Automorphisms of Finite Groups Fixing Every Non-Subnormal Subgroup

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Acknowledgments

Declaratio11

The work in this thesis is my own unless otherwise stated

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Abstract

This thesis is an investigation of automorphisms which act in a way that fixes every non-subnormal subgroup of a finite group. In the first instance this investigation is carried out by establishing some basic properties of such automorphisms, properties which generalise known results about power automorphisms of finite groups.

In the second instance this thesis studies how the properties of an automorphism which fixes every non-subnormal subgroup influences the structure of a finite non-nilpotent group on which it acts. In particular, a detailed investigation is made of the structure of those finite groups which have a non-trivial coprime operator that fixes every non-subnormal subgroup. This is achieved by establishing an extension of a result about the structure of those finite groups which have a coprime operator that acts as a power automorphism. Further restrictions on the group structure are then obtained using some known properties of Frobenius complements and non-soluble groups in which every abelian subgroup is cyclic.
## Contents

Acknowledgments .......................................................... i
Abstract ................................................................ iii
List of Symbols .................................................................. vii

### 1 Introduction

### 2 Preliminaries

2.1 Groups ........................................................................ 5
2.2 Group Actions .......................................................... 6
2.3 Subnormal Subgroups .................................................. 9
2.4 Soluble and Nilpotent Groups ...................................... 11
2.5 The Frattini Subgroup ................................................ 14
2.6 A Property of Iterated Commutators ............................ 14

### 3 Normalising Properties

3.1 The Inner Automorphism Group ................................. 19
3.2 The Automorphism Group .......................................... 22
3.3 Examples .................................................................... 27

### 4 Power Automorphisms of Finite Groups

4.1 Coprime Operators .................................................... 29
4.2 Coprime Action by Power Automorphisms .................. 30

### 5 Restrictions on the Group Structure

5.1 Some Related Results ................................................ 33
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2 Coprime Actions Fixing Non-Subnormal Subgroups</td>
<td>34</td>
</tr>
<tr>
<td>5.3 The Soluble Case</td>
<td>39</td>
</tr>
<tr>
<td>5.4 The Insoluble Case</td>
<td>40</td>
</tr>
<tr>
<td>5.5 Concluding Remarks</td>
<td>43</td>
</tr>
</tbody>
</table>

Bibliography
List of Symbols

$A, G, H, N, Q, X, \ldots$ Groups and sets.
$a, b, \ldots, z$ Elements of a set.
$A \setminus B \{ a \in A \mid a \notin B \}$.
$(X \mid R)$ Group generated by $X$ with the relations $R$.
$|X|, |x|$ Cardinality of $X$, the order of $x$.
$x^g \quad y^{-1}xy$.
$[x, y] \quad x^{-1}x^g$.
$\alpha, \beta, \ldots$ Functions.
$x^\alpha \quad \text{Image of } x \text{ under } \alpha.$
$[x, \alpha] \quad x^{-1}x^\alpha$.
$A \rtimes G$ Semidirect product of the groups $A$ and $G$.
$|G : H|$ Index of $H$ in $G$.
$C_G(X)$ Centraliser of $X$ in $G$.
$N_G(X)$ Normaliser of $X$ in $G$.
$\text{Ker } \theta$ Kernel of $\theta$.
$\text{Aut } G$ Automorphism group of $G$.
$\text{Paut } G$ Power automorphism group of $G$.
$\text{Aut}_{\text{non}} G$ Set of automorphisms of $G$ fixing every non-normal subgroup in $G$.
$\text{Aut}_{\text{non}} G$ Set of automorphisms of $G$ fixing every non-subnormal subgroup in $G$.
$\text{Inn } G$ Inner automorphism group of $G$.
$\kappa(G)$ Norm of $G$.
$\omega(G)$ Wielandt subgroup of $G$.  

vii
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(G)$</td>
<td>Subgroup of $G$ normalising every non-subnormal subgroup in $G$.</td>
</tr>
<tr>
<td>$\gamma_i(G)$</td>
<td>$i$-th term in the lower central series of $G$.</td>
</tr>
<tr>
<td>$\zeta_i(G)$</td>
<td>$i$-th term in the upper central series of $G$.</td>
</tr>
<tr>
<td>$\zeta G, \zeta_\infty G$</td>
<td>Centre of $G$, the hypercentre of $G$.</td>
</tr>
<tr>
<td>Fit $G$</td>
<td>Fitting subgroup of $G$.</td>
</tr>
<tr>
<td>Frat $G$</td>
<td>Frattini subgroup of $G$.</td>
</tr>
<tr>
<td>$O_\pi(G)$</td>
<td>Maximal normal $\pi$-subgroup of $G$.</td>
</tr>
<tr>
<td>$V_u(G)$</td>
<td>Verbal subgroup of $u$ in $G$.</td>
</tr>
<tr>
<td>$M_u(G)$</td>
<td>Marginal subgroup of $u$ in $G$.</td>
</tr>
<tr>
<td>$SL(2,p)$</td>
<td>Special linear group of 2 by 2 matrices over the field of $p$ elements.</td>
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</table>
Chapter 1

Introduction

It is a well known fact that every group of operators on a group $G$ induces an action on the subgroup lattice of $G$ and that this action respects the important structural properties of the subgroup lattice such as containment and normality. This observation leads us to the following two questions:

i. How much is revealed about the structure of the operator group by the action it induces on the subgroup lattice of $G$?

ii. Given an action on the set of subgroups of $G$, what can be said about the structure of $G$ if the action is induced by the action of an operator group on $G$?

We will investigate both questions in the context of a group of operators which acts on a finite group $G$ in a way that fixes every non-subnormal subgroup of $G$.

The earliest work relating directly to the subject of this thesis dates from the first half of the present century, when R. Baer (see [1] and [2]) defined the norm (or Kern in the German) of a group $G$ to be the subgroup of $G$ which normalises every subgroup in $G$. In 1958 E. Schenkman [13] published the important result that each element of the norm of a group $G$ acts trivially on $G$ modulo the centre of $G$. Schenkman’s result implies that the norm of $G$ is always contained in the second centre of $G$ and that if $G$ is either perfect or has trivial centre then the norm of $G$ is trivial.

The results of E. Schenkman were generalised in one direction to the automorphisms of a group by C. Cooper in [4] in 1968 when he showed that the power automorphisms act trivially modulo the centre of the group. Cooper’s result shows of course that if $G$ is a group which is either perfect or has trivial centre then the only automorphism fixing every subgroup of $G$ is the trivial automorphism. An important result on the structure...
of groups of power automorphisms is a theorem of B. Huppert (see [4]) which tells us that if \( G \) is a finite non-abelian \( p \)-group then the power automorphism group of \( G \) is an abelian \( p \)-group.

Further results on these questions are due to R. Brandl and L. Verardi who give in [3] a number of theorems on the structure of the automorphisms of a non-Dedekind group \( G \) which fix every non-normal subgroup of \( G \) and do not fix every normal subgroup of \( G \). In [7] S. Franciosi, F. de Giovanni and H. Heineken show how the existence of an automorphism fixing every non-subnormal subgroup of a non-Dedekind group \( G \) restricts the structure of \( G \). Generalising Schenkman’s result of [13], in [7] the authors show that every automorphism which fixes the non-normal subgroups of a group \( G \) acts trivially on \( G \) modulo the second centre of \( G \).

Our investigation begins in Chapter 3 where we let \( G \) be a finite group and define the subgroup \( \lambda(G) \) to be the subgroup of \( G \) which normalises every non-subnormal subgroup in \( G \). Our interest in this subgroup is motivated primarily by the observation that if \( \alpha \) is an automorphism of \( G \) which fixes every non-subnormal subgroup in \( G \) then \( \alpha \) acts trivially on \( G \) modulo \( \lambda(G) \). This observation, together with the fact that \( \lambda(G) \) is contained in the hypercentre of \( G \) allows us to show that if \( G \) is a finite non-nilpotent group then the set of automorphisms which fix every non-subnormal subgroup of \( G \) is an abelian subgroup of the automorphism group of \( G \). In addition, for a finite group \( G \), the results obtained about the subgroup \( \lambda(G) \) allow us to give sufficient conditions for the subgroup \( \lambda(G) \) to be trivial. We show in particular, that if a finite group \( G \) is either perfect or has trivial centre then \( \lambda(G) \) is trivial and hence, under these assumptions, there is no non-trivial automorphism of \( G \) which fixes every non-subnormal subgroup in \( G \). In the last section of Chapter 3 we give two examples of a finite group \( G \) exhibiting an automorphism which fixes every non-normal subgroup in \( G \) and acts non-trivially on the set of subnormal subgroups of \( G \). In our first example the automorphism has order dividing the order of \( G \) and in our second example, the automorphism has order coprime to the order of \( G \).

In Chapter 4 we introduce some machinery about coprime operators. This allows us to show how the structure of a finite group \( G \) is restricted by the action of a non-trivial power automorphism \( \alpha \) of order coprime to the order of \( G \). The main result of Chapter 4, namely that an abelian non-nilpotent \( p \)-group is necessarily of prime power order, is shown as an application of the machinery introduced in Chapter 4.
though of marginal interest to us in its own right, does play an important part in the work of our final chapter.

In Chapter 5 of this thesis we modify the hypotheses of Chapter 4 to let \( \alpha \) be a non-trivial automorphism of \( G \) and suppose that as well as fixing every non-subnormal subgroup in \( G \), the automorphism \( \alpha \) has order coprime to the order of \( G \). We show in Section 5.2 that the results of Chapter 4 on the structure of a finite group which has a coprime operator acting as a power automorphism may be extended to give a strong restriction on the structure of a finite group \( G \) which has a coprime operator which acts on \( G \) so as to fix every non-subnormal subgroup.

In Sections 5.3 to 5.4 we further restrict the structure of \( G \) using results of Zassenhaus on the structure of soluble and insoluble Frobenius complements and a result of Suzuki which classifies finite insoluble groups in which every abelian subgroup is cyclic. Indeed when \( G \) is insoluble we have used the results Section 5.2 and the theorems of Zassenhaus and Suzuki to account for the structure of \( G \) up to a normal Hall subgroup.

### Commutators and Commutator Subgroups

Let \( G \) be a group and \( g, h \) be elements of \( G \). We define the commutator of \( g \) by \( [g,h] = ghg^{-1}h^{-1} \) and the commutator of \( g_1 \) and \( g_2 \) as \( [g_1, g_2] = g_1g_2g_1^{-1}g_2^{-1} \).

The following identities are basic and may be found in Section 4.1 of [1].

#### 2.1.1 Let \( g, h \) be elements of a group. Then

1. \( [g, h] = [h, g] \) (commutator is commutative)
2. \( [g, 1] = [1, g] = e \) (the identity commutes with every element)
3. \( [g, h^{-1}] = h^{-1}g^{-1}h \) (left inverse)
4. \( [g, h] = [h^{-1}, g^{-1}] \) (right inverse)
Chapter 1. Introduction

In this chapter, we will introduce the concept of a finite nilpotent group and discuss some of its properties. We will start by defining what a nilpotent group is and then explore some of its characteristics.

A nilpotent group is a group that is nilpotent, which means that its derived series terminates in a trivial subgroup. The derived series is a sequence of subgroups of a group, where each subgroup is the commutator subgroup of the previous one.

We will also discuss the significance of nilpotent groups in the context of finite p-groups, which are groups of prime power order. We will show that every finite p-group is nilpotent and that the derived series of a finite p-group terminates in the trivial subgroup.

In addition, we will introduce the concept of a solvable group, which is a group that can be constructed from abelian groups using a series of extensions. We will explore the relationship between nilpotent and solvable groups and provide examples of both.

Furthermore, we will discuss the role of automorphisms in the study of nilpotent groups. An automorphism is an isomorphism from a group to itself, and we will see how automorphisms can be used to classify nilpotent groups.

Finally, we will consider the representation theory of nilpotent groups and how it relates to the structure of the group. We will introduce some basic concepts of representation theory and how they apply to nilpotent groups.

Throughout this chapter, we will provide numerous examples and exercises to help readers understand the concepts and their applications. We hope that this chapter will provide a solid foundation for further study in the field of group theory.
Preliminaries

In this chapter we establish the notation and some basic results to be used in this thesis.

2.1 Groups

Let $G$ be a group. We will denote the identity element of $G$ by $1_G$, or simply 1 if the context is clear. If $X$ is a subset of a group $G$, we will denote by $\langle X \rangle$ the subgroup generated by $X$; the trivial subgroup is of $G$ is the subgroup $\langle 1 \rangle$ and will be written simply as 1.

If $\theta : G \to H$ is a homomorphism of groups, we will write $g^\theta$ for the image of $g \in G$ under $\theta$, and $\text{Ker} \, \theta$ for the kernel of $\theta$. We will write $G \cong H$ to mean $G$ and $H$ are isomorphic groups. We will denote by $\text{Aut} \, G$ the automorphism group of $G$ and by $\text{Inn} \, G$ the inner automorphism group of $G$.

Commutators and Commutator Subgroups

Let $G$ be a group and $g_1, g_2$ be elements of $G$. We define the conjugate of $g_1$ by $g_2$ as $g_1^{g_2} = g_2^{-1}g_1g_2$ and the commutator of $g_1$ and $g_2$ as

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 = g_1^{-1}g_1^{g_2}. $$

The following identities are basic and may be found in Section 5.1 of [12].

