## LIST OF CORRECTIONS

- On Page 115, Hansen Method. Add the following at the end of this paragraph.

Another outstanding question is how the non-standard identification problem on page 59 (Lemma 4.2) can be solved when the high signal to noise ratio assumption is violated.

- In Lemma 4.3, Equation (4.24), when $r_2 = 0$ this leads to the identification of $RU_1$, and when $r_1 = 0$ this leads to the identification of $RV_1$ as opposed to R. As such $U_1$ need not be invertible and V1^-1 may not be stable. Thus a unique determination of R cannot be directly concluded. However by choosing the input signals $r_1 = Nr\alpha$ and $r_2 = Dr\alpha$ the noise free part of $\beta$ is simply $R\alpha$.

- Pg 13, Definition of "Bounded Operator and Gain". Perhaps change the last sentence to:

"A is also a Bounded Input, Bounded Output (BIBO) Operator."

- Change the Title of Chapter 4 to:

  Hansen Method using Right Coprime Factorisations

- Change the Title of Chapter 5 to:

  Hansen Method using Left Coprime Factorisations.

- Insert at the end of Section 2.1 Notation:

$S$ : The set of stable proper transfer functions.

- Remark 4.4

An extension using a nonlinear stabilising controller, similar to that in Chapter 5, is nontrivial. This is due to the difficult issues that are associated with the appearance of the left coprime factors (ie. in (4.5) and (4.6)).

# Identification of Nonlinear Systems

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B.E. Hons. (Systems Engineering) , B. Sc (ANU)

May 1998

*A thesis submitted for the degree of Doctor of Philosophy of*
*The Australian National University*

*Department of Systems Engineering*

*Research School of Information Sciences and Engineering*

*The Australian National University*

# Declaration

These doctoral studies were conducted with Professor Brian D.O. Anderson as supervisor, and Dr Franky De Bruyne and Professor John B. Moore as advisors.

The contents of this thesis are the results of original research, and have not been submitted for a higher degree at any other university or institution.

Natasha Linard

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May 8, 1998
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# Acknowledgements

*" Curiously enough, the only thing that went through the mind of the bowl of petunias as it fell was Oh No, not again." Douglas Adams.*

I would like to thank Brian for being a fantastic supervisor. I've had a challenging, interesting, amusing time here. I think I've learnt more than what is contained in this thesis, and I've almost got those principle and subordinate clauses sorted out (well - maybe not quite :*). Thankyou for everything. *ta'mey Dun, bommey Dun.*

I would like to thank Franky for all his help particularly in all the arduous proof reading he has done. Thankyou, thankyou, thankyou. *qa' wIje'meH maSuv.*

I would like to thank Catrina Waterson for being so supportive when I first arrived and making my time here more enjoyable. *rut yIHmey ghom Hoch.*

I'd like to thank students past and present for their enlivening conversation.
*bomDI' 'IwwIj qaqaw.*

I'd like to thank John for encouraging me to do a PhD.
*rura' pente'Daq church ngevlaH ghaH.*

I'd like to thank Marita and Heather for arranging
heaps of things for me. *pe'vIl mu'qaDmey tIbach.*

Finally, I'd like to thank everyone else in the Department and CRASys.
*pIj monchugh vay' yIvoqQo'.*

# Abstract

This thesis treats the identification of a noise contaminated nonlinear plant operating in closed-loop with a possibly nonlinear stabilising controller. It examines a number of methods of converting the closed-loop identification problem into one of open-loop identification. As two of the identification methods involve the notion of "coprimeness", a discussion of the coprimeness of nonlinear operators is included.

There are two equivalent approaches for defining the coprimeness of nonlinear operators. These are known as the Bezout identity approach and the Set Theoretic approach. This thesis examines the relationship between the Bezout and the Set Theoretic definitions of coprimeness for nonlinear operators. It is shown that left coprimeness in the Set Theoretic sense implies left coprimeness in the Bezout sense. Further, the relationship between different factorisations of a given nonlinear operator is examined. For example, using either definition of coprimeness, two left coprime factorisations of a given nonlinear operator are related by a unit operator.

The first conversion method is referred to as the *Hansen* scheme. Previous work has shown that the set of nonlinear plants stabilised by a known linear controller which also stabilises a linear nominal model of the plant can be parametrised by a stable nonlinear operator known as the Youla-Kucera parameter. This work has been extended by allowing the nonlinearity to also enter through the nominal plant model and the controller. It turns out that identification of the unknown plant is equivalent to identifying the Youla-Kucera parameter associated with the plant. The advantage in using this method is that identification of this parameter is a nonstandard *open-loop* identification problem, which in a low noise situation can be formulated as a standard

open-loop problem. This identification method has been examined using both left and right coprime factorisations of the nominal plant and controller. Note that while the controller is also allowed to be nonlinear for the left coprime case, it remains linear for the right coprime case.

The second conversion method is known as the *Two-Step* method. In the first step the closed-loop operator from the reference input to the plant input is identified through a high order nonlinear model. The estimate of the closed-loop operator is used in the second step to simulate a noise free plant input signal to allow an open-loop like identification of the plant. It is assumed that the plant is open-loop stable, and is operating in an internally stable closed-loop system.

The third method is known as the *Coprime Factor* method; it identifies a pair of right factors of the nonlinear plant through an open-loop-like identification of the filtered sensitivity and complementary sensitivity functions. By introducing auxiliary signals a right *coprime* factorisation of the nonlinear plant can be found.

The fourth identification method is known as *closed-loop identification with a tailor-made parametrisation.* Gradient expressions for a closed-loop parametric identification scheme are presented. The method is based on the minimisation of a standard identification criterion and a parametrisation that is tailored to the closed-loop configuration. It is shown that for both linear and nonlinear plants and controllers, the gradient signals can be computed exactly.

For the first three methods it is assumed that the measurement noise enters the system under a high Signal-to-Noise (SNR) assumption. The fourth method requires a high SNR for consistency.

# Preface

The work in Chapters 2, 3, 4, 5 and 6 was undertaken with Professor Brian Anderson and Dr Franky De Bruyne. The work in Chapter 7 was conducted with Dr Franky De Bruyne, Professor Brian Anderson and Professor Michel Gevers.

## Journal Publications

*Identification of a nonlinear plant under nonlinear feedback using left coprime fractional based representations.*
N. Linard, B.D.O. Anderson and F. De Bruyne.
*Submitted to Automatica*, 1997.

*Coprimeness properties of nonlinear fractional system realizations.*
N. Linard, B.D.O. Anderson and F. De Bruyne.
*Accepted by System and Control Letters*, 1997.

*Iterative controller optimization for nonlinear systems.*
F. De Bruyne, B.D.O. Anderson, M. Gevers and N. Linard.
*Submitted to IEEE Transactions on Automatic Control*, 1998.

*Gradient expressions for a closed loop identification scheme with a tailor-made parametrization.*
F. De Bruyne, B.D.O. Anderson, M. Gevers and N. Linard.
*Submitted to Automatica*, 1998.

## Conference Publications

*Identification of nonlinear plants under linear control using Youla-Kucera parametrizations.*

N. Linard and B.D.O. Anderson.

*Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, pp. 1094-1099, 1996.

*Left coprime fractional based representations of nonlinear systems with a nonlinear nominal plant.*

N. Linard and B.D.O. Anderson.

*European Control Conference, CDRom Proceedings*, Paper 208, FR-A-F-5, 1997.

*Coprimeness properties of nonlinear fractional system realizations.*

N. Linard, B.D.O. Anderson and F. De Bruyne.

*Proceedings of the 2nd Asian Control Conference*, Vol. II, pp. 131-134, 1997.

*A two step method for the closed loop identification of nonlinear systems.*

N. Linard, B.D.O. Anderson and F. De Bruyne.

*Proceedings of Control '97*, Sydney, Australia, pp. 381-385.

*Closed loop identification of nonlinear systems.*

N. Linard, B.D.O. Anderson and F. De Bruyne.

*Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, USA, pp. 2998-3003, 1997.

*Iterative controller optimization for nonlinear systems.*

F. De Bruyne, B.D.O. Anderson, M. Gevers and N. Linard.

*Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, USA, pp 3749-3754, 1997.

*On closed loop identification with a tailor-made parametrization.*

F. De Bruyne, B.D.O. Anderson, M. Gevers and N. Linard.

*Accepted by the American Control Conference*, Philadelphia, USA, 1998.

*The Hansen scheme revisited.*

F. De Bruyne, B.D.O. Anderson and N. Linard.

*Submitted to the 37th Conference on Decision and Control*, Tampa, USA, 1998.

# Contents

# List of Figures

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# Chapter 1

# Introduction

This first chapter provides an introductory discussion of the ideas investigated within this thesis. It outlines the motivation for investigating these ideas, examines their history and describes the progress made in this thesis.

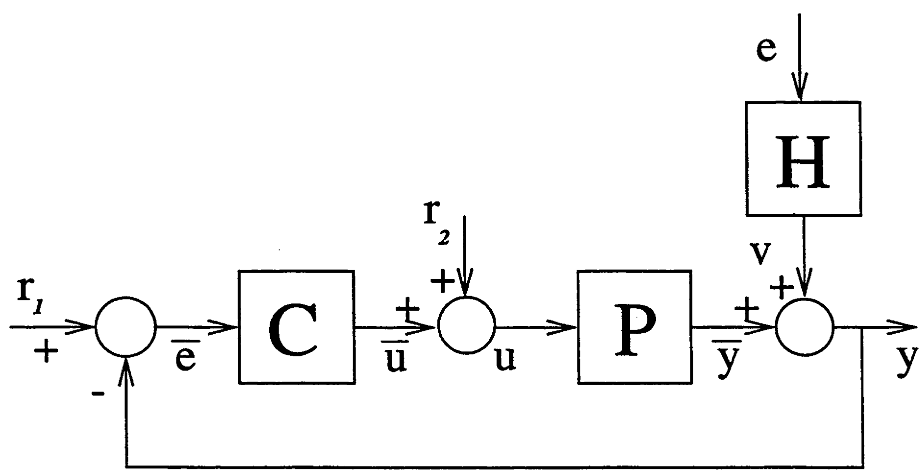



*Figure 1.1: Closed-loop identification setup*

## 1.1 Problem Statement

Consider the setting shown in Figure 1.1 where

- $P$: is a plant to be identified,
- $C$: is a stabilising controller,

- $H$ is a linear stable output measurement noise generating system, driven in turn by the zero mean, white, stationary noise process $e$
- $u$: is the control signal,
- $y$: is the output signal,
- $v$: is the process disturbance signal consisting of filtered white noise, and
- $r_1$, $r_2$: are external reference or setpoint signals that are uncorrelated with the noise.

This setting represents a closed-loop identification problem. For notational convenience, we have dropped the time index of the signals throughout this thesis.

This thesis examines different ways of converting the closed-loop identification problem described above into one of open-loop identification. The identification methods for linear systems have been widely investigated, and this thesis focuses on their extension to a nonlinear setting.

## 1.2 Motivation

Many industrial processes exist that require identification to be performed in the closed-loop setting of Figure 1.1. Often, open-loop identification is easier to perform than closed-loop identification. This section discusses the need to identify in closed-loop as well as the difficulties associated with it. It contrasts the advantages of closed-loop identification and open-loop identification. Various methods exist which allow closed-loop identification problems to be converted into open-loop identification problems. Existence of these methods provide us with the benefits of closed-loop identification with the ease of analysis allowed by open-loop identification.

### 1.2.1 Benefits of open-loop identification

The main attraction of open-loop identification is in the simplicity of the calculations involved. Recall the system arrangement of Figure 1.1. The measurements of $r_1$, $r_2$,

$u$ and $y$ are available; $v$ is a noise process that is independent of $r_1$ and $r_2$, and the closed-loop is stable. Consider the linear system

$$y = Pu + v. \tag{1.1}$$

If the controller $C = 0$ (so that the plant was operating in open-loop) one could cross correlate $y$ with $u$ and then solve for $P$; i.e.

$$\Phi_{yu}(s) \quad = \quad P(s)\Phi_{uu}(s).$$

Most open-loop identification schemes operate somewhat like this (possibly in the time-domain, recursively, and with sample averages).

### 1.2.2 Difficulties with closed-loop identification

In the closed-loop case, the plant input $u$ and the measurement noise $v$ are correlated and what is more this correlation, being dependent on the unknown plant $P$, cannot be determined a priori.

- For linear systems

$$\Phi_{yu}(s) = P(s)\Phi_{uu}(s) - (1 + C^*(s)P^*(s))^{-1}C^*(s)\Phi_{vu}(s). \tag{1.2}$$

  So blind use of an open-loop identification method which implicitly evaluates $\Phi_{yu}(s)\Phi_{uu}^{-1}(s)$ is bound to give bias errors. This is the fundamental reason which makes closed-loop identification difficult.

- Closed-loop identification is hampered by the need to unravel the closed-loop operator to obtain $P$. Even when $P$ and $C$ are linear, $P$ appears in a nonlinear fashion in the closed-loop quantities.

- It can also be the case for linear systems that if an estimate $\hat{P}$ of $P$ is obtained by unraveling an estimate of the closed-loop operator, then $\hat{P}$ and $C$ can have an unstable pole-zero cancellation; see [33] for further details.

- When analysing a closed-loop scenario, software may presuppose the plant $P$ is stable. This is normally a reasonable assumption in the open-loop case but an unwarranted assumption in the closed-loop case as the plant may be coupled with a stabilising controller.

### 1.2.3 Benefits of closed-loop identification

Despite the difficulties involved in closed-loop identification it still has a number of advantages over open-loop identification.

- Often data is collected from a process that is operating in closed-loop. In an industrial situation operating constraints may make it impossible to disconnect these processes to take measurements . For example, if the plant is unstable it is usually necessary to stabilise it first (with a controller creating a closed-loop) as it may be difficult to gather data otherwise.

- Further, there may be situations where it is wiser to identify the plant in closed-loop so that the identified model will capture the dynamical characteristics that are important for control design. We refer the reader to [10, 32] for a discussion of this problem in the linear case.

### 1.2.4 Summary of identification

Ideally we would like to gain the advantages of closed-loop identification with the simplicity of the open-loop calculations. Thus in this thesis we examine a number of methods that convert the closed-loop identification problem into one of open-loop identification. This work has been done in the linear case. The abundance of nonlinearities in everyday identification problems motivates the need to extend the linear theory to deal with nonlinear issues.

### 1.2.5 Motivation for related topics

While the identification problems have been solved when both the plant and controller are linear, difficulties arise when we move to using nonlinear operators. The main problem is that, although the property that $(B+C)A = BA+CA$ holds, the distributivity property $A(B+C) = AB + AC$ is no longer valid. A number of concepts have been adapted from linear theory to mitigate this effect in the nonlinear case.

Firstly, a definition of *differential coprimeness* has been formulated. Secondly, the relationship between the Bezout and Set Theoretic definitions of coprimeness for nonlinear operators has been investigated. Finally, a number of stability results have been found culminating in a nonlinear form of the double Bezout identity. These definitions and stability results are stated and investigated fully in Chapters 2 and 3.

The rest of this chapter is devoted to a discussion of how the research contained in this thesis complements previous work conducted in this area.

While primarily concerned with methods that convert the closed-loop identification problem into an open-loop problem, some related topics are also discussed.

## 1.3 Coprimeness Definitions

Before examining the question of identification we first consider the issue of coprimeness properties of nonlinear operators. The motivation for investigating coprimeness properties is that these properties are a tool for formulating the descriptions of plants and controllers. This type of description is used in some of the identification schemes examined in the later part of this thesis.

### 1.3.1 Definitions of Coprimeness

There are two definitions of coprimeness common in the literature, the *Bezout* definition and the *Set Theoretic* definitions of coprimeness. These definitions are presented in Chapter 2. In the linear case the definitions are equivalent; in the nonlinear case their relationship is less clear.

Identification problems that utilise coprimeness in both the linear and nonlinear cases have been approached predominantly using the Bezout definition of coprimeness. However in [3, 39], the "Set Theoretic" definition of right coprimeness for nonlinear systems has been used. In the linear case it is probably the easier tool to use in testing for coprimeness. The Set Theoretic approach has the advantage that it is not necessary to introduce the additional transfer functions or operators as is required in the Bezout identity approach. Now, as the application of the ideas that utilise coprime factorisations has progressed to nonlinear systems it is of interest to see how the two definitions of coprimeness compare for nonlinear operators. Baños has examined the relationship between the coprimeness definitions for right coprime factors in [2]. In [2], Baños takes the first steps towards a nonlinear version of the double Bezout identity.

This thesis examines the relationship between the "Set Theoretic" and the "Bezout identity" definitions of *left* coprimeness for nonlinear operators.

### 1.3.2 Differential Coprimeness

A notion of differential coprimeness for nonlinear operators is introduced in Chapter 2. For linear operators this concept simply reduces to the Bezout definition of coprimeness.

## 1.4 Stability and Operator Existence

For linear coprime operators we can construct what is known as a double Bezout identity. This relationship is exploited when using the Bezout approach to characterise the set of all plants stabilised by a given controller. It also provides welcome simplifications when using the Hansen approach to convert the closed-loop identification problem to one of open-loop identification; see e.g. [15]. In [2], Baños introduces a generalised "single" nonlinear Bezout identify and discusses how it relates to right coprimeness.

In this thesis we describe a nonlinear form of the double Bezout identity. The nonlinear form is utilised later in this thesis to convert a closed-loop identification problem to one of open-loop identification. The nonlinear form of the double Bezout identity relies on the concept of differential coprimeness.

## 1.5 Closed-loop identification methods

In the "identification for control literature", the problem of identification of a linear system on the basis of data obtained from closed-loop experiments has received considerable attention, see e.g. the survey papers [10, 32] with the many references therein.

One can distinguish three main closed-loop identification procedures in the "linear" literature; see [15, 34, 36] for more details. These techniques have in common the ability to identify approximate models of the open-loop plant on the basis of closed-loop data, while the asymptotic bias distribution of the estimated plant transfer function at each frequency remains independent of the noise and is thus explicitly tunable by the user.

Each method demonstrates a different technique for turning the closed-loop identification problem into an open-loop-like identification problem.

The closed-loop identification of a linear system subjected to a linear controller by minimisation of a closed-loop criterion using a tailor-made parametrisation of the plant is considered in [38]. The method uses knowledge of the controller; it minimises an error between the closed loop transfer functions of the true closed-loop and the model closed-loop. Such an approach had already been mentioned as an exercise in [23]. Further references include [9, 19, 24].

### 1.5.1 The Hansen Method

One use of linear coprime fractional representations has been in the characterisation of the set of all plants that can be stabilised by a given controller. This set of plants is parametrised by the Youla-Kucera parameter. Extensive work has been done in characterising the set of linear plants stabilised by a linear controller; see e.g. [18, 31, 41, 42]. The extension to nonlinear systems has been covered by [12, 26, 39] and a discussion of characterisation in the presence of noise can be found in [5, 14].

The Hansen scheme [15] relies on the ability to parametrise the unknown linear plant in terms of the coprime factors of the linear nominal plant and of the linear controller along with a Youla-Kucera parameter associated with the plant. The controller is assumed to stabilise both the true plant and the nominal plant model. The advantage of this method is that rather than identifying the plant we identify the Youla-Kucera parameter which involves an *open-loop* identification problem.

A number of iterative identification and control procedures based on the Hansen approach can be found in [20, 29]. In [5], the true plant was allowed to be nonlinear, however the nonlinearity entered only via the Youla-Kucera parameter, i.e. the controller and the nominal plant model were still linear. This thesis extends this method to work with a nonlinear model of the plant and a nonlinear controller.

These identification methods have been treated using both left and right coprime factorisations. However, there is much less in the literature about *left* fractional representations of nonlinear systems, $P = D_l^{-1}N_l$, than about *right* fractional representations, $P = N_rD_r^{-1}$. For some further discussion, see [31]. References [26, 27, 28] show that for nonlinear systems a more useful and perhaps fundamental concept than a left realisation is a stable kernel representation. In a sense, left factorisations are a special case of kernel representations.

**The Hansen method using *right* coprime factorisations** As in [5, 15] we have a known linear controller that stabilises an unknown nonlinear plant and a known nominal model of this plant. The method relies on representing the unknown plant in terms of the *right* coprime factors of the plant model and the controller and with the associated Youla-Kucera parameter. This thesis extends [5] by allowing the nonlinearity to enter the plant both via the Youla-Kucera parameter and by allowing the nominal plant model to be nonlinear also.

This method relies on the linearity of the controller to convert the closed-loop identification of the plant into a nonstandard open-loop identification of the Youla-Kucera parameter. In a high SNR situation, this becomes a standard open-loop problem.

**The Hansen method using *left* coprime factorisations** This method is similar to the one outlined above except that we represent the unknown plant in terms of the *left* coprime factors of the plant model and the controller and the associated Youla-Kucera parameter.

The nonlinearity enters through the nominal plant and the controller as well as via the Youla-Kucera parameter. The solution is more involved because of the difficulties with the distributivity problem mentioned above; it also relies on a notion of differential coprimeness. Note that unlike the *right* coprime case we no longer require a linear controller for all the stability results to hold.

### 1.5.2 The Two-Step Method

The Two-Step method for linear systems in [34] identifies the linear sensitivity function of the closed-loop system using a model structure that is flexible enough to contain the dynamics. From this, a noise free estimate of the plant input is used in the second step to identify an estimate of the plant; i.e. the closed-loop identification problem is transformed into an open-loop-like identification problem. The Two-Step method requires the plant to be stable.

In this thesis, we examine a nonlinear version of the Two-Step Method. Again, an estimate of the closed-loop sensitivity operator is identified. This estimate is used in the second step to identify the plant. Both the plant and controller are allowed to be nonlinear and the plant is assumed to be stable. Also it is assumed that measurements of the plant input and output signals are available, see Figure 1.1. Further this method requires measurement of both of the reference signals, or measurement of one of the reference signals and knowledge of the controller.

### 1.5.3 The Coprime Factor Method

In [36], the Coprime Factor method identifies a pair of right factors of the plant. These factors are, respectively, the sensitivity and complementary sensitivity functions which are identified in an open-loop fashion using data collected on the closed-loop system. Auxiliary signals are introduced to identify a pair of right *coprime* factors of the plant. In [36], the additional freedom in the auxiliary signals is used to find a *normalised* right coprime factorisation.

In this thesis, the Coprime Factor method is extended to a nonlinear setting. Similar to the linear case, a pair of right factors of the nonlinear plant are identified and by the introduction of auxiliary signals a nonlinear pair of right *coprime* factors of the plant is found.

### 1.5.4 Tailor-made parametrisations

This thesis uses the same closed-loop matching criterion as in [38] with a tailor-made parametrisation, but it extends the results in two ways. First, in the linear case, we show that the gradient signals of [38] can be generated very simply on closed-loop simulation models. This observation then leads us to show that this simulation method for the computation of the gradients can be extended to the case of nonlinear systems and/or systems with nonlinear controllers.

The ideas in this thesis heavily rely on data-driven model-free control design methods that have recently been proposed in [6, 17, 30]. Indeed, we treat closed-loop identification with a tailor-made parametrisation as a dual of direct controller optimisation.

## 1.6 Thesis Outline

This section gives a brief outline of this thesis.

### Chapter 2

This chapter contains a review of definitions that are used throughout this thesis. A notion of differential coprimeness is introduced and motivation for this definition is discussed. Some novel stability and operator existence results that culminate in a nonlinear form of the Bezout identity are also stated. These stability results are used in Chapter 5, they are also interesting in their own right.

### Chapter 3

Chapter 3 contains a discussion of the properties and relationships between the Bezout and Set Theoretic definitions of coprimeness. This chapter focuses on the relationship between these definitions for nonlinear, left coprime operators. There are also a number of results regarding the relationship between two coprime factorisations of a given nonlinear operator. These relationships are examined for left and right factorisations under both the Bezout and the Set Theoretic approaches to coprimeness.

### Chapter 4

This chapter examines one method to convert the closed-loop identification problem to one of open-loop identification, known as the Hansen method. This is done using *right* fractional descriptions of both the controller and a nominal nonlinear plant stabilised by the controller, as well as a stable operator, known as a Youla-Kucera parameter. After a review of previous work, this chapter examines the noise free nonlinear case and then modifies this description to incorporate non-zero measurement noise into the system. The chapter concludes by depicting how these model characterisations can be used to identify the system under a high Signal-to-Noise Ratio (SNR) assumption.

### Chapter 5

This chapter examines the closed-loop identification problem using the Hansen method, this time using *left* coprime factorisations. The identification task proceeds similar to the right coprime case. It utilises both the idea of differential coprimeness and the double Bezout identity developed in Chapter 2. The chapter concludes with some simulation results of this method.

### Chapter 6

Two closed-loop identification schemes are extended in this chapter to allow the plant and controller to be nonlinear. The first of these is the Two-Step method, the second is the Coprime Factor method. The chapter includes simulation results of the Two-Step method.

### Chapter 7

This chapter presents a closed-loop identification scheme that minimises a closed-loop criterion, using a tailor-made parametrisation of the plant. Gradient signals are shown to be easily generated on closed-loop simulation models, and work done previously in the linear case is extended to nonlinear systems.

### Chapter 8

The final chapter offers concluding remarks and discusses the open problems related to the work contained in this thesis.

# Chapter 2

# Definitions, Stability and Operator Existence

The first section of this chapter states the definitions that are used throughout this thesis. These include the *Bezout* and *Set Theoretic* definitions of coprimeness as well as the notion of "differential coprimeness". The chapter concludes with some stability and operator existence results. These are used in later parts of the thesis. They are interesting in their own right as they form a nonlinear version of the double Bezout identity.

## 2.1 Notation

$L_2^m[0,\infty)$: the vector space of $\mathbb{R}^m$ valued square integrable functions with norm defined by $\|u\|^2 = \int_0^\infty |u(t)|^2 dt$.

$L_2[0,\infty)$: shorthand for $L_2^m[0,\infty)$ where $m$ is an arbitrary positive integer.

$\mathcal{T}_T$: the truncation operator on the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}^m$ ($m$ an arbitrary positive integer). It is defined by $\mathcal{T}_T u(t) = u(t)$ if $t \leq T$, $\mathcal{T}_T u(t) = 0$ if $t > T$.

$L_{2e}^m[0,\infty)$: the vector space of functions $f$ satisfying $\mathcal{T}_T f \in L_2^m[0,\infty)$ for all $T > 0$.

$\mathcal{U}$: is the input space with a suitable norm.

$\mathcal{U}^s$: is the subspace comprising all stable (bounded norm) inputs $u$.

$\mathcal{U}^u$: is the subspace comprising all unstable (unbounded) inputs $u$.

$\mathcal{Y}$: is the output space with a suitable norm.

$\mathcal{Y}^s$: is the subspace comprising all stable (bounded norm) outputs $y$.

$\mathcal{Y}^u$: is the subspace comprising all unstable (unbounded) outputs $y$.

$\mathcal{Z}$: is the "partial state" space with a suitable norm.

$\mathcal{Z}^s$: is the subspace comprising all stable (bounded norm) partial states $z$.

$\mathcal{Z}^u$: is the subspace comprising all unstable (unbounded) partial states $z$.

$I$: the identity operator.

## 2.2 General Definitions

**Well-posedness:** an operator $A : L_{2e}^p[0,\infty) \to L_{2e}^q[0,\infty)$ is well-posed or causal if $\mathcal{T}_T A \mathcal{T}_T = \mathcal{T}_T A$ for all $T > 0$. It is also assumed that $A(0) = 0$.

**Invertible:** an operator A is invertible if for all $z \in \mathcal{Y}$, with $z = Ax$, $x$ can be causally and uniquely determined from $z$.

**Bounded operator and gain:** an operator $A$ is bounded if it is well-posed and the gain

$$\|A\| = \sup_{u \in L_2^p[0,\infty), u \neq 0} \frac{\|A(u)\|}{\|u\|}$$

is finite. $A$ is also termed BIBO stable.

**Unit:** an operator $W$ is a unit if $W^{-1}$ exists, and $W$ and $W^{-1}$ are BIBO stable.

**Right Factorisation:** An operator $A$ has a right factorisation if it can be written as $A = N_r D_r^{-1}$ with $N_r$ and $D_r$ BIBO operators.

**Left Factorisation:** An operator $A$ has a left factorisation if it can be written as $A = D_l^{-1} N_l$ with $N_l$ and $D_l$ BIBO operators.

**Partial state:** Consider the right factorisation $y = N_r D_r^{-1} u$. The partial state $z$ is defined such that $y = N_r z$ and $u = D_r z$. Consider the left factorisation $y = D_l^{-1} N_l u$. The partial state $z$ is defined such that $z = D_l y = N_l u$.

**Weak Lipschitz continuity:** a well-posed operator $A$ is weakly Lipschitz (or weakly Lipschitz continuous) if for every $T > 0$ there exists a finite $\gamma_T$ such that

$$\|\mathcal{T}_T A\|_L \leq \gamma_T$$

where the Lipschitz semi-norm is

$$\|\mathcal{T}_T A\|_L = \sup_{\substack{\mathcal{T}_T x \neq \mathcal{T}_T y, \\ x, y \in L_{2e}[0,\infty)}} \frac{\|\mathcal{T}_T A x - \mathcal{T}_T A y\|}{\|\mathcal{T}_T x - \mathcal{T}_T y\|}$$

and obeys $\|\mathcal{T}_T A\|_L \geq \|\mathcal{T}_T A\|$.

