A Class of Energy Minimisers for the Rotating Drop Problem

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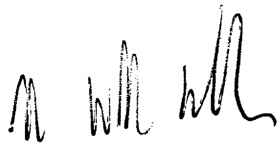
Statement of Authorship

This thesis contains no material which has been accepted for the award of any other degree or diploma. To the best of my knowledge, the work of no other person has been used without due acknowledgement in the text.



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Abstract

This dissertation examines the behaviour of an incompressible, rotating liquid drop withdrawn from the action of gravity and cohered by surface tension. It is composed of two parts, which we summarise below.

In Part 1, we implement classical techniques from the calculus of variations and an implicit function theorem on Banach spaces to derive constraints on angular velocity under which we may verify the existence, boundary regularity, and stability of an energy-minimising family of drops whose boundaries are in $C^{3,\alpha}$ proximity to the unit sphere S^n.

In Part 2, we define, and examine the properties of, a geometric evolution equation corresponding to the problem, in the context of the more contemporary methodology associated with mean curvature flow. Whenever appropriate restrictions are placed on a given initial condition, we obtain unique, global solutions of this *rotating drop flow* which are demonstrated to converge asymptotically to the aforementioned minimisers in infinite time with respect to the topology of $C^\infty(S^n)$. We then establish that the boundary of any such minimiser is uniquely determined in a Lipschitz neighbourhood of S^n.

A Remark on Notation

The notation employed in the body of this thesis is largely self-contained. However, a basic familiarity with the definition and properties of the Hölder, Lipschitz, and Sobolev function spaces will be assumed.

A summary of essential geometric terminology is provided in Section 1 of Appendix A. In addition, we shall stipulate here that the set $\{e_i : i \in \mathbb{N} \;\&\; 1 \leq i \leq n+1\}$ is the standard orthonormal basis for $\mathbb{R}^{n+1}$, while an inner product $\langle \cdot, \cdot \rangle$ appearing without an attached subscript should automatically be perceived as representing the Euclidean scalar product.

Throughout the dissertation, unless otherwise noted, we shall observe the summation convention with respect to repeated raised and lowered indices. Moreover, for integers j and k with $j < k$, we shall specify here that $\sum_{i=k}^{j}(\cdot) \equiv 0$ and $\prod_{i=k}^{j}(\cdot) \equiv 1$.

Contents

Introduction

Investigation of gravitational equilibrium in homogeneous, uniformly rotating masses originated in Newton's treatise on the figure of the earth [**62**], in which he demonstrated that the action of a small rotation on the body about its vertical axis mandates a slight equatorial oblateness. He further deduced that the equilibrium of the figure necessitates a basic proportionality between the effect of rotation, as measured by the 'ellipticity' or mean equatorial dilation, and its cause, which may be quantified by the quotient of the centrifugal acceleration at the equator and the average gravitational acceleration on the surface. These results were refined by Maclaurin [**58**] and inspired work in the 19th and 20th centuries by Jacobi [**50**], Riemann [**72**], Poincaré [**69**], Darwin [**24**], Cartan [**18**], Appell [**7**], and Chandrasekhar [**20**], amongst others.

Contemporaneous to the activities of Jacobi, Riemann, Poincaré, and Darwin was an astonishing series of experimental and theoretical explorations conducted by the Belgian mathematician Plateau [**68**] between 1843 and 1869. These included study of a rotating liquid drop cohered by surface tension, which was modelled through the suspension of oil in a fluid of equivalent density. Despite the manifest cosmogonical implications of Plateau's *rotating drop problem*, the obvious resonance of these two extant considerations appears to have been neglected at the time. It is also important to emphasise that Plateau's work has had extensive mathematical ramifications, most especially in geometry. In particular, his experimental analysis of soap films and bubbles has inspired a vast canon of work on the theory of minimal surfaces and surfaces of constant mean curvature (in the classical setting see [**25, 2, 76, 42**] or [**10**], for instance).

Natural phenomena such as these soap films and the boundary of the rotating drop are but two examples of *capillary surfaces*: free surface interfaces between two immiscible materials, at least one of which is a fluid. Beyond its geometric implications, the study of capillarity has found application in diverse industries, including aeronautical engineering and medicine. Although the subject of contemplation by scientific thinkers as early as da Vinci (see, for example, [**57**]), a more rigorous theory of capillarity did not arise until the 19th century in the writings of Young [**83**] and Laplace [**52**], which were elaborated upon by Gauss [**38**]. A seminal account of both historical and contemporary developments in this field may be found in the text of Finn [**31**].

Part 1

Pursuant to the research of Plateau, analyses of permissible equilibrium configurations for the problem of the rotating drop were performed by Rayleigh [**71**] and Lichtenstein [**53**]. In particular, Hölder [**46**] confirmed in 1926 the existence of a family of star-shaped three-dimensional drops determined by regular boundary, each member of which occurs as the unique critical point of an appropriately defined energy functional indexed by a small angular velocity. Resurgent interest in the question of the rotating drop was subsequently motivated by the study of nuclear fission, perhaps exemplified by the conjecture of Bohr & Wheeler [**11**] that infinitesimal liquid drops under an action of rotation mirror the behaviour of heavy atomic nuclei.

In 1965, Chandrasekhar [**19**] produced a formative description of stability criteria for axisymmetric, three-dimensional rotating drops enclosing the origin. Extended to encompass capillary forces, his principal analytical tool was the *method of the tensor virial* [**20**], an application of the 'method of moments' in mathematical physics to the solution of hydrodynamical problems which account for the gravitational field of the prevailing distribution of matter. Embellished by variational techniques and numerical analysis, this approach elucidated the geometric behaviour of equilibrium configurations for increasing values of the physically determined parameter

$$\Sigma = \frac{\delta\,\omega^2 a^3}{8T}\,.$$

Here ω is the angular velocity, a is the equatorial radius, δ is the density, and T is the interfacial surface tension of the drop. More precisely, it was demonstrated that the initial spheroidal configuration reaches a neutral but stable mode of oscillation at $\Sigma = 0.4587$, while instability occurs subsequently at $\Sigma = 0.8440$ due to overstable oscillations with frequency ω. These findings were extended by Brown & Scriven [**15**], while Auchmuty [**9**], Caffarelli & Friedman [**16**], Friedman & Turkington [**33, 34**], Brulois & Ross [**12**], Brulois [**13**], and Sturzenhecker [**77**] have further illuminated the axisymmetric case in three dimensions. Furthermore, Ross & Smith [**73, 74**] have considered the existence and stability of (potentially) non-axisymmetric drops in $\mathbb{R}^3$.

For rotating drops of arbitrary dimension n with prescribed volume and barycentre, Albano & Gonzalez [**1**] utilised a measure-theoretic approach introduced by De Giorgi [**25**] to establish, among sets of finite perimeter, the existence of connected local energy minimisers which correspond to appropriately small angular velocities. Despite the relatively weak regularity assumptions on its ambient geometric class, such a minimiser was demonstrated to possess $C^{1,\alpha}$ boundary, save a possible singular set of Hausdorff dimension at most $(n-8)$. To account for fixed barycentre in the class of drops under scrutiny, the techniques employed here extend those of Gonzalez, Massari, & Tamanini [**41, 42**] which respectively analyse related capillary and isoperimetric problems. Both the geometric construction and the energy functional introduced in [**1**] are particularly significant to our intended framework of investigation, although we shall require $C^{3,\alpha}$ boundary regularity in our class of rotating drops (here the reader is referred to Definition 1.1 in Chapter 1). Congedo, Emmer, & Gonzalez [23], Congedo [22], and Athanassenas [8] have derived analogous results to those of Albano & Gonzalez in the case of rotating drops with obstacles, where the free boundary formed by intersection with these obstacles necessitates the inclusion of an additional capillarity term in the energy functionals under consideration. While the barycentricity constraint imposed in the literature referenced above connotes a form of symmetry, it is a far less restrictive condition than the axisymmetric formulation prescribed in the purely physical case.

In this first part of the thesis, we verify the existence of a family of energy minimisers for the elliptic problem in a neighbourhood of the unit ball, whenever angular velocity is sufficiently restricted (Theorem 4.9). The essence of this determination is contained in a recently accepted paper [81]. It is achieved both by parametrising the boundary of any compact, star-shaped drop as a (unique) $C^{3,\alpha}$ graph above S^n and, in analogy with the method of Lagrange multipliers, the ensuing analysis of an energy functional augmented by a linear combination of integrals which permit variation of the relevant isoperimetric constraints. After identifying Banach subspaces of $C^{k,\alpha}$ which contain all test functions generating volume- and barycentre-preserving perturbations, we implement an implicit function theorem on Banach spaces to deduce the existence of solutions to the Euler-Lagrange equation in a $C^{3,\alpha}$ neighbourhood of S^n (Theorem 3.1). Higher regularity is then evinced through a suitable 'bootstrapping' argument (Theorem 3.3). This facilitates the calculation of an eigenvalue estimate for the Jacobi operator corresponding to the aforementioned functional (Theorem 4.4). Both the implicit function theorem and the stability analysis rely heavily on the properties of the spherical harmonics of degree two or greater. In particular, proximity to the sphere is crucial in obtaining estimates on expressions arising in the numerator of the Rayleigh quotient associated with the Jacobi operator. An overview of the relevant theory of the spherical harmonics is provided in Appendix C.

Part 2

In the context of geometric measure theory, the *mean curvature flow* was first introduced by Brakke [14], who explored many of the principal attributes of the associated solutions whilst developing a global existence and regularity theory. For a sufficiently regular class of geometric objects, it is instructive to observe that the mean curvature flow may be constructed from the L^2-gradient flow of the energy functional corresponding to surface tension. More precisely, suppose that $(N^n, \tilde{\mathbf{g}})$ and $(M_0^n, \mathbf{g_0})$ are compact, connected Riemannian manifolds without boundary smoothly embedded in $\mathbb{R}^{n+1}$. Further suppose that M_0 may be parametrised by the embedding $X_0 : N \to M_0$ and T is a positive real number. Then the family of manifolds $\{M_t : t \in (0,T)\}$ parametrised by $X : N \times (0,T) \to \mathbb{R}^{n+1}$ evolves by the mean curvature flow if

$$\frac{\partial}{\partial t} X(p,t) = -H(p,t)\nu(p,t) , \qquad \text{(MCF)}$$

with initial condition $X(\cdot,0) = X_0(\cdot)$. Here $\nu(\cdot,t)$ and $H(\cdot,t)$ respectively represent the outward oriented unit normal vectorfield and mean curvature on M_t.

Curvature flows of the type (MCF) (and others) have found application in a multitude of physical and industrial settings, including image processing and as a model for the motion of grain boundaries in an annealing pure metal. However, in accordance with the motivation for this thesis, it is their application to geometric problems with which we are primarily concerned. Indeed, an example of this geometric import may be evidenced by the recent work of Perelman [**64, 65, 66**] which contends a solution to Thurston's geometrisation conjecture[1] [**79**] through the employment of methodology associated with the *Ricci flow.* Pioneering work on the properties of this evolution equation in the framework of dimensions three and four was performed by Hamilton [**44, 45**].

Although the mean curvature flow and its progeny have been analysed from a variety of mathematical perspectives, we shall restrict our attention to those most pertinent to the exposition of this thesis. In 1984, Huisken [**47**] proved that, when confined to strictly convex hypersurfaces, equation (MCF) possesses a smooth solution on a finite time interval while the evolving hypersurface contracts to a 'round' point as the endpoint of the interval is approached. Homothetic expansion about this point then yields a smooth, global solution in the rescaled variables, where the resulting hypersurfaces are demonstrated to converge asymptotically in the topology of C^∞ to a sphere with surface measure $|M_0|$. Andrews [**4**] has subsequently derived analogous results for a more general class of geometric evolution equations. Shortly thereafter, a comparable determination to that of Huisken was verified in the case $n = 1$ by Gage & Hamilton [**36**], while the case of evolving curves has again been studied for a far broader class of curvature flows by Andrews [**5, 6**]. Huisken is also responsible for the celebrated *monotonicity formula* [**49**] describing the asymptotic behaviour of solutions and potential singularities through this procedure of homothetic rescaling. A local version of this result has been deduced by Ecker [**26**].

Of particular relevance to the subject matter of this thesis is another paper of Huisken [**48**], which analyses the *volume-preserving mean curvature flow.* Through the calculation of the steepest descent flow over a class of hypersurfaces characterised by constrained volume, this is constructed by the addition of the quantity $h(t)\nu(p,t)$ to the right hand side of (MCF), where, for each $t \in (0,T)$, $h(t)$ is the global term given by

$$h(t) \;=\; \fint_{M_t} H d\mu_t\,.$$

In contrast to solutions of the original flow, it is then established that this modified evolution equation preserves enclosed volume whilst decreasing surface measure. Furthermore, due to a pinching estimate on the principal curvatures of the arising family of hypersurfaces in conjunction with *a priori* bounds on curvature derivatives of all orders, Huisken

[1] This concerns the complete topological classification of closed three-dimensional manifolds, of which Poincaré's conjecture [**70**] may be interpreted as a subcase.

proved that convexity in the family is maintained and the smooth solutions converge exponentially in infinite time to a sphere with enclosed volume identical to that of the initial condition. This finding followed the analogous result for curves due to Gage [**35**], and has been extended to encompass non-convex hypersurfaces in a $C^{1,\alpha}$ neighbourhood of S^n by Escher & Simonett [**29**], wherein the regularity theory is reliant upon the analytic semigroup treatment of parabolic equations formalised in the texts of Amann [**3**, Chapter III] and Lunardi [**55**, Chapter 9]. Pihan [**67**] and McCoy [**59**] have respectively pursued similar arguments to those of Gage & Hamilton [**35, 36**] and Huisken [**47, 48**] in the case of the *surface area preserving mean curvature flow*, where McCoy [**60, 61**] has extended his original finding to consider a number of *mixed volume preserving curvature flows*. With the exception of the geometric class examined by Escher & Simonett, a crucial distinction to be made between all of these constructions and that which we consider is potential lack of convexity in the rotating drop problem.

The elliptic result of Part 1 is strengthened considerably in this second part of the dissertation to confirm the uniqueness of any minimiser in a Lipschitz neighbourhood of S^n (Theorem 7.13). The substance of this finding has been compiled in a recently submitted paper [**82**]. For a fixed rotational velocity, an initial condition corresponding to a compact, star-shaped drop with $C^{3,\alpha}$ boundary is shown to yield the short time existence of a unique, smooth solution to the rotating drop flow (Theorem 5.5), through the deployment of methods traditional to the analysis of quasilinear parabolic equations. Here we explicitly represent the family of evolving manifolds in the geometry induced from S^n to break the diffeomorphism symmetry intrinsic to such curvature flows. The global terms defined in analogy to the Lagrange multipliers specified in the variational exposition ensure that the evolution equation preserves volume and barycentre whilst decreasing the sum of surface and kinetic energies. Through the application of a generalised form of the maximum principle due to Hamilton [**45**], whereby an ODE inequality describing the behaviour of a spatial maximum is elicited from the corresponding PDE evaluated at such a point, we then derive *a priori* curvature derivative estimates (Theorem 6.11). In contrast to the method of [**28**, Proposition 4.4], for instance, this procedure requires analysis reliant upon a Gagliardo-Nirenberg type inequality [**37, 63**] which bounds supremum norms of tensorfield covariant derivatives by a product of lower and higher order terms, in conjunction with a characterisation of covariant derivatives for position vectorfields and their components on Riemannian manifolds. These are explicated in Appendices B and A respectively, and, to the best of the author's knowledge, are not explicitly contained in the literature.

We subsequently prove asymptotic convergence of the flow to minimisers of the elliptic problem with respect to the C^∞ topology on S^n. We first verify exponential decay on a short time interval in the topology of $L^2(S^n)$ (Theorem 7.4) through an argument which relies on the aforementioned eigenvalue estimate. More precisely, we evolve the square of the L^2 norm of the difference between the two solutions' respective graph characterisations over S^n, where the resulting integral may be interpreted as the inner product of a negative factor of the Euler operator evaluated at the time-dependent solution together with a (non-constant) multiple of this difference. We are therefore able to perform a truncated expansion of the Euler operator in Fréchet derivatives about the minimiser, with the second-order expression evaluated at some intermediate graph. The first-order term dominates the calculation and, after some manipulation, yields a suitable negative factor of the quantity under examination that is proportional to the minimum eigenvalue of the Jacobi operator. The global existence of unique solutions to the parabolic problem (Theorem 7.8) is then implied by Lipschitz proximity to the sphere in infinite time, which we deduce from a corollary of the preceding Gagliardo-Nirenberg inequality and an analogous result which enables the estimation of supremum norms through interpolation between those of L^2 and $C^{0,1}$. Smooth convergence is then determined by successive applications of the original interpolation inequality, where the higher order terms may be appropriately contained by the previously described curvature derivative estimates.

In conclusion, there appears to be no structural impediment to the application of the methodology contained in this thesis to the optimisation of a far broader class of geometric objects, where boundary deformation occurs as a consequence of a forcing process (which may not necessarily have a sound physical basis). More specifically, for any compact, star-shaped domain in $\mathbb{R}^{n+1}$ with prescribed volume, barycentre, and boundary regularity, we might consider functionals in which surface energy is augmented by a parameter-dependent integral over the interior whose integrand is composed of an arbitrary polynomial function in the norm of the position vectorfield or its components. Whether this category of functions may be expanded is a matter of conjecture, but one which warrants investigation.

Part 1

The Elliptic Problem

CHAPTER 1

The Rotating Drop Near the Unit Ball

1. The rotating drop

In the absence of gravity and for $n \geq 2$, we examine the behaviour of a compact, connected liquid drop $E \subset \mathbb{R}^{n+1}$ rotating about the x_{n+1} axis with constant angular velocity $\sqrt{2\Omega}$. We assume that E has fixed measure, barycentre at the origin, and $C^{3,\alpha}$ boundary for some $\alpha \in (0,1)$. We specify this class of *rotating drops* and define an appropriate energy functional, with the aspiration of locating stable energy minimisers for this problem. In the ensuing construction we shall, for mathematical convenience, consider Ω as a parameter over $\mathbb{R}$, before relating our results to the particular physical case.

DEFINITION 1.1. Let the class $\mathscr{E}$ be given by:

$$E \in \mathscr{E} \iff \begin{cases} |E| = \omega_{n+1} \; ; \\ E \subset \mathbb{R}^{n+1} \text{ is compact and connected;} \\ \partial E = M \text{ is of class } C^{3,\alpha}\text{; and} \\ \int_E \langle x, e_i \rangle \, dx = 0 \;\; \forall \, i \in \{1, ..., n+1\}. \end{cases}$$

Then for $E \in \mathscr{E}$ and fixed $\Omega \in \mathbb{R}$, we encapsulate the global action of surface and kinetic energies in the functional:

$$\mathscr{F}_\Omega(E) = |M| + \int_E f_\Omega(x) dx \tag{1.1}$$

with $f_\Omega(x) = -\Omega|\pi_{\mathbb{R}^n} x|^2$. Here $\pi_{\mathbb{R}^n}(\cdot)$ denotes orthogonal projection in $\mathbb{R}^{n+1}$ onto the (hyper)plane $x_{n+1} = 0$.

REMARK 1.2. We observe that $\mathscr{E}$ is non-empty, since it contains the closed unit ball $\overline{B}$ centred at the origin in $\mathbb{R}^{n+1}$. Indeed, it may be demonstrated (see [**2**, Theorem 5] and [**10**, Theorem 1.3], for example) that $\overline{B}$ furnishes a stable, global minimum for the functional $\mathscr{F}_0$.

It is important to emphasise that $\mathscr{F}_\Omega$ remains well-defined for any compact, connected subset of $\mathbb{R}^{n+1}$ with Hausdorff dimension $(n+1)$ and C^1 boundary, and we shall make use of this fact in Chapter 2.

2. The geometry of star-shaped manifolds

In this dissertation, we intend to examine issues of both equilibrium and stability in slowly rotating liquid drops. As a consequence of the symmetries implied by the barycentricity condition imposed on the class $\mathscr{E}$ and the known geometric behaviour of such configurations (see [**1**] or [**19**], for instance), it therefore seems natural to consider candidate drops which are at least star-shaped about the origin. To this end, we shall now introduce an ambient geometric framework for our investigation.

Let $S^{(n)}$ be the n-dimensional unit sphere centred at the origin in $\mathbb{R}^{n+1}$. For arbitrary $k \in \mathbb{N}$, we consider any compact subset F of $\mathbb{R}^{n+1}$ which is star-shaped about the origin and whose boundary N is of class $C^{k,\alpha}$. As a consequence of the induced Euclidean topology on F, N is a compact, connected, Riemannian manifold of dimension n without boundary. For any such manifold, it is clear that the natural projection $\pi_S : N \to S$ determined by

$$\pi_S(x) = \frac{x}{|x|}$$

is a diffeomorphism. We may therefore uniquely represent N as a strictly positive graph $r \in C^{k,\alpha}(S)$, which may be explicitly identified with the magnitude of position on N in the following manner:

$$r(s) = \left|\pi_S^{-1}(s)\right|.$$

DEFINITION 1.3. Suppose $F \subset \mathbb{R}^{n+1}$ is compact and star-shaped about the origin with boundary $\partial F = N$ of class $C^{k,\alpha}$ for some $k \in \mathbb{N}$. Then we define $r \in C^{k,\alpha}(S;\mathbb{R}^+)$ to be the (unique) *parametrising function* for N whenever it may be endowed with the parametrisation $X : S \to N$, given by $X(s) = r(s)s$. For each $k \in \mathbb{N}$, we shall denote the class of such parametrising functions by $\mathcal{R}^k$.

We note that $\mathcal{R}^k$ is an open subset of $C^{k,\alpha}(S)$ for any choice of k. Throughout this thesis, parametrisations X of the type specified above shall be described as *star-shaped about the origin*[2]. In the forthcoming variational analysis, it shall be convenient to suppress the s variable by composing X and other geometric quantities on N with respect to the class of parametrising functions. We may now infer a geometry on any star-shaped N from Definition 1.3, where the reader may refer to Section 1 of Appendix A for precise definitions of the quantities concerned.

LEMMA 1.4. *Suppose $r \in \mathcal{R}^1$ and the indexing sets to be considered are in direct correspondence with the set of integers $\{1, ..., n\}$. Then the following Riemannian geometry on N may be induced from S:*

$$g_{ij}(r) = r^2 g_{ij}(s) + \nabla_i r \nabla_j r \tag{1.2}$$

[2] This description is a clear abuse of definition, since any manifold parametrised in this manner cannot itself be considered as a star-shaped subset of $\mathbb{R}^{n+1}$, but rather as the boundary of such a subset. However, precedence for this terminology is contained in the statement and proof of [**4**, Lemma 3.2].

are the components of the metric on N;

$$g^{ij}(r) = \frac{1}{r^2}\left(g^{ij}(s) - \frac{g^{ik}(s)g^{jl}(s)\nabla_k r \nabla_l r}{r^2 + |\nabla^S r|^2}\right) \tag{1.3}$$

are the components of the inverse metric on N;

$$\nu(r) = \frac{1}{\sqrt{r^2 + |\nabla^S r|^2}}\left(rs - \nabla^S r\right) \tag{1.4}$$

is the outward oriented unit normal vectorfield on N; and

$$\mu_r = r^{n-1}\sqrt{r^2 + |\nabla^S r|^2} \tag{1.5}$$

is the measure on N. If we further stipulate that $r \in \mathcal{R}^2$, then the components of the Riemannian connection on N are given by $\Gamma^k_{ij}(r) = g^{km}(r)\Gamma_{imj}(r)$, where

$$\begin{aligned}\Gamma_{imj}(r) &= \Gamma^l_{ij}(s)\left(r^2 g_{lm}(s) + \nabla_l r \nabla_m r\right) + \nabla_i \nabla^S_j r \nabla_m r \\ &\quad + r\left(g_{jm}(s)\nabla_i r + g_{im}(s)\nabla_j r - g_{ij}(s)\nabla_m r\right).\end{aligned} \tag{1.6}$$

PROOF

It is clear from Definition 1.3 that the inner product of the i^{th} and j^{th} tangent vectors yields (1.2). Furthermore

$$\frac{1}{r^2}\left(g^{ij}(s) - \frac{g^{im}(s)g^{jl}(s)\nabla_m r \nabla_l r}{r^2 + |\nabla^S r|^2}\right) g_{jk}(r) = \delta^i{}_k ,$$

which verifies (1.3). We note that an outward oriented normal vectorfield on N is given by

$$\mathbf{v} = \frac{\partial X}{\partial s_1} \times \ldots \times \frac{\partial X}{\partial s_n} = r^{n-1}\left(r\nu(s) - \nabla^S r\right).$$

Thus we obtain (1.4) and (1.5):

$$\nu(r) = \hat{\mathbf{v}} = \frac{1}{\sqrt{r^2 + |\nabla^S r|^2}}\left(r\nu(s) - \nabla^S r\right)$$

and

$$\mu_r = |\mathbf{v}| = r^{n-1}\sqrt{r^2 + |\nabla^S r|^2}.$$

To confirm (1.6), we observe that

$$\Gamma_{imj}(r) = \left\langle \frac{\partial^2 X}{\partial s_i \partial s_j}, \frac{\partial X}{\partial s_m}\right\rangle$$

$$= \; r^2\Gamma_{imj}(s) \; - \; r\,h_{ij}(s)\nabla_m r \; + \; r\,g_{jm}(s)\nabla_i r \; + \; r\,g_{im}(s)\nabla_j r \; + \; \frac{\partial^2 r}{\partial s_i \partial s_j}\nabla_m r$$

$$= \; \Gamma^l_{ij}(s)\left(r^2 g_{lm}(s) \; + \; \nabla_l r \nabla_m r\right) \; + \; \nabla_i \nabla^S_j r \nabla_m r$$

$$+ \; r\left(g_{jm}(s)\nabla_i r \; + \; g_{im}(s)\nabla_j r \; - \; g_{ij}(s)\nabla_m r\right),$$

as required. Here we have recalled that the metric and the second fundamental form coincide on S, whilst employing the definition of the covariant derivative. ◇

We may now express the parametrisation X introduced in Definition 1.3 with respect to the metric on N.

COROLLARY 1.5. *Suppose $r \in \mathcal{R}^1$. Then the geometry on N allows the representation:*

$$X(r) \; = \; r\left(\sqrt{1 \, - \, |\nabla^N r|^2}\,\nu(r) \; + \; \nabla^N r\right).$$

PROOF

With the assistance of Definition 1.3 and Lemma 1.4, we write X as a linear combination of its normal and tangential components over N:

$$X \;\; = \;\; \langle X, \nu(r)\rangle\,\nu(r) \; + \; g^{ij}(r)\left\langle X, \frac{\partial X}{\partial s_i}\right\rangle \frac{\partial X}{\partial s_j}$$

$$= \;\; \frac{r}{\sqrt{r^2 \, + \, |\nabla^S r|^2}}\left\langle s, rs \; - \; \nabla^S r\right\rangle \nu(r) \; + \; \frac{g^{ij}(r)}{2}\nabla_i(r^2)\frac{\partial X}{\partial s_j}$$

$$= \;\; \frac{r^2}{\sqrt{r^2 \, + \, |\nabla^S r|^2}}\,\nu(r) \; + \; r\nabla^N r\,.$$

Note that we identify the tangential gradient of r on N with $\nabla^N(r \circ X^{-1})$. Moreover, by Lemma 1.4,

$$|\nabla^N r|^2 \; = \; g^{ij}(r)\nabla_i r \nabla_j r \; = \; \frac{|\nabla^S r|^2}{r^2 \, + \, |\nabla^S r|^2}\,.$$

Therefore

$$|\nabla^S r|^2 \; = \; \frac{r^2|\nabla^N r|^2}{1 \, - \, |\nabla^N r|^2} \quad \Longrightarrow \quad \frac{r}{\sqrt{r^2 \, + \, |\nabla^S r|^2}} \; = \; \sqrt{1 \, - \, |\nabla^N r|^2}\,.$$

The desired result now follows after substitution. ◇

Given this formulation for X, we may derive the second fundamental form and related geometric quantities on N. Henceforth we shall work in the induced geometry on N, unless otherwise noted.

LEMMA 1.6. *Suppose $r \in \mathcal{R}^2$. Then the second fundamental form on N may be expressed:*

$$h_{ij}(r) = \frac{1}{r\sqrt{1 - |\nabla^N r|^2}} \left(g_{ij}(r) - \nabla_i r \nabla_j r - r \nabla_i \nabla_j^N r\right). \tag{1.7}$$

PROOF

We first observe from Definition 1.3 that $\nabla_i \nabla_j |X|^2 = 2(\nabla_i r \nabla_j r + r \nabla_i \nabla_j r)$. Conversely,

$$\begin{aligned} \nabla_i \nabla_j |X|^2 &= 2\nabla_i \left\langle \frac{\partial X}{\partial s_j}, X \right\rangle \\ &= 2\left(\left\langle \frac{\partial^2 X}{\partial s_i \partial s_j}, X \right\rangle + g_{ij} - \Gamma_{ij}^k \left\langle \frac{\partial X}{\partial s_k}, X \right\rangle\right) \\ &= 2\left(g_{ij} - \langle X, \nu \rangle h_{ij}\right) \\ &= 2\left(g_{ij} - r\sqrt{1 - |\nabla r|^2}\, h_{ij}\right). \end{aligned}$$

Here we have used the Gauss-Weingarten relations in conjunction with Corollary 1.5. The result then follows through substitution and rearrangement. ◇

We may now calculate the trace and square of the tensorfield norm of $(h_{ij}(r))$ to elicit the ensuing corollary.

COROLLARY 1.7. *Suppose $r \in \mathcal{R}^2$. Then the mean curvature and square of the norm of the second fundamental form on N are respectively given by*

$$H(r) = \frac{1}{r\sqrt{1 - |\nabla^N r|^2}} \left(n - \left|\nabla^N r\right|^2 - r\Delta^N r\right), \tag{1.8}$$

and

$$\begin{aligned} \|A\|^2(r) = \frac{1}{r^2\left(1 - |\nabla^N r|^2\right)} \Big(n + \left|\nabla^N r\right|^4 + r^2 \left\|\left(\nabla_i \nabla_j^N r\right)\right\|^2 \\ - 2\left|\nabla^N r\right|^2 - 2r\Delta^N r + 2r\left(\nabla_i \nabla_j{}^N r\right)\left[\nabla^N r, \nabla^N r\right]\Big). \end{aligned} \tag{1.9}$$

CHAPTER 2

A Variational Approach

1. The first variation of energy

We shall now confine our attention to the star-shaped subclass of $\mathscr{E}$, where we again note that the barycentricity condition prescribed by Definition 1.1 ensures that each such drop encloses the origin. In order to commence our variational analysis, we must recalibrate our functional $\mathscr{F}_\Omega$ to reflect the particular construction explicated in Section 2 of Chapter 1. To simplify this endeavour, we confirm the following lemma.

LEMMA 2.1. *Suppose $F \subset \mathbb{R}^{n+1}$ is compact and star-shaped about the origin with boundary $\partial F = N$ of class C^1. Then, for each $i \in \mathbb{N}$ with $1 \leq i \leq n+1$,*

$$|F| = \frac{1}{n+1} \int_N \langle X, \nu \rangle \, d\mu \, ,$$

$$\int_F \langle x, e_i \rangle \, dx = \frac{1}{n+2} \int_N \langle X, e_i \rangle \langle X, \nu \rangle \, d\mu \, ,$$

and

$$\int_F \langle x, e_i \rangle^2 \, dx = \frac{1}{n+3} \int_N \langle X, e_i \rangle^2 \langle X, \nu \rangle \, d\mu \, .$$

PROOF

In analogy with the preamble to Definition 1.3, we deduce that N is a codimension 1, C^1 Riemannian manifold without boundary. We next observe that $\mathrm{div}_{\mathbb{R}^{n+1}} X = n+1$ for the vectorfield X associated with the position vector $x \in \mathbb{R}^{n+1}$ and apply the divergence theorem in each case, where we have noted that, for any i with $1 \leq i \leq n+1$,

$$\mathrm{div}_{\mathbb{R}^{n+1}} \left(\langle X, e_i \rangle X \right) = \langle D_x \langle X, e_i \rangle , X \rangle + \langle X, e_i \rangle \, \mathrm{div}_{\mathbb{R}^{n+1}} X = (n+2) \langle X, e_i \rangle$$

and

$$\mathrm{div}_{\mathbb{R}^{n+1}} \left(\langle X, e_i \rangle^2 X \right) = \left\langle D_x \langle X, e_i \rangle^2 , X \right\rangle + \langle X, e_i \rangle^2 \mathrm{div}_{\mathbb{R}^{n+1}} X = (n+3) \langle X, e_i \rangle^2 \, . \quad \diamondsuit$$

Under the assumption that our drops are star-shaped, we may now reinterpret Definition 1.1.

DEFINITION 2.2. Let the class $\mathscr{R}$ of parametrising functions for star-shaped M be given by:

$$\rho \in \mathscr{R} \iff \begin{cases} \frac{1}{n+1} \int_M \langle X(\rho), \nu(\rho) \rangle \, d\mu \; = \; \omega_{n+1} \; ; \\ \rho \in \mathcal{R}^3 \; ; \text{ and} \\ \int_M \langle X(\rho), e_i \rangle \langle X(\rho), \nu(\rho) \rangle \, d\mu \; = \; 0 \quad \forall \, i \in \{1, ..., n+1\}. \end{cases}$$

Then for $\rho \in \mathscr{R}$ and fixed $\Omega \in \mathbb{R}$, we may reformulate the energy functional $\mathscr{F}_\Omega$ corresponding to the rotating drop problem:

$$\mathscr{F}_\Omega(\rho) \;\; = \;\; |M| + \frac{1}{n+3} \int_M f_\Omega(\rho) \langle X(\rho), \nu(\rho) \rangle \, d\mu \, . \tag{2.1}$$

Here $f_\Omega(\rho) = -\,\Omega \, |\pi_{\mathbb{R}^n} X(\rho)|^2$ is the restriction of $f_\Omega(x)$ to M.

REMARK 2.3. We observe that $1 \in \mathscr{R}$. Therefore, for each $k \in \{1,2,3\}$, $\mathscr{R}$ is a non-empty subset of $\mathcal{R}^k$. Indeed, in analogy with Remark 1.2, the function $\rho \equiv 1$ provides a stable, global minimum for the functional $\mathscr{F}_0$ (we may again cite [**2**, Theorem 5] and [**10**, Theorem 1.3] to justify this assertion).

Once again, it is important to note that the functional $\mathscr{F}_\Omega$ is well-defined for any strictly positive element of $C^1(S)$. With the intention of implementing a Lagrange multiplier argument, we shall now define a functional on the class $\mathcal{R}^1$ which permits variation of the isoperimetric constraints specified in Definition 2.2.

DEFINITION 2.4. Suppose $r \in \mathcal{R}^1$, Ω is fixed, and the set $\{\beta^i \in \mathbb{R} : i \in \{0, ..., n+1\}\}$ is to be determined. Then we define the functional $\mathscr{G}_\Omega : \mathcal{R}^1 \to \mathbb{R}$ in the following manner:

$$\mathscr{G}_\Omega(r) \;\; = \;\; |N| + \frac{1}{n+3} \int_N f_\Omega(r) \langle X(r), \nu(r) \rangle \, d\mu \; + \; \beta^i \mathscr{C}_i(r) \, . \tag{2.2}$$

Here the functionals $\mathscr{C}_i : \mathcal{R}^1 \to \mathbb{R}$ are given by

$$\mathscr{C}_i(r) \;\; = \;\; \begin{cases} \frac{1}{(n+1)^{\frac{3}{2}}} \int_N \langle X(r), \nu(r) \rangle \, d\mu & i = 0 \; ; \text{ and} \\ \frac{1}{n+2} \int_N \langle X(r), e_i \rangle \langle X(r), \nu(r) \rangle \, d\mu & \forall \, i \in \{1, ..., n+1\}. \end{cases}$$

We shall contextualise our variations of $\mathscr{G}_\Omega$ with respect to Fréchet differentiation in the variable $\rho \in \mathscr{R}$, when evaluated over a class of test functions on S whose regularity will be chosen according to the requirements of the associated operation. In particular, for $m \in \mathbb{N}$ with $\eta \in C^{m,\alpha}(S)$ and $p = \min\{3, m\}$, we consider perturbations of the form $r = \rho + \eta$ which persist in the class $\mathcal{R}^p$. Since the sets $\mathcal{R}^k$ are open, this construction is clearly permissible. To facilitate simplicity of exposition, we shall in general represent these variations with respect to the parametrisation X in the induced geometry on M. While the existence of these Fréchet derivatives is dependent upon the topology of the Hölder spaces, in contrast to the topology-independent formulation intrinsic to classical techniques, we require precisely this methodology at various junctures throughout the dissertation. Despite this distinction in formalism, the two approaches are equivalent when sufficient care is taken to clarify the domain over which our class of test functions is defined.

We recall that our ultimate ambition is to minimise the functional $\mathscr{F}_\Omega$ within the class $\mathscr{R}$. When considered as elements of the set $\mathcal{R}^k$ for $k \in \{1,2,3\}$, variations of $\rho \in \mathscr{R}$ will be denoted *volume-preserving* if they fix volume, or *barycentre-preserving* if they fix barycentre. Clearly, any strictly positive perturbation of ρ determined by an element of $C^{3,\alpha}(S)$ which satisfies both of these conditions remains in the class $\mathscr{R}$.

LEMMA 2.5. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then the Fréchet derivative of $X(\rho)$ may be written:*

$$\partial X(\rho)[\eta] \;=\; \eta s.$$

PROOF

Since $\rho \in C^{1,\alpha}(S)$, it is clear that the identity operator $\partial\rho : C^{1,\alpha}(S) \to C^{1,\alpha}(S)$ given by $\partial\rho[\eta] = \eta$ is the Fréchet derivative of ρ, whenever $\rho + \eta \in \mathcal{R}^1$. The result now follows immediately from Definition 1.3. $\diamondsuit$

We may therefore invoke Lemma 1.4 to compute the normal component of the variation vector $\partial X(\rho)[\eta]$.

COROLLARY 2.6. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then*

$$\langle \partial X(\rho)[\eta], \nu(\rho)\rangle \;=\; \frac{\eta\rho}{\sqrt{\rho^2 + |\nabla^S\rho|^2}} \;=\; \eta\sqrt{1 - |\nabla^M\rho|^2}\,.$$

For convenience, we shall employ prime notation to represent Fréchet differentiation in the general setting.

LEMMA 2.7. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then the Fréchet derivative of the measure on M is given by*

$$\partial\mu_\rho[\eta] \;=\; \mathrm{div}_M\,(\partial X(\rho)[\eta])\,\mu_\rho\,.$$

PROOF

We have

$$(\sqrt{g})' \;=\; \frac{1}{2}\sqrt{g}\,g^{ij}g'_{ij} \;=\; \frac{1}{2}\sqrt{g}\,g^{ij}\left(\left\langle\frac{\partial X'}{\partial s_i},\frac{\partial X}{\partial s_j}\right\rangle + \left\langle\frac{\partial X}{\partial s_i},\frac{\partial X'}{\partial s_j}\right\rangle\right) \;=\; \mathrm{div}\,(X')\,\sqrt{g}\,,$$

from which we infer that $\mu'_\rho = \mathrm{div}\,(X')\,\mu_\rho$, as required. $\diamondsuit$

The ensuing corollary is a consequence of the divergence theorem.

COROLLARY 2.8. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then*

$$\partial\,|M|\,[\eta] \;=\; \int_M H(\rho)\,\langle\partial X(\rho)[\eta],\nu(\rho)\rangle\,d\mu\,.$$

The following technical lemma shall simplify our exposition of the variational process.

LEMMA 2.9. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then*

$$\begin{aligned}\langle X(\rho), \partial\nu(\rho)[\eta]\rangle &= \left\langle \partial X(\rho)[\eta], \nabla^M \langle X(\rho), \nu(\rho)\rangle\right\rangle + n\,\langle \partial X(\rho)[\eta], \nu(\rho)\rangle \\ &\quad - \operatorname{div}_M\left(\langle \partial X(\rho)[\eta], \nu(\rho)\rangle\, X(\rho)\right).\end{aligned}$$

PROOF

We first note that $\nu' \in T_xM$ for each $x \in M$. Hence we may write ν' as a linear combination of tangent vectors, then utilise the orthogonality of the tangent and normal spaces on M together with the Gauss-Weingarten relations to ascertain that

$$\begin{aligned}\nu' &= g^{ij}\left\langle \nu', \frac{\partial X}{\partial s_i}\right\rangle \frac{\partial X}{\partial s_j} \\ &= -g^{ij}\left\langle \nu, \frac{\partial X'}{\partial s_i}\right\rangle \frac{\partial X}{\partial s_j} \\ &= -g^{ij}\left(\nabla_i \langle X', \nu\rangle - \left\langle X', \frac{\partial \nu}{\partial s_i}\right\rangle\right) \frac{\partial X}{\partial s_j} \\ &= g^{ij}h^k{}_i\left\langle X', \frac{\partial X}{\partial s_k}\right\rangle \frac{\partial X}{\partial s_j} - \nabla \langle X', \nu\rangle.\end{aligned}$$

By a further application of the Gauss-Weingarten relations, we then discern that

$$\begin{aligned}g^{ij}h^k{}_i\left\langle X', \frac{\partial X}{\partial s_k}\right\rangle \left\langle X, \frac{\partial X}{\partial s_j}\right\rangle &= g^{kl}h^j{}_l\left\langle X', \frac{\partial X}{\partial s_k}\right\rangle \left\langle X, \frac{\partial X}{\partial s_j}\right\rangle \\ &= g^{kl}\left\langle X', \frac{\partial X}{\partial s_k}\right\rangle \left\langle X, \frac{\partial \nu}{\partial s_l}\right\rangle \\ &= \left\langle X', g^{kl}\nabla_l \langle X, \nu\rangle \frac{\partial X}{\partial s_k}\right\rangle \\ &= \langle X', \nabla \langle X, \nu\rangle\rangle.\end{aligned}$$

Now $\langle \nabla \langle X', \nu\rangle, X\rangle = \operatorname{div}(\langle X', \nu\rangle X) - n\,\langle X', \nu\rangle$. This implies our intended result after substitution. $\diamondsuit$

We now Fréchet differentiate quantities pertinent to volume and barycentricity. Clearly these calculations will permit us to evaluate the first variation of $\mathscr{G}_\Omega$ whilst, at an appropriate juncture, they will also provide motivation for the imposition of analogous conditions on our class of test functions.

COROLLARY 2.10. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then*

$$\partial\mathscr{C}_0(\rho)[\eta] \;=\; \frac{1}{\sqrt{n+1}}\int_M \langle \partial X(\rho)[\eta], \nu(\rho)\rangle\, d\mu\,,$$

and, for each $i \in \mathbb{N}$ with $1 \le i \le n+1$,

$$\partial\mathscr{C}_i(\rho)[\eta] \;=\; \int_M \langle X(\rho), e_i\rangle \langle \partial X(\rho)[\eta], \nu(\rho)\rangle\, d\mu\,.$$

PROOF

By Lemmas 2.1, 2.7, 2.9, and the divergence theorem we may compute that

$$\begin{aligned}
(n+1)^{\frac{3}{2}}\mathscr{C}_0' &= \left(\int_M \langle X,\nu\rangle\, d\mu\right)' \\
&= \int_M \left(\langle X',\nu\rangle + \langle X,\nu'\rangle + \langle X,\nu\rangle\, \mathrm{div}X'\right) d\mu \\
&= \int_M \big(\langle X',\nu\rangle + \langle X',\nabla\langle X,\nu\rangle\rangle + n\langle X',\nu\rangle - \mathrm{div}\left(\langle X',\nu\rangle X\right) \\
&\qquad + \mathrm{div}\left(\langle X,\nu\rangle X'\right) - \langle X',\nabla\langle X,\nu\rangle\rangle\big)\, d\mu \\
&= \int_M \left((n+1)\langle X',\nu\rangle - \langle X',\nu\rangle\langle X,\nu\rangle H + \langle X,\nu\rangle\langle X',\nu\rangle H\right) d\mu \\
&= (n+1)\int_M \langle X',\nu\rangle\, d\mu
\end{aligned}$$

as required. Similarly, for each i with $1 \le i \le n+1$,

$$\begin{aligned}
(n+2)\mathscr{C}_i' &= \left(\int_M \langle X,e_i\rangle\langle X,\nu\rangle\, d\mu\right)' \\
&= \int_M \left(\langle X',e_i\rangle\langle X,\nu\rangle + \langle X,e_i\rangle\left(\langle X',\nu\rangle + \langle X,\nu'\rangle + \langle X,\nu\rangle\,\mathrm{div}X'\right)\right) d\mu
\end{aligned}$$

$$= \int_M \left(\langle X, \nu \rangle \langle X', e_i \rangle + \langle X, e_i \rangle \left((n+1) \langle X', \nu \rangle - \operatorname{div}\left(\langle X', \nu \rangle X \right) + \operatorname{div}\left(\langle X, \nu \rangle X' \right) \right) \right) d\mu$$

$$= \int_M \Big(\langle X, \nu \rangle \langle X', e_i \rangle + (n+1) \langle X, e_i \rangle \langle X', \nu \rangle + \langle X, \nabla \langle X, e_i \rangle \rangle \langle X', \nu \rangle$$

$$- \langle X, \nu \rangle \langle X', \nabla \langle X, e_i \rangle \rangle \Big) \, d\mu \, .$$

Due to the divergence theorem, we have noted here that

$$\int_M \operatorname{div}\left(\langle X, e_i \rangle \langle X', \nu \rangle X \right) d\mu \;=\; \int_M \operatorname{div}\left(\langle X, e_i \rangle \langle X, \nu \rangle X' \right) d\mu$$

$$= \int_M H \langle X, e_i \rangle \langle X, \nu \rangle \langle X', \nu \rangle \, d\mu \, .$$

Now $\nabla \langle X, e_i \rangle = e_i - (e_i)^{\perp}$. This implies our second result after substitution and rearrangement. ◇

Before we calculate the first variation of $\mathscr{G}_\Omega$ over $\mathscr{R}$, we must prove a final lemma.

LEMMA 2.11. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then the Fréchet derivative of $f_\Omega(\rho)$ is given by*

$$\partial f_\Omega(\rho)\,[\eta] \;=\; \langle \partial X(\rho)[\eta], D_x f_\Omega(\rho) \rangle \;=\; -\,2\Omega \,\langle \partial X(\rho)[\eta], \pi_{\mathbb{R}^n} X(\rho) \rangle \, .$$

PROOF

We compute that $f_\Omega{}' = \langle X', D_x f_\Omega \rangle = -\,2\Omega \,\langle X', \pi_{\mathbb{R}^n} X \rangle$, as required. ◇

We now define a matrix of L^2 inner products which shall prove crucial to analysis performed throughout this thesis.

DEFINITION 2.12. Suppose $r \in \mathcal{R}^1$, and $\mathscr{Z}_r = \{\zeta_i(r) \in C^{1,\alpha}(S) : i \in \{0, ..., n+1\}\}$ is the linearly independent set in $L^2(N)$ given by:

$$\zeta_i(r) \;=\; \begin{cases} \frac{1}{\sqrt{n+1}} & i = 0\,; \text{ and} \\ \langle X(r), e_i \rangle & \forall\, i \in \{1, ..., n+1\}. \end{cases}$$

Then, for each $r \in \mathcal{R}^1$, we define $(M_{ij}(r))$ to be the symmetric, invertible $(n+2) \times (n+2)$ matrix whose components are determined by the following inner products:

$$M_{ij}(r) \;=\; \langle \zeta_i, \zeta_j \rangle_{L^2(N)} \, .$$

In the usual manner, the inverse of $(M_{ij}(r))$ shall be denoted by $(M^{ij}(r))$.

REMARK 2.13. In accordance with Theorem C.4 and Lemma C.5 contained in Appendix C, we observe that the mutually orthogonal (in $L^2(S)$) sets $\{\zeta_0(1)\}$ and $\{\zeta_i(1) : 1 \le i \le n+1\}$ comprise respective orthogonal bases for $\mathcal{H}_0^{n+1}$ and $\mathcal{H}_1^{n+1}$, where $\mathcal{H}_l^{n+1}$ is the set of all spherical harmonics of degree $l \in \mathbb{N} \cup \{0\}$. Thus, we may again cite Lemma C.5 to confirm that

$$M_{ij}(1) \;=\; \frac{|S|}{n+1}\,\delta_{ij}\,. \tag{2.3}$$

PROPOSITION 2.14. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{1,\alpha}(S)$. Then the first variation of the functional $\mathscr{G}_\Omega$ may be expressed in the following manner:*

$$\partial\mathscr{G}_\Omega(\rho)[\eta] \;=\; \int_M \big(H(\rho) \,+\, f_\Omega(\rho) \,+\, \beta^i\zeta_i(\rho)\big)\,\langle \partial X(\rho)[\eta], \nu(\rho)\rangle\, d\mu\,.$$

PROOF

We utilise equation (2.2) in conjunction with Corollaries 2.8 and 2.10 to ascertain that

$$\mathscr{G}'_\Omega \;=\; \int_M \big(H \,+\, \beta^i\zeta_i\big)\,\langle X',\nu\rangle\, d\mu + \frac{1}{n+3}\left(\int_M f_\Omega\,\langle X,\nu\rangle\, d\mu\right)'\,.$$

Furthermore, by employing Lemmas 2.7, 2.9, and 2.11, we discover that

$$\begin{aligned}
\left(\int_M f_\Omega\,\langle X,\nu\rangle\, d\mu\right)' \;&=\; \int_M \big(f'_\Omega\,\langle X,\nu\rangle \,+\, f_\Omega\,(\langle X',\nu\rangle \,+\, \langle X,\nu'\rangle \,+\, \langle X,\nu\rangle\,\mathrm{div}X')\big)\, d\mu \\
&=\; \int_M \big(\langle X,\nu\rangle\,\langle X', D_x f_\Omega\rangle \,+\, f_\Omega\,((n+1)\,\langle X',\nu\rangle \,+\, \langle X', \nabla\,\langle X,\nu\rangle\rangle \\
&\qquad - \mathrm{div}\,(\langle X',\nu\rangle\, X) \,+\, \langle X,\nu\rangle\,\mathrm{div}X')\big)\, d\mu \\
&=\; \int_M \big(\langle X,\nu\rangle\,\langle X', D_x f_\Omega\rangle \,+\, (n+1) f_\Omega\,\langle X',\nu\rangle \,+\, f_\Omega\,\langle X', \nabla\,\langle X,\nu\rangle\rangle \\
&\qquad + \langle X, \nabla f_\Omega\rangle\,\langle X',\nu\rangle \,-\, \langle X', \nabla\,(f_\Omega\,\langle X,\nu\rangle)\rangle\big)\, d\mu \\
&=\; \int_M \Big(\langle X,\nu\rangle\,\Big\langle X', \nabla f_\Omega \,+\, (D_x f_\Omega)^{\perp}\Big\rangle \,+\, (n+1) f_\Omega\,\langle X',\nu\rangle \\
&\qquad + \Big\langle X, D_x f_\Omega \,-\, (D_x f_\Omega)^{\perp}\Big\rangle\,\langle X',\nu\rangle \,-\, \langle X,\nu\rangle\,\langle X', \nabla f_\Omega\rangle\Big)\, d\mu
\end{aligned}$$

$$= \quad (n+3)\int_M f_\Omega \langle X', \nu\rangle \, d\mu \, .$$

Here we have observed from the divergence theorem that

$$\int_M \operatorname{div}(f_\Omega \langle X', \nu\rangle X)\, d\mu \;=\; \int_M \operatorname{div}(f_\Omega \langle X, \nu\rangle X')\, d\mu \;=\; \int_M f_\Omega H \langle X, \nu\rangle \langle X', \nu\rangle \, d\mu \, .$$

The result now follows directly through substitution. ◇

For arbitrary $\rho \in \mathscr{R}$, we have thus computed the first variation of $\mathscr{G}_\Omega(\rho)$ with respect to positive $C^{1,\alpha}$ variations which are otherwise unrestricted. In analogy with the method of Lagrange multipliers, we are inspired by [**10**, Proposition 2.7] to establish conditions under which critical points of the functionals $\mathscr{F}_\Omega$ and $\mathscr{G}_\Omega$ coincide over $\mathscr{R}$.

PROPOSITION 2.15. *Suppose $\rho \in \mathscr{R}$ and $\psi, \eta \in C^{1,\alpha}(S)$. Then the following statements are equivalent:*

(1) $\partial\mathscr{F}_\Omega(\rho)[\psi] \;=\; 0$ for each volume- and barycentre-preserving variation of ρ.

(2) $\partial\mathscr{G}_\Omega(\rho)[\eta] \;=\; 0$ for each variation of ρ.

(3) $H(\rho) \;+\; f_\Omega(\rho) \;+\; \beta^i\zeta_i(\rho) \;\equiv\; 0\,.$

PROOF

(1) $\Longrightarrow$ (2) : Suppose $\partial\mathscr{F}_\Omega(\rho)[\psi] = 0$ for any volume- and barycentre-preserving variation of ρ. Moreover, for each $j \in \{0, ..., n+1\}$, let

$$\eta_j \;=\; \frac{\sqrt{\rho^2 \,+\, |\nabla^S \rho|^2}}{\rho}\,\zeta_j(\rho)\, .$$

The $C^{2,\alpha}(S)$ regularity of each η_j ensures that we may find sufficiently small $\delta > 0$ such that the $(n+2)$ variations of ρ determined by the set $\{\delta\eta_j\}$ remain in the class $\mathcal{R}^1$. We may therefore invoke Corollaries 2.6, 2.10, and Definition 2.12, to deduce that, for each i, $j \in \{0, ..., n+1\}$,

$$\partial\mathscr{C}_i(\rho)[\delta\eta_j] \;=\; \delta M_{ij}(\rho)\, .$$

Clearly the matrix $(\partial\mathscr{C}_i(\rho)[\delta\eta_j])$ is non-singular. Since S has no boundary where we have considered only positive $C^{1,\alpha}(S)$ variations of ρ, we may therefore cite [**39**, Theorem 2.1.2] to assert that there exists a set of Lagrange multipliers $\{\gamma^i \in \mathbb{R} : i \in \{0, ..., n+1\}\}$ such that the functional $\mathscr{F}^*_\Omega = \mathscr{F}_\Omega + \gamma^i\mathscr{C}_i$ possesses an unrestricted critical point at ρ. More precisely, for each $\eta \in C^{1,\alpha}(S)$ furnishing a variation of ρ which stays in the class $\mathcal{R}^1$,

$$\partial\mathscr{F}^*_\Omega(\rho)[\eta] \;=\; \partial\mathscr{F}_\Omega(\rho)[\eta] \,+\, \gamma^i\partial\mathscr{C}_i(\rho)[\eta] \;=\; 0\, .$$

Since the set $\{\beta^i\}$ introduced in Definition 2.4 was arbitrary, we may, without loss of generality, take $\mathscr{F}^*_\Omega \equiv \mathscr{G}_\Omega$ to infer our intended result.

(2) $\Longrightarrow$ (3) : Suppose, by way of contradiction, that $\partial\mathscr{G}_\Omega(\rho)[\eta] = 0$ for any variation of ρ, yet there exists $s_0 \in S$ such that $H(\rho(s_0)) + f_\Omega(\rho(s_0)) + \beta^i\zeta_i(\rho(s_0)) \not\equiv 0$. Now, for fixed Ω, $(H(\rho) + f_\Omega(\rho) + \beta^i\zeta_i(\rho)) \in C^{1,\alpha}(S)$. Therefore, by recalling Corollary 2.6, we may again find sufficiently small $\delta > 0$ such that the function $\xi \not\equiv 0 \in C^{1,\alpha}(S)$ given by

$$\xi = \delta(H(\rho) + f_\Omega(\rho) + \beta^i\zeta_i(\rho))\frac{\sqrt{\rho^2 + |\nabla^S\rho|^2}}{\rho}$$

generates a variation of ρ which is strictly positive. Hence we may write

$$0 = \partial\mathscr{G}_\Omega(\rho)[\xi] = \delta\int_M (H + f_\Omega + \beta^i\zeta_i)^2 d\mu > 0\,,$$

from which we derive a contradiction. Thus $H(\rho) + f_\Omega(\rho) + \beta^i\zeta_i(\rho) \equiv 0$.

(3) $\Longrightarrow$ (1) : Suppose $H(\rho) + f_\Omega(\rho) + \beta^i\zeta_i(\rho) \equiv 0$ and $\psi \in C^{1,\alpha}(S)$. Then, by employing (2.1), (2.2), Lemma 2.1, Corollary 2.10, and Definition 2.12 we discern (by abuse of notation) that

$$\partial\mathscr{F}_\Omega(\rho)[\psi] = -\left(\frac{\beta^0}{\sqrt{n+1}}\partial\,|E|\,[\psi] + \beta^i\partial\left(\int_E \langle x, e_i\rangle\, dx\right)[\psi]\right).$$

Clearly, the quantity on the right hand side of this equation vanishes whenever ψ determines a volume- and barycentre-preserving variation of ρ. $\diamondsuit$

For perturbations of $\rho \in \mathscr{R}$ that remain in the class $\mathcal{R}^1$, we now distinguish in notation the critical points of our functional $\mathscr{G}_\Omega$.

DEFINITION 2.16. With respect to variations in the class $\mathcal{R}^1$, we shall denote critical points of the functional $\mathscr{G}_\Omega$ by $\varrho \in \mathscr{R}$, and by $\widetilde{M}$ the manifolds with corresponding parametrisation $X(\varrho)$.

While the Lagrange multiplier theorem cited in the proof of Proposition 2.15 ensures that it is sufficient to consider the set $\{\beta^i\}$ constant with respect to the first variation of energy, we shall require more precision in our evaluation of the second variation. Therefore, we now specify the set of Lagrange multipliers at critical points of $\mathscr{G}_\Omega$.

LEMMA 2.17. *Suppose $\varrho \in \mathscr{R}$ is a critical point of the functional $\mathscr{G}_\Omega$. Then, for each $i \in \mathbb{N}\cup\{0\}$ with $0 \le i \le n+1$,*

$$\beta^i = -M^{ij}(\varrho)\,\langle(H + f_\Omega)\,,\zeta_j\rangle_{L^2(\widetilde{M})}\,.$$

PROOF

By Definition 2.16, $\partial\mathscr{G}_\Omega(\varrho)[\eta] = 0$ for any $\eta \in C^{1,\alpha}(S)$ that furnishes a variation of ϱ in the class $\mathcal{R}^1$. In particular, for any $i \in \{0, ..., n+1\}$, we may find $\delta > 0$ such that the function $\xi^i \not\equiv 0 \in C^{1,\alpha}(S)$ given by

$$\xi^i = \delta M^{ij}(\varrho)\zeta_j(\varrho)\,\frac{\sqrt{\varrho^2 + |\nabla^S\varrho|^2}}{\varrho}$$

permits a strictly positive perturbation of ϱ. We then recall Corollary 2.6 to deduce that

$$\begin{aligned} 0 &= \partial\mathscr{G}_\Omega(\varrho)\left[\xi^i\right] \\ &= \delta M^{ij}(\varrho)\int_{\widetilde{M}}\left(H + f_\Omega + \beta^k\zeta_k\right)\zeta_j d\mu \\ &= \delta\left(M^{ij}(\varrho)\left\langle (H + f_\Omega), \zeta_j\right\rangle_{L^2(\widetilde{M})} + \beta^k M^{ij}(\varrho)M_{kj}(\varrho)\right) \\ &= \delta\left(M^{ij}(\varrho)\left\langle (H + f_\Omega), \zeta_j\right\rangle_{L^2(\widetilde{M})} + \beta^i\right). \end{aligned}$$

This implies our desired result. ◇

We may now combine Proposition 2.15 and Lemma 2.17 to derive the Euler-Lagrange equation for the rotating drop problem.