2.1.1 Let $g_1$, $g_2$ and $g_3$ be elements of a group. Then

(i) $[g_1g_2, g_3] = [g_1, g_3]^{g_2}[g_2, g_3]$ and $[g_1, g_2g_3] = [g_1, g_3][g_1, g_3]^{g_2};$

(ii) $[g_1, g_2]^n = [g_1^n, g_2] = [g_1, g_2^n]$ for all integers $n.$
Chapter 2. Preliminaries

If \( g_1, g_2, \ldots, g_k \) are elements of \( G \) the iterated commutator \([g_1, g_2, \ldots, g_k]\) of weight \( k \geq 2 \) is defined recursively via the relations

\[
[g_1] = g_1 \quad \text{and} \quad [g_1, g_2, \ldots, g_k] = [[g_1, g_2, \ldots, g_{k-1}], g_k].
\]

The above notation is commonly extended to subsets of \( G \) by defining the commutator subgroup of the subsets \( X_1 \) and \( X_2 \) of \( G \) as

\[
[X_1, X_2] = \langle [g_1, g_2] \mid g_1 \in X_1 \text{ and } g_2 \in X_2 \rangle.
\]

Naturally for an integer \( i \geq 2 \) we will write

\[
[X_1, X_2, \ldots, X_i] = [[X_1, X_2, \ldots, X_{i-1}], X_i].
\]

2.2 Group Actions

Let \( G \) be a group. A (right) action of \( G \) on a non-empty set \( X \) is a map \( \rho : X \times G \to X \) which we write \((x, g) \to x^g\), satisfying the conditions \( x^{(g_1 g_2)} = (x^{g_1})^{g_2} \) and \( x^{e_G} = x \). It is easily verified that any such action of \( G \) on \( X \) induces a homomorphism of \( G \) into the symmetric group on \( X \) and vice-versa (see for example, Section 1.6 of [12]). The kernel of this homomorphism is the centraliser of \( X \) in \( G \)

\[
C_G(X) = \{ g \in G \mid x^g = x \text{ for all } x \in X \}
\]

which is clearly a normal subgroup of \( G \). The action of \( G \) on \( X \) is said to be faithful if it has trivial kernel, that is to say, if \( C_G(X) = 1 \). A group \( G \) is said to act transitively on \( X \) if for all \( x, y \in X \) there exists \( g \in G \) such that \( y = x^g \), while the action of \( G \) on \( X \) is said to be semi-regular if \( C_G(x) = 1 \) for all \( x \in X \). A regular group action is an action which is both transitive and semi-regular. Examples of group actions which will be particularly important for our purposes are described below.

Operator Groups

Let \( A \) and \( G \) be groups and suppose that the map \( \rho : A \times G \to G \) given by \((g, a) \mapsto g^a\) is an action of \( A \) on \( G \), with the additional property that for each \( a \in A \) the map \( g \mapsto g^a \) is an automorphism of \( G \); then the group \( A \) is referred to as a group of operators or an
2.2. Group Actions

operator group for $G$. It is easy to see that every homomorphism $\theta : A \to \text{Aut} G$ turns $A$ into an operator group for $G$ via the map $g \mapsto g^{(\theta)}$; $A$ is then said to act on $G$ via $\theta$. If $a \in A$ then a fixed point of $a$ is an element $g \in G$ such that $g^a = g$. An element $a \in A$ is said to be fixed-point-free if $1_G$ is the only fixed point of $a$. More generally the action of $A$ is said to be fixed-point-free if each non-trivial element of $A$ is fixed-point-free.

Given a group of operators $A$ for a group $G$, an important construction is the external semidirect product of $G$ and $A$, which we write $A \ltimes G$. This is the set of ordered pairs $(a, g), a \in A, g \in G$ together with the operation

$$(a_1, g_1)(a_2, g_2) = (a_1a_2, g_1^a g_2).$$

It is readily verified that this operation turns the set of ordered pairs $(a, g), a \in A, g \in G$ into a group with identity $(1_A, 1_G)$ and that $A \ltimes G$ contains subgroups isomorphic to $G$ and $A$, namely $\{(1_A, g) \mid g \in G\}$ and $\{(a, 1_G) \mid a \in A\}$. Conversely, if $G$ is a group, $N$ a normal subgroup of $G$, and $H$ a subgroup of $G$ such that

$$N \cap H = 1 \quad \text{and} \quad G = HN$$

then $H$ acts on $N$ via the map $g^h = h^{-1} g h$ and $G$ is said to be the internal semidirect product of $H$ and $N$. The external semidirect product $A \ltimes G$ may be viewed as an internal semidirect product of the subgroup $\{(1_A, g) \mid g \in G\}$ and the subgroup $\{(a, 1_G) \mid a \in A\}$ isomorphic to $A$. We will therefore avoid making any distinction between internal and external semidirect products.

Let $A$ be group of operators for a group $G$. If $a \in A$ and $g \in G$ the commutator $[g, a]$ has a natural interpretation as $g^{-1} a g$ and we extend this notation to subgroups of $G$ and $A$ by writing $[G, A] = \{[g, a] \mid g \in G, a \in A\}$.

Group Actions on a Subgroup Lattice

Let $G$ be a group and $A$ be an operator group for $G$. The map $H \mapsto H^a$ for $a \in A$ and $H \leq G$ defines an action of $A$ on the set of subgroups of $G$ and the subgroup correspondence theorem (see for example Section 1.4 of [12]) implies that if $H_1$ and $H_2$ are subgroups of $G$ such that $H_1 \leq H_2$ then $H_1^a \leq H_2^a$. Thus an operator group for $G$ acts on the subgroup lattice of $G$ in a structure preserving way.
In the above context, if the operator group is $G$ acting on itself by conjugation, then an
element $g \in G$ is said to normalise a subset $X \subseteq G$ if $X^g = X$. If $X \subseteq G$, the normaliser
of $X$ in $G$ is

$$N_G(X) = \{ g \in G \mid X^g = X \}$$

which is readily seen to be a subgroup of $G$.

**Frobenius Groups**

A Frobenius group is a group $G$ having a subgroup $H$ such that $H \cap H^g = 1$ for every
$g \in G \setminus H$. If $G$ is a group, a subgroup $H$ of $G$ such that $H \cap H^g = 1$ for all $g \in G \setminus H$
is called a Frobenius complement of $G$.

If $G$ is a Frobenius group and $H$ a Frobenius complement of $G$ then the operation
$Hx \mapsto Hxg$ defines a transitive action of $G$ on the set of right cosets of $H$ in $G$. Moreover,
if an element $g \in G$ fixes the distinct cosets $Hx$ and $Hy$ then $Hxg = Hx$ and $Hyg = Hy$
so it follows that $g \in H^x \cap H^y$. But since $xy^{-1} \notin H$ and $g^{y^{-1}} \in H^y H^{-1} \cap H$
we must have $g = 1$. Therefore, with this action of $G$ on the set of right cosets of $H$ in $G$ no non-trivial
element of $G$ can have more than one fixed point. Conversely, suppose that $G$ is a group
which acts transitively but non-regularly on a set $X$ and that each non-trivial element
of $G$ acts with at most one fixed point in $X$. If $x \in X$ and $H = C_G(x)$ then for all
$g \in G \setminus H$ the subgroup $H^g \cap H$ fixes the distinct elements $x$ and $x^g$ and is therefore
trivial. This shows $G$ to be a Frobenius group.

The next result is due to Frobenius and the reader will find a proof in Section 8.5 of [12].

**2.2.1** If $G$ is a finite group with a subgroup $H$ such that $H^g \cap H = 1$ for all $g \in G \setminus H$,
then $N = G \setminus \bigcup_{g \in G} (H^g \setminus 1)$ is a normal subgroup of $G$ such that $G = NH$ and $H \cap N = 1$.

If $G$ is finite Frobenius group with Frobenius complement $H$, then $G$ satisfies the hyp-
theses of Theorem 2.2.1 and so has a normal subgroup $N$ called the Frobenius kernel
of $G$, such that $G = H \ltimes N$.

**2.2.2** If $G$ is a finite Frobenius group with Frobenius complement $H$ and Frobenius kernel
$N$ then the action of $H$ on $N$ by conjugation is fixed-point-free. Conversely if $G = H \ltimes N$
is a finite group and the action of $H$ on $N$ is fixed-point-free, then $G$ is a Frobenius group
with Frobenius kernel $N$ and Frobenius complement $H$. 
2.3. Subnormal Subgroups

The next theorem is to be found in Section 10.5 of [12].

2.2.3 Let $A$ be a fixed-point-free operator group for a finite group $G$. Then every subgroup of $A$ with order $pq$ where $p$ and $q$ are primes, is cyclic. The Sylow $p$-subgroups of $A$ are cyclic if $p$ is odd and cyclic or generalised quaternion if $p = 2$.

Using Theorem 2.2.3 and the characterisation of Frobenius groups in Theorem 2.2.2 we obtain the structure of the Sylow $p$-subgroups of finite Frobenius complements. In particular, since every abelian subgroup of a generalised quaternion group is cyclic, it may be seen that

2.2.4 If $H$ is a finite Frobenius complement then each abelian subgroup of $H$ is cyclic.

2.3 Subnormal Subgroups

A subgroup $H$ of a group $G$ is said to be subnormal in $G$ if there is a finite sequence of subgroups $\{G_i\}_{i=0}^{n}$ such that

$$H = G_n \trianglelefteq G_{n-1} \trianglelefteq G_{n-2} \trianglelefteq \cdots \trianglelefteq G_0 = G;$$

otherwise $H$ is said to be non-subnormal in $G$. We will refer to (2.1) as a subnormal series of length $n$ between $G$ and $H$. The defect of $H$ in $G$ is the length of the shortest such series between $G$ and $H$. We will often write $H \text{ sn } G$ to denote the fact that $H$ is subnormal in $G$ and $H \text{ nsn } G$ to mean that $H$ is non-subnormal in $G$. We record the following basic properties of subnormality (see for example Section 3.3 of [16]).

2.3.1 Let $H$ be a subnormal subgroup of a group $G$. Then

(i) If $K$ is a subnormal subgroup of $H$ then $K$ is subnormal in $G$;

(ii) For any subgroup $K$, we have $K \cap H$ is subnormal in $K$;

(iii) If $H$ is contained in a subgroup $K$ then $H$ is subnormal in $K$.

We will also find need for the following theorem which we obtain by the subgroup correspondence theorem.
2.3.2 Let \( G \) be a group and \( \theta \) a homomorphism of \( G \). Then the subnormal subgroups of \( G^\theta \) are of the form \( H^\theta \) where \( H \) is a subnormal subgroup of \( G \) containing \( \text{Ker} \theta \). A corresponding statement holds for non-subnormal subgroups.

**Proof.** Let \( N \) denote the kernel of \( \theta \). A subnormal subgroup \( H \) of \( G \) containing \( N \) clearly gives rise to a subnormal subgroup \( H/N \) of \( G/N \). We therefore show the converse.

By the subgroup correspondence theorem each subnormal subgroup of \( G/N \) may be written in the form \( H/N \) where \( H \) is a subgroup of \( G \) containing \( N \). Since \( H/N \) is subnormal in \( G/N \), we have, by the subgroup correspondence theorem, a subnormal series of the form \[ H/N = G_n/N \trianglelefteq G_{n-1}/N \trianglelefteq G_{n-2}/N \trianglelefteq \cdots \trianglelefteq G_0/N = G/N. \]

But the correspondence theorem once more implies that \( H = G_i \) is normal in \( G_{i-1} \) for \( i = 1, \ldots, n \) and hence that \( H \) is subnormal in \( G \).

If \( G \) is a group, the *normal closure* of a subset \( X \subseteq G \) is defined to be the intersection of all normal subgroups of \( G \) containing \( X \). The normal closure of \( X \) in \( G \) is itself normal in \( G \) and is therefore the smallest normal subgroup of \( G \) containing \( X \). We will denote the normal closure of \( X \) in \( G \) by

\[ X^G \]

and clearly \( X^G = \langle x^g \mid x \in X, g \in G \rangle \). If \( H \) is a subgroup of \( G \), the series of successive normal closures of \( H \) in \( G \) is the sequence of subgroups

\[ H^{G,0} = G \quad \text{and} \quad H^{G,i+1} = H^{H^{G,i}} \]

which form a subnormal series

\[ G = H^{G,0} \geq H^{G,1} \geq \cdots \]

Our main interest in this series arises from the following theorem (a proof of this result may be found in Section 13.1 of [12]).

2.3.3 Let \( H \) be \( G \) and suppose that \( H = G_n \trianglelefteq G_{n-1} \trianglelefteq G_{n-2} \trianglelefteq \cdots \trianglelefteq G_0 = G \) is a subnormal series between \( G \) and \( H \). Then \( H^{G,i} \leq G_i \) and hence \( H = H^{G,n} \).

It follows from Theorem 2.3.3 that a subgroup \( H \) is subnormal in \( G \) precisely when the series \( S = \{ H^{G,i} \} \) of successive normal closures of \( H \) in \( G \) terminates at \( H \). Moreover, if the series \( S \) terminates at \( H \) then the length of \( S \) is the defect of \( H \) in \( G \).
The Join of Subnormal Subgroups

If $G$ is a group and $H$ and $K$ are subgroups of $G$, the join of $H$ and $K$ is defined to be the subgroup $\langle H, K \rangle$. In 1939 H. Wielandt showed that if $G$ is a finite group then the join of any two subnormal subgroups of $G$ is also subnormal in $G$. This result was later extended to groups satisfying the maximal condition on subnormal subgroups, that is, groups with no infinite strictly ascending chains of subnormal subgroups. Wielandt’s proof and its extensions may be found in [9].

2.3.4 (H. Wielandt) Let $H$ and $K$ be subnormal subgroups of a finite group $G$. Then $\langle H, K \rangle$ is subnormal in $G$.

