Motion of Hypersurfaces by Curvature

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Declaration

The work contained in this thesis is my own, except where otherwise indicated.

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Abstract

We study the behaviour of (non-convex) solutions of a large class of fully non-linear curvature flows; specifically, we consider the evolution of closed, immersed hypersurfaces of Euclidean spaces whose pointwise normal speed is prescribed by a monotone function of their curvature which is homogeneous of degree one.

It is well-known that solutions of such flows necessarily suffer finite time singularities. On the other hand, under various natural conditions, singularities are characterised by a curvature blow-up. Our first main area of study concerns the asymptotic behaviour of the curvature at a singularity. We first prove a quantitative convexity estimate for positive solutions (that is, solutions moving with inward normal speed everywhere positive) under one of the following additional assumptions: either the evolving hypersurfaces are of dimension two, or the flow speed is a convex function of the curvature. Roughly speaking, the convexity estimate states that, for positive solutions, the normalised Weingarten curvature operator is asymptotically non-negative at a singularity. We then prove a family of cylindrical estimates for flows by convex speed functions. Roughly speaking, these estimates state that, for $(m+1)$-positive solutions (that is, solutions with $(m+1)$-positive Weingarten curvature), the Weingarten curvature is asymptotically m-cylindrical at a singularity unless it becomes m-positive. The convexity and cylindrical estimates yield a detailed description of the possible singularities which may form under surface flows and flows by convex speeds. Moreover, they are uniform across the class of solutions with given dimension, flow speed, and initial volume, diameter and curvature hull, which should make them useful for applications such as the development of flows with surgeries.

Our second main area of study concerns the development, in the fully non-linear setting, of the recently discovered *non-collapsing* phenomena for the mean curvature flow; namely, we prove that embedded solutions of flows by concave speeds are *interior non-collapsing*, whilst embedded solutions of flows by convex or inverse-concave speeds are *exterior non-collapsing*. The non-collapsing results complement the above curvature estimates by ruling out certain types of asymptotic behaviour which the curvature estimates do not. (This is mainly due to the non-local nature of the non-collapsing estimates.) As a particular application, we show how non-collapsing gives rise to a particularly efficient proof of the Andrews–Huisken theorem on the convergence of convex hypersurfaces to *round points* under such flows.

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1. Introduction

One of the oldest and most natural questions in Riemannian geometry asks how the curvature of a manifold, a local invariant, is related to its topology, a global invariant. Analogously, one can ask how the extrinsic curvature of a submanifold is related to its topology, and the topology of its ambient space. A natural approach to studying such questions is to allow submanifolds to change shape with velocity prescribed in some way by their curvature. One motivation for doing so is that, in many cases, the resulting evolution equations (referred to as *curvature flows*) are parabolic; this guarantees that solutions exist and that the evolving submanifold 'improves': diffusion tends to make it smoother and homogenize its curvature (at least for a short time). One then hopes that the solution exists long enough to converge to something well-understood, and thereby provide information about the initial submanifold and the ambient space. Unfortunately, however, singularities will generally occur before this can happen. Understanding and overcoming singularity formation is the primary challenge in carrying out this program.

In this thesis, we will consider the evolution of hypersurfaces of Euclidean spaces. More precisely, given a smooth, connected manifold $\mathscr{M}^n$ of dimension n, we study smooth one-parameter families of smooth, complete immersions $\mathscr{X}(\cdot,t):\mathscr{M}^n\to\mathbb{R}^{n+1}$ which solve an equation of the form

$$\frac{\partial \mathscr{X}}{\partial t} = -F(x,t)\nu(x,t)\,, \tag{CF}$$

where $\nu(\cdot,t)$ is a unit normal field for $\mathscr{X}(\cdot,t)$ and F is determined by

$$F(x,t) = f(\vec{\kappa}(x,t))$$

for some smooth function $f:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$ of $\vec{\kappa}(\cdot,t)=(\kappa_1(\cdot,t),\ldots,\kappa_n(\cdot,t))$, the n-tuple of principal curvatures of $\mathscr{X}(\cdot,t)$.

We will generally require that the speed defining function $f:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$ satisfy additional conditions, the most important of which are as follows:

Conditions 1 (Admissibility Conditions)**.**

(i) Symmetry*: f is a symmetric function.*

(ii) Parabolicity*: f is monotone increasing in each variable.*

(iii) Homogeneity*: f is homogeneous of degree one.*

(iv) Positivity*: $f>0$.*

We will refer to a smooth function $f : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ satisfying Conditions 1 (i)–(iii) as an *admissible speed function for* (CF) (or simply an *admissible speed*). If, in addition, (iv) holds, then f will be deemed a *positive admissible speed function for* (CF). We shall discuss the purpose of these conditions in §2.2.

Convex hypersurfaces

Perhaps the first curvature flow to receive significant attention was the *mean curvature flow*, under which the velocity at each point of the submanifold is given by the mean curvature vector at that point. Huisken (1984) proved that smooth, compact, convex initial hypersurfaces (of dimension at least two) of Euclidean spaces admit a unique solution to this flow, which shrinks to a point in finite time, becoming asymptotically round in the process. He then showed that an appropriate rescaling of the solution about the final point yields a family of hypersurfaces which converges smoothly to a round sphere. The corresponding result for smooth, closed, embedded, convex curves in the plane was proved soon after by Gage and Hamilton (1986) (note that, without the embeddedness condition, the Gage–Hamilton statement is false).

After Huisken's result appeared, a variety of other curvature flows were studied, and similar behaviour was observed: Chow (1985), building on work of Chou (Tso 1985), showed that convex hypersurfaces of $\mathbb{R}^{n+1}$ evolving by the n-th root of the Gauß curvature also shrink to points in finite time, and become asymptotically round in the process. He also observed this behaviour for flows by the square root of the scalar curvature, so long as the initial hypersurface is already sufficiently round (in the sense of an explicit pinching condition for the principal curvatures) (Chow 1987). These results were later generalized and improved by Andrews (1994a; 2007), who studied flows by degree one homogeneous functions of the curvature which satisfy one of a list of natural additional conditions. Moreover, Andrews later found that flows of surfaces (that is, when the spatial dimension is two) by *any* parabolic, degree one homogeneous speed function will contract convex initial data to round points (Andrews 2010). Andrews, McCoy, and Zheng (2013) showed that this latter result fails in higher dimensions. Moreover, by constructing and studying counterexamples, they were able to formulate rather sharp conditions under which the behaviour described by Huisken's theorem should hold.

Non-convex hypersurfaces

If the initial hypersurface is not convex, much less is known about the long term behaviour of solutions of equation (CF); although, the one dimensional case is now well-understood: Every smooth, closed, embedded curve evolves to become convex, thereafter shrinking to a round point according to the Gage–Hamilton Theorem (Grayson 1987) (see also the more recent proofs by Hamilton (1995c), Huisken (1998), and Andrews and Bryan (2011)). In higher dimensions, the situation is not so straightforward, since, in general, the solution hypersurfaces will become singular before becoming convex (of course, this must be the case, since in higher dimensions there are non-convex hypersurfaces which are not diffeomorphic to a sphere). On the other hand, Huisken and Sinestrari (1999b; 1999a; 2009) and White (2000; 2003) have developed a detailed structure theory for solutions of *mean-convex* mean curvature flow; that is, mean curvature flow of hypersurfaces with positive mean curvature. In particular, Huisken and Sinestrari have developed a surgery program to allow the continuation of compact, mean convex mean curvature flows through singularities. This program was implemented for 2-convex mean curvature flows (that is, mean curvature flows of hypersurfaces with smallest two principal curvatures summing everywhere to a non-negative value) of dimension $n \geq 3$. In addition, Brendle and Huisken (2013) have recently announced a construction of mean curvature flow with surgery for flows of surfaces. As a consequence, Huisken and Sinestrari prove that any compact, 2-convex hypersurface of $\mathbb{R}^{n+1}$, $n \geq 3$, is diffeomorphic either to a sphere S^n bounding a smooth $(n+1)$-dimensional disk, or to a connected sum of tori $S^{n-1} \times S^1$ bounding a smooth $(n+1)$-dimensional handlebody (Huisken and Sinestrari 2009).

In order to define a flow-with-surgery, one requires a detailed knowledge of the geometry of the hypersurface in a neighbourhood of a singular point, close to the singular time. The main ingredients here are the 'convexity estimate', (Huisken and Sinestrari (1999a, Theorem 1.1). Cf. White (2003, Theorem 1)), and the 'cylindrical estimate' (Huisken and Sinestrari 2009, Theorem 1.5), as well as a gradient estimate (Huisken and Sinestrari 2009, Theorem 1.6) which depends only locally on the value of the mean curvature.

The convexity estimate

Roughly speaking, the convexity estimate asserts that the hypersurface becomes (weakly) convex at any point at which the mean curvature becomes large. In particular, it implies that any limit of rescalings of the flow about a singularity must be a convex solution of the mean curvature flow. This, together with the monotonicity formula of Huisken (1990) and the Harnack inequality of Hamilton (1995b) gives rise to a rather complete description of singularities in the positive mean curvature case.

Cylindrical estimates

The Huisken–Sinestrari cylindrical estimate applies to 2-convex solutions of the mean curvature flow, and states that the hypersurface becomes either convex, or cylindrical (in that the Weingarten map is close to the Weingarten map of a cylinder $\mathbb{R} \times S^{n-1}$ of small radius) at any point at which the mean curvature becomes large. This estimate refines the classification of singular profiles described above, such that the only possibilities are either a shrinking sphere, S^n, a strictly convex translating solution, or a shrinking cylinder, $S^{n-1} \times \mathbb{R}$.

Motivated by these results, our main goal is to investigate the singular behaviour of (non-convex) solutions of more general curvature flows. This is an important step towards extending the Huisken–Sinestrari program to allow a larger class of evolution equations, which is of great interest since it would increase the geometric and topological applications.

Let us conclude with an overview of the contents of this thesis, including a brief description of our main results.

Overview

We begin, in §2, with some background material. In particular, we prove some basic results about *curvature functions*, which provide the natural class of flow speeds, and *time-dependent immersions*, which provide the natural class of solutions.

In §3, we will study the local properties of solutions of (CF). We shall see that the parabolicity condition, Conditions 1 (ii), guarantees the initial value problem for (CF) is well-posed (Theorem 3.7) and gives rise to most of the tools we shall employ to study solutions of (CF), especially the maximum principle.

In §4, we study the global properties of solutions. In general, this problem depends on the form of the speed function F; however, the homogeneity condition, Conditions 1 (iii), already ensures that solutions share some global properties. For example, it ensures that convex hypersurfaces contract. Homogeneity also ensures that the flow is invariant under parabolic rescaling (see §3.1.5), which is a useful tool for studying the behaviour of solutions at a singularity. Finally, by Euler's theorem for homogeneous functions, the degree one homogeneity of F implies that its derivative is homogeneous of degree zero. In a certain sense, this means that diffusion is equally effective at all scales.

These properties place some restrictions on the long-time behaviour of solutions; however, it is still possible for solutions to behave quite badly. For example, there are flows by admissible speed functions which can evolve hypersurfaces to lose higher regularity, or for which the speed F can blow-up whilst the hypersurface remains regular (See Andrews, McCoy, and Zheng 2013, §§5–6). In order to rule this behaviour out, we introduce additional conditions on the speed function F. We will, at various points, make use of one or more of the following conditions:

Conditions 2 (Auxiliary conditions)**.**

(v) Surface flows*: $n = 2$.*

(vi) Concavity*: f is concave.*

(vii) Convexity*: f is convex.*

(viii) Inverse-concavity*: f is inverse-concave.*

(ix) Preserved cones*: The flow* (CF) *admits preserved cones.*

Note that, although Conditions 1 (i)–(iii) will usually be assumed, the Auxiliary Conditions (v)–(ix) will only be assumed as need arises. See §2.2 for a discussion of these conditions, where we also provide some examples of speeds satisfying them.

The purpose of the auxiliary conditions is two-fold: First, some such condition is needed in the (scalar and tensor) maximum principle arguments of §4 to show that some form of initial curvature pinching is preserved under the flow. This ensures that, whilever the curvature is bounded, the principal curvature n-tuple remains in a compact subset of the cone of definition of the speed. Second, except in two space dimensions (Andrews 2004), some concavity condition is required in order to deduce Hölder continuity of the Weingarten curvature using the estimates of Krylov (1982) and Evans (1982), which is the bootstrap for the Schauder estimates to deduce higher regularity whilever the curvature is bounded (see appendix A).

In §5, we will study the asymptotic behaviour of solutions where the curvature blows-up. Since the results of Andrews, McCoy, and Zheng (2013) already tell a rather complete story for flows of *convex* hypersurfaces by degree one homogeneous speeds, we focus our attention on the behaviour of non-convex hypersurfaces, the understanding of which is far less developed. The main result of §5 is an asymptotic estimate on the principal curvatures which shows, for a large class of flows, that the curvature approaches an asymptotically optimal set near a singularity. More precisely, we shall prove, in §5.2, a *convexity estimate* (Theorem 5.2), which shows that, at a singularity, the solution is becoming weakly convex, and, in §5.3, a family of *cylindrical estimates* (Theorem 5.15), which show that, if the solution is already $(m+1)$-convex, then, at a singularity, it is either becoming m-cylindrical or strictly m-convex. These estimates lead, in particular, to a detailed infinitesimal description of singularities for positive solutions of (CF).

Finally, in §6, we study two new extrinsic quantities related to embedded hypersurfaces: the *interior* and *exterior ball curvatures*. The interior ball curvature is defined at each point of a (compact) hypersurface as the curvature of the largest ball which is enclosed by the hypersurface and touches it at that point. The exterior ball curvature is defined similarly by considering enclosing regions. We will prove that flows by concave speeds preserve the ratio of the interior ball curvature to the speed, and flows by convex or inverse-concave speeds preserve the ratio of exterior ball curvature to the speed so long as the latter is positive. These estimates provide useful information about the formation of singularities in embedded solutions which complements the curvature estimates described above. In particular, we are able to give a new, and rather short, proof of a theorem of Andrews (2007) on the convergence to round points of convex hypersurfaces under flows by concave, inverse-concave speeds (Theorem 6.24).

2. Some background material

Before we begin our study of curvature flows, let us pause to develop a little machinery (and notation) which allows us to talk more rigorously about solutions of equation (CF). In §2.1, we study *time-dependent immersions*; these are simply smooth maps of the product of a manifold $\mathscr{M}$ with an interval I into Euclidean space (or, more generally, any Riemannian manifold) such that fixing the 'time' parameter $t \in I$ defines an immersion. When we consider solutions of (CF), we will always mean a smooth time-dependent immersion. It will be convenient to develop a 'time-dependent' hypersurface geometry associated to such maps; this is the primary purpose of §2.1. In the direction of 'spatial' tangent vectors, this geometry is simply the standard hypersurface geometry for 'stationary' immersions; however, there are useful additional identities for the 'temporal' direction which may be understood as evolution equations. We shall only consider the case that the ambient space is Euclidean and the codimension is one, since that is the setting in which we study the equation (CF). A more detailed discussion of time-dependent hypersurfaces, which covers the general setting, is developed in the thesis of Baker (2010). In §2.2, we study *curvature functions*. Given an (time-dependent) immersion, a curvature function F is a smooth, symmetric function of the principal curvatures. Due to Theorems of Glaeser (1963) and Schwarz (1975), this is equivalent to prescribing a smooth (base-point independent) function of the Weingarten curvature. The relationship between the function F considered as a function of the Weingarten curvature $\mathcal{W}$ and its eigenvalues κ_i will be studied. We will conclude by giving precise definitions of Conditions 1–2 and providing some examples of speed functions satisfying them.

We assume the reader is already familiar with basic concepts from Riemannian geometry, for which there are many excellent expositions; for example, the book of Chavel (1993). We also recommend the thesis of Baker (2010) and the book of Andrews and Hopper (2011) for expositions with curvature flows in mind. For some background on the theory of vector bundles, we recommend the book of Hirsch (1994).

A reader already comfortable with this material may fearlessly skip to Section 3, returning here as need dictates.

2.1 Time-dependent hypersurfaces

Let $\mathscr{M}^n$ be a smooth n-dimensional manifold, $I \subset \mathbb{R}$ an interval, and $\mathscr{X} : \mathscr{M}^n \times I \to \mathbb{R}^{n+1}$ a smooth *time-dependent* immersion; that is, $\mathscr{X}$ is a smooth map such that $\mathscr{X}_t := \mathscr{X}(\cdot, t)$

is an immersion for each $t \in I$. Observe that the tangent bundle $T(\mathscr{M}^n \times I)$ splits into a direct sum of the *spatial tangent bundle* $\mathfrak{S} := \{\xi \in T(\mathscr{M} \times I) : dt(\xi) = 0\}$ with the line bundle $\mathbb{R}\,\partial_t$. We will subsequently abuse notation by denoting $T\mathscr{M} = \mathfrak{S}$ but, for clarity, we will continue to use the notation $\mathfrak{S}$ in this section. The spatial tangent bundle admits an induced geometry akin to the standard sub-Riemannian geometry of a 'stationary' immersion, but possesses additional information about the evolution of the immersions $\mathscr{X}_t := \mathscr{X}(\cdot, t)$. It will be useful to spend some time developing this 'time-dependent submanifold geometry' since it will be ubiquitous throughout this thesis. This will serve both to set down some basic machinery for subsequent ease of application, and to familiarize the reader with our perhaps unfamiliar notation.

We begin by equipping the pullback bundle $\mathscr{X}^*T\mathbb{R}^{n+1}$ with the pullback metric ${}^{\mathscr{X}}\langle \cdot\,,\,\cdot\rangle$, which is defined on each fibre by

$$ {}^{\mathscr{X}}\big\langle (x,t,u),(x,t,v)\big\rangle = \big\langle (\mathscr{X}(x,t),u),(\mathscr{X}(x,t),v)\big\rangle\,, $$

where $\langle \cdot\,,\,\cdot\rangle$ is the metric on $T\mathbb{R}^{n+1}$. Here (x,t,u) and (x,t,v) are elements of $(\mathscr{X}^*T\mathbb{R}^{n+1})_{(x,t)}$, so that, by definition of the pullback bundle, $(\mathscr{X}(x,t),u)$ and $(\mathscr{X}(x,t),v)$ are elements of $T_{\mathscr{X}(x,t)}\mathbb{R}^{n+1}$.

The pullback bundle also inherits the pullback connection, ${}^{\mathscr{X}}\!D$, defined by

$$ {}^{\mathscr{X}}\!D_\xi \mathscr{X}^*V = D_{\mathscr{X}_*u}V $$

for every $\xi \in T(\mathscr{M}^n \times I)$ and every pulled-back section $\mathscr{X}^*V \in \Gamma(\mathscr{X}^*T\mathbb{R}^{n+1})$. This extends to all sections of $\Gamma(\mathscr{X}^*T\mathbb{R}^{n+1})$ via the Leibniz rule, as we can always form a local basis of pulled-back sections. Here $\mathscr{X}_* : T(\mathscr{M} \times I) \to \mathscr{X}^*T\mathbb{R}^{n+1}$ denotes the push-forward of $\mathscr{X}$.

The push-forward $\mathscr{X}_*\mathfrak{S}$ of the spatial tangent bundle is a sub-bundle of $\mathscr{X}^*T\mathbb{R}^{n+1}$ of rank n. Its orthogonal complement in $\mathscr{X}^*T\mathbb{R}^{n+1}$ with respect to ${}^{\mathscr{X}}\langle \cdot\,,\,\cdot\rangle$ is a sub-bundle of rank 1, which we denote by $\mathfrak{N}$ and refer to as the *normal bundle* of $\mathscr{X}$. Subsequently, we will denote $N\mathscr{M} = \mathfrak{N}$—which is the common notation for the normal bundle of a stationary immersion—but for this section we continue to use $\mathfrak{N}$ in order to distinguish the two. Thus $\mathscr{X}^*T\mathscr{M}^n = \mathscr{X}_*\mathfrak{S} \oplus \mathfrak{N}$. We refer to the orthogonal projections (with respect to the pullback metric) $\overset{\top}{\pi}\colon \mathscr{X}^*T\mathbb{R}^{n+1} \to \mathscr{X}_*\mathfrak{S}$ and $\overset{\perp}{\pi}\colon \mathscr{X}^*T\mathbb{R}^{n+1} \to \mathfrak{N}$ as, respectively, the *tangential* and *normal projection*.

The spatial tangent bundle $\mathfrak{S}$ inherits an *induced metric* $g \in \Gamma(\mathfrak{S}^* \odot \mathfrak{S}^*)$ and an *induced connection* ∇ from the pullback metric and connection. These are defined by

$$ g\,(u,v) := {}^{\mathscr{X}}\langle \mathscr{X}_*u\,,\,\mathscr{X}_*v\rangle\,, \quad u,v \in \mathfrak{S}_{(x,t)},\ (x,t) \in \mathscr{M} \times I\,, $$

and

$$ \mathscr{X}_*\,\nabla_\xi\,V := \overset{\top}{\pi}\left({}^{\mathscr{X}}\!D_\xi \mathscr{X}_*V\right), \quad V \in \Gamma(\mathfrak{S})\,,\ \ \xi \in T(\mathscr{M}^n \times I) $$

respectively.

For each $t \in I$, the restrictions of $\mathfrak{S}$ and $\mathfrak{N}$ to $\mathscr{M}^n \times \{t\}$ may be canonically identified with the tangent and normal bundles of the immersion $\mathscr{X}_t$. Under this identification, g is the induced metric of the immersion $\mathscr{X}_t$ and, in spatial directions $\xi \in \mathfrak{S}$, ∇ is the induced connection on $\mathscr{X}_t$ (which is the Levi-Civita connection of the induced metric). In fact, by the same straightforward computation as in the stationary setting, we find that ∇ satisfies the Levi-Civita conditions:

$$\nabla_u g = 0\,, \quad u \in \mathfrak{S}\,, \tag{2.1}$$

and

$$\nabla_U V - \nabla_V U - [U, V] = 0\,, \quad U, V \in \Gamma(\mathfrak{S})\,. \tag{2.2}$$

Note that the connection on $\mathfrak{S}^* \otimes \mathfrak{S}^*$ (and similarly for the full tensor algebra of $\mathfrak{S}$) is defined, as usual, by commuting with contractions:

$$\xi g(U, V) = \nabla_\xi g(U, V) + g(\nabla_\xi U, V) + g(U, \nabla_\xi V)$$

for all $\xi \in T(\mathscr{M}^n \times I)$ and all $U, V \in \Gamma(\mathfrak{S})$.

The normal part of the pullback connection, restricted to $\mathfrak{S}$, is a (normal bundle valued) symmetric two-tensor, which we call the *vector second fundamental form* of $\mathscr{X}$, and denote by $\vec{I\!I}$. More precisely, $\vec{I\!I} \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S}^* \otimes \mathfrak{N})$ is defined by

$$\vec{I\!I}(u, v) := \overset{\perp}{\pi} \left({}^{\mathscr{X}}\!D_u \mathscr{X}_* v \right) . \tag{2.3}$$

That this is actually a well-defined tensor follows, just as in the stationary setting, from the calculation

$$\overset{\perp}{\pi} \left({}^{\mathscr{X}}\!D_u \mathscr{X}_* f V \right) = \overset{\perp}{\pi} \left({}^{\mathscr{X}}\!D_u f \mathscr{X}_* V \right) = \overset{\perp}{\pi} \left((uf)\mathscr{X}_* V + f\, {}^{\mathscr{X}}\!D_u \mathscr{X}_* V \right) = f\, \overset{\perp}{\pi} \left({}^{\mathscr{X}}\!D_u \mathscr{X}_* V \right) ,$$

which holds for any $u \in \mathfrak{S}$, $f \in C^\infty(\mathscr{M}^n \times I)$, and $V \in \Gamma(\mathfrak{S})$. Symmetry of $\vec{I\!I}$ follows (just as in the stationary setting) from the orthogonal decomposition

$${}^{\mathscr{X}}\!D_U \mathscr{X}_* V = \mathscr{X}_* \nabla_U V + \vec{I\!I}(U, V)$$

and the Levi-Civita condition 2.2. Since $\vec{I\!I}$ is a symmetric two-tensor (taking values in the normal bundle), for every normal vector there is an associated g-self-adjoint endomorphism. This is encoded in the *vector Weingarten tensor* $\vec{\mathcal{W}} : \Gamma(\mathfrak{N}) \to \Gamma(\mathfrak{S}^* \otimes \mathfrak{S})$. Just as in the stationary setting, $\vec{\mathcal{W}}$ is given explicitly by (see Proposition 2.1)

$$\mathscr{X}_* \vec{\mathcal{W}}_\nu(u) = {}^{\mathscr{X}}\!D_u \nu\,, \quad \nu \in \Gamma(\mathfrak{N}),\ u \in \mathfrak{S}\,. \tag{2.4}$$

If $\mathscr{M}^n$ is orientable, we may choose a global unit normal field $\nu \in \Gamma(S\mathfrak{N})$, with respect to which we may define the *second fundamental form*, $I\!I$, by $\vec{I\!I}(u, v) = -I\!I(u, v)\nu$ and the *Weingarten tensor*, $\mathcal{W}$, by $\mathscr{X}_* \mathcal{W}(u) = {}^{\mathscr{X}}\!D_u \nu$. The eigenvalues of $\mathcal{W}$ are the *principal* *curvatures* of $\mathscr{X}$ (with respect to ν), which we denote by $\kappa_1, \ldots, \kappa_n$.

In the presence of a metric, we will generally not distinguish between tensors related by the metric induced isomorphism of tensor bundles. We shall therefore (apart from in the present section) denote both the (vector) second fundamental form and the (vector) Weingarten tensor of a time-dependent immersion using the symbol, $(\vec{\mathcal{W}})$ $\mathcal{W}$.

The above constructions yield a time-dependent intrinsic and extrinsic geometry for $\mathscr{X}$ which at each time $t \in I$ reduces to the intrinsic and extrinsic geometry of the immersion $\mathscr{X}_t$. So we do not appear to have gained anything as yet; however, we have only considered the information coming from the spatial tangent bundle. There is additional information to be obtained from the temporal direction. This 'temporal' information gives an invariant characterization of the evolution of the geometry of $\mathscr{X}_t$.

First, we exhaust the temporal information contained in the pullback metric by assuming that the velocity of $\mathscr{X}$ is in its normal direction; that is, we assume $\mathscr{X}_* \partial_t = -F\nu$ for some scalar F. Note that this can be achieved for any time-dependent immersion by introducing a 'time-dependent diffeomorphism' of $\mathscr{M}^n$: Let $\mathscr{X} : \mathscr{M}^n \times I \to \mathbb{R}^{n+1}$ be a time-dependent immersion, and let $\phi : \mathscr{M}^n \times I \to \mathscr{M}^n$ be the flow of the negative of the tangential part of the velocity of $\mathscr{X}$; that is,

$$\frac{d\phi}{dt} = -T \circ \phi \,,$$

where $\mathscr{X}_* T = \overset{\top}{\pi} (\mathscr{X}_* \partial_t)$. Then the velocity of the time-dependent immersion $\mathscr{Y}$ defined by $\mathscr{Y}(y,t) := \mathscr{X}(\phi(x,t),t)$ is

$$\mathscr{Y}_* \partial_t = \mathscr{X}_{*\phi} \frac{d\phi}{dt} + \mathscr{X}_{*\phi} \partial_t = \overset{\perp}{\pi} (\mathscr{X}_{*\phi} \partial_t) \ .$$

Thus, modulo a time-dependent diffeomorphism of $\mathscr{M}^n$, we lose no information by assuming the velocity is in the normal direction.

Next consider the temporal Lie derivative of the metric: Computing in a coördinate basis, we find

$$\begin{aligned} \mathcal{L}_t g_{ij} = \partial_t g_{ij} &= \left\langle {}^{\mathscr{X}}\!D_i \mathscr{X}_* \partial_t \,, \mathscr{X}_* \partial_j \right\rangle + \left\langle \mathscr{X}_* \partial_i \,, {}^{\mathscr{X}}\!D_j \mathscr{X}_* \partial_t \right\rangle \\ &= -\left\langle {}^{\mathscr{X}}\!D_i (F\nu) \,, \mathscr{X}_* \partial_j \right\rangle - \left\langle \mathscr{X}_* \partial_i \,, {}^{\mathscr{X}}\!D_j (F\nu) \right\rangle \\ &= -2F\mathcal{W}_{ij} \,. \end{aligned} \tag{2.5}$$

Finally, we consider the temporal information in the pullback connection: Given $V \in \mathfrak{S}$, we have

$$\begin{aligned} {}^{\mathscr{X}}\!D_t \mathscr{X}_* V &= {}^{\mathscr{X}}\!D_V \mathscr{X}_* \partial_t + \mathscr{X}_*[\partial_t, V] \\ &= -{}^{\mathscr{X}}\!D_V (F\nu) + \mathscr{X}_*[\partial_t, V] \\ &= -dF(V)\nu - F\mathscr{X}_* \mathcal{W}(V) + \mathscr{X}_*[\partial_t, V] \,. \end{aligned} \tag{2.6}$$

The tangential part of (2.6) yields the 'temporal component' of the connection ∇ (cf. Andrews and Hopper 2011, Theorem 5.1):

$$\nabla_t V := \mathcal{L}_{\partial_t} V - F\mathcal{W}(V) .$$

It follows that

$$\nabla_t\, g(u,v) = \mathcal{L}_t g(u,v) + g((\mathcal{L}_t - \nabla_t)u, v) + g(u, (\mathcal{L}_t - \nabla_t)v) = 0 .$$

The normal component of (2.6) yields the 'temporal second fundamental form':

$$\vec{I\!I}_t(u) := \overset{\perp}{\pi}\left({}^{\mathscr{X}}\!D_t \mathscr{X}_* u\right) = -dF(u)\nu . \tag{2.7}$$

In this way, we may consider $\vec{I\!I}$ as a section of $T^*(\mathscr{M}^n \times I) \otimes \mathfrak{S}^* \otimes \mathfrak{N}$. Similarly, we may consider $\vec{\mathcal{W}}_\nu$ as a section of $T^*(\mathscr{M}^n \times I) \otimes \mathfrak{S}$ by setting

$$\vec{\mathcal{W}}_\nu(\partial_t) := {}^{\mathscr{X}}\!D_t \nu . \tag{2.8}$$

Let us note that, when we consider the temporal component of the time-dependent connection ∇, we will always be explicit; that is, we reserve the notation ∇V for the *spatial* covariant differential of V.