COROLLARY 2.18. *Suppose $\varrho \in \mathscr{R}$ is a critical point of the functional $\mathscr{G}_\Omega$. Then the Euler-Lagrange equation for the rotating drop problem is given by*

$$H(\varrho) + f_\Omega(\varrho) = M^{ij}(\varrho)\left\langle (H + f_\Omega), \zeta_j\right\rangle_{L^2(\widetilde{M})}\zeta_i(\varrho). \tag{2.4}$$

The existence and higher regularity of solutions to the Euler-Lagrange equation shall be addressed in Chapter 3.

2. The second variation of energy

To compute the second variation of the functional $\mathscr{G}_\Omega$, we shall require test functions in the higher regularity class $C^{2,\alpha}(S)$. We must first prove the following result.

LEMMA 2.19. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{2,\alpha}(S)$. Then the Fréchet derivative of $H(\rho)$ may be written:*

$$\partial H(\rho)[\eta] = -\Delta^M\left\langle \partial X(\rho)[\eta], \nu(\rho)\right\rangle - \left\langle \partial X(\rho)[\eta], \nu(\rho)\right\rangle \|A\|^2(\rho) + \left\langle \partial X(\rho)[\eta], \nabla^M H(\rho)\right\rangle.$$

PROOF

We first observe that

$$\begin{aligned} H' &= \left(g^{ij}\right)' h_{ij} + g^{ij}h'_{ij} \\ &= -g^{ik}g^{jl}g'_{kl}h_{ij} + g^{ij}\left(\left\langle \frac{\partial X'}{\partial s_i}, \frac{\partial \nu}{\partial s_j}\right\rangle + \left\langle \frac{\partial X}{\partial s_i}, \frac{\partial \nu'}{\partial s_j}\right\rangle\right) \\ &= -2g^{ij}\left\langle \frac{\partial X'}{\partial s_i}, \frac{\partial \nu}{\partial s_j}\right\rangle + g^{ij}\left(\left\langle \frac{\partial X'}{\partial s_i}, \frac{\partial \nu}{\partial s_j}\right\rangle + \left\langle \frac{\partial X}{\partial s_i}, \frac{\partial \nu'}{\partial s_j}\right\rangle\right) \end{aligned}$$

$$= g^{ij}\left(\left\langle\frac{\partial X}{\partial s_i},\frac{\partial \nu'}{\partial s_j}\right\rangle - \left\langle\frac{\partial X'}{\partial s_i},\frac{\partial \nu}{\partial s_j}\right\rangle\right).$$

As in the proof of Lemma 2.9, we have $\nu' = -g^{ij}\left\langle \nu, \frac{\partial X'}{\partial s_i}\right\rangle \frac{\partial X}{\partial s_j}$. Thus

$$\begin{aligned}
\left\langle\frac{\partial X}{\partial s_i},\frac{\partial \nu'}{\partial s_j}\right\rangle &= \frac{\partial}{\partial s_j}\left\langle\frac{\partial X}{\partial s_i},\nu'\right\rangle - \left\langle\frac{\partial^2 X}{\partial s_i \partial s_j},\nu'\right\rangle \\
&= -\frac{\partial}{\partial s_j}\left\langle\frac{\partial X'}{\partial s_i},\nu\right\rangle + g^{lm}\left\langle\nu,\frac{\partial X'}{\partial s_m}\right\rangle\left\langle\frac{\partial^2 X}{\partial s_i\partial s_j},\frac{\partial X}{\partial s_l}\right\rangle \\
&= -\nabla_j\left\langle\frac{\partial X'}{\partial s_i},\nu\right\rangle - \Gamma^m_{ij}\left\langle\frac{\partial X'}{\partial s_m},\nu\right\rangle + \Gamma^m_{ij}\left\langle\frac{\partial X'}{\partial s_m},\nu\right\rangle \\
&= -\nabla_j\left(\nabla_i\left\langle X',\nu\right\rangle - \left\langle X',\frac{\partial \nu}{\partial s_i}\right\rangle\right) \\
&= -\nabla_i\nabla_j\left\langle X',\nu\right\rangle + \nabla_j\left({h^m}_i\left\langle X',\frac{\partial X}{\partial s_m}\right\rangle\right) \\
&= -\nabla_i\nabla_j\left\langle X',\nu\right\rangle + g^{mk}\nabla_j\left(h_{ki}\right)\left\langle X',\frac{\partial X}{\partial s_m}\right\rangle \\
&\quad + {h^m}_i\left(\frac{\partial}{\partial s_j}\left\langle X',\frac{\partial X}{\partial s_m}\right\rangle - \Gamma^k_{jm}\left\langle X',\frac{\partial X}{\partial s_k}\right\rangle\right) \\
&= -\nabla_i\nabla_j\left\langle X',\nu\right\rangle + g^{mk}\nabla_k\left(h_{ij}\right)\left\langle X',\frac{\partial X}{\partial s_m}\right\rangle \\
&\quad + {h^m}_i\left(\left\langle\frac{\partial X'}{\partial s_j},\frac{\partial X}{\partial s_m}\right\rangle + \left\langle X',\frac{\partial^2 X}{\partial s_m\partial s_j}\right\rangle - \Gamma^k_{jm}\left\langle X',\frac{\partial X}{\partial s_k}\right\rangle\right) \\
&= -\nabla_i\nabla_j\left\langle X',\nu\right\rangle + \left\langle X',\nabla h_{ij}\right\rangle + \left\langle\frac{\partial X'}{\partial s_j},\frac{\partial \nu}{\partial s_i}\right\rangle - {h^m}_i h_{jm}\left\langle X',\nu\right\rangle.
\end{aligned}$$

Here we have utilised the Gauss-Weingarten relations in conjunction with the Codazzi equations. We then derive our intended result after substitution and contraction over the inverse metric. ♢

In the context of our particular parametrisation for star-shaped M, we now make explicit a class of variational test functions whose properties may be deduced from the geometry of the class $\mathscr{R}$.

DEFINITION 2.20. Suppose $\rho \in \mathscr{R}$ and $k \in \mathbb{N} \cup \{0\}$. Then we define the class $\mathscr{N}^k_{M}$ on M by:

$$\eta \,\dot{\in}\, \mathscr{N}^k_{M} \iff \begin{cases} \eta \in C^{k,\alpha}(M); \text{ and} \\ \\ \langle \eta, \zeta_i \rangle_{L^2(M)} \;=\; 0 \;\; \forall\, \zeta_i(\rho) \in \mathscr{Z}_\rho\,. \end{cases}$$

REMARK 2.21. With respect to the $L^2(S)$ topology, we ascertain from the characterisation of the spherical harmonics that $\mathscr{N}^k_{S}$ lies in the orthogonal complement of $\mathcal{H}^{n+1}_0 \cup \mathcal{H}^{n+1}_1$ (Theorem C.4 and Lemma C.5 in Appendix C). More generally, for $k \in \{0, 1, 2, 3\}$ we deduce that $\mathscr{N}^k_{M}$ is non-trivial for each $\rho \in \mathscr{R}$, since $C^{k,\alpha}(M)$ is a dense, infinite-dimensional subspace of $L^2(M)$ in which we can certainly locate non-zero functions which are orthogonal to the span of $\mathscr{Z}_\rho$. Indeed, for any $\psi \in C^{k,\alpha}(M)$ not contained in the span of $\mathscr{Z}_\rho$, we may employ the Gram-Schmidt process to assert that the function

$$\eta \;=\; \psi \;-\; M^{ij}(\rho)\, \langle \psi, \zeta_i \rangle_{L^2(M)}\, \zeta_j(\rho)$$

is an element of $\mathscr{N}^k_{M}$.

It is evident from inspection that $\mathscr{N}^k_{M}$ is a closed, linear subspace of $C^{k,\alpha}(M)$. We may therefore propound the following lemma.

LEMMA 2.22. *For each $k \in \mathbb{N} \cup \{0\}$, $\mathscr{N}^k_{M}$ is a Banach subspace of $C^{k,\alpha}(M)$.*

We note that, for $k \in \{1, 2\}$ and any $\eta \in C^{k,\alpha}(S)$ generating a star-shaped variation of $X(\rho)$, $\langle \partial X(\rho)[\eta], \nu(\rho) \rangle \in C^{k,\alpha}(M)$ as a consequence of Corollary 2.6 and the relationship between the geometries of S and M connoted by Lemma 1.4. Hence, we obtain the ensuing lemma from Corollary 2.10 and Definition 2.12.

LEMMA 2.23. *Suppose $\rho \in \mathscr{R}$ and $\eta \in C^{k,\alpha}(S)$ for some $k \in \{1, 2\}$. Then η determines a volume- and barycentre-preserving variation of ρ precisely when $\langle \partial X(\rho)[\eta], \nu(\rho) \rangle \in \mathscr{N}^k_{M}$.*

Within this setting, we may therefore evaluate the second variation of the functional $\mathscr{G}_\Omega$ at solutions of the Euler-Lagrange equation.

PROPOSITION 2.24. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation* (2.4) *and $\eta \in C^{2,\alpha}(S)$ with $\langle \partial X(\varrho)[\eta], \nu(\varrho) \rangle \in \mathscr{N}^2_{\widetilde{M}}$. Then, whenever we adhere to the convention that $e_0 = \mathbf{0}$,*

$$\partial^2 \mathscr{G}_\Omega(\varrho)[\eta, \eta] \;=\; \int_{\widetilde{M}} \left| \nabla^{\widetilde{M}} \langle \partial X(\varrho)[\eta], \nu(\varrho) \rangle \right|^2 d\mu \;-\; \int_{\widetilde{M}} \Big(\|A\|^2(\varrho) \;+\; 2\Omega \, \langle \pi_{\mathbb{R}^n} X(\varrho), \nu(\varrho) \rangle$$

$$+\, M^{ij}(\varrho)\, \langle (H \;+\; f_\Omega)\,, \zeta_j \rangle_{L^2(\widetilde{M})}\, \langle \nu(\varrho), e_i \rangle \Big)\, \langle \partial X(\varrho)[\eta], \nu(\varrho) \rangle^2 \, d\mu\,.$$

PROOF

By Proposition 2.14 and Lemma 2.17, we discern that for any variation of ϱ which remains star-shaped about the origin,

$$\mathscr{G}'_\Omega(\varrho) = \int_{\widetilde{M}} \left(H + f_\Omega - M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(\widetilde{M})} \zeta_i\right) \langle X', \nu \rangle \, d\mu \, .$$

More generally, we observe that $X'' = \mathbf{0}$ for any variation of $\rho \in \mathscr{R}$ while employing Lemmas 2.7 and 2.19 to ascertain that

$$\left(\int_M \left(H + f_\Omega - M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(M)} \zeta_i\right) \langle X', \nu \rangle \, d\mu\right)'$$

$$= -\int_M \left(\Delta \langle X', \nu \rangle + \langle X', \nu \rangle \|A\|^2 - \langle X', \nabla H \rangle - \langle X', D_x f_\Omega \rangle\right) \langle X', \nu \rangle \, d\mu$$

$$- \left(M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(M)}\right)' \int_M \zeta_i \langle X', \nu \rangle \, d\mu - M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(M)} \int_M \zeta_i' \langle X', \nu \rangle \, d\mu$$

$$+ \int_M \left(H + f_\Omega - M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(M)} \zeta_i\right) \left(\langle X', \nu' \rangle + \langle X', \nu \rangle \operatorname{div}(X')\right) d\mu \, .$$

In particular, at the critical point ϱ we utilise Lemma 2.11, Corollary 2.18, Definition 2.20, and the divergence theorem to compute that, whenever $\langle X', \nu \rangle \in \mathscr{N}^2_{\widetilde{M}}$,

$$\mathscr{G}''_\Omega(\varrho) = \int_{\widetilde{M}} |\nabla \langle X', \nu \rangle|^2 \, d\mu - \int_{\widetilde{M}} \left(\langle X', \nu \rangle \|A\|^2 - \langle X', D_x f_\Omega - \nabla f_\Omega \rangle\right) \langle X', \nu \rangle \, d\mu$$

$$- M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(\widetilde{M})} \int_{\widetilde{M}} \left(\zeta_i' - \langle X', \nabla \zeta_i \rangle\right) \langle X', \nu \rangle \, d\mu$$

$$= \int_{\widetilde{M}} |\nabla \langle X', \nu \rangle|^2 \, d\mu - \int_{\widetilde{M}} \left(\|A\|^2 + 2\Omega \langle \pi_{\mathbb{R}^n} X, \nu \rangle\right) \langle X', \nu \rangle^2 \, d\mu$$

$$- M^{ij}\left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(\widetilde{M})} \int_{\widetilde{M}} \left\langle X', (e_i)^\perp \right\rangle \langle X', \nu \rangle \, d\mu$$

$$= \int_{\widetilde{M}} |\nabla \langle X', \nu \rangle|^2 \, d\mu - \int_{\widetilde{M}} \left(\|A\|^2 + 2\Omega \langle \pi_{\mathbb{R}^n} X, \nu \rangle \right) \langle X', \nu \rangle^2 \, d\mu$$

$$- M^{ij} \langle (H + f_\Omega), \zeta_j \rangle_{L^2(\widetilde{M})} \int_{\widetilde{M}} \langle \nu, e_i \rangle \langle X', \nu \rangle^2 \, d\mu \, .$$

Subject to the stipulation that $e_0 = \mathbf{0}$, we have again noted here that $\nabla \zeta_i = e_i - (e_i)^\perp$ for each $i \in \{0, ..., n+1\}$. $\diamondsuit$

CHAPTER 3

Solutions of the Euler-Lagrange Equation

1. Existence of solutions

We may now invoke an implicit function theorem on Banach spaces to determine the existence of solutions to the Euler-Lagrange equation in a neighbourhood of $\rho \equiv 1$. In the statement of the theorem and throughout its proof we shall unsuppress the parameter Ω attached to the forcing term f_Ω.

THEOREM 3.1. *Suppose the functional* $G : \mathscr{R} \times \mathbb{R} \to C^{1,\alpha}(S)$ *is given by*

$$G(\rho, \Omega) \;=\; H(\rho) \;+\; f(\rho, \Omega) \;-\; M^{ij}(\rho) \left\langle (H(\rho) \;+\; f(\rho, \Omega)) \,, \zeta_j(\rho) \right\rangle_{L^2(M)} \zeta_i(\rho) \,.$$

Then there exist a neighbourhood $\mathscr{O}$ *of* 0 *in* $\mathbb{R}$ *and an* $\varepsilon > 0$ *such that the equation* $G(\rho, \Omega) = 0$ *is solvable for each* $\Omega \in \mathscr{O}$ *with solution* $\rho = \varrho \in \overline{B}_\varepsilon(1) \subset \mathscr{R}$.

PROOF

We show that G satisfies the hypotheses of an implicit function theorem on Banach spaces (see, for example, [**40**, Theorem 17.6] or [**51**, Theorem 14.2.1]) at the point $(\rho, \Omega) = (1, 0)$. In particular, we claim that:

(1) We may parametrise $\mathscr{R}$ as a Banach manifold over $\mathscr{N}^3_S$;
(2) $G(1, 0) \equiv 0$;
(3) G is continuously Fréchet differentiable at $(1, 0)$;
(4) The partial Fréchet derivative of G with respect to ρ is invertible at $(1, 0)$.

We prove each of these conditions separately:

(1) We assert that there exists a local C^1 diffeomorphism between $\mathscr{R}$ and the open ball of radius 1 centred at 0 in $\mathscr{N}^3_S$, which (in the usual manner) we shall denote by $B_1(0)$. For convenience, we may now utilise Lemma 1.4 and Corollary 1.5 in conjunction with Definitions 1.3 and 2.2 to express the class $\mathscr{R}$ with respect to the geometry on S:

$$\rho \in \mathscr{R} \iff \begin{cases} \int_S \rho^{n+1} \zeta_0(1) d\sigma \;=\; \frac{|S|}{\sqrt{n+1}}; \\[2ex] \rho \in \mathcal{R}^3 \text{ ; and} \\[2ex] \int_S \rho^{n+2} \zeta_i(1) d\sigma \;=\; 0 \;\; \forall\, i \in \{1, ..., n+1\}. \end{cases}$$

We define the map $F : B_1(0) \to \mathcal{R}^3$ in the following manner:

$$F(\psi) \;=\; \left(\fint_S (\psi + 1)^{\frac{n+1}{n+2}}\, d\sigma\right)^{\frac{-1}{n+1}} (\psi + 1)^{\frac{1}{n+2}}\,. \tag{3.1}$$

We observe that $F(0) = 1$. More generally, the properties of both $\mathcal{N}_S^3$ determined by Definition 2.20 and the spherical harmonics (of degree 0 and 1) specified by Theorem C.4 and Lemma C.5 in Appendix C confirm that the image of F is contained in $\mathscr{R}$. Now, for each $\psi \in B_1(0)$, the Fréchet derivative $\partial F(\psi) : \mathcal{N}_S^3 \to C^{3,\alpha}(S)$ is given by:

$$\begin{aligned}\partial F(\psi)[\xi] \;&=\; \frac{1}{(n+2)}\left(\left(\fint_S (\psi + 1)^{\frac{n+1}{n+2}}\, d\sigma\right)^{\frac{-1}{n+1}} (\psi + 1)^{\frac{-(n+1)}{n+2}}\, \xi\right.\\ &\qquad \left. -\, (\psi + 1)^{\frac{1}{n+2}} \left(\fint_S (\psi + 1)^{\frac{n+1}{n+2}}\, d\sigma\right)^{\frac{-(n+2)}{n+1}} \fint_S (\psi + 1)^{\frac{-1}{n+2}}\, \xi d\sigma\right)\\ &=\; \frac{1}{(n+2)} \fint_S (F(\psi))^{n+2}\, d\sigma \left(\frac{\xi}{(F(\psi))^{n+1}} \,-\, F(\psi) \fint_S \frac{\xi}{F(\psi)}\, d\sigma\right). \end{aligned} \tag{3.2}$$

Here we have employed (3.1) in conjunction with Definition 2.20. The coefficients of the linear operator are well-defined, as F is strictly positive and $\psi \in C^{3,\alpha}(S)$. We then deduce that, for any $\xi \in \mathcal{N}_S^3$ with $\delta = \inf_S F(\psi)$,

$$\|\partial F(\psi)[\xi]\|_{C^{3,\alpha}(S)} \;\le\; C\left(n, \delta, \|\psi\|_{C^{3,\alpha}(S)}\right) \|\xi\|_{C^{3,\alpha}(S)}\,.$$

Hence F is a C^1 map.

By recalling that $F(\psi) \in \mathscr{R}$, it is a simple matter to demonstrate that the image of $\partial F(\psi)$ maps to the following Banach subspace of $C^{3,\alpha}(S)$:

$$\eta \in \mathcal{B}^3_{F(\psi)} \iff \begin{cases} \int_S \eta\, (F(\psi))^n\, \zeta_0\, (F(0))\, d\sigma \;=\; 0; \\[2ex] \eta \in C^{3,\alpha}(S)\ ; \text{ and} \\[2ex] \int_S \eta\, (F(\psi))^{n+1}\, \zeta_i\, (F(0))\, d\sigma \;=\; 0 \quad \forall\, i \in \{1, \ldots, n+1\}. \end{cases}$$

Upon examination of the preceding reformulation of $\mathscr{R}$, it is evident that $\mathcal{B}^3_{F(\psi)}$ contains all test functions $\eta \in C^{3,\alpha}(S)$ corresponding to volume- and barycentre-preserving variations of $\rho = F(\psi)$ (as we would expect). It is now sufficient to prove that, for each $\psi \in B_1(0)$, $\partial F(\psi) : \mathcal{N}_S^3 \to \mathcal{B}^3_{F(\psi)}$ is an isomorphism. Suppose, by way of contradiction, that $\xi \in$

$\ker(\partial F(\psi))$, yet $\xi \not\equiv 0$. Then, by (3.2),

$$\xi = (F(\psi))^{n+2} \fint_S \frac{\xi}{F(\psi)} \, d\sigma ,$$

which implies that

$$\int_S \frac{\xi}{F(\psi)} \, d\sigma \neq 0 .$$

However, $\xi \in \mathscr{N}_S^3$ where F is strictly positive on $B_1(0)$, and we may again utilise Definition 2.20 to compute that

$$0 = \int_S \xi d\sigma = \left(\int_S (F(\psi))^{n+2} \, d\sigma \right) \left(\fint_S \frac{\xi}{F(\psi)} \, d\sigma \right) \neq 0 .$$

Thus we derive a contradiction, from which we infer that $\ker(\partial F(\psi))$ is trivial and $\partial F(\psi)$ is injective. Suppose now that $\eta \in \mathcal{B}^3_{F(\psi)}$ and define $\xi \in C^{3,\alpha}(S)$ by

$$\xi = \frac{(n+2)\,(F(\psi))^{n+1}}{\fint_S (F(\psi))^{n+2} \, d\sigma} \left(\eta - \frac{F(\psi)}{\int_S (F(\psi))^{n+2} \, d\sigma} \int_S \eta \, (F(\psi))^{n+1} \, d\sigma \right) .$$

Then $\xi \in \mathscr{N}_S^3$ with $\partial F(\psi)[\xi] = \eta$, and $\partial F(\psi)[\xi]$ is surjective. By the inverse mapping theorem (see, for example, [**51**, Theorem 14.1.2] or [**80**, Theorem 4.1]), F is a local diffeomorphism of class C^1 at each $\psi \in B_1(0)$, from which we deduce that $B_1(0)$ is locally diffeomorphic to $\mathscr{R}$. We may therefore explicitly parametrise $\mathscr{R}$ as a Banach manifold over $\mathscr{N}_S^3$ through composition with the Banach manifold $B_1(0)$.

(**2**) With regard to the variable $\rho \in \mathscr{R}$, we now compose the functional G and the geometry on M with the diffeomorphism F. We recollect that $F(0) = 1$, where $H(1) = n$ and $f(1,0) = 0$. Thus

$$\begin{aligned} G(F(0),0) &= n\left(1 - \sqrt{n+1}\, M^{ij}(F(0))\, \langle \zeta_0, \zeta_j \rangle_{L^2(S)}\, \zeta_i(F(0))\right) \\ &= n\left(1 - \sqrt{n+1}\, \zeta_0(F(0))\right) = 0 . \end{aligned}$$

Here we have employed the properties of the matrix $(M_{ij}(1))$ whilst observing that $\zeta_0(1) = \frac{1}{\sqrt{n+1}}$, in accordance with Definition 2.12.

(**3**) In the following exposition we shall denote partial Fréchet differentiation in the variables $\psi \in B_1(0)$ and $\Omega \in \mathbb{R}$ by the respective subscripts 1 and 2. Suppose $\psi \in B_1(0)$ and $\xi \in \mathscr{N}_S^3$. We note that the analysis of Chapter 2 was contingent only upon the computation of the variation vector corresponding to the parametrisation X. If we replace this quantity by $\partial_1 X(F(\psi))[\xi]$, then we may, without loss of generality, use Lemmas 2.11 and 2.19 to compute the partial Fréchet derivative of $G(F(\psi),\Omega)$ with respect to ψ. Here all geometric quantities are calculated with respect to the metric on M and, by abuse of notation, we shall suppress the composition to enable a more concise explication:

$$\partial_1 G(F(\psi),\Omega)[\xi] \;=\; G_1 \;=\; -\Delta\langle X_1,\nu\rangle - \langle X_1,\nu\rangle\|A\|^2 + \langle X_1,\nabla H\rangle + \langle X_1, D_x f\rangle$$

$$- \left(M^{ij}\langle (H+f),\zeta_j\rangle_{L^2}\zeta_i\right)_1 . \tag{3.3}$$

To calculate the partial Fréchet derivative of the final quantity we first observe that $M^{ij} = M^{ik}M^{jl}M_{kl}$. Hence

$$\left(M^{ij}\right)_1 \;=\; \left(M^{ik}M^{jl}M_{kl}\right)_1 \;=\; 2\left(M^{ij}\right)_1 + M^{ik}M^{jl}\left(M_{kl}\right)_1,$$

from which we infer that

$$\left(M^{ij}\right)_1 \;=\; -M^{ik}M^{jl}\left(M_{kl}\right)_1 . \tag{3.4}$$

In the ensuing calculations, we shall again adhere to the convention that $e_0 = \mathbf{0}$. We may now proceed by employing Definition 2.12, Lemma 2.7, and the divergence theorem:

$$\begin{aligned}
\left(M_{kl}\right)_1 &= \int_M \left((\zeta_k)_1\zeta_l + \zeta_k(\zeta_l)_1 + \zeta_k\zeta_l\,\mathrm{div}X_1\right)d\mu \\
&= \int_M \left((\zeta_k)_1\zeta_l + \zeta_k(\zeta_l)_1 + \langle X_1,\nu\rangle H\zeta_k\zeta_l - \langle X_1,\nabla(\zeta_k\zeta_l)\rangle\right)d\mu \\
&= \int_M \left(\zeta_k\langle\nu,e_l\rangle + \zeta_l\langle\nu,e_k\rangle + H\zeta_k\zeta_l\right)\langle X_1,\nu\rangle\,d\mu\,.
\end{aligned} \tag{3.5}$$

Furthermore, we may again cite Definition 2.12, Lemmas 2.7, 2.11, and 2.19 in conjunction with the divergence theorem to determine that

$$\begin{aligned}
\left(\langle (H+f),\zeta_j\rangle_{L^2}\right)_1 &= -\int_M \left(\Delta\langle X_1,\nu\rangle + \langle X_1,\nu\rangle\|A\|^2 - \langle X_1,\nabla H\rangle - \langle X_1,D_x f\rangle\right)\zeta_j\,d\mu \\
&\quad + \int_M (H+f)\left((\zeta_j)_1 + \zeta_j\,\mathrm{div}X_1\right)d\mu \\
&= -\int_M \left(\Delta\langle X_1,\nu\rangle + \langle X_1,\nu\rangle\|A\|^2 - \langle X_1,\nabla H\rangle - \langle X_1,D_x f\rangle\right)\zeta_j\,d\mu \\
&\quad + \int_M \left((H+f)\left((\zeta_j)_1 + \langle X_1,\nu\rangle H\zeta_j\right) - \langle X_1,\nabla\left((H+f)\zeta_j\right)\rangle\right)d\mu
\end{aligned}$$

$$= -\int_M \left(\Delta\,\zeta_j + \|A\|^2\zeta_j - \langle D_x f, \nu\rangle\,\zeta_j - (H + f)\left(\langle \nu, e_j\rangle + H\zeta_j\right)\right)\langle X_1, \nu\rangle\, d\mu\,. \quad (3.6)$$

We combine (3.3)-(3.6) to discern that

$$\begin{aligned} G_1 \;=\; & -\Delta\,\langle X_1,\nu\rangle - \langle X_1,\nu\rangle\,\|A\|^2 + \langle X_1,\nabla H\rangle + \langle X_1, D_x f\rangle \\ & + M^{ik}M^{jl}\left(\int_M \left(\zeta_k\,\langle \nu, e_l\rangle + \zeta_l\,\langle \nu, e_k\rangle + H\zeta_k\zeta_l\right)\langle X_1,\nu\rangle\, d\mu\right)\langle (H + f)\,,\zeta_j\rangle_{L^2}\,\zeta_i \\ & + M^{ij}\Bigg(\bigg(\int_M \left(\Delta\,\zeta_j + \|A\|^2\zeta_j - \langle D_x f,\nu\rangle\,\zeta_j\right)\langle X_1,\nu\rangle\, d\mu \\ & - \int_M (H + f)\left(\langle \nu, e_j\rangle + H\zeta_j\right)\langle X_1,\nu\rangle\, d\mu\bigg)\,\zeta_i - \langle (H + f)\,,\zeta_j\rangle_{L^2}\,(\zeta_i)_1\Bigg)\,. \quad (3.7) \end{aligned}$$

By Lemma 2.5, we discover that $\partial_1 X(F(\psi))[\xi] = \partial F(\psi)[\xi]\, s$. In particular, we deduce from Definition 2.20 and (3.2) that

$$\partial_1 X(F(0))[\xi] = \frac{\xi\, s}{n+2}\,.$$

In the following calculation we shall suspend our use of the summation convention. We now recall the properties of the matrix $(M_{ij}(1))$ prescribed by Definition 2.12 and our earlier evaluations of $f(1,0)$ and $H(1)$, along with the subsequent observations that $D_x f(1,0) = \mathbf{0}$ and $\|A\|^2(1) = n$, to assert that $\partial_1 G(F(0),0) : \mathscr{N}_S^3 \to C^{1,\alpha}(S)$ is given by:

$$\begin{aligned} & \partial_1 G(F(0),0)[\xi] \\ & = \frac{-1}{n+2}\Bigg(\Delta^S\xi + n\,\xi - \sum_{i=0}^{n+1}\Bigg(\sum_{j=0}^{n+1}\Bigg(M^{ij}(1)\left\langle \Delta^S\zeta_j + n(1-n)\zeta_j,\xi\right\rangle_{L^2(S)} \\ & + n\sqrt{n+1}\left(\sum_{k=0}^{n+1}\sum_{l=1}^{n+1} + \sum_{k=1}^{n+1}\sum_{l=0}^{n+1} + n\sum_{k,l=0}^{n+1}\right) M^{ik}(1)M^{jl}(1)\,\langle \zeta_k\zeta_l,\xi\rangle_{L^2(S)}\,\langle \zeta_0,\zeta_j\rangle_{L^2(S)}\Bigg) \\ & + n\sum_{j=1}^{n+1} M^{ij}(1)\,\langle \zeta_j,\xi\rangle_{L^2(S)}\Bigg)\,\zeta_i(1) + n\sqrt{n+1}\sum_{i=1}^{n+1}\sum_{j=0}^{n+1} M^{ij}(1)\,\langle \zeta_0,\zeta_j\rangle_{L^2(S)}\,\zeta_i(1)\,\xi\Bigg) \end{aligned}$$

$$= \frac{-1}{n+2}\left(\Delta^S\xi + n\xi + n\sqrt{n+1}\sum_{i=1}^{n+1}\delta^i{}_0\,\zeta_i(1)\,\xi + n\sum_{i=0}^{n+1}\left(\sum_{j=1}^{n+1}M^{ij}(1)\,\langle\zeta_j,\xi\rangle_{L^2(S}\right.\right.$$

$$\left.\left. - \sqrt{n+1}\left(\sum_{k=0}^{n+1}\sum_{l=1}^{n+1} + \sum_{k=1}^{n+1}\sum_{l=0}^{n+1} + n\sum_{k,l=0}^{n+1}\right)M^{ik}(1)\delta^l{}_0\,\langle\zeta_k\zeta_l,\xi\rangle_{L^2(S)}\right)\zeta_i(1)\right)$$

$$= \frac{-1}{n+2}\left(\Delta^S\xi + n\xi - n\sum_{i=0}^{n+1}\left(\sum_{k=1}^{n+1} + n\sum_{k=0}^{n+1}\right)M^{ik}(1)\,\langle\zeta_k,\xi\rangle_{L^2(S)}\,\zeta_i(1)\right)$$

$$= \frac{-1}{n+2}\left(\Delta^S\xi + n\xi\right). \tag{3.8}$$

Here we have utilised the $L^2(S)$ orthogonality of the set $\mathscr{Z}_1$ and the class $\mathscr{N}_S^3$ stipulated by Definition 2.20, in addition to the fact that spherical harmonics are eigenfunctions of the Laplace-Beltrami operator on S (Lemma C.5 and Theorem C.10 in Appendix C). It is clear that $\partial_1 G(F(0),0)$ is continuous, since, for each $\xi \in \mathscr{N}_S^3$,

$$\|\partial_1 G(F(0),0)[\xi]\|_{C^{1,\alpha}(S)} \le C(n)\,\|\xi\|_{C^{3,\alpha}(S)}\,.$$

Moreover, where we resume our adherence to the summation convention, the partial Fréchet derivative of G with respect to Ω accords with the usual partial derivative:

$$\partial_2 G(F(\psi),\Omega) = -\,|\pi_{\mathbb{R}^n}X(F(\psi))|^2 + M^{ij}(F(\psi))\,\langle|\pi_{\mathbb{R}^n}X|^2,\zeta_j\rangle_{L^2(M)}\,\zeta_i(F(\psi))$$

$$= -\,(F(\psi))^2\left(1 - \langle s,e_{n+1}\rangle^2\right)$$

$$+ M^{ij}(F(\psi))\left\langle F^2\left(1 - \langle s,e_{n+1}\rangle^2\right),\zeta_j\right\rangle_{L^2(M)}\zeta_i(F(\psi))\,.$$

Here we have employed the definition of X (under composition with F) characterised by Definition 1.3. Since every term in this expression is of regularity at least $C^{3,\alpha}(S)$, it is clear that $\partial_2 G(F(\psi),\Omega) : \mathbb{R} \to C^{1,\alpha}(S)$ is continuous for any choice of $\Omega \in \mathbb{R}$.

(4) Suppose $\xi \in \mathscr{N}_S^3$. Then we may cite (3.8), Definitions 2.12 and 2.20 together with Theorem C.10 in Appendix C and the divergence theorem to deduce that, for each $i \in \mathbb{N}$ with $0 \le i \le n+1$,

$$\langle\partial_1 G(F(0),0)[\xi],\zeta_i\rangle_{L^2(S)} = 0\,.$$

Thus we deduce from Definition 2.20 that the image of $\partial_1 G(F(0),0)$ is contained in $\mathscr{N}_S^1$.

In order to verify the invertibility of the operator $\partial_1 G(F(0),0)$, it is sufficient to ensure the existence of a unique solution $\xi \in \mathscr{N}_S^3$ to the following inhomogeneous problem for each $\varphi \in \mathscr{N}_S^1$:

$$L[\xi] \;=\; \Delta^S \xi + n\xi \;=\; \varphi\,. \tag{3.9}$$

We observe that the operator L is strictly elliptic with smooth coefficients. We intend to apply a form of the Fredholm alternative contained in [**40**, Theorem 6.15] which requires that the solution space of the corresponding homogeneous problem is trivial. Suppose then, by way of contradiction, that $\xi \not\equiv 0 \in \mathscr{N}_S^3$, yet $L[\xi] = 0$. We recall Remark 2.21 to ascertain that $\xi \in L^2(S) \setminus \{\mathcal{H}_0^{n+1} \cup \mathcal{H}_1^{n+1}\}$, and we may employ Theorem C.7 and Definition C.8 to obtain the (truncated) condensed harmonic expansion:

$$\xi \;\sim\; \sum_{k=2}^{\infty} Q_k\,;$$

where, by Theorem C.9,

$$\|\xi\|_{L^2(S)}^2 \;=\; \sum_{k=2}^{\infty} \|Q_k\|_{L^2(S)}^2\;;$$

and, by Corollary C.11,

$$\begin{aligned}\left\|\nabla^S \xi\right\|_{L^2(S)}^2 \;&=\; \sum_{k=2}^{\infty} k(k+n-1)\,\|Q_k\|_{L^2(S)}^2 \\ &\geq\; 2(n+1)\sum_{k=2}^{\infty} \|Q_k\|_{L^2(S)}^2 \\ &=\; 2(n+1)\|\xi\|_{L^2(S)}^2\,.\end{aligned}$$

However, by (3.9) and assumption,

$$\left\|\nabla^S \xi\right\|_{L^2(S)}^2 \;=\; -\left\langle \xi, \Delta^S \xi\right\rangle_{L^2(S)} \;=\; n\,\|\xi\|_{L^2(S)}^2\,,$$

from which we derive a contradiction. Hence $\xi \equiv 0$ and the homogeneous equation $L[\xi] = 0$ possesses only the trivial solution. The proof of the aforementioned [**40**, Theorem 6.15] proceeds unmodified in the current setting but for the calculation of local Schauder estimates on the compact manifold S, which may be derived in a similar fashion to their Euclidean analogues. Therefore, for each $\varphi \in \mathscr{N}_S^1$, we obtain the existence of a unique solution $\xi \in \mathscr{N}_S^3$ to (3.9). Thus the operator $\partial_1 G(F(0),0)$ is invertible.

In conclusion, we may invoke the previously cited [**40**, Theorem 17.6] in conjunction with the diffeomorphism (3.1) to deduce that there exist a neighbourhood $\mathscr{O}$ of 0 in $\mathbb{R}$ and an $\varepsilon > 0$ such that the equation $G(\rho, \Omega) = 0$ is solvable for each $\Omega \in \mathscr{O}$ with solution $\rho = \varrho \in \overline{B}_\varepsilon(1) \subset \mathscr{R}$. $\diamondsuit$

REMARK 3.2. We have determined the existence of solutions ϱ to the Euler-Lagrange equation with $\|\varrho - 1\|_{C^{3,\alpha}(S)} \leq \varepsilon$ where $\varepsilon \to 0$ as $\Omega \to 0$ in $\mathscr{O}$. Furthermore, we discern from [**51**, Theorem 14.2.1] that whenever $\mathscr{O}$ is taken to be a sufficiently small interval, there is a unique correspondence between each $\Omega \in \mathscr{O}$ and $\varrho \in \overline{B}_\varepsilon(1)$.

2. Regularity of solutions

We now address the higher regularity of solutions to the Euler-Lagrange equation.

THEOREM 3.3. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation* (2.4). *Then $\varrho \in C^\infty(S)$.*

PROOF

Suppose $\mathscr{I}$ is a (possibly uncountable) indexing set and $\mathscr{D} = \{D_i : i \in \mathscr{I}\}$ is an open cover for S. If $\varrho \in \mathscr{R}$ is a solution to the Euler-Lagrange equation, then we ascertain by Lemma 2.17 and Corollary 2.18 that, on each D_i,

$$H(\varrho) + f_\Omega(\varrho) + \beta^k \zeta_k(\varrho) = 0 . \tag{3.10}$$

We now utilise Lemma 1.4 and (1.8) to compute:

$$\begin{aligned} H(\varrho) &= \frac{1}{\varrho\sqrt{1 - |\nabla^{\widetilde{M}}\varrho|^2}} \left(n - |\nabla^{\widetilde{M}}\varrho|^2 - \varrho \Delta^{\widetilde{M}} \varrho \right) \\ &= \frac{1}{\varrho\sqrt{\varrho^2 + |\nabla^S \varrho|^2}} \left(n\varrho + \frac{\varrho \left|\nabla^S \varrho\right|^2}{\varrho^2 + \left|\nabla^S \varrho\right|^2} \right. \\ &\qquad \left. - \left(g^{kl}(s) - \frac{g^{km}(s) g^{lp}(s) \nabla_m \varrho \nabla_p \varrho}{\varrho^2 + \left|\nabla^S \varrho\right|^2} \right) \nabla_k \nabla^S_l \varrho \right) \end{aligned} \tag{3.11}$$

and

$$f_\Omega(\varrho) = -\Omega \left(1 - \langle s, e_{n+1} \rangle^2\right) \varrho^2 . \tag{3.12}$$

Despite its quasilinear nature, for the purposes of regularity analysis we may 'linearise' (3.10). We combine (3.10) - (3.12) with Definition 2.12 and rearrange to deduce that, on each D_i, ϱ is a solution of the homogeneous equation:

$$L_\Omega[v] = a^{kl}(s) \nabla_k \nabla^S_l v + b^k(s) \nabla_k v + c_\Omega(s)\, v = 0 ; \tag{3.13}$$

where

$$a^{kl}(s) = g^{kl}(s) - \frac{g^{km}(s)g^{lp}(s)\nabla_m\varrho\nabla_p\varrho}{\varrho^2 + |\nabla^S\varrho|^2};$$

$$b^k(s) = -\frac{\varrho g^{km}(s)\nabla_m\varrho}{\varrho^2 + |\nabla^S\varrho|^2};$$

and

$$c_\Omega(s) = \sqrt{\varrho^2 + |\nabla^S\varrho|^2}\left(\Omega\left(1 - \langle s, e_{n+1}\rangle^2\right)\varrho^2 - \beta^0\zeta_0 - \beta^k\langle s, e_k\rangle\varrho\right) - n .$$

We assert that the operator L_Ω given in (3.13) is uniformly elliptic. We first observe that the eigenvalues of the matrix $(a^{ij}(s))$ are given by:

$$\kappa_m = \begin{cases} \frac{\varrho^2}{\varrho^2 + |\nabla^S\varrho|^2} & m = 1 \text{ ; and} \\ 1 & 2 \le m \le n . \end{cases}$$

Moreover, pursuant to Definition 2.2, we may find $\delta_i > 0$ such that $\inf_{D_i}\varrho \ge \delta_i$. Thus

$$\frac{\delta_i^2}{C(n)\,\|\varrho\|^2_{C^1(D_i)}} \le \kappa_1 \le 1$$

and the ratio $\frac{\kappa_{\max}}{\kappa_{\min}}$ is bounded on each D_i.

The coefficients of L_Ω are in $C^{2,\alpha}(D_i)$, by hypothesis and the smoothness of S. Therefore, standard interior regularity theory (contained in [**32**, Theorem 3.20] or [**40**, Theorem 6.17], for example) implied by the local Schauder estimates for linear elliptic equations (these may again be obtained in a manner analogous to the Euclidean case) guarantees that $\varrho \in C^{4,\alpha}(D_i)$ on each D_i. We now proceed to prove the statement

$$P(k) \quad : \quad \varrho \in C^{k+2,\alpha}(D_i)$$

for each $k \in \mathbb{N}\setminus\{1\}$ by induction on k, where $P(2)$ was established above. Suppose there exists a particular $k \in \mathbb{N}\setminus\{1\}$ such that $P(k)$ holds. Then the coefficients of L_Ω are in $C^{k+1,\alpha}(D_i)$ and we may again invoke our interior regularity theory to deduce that $\varrho \in C^{k+3,\alpha}(D_i)$, which confirms $P(k+1)$. Thus $P(k)$ holds for each $k \in \mathbb{N}\setminus\{1\}$ by the principle of mathematical induction, and as $k \to \infty$ we ascertain from the smoothness of the coefficients of L_Ω that $\varrho \in C^\infty(D_i)$ on each D_i.

To conclude, we observe that since S is compact there exists a finite subcover for S contained in $\mathscr{D}$. Hence $\varrho \in C^\infty(S)$, as required. ◊

CHAPTER 4

Stability Analysis

In this chapter we prove that the solutions of the Euler-Lagrange equation established by Theorem 3.1 correspond to stable energy minimisers for the rotating drop problem whenever Ω is of sufficiently small magnitude.

1. A Rayleigh quotient formulation

We shall now define a linear operator $\mathscr{L}_\Omega$ over the class $\mathscr{N}^2_{\widetilde{M}}$ which is self-adjoint with respect to the $L^2(\widetilde{M})$ topology. Subsequently, we shall utilise a Rayleigh quotient to determine its linear stability where the relationship between $\mathscr{L}_\Omega$ and the Jacobi operator corresponding to $\mathscr{G}_\Omega$ will then establish the existence of stable energy minimisers.

DEFINITION 4.1. Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation (2.4). Then, whenever we adhere to the convention that $e_0 = \mathbf{0}$, we define the linear operator $\mathscr{L}_\Omega$: $\mathscr{N}^2_{\widetilde{M}} \to \mathscr{N}^0_{\widetilde{M}}$ by

$$\mathscr{L}_\Omega[\eta] \quad = \quad -\pi_{\mathscr{N}^0_{\widetilde{M}}}\Big(\Delta^{\widetilde{M}}\eta + \big(\|A\|^2(\varrho) + 2\Omega\,\langle \pi_{\mathbb{R}^n} X(\varrho), \nu(\varrho)\rangle$$

$$+ M^{ij}(\varrho)\,\langle (H + f_\Omega)\,, \zeta_j\rangle_{L^2(\widetilde{M})}\,\langle \nu(\varrho), e_i\rangle\big)\,\eta\Big)\,.$$

We now clarify the relationship between $\mathscr{L}_\Omega$ and the Jacobi operator corresponding to $\mathscr{G}_\Omega$.

LEMMA 4.2. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation (2.4) and $\eta \in C^{2,\alpha}(S)$ with $\langle \partial X(\varrho)[\eta], \nu(\varrho)\rangle \in \mathscr{N}^2_{\widetilde{M}}$. Then the Jacobi operator $\mathscr{J}_\varrho$ corresponding to $\mathscr{G}_\Omega$ may be expressed:*

$$\mathscr{J}_\varrho[\eta] \;=\; \mathscr{L}_\Omega\left[\langle \partial X(\varrho)[\eta], \nu(\varrho)\rangle\right].$$

PROOF

Throughout the proof, all geometric quantities shall be calculated with respect to the metric on $\widetilde{M}$, and we shall again observe the convention that $e_0 = \mathbf{0}$. By definition, the Jacobi operator corresponding to $\mathscr{G}_\Omega$ is the linearisation of the Euler operator evaluated at ϱ. We may therefore cite Corollary 2.18 and, without loss of generality, recall the calculation of (3.7) in the proof of Theorem 3.1 to verify that, for any variation of ϱ which remains in the class $\mathcal{R}^2$,

$$\begin{aligned}
\mathscr{J}_{\varrho}[\eta] &= -\Delta\langle X',\nu\rangle - \langle X',\nu\rangle\|A\|^2 + \langle X',\nabla H\rangle + \langle X', D_x f_\Omega\rangle \\
&+ M^{ik}M^{jl}\langle (H + f_\Omega),\zeta_j\rangle_{L^2}\left(\int_{\widetilde{M}}(\zeta_k\langle\nu,e_l\rangle + \zeta_l\langle\nu,e_k\rangle + H\zeta_k\zeta_l)\langle X',\nu\rangle\, d\mu\right)\zeta_i \\
&+ M^{ij}\Bigg(\Bigg(\int_{\widetilde{M}}(\Delta\langle X',\nu\rangle + \|A\|^2\langle X',\nu\rangle - \langle D_x f_\Omega,\nu\rangle\langle X',\nu\rangle)\,\zeta_j d\mu \\
&- \int_{\widetilde{M}}(H + f_\Omega)\left((\langle\nu,e_j\rangle + H\zeta_j\right)\langle X',\nu\rangle\, d\mu\Bigg)\zeta_i - \langle (H + f_\Omega),\zeta_j\rangle_{L^2}\zeta_i'\Bigg) \\
&= -\Delta\langle X',\nu\rangle - \left(\|A\|^2 + 2\Omega\langle \pi_{\mathbb{R}^n}X,\nu\rangle + M^{ij}\langle (H + f_\Omega),\zeta_j\rangle_{L^2}\langle\nu,e_i\rangle\right)\langle X',\nu\rangle \\
&+ M^{ik}M^{jl}\langle (H + f_\Omega),\zeta_j\rangle_{L^2}\left(\int_{\widetilde{M}}(\zeta_k\langle\nu,e_l\rangle + \zeta_l\langle\nu,e_k\rangle + H\zeta_k\zeta_l)\langle X',\nu\rangle\, d\mu\right)\zeta_i \\
&+ M^{ij}\left(\int_{\widetilde{M}}(\Delta\langle X',\nu\rangle + \|A\|^2\langle X',\nu\rangle - \langle D_x f_\Omega,\nu\rangle\langle X',\nu\rangle)\,\zeta_j d\mu\right)\zeta_i \\
&- M^{ij}M^{kl}\langle (H + f_\Omega),\zeta_l\rangle_{L^2}\left(\int_{\widetilde{M}}(\zeta_k\langle\nu,e_j\rangle + H\zeta_j\zeta_k)\langle X',\nu\rangle\, d\mu\right)\zeta_i \\
&= -\pi_{\mathscr{N}^0_{\widetilde{M}}}\Big(\Delta\langle X',\nu\rangle + \Big(\|A\|^2 + 2\Omega\langle\pi_{\mathbb{R}^n}X,\nu\rangle \\
&\qquad + M^{ij}\langle (H + f_\Omega),\zeta_j\rangle_{L^2}\langle\nu,e_i\rangle\Big)\langle X',\nu\rangle\Big).
\end{aligned}$$

In particular, whenever $\langle X',\nu\rangle \in \mathscr{N}^2_{\widetilde{M}}$, we obtain our intended result. $\diamondsuit$

Moreover, we may integrate by parts whilst invoking Definition 2.20 and Proposition 2.24 to propound the following lemma.

LEMMA 4.3. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation* (2.4) *and* $\eta \in C^{2,\alpha}(S)$ *with* $\langle \partial X(\varrho)[\eta], \nu(\varrho)\rangle \in \mathscr{N}^2_{\widetilde{M}}$. *Then*

$$\langle \mathscr{L}_\Omega \left[\langle \partial X(\varrho)[\eta], \nu(\varrho)\rangle\right], \langle \partial X(\varrho)[\eta], \nu(\varrho)\rangle\rangle_{L^2(\widetilde{M})} \;=\; \partial^2 \mathscr{G}_\Omega(\varrho)[\eta,\eta]\,.$$

We now state the principal result of this section.

THEOREM 4.4. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation* (2.4) *and $\varepsilon > 0$ is such that $\|\varrho - 1\|_{C^{2,\alpha}(S)} \le \varepsilon$. Then the spectrum of $\mathscr{L}_\Omega$ consists of a countably infinite sequence $\Lambda = \{\lambda_i : i \in \mathbb{N}\}$ of real eigenvalues whose eigenfunctions span $\mathscr{N}^2_{\widetilde{M}}$. In particular, for sufficiently small ε, $\Lambda \subset \mathbb{R}^+$ whenever $|\Omega| < \frac{(n+2)}{2}\,(1 - \varepsilon\, C_1(n))$.*

The ensuing technical lemma will illuminate the proof of Theorem 4.4.

LEMMA 4.5. *Suppose $\rho \in \mathscr{R}$, $\eta \in \mathscr{N}^1_M$, and $\varepsilon > 0$ is such that $\|\rho - 1\|_{C^{1,\alpha}(S)} \le \varepsilon$. Then, for sufficiently small ε,*

$$\left\|\nabla^M \eta\right\|^2_{L^2(M)} \;\ge\; 2(n+1)\,(1 - \varepsilon\, C_2(n))\, \|\eta\|^2_{L^2(M)}\,.$$

PROOF

We commence by noting that $\mathscr{N}^1_M \subset C^{1,\alpha}(S)$ due to the interrelated geometries of S and M established by Lemma 1.4. Thus we may project any $\eta \in \mathscr{N}^1_M$, with respect to the $L^2(S)$ topology, onto the class $\mathscr{N}^1_S$:

$$\psi \;=\; \pi_{\mathscr{N}^1_S}\, \eta \;=\; \eta \;-\; M^{ij}(1)\, \langle \eta, \zeta_i\rangle_{L^2(S)}\, \zeta_j(1)\,.$$

We recall that $\mathscr{N}^1_S \subset L^2(S)\backslash\left\{\mathcal{H}^{n+1}_0 \cup \mathcal{H}^{n+1}_1\right\}$ where, in analysis identical to that performed during the proof of step (4) in Theorem 3.1, we may truncate the corresponding condensed harmonic expansion of ψ to include only harmonics of degree $k \ge 2$, and obtain the estimate:

$$\left\|\nabla^S \psi\right\|^2_{L^2(S)} \;\ge\; 2(n+1)\|\psi\|^2_{L^2(S)}\,. \tag{4.1}$$

Now

$$\psi^2 \;=\; \eta^2 \;-\; 2M^{ij}(1)\, \langle \eta, \zeta_i\rangle_{L^2(S)}\, \eta\zeta_j(1)$$

$$+\; M^{ij}(1) M^{kl}(1)\, \langle \eta, \zeta_i\rangle_{L^2(S)}\, \langle \eta, \zeta_k\rangle_{L^2(S)}\, \zeta_j(1)\zeta_l(1)\,, \tag{4.2}$$

and

$$|\nabla^S \psi|^2 \;=\; |\nabla^S \eta|^2 \;-\; 2M^{ij}(1)\, \langle \eta, \zeta_i\rangle_{L^2(S)} \left\langle \nabla^S \eta, \nabla^S \zeta_j(1)\right\rangle$$

$$+\; M^{ij}(1) M^{kl}(1)\, \langle \eta, \zeta_i\rangle_{L^2(S)}\, \langle \eta, \zeta_k\rangle_{L^2(S)} \left\langle \nabla^S \zeta_j(1), \nabla^S \zeta_l(1)\right\rangle. \tag{4.3}$$

Hence, we utilise the characterisation of the matrix $(M_{ij}(1))$ given by (2.3) in conjunction with (4.2) and Definition 2.20 to discern that

$$\begin{aligned}\|\psi\|^2_{L^2(S)} &= \|\eta\|^2_{L^2(S)} - M^{ij}(1)\,\langle\eta,\zeta_i\rangle_{L^2(S)}\,\langle\eta,\zeta_j\rangle_{L^2(S)} \\ &= \|\eta\|^2_{L^2(S)} - \frac{(n+1)}{|S|}\sum_{i=0}^{n+1}\langle\eta,\zeta_i\rangle^2_{L^2(S)}\ .\end{aligned}$$

Similarly, by (2.3), (4.3), the divergence theorem, and Theorem C.10 in Appendix C,

$$\begin{aligned}\left\|\nabla^S\psi\right\|^2_{L^2(S)} &= \left\|\nabla^S\eta\right\|^2_{L^2(S)} + 2M^{ij}(1)\,\langle\eta,\zeta_i\rangle_{L^2(S)}\left\langle\eta,\Delta^S\zeta_j\right\rangle_{L^2(S)} \\ &\quad - M^{ij}(1)M^{kl}(1)\,\langle\eta,\zeta_i\rangle_{L^2(S)}\,\langle\eta,\zeta_k\rangle_{L^2(S)}\left\langle\zeta_j,\Delta^S\zeta_l\right\rangle_{L^2(S)} \\ &= \left\|\nabla^S\eta\right\|^2_{L^2(S)} - \frac{n(n+1)}{|S|}\sum_{i=1}^{n+1}\langle\eta,\zeta_i\rangle^2_{L^2(S)}\ .\end{aligned}$$

Therefore, we deduce from (4.1) that

$$\begin{aligned}\left\|\nabla^S\eta\right\|^2_{L^2(S)} &\geq 2(n+1)\left(\|\eta\|^2_{L^2(S)} - \frac{(n+1)}{|S|}\langle\eta,\zeta_0\rangle^2_{L^2(S)}\right) \\ &\quad - \frac{(n+1)(n+2)}{|S|}\sum_{i=1}^{n+1}\langle\eta,\zeta_i\rangle^2_{L^2(S)}\ . \qquad (4.4)\end{aligned}$$

By utilising our two constraints on the class $\mathscr{N}^1_M$, the Cauchy-Schwarz inequality, and Definition 2.20 we may assert that

$$\begin{aligned}(n+1)\,\langle\eta,\zeta_0\rangle^2_{L^2(S)} &= \left(\int_S \eta\,(1-\mu_\rho)\,d\sigma\right)^2 \\ &\leq \sup_S\,(1-\mu_\rho)^2\,|S|\,\|\eta\|^2_{L^2(S)} \\ &\leq \varepsilon\,K_1(n)\,|S|\,\|\eta\|^2_{L^2(S)} \qquad (4.5)\end{aligned}$$

and, by applying Lemma C.5, for each $i \in \{1, ..., n+1\}$,

$$\langle\eta,\zeta_i\rangle^2_{L^2(S)} = \left(\int_S \eta\zeta_i\,(1-\rho\mu_\rho)\,d\sigma\right)^2$$

$$\leq \sup_S (1 - \rho\mu_\rho)^2 \|\zeta_i\|^2_{L^2(S)} \|\eta\|^2_{L^2(S)}$$

$$\leq \frac{\varepsilon K_2(n)\,|S|}{n+1} \|\eta\|^2_{L^2(S)} . \tag{4.6}$$

We now derive a correspondence between our integrals over S and their representation over M. For sufficiently small ε,

$$\|\eta\|^2_{L^2(S)} \geq \inf_S \left(\frac{1}{\mu_\rho}\right) \|\eta\|^2_{L^2(M)} \geq (1 - \varepsilon K_3(n)) \|\eta\|^2_{L^2(M)} . \tag{4.7}$$

Moreover, by Lemma 1.4 and the Cauchy-Schwarz inequality,

$$|\nabla^M \eta|^2 = \frac{1}{\rho^2}\left(|\nabla^S \eta|^2 - \frac{\langle \nabla^S \eta, \nabla^S \rho\rangle^2}{\rho^2 + |\nabla^S \rho|^2}\right) \geq \frac{|\nabla^S \eta|^2}{\rho^2 + |\nabla^S \rho|^2} .$$

Therefore, we may again cite Lemma 1.4 to ascertain that

$$\|\nabla^S \eta\|^2_{L^2(S)} \leq \sup_S \left(\frac{\sqrt{\rho^2 + |\nabla^S \rho|^2}}{\rho^{n-1}}\right) \|\nabla^M \eta\|^2_{L^2(M)}$$

$$\leq (1 + \varepsilon K_4(n)) \|\nabla^M \eta\|^2_{L^2(M)} . \tag{4.8}$$

By combining (4.4) - (4.8), we conclude that, for sufficiently small ε,

$$\|\nabla^M \eta\|^2_{L^2(M)} \geq \frac{2(n+1)}{(1 + \varepsilon K_4)} (1 - \varepsilon K_3)\left(1 - \varepsilon\left(K_1 + \frac{(n+2)K_2}{2}\right)\right) \|\eta\|^2_{L^2(M)}$$

$$\geq 2(n+1)(1 - \varepsilon C_2(n)) \|\eta\|^2_{L^2(M)} ,$$

as required. ◇

We must now derive estimates on the Lagrange multiplier terms which appear in $\mathscr{L}_\Omega$. We first examine the behaviour of the matrix $(M^{ij}(\rho))$ whenever ρ remains close to 1 with respect to the $C^{1,\alpha}$ norm.

