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Containment Control of Singular Heterogeneous Multi-agent Systems

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Abstract

In this paper, we first consider the containment control problem of singular heterogeneous multi-agent systems, where all the followers converge to the convex hull spanned by the leaders. To solve this problem, we propose two distributed control laws: one is based on the state feedback control framework, which is suitable for the case that the full state information of each follower is accessible; and the other is based on the output regulation framework, where each follower only can access to its output. Furthermore, the distributed observers are designed for every follower to estimate the convex combination of the leader states which is determined by the communication graph. It should be noted that our results can also regard the non-singular multi-agent systems' containment control problem as a special case. Finally, simulation results corroborate the effectiveness of our analytical results.

Keywords: Containment control, singular system, distributed observer, output regulation, heterogeneous multi-agent systems

1. Introduction

In recent years, large amounts of cooperative problems of multi-agent systems have been extensively studied (see [1]-[25]). These studies can be classified into leaderless consensus problem (e.g., [8, 9, 13]), leader-following consensus problem (e.g., [6, 22, 23]), and containment control problem (e.g., [7, 25]). Unlike the leaderless/leader-following consensus problem that all the agents/followers achieve a common state, the containment control problem requires all the followers converge to a convex hull formed by multiple leaders. Also, we can regard the leader-following consensus problem as a special case of the containment control problem, since the convex hull of one leader's state is exactly the state itself.

Because of the multiple leaders' setting, studying the containment control problem becomes more difficult which attracts more recent attentions than the leader-following consensus problem. Reference [26] investigated the problem of distributed containment control of a group of mobile autonomous agents with multiple stationary or dynamic leaders under both fixed and switching directed network topologies. In [27],

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the distributed containment control problem was studied for a group of autonomous vehicles modeled by double-integrator dynamics with multiple dynamic leaders. Reference [28] investigated containment control problems for networked multi-agent systems with multiple stationary or dynamic leaders. Reference [29] considered the distributed containment control for a second-order multi-agent system guided by multiple leaders with random switching topologies. In [30], the containment control problems were investigated for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies. Reference [31] studied the containment control problem for multi-agent systems consisting of multiple leaders and followers connected as a network. Reference [32] considered the containment control problem for a group of autonomous agents modeled by heterogeneous dynamics. In [33], containment control problems were investigated for linear multi-agent systems with input saturation on switching topologies. Reference [34] studied the containment control problem of heterogeneous linear multi-agent systems based on output regulation framework. Reference [35] investigated the containment control problems over directed random graphs. However, to the best of our knowledge, the containment control problem for singular heterogeneous multi-agent systems has not been investigated.

Note that the singular system, also called the descriptor system [36], is a very important class of control systems. It can describe more general physical systems than standard (non-singular) linear differential equations. When an agent in multi-agent systems contains some algebraic constraints in its states, the multi-agent systems cannot be described by non-singular models. Some typical examples are the three-link planar manipulator networks [37], multiple UAVs moving in some subspaces due to mission requirements, etc. Therefore, it is necessary to study the analysis and synthesis of the singular multi-agent systems. Since the late 1970s, singular systems have attracted the attention of many researchers (see [36, 38, 39, 40]). Reference [41] investigated the problem of output regulation for linear singular systems when the measurement output is the same as the vector to be regulated. Reference [42] considers the problems of robust stability and stabilization for uncertain continuous singular systems with state delay. Reference [43] presented generalized Lyapunov equation for singular systems. Reference [37] proposed necessary and sufficient conditions of consensusability with respect to a set of admissible consensus protocols using the tools of algebra, graph and descriptor linear system theory. Reference [45] presented the swarm stability problem of high-order linear time-invariant (LTI) singular multi-agent systems. Reference [46] investigated Guaranteed-cost consensus problems for high-order singular multi-agent systems with switching topologies. Reference [47] considered the containment problem of heterogeneous linear multi-agent systems with directed graphs via dynamic compensator techniques. Reference [48] studied containment analysis and design problems for high-order linear time-invariant singular multi-agent systems on directed graphs with time delays. However the agent's system is identical. Reference [49] studied containment analysis and design problems for general high-order linear time-invariant multi-agent systems with time-varying delays. Reference [50] investigated the leader-following consensus problem of singular heterogeneous multi-agent systems with output feedback control, where each follower must have a directed path from the leader. We can see that the singular multi-agent systems are

meaningful and interesting to consider, and studying the containment control problem for singular heterogeneous multi-agent systems is important.