We now state the structure equations for the time-dependent immersion $\mathscr{X}$. In the spatial directions, these reduce to the standard structure equations for a stationary immersion but yield additional 'evolution' identities for the temporal direction.

Proposition 2.1 (Structure equations for time-dependent hypersurfaces). *Let $\mathscr{X} : \mathscr{M}^n \times I \to \mathbb{R}^{n+1}$ be an oriented*[1] *time-dependent immersion with unit normal $\nu \in \Gamma(S\mathfrak{N})$, and velocity $\mathscr{X}_*\,\partial_t = -F\nu$. Then,*

- The Weingarten equation*: For any $\xi \in T(\mathscr{M} \times I)$, and any $u \in \mathfrak{S}$, we have*

$$g\left(\mathcal{W}(\xi), u\right) = -I\!I(\xi, u) . \tag{2.9}$$

This reduces to the standard Weingarten equation when $\xi \in \mathfrak{S}$. Setting $\xi = \partial_t$ yields the additional temporal Weingarten equation

$${}^{\mathscr{X}}\!D_t \nu = \mathscr{X}_* \operatorname{grad} F . \tag{2.10}$$

- The Gauß equation*: For any $\xi \in T(\mathscr{M} \times I)$ and any $u, v \in \mathfrak{S}$, we have*

$$R(\xi, u)v = \mathcal{W}(\xi) I\!I(u,v) - \mathcal{W}(u) I\!I(\xi, v) , \tag{2.11}$$

where $R(\xi,u)v = \nabla_\xi(\nabla_u v) - \nabla_u(\nabla_\xi v) - \nabla_{[\xi,u]} v$ is the curvature of $(\mathfrak{S}, \nabla)$. This reduces to the standard Gauß equation when $\xi \in \mathfrak{S}$. Setting $\xi = \partial_t$ yields the

[1] This assumption is not strictly necessary; we have merely assumed it for clarity of exposition.

additional temporal Gauß equation

$$R(\partial_t, u)v = \mathit{I\!I}(u,v)dF - dF(v)\mathcal{W}(u)\,. \tag{2.12}$$

– The Codazzi equation*: For any $\xi \in T(\mathscr{M} \times I)$ and any $u \in \mathfrak{S}$, we have*

$$\nabla_\xi \left(\mathcal{W}(u)\right) - \nabla_u \left(\mathcal{W}(\xi)\right) = \mathcal{W}\left([\xi, u]\right)\,. \tag{2.13}$$

This reduces to the standard Codazzi equation

$$\nabla_\xi \mathcal{W}(u) - \nabla_u \mathcal{W}(\xi) = 0$$

when $\xi \in \mathfrak{S}$. Setting $\xi = \partial_t$ yields the additional temporal Codazzi equation

$$\nabla_t \mathcal{W} = \nabla \operatorname{grad} F + F\mathcal{W}^2\,. \tag{2.14}$$

Proof. The proofs are the same as in the stationary setting. The temporal equations follow easily. □

Remark 2.1. From now on, the (vector) second fundamental form ($\vec{\mathit{I\!I}}$) $\mathit{I\!I}$ and (vector) Weingarten tensor ($\vec{\mathcal{W}}$) $\mathcal{W}$ will refer only to the 'spatial' tensors introduced in 2.3 and 2.9, and we shall use the definitions (2.7) and (2.8) explicitly when we require the temporal components.

We now recall the fundamental commutation formula of Simons for the (spatial) Hessian of the second fundamental form Simons 1968.

Proposition 2.2. *Given any spatial tangent vectors $u, v, w, z \in \mathfrak{S}$, we have*

$$\nabla_u \nabla_v \mathit{I\!I}(w,z) - \nabla_w \nabla_z \mathit{I\!I}(u,v) = \mathit{I\!I}(u,v)\mathit{I\!I}^2(w,z) - \mathit{I\!I}(w,z)\mathit{I\!I}^2(u,v)\,, \tag{2.15}$$

where $\mathit{I\!I}^2$ is the symmetric two-tensor corresponding to $\mathcal{W}^2$; that is, $\mathit{I\!I}^2(u,v) = g\left(\mathcal{W}^2(u), v\right) = g\left(\mathcal{W}(u), \mathcal{W}(v)\right)$.

Proof. We compute on a set of arbitrary commuting vectors $u, v, w, z \in \mathfrak{S}$. We first invoke the symmetry of $\nabla \mathcal{W}$ coming from the Codazzi equation (2.13) and then commute covariant derivatives using the definition of the curvature tensor R:

$$\begin{aligned}
\nabla_u \nabla_v \mathit{I\!I}(w,z) &= \nabla_u \nabla_w \mathit{I\!I}(v,z) \\
&= \nabla_w \nabla_u \mathit{I\!I}(v,z) + (R(u,w)\mathit{I\!I})(v,z) \\
&= \nabla_w \nabla_u \mathit{I\!I}(v,z) - \mathit{I\!I}(R(u,w)v,z) - \mathit{I\!I}(v,R(u,w)z)\,.
\end{aligned}$$

Using the Gauß equation (2.11) and once more the Codazzi equation 2.13, we obtain

$$\begin{aligned}\nabla_u \nabla_v \mathit{I\!I}(w,z) = {} & \nabla_w \nabla_u \mathit{I\!I}(v,z) - \mathit{I\!I}(z, \mathit{I\!I}(w,v)\mathcal{W}(u) - \mathit{I\!I}(u,v)\mathcal{W}(w)) \\ & - \mathit{I\!I}(v, \mathit{I\!I}(w,z)\mathcal{W}(u) - \mathit{I\!I}(u,z)\mathcal{W}(w)) \\ = {} & \nabla_w \nabla_z \mathit{I\!I}(u,v) + \mathit{I\!I}(u,v)\mathit{I\!I}^2(w,z) - \mathit{I\!I}(w,z)\mathit{I\!I}^2(u,v)\,.\end{aligned}$$

□

The trace of equation (2.15) plays an important role in minimal surface theory and mean curvature flow. We will make similar use of it in the fully non-linear setting (see Lemma 4.1 and the remarks thereafter).

We also note the following evolution equation for the induced Riemannian measure:

Proposition 2.3. *Let $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ be a time-dependent immersion satisfying $\mathscr{X}_* \partial_t = -F\nu$. Then the induced Riemannian measure μ of $\mathscr{X}$ satisfies*

$$\frac{d}{dt} d\mu = -FH d\mu\,, \tag{2.16}$$

where $H := \mathrm{tr}(\mathcal{W})$ is the mean curvature; that is, for any compact $K \subset \mathscr{M}$, it holds that

$$\frac{d}{dt}\int_K d\mu = -\int_K FH d\mu\,.$$

Proof. Since the velocity of $\mathscr{X}$ is normal, this is simply the standard computation for the first variation of area: Assume that K lies in a single coördinate chart $x^{-1} : U \subset \mathbb{R}^n \to \mathscr{M}$. Then

$$\begin{aligned}\frac{d}{dt}\int_K d\mu &= \frac{d}{dt}\int_{x(K)} \sqrt{\det(x^* g)}\, dx^1 \ldots dx^n \\ &= \int_{x(K)} \frac{\partial}{\partial t}\sqrt{\det(x^* g)}\, dx^1 \ldots dx^n\,.\end{aligned}$$

Now compute

$$\begin{aligned}\partial_t \sqrt{\det(x^* g)} &= \frac{1}{2}\left(\det(x^* g)\right)^{-\frac{1}{2}} \partial_t \det(x^* g) \\ &= \frac{1}{2}\left(\det(x^* g)\right)^{-\frac{1}{2}} \cdot \left[\det(x^* g)\,\mathrm{tr}\left((x^* g)^{-1}\,\partial_t (x^* g)\right)\right] \\ &= \frac{1}{2}\left(\det(x^* g)\right)^{-\frac{1}{2}} \cdot \left[\det(x^* g)\,\mathrm{tr}\left((x^* g)^{-1} x^* \mathcal{L}_t g\right)\right] \\ &= -\left(\det(x^* g)\right)^{\frac{1}{2}} x^*(FH)\,,\end{aligned}$$

where we used (2.5) in the last line. Thus,

$$\frac{d}{dt}\int_K d\mu = -\int_{x^{-1}(K)} x^*(FH)\sqrt{\det(x^* g)}\, dx^1 \ldots dx^n = -\int_K FH d\mu$$

as required. The general case follows by the same computation on each chart of a covering of K by an atlas with a subordinate partition of unity (see, for example, Chavel (1993, §III.3)). □

2.2 Curvature functions

Recall that we are interested in the behaviour of a hypersurface which is moved with normal speed F determined by

$$F = f(\kappa_1, \ldots, \kappa_n)\,.$$

Since the ordering of the principal curvatures is arbitrary, it is natural to assume that f is symmetric under permutations of its variables. Two questions naturally arise: First, since we want the operator F to be smooth, how are the differentiability properties of F related to those of f? And, second, which invariants can be realized in this way? To answer the first question, we need some facts about symmetric functions.

2.2.1 Symmetric functions and their differentiability properties

Definition 2.4 (Symmetric functions)**.** *A function $q : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is* P_n-invariant *(or simply* symmetric*) if $(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) \in \Omega$ and $q(z_1, \ldots, z_n) = q(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ for all $\sigma \in P_n$, where P_n is the group of permutations of the set $\{1, \ldots, n\}$.*

Let $\mathrm{Sym}(n)$ *denote the set of symmetric $n \times n$ matrices. Then a function $q : O \subset$* $\mathrm{Sym}(n) \to \mathbb{R}$ *is* $GL(n)$-invariant *(or simply* symmetric*) if $\Sigma^{-1} \cdot Z \cdot \Sigma \in O$ and $q(Z) = q(\Sigma^{-1} \cdot Z \cdot \Sigma)$ for all $\Sigma \in GL(n)$, where $GL(n)$ is the general linear group of degree n.*

The two types of symmetric function defined above are clearly related: Denote by $\lambda : \mathrm{Sym}(n) \to \mathbb{R}^n$ the *eigenvalue map*; that is, the multi-valued map which assigns to a symmetric matrix Z the set of n-tuples with components given by its eigenvalues, which we denote in no particular order by $\lambda_1(Z), \ldots, \lambda_n(Z)$. Then, given any P_n-invariant function q, we obtain a $GL(n)$-invariant function $\hat{q}$ by setting $\hat{q}(Z) = q(z)$ for any $z \in \lambda(Z)$. Since q is P_n-invariant, it takes the same value on any choice of $z \in \lambda(Z)$, hence $\hat{q}$ is well-defined. Conversely, we obtain a P_n-invariant function q from any $GL(n)$-invariant function $\hat{q}$ by setting $q(z) = \hat{q}(Z)$, for any $Z \in \lambda^{-1}([z])$, where $[z]$ is the orbit of z under the P_n-action. Since $\hat{q}$ is $GL(n)$-invariant, it takes the same value on any two matrices with equal eigenvalues; hence q is well-defined. Thus every P_n-invariant function gives rise to a canonical $GL(n)$-invariant function and vice versa. We will henceforth make the notational abuse of using the same letter (q, say) to denote any two functions related in the above way, and speak of q either 'as a function of matrix variables' or 'as a function of eigenvalue variables'. We will refer to $O \subset \mathrm{Sym}(n)$ as the *matrix domain* of q and $\Omega \subset \mathbb{R}^n$ as the *eigenvalue domain* of q.

Since the eigenvalues of a symmetric matrix are not smooth at points of multiplicity, we might expect that a symmetric function is less regular with respect to the matrix variables than with respect to the eigenvalue variables; however, the following theorem shows that the eigenvalue map behaves with respect to smooth, symmetry preserving compositions as if it were a smooth function:

Theorem 2.5 (Glaeser (1963) and Schwarz (1975))**.** *Let q be a symmetric function. Then q is smooth with respect to the matrix variables if and only if it is smooth with respect to the eigenvalue variables. Moreover, the first and second derivatives are related by the following formulae:*

For any diagonal matrix Z in the matrix domain of q with eigenvalue n-tuple $z \in \{\lambda(Z)\}$, we have

$$\dot{q}_Z^{kl} = \dot{q}_z^k \delta^{kl}\,, \tag{2.17}$$

and, if the eigenvalues are all distinct, we have

$$\ddot{q}_Z^{pq,rs} V_{pq} V_{rs} = \ddot{q}_z^{pq} V_{pp} V_{qq} + 2 \sum_{p>q} \frac{\dot{q}_z^p - \dot{q}_z^q}{z_p - z_q} \left(V_{pq}\right)^2 . \tag{2.18}$$

for any $V \in \mathrm{Sym}(n)$, where we denote

$$\dot{q}_z^i v_i := \left.\frac{d}{ds}\right|_{s=0} q(z+sv)\,, \qquad \ddot{q}_z^{ij} v_i v_j := \left.\frac{d^2}{ds^2}\right|_{s=0} q(z+sv)$$

for z in the eigenvalue domain of q and $v \in \mathbb{R}^n$, and

$$\dot{q}_Z^{ij} V_{ij} := \left.\frac{d}{ds}\right|_{s=0} q(Z+sV)\,, \qquad \ddot{q}_Z^{pq,rs} V_{pq} V_{rs} := \left.\frac{d^2}{ds^2}\right|_{s=0} q(Z+sV)$$

for Z in the matrix domain of q and $V \in \mathrm{Sym}(n)$.

Proof. The proof the 'if' implication is due to Glaeser (1963), and the proof of the 'only if' implication is due to Schwarz (1975). Proofs of the relations (2.17) and (2.18) are given by Gerhardt (1996). See also Gerhardt (2006, Lemma 2.1.14) and Andrews (2007, Theorem 5.1). □

In fact, analogues of Theorem 2.5 hold under much weaker regularity requirements (Ball (1984). See also Gerhardt (2006, Chapter 2)).

Unless otherwise stated, we will henceforth assume all symmetric functions are smooth.

2.2.2 Curvature functions

Now suppose that we are in possession of a time-dependent immersion $\mathscr{X}$ with principal curvature n-tuple $\vec{\kappa} := (\kappa_1, \ldots, \kappa_n)$ and a symmetric function q. Then, so long as the eigenvalue domain of q contains the image of the principal curvatures of $\mathscr{X}$, we can form the function $Q(x,t) := q(\vec{\kappa}(x,t))$. We shall refer to a function so defined as a *curvature function*.

Denote by $\mathscr{W}$ the bundle of endomorphisms of $T\mathscr{M}$ which are self-adjoint with respect to the induced metric. Note that a choice of (time-dependent) smooth local orthonormal frame $\{e_i\}_{i=1}^n$ provides a local identification of the fibres of $\mathscr{W}$ with $\mathrm{Sym}(n)$. This allows us to write Q (locally) as $Q = q(W)$, where W is the local chart for $\mathscr{W}$ which takes a self-adjoint endomorphism of $T\mathscr{M}$ to its matrix of components with respect to the chosen frame (and, as usual, we are using the same letter q to denote the symmetric function as a function of either eigenvalue or matrix variables). We will therefore, by a further slight abuse of notation, often write $Q = q(\mathcal{W})$. It then follows from Theorem 2.5 that Q is smooth. In fact, we can obtain explicit, invariant formulae for the derivatives of Q in terms of derivatives of q and the covariant derivatives of $\mathcal{W}$: Computing at a point (x_0, t_0), we may choose $\{e_i\}_{i=1}^n$ such that $\mathcal{W}$ is diagonalized at (x_0, t_0), so that

$$\partial_k W_{ij} = \nabla_k \mathcal{W}_{ij} + \Gamma_{kij}(\kappa_j - \kappa_i)\,,$$

where $\Gamma_{kij} := g(\nabla_k e_i, e_j)$ are the (anti-symmetric) connection coefficients. Similarly, we obtain

$$\partial_t W_{ij} = \nabla_t \mathcal{W}_{ij} + \Gamma_{tij}(\kappa_j - \kappa_i)\,,$$

where $\Gamma_{tij} := g(\nabla_t \partial_i, \partial_j)$. Since, by identity (2.17) of Theorem 2.5, $\dot{q}^{ij}$ is diagonalized at (x_0, t_0), we obtain the invariant formulae

$$\begin{aligned}
\nabla_i Q &= \dot{Q}^{kl}\,\nabla_i \mathcal{W}_{kl} = \dot{Q}(\nabla_i \mathcal{W})\,,\\
\nabla_t Q &= \dot{Q}^{kl}\,\nabla_t \mathcal{W}_{kl} = \dot{Q}(\nabla_t \mathcal{W})\,,\\
\nabla_i \nabla_j Q &= \dot{Q}^{kl}\,\nabla_i \nabla_j \mathcal{W}_{kl} + \ddot{Q}^{pq,rs}\,\nabla_i \mathcal{W}_{pq}\,\nabla_j \mathcal{W}_{rs}\\
&= \dot{Q}(\nabla_i \nabla_j \mathcal{W}) + \ddot{Q}(\nabla_i \mathcal{W}, \nabla_j \mathcal{W})\,,
\end{aligned}$$

etc, where we are denoting the derivatives of Q with respect to the curvature using dots, just as for q; for example, $\dot{Q} \in \Gamma(\mathscr{W}^*)$ is the tensor defined by

$$\begin{aligned}
\dot{Q}_{(x,t)}(\mathcal{A}) &= \left.\frac{d}{ds}\right|_{s=0} q(W + sA)\\
&= \left.\frac{d}{ds}\right|_{s=0} q(\vec{\kappa} + s\alpha)\,,
\end{aligned}$$

where W and A are the component matrices of $\mathcal{W}_{(x,t)}$ and $\mathcal{A} \in (T^*\mathscr{M} \otimes T\mathscr{M})_{(x,t)}$ with respect to some local frame, and α denotes the eigenvalue n-tuple of $\mathcal{A}$. In particular, with respect to an orthonormal frame, we have the expressions

$$\begin{aligned}
\dot{Q}^{kl} &= \dot{q}_W^{kl}\,, \quad \ddot{Q}^{pq,rs} = \ddot{q}_W^{pq,rs}\,,\\
\dot{Q}^{k} &= \dot{q}_\kappa^{k}\,, \quad \ddot{Q}^{pq} = \ddot{q}_\kappa^{pq}\,,
\end{aligned}$$

etc.

Remark 2.2. In the following sections, when we consider curvature functions along solutions of (CF), it will be much more convenient to use the same letter (the roman capital

Q, say) to denote a symmetric function $q : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ and its corresponding curvature function $Q = q(\mathcal{W})$.

2.2.3 The admissibility conditions

We now summarize the admissibility conditions (Conditions 1) for the flow (CF).

Symmetry

The symmetry condition (Conditions 1 (i)) requires that the flow speed F be given by a smooth symmetric function $f : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ of the principal curvatures. This ensures that F can also be written as a smooth, basis invariant function of the components of the Weingarten tensor. Theorem 2.5 shows that F is smooth with respect to space and time and provides explicit expressions for its derivatives in terms of those of f and $\mathcal{W}$.

Parabolicity

The parabolicity condition (Conditions 1 (ii)) requires that the symmetric function f defining the flow speed be monotone increasing in each variable. Note that, by Theorem 2.5, monotonicity of f with respect to the eigenvalues is equivalent to monotonicity with respect to the symmetric matrices (and their canonical partial ordering). Thus, since f is smooth, the monotonicity condition requires, equivalently, that $\dot{f}^k > 0$ for each k or that the matrix $\dot{f}^{kl}$ is positive definite. In particular, this implies that $\dot{F}^k > 0$ for each k and $\dot{F}^{kl} > 0$ along any solution of the flow.

Homogeneity

The homogeneity condition (Conditions 1 (iii)) requires that the symmetric function f defining the flow speed be homogeneous of degree one. Clearly homogeneity with respect to the eigenvalue variables is equivalent to homogeneity with respect to the matrix variables. In particular, since the curvature of a hypersurface scales inversely with distance, this gives rise to invariance of (CF) under parabolic rescaling (see §3.1.5).

A further useful property of homogeneous functions is provided by Euler's Theorem:

Proposition 2.6 (Euler's theorem for homogeneous functions)**.** *Let E be a finite dimensional normed linear space and suppose that $q : C \subset E \to \mathbb{R}$ is a smooth degree α homogeneous function. Then*

$$Df\big|_z(z) = \alpha q\,,$$

where $Df\big|_z$ is the derivative of f at z.

Proof. Suppose that q is homogeneous of degree α. Then

$$\alpha q(z) = \frac{d}{ds}\bigg|_{s=0}(1+s)^\alpha q(z) = \frac{d}{ds}\bigg|_{s=0} q(z+sz) = Df\big|_z(z)\,.$$

□

Positivity

The positivity condition (Conditions 1 (iv)) requires that the symmetric function f which defines the flow speed be positive. This ensures that solutions of (CF) always move in their 'inwards' normal direction with positive speed. Moreover, since F is a solution of the linearization of (CF) (see §3.3), positivity of F is very useful in comparison arguments, as many natural quantities can be shown to be sub- or supersolutions of the linearized equation.

Note that, by Euler's theorem, any admissible flow speed is automatically positive (negative) at points in the positive cone $\Gamma^n_+ := \{z \in \mathbb{R}^n : z_i > 0 \text{ for each } i\}$ (negative cone $\Gamma^n_- := \{-z : z \in \Gamma_+\}$).

2.2.4 The auxiliary conditions

Next, we consider the Auxiliary Conditions (Conditions 2).

Surface flows

The first of the Auxiliary Conditions (Conditions 2 (v)) requires that the spatial dimension of the evolving hypersurface be two[2]. The derivatives of f (equations (2.17) and (2.18)) take a somewhat simpler form in this case. Combining this with the symmetry of $\nabla\mathcal{W}$ and the fact that, in two dimensions, $\nabla\mathcal{W}$ can have no totally off-diagonal components (such as $\nabla_1\mathcal{W}_{23}$) allows us, in some cases, to obtain results without the need for any of the other auxiliary conditions. These results are special to two dimensions.

Concavity

The convexity and concavity conditions (Conditions 2 (vii) and (vi)) require that f be, respectively, a convex or concave function of the eigenvalue variables[3]. We shall now prove that this is equivalent to convexity, respectively concavity, with respect to the matrix variables:

Lemma 2.7 (Cf. Ecker and Huisken (1989) and Andrews (1994a)). *Let $q : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth, symmetric function. If q is concave, then, for all $z \in \Omega$ with z_k pairwise distinct, it holds that*

$$\frac{\dot{q}^i_z - \dot{q}^j_z}{z_i - z_j} \leq 0 \tag{2.19}$$

for all $i \neq j$.

[2]We note that, by the homogeneity condition, up to scaling, the only admissible flow in one spatial dimension is the curve shortening flow.

[3]In fact, we shall assume a slightly weaker definition of concavity than the usual one, which allows the set Γ to be non-convex (see Remark 2.4).

Proof. Suppose that q is concave. Then, for any $v \in \mathbb{R}^n$ and $s \geq 0$ such that $z + sv \in \Omega$, we have

$$0 \geq \frac{d^2}{ds^2} q(z + sv) = \frac{d}{ds} \dot{q}^i (z + sv) v_i \,, \tag{2.20}$$

so that

$$\dot{q}^i (z + sv) v_i \leq \dot{q}^i (z) v_i \,.$$

Setting $v = -(e_i - e_j)$, where e_i is the basis vector in the direction of the i-th coördinate, we obtain

$$\left(\dot{q}^i - \dot{q}^j\right)\big|_z \leq \left(\dot{q}^i - \dot{q}^j\right)\big|_{z - s(e_i - e_j)} \,.$$

We may assume $z_i \geq z_j$. Then there is some $s_0 \geq 0$ such that

$$\left(z - s_0(e_i - e_j)\right)_i = \left(z - s_0(e_i - e_j)\right)_j \,.$$

By symmetry and convexity, $z - s_0(e_i - e_j) \in \Omega$ (this point lies on the line joining z and the point obtained from z by switching its i-th and j-th coordinates). Since q is symmetric, $\dot{q}^i = \dot{q}^j$ at this point and the claim follows. □

Remark 2.3. Note that, if strict inequality holds in 2.20, then strict inequality also holds in 2.19.

Corollary 2.8. *Let $q : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth symmetric function. Then q is concave (convex) with respect to the eigenvalues if and only if it is concave (convex) with respect to the matrix components.*

Proof. From the identity (2.18) of Theorem 2.5, we have, for any symmetric matrix V,

$$\ddot{q}^{ij,kl} V_{ij} V_{kl} = \ddot{q}^{ij} V_{ii} V_{jj} + 2 \sum_{i > j} \frac{\dot{q}^i - \dot{q}^j}{z_i - z_j} (V_{ij})^2$$

at any diagonal matrix Z with distinct eigenvalues z_i. So suppose that Z is a diagonal matrix with distinct eigenvalues $z_i = \lambda_i(Z)$. Clearly concavity of q at Z with respect to the matrix components implies concavity of q at z with respect to the eigenvalues. The converse follows from Lemma 2.7.

To see that the claim holds at any diagonal matrix Z, we need only observe that this is the limiting case along a sequence $Z^{(k)}$ of diagonal matrices with distinct eigenvalues which limits to Z.

Finally, the general case follows from the invariance of q with respect to similarity transformations.

□

Remark 2.4. Note that Lemma 2.7 and Corollary 2.8 also hold in some cases when the set Ω is not convex: If Ω is not convex, but $q : \Omega \to \mathbb{R}$ is smooth and locally convex, then, whenever q has a smooth, convex extension to a convex set containing Ω, the proof of Lemma 2.7 goes through unchanged. In fact, we do not need to assume that the extension is smooth, since, outside of Ω, derivatives of q may be replaced by difference quotients (See Andrews, Langford, and McCoy 2014b, Lemma 2.2). For this reason, given any open, symmetric set $\Omega \subset \mathbb{R}^n$, we will say that a smooth, symmetric function $q : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *concave* if q is locally concave and either Ω is convex or q extends to a concave function on a convex set containing Ω (and similarly for convexity of q). The reason for this definition is that a speed function may be concave on a large convex set but only parabolic (or positive, or smooth) on a non-convex subset.

It will also be useful to consider curvature functions (not necessarily speed functions) which possess some strict concavity; observe, though, that the Hessian of a degree one (or zero) homogeneous function q is always degenerate in the radial direction (that is, the direction of the argument), since Euler's Theorem implies that

$$\ddot{q}_z^{ij} z_i z_j = 0\,.$$

We will call a symmetric function q *strictly concave (convex) in non-radial directions* if

$$\ddot{q}_z^{ij} v_i v_j < 0\,(> 0)$$

for all z and all vectors v transverse to z; that is, $v \in \mathbb{R}^n \setminus \{kz : k \in \mathbb{R}\}$. By Lemma 2.7 (see Remark 2.3) this is equivalent to requiring

$$\ddot{q}_Z^{ij,kl} V_{ij} V_{kl} < 0\,(> 0)$$

for all Z and all $V \in \mathrm{Sym}(n) \setminus \{kZ : k \in \mathbb{R}\}$. Similarly, we call a curvature function Q strictly concave in non-radial directions if its defining symmetric function possesses the corresponding property.

Example 2.1. Consider the symmetric function n which gives the norm of a non-zero symmetric matrix:

$$n(A)^2 = \mathrm{tr}(AA^T) = \mathrm{tr}(A^2) = \left(\sum_{i=1}^n \lambda_i(A)^2\right)\,.$$

With respect to the eigenvalue coördinates $\{z_i\}_{i=1}^n$, we have

$$\dot{n}^k = \frac{z_k}{n}$$

and

$$\ddot{n}^{ij} = \frac{\delta^{ij}}{n} - \frac{z_i z_j}{n^3}\,.$$

Thus,

$$\ddot{n}^{ij} v_i v_j = \frac{1}{n^3} \left(|v|^2 n^2 - (z \cdot v)^2 \right) .$$

By the Cauchy-Schwarz inequality, this is non-negative, and strictly positive if v is non-radial. It follows that n is strictly convex in non-radial directions.

Inverse-concavity

We next consider the inverse-concavity condition (Conditions 2 (viii)). This is defined as follows.

Definition 2.9 (Inverse-concavity)**.** *Suppose that $q : \Gamma_+^n \subset \mathbb{R}^n \to \mathbb{R}$ is a positive P_n-invariant function, where $\Gamma_+^n := \{z \in \mathbb{R}^n : z_i > 0 \text{ for each } i\}$ is the positive cone. Then q is* inverse-concave *if the function $q_* : \Gamma_+^n \to \mathbb{R}$ defined by*

$$q_* \left(z_1^{-1}, \ldots, z_n^{-1} \right) := q(z_1, \ldots, z_n)^{-1}$$

is concave.

Similarly, let $q : \mathrm{Sym}_+(n) \to \mathbb{R}$ be a positive $GL(n)$-invariant function, where $\mathrm{Sym}_+(n) := \{Z \in \mathrm{Sym}(n) : Z > 0\}$. Then q is inverse-concave *if the function $q_* : \mathrm{Sym}_+(n) \to \mathbb{R}$ defined by*

$$q_* \left(Z^{-1} \right) := q(Z)^{-1}$$

is concave.

Note that, by Corollary 2.8, a symmetric function is inverse-concave with respect to the eigenvalue variables if and only if it is inverse-concave with respect to the matrix variables.

Let us now prove some useful characterizations of inverse-concavity.

Lemma 2.10. *Let q be a positive symmetric function. Then q is inverse-concave if and only if the quadratic form $Q_Z : \mathrm{Sym}(n) \times \mathrm{Sym}(n) \to \mathbb{R}$ defined by*

$$Q_Z(V,V) := \ddot{q}_Z(V,V) - \frac{2\dot{q}_Z(V)\dot{q}_Z(V)}{q(Z)} + 2\dot{q}_Z(VZ^{-1}V)$$

is non-negative definite for all $Z \in \mathrm{Sym}_+(n)$, where juxtaposition of matrix variables denotes matrix multiplication. Equivalently, q is inverse-concave if and only if the quadratic form $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q_z(v,v) := \ddot{q}_z(v,v) - \frac{2\dot{q}_z(v)\dot{q}_z(v)}{q(z)} + 2\dot{q}_z(vz^{-1}v)$$

is non-negative definite for all $z \in \Gamma^n_+$ and it holds that

$$\frac{\dot{q}^i - \dot{q}^j}{z_i - z_j} + \frac{\dot{q}^i}{z_j} + \frac{\dot{q}^i}{z_j} \geq 0$$

for each $i \neq j$ wherever the eigenvalues z_k are pairwise distinct, where $z^{-1} := (z_1^{-1}, \dots, z_n^{-1})$ and juxtaposition of eigenvalue variables denotes component-wise multiplication.