LEMMA 4.6. *Suppose $\rho \in \mathscr{R}$ and $\varepsilon > 0$ is such that $\|\rho - 1\|_{C^{1,\alpha}(S)} \leq \varepsilon$. Then each component of the matrix $(M^{ij}(\rho))$ satisfies an estimate of the form:*

$$\left|M^{ij}(\rho) - M^{ij}(1)\right| \leq \varepsilon C^{ij}(n)\,|S|^{-1} .$$

PROOF

Throughout the proof, we shall repeatedly invoke Definitions 1.3 and 2.12 in conjunction with Lemma 1.4. We begin by establishing preliminary estimates on components of the (symmetric) matrix $(M_{ij}(\rho))$. In general, we may assert that

$$|M_{ij}(\rho) - M_{ij}(1)| = \left|\int_S (\zeta_i(\rho)\zeta_j(\rho)\mu_\rho - \zeta_i(1)\zeta_j(1))\, d\sigma\right| .$$

We now examine the particular cases. When $i = j = 0$, we compute that

$$\begin{aligned} |M_{00}(\rho) - M_{00}(1)| &= \left|\int_S (\mu_\rho - 1)\, \zeta_0^2(1)\, d\sigma\right| \\ &\leq \sup_S |\mu_\rho - 1|\, \|\zeta_0\|^2_{L^2(S)} \\ &\leq \varepsilon\, C_{00}(n) \frac{|S|}{n+1} . \end{aligned} \tag{4.9}$$

Furthermore, for each $j \in \{1, ..., n+1\}$, we deduce that

$$\begin{aligned} |M_{0j}(\rho) - M_{0j}(1)| &= \left|\int_S (\rho\mu_\rho - 1)\, \zeta_0(1)\zeta_j(1)\, d\sigma\right| \\ &\leq \sup_S |\rho\mu_\rho - 1|\, \|\zeta_0\|_{L^2(S)}\, \|\zeta_j\|_{L^2(S)} \\ &\leq \varepsilon\, C_{0j}(n) \frac{|S|}{n+1} . \end{aligned} \tag{4.10}$$

Here we have exploited the properties of the spherical harmonics (of degree 1) prescribed by Lemma C.5. To evaluate the remaining components, which correspond to $i, j \in \{1, ..., n+1\}$, we may again cite Lemma C.5:

$$\begin{aligned} |M_{ij}(\rho) - M_{ij}(1)| &= \left|\int_S (\rho^2\mu_\rho - 1)\, \zeta_i(1)\zeta_j(1)\, d\sigma\right| \\ &\leq \sup_S |\rho^2\mu_\rho - 1|\, \|\zeta_i\|_{L^2(S)}\, \|\zeta_j\|_{L^2(S)} \\ &\leq \varepsilon\, C_{ij}(n) \frac{|S|}{n+1} . \end{aligned} \tag{4.11}$$

Therefore, we may combine (4.9)-(4.11) to verify that each component of $(M_{ij}(\rho))$ satisfies an estimate of the form:

$$|M_{ij}(\rho) - M_{ij}(1)| \leq \varepsilon\, C_{ij}(n) \frac{|S|}{n+1} . \tag{4.12}$$

Since $(M_{ij}(\rho))$ is symmetric and invertible, we may find a non-singular, diagonal matrix **D** composed of its eigenvalues $\{\kappa_m(\rho) : m \in \{0, ..., n+1\}\}$, and an orthogonal matrix **O** whose columns $\{\mathbf{o}_m : m \in \{0, ..., n+1\}\}$ comprise an orthonormal basis for the corresponding eigenvectors such that

$$(M_{ij}(\rho)) \;=\; \mathbf{ODO}^T .$$

We again recall the characterisation of the matrix $(M_{ij}(1))$ given by (2.3) to deduce from (4.12) that, for each $m \in \{0, ..., n+1\}$,

$$|\kappa_m(\rho) \,-\, \kappa_m(1)| \;\leq\; \varepsilon \left|\mathbf{o}_m^T \left(C_{kl}(n)\right) \mathbf{o}_m\right| \frac{|S|}{n+1} \;\leq\; \varepsilon\, K_m(n) \frac{|S|}{n+1} .$$

Thus the corresponding eigenvalues of the two matrices are arbitrarily close, and, for sufficiently small ε, we may infer our desired estimate. ◊

Proposition 4.7. *Suppose $\rho \in \mathscr{R}$ and $\varepsilon > 0$ is such that $\|\rho \,-\, 1\|_{C^{2,\alpha}(S)} \leq \varepsilon$. Then, for each $i \in \mathbb{N}$ with $1 \leq i \leq n+1$,*

$$\left|M^{ij}(\rho)\, \langle (H \,+\, f_\Omega)\,, \zeta_j \rangle_{L^2(M)}\right| \;\leq\; \varepsilon\, C^i(n)\, (1 \,+\, |\Omega|) .$$

Proof

Throughout the proof, we shall repeatedly employ Lemma 1.4 to illuminate the relationship between the geometries of S and M. We commence by deriving estimates on the $(n+2)$ inner products $\langle (H \,+\, f_\Omega)\,, \zeta_j \rangle_{L^2(M)}$. In a similar manner to the computation of (3.11) in the proof of Theorem 3.3, we deduce that

$$\begin{aligned} H \;&=\; \frac{1}{\rho\sqrt{\rho^2 \,+\, |\nabla^S \rho|^2}} \left[n\rho + \frac{\rho|\nabla^S\rho|^2}{\rho^2 \,+\, |\nabla^S\rho|^2} \,-\, \Delta^S \rho + \frac{\left(\nabla_i \nabla_j^S \rho\right) \left[\nabla^S\rho, \nabla^S\rho\right]}{\rho^2 \,+\, |\nabla^S\rho|^2} \right] \\ &=\; \frac{1}{\rho\sqrt{\rho^2 \,+\, |\nabla^S \rho|^2}} \left[n\rho \,-\, \Delta^S \rho + \left\langle \nabla^S \rho, \nabla^S \ln\left(\sqrt{\rho^2 \,+\, |\nabla^S\rho|^2}\right)\right\rangle\right] . \qquad (4.13) \end{aligned}$$

Hence, for $j = 0$,

$$\begin{aligned} \sqrt{n+1}\, \langle H, \zeta_0 \rangle_{L^2(M)} \;&=\; \int_S \rho^{n-2} \left[n\rho \,-\, \Delta^S \rho + \left\langle \nabla^S \rho, \nabla^S \ln\left(\sqrt{\rho^2 \,+\, |\nabla^S\rho|^2}\right)\right\rangle\right] d\sigma \\ &=\; \int_S \rho^{n-3} \Big[n\rho^2 \,+\, (n-2)|\nabla^S\rho|^2 \left(1 \,-\, \ln\left(\sqrt{\rho^2 \,+\, |\nabla^S\rho|^2}\right)\right) \\ &\qquad \,-\, \ln\left(\sqrt{\rho^2 \,+\, |\nabla^S\rho|^2}\right) \rho \Delta^S \rho \Big]\, d\sigma \\ &\leq\; \left(n \,+\, \varepsilon\, B_0(n)\right) |S| . \end{aligned}$$

Conversely,

$$\sqrt{n+1}\,\langle H, \zeta_0\rangle_{L^2(M)} \;\geq\; \Big(n \,-\, \varepsilon\,\tilde{B}_0(n)\Big)\,|S|\,.$$

Here we have integrated by parts and we note that these estimates remain valid in the particular case $n = 2$. Furthermore, we ascertain from the divergence theorem that, for each $k, l \in \{1, ..., n+1\}$,

$$\int_M \langle \zeta_k(\rho)\, e_l, \nu\rangle\, d\mu \;=\; \int_E \operatorname{div}_{\mathbb{R}^{n+1}}\left(\langle x, e_k\rangle\, e_l\right) dx \;=\; \delta_{kl}\,|E| \;=\; \frac{|S|}{n+1}\,\delta_{kl}\,.$$

Thus we may employ the definition of f_Ω to compute that

$$\begin{aligned}
&\sqrt{n+1}\left|\langle f_\Omega, \zeta_0\rangle_{L^2(M)} \,+\, \frac{n\,\Omega\,|S|}{(n+1)^{\frac{3}{2}}}\right| \\
&= \left|\Omega \int_M \langle \pi_{\mathbb{R}^n} X, \nu \,-\, X\rangle\, d\mu\right| \\
&= |\Omega|\left|\int_S \rho^n \left\langle s \,-\, \zeta_{n+1}(1) e_{n+1}, \rho s\left(1 \,-\, \sqrt{\rho^2 \,+\, |\nabla^S \rho|^2}\right) \,-\, \nabla^S \rho\right\rangle d\sigma\right| \\
&\leq\; \varepsilon\,|\Omega|\, D_0(n)\,|S|\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\langle (H \,+\, f_\Omega)\,, \zeta_0\rangle_{L^2(M)} \,-\, \frac{n}{\sqrt{n+1}}\left(1 \,-\, \frac{\Omega}{(n+1)}\right)|S|\right| \\
&\leq\; \left|\langle H, \zeta_0\rangle_{L^2(M)} \,-\, \frac{n}{\sqrt{n+1}}\,|S|\right| \,+\, \left|\langle f_\Omega, \zeta_0\rangle_{L^2(M)} \,+\, \frac{n\,\Omega}{(n+1)^{\frac{3}{2}}}\;|S|\right| \\
&\leq\; \varepsilon\left(\max\{B_0, \tilde{B}_0\} \,+\, |\Omega|\, D_0\right)\frac{|S|}{\sqrt{n+1}} \\
&\leq\; \varepsilon\, K_0(n)\,(1 \,+\, |\Omega|)\,\frac{|S|}{\sqrt{n+1}}\,.
\end{aligned} \tag{4.14}$$

We again integrate by parts and utilise (4.13) in conjunction with Definition 2.12 to evaluate the remaining terms, which correspond to $j \in \{1, ..., n+1\}$:

$$
\begin{aligned}
&\left|\langle H, \zeta_j\rangle_{L^2(M)}\right| \\
&= \left|\int_S \rho^{n-1}\left[n\rho - \Delta^S \rho + \left\langle \nabla^S \rho, \nabla^S \ln\left(\sqrt{\rho^2 + |\nabla^S \rho|^2}\right)\right\rangle\right] \zeta_j(1) d\sigma\right| \\
&= \left|\int_S n\left(\rho^n - 1\right) \zeta_j(1) d\sigma - \int_S \ln\left(\sqrt{\rho^2 + |\nabla^S\rho|^2}\right) \rho^{n-1} \zeta_j(1) \Delta^S \rho \, d\sigma\right. \\
&\quad \left. + \int_S \rho^{n-2}\left((n-1)|\nabla^S\rho|^2 \zeta_j(1) + \rho\left\langle \nabla^S \rho, \nabla^S \zeta_j(1)\right\rangle\right)\left(1 - \ln\left(\sqrt{\rho^2 + |\nabla^S\rho|^2}\right)\right) d\sigma\right| \\
&= \left|\int_S n\left(\rho^n - 1\right) \zeta_j(1) d\sigma - \int_S \ln\left(\sqrt{\rho^2 + |\nabla^S\rho|^2}\right) \rho^{n-1} \zeta_j(1) \Delta^S \rho \, d\sigma\right. \\
&\quad \left. + \int_S \rho^{n-2}\left((n-1)|\nabla^S\rho|^2 \zeta_j(1) + \rho\left\langle \nabla^S \rho, e_j\right\rangle\right)\left(1 - \ln\left(\sqrt{\rho^2 + |\nabla^S\rho|^2}\right)\right) d\sigma\right| \\
&\leq \varepsilon B_j(n) \, |S| \, .
\end{aligned}
$$

Here we have utilised Lemma C.5 and noted that $\nabla^S \zeta_j(1) = e_j - (e_j)^\perp$ (where this orthogonality is interpreted with respect to the tangent space on S). In order to obtain analogous estimates on the terms $\langle f_\Omega, \zeta_j\rangle_{L^2(M)}$, we recall the barycentricity condition alternatively prescribed by Definitions 1.1 and 2.2 to first infer from the divergence theorem that

$$
\int_M \langle \pi_{\mathbb{R}^n} X, \nu\rangle \, \zeta_j(\rho) d\mu \;=\; 0 \, .
$$

Hence,

$$
\begin{aligned}
\left|\langle f_\Omega, \zeta_j\rangle_{L^2(M)}\right| &= \left|-\Omega \int_M \langle \pi_{\mathbb{R}^n} X, X\rangle \, \zeta_j(\rho) d\mu\right| \\
&= \left|\Omega \int_M \langle \pi_{\mathbb{R}^n} X, \nu - X\rangle \, \zeta_j(\rho) d\mu\right|
\end{aligned}
$$

$$= \; |\Omega| \left| \int_S \rho^{n+1} \left\langle s \,-\, \zeta_{n+1}(1)\, e_{n+1}, \rho s \left(1 \,-\, \sqrt{\rho^2 \,+\, |\nabla^S \rho|^2}\right) \,-\, \nabla^S \rho \right\rangle \zeta_j(1)\, d\sigma \right|$$

$$\leq \; \varepsilon\, |\Omega|\, D_j(n)\, |S|\,.$$

Therefore, for each $j \in \{1, ..., n+1\}$,

$$\begin{aligned} \left| \langle (H \,+\, f_\Omega)\,, \zeta_j \rangle_{L^2(M)} \right| \;&\leq\; \left| \langle H, \zeta_j \rangle_{L^2(M)} \right| \,+\, \left| \langle f_\Omega, \zeta_j \rangle_{L^2(M)} \right| \\ &\leq\; \varepsilon\, (B_j \,+\, |\Omega|\, D_j)\, |S| \\ &\leq\; \varepsilon\, K_j(n)\, (1 \,+\, |\Omega|)\, |S|\,. \end{aligned} \tag{4.15}$$

In the ensuing calculation we shall suspend our use of the summation convention. By again invoking the characterisation of the diagonal matrix $(M_{ij}(1))$ determined by (2.3), we now combine (4.14) and (4.15) with Lemma 4.6 to calculate that, for each $i \in \{1, ..., n+1\}$,

$$\begin{aligned} \sum_{j=0}^{n+1} M^{ij}(\rho)\, \langle (H \,+\, f_\Omega)\,, \zeta_j \rangle_{L^2(M)} \;&\leq\; \varepsilon\, \frac{C^{i0}}{\sqrt{n+1}} \left(n \left(1 \,-\, \frac{\Omega}{(n+1)} \right) \,+\, \varepsilon\, K_0\, (1 \,+\, |\Omega|) \right) \\ &\quad +\, \varepsilon \sum_{j=1}^{n+1} \left((n+1)\delta^{ij} \,+\, \varepsilon\, C^{ij} \right) K_j\, (1 \,+\, |\Omega|) \\ &\leq\; \varepsilon\, K^i(n)\, (1 \,+\, |\Omega|)\,. \end{aligned}$$

Conversely, where we resume our adherence to the summation convention,

$$M^{ij}(\rho)\, \langle (H \,+\, f_\Omega)\,, \zeta_j \rangle_{L^2(M)} \;\geq\; -\, \varepsilon\, B^i(n)\, (1 \,+\, |\Omega|)\,.$$

This implies our intended estimate if we take $C^i(n)$ to be the maximum of $\{K^i, B^i\}$. ◇

PROOF OF THEOREM 4.4

Since $\widetilde{M}$ has no boundary and the matrix $(g^{ij}(\varrho))$ is positive definite on $T^*\widetilde{M} \times T^*\widetilde{M}$, we may adapt the argument preceding the statement of [**40**, Theorem 8.37] to our particular setting and verify that the spectrum of $\mathscr{L}_\Omega$ is comprised of a countably infinite sequence Λ of real eigenvalues whose eigenfunctions span $\mathscr{N}^2_{\widetilde{M}}$. To establish conditions under which $\Lambda \subset \mathbb{R}^+$, we examine the Rayleigh quotient of $\mathscr{L}_\Omega$ over $\mathscr{N}^2_{\widetilde{M}} \setminus \{0\}$:

$$J(\eta) \;=\; \frac{\langle \mathscr{L}_\Omega[\eta], \eta \rangle_{L^2(\widetilde{M})}}{\|\eta\|^2_{L^2(\widetilde{M})}}$$

where the minimum eigenvalue satisfies $\lambda_1 = \inf_{\mathscr{N}^2_{\widetilde{M}}\setminus\{0\}} J$. We discern that

$$\langle \mathscr{L}_\Omega[\eta], \eta\rangle_{L^2(\widetilde{M})} = \int_{\widetilde{M}} |\nabla\eta|^2 \, d\mu - \int_{\widetilde{M}} \Big(\|A\|^2 + 2\Omega \left\langle \pi_{\mathbb{R}^n} X, \nu \right\rangle$$

$$+ M^{ij} \left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(\widetilde{M})} \langle \nu, e_i \rangle \Big) \eta^2 d\mu \, .$$

Here we have integrated by parts. We may now employ (1.9) to estimate:

$$\begin{aligned} \|A\|^2(\varrho) &= \frac{1}{\varrho^2 \left(1 - |\nabla\varrho|^2\right)} \left(n + |\nabla\varrho|^4 + \varrho^2 \|(\nabla_i\nabla_j\varrho)\|^2 - 2|\nabla\varrho|^2 - 2\varrho\Delta\varrho \right. \\ &\quad \left. + 2\varrho\left(\nabla_i\nabla_j\varrho\right)[\nabla\varrho, \nabla\varrho]\right) \\ &\leq \frac{1}{\varrho^2 \left(1 - |\nabla\varrho|^2\right)} \left(n + \varrho^2 \|(\nabla_i\nabla_j\varrho)\|^2 + \varrho K(n)\left(1 + |\nabla\varrho|^2\right) \|(\nabla_i\nabla_j\varrho)\|\right). \end{aligned}$$

Here we have noted that our star-shaped construction mandates the condition $|\nabla\varrho|^2 < 1$. Now, we infer from Lemma 1.4 that

$$\nabla_i\nabla_j\varrho = \frac{\varrho}{\left(\varrho^2 + |\nabla^S\varrho|^2\right)} \left(\varrho\nabla_i\nabla_j^S\varrho - 2\nabla_i\varrho\nabla_j\varrho + |\nabla^S\varrho|^2 g_{ij}(s)\right).$$

By Lemma 1.4 and our hypothesis we subsequently compute that

$$\begin{aligned} &\|(\nabla_i\nabla_j\varrho)\|^2 \\ &= \frac{1}{\varrho^2\left(\varrho^2 + |\nabla^S\varrho|^2\right)^2} \left(\varrho^2 \left\|(\nabla_i\nabla_j^S\varrho)\right\|^2 + \left(n - \frac{\left(2\varrho^2 + |\nabla^S\varrho|^2\right)|\nabla^S\varrho|^2}{\left(\varrho^2 + |\nabla^S\varrho|^2\right)^2}\right)|\nabla^S\varrho|^4 \right. \\ &\quad - \frac{2\varrho}{\left(\varrho^2 + |\nabla^S\varrho|^2\right)^2}\left(2\varrho^3 + \left(2\varrho^2 + |\nabla^S\varrho|^2\right)|\nabla^S\varrho|^2\right)\left(\nabla_i\nabla_j^S\varrho\right)\left[\nabla^S\varrho, \nabla^S\varrho\right] \\ &\quad + 2\varrho|\nabla^S\varrho|^2\Delta^S\varrho - \frac{2}{\left(\varrho^2 + |\nabla^S\varrho|^2\right)}\left(\nabla_i\nabla_l^S\varrho\nabla^l\nabla_j^S\varrho\right)\left[\nabla^S\varrho, \nabla^S\varrho\right] \\ &\quad \left. + \frac{\varrho^2}{\left(\varrho^2 + |\nabla^S\varrho|^2\right)^2}\left(\left(\nabla_i\nabla_j^S\varrho\right)\left[\nabla^S\varrho, \nabla^S\varrho\right]\right)^2 \right). \end{aligned} \tag{4.16}$$

Hence

$$
\begin{aligned}
\|(\nabla_i\nabla_j\varrho)\|^2 \;\le\; & \frac{1}{\varrho^6}\Big(\varrho^2\,\big\|(\nabla_i\nabla^S_j\varrho)\big\|^2 + n|\nabla^S\varrho|^4 + 2\sqrt{n}\,\varrho|\nabla^S\varrho|^2\,\big\|(\nabla_i\nabla^S_j\varrho)\big\| \\
& + \frac{2}{\varrho^2}\,D_1(n)\,\big|\nabla^S\varrho\big|^2\,\big\|(\nabla_i\nabla^S_j\varrho)\big\|^2 + \frac{1}{\varrho^2}\,D_2(n)\,\big|\nabla^S\varrho\big|^4\,\big\|(\nabla_i\nabla^S_j\varrho)\big\|^2 \\
& + \frac{2}{\varrho^3}\Big(2\varrho^3 + \big(2\varrho^2 + |\nabla^S\varrho|^2\big)\,\big|\nabla^S\varrho\big|^2\Big)\,D_3(n)|\nabla^S\varrho|^2\,\big\|(\nabla_i\nabla^S_j\varrho)\big\|\Big) \\
\le\; & \varepsilon^2 D_4(n)\,.
\end{aligned}
$$

Here the tensorfield norm $\big\|(\nabla_i\nabla^S_j\varrho)\big\|$ and associated contractions are evaluated with respect to the metric on S. Thus we may once more cite Lemma 1.4 to assert that, for sufficiently small ε,

$$
\|A\|^2(\varrho) \;\le\; n + \varepsilon K_1(n)\,. \tag{4.17}
$$

Furthermore,

$$
2\Omega\,\langle \pi_{\mathbb{R}^n}X(\varrho),\nu(\varrho)\rangle \;\le\; 2\,|\Omega|\,\varrho \;\le\; 2\,|\Omega|\,(1+\varepsilon)\,. \tag{4.18}
$$

To conclude the analysis of the coefficients attached to η^2 which appear in the integrand corresponding to the inner product $\langle \mathscr{L}_\Omega[\eta],\eta\rangle_{L^2(\widetilde{M})}$, we employ Proposition 4.7 to verify that

$$
\begin{aligned}
M^{ij}\,\langle (H+f_\Omega)\,,\zeta_j\rangle_{L^2(\widetilde{M})}\,\langle \nu, e_i\rangle \;\le\; & \varepsilon\, C^i(n)\,(1 + |\Omega|)\,|\langle\nu,e_i\rangle| \\
\le\; & \varepsilon\, K_2(n)\,(1 + |\Omega|)\,.
\end{aligned} \tag{4.19}
$$

By combining (4.17)-(4.19) and Lemma 4.5, we deduce that

$$
\begin{aligned}
& \langle \mathscr{L}_\Omega[\eta],\eta\rangle_{L^2(\widetilde{M})} \\
& \ge\; \big(2(n+1)\,(1-\varepsilon\, C_2) \;-\; (n + \varepsilon\,(K_1+K_2) + |\Omega|\,(2+\varepsilon\,(2+K_2)))\big)\,\|\eta\|^2_{L^2(\widetilde{M})} \\
& \ge\; \big((n+2) \;-\; \varepsilon\, K_3(n) \;-\; 2\,|\Omega|\,(1+\varepsilon\, K_4(n))\big)\,\|\eta\|^2_{L^2(\widetilde{M})}\,.
\end{aligned}
$$

Therefore,

$$
\lambda_1 \;\ge\; \big((n+2) \;-\; \varepsilon\, K_3 \;-\; 2\,|\Omega|\,(1+\varepsilon\, K_4)\big)\,. \tag{4.20}
$$

For sufficiently small ε, this implies our intended result. $\diamondsuit$

2. The existence of stable energy minimisers

The following corollary of Theorem 4.4 establishes criteria under which the functional $\mathscr{F}_\Omega$ admits stable minimisation.

COROLLARY 4.8. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation* (2.4) *and $\varepsilon > 0$ is such that $\|\varrho - 1\|_{C^{2,\alpha}(S)} \leq \varepsilon$. Moreover, for each $k \in \mathbb{N}$, we define the following Banach subspace of $C^{k,\alpha}(S)$:*

$$\eta \in \mathcal{B}^k_\varrho \iff \begin{cases} \int_S \eta \varrho^n \zeta_0(1) d\sigma = 0; \\ \eta \in C^{k,\alpha}(S) \text{ ; and} \\ \int_S \eta \varrho^{n+1} \zeta_i(1) d\sigma = 0 \quad \forall\, i \in \{1, ..., n+1\}. \end{cases}$$

Then, whenever $|\Omega| < \frac{(n+2)}{2}(1 - \varepsilon\, C_3(n))$, the spectrum of the Jacobi operator $\mathscr{J}_\varrho$ is strictly positive on $\mathcal{B}^2_\varrho$. In particular, ϱ is a strict weak minimiser of both the functionals $\mathscr{G}_\Omega$ and $\mathscr{F}_\Omega$.

PROOF

We discern from Lemma 1.4 and Corollary 2.6 that $\mathcal{B}^2_\varrho$ corresponds to all functions $\eta \in C^{2,\alpha}(S)$ with $\langle X', \nu \rangle \in \mathscr{N}^2_{\widetilde{M}}$. Thus, we may again employ Lemma 1.4 and Corollary 2.6 in conjunction with Lemma 4.3, Theorem 4.4, and (4.20) to compute that, for any $\eta \in \mathcal{B}^2_\varrho$,

$$\begin{aligned} \mathscr{G}''(\varrho) &= \langle \mathscr{L}_\Omega [\langle X', \nu \rangle], \langle X', \nu \rangle \rangle_{L^2(\widetilde{M})} \\ &\geq \lambda_1 \|\langle X', \nu \rangle\|^2_{L^2(\widetilde{M})} \\ &= \lambda_1 \int_S \frac{\eta^2 \varrho^{n+1}}{\sqrt{\varrho^2 + |\nabla^S \varrho|^2}} d\sigma \\ &\geq \lambda_1 \left(\inf_S \left(\frac{\varrho^{n+1}}{\sqrt{\varrho^2 + |\nabla^S \varrho|^2}} \right) \right) \|\eta\|^2_{L^2(S)} \\ &\geq \lambda_1 (1 - K_1(n)\, \varepsilon) \|\eta\|^2_{L^2(S)} \\ &\geq ((n+2) - \varepsilon\, K_2(n) - 2\,|\Omega|\,(1 + \varepsilon\, K_3(n))) \|\eta\|^2_{L^2(S)} \,. \end{aligned}$$

When considered as an operator on $\mathcal{B}^2_\varrho$ with respect to the metric on S, we deduce from Lemmas 1.4, 4.2, Corollary 2.6 and Definition 4.1 that the coefficient matrix attached to the second-order term $-\nabla_i \nabla_j^S(\cdot)$ appearing in $\mathscr{J}_\varrho$ is merely

$$\frac{\varrho}{\sqrt{\varrho^2 + |\nabla^S \varrho|^2}} \left(g^{ij}(\varrho)\right),$$

which is positive definite on $T^*S \times T^*S$. Since S has no boundary where we restrict our attention to a Banach subspace of $C^{2,\alpha}(S)$, we may cite [**39**, Theorem 5.1.1] and the estimate above to conclude that the minimum eigenvalue of $\mathscr{J}_\varrho$ is positive on $\mathcal{B}^2_\varrho$, whenever $|\Omega| < \frac{(n+2)}{2}(1 - \varepsilon\, C_3(n))$. Thus we may again invoke [**39**, Theorem 5.1.1] to assert that there exists a $\gamma > 0$ such that, for each $\eta \in \mathcal{B}^1_\varrho$,

$$\mathscr{G}''(\varrho) \;\geq\; \gamma \int_S \left(|\eta|^2 + \left|\nabla^S \eta\right|^2\right) d\sigma\,,$$

and, by [**39**, Theorem 5.1.3], ϱ is a strict weak minimiser of $\mathscr{G}_\Omega$. Furthermore, by (2.1), (2.2), and (2.4), we ascertain that the second variations of $\mathscr{F}_\Omega$ and $\mathscr{G}_\Omega$ coincide for each $\eta \in C^{2,\alpha}(S)$ furnishing a volume- and barycentre-preserving variation of ϱ. Hence we infer from Proposition 2.15 and Lemma 2.23 that ϱ is a also strict weak minimiser of the energy functional $\mathscr{F}_\Omega$. ♢

We conclude Part 1 by evincing the following existence theorem for energy minimisers of the rotating drop problem in the class $\mathscr{E}$.

THEOREM 4.9. *There exists a class of stable energy minimisers for the rotating drop problem determined by smooth boundary in an appropriately chosen neighbourhood of the closed unit ball* $\overline{B}$.

PROOF

Subject to appropriate constraints on Ω and ε, Theorems 3.1, 3.3, and Corollary 4.8 establish the existence of a family of smooth solutions to the Euler-Lagrange equation, $\mathscr{M} = \{\varrho \,:\, \varrho \in B_\varepsilon(1) \subset \mathscr{R}\}$, that admits stable (weak) minimisation of the functional $\mathscr{F}_\Omega$. We recall from Definition 1.3 (and the discussion preceding its statement) that there is a unique correspondence between each member of $\mathscr{M}$ and a star-shaped liquid drop, $\widetilde{E} \in \mathscr{E}$, defined by smooth boundary $\widetilde{M}$. In particular, for sufficiently small $\Omega > 0$, such an $\widetilde{E}$ permits stable energy minimisation of the rotating drop problem. ♢

Part 2

The Parabolic Problem

CHAPTER 5

The Rotating Drop Flow

We now state the initial value problem corresponding to the L^2-gradient (or steepest descent) flow for the rotating drop problem. The derivation of this flow is comparable to the calculation of the first variation of energy exposed in Part 1, but where perturbations of elements in our class $\mathscr{E}$ are considered in a more general setting.

1. The initial value problem

For $n \geq 2$, let $E_0 \in \mathscr{E}$ be a star-shaped rotating drop with boundary M_0. We shall refer to M_0 as the *initial boundary* and to E_0 as the *initial drop*, the global energy of which is to be minimised subject to the constraint of fixed volume and barycentre in the evolution. In correspondence with the formalism of Part 1, we shall interpret Ω as a parameter over $\mathbb{R}$ in advance of our particular physical result.

Suppose that $\Omega \in \mathbb{R}$ is fixed and $T > 0$ is a time to be established later by the existence of solutions to the subsequent problem. We assume that $(N^n, \tilde{\mathbf{g}})$ is a compact, connected Riemannian manifold without boundary smoothly embedded in $\mathbb{R}^{n+1}$, and that the $C^{3,\alpha}$ embedding $X_0 : N \to M_0$ parametrises M_0. Then the family of manifolds $\{M_t : t \in (0,T)\}$ parametrised by $X : N \times (0,T) \to \mathbb{R}^{n+1}$ evolves by the *rotating drop flow* if

$$\left(\frac{\partial}{\partial t} - \Delta^{M_t}\right) X(p,t) \;=\; \left(h^i(t)\zeta_i(p,t) - f_\Omega(p,t)\right) \nu(p,t)\,, \tag{5.1}$$

with initial condition $X(\cdot,0) = X_0(\cdot)$. Here, in analogy with the construction of Part 1, we define the function $f_\Omega : N \times (0,T) \to \mathbb{R}$ by

$$f_\Omega(p,t) \;=\; -\,\Omega\,|\pi_{\mathbb{R}^n} X(p,t)|^2\,. \tag{5.2}$$

Furthermore, $\mathscr{Z}_{M_t} = \{\zeta_i(p,t) : i \in \{0,...,n+1\}\}$ is the linearly independent set in $L^2(M_t)$ given by:

$$\zeta_i(p,t) \;=\; \begin{cases} \frac{1}{\sqrt{n+1}} & i = 0\,;\ \text{and} \\ \langle X(p,t), e_i\rangle & \forall\, i \in \{1,...,n+1\}. \end{cases}$$

For each $t \in (0,T)$, we then formulate the matrix $(\mathcal{M}_{ij}(t))$ and its inverse $(\mathcal{M}^{ij}(t))$ in accordance with Definition 2.12. For each $i \in \{0,...,n+1\}$, the global terms $h^i : (0,T) \to \mathbb{R}$ are consequently determined by

$$h^i(t) \;=\; \mathcal{M}^{ij}(t)\,\langle (H + f_\Omega)\,, \zeta_j\rangle_{L^2(M_t)}\,. \tag{5.3}$$

In conclusion, we shall denote by E_t the closed subset of $\mathbb{R}^{n+1}$ bounded by M_t for each $t \in (0,T)$, where the set $\{E_t : t \in (0,T)\}$ is the associated family of *evolving drops.*

2. Short time existence of unique solutions

We are now able to verify the existence, uniqueness, and higher regularity of star-shaped solutions to the rotating drop flow (5.1) on a short time interval. We first introduce a quasilinear elliptic operator on the class $\mathcal{R}^2$ which shall prove crucial to the remaining analysis of this thesis.

DEFINITION 5.1. Suppose that $\mathscr{V} := S\times\mathbb{R}^+\times TS$ and $\mathscr{S}_2{}^0(S) \subset \mathscr{T}_2{}^0(S)$ is the tensor bundle of symmetric, covariant 2-tensorfields over S. Further suppose that $(s,z,\tau,\vartheta) \in \mathscr{V}\times\mathscr{S}_2^0(S)$ and $\{\zeta_i(s,z) : 0 \leq i \leq n+1\}$ is defined in analogy with the set $\mathscr{Z}_r$ introduced in Definition 2.12:

$$\zeta_i(s,z) \;=\; \begin{cases} \frac{1}{\sqrt{n+1}} & i = 0\ ;\ \text{and} \\ \langle s, e_i\rangle\, z & \forall\, i \in \{1,...,n+1\}. \end{cases}$$

Then we define $Q_\Omega : \mathcal{R}^2 \to C^\alpha(S)$ to be the quasilinear elliptic operator

$$Q_\Omega[r] \;=\; a^{ij}\left(s,r,\nabla^S r\right)\nabla_i\nabla_j^S r \;+\; b^i\left(s,r,\nabla^S r\right)\nabla_i r \;+\; c_\Omega\left(s,r,\nabla^S r\right) r\ ,$$

with coefficients given by

$$a^{ij}\left(s,z,\tau\right) \;=\; \frac{1}{z^2}\left(g^{ij}(s) \;-\; \frac{\tau^i\tau^j}{z^2 \,+\, |\tau|^2}\right),$$

$$b^i\left(s,z,\tau\right) \;=\; -\,\frac{\tau^i}{z\left(z^2 \,+\, |\tau|^2\right)}\ ,$$

and

$$c_\Omega\left(s,z,\tau\right) \;=\; \frac{1}{z^2}\left(\sqrt{z^2 + |\tau|^2}\left(\Omega\left(z^2 - \zeta_{n+1}^2(s,z)\right) \,+\, h^k(z)\zeta_k(s,z)\right) - n\right).$$

Here, for each $i \in \{0,...,n+1\}$, the global terms $h^i : \mathbb{R}^+ \to \mathbb{R}$ may be characterised in the following manner. Suppose that, for $i,j \in \{0,...,n+1\}$,

$$\psi_{ij}(s,z,\tau) \;=\; \zeta_i(s,z)\zeta_j\left(s,z\right)z^{n-1}\sqrt{z^2 \,+\, |\tau|^2}$$

and

$$\left(\psi_\Omega\right)_i\left(s,z,\tau,\vartheta\right) \;=\; \Big(n \,-\, \Omega\sqrt{z^2 + |\tau|^2}\left(z^2 - \zeta_{n+1}^2(s,z)\right)$$

$$-\; z\left(a^{kl}(s,z,\tau)\vartheta_{kl} \,+\, b^k(s,z,\tau)\tau_k\right)\Big)\zeta_i(s,z)z^{n-1}\ .$$

Then, for $r \in \mathcal{R}^2$, we define

$$h^i(r) \;=\; M^{ij}(r)\left(\int_S (\psi_\Omega)_j \left(s, r, \nabla^S r, \left(\nabla_i \nabla_j^S r\right)\right) d\sigma\right),$$

where, in accordance with Definition 2.12, $(M^{ij}(r))$ is the inverse of the symmetric matrix $(M_{ij}(r))$ with components prescribed by

$$M_{ij}(r) \;=\; \int_S \psi_{ij}(s, r, \nabla^S r) d\sigma \,.$$

REMARKS 5.2.

(1) While it is clear that the global terms $\{h^i\}$ prescribed by Equation (5.3) are independent of parametrisation, we shall represent them in the particular manner introduced in Definition 5.1 whenever we examine the operator Q_Ω.

(2) In the context of the metric on S, it is natural to define differentiation in the variables $(\tau, \vartheta) \in TS \times \mathscr{S}_2^0(S)$ by $D_{\tau^i}\tau^j = g^{ij}(s)$ and $D_{\vartheta_{ij}}\vartheta_{kl} = g_{ik}(s)g_{jl}(s)$.

The following lemma is an adaptation of the analogous result contained in [4, Lemma 3.2], where the proof proceeds essentially unmodified. Its principal purpose is to break the tangential diffeomorphism symmetry intrinsic to curvature flows of the type under consideration, whilst reducing the problem to a scalar equation whose spatial domain is the unit sphere.

LEMMA 5.3. *Suppose Q_Ω is the operator prescribed by Definition 5.1. Then there is an injective correspondence between smooth solutions X of equation* (5.1) *which are star-shaped about the origin and smooth positive solutions ρ of the scalar parabolic equation*

$$\frac{\partial}{\partial t}\rho(s,t) \;=\; Q_\Omega[\rho(s,t)]\,, \tag{5.4}$$

with initial condition $\rho(s,0) = \left|\left(X_0 \circ \pi_0^{-1}\right)(s)\right|$. Here $\pi_t : N \to S$ is the natural projection determined by

$$\pi_t(p) \;=\; \frac{X(p,t)}{|X(p,t)|}\,.$$

PROOF

If X is star-shaped about the origin, then π_t is a diffeomorphism. For an arbitrary time $t_0 \in (0,T)$ and $s \in S$, let $\pi_{t_0}^{-1}(s) = q$. Then $X(q,t_0) = \rho(s,t_0)s$ and we may utilise the geometric construction of Chapter 1, in particular Lemma 1.4, to derive the ensuing equations at t_0 by decomposing the evolution equation (5.1) into components orthogonal and tangential to S at s:

$$\frac{\partial}{\partial t}\rho\left(\pi_{t_0}(q)\right) \;=\; \left[\frac{\rho^2\, Q_\Omega[\rho]}{\rho^2 + |\nabla^S \rho|^2}\right]\left(\pi_{t_0}(q)\right);$$

and

$$\frac{\partial}{\partial t}\pi_{t_0}(q) = -\left[\frac{Q_\Omega[\rho]\,\nabla^S\rho}{\rho^2+|\nabla^S\rho|^2}\right](\pi_{t_0}(q)).$$

In accordance with Definition 1.3, Lemma 1.4, (1.8), and (5.2), we have employed a parallel construction to that exposed in the proof of Theorem 3.3 to observe that

$$\left[h^k(t_0)\zeta_k(q,t_0) - H(q,t_0) - f_\Omega(q,t_0)\right] = \left[\frac{\rho\, Q_\Omega[\rho]}{\sqrt{\rho^2+|\nabla^S\rho|^2}}\right](\pi_{t_0}(q)).$$

We may now calculate the evolution equation for ρ at the point (s,t_0):

$$\begin{aligned}\frac{\partial}{\partial t}\rho(s,t_0) &= \frac{\partial}{\partial t}\rho(\pi_{t_0}(q)) - \left\langle\nabla^S\rho, \frac{\partial}{\partial t}\pi_{t_0}(q)\right\rangle \\ &= \left[\frac{\rho^2 Q_\Omega[\rho]}{\rho^2+|\nabla^S\rho|^2} + \frac{|\nabla^S\rho|^2 Q_\Omega[\rho]}{\rho^2+|\nabla^S\rho|^2}\right](s,t_0) \\ &= Q_\Omega[\rho(s,t_0)].\end{aligned}$$

Conversely, a smooth solution of (5.4) generates a smooth solution of (5.1) in the following manner:

Suppose $\tilde{X} : S\times(0,T)\to\mathbb{R}^{n+1}$ is a family of embeddings given by $\tilde{X}_t(s) = \rho(s,t)s$, and that $\psi : N\times(0,T)\to S$ are the diffeomorphisms obtained by solving the ODE:

$$\frac{\partial}{\partial t}\psi_t(p) = -\left[\frac{Q_\Omega[\rho]\,\nabla^S\rho\,\cdot}{\rho^2+|\nabla^S\rho|^2}\right](\psi_t(p)), \tag{5.5}$$

with initial condition $\psi_0(p) = \pi_0(p)$. Then the embedding $X : N\times(0,T)\to\mathbb{R}^{n+1}$ given by $X(p,t) = \left(\tilde{X}_t\circ\psi_t\right)(p)$ is a smooth solution to (5.1). The correspondence is injective since the solution to (5.5) is unique. ◇

REMARK 5.4. It is important to emphasise that we may impose the geometry of Chapter 1 on star-shaped solutions of (5.1), up to tangential diffeomorphism. If we choose $X(p,t) = \rho(s,t)s$ as above, then we locate stationary solutions of the flow whenever $\rho(s,t)\equiv\varrho(s)$, since $Q_\Omega[\varrho]\equiv 0$ on S.

THEOREM 5.5. *Suppose X_0 is a $C^{3,\alpha}$ embedding of M_0. Then there exists a time $T>0$ for which equation* (5.1) *possesses a unique, smooth, star-shaped solution X on $N\times(0,T)$.*

PROOF

Throughout the proof, all geometric quantities shall be computed with respect to the metric on S. By Lemma 5.3, it is sufficient to prove the local existence of a unique, smooth solution to the scalar problem posed by (5.4) through an appropriate application of a fixed point theorem on Banach spaces.

We employ the notation of [**54**, Section 4.1] pertaining to parabolic Hölder spaces and deduce from [**54**, Theorem 8.2] that there exists a time $T > 0$ such that (5.4) possesses a solution $\rho \in H_{2+\alpha}(S \times (0,T))$. Moreover, in analogy with the argument propounded in the proof of Theorem 3.3, we exploit an inductive argument to ascertain from [**32**, Theorem 3.11 & Corollary 3.11.2] that any such $\rho \in C^{\infty}(S \times (0,T))$. In addition, it shall be critical to forthcoming analysis to note that $\rho \in H_{3+\alpha}(S \times [0,T))$ as a consequence of [**32**, Theorem 3.12]. Thus, up to order three, spatial covariant derivatives of our solution are Hölder continuous (exponent $\frac{\alpha}{2}$) in time at $t = 0$. Once again, in the application of this theory it is important to emphasise that the interior and boundary Schauder estimates on $S \times [0,T)$ may be derived in a similar manner to their Euclidean analogues.

To determine uniqueness, we shall argue by contradiction. Suppose ρ_1 and ρ_2 are two distinct smooth solutions to (5.4) on $(0,T)$ with $\rho_1(\cdot,0) \equiv \rho_2(\cdot,0)$. We first extend the domain of the operator Q_Ω introduced in Definition 5.1 to encompass all of $C^{2,\alpha}(S;\mathbb{R})$. To facilitate notational simplicity, we shall merely attach the subscript corresponding to the fixed parameter Ω to the coefficient c_Ω. For $r \in \mathcal{R}^2$, let $Q_r : C^{2,\alpha}(S) \to C^{\alpha}(S)$ be the uniformly elliptic operator given by

$$Q_r[\cdot] \;=\; a^{ij}(s,r,\nabla r)\,\nabla_i\nabla_j^S(\cdot) \;+\; b^i(s,r,\nabla r)\,\nabla_i(\cdot) \;+\; c_\Omega(s,r,\nabla r)\,(\cdot)\,. \tag{5.6}$$

We now consider the evolution of the function $v = (\rho_1 - \rho_2)^2$. For $\lambda \in [0,1]$, we then set $\rho_\lambda = \lambda\,\rho_1 + (1-\lambda)\,\rho_2$. Clearly $\rho_\lambda \in C^{\infty}(S \times (0,T))$ and, in particular, $\rho_\lambda \in \mathcal{R}^k$ for each $k \in \mathbb{N}$ and $t \in (0,T)$. Since each of the coefficients of Q_Ω is at least C^1 in its argument $(s,z,\tau) \in S \times \mathbb{R}^+ \times TS$, we may therefore assert that

$$\frac{\partial v}{\partial t} \;=\; 2\,(\rho_1-\rho_2)\,(Q_\Omega[\rho_1] \;-\; Q_\Omega[\rho_2]) \;=\; 2\,(\rho_1-\rho_2)\int_0^1 \frac{d}{d\lambda}\,(Q_\Omega\,[\rho_\lambda])\,d\lambda\,. \tag{5.7}$$

In the general setting, we now examine the behaviour of the operator Q_r. If $\eta,\xi \in C^2(S;\mathbb{R})$, then we find that

$$Q_r[\eta\,\xi] \;=\; \xi\,Q_r[\eta] \;+\; \eta\,Q_r[\xi] \;+\; 2a^{ij}(s,r,\nabla r)\,\nabla_i\eta\nabla_j\,\xi \;-\; c_\Omega(s,r,\nabla r)\,\eta\,\xi\,. \tag{5.8}$$

In addition, if we take $\mathbf{r}_\lambda = (s,\rho_\lambda,\nabla\rho_\lambda)$, then

$$\frac{d}{d\lambda}\,\mathbf{r}_\lambda \;=\; \mathbf{r}'_\lambda \;=\; (0,(\rho_1-\rho_2)\,,\nabla\,(\rho_1-\rho_2))\,.$$

Hence we confirm from (5.6)-(5.8) that

$$\frac{\partial v}{\partial t} = \int_0^1 \left(Q_{\rho_\lambda}[v] - 2a^{ij}(\mathbf{r}_\lambda)\nabla_i(\rho_1-\rho_2)\nabla_j(\rho_1-\rho_2) + c_\Omega(\mathbf{r}_\lambda)v\right.$$

$$+ 2(\rho_1-\rho_2)\left(\left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathscr{V}} \nabla_i\nabla_j\rho_\lambda + \left\langle Db^i|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathscr{V}} \nabla_i\rho_\lambda\right.$$

$$\left.\left. + \langle Dc_\Omega|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}}\,\rho_\lambda\right)\right) d\lambda\,. \tag{5.9}$$

In our consideration of (5.9), we must clarify our interpretation of the derivatives of the global terms $\{h^i\}$. We determine from Definition 5.1 that

$$\left\langle Dh^i|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathscr{V}} = -M^{ik}(\rho_\lambda)M^{jl}(\rho_\lambda)\left(\int_S \langle D\psi_{kl}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}}\, d\sigma\right)\left(\int_S (\psi_\Omega)_j|_{(\mathbf{r}_\lambda, \nabla^2\rho_\lambda)} d\sigma\right)$$

$$+ M^{ij}(\rho_\lambda)\int_S \left\langle D(\psi_\Omega)_j\Big|_{(\mathbf{r}_\lambda,\nabla^2\rho_\lambda)}, \left(\mathbf{r}'_\lambda, \nabla^2(\rho_1-\rho_2)\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma\,. \tag{5.10}$$

We then integrate by parts to discover that, for each $j,k,l \in \{0,...,n+1\}$,

$$\int_S \langle D\psi_{kl}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}}\, d\sigma = \int_S (\rho_1-\rho_2)\left(D_z - \nabla_i D_{\tau i}\right)\psi_{kl}|_{\mathbf{r}_\lambda} d\sigma\,, \tag{5.11}$$

and

$$\int_S \left\langle D(\psi_\Omega)_j\Big|_{(\mathbf{r}_\lambda,\nabla^2\rho_\lambda)}, \left(\mathbf{r}'_\lambda, \nabla^2(\rho_1-\rho_2)\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma$$

$$= \int_S (\rho_1-\rho_2)\left(D_z - \nabla_i D_{\tau i} + \nabla^i\nabla^m D_{\vartheta_{im}}\right)(\psi_\Omega)_j|_{(\mathbf{r}_\lambda,\nabla^2\rho_\lambda)} d\sigma\,. \tag{5.12}$$

We next recall that $(M_{ij}(\rho_\lambda))$ is a symmetric, invertible matrix and, in analogy with the argument expounded in the proof of Lemma 4.6, for each $i,j \in \{0,...,n+1\}$, we may certainly derive an estimate of the form:

$$\left|M^{ij}(\rho_\lambda)\right| \leq K^{ij}\left(n, \delta_1, \delta_2, \|\rho_1\|_{H_{1+\alpha}}, \|\rho_2\|_{H_{1+\alpha}}\right)|S|^{-1}\,,$$

where $\delta_m = \inf_{S\times[0,T)}\rho_m$ for $m \in \{1,2\}$ and the $H_{k+\alpha}$ norms are evaluated over the domain $S\times[0,T)$. We observe from the structure of the functions ψ_{kl} and $(\psi_\Omega)_j$ introduced in Definition 5.1 that the integrands arising in (5.10)-(5.12) contain at worst third-order spatial covariant derivatives of ρ_1 and ρ_2. Therefore, we deduce from (5.10)-(5.12) that, for each $i \in \{0,...,n+1\}$ and $t \in (0,T)$,

$$\left|\left\langle Dh^i|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathscr{V}}\zeta_i(\rho_\lambda)\right| \leq K_1\left(n, |\Omega|, \delta_1, \delta_2, \|\rho_1\|_{H_{3+\alpha}}, \|\rho_2\|_{H_{3+\alpha}}\right)\|(\rho_1-\rho_2)\|_{L^\infty(S)}\,. \tag{5.13}$$

We recognise that the remaining terms arising in $\langle Dc_\Omega|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle$ may be evaluated in parallel with those occurring in the contractions involving the derivatives of a^{ij} and b^i. When multiplied by the factor $2(\rho_1 - \rho_2)$, we deduce from Definition 5.1 that these quantities may be factorised into either v or ∇v contracted over coefficients which are smooth on $S \times (0,T)$. Moreover, the matrix $(a^{ij}(\mathbf{r}_\lambda))$ is positive definite for any choice of $\lambda \in [0,1]$ and $t \in (0,T)$. Hence we discern from (5.9) and (5.13) that v satisfies the following differential inequality on $S \times (0,T)$:

$$\frac{\partial v}{\partial t} \leq \beta^{ij}(s,\rho_1,\rho_2,\nabla\rho_1,\nabla\rho_2)\nabla_i\nabla_j v + \gamma^i_\Omega\left(s,\rho_1,\rho_2,\nabla\rho_1,\nabla\rho_2,\nabla^2\rho_1,\nabla^2\rho_2\right)\nabla_i v$$
$$+ K_2\left(n,|\Omega|,\delta_1,\delta_2,\|\rho_1\|_{H_{3+\alpha}(S\times[0,T))},\|\rho_2\|_{H_{3+\alpha}(S\times[0,T))}\right)\left(v + \sup_S v\right). \quad (5.14)$$

Clearly the coefficients $\beta^{ij}, \gamma^i_\Omega \in C^\infty(S \times (0,T))$, where the matrix (β^{ij}) is positive definite on $S \times (0,T)$ and the constant K_2 is independent of time. We have further observed here that, for each $\lambda \in [0,1]$ and $t \in (0,T)$,

$$a^{ij}(\mathbf{r}_\lambda)\nabla_i(\rho_1-\rho_2)\nabla_j(\rho_1-\rho_2) \geq 0\,.$$ [3]

We may now apply a form of the maximum principle for parabolic equations due to Hamilton [**45**, Lemma 3.5]. If $S(t) = \{s \in S : v(s,t) = (\sup_S v)(t)\}$, then

$$\frac{d}{dt}\left(\sup_S v\right) \leq \sup_{S(t)}\left(\frac{\partial v}{\partial t}\right).$$

We note that since S has no boundary, for each $t \in (0,T)$ the supremum of the quantity v coincides with an interior spatial maximum on S. We therefore deduce from (5.14) that

$$\frac{d}{dt}v_{\max} \leq 2K_2 v_{\max} \implies v_{\max}(t) \leq v_{\max}(0)\,e^{2K_2 t} = 0\,.$$

Hence $\rho_1 \equiv \rho_2$ and we derive a contradiction. Thus any smooth, star-shaped solution to the initial value problem posed by (5.1) is unique on $(0,T)$. ♢

3. The evolution equations

In order to consider the evolution of various geometric quantities on our family of evolving manifolds, it will be convenient to represent the rotating drop flow (5.1) in the more general form:

$$\frac{\partial}{\partial t}X(p,t) = \eta(p,t)\nu(p,t) \quad (5.15)$$

where $\eta(p,t) = (h^i(t)\zeta_i(p,t) - H(p,t) - f_\Omega(p,t))$. The following results are essentially known in the general setting of mean curvature flow and its variants (see, for example, [**4, 27, 47, 48**] or [**61**]), but we shall present them here for the sake of completeness.

[3] In fact, it is possible to demonstrate that at a spatial maximum of v, $\nabla_i(\rho_1-\rho_2) = 0$ for each $i \in \{1,...,n\}$, otherwise $v_{\max} \equiv 0$.

LEMMA 5.6. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by* (5.15). *Then the components of the metric on* M_t *satisfy the evolution equations:*

$$\frac{\partial}{\partial t}\, g_{ij}(p,t) = 2\eta(p,t)h_{ij}(p,t)\,.$$

PROOF

We compute that

$$\begin{aligned}
\frac{\partial}{\partial t}\, g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial X}{\partial p_i}, \frac{\partial X}{\partial p_j}\right\rangle \\
&= \left\langle \frac{\partial}{\partial p_i}\left(\frac{\partial X}{\partial t}\right), \frac{\partial X}{\partial p_j}\right\rangle + \left\langle \frac{\partial X}{\partial p_i}, \frac{\partial}{\partial p_j}\left(\frac{\partial X}{\partial t}\right)\right\rangle \\
&= \left\langle \frac{\partial}{\partial p_i}(\eta\nu), \frac{\partial X}{\partial p_j}\right\rangle + \left\langle \frac{\partial X}{\partial p_i}, \frac{\partial}{\partial p_j}(\eta\nu)\right\rangle \\
&= \eta\left\langle \frac{\partial \nu}{\partial p_i}, \frac{\partial X}{\partial p_j}\right\rangle + \eta\left\langle \frac{\partial X}{\partial p_i}, \frac{\partial \nu}{\partial p_j}\right\rangle \\
&= 2\eta h_{ij}\,.
\end{aligned}$$

Hence we obtain our desired result. ◇

COROLLARY 5.7. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by* (5.15). *Then the components of the inverse metric and the measure on* M_t *satisfy the respective evolution equations:*

$$\frac{\partial}{\partial t}\, g^{ij}(p,t) = -2\eta(p,t)h^{ij}(p,t)$$

and

$$\frac{\partial}{\partial t}\sqrt{g(p,t)} = \eta(p,t)H(p,t)\sqrt{g(p,t)}\,.$$

In particular,

$$\frac{d}{dt}\,|M_t| = \int_{M_t} \eta(p,t)H(p,t)d\mu_t\,.$$

PROOF

To calculate the evolution of each g^{ij}, recall that we may write $g^{ij} = g^{ik}g^{jl}g_{kl}$. Then, in analogy with the derivation of (3.4) in the proof of Theorem 3.1 and by Lemma 5.6, we have

$$\frac{\partial}{\partial t}\, g^{ij} \;=\; -\,g^{ik}g^{jl}\frac{\partial}{\partial t}\,g_{kl} \;=\; -\,2\eta g^{ik}g^{jl}h_{kl} \;=\; -\,2\eta h^{ij},$$

from which we infer our desired result. Furthermore, we may again employ Lemma 5.6 to discern that

$$\frac{\partial}{\partial t}\sqrt{g} \;=\; \sqrt{g}\,g^{ij}\frac{\partial}{\partial t}\,g_{ij} \;=\; \sqrt{g}\,g^{ij}\,(\eta h_{ij}) \;=\; \eta H\sqrt{g}\,.$$

Thus

$$\frac{d}{dt}\,|M_t| = \int_{M_t} \eta H d\mu_t\,,$$

as required. ◇

We may now determine the behaviour under evolution of the unit normal vectorfield.

LEMMA 5.8. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by* (5.15). *Then the outward unit normal vectorfield on* M_t *evolves in the following manner:*

$$\frac{\partial}{\partial t}\,\nu(p,t) \;=\; -\,\nabla^{M_t}\eta(p,t)\,.$$

PROOF

We first observe that $\frac{\partial \nu}{\partial t} \in T_xM_t$ for each $x = X(p,t) \in M_t$. Hence we may express $\frac{\partial \nu}{\partial t}$ as a linear combination of tangent vectors to assert that

$$\begin{aligned}
\frac{\partial \nu}{\partial t} \;&=\; g^{ij}\left\langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial p_i}\right\rangle \frac{\partial X}{\partial p_j} \\
&=\; -\,g^{ij}\left\langle \nu, \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial p_i}\right)\right\rangle \frac{\partial X}{\partial p_j} \\
&=\; -\,g^{ij}\left\langle \nu, \frac{\partial}{\partial p_i}\left(\frac{\partial X}{\partial t}\right)\right\rangle \frac{\partial X}{\partial p_j} \\
&=\; -\,g^{ij}\left\langle \nu, \frac{\partial}{\partial p_i}\,(\eta\nu)\right\rangle \frac{\partial X}{\partial p_j}
\end{aligned}$$

$$= \; - g^{ij} \nabla_i \eta \frac{\partial X}{\partial p_j}$$

$$= \; - \nabla \eta \, .$$

Here we have invoked the orthogonality of the tangent and normal spaces on M_t. ◇

In a manner analogous to the proof of Lemma 2.11, we elicit the ensuing lemma.

LEMMA 5.9. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by* (5.15). *Then the function* f_Ω *given by* (5.2) *evolves according to the equation:*

$$\frac{\partial}{\partial t} f_\Omega(p,t) \; = \; - 2\Omega \, \eta(p,t) \, \langle \pi_{\mathbb{R}^n} X(p,t), \nu(p,t) \rangle \, .$$

We are now in a position to examine the evolution of the second fundamental form.

LEMMA 5.10. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by the rotating drop flow* (5.15). *Then the components of the second fundamental form on* M_t *satisfy the evolution equations:*

$$\frac{\partial}{\partial t} h_{ij}(p,t) \; = \; - \nabla_i \nabla_j{}^{M_t} \eta(p,t) \; + \; \eta(p,t) h^m{}_i(p,t) h_{mj}(p,t) \, .$$

PROOF

We may again cite the orthogonality of the normal and tangent spaces on M_t to ascertain that

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} \; &= \; \frac{\partial}{\partial t} \left\langle \frac{\partial X}{\partial p_i}, \frac{\partial \nu}{\partial p_j} \right\rangle \; = \; - \frac{\partial}{\partial t} \left\langle \frac{\partial^2 X}{\partial p_i \partial p_j}, \nu \right\rangle \\
&= \; - \left\langle \frac{\partial^2}{\partial p_i \partial p_j} \left(\frac{\partial X}{\partial t} \right), \nu \right\rangle \; - \; \left\langle \frac{\partial^2 X}{\partial p_i \partial p_j}, \frac{\partial \nu}{\partial t} \right\rangle \\
&= \; - \left\langle \frac{\partial^2}{\partial p_i \partial p_j} (\eta \nu), \nu \right\rangle \; + \; \left\langle \frac{\partial^2 X}{\partial p_i \partial p_j}, \nabla^{M_t} \eta \right\rangle \\
&= \; - \eta \left\langle \frac{\partial}{\partial p_j} \left(\frac{\partial \nu}{\partial p_i} \right), \nu \right\rangle \; - \; \frac{\partial^2 \eta}{\partial p_i \partial p_j} \; + \; \Gamma^k_{ij} \nabla_k \eta \\
&= \; - \eta \left\langle \frac{\partial}{\partial p_j} \left(h^m{}_i \frac{\partial X}{\partial p_m} \right), \nu \right\rangle \; - \; \nabla_i \nabla_j \, \eta
\end{aligned}$$

$$= \; -\nabla_i \nabla_j \eta \; - \; \eta h^m{}_i \left\langle \frac{\partial^2 X}{\partial p_m \partial p_j}, \nu \right\rangle$$

$$= \; -\nabla_i \nabla_j \eta \; + \; \eta h^m{}_i h_{mj} \, .$$

Here we have utilised the Gauss-Weingarten relations and the definition of the covariant derivative. ◇

Lemma 5.10 then evinces the following corollary.

