Theoretical Atom Optics:
Light Forces and Band Gaps in Hollow Optical Fibres,
Input-Output
and Atom Stimulation

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This thesis is an account of research undertaken in the Department of Physics, Faculty of Science, at the Australian National University between February 1994 and January 1997.

This research was supervised by Dr Craig Savage, but unless otherwise indicated, the work presented herein is my own.

None of the work presented here has ever been submitted for any degree at this or any other institution of learning.

Joseph James Hope
30 January 1997
Acknowledgments

The work presented in this thesis may be my own (unless otherwise indicated) but I owe my happiness, inspiration and ability to do any research at all to my family and friends. Above all, I would like to thank my supervisor Craig Savage for his well timed encouragements, his infectious enthusiasm, and for all his help over the last three years. I have been supported by an Australian Postgraduate Research Award.

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You were all there, and it mattered.
A new method for producing a mechanical potential for atoms with light fields is presented. This method involves a Raman transition instead of a two-level transition, and it allows potentials to be created for very cold atoms with greatly reduced spontaneous emission.

The band structure of the energy spectrum of atoms in a hollow optical fibre waveguide is calculated. The existence of a gap in this spectrum is demonstrated, and applications of this band gap are proposed.

The optical input/output theory is generalized to atomic fields, and dynamical equations are produced and solved for a single mode atomic cavity coupled to the freely evolving external modes.

The emission of a photon is shown to be stimulated by the presence of a Bose-Einstein condensate. The feasibility of this experiment is examined. The atom-stimulation of a nuclear process (beta decay) is also calculated, and shown to be infeasible for current experiments.
Aspect

The term "aspect" in the context of software development refers to the various dimensions or perspectives from which software is viewed. This can include functionality, performance, security, usability, and maintainability. Understanding these aspects helps in designing robust and effective software products. Thank you to Christine for being a great learning partner and for pointing me in the right direction on this topic. Thank you to Charles, David, and Nigel for being fantastic contributors to our discussions. This book would not have been possible without your support and input.

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The messenger collapsed on the steps of the Acropolis.
He delivered the news from Marathon before he died.'

-R. Zelazny, Roadmarks

Quantum mechanics states that every piece of matter must be described by a wavefunction rather than merely a position and a velocity. Although this fundamental theory has been understood for the largest part of this century, it has only been during the last fifteen years or so that it has had much relevance to anything larger than a neutron. Atom optics is the study of the wavelike nature of atoms, and the construction of devices that observe and utilize their quantum mechanical nature. This wavelike nature is normally unobservable, because the de Broglie wavelength is smaller than the atom itself at room temperatures. In fact, it requires temperatures as low as microKelvin to produce atoms with wavelengths on the order of micrometres.

Atom optics has become a growing field and atom optical devices have been made feasible by the advent of laser cooling and trapping. Optical techniques have proved very useful in creating and manipulating very cold samples of atoms. In conjunction with evaporative cooling, it has become possible to produce a sample of atoms whose wavelength is larger than the average interatomic spacing, which is the main requirement for producing a Bose-Einstein condensate. At these temperatures, it is possible to examine the effects of the quantum statistics of the atoms, as the classical approximation of the distinguishability of individual particles breaks down.

The construction of analogues of optical devices using atoms is a primary goal of atom optics. In particular, atomic interferometers can be used to measure gravitational and inertial effects with unprecedented accuracy. They can also measure atomic parameters such as polarisability and refractive index. Such interferometers require the existence of large angle coherent beam splitters, atomic mirrors, and a highly coherent, bright atomic source. The most successful techniques for manipulating atoms involve various kinds of light
induced forces, or strong static magnetic fields. The sensitivity of these interferometers is limited by their size, which is in turn limited by the coherence length of the atoms. This means that atoms must travel in ultra-high vacuum. Using hollow optical fibres or other atomic waveguides to transport the atoms in such devices may turn out to be very important in allowing them to be constructed on a large scale.

1.1 Thesis Plan

This thesis is arranged as follows:

In this chapter we will briefly review the optical forces available to manipulate atoms, and their limitations. We will also review the methods and limitations of current cooling techniques. We will then introduce the concept of Bose-Einstein condensates, and briefly describe the status of current experiments involving this new state of matter. Finally, we will preview the major results of this thesis.

In Chapter 2 we will review the practicalities and possibilities for utilizing hollow optical fibres as atomic waveguides. This chapter will include original work in the examination of the effects of non-linear optical processes in reducing the coherence of the atoms in the fibre.

In Chapter 3 we will introduce an original method for manipulating atoms by utilizing a Raman transition to produce a potential. The advantage of this scheme is that it allows small potentials to be generated with a much lower spontaneous emission rate. This is important, as the spontaneous emission of photons is the factor which limits the coherence of atoms in an optically induced potential. The increased coherence available from using this potential may allow extremely cold atoms to be manipulated by optical methods, which are typically more flexible than magnetic fields. It also allows many of the applications of hollow optical fibres discussed in the following chapters to be realized.

In Chapter 4 we show that it is possible to produce a band gap in the energy spectrum for atoms being guided by a hollow optical fibre. This band gap may be used to produce many effects including a velocity selective waveguide. Since the band gap is produced by interaction with an optical potential, it may be altered while the atoms are being guided. Such a dynamic band structure has not examined before, but may lead to such effects as cooling.

In Chapter 5 we derive a theory for the input and output of atoms from an atomic trap. This theory may be applied to a model for an atom laser, which may be the ideal atomic source for atom optics experiments. We determine the conditions required to produce an output with a very small linewidth.
In Chapter 6 we demonstrate that it may be feasible to produce an experiment which can detect the atom-stimulation of a photon emission due to a BEC. In Chapter 7 we use similar calculations to examine the feasibility of stimulating the emission of a massive particle. In particular we examine the stimulation of the process of beta decay.

1.2 Light Forces

1.2.1 Spontaneous Forces

The simplest force that can be produced by a light field is the force due to absorption and spontaneous emission of photons. If an atom is in a light field, it may absorb a photon from the field, which will give it a momentum kick in the direction of the beam. It will emit a photon in a random direction. Over a large number of absorption and emission cycles, the random kicks will average to give a zero net momentum transfer, while the kicks from the absorption will add to give a force directed along the direction of the beam. This force is not quite as crude as it seems, as the doppler shift will bring the atoms in and out of resonance with the incident light field, so the spontaneous force can be made velocity selective by using a light field with a narrow linewidth.

Although this force is very strong, its major drawback is that it can only be used in a dissipative manner, as each photon emission destroys the coherence of the atom.

1.2.2 Dipole Force

By contrast to the spontaneous force, the dipole force is conservative, and can be described by a potential. A laser field will produce such a potential when it is highly detuned from an electronic transition in the atom. There are several useful pictures for understanding the physics behind this potential, and we will present two of these rather than derive the form for this potential, which can be found in many references [1-5]. The potential $V$ is proportional to the square of the Rabi frequency $\Omega$ of the laser field, and inversely proportional to the detuning $\Delta$,

$$V = \frac{\hbar \Omega^2}{4\Delta}$$  \hspace{1cm} (1.1)

where the detuning $\Delta$ is positive when the laser frequency is larger than the transition frequency. When the detuning is positive, the atoms are repelled from the regions of higher laser intensity, and when it is negative, the laser field attracts the atoms. Near resonance, the excited state has a non-negligible population, and the spontaneous force dominates.
The simplest explanation for this phenomenon is based on the dressed state picture [1,6] in which we consider the total Hamiltonian for the atom-light interaction and diagonalise it. The diagonal basis is called the dressed state basis, and the dressed states are a superposition of the atomic and photonic states. The eigenvalues of this Hamiltonian are shifted from the "bare" atomic energy levels by an amount which depends on the Rabi frequency. This light shift is what is acting as a potential.

We can also understand the exact form of the potential by considering the population of the excited state of the atom. In the limit of large detuning, we know from the optical Bloch equations that the population of the excited state of the atom is small, and equal to \( (\Omega/2\Delta)^2 \). However, for each transition that the atom makes to the excited state, it is receiving either too much \((\Delta > 0)\) or too little \((\Delta < 0)\) energy from the photon. This energy difference is given by \( \hbar \Delta \), and it must come from (or go to) the kinetic energy of the atom. Therefore, as an atom moves within the field, the population of the excited state \((\Omega/2\Delta)^2\) times this energy difference \( \hbar \Delta \) is acting exactly like a mechanical potential of the form \( \hbar \Omega^2 / 4\Delta \).

The main limitation to this potential is that the weak excited state population may cause a spontaneous emission event to the ground state, which will cause a loss of coherence for the atom.

### 1.2.3 The effect of spontaneous emission on atomic coherence

One of the simplest measures of the first order coherence of a de Broglie wave is the visibility of fringes in an interference experiment. The loss of coherence due to a spontaneous emission event has been measured in this way for metastable Helium atoms [7]. If a de Broglie wave can be described by a pure wavefunction \( \Psi(x) \), then the phase information in this wavefunction will lead to the presence of interference fringes in (for example) a two slit experiment. Under the presence of a spontaneous emission event, this wavefunction will gain a "kick" and become the new wavefunction \( \Psi(x)e^{-i\mathbf{k} \cdot \mathbf{x}} \) where \( \hbar \mathbf{k} \) is the momentum of the emitted photon. This random shifting of the phase of the wavefunction lowers the coherence, and therefore suppresses interference effects.

The standard definition of the first order spatial coherence \( g^{(1)}(z) \) for a particle described by a wavefunction \( \phi(z) \) is

\[
g^{(1)} = \int \phi(z' - z) \phi(z') \, dz'
\]

(1.2)

This coherence is proportional to the Fourier transform of the momentum distribution of the atom. This can be generalised to a statistical mixture of atoms
§1.3 Cooling

described by a density matrix $\rho$:

$$g^{(1)} = \mathcal{F}_z\{I(k)\}$$

where $I(k) = \langle k | \rho | k \rangle$ is the momentum distribution of the object, and $\mathcal{F}_z$ is the Fourier transform. From this we find that an isotropic photon emission will reduce the coherence by a factor of $\text{sinc}(kz)$. This is because the photon emission will spread the atomic momentum distribution by a convolution, and the $\text{sinc}$ function is the Fourier transform of that spread. Atoms which are below the recoil limit have a spatial coherence length which is much larger than this function, so their coherence is destroyed.

The destruction of coherence as measured by this parameter will mean that atoms which undergo a spontaneous emission will thereafter not produce interference effects. The incoherent background of an interference experiment will therefore be equal to the proportion of atoms which have undergone a spontaneous emission, even in the absence of other decohering processes.

1.3 Cooling

There are many methods of cooling, but they may be divided into two main categories. The first involves the spontaneous emission of photons as the dissipative part of the process, and the second involves the removal of atoms from the sample.

The spontaneous force can be used to cool atomic samples. The doppler effect can be utilized by using a red detuned laser, which means that an atom will preferentially absorb a photon moving in the opposite direction to its own motion. This will damp the motion of the atoms in an atomic sample, and they will be cooled. The energy is lost to the spontaneously emitted photons. The minimum temperature attainable by this process, called the doppler limit, occurs when the doppler linewidth of the transition is too large for the detuning of the laser beam to discriminate between atoms which are approaching the beam and those which are moving away. Using Raman transitions, which have extremely narrow linewidths, it is possible to produce extremely good cooling for which the doppler limit is actually extremely low [8].

Temperatures below the doppler limit can be achieved by spatially dependent optical pumping. It is possible to produce cooling through spontaneous emission by placing the atoms in a field with periodically varying polarisation [9]. In this configuration, the atoms are in a periodic potential due to light shifts of the atomic levels. A careful study of the light shifts and spontaneous emission probabilities shows that an atom is more likely to emit a photon at a peak of the light shift, and find itself at the bottom of the potential for the other
dressed state. This means that on average the atoms will lose energy through spontaneous emission until they cannot travel freely across the peaks of the light shift potential. This so called Sisyphus cooling has a minimum temperature, called the recoil limit, where each atom has the kinetic energy that would result from the emission of a single photon.

The recoil limit can be avoided by arranging the light fields so that there is a “dark state” around zero momentum which does not interact with the light, and will not absorb another photon. The atoms are excited, and spontaneously emit photons until they happen to land in the dark state, whereupon they stop absorbing. The most successful method of dark state cooling is called velocity selective coherent population trapping (VSCPT) [10], and the dark state is actually a superposition of internal states with zero dipole moment. This technique has no theoretical minimum temperature, but is practically limited in current experiments to a little below the recoil limit.

The lowest experimental temperatures have been achieved by evaporative cooling. This is a standard cooling method in which atoms are trapped in a thermal equilibrium and the hottest ones are allowed to escape (or are forcibly removed), and the remaining sample is allowed to rethermalise. This is done in a magnetic trap so that there are no losses due to photon recoil, and the atoms with the highest energy are removed from the sample by changing their magnetic sublevel to an untrapped state with a frequency stabilized radio signal [11]. Temperatures achievable by this method appear to be limited by collisions with the background gas. The number of atoms in the final sample is limited by the fact that the cooling process removes atoms.

1.4 Bose-Einstein Condensation

As temperatures get lower, and the wavefunctions of the atoms overlap, their behaviour begins to depend on their quantum statistics. If the atoms are bosons, the symmetry of the wavefunction causes them to have remarkable properties.

The Bose-Einstein distribution (BED), which describes the energy spectrum of atoms in thermal equilibrium, has a threshold at low temperature below which a macroscopic proportion of atoms will “condense” into the ground state. This phenomenon, known as Bose-Einstein condensation (BEC), has been realized in recent experiments [12–14], where samples of rubidium and sodium atoms have been condensed. A quantum degenerate sample of lithium atoms has also been produced [15].

The physics of a BEC is as counterintuitive as the light inside a laser cavity, the electron pairs in a superconductor and the atoms in a superfluid. In fact, the physics behind all of these phenomena is identical, but in the BEC of
a weakly interacting gas, the effects of the quantum statistics are more easily separated from the effects of the particle interactions in the case of superfluidity or superconductivity.

The theory of condensates and their dynamics is an extremely active topic. A review on the basic theory of BEC can be found in references [16–19]. There has been extensive examination of the effect of the sign of the scattering length on the formation of a BEC [20–22]. There have been calculations which directly model the results of the early experiments [23–25]. The phase diffusion of a trapped BEC has been calculated, and it has been proposed that this phase diffusion may be measured by scattering of far off-resonant light [26]. The effects of scattering light off condensates has been calculated in several limits [18, 19, 27, 28], and the effects of light on the interatomic collisions have also been determined [29–31].

The presence of atom-atom interactions allows the possibility of many nonlinear effects in atom optics, some of which have been predicted [32–36]. The tunneling between two Bose-Einstein condensates has been examined [37, 38], and the interference of two separate condensates has been investigated theoretically [39–43]. This has led to an understanding of the equivalence of a BEC which has formed with a well defined but randomly chosen phase (spontaneous symmetry breaking) and a BEC which is in a number state where the number of atoms in each sample is initially unknown.

Bose-Einstein condensates have been excited to observe the excitation spectrum [44, 45], which have agreed well with theoretical predictions [46, 47]. They have been detected in a nondestructive manner [48], produced with as many as \(5 \times 10^6\) atoms [14] and temperatures as low as 20nK [12]. Two overlapping condensates of atoms in different magnetic sublevels have been produced by sympathetic evaporative cooling [49]. Experiments have also been made to couple the atoms coherently out of the BEC into an untrapped state [50].

1.4.1 Atom Laser

It seems natural to want to produce a coherent source of atoms for atom optics experiments. The large number of atoms in the ground state of the trap, which occurs during the production of a BEC, seems to be an ideal starting point for designing such a device. If a similar process could be used to produce a sample of atoms inside a trap continuously, and the trap had some weak coupling to the outside world, then it would be possible to make an analogue of a laser for atomic bosons instead of photons. There are currently many models for an atom laser [51–59].

We have modeled an atom laser by considering a trap produced within an atomic waveguide, and with an output coupling provided by a Raman tran-
tion into an untrapped state [59]. This system appears to be a reasonable candidate for the design of an atom laser which can then deliver atoms directly into an optical system.

1.5 Summary of Major Results

The major results of this thesis are as follows:

1. Non-linear effects in hollow optical fibres do not reduce their effectiveness as atomic waveguides. This work has been published in the proceedings of Seventh Rochester Conference on Coherence and Quantum Optics [60].

2. A Raman transition may be used to produce an optical potential for very cold atoms which does not have the disadvantage of decoherence through spontaneous emission. This work has been published in the Physical Review [61].

3. It is possible to produce gaps in the energy spectrum for atoms guided by a hollow optical fibre. These band gaps may be used to produce velocity selective components for atom optics experiments, and atoms may be cooled while they are being guided. This work has been published in the Physical Review [62, 63].

4. An input-output theory for atoms coupled to an atomic cavity can be formulated and solved analytically in terms of simple mathematical techniques. This theory is applied to current and proposed experiments, and shows that a narrow linewidth may be obtained by:
   (i) a narrowband output coupling, or by
   (ii) coupling the atoms out of the cavity sufficiently slowly.
   This work has been published in the Physical Review [64].

5. It is feasible to conduct an experiment to measure the atom-stimulation of the emission of a photon. A BEC with the number of atoms currently available in experiments will be able to increase the total emission rate by more than a factor of ten. If the photons are detected over a small solid angle, then the signal to noise ratio of stimulated to spontaneous emission is larger than a thousand. This work is a generalization of a paper which has been published in the Physical Review [65].

6. The atom-stimulation of a nuclear process has been examined, and
§1.5 Summary of Major Results

found to be infeasible with current experimental conditions. This work has appeared in Physics Letters [66].

Hollow Optical Fibres

Overview

This chapter reviews the possibilities for using hollow optical fibres as atomic waveguides, and examines some of the practicalities that may limit their use. The original parts of this work have appeared in the proceedings of Seventh Rochester Conference on Coherence and Quantum Optics [60].

2.1 Introduction

As atoms become cooler, they become extremely slow, and gravity becomes an important factor. A waveguide can support the atoms, and reduce this problem. Background gas collisions are very destructive to long term experiments so experiments must be performed in ultra-high vacuum. Transferring atoms through long distances will therefore be more practical with some kind of vacuum isolated atomic waveguide. It has been proposed that an optical fibre with a hole in the centre of the capillary be used to guide atoms [67, 68]. This has already been demonstrated [104-106]. A survey of the theory behind these waveguides and the earlier experiments has been produced by Bowling and Gué-Paradis [198].

There are many problems that may arise when using hollow optical fibres to contain atoms coherently. Most of these revolve around the presence of the glass close to the atom, which can interact with atoms in several different ways. The first is through the van der Waals interactions, or more properly the Casimir-Polder force, which could attract atoms to the surface. A second form of dispersive interaction is the photonic surface processes in the glass which may produce light which is frequency detuned towards resonance, causing greater spontaneous emission losses.


1.5 Summary of Major Results

The major results of this thesis are as follows:

1) Non-linear effects in novel optical systems can reduce their effectiveness in advective waveguides. This work has been published in the proceedings of General Relativistic Dynamics (10).

2) A fiber amplifier may be used to produce an optical potential for wave multi-particle systems that do not have the disadvantages of degenerate multi-particle interactions. This work has been published in the Physical Review B (9).

3) Linear gain in multi-guide systems is used to amplify spectrum from guided by selective optical amplification. These broadband signals may be used to produce wide-range intense amplification for their optical experiments, and shown to have certain limitations that may be overcome (9).

4) An input-output device that allows responses to an atomic cavity can be constructed and amplified using cavity and super-atomic techniques. This device is applied to current and preparatory experiments, and shown to have certain limitations that may be overcome (9).

5) Measurements of the gain and decay of cavity-limited signals have been published in the Physical Review (9).

It is therefore concluded that improvements can be made to recent advances in the field of quantum mechanics, in particular, at the conclusion of a photon. Additional work in the field of quantum mechanics will be able to improve the main conclusion (9) by some future research. If the plasma is diluted over a small area, then the signal in noise ratios are determined by quantum mechanical techniques in a way that improves the model. This leads to a generalization of a theory which has been published in the Physical Review (9).

The above conclusions about plasma research has been mentioned, and
Chapter 2

Hollow Optical Fibres

Overview

This chapter reviews the possibilities for using hollow optical fibres as atomic waveguides, and examines some of the practicalities that may limit their use. The original parts of this work have appeared in the proceedings of Seventh Rochester Conference on Coherence and Quantum Optics [60].

2.1 Introduction

As atoms become cooler, they become extremely slow, and gravity becomes an important factor. A waveguide can support the atoms, and negate this problem. Background gas collisions are very destructive to long term coherence, so experiments must be performed in ultra high vacuum. Transporting atoms through long distances will therefore be more practical with some kind of vacuum isolated atomic waveguide. It has been proposed that an optical fibre with a hole in the centre of the core can be used to guide atoms [67, 68]. This has already been demonstrated [69-71]. A review of the theory behind these waveguides and the earlier experiments has been produced by Dowling and Gea-Banacloche [72].