Proof. Differentiating $q_*(Z^{-1})$ with respect to $Z \in \mathrm{Sym}_+(n)$ in the direction of $V \in \mathrm{Sym}(n)$ yields

$$\dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}) = q(Z)^{-2}\dot{q}_Z(V). \tag{2.21}$$

Differentiating once more yields

$$\ddot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}, Z^{-1}VZ^{-1}) + 2\dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}VZ^{-1}) = \frac{(\dot{q}_Z(V))^2}{q(Z)^3} - \frac{\ddot{q}_Z(V,V)}{q(Z)^2}.$$

Applying 2.21 yields

$$\begin{aligned} -\ddot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1},&Z^{-1}VZ^{-1}) \\ &= q(Z)^{-2}\left(\ddot{q}_Z(V,V) - 2\frac{(\dot{q}_Z(V))^2}{q(Z)} + 2\dot{q}_Z(VZ^{-1}V)\right). \end{aligned} \tag{2.22}$$

The first claim follows.

For the second claim, we differentiate $q_*(z^{-1})$ with respect to z to obtain, for any $z \in \Gamma^n_+$ and any $v \in \mathbb{R}^n$,

$$\dot{q}_{*z^{-1}}(z^{-1}vz^{-1}) = q(z)^{-2}\dot{q}_z(v)\,. \tag{2.23}$$

Differentiating once more and applying (2.23), we obtain

$$-\ddot{q}_{*z^{-1}}(z^{-1}vz^{-1}, z^{-1}vz^{-1}) = q(z)^{-2}\left(\ddot{q}_z(v,v) - 2\frac{(\dot{q}_z(v))^2}{q(v)} + 2\dot{q}_z(vz^{-1}v)\right).$$

Next, consider

$$\begin{aligned} \frac{\dot{q}^i_{*z^{-1}} - \dot{q}^j_{*z^{-1}}}{z_i - z_j} &= \frac{1}{q(z)^2(z_i - z_j)}(\dot{q}^i_z z_i^2 - \dot{q}^j_z z_j^2) \\ &= \frac{z_i z_j}{q(z)^2}\left(\frac{\dot{q}^i_z - \dot{q}^j_z}{z_i - z_j} + \frac{\dot{q}^i_z}{z_j} + \frac{\dot{q}^j_z}{z_i}\right). \end{aligned}$$

The second claim now follows from Lemma 2.7. ☐

For admissible flow speeds, the local characterization of inverse-concavity is simplified:

Lemma 2.11. *Let $q : \Gamma^n_+ \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed. Then q is inverse-concave if and only if the quadratic form $\widehat{Q}_Z : \mathrm{Sym}(n) \times \mathrm{Sym}(n) \to \mathbb{R}$ defined by*

$$\widehat{Q}_Z(V,V) := \ddot{q}_Z(V,V) + 2\dot{q}(VZ^{-1}V)$$

is non-negative definite for every $Z \in \mathrm{Sym}_+(n)$.

Equivalently, q is inverse-concave if and only if the quadratic form $\widehat{Q}_z : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\widehat{Q}_z(v,v) := \ddot{q}_z(v,v) + 2\dot{q}_z(vz^{-1}v)$$

is non-negative definite for each $z \in \Gamma^n_+$ and it holds that

$$\frac{\dot{q}^i_z - \dot{q}^j_z}{z_i - z_j} + \frac{\dot{q}^i_z}{z_j} + \frac{\dot{q}^j_z}{z_i} \geq 0\,.$$

for each $p \neq q$ wherever the eigenvalues z_i are distinct.

Proof. Since q is homogeneous of degree one, q_* is homogeneous of degree one. Thus, recalling 2.22, Euler's theorem implies $Q_Z(V,\cdot) = 0$ whenever $V \propto Z$. Thus, Q_Z is non-negative definite if and only if it is non-negative definite on the transversal subspace $S_Z := \{V \in \mathrm{Sym}(n) : \dot{q}|_Z(V) = 0\}$. But, given $V \in S_Z$,

$$Q_Z(V,V) = \ddot{q}_Z(V,V) + 2\dot{q}_Z(VZ^{-1}V)\,.$$

This implies the first claim. The second claim follows similarly.

□

Lemma 2.12. *Let $q : \Gamma^n_+ \to \mathbb{R}$ be an admissible flow speed. Then q is inverse-concave if and only if the symmetric function $\chi : \Gamma^n_+ \to \mathbb{R}$ defined by*

$$\chi(z^{-1}) = -q(z)$$

satisfies

$$\ddot{\chi} \leq 2\frac{\dot{\chi} \otimes \dot{\chi}}{\chi}$$

Proof. This follows from Lemma 2.10 since, with respect to the matrix variables, we have

$$\dot{\chi}_{Z^{-1}}(V) = \dot{q}_Z(ZVZ)$$

and

$$-\ddot{\chi}_{Z^{-1}}(V,V) = \ddot{q}_Z(ZVZ, ZVZ) + 2\dot{q}_Z(ZVZVZ)\,.$$

□

The flow admits preserved cones

The final auxiliary condition requires that the flow (with speed $f:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$) admit preserved cones[4] (Conditions 2 (ix)); that is, given any solution $\mathscr{X}:\mathscr{M}^n\times[0,T)\to\mathbb{R}^{n+1}$ for (CF) there exists a cone $\Gamma_0\subset\mathbb{R}^n$ satisfying $\overline{\Gamma}_0\subset\Gamma\backslash\{0\}$ such that $\vec{\kappa}(\mathscr{M}\times[0,T))\subset\Gamma_0$. This condition functions as a uniform parabolicity condition since it ensures that the curvature of the solution stays away from the boundary of Γ. Existence of a preserved cone and the presence of one of the auxiliary conditions (v)–(vii) are the crucial components of the long-time existence theorem (Theorem 4.29). In §4.2, we will see that many admissible flow speeds automatically admit preserved cones; in particular, surface flows with positive speed, flows by positive, convex speeds $f:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$ such that $\Gamma_+\subset\Gamma$, and flows by inverse-concave speeds admit preserved cones. Flows by concave speeds $f:\Gamma\to\mathbb{R}$ will also admit preserved cones when restricted to an explicit 'small' cone (determined by the speed function) or if $f|_{\partial\Gamma}=0$; however, in general, flows by concave speeds may not admit preserved cones (see Andrews, McCoy, and Zheng 2013, §5).

2.2.5 Examples

We now describe some examples of curvature functions which define admissible flow speeds, and discuss subsets of these which satisfy each of the auxiliary conditions. The cases for which no proof or reference is given are easily checked.

Let us first recall that the *elementary symmetric polynomials* (in n-variables) are the functions $S_k:\mathbb{R}^n\to\mathbb{R}$, $k=0,\dots,n$ defined by

$$\begin{aligned}S_k(z_1,\dots,z_n)&:=\binom{n}{k}^{-1}\sum_{1<i_1<\dots\leq i_k\leq n} z_{i_1}\dots z_{i_k}\,,\ \text{for } k=1,\dots,n\,,\\ S_0(z_1,\dots,z_n)&:=1\,.\end{aligned}$$

We note that, along an immersion, the elementary symmetric polynomials give rise to several well-known curvature invariants, such as the mean curvature, $H=nS_1(\vec{\kappa})$, the scalar curvature, $\mathrm{Scal}=n(n-1)S_2(\vec{\kappa})$, and the Gauß curvature, $K=S_n(\vec{\kappa})$.

Example 2.2 (Admissible flow speeds)*.* The following symmetric functions define admissible flow speeds:

1. *The curve shortening flow*: Up to a rescaling of the time parameter, the only admissible flow speed for the flow (CF) in one space dimension is $f(z)=z$. The corresponding flow is called the *curve shortening flow*.

2. *The arithmetic mean*,

$$f(z)=S_1(z)\,,$$

[4]See Definition 4.4.

defines an admissible flow speed on all of $\mathbb{R}^n$. It is positive on the positive mean half-space, $\Gamma_1 := \{z \in \mathbb{R}^n : z_1 + \cdots + z_n > 0\}$. The corresponding curvature function is the (normalized) mean curvature and the corresponding flow is (up to a rescaling of the time parameter) the well-known *mean curvature flow*.

3. *The power means*,

$$H_r(z) := \left(\frac{1}{n}\sum_{i=1}^{n} z_i^r\right)^{\frac{1}{r}}, \text{ if } r \neq 0\,,$$
$$H_0(z) := \left(\prod_{i=1}^{n} z_i\right)^{\frac{1}{n}},$$

define positive admissible flow speeds on the cone $\Gamma_r := \{z \in \mathbb{R}^n : \sum_{i=1}^n z_i^r > 0\,, z_i^{r-1} > 0 \text{ for each } i\}$. Note that Γ_r contains the positive cone $\Gamma_+ := \{z \in \mathbb{R}^n : z_i > 0 \text{ for each } i\}$. The corresponding curvature functions include (up to normalization) the mean curvature ($r = 1$), the harmonic mean curvature ($r = -1$), the magnitude of the second fundamental form ($r = 2$), and the n-th root of the Gauß curvature ($r = 0$).

4. *Ratios of consecutive elementary symmetric polynomials*: The functions

$$f = \frac{S_k}{S_{k-1}}\,,\ k = 1, \ldots, n$$

define positive admissible speeds on the cone $\Gamma_k := \{z \in \mathbb{R}^n : S_l(z) > 0 \text{ for } l \leq k\}$ (see, for example, Lieberman 1996, Chapter XV). Note that Γ_k contains the positive cone Γ_+; in fact, $\Gamma_1 \supset \cdots \supset \Gamma_n = \Gamma_+$ (see, for example, Huisken and Sinestrari 1999a, Proposition 2.6). The corresponding curvature functions include (up to normalization) the mean curvature ($n = 1$) and the harmonic mean curvature ($k = n$).

5. *Roots of the elementary symmetric polynomials*: The functions

$$f = S_k^{\frac{1}{k}}\,, k = 1, \ldots, n$$

define positive admissible speeds on the cone $\Gamma_k := \{z \in \mathbb{R}^n : S_l(z) > 0 \text{ for } l \leq k\}$ (see, for example, Lieberman 1996, Chapter XV, or Example 8 below). The corresponding curvature functions include (up to normalization) the mean curvature ($n = 1$), the square root of the scalar curvature ($n = 2$), and the n-th root of the Gauß curvature ($k = n$).

6. *Positive linear combinations*

$$f = \sum_i \omega_i f_i \quad \text{(such that } \omega_i > 0 \text{ for each } i)\,,$$

of (positive) admissible flow speeds $f_i : \Gamma \to \mathbb{R}$ define (positive) admissible flow speeds $f : \Gamma \to \mathbb{R}$.

7. *Weighted geometric means*

$$f = \prod_{i=1}^{N} f_i^{\omega_i} \quad \left(\text{such that } \omega_i \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^{N} \omega_i = 1\right),$$

of positive admissible flow speeds $f_i : \Gamma \to \mathbb{R}$ define positive admissible flow speeds $f : \Gamma \to \mathbb{R}$.

8. *Roots of ratios of elementary symmetric polynomials*: The function

$$f = \left(\frac{S_k}{S_l}\right)^{\frac{1}{k-l}}, \ 0 \leq l < k \leq n,$$

is the geometric mean of $f_i = \frac{S_i}{S_{i-1}}$ for $i = l+1, \ldots, k$, and hence defines a positive admissible speed function on the cone $\Gamma_k := \{z \in \mathbb{R}^n : S_i(z) > 0 \text{ for each } i \leq k\}$.

9. *Homogeneous functions of admissible speeds*: If $f_i : \Gamma \to \mathbb{R}$, $i = 1, \ldots, N$ are admissible speeds and $\phi : \oplus_{i=1}^{N} f_i(\Gamma) \subset \mathbb{R}^N \to \mathbb{R}$ is a smooth, degree one homogeneous (positive) function which is monotone increasing in each variable, then

$$f := \phi(f_1, \ldots, f_N)$$

is a (positive) admissible speed.

We next consider flow speeds which satisfy one of the auxiliary conditions.

Surface flows

Example 2.3 (Admissible surface flows). The following symmetric functions define admissible speeds for surface flows:

1. *Admissible speeds*: All of the examples from Example 2.2 (with $n = 2$).

2. *A general construction for positive admissible speeds:* Write z_1, z_2 in polar coordinates (r, θ) with angle measured anti-clockwise from the positive ray; that is,

$$r = \sqrt{z_1^2 + z_2^2}, \quad \cos\theta = \frac{z_1 + z_2}{\sqrt{2(z_1^2 + z_2^2)}}, \quad \sin\theta = \frac{z_2 - z_1}{\sqrt{2(z_1^2 + z_2^2)}}.$$

Then, writing $f = r\phi(\theta)$ for some $\phi : (-\theta_0, \theta_0) \to \mathbb{R}$, Conditions 1 (i)–(iv) become: $\theta_0 < 3\pi/4$, $\phi > 0$, and

$$A(\theta) < \frac{\phi'(\theta)}{\phi(\theta)} < B(\theta),$$

where

$$A(\theta) := \begin{cases} -\infty\,, & \theta \in (-3\pi/4, -\pi/4]\,; \\ \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\,, & \theta \in (-\pi/4, 3\pi/4)\,; \end{cases}$$

and

$$B(\theta) := \begin{cases} \frac{\cos\theta + \sin\theta}{\sin\theta - \cos\theta}\,, & \theta \in (-3\pi/4, \pi/4)\,; \\ +\infty\,, & \theta \in [\pi/4, 3\pi/4)\,. \end{cases}$$

In particular, given any smooth, odd function $\psi : (-\theta_0, \theta_0) \to \mathbb{R}$, with $0 < \theta_0 \leq 3\pi/4$, satisfying $A(\theta) < \psi(\theta) < B(\theta)$, the function

$$f = r \exp\left(\int_0^\theta \psi(\sigma) d\sigma\right)$$

is a positive admissible speed function on the cone $\Gamma_{\theta_0} := \{z \in \mathbb{R}^2 : \theta(z) \in (-\theta_0, \theta_0)\}$.

Flows by concave speeds

Example 2.4 (Concave admissible flow speeds)*.* The following symmetric functions define concave admissible flow speeds:

1. *The power means* H_r with $r \leq 1$ define positive concave admissible flow speeds.
2. *The consecutive ratios of the elementary symmetric polynomials,* $\frac{S_k}{S_{k-1}}$, $k = 1, \ldots, n$ define positive concave admissible flow speeds (see, for example, Lieberman 1996, Chapter XV).
3. *Concave combinations*: If $f_i : \Gamma \to \mathbb{R}$, $i = 1, \ldots, N$ define concave admissible speeds, and $\phi : \oplus_{i=1}^N f_i(\Gamma) \subset \mathbb{R}^N \to \mathbb{R}$ is a smooth (positive) concave, degree one homogeneous function, then the function

   $$f := \phi\left(f_1, \ldots, f_N\right)$$

   defines a (positive) concave admissible flow speed. In particular, (positive) linear combinations of (positive) concave admissible speeds are (positive) concave admissible speeds and geometric means of positive admissible speeds are positive admissible speeds.
4. *The roots of ratios of the elementary symmetric polynomials,* $\left(\frac{S_k}{S_l}\right)^{\frac{1}{k-l}}$, $0 \leq l < k \leq n$, define positive concave admissible flow speeds.

Flows by convex speeds

Example 2.5 (Convex admissible flow speeds)*.* The following symmetric functions define convex admissible flow speeds:

1. *The power means*: H_r for $r \geq 1$ on the cone $\Gamma_r := \{z \in \mathbb{R}^n : H_r(z) > 0\,, z_i^{r-1} > 0 \text{ for each } i\}$.

2. *Positive linear combinations* of positive, convex admissible flow speeds define positive, convex admissible flow speeds. For example, the functions of the form $f := \sum_r \omega_r H_r$, $\omega_r > 0$, define positive, convex, admissible flow speeds on cones containing Γ_+. In particular, $f_\varepsilon := H_1 + \frac{\varepsilon}{\sqrt{n}} H_2$, $\varepsilon \in (0,1)$, defines a positive, convex admissible flow speed on the round cone $\Gamma_\varepsilon := \{z \in \mathbb{R}^n : H_1(z) + \frac{\varepsilon}{\sqrt{n}} H_2(z) > 0\}$. We note that Γ_ε contains the positive mean half-space.

3. *Convex combinations*: If $f_i : \Gamma \to \mathbb{R}$, $i = 1, \ldots, N$ define convex admissible speeds, and $\phi : \oplus_{i=1}^N f_i(\Gamma)$ is a smooth (positive) convex, degree one homogeneous function, then the function
$$f := \phi\,(f_1, \ldots, f_N)$$
defines a (positive) convex admissible flow speed. For example, the function $f_\varepsilon(z_1, \ldots, z_n) = H_r(z_1 + \varepsilon H, \ldots, z_n + \varepsilon H)$, $r \geq 1$ on the cone $\Gamma_\varepsilon := \{z \in \mathbb{R}^n : z_i + \varepsilon H > 0 \text{ for each } i\}$ defines a convex admissible speed.

4. *Concave functions:* If $g : \Gamma \to \mathbb{R}$ is a concave admissible speed, then the function $f := H - \varepsilon g$ on $\Gamma_\varepsilon := \{z \in \Gamma : \dot{g}^i < \frac{1}{\varepsilon} \text{ for each } i, \ (H > \varepsilon g)\}$ defines a (positive) convex admissible speed.

Flows by inverse-concave speeds

Example 2.6 (Inverse-concave admissible flow speeds (cf. Andrews (2007) and Andrews, McCoy, and Zheng (2013))). The following symmetric functions define inverse-concave admissible flow speeds:

1. *Convex admissible speeds* $f : \Gamma_+ \to \mathbb{R}$ are inverse-concave (this follows from Lemma 2.11). In particular, the power means, H_r with $r \geq 1$ are inverse-concave.

2. *Concave admissible speeds*: If $f : \Gamma_+ \to \mathbb{R}$ is a concave admissible speed, then $f_* : \Gamma_+ \to \mathbb{R}$ is an inverse-concave admissible speed. For example, the harmonic mean $H_{-1} = (H_1)_*$ is inverse-concave.

3. If $f : \Gamma_+ \to \mathbb{R}$ is an inverse-concave admissible speed and $r \in [0,1]$, then the function $f_r : \Gamma_+ \to \mathbb{R}$ defined by
$$f_r(z_1, \ldots, z_n) := \left(f(z_1^r, \ldots, z_n^r)\right)^{\frac{1}{r}} \tag{2.24}$$
defines an inverse-concave admissible speed function (see Andrews 2007, Theorem 3.2).

4. If $f : \Gamma_+ \to \mathbb{R}$ is a concave admissible speed and $r \in [-1, 0]$, then the function $f_r : \Gamma_+ \to \mathbb{R}$ defined by (2.24) defines an inverse-concave admissible speed function (see Andrews 2007, Theorem 3.2).

5. *The power means* H_r with $r \in [-1,1]$ are therefore concave, inverse-concave admissible speeds.

6. *The ratios of consecutive elementary symmetric polynomials*, $f := \frac{S_k}{S_{k-1}}$, $0 < k \leq n$, are concave, inverse-concave admissible speeds, since f is concave and $f_* = \frac{S_{n-k+1}}{S_{n-k}}$ is of the same type.

7. *Inverse-concave combinations*: If f_i, $i = 1, \ldots, N$ are (concave) inverse-concave admissible speeds, and $\phi : \Gamma_+^N \to \mathbb{R}$ is a strictly monotone increasing, degree one homogeneous (concave) inverse-concave function, then the function

$$f := \phi(f_1, \ldots, f_n)$$

is a (concave) inverse-concave admissible speed. In particular, positive linear combinations and weighted geometric means of (concave) inverse-concave speeds are (concave) inverse-concave speeds.

8. *The roots of ratios of the elementary symmetric polynomials*, $f := \left(\frac{S_k}{S_l}\right)^{\frac{1}{k-l}}$, $0 \leq l < k \leq n$, are concave, inverse-concave admissible speeds.

Flows which admit preserved cones

Example 2.7 (Admissible speeds whose flows admit preserved cones). The following symmetric functions define admissible speeds which give rise to flows that admit preserved cones:

1. *Surface flows* by positive admissible speeds admit preserved cones (Corollary 4.15).

2. *Flows by positive, convex admissible speeds* $f : \Gamma \to \mathbb{R}$ satisfying $\Gamma_+ \subset \Gamma$ admit preserved cones (Corollary 4.19).

3. *Flows by inverse-concave admissible speeds* admit preserved cones (this follows, for example, from Theorem 6.1. See also Andrews (2007)).

4. Admissible flow speeds $f : \Gamma \to \mathbb{R}$ for which $\overline{\Gamma}_{f>0} \subset \Gamma \setminus \{0\}$, where $\Gamma_{f>0} := \{z \in \Gamma : f(z) > 0\}$ preserve the cone $\Gamma_{f>0}$ (Proposition 4.5); for example, this holds for the speed $f := H_1 - \frac{1}{\sqrt{n}} H_2$, and many similar speeds which are admissible on $\Gamma = \mathbb{R}^n$.

5. *Concave admissible speeds* $f : \Gamma \to \mathbb{R}$ for which $\liminf_{\lambda \to \partial\Gamma} \frac{H_1}{f} > C$ preserve the cone $\Gamma_C := \{\lambda \in \Gamma : H_1(\lambda) \leq Cf(\lambda)\}$ (Proposition 4.12).

6. *Concave admissible speeds* $f : \Gamma \to \mathbb{R}$ such that $f = 0$ on $\partial\Gamma$. This is a special case of the previous example. It holds, for example, for the speeds $H_r : \Gamma_+ \to \mathbb{R}$, $r \leq 0$.

3. Short-time behaviour

In this section we will derive several results about the flow equation (CF) and its solutions. We begin by describing some invariance properties, and use these to construct some special solutions of the flow. Next, we introduce the linearized flow equation, and use the invariance properties of (CF) to construct some special solutions of the linearized equation. We then prove local existence of solutions of the initial value problem for (CF), which we do by reducing the flow equation to an equivalent scalar equation, and then appealing to a known existence result for (fully non-linear) scalar parabolic equations.

3.1 Invariance properties

We begin by deriving some invariance properties of the equation (CF), which allow us to generate new solutions from old.

3.1.1 Time translation

The simplest invariance property is invariance under time translation: Let $\mathscr{X} : \mathscr{M} \times (t_1, t_2) \to \mathbb{R}^{n+1}$ be a solution of (CF). Then the family $\mathscr{X}_\tau : \mathscr{M} \times (t_1 - \tau, t_2 - \tau) \to \mathbb{R}^{n+1}$ defined by $\mathscr{X}_\tau(x,t) := \mathscr{X}(x, t+\tau)$ also solves (CF), since $\partial_t \mathscr{X}_\tau(x,t) = \partial_t \mathscr{X}(x, t+\tau)$ and $\mathcal{W}_\tau(x,t) = \mathcal{W}(x, t+\tau)$, where $\mathcal{W}_\tau$ is the Weingarten map of $\mathscr{X}_\tau$.

3.1.2 Ambient isometries

Since (CF) is defined in terms of the induced geometry of $\mathscr{X}$, we expect that it should be invariant under isometries of the ambient space, and indeed this is the case, so long as the isometry is orientation preserving[1]: Let $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ be a solution of (CF) and let Φ be an isometry of $\mathbb{R}^{n+1}$. Then the family $\mathscr{X}_\Phi : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ defined by $\widetilde{\mathscr{X}}(x,t) = \Phi(\mathscr{X}(x,t))$ is also a solution of (CF). This is because Φ is affine (and hence its second derivative vanishes) and the induced Weingarten map is invariant under ambient isometries: First note that

$$\partial_t \mathscr{X}_\Phi(x,t) = \Phi_* \, \partial_t \mathscr{X}(x,t) = -F(x,t)\Phi_* \nu(x,t) \, .$$

[1]Orientation reversing isometries leave the flow invariant if F is given by an odd function of the principal curvatures (See §3.1.6).

Next we compute, with respect to some local coördinates,

$$\partial_i \mathscr{X}_\Phi = \Phi_* \, \partial_i \mathscr{X} \, .$$

In particular, the induced metric and normal for $\mathscr{X}_\Phi$ are given by ${g^\Phi}_{ij} = g_{ij}$, and $\nu_\Phi = \Phi_* \nu$. Now, since Φ is affine, we obtain

$$\partial_i \, \partial_j \, \mathscr{X}_\Phi = \Phi_* \, \partial_i \, \partial_j \, \mathscr{X} \, .$$

It follows that the Weingarten map $\mathcal{W}_\Phi$ of $\mathscr{X}_\Phi$ satisfies ${\mathcal{W}^\Phi}_{ij} = \mathcal{W}_{ij}$, so that $F_\Phi := F(\mathcal{W}_\Phi)\nu_\Phi = F(\mathcal{W})\Phi_* \nu = F\Phi_* \nu$ as required.

3.1.3 Reparametrization

Let $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ be a solution of (CF) and let ϕ be a diffeomorphism of $\mathscr{M}$. Then the time-dependent immersion $\mathscr{X}_\phi : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ defined by $\mathscr{X}_\phi(x,t) = \mathscr{X}\big(\phi(x),t\big)$ satisfies

$$\partial_t \mathscr{X}_\phi(x,t) = \partial_t \mathscr{X}(\phi(x),t) = -F(\mathcal{W}(\phi(x),t))\nu(\phi(x),t) = -F(\mathcal{W}_\phi(x,t))\nu_\phi(x,t) \, ,$$

where $\mathcal{W}_\phi$ and ν_ϕ are, respectively, the Weingarten map and normal of $\mathscr{X}_\phi$. Thus $\mathscr{X}_\phi$ is also a solution of (CF).

3.1.4 Time-dependent reparametrization

We observe that the previous calculation does not, in general, work if the reparametrization depends on time: Let $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ be a solution of (CF) and let $\varphi : \mathscr{M} \times I \to \mathscr{M}$ be a time-dependent diffeomorphism (a smooth one parameter family of diffeomorphisms $\varphi(\cdot, t)$). Then the new time-dependent immersion $\mathscr{X}_\varphi : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ defined by $\mathscr{X}_\varphi(x,t) := \mathscr{X}\,(\varphi(x,t),t)$ satisfies

$$\partial_t \mathscr{X}_\varphi(x,t) = \mathscr{X}_{*(\varphi(x,t),t)} \, \partial_t \, \varphi(x,t) - F(\varphi(x,t),t)\nu(\varphi(x,t),t) \, .$$

So $\partial_t \mathscr{X}_\varphi$ has an extra tangential term, $\mathscr{X}_* \, \partial_t \, \varphi$.

Thus, (CF) is *not* invariant under time-dependent diffeomorphisms of $\mathscr{M}$; however, this calculation has a useful converse: Suppose that $\mathscr{Y} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ satisfies

$$\langle \partial_t \mathscr{Y} \, , \, \nu \rangle = -F \, .$$

Then, if we set $\mathscr{X}(x,t) := \mathscr{Y}\,(\varphi(x,t),t)$ for some time-dependent diffeomorphism φ, we obtain

$$\partial_t \mathscr{X} = \mathscr{Y}_* \, \partial_t \, \varphi + T - F\nu \, ,$$

where the vector field $T \in \Gamma(\mathscr{X}^* T\mathbb{R}^{n+1})$ is the component of $\partial_t \mathscr{Y}$ tangent to the image hypersurface. If we now let φ be the solution of the ordinary differential equation

$$\begin{cases} \mathscr{Y}_* \, \partial_t \, \varphi = -T \\ \varphi(\cdot, 0) = \mathrm{id}\,, \end{cases}$$

then we obtain

$$\partial_t \mathscr{X} = -F\nu\,.$$

Therefore, any solution of the equation $\langle \partial_t \mathscr{Y}\,,\, \nu\rangle = -F$ gives rise to a solution of (CF) via a (unique) time-dependent reparametrization.

3.1.5 Space-time rescaling

Homogeneity of the speed implies a further useful invariance property: Observe that dilation of a hypersurface by a factor $\lambda > 0$ rescales the Weingarten curvature, and, due to homogeneity, the speed, by a factor λ^{-1}. This factor can be compensated by rescaling the time variable by a factor λ^{-2}: Let $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ be a solution of (CF) and suppose $\lambda > 0$. Define $\mathscr{X}_\lambda(x,t) := \lambda \mathscr{X}(x, \lambda^{-2}t)$. Then

$$\begin{aligned} \partial_t \mathscr{X}_\lambda(x,t) &= \lambda^{-1}\, \partial_t \mathscr{X}(x, \lambda^{-2}t) \\ &= -\,\lambda^{-1} F(\mathcal{W}(x, \lambda^{-2}t))\nu(x, \lambda^{-2}t) \\ &= -\,\lambda^{-1} F(\lambda \mathcal{W}_\lambda(x,t))\nu_\lambda(x,t) \\ &= -\,F(\mathcal{W}_\lambda(x,t))\nu_\lambda(x,t)\,, \end{aligned}$$

where ν_λ and $\mathcal{W}_\lambda$ are the normal and corresponding Weingarten map of $\mathscr{X}_\lambda$.

3.1.6 Orientation reversal

If the speed function is an odd function of the curvature, then the flow is also invariant under orientation reversals, since in that case

$$-F(\mathcal{W}_\nu)\nu = F(-\mathcal{W}_{-\nu})(-\nu) = -F(\mathcal{W}_{-\nu})(-\nu)\,,$$

where $\mathcal{W}_\nu$ denotes the Weingarten map of ν and $\mathcal{W}_{-\nu}$ the Weingarten map of $-\nu$.