The main contributions of this work are: We first solve the containment control problem of singular heterogeneous swarms system by the distributed feedback control. To be more specific, the distributed observer is designed for each follower to estimate the convex combination of the leader states which is determined by the communication graph. Then, we strictly prove the convergence of the distributed observer. Based on the convergent distributed observers, the containment control becomes a tracking control problem. Thus, we design the distributed dynamic state feedback control law and the distributed output feedback control law to track the distributed observers' states for two different cases that: 1) the full state information is accessible to each follower; and 2) the output information is available to each follower. Then, the convergence of the containment control problem is rigorously proved if a sufficient condition for the control gains holds. Finally, we propose a method to design the control gains for our proposed two controllers, and this method can always guarantee the control gains satisfying the sufficient condition. This means our methods successfully solve the containment control problem for singular heterogeneous multi-agent systems.

The rest of this paper is organized as follows. In Section 2, the singular heterogeneous multi-agent systems with multiple leaders are modeled and its containment control problem is formulated. To solve this containment control problem, we propose two distributed control laws in Section 3. In Section 4, simulations are presented to illustrate the effectiveness of our designed control laws.

Notation: For $E, A \in \mathbb{R}^{n \times n}$, we denote $\sigma(E, A) = \{\lambda: \det(\lambda E - A) = 0\}$ as the generalized spectrum of matrix A w.r.t. matrix E . If E is the identity matrix, then $\sigma(E, A) = \sigma(A)$ is the spectrum of matrix A . We define $\mathbb{C}^- = \{\lambda \in \mathbb{C}: \text{Re}(\lambda) < 0\}$, and pair (E, A) is said to be stable if $\sigma(E, A) \subset \mathbb{C}^-$. For $B \in \mathbb{R}^{n \times m}$, triplet (E, A, B) is stabilizable if there exists K such that $(E, A + BK)$ is stable. For $C \in \mathbb{R}^{d \times n}$, (E, A, C) is detectable if there exists L such that $(E, A + LC)$ is stable. Pair (E, A) is said to be regular if $\det(sE - A) \neq 0$. Pair (E, A) is said to be impulse free if (E, A) is regular and $\deg(\det(sE - A)) = \text{rank}(E)$, where $\deg(\cdot)$ returns the degree of a given polynomial. $\text{Re}(\cdot)$ returns the real part of a given complex number.

2. System Model and Problem Formulation

Consider the multi-agent systems of N followers and M leaders, where agents $1, \dots, N$ are followers, and agents $N+1, \dots, N+M$ are leaders. Each follower $i \in \{1, \dots, N\}$ is governed by the singular linear system as follows:

$$E_i \dot{x}_i = A_i x_i + B_i u_i, \quad (1a)$$

$$y_i = C_i x_i, \quad (1b)$$

where $E_i, A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ (B_i has full column rank), and $C_i \in \mathbb{R}^{d \times n}$. In (1), $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^d$, and $u_i \in \mathbb{R}^m$ are the state, output, and control input of the i^{th} follower, respectively. Mathematically speaking,

system (1) is singular, if $E_i \in \mathbb{R}^{n \times n}$ is with $\text{rank}(E_i) < n$. In our work, we assume that there is at least one follower $i \in \{1, \dots, N\}$ such that $\text{rank}(E_i) < n$.

The leaders are described by the following linear systems:

$$\dot{w}_k = Sw_k, \quad k = N+1, \dots, N+M, \quad (2)$$

where $w_k \in \mathbb{R}^n$ is the exogenous state of the $(k-N)^{\text{th}}$ leader.

Define a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ whose vertex set is $\mathcal{V} = \{1, 2, \dots, N+M\}$, and the edge set is $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge from node i to node j is denoted by (i, j) , and node i is called the neighbor of node j . If the digraph $\mathcal{G}$ contains a sequence of edges with the form $(i_1, j_2), (i_2, j_3), \dots, (i_k, j_{k+1})$, then set $\{(i_1, j_2), (i_2, j_3), \dots, (i_k, j_{k+1})\}$ is called a directed path of $\mathcal{G}$ from i_1 to i_{k+1} , and node i_{k+1} is said to be reachable from node i_1 . A digraph is said to contain a spanning tree if there exists a node i such that any other node is reachable from this node which is called the root of the spanning tree. We use $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{(N+M) \times (N+M)}$ to denote the adjacent matrix of graph $\mathcal{G}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Let $\mathcal{L}$ be the Laplacian of $\mathcal{G}$ corresponding to $\mathcal{A}$. Since any leader does not receive information from other agents (including other leaders and the followers), the Laplacian matrix of $\mathcal{G}$ can be partitioned as follows (see also [26])

$$\begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ 0_{M \times N} & 0_{M \times M} \end{bmatrix}. \quad (3)$$

Assumption 1. For each follower, there exists at least one leader having a directed path to the follower.

Remark 1. Assumption 1 guarantees that the containment control problem is well posed. This is because without this assumption, there will be at least one isolated follower that cannot have obtain any information from any leader.