Counterexamples due to H. Zassenhaus and P. Hall (see for example [9], Section 1.5) show that this result need not hold for an infinite group.

2.4 Soluble and Nilpotent Groups

A group $G$ is said to be soluble if it has an abelian series, that is to say, a series

$$1 = G_0 \unlhd G_1 \unlhd G_2 \unlhd \cdots \unlhd G_n = G$$

such that, for $i = 1, 2, \ldots, n$, each of the factors $G_i/G_{i-1}$ is abelian. The derived series of $G$ is the series

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$$

defined by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. Each term in the derived series of $G$ is characteristic in $G$. Of course if $G$ is a finite group the derived series of $G$ necessarily terminates, though it need not reach 1. If $G$ is soluble however, the next theorem (see Section 5.1 of [12]) shows that the derived series descends faster than any other abelian series for $G$.

2.4.1 If $G$ is a soluble group and $1 = G_0 \unlhd G_1 \unlhd G_2 \unlhd \cdots \unlhd G_n = G$ is an abelian series of $G$, then $G^{(1)} \unlhd G_{n-1}$ and thus $G^{(n)} = 1$.

A group $G$ is called nilpotent if it has a central series, that is to say, a series

$$1 = G_0 \unlhd G_1 \unlhd G_2 \unlhd \cdots \unlhd G_n = G$$
such that \( G_i \leq G \) and \( G_i/G_{i-1} \leq \zeta(G_i/G_{i-1}) \) for \( i \geq 1 \). The nilpotency class of a nilpotent group \( G \) is the length of the shortest such series of \( G \). An important source of finite nilpotent groups is the groups of prime power order. In particular,

2.4.2 Every finite \( p \)-group is nilpotent.

A proof of Theorem 2.4.2 may be found in Section 5.1 of [12]. The next theorem gives a characterisation of finite nilpotent groups and is given in Section 5.2 of [12].

2.4.3 Let \( G \) be a finite group. Then the following properties are equivalent:

(i) \( G \) is nilpotent;
(ii) every subgroup of \( G \) is subnormal;
(iii) \( G \) is the direct product of its Sylow subgroups.

The Fitting Subgroup

The Fitting subgroup of a group \( G \) is the subgroup generated by the normal nilpotent subgroups of \( G \). We will denote the Fitting subgroup of \( G \) by

\[ \text{Fit} \ G. \]

Since the image of a normal subgroup of \( G \) under an automorphism of \( G \) is itself normal and nilpotent, \( \text{Fit} \ G \) is a characteristic subgroup of \( G \). Moreover if \( G \) is a finite group then \( \text{Fit} \ G \) is the maximal normal nilpotent subgroup of \( G \).

2.4.4 Let \( G \) be a group and \( H \) a nilpotent subnormal subgroup of \( G \). Then \( H \) is contained in \( \text{Fit} \ G \).

Proof. We argue by induction on the defect \( n \) of \( H \). If \( n = 1 \) then \( H \leq \text{Fit} \ G \) so let \( H = H_n \leq H_{n-1} \leq \cdots \leq H_0 = G \) be a subnormal series for \( H \). Since \( H = H_n \) is a normal nilpotent subgroup of \( H_{n-1} \) it follows that \( H_n \leq \text{Fit} \ H_{n-1} \). But \( \text{Fit} \ H_{n-1} \) is characteristic in \( H_{n-1} \) and so normal in \( H_{n-2} \). In particular \( \text{Fit} \ H_{n-1} \) is a subnormal nilpotent subgroup of \( G \) with defect at most \( n-1 \) and so by induction \( \text{Fit} \ H_{n-1} \leq \text{Fit} \ G \). Thus \( H = H_n \leq \text{Fit} \ G \) which is what we were required to show. \( \square \)
The Lower and Upper Central Series

We conclude this section with a brief description of the lower central series and the upper central series of a group. The lower central series of a group $G$ is the series

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots$$

defined by $\gamma_1 = G$ and $\gamma_{i+1} = [\gamma_i, G]$ for $i \geq 1$. Each term in the lower central series of $G$ is characteristic in $G$, and if $G$ is a finite group the series necessarily terminates, though it need not in general reach $1$. It is also evident that $\gamma_i G / \gamma_{i+1} G$ lies in the centre of $G / \gamma_{i+1} G$ for $i \geq 1$. The upper central series of a group $G$ is the series

$$1 = \zeta_0 G \leq \zeta_1 G \leq \zeta_2 G \leq \cdots$$

defined by $\zeta_0 = 1$ and $\zeta_{i+1} G / \zeta_i G = \zeta_i (G / \zeta_i G)$ for $i \geq 0$. Each term in the upper central series of a group $G$ is evidently a characteristic nilpotent subgroup of $G$. If $G$ is a finite group the upper central series terminates at a subgroup of $G$ called the hypercentre of $G$ and which we will denote by $\zeta_\infty G$. The two following results may be found in Section 5.1 of [12].

2.4.5 Let $1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$ be a central series in a nilpotent group $G$. Then:

(i) $\gamma_i G \leq G_{n-i+1}$, for all $i = 1, \ldots, n$, and hence $\gamma_n G = 1$.

(ii) $G_i \leq \zeta_i G$, for all $i = 1, \ldots, n$, and hence $G = \zeta_n G$.

(iii) The length of the upper central series is equal to the length of the lower central series which is equal to the nilpotency class of $G$.

2.4.6 Let $G$ be a group and $i$ and $j$ be positive integers. Then:

(i) $\zeta_i (G / \zeta_j G) = \zeta_{i+j} G / \zeta_j G$.

(ii) $[\gamma_i G, \zeta_j G] \leq \zeta_{i-j} G$ if $i \leq j$.

(iii) $[\gamma_i G, \gamma_j G] \leq \gamma_{i+j} G$. 

Chapter 2. Preliminaries

2.5 The Frattini Subgroup

The Frattini subgroup of a finite group $G$ is defined to be the intersection of the maximal subgroups of $G$. We will denote the Frattini subgroup of a group $G$ by

$\text{Frat } G$.

The Frattini subgroup of $G$ is clearly a characteristic subgroup of $G$. The purpose of introducing the Frattini subgroup here is to allow the following results to be stated.

2.5.1 If $G$ is a finite $p$-group then $\text{Frat } G = [G, G](g^p \mid g \in G)$.

The above result may be found in Section 9 of Chapter A in [5]. The following result is due to P. Hall (see for example Section 2.1 of [15]).

2.5.2 If $G$ is a finite $p$-group and $\alpha$ is an automorphism of $G$ which acts trivially on $G/\text{Frat } G$, then $\alpha$ has $p$-power order.

2.6 A Property of Iterated Commutators

Verbal and marginal subgroups have been studied by P. Hall in [8] where he showed how they could be applied in the context of proving a basic property of iterated commutators. In [8] Hall states two rather technical results without proof; since it has not been possible to find a proof of the results of [8] in the literature and since these results will be required in a later chapter, a proof of the results of [8] is given below.

We begin by introducing some fairly standard notation and terminology. Let $X$ be a non-empty set of symbols and let $X^{-1}$ denote the set of symbols $\{x^{-1} \mid x \in X \}$. A word in $X$ is defined to be any finite sequence of symbols from $X \cup X^{-1}$. If $u$ is a word in $X$, it will be convenient to write

$u = x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}$

where $k \geq 0$ and for $i = 1, \ldots, k$, $x_i \in X$ and $e_i = \pm 1$; if $k = 0$ the word $u$ is said to be trivial. In this context we will denote the trivial word in $X$ by 1. The product of two words $u = x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}$ and $w = y_1^{f_1} y_2^{f_2} \ldots y_m^{f_m}$ in $X$ is defined to be the concatenation of $u$ by $v$; we will write

$uw = x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k} y_1^{f_1} y_2^{f_2} \ldots y_m^{f_m}$.
for the product of \( u \) and \( w \), and adopt the convention that \( uw = vu = u \) if \( w \) is the trivial word. The inverse of the word \( u = x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k} \) is defined to be \( u^{-1} = x_k^{-e_k} x_{k-1}^{-e_{k-1}} \ldots x_1^{-e_1} \).

Two words \( u \) and \( v \) in \( X \) are said to be equivalent if it is possible to pass from \( u \) to \( v \) by a finite sequence of the operations

(i) insertion of a pair of consecutive symbols of the form \( x_i^{e_i} \) and \( x_i^{-e_i} \), or

(ii) deletion of a pair of consecutive symbols of the form \( x_i^{e_i} \) and \( x_i^{-e_i} \).

These operations define an equivalence relation on the words in \( X \) and the equivalence classes, together with the operation of concatenation form a group called the free group on \( X \). A more general discussion of free groups may be found in Chapter 2 of [12].

Let \( G \) be a group and \( X \) be a non-empty set of symbols. If \( u = u(x_1, x_2, \ldots, x_k) \) is a non-trivial word in the symbols \( x_1, x_2, \ldots, x_k \in X \) and \( g_1, g_2, \ldots, g_k \in G \) we will denote by \( u(g_1, g_2, \ldots, g_k) \) the word in \( G \) obtained by substituting \( g_i \) for \( x_i \) at each occurrence of the symbol \( x_i \) in \( u \) and \( g_i^{-1} \) for \( x_i^{-1} \) at each occurrence of the symbol \( x_i^{-1} \) in \( u \) for \( i = 1, \ldots, k \). (We will adopt the convention that if \( u \) is the empty word in \( X \) then \( u(g_1, g_2, \ldots, g_k) = 1_G \) for all \( g_1, g_2, \ldots, g_k \in G \).) An element \( g \in G \) is said to be a value of \( u(x_1, x_2, \ldots, x_k) \) in \( G \) if there exist \( g_1, g_2, \ldots, g_k \in G \) such that \( g = v(g_1, g_2, \ldots, g_k) \).

The verbal subgroup of \( u \) in \( G \) is the subgroup generated by the values of \( u \) in \( G \) and will be denoted by

\[
V_u(G).
\]

The verbal subgroup of \( u \) in \( G \) is clearly a normal subgroup of \( G \). Dual to the verbal subgroup of \( u(x_1, x_2, \ldots, x_k) \) is the set of elements \( h \in G \) such that for all choices of \( g_1, g_2, \ldots, g_k \) in \( G \)

\[
u(g_1, g_2, \ldots, g_k) = u(g_1 h, g_2, \ldots, g_k) = u(g_1, g_2 h, \ldots, g_k) = \cdots = u(g_1, g_2, \ldots, g_k h).
\]

The set of elements \( h \) satisfying this condition forms a subgroup of \( G \) known as the marginal subgroup of \( u \) in \( G \) which we will denote by

\[
M_u(G).
\]

If \( u, w \) are words in \( X \) and \( G \) a group, we are interested in determining the marginal subgroup of the word \([u, v] = u^{-1} v^{-1} u v \) and to this end if \( u \) and \( v \) are words in \( X \) and
Chapter 2. Preliminaries

Let $X$ be a non-empty set of symbols and $G$ a group. If $u$ and $v$ are words in $X$ containing no common symbols, then $M_{[u,v]}(G) = M_{u,v}(G) \cap M_{v,u}(G)$.

**Proof.** Let $u = u(x_1, x_2, \ldots, x_k)$, $v = v(x_{k+1}, x_{k+2}, \ldots, x_{k+m})$ and $w = [u, v]$. If $h$ is an element of $M_{u,v}(G) \cap M_{v,u}(G)$ then, by the remarks following the definition of $M_{v,u}(G)$, for all choices of $g_1, g_2, \ldots, g_k$ in $G$ there exist $h_1, h_2, \ldots, h_k$ in $C_G(V_u(G))$ such that

$$u(g_1, g_2, \ldots, g_k) = h_1u(g_1h, g_2, \ldots, g_k) = \cdots = h_ku(g_1, g_2, \ldots, g_kh)$$

and for all choices of $g_{k+1}, g_{k+2}, \ldots, g_{k+m}$ in $G$, there exist $h_{k+1}, h_{k+2}, \ldots, h_{k+m}$ in $C_G(V_u(G))$ such that

$$v(g_{k+1}, g_{k+2}, \ldots, g_{k+m}) = h_{k+1}v(g_{k+1}h, g_{k+2}, \ldots, g_{k+m})$$

$$= \cdots = h_{k+m}u(g_{k+1}, g_{k+2}, \ldots, g_{k+m}h).$$

Since $h_1 \in C_G(V_u(G))$ it follows that for all choices of $g_1, g_2, \ldots, g_{k+m}$ in $G$

$$w(g_1, g_2, \ldots, g_{k+m})$$

$$= u(g_1, g_2, \ldots, g_k) \cdot v(g_{k+1}, \ldots, g_{k+m})^{-1}u(g_1h, g_2, \ldots, g_k)v(g_{k+1}, \ldots, g_{k+m})$$

$$= u(g_1, g_2, \ldots, g_k) \cdot h_1^{-1}v(g_{k+1}, \ldots, g_{k+m})^{-1}h_1u(g_1h, g_2, \ldots, g_k)v(g_{k+1}, \ldots, g_{k+m})$$

$$= u(g_1, g_2, \ldots, g_k) \cdot v(g_{k+1}, \ldots, g_{k+m})^{-1}u(g_1, g_2, \ldots, g_k)v(g_{k+1}, \ldots, g_{k+m})$$

$$= w(g_1, g_2, \ldots, g_{k+m}).$$
and similarly, it may be shown that for all choices of \( g_1, g_2, \ldots, g_{k+m} \) in \( G \),

\[
w(g_1, g_2, \ldots, g_{k+m}) = w(g_1, g_2 h, \ldots, g_{k+m}) = \cdots = w(g_1, g_2, \ldots, g_{k+m} h).
\]