3.2 Generating solutions from symmetries

Let us now introduce the *soliton solutions* of (CF). Broadly speaking, a *soliton solution* of an evolution equation is a solution whose image evolves purely by a one-parameter family of symmetries of the equation. Such solutions are, in a sense, 'stationary' solutions of the flow. We will see in §5.4 that they arise as limits of dilations of singularities.

3.2.1 Solutions generated by ambient isometries

As we have seen in §3.1.2, the orientation preserving isometries of $\mathbb{R}^{n+1}$ generate symmetries of (CF). These form the Lie group $\mathrm{isom}(n+1)$, which is generated by rotations $O \in SO(n+1)$ and translations $T \in \mathbb{R}^{n+1}$. The Lie algebra of $\mathrm{isom}(n+1)$, denoted by $\mathfrak{isom}(n+1)$, is generated by the infinitesimal rotations (antisymmetric matrices) $A \in \mathfrak{so}(n+1)$ and translations $T \in \mathbb{R}^{n+1}$.

Given a Killing vector field $K \in \Gamma(\mathfrak{isom}(n+1))$, let us refer to a solution $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ of (CF) as a *K-soliton* if it is generated by the flow of K; that is, if $\mathscr{X}(\varphi(x,t),t) = \Phi(t, \mathscr{X}_0(x))$ for some time-dependent reparametrization $\varphi : \mathscr{M} \times I \to \mathscr{M}$ satisfying $\varphi(\cdot, 0) = \mathrm{id}$, where $\Phi : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the flow of K (cf. Hungerbühler and Smoczyk 2000).

Let $\mathscr{X}$ be a K-soliton solution. Then differentiation of the defining relation yields

$$\mathscr{X}_{*(\varphi(x,t),t)} \frac{d\varphi}{dt}\Big|_{(x,t)} + \frac{\partial \mathscr{X}}{\partial t}\Big|_{(\varphi(x,t),t)} = \frac{d\Phi}{dt}\Big|_{(t,\mathscr{X}_0(x))}$$

Setting $t = 0$ and using the fact that $\varphi(\cdot, 0)$ is the identity, we obtain

$$-F_0(x) = \langle K(\mathscr{X}_0(x)) \, , \, \nu_0(x) \rangle \, . \tag{3.1}$$

Thus, a solution of the stationary equation (3.1) determines a K-soliton solution of the flow, since the subsequent (and antecedent) evolution of the initial immersion is determined by the flow of K.

Translating solutions

We shall refer to soliton solutions generated by translation as *translating solutions*. The infinitesimal translations are just the constant vector fields $T \in \mathbb{R}^{n+1}$; thus, from equation (3.1), we find that the translating solutions must satisfy

$$-F_0(x) = \langle T \, , \, \nu_0(x) \rangle \, .$$

The resulting solution is then given at other times, up to a time-dependent reparametrization φ, by applying the translation $\tau_t(X) = X + tT$:

$$\mathscr{X}\big(\varphi(x,t), t\big) = \mathscr{X}_0(x) + tT \, .$$

We note that translating solutions are *eternal*; that is, they exist for all times $t \in \mathbb{R}$.

Proposition 3.1. *The* Grim Reaper *curve* $\Gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to \mathbb{R}^2$ *defined by*

$$\Gamma(x,t) := (x, -\log\cos x + t)$$

is, up to a time-dependent reparametrization, a translating solution of the curve shortening flow.

More generally, if F is homogeneous of degree one, then the time-dependent immersion $\Gamma^n : \big((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}^{n-1}\big)\times\mathbb{R}\to\mathbb{R}^{n+1}$ *defined by*

$$\Gamma^n(x,t) := (x, -\log\cos x_1 + t)$$

is, up to a time-dependent reparametrization, a translating solution of (CF)*.*

Proof. We will show that Γ_0, the graph of $-\log\cos x$, satisfies $\langle T\,,\,\nu_0\rangle = -F_0$, where $T := (0,1)$. We have

$$\Gamma_0'(x) = (1,\tan x)$$

so that

$$\nu_0(x) = \frac{1}{1+\tan^2 x}(\tan x,-1) = (\sin x, -\cos x)$$

is the 'downward' normal. Differentiating the tangent vector, we obtain

$$\Gamma_0''(x) = (0,\sec^2 x)\,,$$

so that

$$\kappa_0(x) = -\frac{\langle\Gamma_0''(x)\,,\,\nu_0(x)\rangle}{\langle\Gamma_0'(x)\,,\,\Gamma_0'(x)\rangle} = \cos x$$

is the curvature of Γ_0. Therefore,

$$F_0(x) = F(\kappa_0(x)) = \cos^2 x = -\langle T\,,\,\nu_0(x)\rangle$$

as required.

The higher dimensional result follows similarly, with $T = e_{n+1}$, since the only component of the curvature is in the e_1 direction. □

Rotating solutions

We shall refer to soliton solutions generated by rotations as *rotating solutions.* The rotation generators are the vector fields $R_A : X \mapsto A(X)$, where $A\in\mathfrak{so}(n)$ is an anti-self-adjoint endomorphism of $\mathbb{R}^{n+1}$. From (3.1), we find that the rotating solutions must satisfy

$$-F_0 = \langle A(\mathscr{X}_0)\,,\,\nu_0\rangle\ .$$

The resulting solution $\mathscr{X}$ is then given at other times, up to a time-dependent reparametrization φ, by applying the rotation $\rho_t(X) = \exp(tA)X$:

$$\mathscr{X}\big(\varphi(x,t),t\big) = \exp(tA)\mathscr{X}_0(x)\,.$$

We observe that rotating solutions are also eternal solutions. Rotating solutions of the mean curvature flow have been studied by Hungerbühler and Smoczyk (2000).

3.2.2 Solutions generated by parabolic dilations

Since the flow speed is homogeneous of degree 1, one-parameter families of parabolic dilations δ_λ generate symmetries of (CF) (see §3.1.5). Recall that these are given by

$$\delta_\lambda \mathscr{X}(x,t) = (1+\lambda)\mathscr{X}\left(x,(1+\lambda)^{-2}t\right)$$

for $\lambda \in (-1,\infty)$.

Expanding solutions

We refer to a solution $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ of (CF) as an *expanding solution* if $\mathscr{X}$ is generated by positive dilations; that is, if, up to a time translation,

$$\mathscr{X}(\varphi(x,t),t) = \delta_t \mathscr{X}(x,0) = (1+t)\mathscr{X}(x,0)$$

for some time-dependent reparametrization φ satisfying $\varphi(\cdot,0) = \mathrm{id}$.

Differentiation yields

$$\mathscr{X}_{*(\varphi(x,t),t)} \frac{d\varphi}{dt}\bigg|_{(x,t)} - F(\varphi(x,t),t)\nu(\varphi(x,t),t) = \mathscr{X}(x,0) = \frac{1}{1+t}\mathscr{X}(\varphi(x,t),t)$$

Thus, $\mathscr{X}$ satisfies

$$F(\varphi(x,t),t) = -\frac{1}{1+t}\left\langle \mathscr{X}(\varphi(x,t),t)\,,\, \nu(\varphi(x,t),t)\right\rangle\,.$$

In particular, at time $t=0$, $\mathscr{X}$ must satisfy the stationary equation

$$F_0 = -\left\langle \mathscr{X}_0\,,\, \nu_0\right\rangle\,. \tag{3.2}$$

Conversely, it is easily checked that any solution $\mathscr{X}_0 : \mathscr{M} \to \mathbb{R}^{n+1}$ of (3.2) gives rise (up to a time-dependent reparametrization φ) to an expanding solution: $\mathscr{X}(x,t) := (1+t)\mathscr{X}(\varphi(x,t),0)$.

Observe that expanding solutions are *immortal*: they may be defined for $t \to \infty$.

Expanding solutions of the mean curvature flow (and their stability) have been studied by Clutterbuck and Schnürer (2011).

Shrinking solutions

We refer to a solution $\mathscr{X} : \mathscr{M} \times (-T,0] \to \mathbb{R}^{n+1}$ of (CF) as a *shrinking solution* if $\mathscr{X}$ is generated by negative dilations; that is, if, up to a time translation,

$$\mathscr{X}(\varphi(x,t),t) = \delta_t \mathscr{X}(x,0) = (1-t)\mathscr{X}(x,0)$$

for some time-dependent reparametrization φ satisfying $\varphi(\cdot, 0) = \mathrm{id}$.

Differentiation yields

$$\mathscr{X}_{*(\varphi(x,t),t)} \frac{d\varphi}{dt}\Big|_{(x,t)} - F(\varphi(x,t),t)\nu(\varphi(x,t),t) = -\mathscr{X}(x,0) = -\frac{1}{1-t}\mathscr{X}(\varphi(x,t),t)$$

Thus, $\mathscr{X}$ satisfies

$$F(\varphi(x,t),t) = \frac{1}{1-t}\left\langle \mathscr{X}(\varphi(x,t),t)\,,\,\nu(\varphi(x,t),t)\right\rangle\,.$$

In particular, at time $t = 0$, $\mathscr{X}$ must satisfy the stationary equation

$$F_0 = \langle \mathscr{X}_0\,,\,\nu_0\rangle\,. \tag{3.3}$$

Conversely, it is easily checked that any solution $\mathscr{X}_0 : \mathscr{M} \to \mathbb{R}^{n+1}$ of (3.3) gives rise (up to a time-dependent reparametrization φ) to a shrinking solution $\mathscr{X}(x,t) = (1-t)\mathscr{X}_0(\varphi(x,t),0)$.

We observe that shrinking solutions are *ancient*: they may be defined for $t \to -\infty$.

Proposition 3.2 (The shrinking sphere)**.** *Let $F : \Gamma \to \mathbb{R}$ be an admissible flow speed and let $\mathscr{X}_0 : S^n \to \mathbb{R}^{n+1}$ be the inclusion of $S^n(r_0)$, the sphere of radius $r_0 := c_0^{\frac{1}{2}}$, where $c_0 := F(1,\ldots,1)$. Then,*

$$\begin{aligned}\mathscr{X} : S^n \times (-\infty,1) &\to \mathbb{R}^{n+1}\\ (x,t) &\mapsto (1-t)\mathscr{X}_0(x)\end{aligned}$$

is a shrinking solution of (CF)*.*

Proof. Since $\mathscr{X}_0$ is the sphere of radius r_0, we have $\langle \mathscr{X}_0\,,\,\nu_0\rangle \equiv r_0$ and $F_0 \equiv F(r_0^{-1},\ldots,r_0^{-1}) = r_0^{-1}F(1,\ldots,1) = r_0$. Therefore $\mathscr{X}_0$ satisfies (3.3). The claim follows. □

A similar observation yields the following more general statement:

Proposition 3.3 (Shrinking cylinders)**.** *Let $F : \Gamma \to \mathbb{R}$ be an admissible flow speed defined on $\overline{\Gamma}_+ \setminus \{0\}$ and, for each $k \in \{0,\ldots,n-1\}$, let $\mathscr{X}_0 : \mathbb{R}^k \times S^{n-k} \to \mathbb{R}^{n+1}$ be the inclusion of $\mathbb{R}^k \times S^{n-k}(r_k)$, the round orthogonal cylinder of radius $r_k := c_k^{\frac{1}{2}}$, where $c_k := F(\underbrace{0,\ldots,0}_{k\text{-times}},1,\ldots,1)$. Then,*

$$\begin{aligned}\mathscr{X} : \mathbb{R}^k \times S^{n-k} \times (-\infty,1) &\to \mathbb{R}^{n+1}\\ (x,t) &\mapsto (1-t)\mathscr{X}_0(x)\end{aligned}$$

is (modulo a time-dependent reparametrization) a shrinking solution of (CF)*.*

Proof. The proof is similar to the proof of Proposition 3.2. □

Mean convex shrinking solutions of the mean curvature flow have been classified by Huisken (1990; 1993) and Abresch and Langer (1986). In the embedded case, the only possibilities are the cylinders $\mathbb{R}^k \times S^{n-k}$. There also exist non-mean convex shrinking solutions of the mean curvature flow, such as Angenent's torus (Angenent 1992). Moreover, Huisken's classification results have been extended in several ways to the fully non-linear setting by McCoy (2011). Finally, we mention that Halldorsson (2012) has classified all self-similar solutions of the curve shortening flow.

3.3 The linearized flow

An important equation related to the curvature flow (CF) is the linear equation

$$\partial_t u = \mathscr{L} u + \dot{F}(\mathcal{W}^2)u\,, \tag{LF}$$

where $\mathscr{L}$, the *linearization of* F, is the operator which acts by contracting the covariant Hessian with $\dot{F}$ (thus, in a local orthonormal frame, $\mathscr{L} = \dot{F}^{ij}\,\nabla_i\,\nabla_j$), and $\dot{F}(\mathcal{W}^2)$ denotes the contraction of $\mathcal{W}^2$ with $\dot{F}$ (thus, in a local orthonormal frame of eigenvectors of $\mathcal{W}$, $\dot{F}(\mathcal{W}^2) = \dot{F}^{kl}\mathcal{W}^2_{kl} = \dot{F}^k\kappa_k^2$).

Given a curvature flow (CF), we will refer to (LF) as the corresponding *linearized flow.* This equation arises naturally as the equation satisfied by the normal variation of a smooth family of solutions of (CF):

Lemma 3.4. *Let $\mathscr{X} : \mathscr{M} \times I \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{n+1}$ be a smooth family of solutions of* (CF) *with $\mathscr{X}|_{\varepsilon=0} =: \mathscr{X}_0$. Then the normal component,*

$$v := \langle \mathscr{X}_*\,\partial_\varepsilon, \nu\rangle|_{\varepsilon=0}\ ,$$

of the variation is a solution of the linearized flow (LF)*.*

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame of eigenvectors of $\mathcal{W}$. Observing that ${}^{\mathscr{X}}D_\varepsilon\nu \perp \nu$, a short computation yields

$$\nabla_i\, v = \left(\left\langle {}^{\mathscr{X}}D_\varepsilon \mathscr{X}_* e_i, \nu\right\rangle + \left\langle \mathscr{X}_*\,\partial_\varepsilon, {}^{\mathscr{X}}D_i\nu\right\rangle\right)\Big|_{\varepsilon=0}\ ,$$

and

$$\begin{aligned}\operatorname{Hess} v_{ij} = {}& -\left.\frac{\partial}{\partial\,\varepsilon}\right|_{\varepsilon=0}\mathcal{W}_{ij} + \langle \mathscr{X}_*\,\partial_\varepsilon\,,\,\nabla\,\mathcal{W}_{ij}\rangle|_{\varepsilon=0}\\ &+ \left\langle {}^{\mathscr{X}}D_\varepsilon \mathscr{X}_* e_i, {}^{\mathscr{X}}D_j\nu\right\rangle\Big|_{\varepsilon=0} + \left\langle {}^{\mathscr{X}}D_\varepsilon \mathscr{X}_* e_j, {}^{\mathscr{X}}D_i\nu\right\rangle\Big|_{\varepsilon=0} - \mathcal{W}^2{}_{ij}\big|_{\varepsilon=0}\,v\,.\end{aligned}$$

The time derivative of v is

$$\frac{\partial}{\partial\,t}v = -\left.\frac{\partial}{\partial\,\varepsilon}\right|_{\varepsilon=0} F + \langle \mathscr{X}_*\,\partial_\varepsilon\,,\,\operatorname{grad} F\rangle|_{\varepsilon=0}\ ,$$

where we invoked the identity (2.10) (Proposition 2.1). It follows that

$$(\partial_t - \mathscr{L})v = v\,\dot F(\mathcal{W}^2)\Big|_{\varepsilon=0}$$

since

$$\begin{aligned}\dot F^{ij}\left(\left\langle {}^{\mathscr{X}}\!D_\varepsilon \mathscr{X}_* e_i , {}^{\mathscr{X}}\!D_j \nu\right\rangle + \left\langle {}^{\mathscr{X}}\!D_\varepsilon \mathscr{X}_* e_j , {}^{\mathscr{X}}\!D_i \nu\right\rangle\right)\Big|_{\varepsilon=0} &= 2\,\dot F^i \kappa_i \left\langle {}^{\mathscr{X}}\!D_\varepsilon \mathscr{X}_* e_i\,,\, \mathscr{X}_* e_i\right\rangle\Big|_{\varepsilon=0}\\ &= \dot F^i \kappa_i\, \partial_\varepsilon \left\langle \mathscr{X}_* e_i\,,\, \mathscr{X}_* e_i\right\rangle\Big|_{\varepsilon=0}\\ &= 0\,.\end{aligned}$$

□

Corollary 3.5. *Let $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ be a solution of* (CF)*. Then the following scalars solve the linearized flow* (LF)*:*

1. *The speed function F.*
2. *The functions defined by*

$$u(x,t) := \langle \nu(x,t)\,,\, T\rangle\,,\quad T \in \mathbb{R}^{n+1}\,.$$

3. *The functions defined by*

$$u(x,t) := \langle \nu(x,t)\,,\, A\mathscr{X}(x,t)\rangle\,,\quad A \in \mathfrak{so}(n+1)\,.$$

4. *The function defined by*

$$u(x,t) := \langle \nu(x,t)\,,\, \mathscr{X}(x,t)\rangle + 2tF(x,t)\,.$$

5. *Linear combinations of the above examples.*

Proof. These functions arise from one parameter families of solutions of (CF) constructed from the invariance properties described in §3.1. □

By the maximum principle, the minimum of any initially positive supersolution of (LF) is non-decreasing. In particular, the inequality $F > 0$ is preserved.

Moreover, any subsolution (supersolution) u of (LF) may be compared from above (below) with any positive solution v, since

$$\begin{aligned}(\partial_t - \mathscr{L})\frac{u}{v} &= \frac{1}{v}(\partial_t - \mathscr{L})u - \frac{u}{v^2}(\partial_t - \mathscr{L})v - \frac{2}{v}\left\langle \nabla\frac{u}{v}\,,\,\nabla v\right\rangle\\ &\le (\ge) \frac{2}{v}\left\langle \nabla\frac{u}{v}\,,\,\nabla v\right\rangle\,.\end{aligned}$$

Thus, it is useful to have a positive solution of (CF) at our disposal.

Remarks 3.1. – Part 2. of Corollary 3.5 may be used to show that any inequality of the form $\langle \nu\,,\,T\rangle < 0$, $T\in\mathbb{R}^{n+1}$, is preserved; that is, if the image of the Gauß map of a (non-compact) solution of (CF) lies in a hemisphere at time $t=0$, it continues to lie in this hemisphere at later times.

– Nowhere in the proof of Lemma 3.4 did we use the fact that F is homogeneous. Homogeneity of F was used in Corollary 3.5 only to derive the fourth claim.

– Positivity of F on the initial data is necessary for our main results (Theorems 5.2, 5.15 and 6.1); however, in light of the preceding observation, we note that it is possible to obtain similar estimates by working with different positive solutions of the linearized flow. One situation where this works is when the initial datum is star-shaped, for in that case there is some $p\in\mathbb{R}^{n+1}$ such that $\langle \mathscr{X}_0 - p\,,\,\nu_0\rangle > 0$. Thus, by the maximum principle, the function $u(x,t) = \langle \mathscr{X}(x,t)-p\,,\,\nu(x,t)\rangle + 2tF(x,t)$ is a positive solution of the linearized flow. Smoczyk has made use of this observation to obtain, in particular, a convexity estimate for star-shaped surfaces evolving by mean curvature flow (Smoczyk 1998).

3.4 Evolving graphs

We now consider solutions of (CF) which may be written as graphs, either over a hyperplane or over some other fixed hypersurface of $\mathbb{R}^{n+1}$. Of course, such parametrizations always exist locally, for a short time.

3.4.1 Graphs over a hyperplane

Let Ω^n be a domain in $\mathbb{R}^n$ and consider a function $u:\Omega^n\times I\to\mathbb{R}$ for some time interval I. Consider the 'time-dependent graph' of u:

$$\begin{aligned}\mathrm{G}_u : \Omega^n\times I &\to \mathbb{R}^n\times\mathbb{R}\cong\mathbb{R}^{n+1}\\ (x,t)&\mapsto (x,u(x,t))\,,\end{aligned}$$

where we have identified $\mathbb{R}^n$ with the hypersurface $\{x\in\mathbb{R}^{n+1} : x^{n+1}=0\}$ of $\mathbb{R}^{n+1}$. Then, with respect to the induced Euclidean coördinates $\{x^i\}_{i=1}^n$, we obtain

$$\partial_i\,\mathrm{G}_u = \partial_i + u_i\,\partial_{n+1}\,,$$

where we are denoting $u_i := \partial_i\,u$. Thus,

$$\nu = \frac{1}{\sqrt{1+||Du||^2}}\,(Du-\partial_{n+1}) \tag{3.4}$$

is the 'downward' normal to G_u, where $||\cdot||$ and D denote the (fixed) Euclidean norm and derivative on $\mathbb{R}^{n+1}$. It follows that the induced metric components are given by

$$\begin{aligned} g_{ij} &= \delta_{ij} + u_i u_j \\ \text{and} \quad g^{ij} &= \delta^{ij} - \frac{u_i u_j}{1+||Du||^2} \,. \end{aligned}$$

The second fundamental form is therefore given by

$$\mathcal{W}_{ij} = -\left\langle \partial_i\, \partial_j\, \mathrm{G}_u\,,\, \nu \right\rangle = \frac{u_{ij}}{\sqrt{1+||Du||^2}} \,.$$

Thus, the Weingarten map is given by

$$\begin{aligned} \mathcal{W} = \frac{g^* D^2 u}{\sqrt{1+||Du||^2}} &= \frac{1}{\sqrt{1+||Du||^2}} \left(I - \frac{Du \otimes Du}{1+||Du||^2} \right) D^2 u \\ &= \frac{1}{\sqrt{1+||Du||^2}} \left(D^2 u - \frac{D^2 u (Du) \otimes Du}{1+||Du||^2} \right) , \end{aligned}$$

where g^* is the map $T^*\mathscr{M} \otimes T^*\mathscr{M} \to T^*\mathscr{M} \otimes T\mathscr{M} \cong \mathrm{End}(T\mathscr{M})$ defined by 'raising an index' with the inverse metric; that is,

$$g\left(g^* S(u), v\right) := S(u,v) \,.$$

It will be convenient to rewrite this in the form

$$\mathcal{W} = \frac{P^T D^2 u P}{\sqrt{1+||Du||^2}} \,, \tag{3.5}$$

where P is a square root of the inverse metric and juxtaposition denotes matrix multiplication (Urbas 1991). Writing $P = I - \lambda^{-1} Du \otimes Du$, it is not difficult to compute P explicitly. In fact, we find λ solves

$$\lambda^2 - 2\left(1+||Du||^2\right)\lambda + ||Du||^2\left(1+||Du||^2\right) = 0$$

so we may take

$$\lambda = 1 + ||Du||^2 + \sqrt{1+||Du||^2}\,;$$

that is,

$$P = I - \frac{Du \otimes Du}{\sqrt{1+||Du||^2}\left(1+\sqrt{1+||Du||^2}\right)} \,.$$

Since the time derivative of G_u is just $\partial_t\, \mathrm{G}_u = u_t\, \partial_{n+1}$, we see that G_u gives rise to a solution of (CF) (up to a unique time-dependent reparametrization) if u is a solution of

$$\begin{aligned} u_t &= \sqrt{1+||Du||^2} F\left(\frac{1}{\sqrt{1+||Du||^2}}\left[D^2u - \frac{D^2u(Du)\otimes Du}{1+||Du||^2}\right]\right) \\ &= \sqrt{1+||Du||^2} F\left(\frac{P^T D^2 u P}{\sqrt{1+||Du||^2}}\right) . \end{aligned} \tag{3.6}$$

We note the converse only holds locally, since not all hypersurfaces can be represented as graphs, and an evolving graph will not necessarily remain a graph.

3.4.2 Graphs over a hypersurface

More generally, we may consider time-dependent graphs over any fixed oriented hypersurface $\mathscr{X}_0 : \mathscr{M} \to \mathbb{R}^{n+1}$ (cf. Huisken and Polden 1999): First note that, since $\mathscr{X}_0$ is an immersion, we may choose ε_0 sufficiently small that the map

$$\begin{aligned} X : \mathscr{M} \times (-\varepsilon_0, \varepsilon_0) &\to \mathbb{R}^{n+1} \\ (x,h) &\mapsto \mathscr{X}_0(x) + h\nu_0(x) \end{aligned}$$

is itself an immersion, where ν_0 is a choice of unit normal field for $\mathscr{X}_0$. Let $\overline{g}$ denote the metric induced on $\overline{\mathscr{M}} := \mathscr{M} \times (-\varepsilon_0, \varepsilon_0)$ by X. Then, by the Gauß lemma, $\overline{g}$ admits the decomposition

$$\overline{g} = \overline{g}_h + dh \otimes dh ,$$

where $\overline{g}_h$ is the metric induced on the hypersurface $\mathscr{M} \times \{h\}$ by the immersion $X_h := X(\cdot, h)$.

Now consider the time-dependent graph

$$\begin{aligned} \mathrm{G}_u : \mathscr{M} \times I &\to \mathscr{M} \times \mathbb{R} \\ (x,t) &\mapsto (x, u(x,t)) \end{aligned}$$

of a smooth function $u : \mathscr{M} \times I \to \mathbb{R}$. Then, if u satisfies $\sup_{\mathscr{M}\times I} |u| < \varepsilon_0$, G_u is a time-dependent immersion, with (time-dependent) pullback metric given by

$$\gamma := (\mathrm{G}_u^* \overline{g}) = \overline{g}_u + du \otimes du .$$

The inverse of γ is given by

$$\gamma^* = \overline{g}_u^* - \frac{\overline{\mathrm{grad}}_u u \otimes \overline{\mathrm{grad}}_u u}{1 + |du|^2_{\overline{g}_u}} ,$$

where $\overline{\mathrm{grad}}_u$ is the gradient operator, and $|\cdot|_{\overline{g}_u}$ the norm, induced by the metric $\overline{g}_u$ (in coördinates, $\overline{\mathrm{grad}}_u u$ is given by $\overline{\mathrm{grad}}_u u^i = \overline{g}_u{}^{ik} u_k$, where $u_j = \partial_j\, u$).

It is also straightforward to compute a unit normal vector to G_u; we find

$$\vec{n}_u = \frac{1}{N}\left(\overline{\mathrm{grad}}_u u - \partial_h\right)\,,$$

where

$$N := \sqrt{1 + \overline{g}_u\,(du, du)}\,.$$

The second fundamental form of G_u is therefore given by

$$\begin{aligned}\mathcal{W}_{ij} &= -\,\overline{g}\left({}^{\mathrm{G}_u}\overline{D}_i \mathrm{G}_{u*}\,\partial_j, \vec{n}_u\right)\\ &= -\,\frac{1}{N}\overline{g}\left(\overline{D}_{\partial_i}\,\partial_j + u_{ij}\,\partial_h, \overline{\mathrm{grad}}_u u - \partial_h\right)\\ &= \frac{1}{N}\left(u_{ij} - \overline{g}_u\left(\overline{D}_i\,\partial_j, \overline{\mathrm{grad}}_u u\right)\right)\\ &= \frac{1}{N}\overline{D}_i\overline{D}_j u\,,\end{aligned}$$

where $\overline{D}$ is the connection induced on $\mathscr{M} \times (-\varepsilon_0, \varepsilon_0)$ by X and ${}^{\mathrm{G}_u}\overline{D}$ is the pullback of $\overline{D}$ to $\mathscr{M} \times I$ by G_u. Since the bundle of connections over $\mathscr{M}$ is affine, we may rewrite this in terms of the connection ∇^0 induced by the initial immersion $\mathscr{X}_0$ as

$$\mathcal{W}_{ij} = \nabla^0_i\,\nabla^0_j\,u + \sigma_{ij}\,,$$

where σ is the tensor defined by

$$\sigma(Y, Z) := du\left({}^{\mathrm{G}_u}\overline{D}_Y Z - \nabla^0_X\,Z\right)\,.$$

Importantly for the following section, we note that σ depends on u only up to first order.

Thus, the components of the Weingarten curvature are

$$\mathcal{W}_i{}^j = \gamma^{jk}\mathcal{W}_{ik} = \left(\overline{g}^{jk} - \frac{\overline{g}^{jp}u_p\overline{g}^{qk}u_q}{N^2}\right)\left(\nabla^0_i\,\nabla^0_j\,u + \sigma_{ij}\right)\,.$$

Next, we compose X with G_u to obtain the time-dependent immersion $\mathscr{X}(x,t) = \mathscr{X}_0(x) + u(x,t)\nu_0(x)$. Note first that, since X is an isometric embedding, the curvature of $\mathscr{X} \circ \mathrm{G}_u$ agrees with that of G_u. Moreover, the normal part of the time derivative of $\mathscr{X} \circ \mathrm{G}_u$ is

$$\langle \partial_t \mathscr{X}\,,\,\nu\rangle = u_t\,\langle \nu_0\,,\,X_*\vec{n}_u\rangle = \frac{1}{N}u_t\,.$$

We conclude that $\mathscr{X}$ solves (CF) (up to a unique time-dependent reparametrization) if and only if u solves

$$u_t = \sqrt{1+|du|^2_{\overline{g}_u}}\, F\left(\frac{1}{\sqrt{1+|du|^2_{\overline{g}_u}}}\left(\overline{g}^* - \frac{\overline{\mathrm{grad}}_u u \otimes \overline{\mathrm{grad}}_u u}{1+|du|^2_{\overline{g}_u}}\right)\cdot\left(\nabla^2 u + \sigma\right)\right), \tag{3.7}$$

where the symbol '$\cdot$' denotes the tensor contraction $(U\cdot V)_i{}^j = U^{jk}V_{ik}$.

3.5 Local existence of solutions

The parabolicity condition (Conditions 1 (ii)) ensures that (CF) is locally a (fully non-linear) parabolic system of partial differential equations. However, the invariance of (CF) under reparametrization ensures that this system is degenerate in tangential directions (cf. Hamilton 1982, §4), so that existence of solutions is not readily obtained from the literature. Such issues are common to geometric partial differential equations, and the degeneracy problems may be removed by fixing some special coördinates in such a way that the degeneracies in the highest order term vanish, leaving a non-degenerate system, to which the parabolic theory applies (Fourès-Bruhat 1952; DeTurck 1981/82; DeTurck 1983; Huisken 1984; Baker 2010). These methods are also applicable in our setting; however, we have chosen to take a different approach: By considering time-dependent immersions which may be written as graphs over the initial immersion, we showed in the previous section that (CF) is equivalent (for a short time) to a strictly parabolic scalar equation (cf. Urbas 1991; Giga and Goto 1992; Andrews 1994a; Huisken and Polden 1999). We then need only appeal to the local existence theory for *scalar* parabolic equations, which we have documented in Appendix A.

