising children full-time, and falls in the second phase where women engage in the labor force. Whilst fertility declines unambiguously at the beginning of the second phase, as women enter the labor force, it may bounce-back as income effects start to dominate.

Population and Endogenous Growth introduces a general growth model comprising three sectors (final production, R&D and human capital formation). When population is growing, strictly positive, balanced growth that does not essentially depend on population growth may arise if either growth in human capital or growth in ideas asymptotes to a strictly positive constant. CES technology in human capital accumulation illustrates. In the asymptotic limit, the matrix of structural elasticities is singular ($|A| = 0$) and yet the long run growth rate in per capita growth is non-scale. Our third paper, *Conditions for Non-Scale Growth*, investigates the general conditions for non-scale growth. When population growth is zero, $|A| = 0$ is the necessary condition for strictly positive, balanced growth. In turn, sectoral linearity is sufficient, but not necessary for $|A| = 0$. Our fourth paper, *Lab Equipment Models and Creative Ignorance*, formalizes sufficient conditions for $|A| = 0$ and applies the condition of linearly dependent columns in a lab equipment framework.

Conditions for Non-Scale Growth extends Eicher & Turnovsky's (1999) general non-scale growth model by relaxing the assumptions that all factors are necessary for production in all sectors and $|A| \neq 0$ and by introducing a second dimension of knowledge, which may be embodied or disembodied. Single input linearity in one dimension of knowledge accumulation is one of the conditions sufficient for positive non-scale growth that does not essentially depend on population growth. Eicher & Turnovsky's (1999) conditions are, in general, sufficient but not necessary for strictly positive, non scale growth. The notions of non-linear knowledge accumulation and non-scale growth are logically independent.

Lab Equipment Models of Research and Creative Ignorance extends Dalgaard & Kreiner (2001) by introducing increasing returns to scale in R&D and decreasing returns to scale in human capital formation. The balanced growth equilibrium is characterized by creative ignorance. Ideas and human capital increase in a virtuous circle, but the frontier of ideas grows faster than the knowledge embodied in individuals. Individuals become relatively ignorant over time in the absence of population growth.

To my parents,

Dorothy and Brian,

and to my son,

James, the blue in my sky.

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Acknowledgments

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The first essay of this thesis, *The Dynamics of Fertility and Growth: Baby Boom, Bust and Bounce-Back*, was published in The Berkeley Electronic Journals of Macroeconomics (Day 2004). Comments from Charles Jones and two anonymous referees proved very useful in preparing the final version of this paper.

The second essay, *Population and Endogenous Growth*, was accepted by Dynamics, Economic Growth and International Trade (DEGIT) for Conference XII, 2007 and the third essay, *Conditions for Non-Scale Growth*, will be presented at the PhD Conference, 2007, at University of Western Australia, and at a workshop at the University of California (Davis). I am grateful to Professor Alan Taylor and Dr. Giovanni Peri at the University of California (Davis) for helping to arrange my visit. I would also like to thank Professor Rohan Pitchford at the University of Sydney for inviting me to present the fourth essay, *Lab Equipment Models of Research and Creative Ignorance*, at a seminar.

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Part I

Population, R&D and Economic Growth

Chapter 1

Overview

1.1 Introduction

Imagine life as a typical American child in 1955. You are playing catch in the backyard with your three siblings. Dad arrives home from work, just as Mom is taking a pot roast out of her new oven. Now, span forward to 2000. You are an only child. Mom and Dad have finished work and are picking you up from child care. Their real income is approximately three times that of their 1950's counterparts, reflecting trend growth of approximately two percent per annum.[1] What will the population of the United States and other developed countries be in 2050? Will per capita income continue to grow in the long run?

Referring to Figure 1.1, total population of the OECD today may be one and a half times what it was in 1950, but it is projected to remain stagnant for the next

[1] This rough calculation is based on Gross Domestic Product (GDP) per capita (1990 International Geary Khamis dollars) of $9,561 and $28,403 in 1950 and 2000, respectively. Source: Maddison (2001).

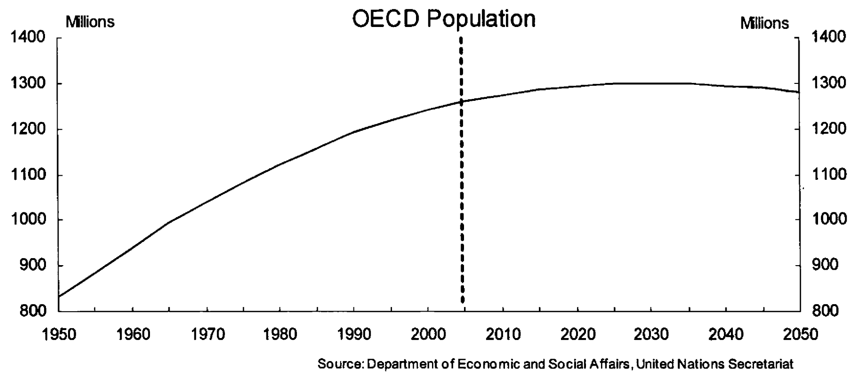

Figure 1.1: OECD Population (actual and projected)

fifty years. The projection of zero population growth[2] is based on a current trend of below replacement fertility. Put simply, even taking into account immigration, the number of people arriving in the OECD will exactly offset the number of people departing the OECD. These developments are quite pervasive - the United Nations (2005) projects global *de*population may occur after 2040.

This projection is supported by a surge in theoretical literature linking changes in fertility[3] to economic growth. Existing models of endogenous fertility (Barro & Becker (1988), Becker, Murphy & Tamura (1990), Erlich & Lui (1991), Galor & Weil (1996) and Galor & Weil (2000)) typically predict a negative relationship between fertility and per capita income in developed economies that have progressed well beyond the Malthusian poverty trap where subsistence incomes limit fertility. We observe that fertility has followed

[2]Curiously, Zero Population Growth (Z. P. G.) was the title of a kitsch 1970's science fiction movie, which depicted an affluent world in which couples cared for robotic children whilst awaiting permission to have a natural child.

[3]Let L_t, denote population at time t. $dL_t/dt = \left(n_t^f + n_t^m - d_t\right) L_t$ where n_t^f, n_t^m and d_t denote the rate of fertility, immigration and mortality, respectively. We focus on declining fertility in most OECD countries over the last century as the main source of declining population growth.

a non-monotonic path in the United States and other developed economies over the past fifty years. Can we develop a model that explains both past and possible future movements in fertility in a developed economy?

Juxtaposed against two centuries of population growth, innovation and improving living standards, the prospect of zero population growth is causing alarm. One source of this alarm is the prediction of early non-scale growth models (Jones (1995*a*), Kortum (1997) and Segerstrom (1998)) that the long run growth rate of the economy is driven by the rate of innovation, which in turn is ultimately driven by population growth. The concern is that if we replenish, but fail to increase, the world's population, we will lose a future Thomas Edison or Bill Gates.

This concern has generated a flurry of research and development (R&D)-based growth models investigating feasibility of long run economic growth in the absence of population growth. Modelling two aspects of R&D, Young (1998), Dinopolous & Thompson (1998), Peretto (1998) and Li (2000) predict that strictly positive population growth is conducive but non-essential to long run economic growth. Modelling R&D and the accumulation of embodied knowledge, Dalgaard & Kreiner (2001), Funke & Strulik (2000) and Strulik (2005) predict that strictly positive population growth is not only non-essential but also detrimental to long run economic growth.

In a decentralized setting, the assumptions that ensure strictly positive equilibrium growth are obscured by the intricacy of these models. For instance, Strulik (2005) comprises 48 equations, not including those contained in the appendix. A clear understanding of the general conditions for strictly positive equilibrium growth that does not essentially depend on population growth is needed to assess whether existing assumptions are necessary and actually met in practice. Can we develop a simple, unified framework within which to analyze thee sector R&D-based growth models where equilibrium growth may be endogenous or semi-endogenous, scale or non-scale?

The new generation R&D-based growth models are largely derived from non-scale growth models that assume population growth is strictly positive. The precedent Romer (1990) type models assume population is constant and would therefore seem pertinent to the current theoretical challenge. However, such models are criticized for assuming linearity in the accumulation of knowledge.[4] Are the notions of non-linear knowledge accumulation and non-scale growth logically independent?

According to existing models, zero population growth implies that ideas and the knowledge embodied in individuals grow at the same rate in the long run. We observe, however, that the frontier of ideas grows faster than the knowledge embodied in individuals. Can we develop a model that explains how growth in ideas may outstrip growth in human capital, even if population ceases to grow?

This thesis explores the theoretical relationship between population growth and economic growth. We investigate both directions of causality, focusing on endogenous fertility and R&D-based growth. This thesis comprises four essays, each addressing one of the four questions broadly defined above.

In addition to motivating the topic, this chapter reviews the theoretical literature, overviews the four essays, showing how they are both interrelated and distinct, and sum-

[4] *Linearity* implies that the output of new knowledge will double whenever we double the existing stock of knowledge.

marizes the key findings. This chapter clarifies both how this thesis contributes to the literature and how our key findings pertain to the projection of zero population growth.

1.2 Review of the literature

1.2.1 Neoclassical Origins

The neoclassical growth model of Solow (1956) and Swan (1956) highlights that the long run level of per capita output depends negatively on population growth, and that long run growth in per capita output requires technological change. According to the neoclassical growth model, declining population growth boosts physical capital per worker, thereby raising the growth rate of the economy, at least until the economy reaches a new long run equilibrium.

The literature on endogenous fertility explores the reverse causality. Endogenous fertility models void of human capital accumulation or technological progress (for example, Galor & Weil (1996)) retain the neoclassical prediction that per capita output ceases to grow in the long run. However, as we will see, analysis of transitional dynamics can generate rich results.

The R&D-based growth literature explains technological change. Earlier models of endogenous growth assume linearity in the accumulation of physical capital or human capital (Lucas 1988). As we discuss, more recent complex models of R&D-based growth models also use linearity to obtain endogenous growth (or long run growth in per capita that does not depend on population growth). We explore the possibility of endogenous growth under alternative assumptions.

On this point, Pitchford (1960) and Jones & Manuelli (1990) establish that endogenous growth may arise in the standard neoclassical growth model when the marginal product is bounded below by a positive constant. Linearity may be an asymptotic requirement rather than something that must hold at all points in time - a concept that can be applied to the recent endogenous growth literature.

1.2.2 Endogenous Fertility

The diversity of endogenous fertility models lies in how children enter into parental utility. Self-interested parents may view children as a consumption item (Galor & Weil 1996) or as a source of income support in old age (Erlich & Lui (1991), Cigno & Rosati (1996)). Alternatively, parents may be altruistic (Barro & Becker (1988), Becker et al. (1990)). In each case, parental utility is increasing in the number of children.[5]

Common to endogenous fertility models is the idea that growth in per capita income raises the opportunity cost of time spent raising children. On the other hand, growing incomes make children more affordable. In a developed economy, the substitution effect dominates the income effect, so that fertility declines. There is scope for contributing to the literature through more sophisticated modelling of the cost of raising children.

Non-monotonic path

Evidence from the United States and other G5 countries suggest that fertility in developed economies follows a non-monotonic path as per capita income rises.[6] In the

[5]Becker et al. (1990) and Erlich & Lui (1991) introduce a child quantity - quality trade-off, in which case, parents weigh the marginal utility of number of children against the marginal utility of children's educational attainment.

[6]See Figure 2.1 in Chapter 2.

United States, the average number of births per woman in 1950 was 3.1. Fertility rose to a peak of 3.8 in 1957 and fell steadily thereafter before plateauing at 1.8 in 1974. Fertility began to rise again in 1987, reaching 2.1 in 2000.

Becker et al. (1990), Erlich & Lui (1991), Barrro & Sala-i-Martin (1999) and Galor & Weil (2000) model a positive relationship between income and fertility rates, but only in the context of Malthusian models of economies at very low levels of capital per person. Surprisingly, there is a dearth of economic theories explaining the twentieth century baby boom. Two notable exceptions are Easterlin (2000) and Greenwood, Seshandri & Vandebroucke (2004), who attribute the baby boom to modest parental consumption habits that were formed as children in the Great Depression and technological progress in the household sector, respectively. Unfortunately, neither model is able the explain why the baby bust levelled out or why we may be witnessing or expect to witness an upturn in fertility - a baby bounce-back. Like their Malthusian counterparts, fertility rates follow an inverted U-shape. Once fertility decline begins, it will continue so long as per capita incomes rise over time.

Extending the literature

Galor & Weil (1996) make an important contribution to the literature by capturing one of the most dramatic developments in advanced economies - the rise of female labor force participation. They define two distinct phases: women at home and raising children full time; women entering the work force and raising children part time. Their model predicts that fertility is constant and monotonically decreasing in the first and second phase, respectively. We observe that this prediction follows directly from the assumption that a fixed fraction of maternal time is the only input into child rearing.[7] Could the existence of an alternative input to maternal time explain rising fertility in the first phase and a non-monotonic path in fertility in the second phase? Could the optimal mix of inputs change over time in response to rising female relative wages?

1.2.3 R&D-based Growth

The theoretical prediction that the long run economic growth either does not depend or does not essentially depend on population growth pertains to a balanced growth equilibrium. We append this chapter with definitions of both balanced growth and phrases italicized in the following discussion.

Let y denote income or output per capita, L the total population, H the stock of human capital and let g_x denote the long run growth rate of any variable x. We review the literature as it pertains to the projection of zero population growth and classify models of R&D-based growth into two broad types: those that assume population is constant and those that allow for population to grow over time.

[7] The assumption that parents derive direct utility from the number of children simplifies the analysis but does not alter the prediction that fertility declines in the second phase. After making the necessary extensions, this author finds that economic growth, via rising female relative wages, generates a fertility decline in all three models of parental utility.

Constant Population

Models of the first type, exemplified by Romer (1990) and Aghion & Howitt (1992), predict[8]:

$$g_y = a.H \tag{1.1a}$$

where all constant exogenous and endogenous parameters are summarized by the term, $a > 0$.[9] Endogenous parameters include the fraction of the labor force engaged in R&D. Thus, long run growth per capita growth is proportional to the skill employed in R&D. To the extent that the skill is embodied in the population, long run growth of the economy is still proportional to the size of the population. The implication that the growth rate of the economy will rise exponentially over time should population grow at a constant rate is not supported by evidence in Jones (1995*b*).

Current models are derivative of these seminal models of R&D-based growth. The stock of labor in these models can be homogenized into either the stock of human capital or total population. Most literature stems from the latter assumption. However, an example of a model that assumes the former is Funke & Strulik (2000). They retain the assumption that population is constant and are therefore a first-type model. By endogenizing the accumulation of human capital, they remove the scale effect from the long run growth rate of the economy:

$$g_y = a \tag{1.1b}$$

[8]The seminal R&D-based growth models (Romer (1990), Aghion & Howitt (1992) and Grossman & Helpman (1991)) are widely criticized for their scale effect: long run per capita output growth is proportional to population size. It is, in fact, a slight misrepresentation of Romer (1990) and Aghion & Howitt (1992) to say that g_y is proportional to L, as predicted by Grossman & Helpman (1991), since both allow for heterogeneous labor.

[9](1.1a) is a generalized expression. Romer (1990) assumes $a = \delta(1 - l_Y)$, where δ is an exogenous productivity parameter and l_Y is the endogenously determined fraction of labor allocated to final production. For $g_y > 0$, H must be sufficiently high that the non-negativity constraint of $(1 - l_Y)H$ is not binding.

All these first type models share the common feature of sectoral linearity in a knowledge accumulation equation, whether knowledge be non-rivalrous ideas or rivalrous human capital. And so, *sectoral linearity* has become synonymous with *endogenous growth* models that treat population as an exogenous constant. Like the *scale* effect, linearity in the accumulation of knowledge is criticized. Jones (2001) argues that, with the exception of the population equation, the assumption of linearity is ad hoc. This brings us to R&D-based growth models of the second type, that introduce a linear population equation.

Growing Population

Early examples of second-type models are Jones (1995*a*), Kortum (1997) and Segerstrom (1998). Their common feature is diminishing marginal returns to ideas (or knowledge spillovers of degree less than one) in the creation of new ideas. Diminishing marginal returns in the stock of ideas requires increasing effort to create an idea. This increasing effort can come from more researchers. Since the fraction of the labor force engaged in R&D is constant in steady state, strictly positive population growth satisfies the increasing efforts needed for strictly positive growth in technology and the overall economy.

In these models, R&D remains the engine of long run economic growth, but population growth is the fuel:

$$g_y = c g_L \tag{1.1c}$$

where all constant exogenous parameters are summarized by the multiplicative term $c > 0$. On the flip side, strictly positive population growth is essential for long run economic growth. To establish feasibility of long run economic growth in the absence of population growth, the most recent literature adapts second-type models. Three main branches have emerged.

The first new branch of second type R&D-based models assumes two aspects of R&D. Examples are Young (1998), Dinopolous & Thompson (1998), Peretto (1998) and Li (2000). In brief, R&D may involve either the creation of new products, so that technological improvement is measured by increased variety of intermediate goods (Romer 1990) or the improvement of existing products as in Aghion & Howitt's (1992) quality-ladder model. We refer to these two aspects as simply variety R&D and quality R&D. Li (2000) shows that if there are no knowledge spillovers in variety R&D and spillovers of degree one (or linearity) in quality R&D then the long run growth rate of the economy is an additively separable function of population growth and a constant term:

$$g_y = b + cg_L \tag{1.1d}$$

where all constant parameters are summarized in the terms b and c. Since b summarizes endogenous parameters, the policy variance result of first type models is restored. The absence of knowledge spillovers in variety R&D implies a one-to-one correspondence between variety growth and population growth. This explains the second term of equation (1.1d). If population is static, the variety of intermediate goods stays constant. However, endogenous technological change is still possible through improving existing products, since linearity in quality R&D implies quality growth is proportional to the population size. This explains the first term of equation (1.1d). Consequently, strictly positive population growth is non-essential to long run economic growth.

A second branch assumes two dimensions of knowledge and models both R&D and human capital formation. Dalgaard & Kreiner (2001) and Strulik (2005) predict

$$g_y = b - cg_L \tag{1.1e}$$

so that strictly positive population growth is not only nonessential but also detrimental to long run economic growth. Linearity in the accumulation of human capital implies population growth is not necessary for long run growth in per capita output, as per the first term of equation (1.1e). The embodiment of human capital, and the associated congestion or capital dilution effect, implies population growth is not conducive for long run growth in per capita output, as per the second term of equation (1.1e).

A third branch endogenizes population (Jones 2001) or both population and human capital (Galor & Weil 2000). Just as Funke & Strulik (2000) removes the *"strong" scale effect* from the early endogenous growth models, these models remove the *"weak" scale effect* from *semi-endogenous growth* models. They predict a long run rate of growth in the economy:

$$g_y = d \tag{1.1f}$$

where d is a constant term summarizing, for example, exogenous efficiency parameters.

Extending the literature

Endogenous Growth Long run economic growth that does not depend on exogenous growth in the population is endogenous. The long run growth rates of (1.1a), (1.1b) and (1.1f) are endogenous and the long run growth rates of (1.1d) and (1.1e) comprise an endogenous component, encapsulated by the summary parameter b.

Recent literature establishing endogenous growth is overwhelmingly derivative of the early semi-endogenous growth models that predict (1.1c), but this is not to say that the assumption of diminishing marginal returns prevails. In a decentralized setting, recent models are intricate. It is not clear what assumptions are necessary for endogenous growth.

For the branch of the literature predicting (1.1d), Li (2000) establishes that linearity in quality R&D implies endogenous growth.[10] Is linearity in the accumulation of knowledge, in general, necessary for endogenous growth?

We observe that each of the models predicting (1.1b), (1.1d) and (1.1e) add a second dimension of knowledge accumulation to a seminal R&D-based growth model. Could we develop a generalized three sector framework for which (1.1a) - (1.1e)[11] are special cases? Using this framework, what are the general conditions for endogenous growth when population growth is assumed strictly positive and zero, respectively?

There is also scope to contribute to the literature by developing a specific model. Applying CES technology to R&D, Dalgaard & Kreiner (2003) predict (1.1c) or (1.1d), depending on the degree of substitutability between ideas and researchers. A priori, however, we would expect a high degree of complementarity between the two R&D inputs. Could a more plausible application of CES technology to the formation of human capital generate (1.1d)?

Non-Scale Growth Eicher & Turnovsky (1999) make an important contribution to our understanding of the conditions for (1.1c) by abstracting from the microeconomic foundations of R&D and modelling the decision making of a central planner in a generalized two-sector non-scale growth model. Eicher & Turnovsky (1999) assume the matrix of structural elasticities is non-singular. Under this restriction, they establish that a strictly

[10]Li (2000) shows that growth is endogenous when the determinant of the coefficient matrix is singular. He proposes that this in turn requires that the return to variety and the return to quality is equated across variety and quality R&D. Zero returns to variety and constant returns to quality satisfy this necessary condition.

[11]We extend the endogenous fertility literature in the first paper and place (1.1f) outside the scope of the last three essays.

positive determinant, diminishing returns to physical capital in final production and diminishing returns to the existing stock of ideas in R&D are necessary and sufficient for strictly positive non-scale growth.

We observe that (1.1d) and (1.1e) are also examples of *non-scale growth* and that these second generation non-scale growth models all work in a similar way. Each model adds a second dimension of knowledge accumulation to a seminal non-scale model of R&D-based growth. The second type of knowledge may be embodied or disembodied. Could we extend Eicher & Turnovsky (1999) by both introducing a third sector of knowledge accumulation and relaxing their restriction? What are the general conditions for strictly positive, non-scale growth? Are the conditions for strictly positive growth in the absence of population growth met in practice? Are the notions of non-scale growth and non-linear knowledge accumulation logically independent?

Creative Ignorance Existing models predict that the aggregate stocks of endogenous factors grow at the same rate along a balanced growth path. In a three sector model, $g_K = g_A = g_H$, where K, A and H denote the aggregate stock of physical capital, ideas and human capital, respectively. It follows that models with embodied knowledge, Dalgaard & Kreiner (2001) and Strulik (2005), predict $g_h = g_A - n$ where h denotes human capital per person. Zero population growth will imply $g_h = g_A$, that is, individuals cease to become relatively ignorant over time.

Dalgaard & Kreiner (2001) is particularly interesting because it applies the lab equipment framework. Few endogenous growth models assume that the formation of knowledge employs the same inputs as final production. Those that do, Rivera-Batiz & Romer

(1991) and, more recently, Dalgaard & Kreiner (2001), assume that knowledge forms using the same input proportions as final production.

Mulligan & Sala-i-Martin (1993) make an important contribution to our understanding of the conditions for endogenous growth when knowledge accumulation features the same inputs as final production. However, they confine knowledge to human capital. What are the general conditions for endogenous growth when both R&D and human capital formation feature the same inputs as final production?

We observe that when stocks accumulate using the same inputs as final production, the prediction that aggregate stocks grow at the same rate follows directly from the assumption that all sectors are subject to constant returns to scale. Could increasing returns to scale in R&D explain why growth in ideas may outstrip growth in human capital per person in the long run, even if population ceases to grow?

1.3 Review of the thesis

This thesis comprises four essays. The first essay develops an overlapping generations model with endogenous fertility. The analysis of transitional dynamics reveals a baby boom, bust, bounce-back sequence. The next three essays make use of a generalized growth framework comprising three sectors producing final output, disembodied knowledge (ideas) and a second dimension of knowledge, which may be embodied or disembodied, although each essay addresses a distinct question. Since the respective questions relating to endogenous growth, non-scale growth and creative ignorance pertain to the long run, these essays focus on the analysis of a balanced growth equilibrium, although dynamic analysis is provided where relevant.

1.3.1 Essay 1 - Dynamics of Fertility and Growth: Baby Boom, Bust and Bounce-Back

This essay examines the dynamic interplay between economic growth and fertility as a developed economy moves through two distinct phases: women at home and raising children full time; women entering the work force and raising children part time. Women's relative wages rise with economic growth, as per Galor & Weil (1996). Higher wages make children more affordable. On the other hand, children are more costly when maternal time, used in child rearing, could be supplied to the labor market. We extend Galor & Weil (1996) by introducing goods and services as a child rearing input. A Constant Elasticity of Substitution (CES) production function for child rearing allows for varying degrees of substitutability between goods and time. The existence of an alternative input to maternal time generates a baby boom-bust cycle: fertility rises in the first phase and falls in the second. Whilst fertility declines unambiguously at the beginning of the second phase, as women enter the labor force, it may bounce-back as income effects start to dominate.

1.3.2 Essay 2 - Population and Endogenous Growth

This essay introduces a general growth model comprising three sectors (final production, R&D and human capital formation). The framework allows us to derive the central planner solution to models of endogenous, semi-endogenous, scale and non-scale growth. Using this framework, we establish general conditions for positive growth in per capita output along a balanced growth path under the alternative assumptions of a growing and stagnant population.

Under the assumption of strictly positive population growth, we establish that strictly positive, balanced growth that does not essentially depend on population growth may arise if either growth in human capital or growth in ideas asymptotes to a strictly positive constant. To illustrate this, we introduce CES technology to human capital accumulation. If the elasticity of substitution between human capital and physical capital exceeds one, long run per capita growth asymptotes to an additively separable function of population growth.

A question arises when solving the illustrative example of asymptotic linearity. In the asymptotic limit, the matrix of structural elasticities is singular ($|A| = 0$) and yet the long run growth rate in per capita growth is non-scale. This paper takes the general conditions for non-scale growth as given. For a general two sector model of non-scale growth, Eicher & Turnovsky (1999) establish that $|A| > 0$ is one of the conditions necessary and sufficient for strictly positive growth in output, capital, consumption and technology. What are the conditions for strictly positive growth in a general three sector model of non-scale growth? Our third essay, *Conditions for Non-Scale Growth*, addresses this specific question.

Under the assumption of zero population growth, we establish that $|A| = 0$ is the necessary condition for strictly positive growth in output, capital, consumption, technology and human capital. In turn, sectoral linearity is sufficient, but not necessary for $|A| = 0$. Linearly dependent columns is also sufficient for $|A| = 0$. We illustrate, by way of a numerical example, how increasing returns to scale in one sector offset by decreasing returns to scale in another sector provides linearly dependent columns.

This is an important result that needs developing. Our fourth essay, *Lab Equipment Models and Creative Ignorance*, formalizes sufficient conditions for $|A| = 0$ and applies the condition of offsetting returns to scale in a lab equipment model to explain "creative ignorance".

1.3.3 Essay 3 - Conditions for Non-Scale Growth

This essay extends Eicher & Turnovsky's (1999) general non-scale growth model by relaxing the assumptions that all factors are necessary for production in all sectors and $|A| \neq 0$ and by introducing a second dimension of knowledge, which may be embodied or disembodied. We establish that single input linearity in one dimension of knowledge accumulation is one of the conditions sufficient for positive non-scale growth that does not essentially depend on population growth. Eicher & Turnovsky's (1999) conditions are, in general, sufficient but not necessary for strictly positive, non scale growth. We show that the notions of non-linear knowledge accumulation and non-scale growth are logically independent. We review the empirical and policy implications of this result. Both intuition and, in the case of disembodied knowledge, existing econometric evidence support diminishing marginal returns to the existing stock of knowledge in the creation of new knowledge.

1.3.4 Essay 4 - Lab Equipment Models of Research and Creative Ignorance

This essay assumes Cobb-Douglas production technology for a general three sector growth framework. We establish three sufficient conditions for $|A| = 0$, as required for endogenous growth. One condition is that all sectors feature constant returns to scale to endogenously accumulating factors, in which case all variables grow at a common rate. We show that identical production technology in existing lab equipment models renders constant returns to scale necessary for balanced growth.

Another condition is that increasing returns to scale to endogenously growing factors in one sector offset by decreasing returns to scale in other sectors such that the columns of A are linearly dependent. We extend Dalgaard & Kreiner (2001) by introducing increasing returns to scale in R&D and decreasing returns to scale in human capital formation. This extension generates a balanced growth equilibrium characterized by creative ignorance. Ideas and human capital increase in a virtuous circle, but the frontier of ideas grows faster than the knowledge embodied in individuals. This characteristic is robust to the absence of population growth. In contrast to existing lab equipment models, we predict that raising the share of resources used in R&D has an unambiguously positive effect on long run economic growth.

1.3.5 Key findings

We summarize the key findings of this thesis that are pertinent to the United Nations's (2005) projection of zero population growth for the next fifty years.

Dynamics of Fertility and Growth: Baby Boom, Bust and Bounce-Back predicts

1. Whilst fertility in a developed economy declines unambiguously at the beginning of the phase where women enter the labor force, it may cease to decline if maternal time becomes a sufficiently small part of the cost of child rearing;

2. Once fertility ceases to decline, any further wages growth, ceteris paribus, induces households to substitute child rearing goods and services for maternal time, enabling a rise in fertility or a *baby bounce-back.*

The implication of a possible upturn in fertility in developed economies is that zero population growth in the OECD may not eventuate.

Population and Endogenous Growth establishes:

1. When population is growing, asymptotically linear accumulation of knowledge (either ideas or human capital) is sufficient for endogenous growth. To illustrate, if human capital is sufficiently substitutable for physical capital in the formation of human capital, long run growth in the economy does not essentially depend on population growth;

2. When population growth is constant, singularity of the matrix of structural elasticities ($|A| = 0$) is necessary for endogenous growth. Linearity, either in the form of constant returns to scale to growing factors or constant returns to the existing stock of knowledge, in the accumulation of knowledge is sufficient but not necessary for $|A| = 0$.

The implication is that if the projection of zero population growth proves correct, long run economic growth is feasible without imposing the restriction of linearity in knowledge accumulation.

Conditions for Non-scale Growth establishes

1. $|A| > 0$, diminishing marginal returns to physical capital in final production ($\sigma_K < 1$) and diminishing marginal returns to knowledge in the accumulation of knowledge ($\eta_A < 1$ in a two sector model; $\omega_Q < 1$ in a three sector model) are sufficient but not necessary for positive growth of output, capital, consumption and knowledge in a non-scale growth model. The non-scale growth equilibrium implied by these conditions depends essentially on population growth;

2. $|A| = 0$, $\sigma_K < 1$, $\eta_A < 1$ and a linear knowledge accumulation equation ($\omega_Q = 1$ and $\omega_i = 0 \forall i \neq Q$, holding asymptotically or at all points in time) is sufficient for positive growth of output, capital, consumption and knowledge in a non-scale growth model. The non-scale growth equilibrium implied by these conditions does not essentially depend on population growth. Equilibrium growth may even decrease with population growth if knowledge (Q) is embodied;

3. The notions of non-linear knowledge accumulation and non-scale growth are logically independent.

Intuition and existing empirical estimates support the assumption that the marginal return to existing knowledge in the formation of knowledge diminishes. By the third proposition, diminishing marginal returns may imply equilibrium growth that is either scale or non-scale. When population growth is zero, scale growth is positive, whereas non-scale growth is only positive if knowledge accumulation is asymptotically linear. The implication is that the prediction of long run economic growth in the absence of population growth is more robust in models of scale growth, an underdeveloped branch of the literature.

Lab Equipment Models of Research and Creative Ignorance establishes

1. Endogenous growth, with or without scale effects, is impossible unless the condition $|A| = 0$ is met.

2. Sufficient conditions for $|A| = 0$ include:

   (a) All sectors feature constant returns to scale to endogenously accumulating factors, in which case all real variables grow at a common rate;

   (b) Increasing returns to scale to endogenously accumulating factors in one sector, offset by decreasing returns to scale to endogenously accumulating factors in other sectors such that the columns of matrix A are linearly dependent, in which case real variable grow at different rates.

3. In a three sector Cobb-Douglas lab equipment economy, increasing returns to scale to A and Q (ideas and aggregate human capital, respectively) in R&D offset by decreasing returns to scale to A and Q in human capital formation (such that $|A| = 0$) generates a balanced growth path along which $g_A = kg_Q$ where $k > 1$. We refer to this characteristic as *creative ignorance*, meaning ideas and aggregate human capital increase in a virtuous circle, but the frontier of ideas grows faster than the knowledge embodied in individuals.

In the absence of population growth, $g_h = g_Q$, where h denotes human capital per person. Thus, the creative ignorance result implies that individuals may become increasingly ignorant over time even if zero population growth eventuates.

Part II

Endogenous Fertility - Essay 1

Chapter 2

The Dynamics of Fertility and Growth: Baby Boom, Bust and Bounce Back

2.1 Introduction

This paper explores reasons why the developed economy 'baby-boom' of the 1950's was characterized not only by a high level of fertility but also by rising fertility, why the boom was followed by a sharp decline in fertility, why the 'baby-bust' then leveled out and why we may be witnessing or expect to witness a rise in fertility rates – a 'baby bounce-back'.

There has been a recent surge in the theoretical literature linking changes in fertility to economic growth. The existing literature predicts a negative relationship between fertility and income *per capita* in developed economies that have progressed beyond the

Malthusian poverty trap where subsistence incomes limit fertility. By contrast, the model developed here is capable of explaining the non-monotonic path of fertility over the past fifty years.

Much of the current literature is motivated by the well publicized decline in fertility that occurred in most OECD countries in the latter half of the 20^{th} century. The fact that fertility is currently below replacement level in all G5 countries, bar the United States, has perhaps biased developments in theory in favor of explanations for fertility decline. Just as we would not want theory of the 1960's to have been unduly influenced by the preceding baby boom, we do not want recent experience to blind us to the possibility of a resurgence of fertility in developed economies.

Common to recent models of endogenous fertility is the idea that growth in income per capita raises the opportunity cost of time spent raising children. In a developed economy, the associated substitution effect dominates the income effect (due to increased affordability of children), so that fertility declines. As a slight variation on this theme, Becker et al. (1990) and Erlich & Lui (1991) introduce a child quantity – quality tradeoff. In this case, fertility declines with economic growth because the opportunity cost of time spent raising children increases relative to the opportunity cost of time spent educating children.

The model of Galor & Weil (1996) has several advantages over its counterparts. Firstly, parents derive direct utility from the number of children, simplifying the analysis without altering any empirically valid predictions.[1] Secondly, a rise in female relative wages

[1]Models endogenising fertility may be differentiated according to how children enter into parental utility. Self-interested parents may view children as a consumption item (Galor & Weil 1996) or as a source of income support in old age (Erlich & Lui (1991), Cigno & Rosati (1996)). Alternatively, parents may be altruistic (Barro & Becker (1988), Becker et al. (1990)). After making the necessary extensions, economic growth, via rising female relative wages, generates a fertility decline in all three models of parental utility.

drives the fertility decline.[2] Thirdly, it captures one of the most dramatic developments in the recent economic history of the advanced economies – the rise in female labor force participation.

Galor & Weil (1996) define two distinct phases of demographic development: in phase one, women are at home full time; in the second phase, women participate in the paid labor force. Throughout the initial phase, fertility is high but constant. Once women enter the labor force, fertility declines monotonically with growth in income per capita. These predictions contrast with the evidence from the US and the other G5 economies in two respects – see Figure 2.1.[3]

Firstly, fertility was actually rising over the "baby boom" phase - whereas Galor & Weil (1996) predict *constant* fertility throughout a phase when married women allocate all of their time endowment to child rearing. This prediction follows directly from their assumption that a fixed fraction of maternal time is the only input into child rearing. Secondly, fertility decline in the leading developed economies has leveled off. In the US, there is clear evidence of an upturn in fertility since 1985.

This raises the question whether the existence of other inputs into child-rearing would enable the model to explain non-monotonicities in the relationship between fertility and income in developed economies.[4] This question has been partially dealt with by

[2]Cigno & Rosati (1996) present evidence for US, UK, Germany and Italy that the recent decline in fertility corresponded to a steady rise in the female-male wage ratio. The gender wage gap has fallen, since 1979, in the US and, since 1970, in other G5 countries. See Blau & Khan (2000) and Harkness & Waldfogel (1999).

[3]Figure 2.1 sourced from *World Bank Tables; Historical Statistics of the US.*

[4]Becker et al. (1990), Erlich & Lui (1991), Barrro & Sala-i-Martin (1999) and Galor & Weil (2000) model a positive relationship between income and fertility rates, but only in the context of Malthusian models of economies at very low levels of capital per person. More relevant to Figure 2.1, is a model that tracks an advanced economy's transition through two phases distinguished by female labor force participation.

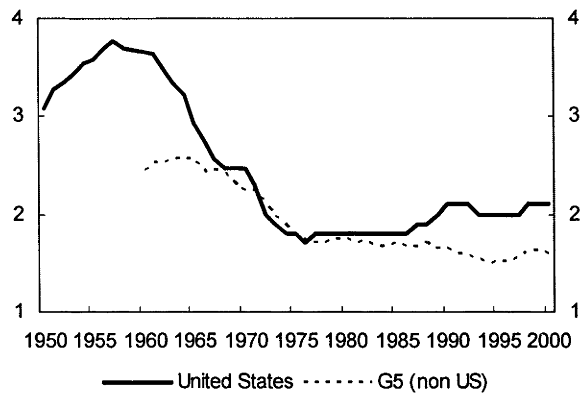

Figure 2.1: Total Fertility Rate (births per woman)

a footnote in Galor & Weil (1996) which deals with the second phase of demographic development. If a fixed quantity of goods is required in addition to parental time, then the fertility decline may be eradicated if the goods input is sufficiently large. This raises the question: what factors determine the goods-time input mix? Whilst Apps & Rees (2001) extend the Galor and Weil approach by introducing a generic production function for child rearing, neither paper analyses the dynamics of fertility in the context of a child-rearing production function, nor do they deal with rising fertility during phase one. This is exactly what this paper sets out to do.

This paper extends Galor & Weil (1996) by introducing goods and services as a child rearing input and allowing the household to choose the optimal goods-time input mix. A Constant Elasticity of Substitution (CES) production function for child rearing allows analysis of the degree of complementarity between goods and time.

2.2 Basic structure of the model

An overlapping generations model in which people live for three periods describes the economy of Galor & Weil (1996). People spend the first period of life as children, consuming time, as well as goods and services, from their parents; the second period of life involves the supply of labor to the market and the raising of children; during the third period, people are retired from the labor force and consume the proceeds of their savings from the previous period. Men and women differ in their wage earning ability because of different labor endowments. The closed economy identity of savings and investment provides the link with growth in the capital stock, productivity and wages, which in turn influence fertility.

Intuitively, when children are a consumption item, higher wages have both an income effect and a substitution effect when labor is used to rear children. These effects work in opposite directions. By construct, the substitution effect dominates. Specifically, if maternal time is the only child rearing input and female wages are a fraction of male wages, fertility unambiguously declines as female relative wages rise. Rising female relative wages are a consequence of economic growth. Women supply only one type of labor that is complementary to capital. Both male and female wages rise over time with the accumulation of capital per worker, but female wages rise relatively more. Combined with the neoclassical capital intensity effect, this final feature generates a feedback loop between growth in output per worker and declining fertility. Growth in capital per worker raises female relative wages, inducing a decline in fertility that, in turn, boosts capital per worker. Thus, Galor & Weil (1996) encapsulate in one model fertility decline as both a symptom and a cause of economic growth.

2.2.1 Production of final output

Physical capital (K), physical labor (L_t^p) and mental labor (L_t^m) are factors of production, all with non-increasing marginal products. The greater the capital-labor ratio in the economy, the more highly rewarded is mental labor relative to physical labor. This is consistent with the relative rise in rewards to mental labor in developed countries. Intuitively, capital does a better job of replacing human strength than human thinking.

The production function is given by

$$Y_t = A\left[\alpha K_t^\rho + (1-\alpha)(L_t^m)^\rho\right]^{1/\rho} + bL_t^p \quad \rho \neq 0 \tag{2.2}$$

where A >0, b >0, $\alpha \in (0,1)$ and $\rho \in (-\infty, 1)$. The separability of the production function captures the assumption that, whereas, capital complements mental labor, physical labor is neither a complement nor a substitute for capital or mental labor.

The household production function is

$$y_t = Y_t/L_t^p = A\left[\alpha k_t^\rho + (1-\alpha)m_t^\rho\right]^{1/\rho} + b \tag{2.3}$$

where k_t and m_t are the per household supplies of capital and mental labor, respectively. Men supply inelastically 1 unit of physical labor and 1 unit of mental labor together. Women supply between 0 and 1 units of mental labor, so mental labor per working age household takes values: $1<m_t<2$, where $(m_t - 1)$ measures female labor force participation.[5]

Profit maximization and competitive markets imply

$$w_t^m = A(1-\alpha)m_t^{\rho-1}\left[\alpha k_t^\rho + (1-\alpha)m_t^\rho\right]^{1-\rho/\rho} \tag{2.4}$$

[5]The underlying assumption here is that men and women are equally endowed with "brains" but only men have "brawn".

$$w_t^p = b \tag{2.5}$$

An increase in capital intensity will therefore raise the wage[6] for mental labor (w_t^m) while the wage for physical labor (w_t^p) is constant. Men earn a wage of $w_t^m + w_t^p$; women earn a wage of w_t^m. It follows that growth in the capital stock over time will increase female wages proportionately more than male wages, thus reducing the gender wage gap.