COROLLARY 5.11. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by* (5.15). *Then the components of the Weingarten map, the mean curvature, and the square of the norm of the second fundamental form on* M_t *satisfy the respective evolution equations:*

$$\frac{\partial}{\partial t} \, h^i{}_j(p,t) \;=\; -g^{ik}(p,t)\nabla_k \nabla_j{}^{M_t}\eta(p,t) \; - \; \eta(p,t) h^i{}_m(p,t) h^m{}_j(p,t) \, ;$$

$$\frac{\partial}{\partial t} \, H(p,t) \;=\; -\Delta^{M_t}\eta(p,t) \; - \; \|A(p,t)\|^2 \eta(p,t) \, ;$$

and

$$\frac{\partial}{\partial t} \, \|A(p,t)\|^2 \;=\; -2\left(h^{ij}(p,t)\nabla_i \nabla_j{}^{M_t}\eta(p,t) \; + \; \eta(p,t) h^i{}_k(p,t) h^j{}_i(p,t) h^k{}_j(p,t)\right).$$

PROOF

We first employ Corollary 5.7 and Lemma 5.10 to calculate that

$$\frac{\partial}{\partial t} \, h^i{}_j \;=\; \frac{\partial}{\partial t} \left(g^{ik} h_{kj}\right)$$

$$= \; g^{ik} \frac{\partial}{\partial t} \left(h_{kj}\right) \; + \; h_{kj} \frac{\partial}{\partial t} \left(g^{ik}\right)$$

$$= \; g^{ik} \left(\eta h^m{}_k h_{mj} \; - \; \nabla_k \nabla_j \eta\right) \; - \; 2\eta h^{ik} h_{kj}$$

$$= \; -g^{ik} \nabla_k \nabla_j \eta \; - \; \eta h^i{}_m h^m{}_j \, .$$

This result implies the desired evolution equation for H, and we further compute that

$$\frac{\partial}{\partial t} \, \|A\|^2 \;=\; \frac{\partial}{\partial t} \left(h^i{}_k h^k{}_i\right)$$

$$= \; h^i{}_k \frac{\partial}{\partial t} \left(h^k{}_i\right) \; + \; h^k{}_i \frac{\partial}{\partial t} \left(h^i{}_k\right)$$

$$= \; -h^i{}_k\left(g^{kj}\nabla_j\nabla_i\eta + \eta h^k{}_j h^j{}_i\right) - h^k{}_i\left(g^{ij}\nabla_j\nabla_k\eta + \eta h^i{}_j h^j{}_k\right)$$

$$= \; -2\left(h^{ij}\nabla_i\nabla_j\eta + \eta h^i{}_k h_i{}^j h^k{}_j\right),$$

from which we obtain our final result. ◇

We now employ (5.15) and present the preceding results in the particular context of the rotating drop flow (5.1).

PROPOSITION 5.12. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of manifolds evolving by the rotating drop flow* (5.1). *Then the geometric evolution equations on* M_t *may be characterised in the following manner:*

$$\frac{\partial}{\partial t}\, g_{ij}(p,t) \;=\; 2\left(h^k(t)\zeta_k(p,t) - H(p,t) - f_\Omega(p,t)\right) h_{ij}(p,t)\,; \tag{5.16}$$

$$\frac{\partial}{\partial t}\, g^{ij}(p,t) \;=\; -2\left(h^k(t)\zeta_k(p,t) - H(p,t) - f_\Omega(p,t)\right) h^{ij}(p,t)\,; \tag{5.17}$$

$$\frac{\partial}{\partial t}\sqrt{g(p,t)} \;=\; \left(h^k(t)\zeta_k(p,t) - H(p,t) - f_\Omega(p,t)\right) H(p,t)\sqrt{g(p,t)}\,; \tag{5.18}$$

$$\frac{\partial}{\partial t}\,\nu(p,t) \;=\; \Delta^{M_t}\nu(p,t) + \|A(p,t)\|^2\,\nu(p,t) - h^k(t)\nabla^{M_t}\zeta_k(p,t) + \nabla^{M_t} f_\Omega(p,t)\,; \tag{5.19}$$

$$\begin{aligned}\frac{\partial}{\partial t}\, f_\Omega(p,t) \;=\; & \Delta^{M_t} f_\Omega(p,t) + 2\Omega\Big(n - \left|\nabla^{M_t}\zeta_{n+1}(p,t)\right|^2 \\ & - \left(h^k(t)\zeta_k(p,t) - f_\Omega(p,t)\right)\langle \pi_{\mathbb{R}^n} X(p,t), \nu(p,t)\rangle\Big)\,;\end{aligned} \tag{5.20}$$

$$\begin{aligned}\frac{\partial}{\partial t}\, h_{ij}(p,t) \;=\; & \Delta^{M_t} h_{ij}(p,t) + \left(\|A(p,t)\|^2 + h^k(t)\,\langle \nu(p,t), e_k\rangle\right) h_{ij}(p,t) \\ & + \left(h^k(t)\zeta_k(p,t) - 2H(p,t) - f_\Omega(p,t)\right) h_{im}(p,t) h^m{}_j(p,t) \\ & + \nabla_i\nabla_j^{M_t} f_\Omega(p,t)\,;\end{aligned} \tag{5.21}$$

$$\frac{\partial}{\partial t} h^i{}_j(p,t) = \Delta^{M_t} h^i{}_j(p,t) + \left(\|A(p,t)\|^2 + h^k(t)\,\langle \nu(p,t), e_k\rangle\right) h^i{}_j(p,t)$$

$$- \left(h^k(t)\zeta_k(p,t) - f_\Omega(p,t)\right) h^i{}_m(p,t) h^m{}_j(p,t)$$

$$+ g^{im}(p,t)\nabla_m \nabla_j^{M_t} f_\Omega(p,t)\,; \qquad (5.22)$$

$$\frac{\partial}{\partial t} H(p,t) = \Delta^{M_t} H(p,t) + h^k(t)\,\langle \nu(p,t), e_k\rangle\, H(p,t) + \Delta^{M_t} f_\Omega(p,t)$$

$$- \left(h^k(t)\zeta_k(p,t) - H(p,t) - f_\Omega(p,t)\right) \|A(p,t)\|^2\,; \qquad (5.23)$$

and

$$\frac{\partial}{\partial t} \|A(p,t)\|^2 = \Delta^{M_t} \|A(p,t)\|^2 + 2\left(\left(\|A(p,t)\|^2 + h^k(t)\,\langle \nu(p,t), e_k\rangle\right) \|A(p,t)\|^2\right.$$

$$- \left\|\nabla^{M_t} A(p,t)\right\|^2 - \left(h^k(t)\zeta_k(p,t) - f_\Omega(p,t)\right) \tilde{C}(p,t)$$

$$\left. + h^{ij}(p,t)\nabla_i \nabla_j^{M_t} f_\Omega(p,t)\right). \qquad (5.24)$$

Here $\tilde{C}(p,t) = h^i{}_m(p,t)h^j{}_i(p,t)h^m{}_j(p,t)$ *and we shall continue to adopt the convention that* $e_0 = \mathbf{0}$.

PROOF

We recall from (5.15) that $\eta = (h^k\zeta_k - H - f_\Omega)$. Therefore, (5.16) - (5.18) are a direct consequence of Lemma 5.6 and Corollary 5.7 respectively. Due to the Codazzi equations, we determine that

$$\Delta\,\nu = -\,\|A\|^2\,\nu + \nabla H\,.$$

Thus we infer (5.19) from Lemma 5.8. By recollecting the definition of $\mathscr{Z}_{M_t}$ and adhering to the convention that $e_0 = \mathbf{0}$, we ascertain that, for each $k \in \{0, ..., n+1\}$,

$$\nabla_i \nabla_j \zeta_k = -\,h_{ij}\,\langle \nu, e_k\rangle\,.$$

We may then derive (5.23) from Corollary 5.11. To confirm (5.20), we employ (5.2) and the Gauss-Weingarten relations to calculate:

$$\Delta f_\Omega = -\Omega\,\Delta\left(|X|^2 - \zeta_{n+1}^2\right)$$

$$= \; -2\Omega\, g^{ij} \left(g_{ij} \; - \; h_{ij} \left\langle X, \nu \right\rangle \; - \; \nabla_i \zeta_{n+1} \nabla_j \zeta_{n+1} \; + \; h_{ij} \zeta_{n+1} \left\langle \nu, e_{n+1} \right\rangle \right)$$

$$= \; -2\Omega \left(n \; - \; |\nabla \zeta_{n+1}|^2 \; - \; H \left\langle \pi_{\mathbb{R}^n} X, \nu \right\rangle \right).$$

We then obtain the desired result after the application of Lemma 5.9. To confirm (5.21), we invoke Lemma 5.10 whilst observing by Simons' identity (see Section 1 of Appendix A) that

$$\nabla_i \nabla_j H \; = \; \Delta h_{ij} \; + \; \|A\|^2 h_{ij} \; - \; H h_{im} h^m{}_j \, .$$

We further deduce from Simons' identity that

$$g^{im} \nabla_m \nabla_j H \; = \; \Delta h^i{}_j \; + \; \|A\|^2 h^i{}_j \; - \; H h^i{}_m h^m{}_j \, ,$$

and

$$2h^{ij} \nabla_i \nabla_j H \; = \; \Delta \|A\|^2 \; + \; 2 \left(\|A\|^4 \; - \; \|\nabla A\|^2 \; - \; H\tilde{C} \right).$$

We may then establish (5.22) and (5.24) after successive applications of Corollary 5.11 and rearrangement. ♢

4. Properties of the flow

We may now verify that the rotating drop flow legitimately preserves volume and barycentre whilst decreasing the sum of surface and kinetic energies. For the sake of brevity, we shall continue to employ (5.15) .

PROPOSITION 5.13. *Volume and barycentre are preserved under the rotating drop flow* (5.1)*. In particular, the family of evolving drops,* $\{E_t : t \in (0,T)\}$*, is a subset of the class* $\mathscr{E}$*.*

PROOF

By employing (5.15) in conjunction with Lemmas 2.1, 5.8, Corollary 5.7, and the divergence theorem, we ascertain that, for each $t \in (0,T)$,

$$(n+1)^{\frac{3}{2}} \frac{d}{dt} \, |E_t| \; = \; \frac{d}{dt} \int_{M_t} \left\langle X, \nu \right\rangle \zeta_0 \, d\mu_t$$

$$= \; \int_{M_t} \left(\eta H \left\langle X, \nu \right\rangle \; + \; \left\langle \frac{\partial X}{\partial t}, \nu \right\rangle \; + \; \left\langle X, \frac{\partial \nu}{\partial t} \right\rangle \right) \zeta_0 \, d\mu_t$$

$$= \; \int_{M_t} \left(\operatorname{div} \left(\eta X^{\perp} \right) \; + \; \eta \; - \; \left\langle X, \nabla^{M_t} \eta \right\rangle \right) \zeta_0 \, d\mu_t$$

$$= \int_{M_t} \left(\operatorname{div}(\eta X) + \eta - \langle X, \nabla^{M_t}\eta\rangle\right) \zeta_0 \, d\mu_t$$

$$= (n+1)\langle \eta, \zeta_0\rangle_{L^2(M_t)} \, .$$

Here we have recalled that ζ_0 is constant. In an analogous manner, for each $i \in \{1, ..., n+1\}$ we compute that

$$(n+2)\frac{d}{dt}\int_{E_t} \langle x, e_i\rangle \, dx$$

$$= \frac{d}{dt}\int_{M_t} \langle X, \nu\rangle \, \zeta_i \, d\mu_t$$

$$= \int_{M_t} \left(\left(\eta H \langle X, \nu\rangle + \left\langle \frac{\partial X}{\partial t}, \nu\right\rangle + \left\langle X, \frac{\partial \nu}{\partial t}\right\rangle\right) \zeta_i + \langle X, \nu\rangle \frac{\partial \zeta_i}{\partial t}\right) d\mu_t$$

$$= \int_{M_t} \left(\operatorname{div}(\eta \, \zeta_i X^{\perp}) + (\eta - \langle X, \nabla^{M_t}\eta\rangle) \, \zeta_i + \eta \, \langle X, \nu\rangle \, \langle \nu, e_i\rangle\right) d\mu_t$$

$$= \int_{M_t} \left(\operatorname{div}(\eta \, \zeta_i X) + (\eta - \langle X, \nabla^{M_t}\eta\rangle) \, \zeta_i + \eta \left\langle X, (e_i)^{\perp}\right\rangle\right) d\mu_t$$

$$= \int_{M_t} \eta \left((n+1)\zeta_i + \left\langle X, \nabla\zeta_i + (e_i)^{\perp}\right\rangle\right) d\mu_t$$

$$= (n+2)\langle \eta, \zeta_i\rangle_{L^2(M_t)} \, .$$

Here we have again observed that $\nabla\zeta_i = e_i - (e_i)^{\perp}$, whenever we adhere to the convention that $e_0 = \mathbf{0}$. By recollecting (5.3) and (5.15), we then assert that, for each $i \in \{0, ..., n+1\}$,

$$\langle \eta, \zeta_i\rangle_{L^2(M_t)} = h^k \langle \zeta_k, \zeta_i\rangle_{L^2(M_t)} - \langle (H + f_\Omega), \zeta_i\rangle_{L^2(M_t)}$$

$$= \mathcal{M}^{kj} \langle (H + f_\Omega), \zeta_j\rangle_{L^2(M_t)} \mathcal{M}_{ki} - \langle (H + f_\Omega), \zeta_i\rangle_{L^2(M_t)} = 0 \, .$$

Since $E_0 \in \mathscr{E}$, we infer our intended result. ◊

We may therefore elicit the following corollary from Lemma 5.3 and Proposition 5.13.

COROLLARY 5.14. *Suppose ρ is a solution of (5.4). Then, for each $t \in (0, T)$, $\rho(\cdot, t) \in \mathscr{R}$.*

Subject to the representation connoted by Lemma 2.1, we are now able to demonstrate that the energy of our family of evolving drops decreases in time with respect to the energy functional $\mathscr{F}_\Omega$ for the rotating drop problem given by (1.1).

PROPOSITION 5.15. *The global energy of the family* $\{E_t : t \in (0,T)\}$ *does not increase under the rotating drop flow* (5.1). *More precisely, on the time interval* $(0,T)$,

$$\frac{d}{dt}\mathscr{F}_\Omega(E_t) \leq 0 .$$

PROOF

We recall the equivalent definitions of $\mathscr{F}_\Omega$ given by (1.1) and (2.1) to discern that

$$\mathscr{F}_\Omega(E_t) \equiv |M_t| + \frac{1}{n+3}\int_{M_t} f_\Omega \langle X,\nu\rangle \, d\mu_t .$$

By (5.15), Lemmas 5.8, 5.9, Corollary 5.7, and the divergence theorem, we have for each $t \in (0,T)$,

$$\begin{aligned}
&\frac{d}{dt}\int_{M_t} f_\Omega \langle X,\nu\rangle \, d\mu_t \\
&= \int_{M_t}\left(f_\Omega\left(\eta H\langle X,\nu\rangle + \left\langle \frac{\partial X}{\partial t},\nu\right\rangle + \left\langle X,\frac{\partial \nu}{\partial t}\right\rangle\right) + \frac{\partial f_\Omega}{\partial t}\langle X,\nu\rangle\right) d\mu_t \\
&= \int_{M_t}\left(\operatorname{div}\left(\eta f_\Omega X^\perp\right) + f_\Omega(\eta - \langle X,\nabla\eta\rangle) + \langle D_x f_\Omega,\nu\rangle\langle X,\nu\rangle\right) d\mu_t \\
&= \int_{M_t}\left(\operatorname{div}(\eta f_\Omega X) + f_\Omega(\eta - \langle X,\nabla\eta\rangle) + \left\langle X,(D_x f_\Omega)^\perp\right\rangle\right) d\mu_t \\
&= \int_{M_t}\eta\left((n+1)f_\Omega + \left\langle X,(D_x f_\Omega)^\perp + \nabla f_\Omega\right\rangle\right) d\mu_t \\
&= (n+3)\int_{M_t}\eta f_\Omega d\mu_t .
\end{aligned}$$

Here we have observed that $\nabla f_\Omega = D_x f_\Omega - (D_x f_\Omega)^\perp$. We may therefore invoke (5.3), (5.15), and Corollary 5.7, to discover that

$$\frac{d}{dt}\mathscr{F}_\Omega(E_t) = \langle (H + f_\Omega), \eta\rangle_{L^2(M_t)}$$

$$= \quad h^i \left\langle (H + f_\Omega), \zeta_i \right\rangle_{L^2(M_t)} \; - \; \|H + f_\Omega\|^2_{L^2(M_t)}$$

$$= \quad \mathcal{M}^{ij} \left\langle (H + f_\Omega), \zeta_j \right\rangle_{L^2(M_t)} \left\langle (H + f_\Omega), \zeta_i \right\rangle_{L^2(M_t)} \; - \; \|H + f_\Omega\|^2_{L^2(M_t)}$$

$$= \quad \left\| \pi_{\mathscr{X}_{M_t}} (H + f_\Omega) \right\|^2_{L^2(M_t)} \; - \; \|H + f_\Omega\|^2_{L^2(M_t)} \;\; \leq \;\; 0 \, .$$

Here we interpret the projection onto $\mathscr{X}_{M_t}$ with respect to the topology of $L^2(M_t)$. ◊

5. The support function

We shall now define the support function on M_t, which transpires to be a useful quantity in the analysis of the next chapter.

DEFINITION 5.16. Suppose $\{M_t : t \in (0,T)\}$ is a family of manifolds evolving by the rotating drop flow (5.1). Then we define the *support function* $u : N \times (0,T) \to \mathbb{R}$ on M_t by $u(p,t) = \langle X(p,t), \nu(p,t) \rangle$.

REMARK 5.17. We recall the unique graph characterisation of star-shaped M_t above S whose evolution is described by (5.4) to deduce from Lemma 1.4 that

$$u(p,t) \;=\; \frac{\rho^2(s,t)}{\sqrt{\rho^2(s,t) \,+\, |\nabla^S \rho(s,t)|^2}} \, . \tag{5.25}$$

Thus the support function is strictly positive for any family of star-shaped manifolds evolving by (5.1) and may be interpreted geometrically as the orthogonal distance of the tangent plane $T_x M_t$ from the origin in $\mathbb{R}^{n+1}$.

We may now establish an evolution equation for u.

PROPOSITION 5.18. *Suppose $\{M_t : t \in (0,T)\}$ is a family of manifolds evolving by the rotating drop flow (5.1). Then the support function on M_t evolves in the following manner:*

$$\frac{\partial}{\partial t} u(p,t) \;\; = \;\; \Delta^{M_t} u(p,t) \,+\, h^0(t)\zeta_0(p,t) \,-\, 2H(p,t) \,+\, f_\Omega(p,t)$$

$$+ \, \left(\|A(p,t)\|^2 \,+\, h^i(t) \left\langle \nu(p,t), e_i \right\rangle \,+\, 2\Omega \left\langle \pi_{\mathbb{R}^n} X(p,t), \nu(p,t) \right\rangle \right) u(p,t) \, .$$

PROOF

We may utilise (5.1), (5.15), Lemma 5.8, and Definition 5.16 to calculate:

$$\frac{\partial}{\partial t} u \;\; = \;\; \frac{\partial}{\partial t} \langle X, \nu \rangle$$

$$= \left\langle \frac{\partial X}{\partial t}, \nu \right\rangle + \left\langle X, \frac{\partial \nu}{\partial t} \right\rangle$$

$$= h^i \zeta_i - H - f_\Omega + \left\langle X, \nabla f_\Omega + \nabla H - h^i \nabla \zeta_i \right\rangle$$

$$= h^i \left(\zeta_i - \langle X, \nabla \zeta_i \rangle \right) - H - f_\Omega + \left\langle X, D_x f_\Omega - (D_x f_\Omega)^\perp + \nabla H \right\rangle$$

$$= h^0 \zeta_0 + h^i \left\langle X, (e_i)^\perp \right\rangle - H + f_\Omega + 2\Omega \left\langle \pi_{\mathbb{R}^n} X, \nu \right\rangle u + \langle X, \nabla H \rangle$$

$$= h^0 \zeta_0 - H + f_\Omega + \left(h^i \langle \nu, e_i \rangle + 2\Omega \left\langle \pi_{\mathbb{R}^n} X, \nu \right\rangle \right) u + \langle X, \nabla H \rangle .$$

Furthermore,

$$\nabla_i \nabla_j u = \nabla_i \nabla_j \langle X, \nu \rangle$$

$$= \nabla_i \left\langle X, \frac{\partial \nu}{\partial p_j} \right\rangle$$

$$= \nabla_i \left(h^k{}_j \left\langle X, \frac{\partial X}{\partial p_k} \right\rangle \right)$$

$$= g^{km} \nabla_i (h_{mj}) \left\langle X, \frac{\partial X}{\partial p_k} \right\rangle + h^k{}_j \left(\frac{\partial}{\partial p_i} \left\langle X, \frac{\partial X}{\partial p_k} \right\rangle - \Gamma^m_{ik} \left\langle X, \frac{\partial X}{\partial p_m} \right\rangle \right)$$

$$= \left\langle X, g^{km} \nabla_m (h_{ij}) \frac{\partial X}{\partial p_k} \right\rangle + h^k{}_j \left(g_{ik} - h_{ik} \langle X, \nu \rangle \right)$$

$$= \langle X, \nabla h_{ij} \rangle + h_{ij} - h_{ik} h^k{}_j u .$$

Here we have employed the Gauss-Weingarten relations, the orthogonality of the normal and tangent spaces on M_t, and the Codazzi equations. Thus $\Delta u = \langle X, \nabla H \rangle + H - \|A\|^2 u$, from which we derive the intended result after substitution and rearrangement. ◇

CHAPTER 6

Interior Regularity Estimates

For a given family of star-shaped manifolds evolving by (5.1) that remain arbitrarily close to S with respect to the Lipschitz topology, we may now demonstrate *a priori* that curvature derivatives of all orders are bounded interior in time. In Chapter 7, we shall verify that there actually exists a short time interval for which this assumption of proximity to the sphere may be justified. Throughout the chapter, all geometric quantities will continue to be evaluated with respect to the metric on M_t, unless otherwise noted.

1. Estimates on $h^i(t)$

We commence by establishing estimates on the global terms $h^i(t)$ introduced in Section 1 of Chapter 5. Although $C^{0,1} \equiv W^{1,\infty}$, the ensuing technical lemma will demonstrate that the Lipschitz seminorm of any scalar-valued function on S of class $W^{1,\infty}$ coincides with the supremum norm of its gradient (interpreted in the weak sense).

LEMMA 6.1. *Suppose $\eta \in W^{1,\infty}(S;\mathbb{R})$. Then*

$$\sup_S \left|\nabla^S \eta\right| \;=\; [\eta]_{C^{0,1}(S)}\,.$$

PROOF

For $\eta \in W^{1,\infty}(S)$, we recall that the Lipschitz seminorm is given by

$$[\eta]_{C^{0,1}(S)} \;=\; \sup_{s_1 \neq s_2 \in S} \frac{|\eta(s_1) \,-\, \eta(s_2)|}{d_S\,(s_1, s_2)}\,. \tag{6.1}$$

Clearly $|\nabla\eta|$ attains its supremum at some point $s_0 \in S$. Moreover, we may interpret $|\nabla\eta(s_0)|$ in the following manner:

$$|\nabla\eta(s_0)| \;=\; \sup_{\substack{\mathbf{v}\in T_{s_0}S \\ |\mathbf{v}|=1}} \langle \nabla\eta(s_0), \mathbf{v}\rangle\,.$$

Suppose now that $\mathscr{I}$ is an (uncountable) indexing set where $\gamma_i : \mathscr{I} \times [0,1] \to S$ is the family of all smooth, unit speed geodesics on S with $\gamma_i(0) = s_0$. Then there exists $m \in \mathscr{I}$ with

$$|\nabla\eta(s_0)| \;=\; \langle \nabla\left(\eta\circ\gamma_m(0)\right), \gamma_m'(0)\rangle \;=\; \left(\eta\circ\gamma_m\right)'(0)\,.$$

We then deduce from (6.1) that

$$[\eta]_{C^{0,1}(S)} \;\geq\; \lim_{h\searrow 0}\left(\frac{\eta\circ\gamma_m(h) \,-\, \eta\circ\gamma_m(0)}{h}\right) \;=\; \left(\eta\circ\gamma_m\right)'(0) \;=\; \sup_S |\nabla\eta|\,.$$

Conversely, if s_1 and s_2 are the two points which determine (6.1), then, for some $\delta \in (0, \pi)$, there certainly exists a unit speed, length-minimising geodesic $\gamma : [0, \delta] \longrightarrow S$ with $\gamma(0) = s_1$ and $\gamma(\delta) = s_2$. We may then invoke the mean value theorem to ascertain that there exists $\beta \in (0, \delta)$ such that

$$[\eta]_{C^{0,1}(S)} \;=\; \left|(\eta \circ \gamma)'(\beta)\right| \;=\; \left|\langle \nabla(\eta \circ \gamma(\beta)), \gamma'(\beta)\rangle\right| \;\leq\; \sup_S |\nabla \eta| .$$

Thus we obtain our desired result. $\diamondsuit$

COROLLARY 6.2. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of star-shaped manifolds evolving by the rotating drop flow* (5.1), *and there exists an* $\varepsilon \in (0, \frac{1}{2})$ *such that* $\||X| - 1\|_{C^{0,1}(S)} \leq \varepsilon$ *on* $(0,T)$. *Then, for each* $t \in (0,T)$,

$$\sup_{M_t} \left|\nabla^{M_t} |X|\right| \;<\; 2\varepsilon , \tag{6.2}$$

and, for each $(p,t) \in N \times (0,T)$,

$$(1 - 2\varepsilon) \;<\; u(p,t) \;\leq\; (1 + \varepsilon) . \tag{6.3}$$

PROOF

Without loss of generality, we may employ the geometric construction introduced in Chapter 1 and prove the result with respect to the time-dependent parametrising function, $\rho = |X|$, introduced in the statement of Lemma 5.3. By hypothesis, we have

$$\|\rho \,-\, 1\|_{C^0(S)} \,+\, [\rho]_{C^{0,1}(S)} \;\leq\; \varepsilon .$$

Therefore, by Lemma 6.1,

$$\sup_S \left|\nabla^S \rho\right|^2 \;\leq\; \varepsilon^2 .$$

We may then employ Lemma 1.4 and our hypothesis to prove (6.2):

$$\sup_{M_t} \left|\nabla^{M_t} \rho\right|^2 \;=\; \sup_S \left(\frac{\left|\nabla^S \rho\right|^2}{\rho^2 \,+\, |\nabla^S \rho|^2}\right) \;\leq\; \frac{\varepsilon^2}{(1 - \varepsilon)^2} \;<\; 4\,\varepsilon^2 ,$$

whenever $\varepsilon < \frac{1}{2}$. Finally, we utilise (5.25) and (6.2) to prove (6.3):

$$(1 + \varepsilon) \;\geq\; u \;\geq\; \frac{(1 - \varepsilon)^2}{\sqrt{(1 + \varepsilon)^2 \,+\, \varepsilon^2}} \;>\; (1 - 2\varepsilon) ,$$

whenever $\varepsilon < \frac{1}{2}$. $\diamondsuit$

Since the $L^2(M_t)$ inner products which define the matrix $(\mathcal{M}_{ij}(t))$ are independent of parametrisation, the proof of the following lemma proceeds unmodified from that of Lemma 4.6.

LEMMA 6.3. *Suppose $\{M_t : t \in (0,T)\}$ is a family of star-shaped manifolds evolving by the rotating drop flow* (5.1), *and $\varepsilon > 0$ is such that $\| |X| - 1 \|_{C^{0,1}(S)} \le \varepsilon$ on $(0,T)$. Then, for each $t \in (0,T)$, the components of the matrix $(\mathcal{M}^{ij}(t))$ satisfy estimates of the form:*

$$\left|\mathcal{M}^{ij}(t) \;-\; M^{ij}(1)\right| \;\le\; \varepsilon\, C^{ij}(n)\, |S|^{-1} \;.$$

In analogy with the results of Proposition 4.7, we may now derive bounds on the terms $h^i(t)$ which occur in the evolution equation (5.1).

PROPOSITION 6.4. *Suppose $\{M_t : t \in (0,T)\}$ is a family of star-shaped manifolds evolving by the rotating drop flow* (5.1), *and $\varepsilon > 0$ is such that $\| |X| - 1 \|_{C^{0,1}(S)} \le \varepsilon$ on $(0,T)$. Then, for each $t \in (0,T)$, the functions $h^i(t)$ given by* (5.3) *satisfy the following estimates:*

$$\left| h^0(t) \;-\; n\sqrt{n+1}\left(1 \;-\; \frac{\Omega}{(n+1)}\right)\right| \;\le\; \varepsilon\, C^0(n)\left(1 \;+\; |\Omega| \;+\; \sup_{M_t} \|A\|\right);$$

and, for each $i \in \mathbb{N}$ with $1 \le i \le n+1$,

$$\left| h^i(t) \right| \;\le\; \varepsilon\, C^i(n)\left(1 \;+\; |\Omega| \;+\; \sup_{M_t} \|A\|\right).$$

PROOF

Once again, we note that the quantities $h^i(t)$ are independent of our choice of parametrisation and, without loss of generality, we may employ the specific representation $X(s,t) = \rho(s,t)s$ given by Definition 1.3. But for terms involving second-order covariant derivatives of ρ, we are therefore able to perform analysis completely analogous to that exposed in the proof of Proposition 4.7. We first verify that

$$\begin{aligned}
\sqrt{n+1}\,\langle H, \zeta_0\rangle_{L^2(M_t)} \;&=\; \int_S \rho^{n-3}\Big[n\rho^2 \;+\; (n-2)|\nabla^S\rho|^2\left(1 \;-\; \ln\left(\sqrt{\rho^2 \;+\; |\nabla^S\rho|^2}\right)\right) \\
&\qquad\qquad -\; \ln\left(\sqrt{\rho^2 \;+\; |\nabla^S\rho|^2}\right)\rho\Delta^S\rho\Big]\, d\sigma \\
&\le\; \left(n \;+\; \varepsilon\, K(n)\left(1 + \sup_S \left|\Delta^S\rho\right|\right)\right)|S| \\
&\le\; \left(n \;+\; \varepsilon\, K_1(n)\left(1 + \sup_{M_t} \left\|(\nabla_i\nabla_j^{M_t}\rho)\right\|\right)\right)|S| \;.
\end{aligned}$$

Conversely,

$$\sqrt{n+1}\,\langle H, \zeta_0\rangle_{L^2(M_t)} \;\ge\; \left(n \;-\; \varepsilon\, K_2(n)\left(1 + \sup_{M_t} \left\|(\nabla_i\nabla_j^{M_t}\rho)\right\|\right)\right)|S| \;.$$

In both cases, we have integrated by parts, and invoked the relationship between the geometries of S and M_t connoted by Lemma 1.4. In particular, we have recollected the calculation of (4.16) contained in the proof of Theorem 4.4 to deduce that

$$\left\| \left(\nabla_i \nabla_j^{M_t} \rho \right) \right\|^2 \;\geq\; (1 - \varepsilon K(n)) \left\| \left(\nabla_i \nabla_j^{S} \rho \right) \right\|^2 .$$

We further note that these estimates remain valid in the particular case $n = 2$. Hence, if we take $B_0(n)$ to be the maximum of $\{K_1, K_2\}$,

$$\left| \langle H, \zeta_0 \rangle_{L^2(M_t)} - \frac{n}{\sqrt{n+1}} |S| \right| \;\leq\; \varepsilon B_0 \left(1 + \sup_{M_t} \left\| \left(\nabla_i \nabla_j^{M_t} \rho \right) \right\| \right) \frac{|S|}{\sqrt{n+1}} . \tag{6.4}$$

Moreover, we may cite analysis contained in the proof of Proposition 4.7 to compute for each $j \in \{1, ..., n+1\}$:

$$\begin{aligned} &\left| \langle H, \zeta_j \rangle_{L^2(M_t)} \right| \\ &= \left| \int_S n \left(\rho^n - 1 \right) \zeta_j(1) d\sigma - \int_S \ln \left(\sqrt{\rho^2 + |\nabla^S \rho|^2} \right) \rho^{n-1} \zeta_j(1) \Delta^S \rho \, d\sigma \right. \\ &\quad \left. + \int_S \rho^{n-2} \left((n-1) |\nabla^S \rho|^2 \zeta_j(1) + \rho \langle \nabla^S \rho, e_j \rangle \right) \left(1 - \ln \left(\sqrt{\rho^2 + |\nabla^S \rho|^2} \right) \right) d\sigma \right| \\ &\leq \varepsilon B_j(n) \left(1 + \sup_{M_t} \left\| \left(\nabla_i \nabla_j^{M_t} \rho \right) \right\| \right) |S| . \end{aligned} \tag{6.5}$$

Furthermore, we may derive the estimates on the inner products $\langle f_\Omega, \zeta_j \rangle_{L^2(M_t)}$ directly from the proof of Proposition 4.7:

$$\left| \langle f_\Omega, \zeta_0 \rangle_{L^2(M_t)} + \frac{n \Omega}{(n+1)^{\frac{3}{2}}} |S| \right| \;\leq\; \varepsilon |\Omega| D_0(n) \frac{|S|}{\sqrt{n+1}} ; \tag{6.6}$$

and, for each $j \in \{1, ..., n+1\}$,

$$\left| \langle f_\Omega, \zeta_j \rangle_{L^2(M_t)} \right| \;\leq\; \varepsilon |\Omega| D_j(n) |S| . \tag{6.7}$$

Now, by (1.7), Corollary 1.5, and Definition 5.16 we observe that

$$\begin{aligned}
&\| \left(\nabla_i \nabla_j^{M_t} \rho\right) \|^2 \\
&= \frac{1}{\rho^2}\left(n + u^2\|A\|^2 + |\nabla\rho|^4 - 2uH - 2|\nabla\rho|^2 + 2A[\nabla\rho, \nabla\rho]\right) \\
&\leq \frac{1}{\rho^2}\left(n + u^2\|A\|^2 + |\nabla\rho|^2\left(|\nabla\rho|^2 - 1\right) + 2u\sqrt{n}\|A\| + 2C(n)\|A\|\,|\nabla\rho|^2\right) \\
&\leq K_3(n)\left(1 + \varepsilon + \|A\|^2\right). \qquad (6.8)
\end{aligned}$$

Here we have employed Cauchy's inequality and Corollary 6.2. In the particular case $i = 0$, we shall suspend our use of the summation convention whilst utilising (6.4)-(6.8), Lemma 6.3, the characterisation of the matrix $(M^{ij}(1))$ given by (2.3), and the triangle inequality to deduce that,

$$\begin{aligned}
h^0 &= \sum_{j=0}^{n+1} \mathcal{M}^{0j}(t)\,\langle (H + f_\Omega), \zeta_j\rangle_{L^2(M_t)} \\
&\leq \left((n+1) + \varepsilon\, \mathcal{C}^{00}\right)\frac{1}{\sqrt{n+1}}\left(n\left(1 - \frac{\Omega}{(n+1)}\right)\right. \\
&\quad \left. + \varepsilon\, B_0\left(1 + \sqrt{K_3}\left(1 + \varepsilon^{\frac{1}{2}} + \sup_{M_t}\|A\|\right)\right) + \varepsilon\,|\Omega|\,D_0\right) \\
&\quad + \varepsilon^2 \sum_{j=1}^{n+1} \mathcal{C}^{0j}\left(B_j\left(1 + \sqrt{K_3}\left(1 + \varepsilon^{\frac{1}{2}} + \sup_{M_t}\|A\|\right)\right) + |\Omega|\,D_j\right) \\
&\leq n\sqrt{n+1}\left(1 - \frac{\Omega}{(n+1)}\right) + \varepsilon\, K^0(n)\left(1 + |\Omega| + \sup_{M_t}\|A\|\right).
\end{aligned}$$

Conversely,

$$h^0 \geq n\sqrt{n+1}\left(1 - \frac{\Omega}{(n+1)}\right) - \varepsilon\, B^0(n)\left(1 + |\Omega| + \sup_{M_t}\|A\|\right).$$

This implies our desired estimate on h^0 if we take $\mathcal{C}^0(n)$ to be the maximum of $\{K^0, B^0\}$. The cases corresponding to $i \in \{1, ..., n+1\}$ then proceed from similar calculations to those expounded in the proof of Proposition 4.7, after the application of (6.4)-(6.8) and Lemma 6.3. ♢

2. An estimate on $\|A\|^2$

Pursuant to the result of Proposition 6.4, we may now establish an *a priori* bound on the square of the norm of the second fundamental form.

THEOREM 6.5. *Suppose* $\{M_t : t \in (0,T)\}$ *is a family of star-shaped manifolds evolving by the rotating drop flow* (5.1), *and* $\varepsilon > 0$ *is such that* $\||X| - 1\|_{C^{0,1}(S)} \leq \varepsilon$ *on* $(0,T)$. *Then, for each* $t \in (0,T)$, *we may estimate the square of the norm of the second fundamental form in the following manner:*

$$\sup_{M_t} \|A\|^2 \;\leq\; \mathcal{C}_0(n) \max\left\{1,\, \frac{\varepsilon}{t}\right\}$$

whenever $|\Omega| \leq \frac{n}{16}$ *and* $\varepsilon < \min\left\{\frac{1}{3},\, \left(\frac{4}{\mathcal{C}_0}\right)^{\frac{1}{2}}\right\}$.

PROOF

Suppose c is a positive constant strictly less than the infimum of u on $N \times (0,T)$ (this is certainly possible by Corollary 6.2 and hypothesis), and

$$\mathcal{Q} \;=\; \frac{\varepsilon^2 \|A\|^2}{(u-c)^2}\,.$$

By employing Propositions 5.12, 5.18, and Lemma A.1 in Appendix A we may compute the evolution equation for $\mathcal{Q}$:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{Q} \;&=\; \Delta\mathcal{Q} + \frac{2\varepsilon^2}{(u-c)^2}\Big(2\Omega\left(h^{ij}\nabla_i\zeta_{n+1}\nabla_j\zeta_{n+1} \,-\, H\right) \,-\, \|\nabla A\|^2 \,-\, (h^k\zeta_k \,-\, f_\Omega)\tilde{C}\Big) \\
&\quad -\, \frac{2\mathcal{Q}}{(u-c)}\left(h^0\zeta_0 \,-\, 2H \,+\, f_\Omega + c\left(h^k\langle\nu, e_k\rangle \,+\, 2\Omega\langle\pi_{\mathbb{R}^n}X, \nu\rangle\right)\right) \\
&\quad -\, \frac{2c(u-c)}{\varepsilon^2}\mathcal{Q}^2 + \frac{2\mathcal{Q}}{(u-c)^2}|\nabla u|^2 + \frac{2}{(u-c)^2}\left\langle\nabla\mathcal{Q}, \nabla(u-c)^2\right\rangle.
\end{aligned} \tag{6.9}$$

Here we recall that $\tilde{C} \;=\; h^i{}_k h^j{}_i h^k{}_j$ is the quantity introduced in Proposition 5.12. In analogy with the proof of Lemma 1.6, we have observed that

$$\begin{aligned}
h^{ij}\nabla_i\nabla_j f_\Omega \;&=\; 2\Omega h^{ij}\left(h_{ij}\langle\pi_{\mathbb{R}^n}X, \nu\rangle \,-\, g_{ij} \,+\, \nabla_i\zeta_{n+1}\nabla_j\zeta_{n+1}\right) \\
&=\; 2\Omega\left(\langle\pi_{\mathbb{R}^n}X, \nu\rangle\;\|A\|^2 \,-\, H \,+\, h^{ij}\nabla_i\zeta_{n+1}\nabla_j\zeta_{n+1}\right).
\end{aligned}$$

We shall now estimate various of the terms that arise in the evolution equation for $\mathscr{Q}$:

$$\begin{aligned}
\left|h^{ij}\nabla_i\zeta_{n+1}\nabla_j\zeta_{n+1} - H\right| &= \left|g^{ik}g^{jl}\left(\nabla_k\zeta_{n+1}\nabla_l\zeta_{n+1} - g_{kl}\right)h_{ij}\right| \\
&\leq K(n)\,\|(\nabla_i\zeta_{n+1}\nabla_j\zeta_{n+1} - g_{ij})\|\,\|A\| \\
&= K\left(n - 2\left|\nabla\zeta_{n+1}\right|^2 + \left|\nabla\zeta_{n+1}\right|^4\right)^{\frac{1}{2}}\|A\| \\
&= K\left(n - 2\left|e_{n+1} - (e_{n+1})^{\perp}\right|^2 + \left|e_{n+1} - (e_{n+1})^{\perp}\right|^4\right)^{\frac{1}{2}}\|A\| \\
&= K\left(n - 1 + \langle\nu, e_{n+1}\rangle^4\right)^{\frac{1}{2}}\|A\| \\
&\leq K_1(n)\|A\|\,. \qquad (6.10)
\end{aligned}$$

Moreover,

$$|H| \;\leq\; \sqrt{n}\,\|A\|\,. \qquad (6.11)$$

Next, we employ Proposition 6.4 (with associated constants $\mathcal{C}^k$) in conjunction with the definition of $\tilde{C}$ to determine that

$$\begin{aligned}
\left|(h^k\zeta_k - f_\Omega)\tilde{C}\right| &\leq \left(|h^k\zeta_k| + |f_\Omega|\right)\left|\tilde{C}\right| \\
&\leq \Bigg(n + |\Omega| + \varepsilon\,\mathcal{C}^0\left(1 + |\Omega| + \sup_{M_t}\|A\|\right) \\
&\qquad + \varepsilon\sum_{k=1}^{n+1}\mathcal{C}^k\left(1 + |\Omega| + \sup_{M_t}\|A\|\right)(1+\varepsilon) + |\Omega|\,(1+\varepsilon)^2\Bigg)K(n)\|A\|^3 \\
&\leq K_2(n)\left(1 + |\Omega| + \varepsilon\left(1 + |\Omega| + \sup_{M_t}\|A\|\right)\right)\|A\|^3\,. \qquad (6.12)
\end{aligned}$$

By a further application of Proposition 6.4, we discern that

$$\left|h^k\langle\nu, e_k\rangle\right| \;\leq\; \varepsilon\sum_{k=1}^{n+1}\mathcal{C}^k\left(1 + |\Omega| + \sup_{M_t}\|A\|\right) \;\leq\; \varepsilon\,K_3(n)\left(1 + |\Omega| + \sup_{M_t}\|A\|\right). \qquad (6.13)$$

Finally, we utilise the Gauss-Weingarten relations, Definition 5.16, and Corollary 6.2 to ascertain that

$$
\begin{aligned}
|\nabla u|^2 &= g^{ij}\nabla_i \langle X,\nu\rangle \nabla_j \langle X,\nu\rangle \\
&= g^{ij}h^m{}_i h^l{}_j \left\langle X, \frac{\partial X}{\partial s_m}\right\rangle \left\langle X, \frac{\partial X}{\partial s_l}\right\rangle \\
&= |X|^2 g^{ij}h^m{}_i h^l{}_j \nabla_m|X|\nabla_l|X| \\
&\leq K(n)|X|^2\, |\nabla|X||^2\, \|A\|^2 \\
&\leq \varepsilon^2 K_4(n)\|A\|^2 \,. \qquad (6.14)
\end{aligned}
$$

We now analyse the quantity $(u-c)$. If we choose $c = (1-3\varepsilon) > 0$, then we may deduce from Corollary 6.2 that

$$\varepsilon \leq (u-c) \leq 4\varepsilon \,. \qquad (6.15)$$

We then substitute (6.10) - (6.15) into (6.9), and further invoke Proposition 6.4, to describe the evolution of $\mathcal{Q}$ in the form of the inequality:

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{Q} \\
&\leq 4K_1\,|\Omega|\,\mathcal{Q}^{\frac{1}{2}} + \frac{2}{\varepsilon}\left(\varepsilon\left(C^0 + K_3\right)\left(1 + \left(\sup_{M_t}\mathcal{Q}\right)^{\frac{1}{2}}\right) - \frac{n}{4} + |\Omega|\left(4 + \varepsilon K_3\right)\right)\mathcal{Q} \\
&\quad + \frac{4}{\varepsilon}\left(\sqrt{n} + 2\varepsilon K_2\left(1 + |\Omega|\right)(1+\varepsilon) + 8\varepsilon^2 K_2\left(\sup_{M_t}\mathcal{Q}\right)^{\frac{1}{2}}\right)\mathcal{Q}^{\frac{3}{2}} \\
&\quad - \frac{2}{\varepsilon}\left(1 - \varepsilon\left(K_4+3\right)\right)\mathcal{Q}^2 + \frac{2}{(u-c)^2}\left\langle\nabla\mathcal{Q}, \nabla(u-c)^2\right\rangle . \qquad (6.16)
\end{aligned}
$$

We note that since we are considering the evolution of a family of manifolds without boundary, the supremum of the quantity $\mathcal{Q}$ coincides with an interior spatial maximum on M_t for each $t \in (0,T)$. Therefore, for solutions $\mathcal{Q}$ of (6.9) satisfying (6.16), we may again apply [**45**, Lemma 3.5] to obtain the following ODE inequality describing the evolution of

$\mathscr{Q}_{\max}$ for each $t \in (0,T)$:

$$\begin{aligned}\frac{d}{dt}\,\mathscr{Q}_{\max} \;\le\;& 4K_1\,|\Omega|\,\mathscr{Q}_{\max}^{\frac{1}{2}} + \frac{8}{\varepsilon}\left(\frac{\varepsilon\,(\mathcal{C}^0 + K_3\,(1+|\Omega|))}{4} + |\Omega| - \frac{n}{16}\right)\mathscr{Q}_{\max}\\ &+ \frac{2}{\varepsilon}\left(2\sqrt{n} + \varepsilon\left(\mathcal{C}^0 + K_3\right) + 4\varepsilon\,K_2(1+|\Omega|)\,(1+\varepsilon)\right)\mathscr{Q}_{\max}^{\frac{3}{2}}\\ &- \frac{2}{\varepsilon}\,(1 - \varepsilon\,(K_4 + 3 + 16\varepsilon\,K_2))\,\mathscr{Q}_{\max}^2\\ \le\;& \frac{K_5(n)}{\varepsilon}\left(\varepsilon\,\mathscr{Q}_{\max}^{\frac{1}{2}} + \varepsilon\,\mathscr{Q}_{\max} + \mathscr{Q}_{\max}^{\frac{3}{2}} + \varepsilon\,\mathscr{Q}_{\max}^2\right) - \frac{2}{\varepsilon}\,\mathscr{Q}_{\max}^2\end{aligned}$$

whenever $|\Omega| \le \frac{n}{16}$ and $\varepsilon < \frac{1}{3}$. Suppose, by way of contradiction, there exists a $t_1 \in (0,T)$ such that $\mathscr{Q}_{\max}(t_1) \ge \left(\frac{2K_5}{1-2\varepsilon K_5}\right)^2$. Then

$$\begin{aligned}\frac{d}{dt}\,\mathscr{Q}_{\max}(t_1) \;\le\;& \frac{1}{\varepsilon}\left(\frac{\varepsilon}{8{K_5}^2}\left((1-2\varepsilon K_5)^3 + 2K_5\,(1-2\varepsilon K_5)^2\right) - \frac{3}{2}\right)\mathscr{Q}_{\max}^2(t_1)\\ <\;& -\,\frac{\mathscr{Q}_{\max}^2(t_1)}{\varepsilon}\end{aligned}$$

whenever $\varepsilon < \min\left\{\frac{1}{3}, \frac{1}{2K_5}\right\}$ with $K_5 \ge \frac{1}{2}$, and $\mathscr{Q}_{\max}$ is decreasing at t_1. In particular, by pursuing reasoning similar to that expounded in the proof of [**45**, Lemma 3.1],

$$\lim_{k\searrow 0}\frac{\mathscr{Q}_{\max}(t_1) - \mathscr{Q}_{\max}(t_1-k)}{k} \le 0$$

and there exists $\delta > 0$ such that $\mathscr{Q}_{\max} \ge \left(\frac{2K_5}{1-2\varepsilon K_5}\right)^2$ on $(t_1-\delta, t_1]$. Let $(a, t_1]$ be the largest such interval with $a \ge 0$. Then, by continuity, $\mathscr{Q}_{\max} \ge \left(\frac{2K_5}{1-2\varepsilon K_5}\right)^2$ on $[a, t_1]$. Moreover, if $a > 0$, then we may again find $\delta > 0$ such that $\mathscr{Q}_{\max} \ge \left(\frac{2K_5}{1-2\varepsilon K_5}\right)^2$ on $(a-\delta, t_1]$ since

$$\frac{d}{dt}\,\mathscr{Q}_{\max}(a) < -\,\frac{\mathscr{Q}_{\max}^2(a)}{\varepsilon} \quad\Longrightarrow\quad \lim_{k\searrow 0}\frac{\mathscr{Q}_{\max}(a) - \mathscr{Q}_{\max}(a-k)}{k} \le 0$$

whenever $\varepsilon < \min\left\{\frac{1}{3}, \frac{1}{2K_5}\right\}$. Therefore, we deduce that $a = 0$ and $\mathscr{Q}_{\max} \ge \left(\frac{2K_5}{1-2\varepsilon K_5}\right)^2$ on $[0, t_1]$. Hence

$$\frac{d}{dt}\,\mathscr{Q}_{\max} < -\,\frac{\mathscr{Q}_{\max}^2}{\varepsilon} \;\Longrightarrow\; \left[-\,\frac{1}{\mathscr{Q}_{\max}(w)}\right]_{w=0}^{w=t} < -\,\frac{t}{\varepsilon} \;\Longrightarrow\; \mathscr{Q}_{\max}(t) < \frac{\varepsilon}{t}$$

on $[0, t_1]$ and we reach a contradiction whenever $t_1 > \varepsilon \left(\frac{1-2\varepsilon K_5}{2K_5}\right)^2$. Thus, for each $t \in (0,T)$,

$$\sup_{M_t} \mathscr{Q} \;\leq\; \max\left\{ \left(\frac{2K_5}{1-2\varepsilon K_5}\right)^2, \frac{\varepsilon}{t} \right\},$$

from which we infer that

$$\sup_{M_t} \|A\|^2 \;\leq\; \mathcal{C}_0(n) \max\left\{1,\, \frac{\varepsilon}{t}\right\}$$

whenever $|\Omega| \leq \frac{n}{16}$ and $\varepsilon < \min\left\{\frac{1}{3}, \left(\frac{4}{\mathcal{C}_0}\right)^{\frac{1}{2}}\right\}$. Here we have employed the definition of $\mathscr{Q}$ and (6.15) to derive a suitable $\mathcal{C}_0$, given by $\mathcal{C}_0 = 144{K_5}^2$. $\diamondsuit$

The ensuing corollary distinguishes an equivalent estimate on the norm of the Hessian of the time-dependent parametrising function ρ introduced in the statement of Lemma 5.3.

COROLLARY 6.6. *Suppose ρ is a solution of* (5.4) *and $\varepsilon > 0$ is such that $\|\rho - 1\|_{C^{0,1}(S)} \leq \varepsilon$ on $(0,T)$. Then, for each $t \in (0,T)$, we may estimate the square of the norm of $(\nabla_i \nabla_j^S \rho)$ in the following manner:*

$$\sup_{S} \|\, (\nabla_i \nabla_j^S \rho)\, \|^2 \;\leq\; \mathcal{D}_0(n) \max\left\{1,\, \frac{\varepsilon}{t}\right\}$$

subject to the constraints on Ω and ε stipulated by Theorem 6.5.

PROOF

As in the proof of Proposition 6.4, we have $\|\, (\nabla_i \nabla_j \rho)\, \|^2 \;\leq\; K_1(n)\,(1 + \varepsilon + \|A\|^2)$ (with respect to the metric on M_t induced from S). Thus we may utilise Theorem 6.5, with its attendant restrictions on Ω and ε, to discern that for each $t \in (0,T)$,

$$\sup_{M_t} \|\, (\nabla_i \nabla_j \rho)\, \|^2 \;\leq\; K_1 \left(1 + \varepsilon + \mathcal{C}_0 \max\left\{1,\, \frac{\varepsilon}{t}\right\}\right).$$

In an identical calculation to that performed in the proof of Proposition 6.4, we once again recall the determination of (4.16) contained in the proof of Theorem 4.4 to ascertain from hypothesis that $\|(\nabla_i \nabla_j \rho)\|^2 \geq (1 - \varepsilon\, K_2(n))\, \left\|(\nabla_i \nabla_j^S \rho)\right\|^2$. Therefore

$$\sup_{S} \left\|(\nabla_i \nabla_j^S \rho)\right\|^2 \;\leq\; \mathcal{D}_0(n) \max\left\{1,\, \frac{\varepsilon}{t}\right\},$$

as required. $\diamondsuit$

3. The evolution equation for $\nabla^m A$

Given a family of manifolds $\{M_t : t \in (0,T)\}$ evolving by (5.1), we shall now employ an inductive argument to procure an evolution equation for curvature derivatives of arbitrary order. We commence by calculating the time derivative of the Riemannian connection on M_t.

LEMMA 6.7. *Suppose $\{M_t : t \in (0,T)\}$ is a family of manifolds evolving by the rotating drop flow* (5.1). *Then the Riemannian connection on M_t satisfies the evolution equation:*

$$\frac{\partial}{\partial t}\mathbf{\Gamma}(p,t) = h^i(t)\zeta_i(p,t)\nabla^{M_t}A(p,t) + h^i(t)\nabla^{M_t}\zeta_i(p,t)\star A(p,t) + A(p,t)\star\nabla^{M_t}A(p,t)$$

$$+ f_\Omega(p,t)\nabla^{M_t}A(p,t) + A(p,t)\star\nabla^{M_t}f_\Omega(p,t)\,.$$

PROOF

For notational convenience we shall reintroduce (5.15). We may then invoke Corollary 5.7 to compute:

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^k_{ij} &= \frac{\partial}{\partial t}\left(g^{km}\left\langle\frac{\partial^2 X}{\partial p_i\partial p_j},\frac{\partial X}{\partial p_m}\right\rangle\right)\\ &= \frac{\partial}{\partial t}\left(g^{km}\right)\Gamma_{imj} + g^{km}\left(\left\langle\frac{\partial^2}{\partial p_i\partial p_j}\left(\frac{\partial X}{\partial t}\right),\frac{\partial X}{\partial p_m}\right\rangle + \left\langle\frac{\partial^2 X}{\partial p_i\partial p_j},\frac{\partial}{\partial p_m}\left(\frac{\partial X}{\partial t}\right)\right\rangle\right)\\ &= -2\eta h^{km}\Gamma_{imj} + g^{km}\left(\left\langle\frac{\partial^2}{\partial p_i\partial p_j}(\eta\nu),\frac{\partial X}{\partial p_m}\right\rangle + \left\langle\frac{\partial^2 X}{\partial p_i\partial p_j},\frac{\partial}{\partial p_m}(\eta\nu)\right\rangle\right).\end{aligned}$$

Now

$$\begin{aligned}\left\langle\frac{\partial^2}{\partial p_i\partial p_j}(\eta\nu),\frac{\partial X}{\partial p_m}\right\rangle &= \left\langle\frac{\partial}{\partial p_i}\left(h^l{}_j\frac{\partial X}{\partial p_l}\right),\frac{\partial X}{\partial p_m}\right\rangle\eta + h_{jm}\nabla_i\eta + h_{im}\nabla_j\eta\\ &= \left(h^l{}_j\Gamma_{iml} + g_{lm}\left(\nabla_i h^l{}_j + \Gamma^q_{ij}h^l{}_q - \Gamma^l_{iq}h^q{}_j\right)\right)\eta\\ &\quad + h_{jm}\nabla_i\eta + h_{im}\nabla_j\eta\\ &= \left(\nabla_m h_{ij} + \Gamma^l_{ij}h_{ml}\right)\eta + h_{jm}\nabla_i\eta + h_{im}\nabla_j\eta\end{aligned}$$

and

$$\begin{aligned}\left\langle\frac{\partial^2 X}{\partial p_i\partial p_j},\frac{\partial}{\partial p_m}(\eta\nu)\right\rangle &= \left\langle\frac{\partial^2 X}{\partial p_i\partial p_j},h^l{}_m\frac{\partial X}{\partial p_l}\right\rangle\eta - h_{ij}\nabla_m\eta\\ &= \eta h^l{}_m\Gamma_{ilj} - h_{ij}\nabla_m\eta\,.\end{aligned}$$

Here we have used the orthogonality of the normal and tangent spaces on M_t, the Gauss-Weingarten relations, the Codazzi equations, and the definition of the covariant derivative for the tensorfield $(h^i{}_j)$. Thus, by recalling that $\eta = (h^l\zeta_l - H - f_\Omega)$ and again employing the Codazzi equations, we ascertain that

$$
\begin{aligned}
\frac{\partial}{\partial t}\Gamma^k_{ij} &= g^{km}\left(\eta\nabla_m h_{ij} + h_{jm}\nabla_i\eta + h_{im}\nabla_j\eta - h_{ij}\nabla_m\eta\right) \\
&= (h^l\zeta_l - H - f_\Omega)g^{km}\nabla_m h_{ij} + h^k{}_j\nabla_i(h^l\zeta_l - H - f_\Omega) \\
&\quad + h^k{}_i\nabla_j(h^l\zeta_l - H - f_\Omega) - h_{ij}g^{km}\nabla_m(h^l\zeta_l - H - f_\Omega) \\
&= h^l\zeta_l\nabla_i h^k{}_j + h^l\left(h^k{}_j\nabla_i\zeta_l + h^k{}_i\nabla_j\zeta_l - g^{km}h_{ij}\nabla_m\zeta_l\right) \\
&\quad + \left(h_{ij}\nabla_m h^{mk} - h^k{}_j\nabla_m h^m{}_i - h^k{}_i\nabla_m h^m{}_j - H\nabla_i h^k{}_j\right) \\
&\quad - f_\Omega\nabla_i h^k{}_j + \left(g^{km}h_{ij}\nabla_m f_\Omega - h^k{}_j\nabla_i f_\Omega - h^k{}_i\nabla_j f_\Omega\right).
\end{aligned}
$$

We then infer that

$$
\frac{\partial}{\partial t}\mathbf{\Gamma} = h^i\zeta_i\nabla A + h^i\nabla\zeta_i \star A + A\star\nabla A + f_\Omega\nabla A + A\star\nabla f_\Omega\,,
$$

as required. ◇

For any (tangential) tensorfield whose evolution is governed by the rotating drop flow, we now proceed to describe the resultant behaviour of its covariant derivative.