**Global Lipschitz continuity:** a well-posed operator $A$ is globally Lipschitz continuous if there exists a $K$ such that for all $x, y \in L_2^p[0,\infty)$ there holds

$$\|A(x+y) - A(x)\| \leq K\|y\|.$$

This means there exists an operator $\partial A_{(x)}$, (causally) dependent on $x$ and bounded independently of $x$ with

$$\partial A_{(x)}(y) = A(x+y) - A(x).$$

When $A$ is linear, $\partial A_{(x)}$ is independent of $x$.

A nontrivial consequence is that the definition implies $\|A(x+y) - A(x)\| \leq K\|y\|$ for any $x \in L_{2e}^p[0,\infty)$, as opposed to $x \in L_2^p[0,\infty)$. This is shown in the following lemma.

**Lemma 2.1** *Let a well-posed operator $A : L^p_{2e}[0,\infty) \to L^p_{2e}[0,\infty)$ be such that for some $K$ and for all $x, y \in L^p_2[0,\infty)$ we have*

$$\|A(x+y) - A(x)\| \leq K\|y\|.$$

*Then this inequality holds for all $x \in L^p_{2e}[0,\infty)$.*

**Proof.** The proof is contained in Appendix 2.7 at the end of this chapter. •

**Smoothing [40]:** a well-posed operator $A$ is said to be *smoothing*, have no instantaneous direct feedthrough, or have zero uniform instantaneous gain, if it is weakly Lipschitz and for every $T > 0$, $\alpha > 0$, there exists $\delta = \delta(\alpha, T) \in (0, T)$ such that

$$\|\mathcal{T}_{t+\delta}(A\mathcal{T}_{t+\delta} - A\mathcal{T}_t)\|_L \leq \alpha, \ \forall t \in [0, T-\delta].$$

[A linear time-invariant operator has this property if its causal impulse response for $t \geq 0$ is of the form $\sum_{i=1}^N \alpha_i \delta(t - \tau_i) + \beta(t)$, $\tau_i > 0$; in discrete time, the equivalent is a strictly causal operator, i.e. $\mathcal{T}_{(k+1)\tau} A \mathcal{T}_{k\tau} = \mathcal{T}_{(k+1)\tau} A$ where $\tau$ is the sampling interval.]

**Remark 2.1** A system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) + j(x)u, \end{aligned} \tag{2.1}$$

with $f(x)$, $g(x)$, $h(x)$, $j(x)$ smooth functions of $x$, is smoothing if and only if $j(x) = 0$.

For *a closed-loop* system as shown in Figure 1.1, with noise $v$ identically zero:

**The closed-loop is well-posed ([40]):** if

- (a) for each $r_1, r_2 \in L_{2e}[0,\infty)$ there exists a unique $\bar{e}, \bar{u}, u, y \in L_{2e}[0,\infty)$ that depend causally on $r_1$ and $r_2$;

- (b) for each finite $T$, the dependence of $\mathcal{T}_T\bar{e}$, $\mathcal{T}_T\bar{u}$, $\mathcal{T}_Tu$, and $\mathcal{T}_Ty$ on $\mathcal{T}_Tr_1$ and $\mathcal{T}_Tr_2$ is Lipschitz continuous. This means there is a causal weakly Lipschitz closed-loop operator from $r_1, r_2$ to $\bar{e}, \bar{u}, u, y$.

The smoothing concept of [40] is a powerful tool for establishing the well-posedness property for a closed-loop where properties of the individual loop components are specified, and include a smoothing property on one of the components.

**The closed-loop is internally stable** if it is well-posed and the associated operator has finite gain.

If the noise $v$ is non-zero $r_1$ and $r_2$ must be replaced with $r_1$, $r_2$ and $v$ in these definitions.

**Remark 2.2** We can summarise the following relationships from [40].

**1** The sum of two weakly Lipschitz operators is also weakly Lipschitz.

**2** The sum of two smoothing operators is also smoothing.

**3** If the operator $A$ is smoothing and the operator $B$ is weakly Lipschitz then $AB$ is smoothing; $BA$ however may not necessarily be smoothing.

**4** If an operator $A$ is smoothing and $B$ is a delay operator then $BA$ is smoothing.

**5** Refer to Figure 2.1. If both the forward subsystem operator $A$ and the feedback subsystem operator $B$ are weakly Lipschitz and one of them is smoothing then the closed-loop feedback system ($A$, $B$) is well-posed.

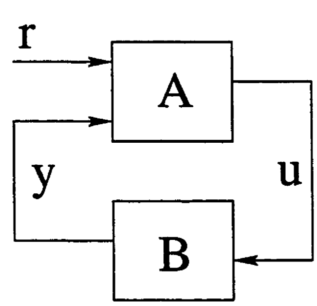



*Figure 2.1: Closed loop configuration for well-posedness.*

**Remark 2.3** The following scenario will be used in Chapter 5. Consider Figure 2.2 where $S_1$ and $S_2$ are smoothing operators. From Item 5 of Remark 2.2 it follows that the closed-loop of Figure 2.2 is well-posed. It follows from our definition of well-posedness that the operator from $r$ to $\bar{e}$ is weakly Lipschitz. Using Item 3 of Remark 2.2, it now follows that the operator from $r$ to $y$ is smoothing.

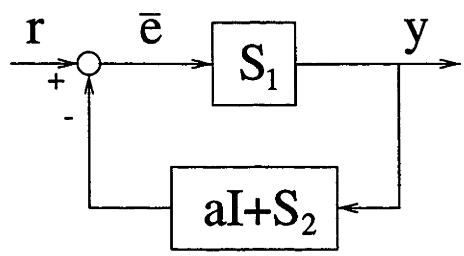



*Figure 2.2: Closed loop configuration for smoothing.*

### 2.2.1 Assumption

We will now state a standing assumption for this thesis. The assumption is common in the literature concerning nonlinear systems, and automatically satisfied in the linear case.

**Assumption 2.1** *Consider an operator $G : \mathcal{U} \to \mathcal{Y}$ and its input-output pairs $(u, y)$ with the output $y \in \mathcal{Y}^{\mathrm{u}}$ . Then the inputs $u$ leading to a given output $y$ are either all stable or all unstable. This assumption is necessary to ensure the existence of a left coprime factorisation for $G$ in a Set Theoretic sense; see [31] for further details.*

### 2.2.2 Initial conditions

As is conventional but not universal in treating BIBO stability of systems, there is no explicit consideration of initial conditions in this thesis. They can be introduced in several ways; by postulating they are established using inputs zero in $t < 0$, and acting over an interval $[0, 1]$ say, to provide the initial condition at time 1; or one can postulate that operators are indexed by an initial condition; or, in some but not all cases, one can replace an initial condition effect by an extra input or disturbance signal.

## 2.3 Definitions of Coprimeness

The definition of coprimeness in nonlinear fractional representations is not universal. The first definitions listed below, based on Bezout identities, have been used in e.g. [3, 4, 39]. However, alternative definitions, based on Set Theoretic ideas, have been used in e.g. [12, 25, 31]. These definitions are applicable for both *left* and *right* coprimeness.

In the linear case, the definitions are equivalent; in the nonlinear case, right coprimeness defined using a Bezout identity is also equivalent to right coprimeness from a Set Theoretic view point; see [2] for further details. The relationship between the two coprimeness definitions for nonlinear operators is discussed in Chapter 3.

### 2.3.1 Bezout Definitions of Coprimeness

**Right coprimeness (linear or nonlinear):** let $N_r$, $D_r$ be a right factorisation for a well-posed $G$, i.e. $G = N_r D_r^{-1}$ where $N_r$ and $D_r$ are BIBO stable. Then $(N_r, D_r)$ is a right coprime factorisation of $G$ if there exists a BIBO operator $\mathcal{L}_l$ for which

$$\mathcal{L}_l \begin{bmatrix} N_r \\ D_r \end{bmatrix} = I. \tag{2.2}$$

The relationship of 2.2 is known as a Bezout identity. A particular case of this which is always true when using linear operators is $\mathcal{L}_l = [X_l \; Y_l]$ with $X_l$ and $Y_l$ BIBO such that

$$X_l N_r + Y_l D_r = W,$$

where $W$ is a unit.

**Left coprimeness (linear or nonlinear):** let $N_l$, $D_l$ be a left factorisation for a well-posed $G$, i.e. $G = D_l^{-1} N_l$, where $N_l$ and $D_l$ are BIBO stable. Then $(N_l, D_l)$ is a left coprime factorisation of $G$ if there exists a BIBO $\mathcal{L}_r$ for which

$$\begin{bmatrix} N_l & D_l \end{bmatrix} \mathcal{L}_r = I. \tag{2.3}$$

Again, (2.3) is a Bezout identity for left coprimeness. In contrast to the fact that $\mathcal{L}_l$ does not have to seperate into $X_l$, $Y_l$ in the nonlinear case, it is obvious that $\mathcal{L}_r$ must have the form $\mathcal{L}_r = [X_r \ Y_r]^T$ for some BIBO operators $X_r$ and $Y_r$. Without loss of generality, the right side of (2.3) can be replaced by an arbitrary unit $W$.

Appendix 2.8 contains a few examples of systems that have left coprime factorisations.

**Normalised Coprime Factors:** The linear right coprime factors $(N_r, D_r)$ are normalised if

$$N_r^* N_r + D_r^* D_r = I. \tag{2.4}$$

See [41] for details. The nonlinear right coprime factors $(N_r, D_r)$ are normalised if

$$\|N_r z\|^2 + \|D_r z\|^2 = \|z\|^2 \ \forall z. \tag{2.5}$$

### 2.3.2 Set Theoretic Definitions of Coprimeness

Consider Figure 1.1, with the plant having either a left or a right coprime factorisation. The Set Theoretic definition of coprimeness roughly means that if there is internal instability it must come from the controller and/or be observed from the outside, and conversely.

**Right coprimeness** With nonlinear $P = N_r D_r^{-1}$ and $N_r$, $D_r$ BIBO, the pair $(N_r, D_r)$ is said to be right coprime in the Set Theoretic sense if the following property holds: if the partial state $z \in \mathcal{Z}^u$, then either $u = D_r z \in \mathcal{U}^{\mathrm{u}}$ or $y = N_r z \in \mathcal{Y}^{\mathrm{u}}$ where $(u, y)$ is an input-output pair.

**Left coprimeness** With nonlinear $P = D_l^{-1} N_l$ and $N_l$, $D_l$ BIBO, the pair $(N_l, D_l)$ is said to be left coprime in the Set Theoretic sense if the following property holds: if $(u, y)$ is an input-output pair with $u \in \mathcal{U}^{\mathrm{u}}$ and $y \in \mathcal{Y}^{\mathrm{u}}$ , then $z = D_l y = N_l u$ is also unstable, i.e. $z \in \mathcal{Z}^u$. (If the input and output are unstable, then the partial state is unstable).

## 2.4 Differential Coprimeness

The notion of differential coprimeness is used in the scheme that converts a closed-loop identification problem to one of open-loop identification, known as the Hansen method.

### 2.4.1 Differential coprimeness

We will now adopt the Bezout definition to coprimeness. Assume the pair $(N_l, D_l)$ is left coprime and globally Lipschitz continuous, then we can write

$$N_l U_r + D_l V_r = W \tag{2.6}$$

where $N_l$, $D_l$, $U_r$ and $V_r$ are all nonlinear and $W$ is a unit.

Then, one can define well-posed operators

$$\partial N_{l(x)}(.) = N_l(x + .) - N_l(x)$$
$$\partial D_{l(z)}(.) = D_l(z + .) - D_l(z)$$

for all signals $x$ and $z \in L_{2e}$.

If $N_l$ and $D_l$ are linear, $\partial N_{l(x)}(.) = N_l(.)\ \forall x$ and $\partial D_{l(z)}(.) = D_l(.)\ \forall z$; then $(\partial N_{l(x)})U_r + (\partial D_{l(z)})V_r$ is a unit, i.e. $W$.

When $N_l$ and $D_l$ are nonlinear, we shall say that they are differentially coprime if and only if the unit property continues to hold, though now the unit will not usually be $W$.

Formally, $N_l$ and $D_l$ are left differentially coprime if for all $x$, $z \in L_{2e}$, there exists BIBO $U_r$ and $V_r$ such that

$$(\partial N_{l(x)})U_r + (\partial D_{l(z)})V_r = W_{xz},$$

where $W_{xz}$ is a unit operator. More motivation for the definition of left differential coprimeness is given in Appendix 2.9 at the end of this chapter. Similar definitions hold for right coprime factorisations.

**Remark 2.4** We say that $N_l$ and $D_l$ are *uniformly* left differentially coprime if there exists $K$ such that $||W_{xz}|| < K$ and $||W_{xz}^{-1}|| < K$ independently of $x$ and $z \in L_{2e}$.

**Remark 2.5** Note that if $N_l$ and $D_l$ are known to be left differentially coprime in the sense that for some bounded $U_r$ and $V_r$, $\partial N_{l(x)} U_r + \partial D_{l(z)} V_r$ is a unit for any $x$ and $z$, then by taking $x = 0$, $z = 0$ we recover the standard coprimeness relation

$$N_l U_r + D_l V_r = W$$

for some unit $W$.

**Remark 2.6** There is an interesting relationship that exists for differentially coprime operators. Indeed, let $z_i$, $i = 1, 2 \in L_{2e}^p$ and set

$$z = z_1 + z_2. \tag{2.7}$$

From (2.7) we can write

$$D_l(z) = D_l(z_1) + \partial D_{l(z_1)}(z_2). \tag{2.8}$$

By rearranging (2.7) we can also write

$$\begin{aligned} D_l(z_1) &= D_l(z) + \partial D_{l(z)}(-z_2) \\ &= D_l(z) + \partial D_{l(z_1+z_2)}(-z_2). \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9) we have:

$$\partial D_{l(z_1)}(z_2) = -\partial D_{l(z_1+z_2)}(-z_2). \tag{2.10}$$

### 2.4.2 A consequence of uniform differential coprimeness

One difficulty in working with nonlinear operators is that $-A(-B)$ does not necessarily equal $AB$. Thus if for some unit $W$, coprime pair $(N_l, D_l)$ and BIBO operators $U_r$, $V_r$, there holds

$$N_l U_r - D_l(-V_r) = W, \tag{2.11}$$

we cannot say this is equivalent to (2.6), or even (2.6) with the unit $W$ replaced by another unit operator. However, with the aid of uniform differential coprimeness, we can show that when $(N_l,\ D_l)$ is uniformly left differentially coprime, so is $(N_l,\ -D_l)$.

**Lemma 2.2** *Let $(N_l,\ D_l)$ be a left coprime pair where $N_l$ and $D_l$ are globally Lipschitz continuous and* uniformly *differentially coprime; i.e. for any $x,\ z,\ \beta \in L^p_{2e}$, we have*

$$(\partial N_{l(x)} U_r + \partial D_{l(z)} V_r)\beta = W_{xz}(\beta) \tag{2.12}$$

*where $\|W_{xz}\| \leq K$, $\|W_{xz}^{-1}\| \leq K$ and we assume that $N_l$ and $U_r$ are smoothing and $D_l$ and $V_r$ are of the form $aI + S$ where $aI$ is the scaled identity operator and $S$ is a smoothing operator.*

*Then $(N_l,\ -D_l)$ is also uniformly left differentially coprime.*

**Proof.** The proof is contained in Appendix 2.10 at the end of this chapter. •

**Remark 2.7** Note that the assumptions on $N_l$, $D_l$, $V_r$ and $U_r$ will be verified when Lemma 2.2 is used in Chapter 5.

**Remark 2.8** Suppose $D_l = aI + S$ where $a$ is a constant, $I$ is the identity operator and $S$ is smoothing. Then

$$\begin{aligned} D_l^{-1} &= (aI + S)^{-1} \\ &= a^{-1}I - a^{-1}S(aI + S)^{-1}. \end{aligned}$$

Now $a^{-1}S(aI + S)^{-1}$ is smoothing, since it results from a loop with $S$ in the forward path, and $a^{-1}$ in a feedback path; see Remark 2.2 for more details. Thus $D_l^{-1}$ is also of the form $\hat{a}I + \hat{S}$ with $\hat{S}$ smoothing. The assumption that $D_l = aI + S$ with $a$ a constant and $S$ smoothing is very reasonable in practice; recall the discussion in Remark 2.1.

The next section outlines some stability and operator existence requirements. These are interesting in their own right as they culminate in a nonlinear version of the Bezout identity.

## 2.5 Stability and operator existence

In order to construct a nonlinear form of the double Bezout identity we shall need several characterisations of stability. We consider that the nominal plant is connected in closed-loop with the stabilising controller. We have a left and a right coprime representation for the controller and a left and a right coprime representation for the nominal plant. In this section we consider the four combinations of these representations to find a nonlinear form of the double Bezout identity. We adopt the following assumptions.

**Assumption 2.2** *A weakly Lipschitz controller $C = U_r V_r^{-1} = V_l^{-1} U_l$ stabilises a smoothing nominal plant model $P_0 = N_r D_r^{-1} = D_l^{-1} N_l$. $U_r, V_r, V_l, U_l, N_r, D_r, D_l$ and $N_l$ are all well-posed, stable, globally Lipschitz operators with $V_r, V_l, D_r, D_l$ invertible. Further, $(N_r, D_r)$ and $(U_r, V_r)$ are differentially coprime pairs and $(N_l, D_l)$ and $(U_l, V_l)$ are uniformly differentially coprime pairs.*

**Combination 2.1** $(P_0, C) = (N_r D_r^{-1}, U_r V_r^{-1})$ *The closed-loop system $(P_0, C)$ is internally stable if and only if*

$$\begin{bmatrix} D_r & -U_r \\ N_r & V_r \end{bmatrix}^{-1} \tag{2.13}$$

*exists and is bounded.*

This result is effectively inherent in the definition, and was established in [39].

**Combination 2.2** $(P_0, C) = (N_r D_r^{-1}, V_l^{-1} U_l)$ *The closed-loop system* $(P_0, C)$ *is internally stable if and only if*

$$(V_l D_r - U_l(-N_r))^{-1}$$

*exists and is bounded.*

**Proof.** The proof is contained in Appendix 2.11 at the end of this chapter. •

The next combination appears to be of independent interest. It is used in this thesis to describe the structure of the set of all plants stabilised by a given controller.

**Combination 2.3** $(P_0, C) = (D_l^{-1} N_l, U_r V_r^{-1})$ *The closed-loop system* $(P_0, C)$ *is internally stable and* $q = V_r^{-1}(r_1 - y)$ *is bounded if and only if*

$$[D_l(-V_r) - N_l U_r]^{-1} \tag{2.14}$$

*exists and is bounded.*

**Proof.** The proof is contained in Appendix 2.12 at the end of this chapter. •

The final stability result is found by constructing a nonlinear form of the double Bezout identity; the actual nonlinear form of the double Bezout identity is displayed in (2.24) of Appendix 2.13.

**Combination 2.4** $(P_0, C) = (D_l^{-1} N_l, V_l^{-1} U_l)$ *The closed-loop system* $(P_0, C)$ *is internally stable if and only if*

$$\begin{bmatrix} V_l & U_l \\ -N_l & D_l \end{bmatrix}^{-1} \tag{2.15}$$

*exists and is bounded.*

**Proof.** The proof is contained in Appendix 2.13 at the end of this chapter. •

## 2.6 Concluding remarks

In this chapter we have stated definitions that will be used throughout this thesis. We have introduced the notion of differential coprimeness and stated some stability and operator existence results. In the next chapter we will build on this preliminary work to examine the relationship between the Bezout and Set Theoretic definitions of coprimeness for nonlinear systems.

# Appendices

These appendices contain the proofs of lemmas contained in this chapter.

## 2.7 Proof of Lemma 2.1

Suppose the conclusion were not true. Let $\bar{x} \in L^p_{2e}[0,\infty)$ and $\bar{y} \in L^p_2[0,\infty)$ be such that

$$\|A(\bar{x}+\bar{y}) - A(\bar{x})\| > (K+\epsilon)\|\bar{y}\|$$

for some $\epsilon > 0$. Then, with $z = A(\bar{x}+\bar{y}) - A(\bar{x})$, there exists $T$ such that

$$[\int_0^T |z(t)|^2 dt]^{\frac{1}{2}} > (K+\frac{\epsilon}{2})\|\bar{y}\|. \tag{2.16}$$

Set $z_T = \mathcal{T}_T z$, $\bar{x}_T = \mathcal{T}_T \bar{x}$, $\bar{y}_T = \mathcal{T}_T \bar{y}$. Then by causality,

$$z_T = \mathcal{T}_T[A(\bar{x}_T + \bar{y}_T) - A(\bar{x}_T)].$$

By hypothesis,

$$\|z_T\| \leq K\|\bar{y}_T\| \leq K\|\bar{y}\|.$$

However, inequality (2.16) implies

$$\|z_T\| > (K+\frac{\epsilon}{2})\|\bar{y}\|$$

which is a contradiction. •

## 2.8 Examples of Left Coprime Factorisations

It has been claimed that there are no plants worth considering that have left coprime factorisations. This appendix contains a few examples of plants that do have left coprime factorisations.

### 2.8.1 Stable Plant

If the plant $P$ is stable we can choose the left fractional representation

$$P = D_l^{-1} N_l$$

where

$$N_l = P \text{ and } D_l = I.$$

### 2.8.2 D.E. representation

Consider

$$\dot{y} + a(y) = bu.$$

Then

$$y = D_l^{-1} N_l u,$$

with

$$D_l = (1 + \frac{1}{s+1}(a(.) - 1)) \text{ and } N_l = \frac{b}{s+1}$$

is a left coprime representation. Similar results hold for matrix differential equations of the type

$$\dot{Y} + A(Y) = BU.$$

### 2.8.3 Input Nonlinearity

Consider an input nonlinearity $\phi$ connected to a linear plant with left factorisation $D_l^{-1} N_l$. We can write this as a left factorisation

$$D_l^{-1}(N_l \phi).$$

Note that if we were to try and write this as a right factorisation we would encounter more difficulties. That is, if we had an input nonlinearity $\phi$ followed by a right factorisation of a linear plant $N_r D_r^{-1}$ we would have

$$N_r(D_r^{-1}\phi).$$

However, we really want

$$\mathcal{D}_r^{-1} = D_r^{-1}\phi.$$

Thus

$$\mathcal{D}_r = \phi^{-1}(D_r^{-1}(.)).$$

This would rule out non-invertible nonlinearities.

## 2.9 Motivation for the definition of differential coprimeness

In this appendix we give a motivation for the definition of differential coprimeness.

### 2.9.1 Left differential coprimeness

Consider Figure 2.3 with $C = U_r V_r^{-1}$ and $P = D_l^{-1} N_l$, where $(N_l, D_l)$ is left coprime.

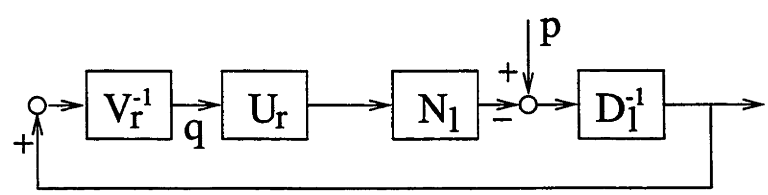



*Figure 2.3: Closed-loop configuration.*

Thus,

$$p - N_l U_r q = D_l V_r q$$

or

$$(N_l U_r + D_l V_r)q = p.$$

As $(N_l, D_l)$ is left coprime, $(N_lU_r + D_lV_r)$ is a unit. So for any bounded input $p$, $q$ is bounded, and conversely.

Now let us introduce the signal $r_2$ into the loop as shown in Figure 2.4 and examine its effect.

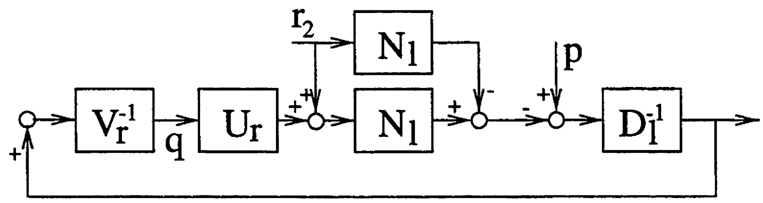



*Figure 2.4: Part left differential coprimeness.*

This yields

$$p - N_l(U_rq + r_2) + N_lr_2 = D_lV_rq$$

or

$$(\partial N_{l(r_2)}U_r + D_lV_r)q = p.$$

Thus, for any $r_2$, we would like $q$ to be bounded if $p$ is bounded. This is the first step towards differential coprimeness. We will further introduce the signal $r_1$ as shown in Figure 2.5.

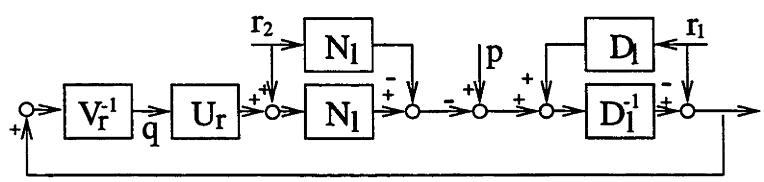



*Figure 2.5: Left differential coprimeness.*

This gives

$$\begin{aligned} p - N_l(U_rq + r_2) + N_lr_2 &= D_l(V_rq + r_1) - D_lr_1 \\ &= \partial D_{l(r_1)}V_rq \end{aligned}$$

or

$$(\partial N_{l(r_2)}U_r + \partial D_{l(r_1)}V_r)q = p.$$

### Summary

**Left differential coprimeness** means that for any inputs $r_1$ and $r_2$, $q$ is bounded if $p$ is bounded, with the gain from $p$ to $q$ depending on $r_1$ and $r_2$.

**Left uniform differential coprimeness** goes further and requires that $q$ be bounded if $p$ is bounded, independently of $r_1$ and $r_2$.

### 2.9.2 Right differential coprimeness

Similarly, consider Figure 2.6 with $C = V_l^{-1}U_l$ and $P = N_r D_r^{-1}$, where $(N_r, D_r)$ is right coprime.

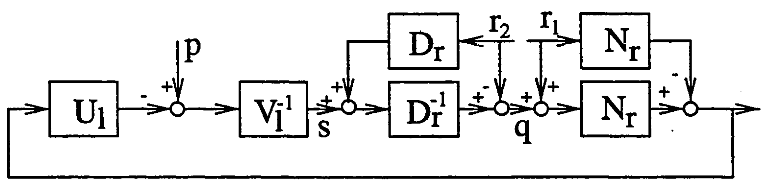



*Figure 2.6: Right differential coprimeness.*

We have

$$q = D_r^{-1}(D_r r_2 + s) - r_2,$$

or equivalently

$$\partial D_{r(r_2)} q = s.$$

Also

$$V_l s = p - U_l(N_r(r_1 + q) - N_r r_1)$$

or

$$V_l \partial D_{r(r_2)} q = p - U_l \partial N_{r(r_1)} q,$$

i.e.

$$(V_l \partial D_{r(r_2)} + U_l \partial N_{r(r_1)})q = p.$$

**Right differential coprimeness** of $(N_r, D_r)$ means that for any inputs $r_1$ and $r_2$, $q$ is bounded if $p$ is bounded, with the gain from $p$ to $q$ depending on $r_1$ and $r_2$.

**Right uniform differential coprimeness** goes further and requires that $q$ be bounded if $p$ is bounded, independently of $r_1$ and $r_2$.

## 2.10 Proof of Lemma 2.2

By applying the relationship in Remark 2.6 to (2.12) one obtains

$$(\partial N_{l(x)} U_r - \partial D_{l(\bar{z})}(-V_r))\beta = W_{xz}(\beta) \tag{2.17}$$

where $\bar{z} = z + V_r\beta$. For arbitrary $x$, $\bar{z}$, $\beta \in L_{2e}^p$, define an operator $\bar{W}_{x\bar{z}}$ by

$$\bar{W}_{x\bar{z}}(\beta) = W_{xz}(\beta) \tag{2.18}$$

where $z = \bar{z} - V_r\beta$. It is trivial to see from this that if $\|W_{xz}\| < K \ \forall\ x,\ z \in L_{2e}$, then $\|\bar{W}_{x\bar{z}}\| < K, \forall\ x,\ \bar{z} \in L_{2e}$.

To show that $(N_l, -D_l)$ is also left differentially coprime we need to show that $\bar{W}_{x\bar{z}}$ is a unit. That is, that $\bar{W}_{x\bar{z}}^{-1}$ is well-posed and bounded independently of $x$ and $\bar{z}$.

Evidently, we have $\bar{W}_{x\bar{z}}^{-1}(\gamma) = \beta$ or $\gamma = \bar{W}_{x\bar{z}}(\beta)$ if and only if $W_{xz}^{-1}(\gamma) = \beta$ or $\gamma = W_{xz}(\beta)$ and $z = \bar{z} - V_r\beta$. Refer to Figure 2.7, where we have represented the operator $\bar{W}_{x\bar{z}}^{-1} : \gamma \to \beta$.

The operator $\bar{W}_{x\bar{z}}^{-1} : \gamma \to \beta$ will be well-posed provided the loop of Figure 2.7 is well-posed. Figure 2.7 can be redrawn as shown in Figure 2.8. Here we have used the results of Appendix 2.9.