2.2.2 Household optimization

Households derive utility directly from the number of children. Children are essentially a durable good. There is no bequest motive.

The household utility function[7] is

$$u_t = \gamma \ln(n_t) + (1-\gamma)\ln(c_{t+1}) \tag{2.6}$$

where c_{t+1} is consumption in retirement and denotes pairs of children (since the couple is the basic unit of analysis), both chosen by the household at time t.[8]

To raise each pair of children, households purchase goods and services and employ a fraction of maternal time, denoted $\hat{x}$ and $\hat{z}$, respectively. Only the wife raises children, $\hat{z}n_t \leq 1$, because the opportunity cost of female labor is lower. Men allocate their labor endowment to paid work only. This paper models the economy's transition through the Baby Boom and Bust during the latter half of the twentieth century. Neither phase saw men withdrawing from the labor force to supplement women at home raising children full time. Even the arrival of "Mr. Mom" in the modern era involved substitution for, rather

[6]This is the real wage, with the price of the aggregate good normalized to 1.

[7]Consumption in the second period of life is assumed to be zero. Galor & Weil (1996) note that if couples had log utility from c_t, the equation of motion would be altered only by a multiplicative constant.

[8]The model structure up to this point corresponds to Galor and Weil (1996). The analysis from this point is the original work of the author.

than supplementation of, maternal time. For the purposes of this paper, therefore, it is assumed[9] the following condition is met:

$$z_t \equiv \hat{z} n_t \leq 1 \tag{2.7}$$

The production function for child-care is of CES form:

$$n_t = \left[(\alpha_1 z_t)^a + (\alpha_2 x_t)^a\right]^{\frac{1}{a}} \quad ; \quad a \neq 0 \tag{2.8}$$

where x and z denote *total* child rearing goods and services and *total* time input, respectively, and a determines the constant elasticity of substitution between time and goods, given by $\varepsilon = 1/(1-a)$. Define $\alpha_1 = \delta^{1/a}$ and $\alpha_2 = (1-\delta)^{1/a}$, where δ is the distribution parameter that measures the relative factor shares in production. The limits of the CES child-rearing production function are:

- as $\delta \longrightarrow 1$, the assumption of Galor & Weil (1996) that child rearing requires only time input;[10]
- as $\varepsilon \longrightarrow 0$, a Leontief technology, where child-care goods and services and maternal time are used in fixed proportions.

Because the child-rearing production function is homogeneous of degree one, the household optimization problem can be solved in two stages. The household first chooses, for a given n_t, the cost minimizing input mix and then chooses n_t, given the efficient input mix, so as to maximize utility subject to a budget constraint.

[9]To ensure that women enter the workforce, Galor & Weil (1996) assume γ <1/2. By implication, the household's optimal choice of fertility satisfies condition (2.7). The approach taken in this paper differs in that parameter values in the household optimization problem are not restricted so as to rule out the scenario where optimizing households would use child-rearing time in excess of the maternal time endowment if they could.

[10]In the limit, $\hat{z} = 1/E$, where E is the level of production technology, analogous to A in (2.2). Replacing the restriction $E = 1$ with E>1 ensures $0 < \hat{z} < 1$.

Cost minimization

Allowing for the possibility of a government subsidy ($0 \leq \beta_t < 1$) per unit of market goods and services used[11], the total cost of rearing children is

$$C_t = w_t^m z_t + (1 - \beta_t)x_t \tag{2.9}$$

There are two cases, constrained (c) and unconstrained (u), depending on whether the maternal time constraint, (2.7), is binding.

The household first chooses the input mix, for a given n_t, so as to minimize (2.9) subject to (2.8), assuming, for the moment, the maternal time constraint (2.7) is not binding. Input demands for time and goods are, respectively,

$$z_t^{*u} = \hat{z}n_t = \left[(w_t^m/\alpha_1)^{a/a-1} + (1 - \beta_t/\alpha_2)^{a/a-1}\right]^{-1/a} (\alpha_1)^{-a/a-1}(w_t^m)^{1/a-1}n_t \tag{2.10}$$

$$x_t^{*u} = \hat{x}n_t = \left[(w_t^m/\alpha_1)^{a/a-1} + (1 - \beta_t/\alpha_2)^{a/a-1}\right]^{-1/a} (\alpha_2)^{-a/a-1}(1 - \beta_t)^{1/a-1}n_t \tag{2.11}$$

The unconstrained per unit cost function is

$$p(w_t^m, \beta_t) = \left[(w_t^m/\alpha_1)^{a/a-1} + (1 - \beta_t/\alpha_2)^{a/a-1}\right]^{a-1/a} \tag{2.12}$$

When (2.7) is binding, $z_t^{*c} = \hat{z}n_t = 1$ and the required goods input, x_t, is derived by inverting equation (2.8).

Utility maximization

When (2.7) is not binding, the household's first period and second period budget constraints are, respectively

$$p(w_t^m, \beta_t)n_t + s_t \leq w_t^p + 2w_t^m \tag{2.13}$$

[11]The price of goods and services is normalized to 1. By implication, the goods and services component of child rearing costs is denominated in terms of goods (final output). We can relax this simplifying assumption without loss of generality. If mental labor is the only input into child rearing services, the goods and services input still becomes relatively cheaper as the wage for mental labor rises. The subsidy is financed by a tax on old age consumption. See second period budget constraint below.

$$c_{t+1} = s_t(1 + r_{t+1})(1 - \tau_{t+1}) \tag{2.14}$$

where r_{t+1} denotes the rate of return on savings, s_t , τ_{t+1} denotes the rate of taxation on old age consumption[12] and $w_t^m(2 - \hat{z}n_t) + w_t^p$ is the couple's income.

The household then chooses n_t and c_{t+1} to maximize (2.6) subject to (2.13) and (2.14), yielding

$$n_t^{*u} = \frac{\gamma(w_t^p + 2w_t^m)}{p(w_t^m, \beta)} \tag{2.15}$$

$$s_t^{*u} = (1 - \gamma)(w_t^p + 2w_t^m) \tag{2.16}$$

When (2.7) is binding, the household's first period budget constraint is

$$(1 - \beta_t)\hat{x}n_t + s_t \leq w_t^m + w_t^p \tag{2.17}$$

Intuitively, when the wife is restricted to using her entire labor endowment to raise children, the husband is the sole income earner. His earnings are then allocated between goods used in child rearing and savings.

Maximizing (2.6) subject to (2.14) and (2.17) yields

$$x_t^{*c} = \hat{x}n_t = \frac{\gamma(w_t^m + w_t^p)}{(1 - \beta_t)} \tag{2.18}$$

$$s_t^{*c} = (1 - \gamma)(w_t^m + w_t^p) = w_t^m + w_t^p - (1 - \beta_t)\frac{\hat{x}}{\hat{z}} \tag{2.19}$$

When $\hat{z}n_t = 1$, substituting from (2.18) into the CES production function for child rearing, each household produces

$$n_t^{*c} = \left[\alpha_1^a + \left(\alpha_2 \frac{\gamma(w_t^m + w_t^p)}{(1 - \beta_t)}\right)^a\right]^{\frac{1}{a}} \tag{2.20}$$

[12] The rate of subsidy and taxation are set so as to satisfy the balanced-budget constraint: $\beta_t \hat{x} n_t \left[n_{t-1}L_{t-1}\right] = s_{t-1}(1 + r_t)\tau_t L_{t-1}$. Although endogenous at the aggregate level, the rate of subsidy and taxation is treated as exogenous by each individual household.

2.2.3 Dynamic system

Capital stock per working age couple fuels growth in this model. The capital stock in each period is determined by the saving of the working age households in the previous period:

$$K_{t+1} = L_t s_t^* \tag{2.21}$$

The number of working age households at time $t+1$ is

$$L_{t+1}^p = L_{t+1} = n_t^* L_t \tag{2.22}$$

Capital stock per household is therefore given by

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{s_t^*}{n_t^*} \tag{2.23}$$

From (2.23), an equation of motion $k_{t+1} = \phi(k_t)$ is obtained, since both household savings and fertility choices are determined by w_t^m, which in turn is a function of k_t.

In Galor & Weil (1996) capital per household evolves through two phases distinguished by female labor force participation. In the initial phase, women allocate their entire labor endowment to child rearing. Once capital per household reaches a sufficiently high level, women enter the paid labor force, marking the transition to the second phase. This paper has similar phases in respect of female labor force participation, but a different predicted path of fertility as the economy moves through the two phases.

The evolution of capital stock per household is governed by two distinct equations of motion, as the economy moves first through

- Phase 1: women are at home, raising children full time ($z_t^{*c} = 1$); and, then
- Phase 2: women participate in the labor force, raising children part time ($z_t^{*u} \leq 1$).

Interdependence of wage for mental labor and capital per couple in Phase 2

The derivation of the equation of motion for Phase 2 is complicated by the fact that both the wage for mental labor and time spent raising children are interdependent. By (2.4), w_t^m is a function of m_t, which in turn is a function of $\hat{z}n_t$. That is, the wage paid to mental labor reflects its marginal product, affected by household supply of mental labor, which in turn depends on the time spent raising children. As previously demonstrated, $\hat{z}n_t$ is a function of w_t^m: time spent raising children falls with the wage for mental labor when women are in the labor force. Thus, we need to obtain an implicit function for $\hat{z}n_t$ when $\hat{z}n_t < 1$.

When $\hat{z}n_t < 1$, noting that

$$m_t = \frac{L_t^m}{L_t^p} = \frac{L_t(2 - \hat{z}n_t)}{L_t} = 2 - \hat{z}n_t \tag{2.24}$$

and substituting from (2.4), (2.15) and (2.24) into (2.10),

$$\hat{z}n_t = f(\hat{z}n_t, k_t) \tag{2.25}$$

Let $G(\hat{z}n_t, k_t) = \hat{z}n_t - f(\hat{z}n_t, k_t) = 0$. Since $G(\hat{z}n_t, k_t) = 0$ has continuous derivatives, by the Implicit Function Theorem, if $G_{\hat{z}n_t} \neq 0$ then there is a differentiable and invertible function $\varphi(k_t)$ such that $\hat{z}n_t = \min(1, \varphi(k_t))$ where

$$\varphi'(k_t) = \frac{-G_{k_t}}{G_{\hat{z}n_t}} = \frac{\partial f/\partial w_t^m . \partial w_t^m/\partial k_t}{[1 - \partial f/\partial w_t^m . \partial w_t^m/\partial m_t . \partial m_t/\partial \hat{z}n_t]} < 0 \quad \text{if} \quad \partial f/\partial w_t^m < 0 \tag{2.26}$$

With the exception of $\partial f/\partial w_t^m$, the signs of the partial derivatives in (2.26) are unambiguous. Assigning a negative value to $\partial f/\partial w_t^m$ is tantamount to assuming that the female labor supply curve is never backward bending. The possibility of a backward bending supply curve arises in this model, since, as we shall see, fertility may rise with female relative wages in Phase 2. A backward bending labor supply curve would occur if the proportionate rise in the number of children exceeds the proportionate fall in the time input per child. For the purposes of this paper, $\partial f/\partial w_t^m < 0$ is assumed.[13]

Thus, by (2.26), as the economy grows in transition to steady state, time spent raising children falls even when an input substitutable for maternal time exists.[14]

The economy enters Phase 2 once a sufficiently high level of capital per couple has been accumulated. Let $k*$ denote the highest level of capital per working age couple for which women raise children full time. That is,

$$\hat{z}n_t = \begin{cases} \varphi(k_t) \in (0,1] & \text{for } k_t \geq k^* \\ 1 & \text{for } k_t \leq k^* \end{cases} \tag{2.27}$$

2.3 Equation of motion

Substituting from (2.15), (2.16) and (2.19), (2.20) into (2.23) and using the definition of k^*, the equation of motion for the system is, therefore,

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{s_t^*}{n_t^*} = \begin{cases} \frac{1-\gamma}{\gamma} p\left(w_t^m, \beta_t\right) & \text{if } k_t \geq k^* \\ \hat{z}\left(w_t^m + w_t^p - (1-\beta_t)\frac{\hat{x}}{\hat{z}}\right) = \frac{1-\gamma}{\gamma}(1-\beta_t)\hat{x} & \text{if } k_t < k^* \end{cases} \tag{2.28}$$

Capital stock per couple evolves from a historically given initial level according to $k_{t+1} = \phi(k_t)$, which is readily derived from (2.28) by substituting for

$$w_t^m = \begin{cases} A(1-\alpha)\left(2-\varphi(k_t)\right)^{\rho-1}\left[\alpha k_t^\rho + (1-\alpha)\left(2-\varphi(k_t)\right)^\rho\right]^{1-\rho/\rho} & \text{if } k_t \geq k^* \\ A(1-\alpha)\left[\alpha k_t^\rho + (1-\alpha)\right]^{1-\rho/\rho} & \text{if } k_t < k^* \end{cases} \tag{2.29}$$

[13] A backward bending female labor supply, although an interesting proposition in itself, adds an unnecessary layer of complication to the model.

[14] When a fixed fraction of time per child is the only input, total time spent raising children necessarily falls as the economy grows. See Galor and Weil (1996).

By (2.29), w_t^m is a non-linear function of k_t. It follows that the equation of motion is a first order non-linear difference equation.[15]

2.3.1 Properties

Curvature

By enabling n_t to change, the introduction of a CES production function for child rearing complicates the analysis of the curvature of the equation of motion.

Recall that capital per household at $t+1$ equals savings per household (s_t) divided by household fertility (n_t). If time is the only input or fixed quantities of time and goods are used in child rearing, fertility is necessarily constant in Phase 1. Consequently, capital per household follows the path of household savings, which increases one for one with the wage for mental labor. Hence, $\phi(k_t)$ is concave (convex) if the wage for mental labor increases at a decreasing (increasing) rate as capital per household accumulates. Referring to the Appendix, if the degree of complementarity between capital and mental labor is relatively low, then w_t^m is increasing and concave in k_t over the interval $(0, k*)$.

When goods are substitutable for time, both fertility and savings increase with the wage for mental labor. Proportionate to the increase in the efficient goods input, fertility rises less than savings rises. Capital per household therefore follows the path of the goods input per pair of children. As a result, the concavity (convexity) of $\phi(k_t)$ also depends on how goods input per pair of children changes with the wage for mental labor.

[15] The log linear specification for household utility function ensures the equation is first order since fertility is a function of its own price. Should fertility be a function of the real interest rate, a higher order equation would be obtained.

Proposition 2.1 *Given a low degree of complementarity between mental labor and capital, a low degree of complementarity between child-rearing goods and time is a sufficient condition for concavity of $\phi(k_t)$ over the interval $(0, k*)$.*

Proof. See Appendix. ■

Regardless of the elasticity of substitution between goods and time, goods input per pair of children rises with the total quantity of goods, which in turn rises with the wage for mental labor. Thus, $\phi'(k_t) = (1-\gamma)\partial\hat{x}/\partial x.\partial x/\partial w_t^m.\partial w_t^m/\partial k_t > 0$. Is the sign of the second order derivative also unambiguous? Given an elasticity of substitution between mental labor and capital in excess of 1, an elasticity of substitution between child-care goods and time exceeding 0.5 (corresponding to a >-1) is a sufficient condition for $\phi''(k_t) < 0$.

The intuition for this result lies in diminishing marginal returns. Growth in k_t raises the wage for mental labor, which in turn boosts the quantity of goods input per child. Input of mental labor is fixed to 1 in Phase 1. The wage (or marginal product) for mental labor increases at a decreasing rate when the degree of complementarity between mental labor and physical capital is relatively low. Similarly, goods input per pair of children increases at a decreasing rate when the degree of complementarity between goods and time is relatively low.

Proposition 2.2 *$\phi(k_t)$ is increasing and concave in k_t over the interval $(k*, \infty)$ regardless of the degree of complementarity between goods and time in child rearing.*

Proof. See Appendix. ■

Referring to (2.28), k_{t+1} is proportional to child rearing costs per pair of children,

$p(w_t^m, \beta)$. By Euler's Theorem, $\partial p / \partial w_t^m = \hat{z} > 0$. Demand for maternal time input, given by (2.10), is downward sloping: $\partial \hat{z} / \partial w_t^m = \partial^2 p / \partial w_t^m 2 < 0 \;\; \forall \;\; a \; \in (-\infty, 1)$.

Thus, the per unit child rearing cost function is increasing and strictly concave in the wage for mental labor, regardless of the degree of complementarity between goods and time. Intuitively, an increase in the price of time input raises the per unit cost of child rearing, albeit at a decreasing rate, as the household uses less and less time input.

As in Galor & Weil (1996), the wage for mental labor is increasing and concave in capital per household. Given (2.26), their proof is applicable.

Discontinuity

The equation of motion is discontinuous at $k*$. Substituting (2.15) into (2.10),

$$\hat{z} n_t = 1 \quad \Rightarrow \quad w_t^p = \tfrac{1-2\gamma}{\gamma} w_t^m + \tfrac{1}{\gamma}(1-\beta_t)\left[\tfrac{(1-\delta) w_t^m}{\delta(1-\beta_t)}\right]^{\varepsilon}$$

k^* is readily obtained, after solving out for the wage for mental labor. Equality of the two equations in (2.28) is not satisfied for $k_t = k^*$. Specifically, equality of the two equations of motion occurs at a lower wage for mental labor: $w_t^p = \frac{1-\gamma}{\gamma} w_t^m + \frac{1}{\gamma}(1-\beta_t)\left[\frac{(1-\delta) w_t^m}{\delta(1-\beta_t)}\right]^{\varepsilon}$, suggesting $\phi(k_t)_{k_t \to k*+} > \phi(k_t)_{k_t \to k*-}$ given concavity of $\phi(k_t)$ throughout both Phase 1 and Phase 2.

For the intuition, first consider why the equation of motion is continuous in the limiting cases, $\delta \to 1$ and $\varepsilon \to 0$. In the case of fixed proportions,

$$\hat{z} n_t = 1 \quad \Rightarrow \quad w_t^p = \tfrac{1-2\gamma}{\gamma} w_t^m + \tfrac{1}{\gamma}(1-\beta_t)\tfrac{\hat{x}}{\hat{z}}$$

In the case of time input only, the last term in the implied equality vanishes. In either case, equality of the two equations analogous to the system in (2.28) is satisfied for $k_t = k^*$.

Intuitively, an optimizing household takes the child-rearing input(s) as given and chooses fertility so as to maximize utility. On the cusp of Phase 2, the wage for mental labor (corresponding to k^*) induces a utility maximizing household to choose fertility such that z^*=1. Continuity of the equation of motion at $k_t = k^*$ is assured by the fact that the time input per child or the goods-time input ratio is, by assumption, fixed and therefore the same in Phase 1 as it is in Phase 2.

Allowing households to substitute child-care goods and services for time introduces another dimension to household decision-making. In addition to choosing fertility so as to maximize utility, an optimizing household chooses the child-rearing input mix so as to minimize cost. Once again, on the cusp of Phase 2, a utility maximizing household chooses fertility implying $z^* = 1$. However, discontinuity in the equation of motion at $k_t = k^*$ arises because the goods-time input ratio throughout Phase 1 differs from the goods-input ratio upon entering Phase 2. If the household cost minimizes throughout, why does this disparity arise? Throughout Phase 1, cost minimization corresponds to a non-tangent corner solution. Growth in the wage for mental labor does not affect the goods-time input mix required for a given number of children. However, total goods used (equivalent to goods-time input ratio when z^*=1) increases with fertility due to a pure (male wage) income effect. In Phase 2, cost minimization corresponds to a tangency point. Growth in the wage for mental labor raises the goods-time input ratio required for any number of children. The effect on fertility depends on competing substitution and (male and female) income effects.

2.3.2 Steady state equilibria

In steady state equilibrium, capital stock per household is stationary: $\overline{k} = \phi\left(\overline{k}\right)$.

Existence

A steady state exists if

- $\phi(0) > 0$; and
- there exists some k_t, such that $\phi(k_t) < k_t$.

The first condition is indeed satisfied. Substituting from (2.18) into the equations for $k_t \in (0, k*)$ of (2.28) and (2.29) implies $\phi(0) = \hat{z}(1-\gamma)\left\{a(1-\alpha)^{1/\rho} + b\right\} > 0$.

Ensuring $\phi(k_t) < k_t$ for some k_t, $\lim_{k_t \to \infty} \phi'(k_t) = \frac{(1-\gamma)}{\gamma} \lim_{k_t \to \infty} \frac{\partial p}{\partial w_t^m} \frac{\partial w_t^m}{\partial k_t} = 0$, since $\frac{\partial p}{\partial w_t^m} = \hat{z}$ is bounded below by zero and the equation for $k_t \in (k*, \infty)$ of (2.29), together with (2.26), imply $\lim_{k_t \to \infty} \frac{\partial w_t^m}{\partial k_t} = 0$.[16]

$\phi(k_t) > k_t \quad \forall \quad k_t \in [0, k*)$ implies a steady state exists in Phase 2 rather than Phase 1. As in Galor & Weil (1996), this paper confines attention to this case. Thus, discontinuity of $\phi(k_t)$ poses no problem for existence provided $\phi(k_t)_{k_t \to k*+} > \phi(k_t)_{k_t \to k*-}$, which is satisfied, as required.

Uniqueness / Multiplicity

Given the conditions for existence are satisfied, multiple steady state equilibria may exist if the Phase 1 equation of motion is convex. Figure 2.2 illustrates such a case. As previously discussed, the convexity of $\phi(k_t)$ over the interval $(0, k*)$ depends not only on the degree of complementarity between capital and mental labor, as found by Galor & Weil (1996), but also the degree of complementarity between child-care goods/services and maternal time. Should $\phi(k_t)$ be convex over the interval $(0, k*)$, a poverty trap (represented

[16]For a formal proof of $\lim_{k_t \to \infty} \frac{\partial w_t^m}{\partial k_t} = 0$, see Galor and Weil (1996).

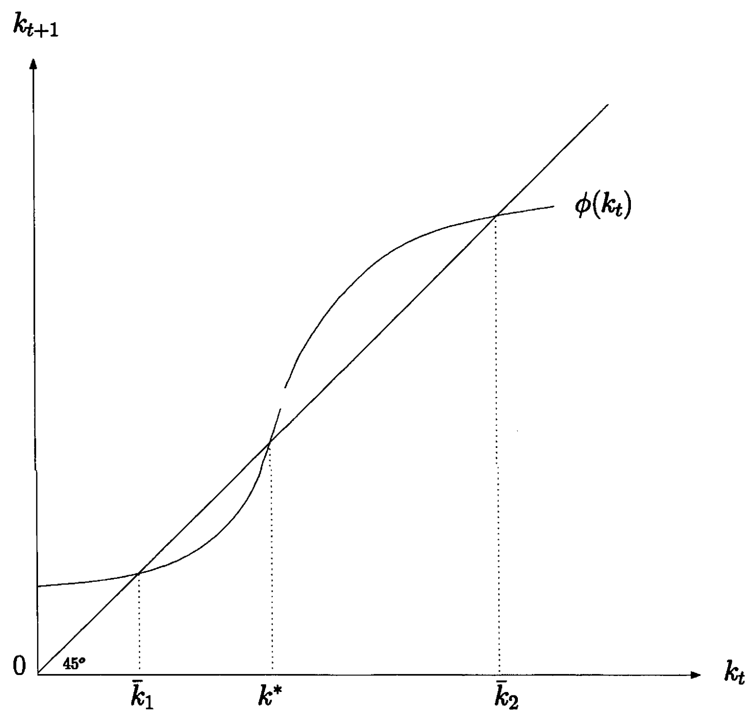

Figure 2.2: Multiple Equilibria

by the stable equilibrium $\overline{k_1}$) may emerge. The subsequent analysis deals with the case where there is no such low level steady state.

2.4 Fertility dynamics

Having established that capital per household (and the wage for mental labor) rise throughout Phase 1 and Phase 2 until eventually converging to steady state in Phase 2, we can explore the dynamics of fertility.

Proposition 2.3 *Throughout Phase 1, fertility necessarily rises with income per household.*

Proof. It follows from (2.20) that rising wages for mental labor boost household fertility, regardless of the degree of complementarity between goods and time. Thus, growth in income per capita generates a baby boom during an era in which women do not participate in the paid labor force. ■

Recall the household's fertility decision comprises 2 stages. The household, in the first stage, chooses the cost minimizing or efficient time-goods input mix given the production possibilities; and then, in the second stage, apportions the husband's income to children and savings given the efficient cost of child rearing.

Figure 2.3 depicts cost minimization for Phase 1 at the point where the isoquant n_1 intersects the vertical constraint, z =1. The slope of the isocost line is $-w_t^m/(1-\beta_t)$. The wage for mental labor is sufficiently small that $z* = 1$. Specifically, $w_t^m < x^{1-a}\,(\delta/1{-}\delta)\,(1-\beta_t)$. The isocost is flatter than the isoquant at the initial corner solution.[17] That is, the house-

[17] If the corner solution were a point of tangency, a rise in the wage for mental labour would imply a move to an interior solution and, therefore, a transition to Phase 2. We want to analyse the implications of a rise in the wage for mental labour while the economy is in Phase 1.

hold would lower the cost of child rearing if it were able to employ time input in excess of the women's total time endowment.

Consider a rise in the wage for mental labor that is not sufficient to induce women to enter the paid workforce. In terms of the diagram, the rotation of the isocost from the relatively flat solid line to the slightly steeper broken line does not yield an interior solution. The efficient goods-time input mix required to rear n_1 pairs of children is unaltered.

At the same time, an increase in the wage for mental labor eases the household's budget constraint. Due to a pure income effect, the demand for children (or, equivalently spending on child rearing goods) increases. An upward shift in the isoquant to n_2 depicts the corresponding scale effect in Figure 2.3.

Recall that for capital per household to accumulate over time, $\hat{x}$ must rise. In addition to illustrating the baby boom in Phase 1, Figure 2.3 provides a diagrammatic proof that $\hat{x}$ rises with the wage for mental labor. When time input is bounded above by 1, an increase in scale can only be met by raising goods input more than proportionate to the rise in n. Hence, $\hat{x}$ necessarily rises.

Proposition 2.4 *As income per household rises throughout Phase 2, fertility may either*

1. *decline monotonically;*

2. *rise monotonically; or*

3. *initially decline and then rise.*

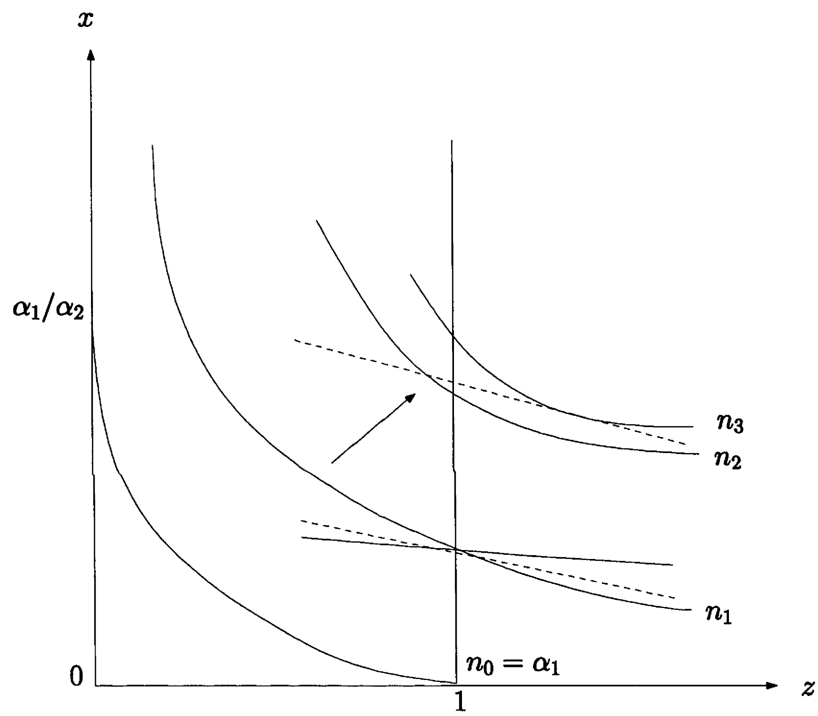

Figure 2.3: The household's child rearing production decision in Phase 1

Proof. Referring to the Appendix, there exists a critical value of the wage for mental labor:

$$w^m ** = \left(\frac{w_t^p}{2}\right)^{1/\varepsilon} \left(\frac{\delta}{1-\delta}\right) (1-\beta_t)^{(\varepsilon-1)/\varepsilon} \tag{2.30}$$

such that $\partial n_t^u / \partial w_t^m < 0 \quad \forall \quad w_t^m < w^m ** \; ; \; \partial n_t^u / \partial w_t^m > 0 \quad \forall \quad w_t^m > w^m ** \cdot$

Let $k**$ denote the level of capital per household, corresponding directly to w^m** (as per (2.29)), sufficient to induce a baby bounce-back in Phase 2.

Three possible cases arise:

1) $k* < \bar{k} < k** \quad : \quad \partial n_t^u / \partial w_t^m < 0 \quad \forall \quad k_t \in (k*, \bar{k})$

Thus, for a sufficiently large $w^m **$, fertility declines monotonically throughout Phase 2. Although the following section examines the determinants of $w^m **$, consider, for the moment, the influence of the production share of maternal time in child-rearing. As $\delta \to 1$, $w^m ** \to \infty$, ensuring that in the limit the above inequality holds, since the steady state value of capital per household is finite. That is, when time is the only input, fertility unambiguously falls with the wage for mental labor throughout Phase 2, confirming the result of Galor and Weil (1996).

2) $k* = k** < \bar{k} \quad : \quad \partial n_t^u / \partial w_t^m > 0 \quad \forall \quad k_t \in (k*, \bar{k})$

Should the wage for mental labor needed to induce a baby bounce-back equal the wage needed to induce women to enter the paid labor force, fertility rises monotonically throughout Phase 2. An instantaneous baby bounce-back, although possible, is unlikely. We can infer from the time input only case that substitution to child-care goods and services is integral to a baby bounce-back eventuating. Market substitutes for maternal time *emerge* in response to continuing cost pressures.

3) $k* < k** < \bar{k}$

In this case, growth in capital (income) per household generates a non-monotonic path in fertility throughout Phase 2:

$$\partial n_t^u / \partial w_t^m < 0 \quad \forall \quad k_t \in (k*, k**) \quad ; \quad \partial n_t^u / \partial w_t^m > 0 \quad \forall \quad k_t \in \left(k**, \bar{k}\right)$$

Although a steep rise in the opportunity cost of maternal time causes an initial decline in fertility, a further rise induces households to substitute out of maternal time, removing the pressure to reduce fertility and allowing the income effect to dominate. This pattern of gradual rather than instantaneous substitution fits with observed trends. ∎

Before investigating the factors underpinning w^m**, equation (2.30) is restated so as to explore the intuition behind Proposition 4.[18]

Corollary 2.1 *As income per household rises throughout Phase 2, fertility declines if and only if the child rearing goods/time input ratio is sufficiently high.*

Proof. Referring to the Appendix, $\partial n_t^u / \partial w_t^m < 0$ if, and only if,[19]

$$\frac{\hat{x}}{\hat{z}} = \left[\frac{w_t^m}{1-\beta_t}\frac{1-\delta}{\delta}\right]^{\varepsilon} < \frac{w_t^p}{2[1-\beta_t]} \tag{2.31}$$

∎

To abstract from the role of wage growth, for the moment, consider the limiting case, $\varepsilon \to 0$. By assumption of fixed proportions, the input ratio (left hand side of (2.31)) remains fixed.

[18]Hereafter, Proposition 4 in the text of each chapter refers to Proposition C.4 where C denotes the chapter number.

[19]Alternatively, this condition can be expressed as $2\gamma < \hat{z}n_t$. Hence the proposition: "If time spent in domestic child care is sufficiently small ... fertility increases with the female wage" (p.7 Apps & Rees (2001)).

This is the assumption made by Galor & Weil (1996) who note briefly that in this case, a rise in female relative wages "reduces fertility if $\hat{x}$ is not too large" (footnote 12, Galor & Weil (1996)).[20] Thus, the introduction of a second child rearing input, albeit in fixed proportions, is sufficient to demonstrate that fertility decline is not an inevitable consequence of rising female relative wages with economic growth.

Nonetheless, the assumption of fixed proportions seems unnecessarily restrictive. Goods and time are unlikely to be perfect complements in the rearing of children. Moreover, only by restoring wages as a determinant of the goods/time ratio can we see why the negative relationship between household income and fertility will unravel over time, as in the third case of Proposition 4.

The intuition lies in a comparison of substitution and income effects. When both husband and wife work, female wages constitute a portion of household income. A proportionate increase in the wage for mental labor results in a less than proportionate rise in household income. To illustrate, if female wages are two thirds of male wages, a 10% rise in the wage for mental labor will increase household income by 8%.

In the limiting case, $\delta \rightarrow 1$, when maternal time is the only child rearing input, a 10% rise in mental wages will increase the cost of raising children by 10%. Hence, given our functional forms for parental utility and for child rearing, the substitution effect dominates the income effect, and fertility necessarily declines. However, when a second input is introduced, the cost of the wife's time is only a portion of the total cost of child rearing, so that the income effect may now dominate.[21]

[20] Clearly, by the corollary to Proposition 4, the conditions of Apps and Rees (2001) and Galor and Weil (1996) are equivalent.

[21] The likely response of fertility to a rise in female relative wages at a point in time can be inferred from the composition of child rearing costs and household income. To illustrate, Haveman & Wolfe (1995) provide

In the limiting case of fixed input proportions, $\varepsilon \to 0$, both potential household income and the per unit cost of child rearing increase at a constant rate with wages growth. Thus, if the substitution effect dominates the income effect at the beginning of Phase 2, further wages growth will not reverse the dominance. However, when Galor & Weil (1996), the per unit cost of child rearing increases at a *decreasing* rate, whereas potential household income increases at a constant rate. It follows that the income effect of rising female relative wages may eventually dominate the substitution effect so that fertility rises instead of declining.

Thus, when $\varepsilon > 0$, the very aspect of economic growth that discourages fertility, rising female relative wages, ultimately remedies fertility decline by raising the goods-time input ratio.

We could simply append the previous discussion with the surmise that the goods-time input mix increases over time as higher wages induce households to switch goods and services for maternal time. However, factors other than higher wages are also at play, such as, households' ability to substitute goods for time and the price of goods and services used in child rearing.

Overlooked in existing literature, a CES production function for child rearing allows the factors underpinning the child rearing input mix to be clearly identified. From this, we can demonstrate how changes in parameter values may either hasten or postpone the onset of a baby bounce-back.

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lower and upper bound estimates of time's share of the total cost of child rearing for the US in 1992: 18% and 73%, respectively. Since the upper bound estimate approximates female earnings as a percentage of male earnings in the same year, we would expect fertility in the US to rise with female relative wages at that time.

Proposition 2.5 *A high rate of subsidy to child-care goods and services prolongs fertility decline in Phase 2, if the degree of complementarity between child-care goods/services and time is relatively high.*

Proof. Consider the case $k* < k** < \bar{k}$.

From (2.30), $\partial w^m ** / \partial \beta_t > 0 \;\Leftrightarrow\; \varepsilon < 1$. Since $w^m **$ is increasing over time, the higher $w^m **$, the later the onset of a baby bounce-back. ■

For the intuition on this result, note that the net price of goods and services used in child rearing appears on both sides of the inequality in (2.31). Accordingly, a subsidy to child-care goods and services affects the fertility response to higher female relative wages in two ways:

1. For a given goods-time input mix, a subsidy raises maternal time's portion of the total cost of child rearing, accentuating the substitution effect that induces fertility decline;

2. A subsidy lowers the net price of goods and services, prompting households to raise the goods-time input ratio (left hand side of (2.31)). A higher goods-time input ratio weakens the substitution effect.

Unless the input mix is highly responsive to changes in relative input prices, the first effect will dominate.

Intuitively, when the degree of complementarity between child-care goods and services and maternal time is relatively high, subsidizing child-care goods and services has a negligible effect on the goods-time input mix. Under these circumstances, a subsidy serves only to boost time's share of the total cost of child rearing. Accordingly, subsidization of child-care goods and services prolongs the decline in fertility due to rising female relative wages.

A priori, we might expect subsidizing child-care goods and services would itself induce a baby bounce-back. Note that Proposition 5 deals specifically with an indirect effect of subsidization. A high rate of subsidy implies a high w^m**, thereby postponing the onset of a naturally occurring baby bounce-back induced by rising wages.

The partial derivatives of w^m** with respect to other parameters are straightforward. In brief, the lower maternal time's share in production (δ) or the lower the wage for physical labor, the earlier the onset of a baby bounce-back.

For the intuition, recall that rising female relative wages boost fertility when the income effect dominates the substitution effect. The lower δ, the smaller maternal time's share of child rearing costs and hence, the weaker the substitution effect. The lower the wage for physical labor, the greater female wages' contribution to household income, amplifying the income effect.

Referring to the Appendix, for given rates of subsidy and a given wage for physical labor, a higher elasticity of substitution hastens the onset of a baby bounce-back.

Proposition 2.6 *In Phase 2, rising female labor force participation is necessary but not sufficient for declining fertility.*

Proof. Recall that female labor supply is given by $1 - \hat{z}n_t$.

Necessity

$\partial n_t^u / \partial w_t^m < 0 \quad \forall \quad k_t \in (k*, k**)$, together with $\partial \hat{z} / \partial w_t^m < 0$

imply $\partial(1 - \hat{z}n_t)/\partial w_t^m > 0$

Sufficiency

By (2.26), $\partial(1-\hat{z}n_t)/\partial w_t^m < 0$, which need not imply declining fertility.

Provided $k** < \bar{k}$, $\partial n_t^u/\partial w_t^m > 0 \quad \forall \quad k_t \in (k**, \bar{k})$. ■

In the limit as $\delta \to 1$, $k** = \infty > \bar{k}$ and the Galor and Weil (1996) result that rising female labor force participation is both necessary and sufficient for declining fertility is obtained. Since time input per pair of children is fixed, the proof that rising female labor supply implies declining fertility is incidental.

2.5 Discussion

2.5.1 Joint evolution of population growth and income per household

Household fertility and income evolve jointly according to the dynamic system explored in Section 2.3. Figure 2.4 depicts the path of fertility in transition to a unique globally stable steady state, corresponding to $\bar{k}$, in Phase 2. t^*, $t**$ and $\bar{t}$ denote the time periods after which women enter the paid labor force, a baby bounce-back commences and the economy reaches steady state, respectively.

Figure 2.4(b) depicts the general results. For contrast, Figure 2.4(a) depicts the limiting cases, $\delta \to 1$ and $\varepsilon \to 0$, when maternal time cannot be substituted for child-care goods and services.

Whilst allowing for the substitution of child-care goods and services for maternal time has striking implications for the path of fertility, a feature common to both Figure 2.4(a) and 2.4(b) is the dramatic change in pace of capital accumulation as the economy enters Phase 2. The pace of capital accumulation slows as the economy nears the end of

Phase 1. On entering Phase 2, a boost to the supply of mental labor fuels an immediate take-off in growth. There is a corresponding steep decline in fertility. The pace of capital accumulation slows again as the economy approaches steady state. This pattern is a consequence of concavity of $\phi(k_t)$ in both phases.

In Figure 2.4(a), the time path for fertility derived for a given goods-time input ratio ($\varepsilon = 0$) follows a similar path to that derived under the assumption of time input only ($\delta = 0$). Overall, allowing for a fixed quantity of goods to be used in child rearing has no dramatic implications for the joint evolution of income per household and fertility.

Compared with the time input only case, household capital and income is lower at any given time period in Phase 1, since spending on child rearing goods and services absorbs some household savings. Accordingly, k^* is lower. That is, women enter the paid workforce at a lower income per capita.[22]

In Phase 2, household savings is the same under either assumption, but the per unit cost of child rearing is higher when goods are an input. Consequently, fertility is lower at any given time period. Less capital dilution implies higher capital (income) per household. Compared with the time input only case, the economy converges to a higher income per capita and lower fertility rate.

Figure 2.4(b) depicts the richer, more general result of a non-monotonic path in fertility. As per Proposition 3, growth in income per household generates a baby boom in Phase 1. Throughout Phase 1, the maternal time constraint is binding. Because household time allocation is at a corner solution, a marginal change in household income has only an income effect and demand for children increases. The rise in fertility becomes less

[22]See Section 2.3 for verification.

pronounced as the economy nears the end of Phase 1, due to the concavity of the equation of motion.