LEMMA 6.8. *Suppose that $\mathcal{T}\in\mathscr{T}^p_q(M_t)$ is a tensorfield of class C^3 evolving by the rotating drop flow* (5.1), *and $\mathcal{W}\in\mathscr{T}^p_q(M_t)$ is a tensorfield of class C^1 such that, for each $t\in(0,T)$,*

$$
\frac{\partial}{\partial t}\mathcal{T}(p,t) = \Delta^{M_t}\mathcal{T}(p,t) + \mathcal{W}(p,t)\,.
$$

Then, for each $t\in(0,T)$, the covariant derivative of $\mathcal{T}$ satisfies an evolution equation of the form:

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^{M_t}\mathcal{T}(p,t) &= \Delta^{M_t}\left(\nabla^{M_t}\mathcal{T}(p,t)\right) + h^i(t)\zeta_i(p,t)\nabla^{M_t}A(p,t)\star\mathcal{T}(p,t) \\
&\quad + h^i(t)\nabla^{M_t}\zeta_i(p,t)\star A(p,t)\star\mathcal{T}(p,t) + A(p,t)\star A(p,t)\star\nabla^{M_t}\mathcal{T}(p,t) \\
&\quad + A(p,t)\star\nabla^{M_t}A(p,t)\star\mathcal{T}(p,t) + f_\Omega(p,t)\nabla^{M_t}A(p,t)\star\mathcal{T}(p,t) \\
&\quad + \nabla^{M_t}f_\Omega(p,t)\star A(p,t)\star\mathcal{T}(p,t) + \nabla^{M_t}\mathcal{W}(p,t)\,.
\end{aligned}
$$

PROOF

We use Lemma 6.7 to compute:

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla\mathcal{T} &= \nabla\left(\frac{\partial}{\partial t}\mathcal{T}\right) + \left(\frac{\partial}{\partial t}\boldsymbol{\Gamma}\right)\star\mathcal{T} \\
&= \nabla\left(\Delta\mathcal{T} + \mathcal{W}\right) + \left(h^i\zeta_i\nabla A + h^i\nabla\zeta_i\star A + A\star\nabla A\right. \\
&\quad \left. + f_\Omega\nabla A + A\star\nabla f_\Omega\right)\star\mathcal{T} \\
&= \nabla\left(\Delta\mathcal{T}\right) + h^i\zeta_i\nabla A\star\mathcal{T} + h^i\nabla\zeta_i\star A\star\mathcal{T} + A\star\nabla A\star\mathcal{T} \\
&\quad + \left(f_\Omega\nabla A + \nabla f_\Omega\star A\right)\star\mathcal{T} + \nabla\mathcal{W}\,.
\end{aligned}$$

By interchanging covariant derivatives, we find that

$$\nabla\left(\Delta\mathcal{T}\right) = \Delta\left(\nabla\mathcal{T}\right) + \mathrm{Rm}\star\nabla\mathcal{T} + \nabla\,\mathrm{Rm}\star\mathcal{T}$$

where $\mathrm{Rm} = (R_{ijkl})$ is the Riemann curvature tensor on M_t. We may invoke the Gauss equations to determine that $\mathrm{Rm} = (h_{ik}h_{jl} - h_{il}h_{jk}) = A\star A$. Therefore,

$$\nabla\left(\Delta\mathcal{T}\right) = \Delta\left(\nabla\mathcal{T}\right) + A\star A\star\nabla\mathcal{T} + A\star\nabla A\star\mathcal{T}\,,$$

which implies the desired result. ◇

Through the deployment of an inductive argument, we are now able to characterise the evolution of any covariant derivative of the second fundamental form.

PROPOSITION 6.9. *Suppose $\{M_t : t \in (0,T)\}$ is a family of manifolds evolving by the rotating drop flow (5.1). Then, for each $t \in (0,T)$ and $m \in \mathbb{N}\cup\{0\}$, the order m covariant derivative of the second fundamental form satisfies an evolution equation of the form:*

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^m A(p,t) &= \Delta^{M_t}\left(\nabla^m A(p,t)\right) + h^l(t)\sum_{i+j+k=m}\nabla^i\zeta_l(p,t)\star\nabla^j A(p,t)\star\nabla^k A(p,t) \\
&\quad + h^l\nabla^{m+2}\zeta_l(p,t) + \sum_{i+j+k=m}\nabla^i A(p,t)\star\nabla^j A(p,t)\star\nabla^k A(p,t) \\
&\quad + \sum_{i+j+k=m}\nabla^i f_\Omega(p,t)\star\nabla^j A(p,t)\star\nabla^k A(p,t) + \nabla^{m+2} f_\Omega(p,t)\,.
\end{aligned}$$

PROOF

We proceed by induction on m. Upon examination of the evolution equation for the second fundamental form given by (5.21), we deduce that

$$\left(\frac{\partial}{\partial t} - \Delta\right) A = h^i \zeta_i \, A \star A + h^i \nabla^2 \zeta_i + A \star A \star A + f_\Omega \, A \star A + \nabla^2 f_\Omega .$$

Here we have observed that, for each $i \in \{0, ..., n+1\}$, $\nabla^2 \zeta_i = A \langle \nu, e_i \rangle$ whenever we adhere to our convention that $e_0 = \mathbf{0}$. Therefore the proposition is valid for $m = 0$. Now suppose that the result holds for a particular $m = l \in \mathbb{N} \cup \{0\}$. Then

$$\frac{\partial}{\partial t} \nabla^l A = \Delta \left(\nabla^l A\right) + h^q \sum_{i+j+k=l} \nabla^i \zeta_q \star \nabla^j A \star \nabla^k A + h^q \nabla^{l+2} \zeta_q$$

$$+ \sum_{i+j+k=l} \nabla^i A \star \nabla^j A \star \nabla^k A + \sum_{i+j+k=l} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A + \nabla^{l+2} f_\Omega .$$

We may therefore apply Lemma 6.8 with $\mathcal{T} = \nabla^l A$ and

$$\mathcal{W} = h^q \sum_{i+j+k=l} \nabla^i \zeta_q \star \nabla^j A \star \nabla^k A + h^q \nabla^{l+2} \zeta_q + \sum_{i+j+k=l} \nabla^i A \star \nabla^j A \star \nabla^k A$$

$$+ \sum_{i+j+k=l} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A + \nabla^{l+2} f_\Omega ,$$

to compute that

$$\frac{\partial}{\partial t} \nabla^{l+1} A = \Delta \left(\nabla^{l+1} A\right) + h^q \zeta_q \nabla A \star \nabla^l A + h^q \nabla \zeta_q \star A \star \nabla^l A + \nabla^{l+3} f_\Omega$$

$$+ A \star A \star \nabla^{l+1} A + A \star \nabla A \star \nabla^l A + \left(f_\Omega \nabla A + \nabla f_\Omega \star A\right) \star \nabla^l A$$

$$+ h^q \sum_{i+j+k=l} \left(\nabla^{i+1} \zeta_q \star \nabla^j A + \nabla^i \zeta_q \star \nabla^{j+1} A\right) \star \nabla^k A + h^q \nabla^{l+3} \zeta_q$$

$$+ \sum_{i+j+k=l} \left(\nabla^{i+1} A \star \nabla^j A + \nabla^{i+1} f_\Omega \star \nabla^j A + \nabla^i f_\Omega \star \nabla^{j+1} A\right) \star \nabla^k A$$

$$= \Delta\left(\nabla^{l+1}A\right) + h^q \sum_{i+j+k=l+1} \nabla^i\zeta_q \star \nabla^j A \star \nabla^k A + h^q \nabla^{l+3}\zeta_q$$

$$+ \sum_{i+j+k=l+1} \nabla^i A \star \nabla^j A \star \nabla^k A + \sum_{i+j+k=l+1} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A + \nabla^{l+3} f_\Omega\,.$$

Thus the result holds for $m = l+1$, and we verify the proposition for each $m \in \mathbb{N} \cup \{0\}$ by the principle of mathematical induction. ◇

For arbitrary $m \in \mathbb{N} \cup \{0\}$, we may now derive our evolution equation for $\|\nabla^m A(p,t)\|^2$ as a consequence of Proposition 6.9.

COROLLARY 6.10. *Suppose $\{M_t : t \in (0,T)\}$ is a family of manifolds evolving by the rotating drop flow* (5.1). *Then, for each $t \in (0,T)$ and $m \in \mathbb{N} \cup \{0\}$, the square of the tensorfield norm of $\nabla^m A$ evolves in the following manner:*

$$\begin{aligned}\frac{\partial}{\partial t}\|\nabla^m A(p,t)\|^2 \;=\;& \Delta^{M_t}\left(\|\nabla^m A(p,t)\|^2\right) + h^l(t)\nabla^{m+2}\zeta_l(p,t) \star \nabla^m A(p,t)\\ &+ h^l(t) \sum_{i+j+k=m} \nabla^i\zeta_l(p,t) \star \nabla^j A(p,t) \star \nabla^k A(p,t) \star \nabla^m A(p,t)\\ &+ \sum_{i+j+k=m} \nabla^i A(p,t) \star \nabla^j A(p,t) \star \nabla^k A(p,t) \star \nabla^m A(p,t)\\ &+ \sum_{i+j+k=m} \nabla^i f_\Omega(p,t) \star \nabla^j A(p,t) \star \nabla^k A(p,t) \star \nabla^m A(p,t)\\ &+ \nabla^{m+2} f_\Omega(p,t) \star \nabla^m A(p,t) - 2\left\|\nabla^{m+1}A(p,t)\right\|^2\,.\end{aligned}$$

PROOF

Upon examination of the evolution equation for the inverse metric given by (5.17), we may assert that

$$\frac{\partial}{\partial t}\mathbf{g}^{-1} = h^i\zeta_i\, A + A \star A + f_\Omega\, A\,.$$

We then employ Proposition 6.9 to compute, for any $m \in \mathbb{N} \cup \{0\}$:

$$\frac{\partial}{\partial t}\|\nabla^m A\|^2 = \frac{\partial}{\partial t}\left\langle \nabla^m A, \nabla^m A\right\rangle_{M_t}$$

$$= 2\left\langle \frac{\partial}{\partial t}\left(\nabla^m A\right), \nabla^m A\right\rangle_{M_t} + \left(h^l \zeta_l A + A \star A + f_\Omega A\right) \star \nabla^m A \star \nabla^m A$$

$$= 2\Bigg\langle \Delta\left(\nabla^m A\right) + h^l \sum_{i+j+k=m} \nabla^i \zeta_l \star \nabla^j A \star \nabla^k A + h^l \nabla^{m+2}\zeta_l + \nabla^{m+2} f_\Omega$$

$$+ \sum_{i+j+k=m} \nabla^i A \star \nabla^j A \star \nabla^k A + \sum_{i+j+k=m} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A, \nabla^m A \Bigg\rangle_{M_t}$$

$$+ \left(h^l \zeta_l A + A \star A + f_\Omega A\right) \star \nabla^m A \star \nabla^m A$$

$$= 2\left\langle \Delta\left(\nabla^m A\right), \nabla^m A\right\rangle_{M_t} + h^l \nabla^{m+2}\zeta_l \star \nabla^m A + h^l \sum_{i+j+k=m} \nabla^i \zeta_l \star \nabla^j A \star \nabla^k A \star \nabla^m A$$

$$+ \sum_{i+j+k=m} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla^m A + \nabla^{m+2} f_\Omega \star \nabla^m A$$

$$+ \sum_{i+j+k=m} \nabla^i A \star \nabla^j A \star \nabla^k A \star \nabla^m A\,.$$

We now observe that $\Delta \|\nabla^m A\|^2 = 2\left(\left\langle \Delta\left(\nabla^m A\right), \nabla^m A\right\rangle_{M_t} + \|\nabla^{m+1} A\|^2\right)$ to obtain the desired result. ◇

4. An estimate on $\|\nabla^m A\|^2$

We may now extend the methodology expounded in the proof of Theorem 6.5 to evince *a priori* estimates on curvature derivatives of any order: these are not optimal but sufficient for the purposes of forthcoming analysis.

THEOREM 6.11. *Suppose $\{M_t : t \in (0,T)\}$ is a family of star-shaped manifolds evolving by the rotating drop flow (5.1), and $\varepsilon > 0$ is such that $\||X| - 1\|_{C^{0,1}(S)} \leq \varepsilon$ on $(0,T)$. Then, for each $t \in (0,T)$ and $m \in \mathbb{N} \cup \{0\}$, the square of the tensorfield norm of $\nabla^m A$ satisfies the following estimate:*

$$\sup_{M_t} \|\nabla^m A\|^2 \leq \mathcal{C}_m(n) \max\left\{1, \frac{1}{t^m}, \frac{\varepsilon}{t^{m+1}}\right\}$$

subject to the constraints on Ω and ε stipulated by Theorem 6.5.

PROOF

We proceed by complete induction on m, where the case $m = 0$ is established by Theorem 6.5. Suppose there exists a particular $m \in \mathbb{N}$ such that the theorem holds for each $l \in \mathbb{N} \cup \{0\}$ with $l < m$. In particular, on $(0, T)$ we have for each $l < m$,

$$\sup_{M_t} \left\|\nabla^l A\right\|^2 \leq \mathcal{C}_l \max\left\{1, \frac{1}{t^l}, \frac{\varepsilon}{t^{l+1}}\right\}. \tag{6.17}$$

On the time interval $[\delta, 2\delta]$, with $0 < \delta < \frac{T}{2}$, we shall examine the quantity

$$\mathscr{Q}^m = \frac{\|\nabla^m A\|^2}{\left(d_m - \|\nabla^{m-1} A\|^2\right)}$$

where $d_m = 2\mathcal{C}_{m-1} \max\left\{1, \frac{1}{\delta^{m-1}}, \frac{\varepsilon}{\delta^m}\right\}$. We then utilise Corollary 6.10 and Lemma A.1 in Appendix A to deduce that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) \mathscr{Q}^m \\
&= \frac{1}{\left(d_m - \|\nabla^{m-1} A\|^2\right)} \Bigg(h^q \nabla^{m+2} \zeta_q \star \nabla^m A + h^q \sum_{i+j+k=m} \nabla^i \zeta_q \star \nabla^k A \star \nabla^j A \star \nabla^m A \\
&\quad + \sum_{i+j+k=m} \nabla^i A \star \nabla^j A \star \nabla^k A \star \nabla^m A + \sum_{i+j+k=m} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla^m A \\
&\quad + \nabla^{m+2} f_\Omega \star \nabla^m A - 2 \left\|\nabla^{m+1} A\right\|^2 \Bigg) + \frac{\mathscr{Q}^m}{\left(d_m - \|\nabla^{m-1} A\|^2\right)} \Big(h^q \nabla^{m+1} \zeta_q \star \nabla^{m-1} A \\
&\quad + h^q \sum_{i+j+k=m-1} \nabla^i \zeta_q \star \nabla^j A \star \nabla^k A \star \nabla^{m-1} A + \sum_{i+j+k=m-1} \nabla^i A \star \nabla^j A \star \nabla^k A \star \nabla^{m-1} A \\
&\quad + \sum_{i+j+k=m-1} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla^{m-1} A + \nabla^{m+1} f_\Omega \star \nabla^{m-1} A \Big) - 2 \left(\mathscr{Q}^m\right)^2 \\
&\quad - \frac{2}{\left(d_m - \|\nabla^{m-1} A\|^2\right)} \left\langle \nabla \mathscr{Q}^m, \nabla \left\|\nabla^{m-1} A\right\|^2 \right\rangle.
\end{aligned} \tag{6.18}$$

For each $l \leq m$, we shall now evaluate terms of the form which arise in the evolution equation for $\mathscr{Q}^m$, where (in the usual manner) $\|\cdot\|_\infty$ shall denote the supremum over M_t.

We commence with the examination of the expression $h^q \nabla^{l+2}\zeta_q \star \nabla^l A$, where we recollect that $\zeta_0 = \frac{1}{\sqrt{n+1}}$. In the particular case $l = 0$, we may cite Proposition 6.4 to find that

$$h^q \nabla^2 \zeta_q \star A \;\leq\; \varepsilon K(n)\,(1 + \|A\|_\infty)\,\|A\|^2 .$$

More generally, if $l \in \mathbb{N}$ is even, then there exists a $p \in \mathbb{N}$ such that $l = 2p$ and we may invoke Lemma A.2 in Appendix A to assert that, for each $q \in \{1, ..., n+1\}$,

$$\nabla^{l+2}\zeta_q \;=\; \nabla^{2(p+1)}\zeta_q \;=\; \langle \nu, e_q\rangle \sum_{j=1}^{p+1} \sum_{\sum_{r=1}^{2j-1} k_r = 2((p+1)-j)} \prod_{r=1}^{2j-1} \circledast \nabla^{k_r} A$$

$$+\; \nabla\zeta_q \star \sum_{j=1}^{p} \sum_{\sum_{r=1}^{2j} k_r = 2(p-j)+1} \prod_{r=1}^{2j} \circledast \nabla^{k_r} A .$$

We may then employ Proposition 6.4 in conjunction with Lemmas A.6 and B.1 (contained in the respective appendices A and B) to ascertain that

$$h^q \nabla^{2(p+1)}\zeta_q \star \nabla^{2p} A$$

$$= h^q \left(\langle \nu, e_q\rangle \left(\sum_{j=2}^{p} \sum_{\sum_{r=1}^{2j-1} k_r = 2((p+1)-j)} \prod_{r=1}^{2j-1} \circledast \nabla^{k_r} A + \nabla^{2p} A + \prod_{r=1}^{2p+1} \circledast A \right) \right.$$

$$\left. + \nabla\zeta_q \star \sum_{j=1}^{p} \sum_{\sum_{r=1}^{2j} k_r = 2(p-j)+1} \prod_{r=1}^{2j} \circledast \nabla^{k_r} A \right) \star \nabla^{2p} A$$

$$= h^q \left(\langle \nu, e_q\rangle \left(\sum_{j=1}^{p-1} \sum_{s=0}^{2j} \prod_{i=1}^{s} \circledast A \star \sum_{\sum_{r=1}^{2j+1-s} (k_r+1) = 2(p-j)} \prod_{r=1}^{2j+1-s} \circledast \nabla^{k_r+1} A + \nabla^{2p} A \right. \right.$$

$$\left. \left. + \prod_{r=1}^{2p+1} \circledast A \right) + \nabla\zeta_q \star \sum_{j=1}^{p} \sum_{s=0}^{2j-1} \prod_{i=1}^{s} \circledast A \star \sum_{\sum_{r=1}^{2j-s} (k_r+1) = 2(p-j)+1} \prod_{r=1}^{2j-s} \circledast \nabla^{k_r+1} A \right) \star \nabla^{2p} A$$

$$
\begin{aligned}
\leq\ & K(2p,n)\sum_{q=1}^{n+1}|h^q|\left(\sum_{j=1}^{p-1}\sum_{s=0}^{2j}\|A\|^s\sum_{\sum_{r=1}^{2j+1-s}(k_r+1)=2(p-j)}\prod_{r=1}^{2j+1-s}\left\|\nabla^{k_r+1}A\right\| + \left\|\nabla^{2p}A\right\|\right.\\
& \left. + \|A\|^{2p+1} + \sum_{j=1}^{p}\sum_{s=0}^{2j-1}\|A\|^s\sum_{\sum_{r=1}^{2j-s}(k_r+1)=2(p-j)+1}\prod_{r=1}^{2j-s}\left\|\nabla^{k_r+1}A\right\|\right)\left\|\nabla^{2p}A\right\|\\
\leq\ & \varepsilon\, K\sum_{q=1}^{n+1}C^q\,(1+|\Omega|+\|A\|_\infty)\left(\|A\|_\infty^{2p+1} + \left\|\nabla^{2p}A\right\|_\infty\right.\\
& + \sum_{j=1}^{p-1}\sum_{s=0}^{2j}\|A\|_\infty^s\sum_{\sum_{r=1}^{2j+1-s}(k_r+1)=2(p-j)}\prod_{r=1}^{2j+1-s}\|A\|_\infty^{1-\frac{(k_r+1)}{2p}}\left\|\nabla^{2p}A\right\|_\infty^{\frac{k_r+1}{2p}}\\
& \left. + \sum_{j=1}^{p}\sum_{s=0}^{2j-1}\|A\|_\infty^s\sum_{\sum_{r=1}^{2j-s}(k_r+1)=2(p-j)+1}\prod_{r=1}^{2j-s}\|A\|_\infty^{1-\frac{(k_r+1)}{2p}}\left\|\nabla^{2p}A\right\|_\infty^{\frac{k_r+1}{2p}}\right)\left\|\nabla^{2p}A\right\|_\infty\\
\leq\ & \varepsilon\, K(2p,n)\,(1+\|A\|_\infty)\left(\|A\|_\infty^{2p+1}\left\|\nabla^{2p}A\right\|_\infty + \left\|\nabla^{2p}A\right\|_\infty^2\right.\\
& \left. + \sum_{j=1}^{p-1}\sum_{s=0}^{2j}\|A\|_\infty^{2j\left(1+\frac{1}{2p}\right)}\left\|\nabla^{2p}A\right\|_\infty^{2-\frac{j}{p}} + \sum_{j=1}^{p}\sum_{s=0}^{2j-1}\|A\|_\infty^{(2j-1)\left(1+\frac{1}{2p}\right)}\left\|\nabla^{2p}A\right\|_\infty^{2-\frac{(2j-1)}{2p}}\right)\\
\leq\ & \varepsilon\, K\,(1+\|A\|_\infty)\left(\frac{1}{2}\|A\|_\infty^{2(2p+1)} + \frac{3}{2}\left\|\nabla^{2p}A\right\|_\infty^2\right.\\
& + \sum_{j=1}^{p-1}(2j+1)\left(\frac{j}{2p}\|A\|_\infty^{2(2p+1)} + \frac{(2p-j)}{2p}\left\|\nabla^{2p}A\right\|_\infty^2\right)\\
& \left. + \sum_{j=1}^{p}2j\left(\frac{(2j-1)}{4p}\|A\|_\infty^{2(2p+1)} + \frac{(4p+1-2j)}{4p}\left\|\nabla^{2p}A\right\|_\infty^2\right)\right)
\end{aligned}
$$

$$\leq \varepsilon K(2p,n)\,(1+\|A\|_\infty)\left(\|A\|_\infty^{2(2p+1)} + \left\|\nabla^{2p}A\right\|_\infty^2\right).$$

Here we have utilised Young's inequality, and through a parallel computation in the case of odd l we discern that, for each $l \leq m$,

$$h^q\nabla^{l+2}\zeta_q \star \nabla^l A \;\leq\; \varepsilon K_1(l,n)\,(1+\|A\|_\infty)\left(\|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2\right). \tag{6.19}$$

We continue by observing that in the particular cases $l = 0$ and $l = 1$, Proposition 6.4 implies that

$$h^q \sum_{i+j+k=0} \nabla^i\zeta_q \star \nabla^j A \star \nabla^k A \star A \;\leq\; K(n)\,(1+|\Omega| + \varepsilon\,(1+\|A\|_\infty))\,\|A\|^3$$

and

$$h^q \sum_{i+j+k=1} \nabla^i\zeta_q \star \nabla^j A \star \nabla^k A \star \nabla A \;\leq\; K(n)\,\big((1+|\Omega| + \varepsilon\,(1+\|A\|_\infty))\,\|A\|\,\|\nabla A\|^2$$

$$+\varepsilon\,(1+\|A\|_\infty)\left(\|A\|^4 + \|\nabla A\|^2\right)\big).$$

Furthermore, we discover that whenever $l \geq 2$,

$$\begin{aligned}
&h^q \sum_{i+j+k=l} \nabla^i\zeta_q \star \nabla^j A \star \nabla^k A \star \nabla^l A \\
&= h^q \sum_{i=0}^{l} \nabla^i\zeta_q \star \sum_{j+k=l-i} \nabla^j A \star \nabla^k A \star \nabla^l A \\
&= h^q \sum_{i=0}^{l} \nabla^i\zeta_q \star \left(A \star \nabla^{l-i}A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1}A \star \nabla^{k+1}A \star \nabla^l A\right) \\
&= h^q \left(\sum_{i=0}^{1} \nabla^i\zeta_q \star \left(A \star \nabla^{l-i}A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1}A \star \nabla^{k+1}A \star \nabla^l A\right)\right. \\
&\quad + \sum_{i=2}^{l-2} \nabla^i\zeta_q \star \left(A \star \nabla^{l-i}A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1}A \star \nabla^{k+1}A \star \nabla^l A\right) \\
&\quad \left. + \sum_{i=0}^{1} \nabla^{l-i}\zeta_q \star A \star \nabla^i A \star \nabla^l A\right).
\end{aligned} \tag{6.20}$$

Clearly the linear combinations of tensor products specified above vanish whenever $(l-2-i)<0$. With the assistance of Proposition 6.4, Lemma B.1, and Young's inequality we evaluate the first term which occurs in this expression:

$$
\begin{aligned}
& h^q \sum_{i=0}^{1} \nabla^i \zeta_q \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right) \\
\leq\ & K(l,n)\, |h^q| \sum_{i=0}^{1} |\nabla^i \zeta_q| \left(\|A\| \, \|\nabla^{l-i} A\| \, \|\nabla^l A\| + \sum_{j+k=l-2-i} \|\nabla^{j+1} A\| \, \|\nabla^{k+1} A\| \, \|\nabla^l A\| \right) \\
\leq\ & K \left(\left(n + |\Omega| + \varepsilon \sum_{q=0}^{n+1} C^q \left(1 + |\Omega| + \|A\|_\infty\right) \right) \left(\|A\|_\infty \left\|\nabla^l A\right\|_\infty^2 \right.\right. \\
& \left. + \sum_{j+k=l-2} \|A\|_\infty^{1-\frac{(j+1)}{l}} \left\|\nabla^l A\right\|_\infty^{\frac{j+1}{l}} \|A\|_\infty^{1-\frac{(k+1)}{l}} \left\|\nabla^l A\right\|_\infty^{\frac{k+1}{l}} \left\|\nabla^l A\right\|_\infty \right) \\
& + \varepsilon \sum_{q=1}^{n+1} C^q \left(1 + |\Omega| + \|A\|_\infty\right) \left(\|A\|_\infty \left\|\nabla^{l-1} A\right\|_\infty \left\|\nabla^l A\right\|_\infty \right. \\
& \left.\left. + \sum_{j+k=l-3} \|A\|_\infty^{1-\frac{(j+1)}{l}} \left\|\nabla^l A\right\|_\infty^{\frac{j+1}{l}} \|A\|_\infty^{1-\frac{(k+1)}{l}} \left\|\nabla^l A\right\|_\infty^{\frac{k+1}{l}} \left\|\nabla^l A\right\|_\infty \right) \right) \\
\leq\ & K(l,n) \left(\left(1 + |\Omega| + \varepsilon \left(1 + \|A\|_\infty\right)\right) \|A\|_\infty \left\|\nabla^l A\right\|_\infty^2 \right. \\
& \left. + \varepsilon \left(1 + \|A\|_\infty\right) \|A\|_\infty^{2-\frac{(l-1)}{l}} \left\|\nabla^l A\right\|_\infty^{1+\frac{(l-1)}{l}} \right) \\
\leq\ & K \left(\left(1 + |\Omega| + \varepsilon \left(1 + \|A\|_\infty\right)\right) \|A\|_\infty \left\|\nabla^l A\right\|_\infty^2 \right. \\
& \left. + \varepsilon \left(1 + \|A\|_\infty\right) \left(\frac{1}{2l} \|A\|_\infty^{2(l+1)} + \frac{(2l-1)}{2l} \left\|\nabla^l A\right\|_\infty^2 \right) \right).
\end{aligned}
$$

We now examine the term

$$h^q \sum_{i=2}^{l-2} \nabla^i \zeta_q \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right).$$

Suppose i is odd. Then there exists a $p \in \mathbb{N}$ such that $i = 2p+1$ where, by again recalling that $\zeta_0 = \frac{1}{\sqrt{n+1}}$ and $e_0 = \mathbf{0}$, we may cite Lemmas A.2 and A.6 to determine that, for each $q \in \{0, ..., n+1\}$,

$$\begin{aligned}
\nabla^{2p+1}\zeta_q &= \sum_{s=1}^{p} \left(\langle \nu, e_q \rangle \sum_{\sum_{r=1}^{2s-1} k_r = 2(p-s)+1} \prod_{r=1}^{2s-1} \circledast \nabla^{k_r} A + \nabla \zeta_q \star \sum_{\sum_{r=1}^{2s} k_r = 2(p-s)} \prod_{r=1}^{2s} \circledast \nabla^{k_r} A \right) \\
&= \sum_{s=1}^{p} \langle \nu, e_q \rangle \sum_{b=0}^{2s-2} \prod_{a=1}^{b} \circledast A \star \sum_{\sum_{r=1}^{2s-1-b} (k_r+1) = 2(p-s)+1} \prod_{r=1}^{2s-1-b} \circledast \nabla^{k_r+1} A \\
&\quad + \sum_{s=1}^{p-1} \nabla \zeta_q \star \sum_{b=0}^{2s-1} \prod_{a=1}^{b} \circledast A \star \sum_{\sum_{r=1}^{2s-b} (k_r+1) = 2(p-s)} \prod_{r=1}^{2s-b} \circledast \nabla^{k_r+1} A + \nabla \zeta_q \star \prod_{r=1}^{2p} \circledast A .
\end{aligned}$$

Since $2p+1 \le l-2$, we observe that the highest order curvature derivative which can appear in the preceding expression is $\nabla^{l-4} A$, and we may utilise Lemma B.1 together with Young's inequality to estimate that, for any odd $i = 2p+1$ with $2 < i \le l-2$,

$$\begin{aligned}
&\nabla^{2p+1}\zeta_q \star \left(A \star \nabla^{l-(2p+1)} A \star \nabla^l A + \sum_{j+k=l-(2p+3)} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right) \\
&\le K(2p+1, l, n) \left(\sum_{s=1}^{p} \sum_{b=0}^{2s-2} \|A\|^b \sum_{\sum_{r=1}^{2s-1-b} (k_r+1) = 2(p-s)+1} \prod_{r=1}^{2s-1-b} \|\nabla^{k_r+1} A\| \right. \\
&\quad \left. + \sum_{s=1}^{p-1} \sum_{b=0}^{2s-1} \|A\|^b \sum_{\sum_{r=1}^{2s-b} (k_r+1) = 2(p-s)} \prod_{r=1}^{2s-b} \|\nabla^{k_r+1} A\| + \|A\|^{2p} \right) \cdot \\
&\quad \left(\|A\| \, \|\nabla^{l-(2p+1)} A\| \, \|\nabla^l A\| + \sum_{j+k=l-(2p+3)} \|\nabla^{j+1} A\| \, \|\nabla^{k+1} A\| \, \|\nabla^l A\| \right)
\end{aligned}$$

$$\leq \; K(2p+1,l,n)\left(\sum_{s=1}^{p}\sum_{b=0}^{2s-2}\|A\|_\infty^b \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)+1}\;\prod_{r=1}^{2s-1-b}\|A\|_\infty^{1-\frac{(k_r+1)}{l}}\left\|\nabla^l A\right\|_\infty^{\frac{k_r+1}{l}}\right.$$

$$\left. + \sum_{s=1}^{p-1}\sum_{b=0}^{2s-1}\|A\|_\infty^b \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)}\;\prod_{r=1}^{2s-b}\|A\|_\infty^{1-\frac{(k_r+1)}{l}}\left\|\nabla^l A\right\|_\infty^{\frac{k_r+1}{l}} + \|A\|_\infty^{2p}\right)\cdot$$

$$\left(\|A\|_\infty^{1+\frac{2p+1}{l}}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2p+1)}{l}} + \sum_{j+k=l-(2p+3)}\|A\|_\infty^{2-\frac{(j+k+2)}{l}}\left\|\nabla^l A\right\|_\infty^{1+\frac{j+k+2}{l}}\right)$$

$$\leq K(2p+1,l,n)\left(\sum_{s=1}^{p}\sum_{b=0}^{2s-2}\|A\|_\infty^{\left(2s-1-\frac{(2(p-s)+1)}{l}\right)}\left\|\nabla^l A\right\|_\infty^{\frac{2(p-s)+1}{l}} + \|A\|_\infty^{2p}\right.$$

$$\left. + \sum_{s=1}^{p-1}\sum_{b=0}^{2s-1}\|A\|_\infty^{\left(2s-\frac{2(p-s)}{l}\right)}\left\|\nabla^l A\right\|_\infty^{\frac{2(p-s)}{l}}\right)\|A\|_\infty^{1+\frac{2p+1}{l}}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2p+1)}{l}}$$

$$\leq \; K\left(\sum_{s=1}^{p}(2s-1)\|A\|_\infty^{2s\left(1+\frac{1}{l}\right)}\left\|\nabla^l A\right\|_\infty^{2-\frac{2s}{l}} + \sum_{s=1}^{p}2s\|A\|_\infty^{(2s+1)\left(1+\frac{1}{l}\right)}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2s+1)}{l}}\right)$$

$$\leq \; K\left(\sum_{s=1}^{p}(2s-1)\left(\frac{s}{l}\|A\|_\infty^{2(l+1)} + \frac{(l-s)}{l}\left\|\nabla^l A\right\|_\infty^2\right)\right.$$

$$\left. + \sum_{s=1}^{p}2s\left(\frac{(2s+1)}{2l}\|A\|_\infty^{2(l+1)} + \frac{(2l-(2s+1))}{2l}\left\|\nabla^l A\right\|_\infty^2\right)\right)$$

$$\leq \; K(2p+1,l,n)\left(\|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2\right).$$

By a further application of Lemma A.2, we derive a similar result through a parallel calculation in the even case. Therefore, we again invoke Proposition 6.4 to verify that

$$h^q \sum_{i=2}^{l-2} \nabla^i \zeta_q \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right)$$

$$\leq \; \varepsilon K(l,n)\,(1 + \|A\|_\infty) \left(\|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2 \right).$$

In analogy with the calculation of (6.19), we may assert the following bound on the final term which appears in (6.20):

$$h^q \sum_{i=0}^{1} \nabla^{l-i} \zeta_q \star A \star \nabla^i A \star \nabla^l A \;\leq\; \varepsilon K(l,n)\,(1 + \|A\|_\infty) \left(\|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2 \right).$$

By collecting the preceding results, we discern that, for each $l \leq m$,

$$h^q \sum_{i+j+k=l} \nabla^i \zeta_q \star \nabla^j A \star \nabla^k A \star \nabla^l A \;\leq\; K_2(l,n) \Big((1 + |\Omega| + \varepsilon\,(1 + \|A\|_\infty))\, \|A\|_\infty \left\|\nabla^l A\right\|_\infty^2$$

$$+ \varepsilon\,(1 + \|A\|_\infty) \left(\|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2 \right) \Big). \qquad (6.21)$$

When $l = 0$, we observe that

$$\sum_{i+j+k=0} \nabla^i A \star \nabla^j A \star \nabla^k A \star \nabla^l A \;\leq\; K(n)\,\|A\|^4 \,.$$

Conversely, when $l \in \mathbb{N}$, we utilise Lemma A.6 to evoke the following characterisation of this quantity:

$$\sum_{i+j+k=l} \nabla^i A \star \nabla^j A \star \nabla^k A \star \nabla^l A \;=\; \left(\sum_{s=0}^{2} \prod_{i=1}^{s} \circledast A \star \sum_{\sum_{r=1}^{3-s}(k_r+1)=l} \prod_{r=1}^{3-s} \circledast \nabla^{k_r+1} A \right) \star \nabla^l A\,.$$

Thus we may again invoke Lemma B.1 and implement analysis similar to that already performed to ascertain that, for each $l \leq m$,

$$\sum_{i+j+k=l} \nabla^i A \star \nabla^j A \star \nabla^k A \star \nabla^l A \;\leq\; K_3(l,n)\, \|A\|_\infty^2 \left\|\nabla^l A\right\|_\infty^2 \,. \qquad (6.22)$$

In the particular case $l = 0$, we cite Cauchy's inequality to find that

$$\nabla^2 f_\Omega \star A \;=\; \Omega \sum_{i=1}^{n} \left(\nabla \zeta_i \star \nabla \zeta_i \,+\, \zeta_i \,\langle \nu, e_i \rangle\, A \right) \star A \;\leq\; |\Omega|\, K(n) \left(1 \,+\, \|A\|^2 \right).$$

We now explore the case for arbitrary $l \in \mathbb{N}$. If l is even, then we may write $l = 2p$ for some $p \in \mathbb{N}$. Hence, by invoking Corollary A.3 and Lemma A.6 in Appendix A, we deduce that for even $l \geq 2$, $\nabla^{l+2} f_\Omega$ is given by

$$
\begin{aligned}
&\nabla^{2p+2} f_\Omega \\
&= \Omega \Bigg[\langle \pi_{\mathbb{R}^n} X, \nu \rangle \sum_{j=1}^{p+1} \sum_{\sum_{i=1}^{2j-1} k_i = 2((p+1)-j)} \prod_{i=1}^{2j-1} \circledast \nabla^{k_i} A \\
&\quad + \sum_{j=1}^{p} \Bigg(\nabla |\pi_{\mathbb{R}^n} X|^2 \star \sum_{\sum_{i=1}^{2j} k_i = 2(p-j)+1} \prod_{i=1}^{2j} \circledast \nabla^{k_i} A \\
&\quad + \sum_{q=1}^{n} \langle \nu, e_q \rangle \nabla \zeta_q \star \sum_{\sum_{i=1}^{2j-1} k_i = 2(p-j)+1} \prod_{i=1}^{2j-1} \circledast \nabla^{k_i} A \\
&\quad + \left(1 - \langle \nu, e_{n+1} \rangle^2 + \sum_{q=1}^{n} \nabla \zeta_q \star \nabla \zeta_q \star \right) \sum_{\sum_{i=1}^{2j} k_i = 2(p-j)} \prod_{i=1}^{2j} \circledast \nabla^{k_i} A \Bigg) \Bigg] \\
&= \Omega \Bigg[\langle \pi_{\mathbb{R}^n} X, \nu \rangle \Bigg(\sum_{j=1}^{p-1} \sum_{s=0}^{2j} \prod_{r=1}^{s} \circledast A \star \sum_{\sum_{i=1}^{2j+1-s} (k_i+1) = 2(p-j)} \prod_{i=1}^{2j+1-s} \circledast \nabla^{k_i+1} A + \prod_{j=1}^{2p+1} \circledast A \\
&\quad + \nabla^{2p} A \Bigg) + \sum_{j=1}^{p} \Bigg(\nabla |\pi_{\mathbb{R}^n} X|^2 \star \sum_{s=0}^{2j-1} \prod_{r=1}^{s} \circledast A \star \sum_{\sum_{i=1}^{2j-s} (k_i+1) = 2(p-j)+1} \prod_{i=1}^{2j-s} \circledast \nabla^{k_i+1} A \\
&\quad + \sum_{q=1}^{n} \langle \nu, e_q \rangle \nabla \zeta_q \star \sum_{s=0}^{2j-2} \prod_{r=1}^{s} \circledast A \star \sum_{\sum_{i=1}^{2j-1-s} (k_i+1) = 2(p-j)+1} \prod_{i=1}^{2j-1-s} \circledast \nabla^{k_i+1} A \Bigg)
\end{aligned}
$$

$$+ \left(1 - \langle \nu, e_{n+1}\rangle^2 + \sum_{q=1}^{n} \nabla\zeta_q \star \nabla\zeta_q \star\right) \left(\prod_{j=1}^{2p} \circledast A\right.$$

$$\left.\left. + \sum_{j=1}^{p-1}\sum_{s=0}^{2j-1}\prod_{r=0}^{s} \circledast A \star \sum_{\sum_{i=1}^{2j-s}(k_i+1)=2(p-j)} \prod_{i=1}^{2j-s} \circledast \nabla^{k_i+1} A\right)\right] .$$

We note that we have isolated the only occurrence of $\nabla^{2p}A$ which appears in the expression above. Consequently, we may utilise Corollary 6.2 and Lemma B.1 in conjunction with Young's inequality to bound the quantity $\nabla^{l+2} f_\Omega \star \nabla^l A$, for each even $l \in \mathbb{N}$ with $l \leq m$:

$$\nabla^{2p+2} f_\Omega \star \nabla^{2p} A$$

$$\leq |\Omega| K(2p,n) \left[\left(\sum_{j=1}^{p-1}\sum_{s=0}^{2j} \|A\|^s \sum_{\sum_{i=1}^{2j+1-s}(k_i+1)=2(p-j)} \prod_{i=1}^{2j+1-s} \|\nabla^{k_i+1} A\| + \|A\|^{2p+1} \right.\right.$$

$$+ \|\nabla^{2p}A\|\Big) + \sum_{j=1}^{p} \left(\left(\sum_{s=0}^{2j-1} \|A\|^s \sum_{\sum_{i=1}^{2j-s}(k_i+1)=2(p-j)+1} \prod_{i=1}^{2j-s} \|\nabla^{k_i+1} A\| \right) \right.$$

$$\left. + \sum_{s=0}^{2j-2} \|A\|^s \sum_{\sum_{i=1}^{2j-1-s}(k_i+1)=2(p-j)+1} \prod_{i=1}^{2j-1-s} \|\nabla^{k_i+1} A\| \right)$$

$$\left. + \left(\|A\|^{2p} + \sum_{j=1}^{p-1}\sum_{s=0}^{2j-1} \|A\|^s \sum_{\sum_{i=1}^{2j-s}(k_i+1)=2(p-j)} \prod_{i=1}^{2j-s} \|\nabla^{k_i+1} A\| \right)\right] \|\nabla^{2p}A\|$$

$$\leq |\Omega| K(2p,n) \left[\sum_{j=1}^{p-1}\sum_{s=0}^{2j} \|A\|_\infty^s \sum_{\sum_{i=1}^{2j+1-s}(k_i+1)=2(p-j)} \prod_{i=1}^{2j+1-s} \|A\|_\infty^{1-\frac{(k_i+1)}{2p}} \left\|\nabla^{2p}A\right\|_\infty^{\frac{k_i+1}{2p}} \right.$$

$$+ \left\|\nabla^{2p}A\right\|_\infty + \sum_{j=1}^{p} \left(\sum_{s=0}^{2j-1} \|A\|_\infty^s \sum_{\sum_{i=1}^{2j-s}(k_i+1)=2(p-j)+1} \prod_{i=1}^{2j-s} \|A\|_\infty^{1-\frac{(k_i+1)}{2p}} \left\|\nabla^{2p}A\right\|_\infty^{\frac{k_i+1}{2p}} \right.$$

$$+ \sum_{s=0}^{2j-2} \|A\|_\infty^s \sum_{\sum_{i=1}^{2j-1-s}(k_i+1)=2(p-j)+1} \prod_{i=1}^{2j-1-s} \|A\|_\infty^{1-\frac{(k_i+1)}{2p}} \left\|\nabla^{2p} A\right\|_\infty^{\frac{k_i+1}{2p}} \Bigg) + \|A\|_\infty^{2p+1}$$

$$+ \|A\|_\infty^{2p} + \sum_{j=1}^{p-1}\sum_{s=0}^{2j-1} \|A\|_\infty^s \sum_{\sum_{i=1}^{2j-s}(k_i+1)=2(p-j)} \prod_{i=1}^{2j-s} \|A\|_\infty^{1-\frac{(k_i+1)}{2p}} \left\|\nabla^{2p} A\right\|_\infty^{\frac{k_i+1}{2p}} \Bigg] \left\|\nabla^{2p} A\right\|_\infty$$

$$\leq |\Omega|\, K \Bigg[\sum_{j=1}^{p-1} (2j+1) \|A\|_\infty^{2j\left(1+\frac{1}{2p}\right)} \left\|\nabla^{2p} A\right\|_\infty^{2-\frac{j}{p}} + (1 + \|A\|_\infty) \|A\|_\infty^{2p} \left\|\nabla^{2p} A\right\|_\infty$$

$$+ \sum_{j=1}^{p} \left(2j \|A\|_\infty^{(2j-1)\left(1+\frac{1}{2p}\right)} + (2j-1) \|A\|_\infty^{\left((2j-1)\left(1+\frac{1}{2p}\right)-1\right)} \right) \left\|\nabla^{2p} A\right\|_\infty^{2-\frac{(2j-1)}{2p}}$$

$$+ \left\|\nabla^{2p} A\right\|_\infty^2 + \sum_{j=1}^{p-1} 2j \|A\|_\infty^{2j\left(1+\frac{1}{2p}\right)} \left\|\nabla^{2p} A\right\|_\infty^{2-\frac{j}{p}} \Bigg]$$

$$\leq |\Omega|\, K \Bigg[\sum_{j=1}^{p-1} (2j+1) \left(\frac{j}{2p} \|A\|_\infty^{2(2p+1)} + \frac{(2p-j)}{2p} \left\|\nabla^{2p} A\right\|_\infty^2 \right) + \frac{1}{2} \left(1 + \|A\|_\infty^2\right) \|A\|_\infty^{4p}$$

$$+ \sum_{j=1}^{p} \left(\frac{(2j-1)}{4p} \left(2j + (2j-1) \|A\|_\infty^{-\frac{4p}{(2j-1)}} \right) \|A\|_\infty^{2(2p+1)} + \frac{(4p-2j+1)}{2p} \left\|\nabla^{2p} A\right\|_\infty^2 \right)$$

$$+ \frac{3}{2} \left\|\nabla^{2p} A\right\|_\infty^2 + \sum_{j=1}^{p-1} 2j \left(\frac{j}{2p} \|A\|_\infty^{2(2p+1)} + \frac{(2p-j)}{2p} \left\|\nabla^{2p} A\right\|_\infty^2 \right) \Bigg].$$

Whenever $1 \leq j \leq p$, we note that $0 \leq 2(2p+1) - \frac{4p}{2j-1} \leq 2(2p+1)$. Regardless of the magnitude of $\|A\|_\infty$, we may then assert that

$$\sum_{j=1}^{p} \frac{(2j-1)^2}{4p} \|A\|_\infty^{2(2p+1)-\frac{4p}{(2j-1)}} \leq K(2p) \left(1 + \|A\|_\infty^{2(2p+1)}\right).$$

Hence we deduce that, for each even $l \leq m$,

$$\nabla^{l+2} f_\Omega \star \nabla^l A \leq |\Omega|\, K(l,n) \left(1 + \|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2\right).$$

Through an analogous application of Corollary A.3 in the case of odd l, we obtain an equivalent estimate on this term. Therefore, we have established that for each $l \leq m$,

$$\nabla^{l+2} f_\Omega \star \nabla^l A \;\leq\; |\Omega|\, K_4(l,n) \left(1 + \|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2\right). \tag{6.23}$$

We now estimate the final term involving covariant derivatives of f_Ω occurring in the evolution equation for $\mathscr{Q}^m$. Notice that in the particular cases corresponding to $l \in \{0, 1, 2\}$, we may use Cauchy's inequality to ascertain that

$$\sum_{i+j+k=0} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star A \;\leq\; |\Omega|\, K(n)\, \|A\|^3 \,,$$

$$\sum_{i+j+k=1} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla A \;\leq\; |\Omega|\, K(n) \left(\|A\|^4 + (1 + \|A\|)\, \|\nabla A\|^2\right) ,$$

and

$$\sum_{i+j+k=2} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla^2 A \;\leq\; |\Omega|\, K(n) \Big(\|A\|_\infty^4 \left(1 + \|A\|_\infty^2\right)$$

$$+ \left(1 + \|A\|_\infty\right) \left\|\nabla^2 A\right\|_\infty^2\Big) .$$

In the final estimate we have invoked Lemma B.1 and again observed that

$$\nabla^2 f_\Omega \;=\; \Omega \sum_{i=1}^{n} \left(\nabla \zeta_i \star \nabla \zeta_i + \zeta_i \left\langle \nu, e_i \right\rangle A\right).$$

In a similar manner to the computation of (6.20), we discern that, for $l \geq 3$,

$$\sum_{i+j+k=l} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla^l A$$

$$= \sum_{i=0}^{2} \nabla^i f_\Omega \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A\right)$$

$$+ \sum_{i=3}^{l-2} \nabla^i f_\Omega \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A\right)$$

$$+ \sum_{i=0}^{1} \nabla^{l-i} f_\Omega \star A \star \nabla^i A \star \nabla^l A \,. \tag{6.24}$$

Once more, it is important to emphasise that the linear combinations of tensor products specified above vanish whenever $(l-2-i)<0$. We may now bound the first expression which occurs in (6.24) through the employment of Corollary 6.2, Lemma B.1 and Young's inequality:

$$\sum_{i=0}^{2} \nabla^i f_\Omega \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right)$$

$$\leq K(l,n) \sum_{i=0}^{2} \|\nabla^i f_\Omega\| \left(\|A\| \, \|\nabla^{l-i} A\| \, \|\nabla^l A\| + \sum_{j+k=l-2-i} \|\nabla^{j+1} A\| \, \|\nabla^{k+1} A\| \, \|\nabla^l A\| \right)$$

$$\leq |\Omega| \, K \left(\left(\|A\|_\infty \left\|\nabla^l A\right\|_\infty^2 + C(l,n) \sum_{j+k=l-2} \|A\|_\infty^{2-\frac{(j+k+2)}{l}} \left\|\nabla^l A\right\|_\infty^{1+\frac{(j+k+2)}{l}} \right) \right.$$

$$+ \left(\|A\|_\infty \left\|\nabla^{l-1} A\right\|_\infty \left\|\nabla^l A\right\|_\infty + C(l,n) \sum_{j+k=l-3} \|A\|_\infty^{2-\frac{(j+k+2)}{l}} \left\|\nabla^l A\right\|_\infty^{1+\frac{(j+k+2)}{l}} \right)$$

$$+ \left(1 + \|A\|_\infty\right) \left(\|A\|_\infty \left\|\nabla^{l-2} A\right\|_\infty \left\|\nabla^l A\right\|_\infty \right.$$

$$\left. \left. + C(l,n) \sum_{j+k=l-4} \|A\|_\infty^{2-\frac{(j+k+2)}{l}} \left\|\nabla^l A\right\|_\infty^{1+\frac{(j+k+2)}{l}} \right) \right)$$

$$\leq |\Omega| \, K(l,n) \left(\|A\|_\infty \left\|\nabla^l A\right\|_\infty^2 + \|A\|_\infty^{2-\frac{(l-1)}{l}} \left\|\nabla^l A\right\|_\infty^{1+\frac{(l-1)}{l}} \right.$$

$$\left. + \left(1 + \|A\|_\infty\right) \|A\|_\infty^{2-\frac{(l-2)}{l}} \left\|\nabla^l A\right\|_\infty^{1+\frac{(l-2)}{l}} \right)$$

$$\leq |\Omega| \, K \left(\|A\|_\infty \left\|\nabla^l A\right\|_\infty^2 + \frac{1}{2l} \|A\|_\infty^{2(l+1)} + \frac{(2l-1)}{2l} \left\|\nabla^l A\right\|_\infty^2 \right.$$

$$\left. + \left(1 + \|A\|_\infty\right) \left(\frac{1}{l} \|A\|_\infty^{l+2} + \frac{(l-1)}{l} \left\|\nabla^l A\right\|_\infty^2 \right) \right)$$

$$\leq \ |\Omega|\, K(l,n) \left(\|A\|_\infty^{2(l+1)} + (1+\|A\|_\infty)\left(\|A\|_\infty^{l+2} + \left\|\nabla^l A\right\|_\infty^2\right)\right).$$

If $i \geq 3$ is odd, then we may find $p \in \mathbb{N}$ such that $i = 2p+1$. Therefore, we may again utilise Corollary A.3 and Lemma A.6 to compute:

$$\nabla^{2p+1} f_\Omega \ = \ \Omega \left[\sum_{s=1}^{p} \langle \pi_{\mathbb{R}^n} X, \nu\rangle \sum_{b=0}^{2s-2} \prod_{a=1}^{b} \circledast A \star \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)+1} \ \prod_{r=1}^{2s-1-b} \circledast \nabla^{k_r+1} A \right.$$

$$+ \sum_{s=1}^{p-1} \left(\nabla |\pi_{\mathbb{R}^n} X|^2 \star \sum_{b=0}^{2s-1} \prod_{a=1}^{b} \circledast A \star \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)} \ \prod_{r=1}^{2s-b} \circledast \nabla^{k_r+1} A \right.$$

$$+ \sum_{q=1}^{n} \langle \nu, e_q\rangle \nabla\zeta_q \star \sum_{b=0}^{2s-2} \prod_{a=1}^{b} \circledast A \star \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)} \ \prod_{r=1}^{2s-1-b} \circledast \nabla^{k_r+1} A$$

$$\left. + \left(1 - \langle \nu, e_{n+1}\rangle^2 + \sum_{q=1}^{n} \nabla\zeta_q \star \nabla\zeta_q \star\right) \sum_{b=0}^{2s-1} \prod_{a=1}^{b} \circledast A \star \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)-1} \ \prod_{r=1}^{2s-b} \circledast \nabla^{k_r+1} A \right)$$

$$\left. + \nabla |\pi_{\mathbb{R}^n} X|^2 \star \prod_{r=1}^{2p} \circledast A + \sum_{q=1}^{n} \langle \nu, e_q\rangle \nabla\zeta_q \star \prod_{r=1}^{2p-1} \circledast A \right].$$

Once again, we note that since $2p+1 \leq l$, $\nabla^{l-2} A$ is the highest order of curvature derivative that can appear in this expression, and we may employ Corollary 6.2 and Lemma B.1 in conjunction with Young's inequality to estimate that, for any odd $i = 2p+1$ with $3 \leq i \leq l-2$,

$$\nabla^{2p+1} f_\Omega \star \left(A \star \nabla^{l-(2p+1)} A \star \nabla^l A + \sum_{j+k=l-(2p+3)} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right)$$

$$\leq \ |\Omega|\, K(2p+1,l,n) \left(\sum_{s=1}^{p} \sum_{b=0}^{2s-2} \|A\|^b \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)+1} \ \prod_{r=1}^{2s-1-b} \left\|\nabla^{k_r+1} A\right\| \right.$$

$$+\left(\sum_{s=1}^{p-1}\sum_{b=0}^{2s-1}\|A\|^b \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)} \prod_{r=1}^{2s-b}\left\|\nabla^{k_r+1}A\right\| + \|A\|^{2p}\right)$$

$$+\left(\sum_{s=1}^{p-1}\sum_{b=0}^{2s-2}\|A\|^b \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)} \prod_{r=1}^{2s-1-b}\left\|\nabla^{k_r+1}A\right\| + \|A\|^{2p-1}\right)$$

$$+\sum_{s=1}^{p-1}\sum_{b=0}^{2s-1}\|A\|^b \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)-1} \prod_{r=1}^{2s-b}\left\|\nabla^{k_r+1}A\right\|\Bigg)\Bigg(\|A\|\left\|\nabla^{l-(2p+1)}A\right\|\left\|\nabla^l A\right\|$$

$$+\sum_{j+k=l-(2p+3)}\left\|\nabla^{j+1}A\right\|\left\|\nabla^{k+1}A\right\|\left\|\nabla^l A\right\|\Bigg)$$

$$\leq \ |\Omega|\, K(2p+1,l,n)\left(\sum_{s=1}^{p}\sum_{b=0}^{2s-2}\|A\|_\infty^b \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)+1} \prod_{r=1}^{2s-1-b}\|A\|_\infty^{1-\frac{(k_r+1)}{l}}\left\|\nabla^l A\right\|_\infty^{\frac{k_r+1}{l}}\right.$$

$$+\|A\|_\infty^{2p}+\|A\|_\infty^{2p-1}+\sum_{s=1}^{p-1}\left(\sum_{b=0}^{2s-1}\|A\|_\infty^b \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)} \prod_{r=1}^{2s-b}\|A\|_\infty^{1-\frac{(k_r+1)}{l}}\left\|\nabla^l A\right\|_\infty^{\frac{k_r+1}{l}}\right.$$

$$+\sum_{b=0}^{2s-2}\|A\|_\infty^b \sum_{\sum_{r=1}^{2s-1-b}(k_r+1)=2(p-s)} \prod_{r=1}^{2s-1-b}\|A\|_\infty^{1-\frac{(k_r+1)}{l}}\left\|\nabla^l A\right\|_\infty^{\frac{k_r+1}{l}}$$

$$\left.\left.+\sum_{b=0}^{2s-1}\|A\|_\infty^b \sum_{\sum_{r=1}^{2s-b}(k_r+1)=2(p-s)-1} \prod_{r=1}^{2s-b}\|A\|_\infty^{1-\frac{(k_r+1)}{l}}\left\|\nabla^l A\right\|_\infty^{\frac{k_r+1}{l}}\right)\right)\cdot$$

$$\left(\|A\|_\infty^{1+\frac{2p+1}{l}}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2p+1)}{l}} + \sum_{j+k=l-(2p+3)}\|A\|_\infty^{2-\frac{(j+k+2)}{l}}\left\|\nabla^l A\right\|_\infty^{1+\frac{j+k+2}{l}}\right)$$

$$\leq \ |\Omega|\, K(2p+1,l,n)\left(\sum_{s=1}^{p}\sum_{b=0}^{2s-2}\|A\|_\infty^{\left(2s-1-\frac{(2(p-s)+1)}{l}\right)}\left\|\nabla^l A\right\|_\infty^{\frac{2(p-s)+1}{l}} + \|A\|_\infty^{2p} + \|A\|_\infty^{2p-1}\right.$$

$$+\sum_{s=1}^{p-1}\left(\sum_{b=0}^{2s-1}\left(\|A\|_\infty^{\left(2s-\frac{2(p-s)}{l}\right)}\left\|\nabla^l A\right\|_\infty^{\frac{2(p-s)}{l}} + \|A\|_\infty^{\left(2s-\frac{(2(p-s)-1)}{l}\right)}\left\|\nabla^l A\right\|_\infty^{\frac{(2(p-s)-1)}{l}}\right)\right.$$

$$\left.\left.+\sum_{b=0}^{2s-2}\|A\|_\infty^{\left(2s-1-\frac{2(p-s)}{l}\right)}\left\|\nabla^l A\right\|_\infty^{\frac{2(p-s)}{l}}\right)\right)\|A\|_\infty^{1+\frac{2p+1}{l}}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2p+1)}{l}}$$

$$\leq \ |\Omega|\, K\left(\sum_{s=1}^{p}(2s-1)\,\|A\|_\infty^{2s\left(1+\frac{1}{l}\right)}\left\|\nabla^l A\right\|_\infty^{2-\frac{2s}{l}} + \sum_{s=1}^{p-1}\left(2s\,\|A\|_\infty^{(2s+1)\left(1+\frac{1}{l}\right)}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2s+1)}{l}}\right.\right.$$

$$\left.+\,2s\,\|A\|_\infty^{\left(2(s+1)\left(1+\frac{1}{l}\right)-1\right)}\left\|\nabla^l A\right\|_\infty^{2-\frac{2(s+1)}{l}} + (2s-1)\,\|A\|_\infty^{\left((2s+1)\left(1+\frac{1}{l}\right)-1\right)}\left\|\nabla^l A\right\|_\infty^{2-\frac{(2s+1)}{l}}\right)$$

$$\left.+\left(\|A\|_\infty^{(2p+1)\left(1+\frac{1}{l}\right)} + \|A\|_\infty^{\left((2p+1)\left(1+\frac{1}{l}\right)-1\right)}\right)\left\|\nabla^l A\right\|_\infty^{2-\frac{(2p+1)}{l}}\right)$$

$$\leq \ |\Omega|\, K\left(\sum_{s=1}^{p}(2s-1)\left(\frac{s}{l}\,\|A\|_\infty^{2(l+1)} + \frac{(l-s)}{l}\left\|\nabla^l A\right\|_\infty^2\right) + \sum_{s=1}^{p-1}\left(2s\left(\frac{(2s+1)}{2l}\,\|A\|_\infty^{2(l+1)}\right.\right.\right.$$

$$\left.+\frac{(2l-(2s+1))}{2l}\left\|\nabla^l A\right\|_\infty^2 + \frac{(s+1)}{l}\,\|A\|_\infty^{\left(2(l+1)-\frac{l}{s+1}\right)} + \frac{(l-(s+1))}{l}\left\|\nabla^l A\right\|_\infty^2\right)$$

$$\left.+\,(2s-1)\left(\frac{(2s+1)}{2l}\,\|A\|_\infty^{\left(2(l+1)-\frac{2l}{2s+1}\right)} + \frac{(2l-(2s+1))}{2l}\left\|\nabla^l A\right\|_\infty^2\right)\right)$$

$$\left.+\,\frac{(2p+1)}{2l}\left(\|A\|_\infty^{2(l+1)} + \|A\|_\infty^{\left(2(l+1)-\frac{2l}{2p+1}\right)}\right) + \frac{(2l-(2p+1)}{l}\left\|\nabla^l A\right\|_\infty^2\right)$$

$$\leq \ |\Omega|\, K(2p+1,l,n)\left(1 + \|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2\right).$$

By a parallel computation in the even case, we discern that

$$\nabla^i f_\Omega \star \left(A \star \nabla^{l-i} A \star \nabla^l A + \sum_{j+k=l-2-i} \nabla^{j+1} A \star \nabla^{k+1} A \star \nabla^l A \right)$$

$$\leq \; |\Omega| \, K(l,n) \left(1 + \|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2 \right).$$

Furthermore, in analogy with the calculation of (6.23), we may bound the final term which appears in (6.24):

$$\sum_{i=0}^{1} \nabla^{l-i} f_\Omega \star A \star \nabla^i A \star \nabla^l A \;\leq\; |\Omega| \, K(l,n) \left(1 + \|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2 \right).$$

We may now collect the preceding results to discover that, for each $l \leq m$,

$$\sum_{i+j+k=l} \nabla^i f_\Omega \star \nabla^j A \star \nabla^k A \star \nabla^l A \;\leq\; |\Omega| \, K_5(l,n) \left(1 + \|A\|_\infty^{2(l+1)} + \left\|\nabla^l A\right\|_\infty^2 \right). \tag{6.25}$$

By recalling the constraints on Ω and ε determined by hypothesis, we may then substitute (6.19), (6.21)-(6.23), and (6.25) into the evolution equation (6.18) to obtain the inequality:

$$\begin{aligned}
\frac{\partial}{\partial t} \mathscr{Q}^m \;\leq\; & \Delta \mathscr{Q}^m + K_6(m,n) \left(\frac{1 + \|A\|_\infty^2}{d_m - \|\nabla^{m-1} A\|^2} \right) \Big(1 + \|A\|_\infty^{2(m+1)} + \|\nabla^m A\|_\infty^2 \\
& + \left(1 + \|A\|_\infty^{2m} + \left\|\nabla^{m-1} A\right\|_\infty^2 \right) \mathscr{Q}^m \Big) - 2 \left(\mathscr{Q}^m\right)^2 \\
& - \frac{2}{\left(d_m - \|\nabla^{m-1} A\|^2\right)} \left\langle \nabla \mathscr{Q}^m, \nabla \left\|\nabla^{m-1} A\right\|^2 \right\rangle \\
\leq \; & \Delta \mathscr{Q}^m + K_7(m,n,C_0,C_{m-1}) \left(\mathcal{A}^2 + \mathcal{A} \left(\mathscr{Q}^m + \sup_{M_t} \left(\mathscr{Q}^m\right) \right) \right) - 2 \left(\mathscr{Q}^m\right)^2 \\
& - \frac{2}{\left(d_m - \|\nabla^{m-1} A\|^2\right)} \left\langle \nabla \mathscr{Q}^m, \nabla \left\|\nabla^{m-1} A\right\|^2 \right\rangle .
\end{aligned} \tag{6.26}$$

Here $\mathcal{A} = \max\left\{1, \frac{\varepsilon}{\delta}\right\}$, where we have deduced from our inductive hypothesis (6.17) that

$$\frac{\|\nabla^{m-1} A\|_\infty^2}{\left(d_m - \|\nabla^{m-1} A\|^2\right)} \;\leq\; 1 \, ,$$

and, on $[\delta, 2\delta]$ for $k \in \mathbb{N} \cup \{0\}$,

$$\frac{\|A\|_\infty^{2(m+k)}}{\left(d_m - \|\nabla^{m-1}A\|^2\right)} \leq \frac{(\mathcal{C}_0\,\mathcal{A})^{m+k}}{\mathcal{C}_{m-1}\max\left\{1, \frac{1}{\delta^{m-1}}, \frac{\varepsilon}{\delta^m}\right\}} \leq K\left(\mathcal{C}_0, \mathcal{C}_{m-1}\right)\mathcal{A}^k .$$

In an argument completely analogous to that posited in the proof of Theorem 6.5, we may again cite [**45**, Lemma 3.5] to derive the following ODE inequality for solutions of (6.18) that satisfy (6.26):

$$\frac{d}{dt}\,\mathcal{Q}^m_{\max} \leq K(m,n)\left(\mathcal{A}^2 + \mathcal{A}\mathcal{Q}^m_{\max}\right) - 2\left(\mathcal{Q}^m_{\max}\right)^2 .$$

Suppose, by way of contradiction, there exists a $t_1 \in [\delta, 2\delta]$ such that $\mathcal{Q}^m_{\max}(t_1) > 2K\mathcal{A}$. Then

$$\frac{d}{dt}\,\mathcal{Q}^m_{\max}(t_1) < \left(\frac{1}{4K} + \frac{1}{2} - 2\right)\left(\mathcal{Q}^m_{\max}(t_1)\right)^2$$

$$< -\left(\mathcal{Q}^m_{\max}(t_1)\right)^2$$

whenever $K > \frac{1}{2}$, and $\mathcal{Q}^m_{\max}$ is decreasing at t_1. By reasoning which parallels that propounded in the proof of Theorem 6.5, we confirm that $\mathcal{Q}^m_{\max} > 2K\mathcal{A}$ on $[\delta, t_1]$. Hence

$$\frac{d}{dt}\,\mathcal{Q}^m_{\max} < -\left(\mathcal{Q}^m_{\max}\right)^2 \implies \left[-\frac{1}{\mathcal{Q}^m_{\max}(w)}\right]_{w=\delta}^{w=t} < -(t-\delta)$$

$$\implies \mathcal{Q}^m_{\max}(t) < \frac{1}{(t-\delta)}$$

on $[\delta, t_1]$, and we reach a contradiction whenever $(t_1 - \delta) \geq \frac{1}{2K\mathcal{A}}$. Thus

$$\mathcal{Q}^m_{\max} \leq \begin{cases} \frac{1}{(t-\delta)} & (t-\delta) < \frac{1}{2K\mathcal{A}} \,. \\ 2K\mathcal{A} & (t-\delta) \geq \frac{1}{2K\mathcal{A}} \,. \end{cases}$$

In particular, for $2\delta \in (0, T)$,

$$\mathcal{Q}^m_{\max}(2\delta) \leq 4K\max\left\{1, \frac{1}{2\delta}\right\},$$

from which we infer that

$$\sup_{M_{2\delta}} \|\nabla^m A\|^2 \leq 4K d_m \max\left\{1, \frac{1}{2\delta}\right\}$$

$$\leq 2^{m+3}K\mathcal{C}_{m-1}\max\left\{1, \frac{1}{(2\delta)^{m-1}}, \frac{\varepsilon}{(2\delta)^m}\right\} \cdot \max\left\{1, \frac{1}{2\delta}\right\}$$

$$\leq \; \mathcal{C}_m(n) \max \left\{ 1, \frac{1}{(2\delta)^m}, \frac{\varepsilon}{(2\delta)^{m+1}} \right\}.$$

This result verifies the theorem for each $l \leq m$. Thus the theorem holds for all $m \in \mathbb{N} \cup \{0\}$ by the principle of mathematical induction. $\diamondsuit$

COROLLARY 6.12. *Suppose ρ is a solution of (5.4) and $\varepsilon > 0$ is such that $\|\rho - 1\|_{C^{0,1}(S)} \leq \varepsilon$ on $(0, T)$. Then, for each $t \in (0, T)$ and $m \in \mathbb{N} \cup \{0\}$, the square of the tensorfield norm of $\nabla^{m+2}\rho$ (with respect to the metric on S) satisfies the following estimate:*

$$\sup_S \|\nabla^{m+2}\rho\|^2 \; \leq \; \mathcal{D}_m(n) \max \left\{ 1, \frac{1}{t^m}, \frac{\varepsilon}{t^{m+1}} \right\}$$

subject to the constraints on Ω and ε stipulated by Theorem 6.5.