Definition 1. (Convex Set and Convex Hull [53]) A set $\mathcal{C} \subseteq \mathbb{R}^N$ is convex if $(1-\lambda)x + \lambda y \in \mathcal{C}$, for any $x, y \in \mathcal{C}$ and any $\lambda \in [0, 1]$. The convex hull $\text{Co}(\mathcal{W})$ of a finite set of points $\mathcal{W} = \{w_1, w_2, \dots, w_q\}$ is the minimal convex set containing all points in $\mathcal{W}$. That is, $\text{Co}(\mathcal{W}) = \left\{ \sum_{i=1}^q \alpha_i w_i : w_i \in \mathcal{W}, \alpha_i \geq 0, \sum_{i=1}^q \alpha_i = 1 \right\}$.

Definition 2. (Containment Control) The singular heterogeneous systems described by (1)-(2) achieve containment if designed control law for each follower can make sure that all followers converge to the convex hull spanned by the dynamic leaders as $t \rightarrow \infty$. That is $\forall i \in \{1, \dots, N\}$, there exists a $\bar{w}_i \in \text{Co}(\mathcal{W})$, where $\mathcal{W} = \{w_{N+1}, w_{N+2}, \dots, w_{N+M}\}$, such that $\lim_{t \rightarrow \infty} \|x_i - \bar{w}_i\| = 0$.

The error vector of the i^{th} follower is defined as

$$e_i = \sum_{j=1}^N a_{ij}(x_i - x_j) + \sum_{k=N+1}^{N+M} a_{ik}(x_i - w_k), \quad (4)$$

Setting $e = [e_1^T, \dots, e_N^T]^T$, $x = [x_1^T, \dots, x_N^T]^T$, and $w = [w_{N+1}^T, \dots, w_{N+M}^T]^T$, we rewrite (4) as

$$e = (\mathcal{L}_1 \otimes I_n)x + (\mathcal{L}_2 \otimes I_n)w = (\mathcal{L}_1 \otimes I_n)[x + (\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_n)w], \quad (5)$$

where I_n is the n -by- n identity matrix.

Lemma 1. ([26]) Under Assumption 1, all the eigenvalues of $\mathcal{L}_1$ have positive real parts, each entry of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ is non-negative, and each row of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has a sum equal to 1.

Remark 2. From (5) and Lemma 1, we know that the multi-agent systems achieve containment if $\lim_{t \rightarrow \infty} e = 0$. This is because: With (5), we get

$$\lim_{t \rightarrow \infty} e = 0 \Rightarrow \lim_{t \rightarrow \infty} x = -(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_n) w.$$

By Lemma 1, $\lim_{t \rightarrow \infty} x_i$ is a convex combination of the leaders' states $w_{N+1}, \dots, w_{N+M}$ for $i = 1, \dots, N$, which means each follower converges to the convex hull formed by the leader.

Assumption 2. (E_i, A_i) is regular, and (E_i, A_i, B_i) is stabilizable, and (E_i, A_i, C_i) is detectable, for $i \in \{1, \dots, N\}$.

Remark 3. Similar to Assumption 1, Assumption 2 also ensures the containment control problem to be well posed. For the regularity of (E_i, A_i) , it guarantees the existence and uniqueness of the solution of the system equations of the followers. The stabilizable (E_i, A_i, B_i) and the detectable (E_i, A_i, C_i) make sure the existence of the distributed containment control law and the local observer.

Assumption 3. The linear matrix equation

$$E_i S = A_i + B_i U_i \quad (6)$$

has at least one solution U_i , for $i = 1, \dots, N$.

Remark 4. Under Assumption 3, there exist matrices U_i such that $E_i S = A_i + B_i U_i$ which implies $E(I_N \otimes S) = A + BU$, where $E = \text{diag}\{E_1, \dots, E_N\}$, $A = \text{diag}\{A_1, \dots, A_N\}$, $B = \text{diag}\{B_1, \dots, B_N\}$, $U = \text{diag}\{U_1, \dots, U_N\}$.

Lemma 2. ([36]) For $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, triplet (E, A, B) is R -controllable if and only if $\text{rank}([sE - A, B]) = n$, $\forall s \in \mathbb{C}$.