This contrasts Figure 2.4(a), where fertility is constant throughout Phase 1. In either limiting case, because the household cannot substitute goods for maternal time, increases in household income are entirely absorbed as additional savings. Since the maternal time constraint is binding, the assumption that maternal time input per pair of children is fixed constrains fertility to a constant, $n_t = 1/\hat{z}$. Thus, the introduction of child rearing goods is, in itself, not sufficient to generate a baby boom in Phase 1.

Phase 2, in Figure 2.4(b), begins with accelerated growth in capital (income) per household and a corresponding steep decline in fertility. As per Proposition 4, the inverse relationship between income per household and fertility may break down before the economy converges to a steady state. Rising female relative wages induce households to substitute child-care goods and services for maternal time. Once the maternal time input in child rearing is sufficiently low, the effect of rising wages on the cost of children is offset by the income effect. Thus, at time $t**$, fertility ceases to decline with growth in income per capita. Any further rise in the wage for mental labor generates a baby bounce-back until the economy reaches steady state and fertility levels off because capital and income per household are no longer rising.

This contrasts Figure 2.4(a) where, so long as income per capita is rising, fertility must decline. Of course, the path in Figure 2.4(a) is derived for a given goods-time input ratio. By the Corollary to Proposition 4, fertility will cease to decline with growth in income per household if goods and services come to dominate child-care inputs. However, in the limiting case $\varepsilon \rightarrow 0$, the likely path of the goods-time ratio over time remains a conjecture insofar as the input mix is an exogenous variable.

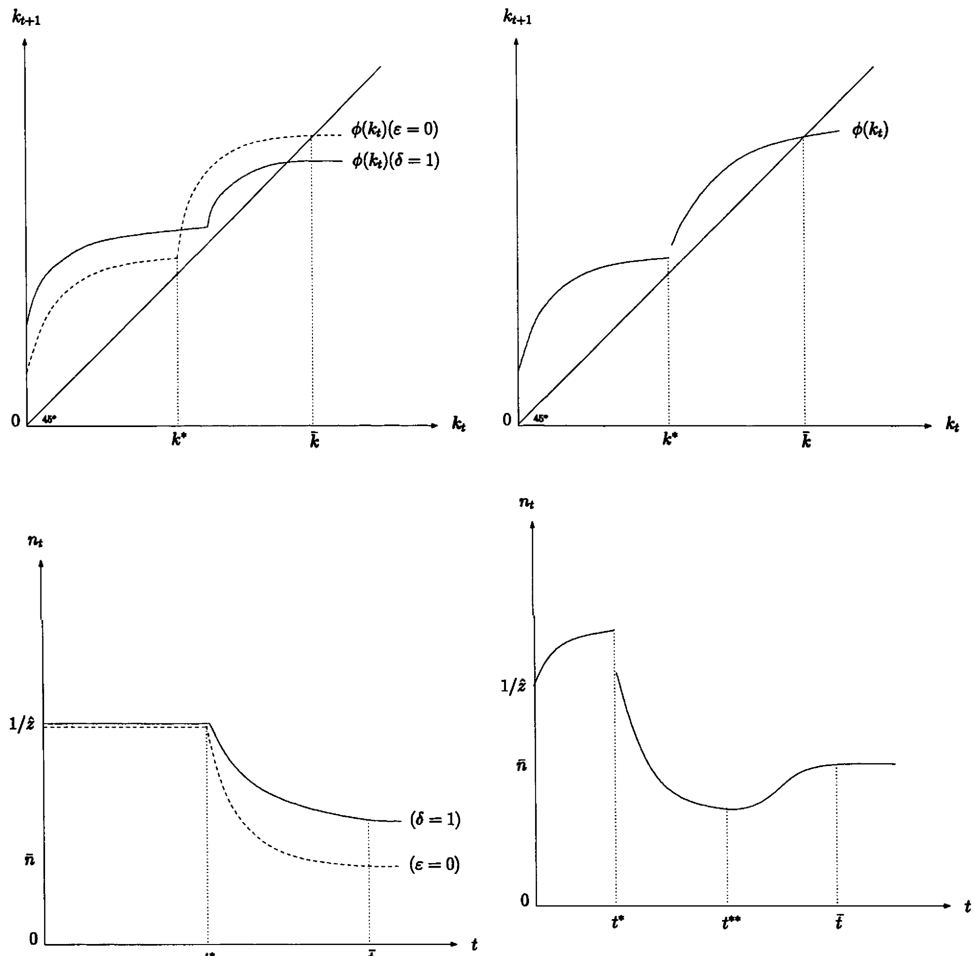
2.5.2 Implications

Given that advanced economies are currently moving through Phase 2, with increasing labor force participation by women, what inferences can we draw for current policy debates?

As income per capita rises, what would the model presented in this paper predict?

1. Female labor force participation rises unambiguously, albeit at a decreasing rate.
2. Rising female labor force participation need not imply declining fertility because purchased goods and services can substitute for maternal time.
3. Fertility may cease to decline before the economy reaches zero growth in per capita income in steady state. Once fertility ceases to decline, any further wages growth generates a baby bounce-back.
4. A high rate of subsidy to child-care goods and services will raise the level of fertility but may postpone the onset of a naturally occurring baby bounce-back.

Forward projections of fertility decline should consider two questions: Will female wages rise relative to male wages to the same extent in the future? Has the wages growth to date caused households to substitute child-care goods and services for maternal time? Galor and Weil (1996) consider the first question in isolation. They present a special case of our model: as $\delta \rightarrow 1$, fertility must decline monotonically as income per capita rises



(a) Fixed quantities of time and goods

(b) CES child rearing function

Figure 2.4: Evolution of capital per household and fertility

in transition to steady state. Once we consider the second question, predicted fertility may assume a non-monotonic path in transition to steady state. This paper identifies the parameters central to answering the second question. From this, we may predict different paths in fertility for different advanced countries. We anticipate that the baby bounce-back, which has been observed in the US since 1985, is likely to occur in other advanced economies where fertility has ceased declining and real wages continue to rise.

2.6 Conclusion

The model presented in this paper generates a non-monotonic path in fertility for a developed economy moving through two phases, distinguished by a dramatic rise in female labor force participation. The introduction of a CES production function for child rearing to Galor and Weil (1996) yields a rich dynamic interplay between fertility and income per capita.

As income per household grows over time, fertility:

- initially *rises* during the first phase when women allocate their total time endowment to child rearing;
- then *declines* as women enter the paid work force; and
- then may *bounce-back*.

The inverse relationship between fertility and per capita income applies only over the initial part of Phase 2, as women are starting to enter the labor force. In contrast to previous models, a positive relationship between fertility and income is predicted to occur in Phase 1 and, possibly, towards the end of Phase 2. Where maternal time is the only input into child-rearing, there is an absolute limit on fertility when women are full-time carers; when we allow for purchased inputs into child-rearing, the limit is removed enabling a baby-boom. As women start to participate in the labor force, the rising opportunity cost of maternal time causes fertility to decline – a feature common to most models. In our model, however, rising female relative wages induce households to substitute child-care goods and services for maternal time. Once maternal time is a sufficiently small part of total child-rearing costs, any further rise in wages causes an upturn in fertility as the income effect starts to dominate.

Some interesting policy implications arise. A concomitant rise in fertility and female relative wages, when women are in the labor force, challenges the belief that reversing current fertility trends necessitates a reduction in female labor force participation. When purchased inputs can substitute for maternal time, as real wages rise and purchased inputs become relatively cheap, fertility may naturally bounce back. Subsidies to child-care inputs will unambiguously raise fertility at any point in time, though they may delay the onset of the fertility bounce-back.

Whilst the resemblance of the generated path in fertility to the G5 data illustrated in Figure 2.1 is compelling, I do not deny the importance of social and political changes in developed economies over this period. Arguably, some of these changes, such as the emancipation of women, are linked to growth in income per capita. Boongarts (1999) offers a novel non-economic explanation for fluctuations in fertility. He argues that sudden changes in the mean age of child bearing distort measured fertility rates. A fall in the mean age of child bearing following World War II meant that births of successive cohorts overlapped in the same period, boosting observed fertility. Conversely, a rise in the mean age of child bearing over recent decades deflated observed fertility. However, after adjusting fertility rates for this distortion, van Imhoff & Keilman (2000) still found a remarkable baby boom-bust sequence. There are some further implications of the model that concern the dynamics of labour force participation and relative wages. These implications are also present in the original model of Galor and Weil (1996). For example, the model predicts that during Phase 2 women's labor force participation and their relative wages are both rising. For some countries this prediction may be counterfactual. Goldin (1986) presents evidence for the United States that the ratio of female to male earnings across all sectors was virtually constant from 1955 through to 1979. Whilst the model broadly captures the Baby Boom and Bust eras, extensions to the model may be able to capture a richer array of stylized facts. One possibility is to incorporate falling costs of child rearing goods and services relative to final consumption goods as a consequence of economic growth.

It would be useful to extend the analysis in several directions, such as recognizing human capital accumulation as central to both the wage story and fertility. Considerable richness and realism would be gained by drawing on the R&D based models of endogenous growth of Romer (1990) and Jones (1995*a*). Both identify population, in level and growth rates, respectively, as the key determinant of technological progress. Galor & Weil (2000) incorporate endogenous fertility into a descriptive model of endogenous growth inspired by Romer (1990). Nesting endogenous fertility within a well-specified model of R&D is a feasible and challenging direction for future research.

Part III

Endogenous Growth - Essay 2

Chapter 3

Population and Endogenous Growth

3.1 Introduction

Total population of the OECD today may be one and half times what it was in 1950, but it is expected to be static for the next fifty years (United Nations 2005). This projection takes into account immigration and moderate fertility assumptions. In fact, depopulation is anticipated if fertility remains constant. The prospect of zero population growth in the world's hub of research and development (R&D) has generated a flurry of R&D based growth models investigating feasibility of long run economic growth in the absence of population growth.

This new branch of the literature is largely derived from semi-endogenous growth models that assume strictly positive population growth. The precedent Romer (1990) type models treat population as an exogenous constant and would therefore seem pertinent to the current theoretical challenge. However, such models have been overlooked because they typically assume linearity in the accumulation of knowledge. *Linearity* implies that the output of new knowledge will double whenever we double the existing stock of knowledge. Semi-endogenous growth models feature the more palatable assumption of diminishing marginal returns to knowledge. This paper examines the tenability of this association between (non-)linearity and (semi-)endogenous growth.

The objective of this paper is two fold. First, to establish general conditions for positive growth in output per capita along a balanced growth path under the alternative assumptions of a static population and a growing population. Second, to construct a specific model that delivers long run growth in the economy with or without population growth, using the most realistic application of these general conditions.

Since Romer's (1990) seminal paper, models of R&D-based growth have become increasingly sophisticated. A recent paper (Strulik 2005) comprises 48 equations, not including those contained in the appendix. The new breed of semi-endogenous growth models typically comprise two aspects of R&D or endogenous fertility. In a decentralized setting, the assumptions that are critical to positive and balanced growth are obscured by the intricacy of these models. Often, simplifying assumptions that make these models tractable, for example, the absence of physical capital, are costly in terms of realism. We overcome such difficulties by abstracting from the microeconomic foundations of R&D and modelling the decision making of a central planner.

This paper introduces a general model comprising three sectors (final output, R&D

and human capital accumulation) in order to prove the following assertions. First, in any model, positive growth along a balanced growth path requires restrictions in terms of a matrix of structural elasticities. Second, if strictly positive population growth is assumed, the notions of diminishing marginal returns and semi-endogenous growth are logically independent. Third, if zero population growth is assumed, linearity in the accumulation of knowledge is not necessary for endogenous growth.

Because we also seek to extend previously underdeveloped areas of the literature, the general model comprises human capital formation rather than a second aspect of R&D and assumes exogenous population growth. We allow for all inputs to be productive in all sectors and also allow for heterogenous labor, an assumption which has been absent from the literature since Romer (1990).

Previous generalized models (Eicher & Turnovsky (1999), Christiaans (2004) and Steger (2005)) comprise two sectors, final production and R&D. By including a third sector, human capital accumulation, we obtain richer results. More importantly, this is the first generalized model that allows for either a growing or static population. In doing so, this paper provides a simple, unified treatment of endogenous and semi-endogenous growth models. This paper further contributes by exploring asymptotic linearity in either R&D or human capital accumulation as a general condition for endogenous growth.

The benefits of this work are manifold. This paper contributes to the literature both by establishing conditions for positive economic growth with or without population growth in a general three sector growth model and by constructing a specific model where Constant Elasticity of Substitution (CES) technology describes the accumulation of human capital. This paper also provides a useful framework for summarizing the two main strands of the literature and a methodology for obtaining central planner solutions for endogenous and semi-endogenous growth models alike.

3.2 Background

Let y denote income or output per capita, L the total population, H the stock of human capital and let g_x denote the long run growth rate of any variable x. We classify models of R&D-based growth into two broad types: those that assume population is constant and those that allow for population to grow over time.

Models of the first type, exemplified by Romer (1990), Aghion & Howitt (1992) and Grossman & Helpman (1991), are widely criticized for their scale effect: long run per capita output growth is proportional to population size. The implication that the growth rate of the economy will rise exponentially over time should population grow at a constant rate is not supported by empirical evidence. It is, in fact, a slight misrepresentation of Romer (1990) and Aghion & Howitt (1992) to say that g_y is proportional to L. By allowing for heterogeneous labor, both predict:

$$g_y = a.H \tag{3.1a}$$

where all constant parameters are summarized by the term, $a > 0$.[1] Thus, long run growth in per capita output is proportional to the skill employed in R&D. However, to the extent that the skill is embodied in the population, long run growth of the economy is still

[1] (3.1a) is a generalized expression. Romer (1990) assumes $a = \delta(1 - l_Y)$, where δ is an exogenous productivity parameter and l_Y is the endogenously determined fraction of labor allocated to final production. For $g_y > 0$, H must be sufficiently high that the non-negativity constraint of $(1 - l_Y)H$ is not binding.

proportional to the size of the population.

Regardless, presenting the prediction in its original form shows how recent models are derivative of these seminal models of R&D-based growth. The stock of labor in these models can be homogenized into either the stock of human capital or total population. Most literature stems from the latter assumption. However, an example of a model that assumes the former is Funke & Strulik (2000). They retain the assumption that population is an exogenous constant and are therefore a first-type model. By endogenizing the accumulation of human capital, they remove the empirically inconsistent scale effect from the long run growth rate of the economy:

$$g_y = a \tag{3.1b}$$

All these first type models share the common feature of sectoral linearity in a knowledge accumulation equation, whether knowledge be non-rivalrous ideas or rivalrous human capital. And so, *sectoral linearity* has become synonymous with *endogenous growth* models that treat population as an exogenous constant. Like the scale effect, linearity in the accumulation of knowledge is widely criticized. Jones (2001) argues that, with the exception of the population equation, the assumption of linearity is ad hoc. This brings us to R&D-based growth models of the second type, that introduce a linear population equation.

Early examples of second-type models are Jones (1995*a*), Kortum (1997) and Segerstrom (1998). Their common feature is diminishing marginal returns to ideas (or knowledge spillovers of degree less than one) in the creation of new ideas. Diminishing marginal returns in the stock of ideas requires increasing effort to create an idea. This increasing effort can come from more researchers. Since the fraction of the labor force engaged in R&D is constant in steady state, strictly positive population growth enables the increasing efforts needed for strictly positive growth in technology and the overall economy. This is the intuition behind semi-endogenous growth. Jones (1995*a*) coined the phrase, which basically means technological change is endogenously determined, but long run growth in the economy requires growth in a factor exogenous to the model, population. And so, *diminishing marginal returns* to knowledge has become synonymous with *semi-endogenous growth.*

Population growth is the engine of long run economic growth in these models:

$$g_y = cg_L \tag{3.1c}$$

where all constant parameters are summarized by the multiplicative term $c > 0$. On the flip side, long run per capita growth of the economy depends essentially on population growth. To establish feasibility of long run economic growth in the absence of population growth, recent literature adapts second-type models. Two main branches have emerged.

The first new branch of second type models assume two aspects of R&D in the one model. Examples are Young (1998), Dinopolous & Thompson (1998), Peretto (1998) and Li (2000). In brief, R&D may involve either the creation of new products, so that technological improvement is measured by increased variety of intermediate goods (Romer 1990) or the improvement of existing products as in Aghion & Howitt's (1992) quality-ladder model. We refer to these two aspects as simply variety R&D and quality R&D. Li (2000) shows that if there are no knowledge spillovers in variety R&D and spillovers of degree one (or linearity) in quality R&D then the long run growth rate of the economy is an additively separable function of population growth and a constant term:

$$g_y = b + cg_L \tag{3.1d}$$

where all constant parameters are summarized in the terms b and c. The absence of knowledge spillovers in variety R&D implies a one-to-one correspondence between variety growth and population growth. This explains the second term of equation (3.1d). If population is static, the variety of intermediate goods stays constant. However, endogenous technological change is still possible through improving existing products, since linearity in quality R&D implies quality growth is proportional to the population size. This explains the first term of equation (3.1d). Consequently, the long run growth rate of the economy can be strictly positive without strictly positive population growth. To obtain this result, note that these models move away from diminishing marginal returns to knowledge in the form of quality improvement. Thus, the strong association between diminishing marginal returns and semi-endogenous growth prevails.

The second branch of second-type R&D-based growth models endogenize either population (Jones 2001) or human capital (as in, Strulik (2005) and Dalgaard & Kreiner (2001)) or both (Galor & Weil 2000). Just as Funke & Strulik (2000) removes the "strong" scale effect from the early endogenous growth models, these models remove the "weak" scale effect from semi-endogenous growth models by endogenizing the culpable variable. They predict a long run rate of growth in the economy:

$$g_y = d \tag{3.1e}$$

where d is a constant term summarizing, for example, exogenous efficiency parameters. In a decentralized setting, these models are intricate. Simplifying assumptions prevent the models from being unwieldy. Examples of such assumptions are the absence of physical capital in final production in Dalgaard & Kreiner (2001) and a reduced form specification of R&D in Galor & Weil (2000). There is a trade-off between sophistication and realism.

Thus, to establish feasibility of long run growth in the economy in the absence of population growth, existing literature extends semi-endogenous theory by either modelling two aspects of R&D or endogenizing fertility. To explore the reasoning behind founding this development in semi-endogenous growth theory, this paper establishes conditions for perpetual growth in a generalized setting. To be inclusive of the early endogenous growth theory, we allow for population growth to be either zero or strictly positive. Interested in exploring new ways to establish long run growth in the economy without population growth, we assume one aspect of R&D and exogenize population growth.

3.3 A General Three Sector Growth Model

The model is general in four aspects: Firstly, two types of labor, skilled and unskilled, accumulate.[2] The allowance for both types of labor, albeit as exogenous constants, appears in Romer (1990), but has been absent from the R&D based growth literature since. Secondly, the economy consists of three sectors (final goods, the accumulation of ideas and the accumulation of human capital) enabling us to replicate the features of a wide variety of R&D-based growth models. Only the accumulation of physical labor is exogenized. Thirdly, each factor of production is allowed to be productive in each sector. Finally, non-parameterized general production functions are employed. Restrictions on parameters and

[2]The assumption of heterogeneous labor confers realism to the model. Explored in a follow on paper, a secondary motivation for this assumption is the possibility of rising research intensity along a balanced growth path.

functional forms are introduced only to clarify certain issues.

We model the decision making of a central planner over our three generalized sectors. In doing so we abstract from issues related to the microfoundations of R&D-based growth models, such as household decision making, the patenting of ideas, monopoly power in the intermediate goods sector and perfect competition in final goods sector. In the words of Eicher & Turnovsky (1999) (p.397),

> We make these abstractions, not because we feel that such issues are unimportant, but to facilitate the identification of the characteristics common to alternative approaches.

All the models presented in this paper, whether original or central planner versions of existing models, can be given microfoundations, and in each case the equilibrium growth rates in the corresponding decentralized economy can be derived. It is worth noting that growth rates derived for a corresponding decentralized economy differ only by the absence of terms, such as a monopoly markup, that capture the negative spillovers that a central planner internalizes.

The economy produces final output (Y), change in technology (the stock of which is denoted by A) and change in human capital (the stock of which is denoted by H) and accumulates stocks of physical capital (K) and physical labor (L). H is measured by total, not average, years of education attained by a pool of workers, so that L is measured by a count of people in the labor force.[3] Alternatively, H could refer to the number of skilled workers and L to the number of unskilled workers. Under either interpretation, H can vary separately from L and replicating a given pool of workers requires that both H and L

[3]Physical labor can be thought of as brawn and basic skills that do not need to be taught, such as, eye-hand co-ordination.

double.

Consider the following general three-sector production structure:

$$Y = F(a_Y A, h_Y H, l_Y L, k_Y K) \tag{3.2a}$$

$$\dot{A} = J(a_A A, h_A H, l_A L, k_A K) \tag{3.2b}$$

$$\dot{H} = Q(a_H A, h_H H, l_H L, k_H K) \tag{3.2c}$$

where the sectoral allocations of factor x_i ($x = a, h, l, k;\quad i = Y, A, H, L$) assume values to reflect general assumptions, that are both intuitive and standard in endogenous growth models. We start with the broad assumption that, with the exception of physical labor, which is used only in human reproduction and the production of final output, inputs may be productive in all sectors. If $\sum_i x_i = 1 \forall i$, the respective input is private. If $x_i = 1 \forall i$, the input is non-rivalrous in use. Thus, we distinguish rivalrous private knowledge (H) from non-rivalrous knowledge (A). Letting h denote the average skill level, we note that $H = hL$. Finally, since physical labor is non-rivalrous in its employment in final production and human reproduction, let l_i denote the portion of human capital (or equivalently, the portion of labor with a given average skill level) allocated to sector i.

After imposing the general assumptions, the generalized production structure simplifies to:

$$Y = F(A, (1 - l_A - l_H) H, L, (1 - k_A - k_H) K) \tag{3.3a}$$

$$\dot{A} = J(A, l_A H, k_A K) \tag{3.3b}$$

$$\dot{H} = Q(A, l_H H, k_H K) \tag{3.3c}$$

where $\dot{L} = nL$ and $n \geq 0$. By allowing for either a growing or static population, our generalized framework can be used to analyze the two main branches of existing R&D-based literature.

Eicher & Turnovsky (1999) can be viewed as a special case of (3.3). They analyze a general non-scale growth model comprising (3.3a) and (3.3b) under the assumptions that both sectors employ homogeneous labor, L, and $n > 0$.

The representative agent of the economy derives utility solely from the consumption of the final output good. As is standard in the existing literature, the representative agent of the economy has intertemporal utility of isoelastic form:

$$\int_0^{\infty} e^{-\rho t} \frac{c^{1-\theta}}{1-\theta} dt \quad \rho > 0; \theta > 0$$

where c denotes consumption per capita, to be replaced by aggregate consumption, C, when $n = 0$. In the absence of depreciation, physical capital accumulates as a residual after aggregate consumption needs have been met:

$$\dot{K} = Y - C \tag{3.3d}$$

The central planner chooses consumption, and the fractions of labor and capital employed in each sector so as to maximize intertemporal utility of the representative agent subject to the production and accumulation constraints, equations (3.3a) - (3.3d). For the purposes of this paper, we note that the following discussion is premised on sectoral allocations of factors that are strictly positive and constant, as required for balanced growth.[4]

[4]We can solve for sectoral allocations of factors from the first order optimality conditions provided in the appendix.

3.4 Balanced Growth Equilibrium

Definition 3.1 *A balanced growth path is a path along which all real variables grow at constant, though not necessarily equal, rates.*

The balanced growth rates of the real variables (Y, K, A, H) are obtained by total differentiation of the production functions (3.3a) - (3.3c), noting that constant growth rates requires $g_Y = g_K$,[5] $g_{\dot{A}} = g_A$ and $g_{\dot{H}} = g_H$. The resulting system of equations can be expressed in matrix form:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_H \\ -\eta_K & (1-\eta_A) & -\eta_H \\ -\omega_K & -\omega_A & (1-\omega_H) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \\ g_H \end{bmatrix} = \begin{bmatrix} \sigma_L n \\ 0 \\ 0 \end{bmatrix} \tag{3.4}$$

where $\sigma_i \equiv F_i i/F \geqslant 0$, $\eta_i \equiv J_i i/J \geqslant 0$ and $\omega_i \equiv Q_i i/Q \geqslant 0$; $i = K, A, H$ denote the structural elasticities in the production, technology and human capital sectors, respectively. The structural elasticities are not necessarily constant.

The system of linear equations in (3.4) is non-homogeneous ($Ax = d$ in matrix form) or homogenous ($Ax = 0$) depending on whether the population is growing or static, respectively.

3.4.1 General Conditions for Positive and Balanced Growth with a growing population

First, consider the case where population growth is strictly positive. The system of equations in (3.4) jointly determine the growth rates of real variables as functions of

[5]The growth rate in physical capital, given by $g_K = \frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{Y}\frac{Y}{K}$, is constant if Y,K and C grow at the same rate.

population growth and the structural elasticities:

$$g_K = \frac{\sigma_L\left[\left(1-\eta_A\right)\left(1-\omega_H\right)-\omega_A\eta_H\right]n}{|A|} \tag{3.5a}$$

$$g_A = \frac{\sigma_L\left[\eta_K\left(1-\omega_H\right)+\omega_K\eta_H\right]n}{|A|} \tag{3.5b}$$

$$g_H = \frac{\sigma_L\left[\eta_K\omega_A+\omega_K\left(1-\eta_A\right)\right]n}{|A|} \tag{3.5c}$$

Strictly Positive Growth and Diminishing Marginal Returns

For the present, we assume that physical capital, technology and human capital are necessary for production in all three sectors, so that all elasticities in the coefficient matrix are strictly positive. In Section 3.4.3 below we will examine a special case which allows for elasticities to be zero.

Consider the conditions under which for $\sigma_L > 0$, $n > 0$ the equilibrium growth rates (3.5a) - (3.5c) are positive. System (3.4) is of the form analyzed by Hawkins & Simon (1949) if $|A| = |a_{ij}| \neq 0$ and $a_{ij} < 0$ for all $i \neq j$; $a_{ii} > 0$ for all i. By their theorem for stationary solutions with all variables positive, a necessary and sufficient condition that g_K, g_A and g_H satisfying (3.4) be all positive is that all principal minors of the coefficient matrix in (3.4) be positive. Denoting $|A_i|$ as the i^{th} principal minor of coefficient matrix A, the Hawkins-Simon conditions provide $|A_1| = (1-\sigma_K) > 0$, $|A_2| = (1-\sigma_K)(1-\eta_A) - \eta_K\sigma_A > 0$ and $|A_3| = |A| > 0$. These conditions, in turn, imply restrictions on structural elasticities: $|A_1| > 0 \Leftrightarrow \sigma_K < 1$,[6] $|A_1| > 0$ and $|A_2| > 0$ together imply $\eta_A < 1$, while $|A_2| > 0$ and

[6]On first inspection of (3.5a) - (3.5c), diminishing returns to capital in the production of final output ($\sigma_K < 1$) does not seem a condition for positive growth. As a check of the Hawkins-Simon conditions, $\sigma_K < 1$, together with $\eta_A < 1$, $\omega_H < 1$ and $|A| > 0$, implies and is implied by $g_K > 0$, $g_A > 0$, $g_H > 0$, since $(1-\sigma_K)\left[(1-\eta_A)(1-\omega_H) - \omega_A\eta_H\right] > \sigma_A\left[\eta_K(1-\omega_H)+\omega_K\eta_H\right] + \sigma_H\left[\eta_K\omega_A + \omega_K(1-\eta_A)\right] \Leftrightarrow |A| > 0$.

Note that $|A| > 0$ imposes a relationship between structural elasticities. The inequality $(1-\eta_A)/\eta_H > \omega_A/(1-\omega_H)$ relates returns to scale to technology and human capital across the two sectors. If we have decreasing returns to technology and human capital in the R&D sector, we must also have decreasing returns

$|A| > 0$ together imply $\omega_H < 1$.

We summarize these results with the following proposition:

Proposition 3.1 (Conditions for Positive Growth) *When strictly positive growth in the economy requires strictly positive population growth,* $|A| > 0$ *and* $\sigma_K < 1$, $\eta_A < 1$ *and* $\omega_H < 1$, *together with* $\eta_K > 0$ *and/or* $\omega_K > 0$, *are necessary and sufficient for strictly positive growth in output, capital, consumption, technology and human capital.*

We discuss two points arising from this proposition.

First, the conditions for positive equilibrium growth rates have an intuitive appeal. The conditions $\sigma_K < 1$, $\eta_A < 1$ and $\omega_H < 1$ reflect diminishing marginal returns to capital, human capital and technology in the sector that produces each input, respectively. Sectoral linearity is ruled out when equilibrium growth rates in the economy depend on growth of an exogenous factor, raw labor. In our model, raw labor is only employed in final production. Diminishing marginal returns to capital in the production of final output requires increasing effort for capital (unconsumed final output) to grow over time. A growing raw labor force provides this increasing effort. By similar reasoning, because capital is employed in the accumulation of knowledge, diminishing marginal returns to knowledge is a condition for positive, non-explosive growth in the stock of technology and human capital.

Second, whilst (3.4) is a non-homogeneous system of the form analyzed by Hawkins & Simon (1949), it does not satisfy the stronger assumption $[d_i > 0]$ since some elements of vector d are zero. Hawkins & Simon (1949) state, in a footnote,

> The stronger condition ... $([d_i > 0])$ ... because of the continuity of solutions of

to technology and human capital in the human capital accumulation sector.

these equations with respect to variations of these coefficients, does not involve any essential loss of generality.

Whilst Hawkins & Simon (1949) suggest their theorem applies when some but not all of the d_i are zero, the proof of their theorem seems to assume $[d_i > 0]$. For completeness, we append this paper with a proof of a corollary to Hawkins & Simon's (1949) general theorem.

Under the weaker assumption $[d_i \geq 0]$ with at least one $d_i > 0$, the assumption that all elasticities in the coefficient matrix are strictly positive becomes critical. We include the condition $\eta_K > 0$ and/or $\omega_K > 0$ in Proposition 1[7], since physical capital must be productive in either R&D or human capital, to obtain positive growth in either sector. This condition results from our allowance for heterogeneous labor and the assumption that physical labor is employed only in the production of final output. Exogenous growth in population or raw labor therefore drives growth in final output and physical capital, and, indirectly, R&D and human capital only if physical capital is employed in these sectors.

Introducing a Lucas (1988) specification for human capital accumulation (with diminishing returns) and a Jones (1995*a*) type R&D sector to our three sector model implies zero growth in both types of knowledge, since $\eta_K = 0$ and $\omega_K = 0$ and therefore would be redundant. There is a trade-off between the realism of heterogeneous labor and the simplicity of single input knowledge accumulation equations, such as $\omega_K = 0; \omega_A = 0; \omega_H = 1$. Existing literature opts for the latter and we introduce the restriction of homogeneous labor later in the paper to illustrate these models as special cases of our general model.

[7]Hereafter, Proposition 1 in the text of each chapter refers to Proposition C.1 where C denotes the chapter number.

Corollary 3.1 (to Proposition 3.1) *A further sufficient condition for strictly positive growth in* *per capita* *output and capital is* $\sigma_L > (1 - \sigma_K)$.

From equation (3.5a),[8] the economy grows at the per capita rate:

$$g_y = \{\sigma_L - (1 - \sigma_K)\} \frac{g_K}{\sigma_L} + \sigma_A \frac{g_A}{\sigma_L} + \sigma_H \frac{g_H}{\sigma_L} \tag{3.6}$$

where $\sigma_L > (1 - \sigma_K) \Rightarrow g_y > 0$, given the above conditions for positive growth in output, physical capital, technology and human capital are satisfied. Eicher & Turnovsky (1999) obtain the same condition for a general two sector growth model with homogeneous labor. Thus, the condition has generality. It implies increasing returns to scale in the final output sector, since $\sigma_L + \sigma_K > 1$, together with non-negativity of σ_i, implies $\sum \sigma_i > 1$. Note that this condition is sufficient but not necessary for strictly positive growth in per capita output. That is, growth in per capita output may still be strictly positive if $\sigma_L \leq (1 - \sigma_K)$. For instance, if $\sigma_L + \sigma_K = 1$, growth in per capita output is strictly positive, given strictly positive growth in technology and human capital.

3.4.2 Balanced Growth and Cobb-Douglas or Constant Returns to Scale Technology

While positive growth is provided by Proposition 1, *balanced* growth requires constancy of the growth rates in (3.5a)-(3.5c), which, in turn, requires constant population growth and constant multiplicative terms.

The multiplicative terms in (3.5a)-(3.5c) are constant if the structural elasticities are constant (as for Cobb-Douglas production functions). In this case, the constancy of the

[8] $g_y = g_K - n = \frac{\{\sigma_L - (1-\sigma_K)\}[(1-\eta_A)(1-\omega_H) - \omega_A \eta_H] + \sigma_A[\eta_K(1-\omega_H) + \omega_K \eta_H] + \sigma_H[\eta_K \omega_A + \omega_K(1-\eta_A)]}{|A|} n$

multiplicative terms is independent of the returns to scale, so that output, technology and human capital may grow at different rates. If the structural elasticities are not constant (as for Constant Elasticity of Substitution (CES) production functions), balanced growth requires that production in each sector exhibit constant returns to scale (i.e. $\sum \sigma_i = 1; \sum \eta_i = 1; \sum \omega_i = 1$), in which case the multiplicative terms reduce to unity and all sectors grow at the common rate: $g_Y = g_K = g_A = g_H = n$.[9]

Thus, for a constant rate of population growth, Cobb-Douglas production technology or constant returns to scale in all sectors are sufficient conditions for balanced growth. This result is known, albeit for a two sector generalized growth model (see Eicher & Turnovsky (1999), who also discuss a third condition, that of homogeneously separable forms). Accordingly, this paper does not state these conditions in a formal proposition. However, since these conditions are widely used, but obscured in sophisticated models, we note some of their implications.

Constant returns to scale, particularly in the production of final output, is unlikely. When non-rivalrous knowledge is employed, final production most likely exhibits increasing returns to scale (Romer 1990). Moreover, constant returns to scale in all sectors implies zero growth in per capita output. This may explain why Cobb-Douglas technology most commonly appears in the literature, but this is not without its limitations, since it assumes input shares are exogenous constants. An all or nothing approach is not required. For instance, we may assume Cobb-Douglas technology in one sector, CES technology in another or even CES technology nested within Cobb-Douglas production function.

[9] To aid the reader in verifying this, if $\sum \sigma_i = 1; \sum \eta_i = 1; \sum \omega_i = 1$ then $|A| = \sigma_H \{\omega_K \eta_H + \eta_K (\omega_A + \omega_K)\}$.

Notwithstanding these caveats, the main problem with these sufficient conditions for balanced growth is the implication that (positive) balanced growth seems inextricably dependent on a (positive) constant population growth. We now establish that positive, balanced growth arises without positive population growth if growth in either technology or human capital asymptotes to a positive constant.

Does Strictly Positive and Balanced Growth require Strictly Positive Population Growth?

Differentiating (3.3a) with respect to time, and noting that constant g_K requires that Y and K grow at the same rate, yields:

$$g_Y = g_K = \frac{\sigma_A}{1-\sigma_K} g_A + \frac{\sigma_H}{1-\sigma_K} g_H + \frac{\sigma_L}{1-\sigma_K} n \tag{3.7}$$

When both g_A and g_H depend on g_K, equation (3.7) reduces to equation (3.5a). However, if either g_A or g_H are independent of g_K, then g_Y is an additively separable function of two terms, only one of which is a function of population growth. Hence, strictly positive, balanced growth no longer requires strictly positive population growth.

Assuming that physical capital is not employed in one knowledge sector does not imply g_A or g_H are independent of g_K. To illustrate, if physical capital is used in human capital accumulation but not in R&D, under normal conditions, g_A is still a function of g_K because human capital is used in R&D. This raises the question, under what condition(s) is either g_A or g_H independent of g_K? The answer lies in the asymptotic nature of g_A or g_H. For instance, if either asymptotes to a positive constant that exceeds g_K, then g_Y is an additively separable function, as required.

Equations (3.3b) and (3.3c) describe the accumulation of non-rivalrous ideas and rivalrous human capital, respectively. The accumulation of each type of knowledge is a function of its own stock, the stock of the alternative knowledge and the stock of physical capital. As a result, there are several ways by which asymptotic limits may imply growth in one type of knowledge is independent of growth in physical capital. To simplify, we confine our analysis to the case where knowledge accumulation is a function of its existing stock of knowledge and one other input.[10] In order to generalize the following proposition, we could define two generic types of knowledge. However, this is a redundant exercise if we consider that certain input combinations are implausible. For instance, can innovation occur through the interaction of existing ideas and physical capital? Researchers are most likely an essential input in R&D. Accordingly, from equations (3.3b) and (3.3c), the two types of knowledge grow at the rates:

$$g_A = \frac{J(A, l_A H,)}{A} \equiv j(A, H; \beta_A) \quad (3.8a)$$

$$g_H = \frac{Q(l_H H, k_H K)}{H} \equiv q(H, K; \beta_H) \quad (3.8b)$$

where β_A and β_H are shift parameters encapsulating, respectively, l_A and l_H, k_H. The growth rates asymptote to:

$$\lim_{A\to\infty} j(A, H; \beta_A) = \bar{j}(H; \beta_A) \quad (3.9a)$$

$$\lim_{H\to\infty} q(H, K; \beta_H) = \bar{q}(K; \beta_H) \quad (3.9b)$$

where $\bar{j}$ and $\bar{q}$ are constants, that may depend on the shift parameters.

[10] The following section introduces CES technology to illustrate the implications of positive asymptotic limits. Whilst it is neater to discuss the degree of substitutability between two inputs, we could apply CES technology to three inputs or nest a CES technology (with two inputs) within a Cobb-Douglas production function (with a third input).

Proposition 3.2 (Condition for Positive, Balanced Growth and Population) *Strictly positive balanced growth may arise without population growth if* $\bar{q}(K;\beta_H) > 0$.

Strictly positive balanced growth requires strictly positive, constant population growth if $\bar{q}(K;\beta_H) = 0$, *unless* $\bar{j}(H;\beta_A) > 0$.

The first part of Proposition 2, refers to the case where g_H asymptotes to a positive constant. If this positive constant exceeds g_K, then g_H does not depend, indirectly on n. Thus, the accumulation of human capital features an endogenous stock of knowledge. Curve 1 in Figure 3.1 illustrates such a case. Substituting $\bar{q}$ for g_H in equation (3.7) then, regardless of whether or not g_A is a function of g_H, g_Y is an additively separable function of two terms, one of which does not depend on population growth.

If g_H asymptotes to zero, as in the second part of Proposition 2, then g_H depends on g_K in steady state (see Curve 2 in Figure 3.1). Two possibilities arise. If g_A asymptotes to zero, implying g_A is a function of g_H, then growth in both types of knowledge requires growth in physical capital. Substituting for g_A and g_H in equation (3.7) gives unambiguous semi-endogenous growth: strictly positive, constant growth in output requires strictly positive, constant population growth. However, if g_A asymptotes to a positive constant that exceeds g_H, then g_A is independent of g_H. Substituting $\bar{j}$ for g_A in equation (3.7) yields an additively separable function for g_Y, so that strictly positive balanced growth does not require strictly positive, constant population growth.