In turn, Figure 2.8 can be redrawn as shown in Figure 2.9.

Now $D_l^{-1}$ has the form $aI + S$ for smoothing $S$ means that $\partial D_{l(z)}^{-1} = \partial S_{(z)}$. Now

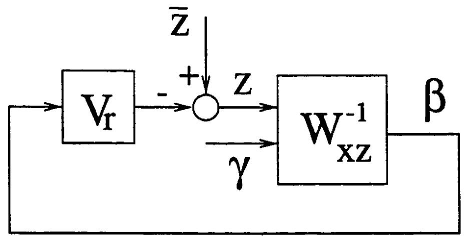



*Figure 2.7: Construction of* $\bar{W}_{x\bar{z}}^{-1}$.

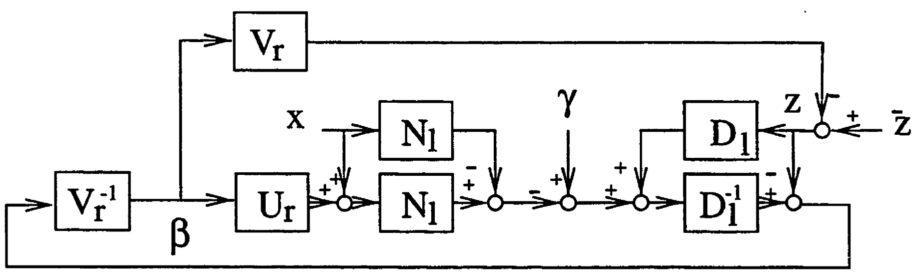



*Figure 2.8: Construction of* $\bar{W}_{x\bar{z}}^{-1}$.

$\partial S_{(z)}(v) = S(v+z) - S(z)$ and for fixed $v$, $\partial S_{(z)}(v)$ is an operator on $z$.

It is clear that for fixed $v$, we can talk about weak Lipschitz continuity with respect to $z$, and indeed the smoothing property; i.e. since $S$ is smoothing, $\partial S_{(z)}(v)$ for fixed $v$ is smoothing in $z$. Similarly, it follows from the assumptions that $\partial N_{l(x)}$ is a smoothing operator.

Using Item 5 of Remark 2.2, it now follows that Figures 2.7, 2.8 and 2.9 are well-posed. Indeed it is clear from Figure 2.9 that there is a smoothing element in each subloop while all elements in these loops are weakly Lipschitz.

Since $W_{xz}^{-1}(.)$ is an operator bounded independently of $x$, $z$, $\bar{W}_{x\bar{z}}^{-1}$ must also be bounded independently of $x$, and $\bar{z}$, i.e. $\bar{W}_{x\bar{z}}$ is a unit uniformly over $x$ and $\bar{z}$. Hence it follows from (2.17) and (2.18) that $(N_l, -D_l)$ is uniformly left differentially coprime. •

## 2.11 Proof of Combination 2.2

Recall the definition of internal stability. We need to show that the closed-loop is well-posed and that the associated operator has finite gain. The closed-loop system is well-posed by assumption. Thus in proving the "if" part of this combination statement, we only have to show that the closed-loop gain is finite.

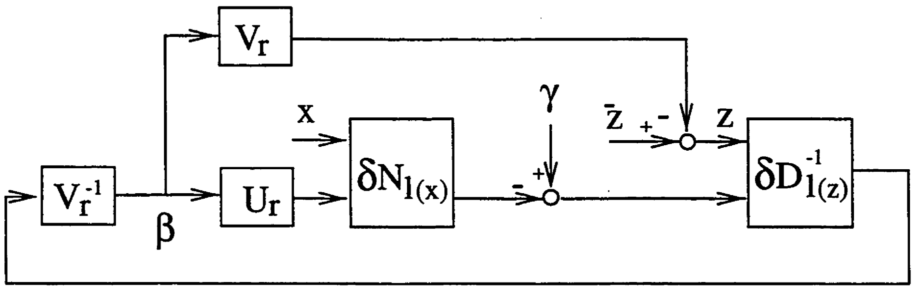



*Figure 2.9: Construction of $\bar{W}_{x\bar{z}}^{-1}$.*

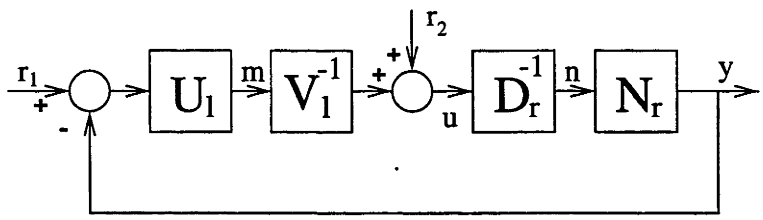



*Figure 2.10: Closed-loop diagram with the nominal plant $P_0 = N_r D_r^{-1}$ and $C = V_l^{-1} U_l$.*

Consider Figure 2.10, then from differential coprimeness

$$\begin{aligned} m &= V_l[D_r n - r_2] \\ &= V_l(D_r n) + \partial V_{l(D_r n)}(-r_2) \end{aligned} \tag{2.19}$$

and also

$$\begin{aligned} m &= U_l[r_1 - N_r n] \\ &= U_l(-N_r)n + \partial U_{l(-N_r n)} r_1. \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20) gives

$$[V_l D_r - U_l(-N_r)]n = \partial U_{l(p)} r_1 - \partial V_{l(q)}(-r_2) \tag{2.21}$$

where $p = -N_r n$ and $q = D_r n$. Here $\partial U_{l(p)}(.)$ and $\partial V_{l(q)}(.)$ are, respectively, bounded independently of $p$ and $q$. These operators exist because of the global Lipschitz continuity of $U_l$ and $V_l$.

If the inverse $[V_l D_r - U_l(-N_r)]^{-1}$ exists and is bounded for all bounded $r_1$, $r_2$, it follows that $n$ is bounded. Then $y = N_r n$ is bounded, as are $\bar{e} = r_1 - y$, $u = D_r n$ and $\bar{u} = u - r_2$. So the system is internally stable.

Conversely, if the system is stable, $y$ and $u$ are bounded for all bounded $r_1$ and $r_2$. By the right coprimeness of $N_r$, $D_r$ there exists a bounded $\mathcal{L}_l$ with

$$\mathcal{L}_l \begin{bmatrix} N_r \\ D_r \end{bmatrix} = I.$$

Now

$$n = \mathcal{L}_l \begin{bmatrix} N_r n \\ D_r n \end{bmatrix} = \mathcal{L}_l \begin{bmatrix} y \\ u \end{bmatrix}.$$

Thus $n$ is always bounded for all bounded $r_1$ and $r_2$ if the system is stable.

As $V_l$ and $U_l$ are left differentially coprime, there exist bounded operators $A_r$ and $B_r$ that satisfy

$$\partial V_{l(p)} B_r - \partial U_{l(q)}(-A_r) = W_{pq}$$

where $W_{pq}$ is a unit. Note that we have used Lemma 2.2. With an arbitrary bounded $r_3$ and with $A_r$ and $B_r$, define bounded signals

$$r_1 = -A_r r_3 \text{ and } r_2 = -B_r r_3.$$

and observe that (2.21) becomes

$$\begin{aligned} & [V_l D_r - U_l(-N_r)]n \\ &= -\partial V_{l(q)}(-r_2) + \partial U_{l(p)} r_1 \\ &= -\partial V_{l(q)} B_r r_3 + \partial U_{l(p)}(-A_r) r_3 \\ &= -\left[\partial V_{l(q)} B_r - \partial U_{l(p)}(-A_r)\right] r_3 \\ &= -W_{pq} r_3. \end{aligned}$$

Since $r_3$ is bounded but arbitrary, $W_{pq}$ is a unit and $n$ is bounded, it follows that the inverse $[V_l D_r - U_l(-N_r)]^{-1}$ exists and is bounded. •

## 2.12 Proof of Combination 2.3

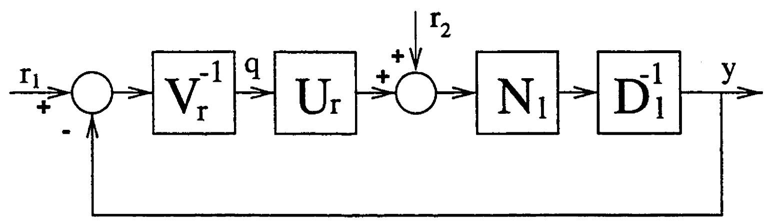



*Figure 2.11: Closed-loop diagram with the nominal plant $P_0 = D_l^{-1}N_l$ and $C = U_rV_r^{-1}$.*

Observe first that the closed-loop system is well-posed by assumption. Thus we only need to show that the closed-loop gain is finite to prove internal stability. Note

$$\begin{aligned} D_l y &= N_l(r_2 + U_r q) \\ &= N_l U_r q + \partial N_{l(p)}(r_2), \end{aligned}$$

where $p = U_r q$ and $\partial N_{l(p)}(.)$ is an operator bounded independently of $q$, existing because of the global Lipschitz continuity of $N_l$. Similarly,

$$y = -V_r q + r_1,$$

so that

$$\begin{aligned} D_l y &= D_l(-V_r q + r_1) \\ &= D_l(-V_r q) + \partial D_{l(t)}(r_1) \end{aligned}$$

where $t = -V_r q$ and $\partial D_{l(t)}(.)$ is an operator bounded independently of $q$. The two expressions for $D_l y$ yield

$$(D_l(-V_r) - N_l U_r)q = \partial N_{l(p)}(r_2) - \partial D_{l(t)}(r_1). \tag{2.22}$$

If $[D_l(-V_r) - N_l U_r]^{-1}$ exists and is bounded, then $q$ is obviously bounded when $r_1$, $r_2$

are bounded. It follows that the error signals $r_1 - y$ and $r_2 + U_r q$ are bounded, i.e. internal stability holds.

To prove the converse, suppose the loop is stable in the sense that for all bounded $r_1$ and $r_2$, $q$ is bounded. We have to prove that $[D_l(-V_r) - N_l U_r]^{-1}$ is also bounded.

Let $r_3$ be an arbitrary bounded signal. Let $L_r$, $M_r$ be bounded operators existing by virtue of the left differential coprimeness property and satisfying

$$\partial D_{l(t)} L_r + \partial N_{l(p)} M_r = W_{pt} \tag{2.23}$$

where $W_{pt}$ is a unit. Using $r_3$, $L_r$ and $M_r$ define bounded signals

$$r_1 = -L_r r_3 \quad \text{and} \quad r_2 = M_r r_3$$

and observe that

$$\begin{aligned} &\partial D_{l(t)}(-r_1) + \partial N_{l(p)}(r_2) \\ &\quad = \partial D_{l(t)} L_r r_3 + \partial N_{l(p)} M_r r_3 \\ &\quad = W_{pt} r_3. \end{aligned}$$

Using (2.22), we obtain

$$[D_l(-V_r) - N_l U_r]\, q = W_{pt} r_3.$$

Now in this equation, $r_3$ is bounded and arbitrary; further, $q$ is bounded (because the boundedness of $r_3$ implies boundedness of $r_1$ and $r_2$, which by hypothesis implies $q$ is bounded). Since $W_{pt}$ is a unit, this means that $[D_l(-V_r) - N_l U_r]^{-1}$ is necessarily a bounded operator. •

## 2.13 Proof of Combination 2.4

Consider the following operator $[p\ q]^T \to [\alpha\ \beta]^T$ :

$$\begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & U_r \\ N_r & -V_r \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

We want to show that the operator in (2.15) is bounded. We shall show that

$$\begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & U_r \\ N_r & -V_r \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} X_{(\beta)} & 0 \\ 0 & Y_{(\alpha)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{2.24}$$

where $X_{(\beta)}$ is a unit that depends on $\beta$, and $Y_{(\alpha)}$ is a unit that depends on $\alpha$.

From the Lipschitz continuity of $N_l$, $D_l$, $U_l$ and $V_l$, (2.24) becomes

$$\begin{aligned} \begin{bmatrix} p \\ q \end{bmatrix} &= \begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r\alpha + U_r\beta \\ N_r\alpha - V_r\beta \end{bmatrix} \\ &= \begin{bmatrix} V_l(D_r\alpha + U_r\beta) - U_l(-N_r\alpha + V_r\beta) \\ -N_l(D_r\alpha + U_r\beta) + D_l(N_r\alpha - V_r\beta) \end{bmatrix} \\ &= \begin{bmatrix} \partial V_{l(U_r\beta)} D_r\alpha - \partial U_{l(V_r\beta)}(-N_r)\alpha \\ -\partial N_{l(D_r\alpha)}(U_r\beta) + \partial D_{l(N_r\alpha)}(-V_r)\beta \end{bmatrix} \\ &= \begin{bmatrix} X_{(\beta)}\alpha \\ Y_{(\alpha)}\beta \end{bmatrix}. \end{aligned} \tag{2.25}$$

Note that to obtain the last equality, we have used $V_lU_r\beta - U_lV_r\beta = 0$ and $-N_lD_r\alpha + D_lN_r\alpha = 0$. These relations are obvious from the definition of $P$ and $C$.

Now suppose the system ($P_0$, $C$) is stable. By Combination 2.3 and as $(U_l, V_l)$ is *uniformly* differentially coprime, $X_{(\beta)}$ is a unit uniformly over $\beta$. By Combination 2.2 and as $(N_l, D_l)$ is *uniformly* differentially coprime, $Y_{(\alpha)}$ is a unit uniformly over $\alpha$.

It follows that for all arbitrary bounded $p$ and $q$, there exists bounded $\alpha$ and $\beta$ such that

$$\begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} D_r & U_r \\ N_r & -V_r \end{bmatrix} \begin{bmatrix} X_{(\beta)}^{-1} & 0 \\ 0 & Y_{(\alpha)}^{-1} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{2.26}$$

The right side is bounded because $X_{(\beta)}^{-1}$ and $Y_{(\alpha)}^{-1}$ have finite gain uniformly over $\alpha$ and $\beta$. Therefore the inverse on the left exists and is bounded.

Conversely, suppose that the inverse exists and is bounded. By assumption the closed-loop system $(P_0, C)$ is well-posed so we only need to demonstrate that if (2.15) exists and is bounded then the closed-loop gain is finite.

Let $p$ be arbitrary but of finite norm and $q = 0$. Then, we have $p = X_{(\beta)}\alpha$ and $q = Y_{(\alpha)}\beta$ which implies $\beta = 0$ and $\alpha = X_{(0)}^{-1}p = X^{-1}p$. It follows that $X_{(0)} = X = [V_l D_r - U_l(-N_r)]$. Let $\mathcal{L}_l$ be such that

$$\mathcal{L}_l \begin{bmatrix} D_r \\ N_r \end{bmatrix} = I.$$

Then for all $p$ from (2.26) we get

$$\mathcal{L}_l \begin{bmatrix} V_l & U_l \\ N_l & D_l \end{bmatrix}^{-1} \begin{bmatrix} p \\ 0 \end{bmatrix} = \mathcal{L}_l \begin{bmatrix} D_r \\ N_r \end{bmatrix} X^{-1}p = X^{-1}p$$

is bounded. Hence $X = V_l D_r - U_l(-N_r)$ is a unit, and so by Combination 2.2, the closed-loop system ($P_0$, $C$) is internally stable. ●

# Chapter 3

# Coprimeness Properties

This chapter contains a discussion on the relationship between the Bezout and Set Theoretic definitions of coprimeness. It is also shown how one coprime factorisation of a given operator relates to other factorisations both when they are coprime and when they are not. It investigates whether some properties for linear left coprime realisations carry over to the nonlinear case.

## 3.1 Coprimeness properties of linear fractional system realisations

There are a number of fundamental results on coprime fractional descriptions of *linear* systems which are of great utility. For example, consider the right fractional descriptions $P(s) = N_r(s)D_r(s)^{-1}$, where $N_r(s)$, $D_r(s) \in \mathbf{M}(\mathbf{S})$ are matrices with entries in $\mathbf{S}$, the ring of proper stable rational transfer functions. Then coprimeness can be defined by the following requirement:

$$\begin{bmatrix} N_r(s_0) \\ D_r(s_0) \end{bmatrix} \text{ has full column rank } \forall s_0 \text{ with } \mathrm{Re}[s_0] \geq 0. \tag{3.1}$$

This is equivalent to the Bezout identity property: there exists $U_l(s)$, $V_l(s) \in \mathbf{M}(\mathbf{S})$ with

$$U_l(s)\,N_r(s) + V_l(s)\,D_r(s) = I. \tag{3.2}$$

A further property is: if $P(s) = N_{r1}(s)\, D_{r1}(s)^{-1} = N_{r2}(s)\, D_{r2}(s)^{-1}$ with $(N_{r1}, D_{r1})$ coprime, then $(N_{r2}, D_{r2})$ is coprime if and only if there exists $W$ a *unit* of $\mathbf{M}(\mathbf{S})$ (i.e. $W(s)$, $W(s)^{-1} \in \mathbf{M}(\mathbf{S})$) such that

$$N_{r2} = N_{r1}W, \;\; D_{r2} = D_{r1}W. \tag{3.3}$$

Finally, if $P(s) = N_{r1}(s)\, D_{r1}(s)^{-1} = N_{r2}(s)\, D_{r2}(s)^{-1}$ with $(N_{r1}, D_{r1})$ coprime, there exists $W \in \mathbf{M}(\mathbf{S})$ such that (3.3) holds.

The above results apply also to left fractional realisations, with obvious changes. Also, in (3.2), the identity matrix can be replaced by any unit without loss of generality.

It is of interest to try to extend these results to coprime descriptions of nonlinear systems.

## 3.2 Nonlinear right fractional realisations

Consider systems defined by a well-posed nonlinear operator $P = N_r D_r^{-1} : \mathcal{U} \to \mathcal{Y}$. Recall the Bezout and Set Theoretic definitions of *right* coprimeness. In the nonlinear case these are equivalent definitions; see [2] for more details. Under the Bezout definition of right coprimeness, we have the following relationship between two right factorisations of a given operator.

**Lemma 3.1** *If $P = N_{r1}\, D_{r1}^{-1} = N_{r2}\, D_{r2}^{-1}$ are two right fractional representations where $N_{r1}$, $D_{r1}$, $N_{r2}$, and $D_{r2}$ are well-posed operators and $(N_{r1}, D_{r1})$ is right coprime in a Bezout sense, then $(N_{r2}, D_{r2})$ is right coprime in a Bezout sense if and only if there exists a unit $W$ such that*

$$N_{r2} = N_{r1}\, W, \;\; D_{r2} = D_{r1}\, W. \tag{3.4}$$

*If $(N_{r2}, D_{r2})$ is not coprime, there exists a well-posed, stable $W$ such that* (3.4) *holds.*

**Proof.** $P = N_{r1}D_{r1}^{-1}$ has a right coprime factorisation if and only if there exists a stable well-posed operator $\mathcal{L}_l$ such that

$$\mathcal{L}_l \begin{bmatrix} N_{r1} \\ D_{r1} \end{bmatrix} = I. \tag{3.5}$$

Now suppose that $W$ is a unit operator and (3.4) holds, then

$$\mathcal{L}_l \begin{bmatrix} N_{r2} \\ D_{r2} \end{bmatrix} = \mathcal{L}_l \begin{bmatrix} N_{r1} \\ D_{r1} \end{bmatrix} W = W. \tag{3.6}$$

Hence

$$(W^{-1}\mathcal{L}_l) \begin{bmatrix} N_{r2} \\ D_{r2} \end{bmatrix} = I, \tag{3.7}$$

i.e. $(N_{r2}, D_{r2})$ is coprime.

Conversely, suppose that $(N_{r2}, D_{r2})$ is a coprime realisation of $P$. Since $P = N_{r1}D_{r1}^{-1} = N_{r2}D_{r2}^{-1}$, then $N_{r2} = N_{r1}D_{r1}^{-1}D_{r2}$. Using (3.5) we have

$$\begin{aligned} \mathcal{L}_l \begin{bmatrix} N_{r2} \\ D_{r2} \end{bmatrix} &= \mathcal{L}_l \begin{bmatrix} N_{r1}D_{r1}^{-1}D_{r2} \\ D_{r2} \end{bmatrix}, &(3.8) \\ &= \mathcal{L}_l \begin{bmatrix} N_{r1} \\ D_{r1} \end{bmatrix} (D_{r1}^{-1}D_{r2}) = D_{r1}^{-1}D_{r2}. &(3.9) \end{aligned}$$

The left side of this equation is well-posed and stable; hence $W = D_{r1}^{-1}D_{r2}$ is also well-posed and stable and satisfies (3.4). The reverse argument shows that $W^{-1} = D_{r2}^{-1}D_{r1}$ is well-posed and stable, i.e. $W$ is a unit operator.

Finally, if $(N_{r2}, D_{r2})$ is not coprime, the argument of the previous paragraph yields the existence of a well-posed and stable $W$ satisfying (3.4). •

## 3.3 Nonlinear left fractional realisations

Consider systems defined by a well-posed operator $P = D_l^{-1}N_l : \mathcal{U} \to \mathcal{Y}$. and recall that a well-posed operator obeys $P(0) = 0$. There is much less in the literature about *left* fractional representations of nonlinear systems, $P = D_l^{-1} N_l$, than about *right* fractional representations. For some key results, see [31]. Notice that while in the linear case it is possible to obtain results for left realisations by transposition of those for right realisations, this is no longer possible in the nonlinear case.

In [26, 27, 28] it is shown that for nonlinear systems a more useful and perhaps more fundamental concept than a left realisation is a stable kernel representation. In a sense, left factorisation representations are a special case of stable kernel representations.

Recall the Bezout and Set Theoretic definitions of *left* coprimeness from Section 2.3. This section explores the relationship between these two definitions. It also describes how one coprime factorisation of a given operator relates to other factorisations that may or may not be coprime.

**Lemma 3.2** *Coprimeness in the Set Theoretic sense implies coprimeness in the Bezout sense.*

**Proof.** By assumption, the partial state $z \in \mathcal{Z}^s$ implies that either the input $u \in \mathcal{U}^s$ or the output $y \in \mathcal{Y}^s$ where $z = N_l u = D_l y$. Equivalently, $z \in \mathcal{Z}^u$ implies that $u \in \mathcal{U}^u$ and $y \in \mathcal{Y}^u$. We wish to use this to show that there exists a well-posed, stable operator $\mathcal{L}_r$ such that

$$\begin{bmatrix} D_l & N_l \end{bmatrix} \mathcal{L}_r = I. \tag{3.10}$$

The proof is constructive: we will consider the three different cases that can occur and, for each of these cases, construct an operator $\mathcal{L}_r$ for which (3.10) holds:

- If $z \in \mathcal{Z}^s$ and $u \in \mathcal{U}^s$ with $N_l u = z$, define

  $$\mathcal{L}_r z = \begin{bmatrix} 0 \\ u \end{bmatrix}. \tag{3.11}$$

  By substituting $\mathcal{L}_r$ in the left hand term of (3.10), we observe that the Bezout identity holds:

  $$\begin{bmatrix} D_l & N_l \end{bmatrix} \mathcal{L}_r z = \begin{bmatrix} D_l & N_l \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} = N_l u = z. \tag{3.12}$$

- If $z \in \mathcal{Z}^s$, $u \in \mathcal{U}^u$ and $y \in \mathcal{Y}^s$ with $D_l\, y = z$, define

$$\mathcal{L}_r\, z = \begin{bmatrix} y \\ 0 \end{bmatrix}. \tag{3.13}$$

By substituting $\mathcal{L}_r$ in the left hand term of (3.10), we observe again that the Bezout identity holds:

$$\begin{bmatrix} D_l & N_l \end{bmatrix} \mathcal{L}_r\, z = \begin{bmatrix} D_l & N_l \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = D_l\, y = z. \tag{3.14}$$

- If $z \in \mathcal{Z}^u$, define

$$\mathcal{L}_r\, z = \begin{bmatrix} D_l^{-1} z \\ 0 \end{bmatrix} \tag{3.15}$$

and substitution into the left hand term of (3.10) yields

$$\begin{bmatrix} D_l & N_l \end{bmatrix} \mathcal{L}_r\, z = \begin{bmatrix} D_l & N_l \end{bmatrix} \begin{bmatrix} D_l^{-1} z \\ 0 \end{bmatrix} = z. \tag{3.16}$$

By collating these results, we obtain a discontinuous construction of $\mathcal{L}_r$ which shows that the definition of Set Theoretic coprimeness implies the Bezout definition of coprimeness. •

**Remark 3.1** The construction is motivated by ideas of both [2] and [31]. The question whether the reverse result holds remains open.

**Remark 3.2** Notice also that the proof of Lemma 3.2 is constructive and the particular pair $U_r$, $V_r$ where $\mathcal{L}_r = [U_r\ V_r]^{-1}$ is constructed so that either $U_r z$ or $V_r z$ is zero for many $z$; in this sense, the construction is quite unlike any construction used in the linear case.

Next, we can address the question of the relationship between left coprime realisations of a given operator using the Set Theoretic definition of coprimeness.

**Lemma 3.3** *Suppose $P = D_{l1}^{-1} N_{l1} = D_{l2}^{-1} N_{l2}$ where $(D_{l1}, N_{l1})$ is coprime in the Set Theoretic sense. Then $(D_{l2}, N_{l2})$ is coprime in the Set Theoretic sense if there exists a unit $W$ with*

$$N_{l2} = W N_{l1}, \ D_{l2} = W D_{l1}. \tag{3.17}$$

*The only if part does not hold, i.e. $(D_{l2}, N_{l2})$ coprime with (3.17) holding does not necessarily imply that $W$ is a unit.*

**Proof.** Refer to Figure 3.1 while reading the following proof.

Define $\gamma_u^u$ as the set of unbounded images of $P$ that have unbounded pre-images. Similarly, $\gamma_s^u$ is defined as the set of unbounded images of $P$ that have bounded pre-images.

Now, we have $y = D_{l1}^{-1} N_{l1} u$, and the partial state $z_1 = D_{l1} y = N_{l1} u$ where $(N_{l1}, D_{l1})$ are coprime in a Set Theoretic sense. If $y \in \gamma_u^u$, then $u \in \mathcal{U}^u$ and by the coprimeness definitions, $z \in \mathcal{Z}^u$. Hence $D_{l1}(\gamma_u^u) \cap \mathcal{Z}_1^s = \emptyset$. Since $W$ is a unit operator, $W(D_{l1}(\gamma_u^u))$ only contains unbounded signals. This in turn implies that $D_{l2}(\gamma_u^u) \cap \mathcal{Z}_2^s = \emptyset$, i.e. $(N_{l2}, D_{l2})$ are coprime in a Set Theoretic sense.

To examine why the "only if" part does not hold suppose that both $(N_{l1}, D_{l1})$ and $(N_{l2}, D_{l2})$ are coprime realisations of $P$. That is, by the Set Theoretic definition of coprimeness we have that $D_{l1}(\gamma_u^u) \cap \mathcal{Z}_1^s = \emptyset$ and $D_{l2}(\gamma_u^u) \cap \mathcal{Z}_2^s = \emptyset$. The operator $W$ in (3.17) is defined by

$$W = D_{l2} D_{l1}^{-1}. \tag{3.18}$$

Note that $W$ is invertible by invertibility of both $D_{l1}$ and $D_{l2}$. Let us define the following sets

$$\begin{aligned} \eta_u &= \mathcal{Y}^u \backslash (\gamma_u^u \cup \gamma_s^u), && (3.19)\\ \pi_1 &= D_{l1}(\eta_u) \cap \mathcal{Z}_1^s\,, && (3.20)\\ \pi_2 &= D_{l2}(\eta_u) \cap \mathcal{Z}_2^s\,. && (3.21) \end{aligned}$$

The stable set $\pi_1$ is not necessarily mapped into the stable set $\pi_2$. It may be partly

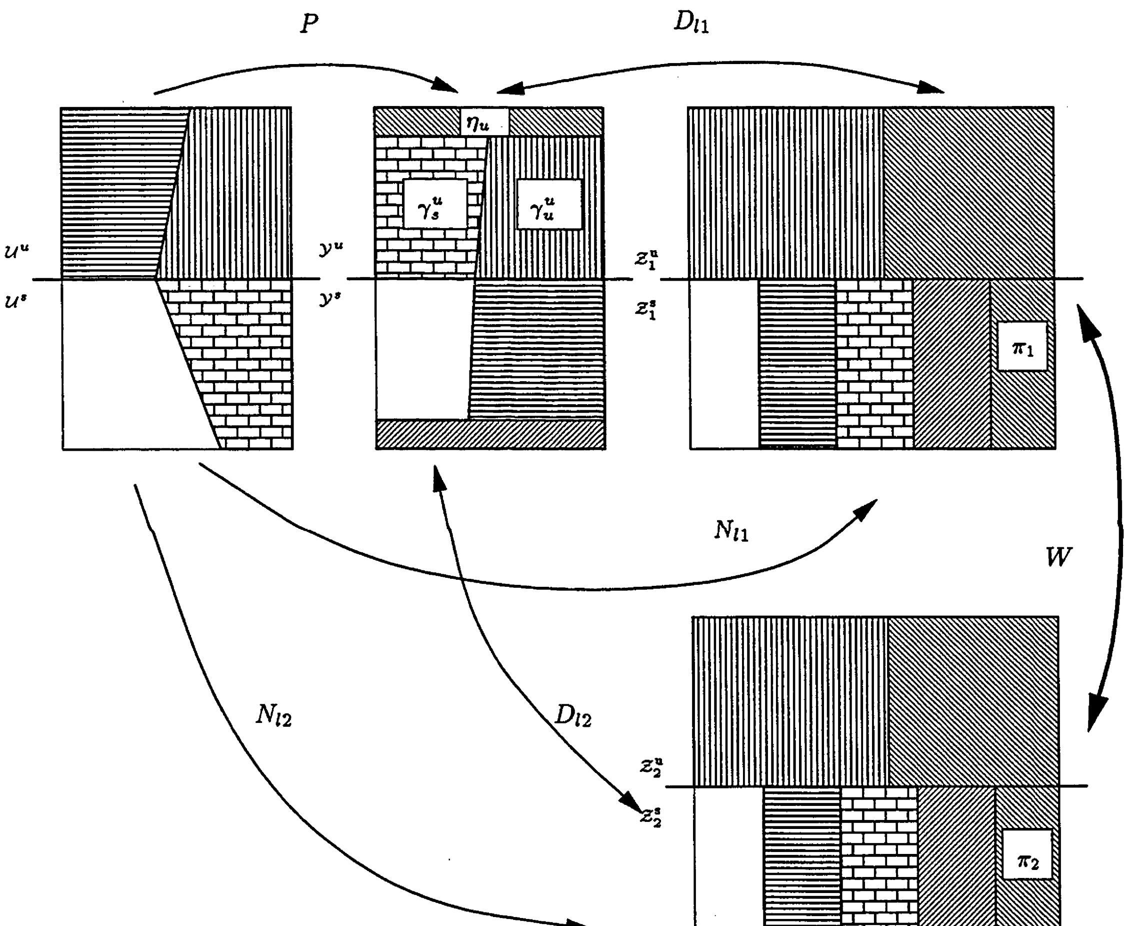



*Figure 3.1: Illustration of the proof of Lemma 3.3*

mapped into the unstable set $\mathcal{Z}_2^u$ . We conclude that $W$ is not necessarily a BIBO operator. Similarly, $W^{-1}$ is not necessarily a BIBO operator. This clearly shows that $W$ above is not necessarily a unit operator and that the only if part does not hold. •

The above lemma is easy to establish. By contrast, the following result is comparatively difficult to establish.