Lemma 3. ([44]) For $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, suppose that (E, A) is regular and impulse free, and (E, A, B) is R -controllable. Then, for any given positive definite matrices Q and R , there exists a positive definite P satisfying the following generalized Riccati equation:

$$A^T P E + E^T P A + E^T Q E - E^T P B R^{-1} B^T P E = 0. \quad (7)$$

3. Distributed Control Laws

We propose two distributed control laws as follows:

1. Distributed dynamic state feedback controller,

$$u_i = K_{1i} x_i + K_{2i} \eta_i \quad (8a)$$

$$\dot{\eta}_i = S \eta_i + \mu \left[\sum_{j=1}^N a_{ij} (\eta_j - \eta_i) + \sum_{k=N+1}^{N+M} a_{ik} (w_k - \eta_i) \right] \quad (8b)$$

2. Distributed dynamic output feedback controller,

$$u_i = K_{1i}\xi_i + K_{2i}\eta_i \quad (9a)$$

$$\dot{\eta}_i = S\eta_i + \mu \left[\sum_{j=1}^N a_{ij}(\eta_j - \eta_i) + \sum_{k=N+1}^{N+M} a_{ik}(w_k - \eta_i) \right] \quad (9b)$$

$$E_i \dot{\xi}_i = A_i \xi_i + B_i u_i + L_i (C_i \xi_i - y_i) \quad (9c)$$

Note that the i^{th} subsystem can access the k^{th} leader if and only if $a_{ik} > 0$. In our design, for any K_{1i} , K_{2i} is given by

$$K_{2i} = U_i - K_{1i}, \quad (10)$$

with U_i satisfying the matrix equation (6).

In the proposed two distributed controllers, (8b) and (9b) are called the distributed observers, where μ is the observer gain.

Remark 5. In our proposed two distributed control laws, the distributed observers (8b) and (9b) are designed to estimate convex combinations of the leader states. The states of distributed observers, i.e., η_i , are exchanged between neighbors, which is different from traditional framework that the state of agents are exchanged. By using the distributed observer, the containment control is decomposed to two parts: observing and tracking. In the observing part, we need to make sure that η_i is a convex combination of the leader states. In the tracking part, each follower just needs to track η_i .

Remark 6. By setting $\eta = [\eta_1^T, \dots, \eta_N^T]^T$, $w = [w_{N+1}^T, \dots, w_{N+M}^T]^T$, equation (8b) or (9b) can be rewritten as

$$\dot{\eta} = [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \eta - \mu(\mathcal{L}_2 \otimes I_n) w. \quad (11)$$

Let $\tilde{w} = \eta + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w$, equation (11) can be transformed into

$$\begin{aligned} \dot{\tilde{w}} &= \dot{\eta} + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) \dot{w} \\ &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \eta - \mu(\mathcal{L}_2 \otimes I_n) w + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) (I_M \otimes S) w \\ &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] [\tilde{w} - (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w] - \mu(\mathcal{L}_2 \otimes I_n) w + (I_N \otimes S) (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w \\ &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \tilde{w}. \end{aligned} \quad (12)$$

The eigenvalues of $(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)$ are from set $\{\lambda: \lambda = \lambda_i(S) - \mu\lambda_j(\mathcal{L}_1), i, j \in \{1, \dots, N\}\}$, and we label the eigenvalue of S with the largest real part as $\lambda_{\text{remax}}(S)$ and the eigenvalue of $\mathcal{L}_1$ with the smallest real part as $\lambda_{\text{remin}}(\mathcal{L}_1)$. If $\mu > \text{Re}(\lambda_{\text{remax}}(S)) / \text{Re}(\lambda_{\text{remin}}(\mathcal{L}_1))$, then all the eigenvalues of $(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)$ have negative real parts. In that case, $\eta \rightarrow -(\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w$ as $t \rightarrow \infty$, which means each distributed observer η_i asymptotically converges to a convex combination of leaders' states.

Theorem 1. Let K_{1i} satisfy $\sigma(E_i, A_i + B_i K_{1i}) \subset \mathbb{C}^-$, and set $K_{2i} = -K_{1i} + U_i$, $i = 1, \dots, N$. Then the multi-agent systems can achieve containment by feedback control law (8) if the observer gain satisfies $\mu > \text{Re}(\lambda_{\text{remax}}(S)) / \text{Re}(\lambda_{\text{remin}}(\mathcal{L}_1))$.

Proof. Under the control law (8), we write the closed-loop system and the error vector of agent i altogether as follows

$$\begin{aligned} E_i \dot{x}_i &= A_i x_i + B_i [K_{1i} x_i + (-K_{1i} + U_i) \eta_i] \\ \dot{\eta}_i &= S \eta_i + \mu \left[\sum_{j=1}^N a_{ij} (\eta_j - \eta_i) + \sum_{k=N+1}^{N+M} a_{ik} (w_k - \eta_i) \right] \\ \dot{w}_k &= S w_k, \quad k = N+1, \dots, M+N \\ e_i &= \sum_{j=1}^N a_{ij} (x_i - x_j) + \sum_{k=N+1}^{N+M} a_{ik} (x_i - w_k), \quad i = 1, \dots, N \end{aligned} \quad (13)$$