This proposition provides a basis for distinguishing endogenous and semi-endogenous growth, often synonymous in the literature with scale and non-scale growth, respectively, which in turn are strongly associated with sectoral linearity and diminishing marginal re-

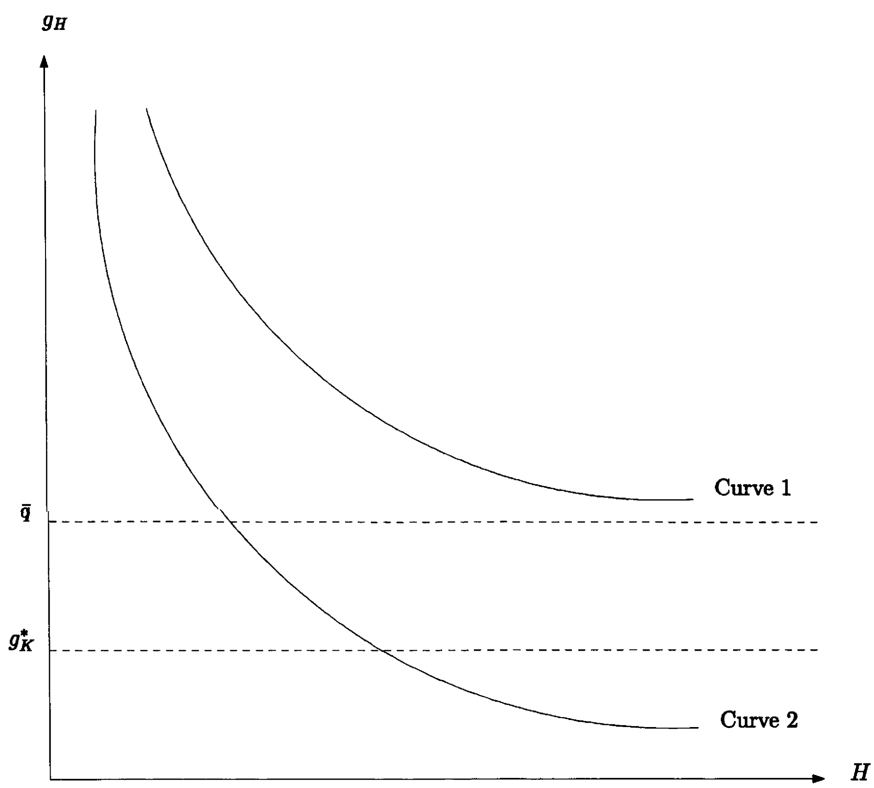


Figure 3.1: Asymptotic Growth in Human Capital

turns, respectively. Endogenous growth differs from semi-endogenous growth in that economic growth does not require strictly positive growth in an exogenous factor, such as population. Proposition 2 reminds us that endogenous growth requires only that growth in knowledge asymptote to a strictly positive constant. Both curves in Figure 3.1 are monotonically decreasing, reflecting diminishing marginal returns to knowledge in the accumulation of knowledge. However, Curve 1 implies endogenous growth, while Curve 2 implies semi-endogenous growth. Thus, the association between diminishing marginal returns and semi-endogenous growth is tenuous.

CES technology - which sector?

Pitchford (1960) and Barrro & Sala-i-Martin (1999) demonstrate the capacity for endogenous growth with CES technology in the standard neoclassical growth model.[11] More recently, this has been extended to a Romer (1990) type R&D-based growth model by Zuleta (2004), where the production function for final output is a CES combination of physical capital and labor. Whereas Zuleta (2004) introduce CES technology to the production of final output, Proposition 2 suggests introducing CES technology to the accumulation of knowledge: the growth in knowledge will asymptote to zero or a positive constant, depending on whether the elasticity of substitution between inputs is less than one or greater than one, respectively. To which knowledge accumulation sector do we introduce CES technology?

Taking inspiration from Dalgaard & Kreiner (2003), we could introduce CES technology to the accumulation of non-rivalrous knowledge, $J(A, l_A H,)$. Because they assume physical labor rather than human capital is employed in the production of new ideas, their

[11] Technically, endogenous growth may arise in the neoclassical growth model when the marginal product of capital is bounded below by a positive constant, as demonstrated by Jones & Manuelli (1990).

application of CES technology addresses a *direct* relationship between g_A and n. Except for the fact that the relationship between g_A and n is indirect in our model, therefore, introducing CES technology to $J(A, l_A H,)$ would effectively replicate existing literature. This is not to suggest that simply by introducing CES technology to an alternative sector we contribute to the literature. In the following section, we introduce CES technology to the accumulation of human capital $Q(l_H H, k_H K)$ not only because it hasn't been done before, but also because it is the more plausible application of CES technology.

Applying l'Hopital's rule[12] to equation (3.8b),

$$\lim_{H\to\infty} q(H, K; \beta_H) = \lim_{H\to\infty} Q_H \tag{3.10}$$

where Q_H is the marginal product of human capital in the generation of new human capital. Thus, another interpretation of Proposition 2 is that endogenous growth arises if the marginal product of knowledge in producing new knowledge tends to a positive constant as the stock of knowledge tends to infinity. This suggests one criterion for introducing CES technology.

Dalgaard & Kreiner (2003) argue that while the marginal product of physical capital most likely tends to zero as the stock of physical capital becomes infinite, the marginal product of knowledge is bounded below by a positive constant. In their words,

> ...why would a new piece of information be completely unproductive in producing new ideas even if there did in fact exist infinitely many other pieces of information?

They make a good point. Ideas are boundless. On this criterion, our application of CES technology to human capital accumulation is closer to Barrro & Sala-i-Martin's

[12] $\lim_{x\to a}\left(\frac{m(x)}{n(x)}\right) = \lim_{x\to a}\left(\frac{m'(x)}{n'(x)}\right)$

(1999) application, since human capital is embodied knowledge. However, this is not the only criterion for introducing CES technology.

Positive asymptotic limits arise with CES technology only when the elasticity of substitution exceeds one. This suggests we should look to define CES technology over inputs that, a priori, we expect may be highly substitutable. In this sense, Barrro & Sala-i-Martin's (1999) application of CES technology has an intuitive appeal that Dalgaard & Kreiner (2003) lacks. There are several real world examples, such as the demise of the typist pool, where physical capital has replaced physical labor in the production of final output. Moreover, empirical evidence supports an elasticity of substitution between physical capital and labor higher than one (Duffy & Papageorgiou 2000).

In contrast, it is hard to think of examples where ideas have replaced researchers in the process of innovation. A scientist works with an existing idea, say $e = mc^2$, to create new ideas, suggesting a high degree of complementarity between the two inputs in R&D. The notion that ideas may be increasingly substituted for physical capital in the process of R&D may be more palatable. Contrast the physical capital requirements of a modern researcher with those of, say, Thomas Edison. Or better still, consider the way the idea to use silicon in a microchip means researchers no longer require computers that take up a floor of a building. There are also examples of physical capital complementing ideas in the process of R&D. However, the notion that knowledge can be substituted for physical capital is more plausible than the notion that knowledge can be substituted for researchers in R&D.

We could broaden Proposition 2 to the case of three inputs and introduce CES technology to the employment of ideas and physical capital in R&D. However, we need not do this if we consider that just as non-rivalrous knowledge (ideas) may replace physical capital in the process of R&D, private knowledge may replace physical capital in the process of learning. For instance, as an economy's stock of human capital accumulates, increased reliance on self-education may be consistent with rivalrous knowledge replacing physical infrastructure. The degree of complementarity between human capital and physical capital in the accumulation of human capital may be high or low, whereas, a priori, the degree of complementarity between human capital and ideas in R&D is high. This reasoning suggests introducing CES technology to $Q(l_H H, k_H K)$ rather than $J(A, l_A H)$.

3.4.3 A Specific Model of Endogenous Growth with or without population growth

Consider an economy comprised of three sectors with the following production technologies:

$$Y = A^{\sigma_A}((1-l_A)H)^{\sigma_H} L^{\sigma_L}((1-k_H)K)^{\sigma_K} \quad ; \quad 0 < \sigma_K < 1; \sigma_i > 0 \forall i \tag{3.11a}$$

$$\dot{A} = A^{\eta_A}(l_A H)^{\eta_H} \quad ; \quad 0 < \eta_A < 1; \eta_H > 0 \tag{3.11b}$$

$$\dot{H} = [(\phi_1 H)^{\rho} + (\phi_2 k_H K)^{\rho}]^{1/\rho} \quad ; \quad -\infty < \rho < 1; \rho \neq 0 \tag{3.11c}$$

where all exponents are constant and ρ (the substitution parameter) determines the constant elasticity of substitution between human capital and physical capital in the accumulation of human capital, given by $\epsilon = 1/(1-\rho) : \epsilon > 0, \epsilon \neq 0$. Define $\phi_1 \equiv \alpha^{1/\rho}$ and $\phi_2 \equiv (1-\alpha)^{1/\rho}$, where $\alpha \in (0,1)$ is the distribution parameter.[13] The parameters satisfy $0 < n \leq g_K < \phi_1$.

[13]When Dalgaard & Kreiner (2003) introduce CES technology to the interaction of A and $l_A L$ in R&D, they assume the equivalent parameter to ϕ_1 is not a function of ρ and α and impose the restriction $\phi_1 \in (0,1)$.

We keep the model as close as possible to the generalized production structure in (3.3). For simplicity, physical capital and the stock of technology are dropped as inputs to R&D and human capital accumulation, respectively, but these assumptions can be relaxed without loss of generality.[14] Also for simplicity, we assume $l_H = 1$, meaning human capital is as a private input allocated to the production of final output and R&D, while at the same time being used in the accumulation of human capital. This assumption is not critical. If we relax the assumption, steady state growth in human capital will be a function of l_H, but the presence of this term is innocuous since sectoral shares of labor and capital are constant along a balanced growth path.

Differentiating g_H with respect to time and recognizing that $\omega_H = (\phi_1/g_H)^\rho$ and $\omega_K = 1 - (\phi_1/g_H)^\rho$ (see Appendix for detail), we obtain:

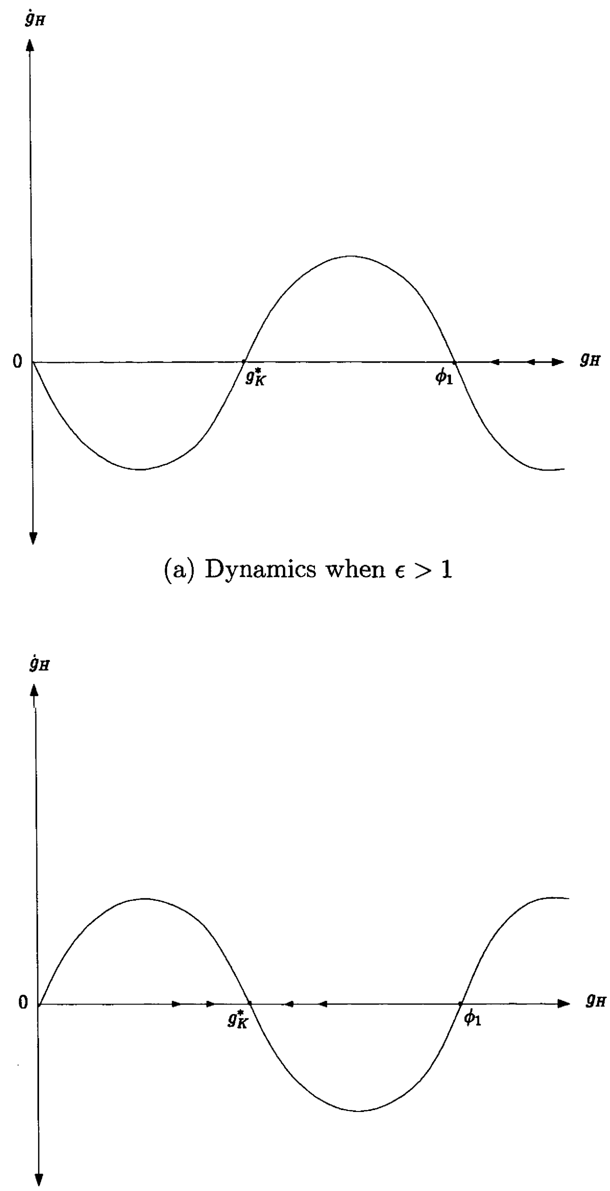
$$\dot{g}_H = g_H\,(g_K - g_H)\left\{1 - \left(\frac{\phi_1}{g_H}\right)^\rho\right\} \tag{3.12}$$

This equation has three steady states: $g_H = 0$, $g_H = g_K$ and $g_H = \phi_1$. Referring to Figure 3.2, g_H converges to either g_K or ϕ_1, depending on whether ϵ is less than one or greater than one, respectively. In the case where $\epsilon > 1$, the growth in human capital is bounded below by ϕ_1.

Thus, when $\epsilon > 1$, human capital grows permanently, independent of and at a higher rate than physical capital. As suggested in the previous section, the intuition for

In a standard CES production function, such as equation (3.11c), $\phi_1 \equiv \alpha^{1/\rho} \gtrless 1$, depending on $\rho \in (-\infty, 1)$; $\rho \neq 1$. A standard CES production is consistent with a growth rate less than 1, since g_K only converges to ϕ_1 when $\rho > 0$, implying $g_K = \phi_1 < 1$.

[14]By dropping physical capital from R&D, our solution for g_K more closely resembles (3.7). Also, this aids derivation of research intensity, measured by l_A. Allowing for physical capital as an input to R&D, the long run growth rate of output is still independent of n when $\epsilon > 1$: $g_Y = \frac{1}{(1-\sigma_K)-\sigma_A\eta_K}\left[\left\{\frac{\sigma_A\eta_H}{(1-\eta_A)} + \sigma_H\right\}\phi_1 + \sigma_L n\right]$.



(a) Dynamics when $\epsilon > 1$

(b) Dynamics when $\epsilon < 1$

Figure 3.2: Dynamics of Growth in Human Capital

this result lies in the fact that human capital is increasingly substituted for physical capital as the accumulation of human capital proceeds.

To solve for the growth rates of the other real variables in the model, we note that when $\epsilon > 1$, we cannot use the solutions given by (3.5a) and (3.5b), obtained by Cramer's Rule, since $g_H = \phi_1$ implies $\omega_H = 1$ and $\omega_K = 0$, which in turn imply $|A| = 0$ for the matrix system (3.4). We therefore proceed with total differentiation of (3.11b) and substitution for g_A and g_H in (3.7). The balanced growth rates in physical capital, output and consumption on the one hand and technology on the other are:

$$\left.\begin{aligned} g_K &= \tau\phi_1 + \upsilon n \\ g_A &= \tfrac{\eta_H}{(1-\eta_A)}\phi_1 \end{aligned}\right\} \text{ if } \epsilon > 1 \qquad (3.13a)$$

$$\left.\begin{aligned} g_K &= \left(\tfrac{\upsilon}{1-\tau}\right) n \\ g_A &= \tfrac{\eta_H}{(1-\eta_A)} g_K \end{aligned}\right\} \text{ if } \epsilon < 1 \qquad (3.13b)$$

where $\tau = \frac{\sigma_A}{(1-\sigma_K)}\frac{\eta_H}{(1-\eta_A)} + \frac{\sigma_H}{(1-\sigma_K)}$ and $\upsilon = \frac{\sigma_L}{1-\sigma_K}$. When $\epsilon < 1$, strictly positive rates of growth requires $(1-\tau) > 0$, as implied by $|A| > 0$.[15] When $\epsilon > 1$, strictly positive growth in output no longer requires strictly positive population growth, n. This is the key result. Note that the restriction $g_K < \phi_1$ requires that σ_i and η_i satisfy $\frac{\upsilon}{1-\tau} < \frac{\phi_1}{n}$. As an illustration, if we assume constant returns to scale in final production and R&D (i.e. $\sum \sigma_i = 1; \sum \eta_i = 1$) this restriction simplifies to $\phi_1 > n$.

[15] $|A| = [(1-\sigma_K)(1-\eta_A) - \sigma_A\eta_H - \sigma_H(1-\eta_A)]\,\omega_K$

The long run *per capita* growth rate of the economy is given by:

$$g_y = \begin{cases} \tau\phi_1 + (\upsilon - 1)\,n & \text{if } \epsilon > 1 \\ \\ \left(\frac{\upsilon}{1-\tau} - 1\right) n & \text{if } \epsilon < 1 \end{cases} \tag{3.14}$$

where $\upsilon > 1$ is *sufficient* for positive per capita growth, as per Corollary 1 to Proposition 1, given $\tau > 0$ and $\phi_1 > 0$, by definition.

Thus, growth in the economy does not require growth in the population, when knowledge is highly substitutable for physical capital in the accumulation of knowledge.

3.4.4 General Conditions for Positive Growth with a static population

Consider now the case where population is static. If $n = 0$, vector d in matrix system (3.4) is a null vector.

Proposition 3.3 (Condition for Positive Growth) *For a static population,* $|A| = 0$ *is necessary for strictly positive growth in output, capital, consumption, technology and human capital.*

It is a well known result of linear algebra that a homogeneous linear system of equations (in matrix form $Ax = 0$) has non trivial solutions iff $|A| = 0$. So the existence of a solution with positive growth rates implies $|A| = 0$. However, $|A| = 0$ does not imply positive growth since a non trivial solution may be one of negative growth. Thus, for a static population, $|A| = 0$ is necessary for strictly positive growth in output, capital, consumption, technology and human capital.

Corollary 3.2 (to Proposition 3.3) *Sectoral linearity is a sufficient but not necessary condition for* $|A| = 0$.

A sufficient condition for $|A| = 0$ is that each of the entries in one or more of the rows or columns in matrix A is zero. Existing models with a static population commonly assume that either (3.3b) or (3.3c) are single input linear equations (as in a Romer (1990) type R&D equation ($\eta_A = 1$) or a Lucas (1988) specification for human capital accumulation ($\omega_H = 1$)). This sectoral linearity assumption implies each of the entries in either the second or third rows of the coefficient matrix in the system (3.4) is zero. Thus, sectoral linearity is introduced to a knowledge accumulation equation in order to solve for strictly positive rates of growth in the real variables of the model. This assumption is widely criticized. To quote Jones (2001) (p.5),

> The linearity in existing models is assumed ad hoc, with no motivation other than that we must have linearity somewhere to generate endogenous growth.

It is therefore worth considering whether we can solve for strictly positive rates of growth under an alternative, more palatable assumption.

Another sufficient condition for $|A| = 0$ is that one row (column) is a linear combination of the other rows (columns) of the matrix. To explore this further, letting v_i denote the i^{th} column vector of the coefficient matrix A, the system $Ax = 0$ can be written as the vector equation:

$$g_K v_1 + g_A v_2 + g_H v_3 = 0$$

where

$$v_1 = \begin{pmatrix} (1-\sigma_K) \\ -\eta_K \\ -\omega_K \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\sigma_A \\ (1-\eta_A) \\ -\omega_A \end{pmatrix}, \quad v_3 = \begin{pmatrix} -\sigma_H \\ -\eta_H \\ (1-\omega_H) \end{pmatrix} \tag{3.15}$$

$|A| = 0$, as required to obtain a strictly positive solution to the system, if the three vectors are linearly dependent:

$$v_1 = av_2 + bv_3 \quad a < 0; b < 0 \tag{3.16}$$

The easiest and most obvious case to consider is that of constant returns to scale to growing factors ($a = b = -1$). Note that, because we have physical labor employed in the production of final output, this case corresponds to increasing returns to scale in final production ($\sum \sigma_i > 1$).

However, we can move away from constant returns to scale to growing factors and still obtain $|A| = 0$. In the case where $a = b = -k$, we have increasing or decreasing returns to scale to growing factors in the final output sector, when k is less than or greater than one, respectively, with ambiguous returns to scale in the other sectors. We can solve for strictly positive growth rates in the physical capital, technology and human capital with varying degrees of returns of scale across sectors, so long as the vectors are linearly dependent.

Example 1 *Consider a Cobb Douglas economy where* $(1-\sigma_K) = \sigma_A = \sigma_H = 0.66$; $\eta_K = \eta_A = \eta_H = 0.25$; $\omega_K = \omega_A = \omega_H = 0.25$. *These values suggest increasing returns to scale in final output and decreasing returns to scale in both knowledge accumulation sectors. The three column vectors are linearly dependent:* $v_1 = -0.5v_2 - 0.5v_3$, *as required to obtain a strictly positive solution to the system.*

To demonstrate that linear dependence and sectoral linearity are both sufficient for $|A| = 0$, as required for strictly positive growth, we can use the generalized setting to describe the decentralized two sector R&D-based growth models of Romer (1990) and Rivera-Batiz & Romer (1991). With no accumulation of human capital and population an exogenous constant, the system of equations which determine positive growth rates reduces to:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A \\ -\eta_K & (1-\eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.17}$$

Both papers assume the same Cobb-Douglas specification for the production of final output, which in a centralized decision making model is given by:

$$Y = \eta^{(\sigma_H+\sigma_L)-1} A^{\sigma_H+\sigma_L} \left(\left(1-l_A\right) H\right)^{\sigma_H} L^{\sigma_L} K^{1-(\sigma_H+\sigma_L)} \tag{3.18a}$$

where η is a constant term that, in a decentralized setting measures the units of foregone consumption (or equivalently, physical capital) required to create one unit of any type of intermediate good.[16] The term l_A is absent in Rivera-Batiz & Romer (1991) since R&D uses units of final output.

The modelling of the R&D sector is the major distinction between the two papers. Romer (1990) assumes neither physical capital nor physical labor are productive in R&D:

$$\dot{A} = \delta\left(l_A H\right) A \tag{3.18b}$$

where δ is a constant efficiency parameter. Rivera-Batiz & Romer (1991), on the other hand, allow for both physical capital and physical labor to be productive inputs in R&D.

[16] In a decentralized setting, A determines the range of intermediate goods that can be produced.

They propose the lab equipment model of R&D:

$$\dot{A} = BA^{\sigma_H+\sigma_L}H^{\sigma_H}L^{\sigma_L}K^{1-(\sigma_H+\sigma_L)} \quad (3.18c)$$

where B is the share of final output invested in R&D.

Substituting for the sectoral elasticities from (3.18b) in (3.17):

$$\begin{bmatrix} (\sigma_H+\sigma_L) & -(\sigma_H+\sigma_L) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.19)$$

where Romer's assumption of sectoral linearity implies $|A| = 0$. We can therefore solve the system for strictly positive rates of growth in physical capital and technology. From (3.19), we have one equation with two unknowns: $g_K = g_A$. Because $|A| = 0$, we employ a different solution method to that used in Section 3.4.1. From (3.18b), $g_A = \delta\,(l_A H)$, where δ and H are exogenous constants and l_A, the portion of human capital allocated to R&D, is constant, as required for balanced growth.[17]

Substituting for the sectoral elasticities from (3.18c) in (3.17):

$$\begin{bmatrix} (\sigma_H+\sigma_L) & -(\sigma_H+\sigma_L) \\ (\sigma_H+\sigma_L)-1 & 1-(\sigma_H+\sigma_L) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.20)$$

where $v_1 = -v_2$, sufficient for $|A| = 0$. Thus, instead of introducing sectoral linearity to the accumulation of knowledge, Rivera-Batiz & Romer (1991) assume diminishing marginal returns to the stock of existing ideas in the creation of new ideas ($\eta_A < 1$). Constant returns to scale to physical capital and ideas, the growing factors, is sufficient for strictly positive rates of growth. From (3.20), we have one equation with two unknowns which again is simply $g_K = g_A$.

[17]An expression for l_A can be derived from the first order conditions.

Thus, we have demonstrated that static population R&D-based growth models can be solved for strictly positive growth without introducing sectoral linearity to the accumulation of knowledge. Note that while Rivera-Batiz & Romer (1991) have assumed constant returns to scale to the growing factors, we know from our discussion of Corollary 2 and the numerical example that we could have varying returns to scale across sectors in a three sector model.

3.5 Homogeneous Labor

In order to evaluate existing R&D based growth models as special cases of our general three sector model, the restriction of homogeneous labor is introduced. We assume all labor is skilled, in order to analyze both endogenous and exogenous labor accumulation.

3.5.1 Endogenous Labor

If the accumulation of skilled labor is endogenized, we obtain a homogeneous form (in matrix algebra, $Ax = 0$) of the system of equations in (3.4).

Illustration of Proposition 3

Funke & Strulik (2000) model the development to an innovative economy. We simplify their model only by removing an exogenous productivity parameter and a distinction between the stock of intermediate goods and physical capital which, in an innovative economy, are one and the same. Their decentralized model is detailed, comprising forty six equations. We can use our generalized model to reveal the salient features of their innovative economy. The production structure is:

$$Y = A^{1-\sigma}\left((1-l_A-l_H)H\right)^{1-\sigma}K^{\sigma}\eta^{-\sigma} \tag{3.21a}$$

$$\dot{A} = \delta\,(l_A H) \tag{3.21b}$$

$$\dot{H} = \xi\,(l_H H) \tag{3.21c}$$

where δ and ξ are constant efficiency parameters.

As per Proposition 3, Funke & Strulik's (2000) assumption of sectoral linearity in the human capital accumulation equation implies positive rates of growth in physical capital, (output, consumption) and technology and human capital, as jointly determined by the system:

$$\begin{bmatrix} (1-\sigma) & -(1-\sigma) & -(1-\sigma) \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_K \\ g_A \\ g_H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.22}$$

Because $|A| = 0$, we employ a different solution method to that used when there is an exogenously growing factor. Sectoral linearity in the human capital accumulation equation gives us two equations with three unknowns, which Funke & Strulik (2000) reduce to one equation with two unknowns by assuming no physical capital is employed in R&D. From (3.22), $g_K = 2\,g_A = 2\,g_H$. Referring to the Appendix, we use the first order optimality conditions to solve. Final output and the two types of knowledge in the economy grow at the rates:

$$g_Y = \tfrac{\xi-\rho}{\theta}; \quad g_A = g_H = \tfrac{\xi-\rho}{2\theta} \tag{3.23}$$

3.5.2 Exogenous Labor

Since $H = hL$, where h is the average skill level, skilled labor accumulates over time due to growth in the average skill level (measured by growth in average educational attainment, e) and/or growth in the labor force (measured by population growth, n). The system of equations which determine positive and balanced growth rates reduces to:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A \\ -\eta_K & (1-\eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma_H (e+n) \\ \eta_H (e+n) \end{bmatrix} \tag{3.24}$$

Despite empirical evidence that a significant portion of income growth (equivalent to g_K in (3.24)) is attributable to growth in educational attainment, e is absent from existing models of R&D-based growth. The reason for this is best articulated by Jones (2001):

> ...roughly 80 percent of post-war US growth is due to increases in human capital investment rates and research intensity and only 20 percent is due to the general increase in population (Jones 2002). However, ... neither educational attainment nor the share of labor force devoted to research can increase forever. So unless there is an ad-hoc Lucas-style linearity in human capital accumulation, population growth remains the only possible source of long run growth...

Introducing the restriction that skilled labor accumulates at the exogenous rate of population growth, n, yields the general two-sector growth model analyzed by Eicher & Turnovsky (1999). Note that the generalized growth rates of real variables, jointly determined by (3.24), are not obtained by setting $\omega_i = 0$ in (3.5a) and (3.5b). However, the rates of growth of physical capital, output and consumption, on the one hand, and knowledge, on the other, are readily obtained (see p. 400 of Eicher & Turnovsky (1999) for a full analysis).

Whilst the general model encompasses several well known non-scale R&D-based growth models as special cases, this paper focuses on the dependency of economic growth on population growth and we discuss an example from the literature that best reflects this focus, namely, a re-parameterization of Jones (1995*a*), suggested by Proposition 1 (Corollary 1).

Illustration of Proposition 1 (especially Corollary 1)

The model of Jones (1995*a*) is well known. He obtains non-scale growth by introducing diminishing marginal returns to the stock of ideas in R&D ($\eta_A < 1$) to a homogenized labor version of Romer (1990). The production structure is:

$$Y = (A(1-l_A)H)^{\sigma} K^{1-\sigma} \tag{3.25a}$$

$$\dot{A} = \delta A^{\eta_A}(l_A H)^{\eta_H} \tag{3.25b}$$

where $\dot{H} = nH$, so that population growth is the only exogenous source of growth in the economy.

As per Proposition 1, diminishing marginal returns implies positive rates of growth of physical capital, output and consumption, on the one hand, and knowledge, on the other, as jointly determined by the system:

$$\begin{bmatrix} \sigma & -\sigma \\ 0 & (1-\eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma_H n \\ \eta_H n \end{bmatrix} \tag{3.26}$$

Technology and *per capita* output, physical capital and consumption grow at a common rate determined by population growth and the shares of labor and stock of knowledge in the R&D sector:

$$g_A = g_y = \psi n \tag{3.27}$$

where $\psi = \eta_H / (1 - \eta_A)$. Constant population growth and Cobb-Douglas technology imply balanced growth.

Of most interest, however, is the implication of Jones's (1995*a*) final production parameter restrictions for the rate of growth in per capita output. In the general two sector growth model, *per capita* rates of growth of physical capital, output and consumption are given by:

$$g_y = \frac{[\{\sigma_H - (1-\sigma_K)\}(1-\eta_A) + (\eta_H + \eta_K)\sigma_A]}{[(1-\sigma_K)(1-\eta_A) - \eta_K \sigma_A]} n \tag{3.28}$$

By introducing the restriction $\sigma_H = (1 - \sigma_K) = \sigma_A = \sigma$, together with $\eta_K = 0$, Jones (1995*a*) obtains a long run growth in per capita output that is determined by relative structural elasticities in the R&D sector, as encapsulated in the parameter ψ.

Jones (2002) provides estimates of ψ for the United States (U.S.) economy, ranging from a low value of 0.05 to a high value of 1/3. These estimates suggest a long run rate of growth of per capita output that is less than a third of the rate of population growth, from which Jones (2002) draws two inferences. Firstly, growth rates in the U.S. for the last century are not indicative of steady state. Secondly, we should anticipate a future slowdown as the economy transits to a long run rate of growth that is *lower* than the rate of population growth.

Would alternative parameter restrictions suggest a less pessimistic outlook for long run growth of the U.S. economy? The restriction $\sigma_H > (1 - \sigma_K) = \sigma_A = \sigma$, by Corollary 1 to Proposition 1, is sufficient for positive *per capita* growth in output:

$$g_y = (\chi + \psi - 1)\, n \tag{3.29}$$

where $\chi = \sigma_H / (1 - \sigma_K)$. Since $\chi > 1$, our re-parameterization of Jones (1995*a*) yields a higher long run rate of growth in per capita output. Moreover, if $(\chi - 1) > (1 - \psi)$, a restriction which does not violate any of the conditions for positive and balanced growth[18], the economy will transit to a long run rate of growth that is *higher* than the rate of population growth.

3.6 Seven Principles for Model Construction

We bring together the formal propositions of this paper in a set of principles for constructing a model of either endogenous or semi-endogenous growth. These principles were derived using a generalized model where physical capital and two types of knowledge grow endogenously and the optimization problem is that of a central planner. Because the principles are general, they can be applied to any R&D-based growth model with specific microfoundations and decentralized decision making. They are equally applicable to models with two or three endogenous factors; homogeneous labor or heterogeneous labor. Recognizing the two broad approaches in the literature, we tailor the principles to the treatment of population as a growing or static factor.

If you want to allow for the possibility that population can grow at an exogenous rate, we propose five principles for constructing a model:

1. To obtain strictly positive growth in output, capital, consumption, technology and human capital that depend essentially on population growth, it is necessary and sufficient to assume diminishing marginal returns to each input in its productive sector and a relationship between the structural elasticities such that $|A| > 0$.

[18] $|A| > 0$ is the only condition that relates relative factor shares in the R&D sector to relative factor shares in the final output sector: $|A| > 0 \Rightarrow \frac{1-\sigma_K}{\sigma_A} > \frac{\eta_K}{1-\eta_A}$.

2. In terms of obtaining a *balanced* growth path, Cobb-Douglas production functions have the benefit of allowing for varying degrees of returns to scale. Constant returns to scale must be assumed if you use CES production functions.

3. For strictly positive *per capita* growth in the economy, it is sufficient to assume increasing returns to scale to physical capital and physical labor in the production of final output.

4. Strictly positive balanced growth may arise without population growth if growth in knowledge (either non-rivalrous ideas or human capital) asymptotes to a positive constant. CES technology is one such production technology for which this is possible.

5. The notions of diminishing marginal returns and semi-endogenous growth are logically independent. Given our fourth principle, diminishing marginal returns to knowledge is consistent with both endogenous growth (i.e. growth in the economy without exogenous population growth) and semi-endogenous growth.

The first three principles are known, albeit for two sector R&D-based growth models, but are often hidden behind the complexity of the decentralized solutions to these models. With the introduction of additional sectors, such models become increasingly complex. This paper demonstrates the generality of these principles to multi-sector growth models, even when we allow for heterogenous labor. Even if these principles are known for two sector R&D-based growth models, they are not always fully utilized. This is demonstrated in the previous section, by applying the third principle to a well-known model to achieve a more general result.

The last two principles are based on new results obtained in this paper. Introducing CES technology to a knowledge accumulation equation to obtain endogenous growth is more than a mathematical peculiarity. We recommend that CES technology be introduced where the inputs are, a priori, highly substitutable. A CES combination of human capital and physical capital in the accumulation of human capital, as analyzed in Section 3.4.3 is such a plausible application.

If you start out with the assumption that population is static, we propose two principles:

1. To obtain strictly positive growth in output, capital, consumption, technology and human capital, it is necessary to assume $|A| = 0$.
2. Diminishing marginal returns to existing knowledge in the accumulation of knowledge is consistent with strictly positive growth. Sectoral linearity is sufficient for $|A| = 0$. However, constant returns to scale to growing factors is also sufficient. Moreover, we can move away from constant returns to scale, so long as the degree of returns to scale vary across sectors such that the column or row vectors of matrix A are linearly dependent.

These two principles are also based on new results in this paper and shed new light on early R&D-based growth models. Being able to construct such models without resorting to sectoral linearity, theorists may rediscover early endogenous growth theory in their endeavour to establish strictly positive long run economic growth with a static population.

3.7 Conclusion

Positive and balanced growth in an economy cannot be obtained without knife edge conditions, whether growth is endogenous or semi-endogenous. By construction of a general three sector growth model, these conditions can be expressed in terms of a matrix of structural elasticities and tailored to the treatment of population as a growing or static factor.

If population grows at a positive, exogenous rate, as in semi-endogenous growth models, diminishing marginal returns and a positive determinant of structural elasticities are necessary and sufficient for positive growth in real variables of the models. However, diminishing marginal returns is consistent with endogenous growth if growth in human capital asymptotes to a positive constant. If population is static, the necessary condition for positive growth is singularity of the matrix, which is achieved by imposing either sectoral linearity, constant returns to scale to growing factors or returns to scale that vary across sectors such that vectors of the elasticity matrix are linearly dependent.

Since our general three sector growth model allows for heterogeneity of knowledge and labor, the conditions are universal. That is, they apply to growth models with either R&D or human capital accumulation or both and either physical labor or human capital augmented labor or both.

The key conditions challenge the convention in growth theory that diminishing marginal returns is synonymous with semi-endogenous growth and linearity is synonymous with endogenous growth.

Semi-endogenous growth models are premised on the possibility that population grows at a positive rate. However, endogenous growth, or growth independent of the growth rate of population, may arise in such models if growth in human capital asymptotes to a positive constant. This is consistent with diminishing marginal returns to human capital as human capital accumulates. Admittedly, the tendency of growth in human capital to a positive constant implies linearity, albeit asymptotic.

Perhaps the more powerful result of this paper pertains to the early endogenous growth models which assume population is static. We establish that diminishing marginal returns to knowledge in the accumulation of knowledge is consistent with singularity of the matrix of structural elasticities, the only necessary condition for strictly positive rates of growth in real variables. Endogenous growth models, premised on the assumption of zero population growth, can be solved for a strictly positive growth rate in the economy without imposing the restriction of sectoral linearity.

Thus, whether population is growing or static, strictly positive long run economic growth, driven by knowledge accumulation, can be obtained under the more palatable assumption of diminishing marginal returns to knowledge in the accumulation of knowledge.

All this suggests that concerns of zero long run economic growth due to forward projections of zero population growth in the world's hub of R&D may be misplaced. Such fears generate pressure on public policy to boost fertility and immigration rates. To the extent that, under reasonable assumptions, the long run growth rates of economies engaging in R&D may be strictly positive without population growth, these policy reforms are, at best, innocuous.

Because the knife edge conditions for positive growth along a balanced growth path are in terms of structural elasticities, we provide a neat and concise framework for analyzing the long run central planner solutions for endogenous and semi-endogenous growth models alike. CES production function are sometimes characterized as cumbersome and difficult to manipulate (Sato 1987). However, using our framework, we solve for the growth rates of all real variables along a balanced growth path in a three sector economy where CES technology describes the accumulation of human capital in less than one page. Similarly, we solve a central planner version of Funke & Strulik (2000) from a single matrix system and four optimality conditions.

A simple, unified framework for analyzing the solutions along a balanced growth path has several benefits. Firstly, we can apply our conditions to well-known models to achieve a more general result, such as, our reparameterisation of Jones (1995*a*). Secondly, we can select the most realistic application of conditions to construct original models to address empirical anomalies, such as, our introduction of CES technology to human capital accumulation to establish the result that growth in the economy does not require growth in population.

All the models presented in this paper, whether original or central planner versions of existing models, can be given microfoundations, and in each case the equilibrium growth rates in the corresponding decentralized economy can be derived. It is worth noting that growth rates derived for a corresponding decentralized economy differ only by the introduction of terms, such as a monopoly markup, that capture the negative spillovers that a central planner internalizes.

We can improve the analysis of our general three sector growth model with a static population by formally defining the relationship between returns to scale across sectors, as implied by equation (3.16). Mulligan & Sala-i-Martin (1993) and Rebelo (1991) analyze a similar relationship for two sector endogenous growth. We may also flesh out the stability of the system by reference to the first order optimality conditions and analyze the transitional dynamics.

As it is straightforward to generalize conditions in terms of determinants to higher dimensions, the model is readily extended along the lines suggested by Papageorgiou (2003), which allows for technological imitation in addition to innovation.

Part IV

Non-Scale Growth - Essay 3

Chapter 4

Conditions for Non-Scale Growth

4.1 Introduction

Diminishing marginal returns to the existing stock of knowledge in the creation of new knowledge is a hallmark of non-scale models of research and development (R&D)-based growth. This paper challenges the conventional wisdom that (non-)linear knowledge accumulation and (non-)scale growth are synonymous.

Romer's (1990) seminal model of R&D-based growth predicts the long run growth rate of the per capita output is increasing in the scale of the economy, typically measured by population size. Jones (1995*a*) observes empirical trends of the past century do not support this prediction and identifies an arbitrary assumption, linearity in R&D, as the source of the prediction. An alternative assumption of diminishing marginal returns to ideas implies, ceteris paribus, the growth rate of ideas will fall over time. In long run equilibrium, an increasing population provides the additional R&D effort needed for the stock of ideas, and output per capita, to grow a positive, constant rate.

Early non-scale models of R&D-based growth (Jones (1995*a*), Kortum (1997) and Segerstrom (1998)) predict strictly positive population growth is essential to long run economic growth. In response to an impending slowdown in population growth[1], a second generation of non-scale growth models establish that the economy may grow independent of population growth. Modelling two aspects of R&D, Young (1998), Dinopolous & Thompson (1998), Peretto (1998) and Li (2000) predict that strictly positive population growth is conducive but non-essential to long run economic growth. Modelling R&D and the accumulation of embodied knowledge, Dalgaard & Kreiner (2001) and Strulik (2005) predict that strictly positive population growth is not only non-essential but also detrimental to long run economic growth.

Hall & van Reenen (1999) estimate an additional dollar of tax benefits stimulates an additional dollar of R&D expenditure. Unlike early non-scale growth models, second generation non-scale growth models maintain the potency of such government policy to promote long run growth.

The empirical and policy relevance of second generation non-scale growth models warrants careful examination and thorough understanding of their general properties. The primary objective of this paper is to establish the general conditions in R&D-based models of non-scale growth that ensure strictly positive long run economic growth does not essentially depend on strictly positive population growth.

The complexity of second generation non-scale growth models in a decentralized setting obscures the conditions for strictly positive equilibrium growth. A recent paper

[1] United Nations (2005) projects population growth in the OECD, the world's hub of R&D, will approximate zero for the next fifty years. In fact, projections indicate negative population growth may arise after 2030.

(Strulik 2005) comprises 48 equations, not including those contained in the appendix. Li (2000) shows that long run economic growth is an additively separable function of population growth if the returns to existing ideas in the two aspects of R&D are zero and constant, respectively. Whilst consideration of Li (2000) and other specific examples of second generation non-scale growth models reveals a non-scale growth model need not feature non-linear knowledge accumulation, a clear cut distinction of these notions is missing, primarily due to the lack of a simple, unified framework within which to analyze these non-scale growth models.

Eicher & Turnovsky (1999) makes an important contribution to our understanding of non-scale models of R&D-based growth. Abstracting from the microeconomic foundations of R&D and modelling the decision making of a central planner in a generalized two-sector non-scale growth model, Eicher & Turnovsky (1999) assume all factors of production are necessary for production in both sectors and the matrix of structural elasticities is non-singular. Under these restrictions, Eicher & Turnovsky (1999) establish diminishing returns to the existing stock of technology in R&D as one of the conditions that are necessary and sufficient for strictly positive equilibrium growth rates in their generalized model.

We aim to establish conditions for strictly positive, non-scale growth that are universal to a class of models broader than that allowed by Eicher & Turnovsky's (1999) restrictions. Since the issue of non-scale growth pertains to the long run, we focus on the balanced growth equilibrium of an economy.