This shows that \( h \in M_{[u,v]}(G) \) and therefore that \( M_{[u,v]}(G) \geq M_{u,v} \cap M_{v,u}(G) \). Conversely, if \( h \in M_{[u,v]}(G) \), then for all choices of \( g_1, g_2, \ldots, g_{k+m} \) in \( G \),

\[
u(g_1 h, g_2, \ldots, g_k)^{-1} v(g_{k+1}, \ldots, g_{k+m})^{-1} u(g_1 h, g_2, \ldots, g_k) v(g_{k+1}, \ldots, g_{k+m})
\]

and hence for all choices of \( g_1, g_2, \ldots, g_{k+m} \) in \( G \),

\[
v(g_{k+1}, g_{k+2}, \ldots, g_{k+m}) = v(g_{k+1}, \ldots, g_{k+m}) (u(g_1, \ldots, g_k) u(g_1, \ldots, g_k)^{-1}).
\]

From this, it follows that for all choices of \( g_1, g_2, \ldots, g_k \) in \( G \),

\[
u(g_1, g_2, \ldots, g_k) \equiv u(g_1 h, g_2, \ldots, g_k) \mod C_G(V_v(G)).
\]

An identical argument may be used to show that for all choices of \( g_1, g_2, \ldots, g_k \) in \( G \),

\[
u(g_1, g_2, \ldots, g_k) \equiv u(g_1 h, g_2, \ldots, g_k) \equiv \cdots \equiv u(g_1, g_2, \ldots, g_k h) \mod C_G(V_v(G))
\]

and that for all choices of \( g_{k+1}, \ldots, g_{k+m} \) in \( G \),

\[
v(g_{k+1}, g_{k+2}, \ldots, g_{k+m}) \equiv v(g_{k+1}, g_{k+2}, \ldots, g_{k+m}) \equiv \cdots \equiv v(g_{k+1}, g_{k+2}, \ldots, g_{k+m} h) \mod C_G(V_a(G)).
\]

This shows that \( M_{[u,v]}(G) \leq M_{u,v} \cap M_{v,u}(G) \) thereby completing the proof of the theorem.

Our main interest in 2.6.1 arises from its applicability to proving some basic facts about the iterated commutators of weight \( k \) in \( G \). Let \( X = \{x_1, x_2, \ldots\} \) be a set of symbols. Defining the the commutator words \( h_i \) in \( X \) by

\[
h_1 = x_1 \quad \text{and} \quad h_{i+1} = [h_i, x_{i+1}],
\]

it is evident that if \( G \) is a group the verbal subgroup of \( h_i \) in \( G \) is in fact the \( i \)-th term in the lower central series of \( G \). The main result of this section follows.

2.6.2 If \( h_1, h_2, \ldots \) are the commutator words in \( X \) defined above, and if \( G \) is a group, then for \( i = 0, 1, \ldots \), the marginal subgroup of \( h_{i+1} \) in \( G \) is \( \zeta_i G \).
Proof. We prove the theorem using induction on $i$, observing firstly that the marginal subgroup of a single letter in $X$ is the trivial subgroup of $G$ and hence $M_{h_1}(G) = 1$; in particular, the hypothesis holds when $i = 1$. Suppose therefore, that for some $i \geq 1$, $M_{h_i}(G) = \zeta_{i-1}G$. By definition,

$$M_{h_i,x_i+1}(G)/C_G(V_{x_i+1}(G)) = M_{h_i}(G/C_G(V_{x_i+1}(G))). \tag{2.2}$$

Since $C_G(V_{x_i+1}(G)) = \zeta G$, it follows from (2.2) and the inductive hypothesis that $M_{h_i,x_i+1}(G)/\zeta G = \zeta_{i-1}(G/\zeta G)$. By (i) in Theorem 2.4.6, $\zeta_{i-1}(G/\zeta G) = \zeta_iG/\zeta G$, so (2.2) becomes

$$M_{h_i,x_i+1}(G)/\zeta G = \zeta_iG/\zeta G$$

from which it may be seen that $M_{h_i,x_i+1}(G) = \zeta_i G$. Using the definition once again,

$$M_{x_i+1,h_i}(G)/C_G(V_{h_i}(G)) = M_{x_i+1}(G/C_G(V_{h_i}(G)))$$

and, since the marginal subgroup of the symbol $x_{i+1}$ in the group $G/C_G(V_{h_i}(G))$ is trivial, it follows that

$$M_{x_i+1,h_i}(G) = C_{G}(V_{h_i}(G))$$

and in particular, since $V_{h_i}(G) = \gamma_i G$, that

$$M_{x_i+1,h_i}(G) = C_G(\gamma_i G). \tag{2.3}$$

From Theorem 2.4.6 (ii) we have $\zeta_i G \leq C_G(\gamma_i G)$ and hence (2.3) shows that

$$\zeta_i G \leq M_{x_i+1,h_i}(G). \tag{2.4}$$

Now by the commutation rule Theorem 2.6.1, we obtain

$$M_{h_i+1}(G) = M_{h_i,x_i+1}(G) \cap M_{x_i+1,h_i}(G). \tag{2.5}$$

Since we have already shown that $M_{h_i,x_i+1}(G) = \zeta_i G$, we observe that together (2.4) and (2.5) imply that $M_{h_{i+1}}(G) = \zeta_i G$. This concludes the proof of the theorem. \qed
Normalising Properties

In this chapter we are concerned with actions of subgroups of the automorphism group of a finite group $G$ on the subgroup lattice of $G$, particularly those subgroups consisting of automorphisms which fix every non-subnormal subgroup in $G$. Recalling a result of E. Schenkman on the norm of a group and a result of H. Wielandt on the Wielandt subgroup of a finite group, we are able to give some basic properties of the intersection of the normalisers of the non-subnormal subgroups of a finite group. These properties offer a convenient means to study the set of automorphisms of $G$ which fix every non-subnormal subgroup in $G$.

3.1 The Inner Automorphism Group

If $G$ is a group and $g$ an element of $G$, the inner automorphism group acts on the subgroup lattice of $G$ via the map which sends each subgroup $H$ of $G$ to its conjugate $H^g$. The kernel of this action is the norm of $G$ which is the subgroup

$$\kappa(G) = \bigcap_{H \leq G} N_G(H). \quad (3.1)$$

The norm of a group $G$ may be seen to consist precisely of those elements $g \in G$ such that $a^g$ is, for every $a \in G$, a power of $a$. If $H$ is a subgroup of $G$ and $\alpha$ an automorphism of $G$ then $\alpha$ maps $N_G(H)$ to $N_G(H^\alpha)$ so it follows from (3.1) that $\kappa(G)$ is a characteristic subgroup of $G$. It is trivial to observe that an alternate definition of the norm of a group $G$ is as the set of elements of $G$ normalising the non-normal subgroups of $G$ and therefore that $G$ is a Dedekind group if and only if $\kappa(G) = G$. The next result is due to E. Schenkman and may be found in [13].

3.1.1 Let $G$ be a group. Then $\kappa(G) \leq \zeta_2G$ and hence $\kappa(G)$ centralises $G' = [G,G]$.  

19
Chapter 3. Normalising Properties

If $H$ is a subnormal subgroup of $G$, it follows from Theorem 2.3.2 that for each element $g \in G$ the subgroup $H^g$ is subnormal in $G$. The group action of $\text{Inn } G$ on the subgroup lattice of $G$ may therefore be restricted to a group action of $\text{Inn } G$ on the lattice of subnormal subgroups of $G$. The kernel of this action is $\omega(G)$, the Wielandt subgroup of $G$ which may be written as

$$\omega(G) = \bigcap_{H \text{ sn } G} N_G(H). \quad (3.2)$$

The Wielandt subgroup of $G$ is therefore a characteristic subgroup of $G$. If $G$ is a finite nilpotent group then $\omega(G)$ coincides with $\kappa(G)$. The following result of Wielandt shows that if $G$ is a finite group then $\omega(G)$ is in fact a non-trivial subgroup of $G$.

**3.1.2** If $N$ is a minimal normal subgroup of a finite group $G$, then $N \leq \omega(G)$.

Wielandt proved Theorem 3.1.2 under the weaker hypothesis that $N$ satisfies the minimal condition on subnormal subgroups. In particular, Wielandt showed that if $N$ is a minimal normal subgroup of a group $G$ and $N$ has no infinite strictly descending chain of subnormal subgroups, then $N \leq \omega(G)$; examples due to D. Robinson [11] show that this condition that $N$ satisfies the minimal condition on subnormal subgroups cannot be dropped here. The reader is referred to Section 4.6 of [9] for a proof of Theorem 3.1.2.

**The Subgroup Normalising Every Non-Subnormal Subgroup**

If $G$ is a finite group, Theorem 2.3.2 implies that the action of $\text{Inn } G$ on the subgroup lattice of $G$ restricts in a natural way to an action of $\text{Inn } G$ on the set of non-subnormal subgroups of $G$. The kernel of this action, which we denote by $\lambda(G)$, is the set of elements of $G$ normalising every non-subnormal subgroup of $G$. Specifically,

$$\lambda(G) = \bigcap_{H \text{ ns } G} N_G(H)$$

from which it is apparent that $\lambda(G)$ is a characteristic subgroup of $G$. We also observe that

$$\kappa(G) = \omega(G) \cap \lambda(G).$$

The remainder of this section is devoted to establishing the basic properties of the subgroup $\lambda(G)$. 


3.1.3 Let $G$ be a group and $N$ be a normal subgroup of $G$. Then $\lambda(G/N) \geq \lambda(G)N/N$.

Proof. By Theorem 2.3.2 each non-subnormal subgroup of $G/N$ may be written as $H/N$ where $H$ is a non-subnormal subgroup of $G$ and $N \leq H$. It follows that if $g$ is an element of $G$ then $gN$ is in $\lambda(G/N)$ precisely when $g$ normalises every non-subnormal subgroup $H$ of $G$ containing $N$. Every element of $\lambda(G)$ satisfies this condition and hence the result is proved.

3.1.4 Let $G$ be a finite group and $M$ be a minimal normal subgroup of $G$. If $M \leq \lambda(G)$ then $M$ is central in $G$.

Proof. Since $M$ is a minimal normal subgroup of $G$ it is contained in $\omega(G)$. It follows that $M \leq \omega(G) \cap \lambda(G)$ and hence that $M \leq \kappa(G)$. Since $\kappa(G) \leq \zeta_2 G$ we have

$$[M, G] \leq [\kappa(G), G] \leq [\zeta_2 G, G] \leq \zeta G.$$  \hspace{1cm} (3.3)

Now $[M, G]$ is a normal subgroup of $G$ which, since $M$ is normal in $G$, is contained in $M$; hence by the minimality of $M$, either $[M, G] = 1$, in which case $M \leq \zeta G$, or $[M, G] = M$. In the latter case (3.3) implies that $M$ is central.

As a corollary to Theorem 3.1.4 we have the following.

3.1.5 If $G$ is a finite group and $G$ has trivial centre then $\lambda(G) = 1$.

Proof. Let $G$ be a finite group. If $\lambda(G)$ is non trivial, Theorem 3.1.4 implies that any minimal normal subgroup of $G$ contained in $\lambda(G)$ is contained in $\zeta G$. But since $\zeta G = 1$, it follows that $\lambda(G) = 1$.

The next result may be viewed as an analogue of Theorem 3.1.1.

3.1.6 If $G$ is a finite group, then $\lambda(G) \leq \zeta \infty G$.

Proof. Let $N = \zeta \infty G$. Since $G/N$ has trivial centre, it follows from Theorem 3.1.5 that $\lambda(G/N) = 1$. From Theorem 3.1.3,

$$\lambda(G/N) \geq \lambda(G)N/N$$

which, since $\lambda(G/N)$ is trivial, shows that $\lambda(G) \leq N$. \hfill \Box
3.2 The Automorphism Group

In this section we consider the action of the automorphism group of a group \( G \) on the subgroup lattice of \( G \). Naturally, if \( \alpha \) is an automorphism of \( G \), the action of \( \alpha \) on a subgroup \( H \) of \( G \) is given by \( H \mapsto H^\alpha \) and, as in the previous section, we appeal to Theorem 2.3.2 to show that the action of \( \text{Aut} \, G \) on the subgroup lattice of \( G \) can be restricted to actions on particular subsets of the subgroup lattice of \( G \).

Power Automorphisms

Let \( G \) be a group and \( H \) be a subgroup of \( G \). The set of automorphisms fixing each subgroup of \( G \) will be denoted

\[ \text{Paut} \, G. \]

Since it is the kernel of the action of \( \text{Aut} \, G \) on the set of subgroups of \( G \), it is clear that \( \text{Paut} \, G \) forms a normal subgroup of \( \text{Aut} \, G \). More particularly it can be observed that \( \text{Paut} \, G \) consists precisely of those automorphisms \( \alpha \) such that \( g^\alpha \in \langle g \rangle \) for every \( g \in G \). For this reason \( \text{Paut} \, G \) is usually referred to as the power automorphism group of \( G \). A power automorphism \( \alpha \in \text{Aut} \, G \) is said to be universal if \( \alpha \) maps each element of \( G \) to the same power, that is to say there exists a fixed integer \( n \) such that \( g^\alpha = g^n \) for every \( g \in G \). The following is a theorem of F. Levi (see Theorem 3.4.1 of [4]).

3.2.1 If \( G \) is a finite abelian group then every power automorphism of \( G \) is universal.

The next theorem is due to B. Huppert (see Corollary 5.1.2 of [4]).