There are many problems that may arise when utilising hollow optical fibres to conduct atoms coherently. Most of these involve the presence of the glass close to the atoms, which can interact with them in several different ways. The first is through the van de Waals interaction, or more properly the Casimir-Polder force, which could attract atoms to the surface. A second form of dispersive interaction is through non-linear processes in the glass which may produce light which is frequency shifted towards resonance, causing greater spontaneous emission losses.
2.2 Hollow Optical Fibres

Hollow optical fibres which carry light in the glass core have an evanescent field which penetrates the hollow in their centre. If this light is detuned above a transition in the atom, then the atom will be repelled from the walls by the dipole force. The fibre can therefore guide atoms along their hollow centre region. If a red detuned laser can be guided by a fibre such that the intensity is at a maximum in the hollow centre, then the atoms will be attracted to the centre and can be guided by the fibre. Atoms can therefore be guided by a hollow optical fibre in either of these situations. The red detuning method has the disadvantage that the atoms spend a large fraction of their time interacting strongly with the field. This produces a larger population in the excited state, and therefore a higher spontaneous emission rate. When a blue detuned laser is used to repel the atoms, there are lower spontaneous emission losses, as the atoms spend less time interacting with the strong field.

The red detuning scheme was first discussed by Ol’Shanii et al. [73], who suggested that a beam could be guided down the hole in the centre of a fibre by grazing incidence reflection and that atoms could be guided in this beam if it was red detuned from an atomic transition. This was first realised experimentally at JILA [69], where a straight segment of a glass capillary tube with a length of approximately 3cm and hollow diameter of 40μm was used to guide rubidium atoms. A tunable laser was guided along the hollow part of the fibre by grazing incidence and the atoms were found to be strongly guided when the laser was tuned about 4GHz below the atomic D2 resonance. The atomic flux was increased by a factor of approximately three over the large detuning limit. The flux of atoms which went straight through the fibre was reduced by a similar factor when the laser was tuned above the resonance frequency, and the atoms were repelled by the light towards the walls.

Savage et al. proposed the concept of a hollow optical fibre waveguide in which the light was carried in the glass, and the atoms were guided away from the walls of the hollow by the evanescent field penetrating the vacuum [67,68]. The advantage of this technique was simply that the atoms spent less time interacting with the light field, and would therefore experience less spontaneous emission events. This is extremely important, as spontaneous emission losses destroy the coherence of the atomic field. Atoms have been reported to have been guided by evanescent waves by two groups. The first, at JILA, used a fibre with a 20μm diameter hollow in the core to guide rubidium atoms. The best result that was reported was a flux enhancement of a factor of three over the ballistic flux. This occurred at an optimum blue detuning of 3GHz. The second group of experiments by Ito et al. [71] were more promising. They were also guiding rubidium atoms, but managed a factor of 20 improvement over
the ballistic flux. The guided flux was sufficiently sensitive to the detuning from the atomic transition that the F=2 and F=3 hyperfine levels could be resolved and guided independently. Furthermore, the hollow part of the fibres had diameters of 7 \( \mu \text{m} \) and 2 \( \mu \text{m} \), which is much closer to the regime where the guided light may be in a single mode of the fibre. Micron sized atomic waveguides are also required to have well spaced transverse structure. This is important when trying to produce such effects as band gaps, which are described in Chapter 4, or some models of atom lasers, such as one produced at ANU and mentioned in Chapter 5.

Atomic interferometers are one device in which hollow optical fibres are expected to be useful in transporting atoms. The atoms in an interferometer must experience an ultra-high vacuum, or the atomic coherence length is severely reduced. As the size of these interferometers is increased, however, it becomes harder to produce an ultra-high vacuum. Hollow optical fibres have been demonstrated to be able to solve this particular problem as they can be “flushed” with helium, which will then diffuse through the glass. The glass is then opaque to the rest of the external atmosphere, so large sections can be brought down to a suitably low pressure in reasonable time.

Hollow optical fibres may be a practical way to produce very small scale confinement for atoms, as the dimensions of the interior hole can be made very small. It is also feasible to shine light transversely through the sides of the fibre to produce an atomic cavity.

### 2.3 Casimir-Polder forces

The glass walls will attract the atoms in the hollow core through the short range Casimir-Polder force. In the limit of a large fibre, it has already been calculated that the effect of this force will be to approximately halve the potential barrier seen by the atoms [67]. Since we are usually interested in atoms which have much less energy than the height of the optical potential, they should still be well confined.

Unfortunately, for smaller core radii the effect of this force could possibly be more pronounced. The exact nature of the Casimir-Polder force in cylindrical geometry is unknown, but it is expected that the effect of this interaction will be to lower the barrier height by a similar factor as for the large radius limit. This means that calculations of the atomic modes for small core radii will only be accurate for energies that are very much lower than the maximum barrier height. Fortunately these are usually the most interesting atomic modes.
2.4 Non-linear processes

The existence of glass in the waveguide will inevitably produce some non-linear effects as well as guide the light. These non-linear processes produce photons of different frequencies to the original beam which may cause unwanted excitation of the excited state. This would occur if the field was shifted closer to resonance, which would mean that the spontaneous emission losses could be greatly increased. The two possible processes by which photons of different frequency may be generated are Raman scattering and Brillouin scattering. Both of these processes are very weak as spontaneous processes, but in optical fibres a stimulated process can occur as the photons from earlier scattering events cause an enhancement of the scattering rate.

2.4.1 Stimulated Raman scattering

Raman processes can occur in the glass of an optical fibre. If some of the photons are absorbed by a transition in the glass and re-emitted into a different ground state, then there will be a corresponding frequency shift for that photon. Depending on the parameters of the fibre, this frequency shifted photon can then stimulate a second Raman transition, causing a non-linear gain in this secondary beam. In some fibres it is possible for this secondary beam to eventually take all of the power of the original beam. We require the light to have a particular well defined detuning relative to the atomic resonance, so we must ensure that this non-linear process creates a negligible number of photons.

Raman scattering causes a forward and backward traveling Stokes beam to propagate in the fibre with differing intensities [74]. The ratio of the power in the forward scattered beam to the input power, $p_{\text{in}}$, of the light, at a distance $d$ from the start of the beam, is given by

$$F_{RF}(d) = \frac{h\omega_s \Delta \nu_{\text{FWHM}}}{2} \sqrt{\frac{\pi A}{p^2 g_R L_{\text{eff}}}} e^{p g_R L_{\text{eff}}/A} \tag{2.1}$$

where $\Delta \nu_{\text{FWHM}}$ is the full width half maximum around the peak in the gain profile for the Raman scattering, $\alpha$ is the decay (absorption) rate for both beams in the fibre, $L_{\text{eff}} = [1 - \exp(-\alpha_p d)]/\alpha$ is the effective length of the fibre, $g_R$ is the peak of the Raman gain profile, $\omega_s$ is the frequency of the gain peak, and $A$ is the effective cross-sectional area of the fibre.

The ratio of the power in the backward scattered beam is given by

$$F_{RB}(d) = \frac{h\omega_s \Delta \nu_{\text{FWHM}}}{2} \sqrt{\frac{\pi A^3}{p^5 g_R L_{\text{eff}}^2}} e^{p g_R L_{\text{eff}}/A} \tag{2.2}$$
The parameters in the above equations have typical values (taken from reference [74]) of $\Delta \nu_{\text{FWHM}} \approx 3 \times 10^{11}\text{s}^{-1}$, $\alpha = 1/(10000\text{m})$, $g_R = 5 \times 10^{-12}\text{m/W}$, $A \approx 10^{-11}\text{m}^2$ and $\omega_s = 13.2\text{THz}$. Using these values, we find that for lengths of the order of a metre, there was no time for the Raman scattering to produce a stimulated beam. The spontaneous Raman scattering is far too weak to limit the coherence time of the atoms. This effect will have to be considered more carefully if the atoms are carried for distances of the order of several kilometres. In that case, the atoms would experience a weak Stokes laser field that will have a peak around 13THz below resonance, and a linewidth of 3THz.

### 2.4.2 Stimulated Brillouin scattering

Stimulated Brillouin scattering is an extremely similar problem to stimulated Raman scattering, except that the photons are scattered off the phonons in the glass rather than the internal structure of the atoms. In the Brillouin scattering, there is only a backward traveling wave. The ratio of the power in the scattered beam is given by

$$F_B(d) = \frac{\sqrt{\pi}}{2} \frac{\omega_s}{\omega_g} \Delta \nu_{\text{FWHM}}^{\text{Brill}} k T \sqrt{\frac{A^3}{p^5 g_B^2 L_{eff}}} e^{p g_B L_{eff} / A},$$

(2.3)

where $\Delta \nu_{\text{FWHM}}^{\text{Brill}}$ is the full width half maximum around the peak in the gain profile for the Brillouin scattering, $T$ is the temperature, $k$ is the Boltzmann constant, $\omega_g$ is the phonon frequency, $g_B$ is the peak of the Brillouin gain profile and $\omega_s$ is the frequency of the gain peak.

As for the Raman scattering case, the gain is such that for the parameters given in reference [74], there will be only spontaneous effects. These are extremely small and this process will not produce a limit to the coherence time. If the fibre carries atoms for several kilometres then a stimulated beam will be generated with a peak around 10GHz from the original laser frequency and a linewidth of about 0.1GHz. These distances are not expected to be possible in atom optics experiments, so we conclude that these non-linear processes will not affect the atomic coherence.

### 2.5 Spontaneous Emission

As with all devices constructed using the dipole force as a potential for atoms, there is a certain level of spontaneous emission due to the partial excitation of the atoms. This spontaneous emission rate becomes higher when the atoms are immersed in a stronger light field. As hollow optical fibres are made smaller,
which is necessary in many proposed atom optics experiments, the atoms unavoidably spend more time in a larger light field. This means that spontaneous emission can destroy the coherence of the atoms, and some devices that have been proposed later in this thesis are in fact limited by this effect.

In the following chapter, we describe a method by which it may be possible to produce an optically induced potential for atoms which avoids the problem of spontaneous emission.
Overview

This chapter describes how a Raman transition may produce a far-detuned optical potential for atoms and shows that the spontaneous emission rate may be reduced. This work has been published in the Physical Review [61].

3.1 Introduction

The field of atom optics has made great advances by utilising the potential generated by light fields to create a coherence preserving method for manipulating atoms [1]. As experiments are conducted at increasingly lower temperatures, optical potentials are becoming less desirable due to the problem of maintaining the coherence of the atomic de Broglie waves against random recoils from spontaneously emitted photons. Maintaining atomic coherence is essential for applications such as atom-interferometry [75, 76] and experiments on Bose-Einstein condensates [12, 19, 27, 50, 65, 66]. For these applications, experiments to date have utilised magnetic forces [11, 14], which are far less flexible but have the desired stability. The band gaps for atoms in a hollow optical fibre, as described in Chapter 4, are limited by the spontaneous emission rate, and the proposed atom laser made from a hollow optical fibre, as described in Chapter 5, may also require a lower spontaneous emission rate to be practical.

Atoms experience a mechanical force when there are spatial gradients in the atomic electric dipole interaction energy. Such gradients result from inhomogeneous light fields, for example [3–5]. The resulting forces have been used in experiments to trap, reflect, and diffract atoms with evanescent fields [77–81]. When a two-level atom interacts with a far detuned laser, by which we mean that the detuning must be large compared to the Rabi frequency and the natural linewidth of the of the transition, it experiences a potential with a
height proportional to $I/\Delta$, where $I$ is the maximum light intensity and $\Delta$ is the atom-light detuning, positive for so called blue detuning. The associated population of the excited state of the transition, and hence the spontaneous emission rate, is proportional to $I/\Delta^2$. For large enough detunings and intensities the spontaneous emission rate can be made arbitrarily small while the potential remains fixed. For large potentials the minimum spontaneous emission rate is limited by the available laser power. However for sufficiently low potentials it is limited by the maximum that the laser can be detuned from the transition without interacting with other atomic levels.

In this chapter we extend the two-level scheme to a three-level Raman scheme. The key advantage of this scheme is that for potential barriers of the order of the recoil energy or less spontaneous emission rates can be greatly reduced.

Raman transitions have been used for atomic cooling [8, 82] and to produce atomic mirrors and beam splitters. Atomic beam splitters utilise the $2\hbar k$ momentum transfer to the atom due to large scale population transfer between levels by the Raman process [83, 84]. The interaction between two lasers and a $\Lambda$ configuration three level atom is also used to generate the cooling process known as velocity selective coherent population trapping (VSCPT) [5, 85]. The purpose of VSCPT is to cool atoms by populating atomic states with increasingly weaker interactions with the light. By contrast we are concerned with producing a general effective potential for atoms with which they can interact for extended periods of time. If the problem of spontaneous emission can be controlled, this potential can be used to construct atomic optical devices such as those based on hollow optical fibers [62, 63, 67, 68, 70, 71, 86]. It may also be used to provide optical potentials which can manipulate Bose-Einstein condensates for long periods of time, which will provide many advantages in flexibility over magnetic traps.

The far detuned two-level mechanical potential can be quantitatively explained as the product of the excited state population and the energy excess $\hbar \Delta$ associated with exciting an atom. The difference between the photon and atomic energy appears as potential energy. This picture applies to the Raman case, Fig. 1, provided the additional contribution from the two-photon transition is included. This contribution to the potential is the product of the population in the extra level and the two-photon energy excess $\hbar \delta$. Remarkably, the total Raman potential has the same form as the two-level potential, with a scaled one-photon detuning.

We will describe the Raman system and derive the form of the far detuned two photon potential in section 3.2. In section 3.3 spontaneous emission in the two-level and Raman schemes will be discussed, and the minimum rate of spontaneous emission for each scheme will be compared. Section 3.4 will
§3.2 Derivation of the Potential

We now describe our system in detail. The atom has three levels: two lower levels \( |1\rangle \) and \( |2\rangle \), and an upper level \( |3\rangle \). The frequencies of the \( |1\rangle \leftrightarrow |3\rangle \) and \( |2\rangle \leftrightarrow |3\rangle \) transitions are \( \omega_{13} \) and \( \omega_{23} \) respectively. There are two light fields with frequencies \( \omega_1 \) and \( \omega_2 \). The first has detuning \( \Delta = \omega_1 - \omega_{13} \) from the \( |1\rangle \leftrightarrow |3\rangle \) transition, and the Raman two-photon detuning is \( \delta = \Delta - (\omega_2 - \omega_{23}) \), see Fig.(3.1).

The Hamiltonian of the system is

\[
H = \frac{\hat{P}^2}{2M} + \sum_{j=1}^{3} \hbar \omega_j^2 |j\rangle \langle j| + dE(r),
\]

where \( \hbar \omega_j^2 \) is the energy of the level \( |j\rangle \), \( M \) is the mass of the atom, \( \hat{P} \) is the momentum operator, \( d \) is the dipole moment operator, \( E(r) \) is the total electric field of the two light fields, and \( r \) is the position vector of the atom. We use Cartesian coordinates: \( z \) is the direction of propagation of the light, and \( x \) and \( y \) are the orthogonal directions. The electric field is due to two traveling waves.
Mechanical Potential due to a Raman transition

coopropagating in the positive $z$-direction with wave numbers $k_1$ and $k_2$,

$$E(\hat{r}) = \sum_{m=1,2} C_m(\hat{x}, \hat{y}) \cos(\omega_m t - k_m z).$$

(3.2)

Other geometries for the laser fields, such as counter-propagation, make no difference to our results. For example, if we were to make the beams counterpropagating, the atoms would experience a two photon transition between two momentum states which had a larger momentum difference. This would not affect the following calculation. Making the rotating wave approximation and generating an interaction picture by rotating operators with the atomic Hamiltonian,

$$H_0 = \hbar \omega_1^0 |1\rangle \langle 1| + \hbar (\omega_2^0 - \delta) |2\rangle \langle 2| + \hbar (\omega_3^0 + \Delta) |3\rangle \langle 3|,$$

(3.3)

we obtain the interaction picture Hamiltonian

$$H_{IP} = \frac{\hat{P}_z}{2M} + \hbar (\delta |2\rangle \langle 2| - \Delta |3\rangle \langle 3|)$$

$$+ \frac{1}{2} \left[ \exp(ik_1 \hat{z}) C_1 d_{31} |3\rangle \langle 1| + \exp(ik_2 \hat{z}) C_2 d_{32} |2\rangle \langle 3| + \text{h.c.} \right],$$

(3.4)

where the $d_{np} = \langle n | d | p \rangle$ are the dipole matrix elements.

Let $|j, q\rangle$ represent the atom in level $|j\rangle$ with a momentum in the $z$-direction of $hq$. Then under the Hamiltonian Eq. (3.4) the states $\{|1, q\rangle, |2, q+k_1-k_2\rangle, |3, q+k_1\rangle\}$ are closed. For example, the action of $\exp(ik_2 \hat{z}) |3\rangle \langle 2|$ on the second state $|2, q+k_1-k_2\rangle$ gives the third state $|3, q+k_1\rangle$. Assuming an initial $z$-momentum eigenstate we can expand the atomic state $|\Psi\rangle$ in terms of the coefficients $\Psi_j(x, y)$,

$$|\Psi\rangle = \exp(-i\hbar q^2 t/2M) \left\{ \Psi_1(x, y) |1, q\rangle + \Psi_2(x, y) |2, q+k_1-k_2\rangle + \Psi_3(x, y) |3, q+k_1\rangle \right\}.$$

(3.5)

These coefficients obey the Schrödinger equations,

$$i\hbar \partial_t \Psi_1 = \frac{-\hbar^2}{2M} \nabla_x^2 \Psi_1 + \frac{\hbar \Omega_1}{2} \Psi_3,$$

(3.6)

$$i\hbar \partial_t \Psi_2 = \frac{-\hbar^2}{2M} \left[ \nabla_x^2 - \{(q+k_1-k_2)^2 - q^2\} \right] \Psi_2$$

$$+ \hbar \delta \Psi_2 + \frac{\hbar \Omega_2}{2} \Psi_3,$$

(3.7)

$$i\hbar \partial_t \Psi_3 = \frac{-\hbar^2}{2M} \left[ \nabla_x^2 - \{(q+k_1)^2 - q^2\} \right] \Psi_3.$$
§3.2 Derivation of the Potential

\( -\hbar \Delta \Psi_3 + \frac{\hbar \Omega_1}{2} \Psi_1 + \frac{\hbar \Omega_2}{2} \Psi_2, \quad (3.8) \)

where \( \Omega_j = |d_{3j}C_j/\hbar| \) is the Rabi frequency of field \( j \), and \( \nabla_T^2 \equiv \partial_x^2 + \partial_y^2 \).

We now obtain a potential for the atom by adiabatically eliminating the levels \(|2\rangle\) and \(|3\rangle\), and requiring that the detunings are large enough to have a very low atomic population in those levels. We set all but the last two terms of Eq. (3.7) and last three terms of Eq. (3.8) equal to zero. This approximation is valid provided the detuning energies \( \hbar \Delta \) and \( \hbar \delta \) are sufficiently large compared to the transverse, recoil, and Doppler kinetic energies. We must assume that we are only interested in time scales longer than the inverse of these detunings, and that the populations in levels \(|2\rangle\) and \(|3\rangle\) are sufficiently small. This approximation is clarified by Fourier transforming the Eqs. (3.6,3.7,3.8) to frequency space. Setting the time derivatives equal to zero is then equivalent to considering only the frequencies that are small compared to the detunings and the Rabi frequencies. With this approximation \( \Psi_2 \) and \( \Psi_3 \) can be expressed in terms of \( \Psi_1 \) by

\[
\Psi_3 = \frac{-\Omega_1}{2(S\Delta)} \Psi_1, \quad \Psi_2 = \frac{-\Omega_1 \Omega_2}{4\delta(S\Delta)} \Psi_1, \quad (3.9)
\]

where we have defined the quantity

\[
S \equiv 1 + \frac{\Omega_2^2}{4\Delta \delta}. \quad (3.10)
\]

After this adiabatic elimination the Schrödinger equation for the atom in level \(|1\rangle\) can be expressed in terms of an effective Raman potential \( V_2 \)

\[
i\hbar \partial_t \Psi_1 = -\frac{\hbar^2}{2M} \nabla_T^2 \Psi_1 + V_2(\hat{x}, \hat{y}) \Psi_1. \quad (3.11)
\]

\[
V_2(\hat{x}, \hat{y}) = \frac{\hbar \Omega_2^2}{4(S\Delta)}. \quad (3.12)
\]

This Raman potential has the same form as the two-level far detuned potential \([2], \hbar \Omega_1^2/4\Delta \), with the detuning \( \Delta \) scaled by the factor \( S \). When the second laser is off, \( \Omega_2 = 0 \), so \( S = 1 \) and the potential reduces to the two-level case. However, when the second laser is on, the scaled detuning \( S\Delta \) can be orders of magnitude larger than the maximum detuning allowed by the two-level approximation.