PROOF

In analogy with the proof of Theorem 6.11, we proceed by total induction on $m \in \mathbb{N} \cup \{0\}$ and first prove the following statement with respect to the metric on M_t induced from S:

$$P(m) \quad : \quad \sup_{M_t} \|\nabla^{m+2}\rho\| \; \leq \; D_m(n) \max \left\{ 1, \left(\frac{1}{t^m}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}} \right\}$$

for each $t \in (0, T)$, where $P(0)$ was established in the proof of Corollary 6.6. Now suppose there exists a particular $m \in \mathbb{N}$ such that $P(l)$ holds for each $l < m$. We note that

$$\nabla^{m+2}\rho^2 \;=\; \sum_{i+j=m+2} \nabla^i \rho \star \nabla^j \rho$$

$$\Longrightarrow \quad \|\nabla^{m+2}\rho^2\| \;=\; \left\| \rho \nabla^{m+2}\rho + \sum_{i+j=m} \nabla^{i+1}\rho \star \nabla^{j+1}\rho \right\|$$

$$\geq \; K(m,n) \left(\rho \, \|\nabla^{m+2}\rho\| - \sum_{i+j=m} \|\nabla^{i+1}\rho\| \, \|\nabla^{j+1}\rho\| \right)$$

$$\Longrightarrow \quad \|\nabla^{m+2}\rho\| \; \leq \; \frac{1}{\rho} \left(\|\nabla^{m+2}\rho^2\| + \sum_{i+j=m} \|\nabla^{i+1}\rho\| \, \|\nabla^{j+1}\rho\| \right), \tag{6.27}$$

whenever $K \geq 1$. We utilise our inductive hypothesis, Corollary 6.2, Theorem 6.11, and Corollary A.4 in Appendix A to estimate the two terms on the right hand side of the inequality:

$$\sum_{i+j=m} \|\nabla^{i+1}\rho\| \, \|\nabla^{j+1}\rho\| \;=\; 2\,|\nabla\rho| \, \|\nabla^{m+1}\rho\| + \sum_{i+j=m-2} \|\nabla^{i+2}\rho\| \, \|\nabla^{j+2}\rho\|$$

$$
\begin{aligned}
&\leq 4\varepsilon D_{m-1}^{\frac{1}{2}} \max\left\{1, \left(\frac{1}{t^{m-1}}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^m}\right)^{\frac{1}{2}}\right\} \\
&\quad + \sum_{i+j=m-2} \left(D_i^{\frac{1}{2}} \max\left\{1, \left(\frac{1}{t^i}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{i+1}}\right)^{\frac{1}{2}}\right\}\right)\left(D_j^{\frac{1}{2}} \max\left\{1, \left(\frac{1}{t^j}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{j+1}}\right)^{\frac{1}{2}}\right\}\right) \\
&\leq K(m,n)\left(\max\left\{1, \left(\frac{1}{t^{m-1}}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^m}\right)^{\frac{1}{2}}\right\} + \sum_{i+j=m-2} \max\left\{1, \left(\frac{1}{t^{i+j+1}}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{i+j+2}}\right)^{\frac{1}{2}}\right\}\right) \\
&\leq K_1(m,n) \max\left\{1, \left(\frac{1}{t^{m-1}}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^m}\right)^{\frac{1}{2}}\right\}. \qquad (6.28)
\end{aligned}
$$

Furthermore, if m is even, then we may find $p \in \mathbb{N} \setminus \{1\}$ such that $m + 2 = 2p$. Now, by Corollary A.4,

$$
\begin{aligned}
\nabla^{m+2}\rho^2 &= \nabla^{2p}\rho^2 \\
&= \sum_{j=1}^{p} u \sum_{\sum_{i=1}^{2j-1} k_i = 2(p-j)} \prod_i^{2j-1} \circledast \nabla^{k_i} A + \sum_{j=1}^{p-1}\left(\nabla\rho^2 \star \sum_{\sum_{i=1}^{2j} k_i = 2(p-j)-1} \prod_i^{2j} \circledast \nabla^{k_i} A\right. \\
&\quad \left. + \sum_{\sum_{i=1}^{2j} k_i = 2(p-j)-2} \prod_i^{2j} \circledast \nabla^{k_i} A\right).
\end{aligned}
$$

Thus we may compute, with the assistance of Corollary 6.2, that

$$
\begin{aligned}
\|\nabla^{2p}\rho^2\| &\leq K(2p,n)\left(\sum_{j=1}^{p} \sum_{\sum_{i=1}^{2j-1} k_i = 2(p-j)} \prod_i^{2j-1} \|\nabla^{k_i} A\|\right. \\
&\quad \left. + \sum_{j=1}^{p-1}\left(\varepsilon \sum_{\sum_{i=1}^{2j} k_i = 2(p-j)-1} \prod_i^{2j} \|\nabla^{k_i} A\| + \sum_{\sum_{i=1}^{2j} k_i = 2(p-j)-2} \prod_i^{2j} \|\nabla^{k_i} A\|\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq K(2p,n)\left(\sum_{j=1}^{p}\sum_{\sum_{i=1}^{2j-1}k_i=2(p-j)}\prod_{i}^{2j-1}(\mathcal{C}_{k_i})^{\frac{1}{2}}\max\left\{1,\left(\frac{1}{t^{k_i}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{k_i+1}}\right)^{\frac{1}{2}}\right\}\right.\\
&+\sum_{j=1}^{p-1}\left(\sum_{\sum_{i=1}^{2j}k_i=2(p-j)-1}\prod_{i}^{2j}(\mathcal{C}_{k_i})^{\frac{1}{2}}\max\left\{1,\left(\frac{1}{t^{k_i}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{k_i+1}}\right)^{\frac{1}{2}}\right\}\right.\\
&\left.\left.+\sum_{\sum_{i=1}^{2j}k_i=2(p-j)-2}\prod_{i}^{2j}(\mathcal{C}_{k_i})^{\frac{1}{2}}\max\left\{1,\left(\frac{1}{t^{k_i}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{k_i+1}}\right)^{\frac{1}{2}}\right\}\right)\right)\\
&\leq K(2p,n)\left(\max\left\{1,\left(\frac{1}{t^{2p-2}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{2p-1}}\right)^{\frac{1}{2}}\right\}+\max\left\{1,\left(\frac{1}{t^{2p-3}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{2p-2}}\right)^{\frac{1}{2}}\right\}\right.\\
&\left.+\max\left\{1,\left(\frac{1}{t^{2p-4}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{2p-3}}\right)^{\frac{1}{2}}\right\}\right)\\
&\leq K(2p,n)\max\left\{1,\left(\frac{1}{t^{2p-2}}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{2p-1}}\right)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

In the case of odd m, we may derive an equivalent expression. Hence,

$$
\left\|\nabla^{m+2}\rho^2\right\| \leq K_2(m,n)\max\left\{1,\left(\frac{1}{t^m}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}}\right\}. \tag{6.29}
$$

By substituting (6.28) and (6.29) into (6.27), we deduce that, for each $t \in (0,T)$,

$$
\sup_{M_t}\left\|\nabla^{m+2}\rho\right\| \leq D_m(n)\max\left\{1,\left(\frac{1}{t^m}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}}\right\}.
$$

Thus $P(l)$ is true for each $l \leq m$ and $P(m)$ holds for each $m \in \mathbb{N}\cup\{0\}$ by the principle of mathematical induction.

To verify the corollary, we again proceed by total induction on $m \in \mathbb{N}\cup\{0\}$ and prove the equivalent statement:

$$
Q(m) \quad : \quad \sup_{S}\|\nabla^{m+2}\rho\| \leq \mathcal{D}_m^{\frac{1}{2}}(n)\max\left\{1,\left(\frac{1}{t^m}\right)^{\frac{1}{2}},\left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}}\right\}
$$

for each $t \in (0,T)$, where $Q(0)$ is established by Corollary 6.6. Now suppose there exists a particular $m \in \mathbb{N}$ such that $Q(l)$ holds for each $l < m$. Throughout the following exposition, we shall employ the notation contained in Section 2 of Chapter 1. In addition, we shall distinguish by subscripts both covariant differentiation and the computation of tensorfield norms with respect to either the metric on S or M_t. We may then exploit the interdependence of the Riemannian geometries on S and M_t implied by Lemmas 1.4 and 5.3 to quantify the relationship between the respective covariant derivatives of ρ:

$$\nabla_S^{m+2}\rho = \nabla_{M_t}^{m+1}\left(\nabla^S\rho\right) + \sum_{p=1}^{m}\sum_{k=1}^{p}\left(\sum_{\sum_{j=1}^{k} q_j = p-k}\prod_{j=1}^{k}\circledast \nabla_S^{q_j}\left(\mathbf{\Gamma}(\rho) - \mathbf{\Gamma}(s)\right)\right)\star\nabla_S^{(m+2)-p}\rho\,.$$

Therefore, by the triangle inequality,

$$\left\|\nabla_S^{m+2}\rho\right\|_S \le B_1(m,n)\left(\sum_{p=1}^{m}\sum_{k=1}^{p}\sum_{\sum_{j=1}^{k} q_j = p-k}\prod_{j=1}^{k}\left\|\nabla_S^{q_j}\left(\mathbf{\Gamma}(\rho) - \mathbf{\Gamma}(s)\right)\right\|_S\left\|\nabla_S^{(m+2)-p}\rho\right\|_S\right.$$

$$\left. + \left\|\nabla_{M_t}^{m+1}\left(\nabla^S\rho\right)\right\|_S\right). \tag{6.30}$$

We further deduce from Lemma 1.4 that

$$\nabla^S\rho = \rho\left(\nabla^{M_t}\rho - \frac{\left|\nabla^{M_t}\rho\right|^2}{\sqrt{1 - |\nabla^{M_t}\rho|^2}}\nu(\rho)\right)$$

and

$$\mathbf{\Gamma}(\rho) - \mathbf{\Gamma}(s) = \left(\frac{\nabla^S\rho}{\rho\left(\rho^2 + |\nabla^S\rho|^2\right)}\right)\star\left(\rho\nabla_S^2\rho + \mathbf{g}(s)\star\left(\rho^2 + \nabla^S\rho\star\nabla^S\rho\right)\right).$$

Hence

$$\nabla_{M_t}^{m+1}\left(\nabla^S\rho\right) = \sum_{i+j+k=m+1}\nabla_{M_t}^{i}\rho\star\nabla_{M_t}^{j}\left(\frac{\left|\nabla^{M_t}\rho\right|^2}{\sqrt{1 - |\nabla^{M_t}\rho|^2}}\right)\star\nabla_{M_t}^{k}\nu(\rho)$$

$$+ \sum_{i+j=m+1}\nabla_{M_t}^{i}\rho\star\nabla_{M_t}^{j+1}\rho\,, \tag{6.31}$$

and, for each $k \in \{0, ..., m-1\}$,

$$\nabla_S^k\left(\mathbf{\Gamma}(\rho) - \mathbf{\Gamma}(s)\right) = \sum_{i+j=k}\nabla_S^i\left(\frac{1}{\rho\left(\rho^2 + |\nabla^S\rho|^2\right)}\right)\star\left(\sum_{p+q+r=j+1}\nabla_S^p\rho\star\left(\nabla_S^q\rho\star\nabla_S^r\rho\right.\right.$$

$$\left.\left. + \nabla_S^{q+1}\rho\star\nabla_S^{r+1}\rho\right)\right). \tag{6.32}$$

Now, for each $j \in \{0, ..., m+1\}$,

$$\nabla^j_{M_t}\left(\frac{|\nabla^{M_t}\rho|^2}{\sqrt{1 - |\nabla^{M_t}\rho|^2}}\right) = \sum_{p+q+r=j} \nabla^{p+1}_{M_t}\rho \star \nabla^{q+1}_{M_t}\rho \star \nabla^r_{M_t}\left(\left(1 - |\nabla^{M_t}\rho|^2\right)^{-\frac{1}{2}}\right), \quad (6.33)$$

where, for each $r \in \{1, ..., m+1\}$,

$$\nabla^r_{M_t}\left(\left(1 - |\nabla^{M_t}\rho|^2\right)^{-\frac{1}{2}}\right)$$

$$= \sum_{q=1}^{r} \left(1 - |\nabla^{M_t}\rho|^2\right)^{-\frac{(2q+1)}{2}} \sum_{\sum_{p=1}^{q} k_p = r-q} \prod_{p=1}^{q} \circledast \left(\sum_{a+b=k_p+1} \nabla^{a+1}_{M_t}\rho \star \nabla^{b+1}_{M_t}\rho\right). \quad (6.34)$$

Similarly, for each $i \in \{1, ..., m-1\}$,

$$\nabla^i_S\left(\left(\rho\left(\rho^2 + |\nabla^S\rho|^2\right)\right)^{-1}\right)$$

$$= \sum_{q=1}^{i} \left(\rho\left(\rho^2 + |\nabla^S\rho|^2\right)\right)^{-(q+1)} \sum_{\sum_{p=1}^{q} k_p = i-q} \prod_{p=1}^{q} \circledast \left(\sum_{a+b+c=k_p+1} \nabla^a_S\rho \star \left(\nabla^b_S\rho \star \nabla^c_S\rho\right.\right.$$

$$\left.\left. + \nabla^{b+1}_S\rho \star \nabla^{c+1}_S\rho\right)\right). \quad (6.35)$$

For $k \in \{1, ..., m+1\}$, we now characterise the terms $\nabla^k_{M_t}\nu(\rho)$ occurring in (6.31). In the odd case there exists $p \in \mathbb{N}$ such that $k = 2p-1$, and we may invoke Lemma A.5 exposed in Appendix A to compute that

$$\nabla^k_{M_t}\nu = \nabla^{2p-1}_{M_t}\nu = \sum_{j=1}^{p} \sum_{\sum_{i=1}^{2j-1} q_i = 2(p-j)} \prod_{i=1}^{2j-1} \circledast \nabla^{q_i}_{M_t} A$$

$$+ \sum_{j=1}^{p-1} \sum_{\sum_{i=1}^{2j} q_i = 2(p-j)-1} \prod_{i=1}^{2j-1} \circledast \nabla^{q_i}_{M_t} A \star \nu\,. \quad (6.36)$$

Similarly, in the even case there exists $p \in \mathbb{N}$ with $k = 2p$ and we discern that

$$\nabla^k_{M_t}\nu \;=\; \nabla^{2p}_{M_t}\nu \;=\; \sum_{j=1}^{p}\left(\sum_{\sum_{i=1}^{2j-1} q_i = 2(p-j)+1}\ \prod_{i=1}^{2j-1} \circledast \nabla^{q_i}_{M_t} A\right.$$

$$\left. + \sum_{\sum_{i=1}^{2j} q_i = 2(p-j)}\ \prod_{i=1}^{2j-1} \circledast \nabla^{q_i}_{M_t} A \star \nu\right). \tag{6.37}$$

We observe from (6.30), (6.32), and (6.35) that every term arising in the expression $\|\nabla^{q_j}_S(\mathbf{\Gamma}(\rho) - \mathbf{\Gamma}(s))\|_S \|\nabla^{(m+2)-p}_S \rho\|_S$ is bounded by either the hypothesis of our corollary or that of our induction. In a manner analogous to calculations already performed in both this proof and that of Theorem 6.11, we may employ (6.32), (6.35), and our two hypotheses to deduce that

$$\sum_{p=1}^{m}\sum_{k=1}^{p}\ \sum_{\sum_{j=1}^{k} q_j = p-k}\ \prod_{j=1}^{k} \left\|\nabla^{q_j}_S(\mathbf{\Gamma}(\rho) - \mathbf{\Gamma}(s))\right\|_S \|\nabla^{(m+2)-p}_S \rho\|_S$$

$$\leq\ B_2(m,n)\max\left\{1, \left(\frac{1}{t^m}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}}\right\}. \tag{6.38}$$

We recall a calculation performed in the proof of Theorem 3.3 to assert that, with respect to the metric on S, the eigenvalues of the matrix $(g^{ij}(\rho))$ are given by

$$\kappa_m \;=\; \begin{cases} \frac{1}{\rho^2 + |\nabla^S \rho|^2} & m = 1\ ;\ \text{and} \\ \frac{1}{\rho^2} & 2 \leq m \leq n\,. \end{cases}$$

We may therefore cite (6.31), (6.33), (6.34), (6.36), and (6.37) in conjunction with Theorem 6.11 and the veracity of the statement $P(m)$ to estimate the last expression appearing in (6.30). The analysis again proceeds in parallel with that already undertaken in both this proof and that of the theorem:

$$\left\|\nabla^{m+1}_{M_t}(\nabla^S \rho)\right\|_S \;\leq\; \kappa_1^{-\left(\frac{m+2}{2}\right)} \left\|\nabla^{m+1}_{M_t}(\nabla^S \rho)\right\|_{M_t}$$

$$\leq\ B_3(m,n)\max\left\{1, \left(\frac{1}{t^m}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}}\right\}. \tag{6.39}$$

We may then substitute (6.38) and (6.39) into (6.30) to confirm that, for each $t \in (0, T)$,

$$\sup_S \|\nabla^{m+2}\rho\| \;\leq\; \mathcal{D}_m^{\frac{1}{2}}(n) \max\left\{1, \left(\frac{1}{t^m}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{t^{m+1}}\right)^{\frac{1}{2}}\right\}.$$

Thus $Q(m)$ is verified for each $l \leq m$ and we establish $Q(m)$ for each $m \in \mathbb{N} \cup \{0\}$ by the principle of mathematical induction. This implies the result of the corollary. $\diamondsuit$

CHAPTER 7

Asymptotic Convergence to Minimisers

In this chapter we shall prove that solutions ρ of (5.4) converge asymptotically to corresponding solutions ϱ of the Euler-Lagrange equation (2.4) determined by Theorem 3.1. Much of the analysis shall be premised on the assumption that a given initial condition $\rho(\cdot, 0) \in \mathscr{R}$ remains appropriately close to ϱ in the Lipschitz topology on S. As a consequence of the diffeomorphism given explicitly by equation (3.1) in the proof of Theorem 3.1, such a choice is clearly possible and we may further restrict this Lipschitz neighbourhood arbitrarily.

1. Short time preservation of Lipschitz proximity to S

Since the results of Chapter 6 were based upon the assumption that M_t remains arbitrarily close to S, we must now verify that there exists a short time interval for which this is actually the case.

PROPOSITION 7.1. *Suppose ρ is a solution of (5.4) and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation (2.4). Further suppose that strictly positive ε_1 and ε_2 are such that $\|\rho(\cdot\,, 0) - \varrho\|_{C^{0,1}(S)} \leq \frac{\varepsilon_1}{2\sqrt{2}}$ and $\|\varrho - 1\|_{C^{3,\alpha}(S)} \leq \varepsilon_2$. Then there exists a constant $C(n)$ such that, for each $t \in (0, C^{-1})$,*

$$\|\rho - \varrho\|_{C^{0,1}(S)} \leq \varepsilon_1$$

whenever Ω and $\varepsilon = \varepsilon_1 + \varepsilon_2$ satisfy the constraints prescribed by Theorem 6.5.

PROOF

Throughout the proof, all geometric quantities shall be computed with respect to the metric on S. By hypothesis and the continuity in time of ρ, there exists a first time $\tilde{t} \in (0, T)$ for which $\|\rho(\cdot\,, \tilde{t}) - \varrho\|_{C^{0,1}(S)} > \varepsilon_1$, otherwise the proposition holds trivially. Therefore, on $(0, \tilde{t})$,

$$\|\rho - \varrho\|_{C^{0,1}(S)} \leq \varepsilon_1 \,. \tag{7.1}$$

Furthermore, we deduce from the triangle inequality and hypothesis that on $(0, \tilde{t})$,

$$\|\rho - 1\|_{C^{0,1}(S)} \leq \varepsilon \,. \tag{7.2}$$

Therefore we infer from Corollary 6.6 that, for each $t \in (0, \tilde{t})$,

$$\sup_S \|\nabla^2 \rho\|^2 \leq \mathcal{D}_0(n) \left\{1, \frac{\varepsilon}{t}\right\} . \tag{7.3}$$

We now consider the evolution of the two functions $v = (\rho - \varrho)^2$ and $w = \left|\nabla^S (\rho - \varrho)\right|^2$. In analogy with the procedure pursued in the proof of Theorem 5.5, we recollect the uniformly elliptic operator Q_r given by (5.6) and set $\rho_\lambda = \lambda\rho + (1-\lambda)\varrho$ for $\lambda \in [0,1]$. By Theorems 3.3 and 5.5, it is evident that $\rho_\lambda \in C^\infty(S \times (0,T))$ and, in particular, $\rho_\lambda \in \mathcal{R}^k$ for each $k \in \mathbb{N}$ and $t \in (0,T)$. Moreover, we infer from our hypothesis conditions and the triangle inequality that, for each $t \in (0,\tilde{t})$ and $\lambda \in [0,1]$,

$$\|\rho_\lambda - 1\|_{C^{0,1}(S)} \le \varepsilon\,. \tag{7.4}$$

For arbitrary $r \in \mathcal{R}^2$, we again note that each coefficient of Q_Ω is at least C^1 in its argument $(s,z,\tau) \in S \times \mathbb{R}^+ \times TS$. By calculations mirroring (5.7) in the proof of Theorem 5.5, we therefore deduce that

$$\frac{\partial v}{\partial t} = 2(\rho-\varrho)\int_0^1 \frac{d}{d\lambda}\left(Q_\Omega\left[\rho_\lambda\right]\right)d\lambda \tag{7.5}$$

and

$$\frac{\partial w}{\partial t} = 2\left\langle \nabla(\rho-\varrho), \nabla\left(\int_0^1 \frac{d}{d\lambda}\left(Q_\Omega\left[\rho_\lambda\right]\right)d\lambda\right)\right\rangle. \tag{7.6}$$

Now, for each $i \in \mathbb{N}$ with $1 \le i \le n$ and $r \in \mathcal{R}^3$, we compute that

$$\nabla_i\left(Q_\Omega[r]\right) = \left(\nabla_i Q_r\right)[r] + Q_r\left[\nabla_i r\right] + a^{kl}(s,r,\nabla r)\left(g_{il}\nabla_k r - g_{kl}\nabla_i r\right). \tag{7.7}$$

Here we interpret $(\nabla_i Q_r)[\cdot]$ as the operator whose coefficients are defined by the first-order covariant derivatives of those corresponding to Q_r, where we have interchanged third-order covariant derivatives of r attached to the coefficients a^{kl}. In particular, we observe that, for any $\eta \in C^3(S;\mathbb{R})$,

$$\left(\nabla_i\nabla_k\nabla_l - \nabla_k\nabla_l\nabla_i\right)\eta = \left(h_{il}h_k{}^m - h_{kl}h_i{}^m\right)\nabla_m\eta = g_{il}\nabla_k\eta - g_{kl}\nabla_i\eta\,,$$

since the metric and the second fundamental form coincide on S. If we now take $\mathbf{r}_\lambda = (s,\rho_\lambda,\nabla\rho_\lambda)$, then

$$\frac{d}{d\lambda}\mathbf{r}_\lambda = \mathbf{r}'_\lambda = \left(0,(\rho-\varrho),\nabla(\rho-\varrho)\right).$$

Hence we infer from the quadratic behaviour of the operator Q_r (as articulated by equation (5.8) in the proof of Theorem 5.5), (7.5), (7.6), and (7.7) that

$$\begin{aligned}\frac{\partial v}{\partial t} = \int_0^1 &\Big(Q_{\rho_\lambda}[v] + c_\Omega(\mathbf{r}_\lambda)\,v - 2a^{ij}(\mathbf{r}_\lambda)\nabla_i(\rho-\varrho)\nabla_j(\rho-\varrho) \\ &+ 2(\rho-\varrho)\Big(\left\langle Da^{ij}\big|_{\mathbf{r}_\lambda},\mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\nabla_j\rho_\lambda + \left\langle Db^i\big|_{\mathbf{r}_\lambda},\mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\rho_\lambda \\ &+ \left\langle Dc_\Omega\big|_{\mathbf{r}_\lambda},\mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\rho_\lambda\Big)\Big)\,d\lambda\,,\end{aligned} \tag{7.8}$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial t} \;=\; & \int_0^1 \Big(Q_{\rho_\lambda}[w] + c_\Omega(\mathbf{r}_\lambda)\, w + 2\left(g^{kl}\nabla_k(\rho-\varrho)\left(\nabla_l Q_{\rho_\lambda}\right)[\rho-\varrho]\right.\\
& + a^{ij}(\mathbf{r}_\lambda)\big(\nabla_i(\rho-\varrho)\nabla_j(\rho-\varrho) - g^{kl}\nabla_i\nabla_k(\rho-\varrho)\nabla_j\nabla_l(\rho-\varrho)\\
& - g_{ij}\left|\nabla(\rho-\varrho)\right|^2\big) + g^{kl}\nabla_k(\rho-\varrho)\nabla_l\Big(\left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\nabla_j\rho_\lambda\\
& + \left\langle Db^{i}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\rho_\lambda + \left\langle Dc_\Omega|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\rho_\lambda\Big)\Big)\Big)\, d\lambda\,. \qquad (7.9)
\end{aligned}
$$

Observe that we may also write $\rho_\lambda = \lambda(\rho-\varrho)+\varrho$. Thus, by extracting the relevant terms from the integrands on the right hand sides of (7.8) and (7.9) whilst again invoking (5.8) and interchanging third-order covariant derivatives, we deduce from the fundamental theorem of calculus that

$$
\begin{aligned}
\frac{\partial v}{\partial t} \;=\; & Q_\rho[v] + c_\Omega(\mathbf{r}_1)\, v - 2a^{ij}(\mathbf{r}_1)\nabla_i(\rho-\varrho)\nabla_j(\rho-\varrho)\\
& + 2(\rho-\varrho)\int_0^1 \Big(\left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\nabla_j\varrho + \left\langle Db^{i}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\varrho\\
& + \left\langle Dc_\Omega|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\varrho\Big)\, d\lambda\,, \qquad (7.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial t} \;=\; & Q_\rho[w] + c_\Omega(\mathbf{r}_1)\, w + 2\left(g^{kl}\nabla_k(\rho-\varrho)\left(\nabla_l Q_\rho\right)[\rho-\varrho]\right.\\
& + a^{ij}(\mathbf{r}_1)\big(\nabla_i(\rho-\varrho)\nabla_j(\rho-\varrho) - g^{kl}\nabla_i\nabla_k(\rho-\varrho)\nabla_j\nabla_l(\rho-\varrho)\\
& - g_{ij}\left|\nabla(\rho-\varrho)\right|^2\big) + g^{kl}\nabla_k(\rho-\varrho)\int_0^1 \Big(\nabla_l\Big(\left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\nabla_j\varrho\\
& \left. + \left\langle Db^{i}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\nabla_i\varrho + \left\langle Dc_\Omega|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}}\varrho\Big)\Big)\, d\lambda\right). \qquad (7.11)
\end{aligned}
$$

We now estimate terms arising in these evolution equations. Clearly

$$
\left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\right\rangle_{\mathcal{V}} \;=\; \left((\rho-\varrho)\, D_z + \nabla_m(\rho-\varrho)\, D_{\tau m}\right) a^{ij}|_{\mathbf{r}_\lambda}\,,
$$

while, for each $l \in \{1, ..., n\}$,

$$\nabla_l \left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda \right\rangle_{\mathscr{V}} = \left(\nabla_l (\rho - \varrho) D_z + (\rho - \varrho) \nabla_l D_z + \nabla_l \nabla_m (\rho - \varrho) D_{\tau m} \right.$$
$$\left. + \nabla_m (\rho - \varrho) \nabla_l D_{\tau m}\right) a^{ij}|_{\mathbf{r}_\lambda} .$$

We recall the characterisation of the coefficients a^{ij} contained in Definition 5.1 to observe that their various mixed derivatives appearing in the expressions above contain at worst linear factors in second-order covariant derivatives of the intermediate graph ρ_λ. Hence we may employ our hypothesis condition on ϱ, (7.1), (7.3), and (7.4) to discern that, on $(0, \tilde{t})$,

$$2 (\rho - \varrho) \left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda \right\rangle_{\mathscr{V}} \nabla_i \nabla_j \varrho = \left(2v D_z + \nabla_m v D_{\tau m}\right) a^{ij}|_{\mathbf{r}_\lambda} \nabla_i \nabla_j \varrho$$
$$\leq \nabla_m v \left(D_{\tau m} a^{ij}\right)|_{\mathbf{r}_\lambda} \nabla_i \nabla_j \varrho + \varepsilon_1^2 \varepsilon_2 K_1(n) , \tag{7.12}$$

and

$$2g^{kl} \nabla_k (\rho - \varrho) \nabla_l \left(\left\langle Da^{ij}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda \right\rangle_{\mathscr{V}} \nabla_i \nabla_j \varrho\right)$$
$$= \left(2w D_z + \nabla_m w D_{\tau m} + 2g^{kl} \nabla_k (\rho - \varrho) \left((\rho - \varrho) \nabla_l D_z\right.\right.$$
$$\left.\left. + \nabla_m (\rho - \varrho) \nabla_l D_{\tau m}\right)\right) a^{ij}|_{\mathbf{r}_\lambda} \nabla_i \nabla_j \varrho + 2g^{kl} \nabla_k (\rho - \varrho) \left((\rho - \varrho) D_z\right.$$
$$\left. + \nabla_m (\rho - \varrho) D_{\tau m}\right) a^{ij}|_{\mathbf{r}_\lambda} \nabla_l \nabla_i \nabla_j \varrho$$
$$\leq \nabla_m w \left(D_{\tau m} a^{ij}\right)|_{\mathbf{r}_\lambda} \nabla_i \nabla_j \varrho + \varepsilon_1^2 \varepsilon_2 K_2(n) \left(1 + \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right) . \tag{7.13}$$

Similarly,

$$2 (\rho - \varrho) \left\langle Db^i|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda \right\rangle_{\mathscr{V}} \nabla_i \varrho \leq \nabla_m v \left(D_{\tau m} b^i\right)|_{\mathbf{r}_\lambda} \nabla_i \varrho + \varepsilon_1^2 \varepsilon_2 K_3(n) , \tag{7.14}$$

and

$$2g^{kl} \nabla_k (\rho - \varrho) \nabla_l \left(\left\langle Db^i|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda \right\rangle_{\mathscr{V}} \nabla_i \varrho\right)$$
$$\leq \nabla_m w \left(D_{\tau m} b^i\right)|_{\mathbf{r}_\lambda} \nabla_i \varrho + \varepsilon_1^2 \varepsilon_2 K_4(n) \left(1 + \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right) . \tag{7.15}$$

Once again, in our consideration of the derivatives of the coefficient c_Ω, we must distinguish our treatment of the global terms $\{h^i\}$. In a computation identical to equation (5.10) from the proof of Theorem 5.5, we ascertain that

$$\langle Dh^i|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}} = -M^{ik}(\rho_\lambda)M^{jl}(\rho_\lambda)\left(\int_S \langle D\psi_{kl}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}}\, d\sigma\right)\left(\int_S (\psi_\Omega)_j|_{(\mathbf{r}_\lambda, \nabla^2\rho_\lambda)} d\sigma\right)$$

$$+ M^{ij}(\rho_\lambda)\int_S \left\langle D\left(\psi_\Omega\right)_j\Big|_{(\mathbf{r}_\lambda,\nabla^2\rho_\lambda)}, \left(\mathbf{r}'_\lambda, \nabla^2(\rho - \varrho)\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma\,. \quad (7.16)$$

In analogy with the calculation of (5.11) and (5.12), we assert that, for each $j, k, l \in \{0, ..., n+1\}$,

$$\int_S \langle D\psi_{kl}|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}}\, d\sigma = \int_S \left((\rho - \varrho)\, D_z + \nabla_i(\rho - \varrho)\, D_{\tau i}\right)\psi_{kl}|_{\mathbf{r}_\lambda} d\sigma\,, \quad (7.17)$$

and, integrating by parts,

$$\int_S \left\langle D\left(\psi_\Omega\right)_j\Big|_{(\mathbf{r}_\lambda,\nabla^2\rho_\lambda)}, \left(\mathbf{r}'_\lambda, \nabla^2(\rho - \varrho)\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma$$

$$= \int_S \left((\rho - \varrho)\, D_z + \nabla^i(\rho - \varrho)\left(g_{im}D_{\tau m} - \nabla^m D_{\vartheta_{im}}\right)\right)(\psi_\Omega)_j|_{(\mathbf{r}_\lambda,\nabla^2\rho_\lambda)} d\sigma\,. \quad (7.18)$$

We note that the result of Lemma 4.6 remains valid in the context of the weaker condition (7.4), and therefore obtain the following estimate on the components of the matrix $(M^{ij}(\rho_\lambda))$:

$$\left|M^{ij}(\rho_\lambda) - M^{ij}(1)\right| \leq \varepsilon K^{ij}(n)|S|^{-1}\,. \quad (7.19)$$

Since the function $(\psi_\Omega)_j$ is linear in the argument $\vartheta \in \mathscr{S}_2^0(S)$ for each $j \in \{0, ..., n+1\}$, we deduce that the integrand occurring in (7.18) again contains at worst linear factors in second-order covariant derivatives of ρ_λ. Moreover, the global terms $\{h^i\}$ are spatially invariant, and we may evaluate the relevant derivatives of the remaining terms arising in the coefficient c_Ω in analogy with those appearing in a^{ij} and b^i. Thus we may collect (7.16)-(7.19) and invoke our hypothesis conditions on ϱ and Ω in conjunction with (7.1), (7.3), (7.4), and Definition 5.1 to demonstrate that

$$2(\rho - \varrho)\langle Dc_\Omega|_{\mathbf{r}_\lambda}, \mathbf{r}'_\lambda\rangle_{\mathscr{V}}\,\varrho \leq \nabla_m v\left(D_{\tau m}c_\Omega - \frac{h^k(z)\zeta_k(s,z)}{z^2}D_{\tau m}\sqrt{z^2 + |\tau|^2}\right)\Bigg|_{\mathbf{r}_\lambda}\varrho$$

$$+ \varepsilon_1^2 K_5(n)\left(1 + \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right), \quad (7.20)$$

and

$$2g^{kl}\nabla_k(\rho-\varrho)\nabla_l\left(\langle Dc_\Omega|_{\mathbf{r}_\lambda},\mathbf{r}'_\lambda\rangle_{\gamma'}\varrho\right)$$

$$\leq \nabla_m w\left(D_{\tau m}c_\Omega - \frac{h^k(z)\zeta_k(s,z)}{z^2}D_{\tau m}\sqrt{z^2+|\tau|^2}\right)\Big|_{\mathbf{r}_\lambda}\varrho + \varepsilon_1^2 K_6(n)\left(1+\left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right). \tag{7.21}$$

We now consider the term involving first-order spatial covariant derivatives of the coefficients $a^{ij}(\mathbf{r}_1)$ which occurs in (7.11). For arbitrary $r \in \mathcal{R}^2$ and $l \in \{1,..,n\}$, we discern from Definition 5.1 that

$$\begin{aligned}\nabla_l a^{ij} &= \nabla_l\left(\frac{1}{r^2}\left(g^{ij}-\frac{\nabla^i r\nabla^j r}{r^2+|\nabla r|^2}\right)\right)\\ &= -\left(\frac{2a^{ij}}{r}\right)\nabla_l r + \frac{1}{r}\left(b^j\nabla^i\nabla_l r + b^i\nabla^j\nabla_l r\right) + 2b^ib^j\left(r\nabla_l r + \nabla^m r\nabla_m\nabla_l r\right).\end{aligned}$$

Therefore, where appropriate, we may add and subtract second-order covariant derivatives of the stationary solution ϱ whilst utilising (7.2), Definition 5.1, and Lemma 6.1 to determine that

$$\begin{aligned}&2g^{kl}\nabla_k(\rho-\varrho)\nabla_l\left(a^{ij}(\mathbf{r}_1)\right)\nabla_i\nabla_j(\rho-\varrho)\\ &= \left(b^i(\mathbf{r}_1)\left(\frac{g^{jk}}{\rho}+b^j(\mathbf{r}_1)\nabla^k\rho\right)\nabla_k w + 2\nabla^k(\rho-\varrho)\left(\frac{1}{\rho}\left(b^i(\mathbf{r}_1)\nabla^i\nabla_k\varrho - a^{ij}(\mathbf{r}_1)\nabla_k\rho\right)\right.\right.\\ &\quad\left.\left. + b^i(\mathbf{r}_1)b^j(\mathbf{r}_1)\left(\rho\nabla_k\rho + \nabla^l\rho\nabla_l\nabla_k\varrho\right)\right)\right)\nabla_i\nabla_j(\rho-\varrho)\\ &\leq \phi^i\left(s,\rho,\nabla\rho,\nabla^2\rho,\nabla^2\varrho\right)\nabla_i w + \varepsilon K_7(n)\left|\nabla(\rho-\varrho)\right|\left\|\left(\nabla_i\nabla_j(\rho-\varrho)\right)\right\|\\ &\leq \phi^i\nabla_i w + \varepsilon K_7\left(\frac{w}{4\delta}+\delta\left\|\left(\nabla_i\nabla_j(\rho-\varrho)\right)\right\|^2\right).\end{aligned} \tag{7.22}$$

Here the coefficients $\phi^i \in C^\infty(S\times(0,T))$ for each $i \in \{1,..,n\}$, and we have applied the Peter-Paul inequality with $\delta > 0$. Now, since $(g^{ij}(\rho)) \equiv (a^{ij}(\mathbf{r}_1))$, we recollect a calculation contained in the proof of Corollary 6.12 to assert that the eigenvalues of the matrix $(a^{ij}(\mathbf{r}_1))$ are given by

$$\kappa_m = \begin{cases}\frac{1}{\rho^2+|\nabla\rho|^2} & m=1\text{ ; and}\\ \frac{1}{\rho^2} & 2\leq m\leq n\,.\end{cases}$$

Hence, where we recall from our hypothesis that $\varepsilon < \frac{1}{3}$,

$$
\begin{aligned}
2a^{ij}(\mathbf{r}_1)\, g^{kl}\nabla_i\nabla_k(\rho-\varrho)\,\nabla_j\nabla_l(\rho-\varrho) &\geq 2\kappa_1 \left\|(\nabla_i\nabla_j(\rho-\varrho))\right\|^2 \\
&\geq \frac{2}{((1+\varepsilon)^2+\varepsilon^2)} \left\|(\nabla_i\nabla_j(\rho-\varrho))\right\|^2 \\
&\geq \left\|(\nabla_i\nabla_j(\rho-\varrho))\right\|^2 .
\end{aligned}
$$

Thus we may choose $\delta = (\varepsilon K_7)^{-1}$ to deduce from (7.1) and (7.22) that

$$
\begin{aligned}
&2\left(g^{kl}\nabla_k(\rho-\varrho)\,\nabla_l\left(a^{ij}(\mathbf{r}_1)\right)\nabla_i\nabla_j(\rho-\varrho) - a^{ij}(\mathbf{r}_1)\, g^{kl}\nabla_i\nabla_k(\rho-\varrho)\,\nabla_j\nabla_l(\rho-\varrho)\right) \\
&\leq \phi^i\nabla_i w + \varepsilon_1^2 K_8(n) . \qquad (7.23)
\end{aligned}
$$

To conclude our analysis of the expressions occurring in equations (7.10) and (7.11), we note that the first-order spatial covariant derivatives of the coefficients $b^i(\mathbf{r}_1)$ and $c_\Omega(\mathbf{r}_1)$ contain at worst linear factors in second-order covariant derivatives of ρ, which may be bounded by (7.3). In addition, we may employ Proposition 6.4, Theorem 6.5, (7.2), and our hypothesis condition on Ω to bound both the parameter and the global terms arising in the gradient of c_Ω. We therefore substitute the results of (7.12)-(7.15), (7.20), (7.21), and (7.23) into the appropriate evolution equations (7.10) and (7.11) to derive the following differential inequalities on $(0,\tilde{t})$:

$$
\begin{aligned}
\frac{\partial v}{\partial t} \leq\; & a^{ij}(\mathbf{r}_1)\,\nabla_i\nabla_j v + \beta^i_\Omega\left(s,\rho,\varrho,\nabla\rho,\nabla\varrho,\nabla^2\varrho\right)\nabla_i v \\
& + \varepsilon_1^2 K_9(n)\left(1+\left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right), \qquad (7.24)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial t} \leq\; & a^{ij}(\mathbf{r}_1)\,\nabla_i\nabla_j w + \gamma^i_\Omega\left(s,\rho,\varrho,\nabla\rho,\nabla\varrho,\nabla^2\rho,\nabla^2\varrho\right)\nabla_i w \\
& + \varepsilon_1^2 K_{10}(n)\left(1+\left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right). \qquad (7.25)
\end{aligned}
$$

Here the coefficients $\beta^i_\Omega, \gamma^i_\Omega \in C^\infty(S\times(0,T))$ for each $i \in \{1,..,n\}$, where we have again cited Proposition 6.4, Theorem 6.5, (7.1), and (7.2), together with our hypothesis to estimate the outstanding terms $2c_\Omega(\mathbf{r}_1)\,v$ and $2c_\Omega(\mathbf{r}_1)\,w$. Since the matrix $(a^{ij}(\mathbf{r}_1))$ is positive definite, we have further observed that, on $(0,\tilde{t})$,

$$
0 \;\leq\; a^{ij}(\mathbf{r}_1)\,\nabla_i(\rho-\varrho)\,\nabla_j(\rho-\varrho) \;\leq\; (1+\varepsilon\, K(n))\,\varepsilon_1^2 ,
$$

while

$$a^{ij}(\mathbf{r}_1)\, g_{ij}\, |\nabla(\rho - \varrho)|^2 \;\geq\; 0\,.$$

Therefore, we may again invoke [**45**, Lemma 3.5] in conjunction with (7.24) and (7.25) to verify that the evolutions of spatial maxima for v and w satisfy the ensuing ODE inequalities on $(0, \tilde{t})$:

$$\frac{d}{dt}\, v_{\max} \;\leq\; \varepsilon_1^2 K_9 \left(1 + \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right)$$

and

$$\frac{d}{dt}\, w_{\max} \;\leq\; \varepsilon_1^2 K_{10} \left(1 + \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right).$$

Furthermore, we discern from Lemma 6.1 and Cauchy's inequality that, on $[0, T)$,

$$(v_{\max} + w_{\max}) \;\leq\; \|\rho - \varrho\|^2_{C^{0,1}(S)} \;\leq\; 2\,(v_{\max} + w_{\max})\,.$$

With the assistance of our hypothesis, we then verify that, on $(0, \tilde{t})$,

$$(v_{\max} + w_{\max}) \;\leq\; \varepsilon_1^2 \left(K_{11}(n)\left(t + 2\,(\varepsilon t)^{\frac{1}{2}}\right) + \frac{1}{4}\right). \tag{7.26}$$

Hence, we deduce from (7.26) that, on $(0, \tilde{t})$,

$$\|\rho - \varrho\|^2_{C^{0,1}(S)} \;\leq\; \varepsilon_1^2 \left(2K_{11}\left(t + 2\,(\varepsilon t)^{\frac{1}{2}}\right) + \frac{1}{2}\right).$$

Under the supposition that $K_{11} \geq 1$ and by recalling from our hypothesis that $\varepsilon < \frac{1}{3}$, we then derive a 'contradiction' whenever $\tilde{t} \leq \frac{3}{100K_{11}^2}$. Therefore, the proposition holds if we take $C(n) = \frac{100K_{11}^2}{3}$. ◇

As a consequence of Proposition 7.1 and the triangle inequality, we may now confirm the existence of a short-time interval for which solutions of (5.1) remain in Lipschitz proximity to S.

COROLLARY 7.2. *Suppose ρ is a solution of* (5.4) *and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation* (2.4)*, both of which satisfy the relevant hypotheses of Proposition 7.1. Further suppose that $C(n)$ is the constant prescribed by Proposition 7.1. Then, for each $t \in (0, C^{-1})$,*

$$\|\rho \;-\; 1\|_{C^{0,1}(S)} \;\leq\; \varepsilon$$

whenever Ω and ε satisfy the constraints prescribed by Theorem 6.5.

REMARK 7.3. Henceforth, in our discussion of the interval $(0, T)$, we shall take T to be the time C^{-1} established by Proposition 7.1, where the *a priori* estimates prescribed by Theorem 6.11 are justified by Corollary 7.2. This enables the extension of the time interval determined by Theorem 5.5, should it be necessary.

2. Exponential decay in $L^\infty(S)$

In this section we demonstrate that the difference between our time-dependent and stationary graph characterisations decays exponentially with respect to the $L^\infty(S)$ topology on the short-time interval $(0,T)$. We first derive an analogous finding in $L^2(S)$, where the exposition shall rely heavily on the results and formalism of Part 1.

THEOREM 7.4. *Suppose ρ is a solution of (5.4) and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation (2.4), both of which satisfy the relevant hypotheses of Proposition 7.1. Then there exists an $\omega(n) > 0$, given explicitly by*

$$\omega^2 \;=\; \frac{7n+16}{8} \;-\; \varepsilon^{\frac{1}{2}} C_4(n)\,,$$

such that, for each $t \in (0,T)$,

$$\|\rho \,-\, \varrho\|_{L^2(S)} \;\leq\; \varepsilon_1 2^{n+2}\,|S|^{\frac{1}{2}}\, e^{-\omega t}$$

whenever $|\Omega| \leq \frac{n}{16}$ and $\varepsilon < \min\left\{\frac{1}{3}, \left(\frac{4}{C_0}\right)^{\frac{1}{2}}, \left(\frac{7n+16}{8C_4}\right)^2\right\}$.