3.2.2 If \( G \) is a finite non-abelian \( p \)-group, then \( \text{Paut} \, G \) is an abelian \( p \)-group.

Theorems 3.2.1 and 3.2.2 together reveal a considerable amount about the power automorphisms of finite \( p \)-groups. In particular we obtain the following.

3.2.3 Let \( G \) be a finite \( p \)-group and \( \alpha \in \text{Paut} \, G \). If \( \alpha \) is non trivial and \( (|G|, |\alpha|) = 1 \) then \( G \) is abelian and \( \alpha \) is universal.

An automorphism of a group \( G \) is said to be central if it acts trivially on \( G/\zeta G \) or equivalently, if \( [g, \alpha] \in \zeta G \) for all \( g \in G \). The following theorem is due to C. Cooper (see Theorem 2.2.1 of [4]).
3.2.4 Let \( G \) be a group and \( \alpha \) a power automorphism of \( G \). Then \( \alpha \) is central.

Theorem 3.2.4 implies that if \( G \) has trivial centre then \( \text{Paut} \ G = 1 \). If \( \alpha \in \text{Paut} \ G \) and 
\( g_1, g_2 \in G \) then \( g_1^\alpha = g_1 z_1 \) and \( g_2^\alpha = g_2 z_2 \) where \( z_1, z_2 \in \zeta G \). We therefore observe that
\[
[g_1, g_2]^\alpha = [g_1^\alpha, g_2^\alpha] = [g_1 z_1, g_2 z_2]
\]
where \( z_1, z_2 \in \zeta G \). Hence by the identities Theorem 2.1.1 (i) it follows that \( \alpha \) acts trivially on the commutator subgroup of \( G \). In particular if \( G \) is perfect then \( \text{Paut} \ G = 1 \).

Automorphisms Fixing Every Non-Normal Subgroup

If \( G \) is a group, the subgroup correspondence theorem tells us that \( \text{Aut} \ G \) has a natural action on the set of normal subgroups of \( G \) as well as a natural action on the set of non-normal subgroups of \( G \). We will denote by
\[
\text{Aut}_{\text{nn}} G
\]
the set of automorphisms of \( G \) which fix every non-normal subgroup of \( G \). Since \( \text{Aut}_{\text{nn}} G \) is the kernel of the action of \( \text{Aut} \ G \) on the set of non-normal subgroups of \( G \), it is clear that \( \text{Aut}_{\text{nn}} G \) forms a normal subgroup of \( \text{Aut} \ G \). Moreover, \( \text{Paut} \ G \leq \text{Aut}_{\text{nn}} G \) and if \( G \) is a Dedekind group then \( \text{Aut}_{\text{nn}} G = \text{Aut} \ G \).

As in [3] an automorphism \( \alpha \) fixing every non-normal subgroup of a group \( G \) will be said to be an anomalous automorphism if \( \alpha \) is not a power automorphism, that is to say if there exists a normal subgroup \( N \leq G \) such that \( N^\alpha \neq N \). Naturally, if we wish to obtain interesting generalisations the properties of \( \text{Paut} \ G \) to \( \text{Aut}_{\text{nn}} G \), it is sufficient to consider anomalous automorphisms of non-Dedekind groups. In [3] R. Brandl and L. Verardi consider the structure of \( \text{Aut}_{\text{nn}} G \) and exhibit properties of \( \text{Paut} \ G \) which carry over to \( \text{Aut}_{\text{nn}} G \) for certain classes of groups. In particular, they prove the following results.

3.2.5 Let \( G \) be a finite non-Dedekind group. Then:

(i) \( \text{Aut}_{\text{nn}} G \) is metabelian.

(ii) If \( G \) is not nilpotent of class 2, then \( \text{Aut}_{\text{nn}} G \) is abelian.

(iii) If \( G \) is not a torsion group, then \( \text{Aut}_{\text{nn}} G \) is abelian.
3.2.6 Let $G$ be a finite non-Dedekind $p$-group. Then:

(i) $\text{Aut}_n G$ is a $p$-group.

(ii) If $p$ is odd, then $\text{Aut}_n G$ is abelian.

In contrast to $\text{Paut}_G$ which is always abelian, Brandl and Verardi show that if $n \geq 3$, the 2-group

$$G = \langle a, b \mid a^{2^n} = b^2 = 1, a^b = a^{1+2^{n-1}} \rangle$$

has two non-commuting anomalous automorphisms, $\alpha$ and $\rho$, which they define by the relations $[b, \alpha] = [b, \rho] = 1$, $a^{\alpha} = a^{-1}b$ and $a^{\rho} = ab^{-1}$.

In [7] S. Franciosi, F. de Giovanni and H. Heineken are largely concerned with how the structure of a non-Dedekind group $G$ is restricted if it has an anomalous automorphism $\alpha \in \text{Aut}_n G$. For our immediate purposes though, we are interested in the following generalisation of Theorem 3.2.4 and a corollary, both given in [7].

3.2.7 Let $G$ be a group. Then $\text{Aut}_n G$ acts trivially on $G/\kappa(G)$. In particular, $\text{Aut}_n G$ acts trivially on $G/\zeta_2 G$ and on $\gamma_3 G$.

3.2.8 Let $G$ be a group which is either perfect or has trivial centre. Then $\text{Aut}_n G = 1$.

In [7] it is shown that if $G$ is a non-abelian group which is not periodic, then every element of $\text{Aut}_n G$ is central (see Theorem 2.4 of [7]). However, a similar statement does not hold for finite groups. In particular, if $p$ is an odd prime, and $G$ is the group

$$G = \langle a, b \mid a^p = b^2 = 1, a^b = a^{1+p} \rangle$$

then the automorphism $\alpha$ of $G$ defined by $a^{\alpha} = ab$ and $a^{\rho} = b$ fixes the non-normal subgroups of $G$ but is not a central automorphism (see [7]).

The aspect of the work of Franciosi, de Giovanni and Heineken in [7], which deals with the structure of those non-Dedekind groups with anomalous automorphisms will be touched upon briefly in the final chapter of this thesis.

Automorphisms Fixing Every Non-Subnormal Subgroup

Our objective in this section is essentially to extend as far as possible the known results about the group $\text{Paut}_G$ to the set of automorphisms which fix every non-subnormal
3.2. The Automorphism Group

subgroup of a finite non-nilpotent group. Since the action of Aut $G$ on the subgroup lattice of $G$ has a natural restriction to the set of non-subnormal subgroups of $G$, the set of automorphisms which fix every non-subnormal subgroup of $G$ is a normal subgroup of Aut $G$. We will use

$$\text{Aut}_{\text{nsn}} G$$

to denote the automorphisms which fix every non-subnormal subgroup of $G$. It is clear that $\text{Paut} G \leq \text{Aut}_{\text{nsn}} G \leq \text{Aut}_{\text{nsn}} \text{G}$ and that if $G$ is a finite nilpotent group then every subgroup of $G$ is subnormal so, in this case at least, $\text{Aut}_{\text{nsn}} G = \text{Aut} G$. Moreover, since $\text{Aut}_{\text{nsn}} G$ maps each non-subnormal cyclic subgroup of $G$ to itself, we use Theorem 2.4.4 to observe that $\text{Aut}_{\text{nsn}} G$ maps every element of $G \setminus \text{Fit} G$ to one of its powers. In particular, $\text{Aut}_{\text{nsn}} G$ acts as a group of power automorphisms of $G/\text{Fit} G$.

3.2.9 Let $G$ be a non-nilpotent group. Then $\text{Aut}_{\text{nsn}} G$ is an abelian subgroup of $\text{Aut} G$.

**Proof.** Let $\alpha$ and $\rho$ be automorphisms of $G$ which fix every non-subnormal subgroup of $G$ and let $g$ be an element of $G \setminus \text{Fit} G$. By Theorem 2.4.4 the subgroup $\langle g \rangle$ is non-subnormal in $G$ and for some integers $m$ and $n$,

$$g^{\alpha \rho} = (g^n)\rho = (g^n)^\rho = g^{nm}\quad \text{and}\quad g^{\alpha \rho} = (g^\alpha)^\rho = (g^\rho)^\alpha = g^{\rho \alpha}.$$

This shows that $[\alpha, \rho]$ centralises $G/\text{Fit} G$. If $h$ is an element of $\text{Fit} G$, then $gh \in G \setminus \text{Fit} G$ so using the identities Theorem 2.1.1 (i),

$$[gh, [\alpha, \rho]] = [g, [\alpha, \rho]]^h[h, [\alpha, \rho]].$$

(3.4)

Since $[g, [\alpha, \rho]] = 1$ and $[gh, [\alpha, \rho]] = 1$, (3.4) shows that $[\alpha, \rho]$ centralises $\text{Fit} G$ and therefore that $[\alpha, \rho]$ centralises $G$.

Using Theorem 3.2.9, we obtain the following.

3.2.10 Let $G$ be a non-nilpotent group. Then $[\lambda(G), \lambda(G)] \leq \zeta G$ and in particular, $\lambda(G)$ is nilpotent of class at most 2.

**Proof.** As $\lambda(G)$ maps to $\text{Aut}_{\text{nsn}} G \cap \text{Inn} G$ under the homomorphism $\theta : G \to \text{Inn} G$, defined by $h^g = h^g$ for all $h, g \in G$, and $\text{Aut}_{\text{nsn}} G$ is abelian, $[\lambda(G), \lambda(G)]$ is contained in $\text{Ker} \theta$. The remainder of the statement now follows, since $\text{Ker} \theta = \zeta G$. 

$\square$
Our next theorem may be viewed as a direct analogue of Theorem 3.2.7 or of Theorem 3.2.4.

3.2.11 Let $G$ be a finite group. Then:

(i) $\text{Aut}_{\text{nsn}} G$ acts trivially on $G/\lambda(G)$ and hence $\text{Aut}_{\text{nsn}} G$ acts trivially on $G/\zeta_\infty G$;

(ii) if $k$ is an integer such that $\zeta_\infty G = \zeta_k G$, then $\text{Aut}_{\text{nsn}} G$ acts trivially on $\gamma_{k+1} G$.

Proof. (i) Let $\alpha$ be an element of $\text{Aut}_{\text{nsn}} G$ and $g$ an element of $G$. If $H$ is a non-subnormal subgroup of $G$, then so is each conjugate of $H$ in $G$ and hence

$$H^g = H^{g^{-1}\alpha g} = ((H^{g^{-1}})^{g^{-1}})^{g^{-1}} = H.$$  

Therefore $[g, \alpha]$ normalises every non-subnormal subgroup of $G$, and belongs to $\lambda(G)$ and so to $\zeta_\infty G$. This shows that $\text{Aut}_{\text{nsn}} G$ acts trivially on $G/\zeta_\infty G$.

(ii) Let $k$ be an integer such that $\zeta_k G = \zeta_{k+1} G$ and let $g = [g_1, g_2, \ldots, g_{k+1}]$ be an element of $\gamma_{k+1} G$. Since $\alpha$ acts trivially on $G/\zeta_k G$, there exist $z_1, z_2, \ldots, z_{k+1}$ in $\zeta_k G$ such that

$$g^\alpha = [g_1, g_2, \ldots, g_{k+1}]^\alpha = [g_1^\alpha, g_2^\alpha, \ldots, g_{k+1}^\alpha] = [g_1 z_1, g_2 z_2, \ldots, g_{k+1} z_{k+1}].$$

By Theorem 2.6.2, for all choices of $g_1, g_2, \ldots, g_{k+1}$ in $G$,

$$[g_1 z_1, g_2 z_2, \ldots, g_{k+1} z_{k+1}] = [g_1, g_2 z_2, \ldots, g_{k+1} z_{k+1}]$$  \hspace{1cm} (3.5)

and proceeding similarly for $1 \leq i \leq k + 1$, we may replace $g_i z_i$ in the iterated commutator (3.5) by $g_i$, thereby observing that for all choices of $g_1, g_2, \ldots, g_{k+1}$ in $G$

$$[g_1 z_1, g_2 z_2, \ldots, g_{k+1} z_{k+1}] = [g_1, g_2, \ldots, g_{k+1}].$$

Since $\gamma_{k+1} G$ is generated by iterated commutators of weight $k + 1$, this completes the proof of the theorem. \hfill \square

The next two results are immediate corollaries of Theorem 3.2.11.

3.2.12 If $G$ is a finite group then $\text{Aut}_{\text{nsn}}(G)$ acts trivially on $G/\text{Fit} G$.

3.2.13 If $G$ is a finite group which is either perfect or has trivial centre then $\text{Aut}_{\text{nsn}} G$ is trivial.
3.3 Examples

We now give examples illustrating some properties of the automorphisms fixing every non-subnormal subgroup of a finite group.

**Example 3.3.1** Our first example is of a group $G$ with an automorphism which has order dividing the order of $G$ and which fixes every non-subnormal subgroup of $G$. Let $A$ be the symmetric group of degree 3 and take the presentation

$$A = \langle a, b \mid a^3 = b^2 = 1, \ a^b = a^2 \rangle.$$  

for $A$. Let $\langle z \rangle$ be a cyclic group of order 3 and let $G = A \times \langle z \rangle$. We define an automorphism $\alpha \in \text{Aut } G$ of order 2 as $[a, \alpha] = [b, \alpha] = 1$, and $z^\alpha = z^2$ and the claim is that $\alpha$ is not a power automorphism of $G$ but $\alpha$ fixes every non-subnormal subgroup of $G$. The non-subnormal 2-subgroups $\langle b \rangle$, $\langle b^2 \rangle$ and $\langle b^3 \rangle$ are fixed by $\alpha$ as are the non-subnormal $\{2, 3\}$-subgroups $\langle b, z \rangle$, $\langle b^2, z \rangle$ and $\langle b^3, z \rangle$. However, while $\alpha$ fixes every non-subnormal subgroup of $G$ $\alpha$ is not a power automorphism since, for example $(za)^\alpha = z^2a$ is not an element of $\langle za \rangle$. □

**Example 3.3.2** In this example we exhibit a group $G$ with an automorphism in $\text{Aut}_{\text{nsm}} G$ of order coprime to the order of $G$.

