Factorization Techniques
in
Robust Controller Design

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B.E. Hons. (JCUNQ)

October 1989

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University

Department of Systems Engineering
The Australian National University

Declaration

These doctoral studies were conducted under the supervision of Professor John B. Moore, and much of the work has been recorded elsewhere in conference proceedings or journals (see [35, 34, 44, 45, 43, 36]).

The content of this thesis, except as otherwise explicitly stated, is the result of original research, and has not been submitted for a degree at any other university or institution.

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Acknowledgements

First of all I wish to thank my supervisor John Moore for allowing me the opportunity to work with him over the past few years, for guidance in academic and personal matters, for sharing his enthusiasm about research, and for making himself available at all times.

Thanks are also due to others in the Department of Systems Engineering for help with academic problems, as well as for their friendship and hospitality. In this respect I would particularly like to mention Bob Bitmead, Brian Anderson, Phillip Musumeci, and Xia Lige. I consider it great fortune to have been part of such a dynamic research environment.

Coming to Canberra has involved many changes, including moving away from my family, but neverless it has been an exciting time. I have been blessed to have made so many friends here. Thanks to Christine, Meredith, and Natalie for sharing a house with me, and to the families at Reid church who have opened their homes to me. A special thanks to my fiancée Heather, who has given me so much love, support, and prayers over this last year.

I dedicate this thesis to my mother and father, whose love and personal sacrifice over the years has enabled me to have so much,

Andrew Telford

Abstract

This thesis is concerned with the design of robust controller algorithms. The mechanics of the design procedures involve factoring the plant and controller transfer functions into stable, proper factors. This has the advantage of allowing the analysis of the stability properties of the control loop, and, for instance, enabling the characterization of the class of all stabilizing controllers for a given plant.

Here results are developed showing that the class of all stabilizing controllers for a given plant can be structured as a state estimate feedback controller, with the state feedback and state estimate gains generalized to be proper transfer functions. This result is also generalized to the case of reduced-order observers. An important by product of the work on reduced order observers is the generation of new state-space realizations of doubly coprime factorizations; these state-space realizations are important both from a theoretical point and a computational point of view.

An arbitrary controller can be organized as a state estimate feedback controller with a constant state feedback and state estimator gain provided that the solution to a particular quadratic matrix equation exists. The insights gained by studying this realization problem lead to an investigation of conditions under which the solutions of of the algebraic Riccati equation exist.

Some related work on the problem of controller reduction follows. A con-

troller reduction scheme is proposed which applies standard model reduction algorithms to augmentations of the controller which arise when working with the class of all stabilizing controllers. Practical issues such as scaling of the plant variables are addressed, and two examples are given to demonstrate the use of the model reduction techniques.

Finally, an algorithm for adaptive resonance suppression is proposed for use in situations where time-varying plants can drift into instability. A simulation study is performed to demonstrate the behaviour of the algorithm. The algorithm appears particularly useful for enhancing existing fixed controller designs.

Preface

The material in this thesis is the result of joint research with my supervisor Professor John Moore. The contributions I particularly identify with are summarized as follows:

- My introduction to research was an involvement with a project initiated by Prof. John Moore and Prof. Keith Glover. This work on factorization theory and frequency-shaped state estimate feedback control appears in Chapter 2. I was responsible for taking the initial ideas, finding and correcting major errors in the theorems, and producing the results in their final form.

- The ideas in Chap. 3, on solution of the nonsymmetric Riccati equation, and Chap. 6, on adaptive resonance suppression, resulted from joint work with Prof. John Moore and myself. I provided many of the theoretical ideas and all of the simulation results.

- The work on controller reduction in Chap. 5 was also done with the Prof. John Moore—In particular, I was responsible for formalizing many of the theoretical aspects of the chapter. Prof. Uy-loi Ly provided realistic aircraft models to test the controller reduction algorithms.

- After initial discussions with Prof. Moore, I carried out almost all of the research in Chap. 4 independently. This work is concerned with doubly coprime factorizations related to reduced order observers.

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Notation

Where further explanation of the symbol is needed, a page number is given.

$\mathbb{C}$	field of complex numbers
$\mathbb{C}_{n,m}$	$n \times m$ matrices over the field of complex numbers
H_∞	a Hardy space, p.34
$J_n(\lambda)$	an $n \times n$ Jordan form, p.46
$\mathcal{L}[\cdot]$	Laplace transform
$\lambda(A)$	set of eigenvalues of the matrix A
L_∞	space of functions with ess. bounded norm
$\mathbb{R}$	field of real numbers
$\mathbb{R}_{n,m}$	$n \times m$ matrices over the field of real numbers
RH_∞	real-rational Hardy space, p.34
R_p	proper transfer functions, p.34
R_{sp}	strictly proper transfer functions, p.34
$\dot{x}(t)$	time derivative of $x(t)$
$\hat{q}$	estimate of the quantity q
$\|\| \ \|\|_\infty$	H_∞ norm, p.34

X^{-R}	a right inverse of X
X^{-L}	a left inverse of X
X'	transpose of X
X^*	Hermitian conjugate of X
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]_{\mathrm{T}}$	state space realization, p.35

Chapter 1

Introduction

Classical controller design and analysis techniques are based on frequency domain concepts and are principally concerned with the control of single-input single-output plants. Such techniques are adequate for a large class of practical engineering design problems: they are simple to understand, do not need complex hardware, and are reliable. In addition, gain/phase margins, Nyquist diagrams, and Bode plots are examples of concepts and tools which can be used to analyse the feedback loop.

There are, however, control problems which cannot be treated using single loop control techniques: a plant may be controlled from more than one input and measurements may be taken from many outputs. Classical control techniques are generally inadequate for use with multivariable plants, although knowledge of classical theory provides a good background for the study of multivariable systems.

The introduction of the idea of the *state* of a linear system into the control literature was a major step forward in the treatment of multivariable systems. State space ideas arise naturally in the mathematical treatment of linear ordinary differential equations, and engineers often model systems using such differential equations. Consider as an example the following state space system $G : u(t) \mapsto y(t)$,

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t); \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \quad (1.1)$$

where for any time t, $A(t) \in \mathbb{R}_{n,n}$, $B(t) \in \mathbb{R}_{n,m}$, and $C(t) \in \mathbb{R}_{p,n}$. The system (1.1) along with initial conditions for $x(t)$ defines the relationship between the $m \times 1$ input vector $u(t)$ and the $p \times 1$ output vector $y(t)$. The $n \times 1$ state vector $x(t)$ is defined by a vector differential equation driven by $u(t)$; the output vector $y(t)$ is a linear combination of $x(t)$ and $u(t)$. Much of what follows is concerned with the class of time-invariant state space systems, where $A(t)$, $B(t)$, and $C(t)$ are restricted to be constant real matrices. In this case Laplace transform techniques provide an alternative representation of G. Assuming zero initial conditions,

$$Y(s) = G(s)U(s) \quad (1.2)$$

$$G(s) = C(sI - A)^{-1}B + D \quad (1.3)$$

$$\text{where} \quad Y(s) = \mathcal{L}[y(t)]; \; U(s) = \mathcal{L}[u(t)]$$

At first (1.1) may not seem like a natural way to describe an engineering system, but it is important to realize that the entries of the state vector may represent physical variables, either measurable or unmeasurable. In some situations it is desirable to obtain on-line estimates of the state variables, either for monitoring unmeasurable physical variables, or for use in the control algorithm. Much of linear control theory is concerned with regulation of the state $x(t)$ to some reference state by appropriate manipulation of the control input. The problem of on-line estimation of the state via *observers* (state estimators) has been an important research topic over the last thirty years, with important initial contributions by Kalman [19] and Luenberger [27].

A natural dual to the problem of state estimation is that of state feedback. While state estimation attempts to estimate the hidden state vector, the aim of state feedback is to control the state via the plant inputs. The two can be combined to form a controller which cascades an observer and a state feedback gain. For the time invariant case, Luenberger [27] shows that there is a separation principle: the design of the observer and the choice of the state feedback gain can be made independently. The closed loop poles of the state estimate feedback scheme can be separated into poles due to the observer and poles due to the choice of the state feedback gain. While a state estimate controller can be designed to assign the closed loop poles in a particular way, there may be alternative design objectives. The state feedback law may be the solution of a *linear quadratic* (LQ) design problem [3]. Similarly in the presence of noise, it may be desirable to use optimal state estimation techniques such as *Kalman filtering* [20]. The LQ controller design and Kalman filtering problems are strict mathematical duals, with a controller formed by the cascade of a Kalman filter and an LQ controller known as a *linear quadratic gaussian* (LQG) controller.

Although LQG control seems like an ideal optimal control strategy, there are some important problems that arise in practice. One such problem is *robustness* to uncertainty in the plant model: the closed loop system may not be stable if the true plant is slightly different from the nominal plant. This is in contrast to the scalar LQ controller, where it can be shown that there is an inherent infinite gain margin and a corresponding 60° phase margin [3]. The problem of robustness to plant uncertainty or small time variations in the plant model is one that becomes more important as control systems become more complex, since plant error can be roughly compared to controller realization error. In the LQG case, techniques such as loop transfer recovery [13] have been proposed to obtain a compromise between the optimality and robustness.

State space realizations such as (1.1) or (1.3) are not the only ways to represent the input-output behaviour of dynamical systems. In Wolovich [48] and [18] there is a treatment of linear time invariant systems represented by ratios of polynomial matrices. Canonical forms and properties such as minimality can be described for polynomial factor representations as has been done for state space systems.

Recently another method has been used to represent the transfer function of a multivariable system. The transfer function is factored into matrices of *stable, proper* transfer functions. Although stable, proper factorizations may seem to be unnecessarily complex, they have the advantage of being useful when analysing the stability of a feedback system. For instance, using stable proper factorizations it is possible to characterize the class of all stabilizing controllers for a given plant, a concept which is very important in this thesis. This factorization approach to controller synthesis and analysis was introduced by Kučera [22] for discrete time and Youla et. al. [49] for continuous time, and later formulated in an axiomatic framework [11, 46]. It provides a framework for research in the area of H_∞ optimal control [15], which is concerned with finding a controller to minimize the L_∞ norm of a disturbance transfer function subject to the constraint that the controller be stabilizing.

Many of these results at first sight seem irrelevant to practical control engineering, because of the abstract algebraic framework which underlies much of the theory. This thesis is concerned principally with using ideas, techniques, and results from the factorization approach in the design of practical robust controllers. Existing knowledge and intuition based on frequency domain ideas is combined with the new theory. A summary of the progression of ideas in the thesis now follows.

State estimate feedback Chapter 2 is concerned with the design of closed-loop systems in which the control signal is a linear function of the state estimate. The work is related to that of Doyle [12], where it is shown that the class of all stabilizing controllers for a given plant can be obtained using a state estimate feedback control structure. The control signal is formed by adding a linear function of the state estimate to a stable filtering $Q(s)r$ of the residuals r, where the residuals is the difference between the true plant output y and estimate $\hat{y}$ of the plant output from the state estimator. The class of all stabilizing controllers for a given plant is obtained by varying the parameter $Q(s)$ over the class of all stable transfer functions. In Doyle's work the state feedback gain and the state estimator gain are constant real matrices, whereas this thesis allows them to be possibly unstable transfer function matrices. We will sometimes refer to the dynamics in the state estimator and state feedback gains as frequency shaping, because the frequency response of the dynamics can sometimes be related to plant or plant noise frequency responses.

In Chap. 2 it is shown that the class of all strictly proper stabilizing controllers for a proper linear plant can be structured as state estimate feedback, with dynamics in the state estimator or in the state estimate feedback law. The place where the dynamics is introduced is at the designers discretion. The parameterization of the controller class can be in terms of an arbitrary proper stable transfer function, with the closed loop system affine in this transfer function. With constant output feedback permitted in addition to the state estimate feedback, the class of all proper stabilizing controllers can be generated in like manner. These results are useful in engineering applications where the states represent physical variables.

Algebraic Riccati equations Following on from this, one can ask when an arbitrary controller can be organized as a state estimate feedback controller with a constant state feedback law and a constant state estimator gain. It was originally observed by Anderson [1] that this is possible only when there exists a nonsingular solution to a particular *nonsymmetric Riccati equation*. Such equations also arise in polynomial factorization theory [9], and although they have been treated at length in the literature, some of their properties are less well understood than for the symmetric case.

Chapter 3 provides contributions to established theory on this fundamental subject. Necessary conditions are shown for solutions of this Riccati equation to exist in terms of controllability and observability of the plant/controller state space realizations. The existence of an inverse of these solutions is given by considering a dual Riccati equation. There is also an alternative proof to that given hitherto, to establish the sufficiency of these conditions for a class of equations associated with certain scalar variable problems. A counterexample is given to the conjecture that the sufficiency conditions can be extended, without modification, to the multivariable case. This leads to generalized conditions for the multivariable case. As a challenge to the reader, it is conjectured that these are also sufficient conditions.

Reduced order observers; doubly coprime factorizations To make use of the results in the factorization approach requires an arbitrary transfer function to be factored into coprime factors in the ring of stable, proper transfer functions. For convenient computation, is it desirable to be able to work with state-space realizations of these transfer functions. In important work by Nett, Jacobson, and Balas [39], explicit state-space realizations of these factorizations are derived using results from state estimation and state feedback theory. These results are based only on *full-order* state estimators, which have realizations with the same McMillan degree n as the plant model.

Asymptotic state estimation can also be achieved by estimators of a lower degree: the degree may be reduced to $n - p$ for a plant with p outputs. This theory on *reduced-order* observers was originally reported by Luenberger [27], and is a generalization of the full-order case. In Chap. 4 new doubly coprime factorizations are developed based on reduced-order observers. Following on from this, various extensions are noted, and it is proved that the class of all stabilizing controllers for a given plant can be generated by dynamic feedback of the state estimate given by the reduced-order observer.

Controller reduction Chapter 5 is concerned with the problem of *controller reduction.* In reducing high order controller designs, such as arise from H_∞ or LQG techniques, to more practical low order ones, a reasonable objective is to preserve the performance and robustness properties. Here standard balanced truncation or Hankel norm approximation methods are applied to augmentations of the controller which emerge when characterizing the class of all stabilizing controllers for a given plant in terms of an arbitrary proper stabilizing transfer function.

In the method, scaling parameters are at the disposal of the engineer to achieve an appropriate compromise between preserving performance for the nominal plant and a certain type of robustness to plant variations. There are a number of unique features of the approach. One feature is that a straightforward re-optimization of a reduced-order controller is possible within the framework of the method. A second feature is that for controllers designed for simultaneous stabilization of a number of plants, the method seeks to preserve the performance and robustness of the reduced-order controller for each plant.

Adaptive resonance suppression The work of Chap. 6 is not concerned directly with stable, proper factorizations, but it is intended that the results be used to complement the work of Tay, Moore, and Horowitz [42]. This related work is concerned with applying adaptive techniques to structures arising when describing the class of all stabilizing controllers for a given plant. Chapter 6 is concerned with control systems that can drift into stability, or less catastrophically, exhibit resonance behaviour. Such resonance phenomena appear in many practical engineering control systems, ranging from relatively slow chemical processes to high performance aircraft controllers.

Fixed controllers may not be robust in the presence of time-varying plant models. One role for adaptive controllers is to learn sufficient information concerning the dominant closed-loop resonant modes so as to apply effective feedback to dampen these modes. In such situations the adaptive loop augments the fixed controller feedback loop. Here an algorithm is presented for *adaptive resonance suppression* and simulation results are provided to study its behaviour in the presence of high-order unmodelled dynamics. The algorithm appears particularly useful for enhancing existing fixed controller designs.

In the final chapter, an overview of the work will be given, and further research possibilities will be discussed.

Chapter 2

All stabilizing controllers as frequency shaped state estimate feedback

2.1 Introduction

Consider the stabilizable and detectable linear time-invariant system with state equations

$$\dot{x} = Ax + Bu, \; y = Cx + Du \tag{2.1}$$

and transfer function $G \in R_p$

$$G = C(sI - A)^{-1}B + D \tag{2.2}$$

The plant $G(s)$ is said to be proper since $|G(\infty)|$ is finite. We formally say that a controller $K(s)$ is *stabilizing* for G (see Fig. 2.1) if the four transfer functions from $[u_1' \; u_2']'$ to $[e_1' \; e_2']'$ are stable.

This chapter is concerned with the structure and properties of state estimate feedback controllers. Unlike much of the work in this area, the state

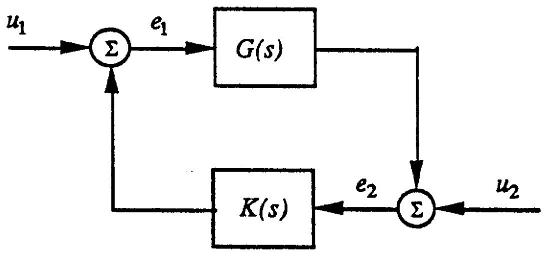

Figure 2.1: Closed loop system

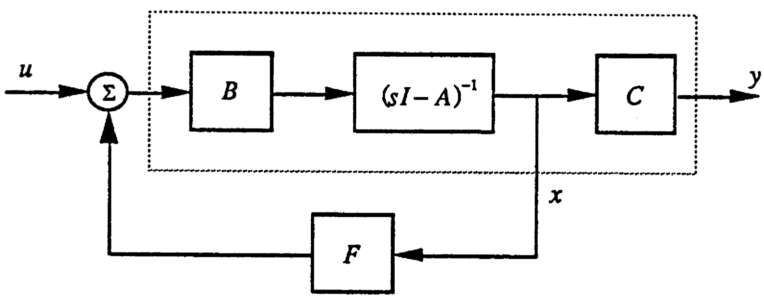

Figure 2.2: Control system with state feedback

feedback gain F and state estimator gain H are here permitted to be proper transfer functions. We recall first what is meant by these terms for the case when F, H are constant. Figure 2.2 shows state feedback for the strictly proper plant $G(s) = C(sI - A)^{-1}B$; the feedback signal is $Fx(t)$. The transfer function from u to y is

$$\frac{Y(s)}{U(s)} = C(sI - A - BF)^{-1}B \quad . \tag{2.3}$$

It well known that by a suitable choice of F, the poles of this transfer function can always be assigned into the stability region, $\mathbb{R}[s] < 0$, if and only if the pair (A, B) is stabilizable. The stabilizability property is equivalent to saying that all unstable plant modes will be controllable. If a constant F is replaced by a possibly unstable $F(s)$, then what choices of $F(s)$ will lead to a stable state feedback controller?

Consider the state estimator in Fig. 2.3, which is the dual of the above case. The transfer function from the input u to the error in the state estimate $x - \hat{x}$ is zero, and furthermore, the transient behaviour of this error, due to non-zero initial conditions, will approach zero if H is chosen such that the eigenvalues of $A+HC$ are in the stability region. This is known to be possible whenever the pair (A, C) is detectable, where detectability is equivalent to saying that all unstable modes of A will be observable. If a constant H is replaced by an arbitrary proper transfer function $H(s)$, then what values of $H(s)$ will give an estimator with an error that tends to zero asymptotically?

Combining state estimation and state feedback, we obtain a state estimate feedback controller. Such a controller, with F, H constant, can be defined by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - H(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du \\ u &= F\hat{x} \end{aligned} \tag{2.4}$$

with transfer function $K \in R_{sp}$

$$K = -F[sI - (A + BF + HC + HDF)]^{-1}H \tag{2.5}$$

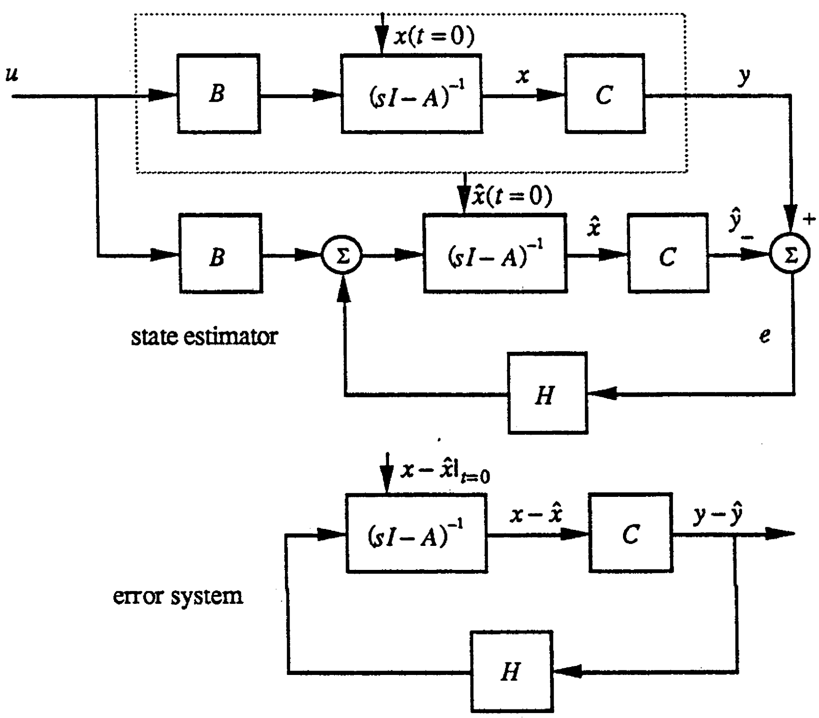

Figure 2.3: An asymptotic state estimator

This controller is known to be stabilizing if and only if

$$[sI - (A + BF)]^{-1},\ [sI - (A + HC)]^{-1} \in RH_\infty \tag{2.6}$$

where RH_∞ is defined to be the class of proper and stable real-rational transfer functions.

With H, F generalized as transfer functions $H(s)$, $F(s) \in R_p$ with stabilizable and detectable state-space realizations, the resulting state estimate feedback controller is known to be stabilizing for the plant $G(s)$, with all states asymptotically stable for arbitrary initial conditions if and only if [33]:

$F(s)$, $H(s)$ stabilize G_F, G_H respectively, where

$$G_F = (sI - A)^{-1}B,\ G_H = C(sI - A)^{-1} \tag{2.7}$$

This result is also shown as a by-product of the theory of this chapter.

In the chapter we show that for the plant G of (2.2) the class of all stabilizing controllers of the form (2.4), parameterized in terms of F, $H \in R_p$ satisfying (2.7), is the entire class of all stabilizing controllers for the plant (2.2). Moreover, the entire class can be generated in terms of a stabilizing $F \in R_p$ for G_F, where $H \in R_p$ is an arbitrary stabilizing controller for G_H with a left inverse $H^{-L} \in RH_\infty$. Likewise, in terms of a stabilizing $H \in R_p$ for G_H, where F is an arbitrary stabilizing controller for G_F with a right inverse $F^{-R} \in RH_\infty$. The existence of a stable proper inverse of H or F is equivalent to a multivariable generalization of the familiar scalar *minimum phase* property, with the additional constraint that the transfer function has relative degree zero.

With constant output feedback permitted in addition to the state estimate feedback, the class of all proper stabilizing controllers can be generated using a mild variation. In addition, the parameterizations can be in terms of arbitrary transfer functions Q_F, $Q_H \in RH_\infty$, with the closed loop transfer functions affine in Q_F or Q_H. The theory developed here is based on results from factorization theory [46, 11], and complements other work which involves modification to standard state estimate feedback [33, 12].

The controller structures of this chapter have the advantage that they are decomposed into a state estimator and a state feedback law, where at the discretion of the designer, generally one or both are frequency shaped. Thus any stabilizing controller can be viewed in terms of filtered feedback of each state estimate, or as direct feedback of each frequency shaped state estimate. This has appeal in engineering situations, where the states represent physical internal variables. For example, knowledge that an effective controller feeds back a low pass filtered velocity or position estimate could be instructive when improving the design by introducing additional sensors, or improving sensor locations. In situations where state estimation is required in addition to control, the results of this chapter give useful implementation possibilities. Gain scheduling could be more systematic in the framework of the state estimate feedback. This is not to say that state estimate feedback is always the best design approach, as illustrated when the frequency shaping in the state estimate feedback cancels out the observer dynamics.

In Sec. 2.2, known theory [12, 11] for the class of all stabilizing controllers is reviewed and extended for use in subsequent sections. In Sec. 2.3, the main results of the chapter are developed. Some useful related results are summarized in Sec. 2.4, and conclusions are drawn in Sec. 2.5.

2.2 Stabilizing controllers for G, G_F, G_H

Employing the notation in Appendix A2.1, the transfer functions G, G_F, G_H from (2.2), (2.7) can be written

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_T, \; G_F = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]_T, \; G_H = \left[\begin{array}{c|c} A & I \\ \hline C & 0 \end{array}\right]_T \tag{2.8}$$

Consider also coprime factorizations over RH_∞

$$\begin{aligned} &G = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \\ &G_F = N_F M_F^{-1} = \tilde{M}_F^{-1}\tilde{N}_F, \\ &G_H = N_H M_H^{-1} = \tilde{M}_H^{-1}\tilde{N}_H, \\ &\text{where } M^{-1}, \tilde{M}^{-1}, M_F^{-1}, \tilde{M}_F^{-1}, M_H^{-1}, \tilde{M}_H^{-1} \in R_p \end{aligned} \tag{2.9}$$

Let us denote K, F, $H \in R_p$ as stabilizing controllers for G, G_F, G_H respectively with coprime RH_∞ factorizations

$$\begin{aligned} &K = UV^{-1} = \tilde{V}^{-1}\tilde{U}, \\ &F = U_F V_F^{-1} = \tilde{V}_F^{-1}\tilde{U}_F, \\ &H = U_H V_H^{-1} = \tilde{V}_H^{-1}\tilde{U}_H, \\ &\text{where } V^{-1}, \tilde{V}^{-1}, V_F^{-1}, \tilde{V}_F^{-1}, V_H^{-1}, \tilde{V}_H^{-1} \in R_p \end{aligned} \tag{2.10}$$

Such stabilizing controllers are known to exist with (A, B) stabilizable and (A, C) detectable.

For what follows, doubly coprime factorizations of $G(s)$ with respect to the ring RH_∞ are required. With the notation above and

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.11}$$

then $NM^{-1} = \tilde{M}^{-1}\tilde{N}^{-1}$ provide doubly coprime factorizations of G. With F, H constant and K equal to the state estimate feedback controller of (2.5), then Nett, Jacobson, and Balas [39] give explicit state space realizations for factorizations of G, K satisfying (2.11).

It is no longer possible to use the doubly coprime factorizations for G of Nett et. al. when F, H are generalized to possibly unstable $F(s), H(s) \in R_p$. Theorem 2.1 overcomes this problem by using suitably modified factorizations.