**Lemma 3.4** *Suppose a well-posed nonlinear operator $P$ has fractional realisations $P = D_{l1}^{-1}\,N_{l1} = D_{l2}^{-1}\,N_{l2}$ and suppose further there exist well-posed stable operators $U_r$ and $V_r$ such that*

$$D_{l1}V_r + N_{l1}U_r = X, \tag{3.22}$$

*for a unit $X$. Define $W$ by*

$$\begin{aligned} N_{l2} &= W\,N_{l1}, \\ D_{l2} &= W\,D_{l1}. \end{aligned} \tag{3.23}$$

*Suppose that $V_r$ is invertible (strict properness of $P$ is effectively sufficient for this), that $W$ is a unit, and that $D_{l1}V_r$, $W$ and $W^{-1}$ are Lipschitz continuous. Then ($D_{l2}$,$N_{l2}$) is coprime in the Bezout sense.*

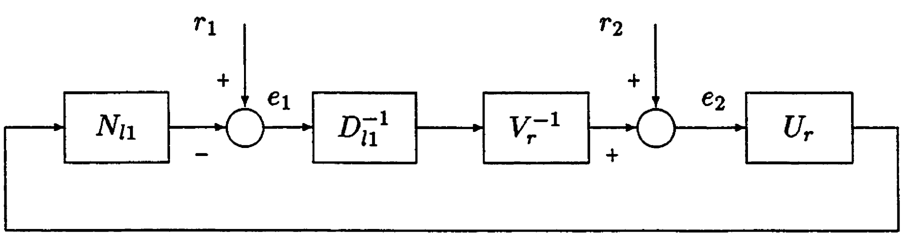



*Figure 3.2: Illustration of Lemma 3.4*

**Proof.** Consider the loop in Figure 3.2.

• **Step 1** We shall show that satisfaction of a Bezout identity involving $N_{l1}$, $D_{l1}$ is necessary and sufficient for the loop to exhibit BIBO behaviour. Suppose first that (3.22) holds. Then

$$e_1 = r_1 - N_{l1}U_r\,e_2 \tag{3.24}$$

$$e_2 = r_2 + V_r^{-1}D_{l1}^{-1}\,e_1 \tag{3.25}$$

or

$$D_{l1}V_r[e_2 - r_2] = e_1 \tag{3.26}$$

Hence

$$r_1 - N_{l1}U_r\, e_2 = D_{l1}V_r[e_2 - r_2]. \tag{3.27}$$

By the Lipschitz continuity of $D_{l1}V_r$, there holds

$$D_{l1}V_r[e_2 - r_2] = D_{l1}V_r\, e_2 + \partial(D_{l1}V_r)_{(e2)}(-r_2). \tag{3.28}$$

It follows from Chapter 2 that $\partial(D_{l1}V_r)_{(e2)}(.)$ is bounded independently from $e_2$. Then from (3.27) and (3.28) we have

$$(D_{l1}V_r + N_{l1}U_r)\, e_2 = r_1 - \partial(D_{l1}V_r)_{(e2)}(-r_2). \tag{3.29}$$

or

$$e_2 = X^{-1}[r_1 - \partial(D_{l1}V_r)_{(e2)}(-r_2)] \tag{3.30}$$

Thus bounded $r_1$, $r_2$ leads to bounded $e_2$ (and by (3.24) also bounded $e_1$).

Conversely, suppose BIBO behaviour is assumed. Take $r_2 = 0$. Then the above calculations show

$$(D_{l1}V_r + N_{l1}U_r)e_2 = r_1 \tag{3.31}$$

(without actually invoking a Lipschitz condition). Since $r_1$ is arbitrary and bounded and $e_2$ is bounded by hypothesis, $(D_{l1}V_r + N_{l1}U_r)^{-1}$ is a BIBO operator.

Now consider the new set-up of Figure 3.3 where $W$ is a unit operator; in effect, $N_{l1}$ and $D_{l1}$ of Figure 3.2 have been replaced by $N_{l1}W$ and $D_{l1}W$ in Figure 3.3.

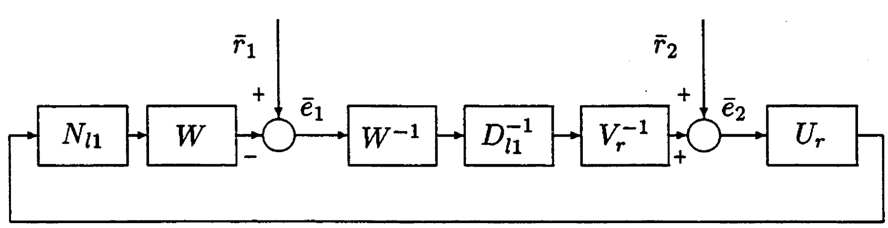



*Figure 3.3: Illustration of Lemma 3.4*

• **Step 2.** We shall show that if the loop of Figure 3.2 is BIBO, then so is that of Figure 3.3 and conversely. To begin, suppose the loop of Figure 3.3 is BIBO, and let $r_1$, $r_2$ be two bounded inputs for Figure 3.2. Define inputs $\bar{r}_1$, $\bar{r}_2$ for the scheme of Figure 3.3 by

$$\begin{aligned} \bar{r}_1 &= W[r_1 - N_{l1}U_r\, e_2] + WN_{l1}U_r\, e_2, \\ \bar{r}_2 &= r_2. \end{aligned} \tag{3.32}$$

Since $W$ is Lipschitz continuous, $\bar{r}_1$ is bounded, irrespective of $e_2$. Now for the loop of Figure 3.3, we have

$$\bar{e}_1 = \bar{r}_1 - WN_{l1}U_r\, \bar{e}_2 = WD_{l1}V_r[\bar{e}_2 - \bar{r}_2]. \tag{3.33}$$

Now use the expression for $\bar{r}_1$, $\bar{r}_2$ of (3.32):

$$W[r_1 - N_{l1}U_r\, e_2] + WN_{l1}U_r\, e_2 - WN_{l1}U_r\, \bar{e}_2 = WD_{l1}V_r[\bar{e}_2 - r_2], \tag{3.34}$$

Compare this with the following consequence of (3.27):

$$W[r_1 - N_{l1}U_r\, e_2] = WD_{l1}V_r[e_2 - r_2]. \tag{3.35}$$

Evidently, $\bar{e}_2 = e_2$ is a solution of (3.34), and by uniqueness, it is the only solution. To summarise, if $r_1$ and $r_2$ are inputs to Figure 3.2, and $\bar{r}_1$, $\bar{r}_2$ are inputs to Figure 3.3 generated using (3.32), there results $e_2 = \bar{e}_2$. Since $r_1$, $r_2$ bounded imply $\bar{r}_1$, $\bar{r}_2$ bounded (as already observed), which implies $\bar{e}_2$ bounded (by hypothesis that Figure 3.3 is BIBO), we have $e_2$ bounded in Figure 3.2, and then $e_1$ bounded i.e. Figure 3.2 is BIBO.

The converse follows by interchanging the roles of $W$ and $W^{-1}$, $r_1$ and $\bar{r}_1$ etc.

• **Step 3.** The proof is completed as follows:

$D_{l1}V_r + N_{l1}U_r = X$ with $X$ a unit operator

$\Leftrightarrow$ the loop of Figure 3.2 is BIBO stable (Step 1),

$\Leftrightarrow$ the loop of Figure 3.3 is BIBO stable (Step 2),

$\Leftrightarrow$ $WD_{l1}V_r + WN_{l1}U_r$ is stably invertible (Step 1 again),

$\Rightarrow (D_{l2}, N_{l2}) = (WD_{l1}, WN_{l1})$ is left coprime. •

**Remark 3.3** The question of whether an "only if" result holds remains open.

**Remark 3.4** There is an apparently simple but actually erroneous approach to proving Lemma 3.4. Suppose $U_r$ and $V_r$ are BIBO operators such that $D_{l1}\,V_r + N_{l1}\,U_r = X$ where $X$ is a unit. It is *not* true that this equation implies $W\,D_{l1}\,V_r + W\,N_{l1}\,U_r = W\,X$ since $W$ is nonlinear. Of course, if this equation were true, it would be an immediate consequence that $(D_{l2} = W\,D_{l1}, N_{l2} = W\,N_{l1})$ is coprime in the Bezout sense, as $WX$ will be a unit when $W$ and $X$ are separately units.

**Remark 3.5** The last comment concerns the incompleteness of yet another result. Suppose $P = D_{l2}^{-1}\,N_{l2}$ with $(D_{l2}, N_{l2})$ coprime in the *Set Theoretic* sense. It is obvious that for any BIBO $W$ with $W^{-1}$ existing (but not necessarily BIBO), $P = D_{l3}^{-1}\,N_{l3}$ where $N_{l3} = W\,N_{l2}, D_{l3} = W\,D_{l2}$. However, it is not true that any realisation gives rise to a $W$ that is necessarily BIBO[1].

In contrast, if $P = D_{l2}^{-1}N_{l2}$ with $(D_{l2}, N_{l2})$ coprime in the *Bezout* sense, then any BIBO $W$ with $W^{-1}$ existing defines another realisation $P = D_{l3}^{-1}N_{l3}$ with $N_{l3} = WN_{l2}$, $D_{l3} = WD_{l2}$. Whether any realisation $P = D_{l3}^{-1}N_{l3}$ implies that $W = D_{l3}D_{l2}^{-1}$ is well-posed and stable is unknown. (In contrast to the Set Theoretic situation, an example with a non-BIBO $W$ is lacking.)

Despite the fact that under the Set Theoretic definition of coprimeness an arbitrary fractional representation of $P$ cannot be related via a BIBO $W$ to a coprime representation that is given *a priori*, we can make the following statement, which is implicit in [31]:

[1]The proof of Lemma 3.3 can be easily varied to establish this claim.

**Lemma 3.5** *Suppose $P = D_{l3}^{-1} N_{l3}$ is a left fractional realisation. Then there exists a left fractional realisation $P = D_{l2}^{-1} N_{l2}$ which is coprime in the Set Theoretic sense, with*

$$N_{l3} = W\, N_{l2}, D_{l3} = W\, D_{l2} \tag{3.36}$$

*where $W$ is BIBO.*

**Proof.** The authors of [31] show how to construct a $W$ such that $P = D_{l2}^{-1} N_{l2}$ is left coprime. Although it is not stated in the proof of [31], it is fairly straightforward to see that this $W$ is also BIBO stable by construction. •

## 3.4 Concluding remarks

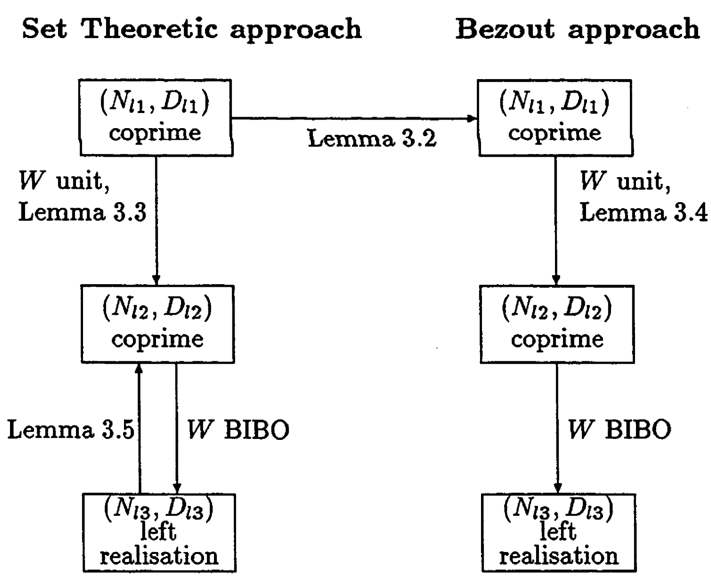



*Figure 3.4: Connections between the "Set Theoretic" and the "Bezout" approaches to left coprimeness.*

Observe Figure 3.4 for a collation of the results presented in this chapter. They show the relationship between the Bezout and Set Theoretic definitions of coprimeness for nonlinear operators. It also summarises how different factorisations of a given operator are related.

# Chapter 4

# The Hansen Method: Part 1

This chapter is concerned with the identification of a nonlinear plant, operating in a closed-loop with a stabilising controller. A Youla-Kucera parametrisation is used to parametrise the unknown plant, and the closed-loop identification is reduced to a conventional open-loop problem. The identification examined in this chapter extends the results of [5] and is referred to as the Hansen method.

Given a *nominal* nonlinear model of a plant $P_0$, and a linear controller $C$ stabilising the nominal model and the true plant $P$, measurements on the closed-loop system involving controller and plant can be used to identify the plant. In this chapter we examine the Hansen method for converting the closed-loop identification problem into one of open-loop identification using *right* coprime factorisations. In Chapter 5 we examine the same problem using *left* coprime factorisations.

Recall the difficulties of closed-loop identification for linear plants discussed in Chapter 1. Section 4.1 serves to put the ideas in context by discussing the resolution of the linear problem by Hansen *et al* in [15, 16]. Section 4.2 contains background on an extension of the idea to nonlinear plants with linear nominal model and with linear controller, for full details see [5]. Section 4.3 discusses new results where the nominal plant model is also allowed to be nonlinear. This problem is considered with and without noise and also in a high SNR situation.

## 4.1 Background: Linear Systems

Consider the arrangement of Figure 1.1 except that the plant and controller are linear. The contribution of [15, 16] serves to replace the closed-loop identification problem by a conventional open-loop identification problem; the tool is to use a Youla-Kucera parameter to describe the plant. This background section contains a review of the multivariable version.

In particular, let $C = U_r V_r^{-1}$ denote a right coprime realisation of the controller; thus $U_r$, $V_r$ are matrices with entries which are stable, proper transfer functions (we shall say $U_r, V_r \in \mathbf{M}(\mathbf{S})$). As $(U_r, V_r)$ are coprime there exist $D_l$, $N_l$ for which the following Bezout identity holds

$$D_l V_r + N_l U_r = I. \tag{4.1}$$

It is well known that one can also find $U_l$, $V_l$, $N_r$, $D_r \in \mathbf{M}(\mathbf{S})$ such that

$$\begin{bmatrix} V_l & U_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -U_r \\ N_r & V_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.2}$$

This double Bezout identity implies that the nominal plant given by

$$P_0 \;=\; D_l^{-1} N_l = N_r D_r^{-1}, \tag{4.3}$$

is stabilised by

$$C \;=\; U_r V_r^{-1} = V_l^{-1} U_l. \tag{4.4}$$

If $C$ and $P_0$ are both prescribed with $C$ stabilising $P_0$, it remains possible to choose fractional representations satisfying (4.2).

Suppose $C$ also stabilises the true plant; then for some $R \in \mathbf{M}(\mathbf{S})$, the true plant has a fractional representation $(N_r + V_r R)(D_r - U_r R)^{-1}$. Figure 4.1 shows the arrangement

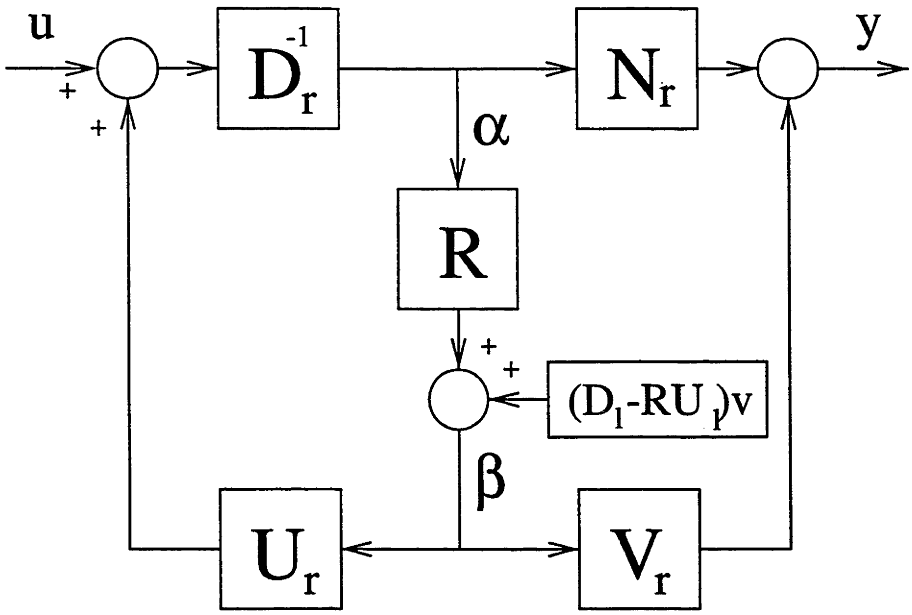



*Figure 4.1: Linear plant representation using controller, nominal model of plant and Youla-Kucera parameter*

of the plant. It can be shown from this figure and (4.2) that

$$y = (N_r + V_r R)(D_r - U_r R)^{-1} u + v.$$

Now it turns out that when the plant is connected within a closed-loop as in Figure 1.1, one can show that the quantities $\alpha$ and $\beta$ in Figure 4.1 are given by

$$\begin{aligned} \alpha &= U_l r_1 + V_l r_2, \\ \beta &= D_l y - N_l u. \end{aligned} \tag{4.5}$$

This means that $\alpha$ and $\beta$ are computable from measured quantities, and $\alpha$ is independent of $v$. Since, as Figure 4.1 implies,

$$\beta = R\alpha + (D_l - RU_l)v, \tag{4.6}$$

with $R$ known to be stable, the identification of $R$ using $\alpha$ and $\beta$ is a standard open-loop identification problem.

## 4.2 Background: Nonlinearity in the Youla-Kucera Parameter

In [5], much of the above linear thinking is carried across to the nonlinear plant case. The nonlinearity enters the plant only via a nonlinear Youla-Kucera parameter.

A linear controller $C$ and linear nominal model $P_0$ for the plant are assumed. This means the Bezout identity (4.2) still holds. Provided the controller $C$ stabilises the true plant $P$, it is known that a Youla-Kucera parameter $R$ exists, which is a stable, nonlinear causal operator. Using $R$, the plant $P$ can be represented as

$$P = (N_r + V_r R)(D_r - U_r R)^{-1}. \tag{4.7}$$

The introduction of noise is less straightforward. The replacement of Figure 4.1 is depicted in Figure 4.2. (The two figures are equivalent if $R$ is linear, but not in general if $R$ is nonlinear).

As shown in the next chapter the use of $N_r, D_r, U_r, V_r$ in the above analysis and Figure 4.1 - all associated with right coprime realisations of linear objects - can be replaced by use of $N_l, D_l, U_l, V_l$ with modest amendments to the equations and the diagram. However, the use of left coprime quantities $D_l$ and $U_l$ associated with the noise cannot be replaced by use of right coprime quantities.

## 4.3 Identification of Nonlinear Plants under Linear Control

In this section, we shall retain the linear controller, but we shall allow the *nominal* plant model to be nonlinear. *The difficult issues to resolve are those associated with the appearance of quantities associated with left coprime realisations - in particular in (4.5) and (4.6).* As it turns out, even the scheme of Figure 4.2 has to be abandoned. We start by formulating our assumptions.

**Assumption 4.1** *There exist $U_r, V_r, U_l, V_l$ all stable, well-posed operators with $V_l, V_r$*

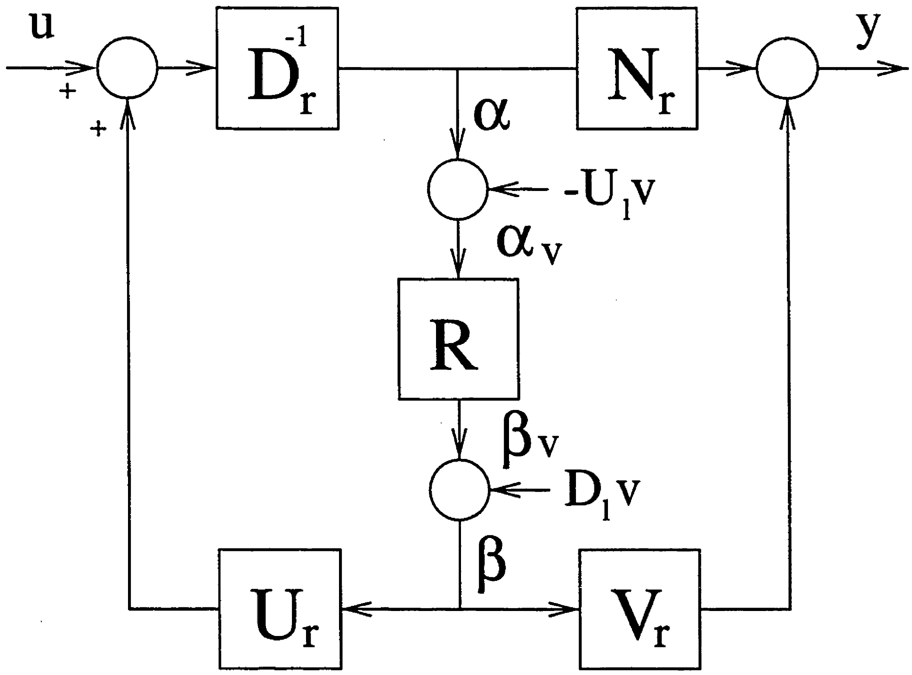



*Figure 4.2: Nonlinear plant representation, using a linear controller, linear nominal model of plant and nonlinear Youla-Kucera parameter*

*both invertible such that the controller is given by*

$$C = U_r V_r^{-1} = V_l^{-1} U_l, \tag{4.8}$$

*where* $(U_r, V_r)$ *are right coprime factors of the controller, and* $(U_l, V_l)$ *are left coprime factors of the controller.*

*The nominal plant model is smoothing. Further, there exist stable well-posed operators* $N_r, D_r$ *with* $D_r$ *invertible, such that*

$$P_0 = N_r D_r^{-1} \tag{4.9}$$

*where* $(N_r, D_r)$ *are right coprime factors of the nominal plant model.*

*The controller* $C$ *is known to stabilise the unknown nonlinear plant* $P$ *and the nonlinear nominal model* $P_0$.

### 4.3.1 Stability and Operator Existence

As the pair $(N_r, D_r)$ is coprime there exists a bounded operator $\mathcal{L}_l$ such that

$$\mathcal{L}_l \begin{bmatrix} D_r \\ N_r \end{bmatrix} = I. \tag{4.10}$$

From Figure 4.3 we can write

$$\begin{bmatrix} D_r & -U_r \\ N_r & V_r \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} \tag{4.11}$$

As $C$ is weakly Lipschitz and $P_0$ is smoothing it follows that the closed-loop $(P_0, C)$ is well-posed; see Remark 2.2 for further details. Further, as the closed-loop operator has finite gain the closed-loop system is internally stable.

Thus from (2.1) in Chapter 2, $\begin{bmatrix} D_r & -U_r \\ N_r & V_r \end{bmatrix}^{-1}$ exists and is bounded.

We begin by obtaining a Bezout identity, but not a double one. This is a simplified version of the nonlinear double Bezout identity in Chapter 2. The simplification occurs because only one set of operators is nonlinear.

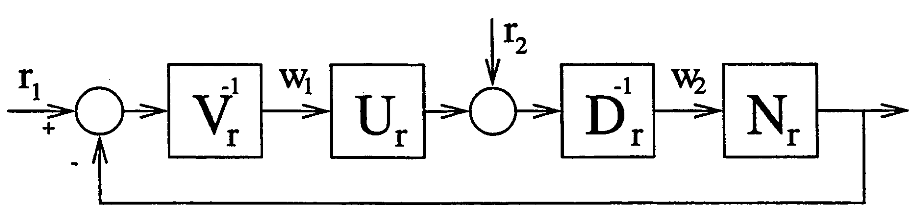



*Figure 4.3: Nominal nonlinear model and linear controller forming a stable closed-loop*

**Lemma 4.1** *Adopt the notation and assumptions above. Then (with abuse of notation regarding $U_l, V_l$ as operators), $W = V_l D_r + U_l N_r$ has a bounded inverse. Further, if $N_r$ and $D_r$ are replaced by $N_r W^{-1}$ and $D_r W^{-1}$ to form a new fractional description of $P_0$ there holds (after this replacement)*

$$V_l D_r + U_l N_r = I. \tag{4.12}$$

**Remark 4.1** One cannot replace $U_l, V_l$ by $W^{-1}U_l, W^{-1}V_l$ without sacrificing the linearity of the operators.

**Proof.** Recalling that $U_l, V_l$ are both linear, apply the operator $[V_l\ U_l]$ to both sides of (4.11). There results

$$(V_l D_r + U_l N_r)w_2 = V_l r_2 + U_l r_1. \tag{4.13}$$

Since $U_l, V_l$ is a coprime pair, there exists $X_r, Y_r \in \mathbf{S}$ such that

$$V_l X_r + U_l Y_r = I. \tag{4.14}$$

Take $r_1 = Y_r r$, $r_2 = X_r r$. Then (4.13) can be rewritten as

$$(V_l D_r + U_l N_r)w_2 = r. \tag{4.15}$$

Stability of the nominal plant model in closed-loop with the controller $C$ implies that for all $r_1, r_2 \in L_2$, $w_2 \in L_2$. Since $r \in L_2$ implies $r_1, r_2 \in L_2$, it follows that for all $r \in L_2$, $w_2 \in L_2$. Thus $(V_l D_r + U_l N_r)$ has a bounded inverse. The remainder of the lemma is trivial.

Let us therefore suppose that (4.12) holds. •

### 4.3.2 Describing the Set of all Plants stabilised by a given Controller

According to [39], there exists a bounded operator $R$ such that $P = (N_r + V_r R)(D_r - U_r R)^{-1}$. Thus neglecting the noise contribution in Figure 4.2, $P$ is expressible in terms of the fractional description of $C$ and $P_0$ via the arrangement of Figure 4.2.

Our immediate goal is to explain how to take noise into account.