(i) Let $A$ be the semidirect product of a cyclic group $\langle a \mid a^3 = 1 \rangle$ with the elementary abelian group $\langle x, y \mid x^5 = y^5 = 1, xy = yx \rangle$. Then $A$ has a presentation

$$A = \langle a, x, y \mid a^3 = x^5 = y^5 = 1, xy = yx, x^a = y, y^a = x^{-1}y^{-1} \rangle.$$  

It may be seen that $A$ has trivial centre and hence by Theorem 3.2.13 we observe that $\text{Aut}_{\text{nsm}} A$ is trivial.

(ii) Let $\langle z \rangle$ be a cyclic group of order 5, $A$ be the group defined in (i) above and let $G = A \times \langle z \rangle$. If we define the automorphism $\alpha \in \text{Aut } G$ by $z^\alpha = z^2$ and $[g, \alpha] = 1$ for all $g \in A$ then $\alpha$ has order 4 and we claim, fixes every non-subnormal subgroup of $G$. In particular, since the Sylow 5-subgroup of $G$ is normal, the non-subnormal subgroups of $G$ are the subgroups of order 3 in $G$, which are of the form $\langle a^{x^i} \rangle$ and the subgroups of order 15 in $G$, which are of the form $\langle a^{x^i}, z \rangle$ where $1 \leq i, j \leq 5$. The definition of $\alpha$ implies that $\alpha$ maps every subgroup of order 15 and every subgroup of order 3 in $G$ to itself. However $\alpha$ does not fix the subgroup $\langle zx \rangle$ since $(zx)^\alpha = z^2x$ cannot be an
element of the subgroup \( \langle zx \rangle \). To see the latter, if \( z^2 x = z^k x^k \) for some integer \( k \), then on the one hand \( k \equiv 1 \mod 5 \) and on the other \( k \equiv 2 \mod 5 \). This is a contradiction. \( \square \)

We will show in the final chapter of this thesis that the salient feature of the Example 3.3.2 (ii) is that \( A/(A \cap \text{Fit } G) \cong \langle a \rangle \) acts fixed-point-freely on \( \langle x, y \rangle \) and that the map \( a \) acts fixed-point-freely on \( \langle z \rangle \).
Chapter 4

Power Automorphisms of Finite Groups

Our main object in this chapter is to present some machinery on coprime operator groups which allows us to show that if $G$ is a finite group with a power automorphism of order coprime to that of $G$ then $G$ is a direct product of two subgroups of coprime order: one consists of the fixed points of the automorphism and the other is central and acted on fixed-point-freely by the automorphism. Although the main result of this chapter is only of marginal interest on its own, we will later show that it has a generalisation which places a strong restriction on the structure of those finite groups $G$ with the property that $\text{Aut } G \neq \text{Paut } G \neq \text{Aut}_{\text{non}} G$.

4.1 Coprime Operators

In what follows we shall require a number of results about coprime operator groups, the known proofs of which depend on the Schur-Zassenhaus theorem. We quote this theorem here and for this purpose it is convenient to use the following terminology. If $G$ is a group and $H$ a subgroup of $G$, a complement of $H$ in $G$ is a subgroup $K \leq G$ such that $K \cap H = 1$ and $G = KH$. If in addition $H$ is normal in $G$ then $G$ is said to split over $H$.

We now state the Schur-Zassenhaus theorem and refer to Section 9.1 of [12] for a proof.

4.1.1 Let $G$ be a finite group, $H$ a normal subgroup of $G$ and suppose that

(i) $(|H|, |G/H|) = 1$, and

(ii) either $G$ or $G/H$ is soluble,

then $G$ splits over $H$ and $G$ acts transitively on the complements of $H$ in $G$. 

29
Chapter 4. Power Automorphisms of Finite Groups

It should be observed that if $(|H|, |G/H|) = 1$ then at least one of $|H|$ and $|G/H|$ is odd so the Feit-Thompson theorem (see [6]) which states that groups of odd order are soluble may be used to remove condition (ii) in Theorem 4.1.1. Indeed by citing the Feit-Thompson theorem here we will for the remainder of this thesis waive condition (ii) of Theorem 4.1.1 in any results which rely on Theorem 4.1.1 in the present and later chapters.

Let $A$ be a group of operators for a group $G$. Then $A$ is said to stabilise a chain of subgroups

$$H_0 \leq H_1 \leq \cdots \leq H_n = G$$

if $A$ fixes every coset of $H_{i-1}$ in $H_i$ for $i = 1, \ldots, n$ or equivalently if $[H_i, A] \leq H_{i-1}$ for $i = 1, \ldots, n$.

The two results quoted below may be found in Section 12 of Chapter A in [5].

4.1.2 Let $A$ be a group of operators for a group $G$ with $(|A|, |G|) = 1$. If $A$ stabilises a chain of subgroups

$$H_0 \leq H_1 \leq \cdots \leq H_n = G$$

in $G$ then $[G, A] \leq H_0$.

4.1.3 If $A$ is a group of operators for a group $G$ with $(|A|, |G|) = 1$ then $G = C_G(A)[G, A]$ and if $G$ is abelian then $G = C_G(A) \times [G, A]$.

4.2 Coprime Action by Power Automorphisms

If $A$ is a coprime operator group for a group $G$, Theorem 4.1.3 shows how the action of $A$ on $G$ restricts the structure of $G$. Our purpose here is to show how the structure of $G$ is further restricted if $A$ acts on $G$ by power automorphisms. In particular we prove the following theorem.

4.2.1 Let $G$ be a finite group and $\alpha$ a power automorphism of $G$ such that $(|G|, |\alpha|) = 1$. Then $G$ can be written as

$$G = [G, \langle \alpha \rangle] \times C_G(\alpha)$$

where $[G, \langle \alpha \rangle]$ and $C_G(\alpha)$ have coprime order and $[G, \langle \alpha \rangle] \leq \langle G \rangle$. 
4.2. Coprime Action by Power Automorphisms

**Proof.** Let \( p \) be a prime dividing \( |G| \) and \( P \) be a Sylow \( p \)-subgroup of \( G \). Using Theorem 4.1.3, we may write

\[
P = [P, \langle \alpha \rangle]C_p(\alpha).
\]

Let \( \alpha_P \) denote the restriction of \( \alpha \) to \( P \). Since \( |\alpha_P| \) divides \( |\alpha| \) it follows that \( \alpha_P \) has order coprime to \( p \) and hence by Theorem 3.2.3 we observe that \( \alpha_P \) is a universal power automorphism of \( P \), say \( g^\alpha = g^n \) for all \( g \in P \). If \( C_P(\alpha_P) \) is non-trivial then \( n \equiv 1 \mod p \) and so, by Theorem 2.5.1 and Theorem 2.5.2, \( \alpha_P \) has \( p \)-power order. We therefore conclude that either \( \alpha_P = 1 \) and \( P = C_P(\alpha) \) or \( C_P(\alpha) = 1 \) and \( P = [P, \langle \alpha \rangle] \).

Moreover, since every power automorphism is central, in the latter case we have

\[
P = [P, \langle \alpha \rangle] \leq \zeta G.
\]

Hence for each prime \( p \) dividing \( |G| \) either all Sylow \( p \)-subgroups of \( G \) lie in \( C_G(\alpha) \) or the (unique) Sylow \( p \)-subgroup of \( G \) is an abelian direct factor of \( G \) and \( \alpha \) acts fixed-point-freely on it.

Conversely it not hard to see that if \( G \) is a finite group and \( H \) is a direct factor of order coprime to the order of its direct complement in \( G \), then every power automorphism of \( H \) extends to a power automorphism of \( G \).

5.1. Some Related Results

In [15], Franchi, de Giovanni and Russo prove the result of which Theorem 4.1.4 is a consequence. They prove both positively and negatively groups satisfying the conclusion of Theorem 3.2.3 exist and determine the structure of \( G \) completely up to a normal subgroup, Hall subgroup.

5.1.4. Let \( G \) be a finite group, and let \( p \) be a prime such that \( G \) contains a \( p \)-subgroup not fixed by \( \text{Aut}_p(G) \). Then every \( p \)-subgroup of \( G \) is normal in \( G \).
Chapter 4. Power Automorphisms of Finite Groups

4.2 Cuppelen Actions for Power Automorphisms

If $A$ is a cuppelen action, then for a group $G$, Equation (1.3) shows how the action of $A$ on $G$ restricts the structure of $A$. Here, one can see how the structure of $G$ is further influenced by the action of $A$ to yield new automorphisms. In particular, we prove the following theorem:

4.2.1 Let $G$ be a finite group and $a$ an automorphism of $G$ such that $(|G|, |a|) = 1$. Then $G$ can be written as

$$G = \langle a \rangle \ast \langle x \rangle$$

where $\langle a \rangle$ and $\langle x \rangle$ are subgroups of $G$ with $\langle a \rangle G = \langle x \rangle G$. 

restrictions on the group structure

We showed in Section 4.2 that if \( \alpha \) is a power automorphism of a finite group \( G \) and \( \alpha \) has order coprime to the order of \( G \) then \( \alpha \) acts as a fixed-point-free power automorphism on a Hall subgroup of \( G \). Moreover, this Hall subgroup is an abelian direct factor of \( G \) whose complement in \( G \) is centralised by \( \alpha \). Under these hypotheses though, little further information is available about the structure of \( G \). By contrast we show in Section 5.2 that if these assumptions are modified so that \( \alpha \) is an element of \( \text{Aut}_{\text{nn}} G \) and \( \alpha \) has order coprime to the order of \( G \) then, if \( \alpha \) is not a power automorphism, \( G \) is essentially the direct product of a Frobenius group, which is centralised by \( \alpha \), with an abelian group on which \( \alpha \) acts as a fixed-point-free power automorphism. In Section 5.3 we assume \( G \) to be soluble and use some results of Zassenhaus on the structure of soluble Frobenius complements to further restrict the structure of \( G \). In Section 5.4 it is shown that if \( G \) is insoluble a classification by Suzuki of the finite groups in which every abelian subgroup is cyclic, together with a result of Zassenhaus on the insoluble Frobenius complements, determines the structure of \( G \) completely up to a normal nilpotent Hall subgroup.

5.1 Some Related Results

In [7] Franciosi, de Giovanni and Heineken study the structure of those groups \( G \) for which \( \text{Paut} G \neq \text{Aut}_{\text{nn}} G \neq \text{Aut} G \) and consider both periodic and non-periodic groups. Among the results obtained by Franciosi, de Giovanni and Heineken about the periodic groups with anomalous automorphisms is the following lemma which shows in particular that if \( G \) is a finite group with an anomalous automorphism then \( G \) is the extension of a nilpotent \( p' \)-group by a \( p \)-group.

5.1.1 Let \( G \) be a finite group, and let \( p \) be a prime such that \( G \) contains a \( p \)-subgroup not fixed by \( \text{Aut}_{\text{nn}} G \). Then every \( p' \)-subgroup of \( G \) is normal in \( G \).
If \( P \) is a normal Sylow \( p \)-subgroup of a finite group \( G \) and \( P \) contains a cyclic subgroup \( \langle g \rangle \) not fixed by some \( \alpha \in \text{Aut}_{sn} G \) then \( \alpha \) is an anomalous automorphism of \( P \). Moreover if \( p \) is an odd prime, we can say the following about those finite \( p \)-groups with anomalous automorphisms (see Theorem 3.6 of [7]).

5.1.2 Let \( p \) be an odd prime, and let \( G \) be a finite \( p \)-group. If \( G \) has an anomalous automorphism then the commutator subgroup of \( G \) is cyclic and \( G \) is nilpotent of class at most 2.

More generally, about those finite groups with anomalous automorphisms, Franciosi, de Giovanni and Heineken prove the following (see Theorem 3.9 of [7]).

5.1.3 Let \( G \) be a non-nilpotent finite group. If \( G \) has an anomalous automorphism, then for every odd prime \( p \), the Sylow \( p \)-subgroups of \( G \) are abelian.

5.2 Coprime Actions Fixing Non-Subnormal Subgroups

Our main object in this section is to study the non-nilpotent finite groups \( G \) having the property that \( 1 \neq \text{Paut} G \neq \text{Aut}_{nsn} G \). In particular, we assume at this stage that \( G \) has a coprime operator which does not act as a power automorphism of \( G \) and which fixes every non-subnormal subgroup of \( G \), and prove an analogue to Theorem 4.2.1. We begin with two lemmas, the first of which may be viewed as a generalisation of Theorem 5.1.1.

5.2.1 Let \( G \) be a finite non-nilpotent group and \( p \) be a prime such that \( G \) contains a cyclic \( p \)-subgroup \( \langle g \rangle \) not fixed by \( \text{Aut}_{nsn} G \). If \( h \) is a \( p' \)-element of \( N_G(\langle g \rangle) \), then \( \langle h \rangle \) is subnormal in \( G \).