Let $x = [x_1^T, \dots, x_N^T]^T$, $\eta = [\eta_1^T, \dots, \eta_N^T]^T$, $e = [e_1^T, \dots, e_N^T]^T$, $u = [u_1^T, \dots, u_N^T]^T$, $w = [w_{N+1}^T, \dots, w_{N+M}^T]^T$, $E = \text{diag}\{E_1, \dots, E_N\}$, $A = \text{diag}\{A_1, \dots, A_N\}$, $B = \text{diag}\{B_1, \dots, B_N\}$, $U = \text{diag}\{U_1, \dots, U_N\}$, $K_1 = \text{diag}\{K_{11}, \dots, K_{1N}\}$, and then we can define an augmented system as follows

$$\begin{aligned} E \dot{x} &= A x + B [K_1 x + (-K_1 + U) \eta] \\ \dot{\eta} &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \eta - \mu(\mathcal{L}_2 \otimes I_n) w \\ \dot{w} &= (I_M \otimes S) w \\ e &= (\mathcal{L}_1 \otimes I_n) x + (\mathcal{L}_2 \otimes I_n) w, \end{aligned} \quad (14)$$

where $K_1 = \text{diag}\{K_{11}, \dots, K_{1N}\}$ is a state feedback gain matrix. We choose K_{1i} in such a way that $\sigma(E_i, A_i + B_i K_{1i}) \subset \mathbb{C}^-$, for $i = 1, \dots, N$, which is equivalent to $\sigma(E, A + B K_1) \subset \mathbb{C}^-$. By Lemma 1 and by adopting the invertible transforms $\bar{\xi} = x + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w$ and $\tilde{w} = \eta + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w$, equation (14) can be rewritten as

$$\begin{aligned} E \dot{\bar{\xi}} &= (A + B K_1) \bar{\xi} + [-A - B U + E(I_N \otimes S)] (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w + B(-K_1 + U) \tilde{w} \\ \dot{\tilde{w}} &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \tilde{w} \\ e &= (\mathcal{L}_1 \otimes I_n) [\bar{\xi} + (I_{Nn} - I_N \otimes I_n) (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w]. \end{aligned} \quad (15)$$

With Remark 4, we can further rewrite it as

$$\begin{aligned} \hat{E} \dot{\varrho} &= \hat{A} \varrho \\ e &= (\mathcal{L}_1 \otimes I_n) \bar{\xi} \end{aligned} \quad (16)$$

where $\varrho = \begin{bmatrix} \bar{\xi} \\ \tilde{w} \end{bmatrix}$, $\hat{E} = \begin{bmatrix} E & \\ & I_{Nn} \end{bmatrix}$ and $\hat{A} = \begin{bmatrix} A + B K_1 & B(-K_1 + U) \\ & (I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n) \end{bmatrix}$.

With Lemma 1, we know that for each follower, there exists at least one leader that has a directed path to the follower, which means the real parts of the eigenvalues of $\mathcal{L}_1$ are all positive. Thus, we can

choose $\mu > \text{Re}(\lambda_{\text{remax}}(S)) / \text{Re}(\lambda_{\text{remin}}(\mathcal{L}_1))$ such that $(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_q)$ is Hurwitz (see Remark 6). With $\sigma(E_i, A_i + B_i K_{1i}) \subset \mathbb{C}^-$, for $i = 1, \dots, N$, we have $\sigma(E, A + BK) \subset \mathbb{C}^-$. Since $\sigma(\hat{E}, \hat{A}) = \sigma(E, A + BK) \cup \sigma((I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_q))$, we get $\sigma(\hat{E}, \hat{A}) \subset \mathbb{C}^-$, which implies $\lim_{t \rightarrow 0} \varrho = 0$, $\forall x(0)$, $\bar{\xi}(0)$, and $x_0(0)$. Therefore, $\lim_{t \rightarrow 0} e = 0$. $\square$

Theorem 2. Let L_i and K_{1i} satisfy $\sigma(E_i, A_i + B_i K_{1i}) \subset \mathbb{C}^-$ and $\sigma(E_i, A_i + L_i C_i) \subset \mathbb{C}^-$, and set $K_{2i} = -K_{1i} + U_i$, $i = 1, \dots, N$. Then the multi-agent systems can achieve containment by distributed dynamic output feedback control law (9) if the observer gain satisfies $\mu > \text{Re}(\lambda_{\text{remax}}(S)) / \text{Re}(\lambda_{\text{remin}}(\mathcal{L}_1))$.