PROOF

Throughout the proof all geometric quantities shall be computed with respect to the metric on S, unless otherwise noted, and we will continue to adopt the convention that $e_0 = \mathbf{0}$. If we take $\eta = (\rho - \varrho)$, then it is evident from Theorems 3.3 and 5.5 that $\eta \in C^\infty(S \times (0,T))$. We first define the following function, which has been chosen to enable direct application of Theorem 4.4 in the forthcoming analysis:

$$\xi \;=\; \eta\varphi^{\frac{1}{2}}\,, \tag{7.27}$$

where

$$\varphi \;=\; \frac{\varrho^{n+1}}{\sqrt{\varrho^2 \,+\, |\nabla\varrho|^2}}\,.$$

Since ϱ is stationary with respect to equation (5.4), we shall subsequently be permitted to relate the evolution of the quantity $\|\xi\|^2_{L^2(S)}$ to that of $\|\eta\|^2_{L^2(S)}$ on $(0,T)$. Furthermore, it is clear from (5.4) that

$$\frac{d}{dt}\,\|\xi\|^2_{L^2(S)} \;=\; 2\,\langle Q_\Omega[\rho], \eta\varphi\rangle_{L^2(S)}\,. \tag{7.28}$$

In analogy with a calculation performed in the proof of Lemma 5.3 and by employing the notation of Chapter 1, we may assert that

$$Q_\Omega[\rho] \;=\; \frac{\sqrt{\rho^2 \,+\, |\nabla\rho|^2}}{\rho}\left(h^k(\rho)\zeta_k(\rho) \,-\, H(\rho) \,-\, f_\Omega(\rho)\right). \tag{7.29}$$

In the general setting, the construction of the L^2-gradient flow generating (5.1) motivates the interpretation of $Q_\Omega : \mathcal{R}^2 \to C^\alpha(S)$ as a non-constant factor of the Euler operator corresponding to the functional $\mathscr{G}_\Omega$ introduced in Definition 2.4. We may therefore expand

$Q_\Omega[\rho]$ in Fréchet derivatives about the minimiser ϱ, where we truncate the expression at the second-order term and recognise that the second Fréchet derivative of Q_Ω is a symmetric bilinear form:

$$Q_\Omega[\rho] \;=\; \partial Q_\Omega[\varrho][\eta] + \int_0^1\int_0^1 \varsigma_1 \partial^2 Q_\Omega\left[\varsigma_1\varsigma_2\rho + (1-\varsigma_1\varsigma_2)\,\varrho\right][\eta,\eta]d\varsigma_1 d\varsigma_2\,. \tag{7.30}$$

We shall now reintroduce the prime notation of Part 1 to distinguish Fréchet differentiation and examine the terms arising in this expression. If the domain of the operator $\mathscr{L}_\Omega$ introduced in Definition 4.1 is extended to encompass all of $C^{2,\alpha}(\widetilde{M})$, then, by recalling the computation of the Jacobi operator contained in the proof of Lemma 4.2 and employing Lemma 1.4, we ascertain from (7.29) that

$$Q'_\Omega[\varrho] \;=\; \frac{-1}{\sqrt{1-|\nabla^{\widetilde{M}}\varrho|^2}}\mathscr{L}_\Omega\left[\langle X'(\varrho),\nu(\varrho)\rangle\right]. \tag{7.31}$$

For the test function η, we then deduce from Corollary 2.6 that

$$\psi \;=\; \langle X'(\varrho),\nu(\varrho)\rangle \;=\; \frac{\eta\varrho}{\sqrt{\varrho^2+|\nabla\varrho|^2}} \;=\; \eta\sqrt{1-|\nabla^{\widetilde{M}}\varrho|^2}\,. \tag{7.32}$$

In the ensuing exposition we shall consider the concepts of orthogonality and tangency with respect to the set $\mathscr{Z}_\varrho$ in the topology of $L^2(\widetilde{M})$. In particular, we note that $\psi^\perp \in \mathscr{N}^2_{\widetilde{M}}$ on $(0,T)$. We further observe from (7.32) that $\eta\varphi = \varrho^n\psi$. We therefore deduce from Lemma 1.4, Definition 4.1, Theorem 4.4, (7.31), and (7.32) that

$$\begin{aligned}
\langle Q'_\Omega[\varrho],\varrho^n\psi\rangle_{L^2(S)} &= -\,\langle\mathscr{L}_\Omega[\psi],\psi\rangle_{L^2(\widetilde{M})} \\
&= -\left(\langle\mathscr{L}_\Omega\left[\psi^\perp\right],\psi^\perp\rangle_{L^2(\widetilde{M})} + \langle\mathscr{L}_\Omega\left[\psi^\top\right],\psi^\perp\rangle_{L^2(\widetilde{M})}\right) \\
&\le -\left(\lambda_1\left\|\psi^\perp\right\|^2_{L^2(\widetilde{M})} + \langle\mathscr{L}_\Omega\left[\psi^\top\right],\psi^\perp\rangle_{L^2(\widetilde{M})}\right) \\
&= -\lambda_1\left(\|\psi\|^2_{L^2(\widetilde{M})} - \left\|\psi^\top\right\|^2_{L^2(\widetilde{M})}\right) - \langle\mathscr{L}_\Omega\left[\psi^\top\right],\psi^\perp\rangle_{L^2(\widetilde{M})}\,.
\end{aligned} \tag{7.33}$$

Here λ_1 is characterised by equation (4.20) contained in the proof of Theorem 4.4: the minimum eigenvalue of $\mathscr{L}_\Omega$ when restricted to its natural domain $\mathscr{N}^2_{\widetilde{M}}$. By employing Lemma 1.4, (7.27), (7.32), and our hypothesis, we note that

$$\|\psi\|^2_{L^2(\widetilde{M})} \;=\; \int_S \eta^2\varphi\,d\sigma \;=\; \|\xi\|^2_{L^2(S)}\,. \tag{7.34}$$

Moreover, since $\psi^\top$ is merely the projection of ψ onto $\mathscr{Z}_\varrho$, it is clear that

$$\psi^\top \;=\; M^{ij}(\varrho)\,\langle\psi,\zeta_i\rangle_{L^2(\widetilde{M})}\,\zeta_j(\varrho)\,. \tag{7.35}$$

We now estimate the inner products $\langle\psi,\zeta_i\rangle_{L^2(\widetilde{M})}$ corresponding to $i \in \{0,...,n+1\}$. We first infer from Proposition 5.13 that $\rho \in \mathscr{R}$ on $(0,T)$. As a consequence of Lemma 1.4 and the isoperimetric constraints specified by Definition 2.2, we then discover that

$$0 \;=\; \int_S \left(\rho^{n+1} - \varrho^{n+1}\right)\zeta_0 d\sigma \;=\; \int_S \eta\left(\sum_{k=0}^{n}\rho^{n-k}\varrho^k\right)\zeta_0 d\sigma\,,$$

while, for each $i \in \{1,..,n+1\}$,

$$0 \;=\; \int_S \left(\rho^{n+2} - \varrho^{n+2}\right)\zeta_i(1) d\sigma \;=\; \int_S \eta\left(\sum_{k=0}^{n+1}\rho^{(n+1)-k}\varrho^k\right)\zeta_i(1) d\sigma\,.$$

Hence, by again employing Lemma 1.4 in conjunction with Corollary 7.2, the Cauchy-Schwarz inequality, (7.32), and hypothesis,

$$\begin{aligned}
\left|\langle\psi,\zeta_0\rangle_{L^2(\widetilde{M})}\right| \;&=\; \left|\int_S \eta\varrho^n\left(1 - \frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{\rho}{\varrho}\right)^{n-k}\right)\zeta_0 d\sigma\right| \\
&\leq\; \sup_S\left|\left(\varrho^{n-1}\sqrt{\varrho^2 + |\nabla\varrho|^2}\right)^{\frac{1}{2}}\left(1 - \frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{\rho}{\varrho}\right)^{n-k}\right)\right| \|\zeta_0\|_{L^2(S)}\,\|\xi\|_{L^2(S)} \\
&\leq\; \varepsilon\, B_0(n)\left(\frac{|S|}{n+1}\right)^{\frac{1}{2}}\|\xi\|_{L^2(S)}\,.
\end{aligned} \tag{7.36}$$

Similarly, where we further utilise Lemma C.5 in Appendix C, for each $i \in \{1,...,n+1\}$,

$$\begin{aligned}
&\left|\langle\psi,\zeta_i\rangle_{L^2(\widetilde{M})}\right| \\
&=\; \left|\int_S \eta\varrho^{n+1}\left(1 - \frac{1}{n+2}\sum_{k=0}^{n+1}\left(\frac{\rho}{\varrho}\right)^{(n+1)-k}\right)\zeta_i(1) d\sigma\right| \\
&\leq\; \sup_S\left|\left(\varrho^{n+1}\sqrt{\varrho^2 + |\nabla\varrho|^2}\right)^{\frac{1}{2}}\left(1 - \frac{1}{n+2}\sum_{k=0}^{n+1}\left(\frac{\rho}{\varrho}\right)^{(n+1)-k}\right)\right| \|\zeta_i\|_{L^2(S)}\,\|\xi\|_{L^2(S)} \\
&\leq\; \varepsilon\, B_i(n)\left(\frac{|S|}{n+1}\right)^{\frac{1}{2}}\|\xi\|_{L^2(S)}\,.
\end{aligned} \tag{7.37}$$

Thus, we infer from Lemma 4.6 and (7.35)-(7.37) that

$$\|\psi^\top\|^2_{L^2(\widetilde{M})} \;=\; M^{ij}(\varrho)\,\langle\psi,\zeta_i\rangle_{L^2(\widetilde{M})}\,\langle\psi,\zeta_j\rangle_{L^2(\widetilde{M})} \;\leq\; \varepsilon^2 K_2(n)\,\|\xi\|^2_{L^2(S)}\,. \tag{7.38}$$

We now evaluate the final term occurring in (7.33) with the assistance of Lemmas 1.4, 4.6, Corollary 1.7, Definition 4.1, Proposition 4.7, (7.34), (7.36), (7.37), and our hypothesis:

$$\begin{aligned} -\left\langle \mathscr{L}_\Omega\left[\psi^\top\right], \psi^\perp\right\rangle_{L^2(\widetilde{M})} &= -M^{kl}(\varrho)\,\langle\psi,\zeta_k\rangle_{L^2(\widetilde{M})}\left\langle \mathscr{L}_\Omega\left[\zeta_l\right], \psi^\perp\right\rangle_{L^2(\widetilde{M})} \\ &\leq \left|M^{kl}(\varrho)\,\langle\psi,\zeta_k\rangle_{L^2(\widetilde{M})}\right| \left\|\mathscr{L}_\Omega\left[\zeta_l\right]\right\|_{L^2(\widetilde{M})} \left\|\psi^\perp\right\|_{L^2(\widetilde{M})} \\ &\leq \varepsilon\, K_3(n)\,\|\xi\|^2_{L^2(S)}\,. \end{aligned} \tag{7.39}$$

Here we have employed the Cauchy-Schwarz inequality on $L^2(\widetilde{M})$ and deduced from the Gauss-Weingarten relations that $\Delta^{\widetilde{M}}\zeta_l(\varrho) = -H(\varrho)\,\langle\nu(\varrho), e_l\rangle$. Thus we may substitute (7.34), (7.38), and (7.39) into (7.33) to assert that

$$\langle Q'_\Omega[\varrho], \eta\varphi\rangle_{L^2(S)} \;\leq\; -\left(\lambda_1\left(1-\varepsilon^2 K_2\right) - \varepsilon\, K_3\right)\|\xi\|^2_{L^2(S)}\,. \tag{7.40}$$

We now account for the second-order term arising in (7.30). Since we do not consider normal perturbations of the class of star-shaped manifolds under examination, it shall be simpler, in contrast to the exposition contained in Part 1, to phrase the ensuing analysis within the context of the invariant geometry on S. For $(\varsigma_1,\varsigma_2)\in[0,1]\times[0,1]$, we set $\rho_\star = (\varsigma_1\varsigma_2\rho + (1-\varsigma_1\varsigma_2)\,\varrho)$ where, by Theorems 3.3 and 5.5, it is clear that $\rho_\star\in C^\infty(S\times(0,T))$ and, in particular, $\rho_\star\in\mathcal{R}^k$ for each $k\in\mathbb{N}$ and $t\in(0,T)$. Moreover, we discern from our hypothesis and Proposition 7.1 that, on $(0,T)$,

$$\|\rho_\star(\cdot,t) - 1\|_{C^{0,1}(S)} \leq \varepsilon\,. \tag{7.41}$$

When composed of the argument $(s,z,\tau)\in S\times\mathbb{R}^+\times TS$, we note that each coefficient of Q_Ω is of class at least C^2. If we take $\mathbf{r}_\star = (s,\rho_\star,\nabla\rho_\star)$, then

$$\partial\,\mathbf{r}_\star[\eta] = \mathbf{r}'_\star = (0,\eta,\nabla\eta)$$

and

$$\partial^2\mathbf{r}_\star[\eta,\eta] = \mathbf{r}''_\star = \mathbf{0}\,.$$

We may therefore compute the second Fréchet derivative of $Q_\Omega[\rho_\star]$ evaluated at η in the following manner:

$$\begin{aligned} (Q_\Omega[\rho_\star])'' &= \left(a^{ij}\right)''\nabla_i\nabla_j\rho_\star + \left(b^i\right)''\nabla_i\rho_\star + c''_\Omega\rho_\star + 2\left(\left(a^{ij}\right)'\nabla_i\nabla_j\eta + \left(b^i\right)'\nabla_i\eta + c'_\Omega\eta\right) \\ &= \left(D^2a^{ij}\right)\Big|_{\mathbf{r}_\star}[\mathbf{r}'_\star,\mathbf{r}'_\star]_{\mathscr{V}}\,\nabla_i\nabla_j\rho_\star + \left(D^2b^i\right)\Big|_{\mathbf{r}_\star}[\mathbf{r}'_\star,\mathbf{r}'_\star]_{\mathscr{V}}\,\nabla_i\rho_\star \\ &\quad + \left(D^2c_\Omega\right)\Big|_{\mathbf{r}_\star}[\mathbf{r}'_\star,\mathbf{r}'_\star]_{\mathscr{V}}\,\rho_\star + 2\left(\left\langle\left(Da^{ij}\right)\big|_{\mathbf{r}_\star},\mathbf{r}'_\star\right\rangle_{\mathscr{V}}\nabla_i\nabla_j\eta\right. \\ &\quad \left. + \left\langle\left(Db^i\right)\big|_{\mathbf{r}_\star},\mathbf{r}'_\star\right\rangle_{\mathscr{V}}\nabla_i\eta + \left\langle(Dc_\Omega)|_{\mathbf{r}_\star},\mathbf{r}'_\star\right\rangle_{\mathscr{V}}\eta\right)\,. \end{aligned} \tag{7.42}$$

We may then integrate by parts and invoke (7.27) to deduce that

$$2\left\langle\left\langle (Da^{ij})\big|_{\mathbf{r}_\star}, \mathbf{r}_\star'\right\rangle_{\mathscr{V}} \nabla_i\nabla_j\eta \,,\, \eta\varphi\right\rangle_{L^2(S)} = \int_S \left(\left(\left(2D_z - \nabla_k(\ln\varphi)\, D_{\tau k}\right) a^{ij}\big|_{\mathbf{r}_\star}\right)\nabla_i\nabla_j\eta \right.$$

$$\left. - \nabla_k\left(\left(D_{\tau k}a^{ij}\big|_{\mathbf{r}_\star}\right)\nabla_i\nabla_j\eta\right)\right)\xi^2 d\sigma\,.$$

We observe, from the structure of the coefficients a^{ij} and b^i of Q_Ω prescribed by Definition 5.1, together with the definitions of η and $\rho_\star$, that the integrand occurring in the expression above contains at worst third-order covariant derivatives in the graph characterisations ρ and ϱ. Therefore, by Corollary 6.12, Proposition 7.1, Corollary 7.2, Lemma B.1 in Appendix B, (7.27), (7.41), the triangle inequality, and our hypothesis, we determine that, for each $t \in (0,T)$,

$$\begin{aligned}
&2\left|\left\langle\left\langle (Da^{ij})\big|_{\mathbf{r}_\star}, \mathbf{r}_\star'\right\rangle_{\mathscr{V}} \nabla_i\nabla_j\eta \,,\, \eta\varphi\right\rangle_{L^2(S)}\right| \\
&\le K(n)\Big(\left\|\nabla^2\eta\right\|_\infty + \varepsilon\left\|\nabla^3\eta\right\|_\infty + \left\|\nabla^2\eta\right\|_\infty\left\|\nabla^2\rho_\star\right\|_\infty\Big)\|\xi\|^2_{L^2(S)} \\
&\le K(n)\left(\|\nabla\eta\|^{\frac12}_\infty\left\|\nabla^3\eta\right\|^{\frac12}_\infty + \varepsilon\left\|\nabla^3\eta\right\|_\infty + \|\nabla\eta\|^{\frac12}_\infty\|\nabla\rho_\star\|^{\frac12}_\infty\left\|\nabla^3\eta\right\|^{\frac12}_\infty\left\|\nabla^3\rho_\star\right\|^{\frac12}_\infty\right)\|\xi\|^2_{L^2(S)} \\
&\le K_4(n)\left(\varepsilon^{\frac12}\max\left\{1, \frac{1}{t^{\frac14}}, \frac{\varepsilon^{\frac14}}{t^{\frac12}}\right\} + \varepsilon\max\left\{1, \frac{1}{t^{\frac12}}, \frac{\varepsilon^{\frac12}}{t}\right\}\right)\|\xi\|^2_{L^2(S)}\,.
\end{aligned}\tag{7.43}$$

Here the supremum norms are evaluated over S at each time $t \in (0,T)$. Similarly,

$$\begin{aligned}
&2\left|\left\langle\left\langle (Db^{i})\big|_{\mathbf{r}_\star}, \mathbf{r}_\star'\right\rangle_{\mathscr{V}} \nabla_i\eta \,,\, \eta\varphi\right\rangle_{L^2(S)}\right| \\
&= \left|\int_S \left(\left(\left(2D_z - \nabla_k(\ln\varphi)\, D_{\tau k}\right) b^{i}\big|_{\mathbf{r}_\star}\right)\nabla_i\eta - \nabla_k\left(\left(D_{\tau k}b^{i}\big|_{\mathbf{r}_\star}\right)\nabla_i\eta\right)\right)\xi^2 d\sigma\right| \\
&\le K(n)\Big(\varepsilon + \varepsilon^2\left\|\nabla^2\rho_\star\right\|_\infty + \left\|\nabla^2\eta\right\|_\infty\Big)\|\xi\|^2_{L^2(S)} \\
&\le K(n)\left(\varepsilon + \varepsilon^2\left\|\nabla^2\rho_\star\right\|_\infty + \|\nabla\eta\|^{\frac12}_\infty\left\|\nabla^3\eta\right\|^{\frac12}_\infty\right)\|\xi\|^2_{L^2(S)} \\
&\le K_5(n)\left(\varepsilon^2\max\left\{1, \left(\frac{\varepsilon}{t}\right)^{\frac12}\right\} + \varepsilon^{\frac12}\max\left\{1, \frac{1}{t^{\frac14}}, \frac{\varepsilon^{\frac14}}{t^{\frac12}}\right\}\right)\|\xi\|^2_{L^2(S)}\,.
\end{aligned}\tag{7.44}$$

Furthermore,

$$\left|\left\langle \left(D^2 a^{ij}\right)\big|_{\mathbf{r}_\star} [\mathbf{r}'_\star, \mathbf{r}'_\star]_\gamma \, \nabla_i \nabla_j \rho_\star \,, \eta\varphi \right\rangle_{L^2(S)}\right|$$

$$= \left|\int_S \left(\left(\left(\eta D_z D_z + 2\nabla_k \eta D_z D_{\tau k} - \frac{1}{2}\nabla_k \left(\ln\varphi\right) \nabla_l \eta D_{\tau k} D_{\tau l}\right) a^{ij}\big|_{\mathbf{r}_\star}\right) \nabla_i \nabla_j \rho_\star \right.\right.$$

$$\left.\left. - \frac{1}{2}\nabla_k \left(\nabla_l \eta \left(D_{\tau k} D_{\tau l} a^{ij}\big|_{\mathbf{r}_\star}\right) \nabla_i \nabla_j \rho_\star\right)\right) \xi^2 d\sigma\right|$$

$$\leq \varepsilon K(n) \left(\left\|\nabla^2 \rho_\star\right\|_\infty + \left\|\nabla^2 \eta\right\|_\infty \left\|\nabla^2 \rho_\star\right\|_\infty + \left\|\nabla^2 \rho_\star\right\|_\infty^2 + \varepsilon \left\|\nabla^3 \rho_\star\right\|_\infty\right) \|\xi\|_{L^2(S)}^2$$

$$\leq \varepsilon K_6(n) \left(\max\left\{1, \frac{\varepsilon}{t}\right\} + \varepsilon \max\left\{1, \frac{1}{t^{\frac{1}{2}}}, \frac{\varepsilon^{\frac{1}{2}}}{t}\right\}\right) \|\xi\|_{L^2(S)}^2 , \tag{7.45}$$

while

$$\left|\left\langle \left(D^2 b^{i}\right)\big|_{\mathbf{r}_\star} [\mathbf{r}'_\star, \mathbf{r}'_\star]_\gamma \, \nabla_i \rho_\star \,, \eta\varphi \right\rangle_{L^2(S)}\right|$$

$$= \left|\int_S \left(\left(\left(\eta D_z D_z + 2\nabla_k \eta D_z D_{\tau k} - \frac{1}{2}\nabla_k \left(\ln\varphi\right) \nabla_l \eta D_{\tau k} D_{\tau l}\right) b^{i}\big|_{\mathbf{r}_\star}\right) \nabla_i \rho_\star \right.\right.$$

$$\left.\left. - \frac{1}{2}\nabla_k \left(\nabla_l \eta \left(D_{\tau k} D_{\tau l} b^{i}\big|_{\mathbf{r}_\star}\right) \nabla_i \rho_\star\right)\right) \xi^2 d\sigma\right|$$

$$\leq \varepsilon K(n) \left(\varepsilon + \left\|\nabla^2 \rho_\star\right\|_\infty + \left\|\nabla^2 \eta\right\|_\infty + \left\|\nabla^2 \rho_\star\right\|_\infty\right) \|\xi\|_{L^2(S)}^2$$

$$\leq \varepsilon K_7(n) \max\left\{1, \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right\} \|\xi\|_{L^2(S)}^2 . \tag{7.46}$$

It is important to emphasise that the derivatives of the global terms $\{h^i\}$ arising in (7.42) again require more delicate examination, since they contain second-order terms. In analogy with the computation of (5.10) in the proof of Theorem 5.5 and by invoking Definition 5.1, we discern that

$$\left\langle Dh^i\big|_{\mathbf{r}_\star}, \mathbf{r}_\star'\right\rangle_{\mathscr{V}} = -M^{ik}(\rho_\star)M^{jl}(\rho_\star)\left(\int_S \langle D\psi_{kl}|_{\mathbf{r}_\star}, \mathbf{r}_\star'\rangle_{\mathscr{V}}\, d\sigma\right)\left(\int_S (\psi_\Omega)_j|_{(\mathbf{r}_\star,\nabla^2\rho_\star)} d\sigma\right)$$

$$+\, M^{ij}(\rho_\star)\int_S \left\langle D\left(\psi_\Omega\right)_j\Big|_{(\mathbf{r}_\star,\nabla^2\rho_\star)}, \left(\mathbf{r}_\star', \nabla^2\eta\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma\,. \tag{7.47}$$

Moreover, where we employ the symmetry of the matrix $(M^{ij}(\rho_\star))$,

$$\left(D^2 h^i\big|_{\mathbf{r}_\star}\right)[\mathbf{r}_\star', \mathbf{r}_\star']_{\mathscr{V}}$$

$$= M^{kp}(\rho_\star)\left(M^{im}(\rho_\star)M^{jl}(\rho_\star) + M^{il}(\rho_\star)M^{jm}(\rho_\star)\right)\cdot$$

$$\left(\int_S \langle D\psi_{mp}|_{\mathbf{r}_\star}, \mathbf{r}_\star'\rangle_{\mathscr{V}}\, d\sigma\right)\left(\int_S \langle D\psi_{kl}|_{\mathbf{r}_\star}, \mathbf{r}_\star'\rangle_{\mathscr{V}}\, d\sigma\right)\left(\int_S (\psi_\Omega)_j|_{(\mathbf{r}_\star,\nabla^2\rho_\star)} d\sigma\right)$$

$$-\, M^{ik}(\rho_\star)M^{jl}(\rho_\star)\left(\left(\int_S \left(D^2\psi_{kl}\right)\big|_{\mathbf{r}_\star}[\mathbf{r}_\star', \mathbf{r}_\star']_{\mathscr{V}}\, d\sigma\right)\left(\int_S (\psi_\Omega)_j|_{(\mathbf{r}_\star,\nabla^2\rho_\star)} d\sigma\right)\right.$$

$$\left. +\, 2\left(\int_S \langle D\psi_{kl}|_{\mathbf{r}_\star}, \mathbf{r}_\star'\rangle_{\mathscr{V}}\, d\sigma\right)\left(\int_S \left\langle D\left(\psi_\Omega\right)_j\Big|_{(\mathbf{r}_\star,\nabla^2\rho_\star)}, \left(\mathbf{r}_\star', \nabla^2\eta\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma\right)\right)$$

$$+\, M^{ij}(\rho_\star)\int_S \left(D^2\left(\psi_\Omega\right)_j\right)\Big|_{(\mathbf{r}_\star,\nabla^2\rho_\star)}\left[\left(\mathbf{r}_\star', \nabla^2\eta\right), \left(\mathbf{r}_\star', \nabla^2\eta\right)\right]_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma\,. \tag{7.48}$$

In calculations which parallel (5.11) and (5.12), we then integrate by parts to discover that, for each $j, k, l \in \{0, ..., n+1\}$,

$$\int_S \langle D\psi_{kl}|_{\mathbf{r}_\star}, \mathbf{r}_\star'\rangle_{\mathscr{V}}\, d\sigma = \int_S \eta\left(D_z - \nabla_i D_{\tau i}\right)\psi_{kl}|_{\mathbf{r}_\star} d\sigma\,, \tag{7.49}$$

and

$$\int_S \left\langle D\left(\psi_\Omega\right)_j\Big|_{(\mathbf{r}_\star,\nabla^2\rho_\star)}, \left(\mathbf{r}_\star', \nabla^2\eta\right)\right\rangle_{\mathscr{V}\times\mathscr{S}_2^0(S)} d\sigma$$

$$= \int_S \eta\left(D_z - \nabla_i D_{\tau i} + \nabla^i\nabla^m D_{\vartheta_{im}}\right)(\psi_\Omega)_j|_{(\mathbf{r}_\star,\nabla^2\rho_\star)} d\sigma\,. \tag{7.50}$$

Similarly,

$$\int_S (D^2\psi_{kl})\big|_{\mathbf{r}_\star} [\mathbf{r}'_\star, \mathbf{r}'_\star]_{\mathscr{V}}\, d\sigma$$

$$= \int_S \eta\,(\eta D_z D_z \,+\, 2\,\nabla_i\eta D_z D_{\tau i} \,-\, \nabla_i\,(\nabla_j\eta D_{\tau i} D_{\tau j}))\,\psi_{kl}|_{\mathbf{r}_\star} d\sigma\,. \tag{7.51}$$

We next observe from Definition 5.1 that the function $(\psi_\Omega)_j$ is linear in the variable $\vartheta \in \mathscr{S}_2^0(S)$. Hence

$$\int_S \left(D^2\,(\psi_\Omega)_j\right)\Big|_{(\mathbf{r}_\star,\nabla^2\rho_\star)} \left[(\mathbf{r}'_\star, \nabla^2\eta)\,,\,(\mathbf{r}'_\star,\nabla^2\eta)\right]_{\mathscr{V}\times\mathscr{S}_2^0(S)}\, d\sigma$$

$$= \int_S \eta\,\big(\eta D_z D_z \,-\, \nabla_i\,(\nabla_l\eta D_{\tau i} D_{\tau l}) \,+\, 2\,\big(\nabla_i\eta D_z D_{\tau i} \,+\, \nabla^i\nabla^k\eta D_z D_{\vartheta_{ik}}$$

$$-\,\nabla_i\left(\nabla^k\nabla^l\eta D_{\tau i} D_{\vartheta_{kl}}\right)\big)\big)\,(\psi_\Omega)_j|_{(\mathbf{r}_\star,\nabla^2\rho_\star)} d\sigma\,. \tag{7.52}$$

Furthermore, by (7.41), the analysis performed in the proof of Lemma 4.6 proceeds unmodified in this context and we ascertain that, on $(0,T)$,

$$\left|M^{ij}(\rho_\star) \,-\, M^{ij}(1)\right| \;\le\; \varepsilon\, K^{ij}(n)\,|S|^{-1}\,. \tag{7.53}$$

We deduce from the form of the functions ψ_{kl} and $(\psi_\Omega)_j$ introduced in Definition 5.1 that the integrands occurring in (7.49)-(7.52) again comprise at worst third-order covariant derivatives in the graph characterisations ρ and ϱ. Moreover, the integrands arising in (7.47) and (7.48) which remain undifferentiated in the variables (s, z, τ, ϑ) include at most second-order covariant derivatives in the quantities ρ and ϱ. We note that the first derivatives in the variables z and τ of the remaining terms in the coefficient c_Ω are respectively contracted over η and $\nabla\eta$. Thus we may invoke Corollary 6.12, Proposition 7.1, Corollary 7.2, (7.27), (7.41), (7.47), (7.49), (7.50), (7.53), the triangle inequality, and our hypothesis to demonstrate that, for each $t \in (0,T)$,

$$2\left|\langle\langle (Dc_\Omega)|_{\mathbf{r}_\star}, \mathbf{r}'_\star\rangle_{\mathscr{V}}\,\eta\,,\,\eta\varphi\rangle_{L^2(S)}\right|$$

$$\le\; \varepsilon\,K(n)\left(1 \,+\, \left\|\nabla^2\rho_\star\right\|_\infty \,+\, \left\|\nabla^2\rho_\star\right\|_\infty^2 \,+\, \varepsilon\left\|\nabla^3\rho_\star\right\|_\infty\right)\|\xi\|^2_{L^2(S)}$$

$$\le\; \varepsilon\,K_8(n)\left(\max\left\{1, \frac{\varepsilon}{t}\right\} \,+\, \varepsilon\max\left\{1, \frac{1}{t^{\frac{1}{2}}}, \frac{\varepsilon^{\frac{1}{2}}}{t}\right\}\right)\|\xi\|^2_{L^2(S)}\,. \tag{7.54}$$

Furthermore, we may integrate by parts and treat the second derivatives in z and τ of the remaining terms in c_Ω similarly to those of a^{ij} and b^i accounted for in the calculations of

(7.45) and (7.46). Thus we may perform analysis parallel to that already undertaken whilst citing Lemma B.1 in Appendix B, (7.27), (7.41), (7.47)-(7.53), and the Cauchy-Schwarz inequality on $L^2(S)$ to compute that

$$\begin{aligned}
&\left|\left\langle (D^2 c_\Omega)\big|_{\mathbf{r}_\star} [\mathbf{r}'_\star, \mathbf{r}'_\star]_{\mathscr{V}\times\mathscr{S}_2^0(S)}\, \rho_\star\,, \eta\varphi\right\rangle_{L^2(S)}\right| \\
&\le\; K(n)\Big(\varepsilon\Big(1 + \left\|\nabla^2\rho_\star\right\|_\infty + \left\|\nabla^2\rho_\star\right\|_\infty^2 + \varepsilon\left\|\nabla^3\rho_\star\right\|_\infty\Big) + \left\|\nabla^2\eta\right\|_\infty \\
&\qquad + \left\|\nabla^2\eta\right\|_\infty \left\|\nabla^2\rho_\star\right\|_\infty\Big)\, \|\xi\|_{L^2(S)}^2 \\
&\le\; K(n)\Big(\varepsilon\Big(1 + \left\|\nabla^2\rho_\star\right\|_\infty + \left\|\nabla^2\rho_\star\right\|_\infty^2 + \varepsilon\left\|\nabla^3\rho_\star\right\|_\infty\Big) + \|\nabla\eta\|_\infty^{\frac{1}{2}} \left\|\nabla^3\eta\right\|_\infty^{\frac{1}{2}} \\
&\qquad + \|\nabla\eta\|_\infty^{\frac{1}{2}} \|\nabla\rho_\star\|_\infty^{\frac{1}{2}} \left\|\nabla^3\eta\right\|_\infty^{\frac{1}{2}} \left\|\nabla^3\rho_\star\right\|_\infty^{\frac{1}{2}}\Big)\, \|\xi\|_{L^2(S)}^2 \\
&\le\; K_9(n)\left(\varepsilon\left(\max\left\{1, \frac{\varepsilon}{t}\right\} + \max\left\{1, \frac{1}{t^{\frac{1}{2}}}, \frac{\varepsilon^{\frac{1}{2}}}{t}\right\}\right) + \varepsilon^{\frac{1}{2}} \max\left\{1, \frac{1}{t^{\frac{1}{4}}}, \frac{\varepsilon^{\frac{1}{4}}}{t^{\frac{1}{2}}}\right\}\right) \|\xi\|_{L^2(S)}^2 .
\end{aligned} \tag{7.55}$$

We now recall the characterisation of λ_1 given by (4.20) and our hypothesis condition on Ω to assert that

$$\lambda_1 \;\ge\; \frac{7n+16}{8} \;-\; \varepsilon_2 K_{10}(n)\,.$$

In addition, it is evident from hypothesis and (7.27) that

$$(1 - \varepsilon_2 K_{11}(n))\, \|\eta\|_{L^2(S)}^2 \;\le\; \|\xi\|_{L^2(S)}^2 \;\le\; (1 + \varepsilon_2 K_{12}(n))\, \|\eta\|_{L^2(S)}^2\;,$$

where the quantity on the left hand side of this inequality is strictly positive for sufficiently small ε_2. Since ϱ is a stationary solution of (5.4), we therefore deduce from (7.28), (7.30), (7.40)-(7.46), (7.54), and (7.55) that, on $(0,T)$,

$$\frac{d}{dt}\, \|\eta\|_{L^2(S)}^2 \;\le\; -2\left(\frac{7n+16}{8} \;-\; \varepsilon^{\frac{1}{2}} K_{13}(n)\left(1 + \frac{1}{t}\right)\right) \|\eta\|_{L^2(S)}^2\,. \tag{7.56}$$

We next deduce from our hypothesis and Proposition 7.1 that, for each $t \in (0,T)$,

$$\|\eta(\cdot,t)\|_{L^2(S)}^2 \;\le\; \varepsilon_1^2\, |S|\,. \tag{7.57}$$

We now choose $K_{13} \equiv C_4(n)$ with $\omega(n)$, Ω, and ε defined in accordance with the statement of the theorem. For an arbitrary time $t \in (0,T)$, we may then integrate the differential inequality (7.56) over the interval $\left(\frac{t}{2}, t\right)$ to ascertain that

$$\left[\ln\left(\|\eta(\cdot,w)\|^2_{L^2(S)}\right)\right]_{w=\frac{t}{2}}^{w=t} \;\leq\; -\omega^2 t + \left(\frac{7n+16}{4}\right)\ln 2\,.$$

Thus, we infer from (7.57) that

$$\|\eta(\cdot,t)\|^2_{L^2(S)} \;\leq\; \varepsilon_1^2 2^{\frac{7n+16}{4}}\,|S|\,e^{-\omega^2 t}\,.$$

This implies our desired result. ◇

COROLLARY 7.5. *Suppose ρ is a solution of* (5.4) *and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation* (2.4)*, both of which satisfy the relevant hypotheses of Proposition 7.1. Then there exists a $\theta_0(n) > 0$ such that, for each $t \in (0,T)$,*

$$\|\rho - \varrho\|_{L^\infty(S)} \;\leq\; 16\pi\varepsilon_1 e^{-\theta_0 t}$$

whenever Ω and ε satisfy the constraints of Theorem 7.4.

PROOF

By Proposition 7.1, it is again clear that, on $(0,T)$,

$$\|\rho - \varrho\|_{C^{0,1}(S)} \;\leq\; \varepsilon_1.$$

Therefore, we may invoke Theorem 7.4 and Lemma B.3 in Appendix B to conclude that, on $(0,T)$,

$$\begin{aligned}
\|\rho - \varrho\|_{L^\infty(S)} \;&\leq\; 4\pi\,|S|^{\frac{-1}{n+2}}\,\|\rho - \varrho\|^{\frac{2}{n+2}}_{L^2(S)}\|\rho - \varrho\|^{\frac{n}{n+2}}_{C^{0,1}(S)} \\
&\leq\; 4\pi\,|S|^{\frac{-1}{n+2}}\left(\varepsilon_1 2^{n+2}\,|S|^{\frac{1}{2}}\,e^{-\omega t}\right)^{\frac{2}{n+2}}(\varepsilon_1)^{\frac{n}{n+2}} \\
&\leq\; 16\pi\varepsilon_1\, e^{-\frac{2\omega t}{n+2}}\,.
\end{aligned}$$

We then choose $\theta_0 = \frac{2\omega}{n+2} > 0$ to verify the proposition. ◇

3. Global existence of unique solutions

We may now verify the global existence of unique solutions to the rotating drop flow (5.1) whenever the initial boundary M_0 of the drop E_0 remains appropriately close to S in the Lipschitz topology. As a consequence of Lemma 5.3, it is again sufficient to prove this result for a given solution of the scalar valued problem (5.4). Due to the interpolative nature of the Gagliardo-Nirenberg type inequalities contained in Appendix B, it shall be necessary to place further restrictions on the initial proximity of any such solution to a corresponding solution of the Euler-Lagrange equation (2.4).

PROPOSITION 7.6. *Suppose ρ is a solution of (5.4), $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation (2.4), and $C(n)$ is the constant determined by Proposition 7.1. Further suppose that strictly positive ε_1 and ε_2 are such that $\|\rho(\cdot\,,0)-\varrho\|_{C^{0,1}(S)} \leq \frac{\varepsilon_1^2}{2\sqrt{2}}$ and $\|\varrho - 1\|_{C^{3,\alpha}(S)} \leq \varepsilon_2$. Then there exists a $\kappa(n)>0$ such that, for each $t \geq \left(\frac{\varepsilon}{C^4}\right)^{\frac{1}{4}}$,*

$$\|\rho - \varrho\|_{C^{0,1}(S)} \;\leq\; \varepsilon_1 C_5(n) e^{-\kappa t}$$

whenever Ω and ε satisfy the constraints of Theorem 7.4.

PROOF

In conjunction with our more restrictive hypothesis condition, we infer from Corollaries 7.2 and 7.5 that, on $(0,T)$, $\|\rho - 1\|_{C^{0,1}(S)} \leq \varepsilon$ and

$$\|\rho - \varrho\|_{L^\infty(S)} \;\leq\; 16\pi\varepsilon_1^2 e^{-\theta_0 t}\,.$$

Therefore, we may employ Corollary 6.12, Corollary B.2 in Appendix B, and the triangle inequality to deduce that, on $(0,T)$,

$$\begin{aligned}
\|\rho - \varrho\|_{C^{0,1}(S)} &\leq K(n)\|\rho-\varrho\|^{\frac{1}{2}}_{L^\infty(S)}\|\rho-\varrho\|^{\frac{1}{2}}_{C^{1,1}(S)} \\
&\leq K\left(16\pi\varepsilon_1^2 e^{-\theta_0 t}\right)^{\frac{1}{2}}\left(\|\rho - 1\|_{C^{1,1}(S)} + \|\varrho - 1\|_{C^{1,1}(S)}\right)^{\frac{1}{2}} \\
&\leq \varepsilon_1 K(n)\, e^{-\frac{\theta_0 t}{2}}\left(\varepsilon + \mathcal{D}_0^{\frac{1}{2}} \max\left\{1, \left(\frac{\varepsilon}{t}\right)^{\frac{1}{2}}\right\}\right)^{\frac{1}{2}} \\
&\leq \varepsilon_1 C_5(n) \max\left\{1, \left(\frac{\varepsilon}{t}\right)^{\frac{1}{4}}\right\} e^{-\frac{\theta_0 t}{2}}.
\end{aligned}$$

Since $C^{1,1} \equiv W^{2,\infty}$, here we have approximated the Lipschitz seminorm of $\nabla\rho$ by $\|\nabla^2\rho\|_\infty$. The result now follows directly if we take $\kappa = \frac{\theta_0}{2}$. ◊

The ensuing corollary is then an immediate consequence of the triangle inequality.

COROLLARY 7.7. *Suppose ρ is a solution of (5.4), $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation (2.4), and $C(n)$ is the constant prescribed by Proposition 7.1. Further suppose that strictly positive ε_1 and ε_2 are such that $\|\rho(\cdot\,,0)-\varrho\|_{C^{0,1}(S)} \leq \frac{\varepsilon_1^2}{2\sqrt{2}C_5^2}$ and $\|\varrho - 1\|_{C^{3,\alpha}(S)} \leq \varepsilon_2$. Then, for each $t \geq \left(\frac{\varepsilon}{C^4}\right)^{\frac{1}{4}}$,*

$$\|\rho - 1\|_{C^{0,1}(S)} \;\leq\; \varepsilon_1 e^{-\kappa t} + \varepsilon_2$$

whenever Ω and ε satisfy the constraints of Theorem 7.4.

Under the hypotheses of Proposition 7.6, we may now confirm the global existence of unique solutions to the problem posed by (5.4).

THEOREM 7.8. *Suppose ρ is a solution of (5.4) and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation (2.4), both of which satisfy the relevant hypotheses of Corollary 7.7. Further suppose that Ω and ε satisfy the constraints of Theorem 7.4. Then $T = \infty$.*

PROOF

By Corollary 7.2, $\|\rho - 1\|_{C^{0,1}(S)} \le \varepsilon$ on $(0, T)$. In particular, we note from our hypothesis condition on ε and Corollary 7.7 that, for each $t \ge (\sqrt[4]{3}\, C)^{-1} \in (0, T)$,

$$\|\rho - 1\|_{C^{0,1}(S)} \;\le\; \varepsilon_1 e^{-\kappa t} + \varepsilon_2. \tag{7.58}$$

Moreover, we may employ Corollary 6.12 to deduce that, for each $m \in \mathbb{N} \cup \{0\}$ and $t \ge (\sqrt[4]{3}\, C)^{-1}$,

$$\sup_S \|\nabla^{m+2}\rho\|^2 \;\le\; \mathcal{D}_m \max\left\{1, (\sqrt[4]{3}\, C)^m, \varepsilon(\sqrt[4]{3}\, C)^{m+1}\right\} \;\le\; K_m(n)\,. \tag{7.59}$$

Thus the interior regularity estimates corresponding to $t \ge (\sqrt[4]{3}\, C)^{-1}$ are independent of time. We now observe that the quantity on the right hand side of inequality (7.58) is decreasing on the interval $[(\sqrt[4]{3}\, C)^{-1}, T)$ and bounded above by ε. Hence we may smoothly extend our solution ρ to encapsulate time T. We may then reapply the argument expounded in the proof of Theorem 5.5 to find $\delta > 0$ such that $\rho \in C^{\infty}(S \times (0, T + \delta))$ uniquely solves (5.4) where the estimates (7.58) and (7.59) remain valid. Successive iterations of the preceding analysis then yield the desired result. ◇

The following corollary is an immediate consequence of Lemma 5.3 and Theorem 7.8.

COROLLARY 7.9. *Suppose X_0 is a $C^{3,\alpha}$ embedding of M_0 and $\varepsilon > 0$ is such that $\||X_0| - 1\|_{C^{0,1}(S)} \le \varepsilon$. Then there exists a unique, smooth, global solution to the rotating drop flow (5.1) whenever Ω and ε satisfy the constraints of Theorem 7.4.*

4. Convergence in $C^{\infty}(S)$

Under the hypotheses of Proposition 7.6, we may now demonstrate that solutions of (5.4) converge asymptotically to corresponding solutions of the Euler-Lagrange equation (2.4) in the topology of $C^{\infty}(S)$.

PROPOSITION 7.10. *Suppose ρ is a solution of (5.4) and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation (2.4), both of which satisfy the relevant hypotheses of Corollary 7.7. Then, for each $j \in \mathbb{N} \cup \{0\}$, there exists a $\theta_j(n) > 0$ such that, for each $t \ge 1$,*

$$\left\|\nabla^j (\rho - \varrho)\right\|_{L^{\infty}(S)} \;\le\; \varepsilon_1^{\frac{2}{j+1}} D_j(n)\, e^{-\theta_j t}$$

whenever Ω and ε satisfy the constraints of Theorem 7.4.

PROOF

The case $j = 0$ with $D_0 = \frac{16\pi}{C_5^2}$ is implied by Corollary 7.5 in conjunction with our more restrictive hypothesis conditions. To verify the remaining cases we first observe from Corollaries 6.12, 7.7, and Theorem 7.8 that, for each $k \in \mathbb{N} \setminus \{1\}$ and $t \geq 1$,

$$\left\|\nabla^k \rho\right\|_\infty^2 \leq \mathcal{D}_{k-2}(n) . \tag{7.60}$$

Furthermore, we infer from Theorem 3.3 that $\left\|\nabla^k \varrho\right\|_\infty$ is certainly finite for each $k \in \mathbb{N} \setminus \{1\}$. Indeed, since ϱ is a stationary solution of (5.4) which is independent of time and satisfies the hypotheses of Corollary 6.12, we may bound its derivatives by the same constants. We may therefore invoke Corollary 7.5 and Lemma B.1 in Appendix B to assert that, for each $j \in \mathbb{N}$ and $t \in \mathbb{R}^+$,

$$\begin{aligned} \left\|\nabla^j (\rho - \varrho)\right\|_\infty &\leq K(n,j) \left\|\rho - \varrho\right\|_\infty^{\frac{1}{j+1}} \left\|\nabla^{j+1} (\rho - \varrho)\right\|_\infty^{\frac{j}{j+1}} \\ &\leq K \left(\frac{16\pi\varepsilon_1^2}{C_5^2} e^{-\theta_0 t}\right)^{\frac{1}{j+1}} \left\|\nabla^{j+1} (\rho - \varrho)\right\|_\infty^{\frac{j}{j+1}} . \end{aligned} \tag{7.61}$$

For arbitrary $(s,t) \in S \times \mathbb{R}^+$, we now examine the tensorfield norm $\|\nabla^{j+1} (\rho - \varrho) (s,t)\|$, where we infer from the inequalities of Cauchy and Cauchy-Schwarz that

$$\left\|\nabla^{j+1} (\rho - \varrho) (s,t)\right\|^2 \leq 2 \left(\left\|\nabla^{j+1} \rho(s,t)\right\|^2 + \left\|\nabla^{j+1} \varrho(s)\right\|^2\right) .$$

Hence, for each $j \in \mathbb{N}$ and $t \geq 1$, we deduce from (7.60) that

$$\left\|\nabla^{j+1} (\rho - \varrho)\right\|_\infty^2 \leq 4\mathcal{D}_{j-1}(n) . \tag{7.62}$$

By substituting (7.62) into (7.61), we obtain our intended result whenever we take $\theta_j = \frac{\theta_0}{j+1}$ and $D_j = K \left(\frac{16\pi}{C_5^2} (4\mathcal{D}_{j-1})^{\frac{j}{2}}\right)^{\frac{1}{j+1}}$. ◊

Thus we have proved the following corollary.

COROLLARY 7.11. *Suppose ρ is a solution of* (5.4) *and $\varrho \in \mathscr{R}$ is a corresponding solution of the Euler-Lagrange equation* (2.4), *both of which satisfy the relevant hypotheses of Corollary 7.7. Then ρ converges asymptotically to ϱ in $C^\infty(S)$ whenever Ω and ε satisfy the constraints of Theorem 7.4.*

5. Energy minimisation in a Lipschitz neighbourhood of S

We are now in a position to prove the principal result of this thesis.

PROPOSITION 7.12. *Suppose $\varrho \in \mathscr{R}$ is a solution of the Euler-Lagrange equation* (2.4) *and $\varepsilon_2 > 0$ is such that $\|\varrho - 1\|_{C^{3,\alpha}(S)} \leq \varepsilon_2$. Then, for sufficiently small $\Omega \in \mathbb{R}$, there exists a neighbourhood of $C^{0,1}(S)$ where ϱ permits stable minimisation of the energy functional $\mathscr{F}_\Omega$. In particular, ϱ is uniquely determined in this neighbourhood.*

PROOF

We first recall from Corollary 4.8, Theorem 3.3, and our hypothesis that $\varrho \in C^\infty(S)$ admits strict (weak) minimisation of the functional $\mathscr{F}_\Omega$ whenever $|\Omega| \leq \frac{n+2}{2}(1 - \varepsilon_2\, C_3(n))$. Suppose now that $\rho \in C^\infty(S)$ is a global solution of (5.4) established by Theorem 7.8. We therefore deduce from Corollary 5.14 that $\rho \in \mathscr{R}$ for all time. Moreover, by the diffeomorphism given explicitly by (3.1) in the proof of Theorem 3.1, we may justify any choice of initial condition $\rho(\cdot\,,0) \in \mathscr{R}$ with

$$\|\rho(\cdot\,,0) \;-\; \varrho\|_{C^{0,1}(S)} \;\leq\; \frac{\varepsilon_1^2}{2\sqrt{2}\,C_5^2} \;=\; \delta\,. \tag{7.63}$$

We then deduce from Proposition 5.15 that, for each $t \in [0, \infty]$,

$$\mathscr{F}_\Omega(\varrho) \;\leq\; \mathscr{F}_\Omega(\rho(\cdot, t))\,.$$

We further note that each distinct choice of such an initial condition yields a unique solution to the parabolic problem (5.4) where, by Corollary 7.11, ρ converges to ϱ in infinite time with respect to the topology of $C^\infty(S)$. Thus we ascertain that ϱ permits stable minimisation of $\mathscr{F}_\Omega$ within the closed ball $\overline{B}_\delta(\varrho) \subset C^{0,1}(S)$ prescribed by (7.63) whenever

$$\varepsilon \;=\; \varepsilon_1 + \varepsilon_2 \;\;<\;\; \min\left\{\frac{1}{3}\,, \left(\frac{4}{C_0}\right)^{\frac{1}{2}}, \frac{1}{C_3}\,, \left(\frac{7n+16}{8C_4}\right)^2\right\},$$

and

$$|\Omega| \;\leq\; \min\left\{\frac{n+2}{2}\,(1 - \varepsilon_2\, C_3(n))\,, \frac{n}{16}\right\}.$$

In conclusion, we observe that the uniqueness of ϱ in the ball is a direct consequence of the minimisation property. $\diamondsuit$

We may therefore combine the results of Theorems 3.1, 3.3, 4.9, and Proposition 7.12 to assert the following local existence and uniqueness theorem for stable energy minimisers of the rotating drop problem in the class $\mathscr{E}$.

THEOREM 7.13. *For each sufficiently small $\Omega > 0$, there exists a stable energy minimiser for the rotating drop problem whose smooth boundary is uniquely determined in a Lipschitz neighbourhood of S.*

APPENDIX A

Differential Geometry

We now provide a summary of the Riemannian geometry which is relevant to the exposition of this thesis. In accordance with the geometric context of the dissertation, we shall merely consider manifolds of codimension one in the Euclidean spaces which are compact, connected, and without boundary. The salient theory is largely contained in either of [**17**] or [**21**], and additional references will be provided where necessary.

1. An overview of Riemannian geometry

Suppose $(N^n, \tilde{\mathbf{g}})$ is a compact, connected Riemannian manifold without boundary smoothly embedded in $\mathbb{R}^{n+1}$. For $k \in \mathbb{N}$ sufficiently large to justify the definitions that follow, further suppose that $M \subset \mathbb{R}^{n+1}$ is a compact, connected set of Hausdorff dimension n without boundary, which may be parametrised by the C^k embedding $X : N \to M$. With respect to geometric quantities on M, all indexing sets to be considered are in direct correspondence with the set of integers $\{1, ..., n\}$

For each $p \in N$, the set of coordinate tangent vectorfields $\{\frac{\partial X}{\partial p_i}(p) : 1 \leq i \leq n\}$ provides a basis for the *tangent space* T_xM of M at $x = X(p)$. Conversely, the dual space of T_xM is called the *cotangent space* of M at x, T_x^*M, which corresponds to the space of all linear functionals $\mathcal{L}(T_xM; \mathbb{R})$. T_x^*M is spanned by the basis of coordinate differential one-forms or covectorfields $\{dX^i(p) : 1 \leq i \leq n\}$. We then define the *[co]tangent bundle* TM $[T^*M]$ of M to be the union of all such [co]tangent spaces:

$$TM = \bigcup_{x \in M} T_xM \quad \left[T^*M = \bigcup_{x \in M} T_x^*M\right].$$

The (unique) *outward unit normal vectorfield* ν on M is orthogonal to TM and may be computed explicitly by the normalised cross product on $\mathbb{R}^{n+1}$:

$$\nu = \frac{\frac{\partial X}{\partial p_1} \times ... \times \frac{\partial X}{\partial p_n}}{\left|\frac{\partial X}{\partial p_1} \times ... \times \frac{\partial X}{\partial p_n}\right|}.$$

For each $x \in M$, the components of the *metric* on M are given by the Euclidean inner products:

$$g_{ij} = \left\langle \frac{\partial X}{\partial p_i}, \frac{\partial X}{\partial p_j} \right\rangle_{\mathbb{R}^{n+1}}.$$

In the classical literature the metric is sometimes referred to as the *first fundamental form.* Endowed with this concept of measurement, $(M, \mathbf{g})$ is a complete metric space. Indeed,

$(M, \mathbf{g})$ is a compact, connected Riemannian manifold of class C^k, which is without boundary and embedded in $\mathbb{R}^{n+1}$.

The metric $\mathbf{g} : TM \times TM \to \mathbb{R}$ is a positive definite, symmetric, bilinear form, and admits an *inverse metric* $(g^{ij}) = \mathbf{g}^{-1}$. Observe that raised indices shall indicate contraction over the inverse metric. If we take $g = \det \mathbf{g}$, then the *surface measure* on M is given by

$$\mu \;=\; \left| \frac{\partial X}{\partial p_1} \times \ldots \times \frac{\partial X}{\partial p_n} \right| \;=\; \sqrt{\frac{g}{\tilde{g}}}\,.$$

We note that we may interpret any function $f \in C^0(M;\mathbb{R})$ as an element of $C^0(N;\mathbb{R})$ through composition with X, and are therefore able to characterise integration on M in the following manner:

$$\int_M f \;=\; \int_M f(x)\, d\mu(x) \;=\; \int_N f \circ X(p)\, \mu(p)\, d\tilde{\mu}(p)\,.$$

The *tangential gradient* of a function $f \in C^1(M;\mathbb{R})$ is defined in local coordinates by

$$\nabla^M f \;=\; g^{ij} \frac{\partial}{\partial p_j} (f \circ X) \frac{\partial X}{\partial p_i} \;=\; g^{ij} \left\langle D_x f, \frac{\partial X}{\partial p_j} \right\rangle_{\mathbb{R}^{n+1}} \frac{\partial X}{\partial p_i} \;=\; g^{ij} \nabla_j f \, \frac{\partial X}{\partial p_i}\,.$$

Here, for each $i \in \{1, \ldots, n\}$, the quantity $\nabla_i f$ is the (first-order) *covariant derivative* of f in the direction of the i^{th} coordinate tangent vectorfield. In this setting it is important to emphasise that we may also express the tangential gradient with respect to the coordinates on M through orthogonal projection of $D_x f$ onto $T_x M$:

$$\nabla^M f \;=\; (D_x f)^{\top} \;=\; D_x f \,-\, \langle D_x f, \nu \rangle_{\mathbb{R}^{n+1}}\, \nu\,.$$

Suppose

$$Y \;=\; Y^j \frac{\partial X}{\partial p_j} \;=\; g^{ij}\, Y_i \frac{\partial X}{\partial p_j}$$

is a tangential vectorfield on M of class C^1. Then we define the components of the *covariant derivative tensorfield* of Y on M in local coordinates by

$$\nabla_i^M Y^j \;=\; \frac{\partial}{\partial p_i} Y^j + \Gamma_{ik}^j Y^k \;=\; g^{jm} \left(\frac{\partial}{\partial p_i} Y_m \,-\, \Gamma_{im}^k Y_k \right).$$

Here $\boldsymbol{\Gamma} = \left(\Gamma_{ij}^k\right)$ is the (induced) *Riemannian* or *Levi-Civita connection*[4] on M whose components, symmetric in their covariant indices, are given in local coordinates by the *Christoffel symbols*:

$$\Gamma_{ij}^k \;=\; g^{km} \left\langle \frac{\partial^2 X}{\partial p_i \partial p_j}, \frac{\partial X}{\partial p_m} \right\rangle_{\mathbb{R}^{n+1}}.$$

More generally, we may express these components with respect to the metric:

$$\Gamma_{ij}^k \;=\; \frac{g^{km}}{2} \left(\frac{\partial}{\partial p_i}\, g_{jm} + \frac{\partial}{\partial p_j}\, g_{im} \,-\, \frac{\partial}{\partial p_m}\, g_{ij} \right).$$

---

[4] It is important to emphasise that the connection is not a tensorfield, since it is not invariant under coordinate transformation.

The *tensor bundle* of (tangential) tensorfields over M identified with type $\{p,q\} \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$ shall be denoted by $\mathscr{T}_q^p(M)$, where

$$\mathscr{T}_q^p(M) \simeq \underbrace{TM \otimes ... \otimes TM}_{p \text{ copies}} \otimes \underbrace{T^*M \otimes ... \otimes T^*M}_{q \text{ copies}} .$$

Note that we take $\mathscr{T}_0{}^0(M) \equiv \mathbb{R}$. If $\mathcal{T} \in \mathscr{T}_q^p(M)$, then we write $\mathcal{T} = (\mathcal{T}_{j_1...j_q}^{k_1...k_p})$.

We now extend our notion of covariant differentiation to tensorfields of arbitrary type. Suppose $\mathcal{T} \in \mathscr{T}_q^p(M)$ is a tensorfield of class C^1. Then the components of the covariant derivative tensorfield of $\mathcal{T}$ are determined explicitly in local coordinates by

$$\nabla_i^M \mathcal{T}_{j_1...j_q}^{k_1...k_p} = \frac{\partial}{\partial p_i} \mathcal{T}_{j_1...j_q}^{k_1...k_p} + \sum_{m=1}^{p} \Gamma_{il}^{k_m} \mathcal{T}_{j_1...j_q}^{k_1...k_{m-1}\, l\, k_{m+1}...k_p} - \sum_{m=1}^{q} \Gamma_{ij_m}^{l} \mathcal{T}_{j_1...j_{m-1}\, l\, j_{m+1}...j_q}^{k_1...k_p} .$$

Here we neglect the undefined terms in the multi-indices at the first and final entries of the sums above. In this case, we deduce that $\nabla^M \mathcal{T} = (\nabla_i^M \mathcal{T}_{j_1...j_q}^{k_1...k_p}) \in \mathscr{T}_{q+1}^p(M)$. In general, for $\mathcal{T} \in \mathscr{T}_q^p(M)$ of class C^k, $\nabla_M^k \mathcal{T} = (\nabla_{i_1}^M ... \nabla_{i_k}^M \mathcal{T}_{j_1...j_q}^{k_1...k_p}) \in \mathscr{T}_{q+k}^p(M)$ shall denote the k^{th} iterated covariant derivative tensorfield of $\mathcal{T}$. Equipped with this definition for the covariant differentiation of tensorfields and that of the connection, we recover the ensuing axiom of Riemannian geometry:

$$\nabla_k^M g_{ij} \equiv 0 ,$$

for each $i, j, k \in \{1, ..., n\}$.

We may now define a (pointwise) *Riemannian inner product* on the tensor bundles over M identified by equivalent type. If both $\mathcal{T}$ and $\mathcal{W}$ are elements of $\mathscr{T}_q^p(M)$, then, at each $x \in M$,

$$\langle \mathcal{T}(x), \mathcal{W}(x) \rangle_M = \prod_{m=1}^{p} \prod_{r=1}^{q} g_{i_m k_m}(x) g^{j_r l_r}(x) \mathcal{T}_{j_1...j_q}^{i_1...i_p}(x) \mathcal{W}_{l_1...l_q}^{k_1...k_p}(x) .$$

This inner product then induces a *norm* on any such tensorfield:

$$\|\mathcal{T}(x)\| = \langle \mathcal{T}(x), \mathcal{T}(x) \rangle_M^{\frac{1}{2}} .$$

We may further define the *tensor product* $\star$ which operates across these tensor bundles[5]. Suppose $\mathcal{T} \in \mathscr{T}_q^p(M)$ and $\mathcal{W} \in \mathscr{T}_s^r(M)$. Then $\mathcal{T} \star \mathcal{W} \in \mathscr{T}_{q+s}^{p+r}(M)$, with components given by

$$(\mathcal{T} \star \mathcal{W})_{j_1...j_{q+s}}^{i_1...i_{p+r}} = \mathcal{T}_{l_1...l_q}^{k_1...k_p} \mathcal{W}_{v_1...v_s}^{m_1...m_r} .$$

[5] It shall occasionally be convenient to abuse this notation and consider the tensor product $\mathcal{T} \star \mathcal{W}$ as a contraction resulting in an element of $\mathscr{T}_{|q-s|}^{|p-r|}(M)$. For example, such a product may have components determined by

$$(\mathcal{T} \star \mathcal{W})_{j_1...j_{|q-s|}}^{i_1...i_{|p-r|}} = \prod_{m=1}^{\min\{p,r\}} \prod_{r=1}^{\min\{q,s\}} g_{i_m k_m} g^{j_r l_r} \mathcal{T}_{j_1...j_q}^{i_1...i_p} \mathcal{W}_{l_1...l_s}^{k_1...k_r} .$$

The context of the argument will generally indicate whether such an interpretation has been employed.

Furthermore, the operation $\circledast$ shall denote the concatenated product of an arbitrary number of tensorfields on M. More precisely, if $\{\mathcal{T}_i \in \mathscr{T}_{q_i}^{p_i} : i,k \in \mathbb{N} \;\&\; 1 \le i \le k\}$, then

$$\prod_{i=1}^{k} \circledast \mathcal{T}_i \;=\; \mathcal{T}_1 \star \ldots \star \mathcal{T}_k \,.$$

Suppose again that Y is a tangential vectorfield on M of class C^1. Then the *tangential divergence* of Y on M may be written

$$\mathrm{div}_M Y \;=\; \nabla_i^M Y^i \;=\; g^{ij}\nabla_i^M Y_j \;=\; \frac{1}{\sqrt{g}}\frac{\partial}{\partial p_i}\left(\sqrt{g}\, g^{ij}\, Y_j\right).$$

Conversely, if the vectorfield $V \in C^1(M;\mathbb{R}^{n+1})$ is not tangential to M, then we may characterise the divergence of V with respect to M in the following manner:

$$\mathrm{div}_M V \;=\; g^{ij}\left\langle \frac{\partial V}{\partial p_i}, \frac{\partial X}{\partial p_j}\right\rangle_{\mathbb{R}^{n+1}}.$$

For any $f \in C^2(M;\mathbb{R})$, we may further define the *Laplace-Beltrami operator* or *Laplacian* on M by

$$\Delta^M f \;=\; g^{ij}\nabla_i\nabla_j^M f \;=\; \mathrm{div}_M\left(\nabla^M f\right).$$

Here $\nabla_M^2 f = (\nabla_i\nabla_j^M f) \in \mathscr{T}_2{}^0(M)$ is called the *Hessian* of f on M. In local coordinates, we therefore compute that

$$\Delta^M f \;=\; g^{ij}\left(\frac{\partial^2 f}{\partial p_i \partial p_j} - \Gamma_{ij}^k \frac{\partial f}{\partial p_k}\right) \;=\; \frac{1}{\sqrt{g}}\frac{\partial}{\partial p_i}\left(\sqrt{g}\, g^{ij}\frac{\partial f}{\partial p_j}\right).$$

We now address the concept of curvature on M. The *second fundamental form* $A = (h_{ij})$ of M is the symmetric, bilinear form on $TM \times TM$ with components determined by

$$h_{ij} \;=\; \left\langle \frac{\partial X}{\partial p_i}, \frac{\partial \nu}{\partial p_j}\right\rangle_{\mathbb{R}^{n+1}}.$$

The *Weingarten map* of M is the operator $h^i{}_j : TM \longrightarrow TM$ with components given by

$$h^i{}_j \;=\; g^{ik}h_{kj}\,.$$

It is a simple matter to demonstrate that the Weingarten map is self-adjoint with respect to the Riemannian inner product on tangential vectorfields specified above. We may then interpret the *principal curvatures* $\{\kappa_i : 1 \le i \le n\}$ of M as the eigenvalues of the Weingarten map with respect to the identity transformation on TM. The *mean curvature* H of M may then be perceived in a variety of ways:

$$H \;=\; \sum_{i=1}^{n}\kappa_i \;=\; h^i{}_i \;=\; g^{ij}h_{ij} \;=\; \mathrm{div}_M \nu\,.$$

For each $i,j \in \{1,\ldots,n\}$, the *Gauss-Weingarten relations* are given by

$$\frac{\partial \nu}{\partial p_i} \;=\; h^k{}_i\frac{\partial X}{\partial p_k} \;=\; g^{kl}h_{li}\frac{\partial X}{\partial p_k}\,,$$

and

$$\frac{\partial^2 X}{\partial p_i p_j} = \Gamma^k_{ij}\frac{\partial X}{\partial p_k} - h_{ij}\,\nu\,.$$

If we take $\overrightarrow{H} = -H\nu$ to be the *mean curvature vectorfield* on M, then we may recover the identity

$$\Delta^M X = \overrightarrow{H}\,.$$

The *Riemann curvature tensor*, $\mathrm{Rm} = (R_{ijkl}) \in \mathscr{T}_4{}^0(M)$, of M has components given by

$$R_{ijkl} = \left\langle \nabla_i \nabla^M_j \left(\frac{\partial X}{\partial p_k}\right) - \nabla_j \nabla^M_i \left(\frac{\partial X}{\partial p_k}\right), \frac{\partial X}{\partial p_l}\right\rangle_{\mathbb{R}^{n+1}}.$$

The *Gauss equations* represent this tensor in terms of the second fundamental form:

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}\,.$$

Through contraction over the inverse metric, we may consequently determine the *Ricci curvature tensor*, $\mathrm{Ric} = (R_{ij}) \in \mathscr{T}_2{}^0(M)$, of M,

$$R_{ij} = g^{kl}R_{ikjl} = Hh_{ij} - h_{im}h^m{}_j\,,$$

and the *scalar curvature* R of M,

$$R = g^{ij}R_{ij} = H^2 - \|A\|^2\,.$$

Here $\|A\|^2 = g^{ik}g^{jl}h_{ij}h_{kl} = h^i{}_j h^j{}_i$ is the square of the norm of the second fundamental form.