Theorem 2.1 *Given any F, $H \in R_p$ stabilizing for G_F, G_H of (2.7) with factorizations (2.10) and (A, B, C) minimal, then coprime factorizations for G_F, G_H, G and K satisfying (2.9), (2.10) are* [1]*:*

$$\begin{aligned} \begin{bmatrix} M_F \\ N_F \end{bmatrix} &= \left[\begin{array}{c|c} A+BF & B\tilde{V}_F^{-1} \\ \hline F & \tilde{V}_F^{-1} \\ I & 0 \end{array}\right]_T \in RH_\infty \\ \begin{bmatrix} -\tilde{N}_F & \tilde{M}_F \end{bmatrix} &= \left[\begin{array}{c|cc} A+BF & -B & BF \\ \hline V_F^{-1} & 0 & V_F^{-1} \end{array}\right]_T \in RH_\infty \end{aligned} \tag{2.12}$$

[1] Here the notation is generalized to allow the four entries of the state realization matrix to be transfer functions as in (2.57).

$$\begin{bmatrix} M_H \\ N_H \end{bmatrix} = \left[\begin{array}{c|c} A+HC & \tilde{V}_H^{-1} \\ \hline HC & \tilde{V}_H^{-1} \\ C & 0 \end{array}\right]_T \in RH_\infty$$

$$\begin{bmatrix} -\tilde{N}_H & \tilde{M}_H \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -I & H \\ \hline V_H^{-1}C & 0 & V_H^{-1} \end{array}\right]_T \in RH_\infty \tag{2.13}$$

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B\tilde{V}_F^{-1} & -U_H \\ \hline F & \tilde{V}_F^{-1} & 0 \\ C+DF & D\tilde{V}_F^{-1} & V_H \end{array}\right]_T \in RH_\infty$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -B-HD & H \\ \hline \tilde{U}_F & \tilde{V}_F & 0 \\ V_H^{-1}C & -V_H^{-1}D & V_H^{-1} \end{array}\right]_T \in RH_\infty \tag{2.14}$$

Moreover, the factorizations satisfy the following double Bezout equations:

$$\begin{bmatrix} \tilde{V}_F & -\tilde{U}_F \\ -\tilde{N}_F & \tilde{M}_F \end{bmatrix} \begin{bmatrix} M_F & U_F \\ N_F & V_F \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.15}$$

$$\begin{bmatrix} \tilde{V}_H & -\tilde{U}_H \\ -\tilde{N}_H & \tilde{M}_H \end{bmatrix} \begin{bmatrix} M_H & U_H \\ N_H & V_H \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.16}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.17}$$

Proof The properties of (2.9), (2.15)–(2.17) can be verified by simple manipulations, as shown in Appendix A2.2. It remains to show that the factors (2.12)–(2.14) are stable, since with (2.15)–(2.17) this implies that the factorizations are coprime in RH_∞. Consider first coprime factorizations $G_F = \mathcal{N}_F\mathcal{M}_F^{-1} = \tilde{\mathcal{M}}_F^{-1}\tilde{\mathcal{N}}_F$. Since F stabilizes G_F, then standard arguments [46] give that

$$(\tilde{\mathcal{M}}_F V_F - \tilde{\mathcal{N}}_F U_F)^{-1}, (\tilde{V}_F \mathcal{M}_F - \tilde{U}_F \mathcal{N}_F)^{-1} \in RH_\infty \tag{2.18}$$

Also from (2.15), $\tilde{M}_F(V_F - G_F U_F) = I$, $(\tilde{V}_F - \tilde{U}_F G_F)M_F = I$ so that

$$\begin{aligned}\left[\tilde{M}_F \; \tilde{N}_F\right] &= (V_F - G_F U_F)^{-1}[I \; G_F] \\ &= (\tilde{\mathcal{M}}_F V_F - \tilde{\mathcal{N}}_F U_F)^{-1}\left[\tilde{\mathcal{M}}_F \; \tilde{\mathcal{N}}_F\right] \in RH_\infty \end{aligned} \quad (2.19)$$

$$\begin{aligned}\begin{bmatrix} M_F \\ N_F \end{bmatrix} &= \begin{bmatrix} I \\ G_F \end{bmatrix}(\tilde{V}_F - \tilde{U}_F G_F)^{-1} \\ &= \begin{bmatrix} \mathcal{M}_F \\ \mathcal{N}_F \end{bmatrix}(\tilde{V}_F \mathcal{M}_F - \tilde{U}_F \mathcal{N}_F)^{-1} \in RH_\infty \end{aligned} \quad (2.20)$$

Analogous proofs for the dual show that N_H, M_H, $\tilde{N}_H$, $\tilde{M}_H \in RH_\infty$. It then follows that N, M, $\tilde{N}$, $\tilde{M} \in RH_\infty$ since

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} I & 0 \\ D & C \end{bmatrix}\begin{bmatrix} M_F \\ N_F \end{bmatrix} \quad (2.21)$$

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} \tilde{N}_H & \tilde{M}_H \end{bmatrix}\begin{bmatrix} B & 0 \\ -D & I \end{bmatrix} \quad (2.22)$$

Finally, since F stabilizes G_F, all four closed loop transfer functions are stable. This implies that

$$[sI - (A + BF)]^{-1}, \; F[sI - (A + BF)]^{-1}B \in RH_\infty$$

$$\Rightarrow F[sI - (A + BF)]^{-1} \in RH_\infty \text{ under } (A, B) \text{ controllable} \quad (2.23)$$

It follows from (2.23) that $U, V \in RH_\infty$. Dual arguments show that $\tilde{U}$, $\tilde{V} \in RH_\infty$.

□

With the factorizations of Theorem 2.1 established, then (2.17) implies that [11]:

$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U} = -F[sI - (A + BF + HC + HDF)]^{-1}H \quad (2.24)$$

will be stabilizing for G. Other standard results can immediately be applied to G, K based on the doubly coprime factorizations of Theorem 2.1. For instance, it is possible to characterize the class of all stabilizing controllers $K(Q)$, that stabilize G, in terms of an arbitrary parameter $Q \in RH_\infty$.

Theorem 2.2 ([11, 12]) *Consider the plant G of (2.8) with coprime factorizations (2.9), (2.14), (2.17) as above. The class of all proper stabilizing controllers can be parameterized in terms of arbitrary $Q \in RH_\infty$ as*

$$\begin{aligned} K(Q) &= U(Q)V(Q)^{-1} = \tilde{V}(Q)^{-1}\tilde{U}(Q) \\ &= K + \tilde{V}^{-1}Q(I + V^{-1}NQ)^{-1}V^{-1} \end{aligned}$$

where

$$\begin{array}{lll} U(Q) = U + MQ; & V(Q) = V + NQ; & |V + NQ| \neq 0 \\ \tilde{U}(Q) = \tilde{U} + Q\tilde{M}; & \tilde{V}(Q) = \tilde{V} + Q\tilde{N}; & |\tilde{V} + Q\tilde{N}| \neq 0 \end{array} \tag{2.25}$$

We finish this section with some remarks:

1. The class of all stabilizing controllers of (2.25) can be depicted as in Fig. 2.4, where

$$J = \begin{bmatrix} K & \tilde{V}^{-1} \\ V^{-1} & -V^{-1}N \end{bmatrix}, T = \begin{bmatrix} \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} & \begin{bmatrix} M \\ N \end{bmatrix} \\ \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} & 0 \end{bmatrix} \tag{2.26}$$

2. The closed loop transfer functions are affine in Q,

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = T_{11} + T_{12}QT_{21} =$$

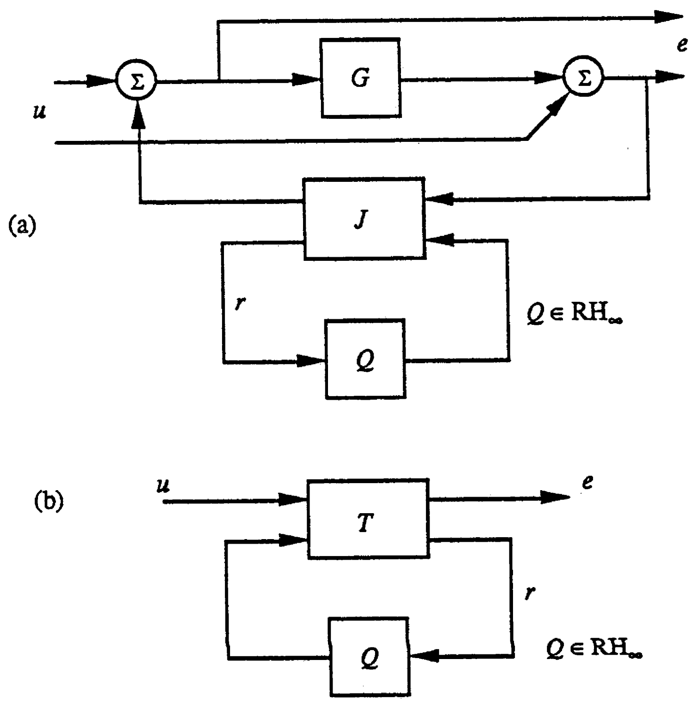

Figure 2.4: Class of all stabilizing controllers

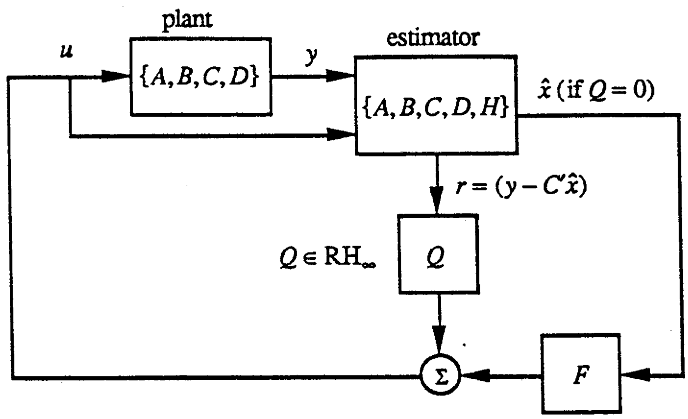

Figure 2.5: Class of all stabilizing controllers: Doyle-Stein form

$$\left[\begin{array}{ccc|cc} A+HC & 0 & 0 & -B-HD & -H \\ B_QV_H^{-1}C & A_Q & 0 & B_QV_H^{-1}D & B_QV_H^{-1} \\ B(F-\tilde{D}C) & B\tilde{V}_F^{-1}C_Q & A+BF & B(I+\tilde{D}D) & B\tilde{D} \\ \hline F-\tilde{D}C & \tilde{V}_F^{-1}C_Q & A+BF & I+\tilde{D}D & \tilde{D} \\ D(F-\tilde{D}C) & D\tilde{V}_F^{-1}C_Q & C+DF & (I+\tilde{D}D)D & I+D\tilde{D} \end{array}\right]_T \tag{2.27}$$

where

$$\tilde{D} = \tilde{V}_F^{-1}D_QV_H^{-1},\ Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]_T$$

The derivation of T is shown in Appendix A2.2. An alternative depiction is in Fig. 2.5; here all stabilizing controllers are be obtained by filtering the residuals $r = y - \hat{y}$ with an arbitrary $Q \in RH_\infty$, and adding this to the controller output.

3. Observe from (2.25) that given an arbitrary proper plant K_1, with arbitrary coprime factorizations

$$K_1 = U_1V_1^{-1} = \tilde{V}_1^{-1}\tilde{U}_1,$$

then $Q_1 \in R_p$ is uniquely determined such that $K_1 = K(Q_1)$, since from simple manipulations:

$$\begin{aligned} Q_1 &= -M^{-1}[I - K_1G]^{-1}(U - K_1V) \\ &= -[\tilde{V}_1M - \tilde{U}_1N)^{-1}[\tilde{V}_1U - \tilde{U}_1V] \\ &= -[\tilde{U}V_1 - \tilde{V}U_1][\tilde{M}V_1 - \tilde{N}U_1]^{-1} \end{aligned} \tag{2.28}$$

A consequence is that MQ_1 and NQ_1 are uniquely determined.

4. Substitution for V, U and $\tilde{V}$, $\tilde{U}$ into (2.17) from (2.25) results in corresponding properties for $V(Q)$, $U(Q)$ and $\tilde{V}(Q)$, $\tilde{U}(Q)$

$$\begin{bmatrix} \tilde{V}(Q) & -\tilde{U}(Q) \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U(Q) \\ N & V(Q) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.29}$$

With the result (2.29), $U_1 = U(Q_1)$, $V_1 = V(Q_1)$, and the duals, then (2.28) simplifies to

$$Q_1 = \tilde{U}(Q_1)V - \tilde{V}(Q_1)U = \tilde{V}U(Q_1) - \tilde{U}V(Q_1) \tag{2.30}$$

5. Note that

$$[K(Q) \in R_{sp}] \Leftrightarrow [Q \in R_{sp}] \tag{2.31}$$

6. Observe that as a consequence of the fact that $T_{22} = 0$, the transfer function from e to r is invariant of Q, and is equal to T_{21}.

7. With F, $H \in RH_\infty$, then without loss of generality, $V_H = I$, $V_F = I$. In this case, the factorizations of (2.12)–(2.14) simplify to the specializations of [39].

8. The above results apply to yield the class of all stabilizing controllers for G_F, G_H in terms of arbitrary Q_F, $Q_H \in RH_\infty$:

$$\begin{aligned} F(Q_F) &= \tilde{V}_F(Q_F)^{-1}\tilde{U}_F(Q_F) \\ &= \tilde{V}_F^{-1}\tilde{U}_F + \tilde{V}_F Q_F(I + V_F^{-1}N_F Q_F)^{-1}V_F^{-1} \end{aligned}$$

$$\begin{gathered} \tilde{V}_F(Q_F) = \tilde{V}_F + Q_F\tilde{N}_F; \quad \tilde{U}_F(Q_F) = \tilde{U}_F + Q_F\tilde{M}_F \\ |\tilde{V}_F + Q_F\tilde{N}_F| \neq 0 \end{gathered} \tag{2.32}$$

and

$$\begin{aligned} H(Q_H) &= U_H(Q_H)V_H(Q_H)^{-1} \\ &= U_H V_H^{-1} + \tilde{V}_H^{-1} Q_H(I + V_H^{-1}N_H Q_H)^{-1}V_H^{-1} \end{aligned}$$

$$\begin{gathered} U_H(Q_H) = U_H + M_H Q_H; \quad V_H(Q_H) = V_H + N_H Q_H \\ |V_H + N_H Q_H| \neq 0 \end{gathered} \tag{2.33}$$

Then the transfer functions $F(Q_F)$, $H(Q_H)$ can be structured as shown in Fig. 2.6, where J_F, J_H are given by

$$J_F = \begin{bmatrix} F & \tilde{V}_F^{-1} \\ V_F^{-1} & -V_F^{-1}N_F \end{bmatrix}, \; J_H = \begin{bmatrix} H & \tilde{V}_H^{-1} \\ V_H^{-1} & -V_H^{-1}N_H \end{bmatrix} \tag{2.34}$$

The main development in this section is to define new state space realizations for doubly coprime factorizations of a plant $G(s)$. They are based on a stabilizing state estimate feedback controller with dynamics in both the state estimator and the state estimate feedback gains. This is a non-trivial extension to the case where only constant gains are permitted. The theory also requires doubly coprime factorizations of the associated 'plants' G_F, G_H

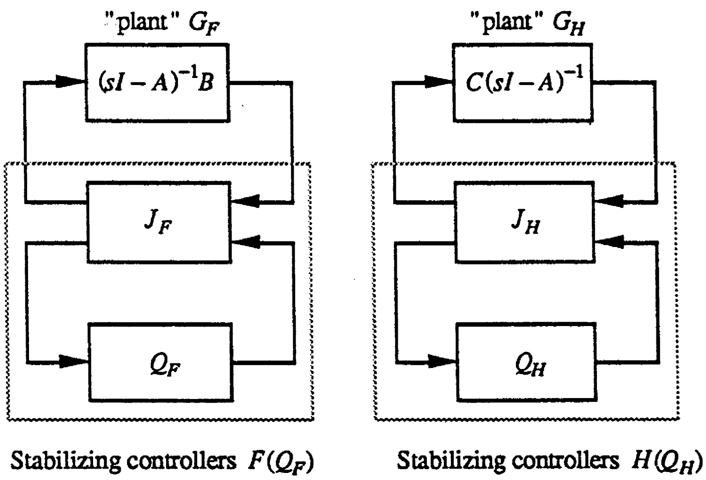

Figure 2.6: Stabilizing controllers $F(Q_F)$, $H(Q_F)$

to be defined. It is shown that once the doubly coprime factorizations are established, then standard results on stability can be applied. In particular, it is possible to parameterize the class of all stabilizing controllers for G, G_F, G_H. In what follows we show that the class of all stabilizing controllers for G can be achieved by a state estimate feedback controller with dynamic state estimator or state estimate feedback gains.

2.3 Stabilizing controllers for G in terms of Q_F, Q_H, $\bar{Q}$

The class of state estimate feedback controllers in terms of F, $H \in R_p$ will be denoted as

$$K[F,H] = \left[\begin{array}{c|c} A+BF+HC+HDF & -H \\ \hline F & 0 \end{array}\right]_T \in R_{sp} \qquad (2.35)$$

Lemma 2.3 *With the definitions (2.35), (2.32), (2.33), and the factorizations of Theorem 2.1, the following classes of strictly proper controllers parameterized by* $Q_H, Q_F \in RH_\infty$ *are stabilizing:*

$$\begin{aligned} K[F, H(Q_H)] &= \mathcal{U}_H(Q_H)\mathcal{V}_H(Q_H)^{-1} \\ \mathcal{U}_H(Q_H) = U + \mathcal{M}Q_H, &\qquad \mathcal{V}_H(Q_H) = V + \mathcal{N}Q_H \end{aligned} \tag{2.36}$$

$$\begin{aligned} K[F(Q_F), H] &= \tilde{\mathcal{V}}_F(Q_F)^{-1}\tilde{\mathcal{U}}_F(Q_F) \\ \tilde{\mathcal{U}}_F(Q_F) = \tilde{U} + Q_F\tilde{\mathcal{M}}, &\qquad \tilde{\mathcal{V}}_F(Q_F) = \tilde{V} + Q_F\tilde{\mathcal{N}} \end{aligned} \tag{2.37}$$

where $\mathcal{M}$, $\mathcal{N}$, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{N}}$ *are given by*

$$\begin{aligned} \begin{bmatrix} \mathcal{M} & U \\ \mathcal{N} & V \end{bmatrix} &= \left[\begin{array}{c|cc} A+BF & -M_H & -U_H \\ \hline F & 0 & 0 \\ C+DF & N_H & V_H \end{array}\right]_T \in RH_\infty \\ \begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix} &= \left[\begin{array}{c|cc} A+HC & -B-HD & H \\ \hline \tilde{U}_F & \tilde{V}_F & 0 \\ -\tilde{M}_F & -\tilde{N}_F & 0 \end{array}\right]_T \in RH_\infty \end{aligned} \tag{2.38}$$

Moreover, the following properties hold

$$G\mathcal{M} = \mathcal{N}, \; \tilde{\mathcal{M}}G = \tilde{\mathcal{N}} \tag{2.39}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix}\begin{bmatrix} \mathcal{M} & U \\ \mathcal{N} & V \end{bmatrix} = \begin{bmatrix} M^{-1}\mathcal{M} & 0 \\ 0 & \tilde{\mathcal{M}}\tilde{M}^{-1} \end{bmatrix} \tag{2.40}$$

$$\begin{aligned} M^{-1}\mathcal{M} &= -\tilde{U}_F[sI - A - HC]^{-1}\tilde{V}_H^{-1} \in RH_\infty \\ \tilde{\mathcal{M}}\tilde{M}^{-1} &= -V_F[sI - A - BF]^{-1}U_H \in RH_\infty \end{aligned} \tag{2.41}$$

Proof See Appendix A2.2

It remains to find conditions under which the class of all strictly proper stabilizing controllers can be structured as $K[F, H(Q_H)]$ or $K[F(Q_F), H]$ for Q_F, $Q_H \in RH_\infty$.

Lemma 2.4 *With the definitions (2.36), (2.37), and (2.25), then*

$$K[F, H(Q_H)] = K(Q) \in R_{sp} \Leftrightarrow \begin{bmatrix} \mathcal{M} \\ \mathcal{N} \end{bmatrix} Q_H = \begin{bmatrix} M \\ N \end{bmatrix} Q \in R_{sp}$$
$$\Leftrightarrow Q = M^{-1}\mathcal{M}Q_H \in R_{sp} \qquad (2.42)$$

For all $Q \in RH_\infty \cap R_{sp}$ *then there exists* $Q_H \in RH_\infty$ *satisfying (2.42) if and only if*

$$F \text{ has a right inverse in } RH_\infty \qquad (2.43)$$

(A dual relationship exists for $K[F(Q_F), H]$ *and* $K(Q)$*, the relationship being* $Q = Q_F\tilde{\mathcal{M}}\tilde{M}^{-1}$. *The dual of (2.43) involves the existence of a left inverse for H.)*

Proof See Appendix A2.2.

Remarks:

1. Condition (2.43) in Lemma 2.4 is equivalent to specifying that F is a minimum phase transfer function with relative degree zero.

2. Since the closed loop transfer functions are affine in Q, and by (2.42), Q_F is linear in Q_H, then the closed loop transfer functions generated by the class of all controllers $K[F, H(Q_H)]$ will be affine in Q_H.

3. The results of (2.42) can be generalized by replacing F by $F(Q_F)$, for some $Q_F \in RH_\infty$. With a fixed $F(Q_F)$ and, for example, writing $K(Q)$ as $K_{F,H}(Q)$ to denote explicitly the state feedback and state estimator gains, the (2.42) becomes

$$K[F(Q_F),\, H(Q_H)] = K_{F(Q_F),H}(Q)$$
$$\Leftrightarrow \quad Q = -\tilde{U}_F(Q_F)[sI - A - HC]^{-1}\tilde{V}_H^{-1}Q_H \qquad (2.44)$$

It follows that the class of all stabilizing controllers can be organized as $K[F(Q_F), H(Q_H)]$ where $F(Q_F)$, $H(Q_H)$ are given by (2.32), (2.33) and Q_F, $Q_H \in RH_\infty$.

The following theorem is now established from Theorem 2.2, Lemma 2.4, and their remarks

Theorem 2.5 *Consider the plants G, G_F, G_H of (2.8) with (A, B) stabilizable and (A, C) detectable, and the state estimate feedback controller $K[F, H]$ of (2.35) for G.*

(i) With F, $H \in R_p$ arbitrary stabilizing controllers for G_F, G_H, respectively, then $K[F, H] \in R_{sp}$ is stabilizing for G and represents the entire class of stabilizing controllers for G.

(ii) With $F_1 \in R_p$ fixed and (strictly) minimum phase as in (2.43), and $H \in R_p$ arbitrary stabilizing proper controllers for G_H, the subclass $K[F_1, H]$ represents the class of all strictly proper controllers for G if

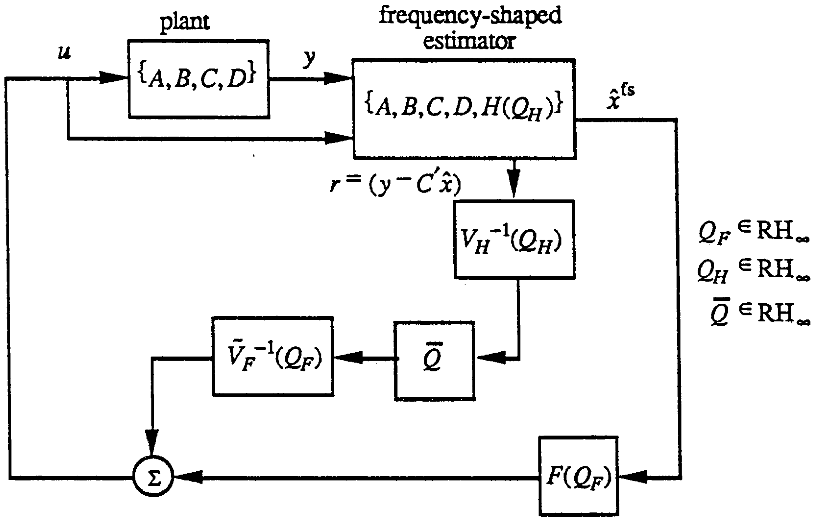

Figure 2.7: Class of all stabilizing state estimate feedback controllers, denoted $K[F(Q_F),\, H(Q_F), \bar{Q}]$

and only if $F_1 \in R_p$ *stabilizes* G_F. *A dual result holds for* $K[F, H_1]$ *when* H_1 *is fixed.*

As an extension, consider a more general class of state estimate feedback controllers $K[F,\, H,\, \bar{Q}]$ as in Fig. 2.7. This is really the class $K_{F,H}(\bar{Q})$, but writing $K[F,\, H,\, \bar{Q}]$ explicitly shows that there are three parameters F, H, and $\bar{Q}$. The class is obtained from the class $K[F,\, H] \in R_{sp}$ by adding to the controls the residuals $(y - \hat{y})$ filtered by $\bar{Q} \in RH_\infty$,

$$u = \bar{Q}(y - \hat{y}) + F\hat{x} \tag{2.45}$$

The results of the previous section can be applied (see Appendix A2.2) to yield the following theorem, which is a generalization of the results in [12] to the case when F, H are dynamic rather than constant.

Theorem 2.6 *With the notation above, $K[F, H, \bar{Q}] \in R_p$ is stabilizing for arbitrary F, H stabilizing for G_F, G_H and arbitrary $\bar{Q} \in RH_\infty$. Moreover, with $\bar{Q}$ an arbitrary constant, then $K[F, H, \bar{Q}]$ represents the entire class of stabilizing controllers in R_p for G.*

Proof See Appendix A2.2

□

Remarks:

1. The controllers $K[F, H, \bar{Q}]$ when $\bar{Q}$ is a constant, are still conveniently viewed as *state* estimate feedback schemes with additional output feedback. With $\bar{Q}$ constant, then the feedback signal u in (2.45) can be formed as a combination of $\hat{x}$ and y:

$$u = (F - \bar{Q}C)\hat{x} + \bar{Q}y \tag{2.46}$$

When $\bar{Q}$ is frequency shaped, there is no ready interpretation as state estimate feedback or even frequency shaped state estimate feedback.

2. The results in this chapter are presented without restricting F, H to be stable, yet in practical controllers the restriction $F, H \in RH_\infty$ is a reasonable one to apply. Of course any implementation of $K[F, H]$ must not include unstable pole/zero cancellations associated with unstable F, H. Such would give instability; a priori cancellation avoids such difficulties.

2.4 Useful relationships

Here several useful formulae will be stated, which relate various transfer functions such as K, Q, J, F, and H of the previous sections. The relationships are verified by algebraic manipulation as was done in Appendix A2.1 for earlier proofs. Consider the plant G with a strictly proper controller $\bar{K} = \bar{C}(sI - \bar{A})^{-1}\bar{B}$ and H, F constant vectors stabilizing G_H, G_F respectively. The factorizations (2.14) can then be specialized with $V_H = I$, $\tilde{V}_H = I$, $V_F = I$, and $\tilde{V}_F = I$. If the control-loop is well posed, then it will be possible to invert (2.25) to obtain a Q such that $K(Q) = \bar{K}$,

$$Q = \left[\begin{array}{cc|c} \bar{A}+\bar{B}D\bar{C} & \bar{B}C & \bar{B} \\ B\bar{C} & A & -H \\ \hline \bar{C} & -F & 0 \end{array}\right]_{\mathrm{T}} \tag{2.47}$$

$$J = \left[\begin{array}{c|cc} A+BF+HC+HDF & -H & B+HD \\ \hline F & 0 & I \\ -C-DF & I & -D \end{array}\right]_{\mathrm{T}} \tag{2.48}$$

The expression for J is stated in [12]. Also from (2.27),

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \left[\begin{array}{cc|cc} \bar{A}+\bar{B}D\bar{C} & \bar{B}C & \bar{B}D & \bar{B} \\ B\bar{C} & A & I & 0 \\ \hline \bar{C} & 0 & I & 0 \\ D\bar{C} & C & D & I \end{array}\right]_{\mathrm{T}} \tag{2.49}$$

Comparing (2.47) and (2.49) show that the modes of Q, perhaps not all controllable or observable, are the same as the modes of the closed-loop transfer functions: this is not unexpected, since $Q \in RH_\infty$ if and only if there is closed loop stability.