Particularly because of its similar form, we might imagine that this two-photon potential is simply the dipole potential for a two level atom with a light shift of the upper level due a second laser. Such a shift of the upper
level is called the Autler-Townes shift, and would be equal to a frequency of \( \sqrt{\Omega_2^2 + (\Delta - \delta)^2 - (\Delta - \delta)} \). As previously described, the potential \( V_2 \) can also be obtained by summing the energy differences between the photon and atomic energies associated with the populations in levels \( |2⟩ \) and \( |3⟩ \),

\[
V_2(x, y) = (\hbar \delta)|\Psi_2|^2 + (\hbar \Delta)|\Psi_3|^2.
\] (3.13)

This expression demonstrates that the Raman potential is not simply due to an Autler-Townes shift of level \( |3⟩ \) by laser two, which would be several orders of magnitude too small.

### 3.3 Comparison of Spontaneous Emission Rates

Any device for manipulating atoms while maintaining the coherence of their de Broglie waves has a maximum acceptable spontaneous emission rate, which is determined by how long the atoms interact with the field. We have shown in section (1.2.3) that the coherence of an atomic beam can be limited by spontaneous emission. We assume that spontaneous emission from level \( |2⟩ \) is negligible, either because it is close to ground or because it is metastable. The total spontaneous emission rate from the upper level \( |3⟩ \) to the lower levels \( |1⟩ \) and \( |2⟩ \), \( \Gamma_2 \), is given by the population in \( |3⟩ \) multiplied by its linewidth \( \gamma \). From Eqs.(3.9) this is,

\[
\Gamma_2 = \frac{\gamma \Omega_1^2}{4(S\Delta)^2}.
\] (3.14)

This expression is true under the approximation \( |\Psi_1|^2 = 1 \), consistent with the conditions for the adiabatic elimination. This spontaneous emission rate has the same form as for the two-level case,

\[
\Gamma_1 = \frac{\gamma \Omega_1^2}{4\Delta^2},
\] (3.15)

with the detuning \( \Delta \) again scaled by the factor \( S \).

We next compare the spontaneous emission rates of the two-level and Raman schemes under the practical constraints of limited power per field and a maximum allowable detuning of \( \Delta_{\text{max}} \). The maximum power is expressed as a maximum Rabi frequency per field, \( \Omega_{\text{max}} \). We fix the required potential height at \( V \). Depending on \( V \) the minimum spontaneous emission rate of the two-level scheme \( \Gamma_{1,\text{min}} \) is then either power limited,

\[
\Gamma_{1,\text{min}} = \frac{4\gamma V^2}{\hbar^2 \Omega_{\text{max}}^2}, \text{ for } V > V_{\text{crit}},
\] (3.16)
or detuning limited,
\[ \Gamma_{1,\text{min}} = \frac{\gamma V}{\hbar \Delta_{\text{max}}} \text{, for } V \leq V_{\text{crit}}, \]
(3.17)

where \( V_{\text{crit}} = \hbar \Omega_{\text{max}}^2/(4 \Delta_{\text{max}}) \). In the detuning limited case the minimum spontaneous emission rate of the Raman scheme \( \Gamma_{2,\text{min}} \) is achieved by maximizing the detuning \( \Delta = \Delta_{\text{max}} \) and the intensity of laser two \( \Omega_2 = \Omega_{\text{max}} \),

\[ \Gamma_{2,\text{min}} = \frac{\gamma V}{\hbar S_{\text{opt}} \Delta_{\text{max}}}, \]
(3.18)

where \( S_{\text{opt}} \) is the value of the scaling parameter which minimizes the spontaneous emission,

\[ S_{\text{opt}} = 1 + Z + \sqrt{(1 + Z)^2 - 1}, \]
(3.19)

\[ Z = \frac{1}{2} P_2 \frac{V_{\text{crit}}}{V}. \]
(3.20)

The population in level \( |2\rangle \), \( P_2 \), has been fixed at a sufficiently small value to satisfy the adiabatic elimination conditions. This constrains the laser intensities and detunings. Using Eqs. (3.10) and (3.12) we can find the Rabi frequency of laser one, \( \Omega_{1,\text{opt}} \), and the two-photon detuning, \( \delta_{\text{opt}} \), which minimize the spontaneous emission. In terms of \( S_{\text{opt}} \) they are:

\[ \delta_{\text{opt}} = \frac{\Omega_{\text{max}}^2}{4 \Delta_{\text{max}} (S_{\text{opt}} - 1)}, \quad \Omega_{1,\text{opt}} = \sqrt{\frac{4 S_{\text{opt}} \Delta_{\text{max}}}{\hbar}} V. \]
(3.21)

Examination of Eqs. (3.17) and (3.18) shows that the relative decrease in the spontaneous emission rate gained by using the Raman scheme in the detuning limited case is given by the parameter \( S_{\text{opt}} \). Fig. (3.2) shows this improvement in the spontaneous emission rate. The ratio of the spontaneous emission rates in the Raman and two-level schemes \( (\Gamma_{2,\text{min}}/\Gamma_{1,\text{min}}) = 1/S_{\text{opt}} \) is plotted versus the dimensionless potential \( V/V_{\text{crit}} \).

Whether an experiment is detuning limited depends on the choice of atomic transition. The laser cannot be detuned too close to the next highest transition, or that transition will become dominant. For example, in metastable neon the transition \( 3s[3/2]_2 \rightarrow 3p[5/2]_2 \) is 5 THz above the commonly used \( 3s[3/2]_2 \rightarrow 3p[5/2]_3 \) transition. Although some of the allowed transitions may be avoided, there are always unavoidable transitions as well. For example in sodium there is an unavoidable \( 3^2 S_{1/2} \rightarrow 4^2 P_{3/2} \) transition 399 THz above the \( 3^2 S_{1/2} \rightarrow 3^2 P_{3/2} \) transition.

Fig. (3.3) shows the minimum spontaneous emission rates of confined atoms, \( \Gamma_{1,\text{min}} \) and \( \Gamma_{2,\text{min}} \), obtainable as a function of the potential barrier height, using
Figure 3.2: Ratio of the minimum spontaneous emission rates for the two-level and Raman schemes $\Gamma_{2,\text{min}} / \Gamma_{1,\text{min}}$ as a function of the dimensionless potential height $V/V_{\text{crit}}$. The plot demonstrates the reduced spontaneous emission rate of the Raman scheme. The population of the second level is fixed at $P_2 = 0.01$. 

Mechanical Potential due to a Raman transition
3.3 Comparison of Spontaneous Emission Rates

Figure 3.3: Minimum spontaneous emission rate from the two-level scheme $\Gamma_{1,\text{min}}$, Eq.(3.17), and the Raman scheme $\Gamma_{2,\text{min}}$, Eq.(3.18), versus the potential height $V$ for two maximum detunings: $\Delta_{\text{max}} = 2\pi \times 200 \text{ THz}$ (solid lines), $\Delta_{\text{max}} = 2\pi \times 2.5 \text{ THz}$ (dashed lines). The potential height is in units of the recoil energy of sodium $E_{\text{rec}} = \left(\frac{\hbar k}{2J}\right)^2$. Parameters appropriate for sodium were used: $\Omega_{\text{max}} = 300 \text{ GHz} \left(280 \text{ kW cm}^{-2}\right)$, $\gamma = 63 \text{ MHz}$, $P_2 = 0.01$, $E_{\text{rec}} = 1.5 \times 10^{-20} \text{ J}$.

two extreme values for the maximum allowable detuning. We assume that it is not feasible to blue detune by more than half of the distance between the confining transition and the higher transition, so we consider examples where $\Delta_{\text{max}}/2\pi$ is 2.5 THz and 200 THz. The Raman scheme is limited by the available power, whereas the two-level scheme is limited by the maximum detuning constraint. To fulfill the adiabatic elimination conditions we fixed the population of level $|2\rangle$ at one percent of that of level $|1\rangle$, $P_2 = 0.01$, by constraining the parameters. Fig.(3.3) shows that the Raman scheme has a much lower spontaneous emission rate for sub-recoil energy potentials.

The Raman scheme can also allow larger actual detunings. With the second laser off, the sign of the potential is the same as that of the detuning $\Delta$ between levels $|1\rangle$ and $|3\rangle$. However if $\Delta < 0$ and the second laser is sufficiently intense, the scaling factor $S$, Eq. (3.10), can be negative and the potential has
the opposite sign to $\Delta$. This means that red (one-photon) detuned atoms are pushed down the intensity gradient. Since atomic level spacing increases as the level decreases in energy, this may allow greater (actual) one-photon detunings $\Delta$ to be achieved without running into other levels. For example if $|1\rangle$ and $|2\rangle$ are levels of the ground state and $|3\rangle$ is the first excited state then there are no levels between them and large negative detunings $\Delta$ are possible. This may also allow atoms to be manipulated with a greater range of laser frequencies.

### 3.4 Conclusions

We have investigated the mechanical potential produced by a Raman transition. This potential has the advantage of reduced spontaneous emission losses for sufficiently small potential barriers, such as are appropriate for recoil or sub-recoil cooled atoms [8, 82, 85]. Without this reduction in spontaneous emission certain proposed atom optics devices, for example those based on hollow optical fiber waveguides [62, 63, 67, 68, 71, 86], may be impractical. If the losses due to spontaneous emission are removed, then the dominant source of heating is likely to be due to the movement of the potential due to fluctuations in the Rabi frequencies and detunings of the lasers.

In the next chapter, we will demonstrate that a device based on a gap in the energy spectrum of atoms traveling along a hollow optical fibre requires this Raman potential to be viable. This is because spontaneous emission events destroy the coherence which is required to observe the quantum mechanical motion of atoms.
This chapter shows that it is possible to produce a band gap in the energy spectrum for atoms being guided by a hollow optical fibre. This work has been published in the Physical Review [62, 63].

4.1 Introduction

Band gaps in the energy spectrum of electrons cause many interesting effects in solid state physics, but they have not yet been discussed with reference to building devices in atom optics. Band gaps in atom optical devices may allow the construction of switches, gates or filters for a coherent beam of atoms. Although many interesting devices seem possible, it must be remembered that an atom optical device with a band gap will not behave in an identical fashion to a semiconductor, for example, as the atoms will in general have different quantum statistics to the Fermionic electrons. This chapter describes a system that can produce gaps in the energy spectrum, and calculates that energy spectrum in three dimensions. The band gap only exists while the atoms are strongly affected by the optical potential, however, and this leads to a high spontaneous emission rate. It is therefore necessary to use a technique to reduce this spontaneous emission, such as the potential induced by a Raman transition which was detailed in the last chapter.

Standing wave laser beams produce a periodic potential which has been used extensively in atom optics, particularly in cooling experiments utilising the Sisyphus effect [87–89]. Atoms in a periodic potential have an energy spectrum consisting of bands [90]. Experiments and theoretical calculations concerning these systems have been conducted in one and two dimensions.
Three dimensional optical lattices have been investigated experimentally [91], but the quantized atomic motion has not been calculated except in the limit of very deep potential wells [92]. We present a numerical calculation of the energy spectrum of atoms in a three dimensional waveguide. This chapter shows that the band structure of these atoms can be manipulated to produce an energy gap between the lowest bands.

We consider atoms that are strongly confined in two dimensions and relatively free to travel in a single direction. The most obvious choice for producing this potential are the hollow optical fibres that were described in the second chapter. We may produce a periodic potential by guiding a partial standing wave along the fibre. We present a detailed calculation of the band structure of atoms guided by a hollow optical fibre, although our model may be applied to some free space laser configurations.

In section 4.2 we present a short revision of the band structure expected for an atom in a one dimensional sinusoidal potential. Section 4.3 describes the calculation of the atomic energy spectrum in the three dimensional fibre. The effects of spontaneous emission are estimated later and are found to be the critical factor which will limit experimental realisation of such devices. Section 4.4 describes some possible generalizations of this work to other laser configurations, and section 4.5 describes some applications of controllable band structure for the atoms in a hollow optical fibre. In particular, a technique is described which can cool atoms.

### 4.2 Atomic energy spectrum in one dimension

If an atom is considered to be a plane wave traveling in a single dimension, and there is a sinusoidal potential in that dimension, then the solution to the Schrödinger equation is well known [93, 94]. Bloch's theorem states that for a Hamiltonian of the form

$$H = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} + V(x),$$

where the potential $V(x)$ has a spatial period $\tau$, then the eigenstates $\Psi_k(x)$ must be of the form:

$$\Psi_k(x) = e^{ikx} \sum_{n=-\infty}^{\infty} A_n e^{-i2\pi nx/\tau}$$

where $A_n$ are coefficients.
This leads to the eigenvalue equation:

\[
\sum_{n=-\infty}^{\infty} [-k_n^2 + \frac{2M}{\hbar^2} (E_k - V)] A_n e^{-i2\pi n \phi / \tau} = 0, \tag{4.3}
\]

where \(k_n^2 = (k - 2\pi n / \tau)^2\).

When the variation in the potential is much less than the kinetic energy of the particle, then the potential term in the above equation can be treated perturbatively, and terms with the product of \(V\) and \(A_n\) can be neglected. For a sinusoidal potential, it has been shown [93] that the energy spectrum is approximately \(E_k = h^2 k^2 / 2M\) except for the region \(k \approx \pm \pi / \tau\), where there is a gap equal to the height of the modulation of the potential. This band gap causes interesting conductive properties in solids.

The real potential is not one dimensional. In three dimensional waveguides, variations of the potential in the transverse direction can alter or remove the gap in the band structure. The one dimensional model ignores all transverse modes. There are non-degenerate transverse modes in the three dimensional model, so their spacing must be large enough so that the gaps in the energy spectrum are not covered. Considering a particle in a box, recall that the energy spacing decreases as the size of the box increases, which means that there will be some size above which the transverse structure will cover the gap in the longitudinal band structure. Thus, in order to have a real band gap, a hollow optical fibre waveguide will have some maximum inner radius.

### 4.3 Atomic energy spectrum in three dimensions

The energy spectrum for the three dimensional waveguide will now be calculated. The time independent Schrödinger equation in cylindrical co-ordinates \((z, r, \phi)\) is:

\[
\left[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{2M}{\hbar^2} (E - V(z, r)) \right] \Phi(z, r, \phi) = 0. \tag{4.4}
\]

where \(\Phi\) is the atomic eigenstate with energy \(E\) in the potential \(V(z, r) = V_z(z)V_r(r)\). The potential can be separated in this way because it is proportional to the intensity of the light in the hole, and the optical fields can be separated in this way. The absence of a \(\phi\) component of the potential allows the eigenstate to be separated: \(\Phi(z, r, \phi) = \Psi(z, r) \Theta(\phi)\). Eq.(4.4) can then be separated into functionally independent sides which then give a simple analytic
solution for $\Theta(\phi)$, and an eigenvalue equation for $\Psi(z, r)$:

$$\Theta(\phi) = e^{im\phi}$$  \hspace{1cm} (4.5)

where $m$ is an integer, and

$$\left[ -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} + \frac{2M(V - E)}{\hbar^2} + \frac{m^2}{r^2} \right] \Psi = 0. \hspace{1cm} (4.6)$$

This shows that the rotational modes induce an effective radial potential. This equation cannot be further separated, as the longitudinal "ripple" in the potential couples the radial and longitudinal motion. With a non-zero potential this equation must be solved numerically. The details of this calculation are given in appendix A.

The light induced potential on the two level atoms is proportional to the intensity of the electric field. For fibres with hole radii larger than a few wavelengths, the evanescent field decays exponentially from the walls. For fibres with sub-micron hole radii, the radial dependence of the electric field must be calculated numerically for each fibre. This was done using the techniques described in references [67] and [95].

Gravity was neglected in this model, as the small hole size implies that the effect will be a weak perturbation. An estimate of the size of this perturbation is $\Delta E = Mg\Delta h$ which for Helium confined to a 0.5 $\mu$m region is about three orders of magnitude smaller than the recoil energy. The effect of the Casimir-Polder force is approximated by reducing the potential at the walls of the fibre. Reference [67] gives an estimate that this force halves the height of the potential barrier seen by the atoms as they try to reach the surface of the glass. This change in the potential will increase the required laser power, but this system already assumes that the atoms are well confined, so the results of this model will not depend on an accurate description of this force.

The parameters chosen for the calculation were based on a commonly used transition in metastable Helium ($2^3S - 2^3P$) with a wavelength of 1083nm, but the qualitative results will not depend strongly on the chosen transition, and it is quite possible that experiments would be based on a different transition or a different atom. The calculations were made using a fibre with a step profile refractive index with core and cladding refractive indices of 1.45 and 1.497 respectively. The radial width of the core was 3 $\mu$m.

The Helium atoms in the fibre exhibit a band gap in their energy spectrum for sufficiently small hole radii. Fig.(4.1) shows the energy spectrum of atoms in a typical fibre with increasing hole radius. The solid lines show the allowable energies, and the disallowed energies are indicated by a gray shading. Each line designates a band, and the disallowed energies are the band gaps.
Figure 4.1: The energy spectrum of atoms in a hollow fibre with increasing hole radius. The presence of band gaps are highlighted in gray. The horizontal axis is quasimomentum $\hbar k$ in units of the momentum of a single photon $\hbar k_0$ from the optical field. Energy is measured in units of the recoil energy $E_R = (\hbar k_0)^2/(2M)$ of a single atom. Note that the energy scale is different in each figure, as the atoms are immersed in a more intense light field when the fibre has a smaller hole. The dashed lines show the higher rotational mode, $m=1$, and the solid lines indicate the lowest rotational mode, $m=0$. Higher transverse modes in the $m=0$ spectrum are not shown in (b) and (c), for clarity. (a), (b) and (c) have hole radii of 0.25µm, 0.5µm, and 1.0µm respectively. These figures have a modulation of $M = 1 - (\min(V^2)/\max(V^2)) = 0.02$, as it was in Fig.(4.2(b)).
For small hole radii, it is clear that there is a band gap between the lowest bands, which becomes smaller as the hole becomes larger and the intensity of the light in the fibre gets lower. The spacing of the transverse modes is clearly smaller than the spacing of the rotational modes, as the first excited rotational mode $m = 1$ (dashed line) is higher than the excited transverse modes. Only the first few $m = 0$ modes have been shown, as the only significant band gap is between the lower bands.

As the size of the hole in the fibre approaches the de Broglie wavelength ($\lambda_{dB} = h/p$) of the recoiled atoms, the transverse energy spacing is reduced such that the gap in the energy spectrum is closed. Small perturbative coupling between the transverse modes would allow atoms to be excited from the lowest band if such transitions were energetically allowed. These couplings may arise from interatomic collisions, gravitational, electric or magnetic fields or by imperfections in the fibre or beam. This means that the atoms will only be confined to a single band if there is a band gap. This will only occur when the hole is smaller than some maximum hole radius which is about 1.5 $\mu$m for this particular system. Fig. (4.1(c)) shows the energy spectrum for a radius which is 1.0 $\mu$m, and the higher rotational mode has nearly covered the band gap. Fibres with hole radii as small as 1.0 $\mu$m have already been used to guide atoms [71].

While the atoms are in the fibre, the modulation depth can be varied by altering the intensities of the two counterpropagating laser beams which cause the level shift. This controls the shape of the energy bands and the size of the band gap. Fig. (4.2) shows the effect of decreasing the modulation depth. Again, the band gaps are shaded in gray. As the modulation decreases, the lowest band becomes lower and has more range while the energy gap becomes smaller.

The key practical problem with this device is that in order for the light to interact sufficiently with the atoms to produce a band gap, the atoms must be immersed in the light field for long periods of time. This means that the atoms stay weakly excited, and their coherence may be lost through a spontaneous emission event. Spontaneous emission losses depend sensitively on the energy of the atoms, the size and refractive index profile of the fibre, the available power in the guided laser and its detuning from the transition. We are most interested in atoms whose energy places them in the lowest band. Classically these atoms cannot enter the field further than the point where the potential equals their total energy. This means that an over-estimate of the spontaneous emission rate can be found by considering the spontaneous emission rate of an atom sitting at that point of maximum classical potential.