Definition 3.6. *Let $F:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$ be an admissible flow speed. Then an* admissible initial datum *for* (CF) *is a smooth immersion $\mathscr{X}_0:\mathscr{M}^n\to\mathbb{R}^{n+1}$ with $\vec{\kappa}(x)\in\Gamma$ for all $x\in\mathscr{M}$.*

Theorem 3.7 (Local existence of solutions)**.** *Let $F:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$, $n\geq 1$, be an admissible flow speed and $\mathscr{X}_0:\mathscr{M}^n\to\mathbb{R}^{n+1}$ an admissible initial datum for* (CF)*. Then there exists $\delta>0$ and a unique, smooth time-dependent immersion $\mathscr{X}:\mathscr{M}\times[0,\delta)\to\mathbb{R}^{n+1}$ satisfying the initial value problem*

$$\begin{cases} \dfrac{\partial\mathscr{X}}{\partial t}(x,t) = -F\left(\vec{\kappa}(x,t)\right)\nu(x,t) & (x,t)\in\mathscr{M}\times(0,\delta) \\ \mathscr{X}(x,0) = \mathscr{X}_0(x) & x\in\mathscr{M}\,. \end{cases}$$

Proof. We saw in the previous section that the statement of the theorem is equivalent to the existence of a solution $u:\mathscr{M}\times[0,\delta)\to\mathbb{R}$ of the initial value problem

$$\begin{cases} u_t(x,t) \;=\; \widehat{F}\left(\nabla^0\nabla^0 u(x,t), \nabla^0 u(x,t), u(x,t), x, t\right) & (x,t)\in\mathscr{M}\times(0,\delta) \\ u(x,0) \;=\; 0 & x\in\mathscr{M}\,, \end{cases} \tag{3.8}$$

where ∇^0 is the connection on $\mathscr{M}$ induced by $\mathscr{X}_0$ and

$$\widehat{F}(r,p,h,x,t) := \sqrt{1+|p|^2_{\overline{g}_h}}\,F\left(\frac{1}{\sqrt{1+|p|^2_{\overline{g}_h}}}\left[\overline{g}^*_h - \frac{\overline{g}^*_h p\otimes \overline{g}^*_h p}{1+|p|^2_{\overline{g}_h}}\right]\cdot[r+\sigma(p,h,x,t)]\right)\,.$$

By Theorem A.1 of Appendix A, it suffices to show that the initial value problem (3.8) is uniformly parabolic. First note that, since $\mathscr{M}^n$ is compact, there is a compact set $\Gamma_0 \subset \Gamma$ such that $\vec{\kappa}^0(\mathscr{M}) \subset \Gamma_0$. Then, since $\dot{F}$ is positive definite on Γ, we have

$$\lambda\,||\xi|| \le \dot{F}^{ij}\big|_{\Gamma_0}\xi_i\xi_j \le \Lambda\,||\xi||$$

for all $\xi \in \mathbb{R}^n$, where $\lambda := \min\{\dot{F}^{ij}(z) : z \in \Gamma_0, 1 \le i,j \le n\} > 0$ and $\Lambda := \max\{\dot{F}^{ij}(z) : z \in \Gamma_0, 1 \le i,j \le n\} < \infty$.

On the other hand, since $u_0 \equiv 0$, a simple computation yields

$$\left.\frac{\partial\,\widehat{F}}{\partial\,r_{ij}}\right|_{\left(\nabla^0\,\nabla^0\,u_0(x),\nabla^0\,u_0(x),u_0(x),x,0\right)} = \dot{F}^{ij}\left(\vec{\kappa}(x,0)\right)\,.$$

This proves the required uniform parabolicity, and hence the theorem.

□

Remark 3.1. Note that homogeneity of F was not used in the proof of Theorem 3.7.

4. Long-time behaviour

Now that we have established short-time existence of solutions for the class of flows admitted by Conditions 1, the next challenge is to understand the long-time change in the shape of solutions, and to characterize their asymptotic behaviour. The present section is concerned with the preservation of certain geometric properties of the initial datum. Our main tool is the maximum principle. We begin by deriving parabolic evolution equations for the Weingarten curvature and scalars constructed from it. We then show that the maximum principle may be applied to conclude that initial curvature cones are preserved by the flow, so long as certain natural auxiliary conditions for the speed function are met. This leads to a global existence theorem, Theorem 4.29: (Assuming the presence of auxiliary conditions) solutions remain smooth until the curvature blows-up. This is achieved by analysing a scalar equation related to a local graphical parametrization and appealing to the scalar parabolic theory. The remainder of the section is concerned with results of a more geometric flavour, including a comparison principle and the preservation of embeddedness.

4.1 Evolution of the curvature

We begin by deriving parabolic evolution equations for the Weingarten curvature and its scalar invariants. We will need the following lemma.

Lemma 4.1 (Simons-type identities)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed. Then, along any solution of* (CF)*, the Weingarten curvature satisfies*

$$\mathscr{L}\mathcal{W}_{ij} = \operatorname{Hess} F_{ij} - \ddot{F}(\nabla_i \mathcal{W}, \nabla_j \mathcal{W}) + \dot{F}(\mathcal{W})\mathcal{W}^2{}_{ij} - \dot{F}(\mathcal{W}^2)\mathcal{W}_{ij}\,. \tag{4.1}$$

Let $G : \Gamma \to \mathbb{R}$ be a curvature function. Then, along any solution of (CF)*, we have*

$$\mathscr{L}G = \dot{G}^{ij}\,\nabla_i\,\nabla_j\,F - \mathscr{Q}_{G,F}(\nabla\,\mathcal{W}, \nabla\,\mathcal{W}) + \mathscr{Z}_{G,F}(\mathcal{W})\,, \tag{4.2}$$

where we have defined

$$\mathscr{Q}_{G,F}(\nabla\,\mathcal{W}, \nabla\,\mathcal{W}) := \left(\dot{G}^{kl}\,\ddot{F}^{pq,rs} - \dot{F}^{kl}\,\ddot{G}^{pq,rs}\right)\nabla_k\,\mathcal{W}_{pq}\,\nabla_l\,\mathcal{W}_{rs} \tag{4.3}$$

and

$$\mathscr{Z}_{G,F}(\mathcal{W}) := \dot{G}^{pq}\dot{F}^{kl}\left(\mathcal{W}_{kl}\mathcal{W}^2_{pq} - \mathcal{W}_{pq}\mathcal{W}^2_{kl}\right) = \left(\dot{G}^{pq}\dot{F}^{kl} - \dot{F}^{pq}\dot{G}^{kl}\right)\mathcal{W}_{kl}\mathcal{W}^2_{pq}\,. \tag{4.4}$$

Proof. We compute locally, beginning with the commutation formula (2.15):

$$\nabla_k \nabla_l \mathcal{W}_{ij} = \nabla_i \nabla_j \mathcal{W}_{kl} + \mathcal{W}_{kl}\mathcal{W}^2{}_{ij} - \mathcal{W}_{ij}\mathcal{W}^2{}_{kl} + \mathcal{W}_{kj}\mathcal{W}^2{}_{il} - \mathcal{W}_{il}\mathcal{W}^2{}_{kj}\,.$$

Contracting this with $\dot{F}$ yields

$$\mathscr{L}\mathcal{W}_{ij} = \dot{F}^{kl}\nabla_i \nabla_j \mathcal{W}_{kl} + \dot{F}^{kl}\mathcal{W}_{kl}\mathcal{W}^2{}_{ij} - \dot{F}^{kl}\mathcal{W}_{ij}\mathcal{W}^2{}_{kl}\,.$$

On the other hand, we know that

$$\nabla_i \nabla_j F = \dot{F}^{kl}\nabla_i \nabla_j \mathcal{W}_{kl} + \ddot{F}^{pq,rs}\nabla_i \mathcal{W}_{pq}\nabla_j \mathcal{W}_{rs}\,,$$

so that

$$\mathscr{L}\mathcal{W}_{ij} = \nabla_i \nabla_j F - \ddot{F}^{pq,rs}\nabla_i \mathcal{W}_{pq}\nabla_j \mathcal{W}_{rs} + \dot{F}^{kl}\mathcal{W}_{kl}\mathcal{W}^2{}_{ij} - \dot{F}^{kl}\mathcal{W}_{ij}\mathcal{W}^2{}_{kl}\,.$$

This proves the first identity.

Now consider a curvature function G. Then

$$\begin{aligned}\mathscr{L}G = \dot{F}^{kl}\nabla_k \nabla_l G &= \dot{F}^{kl}\left(\ddot{G}^{pq,rs}\nabla_k \mathcal{W}_{pq}\nabla_l \mathcal{W}_{rs} + \dot{G}^{pq}\nabla_k \nabla_l \mathcal{W}_{pq}\right)\\ &= \dot{F}^{kl}\ddot{G}^{pq,rs}\nabla_k \mathcal{W}_{pq}\nabla_l \mathcal{W}_{rs} + \dot{G}^{ij}\mathscr{L}\mathcal{W}_{ij}\\ &= \left(\dot{F}^{kl}\ddot{G}^{pq,rs} - \dot{G}^{kl}\ddot{F}^{pq,rs}\right)\nabla_k \mathcal{W}_{pq}\nabla_l \mathcal{W}_{rs} + \dot{G}^{ij}\nabla_i \nabla_j F\\ &\quad + \dot{F}^{kl}\mathcal{W}_{kl}\dot{G}^{pq}\mathcal{W}^2_{pq} - \dot{G}^{kl}\mathcal{W}_{kl}\dot{F}^{pq}\mathcal{W}^2_{pq}\,.\end{aligned}$$

□

Remark 4.1. In fact, the formulae (4.1) and (4.2) of Lemma 4.1 hold for any (time-dependent) immersion and any pair of curvature functions F and G along it. The formulae (4.2) with $F = H$ and $G = |\mathcal{W}|^2$ have been used to prove rigidity results for minimal hypersurfaces (Simons (1968). See also Ecker and Huisken (1989) where non-linear functions of $\mathcal{W}$ are considered). These identities also play an important role in the mean curvature flow, where they are used in arguments to control the asymptotic behaviour of the Weingarten curvature (See Huisken (1984; 1987) and Huisken and Sinestrari (1999b; 1999a; 2009)). In Section 5, we will show that similar estimates for more general flows can be obtained with the help of the identity (4.2). Estimates for the terms $\mathscr{Q}$ and $\mathscr{Z}$ are key.

Proposition 4.2. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed, and $\mathscr{X} : \mathscr{M} \times I \to \mathbb{R}^{n+1}$ a solution of* (CF). *Then the Weingarten curvature of $\mathscr{X}$ satisfies*

$$(\nabla_t - \mathscr{L})\mathcal{W}_{ij} = \ddot{F}(\nabla_i \mathcal{W}, \nabla_j \mathcal{W}) + \dot{F}(\mathcal{W}^2)\mathcal{W}_{ij}\,. \tag{4.5}$$

Proof. Recalling equation (2.14), the claim follows directly from (4.1) and Euler's theorem for homogeneous functions. □

Next, we derive an evolution equation for local scalar invariants constructed from the Weingarten curvature.

Proposition 4.3. *Suppose that G is a curvature function. Then*

$$(\partial_t - \mathscr{L})G = \mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + \dot{G}(\mathcal{W})\dot{F}(\mathcal{W}^2)\,, \tag{4.6}$$

In particular, if G is homogeneous of degree α,

$$(\partial_t - \mathscr{L})G = \mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + \alpha G\dot{F}(\mathcal{W}^2)\,.$$

Proof. Computing locally, we have $\partial_t G = \dot{G}^{kl}\nabla_t \mathcal{W}_{kl}$. The claims now follow by applying equations (2.14) and (4.2), and Euler's theorem for homogeneous functions. □

Remark 4.2. Equations (4.5) and (4.6) are, via the maximum principle, crucial to preserving certain curvature sets, which is a fundamental step towards controlling the long term behaviour of solutions. Note that the degree one homogeneity of F enters here: The 'reaction' term of equation (2.14) is F times the square of $\mathcal{W}$. Due to Euler's theorem, this is cancelled by the term $\dot{F}(\mathcal{W})\mathcal{W}^2$ which arises when we contract $\nabla^2\mathcal{W}$ with $\dot{F}$. Thus, without the homogeneity condition, there is an extra term, $(F - \dot{F}(\mathcal{W}))\mathcal{W}^2$, in the evolution equation for $\mathcal{W}$ (which cannot a priori be controlled). A similar cancellation occurs for the evolution equation for G.

4.2 Preserving curvature cones

In this section we will study conditions under which a given curvature set is preserved by the equation (CF).

Definition 4.4 (Preserved cones)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed. Given a cone $\Gamma_0 \subset \mathbb{R}^n$, let us write $\Gamma_0 \subset\subset \Gamma$ to convey that $\overline{\Gamma}_0 \setminus \{0\} \subset \Gamma$.*

We say that a cone $\Gamma_0 \subset\subset \Gamma$ is preserved by the flow (CF) *if every solution $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ of* (CF) *satisfying $\vec{\kappa}(\mathscr{M} \times \{0\}) \subset \overline{\Gamma}_0$ satisfies $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \overline{\Gamma}_0$. We say that Γ_0 is* strongly preserved by the flow (CF) *if, in addition, either $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \partial\Gamma_0$ or $\vec{\kappa}(\mathscr{M} \times (0,T)) \subset \mathrm{int}(\Gamma_0)$.*

We say that the flow (CF) *by speed F* admits preserved cones *if given any solution $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$, there exists a preserved cone $\Gamma_0 \subset\subset \Gamma$ with $\vec{\kappa}(\mathscr{M} \times \{0\}) \subset \Gamma_0$.*

Preserved cones play a crucial role in controlling the long time behaviour of solutions of (CF).

The maximum principle is the main tool for showing that curvature cones are preserved; the simplest application is the following:

Proposition 4.5. *Suppose $F : \Gamma \to \mathbb{R}$ is an admissible speed such that $\Gamma_0 := \{z \in \Gamma : F(z) \geq 0\} \subset\subset \Gamma$. Then Γ_0 is strongly preserved by* (CF).

Proof. Since F satisfies (LF), the claim follows immediately from the strong maximum principle. □

4.2.1 Cones defined by curvature scalars

It is possible to show that other curvature cones are preserved by applying the maximum principle to the evolution equation (4.6) for a given curvature function G. For example, if G is homogeneous of degree one and $\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) \leq 0$ (at least wherever $\nabla G = 0$) then the cones defined by $G \leq CF$ are preserved by (CF). On the other hand, the expression $\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W})$ is in general rather complicated, so that finding curvature functions which satisfy $\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) \leq 0$ is no easy task. The following lemma provides a useful decomposition.

Lemma 4.6. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ and $G : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be smooth symmetric functions. For any diagonal matrix B with distinct eigenvalue n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma$ and any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, we have*

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\big|_B T_{kpq}T_{lrs} &= (\dot{G}^k\ddot{F}^{pq} - \dot{F}^k\ddot{G}^{pq})\big|_\lambda T_{kpp}T_{kqq} \\ &\quad + 2\sum_{p>q}\frac{(\dot{F}^p\dot{G}^q - \dot{G}^p\dot{F}^q)\big|_\lambda}{\lambda_p - \lambda_q}\Big((T_{pqq})^2 + (T_{qpp})^2\Big) \\ &\quad + 2\sum_{k>p>q}(\vec{G}^{kpq} \times \vec{F}^{kpq})\big|_\lambda \cdot \vec{\lambda}_{kpq}(T_{kpq})^2\,, \end{aligned} \tag{4.7}$$

where '$\times$' and '$\cdot$' are the three dimensional cross and dot product respectively, and the vectors $\vec{F}^{kpq}$, $\vec{G}^{kpq}$ and $\vec{\lambda}_{kpq}$ are defined by

$$\vec{F}^{kpq} := (\dot{F}^k, \dot{F}^p, \dot{F}^q)\,, \quad \vec{G}^{kpq} := (\dot{G}^k, \dot{G}^p, \dot{G}^q)\,,$$

$$\textit{and} \quad \vec{\lambda}_{kpq} := \left(\frac{\lambda_p - \lambda_q}{(\lambda_k - \lambda_p)(\lambda_k - \lambda_q)}\,,\ \frac{\lambda_k - \lambda_q}{(\lambda_k - \lambda_p)(\lambda_p - \lambda_q)}\,,\ \frac{\lambda_k - \lambda_p}{(\lambda_p - \lambda_q)(\lambda_k - \lambda_q)}\right).$$

Proof. Since B is diagonal, the identity (2.18) of Theorem 2.5 yields (suppressing the dependence on B)

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs} &= \sum_{k,p,q}(\dot{G}^k\ddot{F}^{pq} - \dot{F}^k\ddot{G}^{pq})T_{kpp}T_{kqq} \\ &\quad + 2\sum_{\substack{k\geq 1,\\ p>q}}\left(\dot{G}^k\frac{\dot{F}^p - \dot{F}^q}{\lambda_p - \lambda_q} - \dot{F}^k\frac{\dot{G}^p - \dot{G}^q}{\lambda_p - \lambda_q}\right)(T_{kpq})^2 \\ &=: Q_1 + Q_2\,.\end{aligned}$$

We now decompose the second term, Q_2, into the terms with $k = p$, $k = q$, $k > p$, $p > k > q$, and $q > k$ respectively:

$$
\begin{aligned}
Q_2 = {} & \sum_{p>q} \left(\dot{G}^p \frac{\dot{F}^p - \dot{F}^q}{\lambda_p - \lambda_q} - \dot{F}^p \frac{\dot{G}^p - \dot{G}^q}{\lambda_p - \lambda_q} \right) (T_{ppq})^2 \\
& + \sum_{p>q} \left(\dot{G}^q \frac{\dot{F}^p - \dot{F}^q}{\lambda_p - \lambda_q} - \dot{F}^q \frac{\dot{G}^p - \dot{G}^q}{\lambda_p - \lambda_q} \right) (T_{qpq})^2 \\
& + \left(\sum_{k>p>q} + \sum_{p>k>q} + \sum_{p>q>k} \right) \left(\dot{G}^k \frac{\dot{F}^p - \dot{F}^q}{\lambda_p - \lambda_q} - \dot{F}^k \frac{\dot{G}^p - \dot{G}^q}{\lambda_p - \lambda_q} \right) (T_{kpq})^2 \\
=: {} & Q_{21} + Q_{22} + Q_{23}\,.
\end{aligned}
$$

The first two sums add to

$$
Q_{21} + Q_{22} := 2 \sum_{p>q} \frac{\dot{F}^p \dot{G}^q - \dot{G}^p \dot{F}^q}{\lambda_p - \lambda_q} \left((T_{pqq})^2 + (T_{qpp})^2 \right),
$$

and the remaining term may be rewritten as

$$
\begin{aligned}
Q_{23} = {} & \sum_{k>p>q} \Bigg(\dot{G}^k \frac{\dot{F}^p - \dot{F}^q}{\lambda_p - \lambda_q} - \dot{F}^k \frac{\dot{G}^p - \dot{G}^q}{\lambda_p - \lambda_q} + \dot{G}^p \frac{\dot{F}^k - \dot{F}^q}{\lambda_k - \lambda_q} - \dot{F}^p \frac{\dot{G}^k - \dot{G}^q}{\lambda_k - \lambda_q} \\
& \qquad + \dot{G}^q \frac{\dot{F}^k - \dot{F}^p}{\lambda_k - \lambda_p} - \dot{F}^q \frac{\dot{G}^k - \dot{G}^p}{\lambda_k - \lambda_p} \Bigg) (T_{kpq})^2 \\
= {} & \sum_{k>p>q} \Bigg((\dot{G}^p \dot{F}^q - \dot{F}^q \dot{G}^p) \left(\frac{1}{\lambda_k - \lambda_p} - \frac{1}{\lambda_k - \lambda_q} \right) \\
& \qquad - (\dot{G}^k \dot{F}^q - \dot{F}^k \dot{G}^q) \left(\frac{1}{\lambda_p - \lambda_q} + \frac{1}{\lambda_k - \lambda_p} \right) \\
& \qquad + (\dot{G}^k \dot{F}^p - \dot{F}^k \dot{G}^p) \left(\frac{1}{\lambda_p - \lambda_q} - \frac{1}{\lambda_k - \lambda_q} \right) \Bigg) (T_{kpq})^2 \\
= {} & \sum_{k>p>q} (\vec{G}^{kpq} \times \vec{F}^{kpq}) \cdot \vec{\lambda}_{kpq} (T_{kpq})^2\,.
\end{aligned}
$$

□

Convex speeds

The task of finding curvature functions G satisfying $\mathscr{Q}_{G,F} \leq 0$ is made easier if the speed function F is convex (in particular, any convex, monotone decreasing G will do).

First, we note the following simple application of equation (4.6):

Proposition 4.7 (Cf. Andrews (1994a), Theorem 4.1)**.** *Let $F : \Gamma \to \mathbb{R}$ be a convex admissible speed function. Let $G : \Gamma \to \mathbb{R}$ be any smooth, degree one homogeneous curvature function which is convex and monotone decreasing. Suppose that $\Gamma_C := \{\lambda \in \Gamma : G(\lambda) \leq CF(\lambda)\} \subset\subset \Gamma$ for some $C \in \mathbb{R}$. Then Γ_C is strongly preserved by* *the flow* (CF).

Remark 4.3. In particular, if $\Gamma\subset\{z\in\mathbb{R}^n:H(z)>0\}$, where $H(\lambda):=\sum_{i=1}^n\lambda_i$, we can take $G(\lambda):=H(\lambda)\sum_{i=1}^n\varphi\left(\frac{\lambda_i}{H(\lambda)}\right)$ in Proposition 4.7, where $\varphi:\mathbb{R}\to\mathbb{R}_+\cup\{0\}$ is any smooth function which is positive and strictly convex, except on the set $\overline{\mathbb{R}}_+:=\{r\in\mathbb{R}: r\geq 0\}$, where it vanishes identically. This implies that the cones $\{\Gamma_C:=\bigcap_{i=1}^n\{\lambda\in\Gamma: \lambda_i\geq -CF(\lambda)\}$ are preserved, so long as $\Gamma_C\subset\subset\Gamma$. We will generalize this estimate in Proposition 4.8 to flows by convex speeds which may be defined on larger sets than the positive mean half-space $\{H(\lambda)>0\}$.

Proof of Proposition 4.7. This follows immediately by applying the strong maximum principle to the evolution equation (4.6) (for $G-CF$) since the assumptions on G and F ensure $\mathscr{Q}_{G,F}(\nabla\mathcal{W},\nabla\mathcal{W})\leq 0$. □

Next, we will show that flows by positive, convex speeds preserve the pinching ratios $\kappa_1/F\geq-\varepsilon$ for any $\varepsilon>0$. This estimate has a natural geometric interpretation: Observe that the distance from a point $\lambda\in\mathbb{R}^n$ to the positive cone Γ_+ is given by $\mathrm{dist}(\lambda,\Gamma_+)=\max\{-\lambda_{\min},0\}$, where $\lambda_{\min}=\min_i\lambda_i$. Thus, the pinching estimate $\kappa_1\geq-\varepsilon F$ says that the distance of the normalized curvature $\vec{\kappa}/F$ to the positive cone Γ_+ does not decrease under the flow. In Section 5, we will prove that this distance asymptotes to zero at points where F is blowing up.

Proposition 4.8. *Let $F:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$ be a positive, convex admissible speed function, and ε any positive number. Suppose that $\Gamma_0\subset\subset\Gamma$, where $\Gamma_0:=\bigcap_{i=1}^n\{\lambda\in\Gamma:\lambda_i\geq-\varepsilon F(\lambda)\}$. Then Γ_0 is strongly preserved by* (CF).

Proof. We will apply the maximum principle to the evolution equation for a smooth approximation to the function $\max\{-\kappa_1/F,0\}$. To this end, let $\varphi:\mathbb{R}\to\mathbb{R}$ be any smooth function which is positive and strictly convex, except on the set $\overline{\mathbb{R}}_+:=\{r\in\mathbb{R}:r\geq 0\}$, where it vanishes identically. For example, we could take

$$\varphi(r)=\begin{cases} r^4e^{-\frac{1}{r^2}} & \text{if} \quad r<0\,;\\ 0 & \text{if} \quad r\geq 0\,.\end{cases}$$

Now consider the curvature function $G:\Gamma\to\mathbb{R}$ defined by

$$G(\lambda):=F(\lambda)\sum_{i=1}^n\varphi\left(\frac{\lambda_i}{F(\lambda)}\right). \tag{4.8}$$

Observe that G is non-negative and vanishes on (and only on) the set $\overline{\Gamma}_+\cap\Gamma$. Furthermore, G is clearly smooth, symmetric, and homogeneous of degree one. We will show that

$$0\geq\mathscr{Q}_{G,F}\big|_Z(T,T)=(\dot{G}^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}^{pq,rs})\big|_Z T_{kpq}T_{lrs} \tag{4.9}$$

for any $Z\in\mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda\in\Gamma$ and any totally symmetric $T\in\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathbb{R}^n$. In fact, it suffices to prove (4.9) for diagonal Z with distinct eigenvalues, since the general case follows from continuity and $GL(n)$-invariance of $\mathscr{Q}_{G,F}$. In this case, the identity (2.18) of Theorem 2.5 implies (omitting the dependence on Z and λ)

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs} = (\dot{G}^{k}\ddot{F}^{pq} - \dot{F}^{k}\ddot{G}^{pq})T_{kpp}T_{kqq} + 2\sum_{\substack{k\geq 1,\\ p>q}}\left(\dot{G}^{k}\frac{\dot{F}^{p}-\dot{F}^{q}}{\lambda_p-\lambda_q} - \dot{F}^{k}\frac{\dot{G}^{p}-\dot{G}^{q}}{\lambda_p-\lambda_q}\right)(T_{kpq})^2\,.$$

First, observe that

$$\begin{aligned}\dot{G}^{k} &= \dot{F}^{k}\sum_{i=1}^{n}\varphi\left(\frac{\lambda_i}{F}\right) + \sum_{i=1}^{n}\varphi'\left(\frac{\lambda_i}{F}\right)\left(\delta_i^k - \frac{\lambda_i}{F}\dot{F}^{k}\right)\\ &= \varphi'\left(\frac{\lambda_k}{F}\right) + \dot{F}^{k}\sum_{i=1}^{n}\left[\varphi\left(\frac{\lambda_i}{F}\right) - \frac{\lambda_i}{F}\varphi'\left(\frac{\lambda_i}{F}\right)\right].\end{aligned}$$

Next, we compute

$$\begin{aligned}\ddot{G}^{pq} &= \ddot{F}^{pq}\sum_{i=1}^{n}\left[\varphi\left(\frac{\lambda_i}{F}\right) - \frac{\lambda_i}{F}\varphi'\left(\frac{\lambda_i}{F}\right)\right]\\ &\quad + \frac{1}{F}\sum_{i=1}^{n}\varphi''\left(\frac{\lambda_i}{F}\right)\left(\delta_i{}^{p} - \frac{\lambda_i}{F}\dot{F}^{p}\right)\left(\delta_i{}^{q} - \frac{\lambda_i}{F}\dot{F}^{q}\right).\end{aligned}$$

It follows that

$$\begin{aligned}(\dot{G}^{k}\ddot{F}^{pq} - \dot{F}^{k}\ddot{G}^{pq})T_{kpp}T_{kqq} &= \varphi'\left(\frac{\lambda_k}{F}\right)\ddot{F}^{pq}T_{kpp}T_{kqq}\\ &\quad - \frac{\dot{F}^{k}}{F}\sum_{i=1}^{n}\varphi''\left(\frac{\lambda_i}{F}\right)\left(\delta_i{}^{p} - \frac{\lambda_i}{F}\dot{F}^{p}\right)\left(\delta_i{}^{q} - \frac{\lambda_i}{F}\dot{F}^{q}\right)T_{kpp}T_{kqq}\\ &\leq 0\,.\end{aligned}$$

Finally, applying Proposition 2.7, we find

$$\begin{aligned}\sum_{\substack{k\geq 1,\\ p>q}}\left(\dot{G}^{k}\frac{\dot{F}^{p}-\dot{F}^{q}}{\lambda_p-\lambda_q} - \dot{F}^{k}\frac{\dot{G}^{p}-\dot{G}^{q}}{\lambda_p-\lambda_q}\right)(T_{kpq})^2 &= \sum_{k=1}^{n}\varphi'\left(\frac{\lambda_k}{F}\right)\sum_{p>q}\frac{\dot{F}^{p}-\dot{F}^{q}}{\lambda_p-\lambda_q}(T_{kpq})^2\\ &\quad - \sum_{k=1}^{n}\dot{F}^{k}\sum_{p>q}\frac{\varphi'\left(\frac{\lambda_p}{F}\right) - \varphi'\left(\frac{\lambda_q}{F}\right)}{\lambda_p-\lambda_q}(T_{kpq})^2\\ &\leq 0\,.\end{aligned}$$

The maximum principle now implies that upper bounds for G/F are preserved. It follows from the construction of φ that negative lower bounds for κ_1/F are preserved. □

Next, we consider the evolution of $(m+1)$-*convex* hypersurfaces, $m \in \{0,\ldots,n-2\}$; that is, those satisfying $\kappa_1 + \cdots + \kappa_{m+1} \geq 0$ at all points, or, equivalently, $\vec{\kappa} \in \overline{\Gamma}_{m+1}$ at all points, where

$$\Gamma_{m+1} := \bigcap_{\sigma\in P_n} \{\lambda\in\mathbb{R}^n : \lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(m+1)} > 0\}$$

and P_n is the group of permutations of the set $\{1,\dots,n\}$. For convenience, we also define $\overline{\Gamma}_0 := \{k(1,\dots,1) : k > 0\}$.