If a constant H is chosen to stabilize $C(sI-A)^{-1}$ and H has a left inverse H^{-L}, then a controller $K[F(s), H]$ can be realized by using a frequency shaped state estimate feedback, with

$$F(s) = \left[\begin{array}{c|c} \bar{A}+\bar{B}H^{-L}(B+HD)\bar{C} & \begin{array}{c}-[\bar{A}+\bar{B}H^{-L}(B+HD)\bar{C}]\bar{B}H^{-L} \\ +\bar{B}H^{-L}(A+HC)\end{array} \\ \hline \bar{C} & -\bar{C}\bar{B}H^{-L} \end{array}\right]_T \tag{2.50}$$

The dual result exists for $K[F, H(s)]$ in the case when a constant F, which stabilizes $(sI-A)^{-1}B$, has a right inverse

$$H(s) = \left[\begin{array}{c|c} \bar{A}+\bar{B}(C+DF)F^{-R}\bar{C} & \bar{B} \\ \hline (A+BF)F^{-R}\bar{C} - F^{-R}\bar{C}[\bar{A}+\bar{B}(C+DF)F^{-R}\bar{C}] & -F^{-R}\bar{C}\bar{B} \end{array}\right]_T \tag{2.51}$$

Note that existence of a right inverse of the constant H is the same as condition (2.43), but (2.43) considers the case when H is permitted to have dynamics.

2.5 Conclusions

This chapter has demonstrated that the class of all stabilizing controllers can be constructed conveniently from frequency shaped state estimate feedback controllers, with the frequency shaping in the state estimation, in the state estimate feedback, or in both. This result underlines the versatility of controller designs based on frequency shaped estimation and control, and allows elucidation of controllers designed by other approaches in terms of state feedback, albeit frequency shaped.

The proofs rely heavily on the structure of the doubly coprime factorizations introduced in Theorem 2.1, since the derivation of these new factorizations relies on state estimate feedback theory.

A2.1 Some basic definitions

Some function spaces

In this appendix, a few basic concepts will be introduced. The Hardy space H_∞ consists of all complex valued functions $G(s)$ of a complex variable s that are analytic and bounded in the open right half-plane, $\mathbb{R}[s] > 0$. The H_∞ norm of $G(s)$, denoted $||G(s)||_\infty$, is defined by

$$||G(s)||_\infty = \sup_{\mathbb{R}[s]>0} G(s) \tag{2.52}$$

The class RH_∞ is a subset of H_∞, consisting of all rational functions with real coefficients that are bounded in $\mathbb{R}[s] > 0$. Alternatively, for a real-rational $F(s)$, then $F(s) \in RH_\infty$ if and only F is *proper* ($|F(\infty)|$ is finite) and *stable* ($F(s)$ has no poles in the closed right half-plane, $\mathrm{Re}\, s \geq 0$) . The class of proper functions will be denoted R_p; the class of strictly proper functions, denoted R_{sp}, consists of real-rational functions $F(s)$ for which $|F(\infty)| = 0$.

Although spaces such as RH_∞ have been defined as scalar, we will generalize the definition, so that, for example, RH_∞ represents the class of all matrix valued functions, with each entry a real-rational function that is stable and proper.

Working with state-space realizations

Consider a stabilizable and detectable *time-invariant* linear system G with state equations,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.53}$$

A special notation for the transfer function of such a system will be used,

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{\mathrm{T}} \tag{2.54}$$

The horizontal and vertical lines are not partitioning of the block matrices, but indicate that the *matrices* represent transfer functions. Using this convenient notation, some identities will be given: first for the cascade of two systems, and then for the inverse of a system.

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{\mathrm{T}} \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{\mathrm{T}} = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]_{\mathrm{T}} \tag{2.55}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{\mathrm{T}}^{\dagger} = \left[\begin{array}{c|c} A - BD^{\dagger}C & -BD^{\dagger} \\ \hline D^{\dagger}C & D^{\dagger} \end{array}\right]_{\mathrm{T}} \tag{2.56}$$

with † representing a left or right inverse. The notation can also be generalized so that,

$$C(s)[sI - A(s)]^{-1}B(s) + D(s) \triangleq \left[\begin{array}{c|c} A(s) & B(s) \\ \hline C(s) & D(s) \end{array}\right]_{\mathrm{T}} \tag{2.57}$$

Two useful matrix identities follow,

$$X(I + YX)^{-1} = (I + XY)^{-1}X \tag{2.58}$$

$$I + X(I - YX)^{-1}Y = (I - XY)^{-1} \tag{2.59}$$

A2.2 Proofs

Derivation of (2.9)–(2.14)

The factorizations (2.9)–(2.14) can be verified. For example

$$G_F M_F = \left[\begin{array}{cc|c} A & BF & B\tilde{V}_F^{-1} \\ 0 & A+BF & B\tilde{V}_F^{-1} \\ \hline I & 0 & 0 \end{array}\right]_T = \left[\begin{array}{cc|c} A+BF & BF & B\tilde{V}_F^{-1} \\ 0 & A & 0 \\ \hline I & 0 & 0 \end{array}\right]_T = N_F \tag{2.60}$$

Here the second equality follows from a change of basis (second column added to first column and the first row subtracted from the second row). The third equality follows by the deletion of uncontrollable parts. Similarly, (2.39) follows from definition (2.38) and

$$GM_0 = \left[\begin{array}{cc|c} A & BF & 0 \\ 0 & A+BF & -M_H \\ \hline C & DF & 0 \end{array}\right]_T = \left[\begin{array}{cc|c} A & 0 & M_H \\ 0 & A+BF & -M_H \\ \hline C & C+DF & 0 \end{array}\right]_T \tag{2.61}$$

The properties (2.15)–(2.17) and (2.40) can be proved using similar manipulations based on (2.55).

Derivation of closed-loop transfer function

Assume that K has the form (2.25), so that $K(Q) = U(Q)V(Q)^{-1}$ where $U(Q) = U + MQ, V(Q) = V + NQ$. The closed-loop transfer functions from u to e in Fig. 2.4 are

$$e = \begin{bmatrix} I + K(Q)[I - GK(Q)]^{-1}K(Q) & K(Q)[I - GK(Q)]^{-1} \\ [I - GK(Q)]^{-1}G & [I - GK(Q)]^{-1} \end{bmatrix} u$$

$$= \begin{bmatrix} I & -K(Q) \\ -G & I \end{bmatrix}^{-1} u \tag{2.62}$$

Moreover,

$$\begin{aligned}
\begin{bmatrix} I & -K(Q) \\ -G & I \end{bmatrix}^{-1} &= \begin{bmatrix} M & 0 \\ 0 & V(Q) \end{bmatrix} \begin{bmatrix} M & -U(Q) \\ -N & V(Q) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} M & 0 \\ 0 & V(Q) \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \\
&= \begin{bmatrix} M & MQ \\ 0 & V(Q) \end{bmatrix} \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \\
&= \begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} + \begin{bmatrix} 0 & MQ \\ 0 & NQ \end{bmatrix} \begin{bmatrix} \tilde{V} & \tilde{U} \\ \tilde{N} & \tilde{M} \end{bmatrix} \\
&= \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} + \begin{bmatrix} M \\ N \end{bmatrix} Q \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \qquad (2.63)
\end{aligned}$$

The result (2.26) follows from (2.63). The state variable form of the closed-loop transfer functions in (2.27) can be derived by substitution for G, K, N, M, $\tilde{N}$, $\tilde{M}$ into (2.63) from (2.2), (2.5) and (2.17), and with $Q = C_Q(sI - A_Q)^{-1}B_Q + D_Q$.

Proof of Lemma 2.3

(i) Specializing (2.14) with $H = U_H V_H^{-1}$ replaced by

$$H(Q_H) = \mathcal{U}_H(Q_H)\mathcal{V}_H(Q_H)^{-1}$$

of (2.33), and thus U, V replaced by $\mathcal{U}_H(Q_H)$, $\mathcal{V}_H(Q_H)$ we have

$$\begin{bmatrix} \mathcal{U}_H(Q_H) \\ \mathcal{V}_H(Q_H) \end{bmatrix} = \left[\begin{array}{c|c} A + BF & -U_H - M_H Q_H \\ \hline F & 0 \\ C + DF & V_H + N_H Q_H \end{array}\right]_{\mathrm{T}} = \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} \mathcal{M} \\ \mathcal{N} \end{bmatrix} Q_H \tag{2.64}$$

Likewise

$$\begin{aligned}\left[\tilde{V}_F(Q_F) \quad -\tilde{U}_F(Q_F)\right] &= \left[\begin{array}{c|cc} A+HC & -B-HD & H \\ \hline \tilde{V}_F + Q_F\tilde{M}_F & \tilde{U} + Q_F\tilde{N}_F & 0 \end{array}\right]_T \\ &= \left[\begin{array}{cc} \tilde{V} & -\tilde{U} \end{array}\right] + Q_F \left[\begin{array}{cc} \tilde{\mathcal{N}} & -\tilde{\mathcal{M}} \end{array}\right] \end{aligned} \quad (2.65)$$

The class $K[F, H(Q_H)]$ is stabilizing from previous results since F stabilizes G_F and $H(Q_H)$ stabilizes G_H. The dual result follows similarly.

(ii) This follows by direct verification as for the derivation of (2.9) and (2.12) in the beginning of the appendix. To show that $M^{-1}\mathcal{M} \in RH_\infty$, it is necessary to use (2.13)

$$\begin{aligned} & C[sI - A - HC]^{-1}\tilde{V}_H^{-1} \in RH_\infty \cap R_{sp} \\ \Rightarrow\ & C\left\{(A+HC)[sI - A - HC]^{-1} + I\right\}\tilde{V}_H^{-1} \in RH_\infty \text{ (by differentiation)} \\ & = CA[sI - A - HC]^{-1}\tilde{V}_H^{-1} + CM_H \in RH_\infty \\ \Rightarrow\ & CA[sI - A - HC]^{-1}\tilde{V}_H^{-1} \in RH_\infty \text{ (using } M_H \in RH_\infty) \end{aligned}$$

Repeated differentiation leads to

$$\begin{aligned} & \left[C' \ (CA)' \cdots (CA^{n-1})'\right]'[sI - A - HC]^{-1}\tilde{V}_H^{-1} \in RH_\infty \cap R_{sp} \\ \Rightarrow\ & [sI - A - HC]^{-1}\tilde{V}_H^{-1} \in RH_\infty \cap R_{sp} \text{ (Under (A,C) observable)} \\ \Rightarrow\ & M^{-1}\mathcal{M} = -\tilde{U}_F[sI - A - HC]^{-1}\tilde{V}_H^{-1} \in RH_\infty \end{aligned}$$

□

Proof of Lemma 2.4

(i) The second equivalence of (2.42) follows from applying the identities $GM = N$ and $G\mathcal{M} = \mathcal{N}$. For the first equivalence, compare (2.36) and (2.25), note the property (2.31), and exploit the connections between Q and $K(Q)$ as in (2.28) and the corresponding results for Q_H and $K[F, H(Q_H)]$.

(ii) Observe that there exists $Q_H \in RH_\infty$ such that

$$
\begin{aligned}
& Q = M^{-1}\mathcal{M}Q_H \in RH_\infty \cap R_{sp} \\
\Leftrightarrow\quad & \tilde{U}_F[sI - A - HC]^{-1}\tilde{V}_H^{-1}(s+\alpha) \text{ has a right inverse in } RH_\infty \ (\alpha > 0) \\
\Leftrightarrow\quad & \tilde{U}_F \text{ has a right inverse in } RH_\infty \\
\Leftrightarrow\quad & F = \tilde{V}_F^{-1}\tilde{U}_F \text{ has a right inverse in } RH_\infty
\end{aligned}
$$

That is, (2.43) holds.

□

Proof of Theorem 2.6

From (2.14), (2.55), and (2.56)

$$
V^{-1} = \left[\begin{array}{c|c} A+BF & -U_H \\ \hline C+DF & V_H \end{array}\right]_T^{-1} = \left[\begin{array}{c|c} A+BF+HC+HDF & H \\ \hline V_H^{-1}(C+DF) & V_H^{-1} \end{array}\right]_T \tag{2.66}
$$

and

$$
V^{-1}N = \left[\begin{array}{cc|c} A+BF+HC+HDF & -HC-HDF & HD\tilde{V}_F^{-1} \\ 0 & A+BF & -B\tilde{V}_F^{-1} \\ \hline V_H^{-1}(C+DF) & -V_H^{-1}(C_DF) & V_H^{-1}D\tilde{V}_F^{-1} \end{array}\right]_T
$$

$$= V_H^{-1} \left[\begin{array}{c|c} A + BF + HC + HDF & B + HD \\ \hline C + DF & D \end{array} \right]_{\mathrm{T}} \tilde{V}_F^{-1} \tag{2.67}$$

where the last equality follows from a change of basis and the deletion of uncontrolled modes. Likewise for the other terms in J of (2.26), leading to

$$J = \left[\begin{array}{c|cc} A + BF + H(C + DF) & -H & -(B + HD)\tilde{V}_F^{-1} \\ \hline F & 0 & \tilde{V}_F^{-1} \\ -V_H^{-1}(C + DF) & V_H^{-1} & -V_H^{-1}D\tilde{V}_F^{-1} \end{array} \right]_{\mathrm{T}} \tag{2.68}$$

Applying Theorem 2.2, the class of all stabilizing controllers is of the form of Fig. 2.4, with J as in (2.68). It is straightforward to see that this is of the form $K[F, H, \bar{Q}]$ as defined in Sec. 2.3. The Theorem 2.6 result follows.

Derivation of results in Sec. 2.4

The state equations for Q given in (2.47) can be derived by substitution for M, N, U, V from (2.14) into the expression

$$Q = -(M - \bar{K}N)^{-1}(U - KV)$$

derived from (2.28).

The expression for the frequency shaped state estimate feedback, given a desired controller $\bar{K} = \bar{C}(sI - \bar{A})^{-1}\bar{B}$ and constant H with a left inverse H^{-L} was derived as follows. Manipulations with (2.5) lead to

$$\bar{K} = F(s)(sI - A - HC)^{-1}(B\bar{K} - H)$$

or

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array} \right]_{\mathrm{T}} = F(s)(sI - A - HC)^{-1} \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline B\bar{C} & -H \end{array} \right]_{\mathrm{T}} \tag{2.69}$$

It can be verified by direct substitution that one solution for $F(s)$ satisfying (2.69) is

$$F(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right]_{\mathrm{T}} \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline B\bar{C} & -H \end{array}\right]_{\mathrm{T}}^{-L} (sI - A - HC) \qquad (2.70)$$

Then (2.51) follows with the use of (2.55)–(2.56) and the identity

$$sC(sI - A)^{-1}B = C(sI - A)^{-1}AB + CB$$

Chapter 3

On the existence of solutions of nonsymmetric Riccati equations

3.1 Introduction

Consider the matrix Riccati equations

$$AT - T\bar{A} - T\bar{B}CT + B\bar{C} = 0 \quad (3.1)$$

$$\bar{A}Z - ZA - ZB\bar{C}Z + \bar{B}C = 0 \quad (3.2)$$

with $T, Z \in \mathbb{C}_{n,n}$ and $A, \bar{A} \in \mathbb{R}_{n,n}$, $B \in \mathbb{R}_{n,p}$, $\bar{B} \in \mathbb{R}_{n,q}$, $C \in \mathbb{R}_{q,n}$ and $\bar{C} \in \mathbb{R}_{p,n}$.

Although Riccati equations are well studied in the literature there is as yet no theory giving convenient necessary and sufficient conditions for the existence of solutions. Potter [41] characterizes all solutions for the symmetric case when $A = \bar{A}^*$. Other authors [21, 29] give further results for the symmetric case in relation to the optimal control problem. Clements and

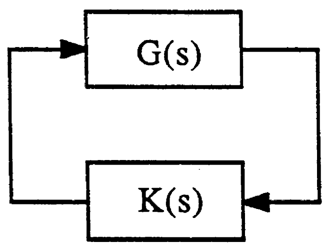

Figure 3.1: Plant/controller feedback pair

Anderson [9] tackle the problem of factoring a polynomial using the Riccati equation, but with $B, \bar{B}, C', \bar{C}'$ rank one vectors and with $A, \bar{A}$ not necessarily of the same dimension. They give sufficient controllability conditions for solutions to exist.

The results of this chapter were motivated by a problem unrelated to optimal control or spectral factorization. Consider the plant/controller pair of Fig. 3.1 with the following minimal transfer functions,

$$G(s) = C(sI - A)^{-1}B \tag{3.3}$$

$$K(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} \tag{3.4}$$

where the coefficient matrices are defined as in (3.1), (3.2).

The problem is to find a state estimator gain H and a state estimate feedback gain F associated with the plant $G(s)$, such that the resulting state estimate feedback controller $-F(sI-A-BF-HC)^{-1}H$ has the same transfer function as $K(s)$. It is known [1] that the existence of F, H is equivalent to the existence of a nonsingular solution of (3.1).

Section 3.2 of the chapter combines the results of Clements and Anderson [9] and a generalization of Mårtensson [29] to give necessary and sufficient conditions for solutions of the Riccati equation to exist for the scalar case, i.e., when $B, \bar{B}, C', \bar{C}'$ are vectors. Further insights can be obtained by considering the state estimate feedback problem described above. The multivariable case will be considered in Sec. 3.3, and it will be seen that the scalar results cannot easily be generalized, as was foreshadowed in [9]. A plant/controller pair with a specific structure is given to provide a counterexample to the conjecture that the sufficiency conditions can be extended, without modification, to the multivariable case.

3.2 The scalar case

The first lemma states clearly the relationship between the Riccati equation and the problem of obtaining an arbitrary controller as a state estimate feedback controller.

Lemma 3.1 *Given minimal $G(s), K(s)$ as in (3.3), (3.4) then the following are equivalent*

(a) There exist real constant F, H such that the state estimate feedback controller $-F(sI - A - BF - HC)^{-1}H$ has the same transfer function as $K(s)$.

(b) There exists a nonsingular real solution T of (3.1).

Proof [1] Starting with F, H satisfying (a), then there must exist a similarity transformation T such that

$$T\bar{A}T^{-1} = A + BF + HC, T\bar{B} = -H, \bar{C}T^{-1} = F \tag{3.5}$$

Substitution with (3.5) leads directly to (3.1). Conversely, if there exists a nonsingular solution T to (3.1), then by defining F, H as

$$H \triangleq -T\bar{B}, F \triangleq \bar{C}T^{-1} \tag{3.6}$$

then (a) will be satisfied.

□

Note: A nonsingular solution T of (3.1) corresponds to a solution $Z = T^{-1}$ of (3.2). This can be seen by premultiplying and postmultiplying (3.1) by T^{-1}.

The following theorem [31], a generalization of the results of Potter, will now be applied to our problem,

Theorem 3.2 *Let* $M = \begin{bmatrix} A & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix} \in \mathbb{R}_{2n,2n}$ *and* $T \in \mathbb{C}_{n,n}$. *The following are equivalent:*

(a) T *is a solution of (3.1).*

(b) there exist $k, a_1, \ldots, a_k \in \mathcal{N}$, *eigenvalues* $\lambda_1, \ldots, \lambda_k$ *of* M *and matrices* $L_\chi \in \mathbb{C}_{2n,a_\chi}$ *for* $\chi = 1, \ldots, k$ *such that*

(i) $\chi \leq \chi'$ *and* $\lambda_\chi = \lambda_{\chi'}$ *imply* $a_\chi \geq a_{\chi'}$ *for* $\chi, \chi' = 1, \ldots, k$,

(ii) $ML_\chi = L_\chi J_{a_\chi}(\lambda_\chi)$ *for* $\chi = 1, \ldots, k$,

(iii) $[L_1 \cdots L_k] = \begin{bmatrix} P \\ Q \end{bmatrix}$ *with* $P, Q \in \mathbb{C}_{n,n}$, Q *nonsingular and* $T = PQ^{-1}$

Notes:

1. The matrix $J_n(\lambda)$ is an $n \times n$ Jordan form with λ on the diagonal, superdiagonal elements equal to one, and zero elsewhere.
2. The columns of the matrices L_χ in the above theorem form a generalized eigenvector associated with the eigenvalue λ_χ of M.
3. Real solutions T of (3.1) can be obtained by choosing the L_χ in complex conjugate pairs.
4. The matrix M is the state transition matrix associated with the closed loop transfer function of the system in Fig. 3.1.

The following theorem is a combination of results from [29] and [9],

Theorem 3.3 *Consider the algebraic Riccati equation (3.1) with* B, $\bar{B}$, C', $\bar{C}'$ *vectors and all solutions* T *obtained by selecting eigenvectors as in Theorem 3.2. A necessary and sufficient condition for all possible eigenvector selections* $(L_1 \cdots L_k)$ *to give a nonsingular* Q *is that* (A, C) *and* $(\bar{A}', \bar{B}')$ *be observable.*

Proof For necessity, the proof is a slight generalization of the results of [29]. Suppose initially that all possible Q are nonsingular. If (A, C) is not observable, then by the Popov-Belevitch-Hautus eigenvector test for observability there exists an eigenvector μ of A with corresponding eigenvalue λ such that

$$A\mu = \lambda\mu, \quad C \cdot \mu = 0 \tag{3.7}$$

$$\Rightarrow \exists \begin{bmatrix} \mu \\ 0 \end{bmatrix} \text{s.t.} \begin{bmatrix} A & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix} \begin{bmatrix} \mu \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \mu \\ 0 \end{bmatrix} \tag{3.8}$$

Since $\begin{bmatrix} \mu \\ 0 \end{bmatrix}$ is an eigenvector of M, including $\begin{bmatrix} \mu \\ 0 \end{bmatrix}$ in $[L_1 \cdots L_k]$ will result in $\begin{bmatrix} P \\ Q \end{bmatrix}$ with a singular Q. This contradicts the initial assumption that all Q are nonsingular, and establishes that the observability of (A, C) is a necessary condition. To establish the necessity of observability of $(\bar{A}', \bar{B}')$, the same analysis can be duplicated, but using a version of (3.1) that is transposed and has T replaced by $-T$. (Alternatively, the proof can use the Popov-Belevitch-Hautus test for controllability of $(\bar{A}, \bar{B})$, and proceed in a dual fashion.) The details are omitted.

The proof of sufficiency of observability of (A, C) and $(\bar{A}', \bar{B}')$ for all Q to be non-singular is given in [9], and is not reproduced here.

□

There is a natural dual to Theorem 3.3. Writing L_χ as $\begin{bmatrix} \ell_{1,\chi} \\ \ell_{2,\chi} \end{bmatrix}$, then from Theorem 3.2,

$$\begin{aligned} \begin{bmatrix} A & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix} \begin{bmatrix} \ell_{1,\chi} \\ \ell_{2,\chi} \end{bmatrix} &= \begin{bmatrix} \ell_{1,\chi} \\ \ell_{2,\chi} \end{bmatrix} J_{a_\chi} \\ \Rightarrow \begin{bmatrix} \bar{A} & \bar{B}C \\ B\bar{C} & A \end{bmatrix} \begin{bmatrix} \ell_{2,\chi} \\ \ell_{1,\chi} \end{bmatrix} &= \begin{bmatrix} \ell_{2,\chi} \\ \ell_{1,\chi} \end{bmatrix} J_{a_\chi} \end{aligned} \tag{3.9}$$

This transposition of the rows and columns of M shows immediately that while solutions of (3.1) are of the form PQ^{-1}, solutions of the dual Riccati equation are of the form QP^{-1}. The dual of Theorem 3.3 is then,

Theorem 3.4 *Consider the algebraic Riccati equation (3.1) with $B, \bar{B}, C', \bar{C}'$ vectors and all solutions T obtained by selecting eigenvectors as in Theorem 3.2. A necessary and sufficient condition for all possible eigenvector selections $[L_1 \cdots L_k]$ to give P nonsingular is that (A, B) and $(\bar{A}', \bar{C}')$ be controllable.*

It is important to note that even if the necessary and sufficient condition of Theorem 3.3 is not satisfied, some eigenvector selections may still lead to a non-singular Q. Consequently, some solutions of the Riccati equation will usually still exist.

Consider again the state estimation problem of Lemma 3.1 in the light of the previous theorems. Lemma 3.1 starts with the assumption that both $G(s)$ and $K(s)$ are minimal and of the same degree. This is equivalent to saying that $(A, B), (\bar{A}, \bar{B})$ are controllable and $(A, C), (\bar{A}, \bar{C})$ are observable. The following result is then immediate.

Theorem 3.5

Consider the algebraic Riccati equation (3.1) where (A, B, C), $(\bar{A}, \bar{B}, \bar{C})$ are scalar, minimal state-space realizations of the same degree. Then all P, Q resulting from eigenvector selections as in Theorem 3.2 will lead to a nonsingular T.

This result can also be proven as follows. Assume that $G(s) = (A, B, C)$ and $K(s) = (\bar{A}, \bar{B}, \bar{C})$ are minimal and of the same degree. The closed-loop poles of the system $G(s)$, $K(s)$ are the eigenvalues of M. The closed-loop poles of the state estimate feedback system are a combination of the observer poles $\lambda(A + HC)$ and the state feedback poles $\lambda(A + BF)$. By controllability of (A, B) and (A', C') it is possible to choose H, F to place $\lambda(A + HC)$, $\lambda(A + BF)$ at the eigenvalues of M. In Appendix A3.1 it is shown that given $G(s)$, there is a one-to-one mapping between the closed-loop poles and the controller transfer function for scalar systems. The choice of H, F as described above will lead to a state estimate controller with the desired closed-loop poles, and thus a transfer function equal to that of $K(s)$. By Lemma 3.1 there exists a corresponding nonsingular solution T of the Riccati equation.

3.3 The multivariable case

A multivariable version of the necessary condition of Theorem 3.3 is,

Theorem 3.6 *Consider the algebraic Riccati equation (3.1) with all solutions T obtained by selecting eigenvectors as in Theorem 3.2. A necessary condition for all possible eigenvector selections $[L_1 \cdots L_k]$ to give nonsingular Q is that (A, C) and $(\bar{A}', \bar{B}')$ be observable.*

Notes:

1. A dual version of Theorem 3.6 also exists.

2. The proof of necessity follows that of Theorem 3.3.

A counterexample to the proposition that observability of (A, C) and $(\bar{A}', \bar{B}')$ is a sufficient condition in the multivariable case will now be presented. Consider $G(s), K(s)$ with state-space realizations,

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.10}$$

and

$$K(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} = \begin{bmatrix} \frac{1}{s+1} + \frac{1}{s+2} & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.11}$$

M has eigenvectors

$$\begin{bmatrix} -0.34 \\ 0.00 \\ 0.25 \\ 0.91 \end{bmatrix} \begin{bmatrix} -0.25 \\ 0.00 \\ 0.91 \\ -0.34 \end{bmatrix} \begin{bmatrix} -0.91 \\ 0.00 \\ -0.34 \\ -0.25 \end{bmatrix} \begin{bmatrix} 0.00 \\ -1.00 \\ 0.00 \\ 0.00 \end{bmatrix}$$

Inspection shows that selecting two eigenvectors of M to form $\begin{bmatrix} P \\ Q \end{bmatrix}$ can result in singular P, Q even though (A, B), $(\bar{A}, \bar{B})$ are controllable and (A, C), $(\bar{A}, \bar{C})$ are observable. Further examination shows that although the state space realizations of $G(s)$, $K(s)$ are minimal, one of the modes of $G(s)$ is not observable from the output of $K(s)G(s)$. This can not occur in the case of scalar $G(s)$, $K(s)$. Examination of the zero entries of the eigenvectors of M

gives a clue as to how the proof techniques of Theorem 3.3 can be used to strengthen Theorem 3.6,

Theorem 3.7 *Consider the algebraic Riccati equation with all solutions T obtained by selecting eigenvectors as in Theorem 3.2. Necessary conditions for all possible eigenvector selections $[L_1 \cdots L_k]$ to give nonsingular Q are that $(A, \bar{B}C)$ be observable and $(\bar{A}, \bar{B}C)$ be controllable.*

Proof When $(A, \bar{B}C)$ is not observable, then there exists an eigenvector μ of A with corresponding eigenvalue λ such that

$$A\mu = \lambda\mu, \quad \bar{B}C \cdot \mu = 0$$

$$\Rightarrow \exists \begin{bmatrix} \mu \\ 0 \end{bmatrix} \text{s.t.} \begin{bmatrix} A & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix} \begin{bmatrix} \mu \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \mu \\ 0 \end{bmatrix} \tag{3.12}$$

Since $\begin{bmatrix} \mu \\ 0 \end{bmatrix}$ is an eigenvector of M, including $\begin{bmatrix} \mu \\ 0 \end{bmatrix}$ in $[L_1 \cdots L_k]$ will result in $\begin{bmatrix} P \\ Q \end{bmatrix}$ with a singular Q. This contradicts the initial assumption that all Q are nonsingular, and establishes that the observability of $(A, \bar{B}C)$ is a necessary condition. The method of establishing the necessity of controllability of $(\bar{A}, \bar{B}C)$ follows in dual fashion. The details are omitted.