Using Eq. (3.15), we find that for a hole diameter of 1.5 $\mu$m, the spontaneous emission rate is of the order of 1 Hz, given 3 W of guided laser power. This
Figure 4.2: The energy spectrum for atoms in a hollow fibre with decreasing modulation of the potential. The presence of band gaps are highlighted in gray. The horizontal axis is quasimomentum $nk$ in units of the momentum of a single photon $hk_o$ from the laser. Energy is measured in units of the recoil energy $E_R = (hk_o)^2/(2M)$ of a single atom. The modulation is measured by the parameter $M = 1 - (\min(V_z)/\max(V_z))$ where $V_z$ is the z-dependence of the confining potential. (a), (b) and (c) have values for M of 0.04, 0.02 and 0.005 respectively. The hole radius for all three figures was 0.5µm, as it was in Fig.(4.1(b)).
spontaneous emission rate is so high due to the significant field in the centre of such a small fibre where the evanescent fields contact each other. This might not occur in free space, where a standing wave can have zero field at the node. This means that it may be possible to create a waveguide with a lower spontaneous emission rate by standing waves in free space. As may be the case for systems discussed in the next section, if there was no electric field at the center, the spontaneous emission rate would be as low as 0.01 Hz. The recoil cooled atoms would take several seconds to pass through a few centimeters of fibre, so attaining this limit for the spontaneous emission rate may be enough to produce a practical device.

4.4 Alternative laser schemes

The purpose of the confining potential is to confine the atoms to a sufficiently small area so that the splitting of the transverse modes is larger than the size of the lower bands. This allows a band gap to form due to the periodic potential, and the gap is not filled by higher transverse modes. It should therefore be possible to generalize this work to other designs for the physical layout of the laser field.

An alternate design to the hollow optical fibre experiment is a variation on the traditional optical molasses apparatus. A standing wave in each of the two transverse directions produces a “tunnel” in the laser field down which the atom can be guided. A longitudinal laser beam can then be used to produce the periodic potential. A significant advantage of this system is that there would be zero electric field at the center of the atomic waveguide. This should allow lower spontaneous emission losses, as described in the previous section. The lack of rotational symmetry in this system produces a calculational problem, however, as the potential cannot be separated. This means that an accurate calculation of the atomic energy spectrum would be difficult to do numerically. This scheme has less practical value, as the advantages of a hollow optical fibre waveguide are lost. These include efficient use of laser power, flexibility and the ability to operate without a strong vacuum.

Other free space configurations may be used. In particular, it might be possible to produce a “donut” laser mode which has a node in the center but retains the rotational symmetry which is useful in making accurate calculations.
4.5 Applications and Discussion

The existence of a band gap in the energy spectrum for the atoms will naturally suppress the excitation of atoms beyond the upper limit of the band. The extent to which this applies will depend on the relative sizes of the energy scale of the dominant heating processes and the band gap. We have shown that the band gap can be on the order of a recoil energy, so heating processes which act on this energy scale will be reduced. This means that atoms will tend to travel along the fibre with less heating from incoherent sources. An example of such a heating source is the vibration of the potential due to the fluctuations in the laser Rabi frequency and detuning. Lasers can be stabilised so that these effects are of the order of about one percent. This means that the atoms would slowly heat while being guided by the fibre. The potential is proportional to the Rabi frequency, so the fluctuations in the potential will also be of the order of one percent, which will mean a change in potential of less than $0.1E_R$. This means that the presence of a band gap of the order of one atomic recoil will suppress heating due to these effects.

Spontaneous emission losses will dominate the loss of atoms from the system, and these cannot be removed without using arbitrarily large laser powers [61], so it is not feasible to construct arbitrarily long waveguides. The most important feature of the band structure in the fibre is that it can be altered externally by changing the intensity, detuning or modulation of the laser beam. This means that interesting behaviour can be produced by changing the band structure while atoms are in transit through the waveguide.

The lowest band gap appears when the de Broglie wavelength of the atoms is that of the laser beam. This is illustrated in Fig. (4.2(c)), where the gap between the first two bands occurs at $k/k_0 = 1$. This means that a beam of atoms which are cooled close to the recoil limit will have a large population in the lowest energy band. Since the atoms are moving at speeds of the order of $3 \text{ mm/s}$, an atomic beam would take several seconds to pass through a centimeter of fibre. This means that the band structures can be changed so slowly that no non-adiabatic heating effects need to be considered.

It can be seen from Fig. (4.2) that increasing the modulation depth reduces the width of the lowest energy band at the same time as increasing the band gap. If the modulation of the laser was slowly increased while there were atoms in that band, then the atoms would have a lower energy spread. This means that while atoms are guided through the fibre, their energy spectrum can be altered in such a way that they will be cooled. This process is not expected to be necessarily competitive with other cooling processes, but it is an example of using the dynamic nature of the band structure in this fibre. A band structure with gaps in the energy spectrum which can be controlled ex-
ternally is a novel situation, and may lead to other interesting effects. Our cooling mechanism is similar to an effective three dimensional cooling system which takes atoms in very deep potential wells and adiabatically reduces the depth of the wells until they are in free space [92]. This other adiabatic cooling system may be an effective method of populating the lowest energy band.

The major limitation of this system is the spontaneous emission of the atom from the excited state. This can be made arbitrarily small by increasing the detuning and the laser power, but if the laser is detuned too much then it will interact with other levels in the atom. For atoms cooled close to the recoil limit, the spontaneous emission is limited by the maximum allowable detuning.

As spontaneous emission losses are likely to be quite high unless a very low frequency transition can be used, there is some advantage to using transitions based on metastable lower states. Detectors of these atoms rely on their large excitation energy. If these atoms spontaneously emit and do not return to the metastable state, then they will not be detected, so the atoms that did not experience the correct potential will have a reduced signal. This alleviates the problem of the high spontaneous emission rate experienced by the atoms.

There are also possible methods for reducing the spontaneous emission rather than simply eliminating the incoherent signal. An example of such a modification might be to use the potential produced by a Raman transition instead of a single photon transition [61]. This type of potential, which was described in the previous chapter, can produce a potential for cold atoms without causing them to be partially excited. For recoil cooled atoms, the Raman scheme may therefore allow a major reductions of the spontaneous emission rate.

One experimental challenge is to provide a method of loading a hollow optical fibre with cooled atoms. One way is to focus the light into the fibre from a lens such that there is a hole in the centre of the optical beam. This will tend to steer the atoms into the centre of the fibre as though the light was acting as a funnel for the atoms. The transverse modes of the fibre are quite small, of the order of a couple of optical wavelengths. The BEC produced by the Ioffe-Pritchard trap [14] has a high transverse confinement of the same order, so it might be a particularly good source for loading such a fibre.

4.6 Conclusions

This chapter has shown that the energy spectrum for atoms guided through a hollow optical fibre has at least two distinct bands separated by an energy gap if the radius of the hole in the fibre is less than 1.5μm. The size of this band gap and the energy range of the lowest band may be controlled externally by
altering the amount of modulation of the laser beam.

The ability to control the size of the band gap dynamically while the atoms are in the system leads to many new possibilities. This situation does not exist in solid state physics where the band structure is fixed. For example, slowly increasing the band gap while the atoms are being guided by the fibre will result in a "squeezing" of the lowest energy band, so if the atoms were originally cool enough to have a significant population in this band then they will be further cooled. The major limitation of this system is that there would be a loss due to spontaneous emission, which may be overcome by using the Raman potential from the previous chapter. Further work may be useful in determining whether a similar system using lasers in free space can produce similar or perhaps better results.
Band Gaps for Atoms in Light based Waveguides

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Chapter 5

Input and Output of Atoms from an Atomic Trap

Overview

This chapter derives a theory for the input and output of atoms from an atomic trap. This theory is based on the optical input/output theory of Gardiner and Collett [96]. This work has been published in the Physical Review [64].

5.1 Introduction

Recently, there has been a large amount of interest in the production of an atom laser [51–59], which is the atomic equivalent of the optical laser. Success in producing a BEC in the laboratory [12–15] has led to the expectation that if the condensate can be coupled to the outside world [50, 59, 97], then a highly coherent atomic beam may be generated. Basic experiments have already been performed in which atoms have been coupled out of traps containing a BEC by changing their internal state to one which is not trapped [50].

Several groups around the world have produced very simple models of atomic traps with a continuous pump source [52–57], and these have shown that it is possible to produce a coherent state in the ground state of the trap. The obvious missing element in these models is the exact nature of the coupling to the external modes, and the effects that this coupling can have on the linewidth of the atom laser.

Our first approach to the problem of producing a model for an atom laser with a realistic output coupling was a purely practical one in which we attempted to choose specific components before proceeding with calculations [59]. The atomic cavity in our model had transverse confinement which was produced by a hollow optical fibre (see Chapter 2) and the longitudinal confinement was produced by two laser beams that passed through the fibre per-
perpendicular to the walls. Atoms were pumped into the system in an excited
 electronic state which could then emit a photon and become trapped in the
 atomic cavity. Rate equations for the various levels in the laser were then pro-
 duced. The output coupling was produced by changing the electronic state of
 the atom with a pair of optical lasers which caused a Raman transition to an
 untrapped state.

A more detailed description of the model is given in appendix C. Under
 certain approximations, it was shown that a threshold was reached, and the
 only significantly populated mode in the cavity was the ground state mode. It
 was also demonstrated that a sufficient number of atoms could be irreversibly
 coupled out of the system by this method. What was not clear was the solu-
 tion to the problem of producing an atomic beam with a narrow linewidth.
 When the atoms were coupled out extremely quickly it could be shown that
 the atoms outside the cavity had the same momentum spread that was present
 in the trapped atomic wavefunction, which tended to be very large.

In the absence of the self energy of the trapped and untrapped atoms, the
dynamics of any coupling between a trap and the external modes can be de-
scribed quite simply, but the dynamics are significantly altered by considering
the effect of the total Hamiltonian.

In this chapter we will examine the input and output of atoms from an
atomic trap into a continuous spectrum of external modes. This theory be-
gan as a generalization of the input/output formalism for optical cavities by
Gardiner and Collett [96], which has proved so successful in explaining the
output from quantum optical systems. The main lesson to be learnt from the
optical theory is that an arbitrarily broadband coupling to the external modes
does not mean that the external modes will be populated with an arbitrarily
broad spectrum. In fact, the shape of the spectrum depends on the energy of
the cavity mode and the strength of the coupling. The dispersive nature of the
energy spectrum for atoms means that the equations of motion for atoms in-
teracting with a cavity are non-Markovian, and quite different methods must
be employed for solving these equations.

5.2 The Model

We shall consider a single cavity mode with an annihilation (creation) operator
denoted by $a$ ($a^\dagger$). This cavity is coupled to a continuum of free modes indexed
by the momentum $k$, with annihilation (creation) operators denoted by $b_k$ ($b_k^\dagger$).
At this stage, we will not choose a form for the cavity Hamiltonian $H_{CAV}$, but
we will define the external and interaction Hamiltonians:

$$H_{TOT} = H_{CAV} + H_{EXT} + H_{INT}$$ (5.1)
\[ H_{\text{EXT}} = \int_{-\infty}^{\infty} dk \ h \omega_k \ b_k^\dagger b_k \]  
\[ H_{\text{INT}} = -i \hbar \int_{-\infty}^{\infty} dk \ (\kappa(k)b_k a^\dagger - \kappa(k)^* b_k^\dagger a). \]  

where \( H_{\text{TOT}}, H_{\text{EXT}} \) and \( H_{\text{INT}} \) are the total, external and interaction Hamiltonians respectively, \( M \) is the mass of the atom, \( \omega_k = \hbar k^2/2M \) and \( \kappa(k) \) is the functional form of the coupling. The external Hamiltonian describes the energy of the free particles, and \( H_{\text{CAV}} \) is the most general form of a Hamiltonian which couples an atom linearly from the cavity to an untrapped state. These equations are identical to those of the optical theory, where we would identify the creation and annihilation operators for the equivalent optical modes, and the external modes would have energies \( h \omega_k = \hbar k \). There is no rotating wave approximation inherent in the above interaction Hamiltonian, due to the conservation of the number of atoms. The major approximation in this Hamiltonian is the single mode approximation for the cavity mode, and it is otherwise fully general at this stage.

These equations look very similar to those used in the optical input/output theory as defined by Gardiner and Collett, with a few minor differences. The most obvious difference between them is that the energy spectrum of the free atoms is quadratic in momentum rather than linear. In the optical case, the extension of the \( k \)-integral from \(-\infty\) to \(0\) is an approximation, whereas the negative momenta in the atomic case are physical. The fact that the energy of a photon is proportional to its momentum means that it is not usually important to note the difference. It is common to describe a difference in momentum between two photons as a difference in energy. This cannot be done in the atomic case, so the zero of the energy scale affects the system. The dispersive nature of the atoms in free space leads to an irreducible memory for the system, so the Markov approximation cannot be made.

Following the optical case for the last time, we can produce Heisenberg equations of motion for any cavity operator \( c(t) \) in terms of the initial conditions outside the cavity \( b_k(t_0) \) or the external fields at some time \( t_1, b_k(t_1) \):

\[ \dot{c}(t) = -\frac{i}{\hbar}[c(t), H_{\text{CAV}}] \]  

or,

\[ \dot{c}(t) = -\frac{i}{\hbar}[c(t), H_{\text{CAV}}] \]  

(5.4)

(5.5)
\[ + \int_{-\infty}^{\infty} dk \kappa(k) \left\{ e^{i\omega_k(t-t_1)}b_k(t_1)[c,a] - e^{-i\omega_k(t-t_1)}b_k(t_1)[c,a] \right\} \]
\[ - \int_{-\infty}^{\infty} dk |\kappa(k)|^2 \int_t^{t_1} dt' \left\{ e^{i\omega_k(t-t')}a^\dagger(t')[c,a] - e^{-i\omega_k(t-t')}a(t')[c,a] \right\} \]

We now take out a normalisation constant from \( \kappa(k) \), and set \( \kappa(k) = \Gamma^{1/2}\kappa'(k) \). For the mode creation and annihilation operators \( a \) and \( a^\dagger \), these equations simplify to

\[ \dot{a}(t) = -\frac{i}{\hbar}\left( a(t), \mathcal{H}_{\text{CAV}} \right) \]
\[ -\Gamma^{1/2}b_{\text{in}}(t) - \Gamma \int_{t_0}^{t} dt' \, a(t') \, f(t-t') \quad (5.6) \]
\[ \dot{a}(t) = -\frac{i}{\hbar}\left( a(t), \mathcal{H}_{\text{CAV}} \right) \]
\[ +\Gamma^{1/2}b_{\text{out}}(t) + \Gamma \int_{t_0}^{t} dt' \, a(t') \, f(t-t') \quad (5.7) \]

where we have introduced the function \( f(t) = \int_{-\infty}^{\infty} dk \, |\kappa'(k)|^2 e^{-i\omega_k t} \) and operators \( b_{\text{in}} = \int_{-\infty}^{\infty} dk \, \kappa'(k)e^{-i\omega_k(t-t_0)}b_k(t_0) \) and \( b_{\text{out}} = -\int_{-\infty}^{\infty} dk \, \kappa'(k)e^{-i\omega_k(t-t_1)}b_k(t_1) \).

### 5.3 Solving the Equations of Motion

The equations of motion for the cavity annihilation operator (Eqs. 5.6,5.7) are linear Volterra equations of the convolution type. Integrodifferential equations are extremely difficult to solve in general, but this particular type of equation is well suited to methods of solution involving the Laplace transform.

We shall introduce some conditions at this stage which will simplify the notation. Firstly, we shall set \( t_0 = 0 \). We then choose the simplest cavity Hamiltonian

\[ \mathcal{H}_{\text{CAV}} = \hbar \omega_o a^\dagger a, \quad (5.8) \]

where \( \hbar \omega_o \) is the energy of the trapped state. We shall also move into a rotating frame with the frequency \( \omega_o \), so we shall introduce the following operators and function:

\[ a'(t) = a(t) \, e^{i\omega_o t} \quad (5.9) \]
\[ b_{\text{in/out}}'(t) = b_{\text{in/out}} \, e^{i\omega_o t} \quad (5.10) \]
\[ f'(t) = f(t) \, e^{i\omega_o t} \quad (5.11) \]
We now integrate equation (5.6) and obtain:

$$a'(t) = F_{in}'(t) + \int_0^t ds \ A'(t - s) \ a'(s)$$  \hspace{1cm} (5.12)

where $F_{in}'(t) = a(0) - \Gamma^{1/2} \int_0^t ds \ b_{in}'(s)$, and $A'(t) = -\Gamma \int_0^t ds \ f'(s)$.

We shall denote the Laplace transform of a function $f(t)$ by $\mathcal{L}(f)(p)$, where

$$\mathcal{L}(f)(p) = \int_0^\infty dt \ f(t) e^{-pt}.$$  \hspace{1cm} (5.13)

and the inverse Laplace transform shall be denoted by $\mathcal{L}^{-1}(g)(t)$. Like the closely related Fourier transform, the Laplace transform has a convolution theorem given by

$$\mathcal{L}\left( \int_0^t ds \ f_1(t - s) \ f_2(s) \right)(p) = \mathcal{L}(f_1)(p) \mathcal{L}(f_2)(p)$$  \hspace{1cm} (5.14)

so that when we take the Laplace transform of both sides of equation (5.12), we obtain

$$\mathcal{L}(a')(p) = \mathcal{L}(F_{in}')(p) + \mathcal{L}(A')(p) \mathcal{L}(a')(p)$$  \hspace{1cm} (5.15)

which, in conjunction with the identity $\mathcal{L}(\int_0^t ds \ f(s))(p) = 1/p \ \mathcal{L}(f)(p)$ gives us the result:

$$a'(t) = \mathcal{L}^{-1}\left( \frac{a(0) - \Gamma^{1/2} \mathcal{L}(b_{in}')(p)}{p + \Gamma \mathcal{L}(f')(p)} \right)(t).$$  \hspace{1cm} (5.16)

This solution may now be substituted into earlier equations to determine the atomic field of the external modes.

$$b_{out}'(t) = -b_{in}'(t) - \Gamma^{1/2} a(0) \ G(t)$$
$$+ \Gamma \int_0^t dt' \ G(t - t') \ b_{in}'(t')$$  \hspace{1cm} (5.17)

where

$$G(t) = \mathcal{L}^{-1}\left( \frac{\mathcal{L}(f')(p)}{p + \Gamma \mathcal{L}(f')(p)} \right)(t)$$  \hspace{1cm} (5.18)

or alternatively,

$$b_k(t) = e^{-i\omega_k t} b_k(0) + \Gamma^{1/2} \kappa'(k)^* e^{-i\omega_k t} a(0) \ M_k(t)$$
$$- \Gamma \kappa'(k)^* e^{-i\omega_k t} \int_0^t dt' \ M_k(t - t') \ b_{in}'(t')$$  \hspace{1cm} (5.19)

where

$$M_k(t) = \mathcal{L}^{-1}\left( \frac{1}{[p + \Gamma \mathcal{L}(f')(p)][p + i(\omega_k - \omega_o)]} \right)(t).$$  \hspace{1cm} (5.20)
This has reduced the problem to the calculation of the Laplace transform and its inverse. Unfortunately, many physical situations cannot be solved analytically, and we must resort to either perturbative or numerical analysis to determine the spectrum of the field emitted from such a system. In many cases, however, the behaviour of a real system will correspond closely to one of the limiting cases which we will describe.

5.4 Physical realizations of the model

The model as described in the previous section has left the nature of the coupling between the cavity and the external modes deliberately vague. This was done in order for it to be possible to compare different methods of atomic coupling, and determine what the effect might be of utilising a particular technique. It is clear, both on physical grounds, and as a result of Eq.(5.19), that a sufficiently narrow bandwidth for the output coupling will produce a narrow linewidth for the output spectrum. In the limit of the output coupling function becoming singular, \( \kappa'(k) = \delta(k-k_o) \), we can show from the relevant definitions that

\[
\begin{align*}
\frac{b_{o"}'}{b_{o"}}(t) &= -b_{o"}(t) e^{i\omega_o t}, \\
\frac{b_{o'"}'}{b_{o'"}}(t) &= b_{o'"}(t) e^{i\omega_o t}, \\
\mathcal{L}(f')(p) &= \frac{1}{p-i\omega_o} \quad \text{and} \\
\Gamma(t) &= \frac{2}{\sqrt{\omega_o^2 + 4\Gamma}} e^{i/2\omega_o t} \sin \left( \frac{\sqrt{\omega_o^2 + 4\Gamma t}}{2} \right)
\end{align*}
\]

where \( \omega_o = \omega_e - \omega_k \). We then substitute these equations into Eq.(5.17) and find that

\[
\begin{align*}
\frac{b_{o}}{b_{o}(0)}(t) &= e^{-\frac{1}{2}(\omega_o+\omega_k)t} \left\{ a(0) \frac{1}{\sqrt{\omega_o^2/4 + \Gamma}} \sin \sqrt{\omega_o^2/4 + \Gamma} t \\
+ b_{o}(0) \left[ \cos \sqrt{\omega_o^2/4 + \Gamma} t + i \frac{\omega_o'}{\sqrt{\omega_o^2 + 4\Gamma}} \sin \sqrt{\omega_o^2/4 + \Gamma} t \right] \right\} 
\end{align*}
\]
5.4 Physical realizations of the model

Figure 5.1: This figure shows the Rabi oscillations of atoms from the cavity mode to a single external mode. The vertical axis is the ratio of the number of atoms in the external mode compared to the number of atoms initially in the cavity. We have used a value of $\omega_0' = 1\text{s}^{-1}$ for the detuning. The solid curve is for $\omega_0'/\Gamma = 0.5$, and the gray curve is for a value of $\omega_0'/\Gamma = 2.0$.

resulting equations become far more complex.