Proof. Under the control law (9), we write the closed-loop system and the error vector of agent i altogether as follows

$$\begin{aligned} E_i \dot{x}_i &= A_i x_i + B_i [K_{1i} \xi_i + (-K_{1i} + U_i) \eta_i] \\ \dot{\eta}_i &= S \eta_i + \mu \left[\sum_{j=1}^N a_{ij} (\eta_j - \eta_i) + \sum_{k=N+1}^{N+M} a_{ik} (w_k - \eta_i) \right] \\ E_i \dot{\xi}_i &= A_i \xi_i + B_i [K_{1i} \xi_i + (-K_{1i} + U_i) \eta_i] + L_i (C_i \xi_i - y_i) \\ \dot{w}_k &= S w_k, \quad i = N+1, \dots, N+M \\ e_i &= \sum_{j=1}^N a_{ij} (x_j - x_i) + \sum_{k=N+1}^{N+M} a_{ik} (w_k - x_i). \end{aligned} \quad (17)$$

Let $x = [x_1^T, \dots, x_N^T]^T$, $\eta = [\eta_1^T, \dots, \eta_N^T]^T$, $\xi = [\xi_1^T, \dots, \xi_N^T]^T$, $e = [e_1^T, \dots, e_N^T]^T$, $u = [u_1^T, \dots, u_N^T]^T$. Furthermore, let $E = \text{diag}\{E_1, \dots, E_N\}$, $A = \text{diag}\{A_1, \dots, A_N\}$, $B = \text{diag}\{B_1, \dots, B_N\}$, $U = \text{diag}\{U_1, \dots, U_N\}$, $K_1 = \text{diag}\{K_{11}, \dots, K_{1N}\}$, $C = \text{diag}\{C_1, \dots, C_N\}$. Then we can define an augmented system as follows

$$\begin{aligned} E \dot{x} &= Ax + B[K_1 \xi + (-K_1 + U) \eta], \\ \dot{\eta} &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \eta - \mu(\mathcal{L}_2 \otimes I_n) w, \\ E \dot{\xi} &= A \xi + B[K_1 \xi + (-K_1 + U) \eta] - L(Cx - C \xi) \\ \dot{w} &= (I_M \otimes S) w, \\ e &= (\mathcal{L}_1 \otimes I_n) x + (\mathcal{L}_2 \otimes I_n) w. \end{aligned} \quad (18)$$

With Lemma 1 by the invertible transformations $\theta = x + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w$, $\tilde{x} = x - \xi$ and $\tilde{w} = \eta + (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w$, we can rewrite the closed loop system (18) as

$$\begin{aligned} E \dot{\theta} &= (A + K_1) \theta - BK_1 \tilde{x} + B(-K_1 + U) \tilde{w} + [-A - BU + E(I_N \otimes S)] (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w, \\ E \dot{\tilde{x}} &= (A + LC_m) \tilde{x}, \\ \dot{\tilde{w}} &= [(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_n)] \tilde{w}, \\ e &= (\mathcal{L}_1 \otimes I_n) [\xi + (I_{nN} - I_N \otimes I_n) (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n) w]. \end{aligned} \quad (19)$$

With Remark 4, we can further rewrite it as

$$\begin{aligned}\hat{M}\dot{\varpi} &= \hat{N}\varpi \\ e &= (\mathcal{L}_1 \otimes I_n)\theta\end{aligned}\tag{20}$$

where $\varpi = \begin{bmatrix} \theta \\ \tilde{x} \\ \tilde{w} \end{bmatrix}$, $\hat{M} = \begin{bmatrix} E & & \\ & E & \\ & & I_{nN} \end{bmatrix}$ and $\hat{N} = \begin{bmatrix} A + K_1 & -BK_1 & B(-K_1 + U) \\ & A + LC & \\ & & (I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_q) \end{bmatrix}$.

With Assumption 1 and Lemma 1, the real parts of the eigenvalues of $\mathcal{L}_1$ are all positive. Thus, we can choose any $\mu > \lambda_{\max}(S)/\lambda_{\min}(\mathcal{L}_1)$ to make sure that $(I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_q)$ is Hurwitz (see Remark 6). With $\sigma(E_i, A_i + B_i K_{1i}) \subset \mathbb{C}^-$ and $\sigma(E_i, A_i + L_i C_i) \subset \mathbb{C}^-$, for $i = 1, \dots, N$, we have $\sigma(E, A + BK) \subset \mathbb{C}^-$ and $\sigma(E, A + LC) \subset \mathbb{C}^-$. Since $\sigma(\hat{M}, \hat{N}) = \sigma(E, A + BK) \cup \sigma(E, A + LC) \cup \sigma((I_N \otimes S) - \mu(\mathcal{L}_1 \otimes I_q))$, we get $\lim_{t \rightarrow 0} \varpi = 0$, $\forall, x(0), \eta(0), \xi(0)$, and $w(0)$. Therefore $\lim_{t \rightarrow 0} e = 0$. $\square$