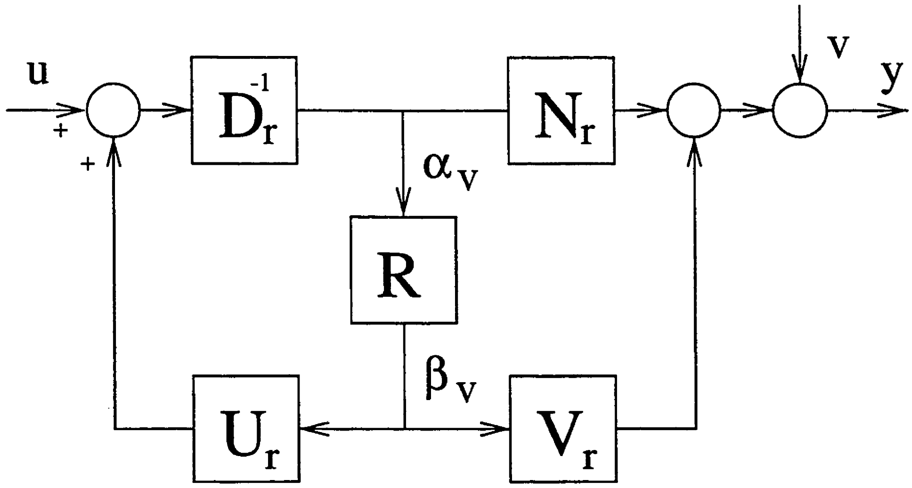



*Figure 4.4: Nonlinear plant represented using a priori data and a Youla-Kucera parameter*

### 4.3.3 Converting to Open-loop Identification

Figure 4.4 depicts the nonlinear plant $P$ using a Youla-Kucera parameter. Observe from Figure 4.4 that

$$\begin{bmatrix} D_r & -U_r \\ N_r & V_r \end{bmatrix} \begin{bmatrix} \alpha_v \\ \beta_v \end{bmatrix} = \begin{bmatrix} u \\ y-v \end{bmatrix}. \tag{4.16}$$

**Lemma 4.2** *Adopt Assumptions 4.1, with $U_l, V_l$ satisfying (4.12). Using the coprimeness of $U_r$ and $V_r$, define linear operators $X_l, Y_l \in$ S such that*

$$X_l V_r + Y_l U_r = I. \tag{4.17}$$

*Then (4.16) can be "solved" as*

$$\alpha_v = U_l r_1 + V_l r_2 - U_l v, \tag{4.18}$$

$$\beta_v = -Y_l u + X_l y + (Y_l D_r - X_l N_r)(U_l r_1 + V_l r_2 - U_l v) - X_l v. \tag{4.19}$$

**Proof.** Operate on the left of (4.16) with the linear operator $[V_l \; U_l]$. Since $V_l^{-1}U_l = U_r V_r^{-1}$ and as the Bezout identity of (4.12) holds, there follows

$$\alpha_v = V_l u + U_l(y-v). \tag{4.20}$$

The linear controller ensures that $u = r_2 + C(r_1 - y)$ or $V_l u + U_l y = U_l r_1 + V_l r_2$. Then (4.18) is immediate.

Next operate on the left of (4.16) by the linear operator $[-Y_l \ X_l]$, and use (4.17). There results

$$\beta_v - Y_l D_r \alpha_v + X_l N_r \alpha_v = -Y_l u + X_l (y - v),$$

or

$$\begin{aligned} \beta_v &= Y_l D_r (U_l r_1 + V_l r_2 - U_l v) \\ &\quad - X_l N_r (U_l r_1 + V_l r_2 - U_l v) - Y_l u + X_l y - X_l v. \end{aligned} \tag{4.21}$$

This is (4.19). •

Lemma 4.2 provides us with a nonstandard open-loop identification problem. For as Figure 4.4 shows, we have

$$\beta_v = R\alpha_v, \tag{4.22}$$

and $\alpha_v$ and $\beta_v$ are composed of measured signals ($r_1, r_2, u$ and $y$), contaminated by noise $v$. Moreover, the noise enters $\alpha_v$ (which is not standard), and enters $\beta_v$ nonlinearly (which is also not standard).

In the high SNR case, a more conventional problem can be obtained.

**Lemma 4.3** *Adopt Assumption 4.1 and suppose that $P$ is modeled as shown in Figure 4.4. Suppose further that $\Delta D_r, \Delta N_r$ and $\Delta R$ represent linearisations of the operators $D_r$, $N_r$ and $R$ around the operating trajectory defined by the input function $U_l r_1 + V_l r_2$. Then neglecting quantities of second order in $v$, there holds*

$$\beta = -Y_l u + X_l y + (Y_l D_r - X_l N_r)(U_l r_1 + V_l r_2) \tag{4.23}$$

$$\quad - (Y_l \Delta D_r - X_l \Delta N_r) U_l v - X_l v, \tag{4.24}$$

$$= R(U_l r_1 + V_l r_2) - \Delta R U_l v,$$

*or*

$$\beta = R\alpha + w, \tag{4.25}$$

*where*

$$\begin{aligned} \alpha &= U_l r_1 + V_l r_2, && (4.26) \\ \beta &= -Y_l u + X_l y + (Y_l D_r - X_l N_r)(U_l r_1 + V_l r_2), && (4.27) \\ w &= [(Y_l \Delta D_r - X_l \Delta N_r)U_l + X_l - \Delta R U_l]v. && (4.28) \end{aligned}$$

**Proof.** By direct calculation based on (4.18), (4.19) and (4.22). •

Notice that (4.26) and (4.28) guarantee $\alpha$ and $w$ are independent; (4.26) and (4.27) guarantee $\alpha$ and $\beta$ are measurable; and (4.25) apart from the nonlinearity of $R$, is an equation defining a standard nonlinear identification problem, (stable or bounded operator, and measurement noise independent of input).

**Remark 4.2** How does this result square up with that of [5]? In [5], it was the case that $P_0 = N_r D_r^{-1}$ was linear, and we could choose a left coprime realisation $D_l^{-1} N_l$ of $P_0$ with the additional property $D_l V_r + N_l U_r = I$ (this equation being part of the double Bezout identity). This means that in Lemma 4.2, we can replace $X_l, Y_l$ with $D_l, N_l$ and then $Y_l D_r - X_l N_r = 0$. This yields major simplifications, i.e.

$$\beta = R(\alpha - U_l v) + D_l v, \tag{4.29}$$

with

$$\begin{aligned} \alpha &= U_l r_1 + V_l r_2 \\ \beta &= D_l y - N_l u \end{aligned} \tag{4.30}$$

In the high SNR case, these equations become

$$\beta = R\alpha + (D_l - \Delta R U_l)v. \tag{4.31}$$

Thus the scheme of Figure 4.2 is indeed recovered.

**Remark 4.3** Note that this method requires both reference signals to be non-zero; see [5] for further details.

The next chapter examines how this identification problem is solved using *left* coprime factorisations.

# Chapter 5

# The Hansen Method: Part 2

In this chapter, using coprime factorisations of the plant and controller the identification of nonlinear time-varying (NLTV) plants operating under nonlinear, possibly time varying feedback is investigated. A model of the plant *based* on the left coprime factors of the nonlinear nominal plant and nonlinear controller, and a Youla-Kucera parameter is constructed. When we say that a plant has a "left coprime factorisation based description", we do not mean that it has a left coprime factorisation ($P = D_l^{-1}N_l$) as such. We mean we can write it in terms of the left coprime factors of the controller and nominal plant model. In the linear case this would reduce to give left coprime factors of the plant.

Identification of the plant is equivalent to identification of the Youla-Kucera parameter, and this observation allows the closed-loop identification problem to be converted to one of open-loop identification.

Section 5.1 contains assumptions that are used throughout the chapter; note that the notion of *differential coprimeness* that was introduced in Chapter 2 is used to help characterise the model set of the plant. Section 5.2 uses these definitions and preliminary results to construct a model of the plant. This is done using left fractional descriptions of both the controller and a nominal nonlinear plant stabilised by the controller, as well as a stable operator, a Youla-Kucera parameter. It covers both the noise free case and then modifies this description to incorporate non-zero measurement noise into the system. The section is concluded by depicting how this model can be used to identify the plant under a high SNR assumption. In general, identification requires $r_1$ and $r_2$ (see Figure 1.1) to be non zero. Section 5.3 contains a numerical illustration of the Hansen method using left coprime factorisations. Section 5.4 offers concluding remarks.

## 5.1 The Hansen method using left coprime factorisations

Consider the setting shown in Figure 1.1, where $P$ is a nonlinear plant to be identified, $C$ is a nonlinear controller, and $H$ is a linear stable output measurement noise generating system, driven in turn by the zero mean, white, stationary noise process $e$. It is assumed that $C$ internally stabilises the unknown plant $P$. While we restrict attention to time-invariant $C$ and $P$, there would seem to be no difficulty in extending the ideas to the time-varying case, as in [5].

In the results that follow in this chapter we will invoke the following assumptions.

**Assumption 5.1** *The nonlinear plant $P$ is weakly Lipschitz and well-posed.*

**Assumption 5.2** *(i) The controller $C$ is weakly Lipschitz and there exists $U_l$, $V_l$, $U_r$, $V_r$ all stable, well-posed operators with $V_l$, $V_r$ invertible such that*

$$C = U_r V_r^{-1} = V_l^{-1} U_l. \tag{5.1}$$

*$(U_r,\, V_r)$ are right coprime factors of the controller that are differentially coprime and globally Lipschitz continuous, and $(U_l,\, V_l)$ are left coprime factors of the controller that are globally Lipschitz continuous and* uniformly *differentially coprime.*

*(ii) The nominal plant model $P_0$ is smoothing, and there exists $N_l$, $D_l$, $N_r$, $D_r$ all stable, well-posed operators with $D_l$, $D_r$ invertible such that*

$$P_0 = N_r D_r^{-1} = D_l^{-1} N_l. \tag{5.2}$$

*$(N_r,\, D_r)$ are right coprime factors of $P_0$ that are differentially coprime and globally Lip-*

*schitz continuous, and $(N_l,\ D_l)$ are left coprime factors of $P_0$ that are globally Lipschitz continuous and* uniformly *differentially coprime.*

*It is further assumed that $N_l$ and $U_l$ are smoothing and $D_l$ and $V_l$ are of the form $aI + S$ where $aI$ is the scaled identity operator and $S$ is a smoothing operator.*

**Remark 5.1** The assumption that $N_l$ is smoothing will be fulfilled if there is at least one integration between the input of $P$ and its output. A system writable in the form of (2.1) with $j(x) = 0$ would have $N_l$ smoothing.

**Remark 5.2** Recall Remark 2.8 concerning the structure of $D_l$.

**Remark 5.3** Since $C$ is weakly Lipschitz and $P_0$ is smoothing it follows that the closed-loop $(P_0, C)$ is well-posed, see Remark 2.2 for further details.

**Assumption 5.3** *The controller $C$ stabilises the nominal plant model $P_0$.*

## 5.2 Characterisation and Identification of nonlinear plants using a left coprime factor based description

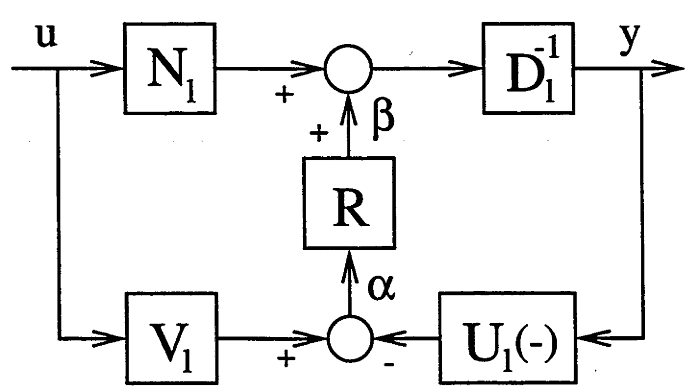



*Figure 5.1: Left coprime factorisation based description of P.*

This section will show that all nonlinear plants stabilised by a nonlinear controller $C$ can be represented by the setting depicted in Figure 5.1, with $R$ a nonlinear BIBO, smoothing, well-posed operator known as the Youla-Kucera parameter. Conversely, if the setup of Figure 5.1 defines a well-posed, smoothing $P$ for some BIBO, smoothing, well-posed $R$, then $P$ is stabilised by $C$.

As this description involves the left coprime factors of the controller, $C$, and the nominal plant, $P_0$, it will be referred to as a left coprime factorisation *based* description. Note that $P$ (as opposed to $P_0$) does not always have a left coprime factorisation due to the nonlinearity of the operators involved. From Figure 5.1, we can write

$$D_l y \;=\; N_l u + R(V_l u - U_l(-y)). \tag{5.3}$$

If $R$, $U_l$ and $N_l$ are linear this reduces to the left coprime factorisation

$$y \;=\; (D_l - RU_l)^{-1}(N_l + RV_l)u.$$

However, no such convenient representation may be written down when $R$, $U_l$ and $N_l$ are allowed to be nonlinear. So we turn to the structure described in Figure 5.1. The theorems of this section argue that this representation depicts the set of all plants stabilised by a given controller and hence shows how the closed-loop identification problem can be converted to an open-loop problem in the presence of noise. Section 5.2 is broken into two parts. The first part treats the noiseless situation, i.e. $v = 0$. The second part treats the case when the noise is no longer zero, and it describes how the disturbance can be incorporated into the identification algorithm.

### 5.2.1 Describing the structure of the set of all plants stabilised by a given controller in a noise free setting.

We will use some of the stability and operator existence results set out in Chapter 2 in proving the following lemma.

**Lemma 5.1** *Adopt the assumptions in Section 5.1. Suppose that $R$ is a well-posed, bounded operator. Then if $R$ is smoothing, $P$ is smoothing. Also the closed-loop of Figure 5.2 is well-posed and internally stable.*

**Proof.** Firstly, we want to show that if $R$ is smoothing, then $P$ is smoothing. Observe from Figure 5.1 that if $R$ is smoothing then, as $U_l$ and $V_l$ are weakly Lipschitz, the

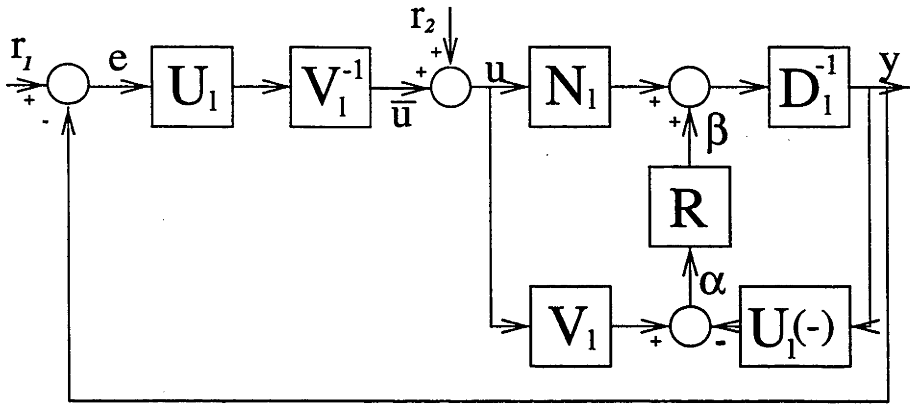



*Figure 5.2: Closed-loop diagram of Figure 1.1 with plant P as in Figure 5.1 and $w = 0$.*

operator $(u, y) \to R(V_l u - U_l(-y))$ is guaranteed to be smoothing; see Remark 2.2 for further details. Since $N_l$ is smoothing then the operator $(u, y) \to N_l u + R\alpha$ is smoothing; i.e. the operator $Z : [u\ y]^T \to z$ is smoothing and Figure 5.1 can be redrawn as in Figure 5.3. Since $D_l^{-1}$ has the form $(aI + S)$ where $S$ is smoothing, it

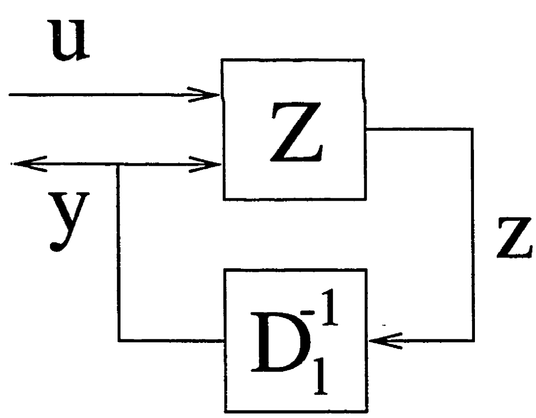



*Figure 5.3: Redrawn Figure 5.1*

follows from Remark 2.3 that $P$ defined in Figure 5.1 is a smoothing operator.

Next we wish to show that the closed-loop of Figure 5.2 is well-posed and internally stable. If we redraw Figure 1.1 as shown in Figure 5.4 then the closed-loop will be well-posed if $A_C$ is weakly Lipschitz and $P$ is smoothing; see Remark 2.2 for more details. We have just shown that $P$ is smoothing, thus we only need to show that $A_C$ is weakly Lipschitz. Note that

$$u = A_C(r_1, r_2, y) = r_2 + C(r_1 - y). \tag{5.4}$$

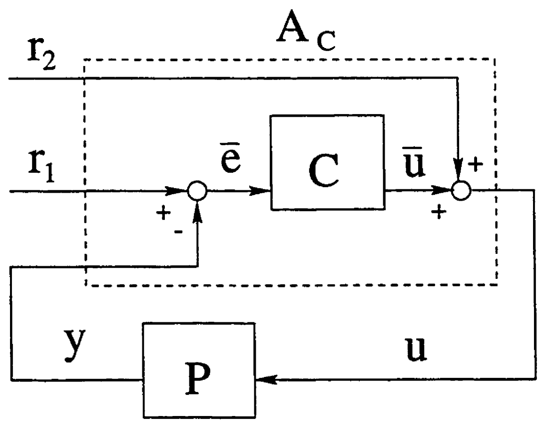



*Figure 5.4: Closed-loop system of Figure 1.1 redrawn as in Figure 2.1 with $e = 0$.*

Since $C$ is weakly Lipschitz it follows that $A_C$ is also weakly Lipschitz. Hence the closed-loop in Figure 5.2 is well-posed.

Finally we wish to show that the closed-loop is internally stable. We have already demonstrated that it is well-posed so we only need to show that the associated loop gain is finite.

Considering Figure 5.1 alone, it is evident that

$$\begin{bmatrix} \alpha \\ R\alpha \end{bmatrix} = \begin{bmatrix} V_l & -U_l(-) \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}. \tag{5.5}$$

Considering Figure 5.2, we see that

$$\begin{aligned} V_l u - U_l(-y) &= V_l(r_2 + \bar{u}) - U_l(-y) \\ &= V_l(\bar{u}) + \partial V_{l(\bar{u})}(r_2) - U_l(-y) \\ &= U_l(r_1 - y) + \partial V_{l(\bar{u})}(r_2) - U_l(-y) \\ &= \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2 \end{aligned} \tag{5.6}$$

i.e. in (5.5), $\alpha = \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2$ is bounded. Since $R$ is bounded, $R\alpha$ is bounded also. By Combination 2.4 of Section 2.5, stability of the closed-loop system $(P_0, C)$ ensures that $u$, $y$ exists and are bounded. Hence $\bar{u} = u - r_2$ and $\bar{e} = r_1 - y$ are also bounded and internal stability follows. •

It now remains to show the converse result, namely that for a stable $(P, C)$ interconnection, there exists an operator $R$ which is bounded.

**Lemma 5.2** *Adopt the assumptions in Section 5.1 and suppose the closed loop in Figure 1.1 is well-posed and internally stable. Then there exists a well-posed, bounded $R$ given by*

$$R = (D_l P - N_l)(V_l - U_l(-P))^{-1}, \tag{5.7}$$

*such that in Figure 5.1*

$$y = Pu.$$

*Further, if $P$ is smoothing then $R$ is smoothing.*

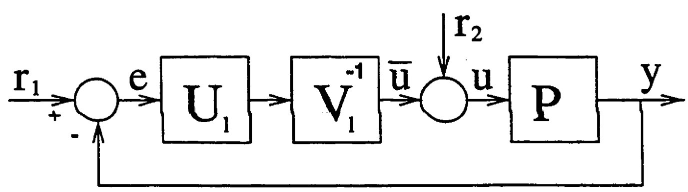



*Figure 5.5: Closed-loop system of Figure 1.1 with true plant $P$ and $w = 0$.*

**Proof.** To show that $R$ is a well-posed operator we first have to show that the operator $(V_l - U_l(-P))^{-1}$ exists. From Figure 5.5, we have

$$V_l(u - r_2) = U_l(r_1 - Pu)$$

whence

$$V_l u + \partial V_{l(u)}(-r_2) = U_l(-Pu) + \partial U_{l(-y)} r_1$$

which implies

$$[V_l - U_l(-P)]\, u = \partial U_{l(-y)} r_1 - \partial V_{l(u)}(-r_2).$$

From Combination 2.2, $[V_l D_r - U_l(-N_r)]^{-1}$ exists. Now

$$[V_l D_r - U_l(-N_r)] = [V_l - U_l(-P)] D_r \tag{5.8}$$

or

$$[V_l D_r - U_l(-N_r)]D_r^{-1} = [V_l - U_l(-P)] \tag{5.9}$$

Hence, $(V_l - U_l(-P))^{-1}$ exists. Thus in (5.7), for each $\alpha \in L_{2e}[0,\infty)$, there exists a unique $R\alpha$ that depends causally on $\alpha$. Further, as $N_l$, $D_l$, $U_l$, $V_l$ and $P$ are well-posed operators, $R$ is also well-posed.

Next, we must show that $R$ is bounded. With $r_1$, $r_2$ as above and from Figure 5.2, we have that $D_l y - N_l u$ is bounded since the closed-loop is internally stable. Further

$$\begin{aligned} D_l y - N_l u &= (D_l P - N_l)u && (5.10)\\ &= (D_l P - N_l)(V_l - U_l(-P))^{-1}\alpha && (5.11)\\ &= R\alpha. && (5.12) \end{aligned}$$

From (5.6) we have

$$\alpha = \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2 \tag{5.13}$$

As $U_l$ and $V_l$ are left uniformly differentially coprime there exist $M_r$, $N_r$ such that

$$\partial U_{l(-y)} M_r + \partial V_{l(\bar{u})} N_r = W_{y\bar{u}} \tag{5.14}$$

where $W_{y\bar{u}}$ is a unit. Thus choosing $r_1 = M_r r$ and $r_2 = N_r r$, (5.13) becomes

$$\alpha = W_{y\bar{u}} r$$

Thus from $r_1$ and $r_2$ we can make arbitrary $\alpha \in L_2$, hence $R\alpha \in L_2$ also, i.e. $R$ is bounded.

It remains to show that if the $R$ defined by (5.7) is inserted in Figure 5.2, the plant $P$ is obtained. To this end, observe from (5.7) that

$$(D_l P - N_l)u = R(V_l - U_l(-P))u$$

or

$$D_l P u = N_l u + R(V_l u - U_l(-P)u). \tag{5.15}$$

In comparison, from Figure 5.2 we have the relationship

$$D_l y \quad = \quad N_l u + R(V_l u - U_l(-y)).$$

As the plant in Figure 5.2 is well-posed, each $u$ must give rise to a unique $y$. From (5.15), $Pu$, constitutes a possible output; hence by uniqueness $y = Pu$.

Lastly, we will show that in (5.7), if $P$ is smoothing, $R$ is smoothing, see Remark 2.2 for more details.

Since $D_l = aI + S$ with $S$ smoothing, $D_l P - N_l$ is smoothing. Also, the operator $[V_l - U_l(-P)]^{-1}$ can be constructed as shown in Figure 5.6. It follows from the assumptions

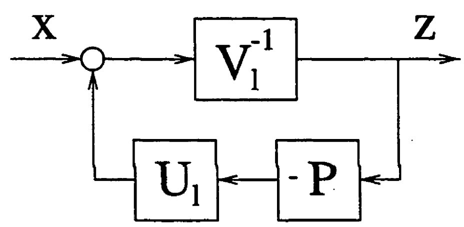



*Figure 5.6: The operator $[V_l - U_l(-P)]^{-1}$.*

and Item 5 of Remark 2.2 that the operator $[V_l - U_l(-P)]^{-1} : x \to z$ is well-posed, i.e. it follows from the definition of well-posedness that the operator $[V_l - U_l(-P)]^{-1} : x \to z$ is weakly Lipschitz.

Therefore it follows from Item 3 of Remark 2.2 that $R$ is smoothing. •

In summary, we now have

**Theorem 5.1** *Adopt the assumptions in Section 5.1. Then the closed-loop in Figure 1.1 is well-posed and internally stable if and only if $P$ has a description of the form of Figure 5.1, with $R$ a well-posed, stable, smoothing operator. Further $P$ is smoothing if and only if $R$ is smoothing.*

**Proof.** Lemmas 5.1 and 5.2 provide the proof for Theorem 5.1. •

### 5.2.2 Conversion to open-loop identification and incorporation of measurement noise

This section demonstrates how the measurement noise can be incorporated in order to enable identification. The conversion to open-loop identification requires a small noise assumption (high SNR) so that $R$ may be linearised around its operating trajectory. As in [5], it is shown that instead of identifying the plant $P$, we can identify the Youla-Kucera parameter, $R$.

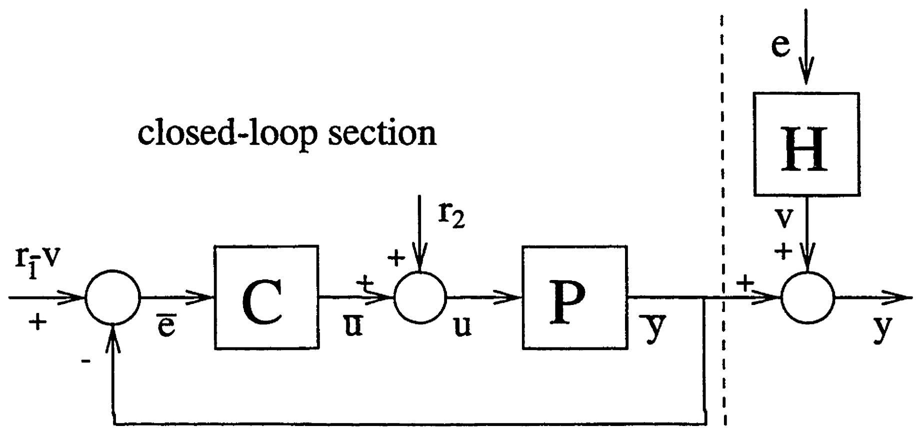



*Figure 5.7: Rearrangement of the closed-loop system of Figure 1.1.*

**The identification method**

The conversion to open-loop identification that is presented in this chapter is similar to the one presented in 4. Refer first to Figure 5.2.

In the noise free case, i.e. with $v = 0$, there holds

$$\begin{aligned} \alpha &= V_l u - U_l(-y) \\ &= \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2. \end{aligned} \tag{5.16}$$

Also, if $\beta = R\alpha$, then

$$\beta = -N_l u + D_l y. \tag{5.17}$$

Stability of the closed-loop ensures that $\alpha$ and $\beta$ are (in principle) computable (boundedly) from $r_1$, $r_2$ and $u$, $y$ respectively. It is now possible to identify $R$ in a standard open-loop fashion. The next paragraph examines how to take measurement noise into account.

**a) Incorporation of measurement noise**

When $v \neq 0$, similar equations hold provided we replace $r_1$ and $y$ by $r_1 - v$ and $y - v$ respectively in determining the input and output of $R$. This can be seen by examining Figure 5.7. Put another way, we now have

$$\beta_v = R\alpha_v$$

with

$$\begin{aligned}
\alpha_v &= V_l u - U_l(-y+v) \\
&= V_l(r_2 + \bar{u}) - U_l(-y+v) \\
&= V_l\bar{u} + \partial V_{l(\bar{u})} r_2 - U_l(-y) - \partial U_{l(-y)} v \\
&= U_l[(r_1 - y)] + \partial V_{l(\bar{u})} r_2 - U_l(-y) - \partial U_{l(-y)} v \\
&= \partial U_{l(-y)} r_1 + \partial V_{l(\bar{u})} r_2 - \partial U_{l(-y)} v \\
&= \alpha - \partial U_{l(-y)} v
\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}
\beta_v &= -N_l u + D_l(y - v) \\
&= -N_l u + D_l y + \partial D_{l(y)}(-v) \\
&= \beta + \partial D_{l(y)}(-v).
\end{aligned} \tag{5.19}$$

As before, $\alpha$ and $\beta$ are given by (5.16) and (5.17) and are effectively measurable. That is, the closed-loop identification problem has been transformed into a nonstandard open-loop identification problem. From $\beta_v = R\alpha_v$ we now obtain Figure 5.8.

As in [5] where the conversion process is considered for a nonlinear plant with a linear nominal plant model and a linear controller, the noise enters the structure in two places. This is opposed to the case where the plant, nominal plant model and controller

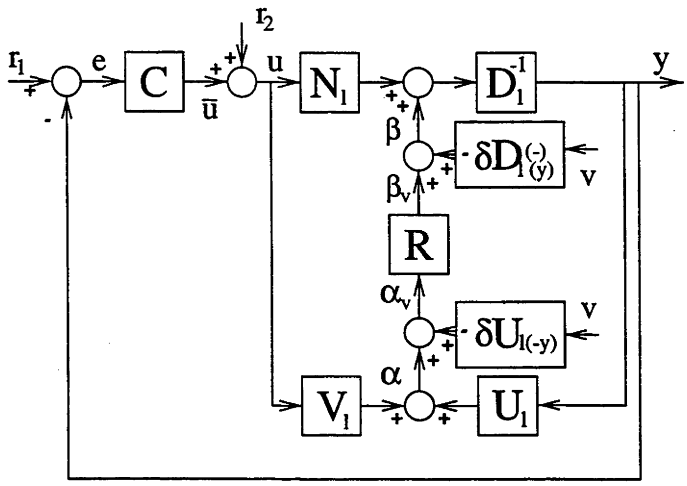



*Figure 5.8: Noise incorporated left coprime factorisation based description and conversion to a nonstandard open-loop identification problem.*

are all linear. In such a case the noise enters in only one place.