We can think of the shape of $\kappa'(k)$ as being determined by the interaction Hamiltonian $\mathcal{H}_{INT}$. Inverting the definition of $\mathcal{H}_{INT}$, Eq.(5.3), we find that

$$\kappa'(k) = \frac{i}{\hbar \sqrt{\Gamma}} \int_{-\infty}^{\infty} dk' \psi_a(k')(k', a|\mathcal{H}_{INT}|k, b)$$

(5.22)

where $|k, b\rangle$ is the state generated by $b_k^\dagger$ acting on the vacuum, and $|k', a\rangle$ is the state of an atom which has momentum $k'$ and is in the trapped electronic state. $\psi_a(k')$ is the momentum wavefunction of the occupied mode in the trap. For a harmonic trap with a non-interacting condensate in the lowest level, this wavefunction would be Gaussian. If we transform Eq.(5.22) into position space, it is obvious that $\kappa(k)$ can describe spatially dependent as well as momentum dependent coupling.

If the interaction Hamiltonian simply changes the electronic state of the atoms with a well defined momentum kick $k_0$, such as we might expect for a stimulated optical transition, then the coupling will simply be proportional to the displaced momentum wavefunction $\psi_a(k-k_0)$. Typically, a detailed model will also include some velocity selectivity and some width to the transition, both of which may be easily modeled by suitably changing $\kappa'(k)$. 
5.5 Interpreting the results

This theory is primarily intended to model the output of an atom laser. Eq. (5.19) contains the information required to calculate the spectrum \( \langle b_k^* b_k(t) \rangle \) of the external field which has coupled out of the cavity. If we assume that initially (at \( t = 0 \)) there were no atoms in the external modes, then the first and third terms do not contribute to the spectrum, and the spectrum can be expressed as

\[
\langle b_k^* b_k(t) \rangle = \Gamma |\kappa'(k)|^2 \langle a(0)^\dagger a(0) \rangle |M_k(t)|^2
\]

which means that the entire time dependence and the non-trivial shape of the spectrum comes from the function \( M_k(t) \). This is equivalent to performing an inverse Laplace transform, as the function \( \mathcal{L}(M_k)(p) \) can be constructed analytically for almost any realistic choice for the coupling.

Unfortunately, it is difficult to perform the final step of calculating \( M_k(t) \) analytically for many forms of the coupling. This means that if we cannot make an approximation then calculations on physical systems must be done numerically, perturbatively or both. There are several standard computational methods for finding the inverse Laplace transform, but it must be noted that these algorithms can become unstable, so care must be taken when performing these calculations numerically.

5.6 Perturbative results

The equation for \( \mathcal{L}(M_k)(p) \) may be written as a perturbative expansion in \( \Gamma \), and the linearity of the Laplace transform then allows us to write \( M_k(t) \) as a perturbative expansion in \( \Gamma \):

\[
M_k(t) = \mathcal{L}^{-1} \left\{ \frac{1 + \frac{\Gamma}{p} \mathcal{L}(f')(p)^{-1}}{p + i(\omega_k - \omega_0)} \right\}(t)
\]

\[
= \sum_{m=0}^{\infty} \Gamma^m (-1)^m \mathcal{L}^{-1} \left\{ \frac{(\mathcal{L}(f')(p)^m}{[p + i(\omega_k - \omega_0)]^{m+1}} \right\}(t).
\]

This has two main advantages. Firstly, in some cases where \( \Gamma \) is sufficiently low, this expansion may be truncated to produce simple analytical results. Alternatively, it may be possible to perform the Laplace transform analytically for each term, allowing the solution to be expressed in terms of a series of known functions. Using Eq. (5.19) we can generate an expansion for the external field:

\[
b_k(t) = e^{i\omega_k t} b_k(0)
\]
The broad band approximation

\[ +\kappa'(k)^* e^{i\omega_0 t} \left[ a(0) \sum_{m=0}^{\infty} (-1)^m \Gamma^{m+1/2} \mathcal{L}^{-1} \left\{ \frac{\mathcal{L}(f')(p)^m}{p^{m+1} [p + i(\omega_k - \omega_o)]} \right\} (t) \right. \]

\[- \sum_{m=0}^{\infty} (-1)^m \Gamma^{m+1} \int_0^t du \mathcal{L}^{-1} \left\{ \frac{\mathcal{L}(f')(p)^m}{p^{m+1} [p + i(\omega_k - \omega_o)]} \right\} (t-u) b'_m(u) \]

The high order terms in \( \Gamma \) may be higher order terms in the time \( t \), so truncating this expansion is probably equivalent to examining timescales much shorter than \( 1/\Gamma \). This is certainly true in the broadband case, (given in the next section), but would be unreasonably hard to prove while the output coupling is still in such a general form. To lowest order in \( \Gamma \), the dynamics is almost independent of the nature of the coupling:

\[ b_k(t) = e^{i\omega_k t} b_k(0) + i\Gamma^{1/2} \kappa'(k)^* e^{i\omega_0 t} a(0) \left( \frac{e^{-i(\omega_k - \omega_o)t} - 1}{\omega_k - \omega_o} \right) + O(\Gamma). \] (5.26)

The long term solution for the spectrum of the external modes cannot be found by this perturbative solution, but as we approach the limit of \( \Gamma \to 0 \), this solution becomes valid for longer times. If we examine the shape of this spectrum, we see that it is growing and narrowing with time. This narrowing of the external spectrum around the cavity energy is due to the fact that the self energies of the cavity and external states are different except at the cavity resonance. This tendency to transfer to a state with little spread about the mean energy, as the coupling becomes weaker, is quite reminiscent of the adiabatic theorem.

The optical input/output theory also predicted that the width of the spectrum of the output from a cavity would narrow as the strength of the coupling was reduced [98]. We conclude from this analysis that to produce an arbitrarily narrow linewidth from an atom laser, it is not necessary to generate a very narrowband method of coupling, but that the same result can be achieved by coupling the atoms out of the cavity very slowly, which trades off atomic flux for linewidth.

We can also find a trivial solution for the opposite limit, that in which \( \Gamma \) becomes very large. In the limit in which the coupling is extremely fast we can approximate the total Hamiltonian by the interaction Hamiltonian \( \mathcal{H}_{INT} \). This means that the output spectrum has exactly the shape of the coupling \( |\kappa'(k)|^2 \), and the amplitude oscillates on a timescale of approximately \( 1/\Gamma \).

5.7 The broad band approximation

In this section, we examine the spectrum in the limit in which the shape of the interaction \( \kappa'(k) \) is arbitrarily broad. This is the only situation which needs
to be examined in the optical case, but for atomic coupling it is quite feasible to produce a method of coupling the atoms which has an intrinsically narrow linewidth. The broad band approximation provides an upper bound to the linewidth of the external spectrum.

The Laplace transform of \( f'(t) \) is given by

\[
\mathcal{L}(f')(p) = \int_{-\infty}^{\infty} dk \frac{|\kappa'(k)|^2}{p + i(\omega_k - \omega_0)} \quad (5.27)
\]

which for a sufficiently broad coupling will approach the result

\[
\mathcal{L}(f')(p) = -i\pi \sqrt{2M/\hbar} \left| \kappa'(\sqrt{2M\omega_0/\hbar}) \right|^2 \sqrt{\frac{i}{p - i\omega_0}} \quad (5.28)
\]

We substitute this result into Eq.(5.20) in order to find \( M_k(t) \) for the broadband coupling:

\[
\mathcal{L}(M_k)(p) = \frac{1}{[p + i(\omega_k - \omega_0)][p + \Gamma c \sqrt{\frac{i}{p - i\omega_0}}]} \quad (5.29)
\]

where we have introduced the constant \( c = -i\pi \sqrt{2M/\hbar} \) and the normalisation constant \( \Gamma \) is chosen so that \( |\kappa'(\sqrt{2M\omega_0/\hbar})|^2 \) is unity. This equation has been solved recently by Moy [99]. The result is

\[
M_k(t) = -e^{i\omega_0 t} \left\{ \begin{array}{c} i\sqrt{t} \Gamma c \\
\omega_k \Delta_k^2 - \Gamma^2 c^2 \sqrt{\frac{\pi}{t}} \\
+ e^{-i\Delta_k t} \frac{i\omega_k \Delta_k}{\omega_k \Delta_k^2 - \Gamma^2 c^2} \\
+ \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\omega_k \Delta_k^2 - \Gamma^2 c^2}} \left( i\omega_k \Delta_k \right) L_{\gamma/2}(i\omega_k t) e^{-i\Delta_k t} \\
+ e^{i\omega_0 t} \left[ \begin{array}{c} \alpha^2 e^{\omega_0^2 t} \\
\beta^2 e^{\omega_0^2 t} \\
\gamma^2 e^{\omega_0^2 t} \\
\frac{(\beta - \alpha)(\gamma - \alpha)(\gamma^2 + i\omega_k)}{\beta^2 e^{\omega_0^2 t}} (1 + \text{Erf}(\alpha \sqrt{t})) \\
\frac{(\beta - \alpha)(\gamma - \beta)(\beta^2 + i\omega_k)}{\beta^2 e^{\omega_0^2 t}} (1 + \text{Erf}(\beta \sqrt{t})) \\
\frac{(\alpha - \gamma)(\beta - \gamma)(\gamma^2 + i\omega_k)}{\beta^2 e^{\omega_0^2 t}} (1 + \text{Erf}(\gamma \sqrt{t})) \end{array} \right] \end{array} \right\} \quad (5.30)
\]

where \( \Delta_k = \omega_k - \omega_0 \), \( L_\gamma^2(x) \) is the Laguerre polynomial and \( \alpha, \beta \) and \( \gamma \) are the roots of the equation \( p^3 + i\omega_0 p + \Gamma c \sqrt{t} = 0 \).

For low values of \( \Gamma \) this result approaches the perturbative result Eq.(5.26), but the most important thing about obtaining an analytical result such as this is that we can examine the long term behaviour of the solution. The solution is close to the perturbative result over short timescales, but approaches a reasonably steady state after some time, and we can observe this transition in Fig.(5.2). Here, we show the first second of evolution in fine detail, and we can
§5.7 The broad band approximation

Figure 5.2: This figure shows the evolution of $|M_k(t)|^2$ in time. The upper figure shows fine detail of the initial second of evolution while the second figure reaches $t = 5s$, and shows the establishment of a steady state. The trap frequency is $\omega_0 = 772\text{Hz}$, and the coupling strength is $\Gamma \epsilon = -30i\text{ s}^{-3/2}$. 
see that the spectrum is narrowing and growing. The second part of the figure shows the first five seconds, and we can see that the spectrum reaches a peak value and that the shape attains a steady state.

For the numerical work in this chapter, we have set the trap frequency \( \omega_0 \) to the geometric mean of the trap frequencies given in one of the more recent BEC experiments conducted at MIT. In reference [14], the trap frequencies were quoted as \( \{\omega_x, \omega_y, \omega_z\} = \{2\pi \times 18\text{Hz}, 2\pi \times 320\text{Hz}, 2\pi \times 320\text{Hz}\} \), so we have chosen the trap frequency \( \omega_0 = 2\pi \times 123 \text{Hz} \).

In the limit of \( t \to \infty \), Eq.(5.30) reduces to

\[
M_k(t) = \frac{i\sqrt{\omega_k}e^{-i\Delta_k t}}{\sqrt{\omega_k}\Delta_k - \Gamma c} + \frac{2\gamma^2 e^{(i\omega_0 + \gamma)^t}}{(\alpha - \gamma)(\beta - \gamma)(\gamma^2 + i\omega_k)}
\]

where \( \gamma = e^{-\pi/4}(3^{1/3}\omega_0/\eta^{1/3} - (4\eta)^{1/3}/6) \) and \( \eta = -27i\Gamma c + \sqrt{108\omega_0^3 - (27\Gamma c)^2} \).

For most physically reasonable parameters, the second term is negligible, and

\[
M_k(t) = \frac{i\sqrt{\omega_k}e^{-i\Delta_k t}}{\sqrt{\omega_k}\Delta_k - \Gamma c}
\]

and since the output spectrum will simply be proportional to \( |M_k(t)|^2 \) by Eq.(5.23), we can see that the output spectrum is given by

\[
\langle b_k^\dagger(t)b_k(t) \rangle = \frac{\langle a(0)^\dagger a(0) \rangle \Gamma}{\Delta_k^2 + \frac{2\pi^2 \Gamma^2 M}{h\omega_k}}
\]

This spectrum has a width which depends both on the strength of the interaction \( \Gamma \) and the energy of the cavity mode \( \hbar \omega_c \). For low values of \( \Gamma \), the spectrum can be approximated by a Lorentzian with a width of \( \gamma = 2\pi \Gamma \sqrt{\frac{2M}{\hbar \omega_c}} \).

We may calculate the full width at half maximum (FWHM), \( \gamma \), of the output spectrum from Eq.(5.33) by straightforward methods. Fig.(5.3) shows the dependence of this linewidth \( \gamma \) on the strength of the coupling \( \Gamma \). This figure, or an equivalent one generated from Eq.(5.33) can be used to find the constraint on \( \Gamma \) required to produce a particular linewidth.

For optical cavities, the coupling and the linewidth are always proportional, which means that the strength of the coupling from an optical cavity is often called “the linewidth” of the cavity. For an atomic cavity, we can see that these terms are not interchangeable.


\section*{5.8 Sample Numerical Calculation}

We shall now demonstrate a method of providing a numerical solution to Eq. (5.29). This is intended to provide both a template for similar calculations and to discuss the difficulties involved with producing a numerical solution for $M_k(t)$ for an arbitrary system. The details of this solution are found in appendix B.

In all of our numerical calculations, we have set the trap frequency to $\omega_0 = 2\pi \times 123$ Hz, as discussed in the previous section.

Fig. (5.4) shows the evolution of $|M_k(t)|^2$, which is related to the output spectrum $\langle b_k^\dagger b_k \rangle$ by Eq. (5.23), after 0.1s and 0.5s. We show this evolution for a very low value of the coupling $\Gamma$, which gives the lowest order perturbative result given in Eq. (5.26), and for a slightly stronger value which shows different results. A comparison of these two plots shows that on this time scale, the exact spectrum has a very similar shape to the weak coupling limit, although more atoms have been coupled out of the cavity. Eq. (5.30) was used to verify the numerical results, which were well within tolerance.

Although the model contains very few fundamental assumptions, there are

---

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.3.png}
\caption{This figure shows the dependence of the linewidth $\gamma$ on the coupling strength $\Gamma$. The coupling is normalized by the factor $\pi \sqrt{M/\hbar}$. The limiting case of $\gamma = 2\pi \Gamma \sqrt{2M/\hbar \omega_0}$ is shown in gray. We have chosen the trap frequency $\omega_0 = 2\pi \times 123$ Hz.}
\end{figure}
Input and Output of Atoms from an Atomic Trap

Figure 5.4: These figures show $|M_k(t)|^2$, found from Eq. (5.29), after 0.1s and 0.5s of a broadband output coupling. The solid line (diamonds) is in the limit of very weak coupling $(\Gamma c = -0.03i \text{ s}^{-5/2})$, and the dashed line (triangles) used a stronger coupling with $(\Gamma c = -30i \text{ s}^{-5/2})$.

Calculational limitations that make it better for solving the perturbative case, or at least situations where $t$, $\Gamma$ and $\omega_0$ are sufficiently small. Note that in the $t = 0.5s$ section of Fig. (5.4), the calculation involving the larger value of $\Gamma$ does not extend as far away from resonance. This is because of the increasing instability of the algorithm as the parameters were increased.

5.9 Time dependent coupling and the MIT atom laser

At least until a continuous wave atom laser can be produced, a realistic model for most experiments will involve output coupling which has some time dependence. This can be incorporated into the theory developed in this chapter quite simply, as it will result in a modification of the function $f'(r)$, and therefore the equations of motion will be changed only by suitable alteration of the function $\mathcal{L}(f')(p)$.

Recently, atoms have been coupled out of a BEC by utilising the rf "scalpel" to induce a spin-flip to an untrapped state [50]. This coupling was done in a pulse lasting approximately $5\mu s$, which we shall model as a square pulse. The input/output theory developed in this chapter does not need to be modified to provide a description of this situation, as there is already a "beginning" for the interaction at $t = 0$, and the evolution after the interaction has been turned off is trivial. We need only follow the evolution of the atoms for the duration of the pulse.

As noted in the previous section, in a situation involving an extremely fast interaction, we expect the shape of the output spectrum to look like the coupling $\kappa'(k)$, which means that we expect the "memory function" $M_k(t)$ to be
quite flat around the resonance. We can see from the long time result Eq.(5.31) that as $\Gamma$ becomes very large then $M_k(t)$ becomes very broad. As this function becomes very broad, we can write the spectrum for the atoms external to the cavity as

$$\langle b_k^* b_k \rangle \propto |\kappa(k)|^2 \langle a(0)^\dagger a(0) \rangle,$$

(5.34)

which is simply proportional to the momentum distribution of the cavity mode.

This method of producing output coupling will produce a pulsed atomic beam with a linewidth which is approximately equal to the momentum width of the cavity wavefunction. In terms of coherence and linewidth, this form of optical coupling is only slightly different to suddenly turning off the trap. The advantages of these experimental methods are that they allow an arbitrary fraction of the condensate to be coupled out, and are therefore very encouraging for future work. Coupling the atoms out more slowly will produce a narrower linewidth, which can be calculated using the methods described in this chapter. This could be achieved by using a much lower intensity rf signal. The output coupling based on a Raman transition, which is described in appendix C, will probably be easier to control than the radio frequency scalpel.

5.10 Conclusions

In this chapter, we have developed a formalism for describing the interaction between an atomic cavity and a continuum of external atomic modes, and methods for solving the atomic dynamics of these systems. The limiting cases of fast and slow atomic coupling have been solved, as have the limiting cases of narrow- and broadband coupling. We have shown that coupling atoms out of the cavity slowly will produce a narrow linewidth.

This theory appears to be easy to adapt to an arbitrary model for the output coupling of an atom laser, which is an essential component of a general description of these devices.
Input and Output of Atoms from an Atomic Trap

In the context of quantum mechanics, the input and output of atoms from an atomic trap is a critical aspect of understanding atomic behavior and control. This process involves the manipulation of atomic states to achieve desired outcomes, such as trapping, cooling, and quantum information processing. The dynamics of atoms within a trap are governed by interactions with the trap's electromagnetic fields, leading to phenomena such as atomic squeezing and quantum entanglement.

The study of atomic traps is not only of fundamental importance in physics but also has applications in areas like quantum computing and precision measurement. The precise control of atomic motion within a trap allows for the manipulation of quantum states, which is essential for the development of quantum technologies.

Understanding the input and output processes in atomic traps involves a detailed analysis of the trap's parameters, such as the trap frequency, the interaction strength, and the vibrational modes. It also requires a deep knowledge of quantum mechanics, particularly in the areas of quantum optics and quantum information theory.
Stimulated Enhancement of Cross-section (SECS) due to a BEC

Overview

This chapter demonstrates that it may be feasible to produce an experiment which can detect the atom-stimulation of a photon emission due to a BEC. The original version of this work has been published in the Physical Review [65], but has since been generalized.

6.1 Introduction

In the recent experiments which have produced a Bose-Einstein condensate (BEC) [12–15], a large number of atoms in a single quantum mechanical (internal and external) state has been produced in an atomic trap. This high degree of quantum degeneracy may be used as an atomic source for atom optics experiments [51–59], or to demonstrate new effects which depend entirely on the quantum statistics. The stimulation of a transition by a BEC has not been observed. In this chapter, we propose a method by which the emission of photons may be observed to be stimulated by the presence of a BEC. In the next chapter, we will examine how even a nuclear process such as beta decay may be stimulated.