We will show that flows of $(m+1)$-convex hypersurfaces by convex speeds preserve the pinching ratios $(\kappa_1 + \cdots + \kappa_{m+1} - c_m^{-1}F)/F \geq -\varepsilon$ for any $\varepsilon > 0$, where

$$c_m := F(\underbrace{0,\dots,0}_{m-\text{times}},1,\dots,1)$$

is the value F takes on the unit cylinder $\mathbb{R}^m \times S^{n-m}$. Let us first provide an interpretation of this estimate: Define the curvature cones

$$\Lambda_m := \bigcap_{\sigma\in P_n} \left\{\lambda\in\Gamma : \lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(m+1)} - c_m^{-1}F(\lambda) \geq 0\right\}.$$

Notice that, by the monotonicity of F, Λ_0 is the positive ray $\{(\lambda,\dots,\lambda) : \lambda > 0\}$; thus, a hypersurface satisfying $\kappa_1 \geq c_0^{-1}F$ is a round sphere. The following lemma shows that, more generally, a hypersurface satisfying $\kappa_1 + \cdots + \kappa_{m+1} \geq c_m^{-1}F$ at all points must be strictly m-convex, $\kappa_1+\cdots+\kappa_m > 0$, wherever it is not '$(n-m)$-umbilic': $\kappa_1+\cdots+\kappa_m = 0$ and $\kappa_{m+1} = \cdots = \kappa_n$.

Lemma 4.9. *The cones Λ_m, $m \in \{0,\dots,n-2\}$, are convex and satisfy $\Lambda_m \subset \overline{\Gamma}_m$. Moreover,*

$$\Lambda_m \cap \partial\,\Gamma_m = \bigcup_{\sigma\in P_n} \left\{\lambda\in\mathbb{R}^n : \lambda_{\sigma(1)} = \cdots = \lambda_{\sigma(m)} = 0,\ \lambda_{\sigma(m+1)} = \cdots = \lambda_{\sigma(n)} > 0\right\}.$$

Proof. Convexity of Λ_m follows from concavity of the defining functions, $G_\sigma := \lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(m+1)} - c_m^{-1}F$.

Next, we note that the point $\vec{\lambda}^m := (\underbrace{0,\dots,0}_{m\text{-times}},1\dots,1)$ satisfies

$$\lambda_1^m + \cdots + \lambda_{m+1}^m - c_m^{-1}F(\vec{\lambda}^m) = 1 - F(\vec{\lambda}^m)^{-1}F(\vec{\lambda}^m) = 0\,.$$

Thus, $\vec{\lambda}^m \in \partial\,\Lambda_m$.

To see that $\Lambda_m \subset \overline{\Gamma}_m$, we will show that the half-space $H_{m+1} := \{z\in\mathbb{R}^n : z_1 + \cdots + z_{m+1} > 0\}$ is a supporting half-space for Λ_m at $\vec{\lambda}^m$ (see Definition 4.16). The claim then follows from convexity of Λ_m and symmetry (see Lemma 4.17, 1.). To this end, note that (by differentiating the defining relations) the inward normal cone to H_{m+1} at $\vec{\lambda}^m$ is generated by the vector

$$\vec{\ell} = (\underbrace{1,\dots,1}_{m\text{-times}},0,\dots,0)$$

and the inward normal cone to Λ_m at $\vec{\lambda}^m$ is generated by the vectors

$$\vec{\ell}_p := \left(1 - c_m^{-1}\dot{F}^1, \dots, 1 - c_m^{-1}\dot{F}^m, -c_m^{-1}\dot{F}^{m+1}, \dots, 1 - c_m^{-1}\dot{F}^p, \dots, -c_m^{-1}\dot{F}^n\right)\Big|_{\vec{\lambda}^m}.$$

Noting that $\dot{F}^1(\vec{\lambda}^m) = \cdots = \dot{F}^m(\vec{\lambda}^m) =: r$ and (using Euler's theorem) that $\dot{F}^{m+1}(\vec{\lambda}^m) = \cdots = \dot{F}^n(\vec{\lambda}^m) = c_m\frac{1}{n-m}$, we find

$$\vec{\ell}_p = \Big(\underbrace{r, \dots, r}_{m\text{-times}}, -\frac{1}{n-m}, \dots, \underbrace{1 - \frac{1}{n-m}}_{p\text{-th position}}, \dots, -\frac{1}{n-m}\Big).$$

It follows that $\sum_{p=m+1}^n \vec{\ell}_p = r(n-m)\vec{\ell}$, so that $\vec{\ell}$ is in the inward normal cone to Λ_m at $\vec{\lambda}^m$. Thus, as claimed, H_{m+1} is a supporting half-space for Λ_m at $\vec{\lambda}^m$.

To prove the final claim, suppose that $\lambda \in \Lambda_m \cap \partial\Gamma_m$. Without loss of generality, assume that $\lambda_1 \leq \cdots \leq \lambda_n$. Then, making use of symmetry, monotonicity, homogeneity, and convexity of F,

$$\begin{aligned}
c_m^{-1}F(\lambda) &\leq \lambda_1 + \cdots + \lambda_{m+1}\\
&= \lambda_{m+1}\\
&= c_m^{-1}F(\underbrace{0, \dots, 0}_{m\text{-times}}, \lambda_{m+1}, \dots, \lambda_{m+1})\\
&\leq c_m^{-1}F(\underbrace{0, \dots, 0}_{m\text{-times}}, \lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_n)\\
&= c_m^{-1}F(\underbrace{\lambda_1 + \cdots + \lambda_m, \dots, \lambda_1 + \cdots + \lambda_m}_{m\text{-times}}, \lambda_{m+1}, \lambda_{m+1}, \dots, \lambda_n)\\
&= c_m^{-1}\frac{1}{n-m}F\Big((\lambda_1, \dots, \lambda_m, \lambda_{m+1}, \dots, \lambda_n) + (\lambda_2, \dots, \lambda_m, \lambda_1, \lambda_{m+1}, \dots, \lambda_n)\\
&\qquad\qquad + \cdots + (\lambda_m, \lambda_1, \dots, \lambda_{m-1}, \lambda_{m+1}, \dots, \lambda_n)\Big)\\
&\leq c_m^{-1}F(\lambda).
\end{aligned}$$

The claim follows since, by strict monotonicity of F, the second inequality is strict unless $\lambda_{m+1} = \cdots = \lambda_n$. □

Thus, the pinching estimate $\kappa_1 + \cdots + \kappa_{m+1} - c_m^{-1}F \geq -\varepsilon F$ says that the distance of the normalized curvature $\vec{\kappa}/F$ to the cone Λ_m does not deteriorate under the flow, so that, in the sense of Lemma 4.9, the hypersurface does not become 'less m-convex'. In Section 5 we will show that the distance of $\vec{\kappa}/F$ to the cone $\Lambda_m \cap \Gamma_+$ asymptotes to zero at points where F is blowing up. This suggests that the hypersurface is becoming convex and either m-cylindrical or strictly m-convex at a singularity (see Theorem 5.15).

Proposition 4.10. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be a convex admissible speed function and m an integer from the set $\{0, \dots, n-2\}$. Suppose that $\Gamma_0 := \bigcap_{\sigma \in P_n}\{\lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(m+1)} - c_m^{-1}F(\lambda) \geq -\varepsilon F(\lambda)\} \subset\subset \Gamma$ for some $\varepsilon \geq 0$. Then Γ_0 is strongly preserved by* (CF).

Remarks 4.1. 1. Taking $\varepsilon = c_m^{-1}$, we find, in particular, that the cones Γ_{m+1} are preserved for each $m \in \{0, \dots n-2\}$.

2. Note also that the $m = 0$ case of the proposition yields the pinching estimates $\kappa_1/F > (c_0^{-1} - \varepsilon)$ for flows of convex hypersurfaces. It follows that every convex admissible flow speed $F : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$ admits preserved cones.

Proof of Proposition 4.10. We will apply the maximum principle to the evolution equation for a smooth approximation to the function $\max\{-(\kappa_1 + \cdots + \kappa_{m+1} - c_m^{-1} F), 0\}$, $m \in \{0, \dots, n-2\}$.

For each $m \in \{0, \dots, n-2\}$, define $G_m : \Gamma \to \mathbb{R}$ by

$$G_m(\lambda) := F(\lambda) \sum_{\sigma \in H_m} \varphi\left(\frac{\sum_{i=1}^{m+1} \lambda_{\sigma(i)} - c_m^{-1} F(\lambda)}{F(\lambda)}\right), \tag{4.10}$$

where H_m is the quotient of P_n by the equivalence relation

$$\sigma \sim \omega \quad \text{if} \quad \sigma\left(\{1, \dots, m+1\}\right) = \omega\left(\{1, \dots, m+1\}\right),$$

and, just as in the proof of Proposition 4.8, $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function which is strictly convex and positive, except on $\overline{\mathbb{R}}_+$, where it vanishes identically.

We will show, using Lemma 4.6, that

$$\mathcal{Q}_{G_m,F}(T,T) = (\dot{G}_m^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}_m^{pq,rs})\big|_Z T_{kpq} T_{lrs} \leq 0$$

for any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ and any symmetric $Z \in \mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda \in \Gamma_0$. As before, by continuity and $GL(n)$-invariance of $\mathcal{Q}_{G_m,F}$, it suffices to prove the estimate for diagonal Z with distinct eigenvalues $\lambda_1 < \cdots < \lambda_n$.

We first compute

$$\begin{aligned}
\dot{G}_m^k &= \dot{F}^k \sum_{\sigma \in H_m} \varphi(r_\sigma) + \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \left({\delta_{\sigma(i)}}^k - \frac{\lambda_{\sigma(i)}}{F} \dot{F}^k \right) \\
&= \dot{F}^k \sum_{\sigma \in H_m} \left(\varphi(r_\sigma) - \varphi'(r_\sigma) \frac{\sum_{i=1}^{m+1} \lambda_{\sigma(i)}}{F} \right) + \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) {\delta_{\sigma(i)}}^k,
\end{aligned}$$

and

$$\begin{aligned}
\ddot{G}_m^{pq} &= \left(\sum_{\sigma \in H_m} \varphi(r_\sigma) - \sum_{\sigma \in H_m} \varphi'(r_\sigma) \frac{\sum_{i=1}^{m+1} \lambda_{\sigma(i)}}{F} \right) \ddot{F}^{pq} \\
&\quad + \sum_{\sigma \in H_m} \frac{\varphi''(r_\sigma)}{F} \sum_{i=1}^{m+1} \left({\delta_{\sigma(i)}}^p - \frac{\lambda_{\sigma(i)}}{F} \dot{F}^p \right) \sum_{i=1}^{m+1} \left({\delta_{\sigma(i)}}^q - \frac{\lambda_{\sigma(i)}}{F} \dot{F}^q \right),
\end{aligned}$$

where we are denoting $r_\sigma(\lambda) := \frac{\sum_{i=1}^{m+1}\lambda_{\sigma(i)} - c_m^{-1}F(\lambda)}{F(\lambda)}$. It follows that

$$\begin{aligned}\dot{G}_m^k\ddot{F}^{pq} - \dot{F}^k\ddot{G}_m^{pq} = &\sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\delta_{\sigma(i)}{}^k\ddot{F}^{pq}\\ &- \dot{F}^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{F}\sum_{i=1}^{m+1}\left(\delta_{\sigma(i)}{}^p - \frac{\lambda_{\sigma(i)}}{F}\dot{F}^p\right)\sum_{i=1}^{m+1}\left(\delta_{\sigma(i)}{}^q - \frac{\lambda_{\sigma(i)}}{F}\dot{F}^q\right).\end{aligned}$$

Fixing the index k and setting $\xi_p = T_{kpp}$, we find

$$\varphi'(r_\sigma)\sum_{i=1}^{m+1}\delta_{\sigma(i)}{}^k\ddot{F}^{pq}\xi_p\xi_q \le 0$$

for each σ, and

$$\begin{aligned}-\dot{F}^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{F}\sum_{i=1}^{m+1}\left(\delta_{\sigma(i)}{}^p - \frac{\lambda_{\sigma(i)}}{F}\dot{F}^p\right)\sum_{i=1}^{m+1}\left(\delta_{\sigma(i)}{}^q - \frac{\lambda_{\sigma(i)}}{F}\dot{F}^q\right)\xi_p\xi_q&\\ = -\dot{F}^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{F}\left(\sum_{i=1}^{m+1}\left(\delta_{\sigma(i)}{}^p - \frac{\lambda_{\sigma(i)}}{F}\dot{F}^p\right)\xi_p\right)^2&\\ \le 0\,.&\end{aligned}$$

Since both inequalities hold for all k, we deduce that

$$\left(\dot{G}_m^k\ddot{F}^{pq} - \dot{F}^k\ddot{G}_m^{pq}\right)T_{kpp}T_{kqq} \le 0\,.$$

Next, consider

$$\begin{aligned}\dot{F}^p\dot{G}_m^q - \dot{G}_m^p\dot{F}^q &= \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\left(\delta_{\sigma(i)}{}^q\dot{F}^p - \delta_{\sigma(i)}{}^p\dot{F}^q\right)\\ &= \left(\sum_{\sigma\in O_q}\varphi'(r_\sigma)\dot{F}^p - \sum_{\sigma\in O_p}\varphi'(r_\sigma)\dot{F}^q\right),\end{aligned}\tag{4.11}$$

where we have introduced the index sets $O_a := \{\sigma \in H_m : a \in \sigma(\{1,\ldots,m+1\})\}$. For $\lambda_p > \lambda_q$, Proposition 2.7 yields

$$\dot{F}^p\dot{G}_m^q - \dot{G}_m^p\dot{F}^q \le \dot{F}^p\left(\sum_{\sigma\in O_q}\varphi'(r_\sigma) - \sum_{\sigma\in O_p}\varphi'(r_\sigma)\right).$$

We now show that the term in brackets is non-positive whenever $\lambda_p > \lambda_q$:

Lemma 4.11. *If $\lambda_p \geq \lambda_q$, then*

$$\sum_{\sigma\in O_p} \varphi'(r_\sigma) - \sum_{\sigma\in O_q} \varphi'(r_\sigma) \geq 0\,.$$

Moreover, equality holds only if either $\lambda_p = \lambda_q$ or $r_\sigma(\lambda) \geq 0$ for all $\sigma \in O_{q,p} := O_q \setminus O_p$.

Proof of Lemma 4.11. First note that

$$\sum_{\sigma\in O_p} \varphi'(r_\sigma) - \sum_{\sigma\in O_q} \varphi'(r_\sigma) = \sum_{\sigma\in O_{p,q}} \varphi'(r_\sigma) - \sum_{\sigma\in O_{q,p}} \varphi'(r_\sigma)\,,$$

where $O_{a,b} := O_a \setminus O_b$. Next, observe that, if $\sigma \in O_{p,q}$, then

$$\lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(m+1)} = \lambda_p + \lambda_{\hat\sigma(i_1)} \cdots + \lambda_{\hat\sigma(i_m)} \tag{4.12}$$

for some $\hat\sigma \in H_{m-2}(p,q) := P_{n-2}(p,q)/\sim$, where $P_{n-2}(p,q)$ is the set of permutations of $\{1,\ldots,n\}\setminus\{p,q\}$, $\{i_1,\ldots,i_m\}$ are a choice of m elements of $\{1,\ldots,n\}\setminus\{p,q\}$, and $\sim$ is defined by

$$\hat\sigma \sim \hat\omega \quad \text{if} \quad \hat\sigma(\{i_1,\ldots,i_m\}) = \hat\omega(\{i_1,\ldots,i_m\})\,.$$

Observe also that the converse holds (that is, (4.12) defines a bijection), so that

$$\begin{aligned}\sum_{\sigma\in O_{p,q}} \varphi'(r_\sigma) - \sum_{\sigma\in O_{q,p}} \varphi'(r_\sigma) = \sum_{\hat\sigma\in H_{m-2}(p,q)} \Bigg[\varphi'\left(\frac{\lambda_p + \sum_{k=1}^m \lambda_{\hat\sigma(i_k)} - c_m^{-1}F}{F}\right) \\ - \varphi'\left(\frac{\lambda_q + \sum_{k=1}^m \lambda_{\hat\sigma(i_k)} - c_m^{-1}F}{F}\right)\Bigg]\,.\end{aligned}$$

The claim now follows from (strict) convexity of φ (where it is positive). $\square$

Thus,

$$\sum_{p>q} \frac{\dot F^p \dot G_m^q - \dot G_m^p \dot F^q}{\lambda_p - \lambda_q}\Big((T_{pqq})^2 + (T_{qpp})^2\Big) \leq 0\,.$$

We now compute

$$\vec G_m^{kpq} = \left(\frac{G_m}{F} - \sum_{\sigma\in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \frac{\lambda_{\sigma(i)}}{F}\right) \vec F^{kpq} + \sum_{\sigma\in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}{}^k, \delta_{\sigma(i)}{}^p, \delta_{\sigma(i)}{}^q\right),$$

so that

$$\begin{aligned}\left(\vec{G}^{kpq}_m \times \vec{F}^{kpq}\right) \cdot \vec{\lambda}_{kpq} &= \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\left[\left({\delta_{\sigma(i)}}^k, {\delta_{\sigma(i)}}^p, {\delta_{\sigma(i)}}^q\right) \times \vec{F}^{kpq}\right] \cdot \vec{\lambda}_{kpq}\\ &= \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\left[\frac{({\delta_{\sigma(i)}}^p\dot{F}^q - {\delta_{\sigma(i)}}^q\dot{F}^p)(\lambda_p-\lambda_q)}{(\lambda_k-\lambda_p)(\lambda_k-\lambda_q)}\right.\\ &\qquad + \frac{({\delta_{\sigma(i)}}^q\dot{F}^k - {\delta_{\sigma(i)}}^k\dot{F}^q)(\lambda_k-\lambda_q)}{(\lambda_k-\lambda_p)(\lambda_p-\lambda_q)}\\ &\qquad \left.+ \frac{({\delta_{\sigma(i)}}^k\dot{F}^p - {\delta_{\sigma(i)}}^p\dot{F}^k)(\lambda_k-\lambda_p)}{(\lambda_k-\lambda_q)(\lambda_p-\lambda_q)}\right].\end{aligned}$$

Removing the positive factor $\alpha_{kpq} := [(\lambda_k-\lambda_p)(\lambda_k-\lambda_q)(\lambda_p-\lambda_q)]^{-1}$ and setting $Q^a := \sum_{\sigma\in O_a}\varphi'(r_\sigma)$, we obtain

$$\begin{aligned}\left(\vec{G}^{kpq}_m \times \vec{F}^{kpq}\right) \cdot \vec{\lambda}_{kpq} = \alpha_{kpq}\Big[&(Q^p\dot{F}^q - Q^q\dot{F}^p)(\lambda_p-\lambda_q)^2 + (Q^q\dot{F}^k - Q^k\dot{F}^q)(\lambda_k-\lambda_q)^2\\ &+ (Q^k\dot{F}^p - Q^p\dot{F}^k)(\lambda_k-\lambda_p)^2\Big].\end{aligned}$$

Applying Lemma 4.11 yields

$$\begin{aligned}&\left(\vec{G}^{kpq}_m \times \vec{F}^{kpq}\right) \cdot \vec{\lambda}_{kpq}\\ &\qquad \leq \alpha_{kpq}\left(Q^q\dot{F}^k - Q^k\dot{F}^q\right)\left[(\lambda_k-\lambda_q)^2 - (\lambda_k-\lambda_p)^2 - (\lambda_p-\lambda_q)^2\right].\end{aligned}$$

Since the term in square brackets is non-negative, applying Lemma 4.11 once more yields

$$\left(\vec{G}^{kpq}_m \times \vec{F}^{kpq}\right) \cdot \vec{\lambda}_{kpq} \leq 0\,.$$

□

Concave speeds

Next, we consider flows by concave speed functions. We first observe the following simple application of equation (4.6):

Proposition 4.12 (Cf. Andrews (1994a), Theorem 4.1)**.** *Let* $F : \Gamma \to \mathbb{R}$ *be a concave admissible speed function. Let* $G : \Gamma \to \mathbb{R}$ *be any smooth, degree one homogeneous curvature function which is convex and monotone increasing. Suppose that* $\Gamma_C := \{\lambda \in \Gamma : G(\lambda) \leq CF(\lambda)\} \subset\subset \Gamma$ *for some* $C \in \mathbb{R}$*. Then* Γ_C *is strongly preserved by the flow* (CF)*.*

Remark 4.4. In particular, we can take $G(\lambda) = H(\lambda) := \sum_{i=1}^n \lambda_i$ in Proposition 4.12. This implies that the cone $\Gamma_C := \{\lambda \in \Gamma : H(\lambda) \leq CF(\lambda)\}$ is preserved, so long as $\Gamma_C \subset\subset \Gamma$. This is the case, for example, if $C < \liminf_{\lambda\to\partial\Gamma}\frac{H}{F}$.

Proof of Proposition 4.12. This follows immediately by applying the strong maximum principle to the evolution equation (4.6) (for $G - CF$) since the assumptions on G and F ensure $\mathscr{Q}_{G,F}(\nabla\mathcal{W}, \nabla\mathcal{W}) \le 0$. $\square$

Next, we consider flows of $(m+1)$-convex hypersurfaces by concave speeds.

Proposition 4.13. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be a concave admissible speed function and m an integer in the set $\{0, \dots, n-2\}$. Suppose that $\Gamma_0 := \bigcap_{\sigma\in P_n}\{\lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(m+1)} - c_m^{-1}F(\lambda) \le \varepsilon F(\lambda)\} \subset\subset \Gamma$ for some $\varepsilon \ge 0$. Then Γ_0 is strongly preserved by* (CF).

Remark 4.5. We will only make use of the case $m = 0$, which yields the pinching estimate $\kappa_n \le (c_0^{-1} + \varepsilon)F$. Note that, if F is concave and $\beta < c_1^{-1}$, then the cone $\Gamma_0 := \{\lambda \in \Gamma : \lambda_i \le \beta F(\lambda)$ for each $i\}$ satisfies $\Gamma_0 \subset\subset \Gamma_+$. Thus, flows by concave speeds preserve sufficiently tight initial curvature pinching. We will show in Section 5 that this estimate improves at a singularity for such flows.

Proof of Proposition 4.13. The proof is similar to Proposition 4.10, replacing the pinching functions defined there with

$$G_m(\lambda) := F(\lambda) \sum_{\sigma\in P_n} \varphi\left(\frac{c_m^{-1}F(\lambda) - \sum_{i=1}^n \lambda_{\sigma(i)}}{F(\lambda)}\right)$$

and arguing from concavity, rather than convexity, of F. $\square$

Surface flows

In two space dimensions, the gradient of the second fundamental form has no totally off-diagonal components. This allows us to isolate the dependence of $\mathscr{Q}_{G,F}$ on the second derivatives of F:

Lemma 4.14 (Cf. Andrews (2010))**.** *Let $F : \Gamma \subset \mathbb{R}^2 \to \mathbb{R}$ be a positive admissible speed function and let $G : \Gamma \to \mathbb{R}$ be any smooth, symmetric, degree zero homogeneous function. Let Z be any diagonal matrix with eigenvalues $\lambda = (\lambda_1, \lambda_2) \in \Gamma$ satisfying $\lambda_1 \le \lambda_2$. Suppose that $\dot G$ is non-degenerate at Z. Then, for every totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, we have*

$$\begin{aligned}
\mathscr{Q}_{G,F}\big|_Z(T,T) = {}& \ddot F^{kl,pq}\left(\dot G^1 T_{k11} + \dot G^2 T_{k22}\right) T_{lpq} \\
&+ \frac{2F\dot G^1}{\lambda_2(\lambda_2 - \lambda_1)}\left[\left(T_{112}\right)^2 + \left(T_{212}\right)^2\right] \\
&- \frac{\dot F^1}{\dot G^1}\frac{\ddot G^{11}}{\dot G^1}\left(\dot G^1 T_{111} + \dot G^2 T_{122}\right)^2 - \frac{\dot F^2}{\dot G^2}\frac{\ddot G^{22}}{\dot G^2}\left(\dot G^1 T_{211} + \dot G^2 T_{222}\right)^2 \\
&+ \left(2\frac{\dot F^1}{\lambda_2} - \frac{\dot F^2 - \dot F^1}{\lambda_2 - \lambda_1}\right)\left(\dot G^1 T_{111} + \dot G^2 T_{122}\right) T_{122} \\
&+ \left(2\frac{\dot F^2}{\lambda_1} - \frac{\dot F^2 - \dot F^1}{\lambda_2 - \lambda_1}\right)\left(\dot G^1 T_{211} + \dot G^2 T_{222}\right) T_{211}\,.
\end{aligned}$$

Remarks 4.2. 1. If G is degenerate at Z, then a careful inspection of the proof of Lemma 4.14 reveals that $\mathscr{Q}_{G,F}\big|_Z \equiv 0$.

2. The significance of Lemma 4.14 is the following observation: If G is evaluated on the curvature of a solution of the flow (CF), then $\nabla_k G = \dot{G}^{pq} \nabla_k \mathcal{W}_{pq} = \dot{G}^1 \nabla_k \mathcal{W}_{11} + \dot{G}^2 \nabla_k \mathcal{W}_{22}$ and Lemma 4.14 yields

$$\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) = \frac{2F\dot{G}^1}{\kappa_2(\kappa_2 - \kappa_1)} \left[\left(\nabla_1 \mathcal{W}_{12} \right)^2 + \left(\nabla_2 \mathcal{W}_{12} \right)^2 \right]$$

at any critical point of G. The maximum principle (applied to equation (4.6)) now implies that sub-level (super-level) sets of any monotone non-increasing (non-decreasing) degree zero homogeneous curvature function G are preserved.

3. The decomposition also plays an important role in Section 5.

Proof. We first show that $\lambda_1 \neq 0$ and $\lambda_2 \neq \lambda_1$ wherever $\dot{G}$ is non-degenerate: First note that, by the identity (2.17) of Theorem (2.5), $\dot{G}^{kl} = \dot{G}^k \delta^{kl}$, so that $\dot{G}^k \neq 0$ for each k. Next, observe that $\dot{G}^1 \lambda_1 + \dot{G}^2 \lambda_2 = 0$ by Euler's theorem for homogeneous functions. Suppose now that $0 \neq \lambda_1 = \lambda_2$. Then $0 = (\dot{G}^1 + \dot{G}^2)\lambda_2$, so that $\dot{G}^1 = -\dot{G}^2$. On the other hand, since G is symmetric, we have $\dot{G}^1 = \dot{G}^2$ whenever $\lambda_2 = \lambda_1$. Thus, $\dot{G}^1 = \dot{G}^2 = 0$, a contradiction. Now suppose that $\lambda_1 = 0$. Then $0 = \dot{G}^2 \lambda_2$. But, since F is positive, Euler's theorem implies $0 < \dot{F}^1 \lambda_1 + \dot{F}^2 \lambda_2 = \dot{F}^2 \lambda_2$. Since F is monotone, we find $\lambda_2 > 0$, and conclude that $\dot{G}^2 = 0$, another contradiction.