We observe that second generation non-scale growth models all work in a similar way. Each model adds a second dimension of knowledge accumulation to a seminal non-scale model of R&D-based growth. The second type of knowledge may be embodied or disembodied.

This paper introduces a generalized three-sector non-scale growth model and extends Eicher & Turnovsky (1999) both by relaxing the assumption that all factors are necessary for production in all sectors and by allowing the matrix of structural elasticities to be either singular or non-singular. With these extensions, we prove the following assertions. First, provided the matrix of structural elasticities is non-singular, Eicher & Turnovsky's (1999) conditions are necessary and sufficient for strictly positive non-scale equilibrium growth even when not all factors are necessary for the production of all goods. Second, single input linearity in one dimension of knowledge accumulation, implying a singular matrix of structural elasticities, is sufficient for strictly positive non-scale equilibrium growth which does not essentially depend on population growth. Third, following from our second assertion, Eicher & Turnovsky's (1999) conditions are, in general, sufficient but not necessary for strictly positive non-scale equilibrium growth. Fourth, the notions of non-linear knowledge accumulation and non-scale growth are logically independent.

The following section outlines the model and summarizes the conditions for strictly positive equilibrium growth rates in a series of propositions. Despite the technical nature of some of these propositions, our results have practical significance. Section 4.3 explores the case of embodied knowledge. The two examples demonstrate how our general framework is useful for gaining insight into existing models and developing new ones. Section 4.4 establishes the logical independence of non-linear knowledge accumulation and non-scale growth. In light of this result, section 4.5 reviews empirical evidence and some conventions in model construction. Section 4.6 concludes.

4.2 General Model of Non-Scale Growth

Growth is non-scale if the long run growth rate of per capita output does not vary with the size of the economy as measured by its population. Diminishing returns to the existing stock of ideas in R&D implies the output of new ideas (R&D) will less than double whenever the existing stock of ideas doubles. In this paper, non-linearity refers to diminishing returns.

Let Y denote the output of the final good, K the stock of physical capital and L the population (labor force), which grows at the exogenous rate $n = \dot{L}/L$. A denotes the stock of non-rivalrous knowledge, as measured by the existing *variety* of intermediate goods. Q denotes either an alternative stock of non-rivalrous knowledge (such as, the existing *quality* of intermediate goods) or rivalrous knowledge (specifically, embodied *human capital*).

Consider the following general three-sector production structure:

$$Y = F\left(A, (1-\alpha_A-\alpha_Q)K, q_Y Q, (1-l_A-l_Q)L\right) \tag{4.1a}$$

$$\dot{A} = J\left(A, \alpha_A K, q_A Q, l_A L\right) \tag{4.1b}$$

$$\dot{Q} = H\left(A, \alpha_Q K, q_Q Q, l_Q L\right) \tag{4.1c}$$

where α_i and l_i $(i = Y, A, Q)$ are the fractions of physical capital and labor, respectively, allocated to sector i. Equation (4.1c) may represent either human capital accumulation or quality R&D. In the case of human capital accumulation, $\sum_i q_i = 1$ if embodied knowledge is rival in use across all three sectors. In the case of R&D, $q_i = 1 \forall i$, since this type of knowledge is non-rivalrous in use.

The representative agent of the economy has intertemporal utility of isoelastic form:

$$\int_0^\infty e^{-\rho t} \frac{c^{1-\theta}}{1-\theta} dt \quad \rho > 0; \theta > 0$$

where c denotes consumption per capita. In the absence of depreciation, physical capital accumulates as a residual after aggregate consumption needs have been met:

$$\dot{K} = Y - C \tag{4.1d}$$

The central planner chooses consumption, and the fractions of labor and capital employed in each sector so as to maximize intertemporal utility of the representative agent subject to the production and accumulation constraints, equations (4.1a) - (4.1d). For the purposes of this paper, we note that the following discussion is premised on sectoral allocations of factors that are strictly positive and constant, as required for a balanced growth equilibrium.

The balanced growth rates of the real variables (Y, K, A, Q) are obtained by total differentiation of the production functions (4.1a) - (4.1c), noting that constant growth rates requires $g_Y = g_K$,[2] $g_{\dot{A}} = g_A$ and $g_{\dot{Q}} = g_Q$. This leads to the following system of three linear equations:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_Q \\ -\eta_K & (1-\eta_A) & -\eta_Q \\ -\omega_K & -\omega_A & (1-\omega_Q) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} \sigma_L n \\ \eta_L n \\ \omega_L n \end{bmatrix} \tag{4.2}$$

[2]The growth rate in physical capital, given by $g_K = \frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{Y}\frac{Y}{K}$, is constant if Y,K and C grow at the same rate.

where $\sigma_i \equiv F_i i/F$, $\eta_i \equiv J_i i/J$ and $\omega_i \equiv H_i i/H$; $i = K, A, Q, L$ denote the structural elasticities in the production, variety R&D and human capital accumulation (or quality R&D) sectors, respectively and $n > 0$. The structural elasticities are not necessarily constant. Non-linearity, or diminishing marginal returns to knowledge, in variety R&D and quality R&D (or human capital accumulation) correspond to $\eta_A < 1$ and $\omega_Q < 1$, respectively.

Dropping from equations (4.1a) - (4.1d), both equation (4.1c) and Q as an input, yields the system of two linear equations analyzed by Eicher & Turnovsky (1999) and Christiaans (2004):

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A \\ -\eta_K & (1-\eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma_L n \\ \eta_L n \end{bmatrix} \tag{4.3}$$

Both systems (4.2) and (4.3) are non-homogeneous, $Ax = d$, in matrix notation, provided $\sigma_L > 0$ and/or $\eta_L > 0$ and/or $\omega_L > 0$.

For a non-homogeneous system of equations

$$\textstyle\sum_{j=1}^{m} a_{ij}x_j = d_i \quad (i = 1, ..., m) \tag{4.4}$$

where $a_{ij} < 0$ for all $i \neq j$; $a_{ii} > 0$ for all i and $|A| = |a_{ij}| \neq 0$, Hawkins & Simon (1949) prove that a necessary and sufficient condition that the x_i satisfying (4.4) be all strictly positive is that all principal minors of the matrix $\|a_{ij}\|$ be strictly positive.

In order to apply the Hawkins-Simon conditions for a strictly positive solution, Eicher & Turnovsky (1999) impose the restrictions $|A| \neq 0$ and $\sigma_i > 0, \eta_i > 0 \forall i$. Under these restrictions, Eicher & Turnovsky (1999) establish $\sigma_K < 1$, $\eta_A < 1$ and $|A| > 0$ as necessary and sufficient conditions for the equilibrium growth rates in (4.3) to be strictly positive, where the two conditions $\sigma_K < 1$ and $|A| > 0$ are provided for by the Hawkins-Simon conditions, and together imply $\eta_A < 1$.

Put simply, Eicher & Turnovsky (1999) establish that, under certain restrictions, non-linearity in R&D and a strictly positive determinant of structural elasticities are essential to solving a non-scale growth model for strictly positive equilibrium growth rates. However, few existing non-scale growth models comply with the certain restrictions.

The restriction $\sigma_i > 0, \eta_i > 0 \forall i$ implies all factors are essential for the production of both goods, final output and R&D. Excepting the Cobb-Douglas hybrid model (Eicher & Turnovsky (1999), p. 410), physical capital does not feature as an R&D input in most non-scale growth models. The restriction $|A| \neq 0$ excludes the linear R&D technology featuring in more recent non-scale growth models.

In the following sections, we relax these restrictions to establish, first, that Eicher & Turnovsky's (1999) conditions apply to first generation non-scale growth models in which physical capital is not an R&D input, and, second, Eicher & Turnovsky's (1999) conditions are sufficient but not necessary for strictly positive equilibrium growth rates in any non-scale growth model.

4.2.1 Hawkins-Simon Conditions when $a_{ij} = 0$ for some $i \neq j$

Consider a non-homogeneous system of two equations, in matrix notation, $Ax = d$:

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{4.5}$$

where $|A| \neq 0$, $a_{ii} > 0$; $[d_i > 0]$ for all i, assumptions underlying the Hawkins-Simon conditions. $a_{12} < 0$ and $a_{21} = 0$ satisfy a weaker assumption of $a_{ij} \leq 0$ for all $i \neq j$.

Corollary 4.1 (to Hawkins & Simon's (1949) Theorem) *A necessary and sufficient condition that the x_i satisfying (4.4) be all strictly positive for any $a_{ij} \leq 0$ for all $i \neq j$ is that all principal minors of the matrix $\|a_{ij}\|$ be strictly positive.*

Proof. Since conditions in terms of determinants are generalizable to higher dimensions, we prove this corollary by reference to the two dimensional system in (4.5). Substituting from the second equation into the first equation in (4.5) yields

$$a_{22}x_2 = d_2 \tag{4.6}$$

$$a_{11}a_{22}x_1 = a_{22}d_1 - a_{12}d_2 \tag{4.7}$$

Suppose (x_1, x_2) satisfying (4.6) and (4.7) are positive, but not all the principal minors of A in (4.5) are positive. Specifically, the first principal minor is positive, but the second principal minor is non-positive: $|A_1| = a_{11} > 0; |A_2| = |A| = a_{11}a_{22} \leq 0$. We show that $a_{11}a_{22} \leq 0$ implies an inconsistency. Together, $a_{11} > 0$ and $a_{11}a_{22} \leq 0$ imply $a_{22} \leq 0$, a contradiction of the underlying assumption $a_{ii} > 0$ for all i. From the solutions in (4.6) and (4.7), if $x_1 > 0$ and $x_2 > 0$, then $a_{22} \leq 0$ contradicts the underlying assumptions $[d_i > 0]$ for all i and $a_{12} < 0$. Thus, that all principal minors of matrix A be positive is a necessary condition for (x_1, x_2) satisfying (4.5) to be all positive.

To prove that all principal minors of matrix A be positive is a sufficient condition for (x_1, x_2) we show that if $|A_1| = a_{11} > 0; |A_2| = |A| = a_{11}a_{22} > 0$ then either $x_1 \leq 0$ or $x_2 \leq 0$ implies a contradiction. Together, $a_{11} > 0$ and $a_{11}a_{22} > 0$ imply $a_{22} > 0$.

For $x_2 \leq 0$, the equality in (4.6) requires $d_2 \leq 0$, contradicting the underlying assumption $[d_i > 0]$ for all i.

■

When $|A| \neq 0$, system (4.3) has the solution

$$g_Y = g_C = g_K = \frac{[\sigma_L(1-\eta_A) + \eta_L\sigma_A]}{|A|}n \tag{4.8a}$$

$$g_A = \frac{[(1-\sigma_K)\eta_L + \eta_K\sigma_L]}{|A|}n \tag{4.8b}$$

In order to apply the Hawkins-Simon conditions for a strictly positive solution, Eicher & Turnovsky (1999)

> ... assume that all three factors of production are necessary for the production of both goods, so that all elasticities are strictly positive. (p.400)

By Corollary 1, we need only assume that structural elasticities are non-negative in order to apply the Hawkins-Simon conditions for a strictly positive solution. Non-negative structural elasticities allow for the possibility that not all factors are used in some sectors.

For special cases, Eicher & Turnovsky (1999) show that their conditions for a strictly positive solution apply when not all factors are used in some sectors. For instance, Jones (1995*a*) assumes $\eta_K = 0$, but is used as a benchmark. In this sense, Corollary 1 and its proof provides completeness. As the number of sectors increase, the more likely not all factors are used in some sectors. By Corollary 1, we need not predicate conditions for a positive solution on the assumption that all four factors are necessary for production in all three sectors of our general model.

We assume $\sigma_i > 0$, $\eta_i \geq 0$ and $\omega_i \geq 0$ $\forall i$, meaning all factors of production are necessary for the production of the *final good*, but some factors of production may not be used in variety R&D and quality R&D (or human capital accumulation). Without the first assumption, modelling the accumulation of factors is inconsequential to the long run growth rate of the economy.

We impose no restrictions on $|A|$, allowing for the coefficient matrix A to be non-singular or non-singular.

4.2.2 Positive Solution when $|A| = |a_{ij}| = 0$

Define an augmented matrix, A_d, comprising A in the first columns and d in the last column. It is a standard result of linear algebra that a necessary and sufficient condition for a linear system of equations to have at least one solution is that the rank[3] of the coefficient matrix equals the rank of the augmented matrix: $Ax = d$ has a solution $\Leftrightarrow r(A) = r(A_d)$. When the common rank of A and A_d equals the number of equations, $|A| \neq 0$. When the common rank A and A_d is less than the number of equations, $|A| = 0$. In either case, the system of equations has a solution.

Two Equation System

Following on from our discussion of Eicher & Turnovsky's (1999) conditions for a positive solution in the previous section, consider the augmented matrix for the two equation system (4.3):

$$A_d = \begin{bmatrix} (1-\sigma_K) & -\sigma_A & \sigma_L n \\ -\eta_K & (1-\eta_A) & \eta_L n \end{bmatrix} \tag{4.9}$$

When $|A| \neq 0$, $r(A) = r(A_d) = 2$ and system (4.3) has a unique solution which, as per the Hawkins-Simon conditions, is strictly positive iff $|A| > 0$, $\sigma_K < 1$ and $\eta_A < 1$, where

[3]The rank of a matrix is equal to the order of the largest minor that is different from 0.

the final condition is implied by the first two conditions. These conditions are *sufficient* for a positive solution to system (4.3). When the stock of ideas is endogenously determined through R&D and the labor force is growing, diminishing marginal returns to the stock of ideas in R&D ($\eta_A < 1$) implies positive growth in the stock of ideas and output.

Intuitively, diminishing returns in the stock of ideas requires increasing effort in R&D for the stock of ideas to grow. This increasing effort can come from other inputs such as the number of researchers. While the fraction of inputs allocated to R&D is constant along a balanced growth path, strictly positive population growth satisfies the increasing R&D effort needed for strictly positive growth in the stock of ideas and the overall economy.

However, the proposition that these conditions are *necessary* for a positive solution to system (4.3) is conditional on the assumption $|A| \neq 0$. System (4.3) also has a solution when $|A| = 0$, provided $r(A) = r(A_d)$. Moreover, we can show that a positive solution exists in this case.

Proposition 4.1 *A solution to a non-scale growth model exists when* $|A| = 0$ *only if a sector features single input linear technology*

Proof. When $r(A) = r(A_d) = 1 < 2$, system (4.3) has a solution although one equation is superfluous to solving the system. Whereas $|A| \neq 0 \Leftrightarrow r(A) = r(A_d) = 2$, $|A| = 0$ is necessary but not sufficient for $r(A) = r(A_d) < 2$. To see this, constant returns to scale to endogenous factors in both sectors ($\sigma_K + \sigma_A = 1$; $\eta_K + \eta_L = 1$) yields $|A| = 0$. All elements of A are non-zero. The minor of A_d obtained by deleting either the first or second column is non-zero. Thus, $r(A) = 1 < r(A_d) = 2$ and the system has no solution. When $|A_2| = |A| = 0$ and $|A_{d2}| = 0$, $r(A) = r(A_d) < 2$ and the system has a solution. For both $|A| = 0$ and $|A_{d2}| = 0$, A_d must contain a row of zeroes. The implication is a single input linear technology.

As it is straightforward to generalize conditions in terms of minors to higher dimensions, this proof is applicable to system (4.2) resulting from our three sector model of non-scale growth. ■

Consider such a linear technology for the R&D sector: $\eta_K = \eta_L = (1 - \eta_A) = 0$. Assuming $\sigma_K < 1$, $r(A) = r(A_d) = 1 < 2$. The second equation in system (4.3) is superfluous, reducing to $g_A = \dot{A}/A = m$, where m is an exogenous positive constant. Differentiating $Y = F(A, (1 - k_A) K, (1 - l_A) L)$ with respect to time, noting that constant g_K requires that Y and K grow at the same rate yields the solution

$$g_Y = g_C = g_K = \frac{\sigma_A}{(1-\sigma_K)} m + \frac{\sigma_L}{(1-\sigma_K)} n \qquad (4.10)$$

$$g_A = m \qquad (4.11)$$

which is strictly positive iff $\sigma_K < 1$. All other variables in the above solution are strictly positive by assumption. This result provides alternative conditions *sufficient* for a positive solution to system (4.3), which we summarize with the following proposition:

Proposition 4.2 *$\sigma_K < 1$, $\eta_A = 1$ and $\eta_i = 0$ $\forall i \neq A$ is sufficient for positive growth of output, capital, consumption and knowledge in a two sector non-scale growth equilibrium.*

Intuitively, linear technology in R&D ($\eta_A = 1$) implies the stock of ideas will grow at a constant positive rate with no increased effort from other inputs of R&D. Recall that system (4.3) was derived under the requirement of constant growth rates. With strictly positive population growth, the number of researchers is increasing. Growth in the stock of ideas will be explosive if increasing amounts of other inputs are used in R&D. Hence, no researchers ($\eta_L = 0$) is an additional condition for a positive solution to system (4.3) in this case. Following the intuition for $\eta_A < 1$ in the previous case, diminishing marginal returns in final production ($\sigma_K < 1$) implies positive growth in output, physical capital and consumption. With positive growth in physical capital, no physical capital in R&D ($\eta_K = 0$) is also a condition for a positive solution to system (4.3).

Corollary 4.2 (to Proposition 4.2) *Iff, further, $\sigma_A + \sigma_K + \sigma_L > 1$ the growth rate of per capita output in this two sector non-scale growth equilibrium will be positive.*

This additional condition follows immediately from (4.10): $g_y = g_Y - n > 0 \Leftrightarrow \sigma_A > [1 - (\sigma_K + \sigma_L)]\, n/m$. As we will discuss, the type of single input linear technology viable for R&D is asymptotic, in which case $m > n$, simplifying the condition to $\sigma_A > 1 - (\sigma_K + \sigma_L) > [1 - (\sigma_K + \sigma_L)]\, n/m$.

By the corollary, increasing returns to scale in the production of final output is necessary and sufficient for positive *per capita* growth of output, given the conditions of Proposition 4.2. For their positive solution, Eicher & Turnovsky (1999) establish increasing returns to scale in the two private factors, physical capital and labor, in the production of final output ($\sigma_K + \sigma_L > 1$), suffices for positive per capita growth of output, independent of further production conditions in the R&D sector. The sufficiency of increasing, or even constant, returns to scale in the two private factors clearly holds for our positive solution. In contrast, the long run growth rate of the economy is positive with decreasing returns to scale in the two private factors, provided the decreasing returns are more than offset by the marginal return to ideas used in production of final output.

The interesting point of the corollary is that positive growth in the economy in a non-scale growth equilibrium provided by a single input linear technology requires less restrictions on individual marginal returns in the production of final output.

Unfortunately, in a two sector R&D-based growth model, the applications of linear technology are limited. Whereas human capital is rivalrous, ideas are a public good. As such, with a simple linear technology, human capital accumulation remains endogenous ($\dot{H} = mh_H H$), but R&D reduces to an exogenous accumulation of ideas ($\dot{A} = mA$). For the R&D process to remain endogenous, we require a technology where growth in the stock of ideas asymptotes to $m > n$. An example of such a technology is Dalgaard & Kreiner's (2003) Constant Elasticity of Substitution (CES) R&D equation with elasticity of substitution greater than one. This example, whilst technically correct, lacks intuitive appeal. Researchers work with existing ideas to create new ideas, and it is hard to conceive that ideas could be substituted for researchers to the point where R&D requires no researchers. This motivates the inclusion of a third sector featuring sectoral linearity.

Three Equation System

Consider the augmented matrix for the three equation system (4.2):

$$A_d = \begin{bmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_Q & \sigma_L n \\ -\eta_K & (1-\eta_A) & -\eta_Q & \eta_L n \\ -\omega_K & -\omega_A & (1-\omega_Q) & \omega_L n \end{bmatrix} \tag{4.12}$$

When $|A| \neq 0 \Leftrightarrow r(A) = r(A_d) = 3$, system (4.2) has a unique solution. As per the Hawkins-Simon conditions, this solution is positive iff $|A| > 0$ and $\sigma_K < 1$, $\eta_A < 1$

and $\omega_Q < 1$, where the final condition is implied by the first three conditions.[4] These conditions are *sufficient* but not necessary for *a* positive solution to system (4.2).

Intuitively, $|A| > 0$ means the three factor accumulation sectors are capable of generating more than their individual growth needs, implying the growth rates produced by this group of sectors are positive. Diminishing returns to each factor in its accumulation sector ($\sigma_K < 1$, $\eta_A < 1$ and $\omega_Q < 1$) implies each endogenous growth rate is proportional to the population growth rate. With diminishing returns, increasing effort from other factors employed in the accumulation sector implies positive growth in the accumulating factor. Along a balanced growth path, sectoral allocations of factors are constant. Accordingly, positive growth in the only exogenous factor, population, provides the increasing effort which implies positive growth in each endogenously accumulating factor.

When $|A_3| = |A| = 0$ and $|A_{d3}| = 0 \Leftrightarrow r(A) = r(A_d) < 3$, system (4.2) has a solution, although there exist one or more superfluous equations. Consider the case where $|A_2| = (1 - \sigma_K)(1 - \eta_A) - \eta_K \sigma_A \neq 0$, implying $r(A) = r(A_d) = 2 < 3$. Consider a production technology for the knowledge accumulation sector, $\dot{Q} = H(.)$, such that $\omega_K = \omega_A = (1 - \omega_Q) = \omega_L = 0$. The third equation in system (4.2) is superfluous, reducing to $g_Q = \dot{Q}/Q = q_Q m$ where m is an exogenous positive constant and $q_Q < 1$ or $q_Q = 1$, depending on whether Q is rival or non-rival in use, respectively. If Q is rival in use across all sectors, as may be the case for human capital, the fraction of human capital allocated to human capital accumulation, q_Q, is constant along a balanced growth path, as required

[4] $|A| = (1 - \sigma_K)(1 - \eta_A)(1 - \omega_Q) - (1 - \sigma_K)\omega_A \eta_Q - \sigma_A[\eta_K(1 - \omega_Q) + \omega_K \eta_Q] - \sigma_H[\eta_K \omega_A + \omega_K(1 - \eta_A)]$

for constant g_Q. System (4.2) reduces to a system of two equations:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A \\ -\eta_K & (1-\eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma_Q q_Q m + \sigma_L n \\ \eta_Q q_Q m + \eta_L n \end{bmatrix} \tag{4.13}$$

which has the solution

$$g_Y = g_C = g_K = \frac{[\sigma_Q(1-\eta_A)+\eta_Q\sigma_A]}{|A_2|}q_Q m + \frac{[\sigma_L(1-\eta_A)+\eta_L\sigma_A]}{|A_2|}n \tag{4.14a}$$

$$g_A = \frac{[(1-\sigma_K)\eta_Q+\eta_K\sigma_Q]}{|A_2|}q_Q m + \frac{[(1-\sigma_K)\eta_L+\eta_K\sigma_L]}{|A_2|}n \tag{4.14b}$$

where the above solution is positive iff $|A_2| = (1-\sigma_K)(1-\eta_A) - \eta_K\sigma_A > 0$ and $\sigma_K < 1$, together implying $\eta_A < 1$. With the exception of η_i, all other variables in the above solution are strictly positive by assumption. No further assumptions are required to obtain positive growth of output, although we may reasonably assume that $\eta_i > 0$ for *any but not necessarily all* $i \neq A$ to ensure positive growth in ideas. This result provides alternative conditions *sufficient* for a positive solution to system (4.2), which we summarize with the following proposition:

Proposition 4.3 *$\sigma_K < 1$, $|A_2| > 0$ (and $\eta_A < 1$), $\omega_Q = 1$ and $\omega_i = 0$ $\forall i \neq Q$ is sufficient for positive growth of output, capital, consumption and knowledge in a three sector non-scale growth equilibrium.*

Intuitively, $|A_2| > 0$ means the physical capital accumulation and variety R&D sectors are capable of generating more than their individual growth needs, implying the growth rates produced by this group of sectors are positive. Linear technology in the third sector ($\omega_Q = 1$ and $\omega_i = 0$ $\forall i \neq Q$) implies all corresponding minors are zero. This means the third sector, quality R&D or human capital accumulation, is just capable of satisfying its growth needs and grows independent of the rest of the group, when the use of Q in accumulating knowledge is non-rival in use, at the rate $m > 0$.

As per the intuition when $|A| > 0$, diminishing returns to physical capital and variety of ideas in their respective sectors ($\sigma_K < 1$ and $\eta_A < 1$) implies both endogenous growth rates are proportional to independent growth in quality ideas (or human capital), on the one hand, and population, on the other.

Corollary 4.3 (to Proposition 4.3) *Iff, further,* $\left[\frac{\eta_Q+\eta_K+\eta_L}{(1-\eta_A)}\right]\sigma_A + \sigma_Q + \sigma_K + \sigma_L > 1$ *the growth rate of per capita output in this three sector non-scale growth equilibrium will be positive.*

This additional condition follows immediately from (4.14a) and the assumption $m > n$.

By the corollary, a relationship between returns to scale in the production of final output and returns to scale in variety R&D is necessary and sufficient for positive *per capita* growth of output. The bracketed term is less than, equal to or greater than one depending on whether variety R&D exhibits decreasing, constant or increasing returns to scale, respectively. Increasing returns to scale in both variety R&D and the production of final output suffice for positive per capita growth, but so to do offsetting returns to scale. For instance, constant or decreasing returns to scale in variety R&D simplifies the condition to increasing returns to scale in the production of final output: $\sigma_A + \sigma_Q + \sigma_K + \sigma_L \geq [.]\,\sigma_A + \sigma_Q + \sigma_K + \sigma_L > 1$.

Constant returns to scale to factors other than variety of ideas in the production of final output ($\sigma_Q + \sigma_K + \sigma_L = 1$) clearly suffices for positive per capita growth of output, independent of further conditions in the variety R&D sector. First and second generation non-scale growth models, alike, typically impose this restriction, which first featured in Romer's (1990) scale growth model.

The interesting point of the corollary is that a non-scale growth model featuring linearity in a third sector generates positive per capita growth with minimal restrictions only on the overall returns to scale in the first two sectors. Constant or decreasing returns to scale in one sector requires increasing returns to scale in the other, but further restrictions within a sector are unnecessary. The implication is that a second generation non-scale growth model could move away from the more stringent parameter restrictions of first generation non-scale growth models and still predict positive long run growth in the economy.

The above propositions state linear technology in knowledge accumulation (and $|A| = 0$) implies a positive non-scale growth equilibrium. $|A| > 0$ and diminishing marginal returns in knowledge accumulation also implies a positive, albeit different, non-scale growth equilibrium. Thus, a positive solution to a non-scale growth model implies neither diminishing marginal returns nor linear technology in knowledge accumulation. We summarize the role of non-linear knowledge accumulation in generating a positive non-scale growth equilibrium with the following proposition:

Proposition 4.4 *$|A| > 0$, $\sigma_K < 1$ and diminishing returns in knowledge accumulation ($\eta_A < 1$ in a two sector model; $\omega_Q < 1$ in a three sector model) are sufficient but not necessary for positive growth of output, capital, consumption and technology in a non-scale growth model.*

The positive non-scale growth equilibrium implied by a linear knowledge accumulation equation has particular appeal because population growth appears as an additively separable term (see (4.14)). This lessens the significance of an impending slowdown in population growth for long run growth in the economy.

The question remaining is which linear production technology describes knowledge accumulation in the third sector. If the knowledge is rival in its use across sectors, knowledge accumulation remains endogenous with a single input linear equation: $\dot{Q} = mq_Q Q$, where $q_Q < 1$. If the knowledge is non-rival in its use across sectors ($q_Q = 1$), knowledge accumulation remains endogenous with a multiple input technology where growth asymptotes to an exogenous rate.

Moreover, whether knowledge is disembodied or embodied has interesting implications for the sign of the additively separable population growth term. Existing R&D-based models of non-scale growth obtain equilibrium growth rates of the additively separable form in (4.14) by allowing for two aspects of R&D, variety of products and quality improvement, where quality R&D features linearity. The partial dependence of the long run growth rate of the economy (as measured by per capita output) is positive. In R&D-based models of non-scale growth featuring embodied knowledge, the long run growth rate of the economy is independent of or negatively dependent on population growth, depending on whether population is assumed constant or growing at a positive rate, respectively.

We explore the accumulation of embodied knowledge, or human capital, further, within our general framework.

4.3 Example: Human Capital Accumulation

4.3.1 Single Input Linear Equation ($\omega_K = \omega_A = \omega_L = 0; \omega_Q = 1$)

Consider an economy comprising three sectors with the following production technologies:

$$Y = A^{\sigma_A} K^{\sigma_K} \left[(1 - q_A - q_Q) Q\right]^{\sigma_Q} L^{\sigma_L} \tag{4.15a}$$

$$\dot{A} = A^{\eta_A} (q_A Q)^{\eta_Q} \tag{4.15b}$$

$$\dot{Q} = \psi q_Q Q \tag{4.15c}$$

where q_Q and q_A denote the fractions of human capital allocated to the accumulation of new human capital and R&D, respectively. All exponents are positive constants. The parameters satisfy $q_Q > n/\psi$. Because human capital is embodied, $Q = hL$, where h denotes human capital *per capita.* For the present we assume homogeneous labor, embodied with human capital, and set $\sigma_L = 0$.

Along a balanced growth path, positive growth rates in output, ideas and aggregate human capital are jointly determined by the system:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_Q \\ 0 & (1-\eta_A) & -\eta_Q \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_K \\ g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.16}$$

where $g_Q = q_Q \psi$. Recognizing q_Q is constant along a balanced growth path, system (4.16) reduces to a system of two equations:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A \\ 0 & (1-\eta_A) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} \sigma_Q q_Q \psi \\ \eta_Q q_Q \psi \end{bmatrix} \tag{4.17}$$

which is of the form (4.13). Notably, because knowledge is embodied in homogeneous labor, the additively separable population growth terms are absent. System (4.17) has the solution:

$$g_Y = g_C = g_K = \frac{\left[\sigma_Q (1-\eta_A) + \eta_Q \sigma_A\right]}{(1-\sigma_K)(1-\eta_A)} q_Q \psi \tag{4.18a}$$

$$g_A = \frac{\eta_Q}{(1-\eta_A)} q_Q \psi \tag{4.18b}$$

where the growth rates are both strictly positive, by Proposition 3, iff $\sigma_K < 1$ and $\eta_A < 1$ and fully endogenous, since q_Q is a choice variable, for which we can solve from the first order optimality conditions.

The long run growth rate of the economy is

$$g_y \equiv g_Y - n = \frac{\left[\sigma_Q (1-\eta_A) + \eta_Q \sigma_A\right]}{(1-\sigma_K)(1-\eta_A)} q_Q \psi - n \tag{4.19}$$

where, by Corollary 3, $\left[\eta_Q / (1-\eta_A)\right] \sigma_A + \sigma_Q + \sigma_K \geq 1$ is sufficient for a strictly positive rate of growth. For instance, with constant returns to scale in both production of final output and R&D, $\sum \sigma_i = 1$ and $\sum \eta_i = 1$, respectively, the long run growth rate of the economy simplifies to $g_y = q_Q \psi - n > 0$.

Interestingly, the long run growth rate of the economy is decreasing in the growth rate of population. Before explaining the intuition, it should be mentioned that this result is sensitive to the specification of household preferences. In our general model, the central planner maximizes intertemporal utility of consumption per capita for a representative household. If the central planner were to use the Benthamite criteria and maximize utility of consumption of all members of a household's dynasty, intertemporal utility would be discounted at the rate, $(\rho - n)$. In the case of Benthamite preferences, the long run growth rate of per capita output is independent of population growth.[5]

The *per capita* human capital accumulation equation provides the growth rate in *per capita* human capital, g_h. Although this equation is not needed to solve for the long run growth rate of the economy, its derivation illuminates why the long growth rate of the economy is decreasing in population growth.

Total differentiation of $Q = hL$ with respect to time yields

$$\dot{h} = \frac{\dot{Q}}{L} - nh = \psi q_Q h - nh \tag{4.20}$$

where the first term, $\psi q_Q h$, conceals a congestion effect in per capita human capital formation, $\dot{Q}/L$, specifically a larger population reduces the human capital acquired by the average person, and the second term, $-nh$, captures a depreciation effect of population growth, associated with bringing the skill level of uneducated newborns up to average level of the existing population. Equation (4.20) is the specification adopted in R&D-based non-scale growth literature (see, for example, Dalgaard & Kreiner (2001) and Strulik (2005)).

Equation (4.20) is reminiscent of the per capita physical capital accumulation equation of the neoclassical growth model. Lucas (1988) reminds us that

> human capital accumulation is a social activity, involving groups of people in a way that has no counterpart in the accumulation of physical capital. (p.19)

The depreciation term is absent in Lucas's (1988) original specification, which may be more appropriate considering inherited skill and passive learning within a family. If the second term of (4.20) is omitted, the growth rate of human capital, g_h, is $q_Q\psi$ instead of $q_Q\psi - n$, although the growth rates of all other variables along a balanced growth path are unaltered.

[5] See Dalgaard & Kreiner (2001) and Strulik (2005) for a detailed discussion of this case.

The intuition for population growth being neither necessary nor conducive for long run economic growth lies in the congestion effect. First, recall that growth in the overall economy is R&D driven. Referring to (4.15b), the aggregate stock of human capital (Q) takes over the role of population size as the source of increasing effort in R&D. As per the intuition accompanying Proposition 3, single input linearity in the endogenous accumulation of human capital implies g_Q, g_A and g_Y are proportional to a constant term, as per the first additively separable term in (4.14). Second, according to the congestion effect, the larger the population to be educated (L), ceteris paribus, the lower the quality of the average student (h). The congestion effect dissipates any scale effects from the size of the population so that growth in the aggregate stock of human capital, g_Q, is unaffected by population growth. Accordingly, the additively separable population growth term is absent from the growth rate of total output in (4.18). Because total output grows at a constant rate, independent of population growth, it follows that growth in *per capita* output is decreasing in population growth.

To sum up, single input linearity in the accumulation of knowledge implies population growth is not necessary for long run growth in per capita output. Embodiment of knowledge, and the associated congestion effect in per capita human capital production, implies population growth is not conducive for long run growth in per capita output.

4.3.2 Asymptotic Linearity ($\omega_Q \to 1$)

Two questions arise from the above analysis of human capital accumulation. First, single input linearity in the accumulation of embodied knowledge is open to the same criticism as Romer's (1990) equation for the accumulation of disembodied knowledge. Could we obtain the result that population growth is nonessential for long run economic growth if the accumulation of embodied knowledge uses multiple inputs and features diminishing marginal returns to knowledge?[6] Second, Dalgaard & Kreiner (2001) and Strulik (2005) note that the result that long run economic growth is decreasing in population growth is sensitive to the specification of household utility. Is the result sensitive to other key assumptions?

Consider an economy comprising three sectors with the following production technologies:

$$Y = A^{\sigma_A} [(1-\alpha_Q) K]^{\sigma_K} [(1-q_A-q_Q) Q]^{\sigma_Q} L^{\sigma_L} \tag{4.21a}$$

$$\dot{A} = A^{\eta_A} (q_A Q)^{\eta_Q} \tag{4.21b}$$

$$\dot{Q} = [(\phi_1 q_Q Q)^{\rho} + (\phi_2 \alpha_Q K)^{\rho}]^{1/\rho} \tag{4.21c}$$

where the constant elasticity of substitution between human capital and physical capital is $\epsilon = 1/(1-\rho) : \epsilon > 0, \epsilon \neq 0$ and the parameters satisfy $\phi_1 q_Q > g_K \geq n > 0$. We relax the assumption that final production employs homogeneous labor and allow $\sigma_L > 0$, where L denotes physical labor or raw skills that do not have to be taught.

Along a balanced growth path:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_Q \\ 0 & (1-\eta_A) & -\eta_Q \\ -(1-\omega) & 0 & (1-\omega) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} \sigma_L n \\ 0 \\ 0 \end{bmatrix} \tag{4.22}$$

[6] In Dalgaard & Kreiner (2001), both ideas and human capital are produced using units of final output which, in turn, comprises inputs, ideas and human capital, and features diminishing returns to human capital. By implication, human capital accumulation also features diminishing returns, but at the cost of restricting final production, R&D and human capital accumualtion to an identical production technology.

where, referring to the appendix, $\omega = [(q_Q\phi_1)/g_Q]^{\rho}$. When $\epsilon > 1$, $g_Q \to q_Q\phi_1$ and $\omega \to 1$. Thus, except for the asymptotic limit, human capital accumulation features diminishing marginal returns to the existing stock of human capital ($\omega < 1$).

The long run growth rate of the economy asymptotes to

$$g_y \to \bar{g}_y = \left[\frac{\sigma_A \eta_Q}{(1-\sigma_K)(1-\eta_A)} + \frac{\sigma_Q}{(1-\sigma_K)}\right] q_Q\phi_1 + \frac{\sigma_L - (1-\sigma_K)}{(1-\sigma_K)} n \tag{4.23}$$

where, by Corollary 3, $[\eta_Q/(1-\eta_A)]\,\sigma_A + \sigma_Q + \sigma_K + \sigma_L \geq 1$ is sufficient for a strictly positive rate of growth.

Thus, when human capital is highly substitutable for physical capital in the accumulation of human capital, long run economic growth asymptotes to a rate that does not essentially depend on population growth. Intuitively, population growth is essential to growth in physical capital. More brawn generates more final output, and thus more physical capital, the residual after consumption needs are met. If human capital is substituted for physical capital in human capital accumulation to the point where physical capital's share tends to zero, then the growth rate of human capital tends to a constant rate that is independent of growth in physical capital, and hence population growth. Growth in aggregate output remains partially dependent on population growth, because raw labor is employed in the manufacture of final output.

However, whether population growth is conducive, irrelevant or detrimental to the asymptotic long run growth rate of the economy depends on returns to scale to L and K in the production of final output are increasing, constant or decreasing, respectively.

Case 1 $(\sigma_L + \sigma_K) > 1 \Leftrightarrow \partial\bar{g}_y/\partial n > 0$

Case 2 $(\sigma_L + \sigma_K) = 1 \Leftrightarrow \partial \bar{g}_y / \partial n = 0$

Case 3 $(\sigma_L + \sigma_K) < 1 \Leftrightarrow \partial \bar{g}_y / \partial n < 0$

Thus, the result of R&D-based growth models with endogenous human capital accumulation that long run economic growth is decreasing in population growth is not robust. In the above model of heterogeneous labor, long run economic growth is decreasing in population growth only if returns to scale to physical capital and raw labor in final production are decreasing.

4.4 Non-Linearity and Non-Scale Growth

Our general three-sector production structure, (4.1a) - (4.1d), comprises the production sectors of several well-known models of non-scale, as well as scale, growth. Among these are the models of Jones (1995*a*), Young (1998) and Romer (1990). Whereas Jones (1995*a*) predicts non-scale growth with diminishing returns in R&D, Young (1998) predicts non-scale growth with linearity in R&D. Our example predicts non-scale growth with both diminishing returns and asymptotic linearity in R&D. Romer (1990) and other models that predict scale growth (notably, Aghion & Howitt (1992) and Grossman & Helpman (1991)) assume linearity or constant returns in R&D. To prove the logical independence of the notions of non-linearity and non-scale growth, we require an example of a model that features non-linearity (diminishing marginal returns to ideas) in R&D and predicts scale growth.

If $n = 0$, system (4.2) is a homogeneous linear system of equations (in matrix form $Ax = 0$). It is a well known result of linear algebra that such a system has a positive solution only if $|A| = 0$. However, whereas a positive solution in our non-homogeneous system implies a row of zeroes, linearly dependent columns suffice for a positive solution to our homogeneous system. The implication is that it is not necessary to assume linear production technology in R&D to solve for a positive scale growth equilibrium. Let v_i denote the i^{th} column vector of the coefficient matrix A. If $v_1 = av_2 + bv_3 \quad a < 0; b < 0$, we can solve for a positive scale growth equilibrium. Constant returns to scale to growing factors $(a = b = -1)$ is an obvious case.