### b) Conversion to a standard open-loop identification problem.

Again assuming a high SNR, there exists a linearisation $\Delta R$ of $R$ around the trajectory produced by the input signal $\alpha$ which yields

$$\begin{aligned} \beta &= R\alpha + \Delta R(-\partial U_{l(-y)}v) - \partial D_{l(y)}(-v) \\ &= R\alpha - (\Delta R \partial U_{l(-y)} + \partial D_{l(y)}(-))v. \end{aligned}$$

This is of the form

$$\text{measured signal} = R(\text{known signal}) + \text{noise}.$$

The closed-loop identification problem has been transformed into a standard nonlinear open-loop identification problem as shown in Figure 5.9. This method requires both reference signals to be non-zero; see [5] for further details.

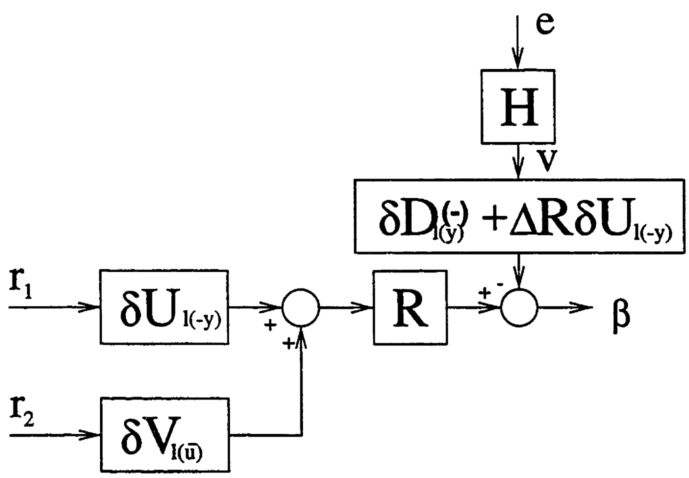



*Figure 5.9: Conversion to a standard open-loop identification problem.*

## 5.3 Simulations

This section contains a discussion of simulations performed using the Hansen method documented in this chapter. We have chosen to illustrate this method with a plant that has a nonlinear input backlash followed by linear dynamics. This simulation identifies a nonlinear plant ($P_{NL}$) connected in closed-loop with a stabilising controller ($C$). This controller also stabilises a *linear* nominal model of the plant ($P_0$). This simulation was implemented in discrete-time.

The nonlinear plant is described by

$$\begin{aligned} y_t &= P_{NL}u_t + v_t, \\ &= \frac{b}{z+a}\phi u_t + v_t. \end{aligned} \tag{5.20}$$

where the disturbance signal $v_t$ is a zero mean white noise signal of variance $\sigma^2$ and $\phi$ is a nonlinear backlash operator defined using the following equations.

$$p_t = BL(q_t) = \begin{cases} q_t - \frac{w}{2} & \text{if } q_t > setpoint_{t-1}, \\ q_t + \frac{w}{2} & \text{if } q_t < setpoint_{t-1} - width, \text{or} \\ p_{t-1} & \text{if } setpoint_{t-1} - width < q_t < setpoint_{t-1} \end{cases} \tag{5.21}$$

$$setpoint_t = \begin{cases} q_t & \text{if } q_t > setpoint_{t-1}, \\ q_t + w & \text{if } q_t < setpoint_{t-1} - width, \text{or} \\ setpoint_{t-1} & \text{if } setpoint_{t-1} - width < q_t < setpoint_{t-1} \end{cases} \tag{5.22}$$

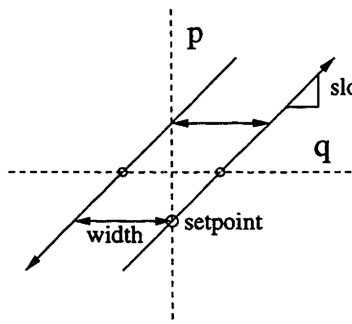



*Figure 5.10: Diagram relating relative movement of input and output*

Figure 5.10 demonstrates the relative movement of the input of the backlash, $q$, and the output of the backlash, $p$. Where

- width ($w$): is the length of the backlash deadzone,
- slope ($s$): is the slope of both curves, here s = 1; i.e. when the input is moving in contact with the output they have the same speed.
- setpoint: a backlash element has memory. The setpoint is chosen to be the point where the current position of the deadzone meets the curve on the right of the backlash.

Here we have taken the following plant parameters $a = 0.2$, $b = 0.5$ and $w = 0.1$.

The stabilising controller $C$ is the one degree of freedom linear controller

$$u_t = -y_t + r_t, \tag{5.23}$$

i.e. in Figure 1.1 of Chapter 1, $r_1 = \frac{r}{2}$, $r_2 = \frac{r}{2}$ and $C = I$. Thus, the left coprime factors of the controller are (without loss of generality), $V_l = I$ and $U_l = I$, i.e.

$$u_t = -V_l^{-1}U_l y_t + r_t. \tag{5.24}$$

The nominal plant model has a left coprime factorisation given by

$$P_0 = D_l^{-1} N_l \tag{5.25}$$

where

$$D_l = \frac{z+d}{z+f} \tag{5.26}$$
$$N_l = \frac{e}{z+f} \tag{5.27}$$

with $|f| < 1$. Here we have chosen $d = 0.1$, $e = 0.6$ and $f = 0.7$. This choice of $(N_l, D_l)$ satisfies the Bezout identity $N_l U_l + D_l V_l = I$ when $f = d + e$. This is a requirement of the Hansen method and implies that $P_0$ is stabilised by $C$.

Of course, $D_l$ and $N_l$ commute, so we have $P_0 = N_l D_l^{-1}$, and in a sense, there is no distinction between left and right fractions. However one cannot express $P_{NL}$ as $(N_p\phi)D_p^{-1}$, or $(\phi N_p)D_p^{-1}$, or something of this character; i.e. for the nonlinear system, the left fractional representation is important with $P_{NL} = D_p^{-1}(N_p\phi)$.

We construct the plant in terms of left coprime factors of the linear controller, the *linear* nominal plant model, and a nonlinear operator known as the Youla-Kucera parameter $(R)$. Identifying this parameter is equivalent to identifying the plant with the advantage that it can be written as an open-loop identification problem.

Recall from Figure 5.1, that one can calculate $\alpha$ and $\beta$ from the data collected on the real plant. These signals are of interest in the open-loop-like identification of the Youla-Kucera parameter. They are given by

$$\alpha = (U_l r_1 + V_l r_2), \tag{5.28}$$
$$\beta = -N_l u + D_l y. \tag{5.29}$$

where $\beta = R\alpha + \Delta R(-\partial U_{l(-y)} v) - \partial D_{l(y)}(-v)$ in a high SNR situation.

The reference signals $r_1 = r_2 = \frac{r}{2}$ were chosen to be known filtered unit variance and zero mean white noise signals independent of the process disturbance signal $v$. Note that this corresponds to an input signal $u$ that is of similar magnitude to the backlash width, $w$. With an input signal of much greater magnitude than $w$ this quantity would be hard to identify, the effect of the nonlinearity being swamped by the signal; if $u$ is of smaller magnitude there will also be a problem.

For identifying the Youla-Kucera parameter we have used the a model structure

$$\hat{\beta}(\theta) = \hat{R}(\theta)\alpha = \hat{R}(\hat{a}, \hat{b}, \hat{w})\alpha, \tag{5.30}$$

where

$$\hat{R}(\theta) = (D_l\hat{P}(\theta) - N_l)(V_l + U_l\hat{P}(\theta))^{-1}, \tag{5.31}$$

and

$$\hat{P}(\theta) = \frac{\hat{b}}{z + \hat{a}}\hat{\phi}(\hat{w}) \tag{5.32}$$

where $\hat{\phi}(\hat{w})$ is the backlash defined in (5.21) with $w$ replaced by $\hat{w}$. Here the parameter vector is

$$\theta = [\hat{a}, \hat{b}, \hat{w}]. \tag{5.33}$$

i.e. three parameters are identified. Note that in (5.31) the Youla-Kucera parameter is parametrised using the parameters of the plant model. Similar issues have been investigated in [7] in a linear context.

We now have $\beta$ from the actual system, and $\hat{\beta}(\theta)$ calculated from $\alpha$ and the latest estimate of the parameter vector $\theta$.

Parameter estimates are found by minimising the cost function given by

$$V_N(\theta) = \frac{1}{2N}\sum(\beta - \hat{\beta}(\theta))^2 \tag{5.34}$$

with respect to the parameter vector $\theta$ using a steepest descent method.

To minimise (5.34) with respect to the model parameter vector $\theta$, it is standard that one can iteratively seek a solution $\theta$ to

$$V_N'(\theta) = -\frac{1}{N}\sum(\beta - \hat{\beta}(\theta))\hat{\beta}'(\theta). \tag{5.35}$$

by taking steps in the negative gradient direction

$$\theta[i+1] = \theta[i] - \gamma_i R_i^{-1} V_N'(\theta). \tag{5.36}$$

It is worthwhile recalling that (5.36) is a batch mode type of adjustment. From (5.35), one can see that it is necessary to compute $\hat{\beta}'(\theta)$. We can rewrite (5.30) as

$$\begin{aligned} \hat{\beta}(\theta) &= (D_l\hat{P}(\theta) - N_l)\hat{z}(\theta), \\ \hat{z}(\theta) &= V_l^{-1}\alpha - C\hat{P}(\theta)\hat{z}(\theta). \end{aligned} \tag{5.37}$$

The $j$th component of the derivative $\hat{\beta}'(\theta)$ for $j = 1, 2, 3$ is given by

$$\hat{\beta}_{\theta j}'(\theta) = -N_l\hat{z}_{\theta j}'(\theta) + D_l(\hat{P}(\theta)\hat{z}_{\theta j}'(\theta) + \hat{P}_{\theta j}'(\theta)\hat{z}(\theta)), \tag{5.38}$$

$$\hat{z}_{\theta j}'(\theta) = -C(\hat{P}(\theta)\hat{z}_{\theta j}'(\theta) + \hat{P}_{\theta j}'(\theta)\hat{z}(\theta)). \tag{5.39}$$

where

$$\hat{P}_{\hat{a}}'(\theta) = \frac{-\hat{b}}{(z+\hat{a})^2}\hat{\phi}, \tag{5.40}$$

$$\hat{P}_{\hat{b}}'(\theta) = \frac{1}{z+\hat{a}}\hat{\phi}, \tag{5.41}$$

$$\hat{P}_{\hat{w}}'(\theta) = \frac{\hat{b}}{z+\hat{a}}\frac{\delta\hat{\phi}}{\delta\hat{w}}. \tag{5.42}$$

with

$$\frac{\delta\hat{\phi}}{\delta\hat{w}} = \begin{cases} -\frac{1}{2} & \text{if } q_t > setpoint_{t-1}, \\ \frac{1}{2} & \text{if } q_t < setpoint_{t-1} - width, \text{or} \\ 0 & \text{if } setpoint_{t-1} - width < q_t < setpoint_{t-1} \end{cases} \tag{5.43}$$

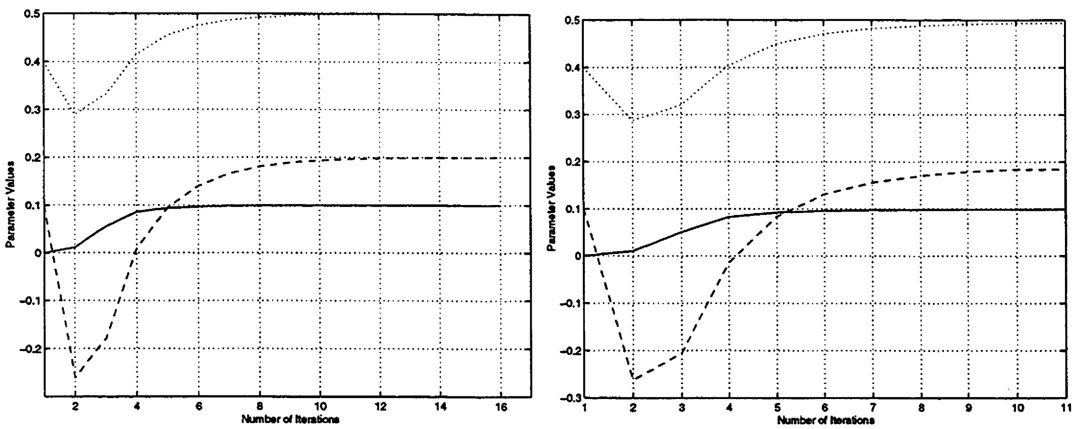



*Figure 5.11: Parameter Estimates versus Number of Iterations when $\sigma^2 = 0$ and $\sigma^2 = 0.000005$ respectively and $w = 0.1$, $a = 0.2$ and $b = 0.5$; $\hat{w}$ (—), $\hat{a}$ (−−), $\hat{b}$ $(\cdots)$.*

Note that there are some simplifications made in these calculations because of the linearity of the controller. Similar results also hold in a nonlinear setting.

Using the previous closed-loop system, we have generated a data set $\{r, u, y\}$ with signals of length $N = 2000$. We started with initial parameter estimates

$$\theta[0] = [\hat{a}[0]\ \hat{b}[0]\ \hat{w}[0]] = [0.1\ 0.4\ 0]. \tag{5.44}$$

When the simulation was run in a noise free situation the parameter estimates converged to the true values. Due to the type of nonlinearity implemented, the identification process was very sensitive to noise. Figure 5.11 shows the results when there is no noise ($\sigma^2 = 0$) and when the noise level is increased slightly ($\sigma^2 = 0.000005$). The accuracy of the final parameter estimates decreases with $\sigma^2$ increasing (SNR decreasing) as predicted earlier in the chapter.

## 5.4 Concluding Remarks

This chapter has considered the identification of a nonlinear plant operating in a closed-loop with a linear controller. Factorisation based structures have previously been derived to help convert the underlying closed-loop identification problem to one that is essentially that of open-loop identification. Many of the results are analogous to those in [1, 5, 21]. In particular, the requirement for high SNR is still present. This chapter extends the forays into this area by allowing the nonlinearity to enter the plant model through the nominal plant model and the controller as well as via the Youla-Kucera parameter.

# Chapter 6

# Two-Step and Coprime Factor Methods

This chapter examines two closed-loop identification methods referred to as the Two-Step method and Coprime Factor method. These methods have been investigated in the linear case, see [10, 34, 35]. This chapter extends the earlier work to encompass nonlinear systems. Recall the setting shown in Figure 1.1, where $P$ is a nonlinear plant to be identified, $C$ is an unknown nonlinear controller.

The idea behind the *Two-Step* method described in [34] is to identify one of the closed-loop operators through a high order linear model in an open-loop fashion and to use it in the second step to simulate a noise free input signal for an open-loop like identification of the plant $P$. This idea is only applicable for a stable plant operating in a stable closed-loop. Section 6.1 extends the *Two-Step identification method* by allowing the plant and controller to be nonlinear.

The *Coprime Factor* method described in [36] identifies a pair of *right* coprime factors of the plant in an open-loop fashion. Unlike the Two-Step method the plant may be unstable. Section 6.2 documents the nonlinear extension of the *Coprime Factor* method of [36].

Notice that both these methods as well as the Hansen scheme in Chapters 4 and 5 deal with noise entering the system under a high SNR assumption. We will also require that the output of the closed-loop system is a smooth function of both the reference signal and the disturbance signal. Without this assumption the analysis is much more involved.

## 6.1 Two-Step method for Nonlinear Systems.

This method first estimates the operator from the external inputs $[r_1\ r_2]^T$ to the input of the plant $u$. This gives rise to a noise free estimate of the plant input, $\hat{u}_r$. In the second step of this method, the operator from $\hat{u}_r$ to the plant output $y$ is estimated to obtain a model of the plant. We will make the following assumption.

**Assumption 6.1** *The closed-loop of Figure 1.1 is internally stable and the closed-loop operators are smooth functions of both reference signals and the disturbance signal.*

**Assumption 6.2** *The plant $P$ is well-posed and stable.*

**Assumption 6.3** *The data is collected under a high SNR assumption while the non-linear controller $C$ is operating.*

**Assumption 6.4** *The data includes one or both of the reference signals $r_1$ and $r_2$ that are measured in addition to $u$ and $y$.*

**Assumption 6.5** *We also have the following restrictions on $C$*

- *If $r_1$ is not available for measurement, $C$ must be a known stably invertible operator to recover $r_1 = y + C^{-1}(u - r_2)$.*
- *If $r_2$ is not measured, $C$ must be a known stable operator in order to reconstruct $r_2 = u - C(r_1 - y)$.*
- *When both $r_1$ and $r_2$ are measured there is no restriction on $C$.*

### 6.1.1 Step 1: Identification of the sensitivity function

In the general case when the reference signals $r_1$ and $r_2$ are non-zero, we have the following relation between the signals $u$, $r_1$, $r_2$ and $v$:

$$u = F_0(r_1, r_2, v) \tag{6.1}$$

where $F_0$ is some stable operator existing by internal stability of the closed-loop system. Under a smoothness assumption on the operator $F_0$ and a small signal assumption on $v$ and with $\Delta F_{0v}(r_1, r_2, 0)$ the linearisation of $F_0$ in response to a perturbation in $v$ around the operating trajectory produced by $r_1$, $r_2$ and $v = 0$, we have that

$$u = F_0(r_1, r_2) + \Delta F_{0v}(r_1, r_2, 0)v \tag{6.2}$$

Since $[r_1 \; r_2]^T$ and $v$ are uncorrelated signals and $u$ and $[r_1 \; r_2]^T$ are available for computation, it follows that we can (in principle) obtain an estimate $\hat{F}_0$ of $F_0$ using a Multiple-Input-Single-Output open-loop identification.

Using $\hat{F}_0$ we can also obtain an estimate $\hat{u}_{r_1,r_2}$ of $u_{r_1,r_2} = F_0(r_1, r_2)$ with

$$\hat{u}_{r_1,r_2} = \hat{F}_0(r_1, r_2) \tag{6.3}$$

Note that by definition $\hat{u}_{r_1,r_2}$ is uncorrelated with the process disturbance signal $v$.

### 6.1.2 Step 2: Open-loop like identification of the Plant

The second step uses the simulated noise free input signal for an open-loop like identification of the plant using the output that was measured in Step 1. From Figure 1.1 we have

$$y = Pu + v. \tag{6.4}$$

By substituting (6.1) into (6.4) we obtain

$$y = PF_0(r_1, r_2, v) + v \tag{6.5}$$

Again, under a smoothness assumption on $PF_0$ and a small signal assumption on $v$ and with $\Delta[PF_0]_v(r_1, r_2, 0)$ the linearisation of $PF_0$ in response to a perturbation in $v$ around the operating trajectory produced by $r_1$, $r_2$ and $v = 0$, we have that

$$y \;=\; PF_0(r_1, r_2) + \Delta[PF_0]_v(r_1, r_2, 0)v + v, \tag{6.6}$$

Substituting the noise free estimate $\hat{u}_r$ of $F_0(r_1, r_2)$ that was found in Step 1 gives

$$y \simeq P\hat{u}_{r_1,r_2} + \Delta[PF_0]_v(r_1, r_2, 0)v + v. \tag{6.7}$$

The last equality follows from stability of $P$ and the smoothness assumption of the nonlinear closed-loop operators. These two assumptions are equivalent to a small signal BIBO stability assumption, i.e. we assume that a small perturbation in the reference (or input) signal produces a small perturbation in the output signal.

Since $\hat{u}_{r_1,r_2}$ and $v$ are uncorrelated and since $y$ and $\hat{u}_{r_1,r_2}$ are available for computation, it is possible to obtain an estimate $\hat{P}$ of $P$ in an open-loop fashion.

**Remark 6.1** This procedure will be greatly simplified when one of the reference signals $r_1$ or $r_2$ equals zero. Note that if $r_1$ equals zero, $F_0$ represents the sensitivity operator $S_0$ of the closed-loop system, i.e. (6.2), (6.3), (6.6) and (6.7) reduce to

$$\begin{aligned} u &= S_0 r_2 + \Delta S_{0v}(r_2, 0)v, \\ \hat{u}_{r_2} &= \hat{S}_0 r_2, \\ y &= PS_0 r_2 + \Delta[PS_0]_v(r_2, 0)v + v, \\ &\simeq P\hat{u}_{r_2} + \Delta[PS_0]_v(r_2, 0)v + v. \end{aligned} \tag{6.8}$$

Similar simplifications occur when $r_2$ equals zero. In such a case $F_0$ represents the complementary sensitivity operator $T_0$ of the closed-loop system.

**Remark 6.2** If the controller $C$ is linear, we can without loss of generality restrict attention to the case with $r_2 = r$ and $r_1 = 0$. If $P$ and $C$ are linear, the linear theory described in [34] is captured.

**Remark 6.3** In the linear case, a high order model structure is used to estimate the sensitivity function in Step 1, so that when this estimate is used in Step 2, the simulated noise free input is as accurate as possible. This might give rise to difficulties in the nonlinear case for computational reasons.

**Remark 6.4** Note that this method can tackle both the situation where one of the signals $r_1$ or $r_2$ is non-zero or where both $r_1$ and $r_2$ are non-zero. This is in contrast with the Hansen method described in Chapters 4 and 5 where both $r_1$ and $r_2$ are required to be non-zero.

**Remark 6.5** Once the identification process has been completed it is wise to include a post identification validation step as in reality the operators may not be satisfactorily linearisable. This involves checking that all the assumptions were satisfied, i.e. that the nonlinearity in the system has not amplified the noise signal in such a way that it would interfere with the previous analysis.

**Remark 6.6** In the previous derivations, we have linearised nonlinear closed-loop operators around their operating trajectory making a small signal assumption on the noise and a smoothness assumption on the closed loop operators. We refer the reader to [8] for more details on such smoothness assumptions and a full treatment of the linearisation problem. Note that similar equations would have been obtained instead by imposing a Lipschitz continuity assumption on these operators.

### 6.1.3 Simulation example

In this section, we illustrate the results of the Two-Step method. We consider a nonlinear system operating in closed-loop with some two degree of freedom linear controller. The nonlinear system is described by

$$y_t = \frac{q^{-1}\,(1 + b_1\,q^{-1})}{1 + a_1\,q^{-1} + a_2\,q^{-2}}\,\mathrm{DZ}(u_t) + v_t \tag{6.9}$$

where DZ is a nonlinear deadzone operator defined using the following equations

$$\mathrm{DZ}(u_t) = \begin{cases} u(t) - d_p & \text{if} \quad u_t \geq d_p \\ 0 & \text{if} \quad -d_m < u_t < d_p \\ u_t + d_m & \text{if} \quad u_t \leq -d_m \end{cases} \tag{6.10}$$

with $|b_1| < 1$, $d_m > 0$ and $d_p > 0$. The disturbance signal is modeled as follows

$$v_t = \frac{1 + c_1\,q^{-1} + c_2\,q^{-1}}{1 + a_1\,q^{-1} + a_2\,q^{-1}}\;e_t \tag{6.11}$$

where $e_t$ is zero mean white noise of variance $\sigma^2$. The linear controller is the optimal two degree of freedom minimum variance controller for the linear system that is obtained from (6.9) by setting both $d_m$ and $d_p$ to zero, i.e.

$$u_t = C_r(q^{-1})\, r_t - C_y(q^{-1})\, y_t, \tag{6.12}$$

$$C_r(q^{-1}) = \frac{1 + c_1\, q^{-1} + c_2\, q^{-1}}{1 + b_1\, q^{-1}}, \tag{6.13}$$

$$C_y(q^{-1}) = \frac{(c_1 - a_1) + (c_2 - a_2)\, q^{-1}}{1 + b_1\, q^{-1}}. \tag{6.14}$$

Here, we have taken the following plant parameters

$$\begin{aligned} b_1 &= -0.9, & a_1 &= -1.5, & a_2 &= 0.7, \\ c_1 &= -1, & c_2 &= 0.2, & d_p &= 0.7, \\ d_m &= 0.2, & \sigma^2 &= 0.3, \end{aligned}$$

but of course these values are not provided to the identification algorithm but rather are to be identified, nor is the identification algorithm provided with the information as to how the controller is designed, though the algorithm is provided with the transfer functions $C_r$ and $C_y$ defining the controller. The reference signal $r_t$ was chosen to be a known unit variance and zero mean white noise signal independent of the process disturbance signal $v_t$. Note that this corresponds to an input signal $u_t$ that is of the same order of magnitude as $d_p$ and $d_m$. With an input signal of much greater magnitude than $d_p$ and $d_m$, these quantities would be hard to identify, the effect of the nonlinearity being swamped by the signal; if $u_t$ is typically of much smaller magnitude, there is obviously also a problem. Using the previous closed-loop system, we have generated a data set $\{r_t, u_t, y_t\}$ with signals of length $N = 2000$.

For the identification of the plant itself, we have used the following model structure

$$\hat{y}_t(\theta) = \theta_1\, \frac{q^{-1}\,(1 + \theta_2\, q^{-1})}{1 + \theta_3\, q^{-1} + \theta_4\, q^{-2}}\, \bar{\mathrm{DZ}}(u_t) \tag{6.15}$$

where $\bar{\mathrm{DZ}}$ is defined as in (6.10) with $d_p$ and $d_m$, respectively, replaced by $\theta_5$ and $\theta_6$.

Estimates of the parameters where obtained by minimising

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}[y_t - \hat{y}_t(\theta)]^2 \tag{6.16}$$

with respect to $\theta$ using a steepest descent method. Using the previously defined data set, we have applied two strategies:

**Strategy I:** direct standard open-loop identification, i.e. we have used the data set $\{u_t, y_t\}$ as if it had been collected in open-loop.

**Strategy II:** modified Two-Step method where $\hat{u}_r$ is obtained by using the sensitivity operator that is obtained by interconnecting the model (6.15) in feedback with the controller (6.12). The feedback loop was simulated with reference signal $r_t$ and with the parameters identified using Strategy I, i.e. there is some approximation involved here.

We have obtained the following results

| | True values | Initial values | Strat. I $\{u, y\}$ | Strat. II $\{\hat{u}_r, y\}$ |
|---|---|---|---|---|
| $\theta_1$ | 1 | 1 | 0.82 | 1.02 |
| $\theta_2$ | -0.9 | -0.2 | -0.88 | -0.87 |
| $\theta_3$ | -1.5 | -1.3 | -1.44 | -1.50 |
| $\theta_4$ | 0.7 | 0.5 | 0.64 | 0.74 |
| $\theta_5$ | 0.7 | 0 | 0.62 | 0.67 |
| $\theta_6$ | 0.2 | 0 | 0.15 | 0.22 |
| $V_N(\theta)$ | | | 0.41 | 0.31 |

*Table 6.1: Identification cost and identified parameters using a one step procedure and a modified Two-Step method.*

Table 6.1 shows the results of estimating $P$ with both procedures. In Figure 6.1, we have compared the magnitude Bode plots of the linear part of the identified models. The results clearly show the degraded performance of the direct identification scheme, i.e. this scheme is unable to produce bias free estimates because the noise dynamics are not modeled. The indirect Two-Step method gives more accurate results for the linear part of the (6.9). Note that the parameters characterising the nonlinear part of the plant could also be identified more accurately with the modified Two-Step methods although the smoothness assumption on $P$ is not satisfied here. The use of a nonlinear model structure for the direct identification of the sensitivity will most probably improve the identification accuracy of the parameters and the applicability of the method in general. Note that the use of complicated model structures like neural nets for the first step of the Two-Step procedure is not a drawback here since their use is only in the generation of a noise free estimate $\hat{u}_r$.

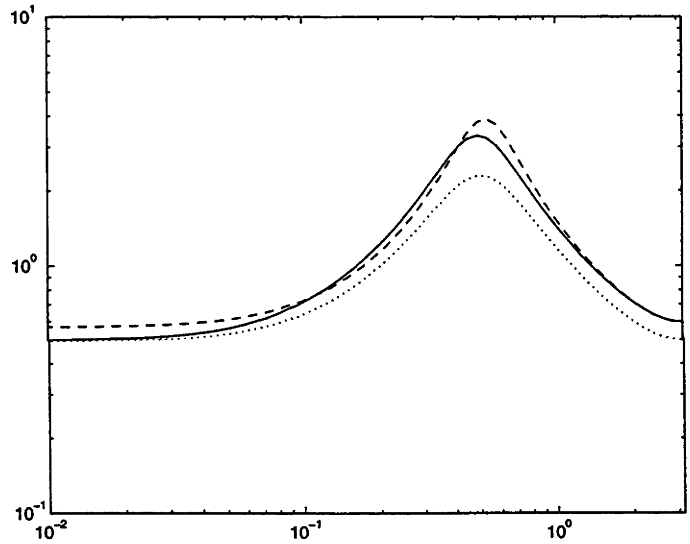



*Figure 6.1: Amplitude Bode plots of the linear part of the nonlinear system (6.9) (—) and estimates of this same transfer function obtained using a one step method (· · ·) and a modified Two-Step method (− −).*

## 6.2 Right Coprime Factor identification for Nonlinear Systems

In the Coprime Factor method we can easily find a right factorisation of the plant. One factor is identified as the operator from $[r_1\ r_2]$ to $u$, the other factor is the operator from $[r_1\ r_2]$ to y. By introducing data filters we can identify right *coprime* factors from the original factorisation.

Consider the system depicted in Figure 1.1. Adopt Assumptions 6.1, 6.3, 6.4. Note that we no longer require the plant to be stable. In addition we will also adopt the following assumption.