□

Notes:

1. The above theorem is an independently derived specialization of a result by Medanic [30], which gives necessary and sufficient conditions for the existence of a Riccati equation solution associated with a *particular* admissible eigenvector selection. Unfortunately the sufficient

conditions given by Medanic can not readily be interpreted as corresponding conditions on the plant/controller pair of Theorem 3.1. Also the sufficient conditions of Medanic can only test one admissible solution at a time, whereas here we are concerned with the existence of any solution.

2. The conditions of Theorem 3.7 imply the observability conditions of Theorem 3.6, because $(A, \bar{B}C)$ observable implies that (A, C) is observable and $(\bar{A}, \bar{B}C)$ controllable implies that $(\bar{A}, \bar{B})$ is controllable. Moreover, this theorem shows that the conditions of Theorem 2.2 do not extend, at least without modification, to the multivariable case.

3. One of the necessary conditions in the above theorem can be interpreted as a condition on observability of the plant states from the controller states. The other condition relates to controllability of the controller states from the plant states.

4. Many combinations of $G(s)$, $K(s)$, with the corresponding solutions of the resulting algebraic Riccati equation, have been studied. So far, all $G(s)$, $K(s)$ satisfying the conditions of Theorem 3.7 have had all P, Q, obtained by eigenvector selection as in Theorem 3.2, nonsingular. It may be that the conditions of Theorem 3.7 are both necessary and sufficient for all possible P, Q to be nonsingular, but a proof of this is elusive—certainly any attempts to generalize the scalar variable

versions of the proofs are fraught with difficulties.

5. The dual result to the above theorem is as follows,

Theorem 3.8 *Consider the algebraic Riccati equation with all solutions T obtained by selecting eigenvectors as in Theorem 3.2. Necessary conditions for all possible eigenvector solutions $[L_1 \cdots L_k]$ to give nonsingular P are that $(A, B\bar{C})$ be controllable and $(\bar{A}, B\bar{C})$ be observable.*

3.4 Conclusions

It is not difficult to use a computer to calculate all solutions of the algebraic Riccati equation using the methods of [31]. It would be desirable to have a simple test giving a priori knowledge of whether or not all solutions exist, or how many exist. Using the connections between the Riccati equation and the problem of finding a state estimate feedback controller gives an intuitive framework in which to study this problem. Theorem 3.7 and its dual extend existing scalar results, and it may be that the necessary conditions are also sufficient.

The approach of this chapter for analysis of the Riccati equation has been similar to that of Potter [41]; the solution of the quadratic matrix equation has been rewritten in terms of eigenvectors of a matrix such as the M of Theorem 3.2. It is surprising that a complete solution to the problem could not be obtained; many different avenues were explored, such as rewriting the nonsymmetric Riccati equation as part of a symmetric Riccati equation with dimensions twice as large.

A3.1 Alternative proof of Theorem 3.5

In the following lemma, the relationship between the closed loop poles and $K(s)$ for the system of Fig. 3.1 is examined, for the case where $G(s), K(s)$ are scalar transfer functions. Consider the representations,

$$G(s) = X(s)Y(s)^{-1}, \quad K(s) = U(s)V(s)^{-1} \tag{3.13}$$

where

$$\begin{aligned} X(s) &= x_1 s^{n-1} + x_2 s^{n-2} + \cdots + x_n \\ Y(s) &= s^n + y_1 s^{n-1} + y_2 s^{n-2} + \cdots + y_n \\ U(s) &= u_1 s^{n-1} + u_2 s^{n-2} + \cdots + u_n, \\ V(s) &= s^n + v_1 s^{n-1} + v_2 s^{n-2} + \cdots + v_n \end{aligned}$$

The closed loop poles are the zeros of

$$\begin{aligned} H(s) &= Y(s)V(s) - X(s)U(s) \\ &= s^{2n} + h_1 s^{2n-1} + h_2 s^{2n-2} + \cdots + h_{2n} \end{aligned} \tag{3.14}$$

Lemma 3.9 *Given a scalar $G(s)$ of degree n and a minimal realization $X(s)Y(s)^{-1}$, there is a one-to-one mapping between the $2n$ closed loop poles and the controller transfer function $K(s)$ of degree n.*

Proof Substituting for $X(s)$, $Y(s)$, $U(s)$, $V(s)$ in (3.14) and collecting coefficients of each power of s one obtains $(2n+1)$ equations,

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ y_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ y_2 & y_1 & 1 & & & x_1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \ddots & 0 & 0 \\ y_n & y_{n-1} & & & 1 & x_{n-1} & & x_1 & 0 \\ 0 & y_n & y_{n-1} & \cdots & y_1 & x_n & & & x_1 \\ & & \ddots & & & & \ddots & & \\ & & & & y_n & & & & x_n \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ \vdots \\ v_n \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ h_1 \\ h_2 \\ \vdots \\ h_n \\ h_{n+1} \\ \vdots \\ h_{2n} \end{bmatrix} \qquad (3.15)$$

The $(2n+1)$ square Sylvester matrix in (3.15) is nonsingular if and only if $X(s), Y(s)$ are coprime [6]. This is true because $\{X(s), Y(s)\}$ is a minimal representation of $G(s)$. Since the Sylvester matrix is nonsingular, the mapping between the controller $U(s)V(s)^{-1}$ and the closed loop poles (ie. the zeros of $H(s)$) is one-to-one.

□

Chapter 4

Doubly coprime factorizations, reduced order observers, and dynamic state estimate feedback

4.1 Introduction

A doubly coprime factorization of the transfer function of a lumped linear time-invariant system is the starting point for many of the powerful results in the factorization approach to multivariable control system analysis and synthesis [46]. In an important paper by Nett, Jacobson, and Balas [39], explicit formulae are given for state-space realizations of the Bezout identity elements. The results of Nett et. al. are based on ideas from the theory of state feedback and state estimation, and use existing computational algorithms, namely pole placement algorithms.

Recently, Hippe [17] has derived modified factorizations which are related to compensators based on reduced-order observers, rather than full-order state observers. One problem with these factorizations is that some of the Bezout identity elements are non-proper, and consequently are not suitable for use with the factorization approach. In Sec. 4.2 of this chapter we derive doubly coprime factorizations related to minimal-order observers, with all Bezout identity elements stable and proper.

In Moore, Glover, and Telford [35], the factorizations of Nett et. al. [39] are generalized to allow for the possibility of dynamic state estimate feedback gains, as well as dynamic state estimator gains. Section 4.3 of this chapter generalizes the factorizations of Sec. 4.2 in a similar manner. To give an example of the utility of the results, it is then proved that all stabilizing controllers for a given plant can be structured as a minimal-order observer, with dynamic state estimate feedback gains. Finally, some dual results are summarized in Sec. 4.4.

4.2 Factorizations related to minimal-order observers

4.2.1 Preliminaries

As in Chap. 2, plant/controller pair $G(s), K(s)$, depicted in Fig. 2.1 will, be said to be well-posed and internally stable if and only if

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \text{ exists and belongs to } RH_\infty. \tag{4.1}$$

This condition corresponds to the transfer functions from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ being stable and proper.

The minimal-order observer for the m-input, p-output plant $G(s)$, with n state controllable and observable state-space realization $C(sI - A)^{-1}B$, will now be briefly reviewed. The treatment is similar to that found in O'Reilly [40]. The observer equations are,

$$\dot{z} = Rz + Sy + TBu \tag{4.2}$$

$$\hat{x} = [\Psi \quad \Theta] \begin{bmatrix} y \\ z \end{bmatrix} \tag{4.3}$$

where

$$C \text{ is full rank,} \tag{4.4}$$

$$[\Psi \quad \Theta] \begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} [\Psi \quad \Theta] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{4.5}$$

$$R = TA\Theta, \quad S = TA\Psi \tag{4.6}$$

A suitable selection of a full row rank matrix T results in $(sI - R)^{-1} \in RH_\infty$, i.e., R is a matrix with all eigenvalues in the open left half-plane $\mathbb{R}[s] < 0$. For such selections, the error in the state estimate $x - \hat{x}$ due to an incorrect initial value of z will approach zero asymptotically.

Figure 4.1 shows the block diagram for an observer-based controller which uses feedback of the state estimate $\hat{x}$ through a constant, real matrix F. The transfer function matrix $K(s)$ of an equivalent controller in the simple

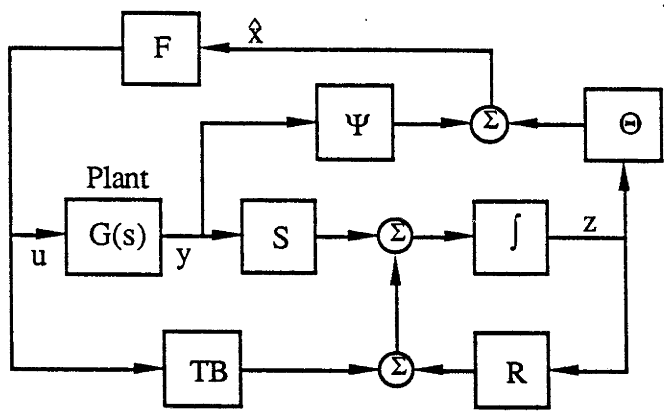

Figure 4.1: Minimal-order observer based control loop

positive feedback configuration of Figure 2.1 is,

$$K(s) = \left[\begin{array}{c|c} R+TBF\Theta & S+TBF\Psi \\ \hline F\Theta & F\Psi \end{array}\right]_T \tag{4.7}$$

4.2.2 Factorizations

The main factorization result will now be stated.

Theorem 4.1 *Consider the plant $G(s) = C(sI - A)^{-1}B$, with (A, B) controllable and (A, C) observable. Choose F, T such that $(sI - A - BF)^{-1}, (sI - R)^{-1} \in RH_\infty$, where R, T are described by the observer equations (4.2)-(4.6). With arbitrary Λ such that $(sI - \Lambda)^{-1} \in RH_\infty$, define*

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & (A+BF-\Psi\Lambda C)\Psi \\ \hline F & I & F\Psi \\ C & 0 & I \end{array}\right]_T \tag{4.8}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A\Theta T + \Psi\Lambda C & -B & -(A - \Psi\Lambda C)\Psi \\ \hline F\Theta T & I & -F\Psi \\ C & 0 & I \end{array}\right]_{\mathrm{T}} \tag{4.9}$$

Then

(i) all transfer function matrices described by (4.8), (4.9) are stable and proper;

(ii) $M, \tilde{M}, V, \tilde{V}$ *have proper inverses;*

(iii) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$;

(iv) $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ *where* K *is the observer-based controller given by (4.7);*

(v)

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.10}$$

Proof Considering (i), the transfer function matrices (4.8),(4.9) are inherently proper. Since F is chosen such that $(sI - A - BF)^{-1} \in RH_\infty$, (4.8) is stable, and furthermore, to see that (4.9) is stable, apply a similarity transformation and use (4.5),

$$A\Theta T + \Psi\Lambda C = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} \Lambda & CA\Theta \\ 0 & R \end{bmatrix} \begin{bmatrix} C \\ T \end{bmatrix} \tag{4.11}$$

Since the similarity transformation leaves the eigenvalues unchanged, the eigenvalues of $A\Theta T + \Psi\Lambda C$ are simply equal to the eigenvalues of Λ, a matrix chosen such that its eigenvalues lie in the left half-plane, together with the eigenvalues of R, which lie in the left half-plane by virtue of the T selection.

It can be deduced from (2.56) that a square proper transfer function matrix has a proper inverse if its direct-feedthrough term D is nonsingular. Considering (ii), it follows that $M, \tilde{M}, V, \tilde{V}$ have proper inverses, because they have unity direct-feedthrough matrices. Application of (2.55), (2.56) shows that (iii), (iv), and (v) hold. As an example of the proof technique, observe that

$$\begin{aligned}
\tilde{M}^{-1}\tilde{N} &= \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda C & -(A-\Psi\Lambda C)\Psi \\ \hline C & I \end{array}\right]_{\mathrm{T}}^{-1} \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda C & B \\ \hline C & 0 \end{array}\right]_{\mathrm{T}} \\
&= \left[\begin{array}{c|c} A & (A-\Psi\Lambda C)\Psi \\ \hline C & I \end{array}\right]_{\mathrm{T}} \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda C & B \\ \hline C & 0 \end{array}\right]_{\mathrm{T}} \quad \text{by (2.56)} \\
&= \left[\begin{array}{cc|c} A & A\Psi C - \Psi\Lambda C & 0 \\ 0 & A\Theta T + \Psi\Lambda C & B \\ \hline C & C & 0 \end{array}\right]_{\mathrm{T}} \quad \text{by (2.55)} \\
&= \left[\begin{array}{cc|c} A\Theta T + \Psi\Lambda C & A\Psi C - \Psi\Lambda C & 0 \\ 0 & A & B \\ \hline 0 & C & 0 \end{array}\right]_{\mathrm{T}} \quad \text{(by change of basis)} \\
&= \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]_{\mathrm{T}} = G \quad \text{(by the removal of unobservable modes)}
\end{aligned}$$

□

Note that these factorizations, like those of Nett et. al. [39], are still n-th order even though they are based on a minimal-order observer design. In this sense they are non-minimal, as are the factorizations of Hippe. We will now attempt to gain more intuition about the results, by comparing their properties with known properties of the full-order factorizations [39, 35].

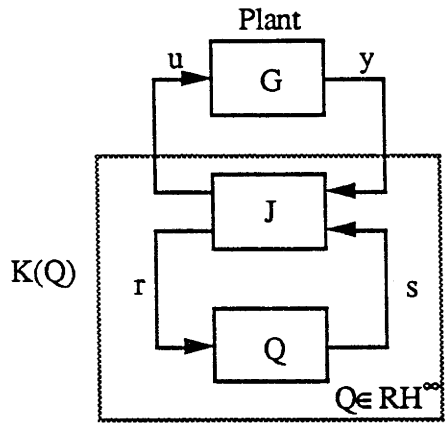

Figure 4.2: Class of all stabilizing controllers for G.

4.2.3 The class of all stabilizing controllers

Once doubly coprime factorizations for the plant $G(s)$ have been found, it is possible to parameterize the class of all proper stabilizing controllers in terms of an arbitrary $Q(s) \in RH_\infty$ [46]. Such a class $\{K(Q)|Q \in RH_\infty\}$ can be written in terms of linear fractional transformations as,

$$\begin{aligned} K(Q) &= (U+MQ)(V+NQ)^{-1} = (\tilde{V}+Q\tilde{N})^{-1}(\tilde{U}+Q\tilde{M}) \\ &= UV^{-1} + \tilde{V}^{-1}Q(I+V^{-1}NQ)^{-1}V^{-1} \end{aligned} \tag{4.12}$$

or diagrammatically as in Fig. 4.2, where, based on the third equality in (4.12),

$$J = \begin{bmatrix} K & \tilde{V}^{-1} \\ V^{-1} & -V^{-1}N \end{bmatrix} \tag{4.13}$$

With the factorizations (4.8), (4.9),

$$J = \left[\begin{array}{cc|cc} \Lambda & C(A+BF)\Theta & C(A+BF-\Psi\Lambda C)\Psi & CB \\ 0 & T(A+BF)\Theta & T(A+BF)\Psi & TB \\ \hline 0 & F\Theta & F\Psi & I \\ -I & 0 & I & 0 \end{array}\right]_T \tag{4.14}$$

The scheme of Fig. 4.2 with J given by (4.14) has an interesting interpretation. To lead us into this, recall that if J is formed according to (4.13), and the doubly coprime factorizations of Nett et. al. [39] are used, then the scheme of Fig. 4.2 can be interpreted as in Fig. 4.3. That is, the class of all stabilizing controllers $\{K(Q)|Q \in RH_\infty\}$ for $G(s)$ can be generated by the use of an observer-based controller, with an additional internal feedback loop involving stable dynamics $Q(s)$ [12]. The residuals $r = (y - \hat{y})$ are filtered by Q to form s, which is added to $F\hat{x}$ to give the control signal u.

A reasonable question to ask is whether, analogously to the full state estimator based scheme of Fig. 4.3, the class of all stabilizing controllers can be obtained with a *minimal-order* observer-based compensator with added stable dynamics. There can not be a direct analogue since the residuals $(y - \hat{y})$, obtained by defining $\hat{y} = C\hat{x}$, are equal to zero, as follows

$$\begin{aligned} r &= y - \hat{y} = y - C\hat{x} = y - C\,[\Psi \quad \Theta]\begin{bmatrix} y \\ z \end{bmatrix} \\ &= y - [I \quad 0]\begin{bmatrix} y \\ z \end{bmatrix} = 0 \quad \text{by (4.5)} \end{aligned}$$

Consider instead residuals $r \triangleq (y - y_e)$, where the estimate y_e of y is, for the case $\Lambda = 0$, the integration of an estimate of the derivative $\dot{y}$. More generally,

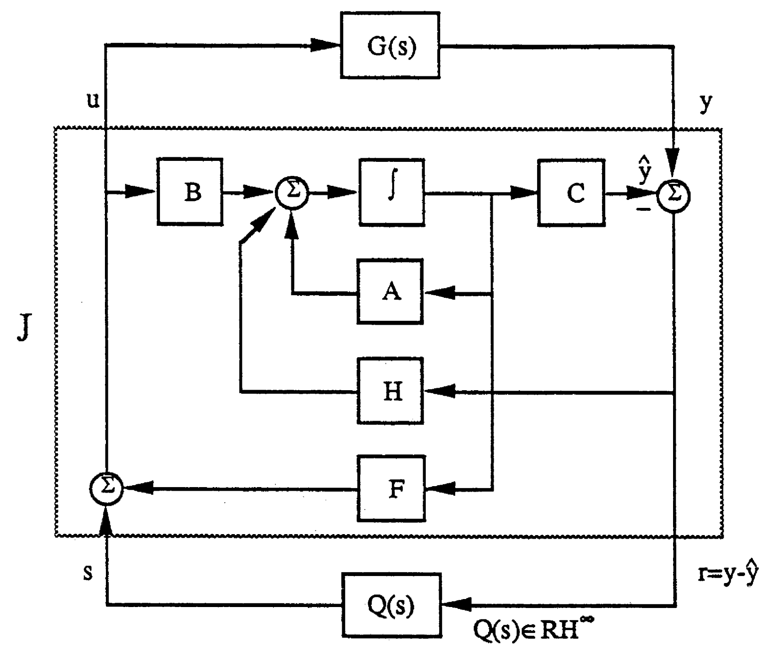

Figure 4.3: Controller class $\{K(Q)|Q \in RH_\infty\}$ based on full-order observer

when Λ is chosen such that its eigenvalues lie in the closed left half-plane, y_e is the solution of

$$\dot{y}_e - \Lambda y_e = C(A\hat{x} + Bu) - \Lambda y = C(A + BF)\hat{x} - \Lambda y \tag{4.15}$$

Here $C(A\hat{x}+Bu)$ is an estimate of $\dot{y}$. From (4.15), we can obtain y_e explicitly by filtering a linear combination of the minimal-order state estimate $\hat{x}$ and the plant output y

$$y_e = (sI - \Lambda)^{-1}[C(A + BF)\hat{x} - \Lambda y] \tag{4.16}$$

With the residuals $r = y - y_e$, and referring to Fig. 4.2, it is reasonable to propose a minimal-order observer-based scheme as in Fig. 4.4. Evaluating the

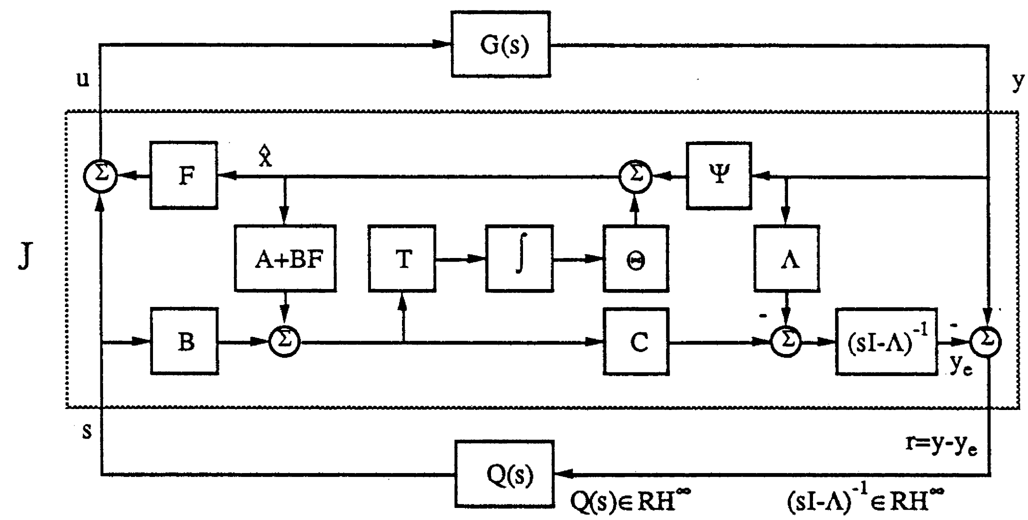

Figure 4.4: Controller class $\{K(Q)|Q \in RH_\infty\}$ based on minimal-order observer

transfer function of the J block defined according to Fig. 4.4 gives precisely the J of (4.14).

In summary we have found a minimal-order observer based compensator, with added stable dynamics, that generates the class of all stabilizing controllers for G as $Q(s)$ varies over RH_∞. Notice that the McMillan degree of J in (4.14) is $(n-p)+p=n$, which is the same as for the J in the full-order scheme of Fig. 4.3.

4.3 Dynamic state estimate feedback

From this point onwards, we will generalize the state estimate feedback gain F to be a proper, rational transfer function matrix, which may possibly be unstable. Assume that a left coprime factorization $F = \tilde{V}_F^{-1}\tilde{U}_F$ has been found: state space realizations for such factorizations are readily available with the use of the doubly coprime factorizations given in Sec. 4.2. It will be necessary to generalize the notation for a state space realization so that, for example,

$$\left[\begin{array}{c|c} A+BF(s) & B\tilde{V}_F^{-1}(s) \\ \hline F(s) & \tilde{V}_F^{-1}(s) \end{array}\right]_{\mathrm{T}} = F(s)[sI-A-BF(s)]^{-1}B\tilde{V}_F^{-1}(s)+\tilde{V}_F^{-1}(s) \tag{4.17}$$

To take account of the dynamic state estimate feedback, new doubly coprime factorizations will be defined.

In Sec. 4.2, a constant F is chosen so that $(sI-A-BF)^{-1} \in RH_\infty$, or equivalently, so that F is a stabilizing controller for the system $(sI-A)^{-1}B$. Generalizing to the case when F is a transfer function matrix, we require $F(s)$ to be a stabilizing controller for the system $G_F \triangleq (sI-A)^{-1}B$.