Now we apply the decomposition (4.7) of Lemma 4.6 to obtain

$$\mathscr{Q}_{G,F}(T,T) = (\dot{G}^k \ddot{F}^{pq} - \dot{F}^k \ddot{G}^{pq}) T_{kpp} T_{lqq} + 2\frac{\dot{G}^1 \dot{F}^2 - \dot{G}^2 \dot{F}^1}{\lambda_2 - \lambda_1} \left((T_{112})^2 + (T_{212})^2 \right) .$$

Consider the terms involving the second derivatives of F:

$$\begin{aligned} Q_1 := \dot{G}^k \ddot{F}^{pq} T_{kpp} T_{kqq} = \ddot{F}^{11} & \left(\dot{G}^1 (T_{111})^2 + \dot{G}^2 (T_{211})^2 \right) \\ & + 2\ddot{F}^{12} \left(\dot{G}^1 T_{111} T_{122} + \dot{G}^2 T_{211} T_{222} \right) \\ & + \ddot{F}^{22} \left(\dot{G}^1 (T_{122})^2 + \dot{G}^2 (T_{222})^2 \right) . \end{aligned}$$

We write this in terms of the 'gradient' $D_kG := (\dot G^1 T_{k11} + \dot G^2 T_{k22})$ as follows:

$$
\begin{aligned}
Q_1 = \ddot F^{11}&\left(T_{111}\left(D_1G - \dot G^2 T_{122}\right) + \frac{\dot G^2}{\dot G^1} T_{211}\left(D_2G - \dot G^2 T_{222}\right)\right)\\
&+ 2\ddot F^{12}\left(\dot G^1 T_{111}(T_{122})^2 + \dot G^2 T_{211}(T_{222})^2\right)\\
&+ \ddot F^{22}\left(\frac{\dot G^1}{\dot G^2} T_{122}(D_1G - \dot G^2 T_{111}) + \dot G^2 T_{222}(D_2G - \dot G^1 T_{211})\right)\\
= &- \dot G^2\left(\ddot F^{11} - \frac{\dot G^1}{\dot G^2}\ddot F^{12} + \left(\frac{\dot G^1}{\dot G^2}\right)^2 \ddot F^{22}\right) T_{111}T_{122}\\
&- \dot G^1\left(\left(\frac{\dot G^2}{\dot G^1}\right)^2 \ddot F^{11} - \frac{\dot G^1}{\dot G^2}\ddot F^{12} + \ddot F^{22}\right) T_{211}T_{222}\\
&+ \ddot F^{11}\left(D_1GT_{111} + \frac{\dot G^2}{\dot G^1} D_2GT_{211}\right) + \ddot F^{22}\left(\frac{\dot G^1}{\dot G^2} D_1GT_{122} + D_2GT_{222}\right) . \qquad (4.13)
\end{aligned}
$$

Now note that, due to Euler's Theorem for homogeneous functions, any smooth, homogeneous degree γ function k of two variables, y_1, y_2, satisfies the following identities:

$$
\begin{aligned}
\dot k^1 y_1 + \dot k^2 y_2 &= \gamma k\,;\\
\ddot k^{11} y_1 + \ddot k^{12} y_2 &= (\gamma-1)\dot k^1\,;\\
\ddot k^{22} y_2 + \ddot k^{12} y_1 &= (\gamma-1)\dot k^2\,;\\
\text{and}\qquad \ddot k^{11}(y_1)^2 + 2\ddot k^{12} y_1 y_2 + \ddot k^{22}(y_2)^2 &= \gamma(\gamma-1)k\,.
\end{aligned}
\qquad (4.14)
$$

The first of these identities implies $\dot G^2/\dot G^1 = -\lambda_1/\lambda_2$. Combining this with the fourth, we observe that the bracketed terms in the first and second lines of (4.13) vanish. Applying the second and third of the identities (4.14) to the remaining terms yields

$$
Q_1 = \ddot F^{11} D_1GT_{111} + \ddot F^{12} D_2GT_{211} + \ddot F^{22} D_2GT_{222} + \ddot F^{12} D_1GT_{122}\,.
$$

Recalling the identity (2.18), we conclude

$$
Q_1 = \ddot F^{klpq} D_kGT_{lpq} - \frac{\dot F^2 - \dot F^1}{\lambda_2 - \lambda_1}\left(D_1GT_{122} + D_2GT_{211}\right) .
$$

We now turn our attention to the terms involving second derivatives of G; a similar computation as in (4.13) yields

$$
\begin{aligned}
Q_2 :=& \dot{F}^k \ddot{G}^{pq} T_{kpp} T_{kqq} \\
=& -\dot{G}^2 \frac{\dot{F}^1}{\dot{G}^1} \left(\ddot{G}^{11} - \frac{\dot{G}^1}{\dot{G}^2} \ddot{G}^{12} + \left(\frac{\dot{G}^1}{\dot{G}^2} \right)^2 \ddot{G}^{22} \right) T_{111} T_{122} \\
& - \dot{G}^1 \frac{\dot{F}^2}{\dot{G}^2} \left(\left(\frac{\dot{G}^2}{\dot{G}^1} \right)^2 \ddot{G}^{11} - \frac{\dot{G}^1}{\dot{G}^2} \ddot{G}^{12} + \ddot{G}^{22} \right) T_{211} T_{222} \\
& + \ddot{G}^{11} \left(\frac{\dot{F}^1}{\dot{G}^1} D_1 G T_{111} + \frac{\dot{F}^2}{\dot{G}^1} D_2 G T_{211} \right) + \ddot{G}^{22} \left(\frac{\dot{F}^1}{\dot{G}^2} D_1 G T_{122} + \frac{\dot{F}^2}{\dot{G}^2} D_2 G T_{222} \right)
\end{aligned}
$$

As above, the first and second lines vanish. We write the third line as

$$
\begin{aligned}
Q_2 =& \frac{\ddot{G}^{11}}{(\dot{G}^1)^2} \left(\dot{F}^1 D_1 G \left(D_1 G - \dot{G}^2 T_{122} \right) + \dot{F}^2 \dot{F}^2 \dot{G}^1 D_2 G T_{211} \right) \\
& + \frac{\ddot{G}^{22}}{(\dot{G}^2)^2} \left(\dot{F}^1 \dot{G}^2 D_1 G T_{122} + \dot{F}^2 D_2 G \left(D_2 G - \dot{G}^1 T_{211} \right) \right) \\
=& \frac{\dot{F}^1}{\dot{G}^1} \left(\dot{G}^{11} - \left(\frac{\dot{G}^1}{\dot{G}^2} \right)^2 \ddot{G}^{22} \right) D_1 G T_{122} + \frac{\dot{F}^2}{\dot{G}^2} \left(\dot{G}^{22} - \left(\frac{\dot{G}^2}{\dot{G}^1} \right)^2 \ddot{G}^{11} \right) D_2 G T_{211} \\
& + \frac{\ddot{G}^{11}}{\dot{G}^1} \frac{\dot{F}^1}{\dot{G}^1} (D_1 G)^2 + \frac{\ddot{G}^{22}}{\dot{G}^2} \frac{\dot{F}^2}{\dot{G}^2} (D_2 G)^2 \, .
\end{aligned}
$$

Now, using the second and third of the identities (4.14), we find

$$
\frac{\ddot{G}^{22} \lambda_2 + \dot{G}^2}{\lambda_1} = -\ddot{G}^{12} = \frac{\ddot{G}^{11} \lambda_1 + \dot{G}^1}{\lambda_2} \, .
$$

It follows that

$$
\dot{G}^{11} - \left(\frac{\dot{G}^1}{\dot{G}^2} \right)^2 \ddot{G}^{22} = \dot{G}^{11} - \left(\frac{\lambda_2}{\lambda_1} \right)^2 \ddot{G}^{22} = \frac{\lambda_2}{\lambda_1} \dot{G}^2 - \frac{\dot{G}^1}{\lambda_1} = -2 \frac{\dot{G}^1}{\lambda_1}
$$

and, similarly,

$$
\dot{G}^{22} - \left(\frac{\dot{G}^2}{\dot{G}^1} \right)^2 \ddot{G}^{11} = -2 \frac{\dot{G}^2}{\lambda_2} \, .
$$

We conclude

$$
Q_2 = \frac{\dot{F}^1}{\dot{G}^1} \frac{\ddot{G}^{11}}{\dot{G}^1} (D_1 G)^2 + \frac{\dot{F}^2}{\dot{G}^2} \frac{\ddot{G}^{22}}{\dot{G}^2} (D_2 G)^2 - 2 \frac{\dot{F}^1}{\lambda_2} D_1 G T_{122} - 2 \frac{\dot{F}^2}{\lambda_1} D_2 G T_{211} \, .
$$

Finally, the coefficient of the remaining term may be rewritten with the help of the first of the identities (4.14) as

$$2\frac{\dot G^1\dot F^2-\dot G^2\dot F^1}{\lambda_2-\lambda_1}=2\frac{\dot G^1\dot F^2\lambda_2-\dot G^2\frac{\lambda_2}{\lambda_1}\dot F^1\lambda_1}{\lambda_2(\lambda_2-\lambda_1)}=2\frac{\dot G^1\dot F^2\lambda_2+\dot G^2\frac{\dot G^1}{\dot G^2}\dot F^1}{\lambda_2(\lambda_2-\lambda_1)}=2\frac{\dot G^1F}{\lambda_2(\lambda_2-\lambda_1)}\,.$$

This completes the proof of the lemma. □

Corollary 4.15. *Let $F:\Gamma\subset\mathbb{R}^2\to\mathbb{R}$ be any positive admissible speed function. Then every curvature cone $\Gamma_0\subset\subset\Gamma$ is strongly preserved by the flow* (CF)*.*

Proof. First note that, in polar coördinates (r,θ) with angle measured from the positive ray (recall Example 2.3), we have $\overline{\Gamma}_0=\{\lambda\in\mathbb{R}^2:-\theta_0\leq\theta(\lambda_1,\lambda_2)\leq\theta_0\}$ for some $\theta_0\in[0,3\pi/4]$. Equivalently, $\overline{\Gamma}_0=\{\lambda\in\mathbb{R}^2:\cos\theta(\lambda_1,\lambda_2)\geq\varepsilon_0\}$ for some $\varepsilon_0\geq-1/\sqrt{2}$. Thus, it suffices to show that the inequalities

$$G(\lambda_1,\lambda_2):=\cos\theta(\lambda_1,\lambda_2)=\frac{\lambda_1+\lambda_2}{\sqrt{2(\lambda_1^2+\lambda_2^2)}}\geq\varepsilon$$

are preserved for each $\varepsilon\geq-1/\sqrt{2}$.

Noting that

$$\sqrt{2}\dot G^1=\frac{\lambda_2(\lambda_2-\lambda_1)}{(\lambda_1^2+\lambda_2^2)^{\frac{3}{2}}}$$

is monotone non-decreasing at any $\lambda\in\Gamma_0$ such that $\lambda_2\geq\lambda_1$, the claim follows from Lemma 4.14 and the maximum principle. □

4.2.2 Cones defined by the Weingarten curvature

It is also possible to obtain preserved cones more directly from the evolution equation for the full Weingarten curvature. In this section, we will derive a useful condition under which a given convex cone of curvatures will be preserved by (CF).

First, we shall need to recall some definitions and results from convex geometry:

Definition 4.16. *Let A be a non-empty, closed, convex subset of $\mathbb{R}^n$.*

1. *A* supporting affine functional *for A at $a\in A$ is an affine function $\ell:\mathbb{R}^n\to\mathbb{R}$ such that $||\nabla\,\ell||=1$, $\ell(z)\geq0$ for all $z\in A$, and $\ell(a)=0$. Given a non-empty, closed, convex set $A\subset\mathbb{R}^n$, we shall denote the set of all supporting affine functionals for A at $a\in A$ by $\mathrm{SAF}_a(A)$, and $\mathrm{SAF}(A):=\cup_{a\in A}\mathrm{SAF}_a(A)$.*

2. *A* supporting half-space *for A is any half-space of the form $H_\ell:=\{z\in\mathbb{R}^n:\ell(z)>0,\ \ell\in\mathrm{SAF}(A)\}$.*

3. *The* signed distance to the boundary of A *is the function $\mathrm{dist}_{\partial A}:\mathbb{R}^n\to\mathbb{R}$ defined by $\mathrm{dist}_{\partial A}(z):=\inf\{\ell(z):\ell\in\mathrm{SAF}(A)\}$.*

4. *The* normal cone to A at $a \in \partial A$ *is the cone* $\mathcal{N}_a A := \{\ell \in (\mathbb{R}^n)^* : \ell(z-a) \geq 0$ *for all* $z \in A\}$. *We also set* $\mathcal{N} A := \sqcup_{a \in A} \mathcal{N}_a A$.

Lemma 4.17. *Let $A \subset \mathbb{R}^n$ be a non-empty, closed, convex set. Then the following statements hold:*

1. *A is the intersection of its supporting half-spaces: $A = \bigcap_{\ell \in \mathrm{SAF}(A)} H_\ell$, where $H_\ell := \{z \in \mathbb{R}^n : \ell(z) \geq 0\}$.*

2. *The signed distance satisfies*

$$\mathrm{dist}_{\partial A}(z) = \begin{cases} \mathrm{dist}(z, \partial A) & \text{if} \quad z \in A \\ -\mathrm{dist}(z, \partial A) & \text{if} \quad z \in \mathbb{R}^n \setminus A\,, \end{cases}$$

where, given a set $B \subset \mathbb{R}^n$, $\mathrm{dist}(z, B) := \inf_{b \in B} ||z - b||$.

3. *If A is a cone (with vertex at the origin), then* $\mathrm{SAF}_a(A) = \{\ell \in \mathcal{N}_a A : ||\ell|| = 1\}$.

Proof. To prove the first claim, we follow Andrews and Hopper (2011, Theorem B1). First note that, by the Hahn–Banach Theorem, $\mathrm{SAF}(A)$ is non-empty. Next, note that the intersection $\bigcap_{\ell \in \mathrm{SAF}(A)} H_\ell$ contains A since, by definition, each of the half-spaces H_ℓ contains A. To prove the reverse inclusion, it suffices to show that, for any $y \notin A$, there is some $\ell \in \mathrm{SAF}(A)$ such that $\ell(y) < 0$. To this end, let x be a closest point of A to y (in fact, there is a unique such point) and define ℓ by $\ell(z) := -\left\langle z - x\,,\, \frac{y-x}{||y-x||} \right\rangle$. Note that ℓ is an affine functional satisfying $||\nabla \ell|| = 1$. We claim that $\ell \in \mathrm{SAF}(A)$; in fact, suppose, to the contrary, that $\ell(w) < 0$ for some $w \in A$. Then, by convexity of A, $x + s(w - x) \in A$ for $0 \leq s \leq 1$, and

$$\frac{d}{ds}\bigg|_{s=0} ||y - (x + s(w-x))||^2 = -2\,\langle w - x\,,\, y - x\rangle = 2\ell(w)\,||y - x|| < 0\,.$$

But this contradicts the fact that x is a closest point of A to y. Since $\ell(x) = 0$, we conclude that $\ell \in \mathrm{SAF}(A)$. The claim follows since $\ell(y) = -\,||y - x|| < 0$.

To prove the second claim, we follow Evans (2010, Lemma 2.3). First consider the case $z \in A$. To prove that $\mathrm{dist}_{\partial A}(z) \geq \mathrm{dist}(z, \partial A)$, note that every $\ell \in \mathrm{SAF}_a(A)$ is of the form $\ell(w) = \langle w - a\,,\, v\rangle$ for some $a \in H_\ell$ and some 'inward pointing normal' $v \in S^n$. Let $\overline{a} := z - \mathrm{dist}(z, \partial H_\ell)v$ be the nearest point of ∂H_ℓ to z. Then $\ell(z) = \langle z - a\,,\, v\rangle = \langle (z - \overline{a}) + (\overline{a} - a)\,,\, v\rangle = \langle z - \overline{a}\,,\, v\rangle = \mathrm{dist}(z, \partial H_\ell)$. Since the line segment joining z and $\overline{a}$ must intersect ∂A, it follows that $\ell(z) \geq \mathrm{dist}(z, \partial A)$. On the other hand, if a is a closest point of ∂A to z, then, for any $\ell \in \mathrm{SAF}_a(A)$, we have $\ell(z) = ||z - a|| = \mathrm{dist}(z, \partial A)$, and so it follows that $\mathrm{dist}_{\partial A}(z) = \ell(z)$. The proof for points $z \in \mathbb{R}^n \setminus A$ is similar.

The final claim follows from the fact that the boundaries of all supporting half-spaces contain the origin. □

Theorem 4.18 (Maximum Principle for the Weingarten curvature)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed and $\Gamma_0 \subset \Gamma$ a convex, symmetric cone. Let $\beta \in \mathbb{R}$ be any* *constant such that* $\Gamma_\beta := \{\lambda \in \Gamma : \mathrm{dist}_{\partial\Gamma_0}(\lambda) \geq \beta F(\lambda)\} \subset\subset \Gamma$. *Suppose that, for every diagonal matrix* W *with eigenvalue* n*-tuple satisfying* $\lambda \in \partial\Gamma_\beta \setminus \{0\}$, *it holds that*

$$\sup_{\Lambda}\left\{\ell^i\left(\ddot{F}^{pq,rs}(W)T_{ipq}T_{irs} + 2\dot{F}^{kl}(W)\left[2\Lambda_{ki}{}^pT_{lip} - \Lambda_{ki}{}^p\Lambda_{li}{}^p(\lambda_p - \lambda_i)\right]\right)\right\} \geq 0 \qquad (4.15)$$

for every $(\lambda_0, \ell) \in \mathcal{N}\Gamma_0$ *such that* $\mathrm{dist}_{\partial\Gamma_0}(\lambda) = \mathrm{dist}(\lambda, \lambda_0)$ *and every totally symmetric* $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ *satisfying* $\ell^i T_{kii} = \beta \dot{F}^i T_{kii}$, *where the supremum is taken over* $\{\Lambda \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n : \Lambda_{kij} + \Lambda_{kji} = 0\}$. *Then* Γ_β *is strongly preserved by* (CF).

Remarks 4.3.

1. In fact, given any degree one homogeneous curvature function G which is a subsolution of (LF), we may replace the cone $\Gamma_\beta := \{\lambda \in \Gamma : \mathrm{dist}_{\partial\Gamma_0}(\lambda) \geq \beta F(\lambda)\}$ in Theorem 4.18 by $\{\lambda \in \Gamma : \mathrm{dist}_{\partial\Gamma_0}(\lambda) \geq G(\lambda)\}$ (replacing also the gradient condition by $\ell^i T_{kii} = \dot{G}^i T_{kii}$).

2. It is instructive to consider the case $\beta = 0$, which is the case we will mostly be interested in.

3. The useful extra terms involving the coefficients Λ are non-trivial: They do not arise in the evolution of scalar functions of the curvature.

4. Theorem 4.18 also has an elliptic analogue.

Proof of Theorem 4.18. The proof is a slight improvement of similar results of Hamilton (1986, Theorem 4.3) and Andrews (2007, Theorem 3.2). Our proof is also influenced by ideas of Evans (2010).

We will show that, given a supporting half-space $H_\ell := \{z \in \mathbb{R}^n : \ell(z) \geq 0\}$ for Γ_0, the function $d_\ell(x,t) := \mathrm{dist}_{\partial H_\ell}(\vec{\kappa}(x,t)) = \ell(\vec{\kappa})$ is a viscosity supersolution[1] of the equation

$$\begin{aligned}(\partial_t - \mathscr{L})f = \dot{F}(\mathcal{W}^2)f + \sup_{\Lambda}\Big\{\ell^i\Big[&\ddot{F}(\nabla_i\mathcal{W}, \nabla_i\mathcal{W}) \\ &+ 2\dot{F}^{kl}\left(2\Lambda_{ki}{}^p\nabla_l\mathcal{W}_{ip} - \Lambda_{ki}{}^p\Lambda_{li}{}^p(\kappa_p - \kappa_i)\right)\Big]\Big\}. \qquad (4.16)\end{aligned}$$

Since Γ_0 is the intersection of its supporting half-spaces, and F satisfies (LF), the claims then follow from the strong maximum principle (see, for example, Da Lio 2004).

In order to show that d_ℓ is a viscosity supersolution of (4.16), we fix an arbitrary point $(x_0, t_0) \in \mathscr{M} \times [0, T)$ and consider an arbitrary lower support function ϕ for d_ℓ at (x_0, t_0); that is, ϕ is smooth on a parabolic ball $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0]$ centred at (x_0, t_0) with $d_\ell \geq \phi$ and equality at (x_0, t_0). Then we need to show that ϕ is a supersolution of (4.16).

Consider the endomorphism $L^0 \in (T^*\mathscr{M} \otimes T\mathscr{M})_{(x_0,t_0)}$ defined by $L^0 := \sum_{i=1}^n \ell^i e^i \otimes e_i$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis of eigenvectors of $\mathcal{W}(x_0, t_0)$. Now extend L^0 to a neighbourhood of (x_0, t_0) by parallel translation with respect to an arbitrary (time-dependent) metric compatible connection $\overline{\nabla}$. The metric compatibility of $\overline{\nabla}$ ensures that

[1] See, for example, Crandall, Ishii, and Lions (1992).

L remains unit length, which, in turn, implies that $d_\ell(x,t) \le L(\mathcal{W}(x,t))$. Since equality holds at (x_0,t_0), we conclude that ϕ is a lower support function for $L(\mathcal{W})$ at (x_0,t_0). But since L is smooth, we have

$$0 = \nabla(L(\mathcal{W}) - \phi) \tag{4.17}$$

and

$$0 \ge (\partial_t - \mathscr{L})(L(\mathcal{W}) - \phi) \tag{4.18}$$

at (x_0,t_0).

On the other hand, a straightforward calculation gives

$$\begin{aligned}(\partial_t - \mathscr{L})\,(L(\mathcal{W})) = {} & (\nabla_t L)\,(\mathcal{W}) + L\,((\nabla_t - \mathscr{L})\mathcal{W}) \\ & - \dot{F}^{kl}\,(2\,\nabla_k L(\nabla_l \mathcal{W}) + \nabla_k \nabla_l L(\mathcal{W}))\,.\end{aligned}$$

Now, since the space of connections is affine, we have $\overline{\nabla} = \nabla + \Lambda$ for some (local) section Λ of $(T^*\mathscr{M} \oplus \mathbb{R}\,\partial_t) \otimes T^*\mathscr{M} \otimes T\mathscr{M}$. Metric compatibility of $\overline{\nabla}$ implies the anti-symmetries $\Lambda_{kij} + \Lambda_{kji} = 0$ and $\Lambda_{tij} + \Lambda_{tji} = 0$, but otherwise Λ_{kij} and Λ_{tij} may take any value at (x_0,t_0).

Next, we compute

$$\nabla_k L = L^{ij}\left(\Lambda_{ki}{}^{p} e_p \otimes e_j + \Lambda_{kj}{}^{q} e_i \otimes e_q\right)\,.$$

Similarly,

$$\nabla_t L = L^{ij}\left(\Lambda_{ti}{}^{p} e_p \otimes e_j + \Lambda_{tj}{}^{q} e_i \otimes e_q\right)\,.$$

In particular, the antisymmetry of Λ in the final two entries implies $\nabla_k L(\mathcal{W}) = \nabla_t L(\mathcal{W}) = 0$. Next, we compute

$$\begin{aligned}\nabla_k \nabla_l L = L^{ij}\big(& \nabla_k \Lambda_{li}{}^{p} e_p \otimes e_j + \Lambda_{li}{}^{p}\Lambda_{kp}{}^{q} e_q \otimes e_j + \Lambda_{li}{}^{p}\Lambda_{kj}{}^{q} e_p \otimes e_q \\ & + \nabla_l \Lambda_{kj}{}^{q} e_i \otimes e_q + \Lambda_{lj}{}^{q}\Lambda_{kq}{}^{p} e_i \otimes e_p + \Lambda_{lj}{}^{q}\Lambda_{ki}{}^{p} e_p \otimes e_q\big)\,.\end{aligned}$$

As above, the gradient terms are killed when contracted with $\mathcal{W}$; thus, recalling (4.18), we obtain

$$\begin{aligned}(\partial_t - \mathscr{L})\phi \ge {} & \dot{F}(\mathcal{W}^2)L(\mathcal{W}) + L^{ij}\Big(\ddot{F}(\nabla_i \mathcal{W}, \nabla_j \mathcal{W}) \\ & \qquad + 2\dot{F}^{kl}\left[2\Lambda_{ki}{}^{p}\,\nabla_l \mathcal{W}_{jp} - \Lambda_{ki}{}^{p}\Lambda_{lj}{}^{q}\mathcal{W}_{pq} - \Lambda_{ki}{}^{q}\Lambda_{lq}{}^{p}\mathcal{W}_{pj}\right]\Big) \\ = {} & \dot{F}(\mathcal{W}^2)\ell(\vec{\kappa}) + \ell^{i}\Big(\ddot{F}(\nabla_i \mathcal{W}, \nabla_i \mathcal{W}) \\ & \qquad + 2\dot{F}^{kl}\left[2\Lambda_{ki}{}^{p}\,\nabla_l \mathcal{W}_{ip} - \Lambda_{ki}{}^{p}\Lambda_{li}{}^{p}(\kappa_p - \kappa_i)\right]\Big)\end{aligned}$$

at (x_0, t_0). Since ℓ satisfies $\ell(\vec{\kappa}) = d_\ell$ at (x_0, t_0), and we may choose $\Lambda_{kij}(x_0, t_0)$ arbitrary (modulo the required antisymmetry), this implies that d_ℓ is a viscosity supersolution of (4.16). Since F is a solution of (LF), it follows that, for any $\beta \in \mathbb{R}$, the function $d_\beta := d_\ell - \beta F$ is a viscosity supersolution of (4.16) (see also Remarks 4.3 1.).

Note, finally, that the gradient identity (4.17) implies that $\nabla_k d_\ell = \ell^i \nabla_k \mathcal{W}_{ii}$ in the viscosity sense; that is, given any (upper or lower) support function ϕ for d_ℓ, we have

$$\nabla_k \phi = L(\overline{\nabla}_k \mathcal{W}) = L(\nabla_k \mathcal{W} + \Lambda_k(\mathcal{W})) = L(\nabla_k \mathcal{W}) = \ell^i \nabla_k \mathcal{W}_{ii}$$

at the point of support.

The claim now follows from the strong maximum principle.

□

Remark 4.6. Since $\kappa_1 = \mathrm{dist}_{\partial\Gamma_+}(\vec{\kappa})$, we obtain the following evolution equation (in the viscosity sense) for the smallest principal curvature:

$$\begin{aligned}(\partial_t - \mathscr{L})\kappa_1 \geq{}& \dot{F}(\mathcal{W}^2)\kappa_1 + \ddot{F}(\nabla_1 \mathcal{W}, \nabla_1 \mathcal{W}) \\ &+ 2 \sup_{\Lambda_{k1}=0} \dot{F}^k \left[2\Lambda_k{}^p \nabla_1 \mathcal{W}_{kp} - (\Lambda_k{}^p)^2(\kappa_p - \kappa_1)\right] . \end{aligned} \tag{4.19}$$

An analogous argument (using the fact that $\kappa_n(x,t) = \sup_{v \in S_{(x,t)}\mathscr{M}} \mathcal{W}(v,v)$ to obtain a lower support function for κ_n) also yields an evolution equation (in the viscosity sense) for the largest principal curvature:

$$\begin{aligned}(\partial_t - \mathscr{L})\kappa_n \leq{}& \dot{F}(\mathcal{W}^2)\kappa_n + \ddot{F}(\nabla_n \mathcal{W}, \nabla_n \mathcal{W}) \\ &+ 2 \inf_{\Lambda_{kn}=0} \dot{F}^k \left[2\Lambda_k{}^p \nabla_n \mathcal{W}_{kp} - (\Lambda_k{}^p)^2(\kappa_p - \kappa_n)\right] . \end{aligned} \tag{4.20}$$

The maximum principle for the Weingarten curvature has several useful consequences. In particular, since the inward normal to the boundary of a cone of the form $\{G \geq 0\}$ is given by $\ell^i = \dot{G}^i$, we can quickly recover the results of the preceding section. On the other hand, since we no longer require G to be smooth, and we have gained a non-trivial extra term, we are able to derive some stronger results.

Convex speeds

We first consider flows by convex speed functions. Our first result shows, in particular, that *every* convex cone $\Gamma_0 \subset\subset \Gamma$ which contains the positive cone is preserved under such flows.

Corollary 4.19. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be a convex admissible speed function. Let Γ_0 be any symmetric, convex cone which contains the positive cone Γ_+. Suppose that $\Gamma_\beta := \{\lambda \in \Gamma : \mathrm{dist}_{\partial\Gamma_0}(\lambda) \geq \beta F(\lambda)\} \subset\subset \Gamma$ for some $\beta \in \mathbb{R}$. Then Γ_β is strongly preserved by* (CF).

Proof. By the maximum principle (Theorem 4.18), it suffices to show that

$$\ell^i \ddot{F}^{pq,rs} T_{ipq} T_{irs} \geq 0$$

at any boundary point $\lambda \in \partial\Gamma_0 \setminus \{0\}$, for any $\ell \in \mathcal{N}_\lambda \overline{\Gamma}_0$ and any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ satisfying $\ell^i T_{kii} = 0$ for each k, where $\mathcal{N}_\lambda \Gamma_0$ is the inward normal cone to Γ_0 at λ. Since F is convex, the claim follows from the following lemma:

Lemma 4.20. *Let Γ_0 be a symmetric, convex cone which contains Γ_+. Then, for any $\lambda \in \partial\Gamma_0$, we have $\mathcal{N}_\lambda \Gamma_0 \subset \overline{\Gamma}_+$.*

Proof. Since $\overline{\Gamma}_+ \subset \overline{\Gamma}_0$, we have $y := \lambda + e_i \in \overline{\Gamma}_0$ for any $\lambda \in \partial\Gamma_0$, and any coördinate direction e_i. Thus, for any $\ell \in \mathcal{N}_\lambda \overline{\Gamma}_0$, we have

$$0 \leq \langle \ell \, , \, y - \lambda \rangle = \langle \ell \, , \, e_i \rangle = \ell^i \, .$$

The claim follows. $\square$

This proves Corollary 4.19. $\square$

For flows of convex hypersurfaces, the strong version of Theorem 4.18 yields the following splitting theorem:

Theorem 4.21 (Splitting theorem for flows by convex speeds)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be a convex admissible flow speed such that $\Gamma_+ \subset\subset \Gamma$, and $\mathscr{X} : \mathscr{M} \times [0, T) \to \mathbb{R}^{n+1}$ a (possibly non-compact) solution of* (CF) *satisfying $\vec{\kappa}(x, 0) \in \overline{\Gamma}_+ \setminus \{0\}$ for all $x \in \mathscr{M}$. Then, either $\mathscr{X}\big|_{\mathscr{M}\times(0,T)}$ is strictly convex or $\mathscr{M}$ splits isometrically off a plane; that is, $\mathscr{M} \cong \mathbb{R}^m \times \Sigma^{n-m}$ for some $1 \leq m \leq n-1$ and there exists $T_m \in (0, T)$ such that $\mathscr{X}\big|_{\mathbb{R}^m \times \{\widetilde{x}\} \times (0,T_m]}$ is flat for each $\widetilde{x} \in \Sigma$ and $\mathscr{X}\big|_{\{0\}\times\Sigma^{n-m}\times(0,T_m]} : \Sigma^{n-m} \times (0, T_m] \to \mathbb{R}^{n-m+1} \cong [\mathscr{X}_* \ker(\mathcal{W})]^\perp$ is a strictly convex solution of* (CF) *in $\mathbb{R}^{n-m+1}$ by the restriction of F to the face $\Gamma_+^{n-m} \cong \{z \in \Gamma_+^n : z_1 = \cdots = z_m = 0, z_{m+1}, \ldots, z_n > 0\}$.*

Proof. Consider, for each $1 \leq i \leq n$, the function $d_i := \mathrm{dist}(\vec{\kappa}, H_i)$ which gives the distance of the curvature n-tuple $\vec{\kappa}$ to the hyperplane $H_i := \{\lambda \in \mathbb{R}^n : \lambda_i = 0\}$. Recalling the proof of Theorem 4.18, by the strong maximum principle, either $d_i\big|_{\mathscr{M}\times(0,T)} > 0$ or $d_i\big|_{\mathscr{M}\times[0,t_i]} \equiv 0$ for some $t_i \in (0, T)$. Thus, either $\mathscr{X}\big|_{\mathscr{M}\times(0,T)}$ is strictly convex or there is some $m \in \{1, \ldots, n-1\}$ and $T_m \in (0, T)$ such that $d_i\big|_{\mathscr{M}\times[0,T_m]} \equiv 0$ for each $i = 1, \ldots, m$ and $d_i\big|_{\mathscr{M}\times(0,T_m]} > 0$ for each $i = m+1, \ldots, n$. In particular (recalling the proof of Theorem 4.18), for each $i \leq m$ we obtain, in an orthonormal frame of eigenvectors of $\mathcal{W}$,

$$0 \equiv \nabla_k \, d_i = \nabla_k \, \mathcal{W}_{ii} \tag{4.21}$$

and

$$0 \equiv (\partial_t - \mathscr{L}) d_i \geq \ddot{F}(\nabla_i \, \mathcal{W}, \nabla_i \, \mathcal{W}) + 2 \sup_{\Lambda} \dot{F}^k \left[2\Lambda_{ki}{}^p \, \nabla_k \, \mathcal{W}_{ip} - (\Lambda_{ki}{}^p)^2 (\kappa_p - \kappa_i) \right] \, .$$

Taking $\Lambda_{kip} = \frac{\nabla_k \mathcal{W}_{ip}}{\kappa_p - \kappa_i}$ when $p > m$ and zero otherwise, we obtain

$$0 \equiv (\partial_t - \mathscr{L}) d_i \geq \ddot{F}(\nabla_i \mathcal{W}, \nabla_i \mathcal{W}) + 2\dot{F}^k \sum_{p=m+1}^{n} \frac{(\nabla_k \mathcal{W}_{ip})^2}{\kappa_p} \geq 0 \tag{4.22}$$

for each $i = 1, \ldots, m$. Since F is convex and strictly monotone, we conclude from (4.21) and (4.22) that $\nabla_v \mathcal{W} = 0$ for all $v \in \ker(\mathcal{W})$. Now consider, for any $v \in \Gamma(\ker(\mathcal{W}))$,

$$0 \equiv \nabla_k(\mathcal{W}(v)) = \nabla_k \mathcal{W}(v) + \mathcal{W}(\nabla_k v) = \mathcal{W}(\nabla_k v)\,.$$

Thus, $\nabla_k v \in \Gamma(\ker(\mathcal{W}))$ whenever $v \in \Gamma(\ker(\mathcal{W}))$; that is, $\ker(\mathcal{W})$ is invariant under parallel translation in space. Since, for any $v \in \Gamma(\ker \mathcal{W})$ and any $u \in T\mathscr{M}$, we have

$${}^{\mathscr{X}}\!D_u \mathscr{X}_* v = \mathscr{X}_* \nabla_u v - \mathcal{W}(u, v)\nu = \mathscr{X}_* \nabla_u v \in \mathscr{X}_* \ker \mathcal{W}\,,$$

this implies that $\mathscr{X}_* \ker \mathcal{W}$ is parallel (in space) with respect to ${}^{\mathscr{X}}\!D$.