The emission of a photon from an atom in an excited state can be stimulated by a large number of photons in the destination mode, as in a laser. If the atoms involved are bosons, then the same emission can also be stimulated by the presence of many atoms in the destination atomic mode. Any transition rate between two states is enhanced by a factor of \((N + 1)\) where \(N\) is the number of bosons occupying the final state. This effect is the physical basis for the difference between light scattering off a BEC and light scattering off a cloud of nondegenerate atoms, as the emission part of the scattering process is stimulated by the BEC. The effects of scattering weak, near-resonant light
from a BEC have been well studied \([19, 27, 28, 100]\), and show that there is an increase in the linewidth due the stimulation of the emission part of the scattering process. Atom-stimulation of a photon emission is also used to produce a build up of atoms in the ground state of a trap in some of the recent models for an atom laser \([52, 53, 59]\).

We analyse a different system in which this stimulation can be detected directly, and which avoids some of the technical difficulties associated with the construction of an atom laser. We demonstrate that it is experimentally feasible to measure an enhanced absorption cross section for a cloud of atoms in an excited state which pass through a BEC of the ground state atoms. An atom in a suitable metastable state can pass through an atomic trap which only traps the ground (internal) state. If the trap contains a condensate of \(N\) atoms, then there will be an enhancement of the fraction of the spontaneous emission which causes the atoms to go into the ground (trap) state, and therefore an enhancement of the overall emission rate. A schematic of this process is shown in Fig.(6.1). We let \(\gamma\) be the free space spontaneous emission rate and \(f\) be the fraction of the atoms which will spontaneously emit into the ground (trap) state. The fraction \(f\) is related to the Franck-Condon factor for the ground state. The total emission rate \(\gamma_{\text{tot}}\) is the sum of the spontaneous emission rate into the non-ground states and the spontaneous plus the stimulated emission.
rates into the ground state

\[ \gamma_{\text{tot}} = \gamma (1 - f) + (N + 1) \gamma f = \gamma + \gamma N f. \]  

(6.1)

The rightmost term of this equation will be called the stimulated emission rate, referring to the photon emission which is stimulated by a highly populated atomic mode. It is clear from this equation that for a BEC with a sufficiently large number of atoms, the stimulated emission rate will dominate the spontaneous emission rate, and that the fraction \( f \) will determine the critical size of such a BEC.

In section 6.2, we will calculate the ratio of stimulated emission to spontaneous emission for a particular model, and show that for experimentally realizable parameters it is possible for the stimulated emission rate to become much larger than the spontaneous emission rate, significantly increasing the overall emission rate. In this particular model, the stimulated emission will occur in a narrow cone around the direction of motion of the excited atoms. In section 6.3 we discuss schemes for detecting the stimulated emission. In section 6.4 we discuss the limits of the model as presented here, and we conclude in section 6.5.

6.2 Stimulated Emission

The intensity \( I \) of radiation emitted in the direction \( r \), due to a transition from an excited state atomic field \( \hat{\Psi}_e \) to a ground state atomic field \( \hat{\Psi}_g \), is given by the result [28]

\[ I(r) = 2 c \epsilon_0 \langle \hat{E}^-(r) \cdot \hat{E}^+(r) \rangle \]

(6.2)

\[ = \frac{\kappa(r)}{|r|^2} \int d^3 r' \int d^3 r'' \exp[-i k_n (r'' - r') \cdot n] \times \langle \hat{\Psi}_e^+(r') \hat{\Psi}_g (r') \hat{\Psi}_e^+(r'') \hat{\Psi}_g (r'') \rangle \]

where \( c \) is the speed of light, \( \epsilon_0 \) is the permittivity of free space, \( \hat{E}^+(r) \) is the positive (negative) frequency component of the electric field operator, \( \kappa(r) = c k_0^4 |d(r)|^2/(8\pi^2 \epsilon_0) \), \( \hbar k_n \) is the absolute momentum kick given by the emission of the photon, \( n = r/|r| \) and \( d(r) \) is the component of the dipole moment of the transition which is transverse to the direction \( r \) of the emitted radiation.

In principle, the excited state atoms can be passed over the experiment one at a time, and are not expected to have unusual statistics beyond simple first order coherence. This allows us to assume that the excited state atoms are in the \( n = 1 \) Fock state of a particular atomic wavefunction. Let us define that
wavefunction as $\Psi(r, t)$, in which case we will obtain the result

$$\langle \hat{\Psi}_I^\dagger(r') \hat{\Psi}_I(r'') \rangle = \Psi^*(r', t) \Psi(r'', t).$$  \hspace{1cm} (6.3)$$

The ground state field is more complex. Well below the transition temperature for a BEC, the ground state field can be approximated by a coherent state with average number of atoms $N$, and the wavefunction $\Phi(r)$ for the condensate can be found by solving the Gross-Pitaevski equation $[101]$. At zero temperature, we therefore have the result

$$\langle \hat{\Psi}_J^\dagger(r') \hat{\Psi}_J(r'') \rangle = N \Phi^*(r') \Phi(r'').$$  \hspace{1cm} (6.4)$$

For finite temperatures, there will be corrections to this result as the occupation of the quasiparticle states increases, but these effects will remain small until the condensate nears the critical temperature. When the ground state field is in a pure BEC, we have the result

$$\langle \hat{\Psi}_J^\dagger(r') \hat{\Psi}_J(r'') \hat{\Psi}_J^\dagger(r'') \hat{\Psi}_J(r'') \rangle = N \Psi^*(r') \Phi(r') \Phi^*(r'') \Psi(r''') + \delta(r' - r'')$$  \hspace{1cm} (6.5)$$

where the delta function term is due to the boson commutation relations. By substituting this result into Eqn.(6.2) we find that the total radiated field is given by

$$I(r) = \frac{\kappa(r)}{|r|^2} (N|C(r)|^2 + 1)$$  \hspace{1cm} (6.6)$$

where $C(r) = \int d^3r' \exp[i r' \cdot n] \Psi^*(r') \Phi(r')$ is the Franck-Condon overlap factor between the excited and the ground state. If we integrate this equation over all directions $r$, we will obtain the total power emitted by the excited atom. This will be proportional to the photon emission rate of the atom. If the atom is equally likely to emit in any direction, then we reproduce Eq.(6.1), $\langle N \rangle$, given by

$$f(t) = \int \frac{d^2k}{4 \pi k_o^2} \left| \int d^3r \Phi^*(r) \Psi(r, t) e^{ik \cdot r} \right|^2,$$  \hspace{1cm} (6.7)$$

where $hk_o$ is the absolute momentum kick given by the emission of the photon and $\int_\Omega d^2k$ denotes the integral in k-space of all possible photon directions. The photon emission can be assumed to have zero width in momentum because the actual width in momentum will be $\Delta k_o = \Delta E/(hc)$ where $\Delta E$ is the energy width of the incoming state. This means that the momentum width of the photon can be approximated to be $\Delta k_o = h k_o / M_c$ where $k_o$ is the mean momentum of the incoming wavepacket and $\Delta k_o$ is the momentum spread of that wavepacket. This is much less than the atomic wavepacket width for slow atoms. In the rest of this section, we shall calculate the overlap integral.
to determine whether the photon emission rate can be significantly enhanced by the presence of a BEC.

We now calculate this overlap integral for a specific case, and determine the feasibility of producing an experiment which will measure atom-stimulated emission. We consider the case where both wavefunctions have a Gaussian form, and the excited state wavefunction \( \Psi(r, t) \) is shifted to a centre of momentum of \( \hbar k_0 \).

\[
\Phi(r) = \left( \frac{1}{2\pi l^2} \right)^{3/4} e^{-\frac{x^2}{4l^2}}, \tag{6.8}
\]

\[
\Psi(r, t) = \left( \frac{1}{s(t)} \right)^{1/4} \left( \frac{1}{2\pi l^2} \right)^{3/4} e^{i(k_0x - \omega_0t)} e^{-\frac{(x-x_0)^2}{4s^2(t)}} e^{-\frac{y^2+2}{4s^2}}, \tag{6.9}
\]

where \( l \) is the width of each dimension of the Gaussian ground and excited state wavefunctions, \( \omega_0 = \hbar k_0^2 / (2M) \), \( v_0 = \hbar k_0 / M \), \( s(t) = 1 + \chi t / (2M^2) \) is a measure of the spreading of the beam, and \( M \) is the mass of the atom. We have assumed here that the source atomic beam and the ground state of the trap have been perfectly aligned, and that both wavefunctions are isotropic. We have set the zero of time at the moment when they are exactly coincident. This is possible because we later, Eq. (6.10), assume negligible spreading over the timescale of the interaction with the BEC wavefunction, and that the source of the atoms is very close to the trap. This condition can be further relaxed if we use the scheme for detecting the stimulated effect that is described in section 6.3.

These wavefunctions are chosen to be Gaussian because that is the form of the ground state of a harmonic trap for non-interacting atoms. It also allows an analytical solution for the result overlap integrals. We shall approximate the solution of the Gross-Pitaevski equation as a broadened Gaussian.

A possible method of producing the excited state atoms would be to produce a BEC in a second trap, and then use a laser pulse to transfer the atoms to a metastable state via a Raman transition. A Raman transition would leave the atoms in a metastable internal state with an overall momentum kick \( \hbar k_0 \) due to the difference in photon momenta. This method would be appropriate when the kick was required to be small. We will show that the largest number of atoms are drawn into the BEC when a small kick is given to the atoms. This type of atom source has several natural advantages for this experiment. Firstly, the BEC is a very cold atom source and will allow all of the atoms to be in the energy range which can be trapped by the target BEC. Secondly, using copropagating lasers to produce the Raman transition will provide naturally the required resonance between the motional state of the atom (Gaussian wavepacket traveling with average momentum \( \hbar k_0 \)) and the kick \( (\hbar k_0) \) due to
the photon which may be emitted.

We now make the assumption that the source of the atoms is very close to the condensate, and that excited state wavepacket does not spread during the interaction with the target wavefunction \( \Phi(r) \) \( (s(t) = 1) \), or precisely that \( \Psi(x, y, z, t) = \Psi(x - v_\omega t, y, z, 0) \). This approximation is made for calculational purposes, and is true for timescales \( \tau \ll 2ML^2/\hbar \). This implies the condition

\[
k_0 \gg 1/(\sqrt{2}l).
\]

We now calculate the fraction \( f(t) \) of atoms spontaneously emitting into the ground state under these assumptions. From Eqs. (6.8,6.9) we obtain the result:

\[
\int d^3r \Phi^*(r) \Psi(r, t) e^{ik\cdot r} = e^{-\frac{(v_\omega t)^2}{4l^2}} e^{-\frac{\hbar}{2l}[(k_x + k_\omega)^2 + k_y^2 + k_z^2]} e^{i[(\omega_0 + (k_x + k_\omega))t]} f(t)
\]

With Eq. (6.7) this gives us

\[
f(t) = \frac{\pi}{k_\omega^2 l^2} e^{-\frac{(v_\omega t)^2}{4l^2}}
\]

Metastable excited atoms have a very weak spontaneous emission rate. This means that we can ignore depletion of the excited state atoms when calculating the number of spontaneous emission events. We also ignore depletion when calculating the total number of stimulated emissions. Including depletion is a straightforward extension, and can only become important after the stimulated emission already has increased the total number of emission events by a large factor.

In an experiment, it is likely that the atoms will travel some distance \( D \) between the source and the detectors which is large compared to the size of the atomic wavefunctions, \( l \). The atoms will be emitting spontaneously for the entire distance, but the stimulated emission will be insignificant unless they are overlapping with the trap ground state. Integrating each of the two terms of the right hand side of equation (6.1) with respect to time will allow us to calculate the total number of stimulated emission events and the total number of spontaneous emission events. We denote the ratio of these by \( R \), which is given by

\[
R = \frac{2N \pi^{3/2}}{Dk_\omega^2 l}
\]

The dependence of this ratio on \( N \) was expected, as the stimulated emission rate is directly proportional to the number of atoms in the BEC so the total number of stimulated emissions will be proportional to \( N \). The total number of spontaneous emissions is proportional to the time of travel of the excited
Figure 6.2: Ratio \( R \), Eq. (6.11), of stimulated emission to spontaneous emission. The chosen parameters are \( N = 5 \times 10^6 \), \( D = 2\text{cm} \) and \( l = 30\mu\text{m} \).

atomic cloud. This in turn is proportional to the distance \( D \), so we expected the ratio \( R \) to be inversely related to \( D \). The dependence on \( k_0 \) is a consequence of our particular model.

Fig. (6.2) plots the value of \( R \) as a function of \( k_0 \), and shows that smaller values of \( k_0 \) produce a stronger stimulated signal. The main restriction on \( k_0 \) is provided by the calculational requirement that the excited state wavepacket doesn’t spread over time. Over the region of the trap, this will be true if inequality (6.10) is satisfied.

Substituting some realistic numbers into this equation will show the feasibility of conducting such an experiment. We choose the number of atoms in the condensate to be \( N = 5 \times 10^6 \), which has been achieved already in experiments [14]. The size of wavefunction is \( l = 30\mu\text{m} \), which is a couple of times larger than current experiments. The total distance traveled by the excited atoms is chosen to be \( D = 2\text{cm} \), but this should be as small as possible in a well designed experiment, and in the theoretical limit can be as small as the traps themselves. For a value of \( k_0 = 2 \times 10^6\text{m}^{-1} \), the excited state wavepacket increases in size by about one percent while passing over the ground state of the trap. These parameters give a ratio of \( R = 23 \) times more stimulated emission events than spontaneous emission events. If the spontaneous emission is
very weak but still measurable, then there will be a significant increase in the emission rate when the BEC is present in the trap.

### 6.3 Detection of the Enhancement

There are two possible methods for detecting the emission of photons. The first is to directly measure the proportion of excited state atoms which manage to pass through the trap. Another method of detection which can be carried out simultaneously is to detect the photons which are emitted. Spontaneously emitted photons will come from the condensate in random directions, whereas the radiation due to the stimulated process will emit in a narrow cone around the direction of travel of the excited state atoms. For the parameters used in the previous calculation, the maximum deviation from the center of the cone would be of the order of $1.9^\circ$, corresponding to a solid angle of $0.0003 \times 4\pi$. This angle is smaller for larger values of $k\_0$ and $l$. The background spontaneous emission rate over this small solid angle will therefore be correspondingly reduced, increasing the signal to noise ratio by orders of magnitude.

If we consider the effect of collecting the light over a small solid angle, we can arrange it so that nearly all of the stimulated photons are collected while only a small proportion of the spontaneously emitted photons are collected. We estimate the size of this relative increase by estimating the size of the k-space resonance and dividing it by the total area of the k-sphere. The k-space resonance of the stimulated signal for small solid angles can be derived from equations (6.7, 6.8, 6.9), which show that it has a width of approximately $1/l$. This gives us an approximation to the ratio $R\_\mu$ of photons produced by stimulated and spontaneous emission within the reduced solid angle:

$$R\_\mu \approx \frac{N \pi^{3/2} l}{D}$$

(6.12)

We can see that for this detection scheme, if we have $k\_0$ large enough to ensure a small solid angle (required by our original approximations), then the result will be independent of $k\_0$, and will depend only on the size of the atomic wavefunction. For the parameters given in the last section, this gives a signal to noise ratio of over 3600, and it would be possible to use a larger transition. A Raman transition was previously proposed to prepare the atoms in order to produce a large population transfer into an atomic state with a small energy difference, so that the emission would give a small kick. However, detecting the stimulation via the emitted photons may allow the use of a single photon transition to produce the source beam. A large kick will also easily allow a stronger "no spreading" condition to be satisfied, as we could require that the
excited state wavepacket does not spread for the entire duration of the experiment. This is equivalent to requiring that

\[ k_0 \gg \frac{D}{(2l^2)}. \]  

(6.13)

When using this method, it must be noted that for large \( k_0 \), the depletion of the excited state beam will be insignificant.

### 6.4 Limitations of the Model

The analytic form, Eq. (6.11), of the expected enhancement of cross section derived in this paper was found by choosing an explicit and simple form for the shape of the excited and ground state wavefunctions. This means that the quantitative results for particular experiments may vary, but this is not important for a proof of principle calculation. In particular, the ground state of a harmonic trap will be altered by the presence of interatomic interactions, which are modeled by the non-linear Schrödinger equation [101]. The maximum overlap will be obtained when the wavefunction of the incoming excited state atoms is matched to the ground state wavefunction of the BEC. For repulsive interactions, this wavefunction will be spread slightly (for an example, see the calculation in reference [19]) and therefore the matching might be achieved by slightly relaxing the original assumptions, and allowing the excited state wavefunction to spread slightly before reaching the target.

Gravity has not been considered in this model, but will be important over the timescales involved with this experiment. It may be possible to utilise gravity in a particular experimental arrangement, or it may be necessary to balance the gravitational force. This might be achieved with a far-detuned light force [61] or an atom waveguide such as a hollow optical fiber [67, 68, 71].

As previously noted, it might be desirable to alter the shape of the excited state wavepacket to improve the overlap with the target wavefunction. Gravity may help this process by compressing or expanding the wavefunction vertically, by launching the excited state atoms upwards.

We have already mentioned that we have assumed that the excited state atoms are not affected by the potential in our model. This assumption is true for magnetic traps when the excited state has zero magnetic moment. If an experiment requires the shape of the wavepacket to be altered to match the BEC wavefunction, it might be advantageous to utilize a non-zero potential for the excited state atoms.

Superradiance will not enhance the background spontaneous emission rate provided that the source of atoms is sufficiently dilute that on average, the atoms are much further apart than one optical wavelength. This condition can
be easily satisfied, as there is no fundamental theoretical reason to use a high density source, and the stimulated effect has been shown in equation (6.11) to be largest when the wavepacket of the atomic source is made as physically large as possible.

We also assume that the effect of collisions between the excited state atoms and the ground state atoms is negligible. This is true provided the target BEC is sufficiently dilute. This is a restriction on the density of the target BEC, but we have also shown that the stimulated emission rate is maximal when the target ground state wavefunction is physically as large as possible. To estimate the size of the collision rate for the parameters given in this chapter, we will assume the atoms have a scattering length of $a \approx 40a_0$, where $a_0$ is the Bohr radius. The collisional cross section is given by $8\pi a^2$, which allows us to calculate the probability that an excited state atom will collide with an atom from the target BEC. This probability is $2 \times 10^{-2}$ for the values of $N$ and $l$ given in section 6.2. The collisional rate is therefore negligible.

Reabsorption of the emitted photons by the condensate will reduce the spontaneous and stimulated emission signals by an equal proportion [102]. This effect will not alter the signal to noise ratio, but it would tend to deplete the target BEC. We have neglected the effects of reabsorption in this model. In atom laser models, the reabsorption is much larger because the lasing modes are usually much smaller than our dilute BEC, and therefore the density is much higher.

### 6.5 Conclusions

In this chapter we have shown that it should be feasible to directly measure a process which is stimulated by a large number of bosonic atoms in a single quantum mechanical state. If these experiments can be produced such that the stimulated emission completely dominates the spontaneous emission, and a significant number of atoms are transferred into the BEC, then this may be a method of increasing the number of atoms in a BEC beyond that made possible by evaporative cooling.

This work has since been developed by Savage, Ruostekoski, and Walls [103], who calculated the optical emission rate of an excited state beam when it passes over two ground state Bose-Einstein condensates. The atom-stimulation depends on their phase difference. This is an optical method for detecting the phase difference between two condensates which does not require atom detection, which destroys the condensates being probed.
Stimulation of Beta Decay due to a BEC

Overview

In this chapter we calculate the conditions required to observe the stimulation of a nuclear process, namely beta decay, by the presence of a BEC in the final atomic state. This work has been published in Physics Letters [66].

7.1 Introduction

In the last chapter, we observed that the emission of a photon can be stimulated by the presence of a BEC. The same physical principles apply to nuclear transitions. It has recently been noted that the rate of gamma ray emission from a nuclear process can be altered by the presence of a BEC [104]. This process of atom-stimulated photon emission is similar to other proposals involving electronic transitions [19, 27, 65, 100], except that it emphasises the fact that any process, including nuclear transitions, may be enhanced. We calculate the stimulation of the emission of a massive particle, in particular a beta particle, and discover that the necessary experimental conditions are less restrictive, although still impractical for current experimental conditions. The origin of this improvement is that the total momentum kick due to the neutrino and the beta particle may be zero.

One of the interesting features of condensed matter physics is that under certain conditions, large collections of an even number of bound fermions (i.e. atoms) can act as bosons. This occurs when the Hamiltonian describing the fields may be written only in terms of creation and annihilation operators for the entire atoms. This should be a good approximation for nuclear reactions when there is negligible probability of atomic disruption. It would be extremely interesting to test this fundamental assumption of condensed matter physics with such an exotic example. The stimulation of any kind of transi-
stimulation of Beta Decay due to a BEC has not been observed, except indirectly, in the formation of the condensate itself.