4.3.1 Factorizations

Theorem 4.2 (Doubly coprime factorizations for $\mathbf{G_F}$) *Given a plant $G_F \triangleq (sI-A)^{-1}B$ with (A,B) controllable, a proper stabilizing controller $F(s)$ with a left coprime factorization $\tilde{V}_F^{-1}\tilde{U}_F$, arbitrary Λ_1 such that $(sI_n - \Lambda_1)^{-1} \in RH_\infty$, and defining,*

$$\begin{bmatrix} M_F & U_F \\ N_F & V_F \end{bmatrix} = \left[\begin{array}{c|cc} A+BF(s) & B\tilde{V}_F^{-1}(s) & A+BF(s)-\Lambda_1 \\ \hline F(s) & \tilde{V}_F^{-1}(s) & F(s) \\ I & 0 & I \end{array}\right]_{\mathrm{T}} \tag{4.18}$$

$$\begin{bmatrix} \tilde{V}_F & -\tilde{U}_F \\ -\tilde{N}_F & \tilde{M}_F \end{bmatrix} = \left[\begin{array}{c|cc} \Lambda_1 & -B & -(A-\Psi\Lambda_1 C) \\ \hline 0 & \tilde{V}_F(s) & -\tilde{U}_F(s) \\ I & 0 & I \end{array}\right]_T \tag{4.19}$$

Then

(i) the transfer matrices defined by (4.18),(4.19) are stable and proper;

(ii) $M, \tilde{M}, V, \tilde{V}$ *have proper inverses;*

(iii) $G_F = N_F M_F^{-1} = \tilde{M}_F^{-1}\tilde{N}_F, \quad F = U_F V_F^{-1};$

(iv)

$$\begin{bmatrix} \tilde{V}_F & -\tilde{U}_F \\ -\tilde{N}_F & \tilde{M}_F \end{bmatrix} \begin{bmatrix} M_F & U_F \\ N_F & V_F \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.20}$$

Proof Statements (ii), (iii) and (iv) can be proved by simple manipulations using (2.55),(2.56). It remains to show that the transfer function matrices (4.18),(4.19) are stable, since together with (iv) this implies that the factorizations are coprime in RH_∞. First note that $\tilde{M}_F, \tilde{N}_F$ are stable, since $(sI - \Lambda_1)^{-1} \in RH_\infty$. Consider then arbitrary stable proper stable factorizations $G_F = \mathcal{N}_F \mathcal{M}_F^{-1}$, $F = \mathcal{U}_F \mathcal{V}_F^{-1}$. Since F stabilizes G_F, then the standard arguments [46] give that

$$(\tilde{V}_F \mathcal{M}_F - \tilde{U}_F \mathcal{N}_F)^{-1},\ (\tilde{M}_F \mathcal{V}_F - \tilde{N}_F \mathcal{U}_F)^{-1} \in RH_\infty$$

Also from (iv) $(\tilde{V}_F - \tilde{U}_F G_F) M_F = I$, $(\tilde{M}_F - \tilde{N}_F F) V_F = I$ so that

$$\begin{bmatrix} M_F \\ N_F \end{bmatrix} = \begin{bmatrix} I \\ G_F \end{bmatrix} (\tilde{V}_F - \tilde{U}_F G_F)^{-1} = \begin{bmatrix} \mathcal{M}_F \\ \mathcal{N}_F \end{bmatrix} (\tilde{V}_F \mathcal{M}_F - \tilde{U}_F \mathcal{N}_F)^{-1} \in RH_\infty \tag{4.21}$$

$$\begin{bmatrix} V_F \\ U_F \end{bmatrix} = \begin{bmatrix} I \\ F \end{bmatrix} (\tilde{M}_F - \tilde{N}_F F)^{-1} = \begin{bmatrix} \mathcal{V}_F \\ \mathcal{U}_F \end{bmatrix} (\tilde{M}_F \mathcal{V}_F - \tilde{N}_F \mathcal{U}_F)^{-1} \in RH_\infty \quad (4.22)$$

□

A generalization of Theorem 4.1 follows,

Theorem 4.3 (Doubly coprime factorizations for G)

Consider a plant $G = C(sI - A)^{-1}B$ with (A, B) controllable and (A, C) observable. Choose T such that $(sI - R)^{-1} \in RH_\infty$, and proper $F(s)$, with arbitrary right coprime factorization $\tilde{V}_F^{-1}\tilde{U}_F$, such that $F(s)$ stabilizes $G_F = (sI - A)^{-1}B$. (The matrices R, T are defined by the observer equations (4.2)–(4.6)). With arbitrary Λ_2 such that $(sI_p - \Lambda_2)^{-1} \in RH_\infty$ define

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A + BF(s) & B\tilde{V}_F^{-1}(s) & (A + BF(s) - \Psi\Lambda_2 C)\Psi \\ \hline F(s) & \tilde{V}_F^{-1}(s) & F(s)\Psi \\ C & 0 & I \end{array}\right]_T \quad (4.23)$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A\Theta T + \Psi\Lambda_2 C & -B & -(A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{U}_F(s)\Theta T & \tilde{V}_F(s) & -\tilde{U}_F(s)\Psi \\ C & 0 & I \end{array}\right]_T \quad (4.24)$$

Then

(i) the transfer functions defined by (4.23), (4.24) are stable and proper;

(ii) $M, \tilde{M}, V, \tilde{V}$ have proper inverses;

(iii) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$;

(iv) $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ where K is the observer based controller given by (4.7);

(v)

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.25}$$

Proof As in previous theorems, (iii)–(v) can be proved by application of (2.55), (2.56). Evaluation of $M^{-1}, \tilde{V}^{-1}$ shows that $M, \tilde{V}$ have proper inverses, because $\tilde{V}_F$ is proper with a proper inverse. Similarly, since $V, \tilde{M}$ have unity direct-feedthrough matrices, $V^{-1}, \tilde{M}^{-1}$ are proper, completing the proof of (ii). It remains only to prove that all of the transfer functions are proper and stable. Consider first $\tilde{V}$,

$$\tilde{V} = \tilde{U}_F \Theta T(sI - A\Theta T - \Psi\Lambda_2 C)^{-1}(-B) + \tilde{V}_F \tag{4.26}$$

Since $(sI - A\Theta T - \Psi\Lambda_2 C)^{-1} \in RH_\infty$, $\tilde{V}$ is formed from the sum and product of stable proper transfer functions. It follows that $\tilde{V}$ is also proper and stable. The same can be seen of $\tilde{U}, \tilde{N}, \tilde{M}$. From the previous theorem, we have stable proper transfer functions M_F, N_F, and

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} M_F \\ N_F \end{bmatrix} \in RH_\infty \tag{4.27}$$

To establish that $\begin{bmatrix} U \\ V \end{bmatrix}$ is stable requires some intermediate results. Since $F(s)$ stabilizes $G_F(s)$,

$$\begin{aligned} \begin{bmatrix} I & -F(s) \\ -G_F(s) & I \end{bmatrix}^{-1} &= \left[\begin{array}{c|cc} A & -B & 0 \\ \hline 0 & I & -F(s) \\ I & 0 & I \end{array}\right]_T^{-1} \\ &= \left[\begin{array}{c|cc} A + BF(s) & B & BF(s) \\ \hline F(s) & I & F(s) \\ I & 0 & I \end{array}\right]_T \in RH_\infty \end{aligned} \tag{4.28}$$

$$\Rightarrow \quad s\begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}B \in RH_\infty \quad \text{(by differentiation)}$$

$$\Rightarrow \quad \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}\{A + BF(s)\}B + \begin{bmatrix} F(s) \\ I \end{bmatrix} B \in RH_\infty$$

$$\Rightarrow \quad \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}AB + \left[\begin{array}{c|c} A + BF(s) & BF(s) \\ \hline F(s) & F(s) \\ I & I \end{array}\right]_T B \in RH_\infty$$

$$\Rightarrow \quad \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}AB \in RH_\infty \text{(by (4.28))}$$

Repeated differentiation leads to

$$\begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}[B \ AB \ A^2B \cdots A^{n-1}B] \in RH_\infty$$

$$\Rightarrow \quad \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1} \in RH_\infty \quad \text{([A,B] controllable)} \tag{4.29}$$

Finally, with the following decomposition,

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\left[\begin{array}{c|c} A + BF(s) & BF(s) \\ \hline F(s) & F(s) \\ I & 0 \end{array}\right]_T \Psi + \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}(A - \Psi\Lambda_2 C)\Psi \tag{4.30}$$

the first and second terms are proper stable transfer functions by virtue of (4.28),(4.29) respectively. Thus $U, V \in RH_\infty$, and the proof of (i) is complete.

□

Observe that when $F(s)$ equals a constant F, then (4.23), (4.24) are identical to (4.8),(4.9), and Theorem 4.3 specializes to Theorem 4.1.

The previous two theorems lead to the following corollary.

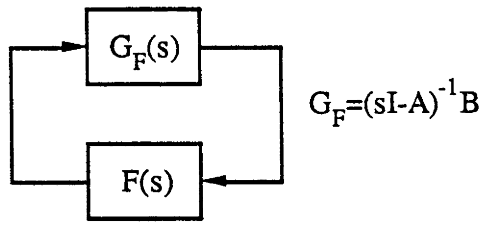

Figure 4.5: Feedback loop containing G_F and F

Corollary 4.4 *Consider a plant* $G = C(sI-A)^{-1}B$ *with* (A, B) *controllable and* (A, C) *observable. Choose* T *such that the corresponding* R *results in a stable observer design. The controller for* G *obtained by dynamic state estimate feedback, via a proper* $F(s)$*, will be stabilizing if* $F(s)$ *is a proper stabilizing controller for* G_F*.*

Proof Start with an arbitrary $F(s)$ which stabilizes G_F (see Fig. 4.5). Choose a left coprime factorization $\tilde{V}_F^{-1}\tilde{U}_F$ for F, and construct the doubly coprime factorizations (4.23), (4.24). A standard result from factorization theory [46] is that $\tilde{V}^{-1}\tilde{U}$ thus obtained will be a stabilizing controller for $\tilde{M}^{-1}\tilde{N}$. Since $G = C(sI-A)^{-1}B = \tilde{M}^{-1}\tilde{N}$, and $K = \tilde{V}^{-1}\tilde{U}$ is the observer based controller given by (4.7) (See Fig. 4.6), the corollary is proved.

□

A natural question to ask is the converse: would the controller of Fig. 4.6 be destabilizing for G if F did not stabilize G_F? The answer to this question is not straight-forward, because the coprime factorizations of Theorem 4.3 rely on F to be stabilizing for G_F. The next section tackles this problem, and demonstrates the utility of the new factorizations at the same time.

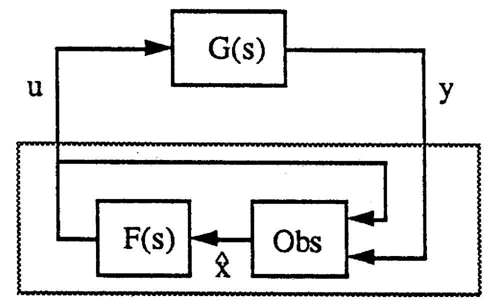

Figure 4.6: Observer based controller with dynamic state estimate feedback.

4.3.2 All stabilizing controllers as minimal-order observer-based controllers

The parameterization of the class of all proper stabilizing controllers for G will now be restated.

$$K(Q) = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}),\ Q \in RH_\infty \tag{4.31}$$

Here $\tilde{M}, \tilde{N}, \tilde{U}, \tilde{V}$ now refer to the factorizations of Theorem 4.3, and can be thought of as functions of $F(s)$. At this point, it is convenient to introduce a new notation—Instead of $K(Q)$ we will write $K[Q, F]$, to note explicitly the dependence of K on the choice of $F(s)$. The controller $K = \tilde{V}^{-1}\tilde{U}$ will be written as $K[0, F]$. Making use of the doubly coprime factorizations of Theorem 4.2, the class of all proper stabilizing controllers for G_F can be written as

$$F(Q_F) = (\tilde{V}_F + Q_F\tilde{N}_F)^{-1}(\tilde{U}_F + Q_F\tilde{M}_F),\ Q_F \in RH_\infty \tag{4.32}$$

What we wish to show is that the class of all proper stabilizing controllers

$\{K[Q,F] | Q \in RH_\infty\}$ is the same as the class of proper observer-based controllers $\{K[0,F(Q_F)] | Q_F \in RH_\infty\}$. The proof of this requires an alternative representation of $K[0,F(Q_F)]$,

Lemma 4.5 *An observer based controller $K[0,F(Q_F)]$ can be restructured as a linear fractional transformation,*

$$K[0,F(Q_F)] = (\tilde{V} + Q_F\tilde{N}_0)^{-1}(\tilde{U} + Q_F\tilde{M}_0) \tag{4.33}$$

where

$$[-\tilde{N}_0 \quad \tilde{M}_0] = \left[\begin{array}{c|cc} A\Theta T - \Psi\Lambda_2 C & B & (A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{M}_F(s)\Theta T & -\tilde{N}_F(s) & \tilde{M}_F(s)\Psi \end{array}\right]_{\mathrm{T}} \tag{4.34}$$

$$[0 \quad \Psi] = [-\tilde{N}_0 \quad \tilde{M}_0]\begin{bmatrix} M & U \\ N & V \end{bmatrix} \tag{4.35}$$

$$\tilde{M}_0\tilde{M}^{-1} = \Psi \tag{4.36}$$

Proof

$$\begin{aligned}
& K[0,F(Q_F)] \\
= \;& \tilde{V}^{-1}\tilde{U}|_{F=F(Q_F)=(\tilde{V}_F+Q_F\tilde{N}_F)^{-1}(\tilde{U}_F+Q_F\tilde{M}_F)} \\
= \;& \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & -B \\ \hline (\tilde{U}_F(s) + Q_F(s)\tilde{M}_F(s))\Theta T & \tilde{V}_F(s) + Q_F(s)\tilde{N}_F(s) \end{array}\right]_{\mathrm{T}}^{-1} \\
& \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & (A + \Psi\Lambda_2 C)\Psi \\ \hline (\tilde{U}_F(s) + Q_F(s)\tilde{M}_F(s))\Theta T & (\tilde{U}_F(s) + Q_F(s)\tilde{M}_F(s))\Psi \end{array}\right]_{\mathrm{T}} \\
= \;& \left\{\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & -B \\ \hline \tilde{U}_F(s)\Theta T & \tilde{V}_F(s) \end{array}\right]_{\mathrm{T}} + Q_F(s)\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & -B \\ \hline \tilde{M}_F(s)\Theta T & \tilde{N}_F(s) \end{array}\right]_{\mathrm{T}}\right\}^{-1}
\end{aligned}$$

$$\left\{ \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & (A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{U}_F(s)\Theta T & \tilde{U}(s)_F\Psi \end{array} \right]_{\mathrm{T}} \right.$$

$$\left. + Q_F(s) \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & (A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{M}_F(s)\Theta T & \tilde{M}_F(s)\Psi \end{array} \right]_{\mathrm{T}} \right\}$$

$$= (\tilde{V} + Q_F\tilde{N}_0)^{-1}(\tilde{U} + Q_F\tilde{M}_0) \quad \text{where } \tilde{N}_0, \tilde{M}_0 \text{ are as defined above.}$$

Finally, (4.35),(4.36) can be proved by application of (2.55), (2.56). □

The main result is then,

Theorem 4.6

The class of proper stabilizing observer-based controllers $\{K[0, F(Q_F)] | Q_F \in RH_\infty\}$ *is the class of all proper stabilizing controllers* $\{K[Q, F] | Q \in RH_\infty\}$ *for* G.

Proof Let us consider $F(Q_F)$, with arbitrary $Q_F \in RH_\infty$. This is an arbitrary stabilizing controller for G_F. Define $Q = Q_F\Psi \in RH_\infty$, then

$$\begin{aligned}
& Q = Q_F\tilde{M}_0\tilde{M}^{-1} \text{ by (4.36)} \\
\Leftrightarrow\ & Q\tilde{M} = Q_F\tilde{M}_0 \\
\Leftrightarrow\ & Q[-\tilde{N} \quad \tilde{M}] = Q_F[-\tilde{N}_0 \quad \tilde{M}_0] \quad \text{(multiplication by } [-G \quad I]) \\
\Leftrightarrow\ & (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) = (\tilde{V} + Q_F\tilde{N}_0)^{-1}(\tilde{U} + Q_F\tilde{M}_0) \text{ by (4.12),(4.33)} \\
\Leftrightarrow\ & K[Q, F] = K[0, F(Q_F)]
\end{aligned}$$

Consequently the observer based controller $K[0, F(Q_F)]$ is a stabilizing controller $K[Q, F]$ for G. Conversely, suppose we have an arbitrary stabilizing controller K for G, and we find $Q \in RH_\infty$ such that $K = K[Q, F]$. Then defining Q_F by

$$Q_F = Q(\Psi^T\Psi)^{-1}\Psi^T$$

it is clear that Q_F satisfies

$$\begin{aligned} & Q = Q_F\Psi \in RH_\infty \\ \Leftrightarrow \quad & K[Q, F] = K[0, F(Q_F)] \quad \text{(as above)} \end{aligned}$$

This completes the proof, by showing that the arbitrary stabilizing controller $K[Q, F]$ can be structured as an observer based controller $K[0, F(Q_F)]$, where $F(Q_F)$ is stabilizing for G_F.

□

4.4 The minimal-order dual observer

The reader may have noticed that Lemma 4.5 and Theorem 4.6 deal primarily with left coprime factorizations of $K[Q, F]$ and $F(Q_F)$. Are there dual results related to right coprime factorizations? In fact, we can exploit the dual minimal-order observer [40]. Whereas the role of the observer is to make full use of the system information in the system outputs, the dual observer takes advantage of the fact that the system can be excited from more than one input. We claim that all of the results of this chapter can be derived in terms of the dual observer. To give an illustration of this, a dual version of Theorem 4.1 will be stated. The dual observer equations are

$$\dot{z} = Dz + \xi Hw, \quad w = y + CSz, \quad u = Lz + \eta Hw \tag{4.37}$$

where

$$B \text{ is full rank} \tag{4.38}$$

$$[S \quad B]\begin{bmatrix}\xi \\ \eta\end{bmatrix} = \begin{bmatrix}\xi \\ \eta\end{bmatrix}[S \quad B] = \begin{bmatrix}I & 0 \\ 0 & I\end{bmatrix} \tag{4.39}$$

$$D = \xi AS, \quad L = \eta AS \tag{4.40}$$

The transfer function matrix $K(s)$ of an equivalent controller for G is

$$K(s) = \left[\begin{array}{c|c} D + \xi HCS & \xi H \\ \hline A + \eta HCS & \eta H \end{array}\right]_{\mathrm{T}} \tag{4.41}$$

Theorem 4.7 *Consider the plant $G(s) = C(sI - A)^{-1}B$ with (A, B) controllable and (A, C) observable. Choose H, S such that$(sI - A - HC)^{-1}, (sI - D)^{-1} \in RH_\infty$ where H, S are described by the observer equations (4.37)–(4.40). With arbitrary Γ such that $(sI - \Gamma)^{-1} \in RH_\infty$, define*

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} S\xi A + B\Gamma\eta & B & -S\xi H \\ \hline -\eta(A - B\Gamma\eta) & I & \eta H \\ C & 0 & I \end{array}\right]_{\mathrm{T}} \tag{4.42}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + HC & -B & H \\ \hline -\eta(A + HC - B\Gamma\eta) & I & -\eta H \\ C & 0 & I \end{array}\right]_{\mathrm{T}} \tag{4.43}$$

Then,

(i) all transfer functions defined by (4.42),(4.43) are stable and proper;

(ii) $M, \tilde{M}, V, \tilde{V}$ have proper inverses;

(iii) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$;

(iv) $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ *where* K *is the observer-based controller given by* *(4.7);*

(v)

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.44}$$

Close comparison of Theorem 4.1 and Theorem 4.7 shows that the corresponding factorizations are natural duals of each other. We can write down a dual to the controller-class of Fig. 4.4, with m integrators required to realize the transfer function $(sI - \Gamma)^{-1}$. The full-order observer based class of Fig. 4.3 has no dual, as can be seen in the inherent symmetry of the block diagram.

4.5 Conclusions

For brevity, the results of the chapter have been obtained in terms of the minimal-order observer, which has McMillan degree $n-p$ $(p \geq 1)$. As shown in Fig. 4.1, the state estimate has an additive term Ψy involving direct-feedthrough of all plant outputs. The results can also be obtained in terms of a reduced-order observer of order $n-\chi$, with $\chi \leq p$. As shown in Fig. 4.7, the reduced-order observer-based controller has direct feedthrough to $\hat{x}$ of only χ plant outputs, the plant outputs being divided as follows,

$$y = Cx = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with a $(p-\chi) \times 1$ vector y_1 and a $\chi \times 1$ vector y_2.

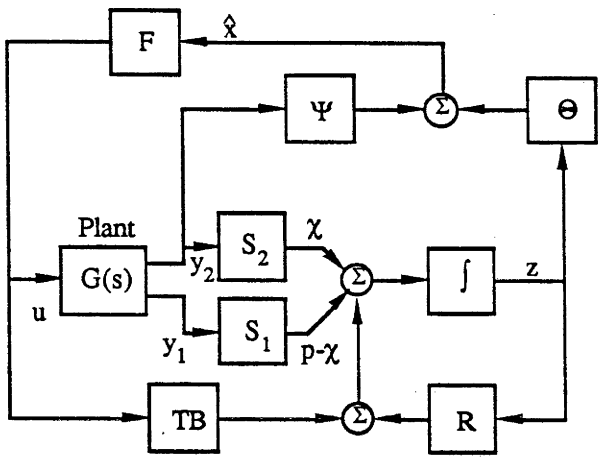

Figure 4.7: Reduced order observer-based control loop

The case $\chi = p$ corresponds to the results of this chapter, while the $\chi = 0$ leads to the results of Nett et. al. [39], with F constant, and the results of Moore et. al. [35], with F dynamic. The requirement that C be full rank is not restrictive, since in practice C can always be made full rank by ignoring certain plant outputs, and deleting the corresponding rows of C.

Finally, it has been shown that an observer-based controller class

$$\{K[0, F(Q_F)] | Q_F \in RH_\infty\}$$

is exactly the class of all proper stabilizing controllers $\{K[Q, F] | Q \in RH_\infty\}$ for G. Trivial extensions show that this is identical to the more general class $\{K[Q, F(Q_F)] | Q, Q_F \in RH_\infty\}$.

Chapter 5

Controller reduction methods maintaining performance and robustness

5.1 Introduction

The process of controller design and implementation can be broken into many stages. One of the first stages is to obtain a model of the plant through knowledge of the physical characteristics, off-line identification, or from existing data. For a large plant with many sensors, actuators, and sub-processes, a complex high-order model may be required to describe accurately the input-output behaviour of the plant. This is usually not a problem, since such high-order models can easily be handled with modern workstations and computers, and the engineer will have a much better chance of designing a satisfactory controller if he starts with a good model of the process. The next stage, assuming that the design objectives have been specified, is to design the actual control algorithm. There are many standard techniques available, with many producing controllers as complex, or even more complex, than the plant model.

Once the design is complete, the engineer must decide how to implement the controller. Some problems that can arise when implementing a high-order controller are numerical instability and inability of the controller to execute the real time algorithm quickly enough. This may necessitate a final *controller reduction* step. Here the high-order controller is replaced by a low-order controller which will give approximately the same closed-loop properties.

The *model reduction* methods of Glover [16] provide *a priori* bounds on reduction errors in terms of L_∞ measures. A simpler technique, termed balanced realization, has guaranteed bounds which are not quite so good [16, 32]. Such techniques are then attractive to achieve controller reduction, but without modification do not take into account the fact that the controller is in a control loop and needs to achieve performance and robustness properties. In the reduction, these techniques without modification weight all frequencies equally.

The notion of a frequency-weighted model reduction based on the techniques of [16, 32] has been explored in by other authors [23, 2, 14]. It is not clear from these results how best to use knowledge of the frequency characteristics of a plant, or closed-loop, to frequency-weight the controller reduction.

Special frequency-weightings based on controller characteristics are studied in [14].

A technique for controller reduction for linear quadratic gaussian designs is given in [24]. This exploits the fact that the innovations process is white (as in the techniques of [10]) and reduces the subsystems of the controller driven from this white noise. In effect there is a particular coprime stable factorization of the controller, and it is proposed that reductions on these be implemented using standard methods (balanced realizations without frequency shaping). A possible disadvantage for this approach is that stability of the original controller design is not guaranteed in the reduction.

In this chapter, a novel controller reduction approach is proposed. It is based on the application of standard model reduction techniques to a system calculated from both plant and controller. The method utilizes theory for the class of all stabilizing controllers [39] based on the work of [49]. Thus referring to Fig. 5.1 with plant $G(s) \in R_p$, controller $K(s) \in R_p$, then the class of all stabilizing controllers is given in terms of $J(K,G) \in R_p$ and arbitrary $Q(s) \in RH_\infty$, where R_p denotes the class of rational proper transfer functions and RH_∞ the class of stable rational proper transfer functions.

The selection of $J(s)$ we consider is where the block $J_{11}(s)$ is in fact the controller $K(s)$, and the other elements $J_{ij}(s)$ are appropriately scaled. Using the circumflex to denote a low order approximation, we propose that $J(s)$ first be approximated by $\hat{J}(s)$ using standard model reduction (possibly

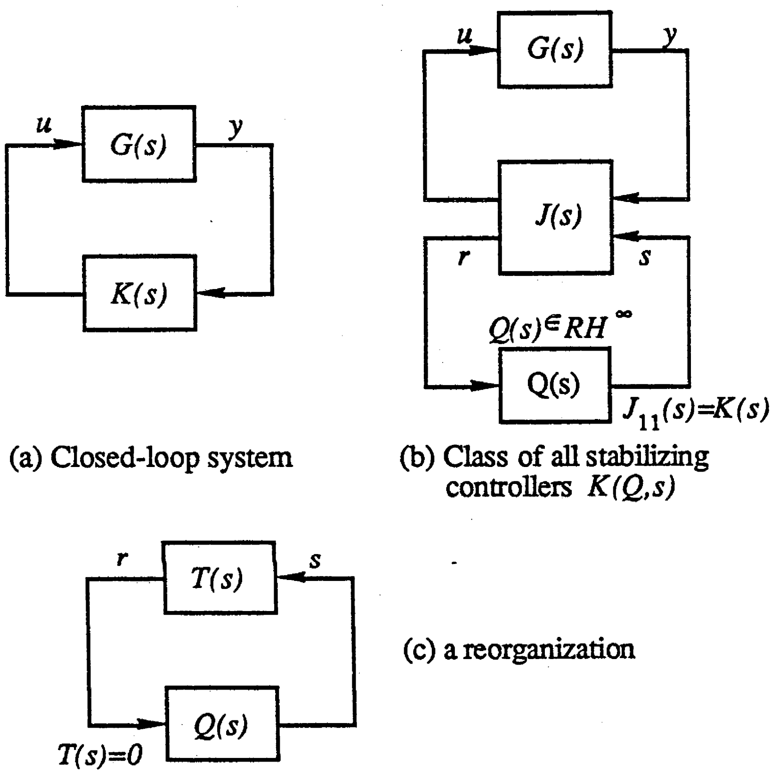

Figure 5.1: Feedback structures based on the class of all stabilizing controllers

frequency-weighted). Then the reduced order controller $\hat{K}(s)$ is taken as the 11-block of this. That is,

$$\hat{K}(s) = [\hat{J}(s)]_{11} \quad \text{where} \quad K(s) = J_{11}(s) \tag{5.1}$$

This contrasts the more direct application of model reduction where only $J_{11}(s) = K(s)$ is approximated, so that $\hat{K}(s) = \hat{J}(s)_{11}$.

An extension of the approach proposed is to work with the class of controllers of Fig. 5.1b with $Q(s)$ constrained as constant. Thus consider the class

$$K(Q,s) = J_{11}(s) + J_{12}(s)Q[I - J_{22}(s)Q]^{-1}J_{21}(s), \;\; Q \text{ constant} \tag{5.2}$$

and its reduced order versions

$$\hat{K}(Q,s) = [\hat{J}(s)]_{11} + [\hat{J}(s)]_{12}Q\left[I - [\hat{J}(s)]_{22}Q\right]^{-1}[\hat{J}(s)]_{21} \tag{5.3}$$

Here $\hat{K}(s)$ of (5.1) is equal to $\hat{K}(Q = 0, s)$. Also note that

$$\text{degree } \hat{K}(Q \mid Q = \text{constant}, s) = \text{degree } \hat{K}(Q = 0, s) \tag{5.4}$$

In this chapter one proposal is that $\hat{K}(Q \mid Q = \text{constant}, s)$ be re-optimized over constant Q in terms of the original (or related) controller robustness/performance objectives.

To maintain performance which penalizes some internal variables,or their estimates e, a refinement of the above method is to modify the $J(s)$ or $K(Q,s)$ blocks in Fig. 5.1 to have an additional output e. Denoting these blocks as $J_e(s)$, $K_e(Q,s)$ we propose the reduction of $K(s)$ via $J_e(s)$ or $K_e(Q,s)$ to maintain performance as well as robustness. Again scaling gives desired trade-off between performance and robustness. When $K(s)$ is designed to achieve simultaneous stabilization of a number of plants, it is proposed to maintain its performance/robustness properties for each of these plants by working with appropriate augmentations of $J(s)$. Details are given in the chapter. A dual version of the method is where the role of $G(s)$ and $K(s)$ are interchanged.

In Sec. 5.2, the controller reduction techniques for preserving robustness are given in details. A rationale for the proposed controller reduction is given in Sec. 5.3, and examples are studied in Sec. 5.4. Conclusions are drawn in Sec. 5.5.

5.2 Details of controller reduction

5.2.1 Definitions

Referring to Fig. 5.1, let us first recall the formulation of $J_{11}(s) = K(s)$ based on the theory for the class of all stabilizing controllers [35]. Let us denote

$$G(s) = C(sI - A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{\mathrm{T}} \tag{5.5}$$

Also, in the first instance let us consider that $K(s)$ belongs to the stabilizing controller class having the form of Fig. 5.2 for the case $Q(s) \triangleq 0$. Thus $K(s)$ is characterized in terms of F, H (see also Sec. 5.2.7) as

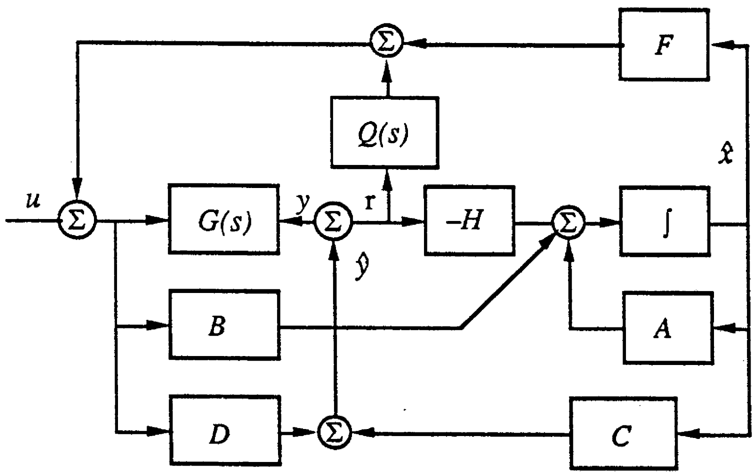

Figure 5.2: Controller class

$$K(s) = \left[\begin{array}{c|c} A_K & -H \\ \hline F & 0 \end{array}\right]_T \in R_{sp}, \quad A_K = A + BF + HC + HDF \tag{5.6}$$

$$[sI - (A + BF)]^{-1}, [sI - (A + HC)]^{-1} \in RH_\infty \tag{5.7}$$

where R_{sp} denotes rational strictly proper.