**Assumption 6.6** *Either one of the reference signals $r_1$ or $r_2$ is zero or the reference signals $r_1$ and $r_2$ are filtered versions of the same signal $r$, i.e. $r_1 = N\,r$ and $r_2 = D\,r$ where $(N,\,D)$ is a coprime pair.*

### 6.2.1 Identification of the right factors of $P$

We will now show that it is possible to generalise the closed-loop identification scheme of [36] to the nonlinear case. Using measurements of $u$ and $y$ together with measurements or reconstructions of $r_1$ or/and $r_2$, we can identify the closed-loop relations in an open loop fashion.

Recall from Step 1 of the Two-Step method that we can write

$$u = F_0(r_1, r_2) + \Delta F_{0v}(r_1, r_2, 0)v. \tag{6.17}$$

Recall also that from Step 2 of the Two-Step method that we can write

$$y \;=\; PF_0(r_1, r_2) + \Delta[PF_0]_v(r_1, r_2, 0)v + v. \tag{6.18}$$

From (6.17) we can obtain an estimate of $F_0(r_1, r_2)$ and from (6.18) we can obtain an estimate of $PF_0(r_1, r_2)$

The corresponding right factorisation of $P$ that can be estimated in this way is the factorisation $(PF_0, F_0)$; i.e. $y = (PF_0)F_0^{-1}$. However, similar to the linear case shown in [36], this is only one of many factorisations of $P$ and there is no guarantee that this factorisation is coprime.

### 6.2.2 Identification of right *coprime* factors of $P$

We will now introduce auxiliary signals that will allow us to find a right *coprime* factorisation of $P$ from the simple right factorisation found in the previous section.

Suppose that

$$\begin{aligned} x_1 &= F_1 r_1, \\ x_2 &= F_2 r_2 \end{aligned} \tag{6.19}$$

with $F_1$, $F_2$ fixed stable and invertible operators; then one can rewrite the system equations of (6.17) and (6.18) as

$$\begin{aligned} u &= F_0(F_1^{-1}x_1, F_2^{-1}x_2) + \Delta F_{0v}(r_1, r_2, 0)\, v, \\ y &= PF_0(F_1^{-1}x_1, F_2^{-1}x_2) + \Delta[PF_0]_v(r_1, r_2, 0)\, v + v. \end{aligned}$$

We shall show below that when $C$ is either

- linear,
- nonlinear but stable, or
- nonlinear but stably invertible,

then the introduction of appropriate filters allows us to identify a right *coprime* factorisation of a nonlinear plant in an open-loop fashion.

### 6.2.3 Design of the data filters $F_1$ and $F_2$

We will now examine the process for designing the data filters when the controller is linear (the first case mentioned above). Suppose that the linear controller $C$ has left coprime factorisation $C = V_l^{-1}U_l$. The plant $P$ is allowed to be nonlinear and has a right coprime factorisation $P = N_r D_r^{-1}$, i.e. there exists a BIBO operator $\mathcal{L}_l$ for which

$$\mathcal{L}_l \begin{bmatrix} N_r \\ D_r \end{bmatrix} = I. \tag{6.20}$$

Consider Figure 6.2. Using the linearity of $U_l$, $V_l$

$$\begin{aligned} m &= V_l[D_r n - r_2] \\ &= V_l D_r n - V_l r_2, \end{aligned} \tag{6.21}$$

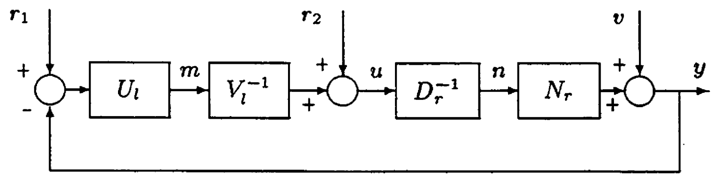



*Figure 6.2: The closed-loop system*

and also

$$\begin{aligned} m &= U_l[r_1 - N_r\, n - v] \\ &= U_l\, r_1 - U_l N_r\, n - U_l\, v. \end{aligned} \tag{6.22}$$

Combining (6.21) and (6.22) yields

$$n = W^{-1}[U_l\, r_1 + V_l\, r_2 - U_l\, v] \tag{6.23}$$

where $W = U_l N_r + V_l D_r$ is a unit by internal stability of the closed loop system.

From Figure 6.2 we have:

$$\begin{aligned} u &= D_r n, \\ y &= N_r n + v. \end{aligned} \tag{6.24}$$

Substituting (6.23) into (6.24) we get

$$\begin{aligned} u &= D_r W^{-1}[U_l\, r_1 + V_l\, r_2 - U_l\, v], \\ y &= N_r W^{-1}[U_l\, r_1 + V_l\, r_2 - U_l\, v] + v. \end{aligned} \tag{6.25}$$

Under a smoothness assumption on the closed-loop system and a small signal assumption on $v$, define $\Delta Y_v(r_1, r_2, 0)$ and $\Delta U_v(r_1, r_2, 0)$ to be the linearisations of the closed-loop operators in response to a perturbation in $v$ around the operating trajectories produced by $r_1$, $r_2$ and $v = 0$. From now on we will use $\Delta Y_v$ as a shortened form of $\Delta Y_v(r_1, r_2, 0)$ and $\Delta U_v$ as a shortened form of $\Delta U_v(r_1, r_2, 0)$. Thus

$$u = D_r W^{-1}[U_l\, r_1 + V_l\, r_2] + \Delta U_v\, v,$$

$$y = N_r W^{-1}[U_l r_1 + V_l r_2] + \Delta Y_v v + v.$$

Introducing the filters $F_1$ and $F_2$ from (6.19) we obtain

$$\begin{aligned} u &= D_r W^{-1}[U_l F_1^{-1} x_1 + V_l F_2^{-1} x_2] + \Delta U_v v, \\ y &= N_r W^{-1}[U_l F_1^{-1} x_1 + V_l F_2^{-1} x_2] + \Delta Y_v v + v. \end{aligned}$$

- Let us consider the case where $r_1 = F_1^{-1} x_1 = 0$. The closed-loop relations reduce to

$$\begin{aligned} u &= D_r W^{-1} (V_l F_2^{-1} x_2) + \Delta U_v v, \\ y &= N_r W^{-1} (V_l F_2^{-1} x_2) + \Delta Y_v v + v \end{aligned}$$

which can be used to provide an open-loop estimate for a right coprime factorisation of $P$. Indeed, it is easy to see that

$$\begin{bmatrix} \bar{N}_r \\ \bar{D}_r \end{bmatrix} = \begin{bmatrix} N_r W^{-1} (V_l F_2^{-1}) \\ D_r W^{-1} (V_l F_2^{-1}) \end{bmatrix} \tag{6.26}$$

is a right factorisation of $P$ if and only if $V_l F_2^{-1}$ is a stable operator. This same factorisation is right *coprime* if and only if

$$F_2 = \hat{W} V_l, \tag{6.27}$$

with $\hat{W}$ a unit operator. Indeed, premultiplication by the stable operator $W\mathcal{L}_l$ of (6.26) yields $V_l F_2^{-1}$. The choice (6.27) implies that this quantity is a unit operator and that the right factorisation of (6.26) is coprime. Conversely, if (6.26) is a right coprime factorisation then exists a unit operator $\tilde{W}$ such

$$\begin{bmatrix} \bar{N}_r \\ \bar{D}_r \end{bmatrix} = \begin{bmatrix} N_r W^{-1} \\ D_r W^{-1} \end{bmatrix} \tilde{W}$$

which implies (6.27) with $\hat{W} = \tilde{W}^{-1}$. Refer to Chapter 3 for a full discussion of the relationship between different factorisations of a given operator.

- A similar reasoning process in the case where $r_2 = F_2^{-1}x_2 = 0$ shows that

$$F_1 = \hat{W}U_l, \tag{6.28}$$

with $\hat{W}$ a unit operator, is a necessary and sufficient condition to produce a right coprime factorisation

$$\begin{bmatrix} N_r W^{-1} U_l F_1^{-1} \\ D_r W^{-1} U_l F_1^{-1} \end{bmatrix}$$

of $P$. Here, we have to restrict attention to the case where $U_l$ is square, i.e. the number of inputs $u$ and number of outputs $y$ are equal.

- Let us now assume that $F_1 = 1$, $F_2 = 1$, $r_1 = A\,r$ and $r_2 = B\,r$. The closed-loop equations reduce to

$$\begin{aligned} u &= D_r W^{-1}[U_l B + V_l A]r + \Delta U_v\, v, \\ y &= N_r W^{-1}[U_l B + V_l A]r + \Delta Y_v\, v + v. \end{aligned}$$

It can easily be seen that if $A$ and $B$ have been chosen such that $U_l B + V_l A$ is a unit and, provided that $u$, $y$ and $r$ are available for computation, one can estimate a right coprime factorisation of $P$ using open-loop identification techniques.

**Remark 6.7** When $C$ is a known *nonlinear* stable operator, one can without loss of generality choose a left coprime factorisation with $U_l = C$ and $V_l = 1$. Then, assuming that $r_1 = 0$, it is easy to show that, under a small signal assumption on $v$ and a smoothness assumption on the closed-loop operators, the equations reduce to

$$\begin{aligned} u &= D_r[D_r - U_l(-N_r)]^{-1}\, r_2 + \Delta U_v\, v, \\ y &= N_r[D_r - U_l(-N_r)]^{-1}\, r_2 + \Delta Y_v\, v + v. \end{aligned}$$

This allows an open-loop like identification of right coprime factors of $P$. The filter $F_2$

can be chosen to be any unit operator. Similarly, if $C$ is a nonlinear, stably invertible operator and $r_2$ equals zero, one can choose $U_l = I$ and $V_l = C^{-1}$ without loss of generality. Again, we have to resrict attention to the case where $C$ is square. Using small signal arguments, one obtains the following closed-loop equations

$$\begin{aligned} u &= D_r[V_l D_r + N_r]^{-1} r_1 + \Delta U_v\, v, \\ y &= N_r[V_l D_r + N_r]^{-1} r_1 + \Delta Y_v\, v + v. \end{aligned}$$

Again, it is possible to estimate a right coprime factorisation of $P$ using open-loop like techniques. We can choose any unit operator for $F_1$.

**Remark 6.8** It is not clear how to select the data filters $F_1$ or $F_2$ in the general case, i.e. when $C$ can be nonlinear, unstable and/or non minimum phase.

**Remark 6.9** In the linear case, the remaining freedom in $F_1$ and $F_2$ can be used to estimate normalised coprime factors, i.e. the liberty in choosing the unit operator $W$ is used to construct a normalised coprime factorisation of $P$. We refer the reader to [36] for more details. Similar ideas could be applied in the nonlinear case. We refer the reader to [37] for more details on normalised coprime factorisations in the nonlinear case.

## 6.3 Summary

This chapter has examined two methods for the approximate identification of a nonlinear open-loop plant on the basis of closed-loop data. The first method is an indirect Two-Step method based on the identification of the sensitivity operator of the closed-loop system. The second method identifies the right coprime factors through an identification of the sensitivity and the complementary sensitivity operator of the closed-loop system. The next chapter discusses another method for closed-loop identification.

# Chapter 7

# Gradient expressions for a closed-loop identification scheme with a tailor-made parametrisation

Another approach to closed-loop identification of linear systems that has received less attention than the linear versions of the methods in Chapters 4, 5 and 6 has been examined in [38]. Such an approach had already been mentioned as an exercise in [23]. Further references include [9, 19, 23]. In [38], Van Donkelaar and Van den Hof consider the closed-loop identification of a linear system subjected to a linear controller by minimisation of a closed-loop criterion, using a tailor-made parametrisation of the plant. The method uses knowledge of the controller; it minimises an error between the closed loop transfer functions of the true closed-loop and the model closed-loop. The main result of this reference is to show that, provided the model order is higher than the order of the controller, the parameter set is connected. This chapter also provides consistency results and gradient expressions.

Here, the same closed-loop matching criterion found in [38] is used with a tailor-made parametrisation, but this chapter extends the results in two ways. First, in the linear case, we show that the gradient signals of [38] can be generated very simply on closed loop simulation models. This observation then leads us to show that this simulation method for the computation of the gradients can be extended to apply to nonlinear systems and/or systems with nonlinear controllers.

The ideas in this thesis heavily rely on data-driven model-free control design methods that have recently been proposed in [6, 17, 30]. Indeed, we treat closed-loop identification with a tailor-made parametrisation as a dual of direct controller optimisation.

The organisation of the chapter is as follows. In Section 7.1, we describe the problem at hand. In Section 7.2 we present expressions of the gradient signals in the linear case. Section 7.3 considers the general case where both the plant and the known controller are possibly nonlinear. Section 7.4 presents consistency results in the nonlinear case. In Section 7.5, we present some numerical simulations. We conclude in Section 7.6.

## 7.1 General problem setting

The true system is the Single-Input-Single-Output (SISO) nonlinear time-invariant system described by

$$\mathcal{S} : y = P(u, v) \tag{7.1}$$

where $P$ is an unknown causal nonlinear operator. The restriction to scalar plants is not essential, but notationally convenient.

- $u$ is the control input signal,
- $y$ is the achieved output signal, and
- $v$ is a process disturbance signal that is allowed to enter the system nonlinearly.

The input signal is determined according to a known controller

$$\mathcal{C} : u = C(r, y). \tag{7.2}$$

- $r$ is an external reference which is assumed to be quasi-stationary and uncorrelated with $v$.
- The controller $C$ is a causal nonlinear operator of both $r$ and $y$.

The closed-loop operator from measured reference signal $r$ to measured output signal $y$, as defined in Figure 7.1, can be written as follows,

$$y = T_o(r, v). \tag{7.3}$$

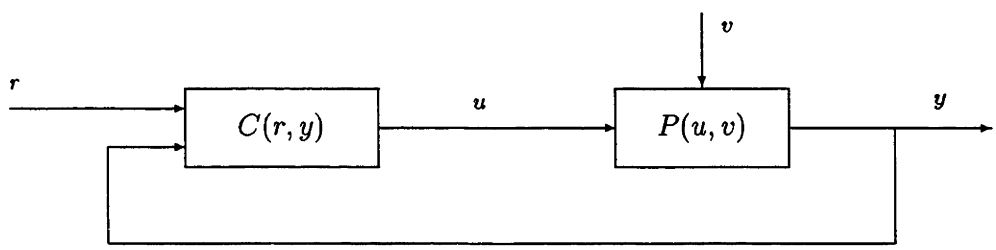



*Figure 7.1: The actual loop*

### 7.1.1 Closed-loop identification with a tailor-made parametrisation

The basic idea is that the closed-loop operator from the reference signal $r$ to the output signal $y$ is identified using a parametrised output predictor

$$\hat{y}(\theta) = \hat{T}(\theta, r) \tag{7.4}$$

obtained from the feedback interconnection of an open-loop plant model

$$\mathcal{M} : \hat{y}(\theta) = \hat{P}(\theta, u) \tag{7.5}$$

for $P$, parametrised by a vector $\theta \in D_\theta \subset \mathbb{R}^n$ where $D_\theta$ is some prescribed domain, and the possibly nonlinear controller $C$ in (7.2). Adopt the following assumption.

**Assumption 7.1** *The output predictor (7.4) or, equivalently, the loop in Figure 7.2 has the BIBO and smoothness properties of the true closed-loop system, for all values of $\theta \in D_\theta$.*

**Remark 7.1** Note that, unless an explicit temporary assumption is made to the contrary, it is not assumed that the true system (even without noise) is in the model set.

**Assumption 7.2** *The closed-loop system of Figure 7.1 is BIBO stable.*

**Assumption 7.3** *The plant, the model, the controller and all closed-loop operators are smooth functions of the reference signal, the input signal, the output signal and the disturbance signal.*

**Remark 7.2** Assumption 7.3 is useful because the work in this chapter makes use of linearisations of some nonlinear operators around their operating trajectories. Assumption 7.3 means that if the closed-loop operator is linearised around any (stable) trajectory, the resulting linear system is BIBO stable. The reader is referred to [8] for more details on such smoothness assumptions and a full treatment of the linearisation problem.

**Remark 7.3** Note that, as opposed to the nonlinear methods described in Chapters 4 and 5, there is no restriction on the Signal-to-Noise-Ration (SNR) when using the method; consistency however may require a high SNR as discussed later. Also, all signals can either be continuous or discrete in time.

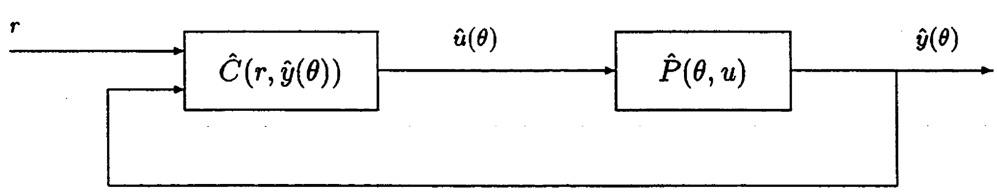



*Figure 7.2: The simulation loop*

Suppose that a data set $\{r, y\}$ has been collected on the actual system of Figure 7.1. The problem that is addressed in this chapter is the one of selecting the model for $P$ in the set (7.5) that best explains this data set in a closed-loop sense.

We make use of the identification criterion

$$V_N(\theta) = \frac{1}{2N} \sum_{t=1}^{N} [L(y - \hat{y}(\theta))]^2. \tag{7.6}$$

Here $L$ can be any causal BIBO stable design operator. Besides the intuitively reasonable aspect of (7.6), it is shown in [38] that this criterion allows a consistent identification of a linear plant under linear feedback, when the input-output dynamics are in the model set. Section 7.3 gives some insight on how this result generalises when both the plant and the controller are allowed to be nonlinear. In any case, the linear consistency result adds greater weight to the selection of the identification criterion (7.6). The reader is referred to [11] for variance considerations in the linear case.

Note that, provided the input signal $u$ is measured, the generalisation to the non-standard identification criterion

$$\mathcal{V}_N(\theta) = \frac{1}{2N} \sum_{t=1}^{N} \left\{ [L_y(y - \hat{y}(\theta))]^2 + \lambda \, [L_u(u - \hat{u}(\theta))]^2 \right\}$$

that was introduced by the authors of [43] is straightforward. Again, $L_y$ and $L_u$ are causal BIBO stable design operators.

The preceding parameter estimation problem is typically solved using gradient search techniques such as Gauss Newton; see [23] for a discussion on initial estimates, convergence, local minima, etc. A discussion on the connectedness of the set of all models (7.5) stabilised by the controller (7.2) in the linear case is contained in [38].

To minimise (7.6) with respect to the model parameter vector $\theta$, it is standard that one can iteratively seek a solution for $\theta$ to

$$V_N'(\theta) = -\frac{1}{N} \sum_{t=1}^{N} \left[ (y - \hat{y}(\theta)) \, \hat{y}'(\theta) \right] = 0 \tag{7.7}$$

by taking steps in the negative gradient direction

$$\theta[i+1] = \theta[i] - \gamma_i \, R_i^{-1} \, V_N'(\theta[i]) \tag{7.8}$$

where $V_N'(\theta)$ and $\hat{y}'(\theta)$, respectively, denote the gradient of $V_N(\theta)$ and $\hat{y}(\theta)$ with respect to $\theta$, and where $R_i$ is some appropriate positive definite matrix, typically an estimate of the Hessian of $V_N$. The update equation (7.8) is a batch mode type of adjustment. We also consider the the following assumption holds.

**Assumption 7.4** *The stability of the closed-loop output predictor (7.4) is preserved while iterating.*

**Remark 7.4** This is a very reasonable assumption since the step size $\gamma_i$ can be used effectively to control how much the model is allowed to change per iteration. Therefore, in practice, the identified model is stabilised by the known controller.

The key technical step in this iterative algorithm is the computation of the gradient $\hat{y}'(\theta)$. Our contribution here is to show that this gradient computation can be performed by feeding the signal $\hat{u}(\theta)$ of Figure 7.2 as the input of a closed loop simulation system. For simplicity, our method is explained first in Section 7.2 for the linear case, in which case our method is a simple alternative to the gradient computation proposed in [38]. The real advantage of our computation method is that it allows a good understanding of the stability issue and a generalisation to a nonlinear setting, as is shown in Section 7.3.

## 7.2 Gradient expressions in the linear case

In this section, we consider the simplified case where both the real system and the controller are linear, i.e. we suppose that (7.1), (7.2) and (7.5) reduce to

$$\mathcal{S}: y = P\,u + v, \quad \mathcal{C}: u = C_r\,r - C_y\,y, \quad \mathcal{M}: \hat{y}(\theta) = \hat{P}(\theta)\,u.$$

Let us first consider the following equations

$$\hat{y}(\theta) = \hat{P}(\theta)\,\hat{u}(\theta) \quad \text{and} \quad \hat{u}(\theta) = C_r\,r - C_y\,\hat{y}(\theta). \tag{7.9}$$

The gradients of these two signals with respect to the $j$-th entry of $\theta$ are, respectively, denoted by $\hat{u}'_{\theta_j}(\theta)$ and $\hat{y}'_{\theta_j}(\theta)$. They are the $j$-th component of the vectors $\hat{u}'(\theta)$ and $\hat{y}'(\theta)$ and they satisfy, for $j = 1, \cdots, n$,

$$\hat{y}'_{\theta_j}(\theta) = \hat{P}'_{\theta_j}(\theta)\,\hat{u}(\theta) + \hat{P}(\theta)\,\hat{u}'_{\theta_j}(\theta) \tag{7.10}$$

$$\hat{u}'_{\theta_j}(\theta) = -C_y\,\hat{y}'_{\theta_j}(\theta) \tag{7.11}$$

where $\hat{P}'_{\theta_j}(\theta)$, the derivative of $\hat{P}(\theta)$ with respect to $\theta_j$, can easily be obtained since $\hat{P}(\theta)$ has a known structure. It now easily follows that each entry of $\hat{u}'(\theta)$ and $\hat{y}'(\theta)$ can be computed as shown in the loop of Figure 7.3.

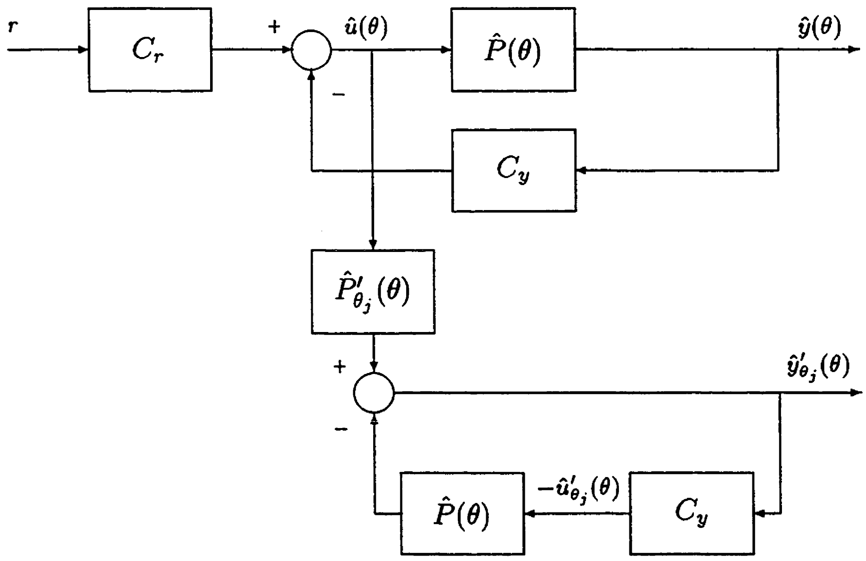



*Figure 7.3: Generation of $\hat{u}'_{\theta_j}(\theta)$ and $\hat{y}'_{\theta_j}(\theta)$*

The scheme in Figure 7.3 can always be implemented in a stable way if $\hat{P}(\theta)$ is stabilised by $C$ as is assumed earlier. Indeed, let

$$\hat{P}(\theta) = [\hat{D}(\theta)]^{-1}\hat{N}(\theta) = \hat{N}(\theta)[\hat{D}(\theta)]^{-1} \tag{7.12}$$

be a stable coprime factorisation of $\hat{P}(\theta)$. Then, one can redraw Figure 7.3 as shown in the loop of Figure 7.4. Note that Figure 7.3 can be used to generate $\hat{y}'_{\theta_j}(\theta)$ if $\hat{P}(\theta)$ is open-loop stable.

The stability of Figure 7.4 follows from the stability of the predictor loop and the verifiable fact that $\hat{D}^2(\theta)\hat{P}'_{\theta_j}(\theta)$ is stable for $j = 1, \cdots, n$ and $\forall\ \theta \in D_\theta$.

It is now straightforward to see that

$$\begin{aligned}
\hat{y}'_{\theta_j}(\theta) &= \frac{\hat{P}'_{\theta_j}(\theta)}{[1+\hat{P}(\theta)C_y]}\hat{u}(\theta) = \frac{\hat{P}'_{\theta_j}(\theta)C_r}{[1+\hat{P}(\theta)C_y]^2}r, && (7.13)\\
\hat{u}'_{\theta_j}(\theta) &= -C_y\,\hat{y}'_{\theta_j}(\theta). && (7.14)
\end{aligned}$$

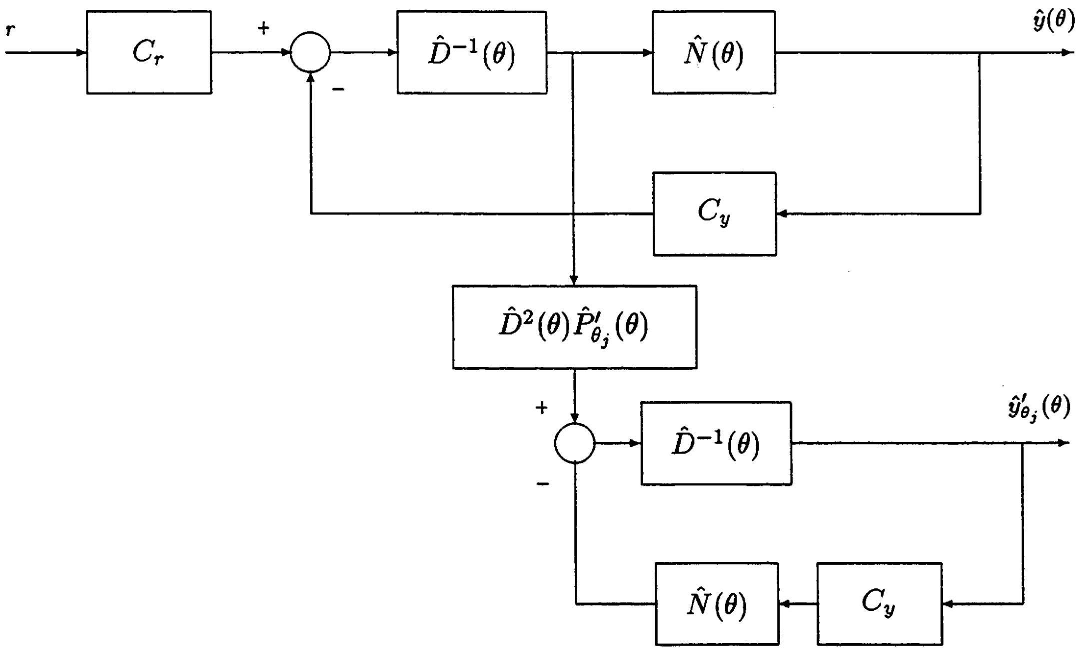



*Figure 7.4: Stable implementation of Figure 7.3*

## 7.3 Gradient expressions in the nonlinear case

Let us now consider the nonlinear case of Section 7.1, i.e. we have the following equations

$$\hat{y}(\theta) = \hat{P}(\theta, \hat{u}(\theta)), \quad \hat{u}(\theta) = C(r, \hat{y}(\theta)). \tag{7.15}$$

As a tool for obtaining the gradient of $V_N$ with respect to $\theta$, we seek the gradients of $\hat{u}(\theta)$ and $\hat{y}(\theta)$ with respect to $\theta_j$. If one of the parameter vector entries, say $\theta_j$, is perturbed by a small $\delta\theta_j$, we obtain

$$\begin{aligned} \hat{u}(\theta_1, \cdots, \theta_j + \delta\theta_j, \cdots, \theta_n) &= C[r, \hat{y}(\theta_1, \cdots, \theta_j + \delta\theta_j, \cdots, \theta_n)], \\ &\simeq C[r, \hat{y}(\theta) + \hat{y}'_{\theta_j}(\theta)\, \delta\theta_j], \\ &\simeq C(r, \hat{y}(\theta)) + \Delta C_y(r, \hat{y}(\theta))\, \hat{y}'_{\theta_j}(\theta)\, \delta\theta_j, \\ &= \hat{u}(\theta) + \Delta C_y(r, \hat{y}(\theta))\, \hat{y}'_{\theta_j}(\theta)\, \delta\theta_j \end{aligned} \tag{7.16}$$

where $\Delta C_y(r, \hat{y}(\theta))$ is the linearisation of $C$ in response to a perturbation in $y$ around the trajectory produced by $r$ and by $\hat{y}(\theta)$, i.e. the trajectory around which $C$ is linearised depends on $\theta$. The derivative of $\hat{y}(\theta)$ with respect to $\theta_j$ is denoted $\hat{y}'_{\theta_j}(\theta)$ and it is the $j$-th component of the vector $\hat{y}'(\theta)$. It is straightforward to see that (7.16) yields

$$\hat{u}'_{\theta_j}(\theta) = \Delta C_y(r, \hat{y}(\theta))\, \hat{y}'_{\theta_j}(\theta) \tag{7.17}$$

where $\hat{u}'_{\theta_j}(\theta)$ is defined in a similar fashion as $\hat{y}'_{\theta_j}(\theta)$. A similar reasoning yields

$$\hat{y}'_{\theta_j}(\theta) = \hat{P}'_{\theta_j}(\theta, \hat{u}(\theta)) + \Delta \hat{P}_u(\theta, \hat{u}(\theta))\, \hat{u}'_{\theta_j}(\theta) \tag{7.18}$$

where $\Delta \hat{P}_u(\theta, \hat{u}(\theta))$ is the linearisation of $\hat{P}(\theta)$ in response to a perturbation in $u$ around the trajectory produced by $\hat{u}(\theta)$. The partial derivative of $\hat{P}(\theta)$ with respect to $\theta_j$ is denoted by $\hat{P}'_{\theta_j}(\theta, \hat{u}(\theta))$. It can easily be obtained since $\hat{P}(\theta)$ has a known structure.