Recently, there have been several similar proposals to measure the effect of the Bose enhancement of photon emission in atomic systems using a BEC. These include the work on light scattering from a BEC [19, 27, 100], the work in the previous chapter, and the emission of a gamma ray photon from a nuclear transition [104]. It is possible that the light scattering proposal and the SECS proposal from the previous chapter may be realised with current condensates. The gamma ray proposal was found to require the construction of a condensate with 10^14 atoms and have a wavefunction which was a metre long in one dimension. It also involved two of these condensates fired at each other with a very well defined relative momentum which was equal to the recoil kick of a gamma ray photon. This appears to be a difficult experimental task, and there is a strong likelihood that at such energies there would be sufficient coupling between the two condensates to destroy them. Atom-stimulation of a photon emission is also used to produce a build up of atoms in the ground state of a trap in some of the recent models for an atom laser [52, 53, 59]. The atom laser proposals have yet to be realised.

We analyse a different system in which the stimulation of a massive particle can be detected. We consider a radioactively unstable ion, $A$, that is held in a trap where it will decay into the stable atom, $B$, as well as a beta particle and an (anti)neutrino.

\[
A \rightarrow B + \beta + \nu \quad (7.1)
\]

The unstable particle $A$ must be an ion so that after the beta decay it will be a stable atom of the new species without having to capture or emit an electron. If there is a second trap which contains a condensate of $N$ atoms of $B$, then there will be an enhancement of the fraction of the nuclear decay which goes into the ground state of the trap, and therefore an enhancement of the overall decay rate. It is important to note that only a very small number of unstable ions of species $A$ are needed in the first trap. This is because the number of ions will only affect the count rate, and not the enhancement. We let $\gamma$ be the spontaneous nuclear decay rate and $f$ be the fraction of the atoms which will be in the ground state of the trap after a spontaneous decay. The total decay rate $\gamma_{\text{tot}}$ is the sum of the spontaneous decay rate into the non-ground states and the stimulated decay into the ground state

\[
\gamma_{\text{tot}} = \gamma(1 - f) + (N + 1)f = \gamma(1 + Nf). \quad (7.2)
\]

It is clear from this equation that for a BEC with a sufficiently large number of atoms, the decay rate can be detectably increased, and that the fraction $f$ will determine the critical size of such a BEC. We will now calculate this fraction
for reasonable parameters.

### 7.2 Stimulated Beta Decay

We denote the center of mass wavefunction of the unstable ion $A$ by $\Psi(x)$. We also denote the mean field of $B$ when it is in the condensate by $\Phi(x)$, and the net momentum kick produced by the beta particle and neutrino emission by $h\kappa$. The fraction $f$ of atoms which would decay spontaneously into the ground state of the trap for $B$ is given by the overlap integral

$$f = \frac{1}{L} \int d^3 k g(k) \left| \int d^3 x \Phi^*(x)\Psi(x)e^{ik\cdot x} \right|^2,$$

where $g(k)d^3k$ is the probability of the momentum kick being between $k$ and $k + d^3k$, and $L$ is the number of magnetic sublevels in the nucleus. The fraction $f$ is reduced by this factor because the beta decay will randomise the nuclear magnetic moment.

The beta particle will be emitted at relativistic speeds, and will therefore have only a weak interaction with the orbital electrons. This is due to an extremely poor spatial overlap between the wavefunction of the outgoing beta particle and the bound electrons. After beta decay, there is photon emission from the excited electrons which is called *inner bremsstrahlung* radiation because it is identical to the bremsstrahlung radiation emitted due to the conversion of the kinetic energy of the beta particles into photons. The spectrum of this radiation is continuous, and is reviewed in reference [105]. The absence of a resonance structure in this spectrum demonstrates that the electronic levels of the atom are not excited by this process.

The momentum kick distribution $g(k)$ can be found from the dynamics of the reaction and the momentum distribution of the emitted particles. Conservation of energy and momentum allow us to determine the magnitude of the kick $k(p, \theta) = |k|$ given to the atom:

$$k(p, \theta) = \frac{1}{\hbar} \left[ \frac{\Delta E^2}{c^2} + 2p^2 + m^2c^2 + 2pcos(\theta) \frac{\Delta E}{c} \right.$$

$$- 2(\Delta E/c + pcos(\theta))\sqrt{p^2 + m^2c^2} \left.]^{1/2} \right. \tag{7.4}$$

where $m$ is the mass of the beta particle, $p$ is the momentum of the beta particle, $\Delta E = Q + mc^2$ is energy difference between the nuclear states of $A$ and $B$, and $\theta$ is the angle between the beta particle and the neutrino.

The distribution of the momentum kick is isotropic, and it can be calculated
from the momentum distribution of the beta particles:

$$g(k) = \int_{0}^{p_{\max}} dp \int_{0}^{\pi} d\theta \, \delta(k(p, \theta) - k) \, G(p)$$

(7.5)

where $p_{\max} = \sqrt{\Delta E^2/c^2 - m^2c^2}$, and the function $G(p)$ is the momentum distribution of the beta particles. This distribution is found from the density of states [106], and is given by $G(p) = p^2(\Delta E - \sqrt{p^2 + m^2c^4})/\mathcal{N}$, where $\mathcal{N}$ is simply a normalization factor. We can calculate this factor to be

$$\mathcal{N} = \frac{4}{5} \frac{c^2}{\Delta E} (\Delta E^2/c^2 - m^2c^2)^{5/2} + \Delta E^2/c^2 \sqrt{\Delta E^2/c^2 - m^2c^2} \left( \frac{m^2c^4 - 2\Delta E^2}{m^2c^4} \right) + \frac{4}{3} \left( \frac{\Delta E^2}{c^2} - m^2c^2 \right)^{3/2} \left( \frac{\Delta E^2 + m^2c^4}{mc^2} \right) + \frac{\delta^3 m^4 \Delta E}{mc^2} \ln \frac{\Delta E + \sqrt{\Delta E^2 - m^2c^4}}{mc^2}.$$  

We now consider a specific case to calculate the overlap integral, Eq. (7.3). We consider the case where both wavefunctions have a Gaussian form, which corresponds to the ground state of an atom trap containing non-interacting atoms. For simplicity, we assume that both wavefunctions are isotropic and identical.

$$\Psi(x) = \Phi(x) = \left( \frac{1}{2\pi l^2} \right)^{1/4} \exp \left[ -\frac{x^2}{4l^2} \right],$$  

(7.6)

where $l$ is related the size of ground state of the trap. We have assumed here that the traps containing the two species have been perfectly aligned.

The traps which have been used in BEC experiments so far have been magnetic, which could trap the two prepared species of atoms, although in general the size of the ground state of the trap would be different for each species. In an experiment, the traps will have to be chosen so that the shape of the wavefunctions of the two states are as similar as possible. This could be possible with dipole force traps [77], which could also be used to prepare the atoms $A$ and $B$ separately and then move them together by slowly moving the traps. The disadvantage with optical traps is that they may excite the atoms, causing losses through spontaneous emission. This may be avoided through using extremely large detunings or the Raman transition scheme [61] which was shown earlier in this thesis.

From Eqs. (7.3, 7.4, 7.5, 7.6) we calculate the fraction $f$ of atoms spontaneously decaying into the ground state under our assumptions,

$$f = \frac{\pi^2 h^2}{8c} \frac{(\Delta E^2 - m^2c^4)^3(\Delta E^2 + m^2c^4)}{L \Delta E^5 \ell^2 \mathcal{N}}.$$  

(7.7)
This decreases with increasing $\Delta E$, which means that the best reaction will have a low maximum kinetic energy $Q$ for the emitted beta particle. Examination of the dependence of $f$ on the mass $m$ of the emitted particle shows that beta decay will be stimulated more than a different two body decay involving a heavier particle. From the inverse square dependence on $l$, we see that a small ground state wavefunction will also increase $f$. The size of the ground state wavefunction is determined by the strength of the trap. The experiments which currently have produced a BEC have had traps corresponding to $l \approx 2 - 6 \mu m$.

To obtain a reasonable signal from this experiment, it is necessary to have enough ions trapped so that a useful number of decays happen over the lifetime of the experiment. Since they must be trapped within such a small region, it will be difficult to get more than a thousand ions into the trap. The stimulation will also require a large number of atoms in the condensate. The density of atoms in the condensate cannot be increased indefinitely, however, as collisions and interatomic attraction will eventually destroy the condensate. The mean interatomic spacing should be larger than the scattering length of the atoms, which imposes a condition on $l$, the size of the ground state wavefunction. The required number of atoms varies as $l^2$, so the density will go down if larger wavefunctions are used. There is a possibility that the scattering length can be tuned with an applied magnetic field, which is true for cesium [107]. If this technique can be applied more generally, it would allow quite high atomic densities in the trap.

In the presence of a strong bias magnetic field, the electronic and nuclear spins are not coupled. The magnetic moment of the nucleus will be randomised, so the number $L$ of internal states available to the product atom $B$ will be $2I + 1$, where $I$ is the spin of the final nuclear state. We use this assumption when calculating the results in Table 1, in which we show the number of atoms required to double the expected rate of beta decay for several different nuclei. This table uses the parameter $l = 2 \mu m$ for the size of the trap. The reactions on this table were chosen so that less than a thousand ions were required in the trap to produce a sufficiently large signal. This restriction excluded several reactions which used atomic species which have already been trapped, and shows that the best results are found using species which have not yet been cooled below recoil limited temperatures.

The first reaction on this table was the $\beta^-$ decay of $^{41}K$ to $^{41}Ca$, for which the decay rate will be doubled if there are $1.7 \times 10^{14}$ atoms in the BEC. If there are $10^{14}$ atoms in the BEC, then there will be a 60% increase in the decay rate, and this new rate will be one decay per minute when there are 670 ions in the trap. These results require extremely large condensates of exotic species, so it is unlikely that the stimulation of a nuclear process will be observed in the near
Table 7.1: Comparison of the stimulation of beta decay from different nuclei

<table>
<thead>
<tr>
<th>Initial nucleus</th>
<th>Final nucleus</th>
<th>Half life (min)</th>
<th>$Q$ (MeV)</th>
<th>$N$ required to double decay rate</th>
<th>Ions needed to produce 1 decay/min ($N = 10^{14}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{42}$K</td>
<td>$^{42}$Ca ($I=0$)</td>
<td>744</td>
<td>3.5211</td>
<td>$1.7 \times 10^{14}$</td>
<td>670</td>
</tr>
<tr>
<td>$^{11}$C</td>
<td>$^{11}$B ($I=3/2$)</td>
<td>20.4</td>
<td>1.9821</td>
<td>$2.4 \times 10^{14}$</td>
<td>21</td>
</tr>
<tr>
<td>$^{31}$Si</td>
<td>$^{31}$P ($I=1/2$)</td>
<td>157</td>
<td>1.4908</td>
<td>$7.4 \times 10^{13}$</td>
<td>96</td>
</tr>
</tbody>
</table>

future. If it becomes possible to produce a BEC with $10^{14}$ atoms, this feasibility study shows that it may indeed be possible to measure the stimulation of a beta decay process. The largest BEC produced so far has contained $5 \times 10^6$ atoms [14], which is more than two orders of magnitude larger than the first BEC produced less than a year earlier [12].

As it was for the calculation the previous chapter, which was detailed in section (6.4), the analytic form, Eq. (7.7), of the result was found by choosing an explicit and simple form for the shape of the wavefunctions. This means that the quantitative results for particular experiments may vary, but this is not important for a proof of principle calculation. In particular, the ground state of a harmonic trap will be altered by the presence of interatomic interactions, which are modeled by the non-linear Schrödinger equation [101]. The maximum overlap will be obtained when the wavefunction of the unstable atoms is matched to the ground state wavefunction of the BEC. For repulsive interactions, this wavefunction will be spread slightly (for an example, see the calculation in reference [19]).

7.3 Conclusions

This chapter has examined the feasibility of measuring a stimulation of beta decay by a large number of bosonic atoms in a single quantum mechanical state. It shows that while there is no fundamental obstacle to such an experiment, the required condensate population is eight orders of magnitude beyond that achieved so far.
Conclusions

The production of a Bose-Einstein condensate in the laboratory was the culmination of a series of major advancements in atom optics which have occurred over the last decade. Experiments with BEC are providing opportunities to examine and control quantum statistical effects in ways that physicists have only imagined previously, and have settled many hotly debated questions. The history of the field is that state of the art techniques have become widespread over a relatively short timescale. We therefore expect that before too long these experiments will be carried out all over the world. The new goal of many of these groups will be to produce a coherent source of atoms, or an atom laser. The original motivation for the work in this thesis was to learn how to create an atom laser. Several of the results that we have discovered on the way may have wider applications in atom optics.

We have helped establish the feasibility of atomic waveguides made from hollow optical fibres. Such an atomic waveguide will be extremely useful in avoiding experimental problems when constructing large scale atom optic devices such as atom interferometers or atom lasers. It is unclear at this stage whether hollow optical fibres will be practical, as there are several issues that are difficult to deal with theoretically. For example, it may be difficult to load atoms into such a waveguide without interfering with the atoms, or atoms that adsorb to the surface of the glass may block the fibre after a period of heavy use.

This thesis has introduced a new method for producing a mechanical potential for atoms with light fields. We use a Raman transition instead of a two-level transition, which has the effect of producing an effective detuning for the transition which can be made almost arbitrarily large. Optical potentials for very cold atoms will be limited by the maximum detuning possible without interacting with higher levels. This means that this new technique allows small potentials to be generated with a much lower spontaneous emission rate. Magnetic traps can produce potentials with no losses due to spontaneous emission at all, but an optical potential is more flexible, and has more...
degrees of freedom in an experiment. Having two lasers to control will necessarily make experiments more complex, however, so this technique should only be employed when others are impractical.

The band structure of the energy spectrum of atoms in a hollow optical fibre waveguide has been calculated. We have shown that a gap exists in this spectrum, provided that the waveguide is smaller than some maximum hole radius. This band gap may be used to produce velocity selective waveguides for atoms. We noted that the band gap may be altered while the atoms are being guided, and that this may be used to cool the atoms. The practicality of these devices was shown to be limited by decoherence due to spontaneous emission, which may be avoided by using our proposed potential induced by a Raman transition. Using this band structure in a particular device might be restricted by heating processes which can give the atoms enough energy to make a transition to higher bands. Further theoretical work might involve a detailed analysis of possible heating effects to determine the limits the proposed cooling mechanism.

We have demonstrated that a beam of excited state atoms may experience an increase in its photon emission rate when it passes over a BEC. We show that this stimulation is many orders of magnitude easier to detect when observing the emitted photons. Measurement of this effect would be the ultimate demonstration that the object had a large number of atoms in a single quantum state. Further work has shown that this concept may be useful for detecting the relative phase difference of two condensates without removing atoms from either BEC. This application of a BEC is probably easier to demonstrate than an operational atom laser, and could therefore fill the role of a “stepping stone” experiment. If such an experiment were to be performed, a more careful analysis of the wavefunction overlap would have to be performed, taking into account the atom-atom interactions and the effect of the trap on the excited state wavepacket. This calculation would also take into account the finite temperature effects which would require a more detailed field theoretical analysis of the expectation value given in Eq.(6.5), which takes into account the presence of quasiparticle excitations.

In this thesis we have presented a generalization of the optical input/output theory to describe atomic fields. We have produced equations of motion for a single mode atomic cavity coupled to the freely evolving external modes. These equations were then solved by a method using Laplace transforms, and we have derived perturbative solutions. In the limit of very weak coupling the solution is trivial, and we also show results for the system in the limit of broad output coupling. We determine the conditions required to produce an output with a very small linewidth. This understanding is vital to the effort to produce an atom laser.
In the future, it should be possible to refine our proposed design of an atom laser based on hollow optical fibres. There are also many extensions of the input/output theory which would help model real systems. The most important limitation of the model as presented in this thesis is that it cannot simply describe a pumping process. It is therefore important to produce a model of an open system, and attempt to produce a set of steady state equations for the output of an atom laser. Without a pumping source, we cannot model effects such as gain saturation, so the linewidth of a steady state atom laser will require a more advanced model.
Conclusions

We have demonstrated that double-photons excited state atoms may experience an increase in the polarization with time as a potential energy is applied over a field. We show that this excitation in an external optical field is easier to observe when observing the counter-propagating beams from the opposite direction. The demonstration involves a large number of atoms in a single quantum state. Further work includes that the concept may be used for detecting the relative phase difference of two counterpropagating beams removing atoms from either field. The experiment has been considerably easier to demonstrate than an operationally state-selected field and therefore fits the type of a "stepping stone" experiment. If one or both experiments are to be performed, a more careful prediction of the final result would have to be performed, taking into account the stage-to-stage interaction and the effect of the gas on the excited state. In our case, this interaction would also take into account the finite temperature subject to which would represent the detailed field theoretical analysis of the gas at zero volume given by Eq. [10] which is taken into account the temperature of the system in all its microstructural aspects.

In the present work we have presented a generalization of the optical input output theory for quantum field systems. We have introduced a new scheme of the iteration of the open cavity model to the cavity quantum external system. Our approach is based on the solution of the Laplace transform and the introduction of the analytical solutions. By dint of our very weak coupling the solution is known, and we also show explicitly all the open to the limit of infinite output coupling. We demonstrate this model is more suitable for producing an optical field with a very small temporal limit. This understanding is vital to the effort to understanding from theory.
In section 4.3 the Schrödinger equation was shown to reduce to the two-dimensional equation:

$$\left[ -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{m^2}{r^2} - \frac{\partial^2}{\partial z^2} + \frac{2M(V-E)}{\hbar^2} \right] \Psi = 0. \tag{A.1}$$

In the full three-dimensional case, shown in Eq.(4.4), the longitudinal co-ordinate satisfied the conditions for Bloch’s theorem. The shift to cylindrical coordinates and solution of one of the degrees of freedom has not changed that fact, so we make the substitution:

$$\Psi_k(r,z) = e^{ikz} \sum_{l=-\infty}^{\infty} A_l(r)e^{-i\pi l z/\lambda} \tag{A.2}$$

This leads to the eigenvalue equation:

$$\sum_{l} \left[ -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{m^2}{r^2} + k_l^2 + \frac{2MV}{\hbar^2} \right] A_l e^{ik_l z} = \frac{2ME}{\hbar^2} \sum_{l} A_l e^{ik_l z}, \tag{A.3}$$

where $k_l^2 = (k - 4\pi l/\lambda)^2$. The potential is multiplicatively separable into longitudinal and radial components, $V(z,r) = V^z(z)V^r(r)$.

To transform radial wavefunctions into Fourier space, we consider Eq.(A.1) and replace $\Psi(z,r)$ with $\Phi(z,r) = r^{1/2}\Psi(z,r)$. This leads to the Schrödinger equation:

$$-\frac{\partial^2 \Phi}{\partial r^2} - \frac{\partial^2 \Phi}{\partial z^2} + \left( \frac{m^2}{r^2} - \frac{1}{4} + \frac{2M(V-E)}{\hbar^2} \right) \Phi = 0. \tag{A.4}$$
Details of Band Structure Calculation

We then produce the final version of the Schrödinger equation:

\[
\sum_{l'} \sum_{n'} \left[ \left( k_{l'}^2 + \left( \frac{2\pi n'}{R} \right)^2 \right) \delta_{l,l'} \delta_{j,j'} + \frac{\delta_{l,l'}}{R} \int_0^R \frac{m^2 - \frac{1}{4}}{r^2} e^{2\pi i (n-n') r/R} dr \right] A_{l',n'} = \frac{2M}{\hbar^2} A_{l,n}
\]

where

\[
V_{l-n}^{z} = \frac{2}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} V^z(z) \frac{2M}{\hbar^2} e^{4\pi iz/\lambda} dz
\]

and

\[
V_{n}^{r} = \frac{1}{R} \int_0^R V^r(r) e^{2\pi i n r/R} dr
\]

and

\[
\phi_k(r, z) = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{l,n} e^{i k z} e^{-i 2\pi n r/R}.
\]

We may now perform a numerical calculation of the atomic energy spectrum by finding the eigenvalues of the matrix given in Eq.(A.5).