As this is an example to show the logical possibility of scale growth with non-linear R&D, we abstract from the central planner's choice of consumption and factor allocations between sectors[7]. Let

$$Y = [(1-\alpha_A)K]^{1-\sigma} A^{\sigma} [(1-l_A)L]^{\sigma}, \quad 0 < \sigma < 1 \tag{4.24a}$$

$$\dot{A} = B[\alpha_A K]^{1-\eta} A^{\eta} [l_A L]^{\eta}, \quad B > 0, 0 < \eta < 1 \tag{4.24b}$$

$$\dot{K} = sY \quad 0 < s < 1 \tag{4.24c}$$

where α_A, l_A, s and B are exogenous, positive constants. Note that in the special case of $\sigma = \eta$, we may drop α_A and l_A, so that B measures the share of total output invested in R&D. Variety R&D features diminishing marginal returns to the existing stock of ideas since $\eta < 1$. The requirement of constant growth rates implies the following system of three linear equations:

$$\begin{bmatrix} \sigma & -\sigma \\ -(1-\eta) & (1-\eta) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.25}$$

where $v_1 = -v_2 \Rightarrow |A| = 0$. Referring to the appendix, the long run equilibrium growth

[7]Both abstractions are innocuous since sectoral allocations of factors are constant in a balanced growth equilibrium and an endogenous savings rate is constant by the Euler condition, $g_C = \frac{1}{\theta}\left[(1-\alpha)\frac{Y}{K} - \rho\right]$.

rate of the economy is

$$g_y = g_A = \left(\beta_K{}^{1-\eta}\beta_A{}^{\sigma}L^{\sigma}\right)^{\lambda} \tag{4.26}$$

where $\beta_K = s\,(1-\alpha_A)^{1-\sigma}(1-l_A)^{\sigma}$, $\beta_A = B\,(\alpha_A)^{1-\eta}(l_A)^{\eta}$ and $\lambda = 1/(\sigma+1-\eta) : 0 < \lambda \leq 1$.

The long run growth rate of the economy is unambiguously increasing in the scale of the economy, as measured by population level:

$$\frac{\partial g_y}{\partial L} = \frac{\sigma}{(\sigma+1-\eta)}\frac{g_y}{L} > 0 \tag{4.27}$$

Not surprisingly, the magnitude of the scale effect is increasing in σ and η, labor's share of final production and R&D, respectively. Interestingly, an increase in labor's share of R&D magnifies the scale effect proportionately more.

The R&D sector features non-linearity or diminishing returns and the long run growth rate of the economy exhibits scale effects. This proves

Proposition 4.5 *The notions of non-linear knowledge accumulation and non-scale growth are logically independent*

4.5 Discussion

4.5.1 Empirical Studies

By Proposition 5, a model of non-scale growth need not feature diminishing returns to knowledge in the accumulation of knowledge. And, vice versa, a model featuring a non-linear knowledge accumulation equation need not predict non-scale growth. In light of this logical independence, it is worth considering which has more empirical support: "the means" of (non-)linear knowledge accumulation or "the end" of (non-)scale growth?

Diminishing Marginal Returns to Knowledge in Knowledge Accumulation

Diminishing marginal returns to knowledge captures the hypothesis that the higher the stock of knowledge, the more difficult it becomes to create new knowledge. As such, the creation of new knowledge increases at a decreasing rate. This is intuitive when knowledge is embodied, but less obvious when knowledge is disembodied.

Disembodied Knowledge Regarding disembodied knowledge, $\eta_A < 0$ and $\eta_A > 0$ correspond to the case of negative and positive external returns in R&D, respectively. With negative returns, the discovery of new ideas decreases with the level of knowledge because the most obvious ideas are discovered first or, in Jones's (2004) terminology, "fished out". In the case of positive returns, the creation of new ideas is increasing in the level of knowledge because researchers "stand on the shoulders of giants". Whereas we may intuit that researchers face diminishing marginal returns to their effort ($\eta_L < 1$), there is nothing innate in the R&D process to indicate whether positive returns to ideas in R&D are diminishing, constant or increasing, $\eta_A < 1$, $\eta_A = 1$ and $\eta_A > 1$, respectively. Having said this, some cases may be less intuitive than others. For instance, $\eta_A = 1$ requires the number of new ideas to exactly double when the stock of existing ideas doubles. Jones (2004) argues

> The case of $\phi = 1$ appears to have little in way of intuition or evidence to recommend it. (p.63)

Empirical estimates of η_A (termed ϕ by Jones (1995*a*)) for the United States (US) and other OECD countries support diminishing marginal returns to the existing stock of knowledge in R&D ($\eta_A < 1$).

Jones (2002) constructs a multifactor productivity index, as a measure of A, for the US economy, 1950 - 1993. Using this index, he obtains an upper bound non-linear least squares estimate of 0.3 for $\gamma \equiv \eta_L / (1 - \eta_A)$, from which an upper bound estimate of $0.1\dot{6}$ for η_A can be inferred for a plausible value of 0.25 for η_L.[8] Separately identified estimates of η_A and η_L reflect Jones's (2002) specification of the R&D equation which assumes $\eta_K = 0$. Notwithstanding this caveat, the upper bound estimate is consistent with estimates obtained in independent empirical studies.

Also using time series data, 1962 - 1996, Gong, Greiner & Semmler (2004) obtain a non-linear least squares estimate of η_A of 0.1 and 0.08 for U.S. and Germany, respectively. Using the perpetual inventory method, they compute the stock of knowledge, A, and the stock of embodied knowledge allocated to R&D, $l_A H$, from total expenditure for R&D and cumulated salaries in R&D, respectively. Whereas Jones (2002) employs calibration techniques, Gong et al. (2004) directly estimate the implied parameters of R&D-based growth models. Their estimate of η_A does reflect a Jones (2002) specification of R&D. To allow for other economic variables, such as physical capital, in R&D, Gong et al. (2004) also regress g_A on $(l_A H)^{\eta_L}$ and a coefficient with an exogenous time trend. The estimated coefficient depends negatively on time, from which they infer evidence that other economic variables, such as, physical capital are relevant for the accumulation of knowledge.

Embodied Knowledge The notion of diminishing marginal returns to embodied knowledge, or human capital, in the accumulation of embodied knowledge has intuitive appeal.

[8] $\hat{\gamma} < 1/3 \Rightarrow \eta_A < 1 - 3\eta_L$. Note that $\eta_L > 1/3 \Rightarrow \eta_A < 0$, suggesting negative returns (or negative spillovers) in R&D.

To cite Jones's (2001) rhetorical question (p.12)

> If a 7th grader and a high school graduate each go to school for 8 hours per day, does the high school graduate learn twice as much?

According to the Mincerian wage regression evidence, each year of schooling appears to raise a worker's wage by a constant percentage. However, this is not evidence of a linear human capital accumulation equation, that is, $\omega_Q = 1$ where Q denotes rivalrous human capital, since Bils & Klenow (2000) shows that a human capital accumulation equation featuring diminishing marginal returns to human capital ($\omega_Q < 1$) captures this evidence.

Empirically, if diminishing marginal returns are present, then, ceteris paribus, we can expect to observe that the growth rate of knowledge capital declines as the stock of knowledge as a level variables rises.[9]

Using a Mincerian equation[10], we compute the stock of human capital per person, h, from average years of schooling of the population aged 15-64 who is not studying. Years of schooling is itself a common proxy for human capital. However, as Cohen & Soto (2001) point out, according to the years of schooling proxy, countries with low initial levels of human capital show the fastest rates of growth in human capital.[11]

Figure 4.3 plots the growth rate of human capital against the level of human capital for G7 countries, 1960 - 2000. With the exception of Italy, the growth rate of human capital declines as the stock of human capital as a level variable rises. We discern two patterns. In the top panel, the growth rate of human capital declines unabated, whereas in the bottom

[9] As in Figure 3.1 in Chapter 3. See also Dalgaard & Kreiner (2003) and Gong et al. (2004).

[10] Specifically, we measure human capital at time t as $h_t = e^{0.07 s_t}$, a special case of the Bils & Klenow (2000) formulation suggested by Jones (2002). s is average years of schooling of the population aged 15-64 who is not studying, as provided by Cohen & Soto (2001).

[11] We confirm this result. Proxying human capital by years of schooling, France's 1960-1970 growth rate in human capital (1.75%) exceeds Germany's (1.57%), which, in turn, approximately equals Italy's (1.53%). Overall, the growth rates in human capital are typically above 1%, higher than those depicted in Figure 4.3.

panel, the growth rate of human capital asymptotes to a constant.

To infer from these two patterns evidence of diminishing marginal returns and asymptotic linearity, respectively, we would need to control for both the fraction of human capital allocated to and other factors employed in human capital accumulation.

Regardless, the patterns in growth rates of human capital depicted in Figure 4.3 are more reliable than those suggested by Barro & Lee (1993), the data set most widely used in econometric studies. For the period, 1960 - 1990, Serrano (2003) show the growth rate of schooling in the OECD is more accurately measured by de la Fuente & Donenech (2000). The source for Figure 4.3, Cohen & Soto (2001) retains the accuracy of de la Fuente & Donenech (2000), whilst providing a longer series, 1960 - 2000.

Thus, other things equal, Figure 4.3 is suggestive of diminishing marginal returns to human capital and, possibly, asymptotic linearity, in human capital accumulation.

Scale Effects

In the empirical literature, the jury is out on whether the scale of the economy, typically captured by population size, affects the long-run growth rate. With the advent of first generation non-scale growth models, came an empirical verdict against strong scale effects in growth[12].

To cite Jones (1995*a*),

> The evidence [against strong scale effects in growth] is compelling ... One might worry about the relevant unit of observation (the world vs. a single country)

[12]Typically, the theoretical literature distinguishes models of scale or non-scale growth, or growth with or without scale effects. The convention in the empirical literature is to refer to evidence of either strong or weak scale effects in growth, meaning the growth rate of the economy is increasing in the size or growth rate of the population, respectively. Thus, strong scale effects in growth is equivalent to scale growth and weak scale effects in growth is equivalent to non-scale growth.

> or the lags associated with R&D, but it should be clear ... that these concerns cannot overturn the rejection of scale effects.

The debate seemed settled until a recent resurgence of evidence supporting strong scale effects in growth (Todo & Miyamoto 2002).

Typically, empirical studies find for or against scale growth depending on whether data is cross-country or time series and whether evidence is historical or modern.

Cross Sectional The ideal cross-sectional evidence would observe two regions at a point in time, one with a larger population, that are otherwise identical. The two regions would not interact via flows of ideas, goods, capital or labor. The only source of ideas in a region would be it's own population. If scale affects growth, then, in the long run, we would expect the region with the larger population to grow more. Of course, regions do interact and things other than population are not equal across regions. Thus, econometric evidence attempts to control for cross-country interaction and differences.

To the extent that a country open to trade is also open to ideas, Backus, Kehoe & Kehoe (1992), Barrro & Sala-i-Martin (1999) and Alcala & Ciccone (2002) find evidence that growth rate in per capita output is affected by the growth rate of the population. Alcala & Ciccone (2002) control for institutions, as well as trade. Whilst in these studies the estimated elasticity of per capita output with respect to population is positive, it is small (0.02 and 0.2 in Barrro & Sala-i-Martin (1999) and Alcala & Ciccone (2002), respectively) and, more importantly, statistically insignificant at the usual critical levels. Also, the estimated coefficient on population growth is not an estimate of the structural parameter in (4.8). A theory of cross-country technology adoption (as suggested by Papageorgiou

(2003)) is needed to make sense of the estimates. Thus, cross-country evidence favors models of non-scale growth, but the evidence is not robust.

An alternative way to control for international knowledge diffusion is to focus on large regions, such that, ideas flow within but not across regions. With modern transportation, telecommunications and, more recently, information technology, ideas may flow to all places, almost instantaneously. No two regions are isolated. Thus, Kremer (1993) looks to historical data for evidence of strong scale effects in growth.

From the end of the ice age in 10,000 B.C. until the advent of larger sailing ships in 1500, five regions were mutually isolated from each other. Ranked by population size in 10,000 B.C., in descending order, the five regions comprise the Eurasian/African continents, the Americas, Australia, Tasmania and Flinders Island. Kremer (1993) observes a region's technological rank 12,000 years later exactly matches it's initial population rank. The larger the initial population size, the higher the long run growth rate of the economy, as predicted by models of scale growth.

Kremer's (1993) cross-sectional evidence of strong scale effects in growth is the most convincing. However, to the extent that each region's initial population size and population growth rate over the period is positively correlated, it is not clear whether this evidence supports scale or non-scale models of R&D-based growth. Moreover, we have no reason to believe that the same processes continue to govern the creation of ideas in the modern era.

Time Series Also using historical data and retaining region as the unit of observation, Todo & Miyamoto (2002) find time series evidence of strong scale effects in growth. Sourcing historical series (1 A.D. to 1998 A.D.) of per capita output and population from Maddison (2001), they focus on a single region, comprising the current seventeen most advanced countries[13].

Before 1870, data for the region's full sample are available for 1 A.D., 1000, 1500, 1700, 1820 and 1850. Todo & Miyamoto (2002) divide yearly data, available from 1870, into periods of thirty and twenty years. The rationale is that such period lengths are sufficient for most regional knowledge developed in the period to be dissipated to all countries by the end of that period. The consequence is that the number of observations for regression analysis is reduced to twelve and the time series are smoothed via ad hoc period selection, rather than a five or ten year moving average.

For the reduced sample size, Todo & Miyamoto (2002) regress growth in per capita output on initial population and growth in population.[14] The estimated coefficient on initial population is positive and statistically significant, whereas the estimated coefficient on population growth is insignificant. They also regress population growth on population size and, since the R-squared from this regression is less than the R-squared from the main regression, they reject multicollinearity.

Prior evidence suggests population growth and initial population size are highly correlated for much of the period studied by Todo & Miyamoto (2002) (see Figure 1 of Kremer (1993)). Although the estimated coefficients on initial population and population growth remain unbiased in the presence of multicollinearity, *interpretation* of each coefficient

[13]Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, United Kingdom, United States. Japan is excluded until 1870. Australia, Canada, New Zealand and United States are excluded until 1820.

[14]After diagnostic tests, Todo & Miyamoto (2002) correct for heteroskedasticity generated by period lengths, autocorrelation and non-stationarity.

becomes difficult. Todo & Miyamoto's (2002) estimated coefficient of initial population implies growth in per capita output (%) is approximately four times initial population size (billion), *other things being equal.* Clearly, other things are not equal if population growth increases whenever initial population increases.

There are other tests for multicollinearity. For instance, if dropping population growth from the regression equation lowers the standard error of population size, multicollinearity will usually be the source of the problem.[15]

However, a plot of population size, growth in population and growth in per capita output, 1 A.D. to 2003, for the same group of seventeen advanced countries, without Todo & Miyamoto's (2002) post 1870 adhoc period aggregation, illuminates the episode of multicollinearity. Referring to Figures 4.1 and 4.2, we divide the series according to the availability of yearly observations in Maddison's (2001) data.

Figure 4.1(a) reveals near perfect collinearity between population size and population growth, 1 A.D. to 1870. This subset of the series comprises seven of the twelve observations used by Todo & Miyamoto (2002)[16]. For this subset, the estimated coefficient on initial population would approximate Todo & Miyamoto's (2002), but would be uninterpretable as evidence of strong scale effects, because of almost perfect collinearity. To illustrate, 1850 - 1870, the average annual growth rate in per capita output is 0.78 (%), approximately four times the initial 1850 population, 0.17 (billion) and also approximately four times the average annual growth rate in population, 0.177 (%). As in Kremer (1993), is a positive estimate of the coefficient on population size for 1 A.D. to 1870 evidence of a

[15]For further explanation and other tests for multicollinearity, see Farrer & Glauber (1967).

[16]Thus, their finding for strong scale effects in growth reflects primarily Figure 4.1 and a selection of only five observations from Figure 4.2.

model of scale or non-scale growth?

After 1870, initial population size and population growth are no longer positively correlated, as shown in Figure 4.2(a). Thus, most econometric studies of scale effects in growth use a modern data series, such as provided in Figure 4.2.

Using modern data, Jones (1995*b*) provides the simplest and most compelling evidence against strong scale effects in growth. For the US economy, 1880 - 1997, he observes that a simple linear trend fits per capita output (in logs) extremely well. The implication is that either nothing has had a large, persistent effect on the growth rate or whatever persistent effects have occurred have been offsetting. According to R&D-based models of scale growth, the exponential trend in the level of the labor force should lead to an exponential trend in per capita output growth. It is difficult to think of any variable(s) that could offset the exponential scale effect.

More rigorous analysis, specifically, a time trend test, an augmented Dickey-Fuller test and a difference of means test omitting the Great Depression confirms his casual observation that US growth fluctuates around a constant mean and exhibits no time trend[17]. For the period 1900 - 1987, he also observes fluctuation in growth about a constant mean in fourteen other OECD countries, although a few countries exhibit a positive mean shift and a post-World War II downward trend. For the period 1871 - 1988, and the same countries, Papageorgiou (2003) confirms Jones's (1995*b*) results, although he reports positive (albeit small), statistically significant coefficients on the time trend test for most countries, 1871 - 1949[18]. Figure 4.2(b) depicts the positive mean shift in growth of per capita output for an

[17]More correctly, the coefficient estimate from the time trend test is statistically insignificant at the normal confidence levels.

[18]For 1900 - 1987, Jones (1995*b*) reports coefficient estimates from the trend test that are statistically

aggregate sample of seventeen OECD countries.

Jones (1995*b*) attributes the positive mean shift and downward trend to the decimation of physical capital in World War II. This explanation draws attention to fact that much of what is observed from modern data series is actually transitional dynamics, calling into question inferences from modern data about the long-run growth rate of the economy. In the words of Temple (2003),

> Empirically, the question of whether or not long-run growth depends on scale is probably unanswerable ... We do not observe the long-run growth rate; even if we could, we cannot test long-run predictions against the data, unless we make some truly heroic assumptions about relevant unobservable variables; and the long run equilibrium may be so distant in the future that we have neither the ability to resolve the debate, nor any practical need to do so.

Recognition of transitional dynamics in modern data does not overturn Jones's (1995*b*) finding against strong scale effects in growth, since in transition, a model of scale growth predicts growth in per capita output outstrips population size. Also, what unobservable variable would offset the three-fold increase in the scale of the economy, depicted in Figure 4.2(a), to generate the relatively stable growth in per capita output for more than a century, depicted in Figure 4.2(b)?

For the purposes of empirical testing, the hypothesis of scale effects could be reformulated in terms of level effects. This would avoid the need to "measure" long-run equilibrium growth. Moreover, whether, other things equal, the level of per capita output is dependent or independent of the level of population level is an empirical test of first generation versus second generation non-scale growth models.

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insignificant at the normal confidence levels for all countries.

Summary

Both intuition and, in the case of disembodied knowledge, econometric evidence support diminishing marginal returns to the existing stock of knowledge in the creation of new knowledge.

Historical evidence reveals long run growth in per capita output is positively correlated with the scale of an economy, as measured by initial population size. However, to the extent that initial population size and long run growth in population are positively correlated in historical series, it is not clear whether this evidence supports an R&D-based models of scale or non-scale growth. Notwithstanding that much of what is observed is transitional dynamics, modern evidence rejects strong scale effects in growth.

The question remaining is whether modern evidence supports first or second generation models of non-scale growth. That is, other things equal, is the long run level of per capita output dependent or independent of the level of population, weak or no scale effects, respectively?

Overall, there is more support for the means (diminishing marginal returns to knowledge in the accumulation) than the end (strong scale effects, weak scale effects or no scale effects).

4.5.2 Implications

The results obtained in this paper, in particular, the clear distinction between non-linearity in the accumulation of knowledge and non-scale growth, have interesting implications for both assessing existing models and constructing new models of R&D-based

growth. We draw on this distinction to dispel some common misconception:

Misconception 1 *The linearity critique is the main argument in favor of models of non-scale growth*

The conventional wisdom is that linearity is an essential feature of a model predicting scale growth. Accordingly, the case for models predicting non-scale growth is built on intuition or evidence of non-linearity (diminishing marginal returns to the stock of knowledge) in R&D.

The example given in the proof to Proposition 5 dispels the critique that if diminishing returns is introduced to the stock of ideas, a scale growth model cannot generate long run growth. Moreover, by Propositions 2 and 3, a model predicting non-scale growth may feature a linear R&D equation. Thus, the linearity critique is an argument neither for nor against scale or non-scale growth.

A more robust argument for models of non-scale growth is empirical evidence against scale effects in the long run growth rate of the economy.

Misconception 2 *(Non-)linearity in knowledge accumulation is the source of policy (in)variance in the long run growth rate of the economy*

To cite Jones (2004),

> if $\phi < 1$ ($\eta_A < 1$ in our model), then the long run growth rate depends on elasticities of production functions and on the rate of population growth. To the extent that these parameters are unaffected by policy, policy changes such as a subsidy to R&D or a tax on capital will have no affect on the long run growth rate. (p.45)

This link between diminishing returns in R&D and policy variance, whilst true for his model, does not generalize.

Jones (2004) discusses how second generation non-scale growth models, specifically, Young (1998), Dinopolous & Thompson (1998) and Peretto (1998)), restore policy variance in the long run growth rate by imposing a knife edge restriction in R&D. It is interesting that whereas the first generation non-scale growth models assume diminishing marginal returns in R&D and predict policy invariant long run growth, the early scale growth models and second generation non-scale growth models assume linearity in R&D and predict policy variant long run growth.

However, two results in this paper run contrary to this association between (non-)linear R&D and policy (in)variance. First, production structure (4.24) features $\eta_A < 1$ and predicts a long run growth of the economy that depends on the fraction of labor and capital allocated to R&D, and is therefore, policy variant. Second, in the example accompanying Proposition 1, despite assuming $\eta_A = 1$, the long run growth rate of the economy implied by (4.10) is policy invariant.

These two results highlight that the primary source of policy variance is the dependence of the long run growth rate on the fraction of inputs allocated to R&D (l_A and/or k_A). Not surprisingly, regardless of whether $\eta_A = 1$ or $\eta_A < 1$, a scale growth model will predict policy variance because the long run growth rate of the economy is, by definition of scale growth, proportional to the total number of researchers ($l_A L$).

Misconception 3 *If an R&D-based growth model assumes strictly positive population growth it is because linearity must feature somewhere and population growth is the least objectionable place to locate a linear differential equation*

For a homogeneous system, Christiaans (2004) shows that a steady state with positive growth rates does not exist unless the matrix of structural elasticities is singular. From this, he discusses how existing models assume a differential equation is linear in its stock variable and, consistent with Jones (2001), argues that with the exception of the population equation, the linearity assumption is ad hoc.

However, as the example proving Proposition 5 illustrates, singularity of the matrix of structural elasticities in a homogenous system does not imply a differential equation is linear in its stock variable.

Population growth may be the least objectionable place to locate a linear differential equation, but a model of R&D-based growth can be solved for a steady state with positive growth rates without featuring a linear differential equation anywhere.

4.6 Conclusion

Recent non-scale models of R&D-based growth introduce a second dimension of knowledge accumulation and predict that long run economic growth, as measured by growth in per capita output, does not essentially depend on population growth. If knowledge is embodied, strictly positive population growth diminishes long run economic growth.

The general three sector model of non-scale growth presented in this paper encompasses both seminal models and second generation models, where the second dimension of knowledge may be embodied or disembodied. We find that, in general, diminishing returns to the existing stock of knowledge in the creation of new knowledge is sufficient but not necessary for strictly positive non-scale growth.

Linearity, or constant returns to the existing stock of knowledge, in the creation of new knowledge is sufficient for positive non-scale growth. In this case, the long run growth rate of the economy does not essentially depend on population growth. Intuitively, linearity implies the stock of knowledge grows at a constant positive rate with no increasing effort from other inputs. New knowledge drives growth in the overall economy. Thus, the long run growth rate of the economy is a function of an additively separable term that does not depend on exogenous growth in the population.

A congestion effect in per capita knowledge production, associated with the embodiment of knowledge, further implies long run economic growth is decreasing in population growth. However, we find this result is not robust. In the case of heterogeneous labor, population growth has an ambiguous effect on the long run growth rate of the economy.

In addition to establishing general conditions for positive, non-scale growth, we show that the notions of non-linear knowledge accumulation and non-scale growth are logically independent.

Our findings have several implications. Primarily, the well-known linearity critique is an argument neither for nor against models of scale or non-scale growth.

Discussion of second generation non-scale growth models focuses on the empirical and policy relevance of the predicted long run growth rate of the economy. The findings of this paper remind us to give due consideration to whether conditions for non-scale growth without population growth are met in practice. Both intuition and, in the case of disembodied knowledge, econometric evidence support diminishing marginal returns to the existing stock of knowledge in the creation of new knowledge. The case of asymptotic linearity may be a more empirically consistent means of achieving an end where long run economic growth tends to a rate where strictly population growth is non-essential, and possibly irrelevant.

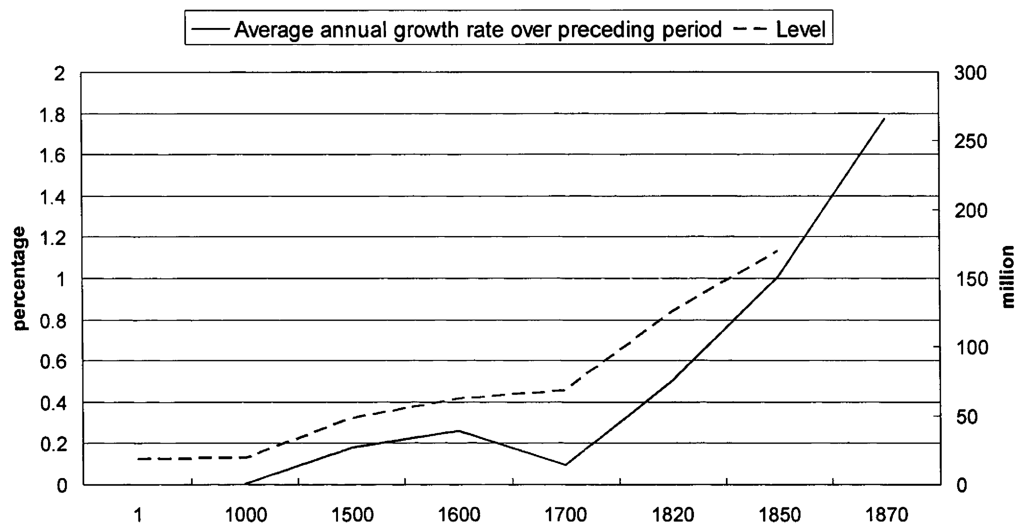
Second generation non-scale growth models follow convention by imposing linearity in knowledge accumulation to obtain the policy relevance result of seminal scale growth models. However, convention may belie the conditions that generate such results. For instance, our general framework reveals that a scale growth model will predict policy variance regardless of linearity or diminishing marginal returns to existing ideas in R&D, and linearity, without rivalry in knowledge, implies policy invariance in a non-scale growth model.

Our general solution for non-scale, long run economic growth illuminates conditions for strictly positive growth but is subject to some caveats, which also guide further research.

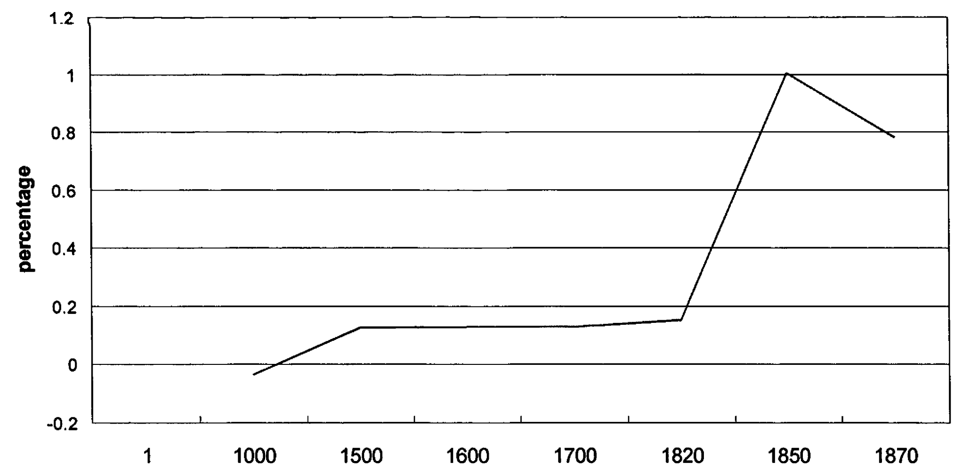
The general conditions provided in this paper are for strictly positive growth in a balanced growth equilibrium, which is characterized by constant growth rates in real variables. For their two sector non-scale growth model, Eicher & Turnovsky (1999) also provide general conditions for balanced growth. We note that constancy of exogenous parameters, specifically, structural elasticities and population growth, is sufficient but not necessary for constant growth rates in real variables. Constant population growth is a standard assumption in models of non-scale growth. Structural elasticities are constant if Cobb-Douglas technology describes the economy. If structural elasticities are not constant, as in the case of CES technology, restrictions on returns to scale imply constant growth rates. We could extend our analysis to include general conditions for balanced growth.

Such an extension is interesting to the extent that the conditions on returns to scale in our three sector non-scale growth model differ from those in the two sector model.

As required for a balanced growth equilibrium, the sectoral allocations of factors are assumed strictly positive and constant. This assumption, which also features in Eicher & Turnovsky (1999), serves the purposes of this paper. Moreover, sectoral allocations of labor and physical capital do not affect long run economic growth in non-scale models and the sectoral allocation of knowledge affects long run economic growth only if knowledge is rivalrous in use across sectors. Nonetheless, we recognize that factor allocations are endogenous. A general solution for sectoral allocations would enrich not only the equilibrium but also discussion of its policy and empirical implications. For instance, a general expression for the allocation of labor to R&D may offer insight into the empirical anomaly of rising research intensity and relatively stable growth in per capita output for more than a century.

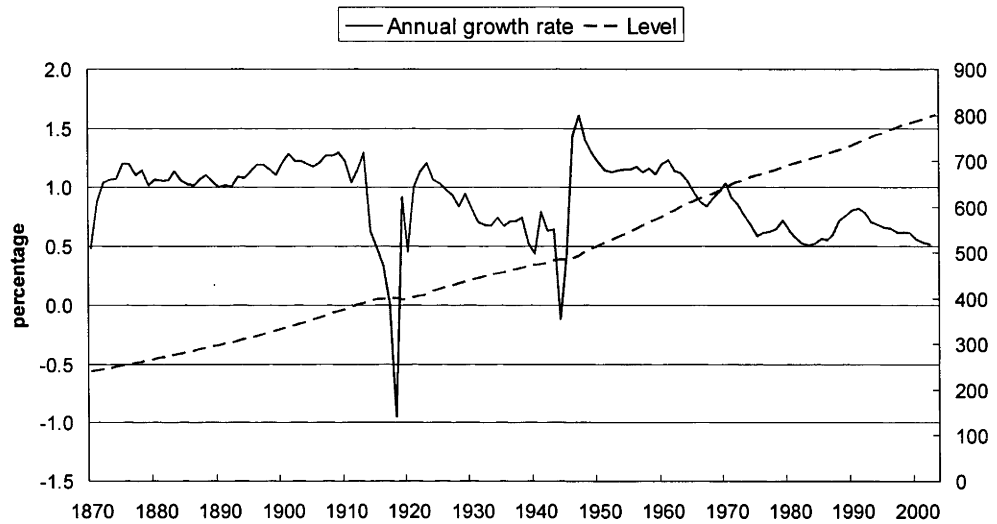


(a) Population

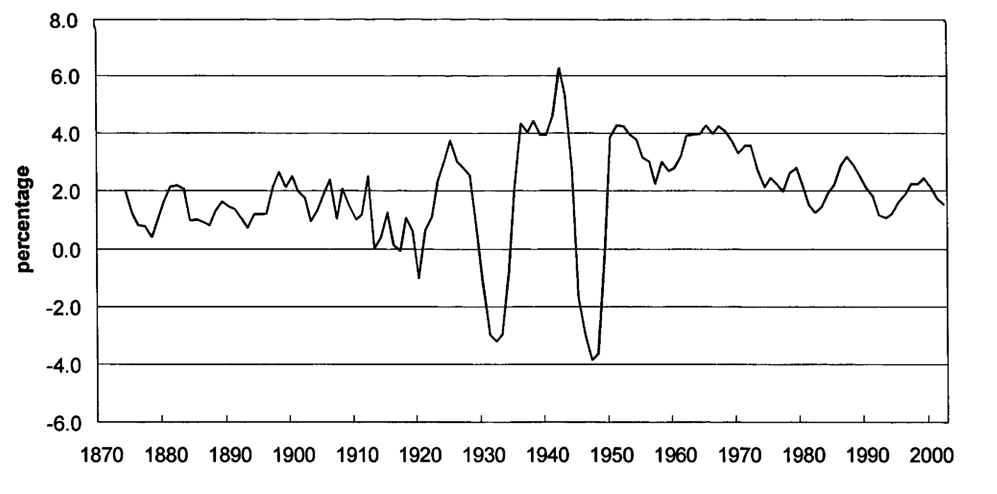


(b) Growth rate in GDP per capita (annual average over preceding period)

Figure 4.1: Population and Economic Growth in Advanced Countries: 1AD - 1870



(a) Population



(b) Annual growth rate in GDP per capita (5 year moving average)

Figure 4.2: Population and Economic Growth in Advanced Countries: 1870 - 2003

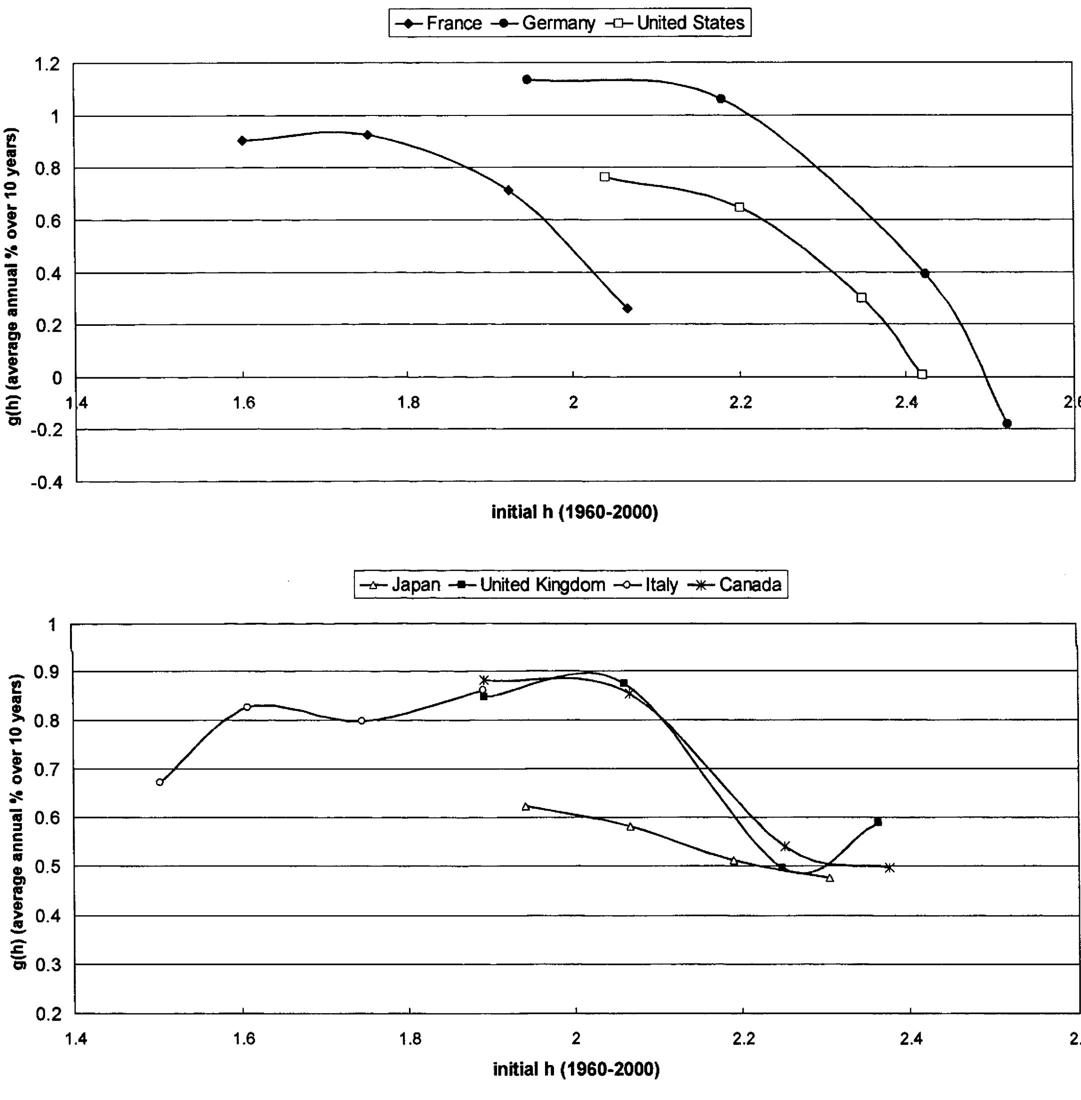


Figure 4.3: Human Capital per person: Growth versus Initial Stock in G7 countries

Part V

Lab Equipment Models of R&D - Essay 4

Chapter 5

Lab Equipment Models of Research and Creative Ignorance

5.1 Introduction

This paper explores the possibility that growth in ideas may outstrip growth in human capital in the long run, the reason why individuals may become relatively ignorant over time and how resources may be reallocated to research and development (R&D) so as to have an unambiguously positive effect on the long run growth rate of the economy.

The endogenous growth literature chiefly originates from Romer (1990) and Lucas (1988), who model long run economic growth as driven by innovation and investment in human capital, respectively. In neither model does long run growth in the economy require growth in an exogenous factor, typically, population growth. An impending worldwide slowdown in population growth has led a resurgence of interest in the literature. Recent models of endogenous growth consolidate R&D and investment-based growth (see, for instance, Funke & Strulik (2000), Dalgaard & Kreiner (2001) and Strulik (2005).

When we consider Thomas Edison's use of three thousand light globes or Bill Gates' employment of a team of programmers and personal computers, it is clear that, in addition to human capital and the existing stock of ideas, innovators may work with factors such as physical capital. Similarly, the acquisition of skill may require an individual to use ideas, encapsulated in the application of skill and, possibly, physical capital.

Interestingly, few endogenous growth models assume that R&D and human capital formation employ the same inputs as final production. This can be attributed to their origins. Lucas (1988), for instance, assumes human capital accumulates only from the existing stock of human capital. A notable exception is Dalgaard & Kreiner (2001), who apply the lab equipment framework of Rivera-Batiz & Romer (1991).

The literal definition of lab equipment is machinery and equipment housed in a room or building for research. In their seminal lab equipment model of R&D, Rivera-Batiz & Romer (1991) assume researchers employ physical capital. However, Dalgaard & Kreiner (2001) describe their model of R&D as applying the lab equipment framework even though no physical capital is used. The commonality of the two models is that R&D uses the same inputs, in the same proportion, as final production. The assumption that R&D uses physical capital and/or other inputs of final production, in the same proportion, as the manufacturing sector potentially contradicts the notion of inputs equipped to meet the specific requirements of R&D. Accordingly, we define a lab equipment model of research to be one in which all sectors employ the same factors, although not necessarily in the same proportion.

In existing lab equipment models of R&D, new ideas and human capital are created using units of final output. This feature has interesting empirical and policy implications. First, since the production of final output features diminishing marginal returns to the stock of ideas, R&D does not attract the well known linearity critique of other endogenous growth models. Recent empirical estimates support diminishing marginal returns to the stock of ideas in R&D (Gong et al. 2004). Second, the long run growth rate of the economy is increasing in the shares of output spent on R&D and education, suggesting implications for government policy.

The existing lab equipment approach suffer two key limitations. First, new knowledge is created using the exact same technology as the production of final output. We may reasonably expect the share of skill in the creation of knowledge to exceed the share of skill in final production. Empirical estimates confirm a priori expectations. Gong et al. (2004) estimate the structural elasticity of human capital in R&D is 0.48 for the United States[1], higher than the estimates of human capital's share of final production provided by Temple (1999), de la Fuente & Donenech (2000) and Hojo (2003). Second, the creation of ideas and skill feature constant returns to scale. The knife edge restriction of linearity prevails.

This paper has two main objectives. First, to understand why existing lab equipment models of research assume identical production and knowledge accumulation technologies and investigate the implications of this assumption. Second, to establish why constant returns to scale in the creation of knowledge is necessary in the two sector lab equipment

[1]Noteably, Gong et al. (2004) control for the possibility that factors other than the existing stock of ideas and researchers affect R&D. Their estimate is therefore relevant to the lab equipment model of research where physical capital is a R&D input.

framework and explore the implications of relaxing the restriction of linearity in a three sector framework.

Mulligan & Sala-i-Martin (1993) make an important contribution to our understanding of the conditions for endogenous growth when knowledge accumulation features the same inputs as final production. They confine their analysis to human capital. Because human capital is embodied in individuals, their production functions are expressed in per capita terms. Recognizing the inherent public good characteristics of disembodied knowledge, we endeavour to establish the knife edge conditions for endogenous growth when both innovation and human capital formation feature the same inputs as final production.