**Proof.** Let \( \alpha \in \text{Aut}_{nsn} G \) be an automorphism such that \( \langle g \rangle^\alpha \neq \langle g \rangle \). Then \( \langle g \rangle \) is subnormal in \( G \). If \( h \) is a \( p' \)-element of \( N_G(\langle g \rangle) \), then \( \langle g \rangle \) is characteristic in \( \langle g, h \rangle \). In particular, \( \alpha \) does not fix \( \langle g, h \rangle \) and so \( \langle g, h \rangle \) is subnormal in \( G \).

Suppose now that \( \langle h \rangle \) is non-subnormal in \( \langle g, h \rangle \). By Theorem 2.3.2 the subgroup \( \langle h^p \rangle \) is non-subnormal in \( \langle g, h \rangle \) and so \( \langle h \rangle \) and \( \langle h^p \rangle \) are fixed by \( \alpha \). It now follows that \( \alpha \) fixes \( \langle h, h^p \rangle \). But Theorem 4.1.3 tells us that \( \langle g \rangle = [\langle g \rangle, \langle h \rangle] \) and hence by the commutator identity (ii) in Theorem 2.1.1, \( \langle g \rangle = \langle [g, h] \rangle \). In particular, since \( \langle [g, h] \rangle \leq \langle h, h^p \rangle \), we observe that \( g \in \langle h, h^p \rangle \) and hence that \( \langle h, h^p \rangle = \langle h, g \rangle \). This is a contradiction since
5.2. Coprime Actions Fixing Non-Subnormal Subgroups

\( \langle h, g \rangle^\circ \neq \langle h, g \rangle \) and \( \langle h, h^x \rangle^\circ = \langle h, h^x \rangle \) cannot hold simultaneously. Consequently \( \langle h \rangle \) is subnormal in \( \langle g, h \rangle \) and also in \( G \). \( \square \)

5.2.2 Let \( G \) be a finite non-nilpotent group and \( \alpha \) be an element of \( \text{Aut}_{\text{nsn}} G \) such that \( (|G|, |\alpha|) = 1 \). Then \( \alpha \) acts trivially on each Sylow subgroup of \( G \) not contained in \( \text{Fit} G \).

**Proof.** If \( g \) is a \( p \)-element not contained in \( \text{Fit} G \) then \( \alpha \) maps \( g \) to one of its powers, and since \( (|g|, |\alpha|) = 1 \), by Theorem 4.2.1, either \( (|g, \alpha|) = 1 \) or \( (|g, \alpha|) = |g| \). But \( g \notin \text{Fit} G \), so the latter contradicts Theorem 3.2.12 and \( \alpha \) must centralise \( g \). If \( h \) is a \( p \)-element in \( \text{Fit} G \) then \( gh \) is a \( p \)-element not contained in \( \text{Fit} G \) and so \( [gh, \alpha] = 1 \). Using the commutator identity (i) in Theorem 2.1.1, \( [gh, \alpha] = [g, \alpha]{h}[h, \alpha] \) and since \( [g, \alpha] = 1 \), it now follows that \( [h, \alpha] = 1 \). \( \square \)

5.2.3 Let \( G \) be a finite non-nilpotent group and \( \alpha \in \text{Aut}_{\text{nsn}} G \) be a non-trivial automorphism of \( G \). If \( (|G|, |\alpha|) = 1 \), then there exist subgroups \( Q \leq G \) and \( A \leq G \) of coprime order such that

(i) \( G = A \rtimes Q \) and \( [A, (\alpha)] = 1 \)

(ii) \( Q = C_Q(A) \times C_Q(\alpha) \)

(iii) \( \alpha \) is a fixed-point-free power automorphism of \( C_Q(A) \).

**Proof.** (i) Let \( \pi \) denote the set of primes dividing \( |G/\text{Fit} G| \). If \( H \) is the maximal \( \pi' \)-subgroup of \( \text{Fit} G \) then \( H \) is normal in \( G \) and \( [G : H] = |G/\text{Fit} G| \cdot |\text{Fit} G : H| \) so \( H \) has order coprime to its index in \( G \). Thus by the Schur-Zassenhaus theorem, there is a subgroup \( K \leq G \) such that \( G = K \rtimes H \) and, since \( K \) is generated by Sylow subgroups of \( G \) not properly contained in \( \text{Fit} G \), the previous lemma implies that \( \alpha \) centralises \( K^G \). If we now let

\[ Q = \langle P \mid P \in \text{Syl}_p(H) \text{ where } p \in \pi' \text{ and } [P, (\alpha)] \neq 1 \rangle \]

and

\[ R = \langle P \mid P \in \text{Syl}_p(H) \text{ where } p \in \pi' \text{ and } [P, (\alpha)] = 1 \rangle \]

it is evident that \( Q \) and \( R \) are characteristic subgroups of \( G \) and that \( H = Q \rtimes R \). Moreover, if we write \( A = K \rtimes R \) then \( G = A \rtimes Q \) and by construction \( Q \) and \( A \) are of coprime order. Since it is clear that \( \alpha \) centralises \( A^G \), this completes the proof of (i).
We now prove (iii). Since $Q$ is the direct product of its Sylow subgroups, it suffices to show that $\alpha$ is a fixed-point-free automorphism of each Sylow $p$-subgroup of $Q$.

First, using Theorem 4.1.3, we may write

$$G = A \times (C_Q(A) [Q, A])$$

and since $\alpha$ centralises $A^G$ and $[Q, A] \leq A^G$, if $\alpha$ is to act non-trivially on $G$ then $\alpha$ must also act non-trivially on $C_Q(A)$. Now suppose that $g \in C_Q(A)$ and let $a \in A \setminus \text{Fit} G$. Then $ga \not\in \text{Fit} G$ so, by Theorem 2.4.4 the subgroup $\langle ga \rangle$ is non-subnormal in $G$ and for some integer $m$,

$$(ga)^g = (ga)^m = g^m a^m$$

and

$$(ga)^g = g^\alpha a^\alpha = g^\alpha a$$

which, equating the two expressions, implies that

$$g^{-m} g^\alpha = a^{m-1}.$$ 

The choices of $g$ and $\alpha$ however were such that $g \in Q$, $a \in A$, where $Q$ and $A$ are subgroups of $G$ with trivial intersection so the above shows that $a^{m-1} \in Q \cap A = 1$ and that $g^\alpha = g^m$. Thus $\alpha$ maps each element of $C_Q(A)$ to one of its powers.

To show that the action of $\alpha$ on $C_Q(A)$ is fixed-point-free we observe that, since $Q$ is nilpotent, we may write $Q$ as a direct product of its Sylow subgroups

$$Q = P_1 \times P_2 \times \cdots \times P_n$$

and that $\alpha$ acts non-trivially on $P_i$ for $i = 1, \ldots, n$. We may also write

$$C_Q(A) = (P_1 \cap C_Q(A)) \times (P_2 \cap C_Q(A)) \times \cdots \times (P_n \cap C_Q(A))$$

where $\alpha$ acts as a non-trivial power automorphism of $P_i \cap C_Q(A)$ for $i = 1, \ldots, n$.

Therefore, by Theorem 4.2.1, each Sylow subgroup of $C_Q(A)$ is an abelian subgroup of $G$ on which $\alpha$ acts as a fixed-point-free power automorphism. From this it follows immediately that $C_Q(A)$ is an abelian subgroup of $G$ and that $\alpha$ acts on $C_Q(A)$ as a fixed-point-free power automorphism. This completes the proof of (iii).

To prove (ii) we observe first of all that $\alpha$ stabilises the chain of subgroups

$$[[Q, \langle \alpha \rangle], \langle \alpha \rangle] \leq [Q, \langle \alpha \rangle] \leq Q$$

(5.2)
so, by Theorem 4.1.2 we may write 
\[ [Q, \langle \alpha \rangle] = [[Q, \langle \alpha \rangle], \langle \alpha \rangle]. \]
Furthermore, if \( g \in Q \), then for all \( a \in A \)
\[ a^{[g, \alpha]} = (g^{-1})^\alpha (gag^{-1})^\alpha = (g^{-1})^\alpha (gag^{-1})^\alpha g^\alpha = (g^{-1}gag^{-1}g)^\alpha = a^\alpha = a \]
and hence \( A \) centralises \([Q, \langle \alpha \rangle]\). Now using (5.2) we obtain
\[ [Q, \langle \alpha \rangle] = [[Q, \langle \alpha \rangle], \langle \alpha \rangle] \leq [C_Q(A), \langle \alpha \rangle] \leq [Q, \langle \alpha \rangle] \]
and it follows that
\[ [C_Q(A), \langle \alpha \rangle] = [Q, \langle \alpha \rangle]. \] (5.3)
Since we have already shown that the action of \( \alpha \) on \( C_Q(A) \) is fixed-point-free, Theorem 4.1.3 implies that \( C_Q(A) = [C_Q(A), \langle \alpha \rangle] \). It now follows from (5.3) that
\[ C_Q(A) = [Q, \langle \alpha \rangle]. \]

We now use a similar argument to show that \( C_Q(\alpha) = [Q, A] \). Since \([A^G, \langle \alpha \rangle]\) = 1 and \([C_Q(A), \langle \alpha \rangle] \leq [A^G, \langle \alpha \rangle]\) it is clear that \([Q, A] \leq C_Q(\alpha)\). Moreover, \( A \) stabilises the chain of subgroups
\[ [[Q, A], A] \leq [Q, A] \leq Q \]
and so using Theorem 4.1.2 once more, \([Q, A] = [[Q, A], A]\). The fact that \( C_Q(\alpha) = [Q, A]\) now implies that
\[ [Q, A] = [[Q, A], A] = [C_Q(\alpha), A] \leq [Q, A] \]
and in particular that \([C_Q(\alpha), A] = [Q, A]\). The action of \( \alpha \) on \( C_Q(A) \) is fixed point free so \( C_Q(A) \cap C_Q(\alpha) = 1 \) and hence if we view \( A \) as an operator group for \( C_Q(\alpha) \) and use Theorem 4.1.2, we may write
\[ C_Q(\alpha) = [C_Q(\alpha), A]. \] (5.4)
We have already shown that \([C_Q(\alpha), A] = [Q, A]\) and so from (5.4) it is evident that \( C_Q(\alpha) = [Q, A]\). Moreover, by (5.1),
\[ Q = C_Q(A)C_Q(\alpha) \]
and since \( C_Q(A) \) and \( C_Q(\alpha) \) are normal subgroups of \( Q \) with trivial intersection, it follows that
\[ Q = C_Q(A) \times C_Q(\alpha). \]
In particular, we observe that \( C_Q(A) \) is central in \( G \). This concludes the proof of (ii). \( \square \)
If $G$ is a finite group which satisfies the hypotheses of Theorem 5.2.3 we may write $G = A \times Q$ where $Q = C_Q(A) \times C_Q(\alpha)$. Moreover, when $C_Q(\alpha)$ is trivial, $Q = C_Q(A)$ and $G$ decomposes as $G = A \times C_Q(A)$ into direct factors of coprime order. Since $\alpha$ acts as a non-trivial power automorphism on one of these direct factors and centralises the other, when $C_Q(\alpha)$ is trivial $\alpha$ acts as a power automorphism of $G$.

Conversely, we show that if $C_Q(\alpha)$ is non-trivial then it is possible to exhibit a subnormal subgroup $H \leq G$ such that $H^\alpha \neq H$. First, for each prime $p$ dividing $|C_Q(\alpha)|$ the Sylow $p$-subgroup $P$ of $Q$ may be written as

$$P = (C_Q(A) \cap P) \times (C_Q(\alpha) \cap P)$$

and since $Q$ was defined as the direct product of those Sylow subgroups not centralised by $\alpha$, it follows that every prime dividing $|C_Q(\alpha)|$ also divides $|C_Q(A)|$. We may therefore choose non-trivial elements $g \in C_Q(A)$ and $h \in C_Q(\alpha)$ of order $p$, where $p$ is any prime dividing $|C_Q(\alpha)|$. Our claim is that $H = \langle gh \rangle$ is a subgroup not fixed by $\alpha$. To see this, observe that $\alpha$ acts as a fixed-point-free automorphism of $C_Q(A)$ and that for some integer $m$, where $m \neq 1 \mod p$, we have $(gh)^\alpha = g^m h$. If $g^m h = g^k h^k$ for some integer $k$, then $m \equiv k \mod p$ and $k \equiv 1 \mod p$. This is a contradiction, so $(gh)^\alpha \notin \langle gh \rangle$ which proves the claim.

It now follows that under the hypotheses of Theorem 5.2.3, $\alpha \in \text{Aut}_{sub} G$ is a power automorphism of $G$ precisely when $C_Q(\alpha)$ is trivial. Moreover, since we are interested in determining the structure of a group $G$ such that $\text{Paut} G \neq \text{Aut}_{sub} G$, we will restrict our attention to the case where $C_Q(\alpha)$ is non-trivial. For the remainder of this chapter our objective is to show how the action of $A$ on $C_Q(\alpha)$ influences the structure of $A$.

5.2.4 Let $G$ be a finite group, assume the hypotheses and the notation of Theorem 5.2.3 and let $A_0 = A/(A \cap \text{Fit} G)$. If $\alpha$ is not a power automorphism of $G$ then:

(i) The kernel of the action of $A$ on $C_Q(\alpha)$ is $A \cap \text{Fit} G$.

(ii) The action of $A_0$ on $C_Q(\alpha)$ is fixed-point-free.