Remark 7. For Theorem 1 and Theorem 2, we solve the generalized Riccati equation in Lemma 3 to find K_{1i} : let $Q = I$ and $R = I$, and then we get $K_{1i} = B_i^T P_i E_i$. The computational complexity of deriving K_{1i} is polynomial. To find K_{2i} , we use the relationship $K_{2i} = U_i - K_{1i}$ in (10): Since B_i has full column rank, the solution of (6) is $U_i = (B_i^T B_i)^{-1} B_i^T E_i S - (B_i^T B_i)^{-1} B_i^T A_i$. Therefore, $K_{2i} = U_i - K_{1i} = (B_i^T B_i)^{-1} B_i^T E_i S - (B_i^T B_i)^{-1} B_i^T A_i - B_i^T P_i E_i$. The computational complexity of calculating K_{2i} is $O(mn^2)$, which is determined by the matrix multiplications.

4. Simulation Results

Figure 1: Topology of the multi-agent systems. Agents 1-5 are followers and agents 6-8 are leaders.

Now, we present the simulation results to corroborate the effectiveness of our theoretical results. Consider a group of 5 followers and 3 leaders, and the digraph associated with our system is shown as Fig. 1. The

adjacent matrix of the multi-agent systems is

$$\mathcal{A} = \begin{bmatrix} 0 & 0.25 & 0.35 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 1 \\ 0.2 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (21)$$

Note that the Laplacian has the form of (3), where $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$\mathcal{L}_1 = \begin{bmatrix} 1.6 & -0.25 & -0.35 & 0 & 0 \\ 0 & 1.4 & -0.4 & 0 & 0 \\ -0.2 & -0.25 & 0.45 & 0 & 0 \\ 0 & -0.4 & -0.2 & 0.6 & 0 \\ 0 & -0.3 & -0.5 & 0 & 1.8 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \quad (22)$$

The dynamics of the followers and the leaders are given by (1) and (2), with the following parameters:

$$\begin{aligned} E_i &= \begin{bmatrix} \beta_i & 0 & 0 & 0 \\ 0 & \alpha_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & \beta_i & 0.3\beta_i & 0.1\beta_i \\ -\alpha_i & 0 & 0.2\alpha_i & -0.1\alpha_i \\ 0 & 4\alpha_i & 10\beta_i & 0 \\ 5\gamma_i & 0 & 0 & 7\alpha_i \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \gamma_i & 0 \\ 0 & \beta_i \end{bmatrix} \\ C_i^T &= \begin{bmatrix} 2 \\ 3\beta_i \\ 7\beta_i \\ 3\alpha_i \end{bmatrix}, \quad D_i = \alpha_i, \quad S = \begin{bmatrix} 0 & 1 & 0.3 & 0.1 \\ -1 & 0 & 0.2 & -0.1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad i = 1, \dots, 5, \quad k = 6, 7, 8, \end{aligned}$$

where $\alpha_1 = 6.4378$, $\alpha_2 = 8.6012$, $\alpha_3 = 4.0188$, $\alpha_4 = 6.3193$, $\alpha_5 = 9.8524$, $\beta_1 = 8.8764$, $\beta_2 = 1.9874$, $\beta_3 = 3.9537$, $\beta_4 = 9.9218$, $\beta_5 = 4.0235$, $\gamma_1 = 5.5948$, $\gamma_2 = 9.3359$, $\gamma_3 = 7.2034$, $\gamma_4 = 4.8404$, and $\gamma_5 = 6.3903$. It can be verified that under Assumption 1–3, Theorems 1 and 2 are all satisfied. The initial states are $x_i(0)$ and $w_k(0)$ whose entries are randomly chosen for $i = 1, \dots, 5$ and $k = 6, 7, 8$. Let $\eta_i(0)$ and $\xi_i(0)$ be zeros, for $i = 1, \dots, 5$. We solve the containment control problem by using our proposed control laws (8) and (9). The design process is given as follows. We choose $K_{1i} = B_i^T P_i E_i$ satisfying $\sigma(E_i, A_i + B_i K_{1i}) \subset \mathbb{C}^-$, $i = 1, \dots, 5$, where P_i is the solution of (7). We also choose $L_i^T = C_i P_{Li} E_i^T$ satisfying $\sigma(E_i, A_i + L_i C_i) \subset \mathbb{C}^-$, $i = 1, \dots, 5$ and $\sigma(E, A + LC) \subset \mathbb{C}^-$, $i = 1, \dots, 5$, where P_{Li} is the solution of (7). Since B_i has full column rank, there exist $U_i = (B_i^T B_i)^{-1} B_i^T E_i S - (B_i^T B_i)^{-1} B_i^T A_i$ for $i = 1, \dots, 5$.