Clearly, the class of LQG controllers is a subset of this controller class. From [35], the class of all stabilizing controllers for $G(s)$ has the form of Fig. 5.2, being parameterized in terms of $Q(s) \in RH_\infty$. Moreover $J(s)$ has the form

$$J(s) = \left[\begin{array}{c|cc} A_K & -H & B + HD \\ \hline F & 0 & I \\ -C - DF & I & -D \end{array}\right]_T, J_{11}(s) = K(s) \tag{5.8}$$

It should be clear that Fig. 5.1b for this case takes the form of Fig. 5.2 with the $Q(s)$ nonzero.

Other relationships of interest are reviewed. Defining

$$\mathcal{X} \triangleq \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -H \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right]_{\mathrm{T}} \in RH_\infty \tag{5.9}$$

$$\tilde{\mathcal{X}} \triangleq \begin{bmatrix} \tilde{V}(s) & -\tilde{U}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -(B+HD) & H \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]_{\mathrm{T}} \in RH_\infty \tag{5.10}$$

then

$$\tilde{\mathcal{X}}(s)\mathcal{X}(s) = \mathcal{X}(s)\tilde{\mathcal{X}}(s) = I \quad \text{(double Bezout)} \tag{5.11}$$

$$G(s) = N(s)M(s)^{-1} = \tilde{M}(s)^{-1}\tilde{N}(s), \quad K(s) = U(s)V(s)^{-1} = \tilde{V}(s)^{-1}\tilde{U}(s) \tag{5.12}$$

Also

$$\begin{aligned} J(s) &= \begin{bmatrix} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{bmatrix} = \begin{bmatrix} K(s) & \tilde{V}(s)^{-1} \\ V(s)^{-1} & -V(s)^{-1}N(s) \end{bmatrix} \\ &= \begin{bmatrix} K(s) & M(s)-K(s)N(s) \\ \tilde{M}(s)-\tilde{N}(s)K(s) & -[\tilde{M}(s)-\tilde{N}(s)K(s)]N(s) \end{bmatrix} \end{aligned} \tag{5.13}$$

and referring to Fig. 5.1c,

$$T(s) = J_{22}(s) + J_{21}(s)G(s)[I - J_{11}(s)G(s)]^{-1}J_{12}(s) = 0 \tag{5.14}$$

5.2.2 Scaling

Before applying any multivariable model reduction technique to $J(s)$ to yield a $\hat{J}(s)$, it makes sense to scale the inputs $y(t)$, $s(t)$ and outputs $u(t)$, $r(t)$ in such a way that they are given appropriate significance. We do not propose an optimal scaling selection. Based on experience we know that scaling can be crucial to a good reduction.. In the examples studied in this chapter, we determine the scalings of the variables $y(t)$, $s(t)$, $u(t)$ and $r(t)$ using their closed-loop auto-covariance responses to realistic stochastic disturbances. This is achieved by solving a steady state Lyapunov equation associated with the closed-loop system. Thus consider the stochastic closed-loop system driven by the process noises $w(t)$ and measurement noise $v(t)$,

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + w(t) \tag{5.15}$$

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) - Hr(t) \tag{5.16}$$

where

$$r(t) = y(t) - [C\hat{x}(t) + Du(t)],\ y(t) = Cx(t) + Du(t) + v(t)$$

$$u(t) = F\hat{x}(t)$$

$$E[w(t)w'(\tau)] = Q_w\delta(t-\tau),\ E[v(t)v'(\tau)] = Q_v\delta(t-\tau)$$

The state/state-estimate auto-covariance matrix P satisfies the following Lyapunov equation,

$$PA_c' + A_cP + \begin{bmatrix} I & 0 \\ 0 & -H \end{bmatrix} Q_n \begin{bmatrix} I & 0 \\ 0 & -H \end{bmatrix}' = 0 \tag{5.17}$$

where

$$P = E\left[\begin{pmatrix} x \\ \hat{x} \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}'\right],\ A_c = \begin{bmatrix} A & BF \\ -HC & A + BF + HC \end{bmatrix}$$

$$Q_n = \begin{bmatrix} Q_w & 0 \\ 0 & Q_v \end{bmatrix}$$

and

$$E[y(t)y'(t)] = [C \;\; DF]P[C \;\; DF]' + [0 \;\; I]Q_n[0 \;\; I]' \tag{5.18}$$

$$E[u(t)u'(t)] = [0 \;\; F]P[0 \;\; F]' \tag{5.19}$$

$$E[r(t)r'(t)] = [C \;\; -C]P[C \;\; -C]' + [0 \;\; I]Q_n[0 \;\; I]' \tag{5.20}$$

It is not possible to calculate a value for $E[s(t)s'(t)]$, because it is dependent on the value of $Q(s)$. Choosing a value that it too large will place too much emphasis on the $s(t)$ input in the reduction. One suggested selection is to choose $E[s(t)s'(t)] = E[u(t)u'(t)]$. We propose that the square roots of the diagonal elements of these matrices be used to generate scaling matrices D_y, D_u, D_r, D_s to scale $J(s)$ as follows

$$J_{\text{scaled}}(s) = \left[\begin{array}{c|cc} A_K & -HD_y & (B+HD)D_s \\ \hline D_u^{-1}F & 0 & D_u^{-1}D_s \\ D_r^{-1}(C+DF) & D_r^{-1}D_y & -D_r^{-1}DD_s \end{array}\right]_{\text{T}} \tag{5.21}$$

In the system $J_{\text{scaled}}(s)$ the variances of the scaled input/output variables in the closed-loop system will be unity.

Now model reduction techniques as in [16] can be applied to $J_{\text{scaled}}(s)$ to yield low-order models,

$$\hat{J}_{\text{scaled}}(s) = \left[\begin{array}{c|cc} \widehat{(A_K)} & -\hat{H}_{\text{scaled}} & * \\ \hline \hat{F}_{\text{scaled}} & \hat{D}_{\text{scaled}} & * \\ * & * & * \end{array}\right]_{\text{T}} \tag{5.22}$$

from which a reduced order controller is taken as

$$\hat{K}(s) = \left[\begin{array}{c|c} \widehat{(A_K)} & -\hat{H}_{\text{scaled}}D_y^{-1} \\ \hline D_u\hat{F}_{\text{scaled}} & D_u\hat{D}_{\text{scaled}}D_y^{-1} \end{array}\right]_{\text{T}} = D_u[\hat{J}_{\text{scaled}}(s)]_{11}D_y^{-1} \tag{5.23}$$

More generally, $\hat{K}(Q,s)$ can be in terms reductions on J and constant Q as in (5.3).

Other scaling possibilities can be envisaged. Observe that at the one extreme with D_u approaching zero, then $[\hat{J}(s)]_{11} \rightarrow \hat{J}_{11}(s)$ and standard controller reduction is achieved. At the other extreme with $D_r \rightarrow 0$, maintaining prediction quality is emphasized—this is linked to maintaining quality of the state estimate feedback. When the prediction errors are white and state estimation is optimal, then with $D_r \rightarrow 0$ these qualities are preserved as much as possible.

Of course, a search procedure over D_u, D_r and D_y may achieve an improved compromise between performance and robustness. Such brute force optimizations are not explored further in this chapter. There is no proof or rationale in this chapter to suggest that a selection $D_u \neq 0$ is always better than a selection $D_u = 0$. However, out experience has certainly shown that it is sometimes better. One scaling technique has been presented above based on certain intuitions which appear to work well. It could be used as the starting point for a search for an improved reduction.

5.2.3 Re-Optimization

Referring to (5.2)–(5.4), it is clear that a class of reduced order controllers having the same dimension can be defined in terms of the sub-blocks of $\hat{J}(s)$ and $Q(s)$, with $Q(s)$ constrained to be constant. These are parameterized in terms of a constant Q matrix having the dimensions of the plant transfer function matrix. A search over all constant Q can lead to improved reduced order controllers over that of the simplest case where $Q = 0$ as in the previous subsection.

Such a search over constant Q is relatively simple computationally compared to a search over the scale factors D_r, D_u, D_y involving repeated application of the balanced realization algorithm.

The search over constant $Q(s)$ can be simplified exploiting the fact that all closed-loop transfer functions are affine in $Q(s)$ when $\hat{J}(s) = J(s)$, so are "close" to affine in $Q(s)$ when $\hat{J}(s)$ is "close" to $J(s)$.

5.2.4 Estimation-based reduction

Control schemes based on state estimate feedback can be viewed as an estimator/controller driven from both the plant inputs $u(t)$ and outputs $y(t)$ with an output $u(t)$. As depicted in Fig. 5.3a, we can think of an augmented plant $G'_a(s) = [G'(s) \;\; I]$ with an augmented output $[y'(t) \;\; u'(t)]'$ driving a controller, denoted $K_a(s)$. Now the corresponding $K_a(s)$ and $J_a(s)$ are given from

$$J_a(s) = \left[\begin{array}{c|ccc} A+HC & [-(B+HD) & H] & 0 \\ \hline -F & [0 & 0] & I \\ C & [-D & I] & 0 \end{array}\right]_{\mathrm{T}} \qquad (5.24)$$

Notice that $J_a(s)$ is stable so that reduced order approximations $\hat{J}_a(s)$, $\hat{K}_a(s) = [\hat{J}_a(s)]_{11}$ are also stable. There appears to be no other *a priori* guideline in selecting between reducing this controller and the conventional one.

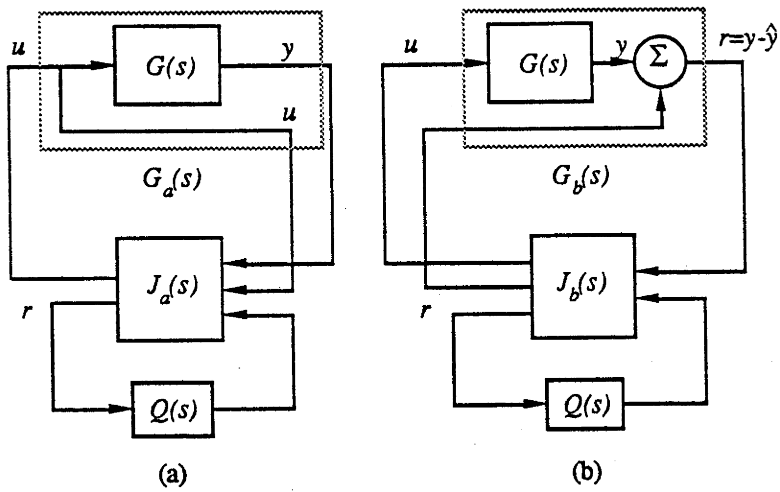

Figure 5.3: Estimation-based reduction

Clearly in any particular application one may be given a "better" reduced order performance and robustness. A dual approach is to view the plant as in Fig. 5.3b, where the plant $G_b(s) = [G(s) \;\; I]$ has an additional input which is added to the output of $G(s)$. The corresponding transfer function for $J_b(s)$ is given by

$$J_b(s) = \left[\begin{array}{c|cc} A+BF & H & -B \\ \hline -F & 0 & I \\ C+DF & 0 & -D \\ 0 & I & 0 \end{array}\right]_{\mathrm{T}} \tag{5.25}$$

Comparison of (5.24), (5.25) with (5.9), (5.10) reveals that

$$\tilde{\mathcal{X}}(s) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} J_a(s) \begin{bmatrix} I & 0 \\ 0 & I \\ -I & 0 \end{bmatrix} \tag{5.26}$$

$$\mathcal{X}(s) = \begin{bmatrix} -I & 0 & 0 \\ 0 & I & -I \end{bmatrix} J_b(s) \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \tag{5.27}$$

This suggests the possibility of reducing $\tilde{\mathcal{X}}$ or $\mathcal{X}(s)$ as another method of controller reduction. The fractional decomposition of the reduced order controller would be found by applying standard model reduction methods to $\tilde{\mathcal{X}}$ or $\mathcal{X}(s)$.

5.2.5 Controller reduction maintaining performance

Consider that a performance objective is to minimize the energy in some internal variable, or its estimate, denoted e. For the controller class (5.6), which can be interpreted as state estimate feedback, it is common for e to be a linear combination of the states of $K(s)$. Thus here we assume that the transfer function from u to e is $E(sI - A_K)^{-1}(-H)$. Now, the augmentation of the $J(s)$ block of (5.8) to incorporate this transfer function is

$$J_e(s) = \left[\begin{array}{c|cc} A_K & -H & B \\ \hline F & 0 & I \\ -(C+DF) & I & -D \\ E & 0 & 0 \end{array}\right]_{\mathrm{T}}, \quad K_e(s) = [J_e(s)]_{11} \qquad (5.28)$$

Scaling of this in terms of D_y, D_u, D_r, D_s and D_e is now a natural extension of the scaling in (5.21). Likewise generalizations of (5.22) to $\hat{J}_{\text{scaled}}(s)$ and $\hat{K}(s) = D_u[\hat{J}_{\text{scaled}}(s)]_{11}D_y^{-1}$ are straightforward. The relative significance of D_e determines the emphasis on performance of the controller in the reduction process, and can be fine tuned by a trial and error procedure.

5.2.6 Frequency shaped reduction

Just as a frequency shaped reduction of $K(s)$ can lead to improved reduced order controllers, so a frequency shaped reduction of $J(s)$ leading to a reduced

$\hat{K}(s)$ can give improvement. It might be that we require robustness in a frequency band only. That is, we require robustness to $Q(s) \in RH_\infty$ in this frequency band. Under such circumstances it makes sense to insert in Fig. 5.1b a stable band pass filter between the residuals $r(t)$ and the input to $Q(s)$, and require robustness to all $Q(s) \in RH_\infty$ as before. The band pass filter can be used as a frequency shaped augmentation of $J(s)$, being in series with $J_{21}(s)$ (or $J_{12}(s)$) and $J_{22}(s)$. Again the augmented $J(s)$ can be reduced and the 11-block extracted as a frequency shaped reduced controller $\hat{K}(s)$. The augmentation increases the degree of $J(s)$, while the following step reduces the degree of $J(s)$. In many cases, the effect of the errors introduced by increasing the degree of $J(s)$ in the intermediate step will be outweighed by the improved robustness of the closed loop controller.

Of course general frequency shapings can be employed based on the closed-loop transfer functions. In fact, it is sometimes impossible to obtain a good reduction of $J(s)$ unless frequency weighted reduction methods are used.

To avoid numerical difficulties when the combined order of $J(s)$ and any frequency shaping is high, it makes sense to first carry out a preliminary unweighted reduction of $J(s)$ and any frequency shaping using balanced truncation. Such a reduction allows a degree reduction with relatively small error.

5.2.7 Generation of F, H

When the plant and controller have the same degree, but a selection F, H to satisfy (5.6) is not known *a priori*, then such selections can be found for generic $K(s)$, $G(s)$ [31]. More precisely

Lemma 5.1 *Consider the plant/controller pair $G(s)$, $\bar{K}(s)$ with minimal nth-order state-space realizations*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{\mathrm{T}}, \quad \bar{K}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right]_{\mathrm{T}} \tag{5.29}$$

The controller can only be represented in the form (5.6) if and only if there exists a real, nonsingular solution Z to the quadratic matrix equation

$$AZ + B\bar{C} - Z\bar{B}CZ - Z(\bar{A} + \bar{B}D\bar{C}) = 0 \tag{5.30}$$

Moreover, when a real, nonsingular Z exists

$$F = \bar{C}Z^{-1}, \quad H = -Z\bar{B} \tag{5.31}$$

Proof: As in Theorem 3.1

In the SISO case solutions F, H always exist [9] under

$$[A, B], [\bar{A}, \bar{B}] \text{ controllable,} \quad [A, C], [\bar{A}, \bar{C}] \text{ observable} \tag{5.32}$$

Remarks:

(i) Sufficient conditions for multivariable $G(s)$, $K(s)$ have been considered in Chapter 3.

(ii) There are in general a class of nonsingular solutions of (5.30), giving rise to a class of $J(s)$, $\hat{J}(s)$ and $\hat{K}(s)$. For each $\hat{J}(s)$ the bounds on $\|J(s) - \hat{J}(s)\|$ will in general be different, and each approximation will have its own inherent frequency shaping. Clearly some selections of $J(s)$ will be better than others. This has been borne out with examples studied, but as yet there is no elegant method to select the best $J(s)$ to use.

5.2.8 Staged reduction

So far the simplest situation has been studied—namely when the degrees of $G(s)$,$K(s)$ are the same. Should $K(s)$ be of a higher degree than $G(s)$, it makes sense to first perform a standard reduction of $K(s)$ until it is the same degree as $G(s)$. Such preliminary reduction can usually be made with negligible errors compared to subsequent reductions to achieve a lower degree estimates $\hat{K}(s)$. The same holds *mutatis mutandis* when $G(s)$ is of a higher degree than $K(s)$.

5.2.9 Simultaneous stabilization

Consider that K is designed to give acceptable performance/robustness for a number of plants G_1, $G_2 \ldots G_N$. Associated with each plant G_i there is a corresponding J_i with $[J_i]_{11} = K$ for each i. By bringing each J_i to the same co-ordinate basis it is possible to define a block $\mathcal{J}(s)$, $\mathcal{Q}(s)$ as in Fig. 5.4 such that

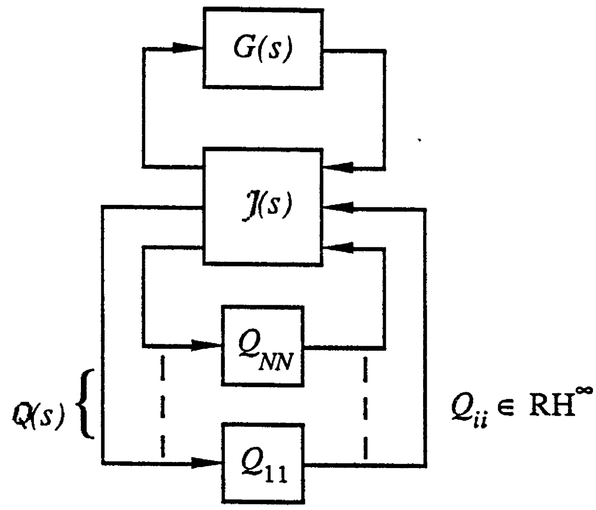

Figure 5.4: New J and Q blocks based on controller designed for simultaneous stabilization of many plants.

$$J_i = \begin{bmatrix} \mathcal{J}_{1,1} & \mathcal{J}_{1,i+1} \\ \mathcal{J}_{i+1,1} & \mathcal{J}_{i+1,i+1} \end{bmatrix} \tag{5.33}$$

By setting $\mathcal{Q}_{kl} = 0$ for $k, l \neq i,\ i = 1, 2 \dots N$, the class of all stabilizing controllers for G_i is characterized in terms of $Q_{ii} \in RH_\infty$. This leads to the following lemma.

Lemma 5.2 *With (5.33) holding, the class of all stabilizing controllers for G_i for $i = 1, 2 \dots N$ is a subset of the class of all controllers of Fig. 5.3 with arbitrary $\mathcal{Q}(s) \in RH_\infty$.*

To achieve a reduced order controller $\hat{K}(s)$ for $K(s)$, we propose the reduction of $\mathcal{J}(s)$ giving $\hat{K} = [\hat{\mathcal{J}}]_{11}$. When $N = 1$, this method reduces to that presented previously. Scaling can be introduced to order the importance of the various plants G_i.

5.2.10 Reduced order plant

It may be that for simulation purposes a reduced order plant is required. In the reduction technique described above it is possible to extract a reduced order plant $\hat{G}(s)$ in addition to the reduced order controller as follows

$$\hat{G} = \left[\begin{array}{c|c} \widehat{A_K} - \hat{B}\hat{F} - \hat{H}\hat{C} - \hat{H}\hat{D}\hat{F} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]_{\mathrm{T}} \tag{5.34}$$

where the estimates $\widehat{A_K}$,$\hat{H}$,$\hat{F}$,$\hat{C}$ and $\hat{D}$ are obtained from $\hat{J}$. One problem is that there is no guarantee that $\hat{K}$ close to K will ensure that $\hat{G}$ will be close to G, or indeed, that K or $\hat{K}$ will stabilize $\hat{G}$. Let us instead propose that a reduced order $\hat{G}$ be obtained from a dual procedure to that giving $\hat{K}$, so that at least $\hat{G}$ is close to G and is stabilized by K. In the dual procedure the roles of K, G are merely interchanged.

5.3 Rationale

5.3.1 Preserving robustness properties

The class of all stabilizing controllers for a plant $G(s) \in R_p$ shall be denoted

$$\mathcal{K}_G \triangleq \{\, K \in R_p \mid H(G,K) \in RH_\infty \,, \det(I - GK) \neq 0\} \tag{5.35}$$

where $H(G,K)$ represents the closed loop transfer functions

$$H(G,K) = \begin{bmatrix} I + K(I-GK)^{-1}G & K(I-GK)^{-1} \\ (I-GK)^{-1}G & (I-GK)^{-1} \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \tag{5.36}$$

Such classes have a parametrization in terms of an arbitrary $Q \in RH_\infty$ and an arbitrary factorization $K = UV^{-1} \in \mathcal{K}_G$

$$\mathcal{K}_G \triangleq \{\, \mathcal{K}_G(Q) = (U + MQ)(V + NQ)^{-1} | \, Q \in RH_\infty, \det(V + MQ) \neq 0 \,\} \tag{5.37}$$

By duality, a controller $K(s) \in R_p$ stabilizes a class of plants $\mathcal{G}_K$, and the reduced order controller $\hat{K}$ stabilizes a class of plants $\mathcal{G}_{\hat{K}}$.

Definition 5.3 *The robustness properties of a stabilizing controller K with respect to a plant class $\mathcal{G}^\star$ are said to be preserved in a controller reduction, yielding $\hat{K}$ when*

$$\mathcal{G}^\star \subset \mathcal{G}_{K,\hat{K}} \triangleq \mathcal{G}_K \cap \mathcal{G}_{\hat{K}} \tag{5.38}$$

Remarks:

(i) A dual definition of preserving robustness is as follows. With $\mathcal{K}_{\hat{G}}$ the class of all stabilizing controllers for a reduced order plant $\hat{G}$, the robustness properties of a plant G with respect to a controller class $K^\star$ are said to be preserved in a plant reduction yielding $\hat{G}$ when

$$\mathcal{K}^\star \subset \mathcal{K}_{G,\hat{G}} \triangleq \mathcal{K}_G \cap \mathcal{K}_{\hat{G}} \tag{5.39}$$

The class of stabilizing controllers for a plant $\hat{G}$ can similarly be parameterized in terms of $Q \in RH_\infty$.

$$\mathcal{K}_{\hat{G}} \triangleq \{\, \mathcal{K}_{\hat{G}}(Q) = (\hat{U}+\hat{M}Q)(\hat{V}+\hat{N}Q)^{-1} \mid Q \in RH_\infty, \det(\hat{V}+\hat{N}Q) = 0 \,\} \tag{5.40}$$

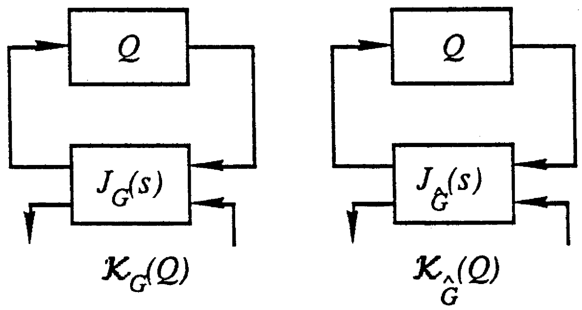

Figure 5.5: Linear fractional maps

where $\hat{K} = \hat{U}\hat{V}^{-1}$ is a stabilizing reduced order controller for the reduced order plant $\hat{G} = \hat{N}\hat{M}^{-1}$.

(ii) A controller or plant reduction that preserves the robustness properties defined in (5.38), (5.39) should maintain $\mathcal{G}_{K,\hat{K}}$ close to $\mathcal{G}_K$ and $\mathcal{K}_{G,\hat{G}}$ close to $\mathcal{K}_G$. In other words, the reduction should give $\hat{K}$ such that

$$\Delta\mathcal{G} \triangleq \mathcal{G}_K \Delta \mathcal{G}_{\hat{K}} \quad \text{or} \quad \Delta\mathcal{K} \triangleq \mathcal{K}_G \,\Delta\, \mathcal{K}_{\hat{G}} \text{ is small }[1] \tag{5.41}$$

(iii) The fractional maps (5.37), (5.40) can be depicted as in Fig. 5.5.

5.3.2 Closeness measures

Standard L_2 or L_∞ norms define measures of closeness of $\mathcal{G}_K(Q)$ to $\mathcal{G}_{\hat{K}}(Q)$ for any specific $Q \in RH_\infty$, with such norms highly Q dependent functions. The controller reduction method based on the reduction of $\tilde{\mathcal{X}}(s)$ or $J_a(s)$

[1]Here the binary set operator Δ is the symmetric difference defined as $A \,\Delta\, B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$.

suggests convenient measures of closeness of the classes $\mathcal{G}_K$,$\mathcal{G}_{\hat{K}}$ being

$$\|\Delta\tilde{\mathcal{X}}(s)\| \text{ or } \|\Delta J_a(s)\| \quad \text{respectively} \tag{5.42}$$

where $\Delta J_a(s) = J_a(s) - \hat{J}_a(s)$ etc.

The following lemma shows that a sufficient condition for the controller reduction objective that $\mathcal{G}_K(Q)$ is close to $\mathcal{G}_{\hat{K}}(Q)$ is that $\|\Delta\tilde{\mathcal{X}}(s)\|$ or $\|\Delta J_a(s)\|$ in (5.42) be small. A dual argument can be developed for the corresponding plant reduction.

Lemma 5.4 *With the definition (5.42) and*

$$\|\Delta\tilde{\mathcal{X}}\| \leq \epsilon \textit{ or } \|\Delta J_a\| \leq \epsilon \tag{5.43}$$

then for generic $Q \in RH_\infty$, as $\epsilon \to 0$

$$\|\mathcal{G}_K(Q) - \mathcal{G}_{\hat{K}}(Q)\| \to 0 \textit{ with } \mathcal{O}(\epsilon) \tag{5.44}$$

Proof Observe that from (5.44)

$$\begin{aligned} \mathcal{G}_{\hat{K}}(Q) - \mathcal{G}_K(Q) &= (\hat{N} + \hat{V}Q)(\hat{M} + \hat{U}Q)^{-1} - (N + VQ)(M + UQ)^{-1} \\ &= \{\mathcal{G}_K(Q)[\Delta M + \Delta UQ] - [\Delta N + \Delta VQ]\}[\hat{M} + \hat{U}Q]^{-1} \end{aligned} \tag{5.45}$$

For generic Q and with $\|\Delta\tilde{\mathcal{X}}\| \leq \epsilon$, as $\Delta M, \Delta N, \Delta U, \Delta V \to 0$, $\|\mathcal{G}_{\hat{K}}(Q) - \mathcal{G}_K(Q)\| \to 0$ with $\mathcal{O}(\Delta M, \Delta N, \Delta U, \Delta V)$ and the result (5.44) follows. Since from (5.25) $\|\Delta J_a\| \leq \epsilon$ implies $\|\Delta\tilde{\mathcal{X}}\| \leq \sqrt{2}\epsilon$, then $\|\Delta J_a\| \leq \epsilon$ implies (5.44) also.