This paper presents a generalized model of endogenous growth with three accumulating stocks, physical capital, ideas and human capital. Following Romer (1996), we make two major simplifications. First, we assume all sectors are generalized Cobb-Douglas production functions. The production functions remain generalized in that, apart from constancy, we impose no further restrictions on the structural elasticities. Second, the fraction of output saved and the fractions of inputs allocated to each sector are exogenous and constant. These assumptions do not affect the main implications of our general model.

We seek to make a general contribution to the endogenous growth literature by establishing knife edge conditions for endogenous growth. We also extend the lab equipment branch of the literature with a specific application that generates an interesting result, which we term *creative ignorance.*

5.2 General Cobb-Douglas Model of Lab Equipment Research

Let Y denote the output of the final good, K the stock of physical capital and L the population (labor force), which is exogenous. A denotes the stock of non-rivalrous knowledge, as measured, in a decentralized economy, by the existing *variety* of intermediate goods. Q denotes either an alternative stock of non-rivalrous disembodied knowledge (such as, the existing *quality* of intermediate goods) or rivalrous embodied knowledge (or *human capital*).

Consider the following general three-sector Cobb-Douglas production structure:

$$Y = A^{\sigma_A}\left[(1-\alpha_A-\alpha_Q)\,K\right]^{\sigma_K}\left[q_Y Q\right]^{\sigma_Q}\left[(1-l_A-l_Q)\,L\right]^{\sigma_L} \quad 0 \le \sigma_i \le 1 \forall i \tag{5.1a}$$

$$\dot{A} = B_A A^{\eta_A}\left[\alpha_A K\right]^{\eta_K}\left[q_A Q\right]^{\eta_Q}\left[l_A L\right]^{\eta_L} \qquad 0 \le \eta_i \le 1\forall i \quad B_A > 0 \tag{5.1b}$$

$$\dot{Q} = B_Q A^{\omega_A}\left[\alpha_Q K\right]^{\omega_K}\left[q_Q Q\right]^{\omega_Q}\left[l_Q L\right]^{\omega_L} \qquad 0 \le \omega_i \le 1\forall i \quad B_Q > 0 \tag{5.1c}$$

where α_i and l_i ($i = Y, A, Q$) are the fractions of physical capital and labor, respectively, allocated to sector i. B_i is an exogenous shift parameter, which in the special case of identical input proportions also measures the share of total output invested in each aspect of knowledge accumulation.[2]

Equation (5.1c) may represent either human capital accumulation or quality R&D. In the case of human capital accumulation, $\sum_i q_i = 1$ if embodied knowledge is rival in use across all three sectors. Because knowledge is embodied, $Q = hL$, where h denotes per capita human capital. If labor is homogeneous, we set $l_i = 0\forall i$, and q_i denotes both the

[2]If $\sigma_i = \eta_i = \omega_i$ for each $i = A, K, Q, L$, the share of output invested in R&D is $\dot{A}/Y = B_A\left[\alpha_A/\alpha_Y\right]^{\sigma_K}\left[q_A/q_Y\right]^{\sigma_Q}\left[l_A/l_Y\right]^{\sigma_L}$ and the share of output invested in human capital formation is $\dot{Q}/Y = B_Q\left[\alpha_Q/\alpha_Y\right]^{\sigma_K}\left[q_Q/q_Y\right]^{\sigma_Q}\left[l_Q/l_Y\right]^{\sigma_L}$. If $\alpha_A = \alpha_Q = \alpha_Y; q_A = q_Q = q_Y; l_A = l_Q = l_Y$, then $\dot{A}/Y = B_A$ and $\dot{Q}/Y = B_Q$.

fraction of human capital and the fraction of labor allocated to sector i. In the case of R&D, $q_i = 1 \forall i$, since this type of disembodied knowledge non-rivalrous in use.

The stock of labor grows at the rate of population growth: $\dot{L} = nL$, where $n \geq 0$. We confine our analysis of $n > 0$ to the case of knowledge embodied in homogeneous labor. If knowledge is disembodied and non-rivalrous, we assume $n = 0$.

In the centralized version of R&D-based growth models, the rate of savings and the fractions of labor, physical capital and human capital are endogenous. The central planner chooses consumption, and the fractions of labor and capital employed in each sector so as to maximize intertemporal utility of the representative agent subject to the production and accumulation constraints, equations (5.1a) - (5.1c) and the constraint that physical capital accumulates as a residual after aggregate consumption and any knowledge accumulation needs are met.

For the purposes of this paper, we assume that, as required for balanced growth, the sectoral allocations of factors and savings rate are strictly positive and constant. The physical capital accumulation equation is given by

$$\dot{K} = sY; \qquad s > 0 \tag{5.1d}$$

where s measures the share of total output invested in physical capital accumulation.

Definition 5.1 *In a balanced growth equilibrium all real variables grow at constant, although not necessarily equal, rates.*

The balanced growth rates of the real variables (Y, K, A, Q) are obtained by total differentiation of the production functions (5.1a) - (5.1c), noting that constant growth rates requires $g_Y = g_K$, $g_{\dot{A}} = g_A$ and $g_{\dot{Q}} = g_Q$. Note also that unless knowledge is embodied in homogenous labor (in which case $l_i = 0 \forall i$) we assume $n = 0$.

A balanced growth equilibrium is therefore characterized by the following system of three linear equations:

$$\begin{bmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_Q \\ -\eta_K & (1-\eta_A) & -\eta_Q \\ -\omega_K & -\omega_A & (1-\omega_Q) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.2}$$

which is homogeneous (in matrix form $Ax = 0$).

It is a well known result of linear algebra that such a system has a solution other than $g_K = g_A = g_Q = 0$ only if $|A| = 0$. From (5.2), the growth rate in output in a balanced growth equilibrium simplifies to:

$$g_Y = g_K = \left[\frac{\sigma_A + (1-\eta_A) - \omega_A}{(1-\sigma_K) + \eta_K + \omega_K}\right] g_A + \left[\frac{\sigma_Q - \eta_Q + (1-\omega_Q)}{(1-\sigma_K) + \eta_K + \omega_K}\right] g_Q \tag{5.3}$$

The solution is one of endogenous growth since growth in the endogenous variables does not depend on growth in an exogenous variable. Having said this, equilibrium growth is dependent on or independent of the exogenous level of population, depending on whether Q denotes disembodied or embodied knowledge, respectively. Thus, $|A| = 0$ is the key condition that must hold if the model is to deliver endogenous growth at positive, constant rates. We summarize this discussion with the following proposition:

Proposition 5.1 (Necessary Condition for Endogenous Growth) *Endogeneous growth, with or without scale effects, is impossible unless the knife edge condition $|A| = 0$ is met*

Most models of endogenous growth obtain $|A| = 0$ by assuming that one of the knowledge accumulation equations in system (5.2) is linear. For instance, a Romer (1990) type R&D equation and a Lucas (1988) type human capital accumulation equation implies zero entries in the second and third rows, respectively.

Despite being widely considered a necessary condition for endogenous growth (see for instance, Jones (2001)), linearity in knowledge accumulation is only one of the conditions sufficient for $|A| = 0$. It is, in fact, $|A| = 0$ that is necessary for endogenous growth.[3]

Also sufficient for $|A| = 0$ is that the columns of A are linearly dependent in such a way that

$$v_1 = av_2 + bv_3 \quad (a < 0; b < 0) \quad \Rightarrow |A| = 0 \tag{5.4}$$

where v_i denotes the i^{th} column vector of the coefficient matrix A. Existing lab equipment models satisfy (5.4) by assuming constant returns to scale in growing factors ($a = b = -1$). We elaborate on this in the following section.[4] For now, we summarize the general conditions that imply balanced growth with the following proposition:

Proposition 5.2 (Sufficient Conditions for $|A| = 0$) $|A| = 0$ *if either:*

1. *In a knowledge accumulation sector, existing knowledge exhibits constant marginal returns and, if $n > 0$, is the sole input or, if $n = 0$, is a joint input with labor.*

2. *All sectors are subject to constant returns to scale to endogenously accumulating factors, in which case all variables grow at a common rate $g_Y = g_K = g_A = g_Q$.*

[3] For a two sector growth model where growth in population is exogenous, Christiaans (2004) also establishes that a singular matrix of structural elasticities in necessary for endogenous growth and/or growth with scale-effects.

[4] For a system of two equations in g_A and g_Q where Q denotes disembodied knowledge in the form of quality improvements, Li (2002) proposes that singularity of the structural elasticities matrix in turn requires, in our notation, $\eta_A = \omega_A$ and $\eta_Q = \omega_Q$. We will see that identical structural elasticities across sectors renders constant returns to scale necessary for balanced growth and is therefore a special case of our second sufficient condition for $|A| = 0$.

3. *Increasing (decreasing) returns to scale to endogenously accumulating factors in final production are offset by decreasing (increasing) returns to scale to growing factors in the knowledge accumulation sectors such that* $kv_1 = -(v_2 + v_3)$*, in which case* $g_Y = g_K = kg_A = kg_Q$, where $k \gtrless 1$.

Proof. Define x_L where $x = \sigma, \eta, \omega$. System (5.2), which characterizes balanced growth, is homogeneous provided $x_L = 0$ or $n = 0$. Each of the following cases implies $|A| = 0$, as required for a non-trivial solution to system (5.2):

1. Zero entries in the second and third row is provided by $\eta_A = 1; \eta_i = 0 \forall i \neq A, L$ *and* $\omega_Q = 1; \omega_i = 0 \forall i \neq Q$, respectively.

2. $\sum \sigma_i = 1; \sum \eta_i = 1; \sum \omega_i = 1 \forall i \neq L \Rightarrow v_1 = -v_2 - v_3$

3. Increasing (decreasing) returns to scale to endogenously accumulating factors in final production correspond to $\sum_{\forall i \neq L} \sigma_i \gtrless 1$, respectively. Let $(1 - \sigma_K) = \sigma$. The relationship $kv_1 = -(v_2 + v_3) \Rightarrow \sum_{\forall i \neq L} \sigma_i = 1 + (k-1)\sigma \gtrless 1 \Leftrightarrow k \gtrless 1$. Similarly, $\sum_{\forall i \neq L} \eta_i \lesseqgtr 1 \Leftrightarrow k \gtrless 1$ and $\sum_{\forall i \neq L} \omega_i \lesseqgtr 1 \Leftrightarrow k \gtrless 1$.

Set $g_A = g_Q$ in (5.3). Substituting $\sum \sigma_i = 1; \sum \eta_i = 1; \sum \omega_i = 1 \forall i \neq L$ gives $g_Y = g_K = g_A = g_Q$. Substituting $(\sigma_A + \sigma_Q) = k(1 - \sigma_K)$, $(\eta_A + \eta_Q) = (1 - k\eta_K)$ and $(\omega_A + \omega_Q) = (1 - k\omega_K)$ gives $g_Y = g_K = kg_A = kg_Q$. ■

5.3 Constant Returns in Two Sector Models

The few applications of the lab equipment framework in the R&D-based growth literature feature restrictive assumptions. Both Rivera-Batiz & Romer's (1991) seminal lab equipment model of research and Dalgaard & Kreiner (2001) assume R&D uses not only the same inputs but also the same input proportions as the final production technology.

We consider a two sector model of R&D-based growth, to illustrate the generality of the above propositions and to explore both why existing models restrict production technology to be identical across sectors and the implications of such a restriction. We also simplify our exposition of the conditions for balanced growth with a phase diagram analysis.

A general two sector lab equipment model is described by equations (5.1a), (5.1b) and (5.1d), setting $\sigma_Q = \eta_Q = \alpha_Q = l_Q = 0$:

$$\dot{K} = sA^{\sigma_A}\left[(1-\alpha_A)K\right]^{\sigma_K}\left[(1-l_A)L\right]^{\sigma_L} \qquad 0<\sigma_i<1\forall i \tag{5.5a}$$

$$\dot{A} = B_A A^{\eta_A}\left[\alpha_A K\right]^{\eta_K}\left[l_A L\right]^{\eta_L} \qquad 0<\eta_i<1\forall i \tag{5.5b}$$

where $\bar{L}$. Corresponding to the two accumulating sectors are the isoclines:

$$\dot{g}_K = -(1-\sigma_K)g_K + \sigma_A g_A = 0 \tag{5.6a}$$

$$\dot{g}_A = \eta_K g_K - (1-\eta_A)g_A = 0 \tag{5.6b}$$

Recalling the definition of balanced growth, the A and K isoclines intersect where both $\dot{A}$ and $\dot{K}$ equal zero. The arrows of motion in a phase diagram give a rough picture of what the trajectories look like and whether they move towards balanced growth. Like system (5.2), the phase diagram does not tell us what balanced growth path the economy converges to[5], but it does reveal the conditions for a unique balanced growth path.

The phase diagram in Figure 5.1 illustrates Proposition 1. Specifically, $(1-\sigma_K)(1-\eta_A)-\eta_K\sigma_A = |A| = 0$ is necessary for balanced growth ($\dot{g}_K = \dot{g}_A = 0$). The top and bottom

[5] As we will see, the economy's growth rate on that path is a complicated function of the parameters.

panels illustrate $|A| > 0$ and $|A| < 0$, respectively. For the intuition of this result, consider the case, conventional in growth literature, where final production features constant returns to scale to growing factors: the K isocline has a slope of unity. The economy exhibits zero or explosive growth depending on whether returns to scale in R&D are decreasing or increasing. The intuition is essentially the same as the intuition for why increasing (decreasing) returns to the existing stock of ideas generates explosive (zero) growth in Romer's (1990) model of R&D-based growth.

Strictly positive, but constant growth in A and K requires that either the A isocline is vertical at some $g_A > 0$ or the A and K isoclines lie directly on top of one another. Figure 5.2 illustrates a vertical A isocline: $(1 - \eta_A) = \eta_K = 0 \Rightarrow |A| = 0$. g_A is constant regardless of its initial situation. In this case, knowledge is just useful enough in producing new knowledge that the level of A has no impact on its growth rate. The figure shows that, regardless of where the economy begins, it converges to a balanced growth path where $g_K = \sigma_A/ (1 - \sigma_K) Bl_A L$. This illustrates case 1 of Proposition 2.

Alternatively, the A and K isoclines lie directly on top of one another: $(1 - \eta_A) / \eta_K = \sigma_A/ (1 - \sigma_K) \Rightarrow |A| = 0$. This illustrates cases 2 and 3 of Proposition 2. That is, $(1 - \eta_A) / \eta_K = \sigma_A/ (1 - \sigma_K) = k$, such that balanced growth is implied by either constant returns to scale in both sectors ($k = 1$) or increasing (decreasing) returns to scale in final production offset by decreasing (increasing) returns to scale in R&D ($k \lessgtr 1$).

In a two sector lab equipment model, the growth rate in output in a balanced growth equilibrium simplifies to

$$g_Y = g_K = \left[\frac{\sigma_A + (1 - \eta_A)}{(1 - \sigma_K) + \eta_K}\right] g_A = k g_A \tag{5.7}$$

where $k \lesseqgtr 1$ denotes decreasing, constant and increasing returns to scale to growing factors in the production of final output, respectively, with offsetting returns to scale to growing factors in R&D and $g_Y \lesseqgtr g_A \Leftrightarrow k \lesseqgtr 1$. Constant returns to scale satisfies the requirement of strictly positive, constant growth rates where the ratio Y/A is also constant in a balanced growth equilibrium.

Thus, as in our three sector model, constant returns to scale in both sectors or increasing returns to scale in one sector offset by decreasing returns in the other are sufficient for a balanced growth equilibrium in a two sector model. Why then do existing two sector lab equipment models of research restrict the production function in both sectors to be identical?

Rivera-Batiz & Romer (1991) draws our attention to constant returns to scale as a key implication of identical production functions.

> ... the output of patents at any date increases in proportion to the resources devoted to R&D. This permits the solution for balanced growth paths using linear equations, but it cannot be a good description of actual research opportunities. Rivera-Batiz & Romer (1991) p. 551

We summarize the relationship between identical production functions and constant returns to scale with the following proposition:

Proposition 5.3 *In a two sector Cobb-Douglas lab equipment model of research where sectors share identical production technology, constant returns to scale is necessary and sufficient for balanced growth*

Proof. If $\eta_A = \sigma_A$ and $\eta_K = \sigma_K$, then, by (5.7), regardless of the degree of returns to scale, $g_Y = g_K = g_A$. The dynamics of growth in output is determined by the dynamics of growth in knowledge, which is given by

$$\dot{g}_A = (\sigma_A - 1)\, g_A + \sigma_K g_K = (\sigma_A + \sigma_K - 1)\, g_A$$

where $\dot{g}_A \gtreqless 0 \Leftrightarrow (\sigma_A + \sigma_K - 1) \gtreqless 0$. $\dot{g}_A \gtrless 0$ implies explosive and ever-decreasing growth, respectively, and is inconsistent with balanced growth. Thus, constant returns to scale is necessary and sufficient for a balanced growth equilibrium: $(\sigma_A + \sigma_K) = 1 \Leftrightarrow \dot{g}_A = \dot{g}_K = 0$. ■

Identical production technology renders constant returns to scale necessary for balanced growth, but constant returns to scale is itself only sufficient for balanced growth. Why then impose identical production functions? As we will see in the following analysis of special cases, solving the centralized versions of lab equipment models with identical production functions is straightforward using our framework.

However, as the second sentence of the above quote suggests, it is the linearity of the equations, provided by constant returns to scale to accumulating factors, that makes lab equipment models easy to solve. To see this, consider (5.5). When each sector features constant returns to scale in accumulating factors ($\sigma_A + \sigma_K = 1; \eta_A + \eta_L = 1$), our analysis of (5.5) simplifies to an analysis of the dynamic evolution of a single variable $\chi = A/K$, where $g_\chi = 0$ in balanced growth. However, apart from the fact that the equilibrium value of χ is dependent on σ and η, solving under constant returns to scale without identical production functions involves little additional difficulty.

Rivera-Batiz & Romer (1991) and Dalgaard & Kreiner (2001) are special cases, with, and without, scale effects, respectively, of our general Cobb-Douglas model of lab equipment research. It is instructive to show how easily the general framework replicates the traditional lab equipment models of research, and to show how their structures and growth rates fit into our framework of propositions.

5.3.1 Ideas and Physical Capital

Rivera-Batiz & Romer's (1991) seminal lab equipment model of research introduces a technology for R&D that uses the same inputs as the manufacturing technology, in the same proportions. The production possibility frontier in the space of new ideas (designs) and final goods (manufactured goods) is a straight line, as opposed to the usual, concave to the origin, shape, in Romer's (1990) model. In the lab equipment model, if the output of final goods is reduced by one unit and the inputs released are transferred to the R&D sector, they yield B_A patents (new ideas). The opportunity cost of producing one unit of final goods is B_A units of new ideas.

Looking at the big picture, Rivera-Batiz & Romer (1991) take the aggregation of sectors to another level. In Romer (1990), since physical capital goods and consumption goods have the same production technology, they are integrated them into a single manufacturing sector (i.e. $Y = C + \dot{K}$). Rivera-Batiz & Romer (1991) aggregates manufacturing and research into a single sector (i.e. $Y = C + \dot{K} + \dot{A}/B_A$).

The centralized version of Rivera-Batiz & Romer (1991) with an exogenous savings rate is described by (5.1a), (5.1b) and (5.1d), setting $\alpha_i = q_i = l_i = 0$:

$$Y = A^{\sigma_Q+\sigma_L} K^{1-(\sigma_Q+\sigma_L)} Q^{\sigma_Q} L^{\sigma_L} \qquad 0 < \sigma_Q < 1; 0 < \sigma_L < 1 \tag{5.8a}$$

$$\dot{A} = B_A Y \tag{5.8b}$$

where B_A measures the share of total output invested in R&D and Q and L denote human capital and unskilled labor, respectively. The heterogeneity of labor does not complicate the balanced growth solution, since both types of labor are assumed to be stagnant: $\bar{Q}$ and $\bar{L}$.

Substituting for the structural elasticities from (5.8) in (5.2):

$$\begin{bmatrix} (\sigma_Q + \sigma_L) & -(\sigma_Q + \sigma_L) \\ -(1 - (\sigma_Q + \sigma_L)) & (1 - (\sigma_Q + \sigma_L)) \end{bmatrix} \begin{bmatrix} g_K \\ g_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.9}$$

where $v_1 = -v_2$, sufficient for $|A| = 0$. Constant returns to scale to physical capital and ideas, the growing factors, is sufficient for strictly positive, balanced growth. From (5.9), we have one equation with two unknowns which implies $g_K = g_A$.

To solve the model, we define $\chi \equiv A/K$. The dynamic evolution of χ_t is subsequently derived from equations (5.8a), (5.8b) and (5.1d):

$$g_\chi \equiv g_A - g_K = \left(B_A \chi_t^{(\sigma_Q+\sigma_L)-1} - s\chi_t^{(\sigma_Q+\sigma_L)} \right) Q^{\sigma_Q} L^{\sigma_L} \tag{5.10}$$

Along a balanced growth path, where g_y is constant, ideas and physical capital have to grow at the same rate, as implied by (5.9). Thus, $g_\chi = 0$ implies a steady state ratio of ideas to physical capital:

$$\chi = \frac{B_A}{s} \tag{5.11}$$

which, by (5.10), is stable.

Using equations (5.1d), (5.8a), (5.8b) and (5.11), the long run *growth rate* of per capita output is

$$g_y = g_Y = g_K = g_A = (B_A)^{(\sigma_Q+\sigma_L)}\, s^{1-(\sigma_Q+\sigma_L)} Q^{\sigma_Q} L^{\sigma_L} \tag{5.12}$$

which is positively dependent on the scale of the economy, as measured by the size of human capital (Q) and/or the size of the population (L) and positively dependent on the share of output invested in R&D and physical capital accumulation, B_A and s, respectively.[6]

5.3.2 Ideas and Human Capital

The centralized version of Dalgaard & Kreiner (2001) is described by (5.1a), (5.1b) and (5.1c), setting $a_K = a_L = 0$ ($a = \sigma, \eta, \omega$) and $q_i = 0$:

$$Y = A^{1-\sigma}Q^{\sigma} \qquad 0 < \sigma < 1 \tag{5.13a}$$

$$\dot{A} = B_A Y \tag{5.13b}$$

$$\dot{Q} = B_Q Y \tag{5.13c}$$

where $Q = hL$ represents embodied knowledge and $n > 0$. B_A and B_Q measure the share of total output invested in R&D and used to produce to human capital, respectively.

Substituting for structural elasticities from (5.13) into (5.2):

$$\begin{bmatrix} \sigma & -\sigma \\ -(1-\sigma) & (1-\sigma) \end{bmatrix} \begin{bmatrix} g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.14}$$

where $v_1 = -v_2$, sufficient for $|A| = 0$. Constant returns to scale to the endogenously growing factors, is sufficient for strictly positive, balanced growth. (5.14) provides one equation with two unknowns which implies $g_A = g_Q$.

To solve the model, we define $\psi \equiv Q/A$. The dynamic evolution of ψ_t is subsequently derived from equations (5.13a), (5.13b) and (5.13c):

$$g_\psi \equiv g_Q - g_A = \left(B_A\psi_t^{\sigma-1} - B_Q\psi_t^{\sigma}\right) \tag{5.15}$$

[6]In Rivera-Batiz & Romer (1991), the savings rate, s, is endogenous, as per a Ramsey consumer, so that the long run growth rate of per capita output is $g_y = \left(\Gamma Q^{\sigma_Q} L^{\sigma_L} - \rho\right)/\sigma$ where $\Gamma = (B_A)^{(\sigma_Q+\sigma_L)}(\sigma_Q+\sigma_L)^{(\sigma_Q+\sigma_L)}(1-(\sigma_Q+\sigma_L))^{2-(\sigma_Q+\sigma_L)}$.

Along a balanced growth path, ideas and aggregate human capital grow at the same rate, as implied by (5.14). Thus, $g_\psi = 0$ implies a steady state ratio of aggregate human capital to ideas:

$$\psi = \frac{B_Q}{B_A} \tag{5.16}$$

which, by (5.15), is stable.

Using equations (5.13a), (5.13b), (5.13c) and (5.16), along a balanced growth path:

$$g_A = g_Q = g_h + n = (B_A)^{1-\sigma} (B_Q)^{\sigma} \tag{5.17}$$

and from (5.13a), the long run *growth rate* of per capita output is

$$g_y = (B_A)^{1-\sigma} (B_Q)^{\sigma} - n \tag{5.18}$$

which, apart from the independence of scale (i.e. independence of L), resembles (5.12). In contrast to Jones (1995*a*) and other non-scale growth models[7], scale effects are also absent from the *level* of per capita output:

$$y_t = \left(\frac{B_A}{B_Q}\right)^{1-\sigma} h_0 e^{g_y t} \tag{5.19}$$

It is Dalgaard & Kreiner's (2001) endogeneity of embodied knowledge, rather than the lab equipment specification, that removes scale effects from both the long run growth rate and the level of per capita output.[8] Aside from scale effects, the similarity of (5.12) and (5.18) is stark. The assumption that R&D and final production are described by identical production functions has implications for the long run growth rate of economy.

[7] Strulik (2005) is a noteable exception.

[8] To see this, replace (5.13c) with a Lucas (1988) specification and solve for the balanced growth equilibrium.

5.3.3 Implications of Identical Production Functions

Referring to (5.12) and (5.18), two features stand out. First, the share of output invested in the accumulation of endogenous factors determines the long run growth rate and level of per capita output. Both Rivera-Batiz & Romer (1991) and Dalgaard & Kreiner (2001) predict that increasing the share of output invested in R&D, B_A, raises the long run growth rate of the economy. The long growth rate of the economy is also increasing in either the share of output invested in the production of physical capital or human capital, s or B_Q, in Rivera-Batiz & Romer (1991) and Dalgaard & Kreiner (2001), respectively. Second, endogenously accumulating factors grow at the same rate in the long run. Rivera-Batiz & Romer (1991) and Dalgaard & Kreiner (2001) predict the ratios K/A and Q/A, respectively, are constant along a balanced growth path.

The first feature has clear policy implications. Economic incentives and policy may affect the long run growth rate of the economy through B_A and s or B_Q, which are chosen by an optimizing central planner or household.

The first implication is that the rate of return on investment in output is an avenue for government policy. In a decentralized setting, households choose the portion of income consumed and invested in the production of both human capital and ideas. Households earn a return on investment in human capital and ideas in the form of income from wages and dividends, respectively. Financed by a lump sum tax, a government can boost B_A either by subsidizing the dividend from R&D investment or through the direct provision of R&D output.

The key point is that the long run growth rates, (5.12) and (5.18), focus policy on the sectoral allocation of production rather than the sectoral allocation of factors, such as physical capital and labor. For instance, the success of a policy program aimed at encouraging R&D would be measured by an increase in R&D spending as a proportion of Gross Domestic Product (GDP) rather than an increase in the proportion of the labor force employed in R&D. This is a direct implication of assuming that R&D uses units of final output rather than unique proportions of inputs.

The second implication is one of a policy trade-off, which we illustrate by reference to (5.18). If the share of output used in R&D, B_A, can only be raised at the expense of the share of output used in human capital formation, B_Q, then the long run growth rate of the economy may fall.

Remark 5.1 *In a lab equipment model of research featuring identical production functions across sectors, an increase in the share of output spent on R&D has an ambiguous effect on the long run growth rate of the economy*

To simplify our exposition of this remark, we assume $n = 0$. The total derivative of (5.18) yields

$$-\frac{\%\Delta B_A}{\%\Delta B_Q} \gtreqless \frac{\sigma}{(1-\sigma)} \Rightarrow \%\Delta g_y \gtreqless 0 \tag{5.20}$$

which implies an increase in B_A that is offset by a proportionate fall in B_Q boosts the long run growth rate of the economy only if $\sigma < 1/2$, that is, if ideas' share of final production exceeds human capital's share of final production.

The suggestion that policy makers face such a trade-off is reasonable. To see this, recognize that $(B_A + B_Q)$ represents total investment as a portion of total output. Suppose that, as in Dalgaard & Kreiner (2001), subsidies to boost B_A and B_Q are financed through a lump sum tax, T, and that the government maintains a balanced budget ($G = T$). The closed economy expenditure identity, $Y \equiv C + I + G = C + \dot{A} + \dot{Q} + T$ yields

$$(1 - c) = (B_A + B_Q) + \tau \tag{5.21}$$

where c denotes C/Y and τ denotes T/Y, measuring the size of government as a portion of total output. If government is neither willing nor able to reduce either τ or c,[9] then an increase in B_A must be offset by a proportional decrease in B_Q.

Dalgaárd & Kreiner (2001) present the implication that economic policy may shape long-run growth through increases B_A and B_Q as interesting, whilst conceding in a footnote

> However, the growth rate is not necessarily increasing in the share of resources used on R&D ... If the share is raised at the expense of resources used for human capital formation growth may decrease (footnote 13, page 202).

The implication of a policy trade-off, analyzed more fully in this paper, reveals an important shortcoming of the existing lab equipment approach. The trade-off between boosting R&D's share of output and boosting human capital accumulation's share of output is a direct implication of assuming that R&D and human capital accumulation use the same input proportions as the final production technology. We therefore relax the assumption of identical production functions across sectors and explore the policy implications.

The second feature, that endogenous accumulating factors grow at the same rate, is a direct implication of constant returns to scale, which identical production functions renders necessary for balanced growth. Recalling the definition, balanced growth requires only that variables grow at constant rates. We have established that non-identical production technology with decreasing or increasing returns to scale in R&D also permits a solution

[9]For instance, demographic pressures in an economy may account for upward pressure on both consumption and government spending, as a percentage of GDP.

for a balanced growth equilibrium, where the endogenously accumulating factors grow at constant, but not equal rates.

A priori reasoning does not suggest returns to scale are constant in all sectors. On the one hand, coordination and other problems at the micro level suggest decreasing returns to scale. On the other hand, positive spillovers in disembodied knowledge may suggest increasing returns to scale in R&D. As the number of sectors increase, the assumption of that all sectors feature constant returns to scale becomes less plausible.

5.4 Varying Returns to Scale

Corresponding to the general three sector lab equipment model described by equations (5.1a) - (5.1d) are the isoclines:

$$\dot{g}_K = -(1-\sigma_K)\, g_K + \sigma_A g_A + \sigma_Q g_Q = 0 \qquad (5.22a)$$

$$\dot{g}_A = \eta_K g_K - (1-\eta_A)\, g_A + \eta_Q g_Q = 0 \qquad (5.22b)$$

$$\dot{g}_Q = \omega_K g_K + \omega_A g_A + (1-\omega_Q)\, g_Q = 0 \qquad (5.22c)$$

where the intersection of all three ($\dot{g}_K = \dot{g}_A = \dot{g}_Q = 0 \Rightarrow |A| = 0$) is necessary for balanced growth. The intersection of all three isoclines in the positive orthogonal requires that all three isoclines lie directly on top of one another in three dimensional (g_K, g_A, g_Q) space. Phase diagram analysis in three dimensional space depicts sufficient conditions for a unique balanced growth equilibrium, but does not tell us to what equilibrium the economy converges.

Figures 5.3 and 5.4 depict the isoclines in two dimensional space. The top and bottom panels of Figure 5.3 depict the K and A isoclines, for a given g_Q, and the K and Q

isoclines, for a given g_A, respectively. Figure 5.4 depicts the Q and A isoclines, for a given g_K.

Viewing the isoclines in this way is useful for two reasons. First, it depicts the proportionality of growth rates in a balanced growth equilibrium, allowing us to illustrate how varying the returns to scale in each sector alters the proportionality. Second, it allows a more detailed analysis of constant marginal returns (or linearity) in knowledge accumulation, as per the first condition of Proposition 2.

Referring to the top panel of Figure 5.3, for a given strictly positive and constant g_Q, strictly positive and constant g_A and g_K requires that the A isocline is steeper than the K isocline in (g_K, g_A) space. Consider the case of constant returns to scale to growing factors in all sectors. The slope of the K isocline must be less than unity $(\sigma_A + \sigma_K < 1)$ and the slope of the A isocline must be exceed unity $(\eta_A + \eta_K < 1)$, in (g_K, g_A) space. For a strictly positive and constant g_Q, the isoclines must cross at $g_K = g_A = g_Q > 0$. Consider a move away from constant returns to scale. Introducing increasing returns to scale in final production and decreasing returns to scale in R&D steepens the K isocline and A isocline, respectively. For instance, if the slope of the K isocline is unity $(\sigma_A + \sigma_K = 1)$, then final production features increasing returns to scale to growing factors $(\sigma_A + \sigma_K + \sigma_Q > 1)$. The isoclines now cross at $g_K = kg_A = kg_Q > 0$, where $k > 0$. This illustrates cases 2 and 3 of Proposition 2.

Consider case 1 of Proposition 2. If $\omega_K = \omega_A = (1 - \omega_Q) = 0$, then $g_Q = B_Q q_Q > 0$. The Q isocline is vertical at $g_Q = B_Q q_Q$ in (g_K, g_Q) and (g_A, g_Q) space. Thus, the top panel of Figure 5.3 provides a complete analysis of the conditions for balanced growth. If final production features constant returns to scale to A and K, the slope of the K isocline is unity. It follows that variety R&D must feature decreasing returns to scale to A and K for g_A and g_K to be strictly positive and constant, as required for balanced growth. The intuition for decreasing returns to A and K in variety R&D when the $\dot{Q}$ equation is linear is similar to the intuition for diminishing marginal returns to A in variety R&D when the $\dot{L}$ equation is linear, as in Jones (1995a).

As an example of varying returns to scale in a three sector lab equipment model, let

$$Y = A^{\sigma}\left[(1-\alpha_A-\alpha_Q)\,K\right]^{1-\sigma}\left[(1-q_Q)\,Q\right]Q^{\sigma}L^{\sigma},\quad 1>\sigma>0 \tag{5.23a}$$

$$\dot{A} = \delta A^{\eta}\left[\alpha_A K\right]^{(1-\eta)/2}L^{\eta},\quad \delta>0, 1>\eta>0 \tag{5.23b}$$

$$\dot{Q} = \xi\left[\alpha_Q K\right]^{(1-\omega)/2}\left[q_Q Q\right]^{\omega}L^{\omega},\quad \xi>0, 1>\omega>0 \tag{5.23c}$$

where $\bar{L}$. Production of final output exhibits increasing returns to scale and both variety and quality R&D exhibit decreasing returns to scale to growing factors. The balanced growth equilibrium solves the following system of three linear equations:

$$\begin{bmatrix} \sigma & -\sigma & -\sigma \\ -\frac{(1-\eta)}{2} & (1-\eta) & 0 \\ -\frac{(1-\omega)}{2} & 0 & (1-\omega) \end{bmatrix}\begin{bmatrix} g_K \\ g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.24}$$

where $v_1 = 0.5v_2 + 0.5v_3 \Rightarrow |A| = 0$. From (5.24), $g_K = 2g_A = 2g_Q$. We can see this diagrammatically. Figure 5.6 depicts the interception of A and Q isoclines in (g_A, g_Q) space at $g_A = g_Q = 1/2g_K$. In (g_K, g_A) and (g_K, g_Q) space, the A and Q isoclines will have $(0,0)$ intercepts and intersect at $g_K = 2g_A = 2g_Q$.

5.5 Application: A Model of Creative Ignorance

5.5.1 A Plausible Balanced Growth Equilibrium?

By relaxing the assumption that production technology is identical across sectors, constant returns to scale is no longer necessary for balanced growth. If increasing (decreasing) returns to scale in final production is offset by decreasing (increasing) returns to scale in both R&D and human capital formation, real variables grow at constant, but unequal rates in a balanced growth equilibrium. We have illustrated a three sector R&D-based growth model which predicts both K/A and K/H rise over the long run.

In their discussion of Mulligan & Sala-i-Martin's (1993) two sector education-based growth model, Barrro & Sala-i-Martin (1999) suggest that only the balanced growth equilibrium implied by constant returns to scale in both final production and human capital formation is plausible:

> Since the alternative in which K/H rises or falls forever seems implausible, we assumed ... that constant returns held in each sector. (p.200)

Extrapolating to a three sector model, we consider whether an equilibrium in which both K/A and K/H rise (or fall) forever, is implausible. We define an implausible equilibrium as one which is not empirically supported and/or lacks intuitive appeal.

For empirical support, we would examine the long run trend in K/A and K/H, controlling for other factors. The measurement of A, the stock of ideas, poses an immediate challenge. A common proxy for the stock of ideas or technology is multi-factor productivity, an index constructed from measured output and measured factors of production. Two key problems arise. First, for the index to measure residual or unaccounted for output, all factors of production must be properly accounted. Second, inferring that residual growth in output measures growth in ideas imposes a theory of R&D-based growth. Theory decides what we observe. Patent counts provides a simple, alternative measure of the number of ideas produced.[10] However, not all ideas are patented. More importantly, not all ideas that are patented are valuable, in the sense that they drive growth in the overall economy. Patented ideas that are valuable are certainly not equally valuable.

In terms of intuitive appeal, we consider the inherent characteristics of physical capital, human capital and ideas. Both physical capital and human capital are private goods. The use of either type of capital in one sector rivals its use in another sector. Capital use within a sector is also rivalrous. The embodiment of human capital implies that additional labor congests its use, as with physical capital. In contrast, ideas have public good characteristics. Ideas are non-rival across sectors and, possibly, within a sector.

Both the amount of physical capital and human capital equipping an individual in their lifetime is finite. In contrast, the number of ideas may grow without bound. Consequently, the suggestion that growth in ideas could outstrip growth in either type of capital seems reasonable.

A three sector model affords us the choice of which two real variables grow at different rates. On balance, the result that K/A falls over the long run is more plausible than the result that K/H falls over the long run. However, the associated fall in Y/A along a balanced growth path may be less palatable. If the creation of new ideas ultimately drives growth in final output, it may be reasonable to predict that growth in output equals

[10] Porter & Stern (2000) and, more recently, Ulku (2004), use aggregate level patent data to measure the stock of disembodied knowledge.

growth in ideas in the long run. Therefore, a balanced growth equilibrium in which A/H rises, but Y/A is constant, may be most plausible.

5.5.2 Model

Consider the following three-sector Cobb-Douglas economy:

$$Y = A^{\sigma_A}\left[(1 - q_A - q_Q)\, Q\right]^{\sigma_Q} Z^{\sigma_Z} \quad 0 < \sigma_i < 1 \forall i \tag{5.25a}$$

$$\dot{A} = B_A A^{\eta_A}\left[q_A Q\right]^{\eta_Q} Z^{\eta_Z} \qquad 0 < \eta_i < 1 \forall i \quad B_A > 0 \tag{5.25b}$$

$$\dot{Q} = B_Q A^{\omega_A}\left[q_Q Q\right]^{\omega_Q} Z^{\omega_Z} \qquad 0 < \omega_i < 1 \forall i \quad B_Q > 0 \tag{5.25c}$$

where Z is a fixed factor of production, say, land, and $Q = hL$. We abstract from the fixed factor of production by normalizing Z to 1. We also assume $n = 0$, so that $g_Q = g_h$.

From (5.25a), the long run growth rate of the economy is given by $g_y = \sigma_A g_A + \sigma_Q g_Q$, which is constant, as required for balanced growth, if both g_A and g_Q are constant. The conditions for a balanced growth equilibrium are therefore provided by the solution to the following system of two linear equations:

$$\begin{bmatrix} (1-\eta_A) & -\eta_Q \\ -\omega_A & (1-\omega_Q) \end{bmatrix} \begin{bmatrix} g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.26}$$

where $|A| = 0$ is necessary for a solution and $v_2 = -kv_1$ is sufficient for $|A| = 0$. In a balanced growth equilibrium, the growth rate of ideas satisfies

$$g_A = \left[\frac{\eta_Q + (1-\omega_Q)}{(1-\eta_A) + \omega_A}\right] g_Q = k g_Q \tag{5.27}$$

where $k \lesseqgtr 1$ denotes decreasing, constant and increasing returns to scale to growing factors in R&D, respectively, with offsetting returns to scale to growing factors in human capital accumulation and $g_A \lesseqgtr g_Q \Leftrightarrow k \lesseqgtr 1$.

5.5.3 Creating Ideas and Ignorance

Consider the introducing the following parameter restrictions to (5.25b) and (5.25c):

$$\eta_A = (1-\eta)\,; \qquad \eta_Q = k\eta \tag{5.28a}$$

$$\omega_A = \omega; \qquad \omega_Q = 1 - k\omega \tag{5.28b}$$

where $1/\omega > k > 1$ implies increasing returns to scale to growing factors in R&D $(1 + (k-1)\,\eta > 1)$ and decreasing returns to scale to growing factors in human capital accumulation $(1 + (1-k)\,\omega < 1)$.