In the nonlinear case, there are no compact expressions for the gradient signals. Indeed, the exact gradient signals can be obtained by feeding the signal $\hat{u}(\theta)$ generated in the loop of Figure 7.2, filtered through $\hat{P}'_{\theta_j}(\theta, \hat{u}(\theta))$, as input of the (linear time-varying) linearised closed loop system of Figure 7.5. A similar observation had already been made in [6, 30] for an iterative feedback tuning scheme. The stability of the lower loop follows from the smoothness assumption on the nonlinear closed-loop operator and the stability of the predictor at each iteration. These two assumptions are equivalent to a small signal BIBO stability assumption, i.e. we assume that a small perturbation in the reference signal produces a small perturbation in the output signal.

The scheme shown in Figure 7.5 can be implemented in a stable way, even if $\hat{P}(\theta, \hat{u}(\theta))$ and thus $\hat{P}'_{\theta_j}(\theta, \hat{u}(\theta))$ are unstable, provided we can construct

$$\begin{aligned} \hat{y}(\theta) &= \hat{N}_r(\theta, \hat{z}_r(\theta)), \quad u = \hat{D}_r(\theta, \hat{z}_r(\theta)) \text{ and} && (7.19) \\ \hat{z}_l(\theta) &= \hat{D}_l(\theta, \hat{y}(\theta)) = \hat{N}_l(\theta, u) && (7.20) \end{aligned}$$

respectively, as stable right and left coprime descriptions of (7.5); see [13] for further details. Then, one can redraw Figure 7.5 as shown in Figure 7.6. Note that Figure 7.5 can be used to generate $\hat{y}'_{\theta_j}(\theta)$ if $\hat{P}(\theta, \hat{u}(\theta))$ is open-loop stable.

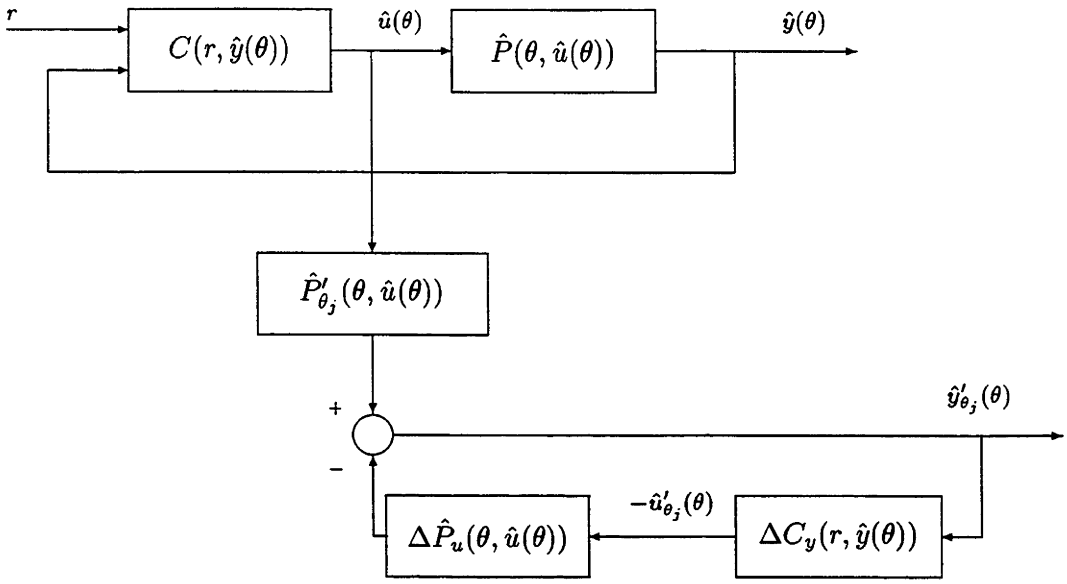



*Figure 7.5: Generation of $\hat{u}'_{\theta_j}(\theta)$ and $\hat{y}'_{\theta_j}(\theta)$ in the nonlinear case*

Here $\Delta\hat{D}_{l_y}(\theta, \hat{y}(\theta))$ and $\Delta\hat{N}_{l_u}(\theta, \hat{u}(\theta))$ are, respectively, the linearisations of $\hat{D}_l(\theta, \hat{y}(\theta))$ and $\hat{N}_l(\theta, \hat{u}(\theta))$ around their trajectory.

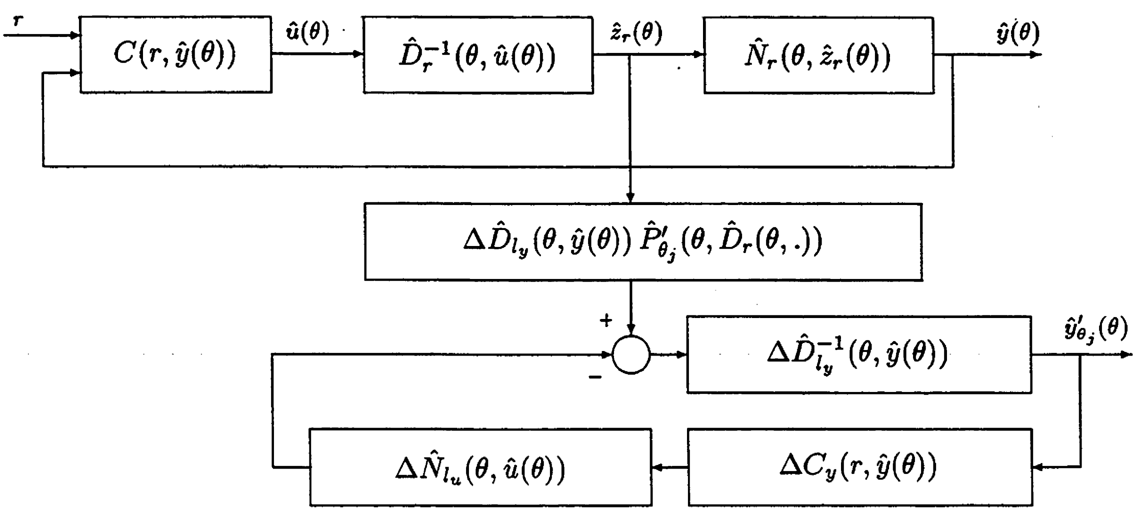



*Figure 7.6: Stable implementation of Figure 7.5*

The stability of Figure 7.6 follows from the stability of the predictor loop, the smoothness assumption on the closed loop system and the fact that

$$\Delta\hat{D}_{l_y}(\theta, \hat{y}(\theta))\, \hat{P}'_{\theta_j}(\theta, \hat{D}_r(\theta, .)) \tag{7.21}$$

is a stable operator for $j = 1, \cdots, n$ and $\forall\ \theta \in D_\theta$. Indeed, it follows from (7.19) and (7.20) that

$$\begin{aligned} \hat{z}'_{l_{\theta_j}}(\theta) &= \hat{N}'_{l_{\theta_j}}(\theta, u) = \hat{D}'_{l_{\theta_j}}(\theta, \hat{y}(\theta)) + \Delta\hat{D}_{l_y}(\theta, \hat{y}(\theta))\, \hat{y}'_{\theta_j}(\theta), && (7.22) \\ \hat{y}'_{\theta_j}(\theta) &= \hat{P}'_{\theta_j}(\theta, u). && (7.23) \end{aligned}$$

Using the preceding equations, one obtains

$$\Delta\hat{D}_{l_y}(\theta, \hat{y}(\theta))\, \hat{P}'_{\theta_j}(\theta, u) = \hat{N}'_{l_{\theta_j}}(\theta, u) - \hat{D}'_{l_{\theta_j}}(\theta, \hat{y}(\theta)). \tag{7.24}$$

It is now straightforward to see that

$$\Delta\hat{D}_{l_y}(\theta, \hat{y}(\theta))\, \hat{P}'_{\theta_j}(\theta, \hat{D}_r(\theta, \hat{z}_r(\theta))) = \hat{N}'_{l_{\theta_j}}(\theta, \hat{D}_r(\theta, \hat{z}_r(\theta))) - \hat{D}'_{l_{\theta_j}}(\theta, \hat{N}_r(\theta, \hat{z}_r(\theta))) \tag{7.25}$$

which shows that (7.21) is a BIBO operator.

## 7.4 Consistency results

In this section, we show that the linear consistency results for the input-output dynamics derived in [38] do not carry over in general in the nonlinear case. See [22, 23] for further details on consistency results. In this section we adopt the following assumption.

**Assumption 7.5** *There exists $\theta_o$ such that the true system without noise lies in the model set, i.e.*

$$P(u, 0) = \hat{P}(\theta_o, u)\ \forall\, u \quad \text{or} \quad T_o(r, 0) = \hat{T}(\theta_o, r)\ \forall\, r.$$

In this situation, we would hope for consistent identification. Rewriting the identification error, we obtain

$$\begin{aligned} y - \hat{y}(\theta) &= [y - \hat{y}(\theta_o)] + [\hat{y}(\theta_o) - \hat{y}(\theta)] \\ &= [T_o(r, v) - \hat{T}(\theta_o, r)] + [\hat{T}(\theta_o, r) - \hat{T}(\theta, r)], \end{aligned}$$

whence

$$\begin{aligned} E\{[y-\hat{y}(\theta)]^2\} &= E\{[[T_o(r,v)-\hat{T}(\theta_o,r)]^2\}+E\{[\hat{T}(\theta_o,r)-\hat{T}(\theta,r)]^2\} \\ &\quad +2\,E\{[T_o(r,v)-\hat{T}(\theta_o,r)][\hat{T}(\theta_o,r)-\hat{T}(\theta,r)]\}\cdot \end{aligned} \tag{7.26}$$

Here, the expected value is taken with respect to the probability distributions of the noise and the reference signals; the earlier assumption that $r$ and $v$ are independent is important. It is clear from (7.26) that a sufficient condition for consistency is given by

$$E\{[T_o(r,v)-\hat{T}(\theta_o,r)][\hat{T}(\theta_o,r)-\hat{T}(\theta,r)]\}=0. \tag{7.27}$$

It is easily established that this condition (not unexpectedly) is not satisfied in general, when $T$ is nonlinear. A sufficient condition for (7.27) to hold is

$$T_o(r,v)-\hat{T}(\theta_o,r)=v^{2k+1}\,R(\theta_o,r) \tag{7.28}$$

for some nonnegative integer $k$ and some noise independent operator $R$; we can isolate several important situations where this holds.

**Remark 7.5** Note that in a small noise situation, (7.28) approximately holds with $R(\theta_o,r)=\Delta T_{ov}(r,0)$ and $k=0$. Here $\Delta T_{ov}(r,0)$ is the linearisation of $T_o$ in response to a perturbation in $v$ around the trajectory produced by $r$ and $v=0$. We conclude that at SNRs where linearisation is valid, one has approximate consistency.

**Remark 7.6** In many industrial processes, although the open-loop system is nonlinear, the controller has been designed in order for the closed loop system $T_o$ to have a quasi-linear behaviour with respect to the reference signal $r$ (and the disturbance signal $v$; at least if the noise signal $v$ is additive). It is clear from (7.28) that consistency approximately holds in such cases.

**Remark 7.7** As is shown in [22], consistency can be recovered using direct identification, i.e. using the data set as if it had been collected in open-loop, if the system (input-output and noise dynamics) can be modeled exactly. However, the number of parameters increases, i.e. one has to estimate the noise dynamics. This can be problematic (because of variance considerations) in the case of small data sets.

**Remark 7.8** Note that in [22] it is implicitly assumed that the noise enters linearly, i.e. the noise is required to be additive. This is a restrictive assumption in a nonlinear context since it is no longer the case that disturbances entering different parts of the plant can be lumped into one additive output term.

If the noise signal $v$ enters the plant nonlinearly, direct identification of input-output and noise dynamics might be difficult to implement. Indeed, in most cases it is not possible to rewrite the system (7.1) in the form

$$y_t = f(y^{t-1}, u^{t-1}, t) + e_t \tag{7.29}$$

as required in [22] for consistency. Here $f$ is deterministic function, $y^{t-1}$ and $u^{t-1}$ are vectors of previous values and $e_t$ is an independent stochastic process; we refer to [22] for further details.

When the disturbance signal enters nonlinearly, our approach offers a valid alternative as is shown using a simulation example in the next section.

## 7.5 Numerical illustration

In this section, we illustrate the closed-loop identification "procedure" presented in this chapter. We consider a nonlinear system operating in closed-loop with a two-degree of freedom linear controller. The nonlinear system is described by

$$\left\{ \begin{array}{rcl} y_t & = & P(u_t, v_t) = \mathrm{DZ}(\bar{y}_t) \\ \bar{y}_t & = & \dfrac{q^{-1}\,(1 + b_1\,q^{-1})}{1 + a_1\,q^{-1} + a_2\,q^{-2}}\, u_t + v_t \end{array} \right. \tag{7.30}$$

where DZ is a nonlinear deadzone operator defined using the following equations

$$z_t = \mathrm{DZ}(x_t) = \begin{cases} x_t - d_p & \text{if} \quad x_t \geq d_p \\ 0 & \text{if} \quad -d_p < x_t < d_p \\ x_t + d_p & \text{if} \quad x_t \leq -d_p \end{cases} \tag{7.31}$$

with $|b_1| < 1$, $d_p > 0$. The disturbance signal affects the system output *nonlinearly* and is modeled as follows

$$v_t = \frac{1 + c_1\, q^{-1} + c_2\, q^{-1}}{1 + a_1\, q^{-1} + a_2\, q^{-1}}\, e_t \tag{7.32}$$

where $e_t$ is zero mean white noise of variance $\sigma^2$. The linear controller is the two-degree of freedom minimum variance controller optimal for the linear system that is obtained from (7.30) by setting $d_p$ to zero, i.e.

$$\begin{aligned} u_t &= C_r(q^{-1})\, r_t - C_y(q^{-1})\, y_t, \\ C_r(q^{-1}) &= \frac{1 + c_1\, q^{-1} + c_2\, q^{-1}}{1 + b_1\, q^{-1}} \quad \text{and} \quad C_y(q^{-1}) = \frac{(c_1 - a_1) + (c_2 - a_2)\, q^{-1}}{1 + b_1\, q^{-1}}. \end{aligned}$$

Here, we have taken the following plant parameters

$$b_1 = -0.5, \; a_1 = -1.5, \; a_2 = 0.7, \; c_1 = -1, \; c_2 = 0.2, \; d_p = 0.5.$$

The reference signal $r_t$ was chosen to be a known unit variance and zero mean white noise signal independent of the process disturbance signal $v_t$. Note that this corresponds to signal $\bar{y}_t$ entering the deadzone that is of the same order of magnitude as $d_p$. With a signal $\bar{y}_t$ of much greater magnitude than $d_p$, this quantity would be hard to identify, the effect of the nonlinearity being swamped by the signal; if $\bar{y}_t$ is of much smaller magnitude, there is obviously an excitation problem. Using the previous closed-loop system, we have generated a data set $\{r_t, u_t, y_t\}$ with signals of length $N = 2000$. For the identification of the plant itself, we have used the following model structure

$$\begin{cases} y_t(\theta) &= \hat{P}(\theta, u_t) = \bar{\mathrm{DZ}}\,(\bar{y}_t(\theta)) \\ \bar{y}_t(\theta) &= \dfrac{q^{-1}\,(\theta_1 + \theta_2\, q^{-1})}{1 + \theta_3\, q^{-1} + \theta_4\, q^{-2}}\, u_t \end{cases} \tag{7.33}$$

where $\bar{DZ}$ is defined as in (7.31) with $d_p$ replaced by $\theta_5$. Here the parameter vector is

$$\theta = [\theta_1 \;\; \theta_2 \;\; \theta_3 \;\; \theta_4 \;\; \theta_5]^T, \tag{7.34}$$

i.e. five parameters are identified. Note that the input-output dynamics can be modeled exactly. In the sequel, we have considered situations with $\sigma^2 = 0.03$ and $\sigma^2 = 0.1$. Using the previously defined data set, we have applied two strategies:

**Strategy I:** direct standard open loop identification, i.e. we have used the data set $\{u_t, y_t\}$ as if it had been collected in open loop.

**Strategy II:** closed-loop identification with the tailor-made parametrisation obtained by interconnecting the models in (7.33) in feedback with the controller in (7.33) using the algorithm based on Figure 7.5.

We have run 20 Monte Carlo simulations for both strategies to obtain estimates of the asymptotic values of the parameters. Here the functional form of the linearisation operators of $\hat{P}(\theta)$ and $C_y$ can be easily computed analytically. We have the following results

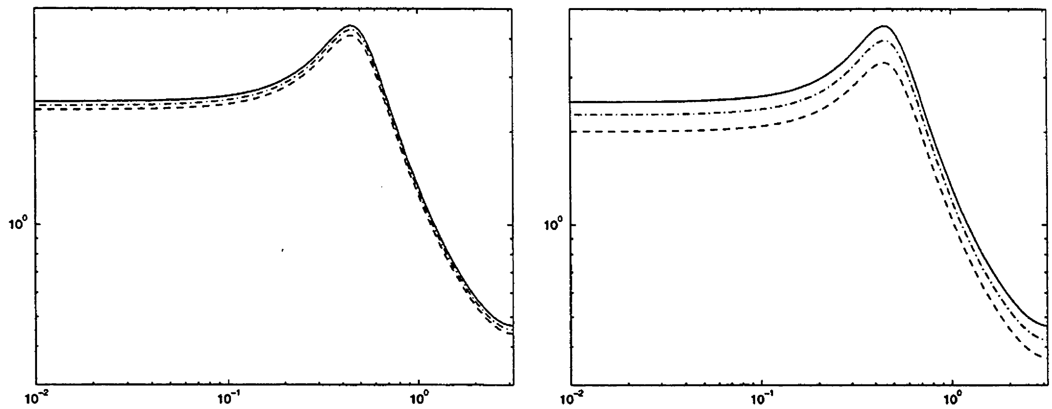
$$\Delta C_y(r, \hat{y}(\theta)) = -C_y \quad \text{and} \quad \Delta \hat{P}_u(\theta, \hat{u}(\theta))(.) = \mathrm{NL}\left(\frac{q^{-1}\,(\theta_1 + \theta_2\, q^{-1})}{1 + \theta_3\, q^{-1} + \theta_4\, q^{-2}}\,(.)\right) \tag{7.35}$$

where NL is a nonlinearity defined using the following equations

$$z_t = \mathrm{NL}(x_t) = \begin{cases} x_t & \text{if} \quad \bar{y}_t(\theta) \geq \theta_5 \\ 0 & \text{if} \quad -\theta_5 < \bar{y}_t(\theta) < \theta_5 \\ x_t & \text{if} \quad \bar{y}_t(\theta) \leq -\theta_5 \end{cases}$$

with $\bar{y}_t(\theta)$ the signal collected while simulating $y_t(\theta)$; see Section 7.3. The initial parameter vector for Strategy I was chosen to be

$$\theta[0] = [0.9 \;\; -0.4 \;\; -1.3 \;\; 0.5 \;\; 0.4]^T.$$




(a) $\sigma^2 = 0.03$ (b) $\sigma^2 = 0.1$

*Figure 7.7: Amplitude Bode plots of the linear part of the true nonlinear system (7.30) (—) and estimates of this same transfer function obtained using Strategy I (− −) and Strategy II (−·).*

This parameter vector corresponds to a model $\hat{P}(\theta[0])$ that is stabilised by $[C_r \quad C_y]$. The parameters identified using Strategy I were used as initial estimates for Strategy II. The parameter estimates are given in Table 7.1. We have compared the magnitude Bode plots of the linear part of the identified models in Figure 7.7.

The results show the degraded performance of the direct open loop identification scheme, i.e. Strategy I is unable to produce bias-free estimates. This follows from the fact that the model structure (7.33) cannot model the system exactly, i.e. the noise dynamics are unmodeled. Since the noise enters the system nonlinearly, it is difficult to use a model structure which incorporates a model for the noise dynamics. In particular, the consistency results in [22] are inapplicable. Indeed, it would be a complex task to produce a predictor for such a model structure.

As shown in the previous section, our procedure is not able always to produce consistent estimates. However as predicted, the estimates are approximately consistent in a high signal to noise situation; see Figure 7.7a. It follows from Figures 7.7a and 7.7b that the bias that affects the parameters is much higher using Strategy I.

It follows from Table 7.1 that the parameters characterising the nonlinear part of the plant are better identified using the closed loop strategy with tailor-made parametrisation, i.e. Strategy II outperforms Strategy I. Note that the quality of the identification of the parameters characterising the nonlinearity is strongly linked with the power of the signal $\bar{y}_t$. The best results are (not surprisingly) obtained with an input signal $\bar{y}_t$ that is comparable in magnitude to $d_p$. This holds for both identification strategies.

| | System | Model | Model | Model | Model |
|---|---|---|---|---|---|
| | | Strategy I | Strategy II | Strategy I | Strategy II |
| | | $\sigma^2 = 0.03$ | $\sigma^2 = 0.03$ | $\sigma^2 = 0.1$ | $\sigma^2 = 0.1$ |
| $\theta_1$ | 1 | 0.939 | 0.970 | 0.786 | 0.902 |
| $\theta_2$ | -0.5 | -0.470 | -0.4852 | -0.380 | -0.4425 |
| $\theta_3$ | -1.5 | -1.497 | -1.500 | -1.482 | -1.494 |
| $\theta_4$ | 0.7 | 0.697 | 0.700 | 0.685 | 0.696 |
| $\theta_5$ | 0.5 | 0.430 | 0.458 | 0.2485 | 0.360 |

*Table 7.1: Parameter estimates.*

The previous simulation results show that the closed identification method with tailor-made parametrisation described in Section 7.3 can be used effectively under a high SNR assumption. Also, the previous simulation example shows that our method offers a valid alternative when the noise enters the system nonlinearly.

## 7.6 Concluding Remarks

In this chapter, we have presented gradient expressions for a closed-loop identification scheme with tailor-made parametrisation. The main advantage of these gradient expressions is that they can easily be extended to non-standard identification criteria and that the plant, the parametric model and the controller are allowed to be nonlinear.

# Chapter 8

# Conclusions

This thesis has examined a number of closed-loop identification methods for nonlinear systems. This final chapter contains a summary of the work achieved in this thesis. It also outlines a few open problems that are related to the current research.

## 8.1 Coprimeness Properties

The relationship between the "Set Theoretic" and the "Bezout" approaches to left coprimeness has been investigated. This research is motivated by dual results for right coprimeness in [2]. In particular, it is shown that left coprimeness in the Set Theoretic sense implies left coprimeness in the Bezout sense. Other key results of this thesis include:

- Under the Bezout definition of right coprimeness, if we have two coprime factorisations of a given operator then the factorisations are related by a unit operator.
- Under the Bezout definition of right coprimeness, if we have one coprime factorisation and one non-coprime factorisation then they are related by a BIBO operator.
- Under either approach to left coprimeness, a coprime factorisation of a given operator can be constructed from another coprime factorisation using a unit operator. Note that the Bezout approach requires some additional assumptions.

- Under the Set Theoretic definition of left coprimeness, if we have one non-coprime factorisation we can find a coprime factorisation that is related by a BIBO operator.

Further, using the notions of differential coprimeness and total differential coprimeness a nonlinear form of the Bezout identity has been developed.

## 8.2 Identification schemes for nonlinear systems

This thesis has considered four schemes for identifying a nonlinear plant operating in closed-loop with a stabilising possibly nonlinear controller.

**Hansen Scheme** In this method the plant is represented in terms of the coprime factors of the nominal plant model and the controller along with a Youla-Kucera parameter. This factorisation based structure has previously been derived to help convert the underlying closed-loop identification problem into one that is essentially that of open-loop identification. This thesis has extended the forays into the nonlinear arena.

**Right coprime factorisations** The nonlinearity enters via the Youla-Kucera parameter and the nominal plant model. The controller remains linear as we use the linearity property to form an additional Bezout identity.

**Left coprime factorisations** The nonlinearity enters via the Youla-Kucera parameter, the nominal plant model and the controller. The results rely on the notion of differential coprimeness.

This scheme requires the system to have both of the reference signals $r_1$ and $r_2$ non-zero (refer to Figure 1.1).

**Two Step Scheme** The Two Step scheme has been adapted from [34] to work for a system that has both a stable nonlinear plant and a nonlinear controller. If there is access to measurements of both the reference signals $r_1$ and $r_2$ we do not need to have knowledge of the controller as long as it stabilises the nonlinear plant.

**Coprime Factor Scheme** This scheme also converts the closed-loop identification problem of Figure 1.1 to an open-loop like identification problem by identifying a pair of right factors of the plant. Introduction of auxiliary signals allows a pair of right *coprime* factors to be identified. Again, the plant and controller may both be nonlinear. There is one limitation in that either one of the reference signals must be zero, or else they must be filtered versions of the same signal.

**Tailor-made Parametrisation Scheme** Gradient expressions for a closed-loop identification scheme with a tailor-made parametrisation have been presented. The main advantage of these gradient expressions is that they can easily be extended to non-standard identification criteria and that the plant, the parametric model and the controller are allowed to be nonlinear.

## 8.3 The continuing mission

This section contains a few problems that are related to the work achieved in this thesis.

**Coprimeness properties** There are still some open questions on the relationship between the Bezout and the Set Theoretic definitions of coprimeness. The current knowledge is depicted in Figure 3.4, which is also an illustration of the open questions in this area.

- It is still unknown whether coprimeness in the Bezout sense implies coprimeness in the Set Theoretic sense for nonlinear left coprime factorisations.
- Under the Bezout definition of coprimeness, if $P = D_{l1}^{-1} N_{l1} = D_{l2}^{-1} N_{l2}$ and $(N_{l1}, D_{l1})$ and $(N_{l2}, D_{l2})$ are left coprime pairs. Then are there two factorisations related by a unit?

  If $P = D_{l2}^{-1} N_{l2}$ with $(N_{l2}, D_{l2})$ coprime in the *Bezout* sense, then any BIBO $W$ with $W^{-1}$ existing defines another realisation $P = D_{l3}^{-1} N_{l3}$ with $N_{l3} = W N_{l2}$, $D_{l3} = W D_{l2}$. It is unknown whether any realisation $P = D_{l3}^{-1} N_{l3}$ implies that $W = D_{l3} D_{l2}^{-1}$ is well-posed and stable. (In contrast to the Set Theoretic situation, an example with a non-BIBO $W$ is lacking.)

**Hansen Method** Consider the Hansen identification method for *right* coprime factorisation representations of the plant. We have examined the problem when the plant, Youla-Kucera parameter, and the nominal plant model are nonlinear while the controller remains linear. One topic that could be investigated is what happens when the controller is also allowed to be nonlinear? Like the similar case using *left* coprime factorisations this would probably invoke the use of differential coprimeness. Further, one could examine the relationship between small nonlinearities in the plant and the Youla-Kucera parameter. Does a small nonlinearity in the plant produce a small nonlinearity in the Youla-Kucera parameter? What is the relationship between the Youla-Kucera parameter R and $(P - \hat{P})$ in a nonlinear setting?

**Coprime Factor Method** Recall that this method estimates a right coprime factorisation of the plant in an open-loop fashion by introducing data filters. In the linear case the remaining freedom in the data filters can be used to construct a normalised factorisation of the plant. It is worth investigating if the same is true in the nonlinear case.

**Windsurfer approach** Another goal would be to devise a nonlinear adaptive control scheme, analogous to the "windsurfer" approach to the adaptive control of linear systems, achieved in fact not by continuous adjustment of the controller, but by an iterative process of controller design and identification. Successive controller designs achieve progressively wider closed-loop bandwidths.

For there to be a nonlinear analog, one would need a scheme for closed-loop identification of a nonlinear plant, given a nonlinear, a priori model and a nonlinear stabilising controller of the plant and model. [One might need a number of other things too, including a scheme for approximation of a high order nonlinear model by a lower order model].

**Stable Kernel Representations** The work of [27] suggests that Youla parametrisations are best obtained for nonlinear systems using stable kernel factorisations, the analog of left coprime factorisations. This may be useful in identification.

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