□

Remark: By appropriate scaling, the controller reduction methods can be specialized to those of [25] involving only $\Delta\tilde{U}, \Delta\tilde{V}$. Clearly the methods proposed take into the account both the plant and the controller dynamics.

5.4 Examples

The method described in Sec. 5.2.1-5.2.2 has been applied to the reduction of a 55th order LQG controller for an advanced active control research aeroplane [38, 26]. Figure 5.6 shows the block diagram of a flutter suppression and gust load alleviation design. The controls used are the elevator and the outboard aileron surfaces. Measurements of pitch rate and wing tip acceleration are used to estimate the aeroplane's rigid and elastic motion. Also shown in parentheses in Fig. 5.6 are the root-mean-square responses, at various points in the control loop, to a 10fts^{-1} vertical Dryden turbulence. These values were used to scale $J_{\mathbf{scaled}}(s)$ of (5.21), and Hankel norm approximation was used to obtain the reduced order $\hat{J}_{\mathbf{scaled}}(s)$. Reduced order controllers of as low as fourth order could give a satisfactory closed loop performance.

Table 5.1 summarizes the results for different controllers, ranging from the original controller to the fourth order controller. Robustness properties have been evaluated based on single loop phase and gain margins, and the worst-case stability margins have been recorded in the table. Note that the margins of stability have been preserved in accordance with the design requirements (gain margins of 6dB and phase margins of 30°). Similarly, the damping

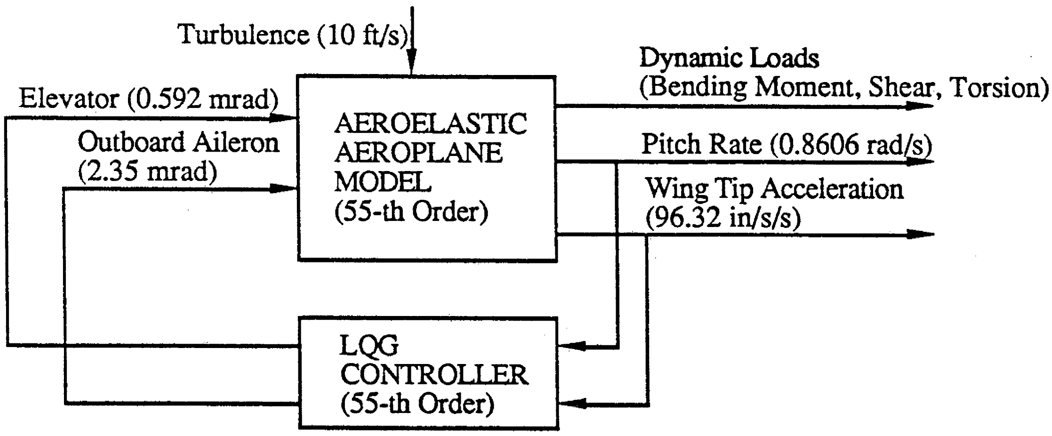

Figure 5.6: Block diagram of the flutter suppression and gust load alleviation system

of the flutter mode always exceed the design requirement of 0.015. Further reduction leads to an unstable closed-loop system. With other controller reduction methods such as modal residualization, the minimum order for the reduced order controller is ten. It is perceived that if the options described in Sec. 5.2.3, 5.2.5–5.2.6 were considered, then further improvements in the controller reduction could be expected. This will be left for future work.

We will now make some remarks on a second example, one which is well studied in the literature [14, 24, 25]. Our aim here is not to demonstrate the superiority of our various methods, since the inbuilt frequency weighting in the reduction technique of [24] turns out to be highly suited to this example; a simple application of our methods does not do as well. Rather, our aim is to be convinced that the methods here can be competitive, depending on the engineering criteria for judging robustness/performance. Indeed, for

Order	Flutter Mode Damping	Stability Margins	Bending Moment (in-lbs)	Shear (lbs)	Torsion (in-lbs)
55	0.074	14.0dB, 58.6°	2.348×10^5	854	4.437×10^4
10	0.034	14.0dB, 59.0°	2.593×10^5	890	4.200×10^4
9	0.039	5.8dB, 70.0°	2.348×10^5	859	4.495×10^4
8	0.032	10.0dB, 69.0°	2.610×10^5	930	4.821×10^4
7	0.032	15.0dB, 38.0°	2.345×10^5	862	4.779×10^4
6	0.027	7.0dB, 28.0°	2.362×10^5	871	4.968×10^4
5	0.016	15.0dB, 81.0°	2.371×10^5	997	7.117×10^4
4	0.016	7.5dB, 70.0°	2.680×10^5	1102	7.877×10^4

Table 5.1: Reduction of a 55th order Flutter Suppression and Gust Load Alleviation Controller[1].

a frequency weighted version of our technique we claim equality with, and perhaps marginal superiority to, some of the methods of [25].

An eighth-order controller is reduced to a fifth-order controller using various controller reduction methods. The original plant and controller are given in [14] (case q=100). The plant has one rigid body mode and three lightly damped structural modes ($\zeta = 0.02$). The command response corresponding to the full order controller does not exhibit any lightly damped structural modes. This is due to the fact that with precisely placed notch filters in the feedback controller, the residues at the structural mode frequencies are negligible. Reduction of the controller order alters the location of the notch filter poles and zeros, and may introduce large residues at the uncontrolled structural modes.

[1]The bending moment, shear force, and torsion are root-mean-square responses to a 10fts^{-1} vertical Dryden turbulence.

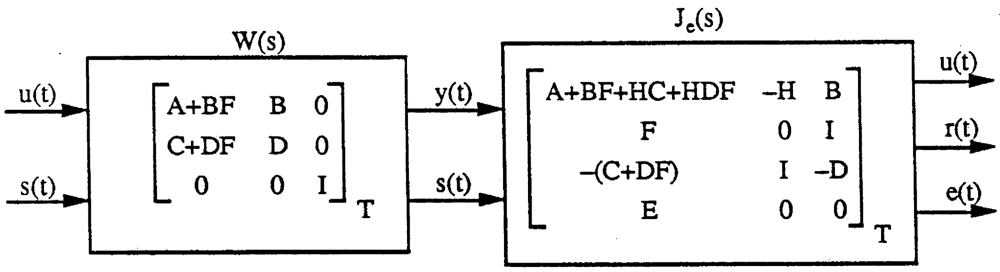

Figure 5.7: Frequency-weighting at the controller inputs

Reduction of a stable right coprime factorization of the controller [24] produces a good approximation in the low frequency region, and introduces residual responses at the second structural mode frequency. Comparison of the frequency weighted balanced truncation method of [14] with other reduction methods is given in [24, 25]. Frequency weighted balanced truncation is here applied to the controller structure $J_e(s)$ described in previously (see Fig. 5.7). The criterion function in the cost function is appended to the outputs of $J_e(s)$ to maintain performance. The scale factors applied at the inputs and outputs of $J_e(s)$ are determined by evaluating the closed-loop covariance responses. This reduction technique yields a reduced order controller with good robustness/performance properties in a systematic fashion, with fewer iterations than the method of [25]. (In [25], an augmented output is also included in the reduction to improve performance.)

A reduced order controller obtained by direct optimization via the SANDY design algorithm [28] has also been studied for comparative purposes. The reduction was based on the same cost objective, process noise, and sensor noise characteristics. The resulting step response agrees closely with the original design. The low order controller has the advantage that it only excites the plant at the first structural mode frequency.

5.5 Conclusions

A class of controller reduction methods have been proposed which preserve the robustness and performance qualities of the controller. The methods can be viewed as consisting of three steps. The first is organizing the plant and controller information. The second is applying standard model reduction techniques, and the third is extracting and re-optimizing (if necessary) a reduced order controller from the second step results using a constant stabilizing controller structure. Trade-offs between performance and robustness can be achieved by scaling, and indeed by certain extreme scalings, other methods in the literature can be recovered.

The proposed methods are, in the first instance, most appropriate for controller designs organized as state estimate feedback schemes. However all stabilizing controller designs can be organized as such [35], and our methods do extend of stabilizing controllers. Simulation studies have supported the rationale for the methods proposed.

Chapter 6

A study on adaptive stabilization and resonance suppression

6.1 Introduction

With precise knowledge of a linear multivariable plant, controller design techniques are effective at achieving a high performance in terms of disturbance response and control energy trade-offs. However, since plants are invariably uncertain objects, possibly drifting in their characteristics with time, performance for a nominal model is usually compromised in a design procedure to achieve robustness to plant uncertainty. Even with a fairly robust design, there is a possibility that the control system can catastrophically drift into instability or approach such a condition by exhibiting resonance behaviour. In such cases, one lightly damped mode often dominates, and if this dominant mode can be dampened by adaptive techniques, and the resonance suppressed without otherwise inordinately influencing the control system, robustness/performance enhancement will result.

One area of application of resonance suppression is in aircraft control. For example, ride quality can deteriorate if a structural resonance is excited by turbulence. Indeed, catastrophic failure can arise in the presence of wing flutter, which can occur in an emergency situation when the flutter speed is exceeded. The work of this chapter is motivated by the need for adaptive resonance suppression, rather than the possibility of devising an elegant solution to such a problem. Clearly, if low order adaptive schemes are applied to high order uncertain plants, such as an aircraft body or wing, there are inevitably unmodelled dynamics. How then can the problem of unmodelled dynamics be overcome. The methods proposed in this chapter are presented more as a challenge for following researchers, rather than to give an optimal approach. We do not seek global convergence results for our methods, because of unmodelled dynamics, and since there are high order unmodelled dynamics, we do not seek to calculate regions of local convergence. Rather we assess our methods by simulation studies with random model selection.

The adaptive scheme proposed here differs from those treated in the literature more in terms of its objectives and orchestration than in terms of its building blocks. It has evolved from case studies such as earlier work [8]. Most adaptive control designs in the literature tend to replace an off-line designed controller with an adaptive version of the off-line design. Thus, instead of a pole assignment or linear quadratic controller, there is an adaptive pole assignment scheme or an adaptive linear quadratic controller. These utilize linear parameter estimation schemes, based often on input-output models with little or no incorporation of *a priori* model information.

The scheme proposed here is to provide an additional loop on an existing off-line designed control system, associated with a nominal model. The *a priori* information is that the control system is stabilizing for the nominal model, but may drift into a resonant condition or instability. Thus, in the simplest case we may reasonably assume that all modes are stable, except for one dominant mode which is near instability or is just unstable. The desired control objective is merely to dampen the dominant mode somewhat. This suggests estimating its frequency and damping, and applying an adaptive pole-assignment scheme to force this mode to a location at the same frequency, but with greater damping. The intention is to achieve this without driving the other modes into instability or exciting other lightly damped modes.

The algorithms are presented in Sec. 6.2 based on low-order plant idealizations and known least-squares identification and pole assignment techniques. In Sec. 6.3, simulation studies are presented for both idealized low order plants and random high order ones, and stabilization and resonance suppression properties are observed. Section 6.4 presents an application of the new algorithm, which allows it to adaptively enhance fixed controller designs. Conclusions are drawn in Sec. 6.5.

6.2 An adaptive algorithm for resonance suppression

Consider now a scalar stochastic plant or closed loop system $G(q^{-1})$, modelled in terms of polynomials in the backwards shift operator q^{-1}, with one dominant resonant mode associated with a pair of poles in the vicinity of the unit circle. The adaptive resonance suppression algorithm presented here seeks to determine the frequency band of the resonant mode via an inner identification loop and exploit this information in an outer adaptive control loop. The algorithm is shown in Fig. 6.1, and is now developed in some detail.

Plant Model It is assumed that the plant $G(q^{-1})$ can be accurately modelled by a high-order moving-average autoregressive model(ARMAX):

$$\begin{aligned} y_k + a_1 y_{n-1} + \cdots + a_{n_0} y_{k-n_0} &= b_1 u_{k-1} + \cdots + b_{m_0} u_{k-m_0} \\ &\quad + c_1 w_{k-1} + \cdots + c_{p_0} w_{k-p_0} \end{aligned} \tag{6.1}$$

where w is a zero mean gaussian noise disturbance process. The parameters of the ARMAX model will be represented by a vector

$$\Theta = [a_1 \cdots a_{n_0}\ b_1 \cdots b_{m_0}\ c_1 \cdots c_{p_1}]' \tag{6.2}$$

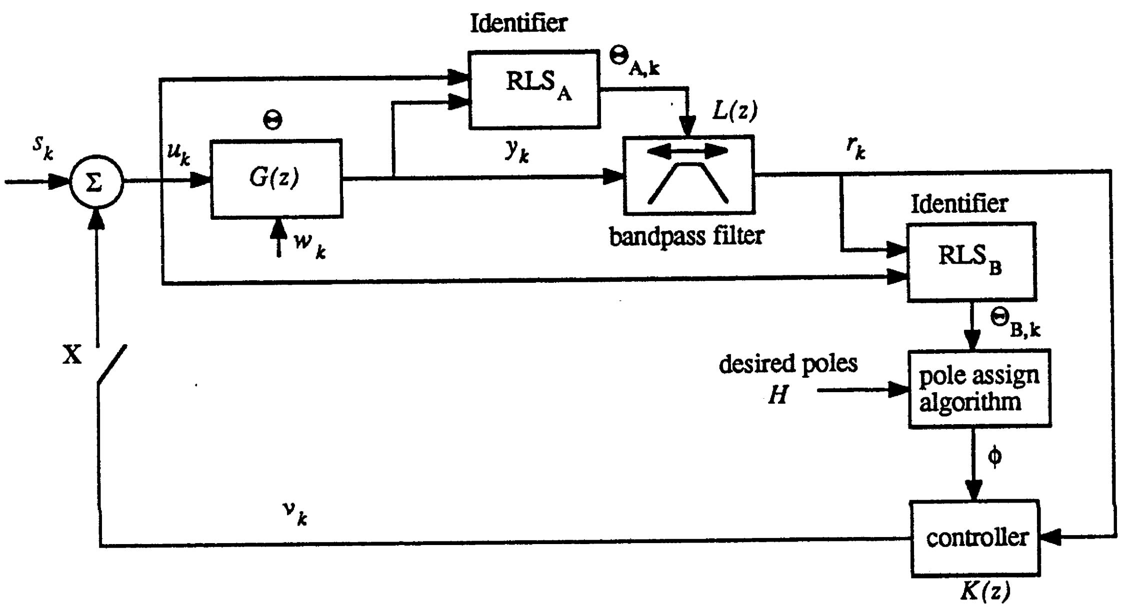

Figure 6.1: Main control system

Inner Loop Identification The recursive least squares identifier RLS_I is based on a low-order ARMA model:

$$y_k + a_1 y_{k-1} + \cdots + a_{n_0} y_{k-n_1} = b_1 u_{k-1} + \cdots + b_{m_1} u_{k-m_1} \tag{6.3}$$

where typically $n_1 = 4$. This identification block has two outputs, $\Theta_{I,k}$ and $\omega_{I,k}$. The value $\Theta_{I,k}$ is a recursive estimate at time k of the true model Θ. The estimate will not in general asymptotically approach any of the true model parameters, because the model is underparametrized. However, in the presence of a dominant resonant mode (or unstable mode) $\Theta_{I,k}$ can allow a crude estimate of $\omega_1 \in [0, \pi]$, the radian frequency, normalized by the sampling frequency, of the least stable pole pair. Here the least stable pole pair is the pole pair belonging to Θ, which is furthest from the origin. The estimate $\omega_{1,k}$ of ω_1 is the frequency of the least stable poles of the ARMA model given by $\Theta_{I,k}$. The idea is that RLS_I will give a rapidly converging estimate of the frequency ω_1 of the dominant resonance, and this information can be used to adjust the bandpass filter at the plant output. Here the dominant resonant mode is always considered to be due to a complex pole pair; the possibility of a nearly stable or unstable pole on the real axis is excluded.

Frequency Shaping Filter Using the frequency information from the inner loop, our approach is for the bandpass filter at the plant output to be adjusted to accentuate signals in the frequency band associated with the resonant mode, and to attenuate signals at other frequencies. With the series band pass filter in place, the outer identification/control loop can concentrate on stabilizing the resonant mode, ignoring the now well damped stable modes. The underlying assumption of this approach is that the augmented plant, formed by the combination of the plant and the frequency shaping filter, can be closely approximated by a lower order plant. Whether this is achieved in practice depends on the nature of the plant and filter. Certainly the frequency shaping attenuates plant modes at a frequencies far from the frequency of the dominant resonant mode.

Outer-Loop Control The outer identification loop consists of a recursive least squares identifier RLS_O and a pole assignment controller. There is no need to be restricted to RLS identification or pole positioning algorithms: these algorithms were chosen in the simulations for simplicity. Notice that the control loop can be broken at the point **X** . It may be necessary initially to run the identification algorithms without feedback applied, to allow the estimates $\Theta_{I,k}$, $\Theta_{O,k}$ to get close to their final values.

Since the frequency of the resonant mode is assumed to be initially unknown, a standard pole assignment controller may attempt to move the resonant mode to an assigned pole location far from its initial position. This will result in a large control energy, and possibly destabilize the closed-loop system.

As an alternative to using standard pole assignment, we propose an algorithm in which the location to which the closed loop poles are assigned is

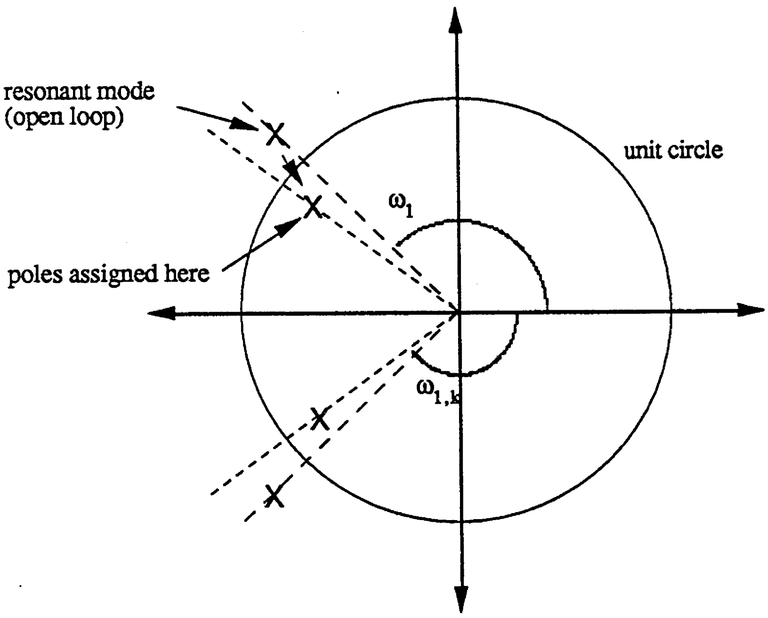

Figure 6.2: Pole assignment with the dominant resonant mode assigned radially inwards

a function of the estimate $\omega_{1,k}$ of the frequency of the resonant mode. An attempt is made to assign the resonant mode radially inwards, so that it has a higher damping than originally, but is still not heavily damped. This is idea is depicted in Fig. 6.2. If $\omega_{1,k}$ is a good estimate of ω_1, then this pole assignment strategy minimizes the distance that the pole is moved, and hence the control energy to achieve more damping.

In the following section, simulation results are discussed, and improvements to the algorithm are suggested.

6.3 A Simulation Study

6.3.1 Preliminary Results with Standard Pole Assignment

In all of the following simulations a time invariant plant $G(q^{-1})$ is used, even though the algorithm is ultimately intended for use when plants have slow time variations. The parameter estimates $\Theta_{I,k}$, $\Theta_{O,k}$ are given arbitrary initial values, and the covariance matrices associated with the RLS estimation are initialized to matrices of the form αI, with α a large real number.

It is also necessary to choose the relative magnitudes of the external excitation s_k, and the noise w_k inherent in the ARMAX model. In most of the following, the magnitude of s_k is ten times the magnitude of $c_1 w_{k-1}$ ($p_O = 1$).

With $n_1 = 4$, and with various different plants and initial conditions, it is observed that $\omega_{1,k}$ is a good estimate of the frequency of the least stable pole ω_1. As one would intuitively expect, increasing n_1 results in $\omega_{1,k}$ being a more accurate estimate: the disadvantage is that the computational effort becomes much greater. As a result of these initial tests, it seemed reasonable to set $n_1, m_1 = 4$ for the remaining simulations. This corresponds to RLS_I identifying a model of the form

$$\frac{b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3} + b_4 q^{-4}}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} + a_4 q^{-4}} \qquad (6.4)$$

The loop filter $L(q^{-1})$ is designed by bilinear transformation of a low pass

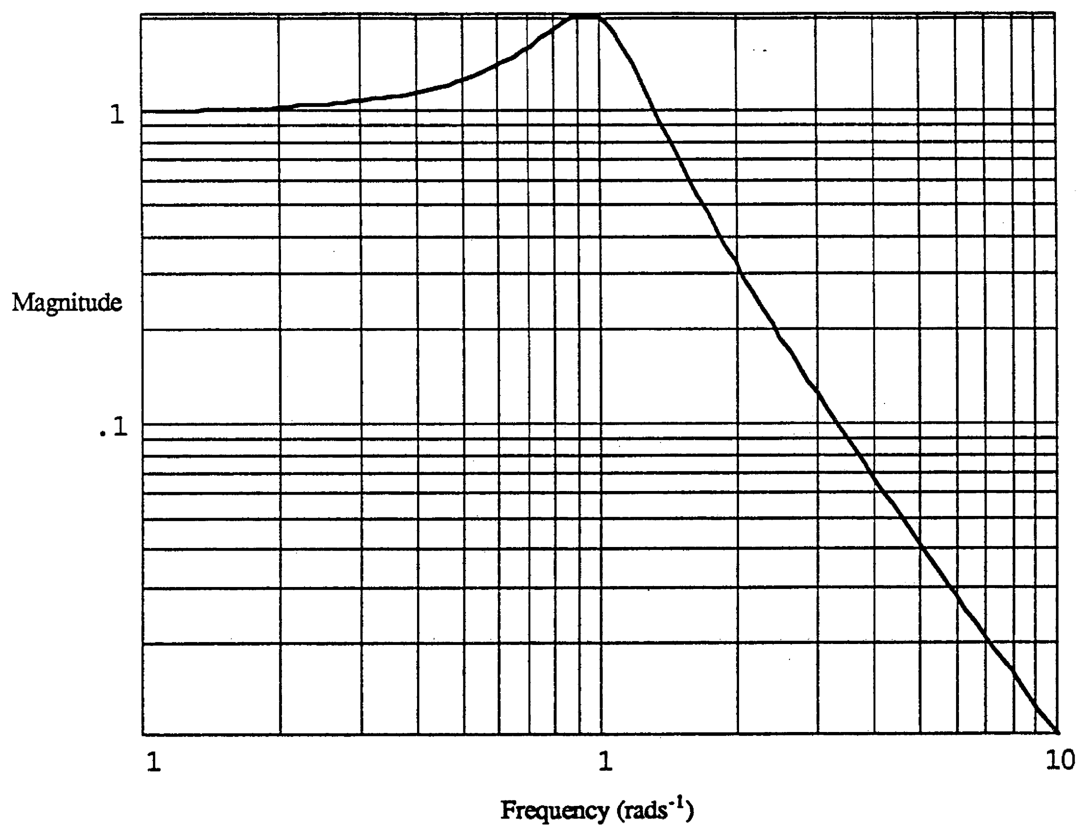

Figure 6.3: Frequency response of second-order lightly damped transfer function

continuous time prototype of the form

$$L(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} \tag{6.5}$$

Figure 6.3 shows the frequency response of such a system. The magnitude of the frequency peak can be increased by reducing the damping coefficient ζ. In the outer loop, a model of the form (6.4) is also identified. In the simulation, the loop is closed at **X** after a fixed number of iterations and trials run with different values of ζ. For simplicity, and to enable comparison with the adaptive pole positioning in the following section, the closed-loop pole assignment is initially made non-adaptive. Two of the closed-loop poles are assigned to $0.5 \pm 0.5j$ and the rest are assigned to the origin. Since maintaining stability is the most important criterion for a successful controller, many simulations must be run, and a check made of how many runs result in closed loop stability.

The results indicate that there is, in general, no advantage in using the loop filter $L(q^{-1})$. There seemed to be little correlation between the damping of $L(q^{-1})$, and hence the corresponding frequency selectivity, and the effectiveness of the controller. It is important to note that for a continuous time plant with a broad spectral response sampled prior to applying adaptive techniques, there is already an in-built pre-filtering which focuses on any frequency band of interest. Our simulations suggest that, in general, there is no merit in a further prefiltering, although there are certainly situations where prefiltering does help.

6.3.2 Closed-loop Poles Adaptively Assigned

We now allow the location to which the closed-loop poles are assigned to be a function of $\omega_{1,k}$. In the simulations, one pair of closed-loop poles is assigned to frequencies of $\pm\omega_{1,k}$ and at a radius of 0.7. The other closed-loop poles are assigned to the origin. The plant models in the simulation have a lightly damped dominant pole with a randomly chosen frequency, and the other poles with random locations close to the origin. The design parameter that

	no. runs	no. successful runs	success rate
No filter	24	18	0.75
$\zeta = 0.6$	11	2	0.18
$\zeta = 0.2$	5	11	0.45

Table 6.1: Simulation results: Adaptive pole positioning

is changed is the damping coefficient ζ associated with the loop filter design. Some simulations are also run with no loop filter: here the only purpose of the inner identifier is to allow the location to which the closed-loop poles are assigned to be adaptive. In all cases, the control algorithm is run in open loop for the first fifty iterations, then closed. The results, based on a small number of simulations, are given in Table 6.1.

Loop Filter The results of Table 3.1 indicate that the use of the loop filter can actually reduce the chance of obtaining a stabilizing controller. One reason for this is that the loop filter introduces additional poles into the control loop, and these must be taken into account in the pole assignment. A refinement of the algorithm is to assign two closed loop pole pairs to a radius of approximately 0.7 and the rest to the origin: one to assign the dominant resonant pole radially inwards, and the second to take into account the filter poles. Simulations show that this assignment of a double pole pair results in a more reliable controller.

In the following, we describe some enhancements to the controller design.

6.3.3 Cautious Control

In the simulations above, the control loop is not closed until after the first fifty iterations. The resulting transient sometimes causes the system to go unstable. To lessen the effect of this startup transient, concepts from Åström's cautious control [4] can be implemented. Indirect adaptive control algorithms employing caution use not only the parameter estimate, but also information about the covariance of the estimate, when designing the control law. In our case, the algorithm is modified by replacing the control signal v_k by a scaled control signal $Q_k v_k$, where Q_k is a nondecreasing positive sequence bounded above by one. Such a sequence can be generated as follows. Given outcomes r_k and a regression vector ϕ_k of past measurements, the RLS algorithm identifies the parameters θ of the best model of the form

$$r_k = \phi_k \theta \tag{6.6}$$

The RLS algorithm also calculates a matrix P_k, which, under certain noise assumptions, can be interpreted as an estimate of the covariance matrix associated with the estimate $\hat{\theta}_k$ of the true system model. We therefore propose Q_k given by (6.7), where k_s is some empirically determined positive constant.

$$Q_k = \frac{1}{1 + k_s/(\theta_k' P_k^{-1} \theta_k)} \tag{6.7}$$

Simulations show that there is little variation in Q_k for different simulation runs. As a result of this Q_k is made independent of θ_k, P_k, and increasing in a linear fashion from zero ($k = 6$) to one ($k = 50$). In general, the simulations show that the use of caution results in a more reliable controller.