Substituting for the structural elasticities from (5.28) into (5.26):

$$\begin{bmatrix} \eta & -k\eta \\ -\omega & k\omega \end{bmatrix} \begin{bmatrix} g_A \\ g_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.29}$$

where $v_2 = -kv_1$ is sufficient for $|A| = 0$, as required for a balanced growth equilibrium.

To solve the model, we define $\phi \equiv A/Q^k$. The dynamic evolution of ϕ_t is subsequently derived from (5.25a), (5.25b) and (5.25c), after imposing (5.28):

$$g_\phi \equiv g_A - kg_Q = \left(\beta_A \phi_t^{-\eta} - k\beta_Q \phi_t^{\omega}\right) \tag{5.30}$$

where $\beta_A = B_A\,(q_A)^{k\eta}$ and $\beta_Q = B_Q\,(q_Q)^{1-k\omega}$. Along a balanced growth path, ideas grow proportional to aggregate human capital, as implied by (5.29). Thus, $g_\phi = 0$ implies a steady state value of:

$$\phi = \left(\frac{\beta_A}{k\beta_Q}\right)^{1/(\omega+\eta)} \tag{5.31}$$

which, by (5.30), is stable.

Using equations (5.25b), (5.25c), with (5.28), and (5.31), along a balanced growth path:

$$g_A = k^{\eta/(\omega+\eta)} (\beta_A)^{\omega/(\omega+\eta)} (\beta_Q)^{\eta/(\omega+\eta)} \tag{5.32a}$$

$$g_Q = g_h = k^{-\omega/(\omega+\eta)} (\beta_A)^{\omega/(\omega+\eta)} (\beta_Q)^{\eta/(\omega+\eta)} \tag{5.32b}$$

where $g_A = kg_Q > g_Q$.

We refer to the result $g_A > g_Q$ as *creative ignorance*, meaning that the frontier of disembodied knowledge (ideas) grows faster than the knowledge embodied in individuals. An individual's average skill level is given by $g_h = g_Q$. Accordingly, even in the absence of population growth, individuals tend to become relatively more ignorant over time, $g_A > g_h$.

The creation of new ideas draws on embodied knowledge, which in turn, accumulates by drawing on new ideas. Creative ignorance is a by-product of a virtuous circle of increasing ideas and human capital.

The notion that human capital continues to grow over time, but cannot keep abreast of innovation is consistent with anecdotal evidence from technology revolutions, including the recent information technology revolution. Innovation is a double-edged sword. Advances in technology drive long run growth in the economy and generate skill obsolescence in the process.

Intuitively, both ideas and human capital are employed in the creation of new ideas and in the production of human capital, but doubling ideas and human capital less than doubles the output of human capital and more than doubles the output of new ideas. If the returns to scale offset each other, ideas and human capital grow at constant rates, as required for a balanced growth equilibrium. However, the increasing returns to R&D implies growth in ideas will outstrip growth in human capital.

From (5.25a), the long run *growth rate* of per capita output is

$$g_y = \left(\sigma_A + \frac{\sigma_Q}{k}\right) g_A = (k\sigma_A + \sigma_Q)\, g_h \tag{5.33}$$

where $g_y \gtreqless g_A \Leftrightarrow \sigma_Q \gtreqless k(1-\sigma_A)$. These parameter restrictions, together with $k > 1$, imply returns to scale in final production. For instance, $\sigma_Q = k(1-\sigma_A)$ implies increasing returns to scale to growing factors in the production of final output $(k + (1-k)\,\sigma_A > 1)$.

5.5.4 Discussion

Dalgaard & Kreiner (2001) also predict that individuals tend to become relatively ignorant over time, but for a fundamentally different reason. Specifically, constant returns to scale implies that the total stock of knowledge embodied in individuals grows at the same rate as the stock of ideas: $g_A = g_Q$. Constant growth in the total stock of human capital, as required for balanced growth, in turn implies that, in the presence of population growth, growth in human capital per person must be less than growth in the aggregate stock of human capital: $g_Q - g_h = n > 0$. It follows that, provided population grows at a strictly positive rate, $g_A > g_h$.

Thus, Dalgaard & Kreiner (2001) attribute the relative ignorance of individuals to a growing population. There is nothing innate in the processes of R&D and human capital production to generate a perpetual lagging of growth in human capital in the absence of population growth.

The implication of their result is that as population growth tends to zero, the gap between growth in ideas and growth in human capital per person vanishes. In contrast, our creative ignorance result is robust to the absence of population growth and has unique implications for policy which we discuss below.

Policy Implications

The balanced growth equilibrium described by (5.32a), (5.32b) and (5.33) has two implications for policy. First, the long run growth in ideas, g_A, is proportionally higher than the long run growth in human capital, g_Q, by the factor $(k-1)$, which reflects the degree of increasing and decreasing returns to scale in R&D and human capital accumulation, respectively. As the returns to scale in both R&D and human capital accumulation tend to constant returns to scale, $k \to 1$, the gap between g_A and g_Q vanishes. However, to the extent that economic incentives cannot alter sectoral returns to scale, the creative ignorance result is impervious to government policy. Thus, in contrast to Dalgaard & Kreiner (2001), we predict that individuals may remain relatively ignorant, even if the government targets zero population growth. No government action may be preferable to misdirected government action.

Second, economic incentives and policy may affect the long run growth rate of the economy through q_A and q_Q, the portion of the labor force employed in R&D and human capital, respectively. This contrasts with existing lab equipment models where the policy target is R&D expenditure as a portion of total output. Importantly, increasing q_A is consistent with increasing q_Q. In contrast to Dalgaard & Kreiner (2001), raising the share of resources used in R&D is not at the expense of resources used for human capital formation.

Remark 5.2 *If a lab equipment model features non-identical production functions, an in-*

crease in the portion of the labor force engaged in R&D has an unambiguously positive effect on the long run growth rate of the economy

To simplify our exposition of this remark, we assume $\sigma_Q = k(1-\sigma_A)$, so that $g_y = g_A$. The total derivative of (5.32a) yields

$$\%\Delta g_y = \frac{\eta k \omega}{(\omega+\eta)}\%\Delta q_A + \frac{\eta(1-k\omega)}{(\omega+\eta)}\%\Delta q_Q \tag{5.34}$$

which is unambiguously positive if both q_A and q_Q increase.

Given a fixed pool of labor, we may raise the share of labor allocated to both R&D and human capital formation $(q_A + q_Q)$ by lowering the share of labor used in final production $(1 - q_A - q_Q)$. An unambiguously positive effect on the long run growth rate of per capita output may at first seem counterintuitive. How can we withdraw resources from the production of final output and achieve higher growth in final output? The simple answer is that we achieve higher growth by *investing* the resources taken from final production in accumulating stocks which ultimately drive growth.

For the full intuition, we distinguish instantaneous, transitional and long run effects. Referring to Figure 5.5, a decrease in the portion of labor allocated to the production of final output $(1 - q_A - q_Q)$, at time t_0, results in an instantaneous drop in the level of per capita output. Intuitively, up to and including at time t_0, the level of per capita output is determined by the initial steady state growth in ideas, g_A. After t_0, the investment of additional labor in R&D and human capital accumulation raises both g_A and g_Q. The economy transits to a new balanced growth path, where both the long run growth rate and level of per capita output are higher, as illustrated by the increased gradient and position of $\ln y_t$.

From (5.25a), the long run level of per capita output is

$$y_t = \left(\frac{\beta_A}{k\beta_Q}\right)^{\sigma/(\omega+\eta)} (1 - q_A - q_Q)^{k(1-\sigma)} L^{(k-1)} (h_0)^k e^{g_A t} \tag{5.35}$$

By (5.35), the exponential effect of a higher g_A overshadows the multiplicative effect of lower $(1 - q_A - q_Q)$, so that an increase in $(q_A + q_Q)$ unambiguously raises both the growth rate and level of output per capita in the long run.

Euler's Theorem and Factor Payments

The creative ignorance model can be given microfoundations and a decentralized equilibrium corresponding to (5.32a), (5.32b) and (5.33) can be derived. A decentralized economy comprises a household sector and firms in the sectors of manufacturing, intermediate goods, R&D and education. The R&D sector produces designs for intermediate goods, which are patentable.

Introducing increasing returns to scale to the R&D sector has important implications for how equilibrium in the R&D sector is decentralized. Existing lab equipment models of R&D assume the output of designs is the same constant returns to scale production function as in the manufacturing sector and/or the education sector. By Euler's Theorem[11], when the production function is homogeneous of degree one, all factors are paid their value of marginal product.

With increasing returns to scale, it is no longer possible for both of the inputs A and H to be paid the value of their marginal product. Since a central requirement of a

[11]If the function $Y(A, H)$ is homogeneous of degree k, then

$$kY = \frac{\partial Y}{\partial A}A + \frac{\partial Y}{\partial H}H$$

competitive equilibrium is that factors are paid their value of marginal product, increasing returns to scale necessitates imperfect competition.

Romer (1990) is a well-known precedent of a decentralized equilibrium for a R&D sector with increasing returns to scale. The production of ideas (in our notation, $\dot{A} = B_A A [q_A Q]$) is homogeneous of degree two. Since it is impossible for both inputs to be paid their value of marginal product, the assumption is made that A receives no compensation. Holders of patents on previous designs have no means of preventing inventors of new intermediate goods from using the ideas encapsulated in existing designs.

However, holders of patents on designs have legal means of preventing the *manufacture* of intermediate goods. Thus, imperfect competition appears in the intermediate goods sector. Firms in this sector own the patents for designs, enabling them to sell intermediate goods at a price greater than marginal cost. However, monopoly profits are extracted by the inventors in the form of compensation for their time spent researching or "searching" for new designs. The framework is one of monopolistic competition. No economic profits accrue: all rents compensate some factor input. In aggregate, physical capital is paid less than its marginal product and the remainder is used to compensate researchers for the creation of new ideas.

Increasing returns to scale to R&D in the creative ignorance model necessitates the same monopolistic competition framework. The lab equipment specification in R&D introduces some additional complexity. Whereas in Romer (1990), R&D is undertaken by researchers, in Rivera-Batiz & Romer's (1991) lab equipment model, we conceive R&D as being undertaken by separate firms that hire physical capital and labor. Whereas in Romer

(1990), the stock of human capital used in R&D is an exogenous constant, in the creative ignorance model, we conceive households investing in human capital.

Thus, the additional complexity appears in the household sector. Firms decide on resources employed in R&D and in the production of intermediate goods. Households decide on investing in human capital. The representative household earns a wage from investing in human capital, which it compares with a dividend from investing in patents. Decreasing returns to scale in human capital formation implies that H is paid more than its value of marginal product. This is consistent with a portion of the monopoly profits in the intermediate goods sector being extracted by households to compensate for investing in human capital.[12] Provided the total return on investment in human capital equals the total return on investment in ideas, the household will invest in both R&D and human capital formation.

Specifying the microfoundations of the creative ignorance model and solving for the decentralized equilibrium is therefore a challenging, but feasible exercise.

5.6 Conclusion

In the words of Solow (1956) p.65

> All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.

Existing lab equipment models of R&D assume researchers use not only the same inputs but also in the same input proportions as the final production technology. Under

[12] We may also consider payment of rent to the fixed factor, Z.

the assumption of identical input proportions, constant returns to scale to growing factors in all sectors becomes a necessary condition for a balanced growth equilibrium. This paper finds that, in general, constant returns is not necessary for balanced growth. Moreover, allowing for different returns to scale in R&D and human capital formation generates richer results.

The model presented in this paper generates a balanced growth equilibrium characterized by creative ignorance. Ideas and human capital increase in a virtuous circle, but the frontier of ideas grows faster than the knowledge embodied in individuals. The result that growth in ideas outstrips growth in human capital in the long run is consistent with a priori reasoning. Contrary to human capital and physical capital, ideas are innately non-rival and may therefore grow without bound.

The balanced growth equilibrium has a number of interesting characteristics. Individuals become relatively ignorant over time. Raising the share of resources used in R&D has an unambiguously positive effect on the long run growth rate of the economy.

These results are important when it comes to assessing future prospects and policy implications. According to existing literature, growth in ideas outstrips growth in human capital per person only if population grows at a strictly positive rate. By implication, zero population growth in the long run remedies the relative ignorance of individuals. In contrast, the model presented in this paper predicts that individuals become relatively ignorant over time, regardless of whether their population grows. In light of the United Nations's (2005) forward projection of zero population growth in the OECD, a source of relative ignorance other than population growth warrants further investigation.

In contrast to existing lab equipment models, raising the share of resources used in R&D has an unambiguously positive effect on the long run growth rate of the economy. When R&D and human capital formation use the same input proportions as final production, an increase in R&D's share of output is at the expense of education's share of output. Allowing for different input proportions, the share of resources invested in both R&D and human capital formation may be raised.

The conclusions of our model must be tempered by two key qualifications. First, increasing returns to scale to growing factors in R&D must be offset by decreasing returns to scale to growing factors in human capital formation in such a way that the matrix of structural elasticities is singular. Endogenous growth requires such knife edge restrictions. Whether the equations that are the engine of growth are homogeneous of a degree other than one is a matter for empirical investigation.

Second, there are many details of R&D at the micro level from which our model abstracts. Importantly, increasing returns to scale in R&D necessitates imperfect competition, for which Romer's (1990) monopolistic framework is a precedent. The equilibrium growth rates in a decentralized economy will resemble those provided in this paper, except for terms that capture the negative spillovers of imperfect competition that a central planner internalizes. A decentralized analysis should confirm and provide insight into our results.

Given these qualifications, our only claim is to have formalized, and we hope illuminated, an effect that is potentially important. There are other explanations, but in future empirical and theoretical work, we argue that creative ignorance induced by increasing returns to scale in R&D is worth considering.

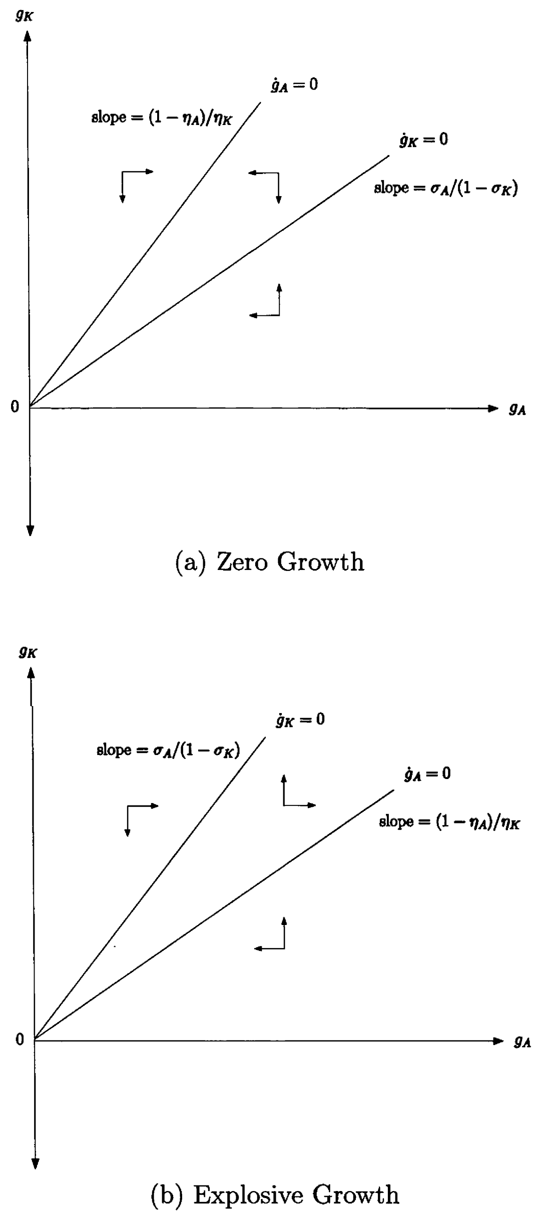


Figure 5.1: Phase Diagram in Growth of Physical Capital and Ideas

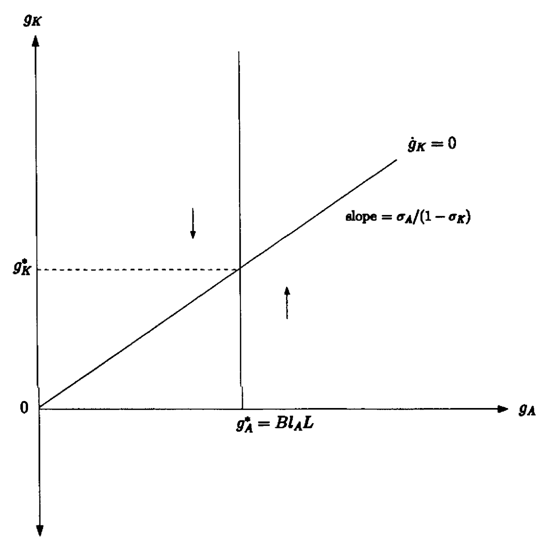


Figure 5.2: Phase Diagram in Growth of Physical Capital for a Romer-type Linear R&D equation

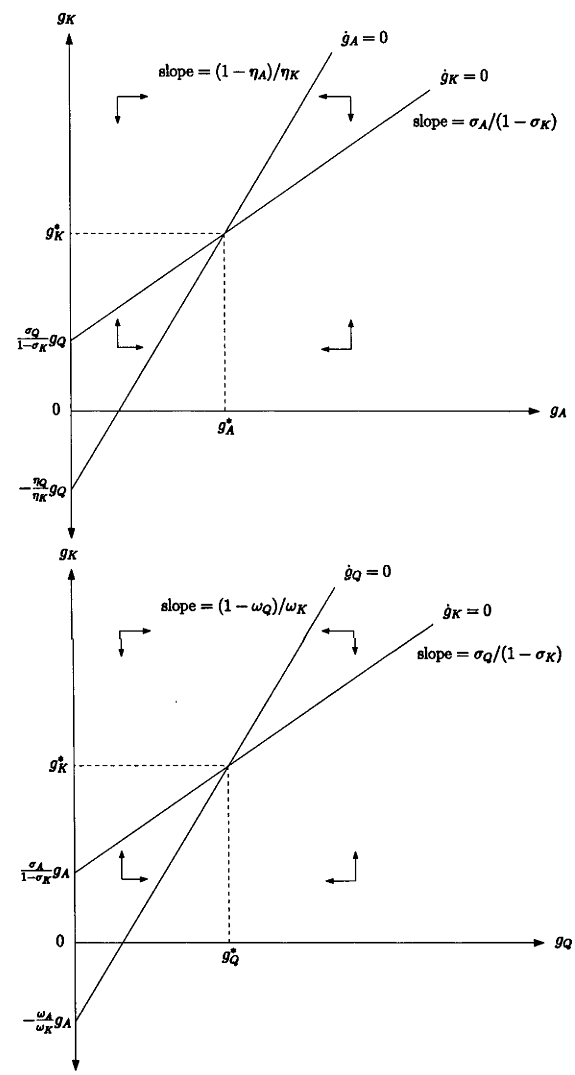


Figure 5.3: Phase Diagram in Growth of Capital and Two Aspects of Knowledge

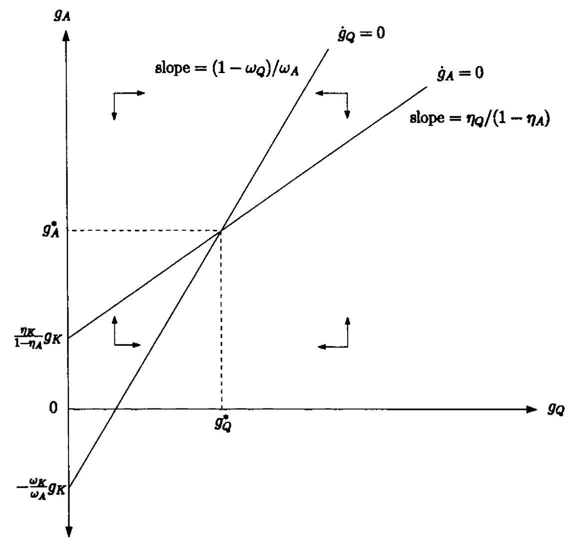


Figure 5.4: Phase Diagram in Growth of Two Aspects of Knowledge

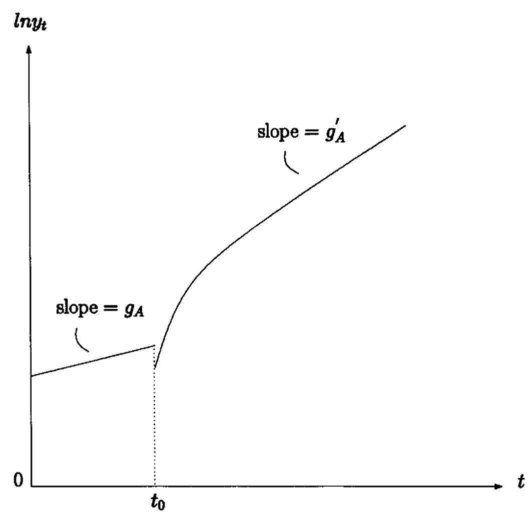


Figure 5.5: Intertemporal effect of an increase in $(q_A + q_Q)$ at t_0

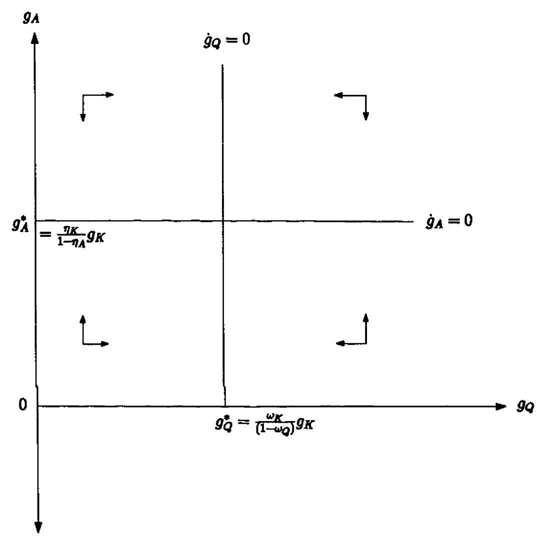


Figure 5.6: Phase Diagram for example (5.24)

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Appendix A

Appendix to Chapter 1

Definition A.1 *A* ***balanced growth equilibrium*** *is an equilibrium in which all real variables grow at constant, though not necessarily equal rates.*

Definition A.2 *A linear differential equation is of the form* $dX/dt \equiv \dot{X} = \_X$ *, where the blank may be endogenous and/or exogenous. Examples from the R&D-based literature include the population equation* $\dot{L} = nL$ *(Jones 1995a), a human capital accumulation equation* $\dot{H} = l_A H$ *(Lucas 1988), and a R&D equation* $\dot{A} = l_A LA$ *(Romer 1990).*

Definition A.3 ***Sectoral linearity*** *in the accumulation of knowledge implies that the output of new knowledge will double whenever we double the existing stock of knowledge.* $\dot{H} = l_A H$ *and* $\dot{A} = l_A LA$ *are examples of sectoral linearity in the human capital formation and R&D sectors, respectively. We distinguish sectoral linearity from linearity, in general, which can take the form of constant returns to scale in models with multiple state variable.*

The following definitions relate to the characteristics of the growth rate of the economy, as measured by the growth in per capita output, in a balanced growth equilibrium.

Definition A.4 *Growth is* ***endogenous*** *if long run growth in the economy is determined within the model, rather than by some exogenously growing variables like population growth or unexplained technological progress.*

Definition A.5 *Growth is* ***semi-endogenous*** *if technological change is endogenously determined, but long run growth in the economy requires growth in a factor exogenous to the model, population. (1.1c) is an example.*

Definition A.6 *Growth is* ***scale*** *or exhibits strong scale effects if both the growth rate and level of per capita output are increasing in the population level. (1.1a) is an example.*

Definition A.7 *Growth is* ***non-scale*** *if the long run growth rate of the economy does not vary with the size of the economy as typically measured by its population. Examples of non-scale growth are (1.1c), (1.1d) and (1.1e). Thus, non-scale growth need not imply semi-endogenous growth. Christiaans (2004) shows that the notions of non-scale growth and semi-endogenous growth are logically independent.*

Definition A.8 *Growth exhibits* ***weak scale effects*** *if the growth rate of per capita output does not vary with population level, but the level of per capita output is increasing in the population level. For example, (1.1c) exhibits weak scale effects; (1.1e) does not exhibit weak scale effects.*

Appendix B

Appendix to Chapter 2

B.1 Proof of Proposition 2.1

$$\phi'(k_t) = \tfrac{1-\gamma}{\gamma}(1-\beta_t)\tfrac{\partial \hat{x}}{\partial x}\tfrac{\partial x}{\partial w_t^m}\tfrac{\partial w_t^m}{\partial k_t} > 0 \quad \forall \quad k_t \in (0, k*)$$

Using $x_t = \hat{x} n_t$ and rearranging from (2.20),

$$\frac{\partial \hat{x}}{\partial x} = \frac{\partial(x/n)}{\partial x} = \alpha_1^a \left[\alpha_1^a + (\alpha_2 x_t)^a\right]^{\frac{-1-a}{a}} > 0$$

$$\frac{\partial x}{\partial w_t^m} = \frac{\gamma}{(1-\beta_t)} > 0$$

$$\frac{\partial w_t^m}{\partial k_t} = \alpha A(1-\alpha)(1-\rho)\left[\alpha k_t^\rho + (1-\alpha)\right]^{1/\rho - 2} k_t^{\rho-1} > 0$$

$$\phi''(k_t) = (1-\gamma)\alpha_1^a \left[\alpha_1^a + (\alpha_2 x_t)^a\right]^{\frac{-1-2a}{a}} \left\{(-1-a)\alpha_2^a x^{a-1} \frac{\gamma}{1-\beta_t}\frac{\partial w_t^m}{\partial k_t} + \left[\alpha_1^a + (\alpha_2 x_t)^a\right]\frac{\partial^2 w_t^m}{\partial k_t^2}\right\}$$

Noting that

$$\frac{\partial^2 w_t^m}{\partial k_t^2} = \alpha A(1-\alpha)(1-\rho)k_t^{\rho-2}\left[\alpha k_t^\rho + (1-\alpha)\right]^{1/\rho-3}\left\{(1-\alpha)(\rho-1) - \alpha\rho k_t^\rho\right\}$$

where

$$\varepsilon_{L^m,K} > 1 \quad \Leftrightarrow \quad \rho \in [0,1) \quad \Rightarrow \quad \partial^2 w_t^m / \partial k_t^2 < 0$$

$$(-1-a) < 0 \quad \Leftrightarrow \quad \varepsilon_{x,z} > 0.5$$

Given $\varepsilon_{L^m,K} > 1, \quad \varepsilon_{x,z} > 0.5 \quad \Rightarrow \quad \phi''(k_t) < 0$

B.2 Proof of Proposition 2.4

Noting that (2.12) is homogeneous of degree 1, $z_t^* = \partial p/\partial w_t^m.n_t$ and $x_t^* = -\partial p/\partial \beta_t.n_t$.

$$\partial n_t * /\partial w_t^m = \frac{\gamma 2\left[w_t^m.\partial p/\partial w_t^m + (1-\beta_t). - \partial p/\partial \beta_t\right] - \gamma(w_t^p + 2w_t^m)\partial p/\partial w_t^m}{\left[p(w_t^m, \beta_t)\right]^2}$$

$$\partial n_t * /\partial w_t^m = \frac{\gamma 2\left[(1-\beta_t). - \partial p/\partial \beta_t\right] - \gamma w_t^p \partial p/\partial w_t^m}{\left[p(w_t^m, \beta_t)\right]^2} < 0$$

$$\partial n_t/\partial w_t^m < 0 \Leftrightarrow \frac{-\partial p/\partial \beta_t}{\partial p/\partial w_t^m} < \frac{w_t^p}{2\left[1-\beta_t\right]}$$

Substituting from (2.10) and (2.11),

$$\partial n_t/\partial w_t^m < 0 \Leftrightarrow \frac{(\alpha_2)^{-a/a-1}(1-\beta_t)^{1/a-1}}{(\alpha_1)^{-a/a-1}\,(w_t^m)^{1/a-1}} < \frac{w_t^p}{2\left[1-\beta_t\right]}$$

$$\partial n_t/\partial w_t^m < 0 \Leftrightarrow \left(\frac{\alpha_1}{\alpha_2}\right)^{a/a-1}\left(\frac{1-\beta_t}{w_t^m}\right)^{1/a-1} < \frac{w_t^p}{2\left[1-\beta_t\right]}$$

Given the definition of α_1, α_2 and ε,

$$\partial n_t/\partial w_t^m < 0 \quad \Leftrightarrow \quad \left[\tfrac{w_t^m}{1-\beta_t}\tfrac{1-\delta}{\delta}\right]^{\varepsilon} < \frac{w_t^p}{2\left[1-\beta_t\right]}$$

$$\partial n_t/\partial w_t^m = 0 \quad \Leftrightarrow \quad w_t^m = w^m ** = \left(\tfrac{w_t^p}{2}\right)^{1/\varepsilon}\left(\tfrac{\delta}{1-\delta}\right)(1-\beta_t)^{(\varepsilon-1)/\varepsilon}$$

B.3 Proof accompanying discussion of Proposition 2.5

$$w^m ** = \frac{\delta}{1-\delta}\left[e^{\ln(1-\beta_t)}\right]^{(\varepsilon-1)/\varepsilon}\left[e^{\ln\left(w_t^p/2\right)}\right]^{1/\varepsilon}$$

$$\frac{\partial w^m **}{\partial \varepsilon} = \frac{\delta}{1-\delta}\frac{1}{\varepsilon^2}\left(1-\beta_t\right)^{(\varepsilon-1)/\varepsilon}\left(\frac{w_t^p}{2}\right)^{1/\varepsilon}\left\{\ln\left(1-\beta_t\right)-\ln\left(\frac{w_t^p}{2}\right)\right\}$$

$$\frac{\partial w^m **}{\partial \varepsilon} < 0 \;\Leftrightarrow\; \frac{w_t^p}{2} > \left(1-\beta_t\right)$$

Appendix C

Appendix to Chapter 3

C.1 The central planner's optimization problem

The central planner's control variables are c, l_A, l_H, k_A and l_H. Utilizing relationships, such as, $F_H = (1 - l_A - l_H)\, F_{(1-l_A-l_H)H}$, the first order optimality conditions can be summarized in terms of F_j, J_j and Q_j $(j = A, K, H)$:

$$c^{-\theta} = \lambda L \tag{C.1a}$$

$$\lambda \frac{F_H}{(1 - l_A - l_H)} = \mu \frac{J_H}{l_A} = \gamma \frac{Q_H}{l_H} \tag{C.1b}$$

$$\lambda \frac{F_K}{(1 - k_A - k_H)} = \mu \frac{J_K}{k_A} = \gamma \frac{Q_K}{k_H} \tag{C.1c}$$

$$\rho - \frac{\dot{\lambda}}{\lambda} = F_K + \frac{\mu}{\lambda} J_K + \frac{\gamma}{\lambda} Q_K \tag{C.1d}$$

$$\rho - \frac{\dot{\mu}}{\mu} = \frac{\lambda}{\mu} F_A + J_A + \frac{\gamma}{\mu} Q_A \tag{C.1e}$$

$$\rho - \frac{\dot{\gamma}}{\gamma} = F_H + \frac{\mu}{\gamma} J_H + Q_H \tag{C.1f}$$

where λ, μ and γ are the shadow values of aggregate physical capital, ideas and human capital, respectively. If $n = 0$, we replace the first condition with $C^{-\theta} = \lambda$.

C.2 Hawkins-Simon Conditions when $[d_i \geq 0]$

Consider a non-homogeneous system of equations:

$$\sum_{j=1}^{m} a_{ij} x_j = d_i \quad (i = 1, ..., m) \tag{C.2}$$

where $a_{ij} < 0$ for all $i \neq j$; $a_{ii} > 0$ for all i and $|A| = |a_{ij}| \neq 0$. System (C.2) expressed in matrix form is $Ax = d$, where A is an $m \times m$ matrix.

Corollary C.1 (to Hawkins & Simon's (1949) Theoreom) *A necessary and sufficient condition that the x_i satisfying (C.2) be all positive for any set $[d_i \geq 0]$ with at least one $d_i > 0$ is that all principal minors of the matrix $\|a_{ij}\|$ be positive.*

Proof. Since conditions in terms of principal minors and determinants are generalizable to higher dimensions, we prove this corollary by reference to a two dimensional system:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} \tag{C.3}$$

where $a_{ij} < 0$ for all $i \neq j$; $a_{ii} > 0$ for all i, assumptions underlying the Hawkins-Simon conditions. $d_1 > 0$ and $d_2 = 0$ satisfy a weaker assumption of $[d_i \geq 0]$ with at least one $d_i > 0$.

Suppose (x_1, x_2) satisfying (C.3) are positive, but not all the principal minors of A in (C.3) are positive. Specifically, the first principal minor is positive, but the second principal minor is non-positive: $|A_1| = a_{11} > 0; |A_2| = |A| = a_{11}a_{22} - a_{21}a_{12} \leq 0$. We show that $|A| \leq 0$ implies an inconsistency. Substituting from the first equation in (C.3) for x_1

in the second equation gives

$$(a_{11}a_{22} - a_{21}a_{12})\, x_2 = -a_{21}d_1 \tag{C.4}$$

For the equality in (C.4) to hold, $x_2 > 0$ together with $(a_{11}a_{22} - a_{21}a_{12}) \leq 0$ implies $-a_{21}d_1 \leq 0$, which can only be true if either $a_{21} \geq 0$ (and $d_1 \geq 0$) or $d_1 \leq 0$ (and $a_{21} \leq 0$), a contradiction of either $a_{ij} < 0$ for all $i \neq j$ and $d_1 > 0$, both underlying assumptions, which together imply $-a_{21}d_1 > 0$. Thus, all principal minors of matrix A are positive is a necessary condition for (x_1, x_2) satisfying (C.3) to be all positive.

To prove that all principal minors of matrix A be positive is a sufficient condition for (x_1, x_2) we show that if $|A_1| = a_{11} > 0; |A_2| = |A| = a_{11}a_{22} - a_{21}a_{12} > 0$ then either $x_1 \leq 0$ or $x_2 \leq 0$ implies a contradiction. Consider $|A| = a_{11}a_{22} - a_{21}a_{12} > 0$, but $x_2 \leq 0$. The equality in (C.4) requires $-a_{21}d_1 \leq 0$, which contradicts either $a_{ij} < 0$ for all $i \neq j$ or $d_1 > 0$, both underlying assumptions. ■

C.3 Derivation of equation (3.12)

$$g_H = \frac{\dot{H}}{H} = \frac{Q\,(H, k_H K)}{H}$$

Differentiating g_H with respect to time,

$$\dot{g}_H = \frac{\left(Q_{k_H}\dot{k}_H + Q_H\dot{H} + Q_K\dot{K}\right)H - Q\,(H, k_H K)\,\dot{H}}{H^2}$$

Noting that $\dot{k}_H = 0$ and $\omega_H \equiv \frac{Q_H H}{Q}$ and $\omega_K \equiv \frac{Q_K K}{Q}$, we obtain:

$$\dot{g}_H = g_H \left\{\omega_H g_H + \omega_K g_K - g_H\right\}$$

Substituting for ω_H and ω_K:

$$\omega_H = \frac{(\phi_1 H)^\rho}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]}$$

$$= \left(\frac{(\phi_1 H)}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]^{1/\rho}} \right)^\rho$$

$$= \left(\frac{\phi_1}{g_H} \right)^\rho$$

$$\omega_K = \frac{(\phi_2 k_H K)^\rho}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]}$$

$$= \frac{(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho - (\phi_1 H)^\rho}{[(\phi_1 H)^\rho + (\phi_2 k_H K)^\rho]}$$

$$= 1 - \left(\frac{\phi_1}{g_H} \right)^\rho$$

gives us equation (3.12).

C.4 Solution to Section 3.5.1

For the production structure (3.21a)-(3.21c), the first order optimality conditions relevant to solving the model are:

$$C^{-\theta} = \lambda \tag{4a'}$$

$$\lambda(1-\sigma)\frac{Y}{(1-l_A-l_H)\,H} = \gamma\xi \tag{4b'}$$

$$\dot{\lambda} = \rho\lambda - \sigma\frac{Y}{K}\lambda \tag{4d'}$$

$$\dot{\gamma} = \rho\gamma - \xi\gamma \tag{4f'}$$

Total differentiation of (4a') with respect to time, after inserting (4d') gives Euler's equation: $g_C = \frac{1}{\theta}\left(\sigma\frac{Y}{K} - \rho\right)$. As shown in the beginning of Section 4, the growth rate in physical capital is constant when $g_K = g_Y = g_C$. Differentiating (4b') with respect to time, we obtain $\dot{\lambda}\,(1-\sigma)\,\frac{Y}{(1-l_A-l_H)H} = \dot{\gamma}\xi$, so that from (4d) and (4f), we get $\sigma\frac{Y}{K} = \xi$. Substituting for $\sigma\frac{Y}{K} = \xi$ in Euler's equation yields:

$$g_Y = g_K = g_C = \frac{\xi - \rho}{\theta}$$

Appendix D

Appendix to Chapter 4

D.1 Derivation of ω in (4.22)

From $\omega_Q \equiv \frac{H_Q Q}{H}$ and $\omega_K \equiv \frac{H_K K}{H}$,

$$\begin{aligned}
\omega_Q &= \frac{(\phi_1 q_Q Q)^\rho}{[(\phi_1 q_Q Q)^\rho + (\phi_2 \alpha_Q K)^\rho]} \\
&= \left(\frac{(\phi_1 q_Q Q)}{[(\phi_1 q_Q Q)^\rho + (\phi_2 \alpha_Q K)^\rho]^{1/\rho}} \right)^\rho \\
&= \left(\frac{\phi_1 q_Q}{g_Q} \right)^\rho \equiv \omega
\end{aligned}$$

$$\omega_K = \frac{(\phi_2 \alpha_Q K)^\rho}{[(\phi_1 q_Q Q)^\rho + (\phi_2 \alpha_Q K)^\rho]}$$

$$= \frac{(\phi_1 q_Q Q)^\rho + (\phi_2 \alpha_Q K)^\rho - (\phi_1 q_Q Q)^\rho}{[(\phi_1 q_Q Q)^\rho + (\phi_2 \alpha_Q K)^\rho]}$$

$$= 1 - \left(\frac{\phi_1 q_Q}{g_Q}\right)^\rho = 1 - \omega$$

Differentiating $g_Q = \frac{\dot{Q}}{Q} = \frac{H(q_Q Q, \alpha_Q K)}{Q}$ with respect to time and noting that $\dot{q}_Q = 0$ and $\dot{\alpha}_H = 0$ along a balanced growth path, and $\frac{H_Q Q}{H} = \omega$ and $\frac{H_K K}{H} = 1 - \omega$, yields

$$\dot{g}_Q = g_Q \left(g_K - g_Q\right) \left\{1 - \left(\frac{\phi_1 q_Q}{g_Q}\right)^\rho\right\}$$

which implies a stable steady state of g_K or $q_Q \phi_1$, depending on whether $\epsilon < 1$ or $\epsilon > 1$, respectively.

D.2 Derivation of (4.26)

We define $\chi = A/K$. The dynamic evolution of χ_t is subsequently derived from equations (4.24a) - (4.24c):

$$g_\chi \equiv g_A - g_K = \beta_A \chi_t^{\eta-1} L^\eta - \beta_K \chi_t^\sigma L^\sigma \quad \text{(D.1)}$$

where $\beta_K = s\ (1 - \alpha_A)^{1-\sigma} (1 - l_A)^\sigma$ and $\beta_A = B\,(\alpha_A)^{1-\eta} (l_A)^\eta$.

Along a balanced growth path, where g_y is constant, ideas and physical capital have to grow at the same rate, as implied by (4.25). Thus, $g_\chi = 0$ implies a steady state ratio of ideas to physical capital:

$$\chi = \left(\frac{\beta_A L^{\eta-\sigma}}{\beta_K}\right)^{1/(\sigma+1-\eta)} \quad \text{(D.2)}$$

which simplifies to $\chi = B/s$ in the special case where final production and R&D have the same production technology ($\eta = \sigma$). Substituting for (D.2) in (4.24a) gives the long run *growth rate* of per capita output:

$$g_y = g_Y = g_K = g_A = \left(\beta_K{}^{1-\eta}\beta_A{}^{\sigma}L^{\sigma}\right)^{1/(\sigma+1-\eta)} \tag{D.3}$$

and, simplified for the case of $\eta = \sigma$, the long run *level* of per capital output:

$$y_t \equiv \frac{Y_t}{L_t} = \left(\frac{s}{BL}\right)^{1-\sigma} A_0 e^{g_y t} \tag{D.4}$$