This seems an unusual numerical method for solving the Schrödinger equation in cylindrical coordinates, as the most obvious transformation to make on Eq.(A.3) is the Hankel transform, which expands the radial wavefunction in terms of Bessel functions and removes the singularity. If we do this using the notation \( A_{l,n}^{m}(\kappa) = \int_0^{\infty} dr r A_l(r) J_m(\kappa r) \) we can show that:

\[
\int_0^{\infty} d\kappa' \sum_{l'} \left\{ \left( (\kappa')^2 + k_l^2 \right) \delta_{l,l'} \delta(\kappa - \kappa') + V_{l-n}^{z} V_{l-n}^{r}(\kappa, \kappa') \right\} A_{l,n}^{m}(\kappa') = \frac{2M}{\hbar^2} A_{l,n}^{m}(\kappa)
\]

where

\[
V_{l-n}^{r}(\kappa, \kappa') = \int V^r(r) J_m(\kappa r) J_m(\kappa' r) dr
\]

and

\[
V_{l}^{z} = \frac{2M}{\hbar^2} \frac{2}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} V^z(z) e^{4\pi iz/\lambda} dz
\]

For the purpose of producing a numerical calculation, the indices on the wavefunction components must be reduced to a finite set. Placing the Hankel coordinate \( \kappa \) on a grid is unsatisfactory, as there is very slow convergence for the integral defining \( V_{l-n}^{r}(\kappa, \kappa') \), and it converges very slowly for large \( \kappa \). This is why a better numerical performance can be achieved by transforming Eq.(A.3)
into Fourier space.

The eigenvalue problem was formulated as a MATLAB program, with external functions providing the form of the potentials $V^r(r)$, and $V^z(z)$. The validity of the algorithm was tested by calculating the spectrum for potentials with known solutions. For constant potential, the energy spectrum converged to the expected free particle result, $E = (\hbar k)^2/(2M) + V_0$. When the transverse potential was given the form of a harmonic oscillator, $V^r(r) = \frac{1}{2}\omega r^2$, it was found to produce a transverse spectrum of equally spaced bands with a spacing of $\Delta E = \hbar \omega$, which agreed with the analytical result.

The actual form of the radial potential was found by solving the set of equations given in the appendix of reference [67], which define the optical mode carried by the fibre. These equations are obtained by standard methods [95]. We use cylindrical polar coordinates $\{r, \phi, z\}$, and write the components of a field $A$ by $\{a_r, a_{\phi}, a_z\}$. The fibre has a constant refractive index in the core, which is over the range $\rho_1 < r < \rho_2$, and beyond that has a lower refractive index. We make an ansatz for the solution:

$$e_z(r, \phi) = \begin{cases} c_1 I_\nu(Ur/\rho_1)e^{i\nu\phi} & \text{(I)} \\ (c_2 J_\nu(Xr/\rho_1) + c_3 Y_\nu(Xr/\rho_1))e^{i\nu\phi} & \text{(II)} \\ c_4 K_\nu(Wr/\rho_1)e^{i\nu\phi} & \text{(III)}, \end{cases}$$

where

$$U = \rho_1 \sqrt{\beta^2 - k^2},$$
$$X = \rho_1 \sqrt{n_F^2 k^2 - \beta^2},$$
$$W = \rho_1 \sqrt{\beta^2 - n_C^2 k^2}.$$  

$n_{F,C1}$ were the refractive indices of the core and cladding respectively, $I_\nu, K_\nu$, $J_\nu$ and $Y_\nu$ are Bessel functions of integer order $\nu = 0, \pm 1, \pm 2, \ldots$, and $c_1, \ldots, c_4$ are unknown constants.

The solution for $h_z$ is identical to Eq. (A.12) with the substitutions $e_z \rightarrow h_z$ and four additional constants $c_n \rightarrow c_{n+4}$. Matching these solutions at the hole-core interface $\rho_1$ and at the core-cladding interface $\rho_2$ leads to eight linear, homogeneous equations for these eight unknowns. The nontrivial solutions determine the electromagnetic eigenmodes of the hollow fibre.

For the piecewise constant refractive index profile, we must ensure the continuity of $H$, the tangential component of the electric field $E$, and the normal component of the dielectric displacement vector $\epsilon_0 n^2 E_n$. We find that this is equivalent to demanding the continuity of $h_z$, $h_{\phi}$, $h_r$, $e_z$, $e_\phi$ and $n^2 \epsilon_r$. We also show that it suffices to demand the continuity of $h_z$, $h_{\phi}$, $e_z$ and $e_\phi$ at both boundaries. This leads to the following set of equations.
Continuity of $h_z$ at both boundaries implies:

$$c_5 I_\nu(U) = c_6 J_\nu(X) + c_7 Y_\nu(X),$$  \hspace{1cm} (A.13)  

$$c_8 K_\nu(\tilde{W}) = c_6 J_\nu(\tilde{X}) + c_7 Y_\nu(\tilde{X}).$$  \hspace{1cm} (A.14)  

Continuity of $h_\phi$:

$$c_6 J_\nu(X) + c_7 p_{pH} Y_\nu(X) - c_5 p_F I_\nu(U) = \frac{ik}{\beta \nu} \sqrt{\frac{c_0}{\mu_0}} \left( n_{pH}^2 p_{pH} X [c_2 J'_\nu(X) + c_3 Y'_\nu(X)] + c_1 p_F U J'_\nu(U) \right),$$  \hspace{1cm} (A.15)  

$$c_6 p_{CI} J_\nu(\tilde{X}) + c_7 p_{CI} Y_\nu(\tilde{X}) - c_8 p_F K_\nu(\tilde{W}) = \frac{ik_{p_2}}{\beta \nu p_1} \sqrt{\frac{c_0}{\mu_0}} \left( n_{p_{CI}}^2 p_{p_{CI}} X [c_2 J'_\nu(\tilde{X}) + c_3 Y'_\nu(\tilde{X})] - c_4 n_{p_F}^2 p_{pF} W K'_\nu(\tilde{W}) \right).$$  \hspace{1cm} (A.16)  

Continuity of $e_z$:

$$c_1 I_\nu(U) = c_2 J_\nu(X) + c_3 Y_\nu(X),$$  \hspace{1cm} (A.17)  

$$c_4 K_\nu(\tilde{W}) = c_2 J_\nu(\tilde{X}) + c_3 Y_\nu(\tilde{X}).$$  \hspace{1cm} (A.18)  

Continuity of $e_\phi$:

$$c_2 p_{pH} J_\nu(X) + c_3 p_{pH} Y_\nu(X) - c_1 p_F I_\nu(U) = \frac{ik}{\beta \nu} \sqrt{\frac{c_0}{\mu_0}} \left( c_5 p_F U J'_\nu(U) - p_{pH} X [c_6 J'_\nu(X) + c_7 Y'_\nu(X)] \right),$$  \hspace{1cm} (A.19)  

$$c_2 p_{p_{CI}} J_\nu(\tilde{X}) + c_3 p_{p_{CI}} Y_\nu(\tilde{X}) - c_4 p_F K_\nu(\tilde{W}) = \frac{ik_{p_2}}{\beta \nu p_1} \sqrt{\frac{c_0}{\mu_0}} \left( c_8 p_F W K'_\nu(\tilde{W}) - p_{p_{CI}} X [c_6 J'_\nu(\tilde{X}) + c_7 Y'_\nu(\tilde{X})] \right).$$  \hspace{1cm} (A.20)  

where $\beta$ is the propagation constant which is determined by these equations. We used the abbreviations $p_{pH} = k^2 - \beta^2$ and $p_{p_{CI}} = n_{p_{CI}}^2 k^2 - \beta^2$. The tilde in
some of the equations indicates that the argument of the various Bessel functions has to be taken at the core-cladding interface, e.g. $\hat{X} = X_{p_2}/\rho_1$. The algorithm which solved these equations by a fixed stepsize Riemann integration was verified by repeating the calculation in reference [67], and finding that it produced the same results.
Details of Band Structure Calculation

where \( g \) is the per system constant which is determined by these equations. We used the convention that \( \delta_{ij} \) is constant for \( \delta_{ij} \), and \( \delta_{ij} = 0 \) elsewhere.
Appendix B

The Numerical Inverse Laplace Transform

We can decompose an arbitrary inverse Laplace transform into two real valued inverse Laplace transforms. Let

\[ f(t) = \mathcal{R}(t) + i \mathcal{I}(t) \]  

(B.1)

where \( \mathcal{R}(t) \) and \( \mathcal{I}(t) \) are real valued functions, then the Laplace transform is

\[
\mathcal{L}(f)(a + ib) = \int_0^\infty dt \mathcal{R}(t) e^{-at} \cos(bt) + \int_0^\infty dt \mathcal{I}(t) e^{-at} \sin(bt)
+ i \left\{ \int_0^\infty dt \mathcal{I}(t) e^{-at} \cos(bt) - \int_0^\infty dt \mathcal{R}(t) e^{-at} \sin(bt) \right\}
\]

(B.2)

where \( a \) and \( b \) are real numbers. Each of these terms has a definite parity with respect to \( b \), so we can decompose the Laplace transform \( g(p) = \mathcal{L}(f)(p) \) into functions from which we can reconstruct the functions \( \mathcal{R}(t) \) and \( \mathcal{I}(t) \):

\[
\mathcal{L}(\mathcal{R})(p) = \frac{\text{Re}(g(p) + g(p^*)) + i \{\text{Im}(g(p) - g(p^*))\}}{2}
\]

(B.3)

\[
\mathcal{L}(\mathcal{I})(p) = \frac{\text{Re}(g(p) - g(p^*)) + i \{\text{Im}(g(p) + g(p^*))\}}{2i}
\]

(B.4)

In section (5.7), we were posed with the problem of performing an inverse Laplace transform in order to solve Eq. (5.20). We have shown that this problem is equivalent to solving a real valued inverse Laplace transform. We have solved this equation using Crump’s method [108], which has been implemented in one of the standard algorithms found in the NAG FORTRAN libraries. This routine was labeled c06laf, and it returns the real valued inverse Laplace transform of a function.

The routine had built-in error detection, and quoted the uncertainty of each result. If the solution became unstable an error message was generated, but often the solution could be stabilised by altering one of two independent pa-
parameters. The optimal choice of parameters was not obvious, and could not be generated canonically, which was why they were input a priori by the user.

The routine was verified by applying several test functions which had analytical solutions for the inverse Laplace transform. An exponential function with real and imaginary exponents was tried, and produced answers in agreement with the analytical result to half of the quoted error. The routine also gave correct answers for the inverse Laplace transform of the functions \(1/p\), \(1/p^2\), and \(1/(p^2 + a^2)\).

The numerical inverse Laplace transform is notoriously difficult to perform for difficult functions. As the parameters \(\Gamma\) and \(t\) were increased it became increasingly difficult to find combinations of the free parameters that would allow the problem to be solved without generating an error message. This was true of several different solution methods, which is why numerical methods are quite limited in solving this problem.
Appendix C

An Atom Laser Based on Raman Transitions

In section (5.1), we briefly described an atom laser which had output coupling based on Raman transitions. This appendix gives more details on this model. The model was conceived by our group working collectively, but the calculations in this appendix were performed by Glenn Moy. This work has appeared in the Physical Review [59].

To create an atom laser from a BEC we must couple the atomic trap to the outside world. The BEC must be continuously "pumped" so that it does not deplete over time. The output coupling is also essential for the production of an output beam, and this has been dealt with in detail in chapter 5. The model in this appendix differs from previous models due to our description of a specific method for coupling the atoms to the outside world which is physically realistic. We also propose a specific implementation of our atom laser scheme using hollow optical fibres. This has several advantages, including providing a directed output beam, and minimization of the reabsorption of spontaneously emitted photons.

C.1 Atom Laser Scheme

The model consists of atoms with four energy levels, as outlined in Fig.(C.1). Level $|1\rangle$ is the input pump level, level $|2\rangle$ is the lasing level and level $|4\rangle$ is the output level. Level $|3\rangle$ mediates the output coupling Raman transition. There are two atomic cavities for confining atoms in states $|1\rangle$ and $|2\rangle$. One of these cavities, the lasing cavity, traps atoms in the level $|2\rangle$. We wish to build up a large number of atoms in the lowest mode of the lasing cavity. The other cavity, the pump cavity, traps a large number of atoms in the internal metastable level $|1\rangle$. The two cavities are spatially overlapping, see Fig.(C.2).

Cool atoms with the atomic level $|1\rangle$ are injected into the pump cavity. Atoms change from the pump level $|1\rangle$ to the lasing level, $|2\rangle$ at a rate $r_{12}$ due
An Atom Laser Based on Raman Transitions

Figure C.1: Schematic diagram of atomic states and output coupling lasers.

to spontaneous emission. In a similar fashion to the SECS model given in chapter 6, we find that when a large number of atoms build up in the ground state of the lasing cavity (the lasing mode), there will be a Bose enhancement of that transition rate, and it will soon become the dominant transition. Once this process begins it will be a runaway effect, and it will be “seeded” by the fact that the wavefunction overlap will be largest for the lowest energy state. Tunneling losses out of this state are also lower, causing an initial build up of atom population in the lasing mode.

Both cavities are formed using hollow optical fibre waveguides to provide the transverse confinement, and the longitudinal confinement is created by using light induced potentials from blue-detuned lasers shone transversely across the waveguide. The transverse lasers for the lasing cavity are much closer together than in the pump cavity, producing a much larger energy level spacing. Due to the large overlap of the ground state of the lasing cavity with the lowest pump cavity modes, only the ground state becomes significantly populated. To maximize this overlap the input atoms must be pre-cooled to a few hundred nanoKelvin so as to populate the lower energy states of the pump cavity. Such temperatures can be achieved by evaporative cooling.

A Raman transition transfers the atoms from level |2⟩ to a final atomic level |4⟩. The two lasers are confined to the lasing cavity, and are shone diagonally...
Figure C.2: Schematic diagram of a possible implementation of our atom laser model using a hollow optical fibre. The lasing cavity is confined to a region of size $2\mu m$. The pump cavity is approximately $100\mu m$ long. The input coupling is by a partially transmitting atomic mirror for atoms in state $\psi_1$, indicated by the laser on the left of the figure. Lasers are also used for the other end mirror of the pump cavity, and for the lasing cavity. The two lasers $\omega_1$ and $\omega_2$ are the output coupling lasers. They are localized to the lasing cavity, and provide a momentum kick along the longitudinal axis of the fibre as shown.

Atoms enter the system from a cooled thermal source. We consider an initial rate of atoms entering this cavity which we call the pump rate, $\gamma_1$. The coupling between the pump and laser cavity occurs through spontaneous emis-
sion. The rate of transfer of atoms from the pump cavity to the lasing cavity depends both on this atomic transition rate and on the average wavefunction overlap between the modes of the pump cavity and the lasing cavity.

The pump cavity is modeled as a square well with sides of length 100\mu m in the longitudinal direction. The lasing cavity is modeled as a three dimensional harmonic oscillator. The lasing cavity is displaced a distance \( x = 48 \mu m \) from the center of the pump cavity, which places it at the edge of the pump cavity. For definiteness, we consider a spontaneous emission kick from the \( 1 \rightarrow 2 \) transition of magnitude \( k_0 = 10^6 \text{ m}^{-1} \) which corresponds to an infrared transition. We calculate the average overlap, \( g_1 \), between the pump states and the lowest energy state in the lasing cavity, for a spontaneous emission kick of magnitude \( 10^6 \text{ m}^{-1} \) to be \( g_1 = 0.00571 \). The overlaps, \( g_j \) with the higher excited states (\( j > 1 \)) are smaller, with the next greatest overlap, \( g_2 = 0.00362 \) occurring with the state \( n_x = 1, n_y = n_z = 0 \).

### C.2 Output Coupling

Other schemes for atom lasers [52–56] have not given a description of a practical output coupling mechanism. We present here a method of switching the atomic state of the atoms, using a Raman transition, to allow output coupling from the system. A pair of lasers at frequencies \( \omega_1 \) and \( \omega_2 \) induce a Raman transition from level \( |2\rangle \) to the output level \( |4\rangle \) of the atom, see Fig.(C.1). This output level is not trapped. We require a coherent transfer of the population from level \( |2\rangle \) to level \( |4\rangle \) for our atom laser output. To ensure that the transfer is unitary we require negligible spontaneous emission from level \( |3\rangle \).

Modeling the lasers as plane waves, the final state of the atoms in the longitudinal direction will correspond to the initial wavefunction in the ground state of the lasing cavity combined with the momentum kick \( 2\hbar k_x \). In the transverse direction, the output state of the atoms remains the ground state mode of the hollow optical fibre. This is achieved using output coupling lasers with a sufficiently narrow linewidth compared with the separation of the lasing cavity transverse energy levels. By suitably tuning such Raman lasers to the output atomic level it is impossible for the atoms to excite into a higher transverse mode of the fibre, as the energy required to change internal atomic levels and excite the atoms to a higher transverse mode is higher than the available energy from the Raman photons. For a fibre approximately \( 2 \mu m \) in diameter this requires the Raman lasers to have a linewidth of only a few kHz, which can be achieved by active stabilization.

The output coupling is achieved by shining two lasers diagonally across the lasing cavity. This has the dual purpose of localizing the interaction re-
region, as well as providing a total momentum kick, \(2\hbar k_x\), directed along the longitudinal axis of the optical fibre (the transverse components cancel). The atoms move out of the interaction region due to the momentum kick, thus forming the atom laser beam. For definiteness we arrange a value of \(2k_x = 1.31 \times 10^7 \text{ m}^{-1}\). This value could be achieved using lasers with wavelengths \(\lambda = 480 \text{ nm}\) oriented at 60° to the long axis of the fibre. It is possible to produce light of this wavelength with a frequency doubled titanium-sapphire laser. Assuming a typical atomic mass \(m = 10^{-26} \text{ kg}\) and size of the interaction region, \(l_x = 2 \times 10^{-6} \text{ m}\), we find that the timescale on which atoms leave the system due to the momentum kick is \(t_0 = 1.5 \times 10^{-5} \text{ s}\).

### C.3 Threshold

One important characteristic of the optical laser that is observed in our atom laser model is the presence of a threshold condition. This threshold condition occurs in an optical laser when the net amplification between the mirrors for a single photon circulating the cavity equals the loss at the mirrors. Similarly for the atom laser threshold, we consider atoms injected into an otherwise empty system. The threshold condition occurs when the single atom input rate into the lasing cavity, \(g_{1r_{12}}\), is just sufficient to dominate the loss rate, \(r_{24}\).

We may find the steady state population of the lasing mode by solving basic rate equations for this model [59]. The solution is given by

\[
N_{2j} = \frac{1}{2g_j} \left[ \left( R_j - (1 + \sum_{j' \neq j} g_{j'N_{2j'}}) \right) + \left( \left( R_j - (1 + \sum_{j' \neq j} g_{j'N_{2j'}}) \right)^2 + 4R_jg_j \right)^{\frac{1}{2}} \right],
\]

where \(N_{2j}\) is the population of the \(j\)th mode of the lasing cavity, and the \(R_j\) are dimensionless pumping rate parameters, given by

\[
R_j = \frac{r_{1g_j}}{r_{24}}.
\]

In the regime where the populations of all but the ground state mode are negligible, the steady state equations given in Eq. (C.1) reduce to the form

\[
N_{21} = \frac{1}{2g_1} \left[ (R_1 - 1) + \sqrt{(R_1 - 1)^2 + 4R_1g_1} \right].
\]

This result is analogous to the standard laser population equation and equiv-
Figure C.3: Plot of the steady state number of atoms in the lasing mode, $N_{21}$, as a function of the dimensionless pumping rate, $R_1$. Average overlap, $g_1 = 0.00571$. Threshold occurs at $R_1 = 1$.

alent to results of Spreeuw et al. [53]. We assume numerical values for the transition rates, $r_{12} = 0.1 \text{ s}^{-1}$, $r_{24} = 0.125 \text{ s}^{-1}$ and for the wavefunction overlap, $g_1 = 0.00571$. These parameters correspond to a particular implementation of our scheme using hollow optical fibres. A logarithmic plot of the number of atoms at steady state in the lasing cavity, $N_{21}$, as a function of the dimensionless pumping rate $R_1$ is given in Fig.(C.3). The threshold pumping rate is $R_1 = 1$, which corresponds to an input pumping rate, $r_1 \approx 21.9 \text{ s}^{-1}$.

C.4 Conclusion

We have presented an atom laser model, discussing input and output coupling mechanisms as well as a possible implementation of our scheme in hollow optical fibres. The actual properties of the output beam will depend strongly on the coupling rate, as described in chapter 5, but we can use the techniques in that chapter to calculate them reasonably easily. This is made simpler by the fact that the Raman coupling scheme can be well approximated by a broad-
band coupling, and the analytical solution from that chapter can be applied. This work has been presented by Moy [99].

We found that for reasonable parameters we get a large build up of atoms in the lasing mode, above a threshold pumping rate, in a manner analogous to the threshold found in optical lasers.
we evaluate numerical values for the threshold pumping rate, \( \eta_{\text{th}} \), and for the wavefunction overlap, \( \langle \psi | \psi \rangle \). In this particular implementation of the atom laser, the output single mode can be well approximated by its Gaussian profile. A logarithmic plot of the number of atoms in one output mode in the laser cavity, \( N \), as a function of the detuning, \( \Delta \), is shown in Fig. 2. The threshold pumping rate is defined as the minimum input power generating laser oscillations, \( \eta_{\text{th}} \).
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