6.3.4 Central Tendency Adaptive Pole Assignment

One problem with adaptive pole assignment schemes is that when the estimated plant has a near pole zero cancellation, the resulting controller can produce excessive control signals. This occurs because the controller design must invert a nearly singular Sylvester matrix. The central tendency adaptive control algorithm [37] chooses controller parameters based on a trade-off between the confidence in the estimated plant parameters and ill-conditioning of the Sylvester matrix, as described below.

Suppose we are given an estimate $G_O(q^{-1})$ by RLS$_O$ of the form

$$G_O(q^{-1}) = B(q^{-1})/A(q^{-1}) \tag{6.8}$$

where

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} && (6.9) \\ B(q^{-1}) &= b_1 q^{-1} + \cdots + b_m q^{-m} && (6.10) \end{aligned}$$

For this plant the pole assignment control scheme is

$$\begin{aligned} E(q^{-1})v_k &= -F(q^{-1})r_k && (6.11) \\ A(q^{-1})E(q^{-1}) + B(q^{-1})F(q^{-1}) &= H(q^{-1}) && (6.12) \end{aligned}$$

where

$$
\begin{aligned}
E(q^{-1}) &= 1 + e_1 q^{-1} + \cdots + e_m q^{-m}, \\
F(q^{-1}) &= f_1 q^{-1} + \cdots + f_n q^{-n}, \\
H(q^{-1}) &= 1 + h_1 q^{-1} + \cdots + h_{n+m} q^{-(n+m)}
\end{aligned}
$$

The polynomial equation (6.12) can be written in terms of a Sylvester equation.

$$
S_{AB} \begin{bmatrix} 1 \\ \psi \end{bmatrix} = h, \quad \psi' = [\bar{e}' \, \bar{f}'], \quad h' = [1 \; \bar{h}'] \tag{6.13}
$$

where

$$
\bar{e}' = [e_1 e_2 \dots e_m], \quad \bar{f}' = [f_1 f_2 \dots f_n], \quad \bar{h}' = [h_1 h_2 \dots h_{n+m}]
$$

$$
S_{AB} = \begin{bmatrix} 1 & & & 0 & & & \\ a_1 & 1 & & 0 & 0 & & \\ a_2 & a_1 & \cdot & b_1 & 0 & \cdot & \\ \cdot & \cdot & \cdot & b_2 & b_1 & \cdot & \cdot \\ & & \cdot & \cdot & & & \end{bmatrix}
$$

The dual form of this relation is

$$
S_{EF} \begin{bmatrix} 1 \\ \theta \end{bmatrix} = h, \quad \theta' = [\bar{a}' \, \bar{b}'] \tag{6.14}
$$

With $\mathcal{F}_k$ the σ-algebra generated by measurements up to and including time k, the central tendency pole assignment algorithm chooses the controller ψ which maximizes the probability density $f[\psi(\theta)|\mathcal{F}_{k-1}]$, where the minimization is carried out over all θ. It is shown in [37] that

$$
f[\psi(\theta)|\mathcal{F}_{k-1}] = K_a |\det \; J^{-1}(\theta)| \exp\{-\frac{1}{2}(\theta - \hat{\theta}_k)' P_k^{-1} (\theta - \hat{\theta}_k)\} \tag{6.15}
$$

where

$$|\det\ J| = \left|\frac{\det\ S_{EF}}{\det\ S_{AB}}\right| \tag{6.16}$$

The minimization above is not possible in practice, because it is impossible to evaluate $f[\psi(\theta)|\mathcal{F}_{k-1}]$ at all values of θ. As a compromise, we instead evaluate this expression only over the set of θ for which $\psi(\theta)$ is of necessity evaluated, that is $\hat{\theta}_k, \hat{\theta}_{k-1}, \ldots, \hat{\theta}_{k-M}$ for some M.

For the simulations above based on random plant parameter selection, there is a low probability of introducing near pole-zero cancellations, so that we do not expect any improvement in an average sense as a result of introducing central tendency modifications. However, as shown in [37], non-generic cases can arise where dramatic improvements to performance can be expected.

6.3.5 Transient Performance Simulation Results

Some simulation results are now presented to show the typical transient behaviour of the controller algorithm. The tenth-order plant is randomly chosen with one unstable pole pair at a radius of 1.1, and four other pole pairs randomly distributed inside a circle of radius 0.7 centred at the origin. The frequency of all of the plant poles is uniformly distributed on $[0, \pi)$, and the radius of the stable plant poles is uniformly distributed on $[0, 0.7)$. It is suggested here that the above class of randomly selected plants be used as a benchmark, enabling comparison of our resonance suppression algorithm with those of other authors. The identifiers RLS$_{\rm I}$ and RLS$_{\rm O}$ identify fourth-order ARMA models. Cautious control and central tendency concepts (with $M = 5$) are used as described in Sec. 6.3.

One pole pair is assigned to a radius of 0.7 and a frequency given by $\omega_{1,k}$. The control loop does not include a loop filter $L(z)$. Figure 6.4 shows the estimation of the frequency of the least stable pole pair; the actual frequency based on the true plant parameters is also marked on the graph. The parameter estimates given by RLS$_{\rm O}$ are shown in Fig. 6.5. The plant output y_k increases initially in an unstable manner until the control system learns the plant parameters, after which time y_k settles down again.

6.4 Preconditioning Methods

The methods proposed here have largely been motivated for use in conjunction with other control methods. As an example, consider the indirect adaptive techniques of [42], based on the theory on the class of all stabilizing controllers [46, 12]. In this work, the real plant is embedded in a control loop, as in Fig. 6.7. The design of J_K is based on a control system with a nominal plant G_0 and a stabilizing controller K_0. With stable proper coprime factorizations

$$G_0 = N_0 M_0^{-1} = \tilde{M}_0^{-1} \tilde{N}_0 \qquad (6.17)$$

$$K_0 = U_0 V_0^{-1} = \tilde{V}_0^{-1} \tilde{U}_0 \qquad (6.18)$$

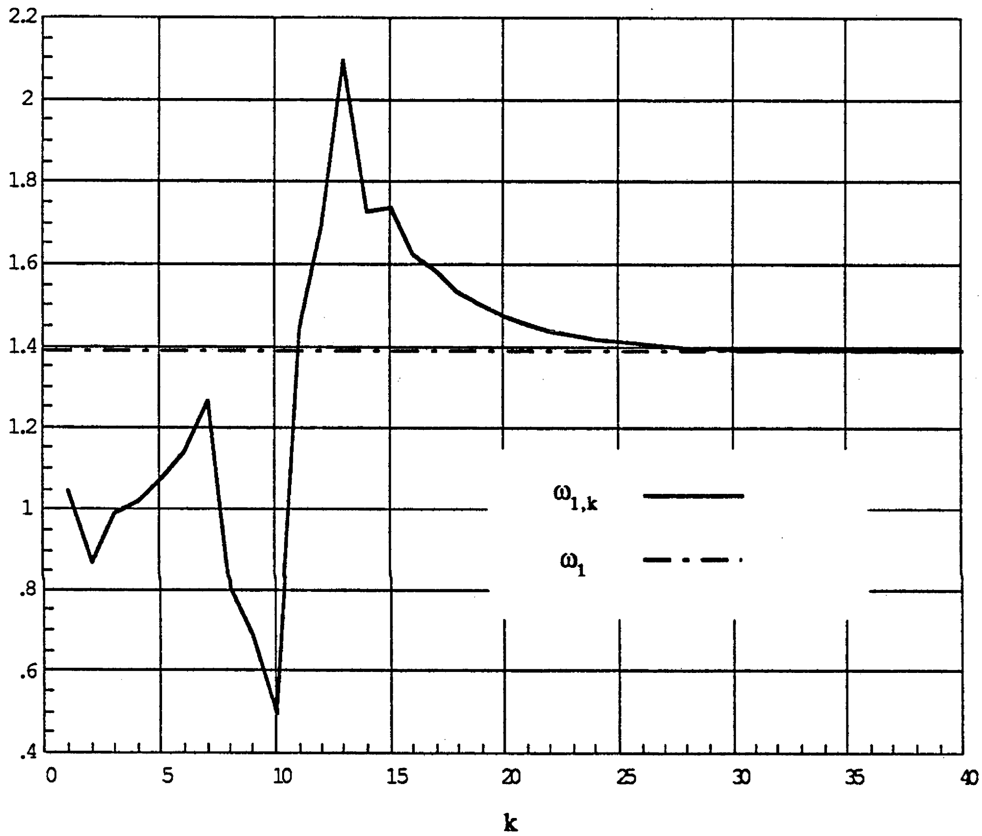

Figure 6.4: Estimate of the frequency ω_1 of the least stable pole pair

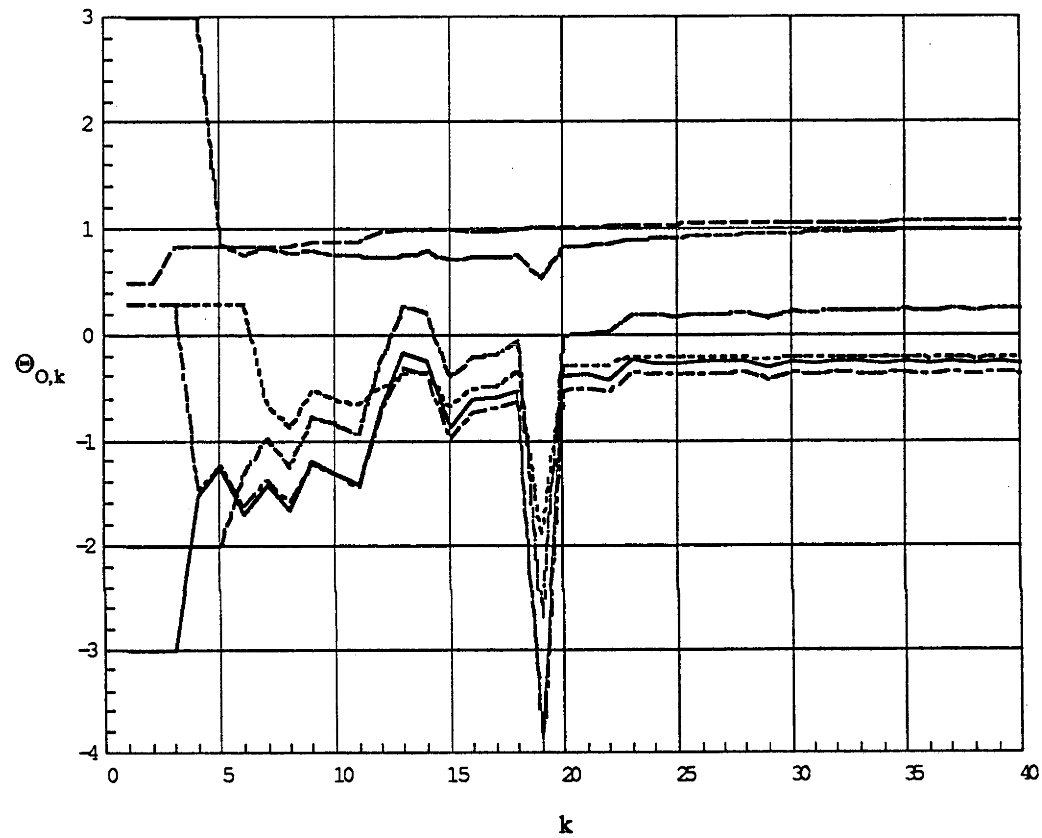

Figure 6.5: Estimate $\Theta_{O,k}$ generated by outer identifier(RLS$_O$)

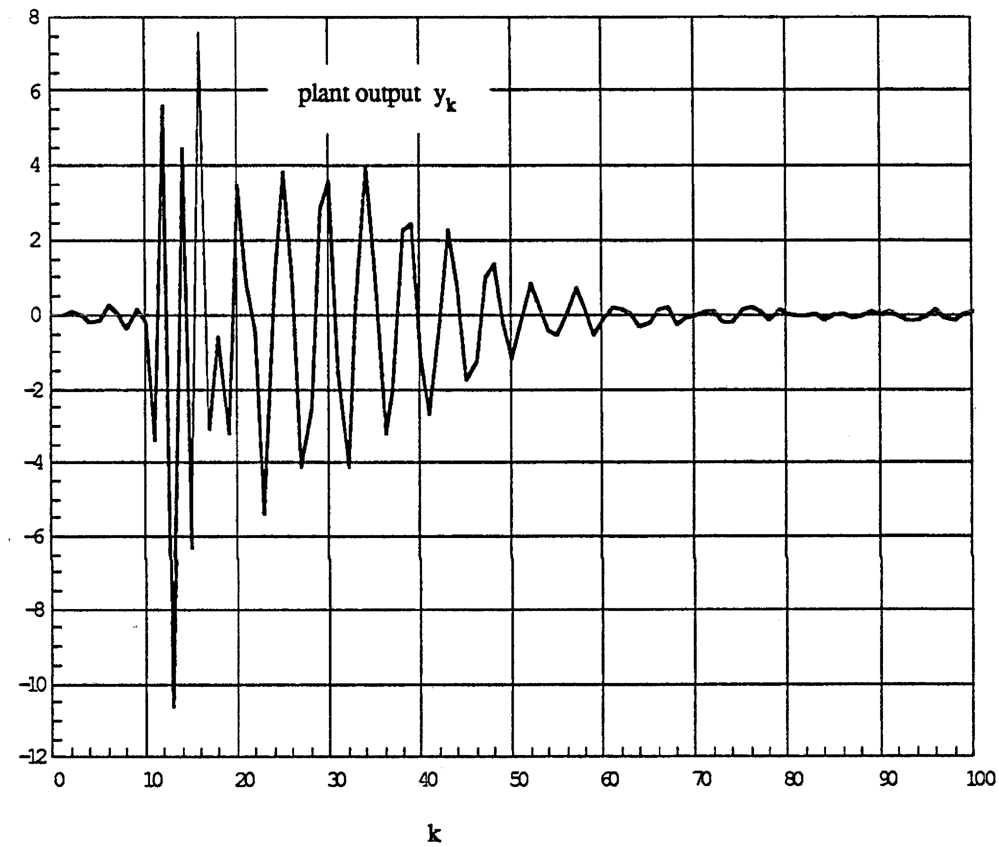

Figure 6.6: Plant output

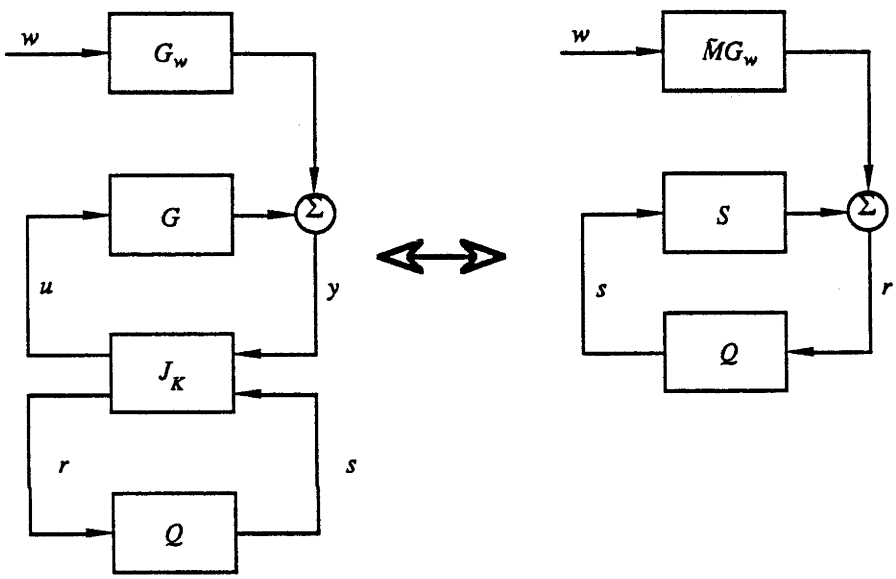

Figure 6.7: Plant/noise model

then

$$J_K = \begin{bmatrix} K_0 & \tilde{V}_0^{-1} \\ V_0^{-1} & -V_0^{-1}N_0 \end{bmatrix} \tag{6.19}$$

With $Q = 0$, J_K forms a stabilizing controller for the nominal plant; such a controller can be designed to achieve specific performance or robustness objectives for the nominal plant. The block Q represents an additional adaptive feedback path over that of the nominal controller. In fact, the nominal plant G_0 is stabilized if and only if Q itself is stable. Furthermore, as Q spans the class of all stable transfer functions, the controller class

$$K(Q) = K_0 + \tilde{V}_0^{-1}Q(I + V_0^{-1}N_0Q)^{-1}V_0^{-1} \tag{6.20}$$

is the class of all stabilizing controllers for G_0.

One result in [42] is that G will be stabilized if and only if Q stabilizes S, where

$$S = \tilde{M}(G - G_0)M_0 \tag{6.21}$$

Here $\tilde{M}, M_0$ provide a natural frequency weighting for $(G - G_0)$ in the frequency bands of interest.

We now consider the problem of finding a suitable adaptive Q to stabilize the augmented plant S. Since S emphasizes frequencies in the passband of the closed loop system (G, K_0), it follows that S may often be a transfer function with a dominant resonant mode. This is an ideal opportunity to utilize the resonance suppression techniques of Sec. 6.2. The proposed scheme is shown in Fig. 6.8.

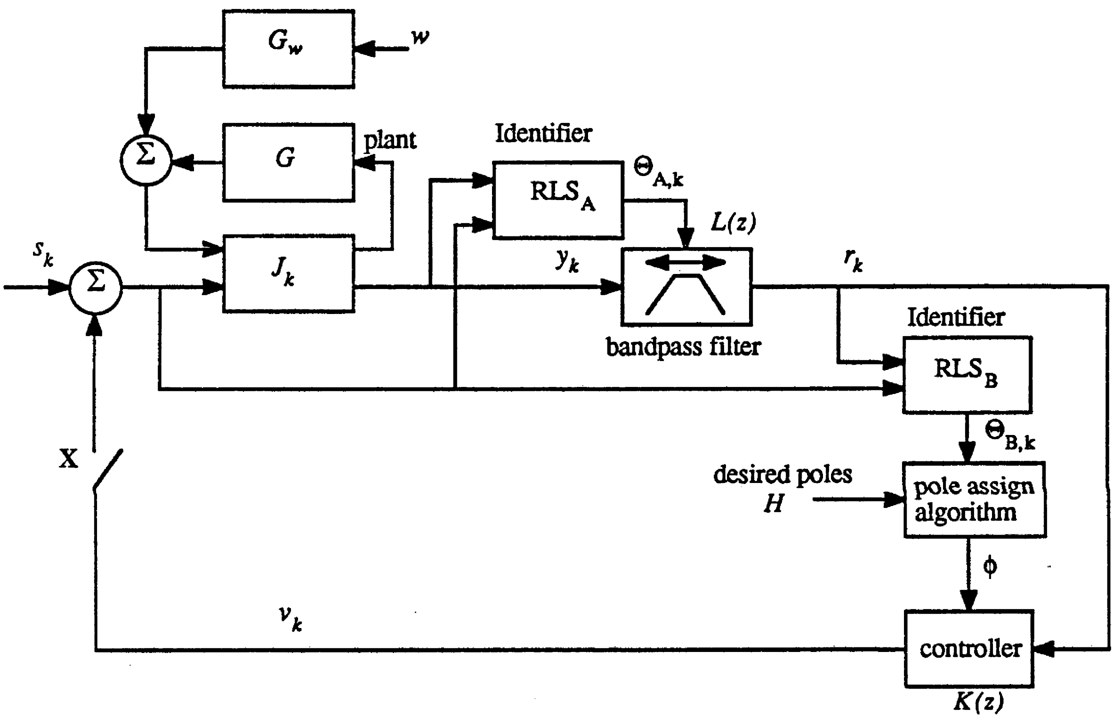

Figure 6.8: Adaptive scheme to enhance fixed controller design

6.5 Conclusions

We present here a preliminary investigation into the problem of designing an adaptive controller when there is *a priori* knowledge that the plant has one dominant lightly damped, or possibly unstable, mode. Such a problem could be tackled without making any use of the knowledge that a dominant mode exists, as is the case when standard adaptive control schemes of are applied.

The algorithm is Sec. 6.2 makes use of an inner underparameterized identification of the plant, which enables fast estimation of the frequency of the resonant mode. This estimate can be used to adjust filters in the control loop, or even to adjust the position to which the closed loop poles are assigned. The latter possibility seems to be particularly attractive, as it prevents a situation where the controller tries to change the frequency of the of the dominant resonant mode by arbitrary pole assignment. The controller can instead simply apply feedback to increase the damping of the resonant mode. In practice, techniques such as central tendency adaptive control [37] and cautious control [4] can be used to improve the robustness of the algorithm.

The problem which is studied is one that we believe is important and that arises in many engineering situations. It is not shown that our algorithm is a universal or optimal resonance suppression algorithm, but instead that *ad hoc* modifications to existing adaptive control algorithms can improve their reliability.

Chapter 7

Conclusions and Further Research

7.1 Conclusions

This thesis has developed new theory and algorithms for multivariable controller design. A basic assumption throughout much of the work is that the input-output behaviour of the plant is linear and time-invariant, and can thus be represented by factors of stable proper transfer functions. One consequence of having an underlying axiomatic framework is that the theory may be extended to more general settings simply by choosing different definitions for the ring of plant transfer functions (rather than R_p), and the subring of *stable* transfer functions (rather that RH_∞). In [11, 7] it is noted that the results may also be applied to time-varying and distributed systems. A simple variation of the theory of this thesis is to consider the case, as shown in Fig. 7.1, where the stability region is defined to be some subset of the complex left half-plane. A design procedure based on factorizations with poles

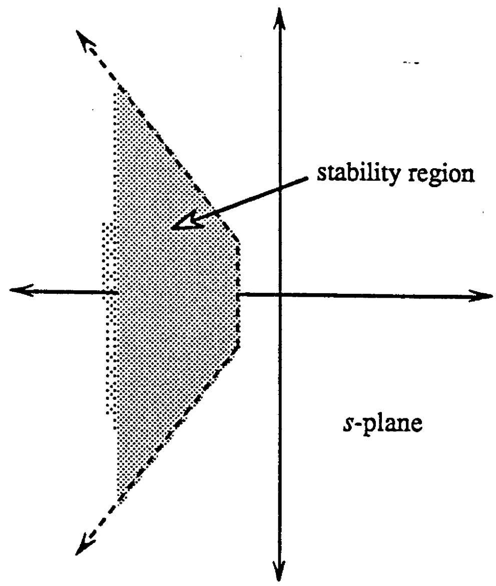

Figure 7.1: Alternative stability region

in this stability region will result in a closed-loop system with guaranteed bounds on the overshoot and settling time [46]. We now summarize some of the important contributions of this thesis:

- There have been theoretical contributions in the area of doubly coprime factorizations; these are right and left coprime factorizations of the plant and controller satisfying double Bezout identities. State space realizations of doubly coprime factorizations are given by Nett, Jacobson, and Balas [39] based on the theory of full-order state estimator. In Chap. 4 new doubly coprime factorizations are derived related to reduced-order observers. These factorizations are important both from a computational and a theoretical point of view. For computation it is convenient to be able to describe transfer functions in state-space form. The theoretical results on observer-based controllers with dynamic state feedback could not have been derived without the use of the new factorizations.

- A second contribution to the theory of doubly coprime factorizations is to generalize the state-space realizations to allow for possibly unstable dynamic estimator and state feedback gains. This nontrivial generalization is achieved both for the full-order and the reduced-order state estimators.

- Furthermore it is shown that any stabilizing controller for a given plant can be structured as a state estimate feedback controller, with the dynamics in either the state estimator or in the state feedback law. Conditions for closed loop stability when the state estimate feedback controller has dynamic state estimate and state estimate feedback gains are given.

- Finally, it is recalled that an arbitrary controller can be organized as a state estimate feedback controller with constant state estimate and state estimate feedback gains if and only if a nonsingular solution of a particular nonsymmetric Riccati equation exists. An example is given to show that plant/controller pairs with certain structural properties can not be reorganized as such. Necessary controllability and observability conditions for the existence of a solution are given; sufficient conditions are as yet not available.

Two new design algorithms are also presented: the first deals with the controller reduction problem, and the second with adaptive resonance suppression.

- The controller reduction problem is tackled by applying standard model reduction algorithms to augmentations of the controller which arise when characterizing the class of all stabilizing controllers, and it is claimed that the method preserves robustness and performance qualities of the controller. The method is especially well suited to state estimate feedback controllers, and specializes to other methods [24] when appropriate scaling is used. Issues such as scaling of input/output variables, maintaining controller performance, and simultaneous stabilization of a class of plants are discussed.

- The adaptive resonance suppression algorithm presented here is intended for use when a plant has an unknown unstable or resonant mode. The algorithm attempts quickly to identify the resonant mode, and to apply a control signal which dampens the resonance. The structure of the algorithm has evolved from previous research results, such as [8], and appears sensible from an engineering point of view. It is certainly not reasonable to expect an elegant analytical analysis of the performance of the controller. A simulation study shows that the algorithm performs well in certain situations, and that the underparameterized inner identification loop does effectively identify the frequency of the resonant mode.

- One important by-product of this work is the proposal in Sec. 6.3.5 of a benchmark enabling comparison of resonance suppression schemes. This benchmark specifies a class of unstable discrete-time plant models, along with the distribution associated with the random plant selection process.

7.2 Further research

We now note some possibilities for further research which arise from results developed in this thesis:

- In the area of doubly-coprime factorizations, Wang and Balas [47] have recently presented an extension of earlier results [39] which provides explicit formulae for doubly coprime factorizations of the transfer function of a *generalized dynamical system*, such as

$$E\dot{x} = Ax + Bu \tag{7.1}$$

$$y = Cx + D \tag{7.2}$$

  Here E is a possibly singular matrix; if E were non-singular, then the system could be replaced by an equivalent system with E replaced by the identity. Further work could include extending the results of this thesis to cover generalized dynamical systems.

- Recall from Lemma 2.4 that given a minimum-phase state estimate feedback gain F, that stabilizes G_F, then any stabilizing controller K for G can be obtained as an equivalent state estimate feedback controller $K[F, H(Q_H)]$ for some $Q_H \in RH_\infty$. More simply stated, subject to the minimum phase condition on F, which stabilizes G_F, any stabilizing controller K for G can be obtained as a state estimate feedback controller $K[F, H]$ for some H which stabilizes G_H. This simple result requires a complicated proof, perhaps indicating that a simpler and more insightful proof could be found.

- It would also be interesting to develop connections between the loop transfer recovery techniques of Doyle and Stein [5, 13] and the frequency shaped state estimate feedback controllers of this thesis. The loop transfer recovery method generates a controller whose closed loop behaviour arbitrarily closely approximates a given *target feedback loop* subject to certain minimum phase requirements. The design method exploits the solution of the linear quadratic regulator problem, and uses a LQG-based controller structure. Perhaps there exists a generalization of existing LTR/LQG method based on controllers with dynamic state estimate and state estimate feedback gains.

- Chapter 3 gives a simple example of a plant/controller pair which can not be organized as a state estimate feedback controller. Further work could define classes of plant/controller pairs which can not be organized as such. The sufficient conditions of Medanic [30], which apply to solutions of the Riccati equation based on only *one* admissible eigenvector selection, might allow such classes to be specified.

- The preconditioning methods for the resonance suppression algorithm of Chap. 6 have not as yet been investigated. This would require more simulation trials, perhaps with actual aircraft flutter models. It would be useful to compare the simulations results here with corresponding results for LQG based adaptive schemes.

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