By setting $\mu = 5$, the state trajectories under the distributed control laws (8) and (9) are shown as Fig. 2

Figure 2: State trajectories under distributed dynamic state feedback controller with $\mu = 5$. (a) state component 1; (b) state component 2; (c) state component 3; (d) state component 4.

and Fig. 3, respectively. We can see that each state component of any follower converges into the envelope determined by the corresponding state component of the leaders.

Note that Fig. 2 and Fig. 3 can just present the convergence of containment control component-wisely. To validate if the followers converge to the 4-dimensional convex hull formed by the leaders, we need to define a distance as follows

$$D(a, \text{Co}(\mathcal{W})) = \min \{ \|a - b\| : b \in \text{Co}(\mathcal{W}) \}, \quad (23)$$

where $a \in \mathbb{R}^n$, and $\text{Co}(\mathcal{W})$ is the convex hull formed by the leaders' states $w_{N+1}, \dots, w_{N+M}$. $D(a, \text{Co}(\mathcal{W}))$ returns the distance between point/state a and convex set $\text{Co}(\mathcal{W})$.¹ We can see that $D(a, \text{Co}(\mathcal{W})) = 0$ if and only if $a \in \text{Co}(\mathcal{W})$. This means that using $D(x_i, \text{Co}(\mathcal{W}))$, we can measure the distance between follower i

¹In (23), the operator is min instead of inf. This is because $\mathcal{W}$ is a finite set and its convex hull $\text{Co}(\mathcal{W})$ is compact.

Figure 3: State trajectories under distributed dynamic output feedback controller with $\mu = 5$. (a) state component 1; (b) state component 2; (c) state component 3; (d) state component 4.

and the convex hull $\text{Co}(\mathcal{W})$. Fig. 4 gives how $D(x_i, \text{Co}(\mathcal{W}))$ changes with time.

From Fig. 4, we can see that $D(x_i, \text{Co}(\mathcal{W}))$ converges to zero for all $i = 1, \dots, 5$, which means all the followers converge to the convex hull formed by the leaders as time goes by.

In our proposed two distributed controllers, the distributed observer plays an important role, which determines each follower's estimation of the corresponding convex combination of the leaders' states. With Theorem 1 and Theorem 2, we know that the containment can be achieved if $\mu > \text{Re}(\lambda_{\text{remax}}(S)) / \text{Re}(\lambda_{\text{remin}}(\mathcal{L}_1)) = 0$. To corroborate this result and also to see the system performances with different μ , we present the simulations for $\mu = 0.01, 0.1, 1, 10$ in Fig. 5, where

$$\bar{D}(x, \text{Co}(\mathcal{W})) = \frac{1}{5} \sum_{i=1}^5 D(x_i, \text{Co}(\mathcal{W})) \quad (24)$$

gives the average distance of $D(x_i, \text{Co}(\mathcal{W}))$ for $i = 1, \dots, 5$.

Figure 4: Distance between follower and the convex hull with $\mu = 5$. (a) under distributed dynamic state feedback control; (b) under distributed dynamic output feedback control.

From Fig. 5, we can see that:

- Under any condition $\mu = 0.01, 0.1, 1, 10$, the followers will converge to the convex hull formed by the leaders (we have verified the cases with $\mu = 0.01, 0.1$ when $t = 200$). This coincides with our theoretical results in Theorem 1 and Theorem 2 that the containment can be achieved if $\mu > \text{Re}(\lambda_{\text{remax}}(S)) / \text{Re}(\lambda_{\text{remin}}(\mathcal{L}_1)) = 0$.
- By increasing μ , the convergence speed becomes larger. Note that μ is a parameter for calculating η_i in each distributed observer, and is not related to any control energy. Thus, we can choose a sufficiently large μ to make sure the convergence of the containment control without knowing any graph information, which means our proposed controllers are in a fully distributed manner.
- For the cases with $\mu = 1$ and $\mu = 10$, there is no big difference. This is because the control gain cannot provide sufficiently fast response for tracking the distributed observer.

5. Conclusion

In this paper, we have studied the containment control problem of singular heterogeneous multi-agent systems based on the output regulation method. The containment control problem has been solved by our proposed two control laws in the output regulation framework: i.e., the distributed dynamic state feedback control law and the distributed output feedback control law. Simulation examples are proposed to illustrate the effectiveness of our analytical results.

6. References

Figure 5: Average distance between follower and the convex hull with $\mu = 0.01, 0.1, 1, 10$. (a) under distributed dynamic state feedback control; (b) under distributed dynamic output feedback control.

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