Proof. Since $Q$ and $A$ have coprime order, and $Q \leq \text{Fit} G$, it follows from (iii) of Theorem 2.4.3 that $A \cap \text{Fit} G$ centralises $Q$. Conversely, let $p$ be a prime dividing $|C_Q(\alpha)|$ and suppose that $g$ is a non-trivial $p$-element of $C_Q(\alpha)$. As is shown in our earlier remarks, we may choose a non-trivial $p$-element $h \in C_Q(A)$, and for each such choice, $\alpha$ does not
5.3. The Soluble Case

fix the subgroup \((gh)\). By Theorem 5.2.1, every \(p'\)-element of \(N_G((gh))\) is subnormal in \(G\) and lies in \(\text{Fit}\ G\). In particular, if \(a \in A\) and \([a, g] = 1\) then \(a\) normalises \((gh)\) and must lie in \(A \cap \text{Fit}\ G\). It follows that each element of \(A \setminus \text{Fit}\ G\) acts fixed-point-freely on \(C_Q(\alpha)\).

We may now make the following observation which is a generalisation of Theorem 5.1.3.

**5.2.5** Let \(G\) be a finite group, assume the hypotheses and the notation of Theorem 5.2.3 and let \(A_0 = A / (A \cap \text{Fit}\ G)\). If \(\alpha\) is not a power automorphism of \(G\) then the group

\[G_0 = A_0 \rtimes C_Q(\alpha)\]

is a Frobenius group. The Frobenius kernel of \(G_0\) is \(C_Q(\alpha)\) and the Frobenius complement of \(G_0\) is \(A_0\). For every odd prime \(p\), the Sylow \(p\)-subgroups of \(A_0\) are cyclic. Each abelian subgroup of \(A_0\) is cyclic.

**Proof.** Since the action of \(A_0\) on \(C_Q(\alpha)\) is fixed-point-free, by Theorem 2.2.2, \(G_0\) is a Frobenius group with Frobenius complement \(A_0\) and Frobenius kernel \(C_Q(\alpha)\). The remainder of the statement now follows from Theorem 2.2.3 and Theorem 2.2.4.

Some further results of Zassenhaus on the soluble and insoluble finite Frobenius complements and a result of Suzuki characterising the insoluble finite groups in which each abelian subgroup is cyclic may now be used to give more information about the structure of \(A / (A \cap \text{Fit}\ G)\).

## 5.3 The Soluble Case

A finite group in which every Sylow subgroup is cyclic is referred to as a \(Z\)-group and the following result of Hölder, Burnside and Zassenhaus gives the structure of these groups.

**5.3.1** If \(G\) is a finite group all of whose Sylow subgroups are cyclic, then \(G\) has the presentation

\[G = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle\]

where \(m\) is odd, \(r^n \equiv 1 \mod m\), and \((m, n(r - 1)) = 1\). Conversely, in a group with such a presentation all Sylow subgroups are cyclic.
Theorem 5.3.1 will be found in Section 10.1 of [12]. The next result and the proof we give are adapted from the more general theorem of Zassenhaus on soluble Frobenius complements given in Section 18 of [10].

5.3.2 Let $G$ be a finite group. Assuming the hypotheses and the notation of Theorem 5.2.3, let $A_0 = A/(A \cap \text{Fit } G)$. If $\alpha$ is not a power automorphism of $G$ and if the Sylow subgroups of $A_0$ are cyclic then $A_0$ (and hence $G$) is soluble and

$$A_0 = \langle a, b \mid a^m = b^n = 1, a^b = a^r \rangle.$$

Here $m$ is odd, $(m, n(r-1)) = 1$ and, if $n'$ denotes the product of the distinct prime factors of $n$, $r^{n/n'} \equiv 1 \mod m$.

**Proof.** Since the Sylow subgroups of $A_0$ are cyclic, $A_0$ is a $Z$-group and so generated by elements $a$ and $b$ such that $a^m = b^n = 1$, and $a^b = a^r$ where $m$ is odd, $r^n \equiv 1 \mod m$, and $(m, n(r-1)) = 1$. If $p$ and $q$ are primes dividing $m$ and $n$ respectively, then the subgroup $\langle a^{m/p}, b^{n/q} \rangle$ has order $pq$ and is cyclic by Theorem 2.2.3. This implies that $b^{n/q}$ centralises the elements of $\langle a \rangle$ of prime order and so, by Theorem 2.5.2, $b^{n/q}$ centralises $\langle a \rangle$. Since this holds for every prime dividing $n$, it follows that $b^{n/n'}$ centralises $\langle a \rangle$, and that $r^{n/n'} \equiv 1 \mod m$. To complete the proof of the theorem we observe that $A$ is an extension of $(A \cap \text{Fit } G)$ by $A_0$ and that $G$ is an extension of $A$ by $Q \leq \text{Fit } G$ so $G$ must be soluble.

It is worth noting that the parametrisation of $A_0$ in terms of the integers $m$, $n$ and $r$ given in Theorem 5.3.2 need not be unique and that a degeneracy where $A_0$ is cyclic may arise; Example 3.3.2 is an instance of the latter.

Zassenhaus’ theorem on the structure of soluble Frobenius complements states that every soluble Frobenius complement $G$ has a normal subgroup $N$ which is a $Z$-group such that $G/N$ is isomorphic to a subgroup of the symmetric group of degree 4 (see Section 18 of [10]). Whilst Zassenhaus’ result does give enough information to complete the classification of the finite soluble Frobenius complements, this exercise would take us outside the scope of this thesis and we do not attempt it here.

5.4 The Insoluble Case

Assuming once more the hypotheses and the notation of Theorem 5.2.3, in this section we give a complete statement of the structure of $A/(A \cap \text{Fit } G)$ when $G$ is insoluble.
5.4. The Insoluble Case

Since each group in which every Sylow subgroup is cyclic is necessarily soluble, if $G$ is a finite insoluble Frobenius complement then $G$ has a non-trivial Sylow 2 subgroup which is a generalised quaternion group. The following result of Zassenhaus (see Section 18 of [10]) gives a partial account of the structure of such groups.

5.4.1 Let $G$ be an insoluble Frobenius complement. Then $G$ has a normal subgroup $N$ of index 1 or 2 such that $N \cong SL(2,5) \times M$ with $M$ a Z-group of order coprime to 2, 3 and 5.

It may be observed that since a subgroup group of a finite Frobenius complement is also a Frobenius complement, the Z-group $M$ in this result will have a presentation of the form given in Theorem 5.3.2. To complete this picture, we need a result of M. Suzuki (see Theorem E of [14]) giving a precise statement of the structure of the insoluble groups in which each abelian subgroup is cyclic. In the notation of [14] we let $e$ be a generator of the multiplicative group of the field of $p$ elements and define an automorphism $\theta$ of the special linear group $SL(2, p)$ by its action on a pair of generators of $SL(2, p)$ as follows:

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & e^{-1} \\ -e & 0 \end{pmatrix}.
\]

We may now quote Theorem E of [14].

5.4.2 Every abelian subgroup of an insoluble group $G$ of finite order is cyclic if and only if either

(i) $G$ is a direct product of a Z-group $M$ and a group isomorphic with the special linear group $SL(2, p)$ for some $p > 3$ and having order relatively prime to $M$ or,

(ii) $G$ is generated by $L \cong SL(2, p)$, a Z-group

\[ M = \langle a, b | a^m = b^n = 1, a^b = a^r \rangle \]

and an element $d$ of order 4 such that for all $h \in L$

\[ d^2 \in \zeta L, \quad h^d = h^\theta, \quad a^d = a^{-1} \]

\[ [b, d] = 1, \quad [a, h] = 1, \quad [b, h] = 1. \]

Here $m, n$ and $r$ are integers such that $(m, n) = 1$, $(mn, p(p^2 - 1)) = 1$ and $r^n \equiv 1 \mod m$. 
Chapter 5. Restrictions on the Group Structure

If $G$ is insoluble, the next result yields the structure $G$ up to a normal nilpotent Hall subgroup.

5.4.3 Let $G$ be an insoluble finite group and, assuming the hypotheses and the notation of Theorem 5.2.3, let $A_0 = A/(A \cap \text{Fit } G)$. If $\alpha$ is not a power automorphism of $G$ then either

(i) $A_0$ is a direct product of a group isomorphic to the special linear group $SL(2,5)$ and a $Z$-group

$$M_1 = \langle a, b \mid a^n = b^r = 1, a^b = a^r \rangle$$

of order coprime to 2, 3 and 5 such that $(m, n(r-1)) = 1$, and if $n'$ denotes the product of the distinct prime factors of $n$, $r^{n/n'} \equiv 1 \pmod{m}$
or,

(ii) $A_0$ is generated by $L \cong SL(2,5)$, a $Z$-group

$$M_2 = \langle a, b \mid a^n = b^r = 1, a^b = a^r \rangle$$

of order coprime to 2, 3 and 5, and an element $d$ of order 4 such that for all $h \in L$

$$d^2 \in \zeta L, \quad h^d = h^r, \quad a^d = a^{-1}$$

$$[b, d] = 1, \quad [a, b] = 1, \quad [b, h] = 1.$$ 

Here $m$, $n$ and $r$ are integers such that $(m, n(r-1)) = 1$, and if $n'$ denotes the product of the distinct prime factors of $n$, $r^{n/n'} \equiv 1 \pmod{m}$.

Proof. By Theorem 5.4.1, $A_0$ has a normal subgroup $N$ of index 1 or 2 such that $N = SL(2,5) \times M$ where $M$ is a $Z$-group of odd order. If $N$ has index 1 in $A_0$ then it is apparent that (i) above holds and we are done.

Supposing therefore that $N$ has index 2 in $A_0$, we see that $N$ must be the smallest normal subgroup of $A_0$ with the property that its factor group is a 2-group: $N$ is therefore characteristic in $A_0$. Moreover, we have that $M = O_2(A_0)$ and, since $SL(2,5)$ is perfect and $M^{(3)}$ is trivial, we see that $A_0^{(3)} \cong SL(2,5)$. Thus we conclude that

$$N = A_0^{(3)} \times O_2(A_0).$$ (5.5)
5.5 Concluding Remarks

But since \( A_0 \) is a finite Frobenius complement the abelian subgroups of \( A_0 \) must be cyclic and we may use (ii) in Theorem 5.4.2 to observe that \( A_0 \) is generated by an element \( d \) of order 4, together with a normal subgroup \( N_p \) such that

\[
N_p \cong SL(2, p) \times M_p
\]

where \( p \) is prime, \( M_p \) is a \( Z \)-group satisfying the conditions of (ii) in Theorem 5.4.2 and \( d \) acts on \( N_p \) according to (ii) in Theorem 5.4.2. The relations given in (ii) of Theorem 5.4.2 imply that \( [A_0, d] \leq N_p \) so we may now observe that \( O_2(A_0) \cong M_p \) and that \( A_0^{(3)} \cong SL(2, p) \): in particular, it follows from (5.5) that

\[
SL(2, p) = SL(2, 5) \quad \text{and} \quad M_p = M.
\]

With these restrictions, our result now follows directly from Theorem 5.4.2.

It should be noted that the caveat about possible non-uniqueness or degeneracy in the parametrisation given in Theorem 5.3.2 also applies to Theorem 5.4.3. For example, since \( SL(2, 5) \) is itself an insoluble Frobenius complement (this is shown in Section 18 of [10]), in Theorem 5.4.3 one should allow for the case where \( A/(A \cap \text{Fit } G) \) is perfect, as well as for degeneracy of the type exhibited in Example 3.3.2.

5.5 Concluding Remarks

Using the basic properties of \( \lambda(G) \) established in Chapter 3, we have been able to obtain some generalisations to \( \text{Aut}_{\text{nns}} G \) of known results about \( \text{Paut } G \) and \( \text{Aut}_{\text{nns}} G \). The results from Chapter 3 of this thesis give moderately detailed information about \( \text{Aut}_{\text{nns}} G \) for finite groups, information which leads us to anticipate the kinds of groups for which \( \text{Aut}_{\text{nns}} G \) will be non-trivial and interesting. Indeed we are able to give two such examples in Section 3.3.

The main outstanding question from Chapter 3 is to determine whether there is finite non-nilpotent groups \( G \) such that \( \kappa(G) < \lambda(G) \).

The results of Chapter 5 give a detailed account of the structure of the finite groups which have non-trivial coprime operators fixing every non-subnormal subgroup. In particular, if \( G \) is a finite group and \( G \) has a non-trivial coprime operator \( \alpha \) fixing every non-subnormal subgroup then Section 5.2 shows that up to a normal nilpotent Hall subgroup, \( G \) is the
direct product of an abelian group with a Frobenius group, where each prime $p$ dividing the order of the Frobenius kernel also divides the order of the abelian direct factor. We also showed that $\alpha$ centralises the Frobenius group and that $\alpha$ acts as a fixed-point-free power automorphism of the abelian direct factor.

The results of the final chapter of this thesis leave a number of interesting questions outstanding. The first and probably the easier of these is to determine the conditions under which a converse to the main results of Section 5.2 holds. That is, given a finite group which can be written a direct product

$$G = A \times Q$$

where $Q$ is abelian, $A = H \ltimes N$ is a Frobenius group with Frobenius complement $H$ and Frobenius kernel $N$ and where each prime dividing the order of $N$ also divides the order of $Q$, it would be useful to determine whether every automorphism of $G$ which centralises $A$ and acts as a fixed-point-free power automorphism of $Q$ fixes every non-subnormal subgroup in $G$.

The second question left outstanding by the work in Chapter 5 is to determine how much can be said about the structure of a finite group $G$ which has an automorphism which is not a coprime operator, and which fixes every non-subnormal subgroup in $G$. At first sight this question does seem rather more difficult than the problem solved in Sections 5.2 to 5.4 of this thesis.

These outstanding problems do leave scope for additional work in this area.
Bibliography