The *Codazzi equations* confirm that the tensorfield containing all first-order covariant derivatives of the second fundamental form, $\nabla^M A = (\nabla_k h_{ij}) \in \mathscr{T}_3{}^0(M)$, is completely symmetric. In particular, for each $i, j, k \in \{1, ..., n\}$,

$$\nabla^M_k h_{ij} = \nabla^M_j h_{ki} = \nabla^M_i h_{jk}\,.$$

We may then infer the following identity from the Codazzi equations:

$$\Delta^M \nu = -\|A\|^2\nu + \nabla^M H\,.$$

We have seen that the concept of the covariant derivative may be extended to encompass tensorfields. Therefore we may compute the Laplacian of a tensorfield in accordance with our pre-existing definition and, by interchanging second-order covariant derivatives of the second fundamental form, whilst invoking the Gauss and Codazzi equations, we obtain *Simons' identity*[6] (see the proof of [**75**, Lemma B.8] for details of this calculation):

$$\Delta^M h_{ij} = \nabla_i\nabla^M_j H - \|A\|^2 h_{ij} + Hh_{im}h^m{}_j\,.$$

Furthermore, through contraction over the quantity $h^{ij} = g^{ik}g^{jl}h_{kl}$, we deduce that

$$\Delta^M\|A\|^2 = 2\left(h^{ij}\nabla_i\nabla^M_j H + \left\|\nabla^M A\right\|^2 + Hh^i{}_k h^j{}_i h^k{}_j - \|A\|^4\right),$$

[6] To avoid potential ambiguity in the derivation of this terminology, we note that this identity is so named as a consequence of calculations which appear in the proof of [**76**, Theorem 4.2.1].

where $\left\|\nabla^M A\right\|^2$ may be computed explicitly in accordance with our definition:

$$\left\|\nabla^M A\right\|^2 = g^{il}g^{jm}g^{kp}\nabla_i^M h_{jk}\nabla_l^M h_{mp} .$$

We shall conclude this section by stating the two forms of the divergence theorem which are pertinent to the explication of this thesis. Their proofs may be found in [**75**].

THE DIVERGENCE THEOREM I. *Suppose that F is a compact subset of $\mathbb{R}^{n+1}$, $\partial F = (M^n, \mathrm{g})$ is a Riemannian manifold of class C^1, and X is a C^1 vectorfield defined on F. Then*

$$\int_F \mathrm{div}_{\mathbb{R}^{n+1}} X\, dx = \int_M \langle X, \nu\rangle_{\mathbb{R}^{n+1}}\, d\mu .$$

THE DIVERGENCE THEOREM II. *Suppose that (M^n, g) is a compact, orientable Riemannian manifold without boundary of class C^2 embedded in $\mathbb{R}^{n+1}$, and X is a C^1 vectorfield defined on M. Then*

$$\int_M \mathrm{div}_M X\, d\mu = -\int_M \left\langle X, \vec{H}\right\rangle_{\mathbb{R}^{n+1}} d\mu = \int_M H\,\langle X, \nu\rangle_{\mathbb{R}^{n+1}}\, d\mu .$$

2. Geometric identities

In this section we derive some geometric identities under covariant differentiation which shall be efficacious to analysis performed throughout the dissertation. While the following lemma is a simple consequence of the product rule, we include it here for direct citation in the calculation of various evolution equations appearing in the work of Chapter 6.

LEMMA A.1. *Suppose (M^n, g) is a Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$ with $\eta, \xi \in C^2(M;\mathbb{R})$. Then, whenever the quotient is well defined,*

$$\Delta^M\left(\frac{\eta}{\xi}\right) = \frac{\xi\Delta^M\eta - \eta\Delta^M\xi}{\xi^2} - \frac{2}{\xi}\left\langle\nabla^M\left(\frac{\eta}{\xi}\right), \nabla^M\xi\right\rangle.$$

PROOF

We compute that

$$\begin{aligned}
\nabla_i\nabla_j\left(\frac{\eta}{\xi}\right) &= \nabla_i\left(\frac{\xi\nabla_j\eta - \eta\nabla_j\xi}{\xi^2}\right) \\
&= \frac{\xi\nabla_i\nabla_j\eta - \eta\nabla_i\nabla_j\xi}{\xi^2} + \frac{\nabla_i\xi\nabla_j\eta - \nabla_i\eta\nabla_j\xi}{\xi^2} - \frac{2}{\xi^3}\left(\xi\nabla_j\eta - \eta\nabla_j\xi\right)\nabla_i\xi \\
&= \frac{\xi\nabla_i\nabla_j\eta - \eta\nabla_i\nabla_j\xi}{\xi^2} + \frac{\nabla_i\xi\nabla_j\eta - \nabla_i\eta\nabla_j\xi}{\xi^2} - \frac{2}{\xi}\nabla_i\xi\nabla_j\left(\frac{\eta}{\xi}\right).
\end{aligned}$$

Thus

$$\Delta\left(\frac{\eta}{\xi}\right) = \frac{\xi\Delta\eta - \eta\Delta\xi}{\xi^2} - \frac{2}{\xi}\left\langle\nabla\left(\frac{\eta}{\xi}\right), \nabla\xi\right\rangle,$$

as required. ◇

The ensuing lemma and its two corollaries characterise covariant derivatives for position vectorfields and their components on oriented Riemannian manifolds. Although we intuitively expect such derivatives of order k to contain curvature derivatives of at worst order $(k-2)$, it is important to formally distinguish these quantities for application in Theorem 6.11 and Corollary 6.12 of Chapter 6. Throughout the remainder of this section recall that, whenever $j < k$, we continue to adopt the conventions that $\sum_{i=k}^{j}(\cdot) \equiv 0$ and $\prod_{i=k}^{j}(\cdot) \equiv 1$.

LEMMA A.2. *Suppose $(M^n, \mathbf{g})$ is an oriented Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$, X is the position vectorfield on M, and $\mathscr{Z} = \{\zeta_i \in C^\infty(M) : i \in \{1, ..., n+1\}\}$ is the set with elements defined by $\zeta_i = \langle X, e_i\rangle$. Then, for each $l \in \mathbb{N}$,*

$$\nabla^{2l}\zeta_i = \langle\nu, e_i\rangle \sum_{j=1}^{l} \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A$$

$$+ \nabla\zeta_i \star \sum_{j=1}^{l-1} \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A\,,$$

and

$$\nabla^{2l+1}\zeta_i = \sum_{j=1}^{l}\left(\langle\nu, e_i\rangle \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A\right.$$

$$\left. + \nabla\zeta_i \star \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A\right).$$

PROOF

We proceed by induction on $\mathbb{N}$ and first verify the case $l = 1$. As a consequence of the Gauss-Weingarten relations, we note that

$$\nabla^2\zeta_i = \langle\nu, e_i\rangle A\,, \tag{A.1}$$

and

$$\nabla\langle\nu, e_i\rangle = \nabla\zeta_i \star A\,. \tag{A.2}$$

Thus

$$\nabla^3\zeta_i = \nabla\left(\nabla^2\zeta_i\right) = \nabla\left(\langle\nu, e_i\rangle A\right) = \langle\nu, e_i\rangle \nabla A + \nabla\zeta_i \star A \star A\,.$$

The lemma is then established for $l = 1$. Suppose now that the lemma holds for a particular $l = p \in \mathbb{N}$. We consider the even case, where we shall repeatedly apply (A.1) and (A.2):

$$
\begin{aligned}
&\nabla^{2(p+1)}\zeta_i \\
&= \nabla^2 \left(\langle \nu, e_i \rangle \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \nabla\zeta_i \star \sum_{j=1}^{p-1} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right) \\
&= \nabla \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \left(\nabla\zeta_i \star A \star \nabla^{k_1} A + \langle \nu, e_i \rangle \nabla^{k_1+1} A \right) \star \prod_{m=2}^{2j-1} \circledast \nabla^{k_m} A \right. \\
&\quad \left. + \sum_{j=1}^{p-1} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \left(\langle \nu, e_i \rangle A \star \nabla^{k_1} A + \nabla\zeta_i \star \nabla^{k_1+1} A \right) \star \prod_{m=2}^{2j} \circledast \nabla^{k_m} A \right) \\
&= \nabla \left(\langle \nu, e_i \rangle \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=1}^{p-1} \sum_{\sum_{m=1}^{2j+1} k_m = 2(p-j)-1} \prod_{m=1}^{2j+1} \circledast \nabla^{k_m} A \right) \right. \\
&\quad \left. + \nabla\zeta_i \star \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A + \sum_{j=1}^{p-1} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right) \right) \\
&= \nabla \left(\langle \nu, e_i \rangle \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=2}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \right) \right. \\
&\quad \left. + \nabla\zeta_i \star \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right) \\
&= \nabla \left(\langle \nu, e_i \rangle \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \nabla\zeta_i \star \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+1} \left(\nabla\zeta_i \star A \star \nabla^{k_1} A \,+\, \langle \nu, e_i\rangle\, \nabla^{k_1+1} A\right) \star \prod_{m=2}^{2j-1} \circledast\, \nabla^{k_m} A \\
&\quad + \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \left(\langle \nu, e_i\rangle\, A \star \nabla^{k_1} A \,+\, \nabla\zeta_i \star \nabla^{k_1+1} A\right) \star \prod_{m=2}^{2j} \circledast\, \nabla^{k_m} A \\
&= \langle \nu, e_i\rangle \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+2} \prod_{m=1}^{2j-1} \circledast\, \nabla^{k_m} A + \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j+1} k_m = 2(p-j)} \prod_{m=1}^{2j+1} \circledast\, \nabla^{k_m} A \right) \\
&\quad + \nabla\zeta_i \star \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)+1} \prod_{m=1}^{2j} \circledast\, \nabla^{k_m} A \\
&= \langle \nu, e_i\rangle \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+2} \prod_{m=1}^{2j-1} \circledast\, \nabla^{k_m} A + \sum_{j=2}^{p+1} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+2} \prod_{m=1}^{2j-1} \circledast\, \nabla^{k_m} A \right) \\
&\quad + \nabla\zeta_i \star \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)+1} \prod_{m=1}^{2j} \circledast\, \nabla^{k_m} A \\
&= \langle \nu, e_i\rangle \sum_{j=1}^{p+1} \sum_{\sum_{m=1}^{2j-1} k_m = 2((p+1)-j)} \prod_{m=1}^{2j-1} \circledast\, \nabla^{k_m} A \\
&\quad + \nabla\zeta_i \star \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2((p+1)-j)-1} \prod_{m=1}^{2j} \circledast\, \nabla^{k_m} A\,.
\end{aligned}
$$

Therefore the lemma holds at $l = p + 1$ in the even case. By a parallel computation in the odd case, we determine that the lemma is valid in generality for $l = p + 1$. Hence we verify the lemma by the principle of mathematical induction. $\diamondsuit$

COROLLARY A.3. *Suppose (M^n, g) is an oriented Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$, X is the position vectorfield on M, and $\mathscr{X} = \{\zeta_i \in C^\infty(M) : i \in \{1, ..., n+1\}\}$ is the set with elements defined by $\zeta_i = \langle X, e_i \rangle$. Then, for each $l \in \mathbb{N} \setminus \{1\}$,*

$$\begin{aligned}\nabla^{2l}\zeta_i^2 &= \zeta_i \langle \nu, e_i \rangle \sum_{j=1}^{l} \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \\ &+ \sum_{j=1}^{l-1} \left(\nabla\zeta_i^2 \star \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A + \langle \nu, e_i \rangle \nabla\zeta_i \star \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \right. \\ &+ \left. \left(\langle \nu, e_i \rangle^2 + \nabla\zeta_i \star \nabla\zeta_i \star\right) \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-2} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right).\end{aligned}$$

While, for each $l \in \mathbb{N}$,

$$\begin{aligned}\nabla^{2l+1}\zeta_i^2 &= \sum_{j=1}^{l} \left(\zeta_i \langle \nu, e_i \rangle \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \right. \\ &+ \left. \nabla\zeta_i^2 \star \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A + \langle \nu, e_i \rangle \nabla\zeta_i \star \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \right) \\ &+ \sum_{j=1}^{l-1} \left(\langle \nu, e_i \rangle^2 + \nabla\zeta_i \star \nabla\zeta_i \star\right) \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A .\end{aligned}$$

PROOF

We shall merely prove the odd case since an entirely analogous calculation yields the formula for derivatives of even order. It is, however, important to observe that the construction fails for $l = 1$ in the even case as

$$\nabla^2\zeta_i^2 = \nabla\zeta_i \star \nabla\zeta_i + \zeta_i \langle \nu, e_i \rangle A .$$

We employ Lemma A.2 to compute that, for each $l \in \mathbb{N}$,

$$\nabla^{2l+1}\zeta_i^2 = \sum_{j+k=2l+1} \nabla^j\zeta_i \star \nabla^k\zeta_i = \zeta_i\nabla^{2l+1}\zeta_i + \nabla\zeta_i \star \nabla^{2l}\zeta_i + \sum_{j+k=2l-3} \nabla^{j+2}\zeta_i \star \nabla^{k+2}\zeta_i$$

$$= \zeta_i \left(\sum_{j=1}^{l} \left(\langle \nu, e_i \rangle \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \nabla \zeta_i \star \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right) \right)$$

$$+ \nabla \zeta_i \star \left(\langle \nu, e_i \rangle \sum_{j=1}^{l} \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \right.$$

$$\left. + \nabla \zeta_i \star \sum_{j=1}^{l-1} \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right) + \sum_{k \le 2l-3} \nabla^{k+2} \zeta_i \star \nabla^{2l-(k+1)} \zeta_i \, . \tag{A.3}$$

We must now evaluate the term $\nabla^{k+2}\zeta_i \star \nabla^{2l-(k+1)}\zeta_i$ which occurs in the final sum of this expression. We note that whenever $l > 1$ the quantity $(2l-3) \in \mathbb{N}$ is odd. Thus we may, without loss of generality, assume that k is odd with $(2l-(k+1))$ even, since the essential structure of the linear combination of tensor products remains unchanged. Hence there exists a $p \in \mathbb{N} \cup \{0\}$ such that $k = 2p+1$. Therefore, we may again cite Lemma A.2 to discern that

$$\sum_{k \le 2l-3} \nabla^{k+2} \zeta_i \star \nabla^{2l-(k+1)} \zeta_i = \sum_{q \le l-2} \nabla^{2q+3} \zeta_i \star \nabla^{2(l-q-1)} \zeta_i$$

$$= \sum_{q \le l-2} \left(\sum_{j=1}^{q+1} \left(\langle \nu, e_i \rangle \sum_{\sum_{m=1}^{2j-1} k_m = 2(q-j)+3} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \right. \right.$$

$$\left. + \nabla \zeta_i \star \sum_{\sum_{m=1}^{2j} k_m = 2(q-j)+2} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right) \star$$

$$\left(\langle \nu, e_i \rangle \sum_{r=1}^{l-q-1} \sum_{\sum_{p=1}^{2r-1} k_p = 2(l-q-r)-2} \prod_{p=1}^{2r-1} \circledast \nabla^{k_p} A \right.$$

$$\left. \left. + \nabla \zeta_i \star \sum_{r=1}^{l-q-2} \sum_{\sum_{p=1}^{2r} k_p = 2(l-q-r)-3} \prod_{p=1}^{2r} \circledast \nabla^{k_p} A \right) \right)$$

$$= \langle \nu, e_i \rangle^2 \sum_{j+r=2}^{l} \sum_{\sum_{m=1}^{2(j+r)-2} k_m = 2(l-(j+r))+1} \prod_{m=1}^{2(j+r)-2} \circledast \nabla^{k_m} A$$

$$+ \langle \nu, e_i \rangle \nabla \zeta_i \star \sum_{j+r=2}^{l-1} \sum_{\sum_{m=1}^{2(j+r)-1} k_m = 2(l-(j+r))} \prod_{m=1}^{2(j+r)-1} \circledast \nabla^{k_m} A$$

$$+ \langle \nu, e_i \rangle \nabla \zeta_i \star \sum_{j+r=2}^{l} \sum_{\sum_{m=1}^{2(j+r)-1} k_m = 2(l-(j+r))} \prod_{m=1}^{2(j+r)-1} \circledast \nabla^{k_m} A$$

$$+ \nabla \zeta_i \star \nabla \zeta_i \star \sum_{j+r=2}^{l-1} \sum_{\sum_{m=1}^{2(j+r)} k_m = 2(l-(j+r))-1} \prod_{m=1}^{2(j+r)} \circledast \nabla^{k_m} A$$

$$= \langle \nu, e_i \rangle^2 \sum_{j=1}^{l-1} \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A + \langle \nu, e_i \rangle \nabla \zeta_i \star \sum_{j=2}^{l} \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A$$

$$+ \nabla \zeta_i \star \nabla \zeta_i \star \sum_{j=2}^{l-1} \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \,. \tag{A.4}$$

We may now substitute (A.4) into (A.3) to obtain our intended result, where we have observed that $\nabla \zeta_i^2 \sim \zeta_i \nabla \zeta_i$. ◊

COROLLARY A.4. *Suppose $(M^n, \mathbf{g})$ is an oriented Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$ and X is the position vectorfield on M. Then, for each $l \in \mathbb{N} \setminus \{1\}$,*

$$\nabla^{2l} |X|^2 = \langle X, \nu \rangle \sum_{j=1}^{l} \sum_{\sum_{m=1}^{2j-1} k_m = 2(l-j)} \prod_{m}^{2j-1} \circledast \nabla^{k_m} A$$

$$+ \sum_{j=1}^{l-1} \left(\nabla |X|^2 \star \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A + \sum_{\sum_{m=1}^{2j} k_m = 2(l-j)-2} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \right).$$

While, for each $l \in \mathbb{N}$,

$$\nabla^{2l+1}|X|^2 = \sum_{j=1}^{l}\left(\langle X,\nu\rangle \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)+1} \prod_{m}^{2j-1} \circledast \nabla^{k_m}A \right.$$

$$\left. + \nabla |X|^2 \star \sum_{\sum_{m=1}^{2j} k_m=2(l-j)} \prod_{m}^{2j} \circledast \nabla^{k_m}A \right) + \sum_{j=1}^{l-1} \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-1} \prod_{m}^{2j} \circledast \nabla^{k_m}A\,.$$

PROOF

For the sake of notational convenience, we shall reintroduce the set $\mathscr{X}$ defined in the statements of Lemma A.2 and Corollary A.3. We then ascertain from Corollary A.3 that, for each $l \in \mathbb{N}\setminus\{1\}$,

$$\nabla^{2l}|X|^2 = \sum_{i=1}^{n+1} \nabla^{2l}\zeta_i^2$$

$$= \sum_{i=1}^{n+1}\left(\sum_{j=1}^{l}\left(\zeta_i \langle \nu, e_i\rangle \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m}A\right)\right.$$

$$+ \sum_{j=1}^{l-1}\left(\nabla\zeta_i^2 \star \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m}A\right.$$

$$+ \langle \nu, e_i\rangle \nabla\zeta_i \star \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m}A$$

$$\left.\left. + \left(\langle \nu, e_i\rangle^2 + \nabla\zeta_i \star \nabla\zeta_i \star\right) \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-2} \prod_{m=1}^{2j} \circledast \nabla^{k_m}A\right)\right)$$

$$= \sum_{j=1}^{l}\left(\langle X,\nu\rangle \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m}A\right)$$

$$+\sum_{j=1}^{l-1}\left(\nabla |X|^2 \star \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A\right.$$

$$+\sum_{i=1}^{n+1}\left(\langle \nu, e_i\rangle \nabla\zeta_i\right) \star \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A$$

$$\left.+\left(1+\sum_{i=1}^{n+1}\nabla\zeta_i \star \nabla\zeta_i \star\right) \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-2} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A\right). \tag{A.5}$$

We may now cite the Gauss-Weingarten relations to assert that

$$\begin{aligned}
\sum_{i=1}^{n+1}\left(\langle \nu, e_i\rangle \nabla\zeta_i\right) &= \sum_{i=1}^{n+1}\left(\nabla\left(\zeta_i \langle \nu, e_i\rangle\right) - \zeta_i \nabla \langle \nu, e_i\rangle\right) \\
&= \nabla \langle X, \nu\rangle + \sum_{i=1}^{n+1} \zeta_i \nabla\zeta_i \star A \\
&\sim \nabla |X|^2 \star A + \sum_{i=1}^{n+1} \nabla\zeta_i^2 \star A \\
&= \nabla |X|^2 \star A \,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{l-1}\left(\sum_{i=1}^{n+1}\left(\langle \nu, e_i\rangle \nabla\zeta_i\right) \star \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A\right) \\
&= \sum_{j=1}^{l-1} \nabla |X|^2 \star A \star \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \\
&= \sum_{j=1}^{l-1} \nabla |X|^2 \star \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-1} \prod_{m=1}^{2j} \circledast \nabla^{k_m} A \,.
\end{aligned} \tag{A.6}$$

Moreover, by again invoking the Gauss-Weingarten relations, we deduce that

$$\sum_{i=1}^{n+1} \nabla\zeta_i \star \nabla\zeta_i \sim \sum_{i=1}^{n+1}\left(\nabla^2\zeta_i^2 - \zeta_i\nabla^2\zeta_i\right) = \nabla^2|X|^2 + \sum_{i=1}^{n+1}\zeta_i \langle \nu, e_i\rangle A = \mathbf{g} + \langle X,\nu\rangle A\,.$$

Thus

$$\sum_{j=1}^{l-1}\left(\sum_{i=1}^{n+1}\nabla\zeta_i\star\nabla\zeta_i\star \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-2}\ \prod_{m=1}^{2j}\circledast\nabla^{k_m}A\right)$$

$$= \sum_{j=1}^{l-1}(\mathbf{g} + \langle X,\nu\rangle A)\star \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-2}\ \prod_{m=1}^{2j}\circledast\nabla^{k_m}A$$

$$= \sum_{j=1}^{l-1}\ \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-2}\ \prod_{m=1}^{2j}\circledast\nabla^{k_m}A + \sum_{j=1}^{l-1}\langle X,\nu\rangle \sum_{\sum_{m=1}^{2j+1} k_m=2(l-j)-2}\ \prod_{m=1}^{2j+1}\circledast\nabla^{k_m}A$$

$$= \sum_{j=1}^{l-1}\ \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-2}\ \prod_{m=1}^{2j}\circledast\nabla^{k_m}A + \sum_{j=2}^{l}\langle X,\nu\rangle \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)}\ \prod_{m=1}^{2j-1}\circledast\nabla^{k_m}A\,. \qquad (A.7)$$

We therefore obtain our desired result for covariant derivatives of even order by substituting (A.6) and (A.7) into (A.5). In an analogous manner, we may derive the result in the odd case. ◇

LEMMA A.5. *Suppose $(M^n, \mathbf{g})$ is an oriented Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$, X is the position vectorfield on M, and ν is the outward oriented unit normal vectorfield on M. Then, for each $l \in \mathbb{N}$,*

$$\nabla^{2l-1}\nu = \sum_{j=1}^{l}\ \sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)}\ \prod_{m=1}^{2j-1}\circledast\nabla^{k_m}A + \sum_{j=1}^{l-1}\ \sum_{\sum_{m=1}^{2j} k_m=2(l-j)-1}\ \prod_{m=1}^{2j-1}\circledast\nabla^{k_m}A\star\nu\,,$$

and

$$\nabla^{2l}\nu = \sum_{j=1}^{l}\left(\sum_{\sum_{m=1}^{2j-1} k_m=2(l-j)+1}\ \prod_{m=1}^{2j-1}\circledast\nabla^{k_m}A + \sum_{\sum_{m=1}^{2j} k_m=2(l-j)}\ \prod_{m=1}^{2j-1}\circledast\nabla^{k_m}A\star\nu\right).$$

PROOF

We again proceed by induction on $\mathbb{N}$ and first confirm the case $l = 1$. By the Gauss-Weingarten relations, we determine that

$$\nabla\nu = A\,, \qquad (A.8)$$

and, through an application of the Codazzi equations,

$$\nabla^2 \nu \;=\; \nabla A + A \star A \star \nu\,. \tag{A.9}$$

Therefore the lemma is verified in the case $l = 1$. Suppose now that the lemma holds for a particular $l = p \in \mathbb{N}$. We evaluate the odd case where we shall repeatedly invoke (A.8) and (A.9):

$$\begin{aligned}
\nabla^{2p+1}\nu \;=\; & \nabla^2 \left(\sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=1}^{p-1} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nu \right) \\
= & \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)+2} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=1}^{p-1} \left(\sum_{\sum_{m=1}^{2j} k_m = 2(p-j)+1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nu \right. \\
& + \left. \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nabla \nu + \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nabla^2 \nu \right) \\
= & \sum_{j=2}^{p+1} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=2}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nu \\
& + \sum_{j=1}^{p-1} \left(\sum_{\sum_{m=1}^{2j} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star A \right. \\
& + \left. \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star (\nabla A + A \star A \star \nu) \right) \\
= & \sum_{j=2}^{p+1} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=2}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nu \\
& + \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nu
\end{aligned}$$

$$= \sum_{j=1}^{p+1} \sum_{\sum_{m=1}^{2j-1} k_m = 2(p-j)} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A + \sum_{j=1}^{p} \sum_{\sum_{m=1}^{2j} k_m = 2(p-j)-1} \prod_{m=1}^{2j-1} \circledast \nabla^{k_m} A \star \nu\,.$$

Thus the lemma is verified at $l = p+1$ in the odd case. Through an analogous calculation in the even case, we deduce that the lemma is valid in generality for $l = p+1$ and we have verified the result by the principle of mathematical induction. ◇

For a (tangential) tensorfield of arbitrary type and sufficient regularity on a Riemannian manifold, the ensuing lemma permits the extraction of the lowest order terms from a linear combination of tensor products determined by its covariant derivatives. This result is particularly relevant to the analysis contained in the proof of Theorem 6.11.

LEMMA A.6. *Suppose (M^n, g) is a Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$, $\mathcal{T} \in \mathscr{T}_q^p(M)$ is a tensorfield of class $W^{k,\infty}(M)$, and $l \in \mathbb{N}$ is fixed with $l \leq k$. Then, for each $j \in \mathbb{N}$,*

$$\sum_{\sum_{i=1}^{j} m_i = l} \prod_{i=1}^{j} \circledast \nabla^{m_i} \mathcal{T} = \sum_{s=0}^{j-1} \prod_{r=1}^{s} \circledast \mathcal{T} \star \sum_{\sum_{i=1}^{j-s}(m_i+1)=l} \prod_{i=1}^{j-s} \circledast \nabla^{m_i+1} \mathcal{T}\,.$$

PROOF

We shall prove the lemma by induction on $j \in \mathbb{N}$, where we commence by observing that the case $j = 1$ holds trivially. Suppose then that there exists a particular $j \in \mathbb{N}$ for which the lemma is true. Therefore

$$\sum_{\sum_{i=1}^{j} m_i = l} \prod_{i=1}^{j} \circledast \nabla^{m_i} \mathcal{T} = \sum_{s=0}^{j-1} \prod_{r=1}^{s} \circledast \mathcal{T} \star \sum_{\sum_{i=1}^{j-s}(m_i+1)=l} \prod_{i=1}^{j-s} \circledast \nabla^{m_i+1} \mathcal{T}\,. \tag{A.10}$$

Now,

$$\begin{aligned}
\sum_{\sum_{i=1}^{j+1} m_i = l} \prod_{i=1}^{j+1} \circledast \nabla^{m_i} \mathcal{T} &= \mathcal{T} \star \sum_{\sum_{i=1}^{j} m_i = l} \prod_{i=1}^{j} \circledast \nabla^{m_i} \mathcal{T} + \sum_{\sum_{i=1}^{j+1} m_i = l-1} \nabla^{m_1+1} \mathcal{T} \star \prod_{i=2}^{j+1} \circledast \nabla^{m_i} \mathcal{T} \\
&= \sum_{s=0}^{j-1} \prod_{r=1}^{s+1} \circledast \mathcal{T} \star \sum_{\sum_{i=1}^{j-s}(m_i+1)=l} \prod_{i=1}^{j-s} \circledast \nabla^{m_i+1} \mathcal{T} \\
&\quad + \sum_{t=0}^{l-1} \nabla^{t+1} \mathcal{T} \star \sum_{\sum_{i=1}^{j} m_i = l-1-t} \prod_{i=1}^{j} \circledast \nabla^{m_i} \mathcal{T}
\end{aligned}$$

$$= \sum_{s=1}^{j}\prod_{r=1}^{s}\circledast\mathcal{T}\star\sum_{\sum_{i=1}^{j-(s-1)}(m_i+1)=l}\ \prod_{i=1}^{j-(s-1)}\circledast\nabla^{m_i+1}\mathcal{T}$$

$$+\sum_{t=0}^{l-1}\nabla^{t+1}\mathcal{T}\star\sum_{s=0}^{j-1}\prod_{r=1}^{s}\circledast\mathcal{T}\star\sum_{\sum_{i=1}^{j-s}(m_i+1)=l-1-t}\ \prod_{i=1}^{j-s}\circledast\nabla^{m_i+1}\mathcal{T}$$

$$= \sum_{s=1}^{j}\prod_{r=1}^{s}\circledast\mathcal{T}\star\sum_{\sum_{i=1}^{(j+1)-s}(m_i+1)=l}\ \prod_{i=1}^{(j+1)-s}\circledast\nabla^{m_i+1}\mathcal{T}$$

$$+\sum_{s=0}^{j-1}\prod_{r=1}^{s}\circledast\mathcal{T}\star\sum_{\sum_{i=1}^{(j+1)-s}(m_i+1)=l}\ \prod_{i=1}^{(j+1)-s}\circledast\nabla^{m_i+1}\mathcal{T}$$

$$= \sum_{s=0}^{j}\prod_{r=1}^{s}\circledast\mathcal{T}\star\sum_{\sum_{i=1}^{(j+1)-s}(m_i+1)=l}\ \prod_{i=1}^{(j+1)-s}\circledast\nabla^{m_i+1}\mathcal{T}\,.$$

Here we have repeatedly employed (A.10). Thus the result is established for $j+1$ and we verify the lemma by the principle of mathematical induction. ◇

APPENDIX B

Inequalities of Gagliardo-Nirenberg Type

The results of this appendix are motivated by the works of Gagliardo [**37**] and Nirenberg [**63**], in which the authors independently derived a number of interpolation inequalities on the Sobolev spaces $W^{k,p}$ defined over open domains in $\mathbb{R}^n$. These were intended to facilitate the analysis of extant problems in the field of elliptic partial differential equations, including that of Dirichlet, and have since inspired a large body of related literature (a partial summary of which is contained in [**78**]). While, in general, such inequalities have been formulated with respect to finite p in the context of the aforementioned Euclidean spaces, the geometric and functional analytic requirements of this dissertation necessitate their extension to the spaces $W^{k,\infty} \equiv C^{k-1,1}$ defined over Riemannian manifolds. Throughout the proofs to follow, we shall assume that the tensorfields under consideration are non-constant, otherwise each corresponding result holds trivially.

The ensuing lemma is especially pertinent to various arguments pursued in Chapters 6 and 7.

LEMMA B.1. *Suppose (M^n, g) is a complete Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$, and $\mathcal{T} \in \mathscr{T}^p_q(M)$ is a tensorfield of class $W^{k,\infty}(M)$. Then, for each $k \in \mathbb{N}\setminus\{1\}$ and $j \in \mathbb{N}$ with $0 < j < k$,*

$$\left\|\nabla^j \mathcal{T}\right\|_{L^\infty(M)} \leq C(n,j,k,p,q) \left\|\mathcal{T}\right\|^{1-\frac{j}{k}}_{L^\infty(M)} \left\|\nabla^k \mathcal{T}\right\|^{\frac{j}{k}}_{L^\infty(M)} .$$

PROOF

We prove the result by induction on $k \in \mathbb{N} \setminus \{1\}$. Suppose that $\mathcal{T} \in \mathscr{T}^p_q(M)$ is of class $W^{2,\infty}(M)$, $x \in M$, and $\mathscr{I}$ is an (uncountable) indexing set where $\gamma_i : \mathscr{I} \times (-1,1) \to M$ is the family of all smooth, unit speed geodesics on M with $\gamma_i(0) = x$. Then for each i we may expand $\mathcal{T}$ about $s = 0$:

$$\mathcal{T}(\gamma_i(s)) = \mathcal{T}(\gamma_i(0)) + \langle \nabla\mathcal{T}(\gamma_i(0)), \gamma_i'(0)\rangle s + \frac{1}{2}\nabla^2\mathcal{T}(\gamma_i(s_1))[\gamma_i'(s_1), \gamma_i'(s_1)] s^2$$

for some s_1 with $|s_1| < |s|$, where we have noted that $\gamma_i''(s) \in \left(T_{\gamma_i(s)}M\right)^\perp$ for each $s \in (-1,1)$. Moreover, by the parallel transport of tangential tensorfields along γ_i we may take $\|\nabla^2\mathcal{T}(\gamma_i(s_1))\| = \|\nabla^2\mathcal{T}(\gamma_i(0))\|$. Therefore, for each i,

$$\|\langle \nabla\mathcal{T}(\gamma_i(0)), \gamma_i'(0)\rangle\| \leq \frac{\|\mathcal{T}(\gamma_i(s)) - \mathcal{T}(\gamma_i(0))\|}{|s|} + C(n,p,q)\left\|\nabla^2\mathcal{T}(\gamma_i(0))\right\| |s| .$$

Now, if $\mathcal{W} \in \mathscr{T}_s^r(M)$, then we may interpret the norm of $\mathcal{W}$ evaluated at any $y \in M$ in the following manner:

$$\|\mathcal{W}(y)\| \;=\; \sup_{\substack{\mathbf{v}^{(*)} \in T_y^{(*)} M \\ \|\mathbf{v}^{(*)}\|=1}} \mathcal{W}(y)[\underbrace{\mathbf{v}^*, ..., \mathbf{v}^*}_{r \text{ factors}}; \underbrace{\mathbf{v}, ..., \mathbf{v}}_{s \text{ factors}}] \,.$$

Consequently, we may find $m \in \mathscr{I}$ such that $\|\nabla \mathcal{T}(x)\| = \|\langle \nabla \mathcal{T}(\gamma_m(0)), \gamma_m'(0)\rangle\|$ and, if we take $|s| = \varepsilon$, we have

$$\|\nabla \mathcal{T}(x)\| \;\leq\; \frac{\operatorname{osc}_{\gamma_m} \mathcal{T}}{\varepsilon} + C\varepsilon \left\|\nabla^2 \mathcal{T}(x)\right\| \,.$$

Thus we may infer on γ_m an estimate of the form:

$$\|\nabla \mathcal{T}\|_\infty \;\leq\; \frac{2\,\|\mathcal{T}\|_\infty}{\varepsilon} + C\varepsilon \left\|\nabla^2 \mathcal{T}\right\|_\infty. \tag{B.1}$$

Furthermore, since M is complete, we may infinitely extend γ_m by the theorem of Hopf-Rinow (see [**21**, Theorem 8.7.2], for example) to infer that the estimate holds everywhere on M. We now minimise the expression on the right hand side of (B.1) with respect to ε:

$$\varepsilon_{\min} \;=\; \left(\frac{2\,\|\mathcal{T}\|_\infty}{C\left\|\nabla^2 \mathcal{T}\right\|_\infty}\right)^{\frac{1}{2}}.$$

Thus

$$\|\nabla \mathcal{T}\|_\infty \;\leq\; 2\left(2C\,\|\mathcal{T}\|_\infty \left\|\nabla^2 \mathcal{T}\right\|_\infty\right)^{\frac{1}{2}} \;\leq\; C(n,p,q)\,\|\mathcal{T}\|_\infty^{\frac{1}{2}} \left\|\nabla^2 \mathcal{T}\right\|_\infty^{\frac{1}{2}}. \tag{B.2}$$

Hence the theorem is established at $k = 2$ for $W^{2,\infty}(M)$ (tangential) tensorfields of any type, since $\{p, q\}$ was arbitrary. Suppose now that there exists a particular $k \in \mathbb{N}\backslash\{1\}$ such that the theorem holds. In particular, if $\mathcal{T} \in \mathscr{T}_q^p(M)$ is a tensorfield of class $W^{k+1,\infty}(M)$, then for each $j \in \mathbb{N}$ with $0 < j < k$,

$$\left\|\nabla^j \mathcal{T}\right\|_\infty \;\leq\; C(n,j,k,p,q)\,\|\mathcal{T}\|_\infty^{1-\frac{j}{k}} \left\|\nabla^k \mathcal{T}\right\|_\infty^{\frac{j}{k}} \,. \tag{B.3}$$

We next employ (B.2) and our inductive hypothesis to deduce that

$$\begin{aligned}
\left\|\nabla^k \mathcal{T}\right\|_\infty &= \left\|\nabla\left(\nabla^{k-1} \mathcal{T}\right)\right\|_\infty \\
&\leq C(n,p,q)\left\|\nabla^{k-1} \mathcal{T}\right\|_\infty^{\frac{1}{2}} \left\|\nabla^{k+1} \mathcal{T}\right\|_\infty^{\frac{1}{2}} \\
&\leq C(n,k,p,q)\left(\|\mathcal{T}\|_\infty^{1-\frac{k-1}{k}} \left\|\nabla^k \mathcal{T}\right\|_\infty^{\frac{k-1}{k}}\right)^{\frac{1}{2}} \left\|\nabla^{k+1} \mathcal{T}\right\|_\infty^{\frac{1}{2}} \,.
\end{aligned}$$

Therefore

$$\left\|\nabla^k \mathcal{T}\right\|_\infty \;\leq\; C(n,k,p,q)\,\|\mathcal{T}\|_\infty^{1-\frac{k}{k+1}} \left\|\nabla^{k+1} \mathcal{T}\right\|_\infty^{\frac{k}{k+1}} \,. \tag{B.4}$$

We may now substitute (B.4) into (B.3) to assert that, for each $j \in \mathbb{N}$ with $0 < j < k$,

$$\begin{aligned}\left\|\nabla^j \mathcal{T}\right\|_\infty &\leq C(n,j,k,p,q)\,\|\mathcal{T}\|_\infty^{1-\frac{j}{k}}\left(\|\mathcal{T}\|_\infty^{1-\frac{k}{k+1}}\left\|\nabla^{k+1}\mathcal{T}\right\|_\infty^{\frac{k}{k+1}}\right)^{\frac{j}{k}}\\ &\leq C\,\|\mathcal{T}\|_\infty^{1-\frac{j}{k+1}}\left\|\nabla^{k+1}\mathcal{T}\right\|_\infty^{\frac{j}{k+1}},\end{aligned}$$

where (B.4) verifies the outstanding case $j = k$. The estimate remains valid for $W^{k+1,\infty}(M)$ (tangential) tensorfields of any type, since again $\{p, q\}$ was arbitrary. Thus the theorem holds for each $k \in \mathbb{N} \setminus \{1\}$ by the principle of mathematical induction. ◇

COROLLARY B.2. *Suppose (M^n, g) is a complete Riemannian manifold smoothly embedded in $\mathbb{R}^{n+1}$ and $\mathcal{T} \in \mathcal{T}_q^p(M)$ is a tensorfield of class $C^{k-1,1}(M)$. Then, for each $k \in \mathbb{N} \setminus \{1\}$ and $j \in \mathbb{N}$ with $0 < j < k$,*

$$\|\mathcal{T}\|_{C^{j-1,1}(M)} \leq C(n,j,k,p,q)\,\|\mathcal{T}\|_{L^\infty(M)}^{1-\frac{j}{k}}\,\|\mathcal{T}\|_{C^{k-1,1}(M)}^{\frac{j}{k}}\,.$$

PROOF

Since $W^{k,\infty} \equiv C^{k-1,1}$, we infer that $\mathcal{T} \in W^{k,\infty}(M)$. Therefore, we may repeatedly employ Lemma B.1 to determine that, for each $k \in \mathbb{N} \setminus \{1\}$ and $j \in \mathbb{N}$ with $0 < j < k$,

$$\begin{aligned}\|\mathcal{T}\|_{W^{j,\infty}} &\leq \|\mathcal{T}\|_\infty + \sum_{l=1}^{j} C_l(n,k,p,q)\,\|\mathcal{T}\|_\infty^{1-\frac{l}{k}}\left\|\nabla^k\mathcal{T}\right\|_\infty^{\frac{l}{k}}\\ &\leq C(n,j,k,p,q)\sum_{l=0}^{j}\|\mathcal{T}\|_\infty^{1-\frac{l}{k}}\,\|\mathcal{T}\|_{W^{k,\infty}}^{\frac{l}{k}}\\ &= C\,\|\mathcal{T}\|_\infty^{1-\frac{j}{k}}\,\|\mathcal{T}\|_{W^{k,\infty}}^{\frac{j}{k}}\sum_{l=0}^{j}\|\mathcal{T}\|_\infty^{\frac{j-l}{k}}\,\|\mathcal{T}\|_{W^{k,\infty}}^{\frac{l-j}{k}}\\ &\leq (j+1)C\,\|\mathcal{T}\|_\infty^{1-\frac{j}{k}}\,\|\mathcal{T}\|_{W^{k,\infty}}^{\frac{j}{k}}\,.\end{aligned}$$

The desired result now follows immediately from the equivalence of the two norms. ◇

We now derive the following inequality which proves critical to the convergence analysis of Chapter 7.

LEMMA B.3. *Suppose $v \in C^{0,1}(S;\mathbb{R})$. Then*

$$\|v\|_{L^\infty(S)} \leq 4\pi\,|S|^{\frac{-1}{n+2}}\,\|v\|_{L^2(S)}^{\frac{2}{n+2}}\,\|v\|_{C^{0,1}(S)}^{\frac{n}{n+2}}\,.$$

PROOF

We may clearly assert that for any $s, s_0 \in S$

$$|v(s) - v(s_0)| \leq \|v\|_{C^{0,1}(S)}\, d_S(s, s_0)\,. \tag{B.5}$$

Now, by Cauchy's inequality,

$$|v(s_0)|^2 \leq 2\,|v(s)|^2 + 2\,|v(s) - v(s_0)|^2\,. \tag{B.6}$$

We now fix s_0 and confine our attention to the open ball of radius $r \in (0, \pi]$ on S, $B_r(s_0) = \{s \in S : d_S(s, s_0) < r\}$. By integrating the inequality (B.6) over B_r and invoking (B.5), we obtain

$$\begin{aligned} |v(s_0)|^2\,|B_r| &\leq 2\left(\int_{B_r} |v(s)|^2\, d\sigma + \int_{B_r} |v(s) - v(s_0)|^2\, d\sigma\right) \\ &\leq 2\left(\|v\|^2_{L^2(S)} + \|v\|^2_{C^{0,1}(S)} \int_{B_r} d_S^2(s, s_0) d\sigma\right) \\ &\leq 2\left(\|v\|^2_{L^2(S)} + r^2\,|B_r|\,\|v\|^2_{C^{0,1}(S)}\right). \end{aligned} \tag{B.7}$$

To evaluate the quantity $|B_r|$, we first consider the case $0 < r \leq \frac{\pi}{2}$. By parametrising B_r above its orthogonal projection onto the n-dimensional equatorial hyperplane determined by s_0 and its antipode, we verify that

$$|B_r| = n\,\omega_n \int_0^r \sin^{n-1}\varphi\, d\varphi\,. \tag{B.8}$$

We now perform some perfunctory analysis to demonstrate that the function

$$f(\varphi) = \frac{2\varphi}{\pi} - \sin\varphi$$

is non-positive on the interval $(0, \frac{\pi}{2}]$. We note that $f(0) = f(\frac{\pi}{2}) = 0$, where

$$f'(\varphi) = \frac{2}{\pi} - \cos\varphi\,.$$

Hence f attains its sole interior critical point at $\varphi_0 = \cos^{-1}(\frac{2}{\pi})$ which, since f'' is strictly positive on the interval under examination, corresponds to a local minimum. We then deduce that f achieves its maximum value on the boundary of the interval $(0, \frac{\pi}{2}]$, and

$$\frac{2\varphi}{\pi} \leq \sin\varphi\,.$$

Thus we ascertain from (B.8) that, on $(0, \frac{\pi}{2}]$,

$$|B_r| \geq \omega_n \left(\frac{2}{\pi}\right)^{n-1} r^n\,.$$

Similarly, for $\frac{\pi}{2} < r \leq \pi$, it is evident that

$$|B_r| \;>\; \frac{|S|}{2} \;=\; n\,\omega_n \int_0^{\frac{\pi}{2}} \sin^{n-1}\varphi\, d\varphi \;\geq\; \frac{\pi\omega_n}{2}\,.$$

We may then combine the results of the two cases to discern that, for $0 < r \leq \pi$,

$$|B_r| \;\geq\; \frac{\pi\omega_n}{2}\left(\min\left\{\frac{2r}{\pi}, 1\right\}\right)^n. \tag{B.9}$$

Moreover, if we take $\alpha = \min\left\{\frac{2r}{\pi}, 1\right\}$, then it is clear that $r \leq \pi\alpha$ on $(0,\pi]$ and we deduce from (B.7) and (B.9) that

$$|v(s_0)|^2 \;\leq\; 2\left(\frac{2\,\|v\|^2_{L^2(S)}}{\pi\omega_n\alpha^n} + (\pi\alpha)^2\,\|v\|^2_{C^{0,1}(S)}\right).$$

We minimise the right hand side of this expression with respect to α to establish that

$$\alpha_{\min} \;=\; \left(\frac{n\,\|v\|^2_{L^2(S)}}{\pi^3\omega_n\,\|v\|^2_{C^{0,1}(S)}}\right)^{\frac{1}{n+2}}.$$

Since s_0 was arbitrary in this formulation, we consequently affirm the multiplicative interpolation inequality for $v \in C^{0,1}(S;\mathbb{R})$:

$$\begin{aligned}\|v\|_{L^\infty(S)} \;&\leq\; 2\left(\frac{n\,\pi^{n-1}}{\omega_n}\right)^{\frac{1}{n+2}} \|v\|^{\frac{2}{n+2}}_{L^2(S)}\,\|v\|^{\frac{n}{n+2}}_{C^{0,1}(S)} \\ &\leq\; 2\left(\frac{n^2\pi^n}{|S|}\right)^{\frac{1}{n+2}} \|v\|^{\frac{2}{n+2}}_{L^2(S)}\,\|v\|^{\frac{n}{n+2}}_{C^{0,1}(S)} \\ &\leq\; 4\pi\,|S|^{\frac{-1}{n+2}}\,\|v\|^{\frac{2}{n+2}}_{L^2(S)}\,\|v\|^{\frac{n}{n+2}}_{C^{0,1}(S)}\,.\end{aligned}$$

In analysis consistent with that performed previously, here we have observed that $|S| \leq n\pi\omega_n$. We have further noted that the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$h(x) \;=\; x^{\frac{2}{x+2}}$$

achieves its global maximum on the interval $(4, 5)$ with $h(5) < h(4) < 2$. Hence we have verified the lemma. ◇

Although the ensuing two corollaries are not cited directly in the body of the thesis, they are provided for the interest of the reader.

COROLLARY B.4. *Suppose $E \subset \mathbb{R}^{n+1}$ is compact and star-shaped about the origin with boundary $\partial E = M$ of class C^∞. Further suppose that $v \in C^{0,1}(M;\mathbb{R})$, X is the position vectorfield on M, and $\varepsilon > 0$ is such that $\||X| - 1\|_{C^{0,1}(S)} \le \varepsilon$. Then*

$$\|v\|_{L^\infty(M)} \le C(n)\,|M|^{\frac{-1}{n+2}}\,\|v\|_{L^2(M)}^{\frac{2}{n+2}}\,\|v\|_{C^{0,1}(M)}^{\frac{n}{n+2}}\,.$$

PROOF

As a consequence of our hypothesis and in the manner of Definition 1.3 contained in Chapter 1, we observe that M may be parametrised by a strictly positive graph $r \in C^\infty(S)$ above S with $|X| = r$. In the following exposition, we shall again employ the equivalence of $C^{0,1}$ and $W^{1,\infty}$. If $v \in W^{1,\infty}(M)$, then we deduce from Definition 1.3 and Lemma 1.4 that the function $w = v \circ X \in W^{1,\infty}(S)$. Indeed

$$\|v\|_{L^\infty(M)} = \|w\|_{L^\infty(S)}\,. \tag{B.10}$$

Moreover, by our hypothesis and Lemma 1.4,

$$\|w\|_{L^2(S)}^2 \le \sup_S\left(\frac{1}{\mu_r}\right)\|v\|_{L^2(M)}^2 \le (1+\varepsilon\,K_1(n))\,\|v\|_{L^2(M)}^2\,, \tag{B.11}$$

while

$$|M| \le \sup_S(\mu_r)\,|S| \le (1+\varepsilon\,K_2(n))\,|S|\,. \tag{B.12}$$

In addition, where we again invoke Lemma 1.4 in conjunction with the Cauchy-Schwarz inequality,

$$\begin{aligned} |\nabla^M v|^2 &= \frac{1}{r^2}\left(|\nabla^S w|^2 - \frac{\langle\nabla^S w, \nabla^S r\rangle^2}{r^2 + |\nabla^S r|^2}\right) \\ &\ge \frac{|\nabla^S w|^2}{r^2 + |\nabla^S r|^2} \\ &\ge \inf_S\left(\frac{1}{r^2 + |\nabla^S r|^2}\right)|\nabla^S w|^2 \\ &\ge (1-\varepsilon\,K(n))|\nabla^S w|^2\,. \end{aligned}$$

Therefore,

$$\|w\|_{W^{1,\infty}(S)} \le \left(1+\varepsilon^{\frac{1}{2}}K_3(n)\right)\|v\|_{W^{1,\infty}(M)}\,. \tag{B.13}$$

Thus we may combine Lemma B.3 with (B.10)-(B.13) to ascertain that, for sufficiently small ε,

$$\begin{aligned}\|v\|_{L^\infty(M)} &= \|w\|_{L^\infty(S)} \leq 4\pi\,|S|^{\frac{-1}{n+2}}\,\|w\|_{L^2(S)}^{\frac{2}{n+2}}\,\|w\|_{W^{1,\infty}(S)}^{\frac{n}{n+2}} \\ &\leq C(n)\,|M|^{\frac{-1}{n+2}}\,\|v\|_{L^2(M)}^{\frac{2}{n+2}}\,\|v\|_{W^{1,\infty}(M)}^{\frac{n}{n+2}}.\end{aligned}$$

The intended result is again a direct consequence of the equivalence of the two norms. ♢

COROLLARY B.5. *Suppose $E \subset \mathbb{R}^{n+1}$ is compact and star-shaped about the origin with boundary $\partial E = M$ of class C^∞. Further suppose that $v \in C^{k-1,1}(M;\mathbb{R})$ for some $k \in \mathbb{N}\setminus\{1\}$, X is the position vectorfield on M, and $\varepsilon > 0$ is such that $\||X| - 1\|_{C^{0,1}(S)} \leq \varepsilon$. Then, for each $k \in \mathbb{N}\setminus\{1\}$ and $j \in \mathbb{N}$ with $0 < j < k$,*

$$\|v\|_{C^{j-1,1}(M)} \leq C(n,j,k)\,|M|^{-\frac{1}{2}(1-\theta)}\,\|v\|_{L^2(M)}^{1-\theta}\,\|v\|_{C^{k-1,1}(M)}^{\theta}$$

where $\theta = \frac{n+2j}{n+2k}$.

PROOF

We may utilise Corollaries B.2 and B.4 to discern that, for sufficiently small ε,

$$\begin{aligned}\|v\|_{j-1,1} &\leq C(n,j,k)\,\|v\|_\infty^{1-\frac{j}{k}}\,\|v\|_{k-1,1}^{\frac{j}{k}} \\ &= C\,\|v\|_\infty^{1-\frac{j}{k}+\beta}\,\|v\|_\infty^{-\beta}\,\|v\|_{k-1,1}^{\frac{j}{k}} \\ &\leq C(n,j,k)\left(|M|^{\frac{-1}{n+2}}\,\|v\|_2^{\frac{2}{n+2}}\,\|v\|_{0,1}^{\frac{n}{n+2}}\right)^{1-\frac{j}{k}+\beta}\|v\|_\infty^{-\beta}\,\|v\|_{k-1,1}^{\frac{j}{k}} \\ &\leq C(n,j,k)\left(\frac{\|v\|_2^2}{|M|}\right)^{\frac{1}{n+2}\left(1-\frac{j}{k}+\beta\right)}\|v\|_\infty^{\left(\frac{n}{n+2}\left(1-\frac{j}{k}+\beta\right)\left(1-\frac{1}{k}\right)-\beta\right)}\,\|v\|_{k-1,1}^{\left(\frac{j}{k}+\frac{n}{n+2}\left(1-\frac{j}{k}+\beta\right)\frac{1}{k}\right)}.\end{aligned}$$

If we choose $\beta = \frac{n(k-j)(k-1)}{k(n+2k)}$, then the exponent of $\|v\|_\infty$ vanishes and we deduce that

$$\begin{aligned}\|v\|_{j-1,1} &\leq C\,|M|^{\frac{(j-k)}{n+2k}}\,\|v\|_2^{\frac{2(k-j)}{n+2k}}\,\|v\|_{k-1,1}^{\left(\frac{j}{k}+\frac{n(k-j)}{k(n+2k)}\right)} \\ &= C\,|M|^{-\frac{1}{2}\left(1-\frac{n+2j}{n+2k}\right)}\,\|v\|_2^{\left(1-\frac{n+2j}{n+2k}\right)}\,\|v\|_{k-1,1}^{\frac{n+2j}{n+2k}},\end{aligned}$$

as required. ♢

APPENDIX C

The Spherical Harmonics

The following summary of the theory of the spherical harmonics is derived entirely from [**43**, Chapter 3], where we shall omit the associated proofs for the sake of brevity. It is not intended to be exhaustive, but merely to illuminate the results of Chapters 3, 4, and 7. In accordance with the methodology of this thesis, we shall assume that $n \geq 2$.

In essence, the spherical harmonics may be interpreted as higher dimensional analogues of Fourier series on S^1. These quantities have a particular application to the geometric analysis of both convex and star-shaped bodies, which was first recognised by Minkowski.

DEFINITION C.1. Suppose $p : \mathbb{R}^{n+1} \to \mathbb{R}$ is a polynomial of degree k. We call p *harmonic* if $\Delta^{\mathbb{R}^{n+1}} p = 0$, and we shall denote the space of all such harmonic polynomials by $\mathcal{Q}_k^{n+1}$.

DEFINITION C.2. A *spherical harmonic* of dimension $(n+1)$ and degree k is the restriction of some $p \in \mathcal{Q}_k^{n+1}$ to S. We shall denote the space of all such spherical harmonics by $\mathcal{H}_k^{n+1}$.

THEOREM C.3. *The dimensions of the spaces $\mathcal{Q}_k^{n+1}$ and $\mathcal{H}_k^{n+1}$ coincide and may be quantified in the following manner:*

$$\dim\left(\mathcal{Q}_k^{n+1}\right) = \dim\left(\mathcal{H}_k^{n+1}\right) = \frac{2k+n-1}{k+n-1}\begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix}.$$

THEOREM C.4. *Suppose $\zeta \in \mathcal{H}_l^{n+1}$ and $\xi \in \mathcal{H}_m^{n+1}$ with $l \neq m$. Then*

$$\langle \zeta, \xi \rangle_{L^2(S)} = 0\,.$$

LEMMA C.5. *The set $\{\langle s, e_i \rangle : 1 \leq i \leq n+1\}$ forms an orthogonal basis for $\mathcal{H}_1^{n+1}$ in $L^2(S)$. Furthermore, for each $i \in \mathbb{N}$ with $1 \leq i \leq n+1$,*

$$\|\langle s, e_i \rangle\|_{L^2(S)}^2 = \frac{|S|}{n+1} = \omega_{n+1}\,.$$

Given any finite set $H \subseteq \mathcal{H}_k^{n+1}$, we may employ the Gram-Schmidt process to obtain an orthogonal basis for H with respect to the topology of $L^2(S)$. We may therefore procure an orthogonal basis corresponding to $\mathcal{H}_k^{n+1}$ for each $k \in \mathbb{N} \cup \{0\}$.

DEFINITION C.6. Suppose $(H_j)_{j \in \mathbb{N} \cup \{0\}}$ is an orthogonal sequence of spherical harmonics which, for each $k \in \mathbb{N} \cup \{0\}$, contains $\dim\left(\mathcal{H}_k^{n+1}\right)$ terms of order k. Then we define (H_j) to be a *standard sequence* of spherical harmonics.

THEOREM C.7. *Every standard sequence of spherical harmonics is complete in $L^2(S)$.*

DEFINITION C.8. Suppose (H_j) is a standard sequence of spherical harmonics and $\eta \in L^2(S)$. Then an *harmonic expansion* of η is given by

$$\eta \sim \sum_{j=0}^{\infty} c_j H_j$$

where, for each $j \in \mathbb{N} \cup \{0\}$,

$$c_j = \frac{\langle \eta, H_j \rangle_{L^2(S)}}{\|H_j\|^2_{L^2(S)}} .$$

Furthermore, if

$$Q_k = \sum_{\chi(H_j)=k} c_j H_j ,$$

then the (unique) *condensed harmonic expansion* of η is given by

$$\eta \sim \sum_{k=0}^{\infty} Q_k .$$

THEOREM C.9. *Suppose that η and $\psi \in L^2(S)$ possess the associated condensed harmonic expansions*

$$\eta \sim \sum_{k=0}^{\infty} Q_k \quad \text{and} \quad \psi \sim \sum_{k=0}^{\infty} R_k .$$

Then

$$\|\eta\|^2_{L^2(S)} = \sum_{k=0}^{\infty} \|Q_k\|^2_{L^2(S)} \quad \text{and} \quad \langle \eta, \psi \rangle_{L^2(S)} = \sum_{k=0}^{\infty} \langle Q_k, R_k \rangle_{L^2(S)} .$$

THEOREM C.10. *Suppose $\zeta \in \mathcal{H}_k^{n+1}$. Then ζ is an eigenfunction of the Laplace-Beltrami operator on S. In particular,*

$$\Delta^S \zeta = -k\,(k+n-1)\,\zeta .$$

COROLLARY C.11. *Suppose that $\eta \in C^2(S)$ possesses the associated condensed harmonic expansion*

$$\eta \sim \sum_{k=0}^{\infty} Q_k .$$

Then

$$\left\|\nabla^S \eta\right\|^2_{L^2(S)} = \sum_{k=0}^{\infty} k(k+n-1)\,\|Q_k\|^2_{L^2(S)} .$$

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