Moreover, using the evolution equation (4.5) for $\mathcal{W}$, we obtain

$$\begin{aligned}
\nabla_t \mathcal{W}(v) &= \mathscr{L}\mathcal{W}(v) + \ddot{F}(\nabla_v \mathcal{W}, \nabla \mathcal{W}) + \dot{F}(\mathcal{W}^2)\mathcal{W}(v) \\
&= \mathscr{L}\mathcal{W}(v) \\
&= \dot{F}^{kl} \nabla_k \nabla_l \mathcal{W}(v) \\
&= \dot{F}^{kl} \left[\nabla_k \left(\nabla_l \mathcal{W}(v)\right) - \nabla_l(\mathcal{W}(\nabla_k v)) - \mathcal{W}(\nabla_l \nabla_k v)\right] \\
&= 0\,,
\end{aligned}$$

so that

$$\mathcal{W}(\nabla_t v) = \nabla_t(\mathcal{W}(v)) - \nabla_t \mathcal{W}(v) = 0\,;$$

that is, $\ker \mathcal{W}$ is also invariant with respect to ∇_t. Since, for any $v \in \Gamma(\ker(\mathcal{W}))$, we have $\nabla_v F = \dot{F}^{kl} \nabla_v \mathcal{W}_{kl} \equiv 0$, this implies that

$${}^{\mathscr{X}}\!D_t \mathscr{X}_* v = \nabla_v F \nu + \mathscr{X}_* \nabla_t v = \mathscr{X}_* \nabla_t v\,,$$

so that $\mathscr{X}_* \ker \mathcal{W}$ is also parallel in time. We conclude that the orthogonal compliment of $\mathscr{X}_* \ker(\mathcal{W})$ is a constant subspace of $\mathbb{R}^{n+1}$.

Now consider any geodesic $\gamma : \mathbb{R} \to \mathscr{M} \times \{t_0\}$, $t_0 \in (0, T_m]$, with $\gamma'(0) \in \ker(\mathcal{W})$. Then, since $\ker(\mathcal{W})$ is invariant under parallel translation, $\gamma'(s) \in \ker(\mathcal{W})$ for all s, so that

$${}^{\mathscr{X}}\!D_s \mathscr{X}_* \gamma' = \mathscr{X}_* \nabla_s \gamma' - \mathcal{W}(\gamma', \gamma')\nu = 0\,.$$

Thus, $\mathscr{X} \circ \gamma$ is geodesic in $\mathbb{R}^{n+1}$. We may now conclude that $\mathscr{X}$ splits off an m-plane: $\mathscr{M} \cong \mathbb{R}^m \times \Sigma^{n-m}$, such that $\mathbb{R}^m$ is flat ($T\mathbb{R}^m$ is spanned by the flat principal directions $\{e_i\}_{i=1}^m$) and Σ^{n-m} is strictly convex ($T\Sigma^{n-m}$ is spanned by the positively curved principal directions $\{e_i\}_{i=n-m+1}^n$) and maps into the constant subspace $(\mathscr{X}_* \ker(\mathcal{W}))^\perp \cong \mathbb{R}^{n-m+1}$. It follows that $\mathscr{X}\big|_{\{0\}\times\Sigma^{n-m}\times(0,T_m]}$ satisfies

$$\partial_t \widetilde{\mathscr{X}}(\widetilde{x},t) = -\widetilde{F}(\widetilde{x},t)\widetilde{\nu}(\widetilde{x},t)\,,$$

for all $(\widetilde{x},t) \in \{0\}\times\Sigma^{n-m}\times(0,T_m]$, where $\widetilde{\nu} = \nu\big|_{\{0\}\times\Sigma\times(0,T_m]}$ and $\widetilde{F}$ is the restriction of F to $\Gamma_+^{n-m} \cong \{z \in \overline{\Gamma}_+ : z_1 = \cdots = z_m = 0, z_{m+1} > 0, \ldots, z_n > 0\}$. $\square$

Surface flows

For surface flows, we have already proved the following result; however, the proof using Theorem 4.18 seems more direct:

Corollary 4.22 (Cf. Andrews (2010))**.** *Let $F : \Gamma \subset \mathbb{R}^2 \to \mathbb{R}$ be a positive admissible speed function for* (CF)*. Then every convex, symmetric cone $\Gamma_0 \subset\subset \Gamma$ is strongly preserved by* (CF)*.*

Proof. First observe that any (closed) convex, symmetric cone of curvatures containing the positive ray may be defined by the inequality $\min\{\lambda_1,\lambda_2\} \geq \varepsilon\max\{\lambda_1,\lambda_2\}$ for some $\varepsilon \in [-1,1]$. In view of Theorem 4.18, let λ be a non-origin boundary point of $\Gamma_0 = \{\min\{\lambda_1,\lambda_2\} \geq \varepsilon\max\{\lambda_1,\lambda_2\}\}$. Without loss of generality, suppose that $\lambda_1 < \lambda_2$. Then the inward normal cone to $\overline{\Gamma}_0$ at λ is generated by $\ell := (1,-\varepsilon)$. Thus, we need to show that

$$\begin{aligned} 0 &\leq \ddot{F}^{pq,rs}\left(T_{1pq}T_{1rs} - \varepsilon T_{2pq}T_{2rs}\right) + 2(1+\varepsilon)\frac{\dot{F}^{kl}}{\lambda_2-\lambda_1}T_{k12}T_{l12} \\ &= \ddot{F}^{pq}\left(T_{1pp}T_{1qq} - \varepsilon T_{2pp}T_{2qq}\right) + \frac{2}{\lambda_2-\lambda_1}(\dot{F}^2+\varepsilon\dot{F}^1)\left((T_{112})^2+(T_{122})^2\right) \end{aligned}$$

for any T such that $T_{k11} = \varepsilon T_{k22}$.

To show that the first term is non-negative, we make use of the gradient and zero order conditions, and the homogeneity of F: Replacing T_{111} by εT_{122} and T_{211} by εT_{222}, we obtain

$$\ddot{F}^{pq}\left(T_{1pp}T_{1qq} - \varepsilon T_{2pp}T_{2qq}\right) = \left(\varepsilon^2\ddot{F}^{11} + 2\varepsilon\ddot{F}^{12} + \ddot{F}^{22}\right)\left((T_{122})^2 - \varepsilon(T_{222}^2)\right)$$

Making use of the zero order condition, $\lambda_1 = \varepsilon\lambda_2$, and noting that λ_2 is positive yields

$$\ddot{F}^{pq}\left(T_{1pp}T_{1qq} - \varepsilon T_{2pp}T_{2qq}\right) = \frac{1}{\lambda_2^2}\left(\lambda_1^2\ddot{F}^{11} + 2\lambda_1\lambda_2\ddot{F}^{12} + \lambda_2^2\ddot{F}^{22}\right)\left((T_{122})^2 - \varepsilon(T_{222})^2\right)\,,$$

which vanishes by Euler's theorem.

Using the zero order condition and Euler's theorem, the remaining term is

$$\frac{2}{\lambda_2-\lambda_1}(\dot{F}^2+\varepsilon\dot{F}^1)\left((T_{112})^2+(T_{122})^2\right) = \frac{2F}{\lambda_2(\lambda_2-\lambda_1)}\left((T_{112})^2+(T_{221})^2\right) \geq 0\,.$$

$\square$

We also obtain a splitting theorem for admissible flows of weakly convex surfaces:

Theorem 4.23 (Splitting theorem for surface flows)**.** *Let $F:\Gamma\subset\mathbb{R}^2\to\mathbb{R}$ be an admissible flow speed such that $\Gamma_+\subset\subset\Gamma$, and $\mathscr{X}:\mathscr{M}\times[0,T)\to\mathbb{R}^3$ a (possibly non-compact) solution of* (CF) *satisfying $\vec\kappa(x,0)\in\overline{\Gamma}_+\setminus\{0\}$ for all $x\in\mathscr{M}$. Then either $\mathscr{X}\big|_{\mathscr{M}\times(0,T)}$ is strictly convex or $\mathscr{M}$ splits isometrically off a line; that is, $\mathscr{M}\cong\mathbb{R}\times\Sigma$ and there exists $T_1\in(0,T)$ such that $\mathscr{X}\big|_{\mathbb{R}\times\{s\}\times(0,T_1]}$ is flat for each $s\in\Sigma$ and $\mathscr{X}\big|_{\{0\}\times\Sigma\times(0,T_1]}:\Sigma\times(0,T_1]\to\mathbb{R}^2\cong[\mathscr{X}_*\ker(\mathcal{W})]^\perp$ is a strictly convex solution of the scaled curve shortening flow*

$$\partial_t\gamma=-c_1\kappa\nu\quad c_1:=F(0,1)\,.$$

Proof. In light of the proof of Corollary 4.22, the proof is similar to the proof of theorem 4.21. □

Inverse-concave speeds

Finally, we consider flows by inverse-concave speeds.

Corollary 4.24 (Andrews (2007))**.** *Let $F:\Gamma\subset\mathbb{R}^n\to\mathbb{R}$ be an admissible speed function such that $\Gamma_+\subset\subset\Gamma$ and $F\big|_{\Gamma_+}$ is inverse-concave. Then Γ_+ is strongly preserved.*

Remarks 4.4.
1. Note that it suffices to assume that the restriction of F to the face $\Gamma_+^{n-1}=\{(0,\lambda_2,\dots,\lambda_n):\lambda_2,\dots,\lambda_n>0\}$ is inverse-concave.
2. Flows by inverse-concave speeds $F:\Gamma_+\to\mathbb{R}$ also preserve the curvature cones $\{\kappa_1\geq\varepsilon H\}$, $\varepsilon>0$ (Andrews (2007), see also Theorem 6.1).

Proof. The proof is essentially that of a similar result of Andrews (2007, Theorem 3.3) since, for speeds defined on Γ_+, Theorem 4.18 reduces to Andrews (2007, Theorem 3.2). We include the proof here as we will consider flows by inverse-concave speeds in Sections 5 and 6.

To apply the maximum principle (Theorem 4.18), we need to show that

$$\sup_\Lambda\left\{\ell^i\left(\ddot F^{pq,rs}T_{ipq}T_{irs}+2\dot F^{kl}\left[2\Lambda_{ki}{}^pT_{lip}-\Lambda_{ki}{}^p\Lambda_{li}{}^p(\lambda_p-\lambda_i)\right]\right)\right\}\geq 0$$

at any boundary point $\lambda\in\partial\Gamma_0\setminus\{0\}$, for any $\ell\in\mathcal{N}_\lambda\overline{\Gamma}_0$ and any totally symmetric $T\in\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathbb{R}^n$ satisfying $\ell^iT_{kii}=0$ for each k.

So fix $\lambda\in\partial\Gamma_+\setminus\{0\}$. Without loss of generality, we may assume that $\lambda_1=0$. Assume further that $\lambda_1<\cdots<\lambda_n$. Then the inward normal cone at λ is generated by $\ell=e_1$. Thus, in this case, we need only show that

$$\begin{aligned}0&\leq\sup_\Lambda\left\{\ddot F^{pq,rs}T_{1pq}T_{1rs}+2\dot F^{kl}\left[2\Lambda_k{}^pT_{l1p}-\Lambda_k{}^p\Lambda_l{}^p(\lambda_p-\lambda_1)\right]\right\}\\&=\sup_\Lambda\left\{\ddot F^{pq,rs}T_{1pq}T_{1rs}+2\sum_{k\geq1,p\geq2}\left[\frac{\dot F^k}{\lambda_p}(T_{k1p})^2-\dot F^k\lambda_p\left(\Lambda_k{}^p-\frac{T_{k1p}}{\lambda_p}\right)^2\right]\right\}\,,\end{aligned}$$

where the supremum is now taken over $\Lambda_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^n$. The supremum is clearly attained with the choice

$$\Lambda_{ij} = \frac{T_{i1j}}{\lambda_j}\,.$$

Applying the derivative identity (2.18), we obtain

$$\begin{aligned}
\ddot{F}^{pq,rs}T_{1pq}T_{1rs}{+}2\sum_{k\geq 1,p\geq 2}\frac{\dot{F}^k}{\lambda_p}(T_{k1p})^2 \\
&= \ddot{F}^{pq}T_{1pp}T_{1qq} + 2\sum_{p>q}\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}(T_{1pq})^2 + 2\dot{F}^k\sum_{p=2}^{n}\frac{(T_{1kp})^2}{\lambda_p}\\
&= \ddot{F}^{pq}T_{1pp}T_{1qq} + 2\sum_{p>q>1}\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}(T_{1pq})^2 + 2\sum_{p>1}\frac{\dot{F}^p-\dot{F}^1}{\lambda_p}(T_{1p1})^2\\
&\quad + 2\sum_{p>1}^{n}\frac{\dot{F}^1}{\lambda_p}(T_{11p})^2 + 2\sum_{p>1}^{n}\frac{\dot{F}^p}{\lambda_p}(T_{1pp})^2 + 2\sum_{p>q>1}^{n}\left(\frac{\dot{F}^q}{\lambda_p}+\frac{\dot{F}^p}{\lambda_q}\right)(T_{1pq})^2\\
&= \sum_{p,q\neq 1}\left(\ddot{F}^{pq}+2\frac{\dot{F}^p\delta^{pq}}{\lambda_p}\right)T_{1pp}T_{1qq}\\
&\quad + 2\sum_{p>q>1}\left(\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}+\frac{\dot{F}^q}{\lambda_p}+\frac{\dot{F}^p}{\lambda_q}\right)(T_{1pq})^2\,.
\end{aligned}$$

The claim now follows from Lemma 2.10.

The general case, for which λ_i may not be distinct, follows as a limiting case: For any $\lambda \in \Gamma_+ \setminus \{0\}$, there exists a sequence of points $\lambda^{(m)} \in \partial\,\Gamma_+ \setminus \{0\}$ with $\lambda_i^{(m)}$ pairwise distinct which converges to λ. The claim follows since the term we wish to estimate is upper semi-continuous in λ, and non-negative along the sequence. □

We expect that Theorem 4.18 will have further applications to flows of non-convex hypersurfaces by concave speeds.

4.2.3 Estimating homogeneous functions of the curvature

An important consequence of the existence of a preserved cone is the following simple observation, which will be used extensively in Section 5.

Lemma 4.25. *Let $Z : \Gamma \to \mathbb{R}$ be a symmetric function which is homogeneous of degree zero. Suppose that $\Gamma_0 \subset\subset \Gamma$. Then the extrema $\sup\{Z(\lambda) : \lambda \in \Gamma_0\}$ and $\inf\{Z(\lambda) : \lambda \in \Gamma_0\}$ are attained. Moreover, if $Z > 0$ on Γ, then the infimum $\inf\{Z(\lambda) : \lambda \in \Gamma_0\}$ is positive.*

Proof. Since Z is degree zero homogeneous, its value at a point λ agrees with its value at the projection of λ onto the unit sphere. The claims now follow from compactness of $\Gamma_0 \cap S^n$. □

As a particular application, we obtain the following uniform parabolicity estimate:

Proposition 4.26 (Uniform parabolicity). *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible speed, and $\Gamma_0 \subset\subset \Gamma$ a curvature cone. Then there exists a constant C (depending only on F and Γ_0) such that along any solution $\mathscr{X} : \mathscr{M} \times [0, T) \to \mathbb{R}^{n+1}$ of* (CF) *satisfying $\vec{\kappa}(\mathscr{M} \times [0, T)) \subset \Gamma_0$ it holds that*

$$C^{-1} g^{ij}_{(x,t)} v_i v_j \le \dot{F}^{ij}_{(x,t)} v_i v_j \le C g^{ij}_{(x,t)} v_i v_j \tag{4.23}$$

for all $(x, t) \in \mathscr{M} \times [0, T)$ and all $v \in T_x\mathscr{M}$.

Proof. In an orthonormal frame of eigenvectors of the Weingarten map, we have, by (2.17), that $\dot{F}^{kl} v_k v_l = \dot{F}^k (v_k)^2$. Since F is strictly monotone and homogeneous of degree one, the claim follows from the preceding lemma. □

It will be useful to introduce the following notation:

$$\langle u, v\rangle_F := \dot{F}^{kl} u_k v_l$$

and

$$|v|^2_F := \dot{F}^{kl} v_k v_l \,.$$

By Proposition 4.26, $\langle \cdot\,,\ \cdot\rangle_F$ and $|\cdot|_F$ define an inner product and norm on $T\mathscr{M}$ uniformly equivalent to g and its induced norm whenever the flow admits a preserved cone.

4.3 Global existence

Having established the existence of solutions of (CF) for a short time, we now examine their long time regularity. By writing solutions locally as graphs over a tangent plane, we will be able to make use of the existing regularity theory for fully non-linear parabolic scalar equations. We have collected the required results in Appendix A.

Proposition 4.27. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, be an admissible flow speed. Suppose that one of the following holds:*

1. *$n = 2$; or*
2. *F is convex; or*
3. *F is concave.*

Then, given a curvature bound, $C_0 < \infty$, and a curvature cone, $\Gamma_0 \subset\subset \Gamma$, the following estimate holds: For every $m \in \mathbb{N}$ there exists $C_m < \infty$, depending only on n, F, C_0, and Γ_0, such that every smooth solution $\mathscr{X} : \mathscr{M}^n \times (0, T] \to \mathbb{R}^{n+1}$ of (CF) *satisfying $\sup_{\mathscr{M}\times[T/2,T]} |\mathcal{W}| \le C_0$ and $\vec{\kappa}(x, t) \in \Gamma_0$ for all $(x, t) \in \mathscr{M} \times [T/2, T]$ satisfies*

$$\sup_{\mathscr{M}\times[T/2,T]} |\nabla^m \mathcal{W}| \le C_m \,.$$

Remarks 4.5. 1. We have only assumed the speed conditions 1–3 in order that (CF) admits Hölder estimates for the curvature; that is, so that, for any compact subset $K \subset \Gamma$ and any $T > 0$, there exist $\alpha \in (0, 1]$ and $C > 0$ (depending only on K and T) such that for any solution $\mathscr{X} : \mathscr{M}^n \times [0, T] \to \mathbb{R}^{n+1}$ of (CF) with $\vec{\kappa}(\mathscr{M} \times [0, T]) \subset K$ it holds that $||\mathcal{W}||_{C^{0,\alpha}(\mathscr{M}\times[T/2,T])} \leq C$.

2. Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible speed function. Then (by Proposition 4.13, and Corollaries 4.19, 4.15, and 4.24) the assumption $\vec{\kappa}(\mathscr{M} \times [T/2, T]) \subset \Gamma_0 \subset\subset \Gamma$ automatically holds for some cone Γ_0 (which depends only on F and the immersion $\mathscr{X}_{T/2}$) if F is concave and $\Gamma_0 = \{\kappa_n \leq \beta F\}$ for some $\beta < c_1^{-1}$ (see Remark 4.5), or $\Gamma_+ \subset \Gamma$ and F is convex, or $n = 2$, or $\Gamma_0 = \Gamma_+$ and F is concave and inverse-concave.

Proof. In order to make use of the estimates of Appendix A, we write the solution locally as an evolving graph: Fix any $(x_1, t_1) \in \mathscr{M} \times [T/2, T]$; up to an ambient isometry, we may assume that $\mathscr{X}(x_1, t_1)$ is the origin and $\nu(x_1, t_1) = e_{n+1}$. An application of the rank theorem implies the existence of neighbourhoods U of x_1 and I of t_1, an open set $V \in \mathbb{R}^n$, a diffeomorphism $\varphi : V \to U$, and a smooth function $u : U \times I \to \mathbb{R}$ such that $\mathscr{X}\left(\varphi^{-1}(y), t\right) = (y, u(y, t))$ for all $y \in U$ and $t \in I$.

Observe that the Weingarten curvature of $\mathscr{X}$ and its derivatives are controlled near (x_1, t_1) by u and its derivatives near (y_1, t_1), where $y_1 = \varphi(x_1)$: We computed, in §3.4.1,

$$\mathcal{W}_{ij} = \frac{u_{ij}}{\sqrt{1 + ||Du||^2}}\,,$$

and

$$g^{ij} = \delta^{ij} - \frac{u_i u_j}{1 + ||Du||^2}\,.$$

Note also that the connection coefficients are given in the graphical coördinates by

$$\Gamma_{ijk} = \left\langle \frac{\partial^2 \mathscr{X}}{\partial y^i\, \partial y^j}\,, \frac{\partial \mathscr{X}}{\partial y^k} \right\rangle = u_{ij} u_k.$$

It follows that, with respect to the graphical coördinates, the components $\nabla^m_k \mathcal{W}_{ij}$ of the m-th covariant derivative of $\mathcal{W}$ are given by an expression which depends on $D^k u$ for $1 \leq k \leq m + 2$. Since the induced metric depends only on first derivatives of u, we conclude that $|\nabla^m \mathcal{W}|$ is bounded once the derivatives of u up to order $m+2$ are bounded.

Thus, to estimate $|\nabla^m \mathcal{W}|$ at (x_1, t_1), it suffices to obtain estimates for the derivatives of u at (y_1, t_1) up to order $m + 2$. We first obtain a local bound for $||Du||$ depending only on the bound for $\sup_{\mathscr{M}\times[T/2,T]} |\mathcal{W}|$: Observe that the Weingarten curvature is given locally by

$$\frac{\partial}{\partial y^i}\left(\frac{u_j}{\sqrt{1 + ||Du||^2}}\right) = -\mathcal{W}_i{}^j\,.$$

Integrating the trace elements, we obtain

$$-\sqrt{C_0}y^i \leq \frac{u_i}{\sqrt{1+||Du||^2}} \leq \sqrt{C_0}y^i\,.$$

It follows that

$$\frac{||Du||^2}{1+||Du||^2} \leq C_0\,||y||^2\,.$$

Thus, $||Du||^2 \leq 1$ on $B_{1/\sqrt{2C_0}}(0) \times I$.

We next prove a $C^{2,\alpha}$-estimate for u: Recall from §3.4.1 that u satisfies

$$\partial_t\, u = \widehat{F}(D^2u, Du) := F(PD^2uP)\,, \tag{4.24}$$

where

$$P = I - \frac{Du \otimes Du}{\sqrt{1+||Du||^2}\left(1+\sqrt{1+||Du||^2}\right)}\,.$$

Note that $\widehat{F} \equiv \widehat{F}(r,p)$ is smooth (and therefore, in particular, Lipschitz) in each argument, and satisfies

$$\left.\frac{\partial\,\widehat{F}}{\partial\,r_{kl}}\right|_{(D^2u,Du)} = \dot{F}^{pq}P_p{}^kP_q{}^l\,.$$

Since the cone Γ_0 is preserved, it follows from Proposition 4.26 that equation (4.24) is uniformly parabolic (with constants depending only on C_0, F, and Γ_0). Moreover, since $F(PD^2uP) = \sqrt{1+||Du||^2}F(\mathcal{W}) = \sqrt{1+||Du||^2}\dot{F}^{kl}\mathcal{W}_{kl}$, we have that $\partial_t\,u$ is bounded by a constant that depends only on C_0, F, and Γ_0.

If $n = 2$, then a parabolic $C^{2,\alpha}$ estimate (with constants depending only on C_0, F, and Γ_0) for u now follows from Theorem A.3.

For the remaining cases, observe that

$$\left.\frac{\partial^2\,\widehat{F}}{\partial\,r_{kl}\,\partial\,r_{mn}}\right|_{(D^2u,Du)} M_{kl}M_{mn} = \ddot{F}^{pq,rs}\big|_{PD^2uP}P_p{}^kP_q{}^lM_{kl}P_r{}^kP_s{}^lM_{kl}\,.$$

If F is concave, we obtain a $C^{2,\alpha}$ estimate from Theorem A.2 (with constants depending only on n, C_0, F, and Γ_0). If F is convex, we similarly obtain a $C^{2,\alpha}$ estimate for $-u$, since $v := -u$ satisfies the equation

$$v_t = \widehat{F}_*(D^2v, Dv)\,,$$

where $\widehat{F}_*(r,p) = -\widehat{F}(-r,-p)$.

Next, we consider the evolution equations for the spatial derivatives of u: Set $v = u_i$.

Then v satisfies the equation

$$v_t = a^{ij} v_{ij} + b^i v_i \,, \tag{4.25}$$

where

$$a^{ij} = \left.\frac{\partial \widehat{F}}{\partial r_{ij}}\right|_{(D^2u,Du)} = \dot{F}^{pq} P_p{}^i P_q{}^j$$

and

$$b^i = \left.\frac{\partial \widehat{F}}{\partial p_i}\right|_{(D^2u,Du)} = -\frac{1}{1+||Du||^2} \dot{F}^{kl} \left(u_{ik} u_l + u_p u_{pk} \delta_{il} - \frac{2u_i u_p u_{pk} u_l}{1+||Du||^2} \right) .$$

In particular, the coefficients a^{ij} and b^i are bounded by constants that depend only on bounds for $\left|\left|D^2u\right|\right|$ and $||Du||$ (which we have seen depend only on C_0). Furthermore, we have seen that a^{ij} is uniformly positive definite (with constants depending only on C_0, F and Γ_0). Finally, the $C^{2,\alpha}$ estimate for u implies that the coefficients of (4.25) are Hölder continuous. We may therefore apply Theorem A.4, yielding $C^{2,\alpha}$ bounds for v. Higher regularity follows by applying Theorem A.4 inductively: The higher derivatives of u satisfy equations similar to (4.25). Hölder continuity of the m-th spatial derivatives of u permits us to apply Theorem A.4 to the evolution equation for the $(m-1)$-st spatial derivatives, yielding Hölder continuity of the $(m+1)$-st derivatives. By induction, this yields bounds on all spatial derivatives of u (which depend only on n, C_0, F, and Γ_0). □

Definition 4.28. *Let $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ (T possibly infinite) be a solution of* (CF). *Then $\mathscr{X}$ is* maximal *if for every solution $\mathscr{Y} : \mathscr{M} \times [0,S) \to \mathbb{R}^{n+1}$ satisfying $\mathscr{Y}(x,t) = \mathscr{X}(x,t)$ for all $(x,t) \in \mathscr{M} \times ([0,T) \cap [0,S))$ we have $S \leq T$.*

Theorem 4.29 (Global existence)**.** *Let $F : \Gamma \subset \mathbb{R}^n$, $n \geq 1$ be an admissible speed. Suppose that one of the following conditions hold:*

1. *$n = 2$; or*
2. *F is convex; or*
3. *F is concave.*

Let $\Gamma_0 \subset\subset \Gamma$ be a curvature cone and $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ a maximal (compact) solution of (CF) *such that $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \Gamma_0$. Then $T < \infty$, and* $\limsup_{t\to T} \sup_{\mathscr{M}\times\{t\}} |\mathcal{W}| = \infty$.

Proof. That $T < \infty$ follows (for compact solutions of flows by any admissible speed) from the avoidance principle (Theorem 4.32 below) since the initial hypersurface $\mathscr{M}_0 := \mathscr{X}(\mathscr{M}, 0)$ may be enclosed by some sufficiently large sphere, which shrinks homothetically under (CF) to a point after finite time.

The remaining claim follows easily from Proposition 4.27 and Theorem 3.7: Suppose, contrary to the conclusion of the theorem, that $C_0 := \sup_{\mathscr{M}\times[0,T)} |\mathcal{W}| < \infty$. We will show that the solution $\mathscr{X}(\cdot,t)$ converges to a smooth limit immersion as $t \to T$. This limit can be used as initial data for (CF), which, by the short time existence theorem, may be flowed for a small time, yielding a contradiction to the maximality of $\mathscr{X}$.

We first derive a C^0-limit (cf. Huisken 1984): Since $|\mathcal{W}|$ is bounded and the flow admits preserved curvature cones, we have a speed bound $F \le C$. Thus, for any $x \in \mathscr{M}$ and $t_1 \le t_2 \in [0,T)$,

$$\begin{aligned}
|\mathscr{X}(x,t_2) - \mathscr{X}(x,t_1)| &= \left|\int_{t_1}^{t_2} \partial_t \mathscr{X}(x,t)dt\right| \\
&\le \int_{t_1}^{t_2} |\partial_t \mathscr{X}(x,t)|\,dt \\
&= \int_{t_1}^{t_2} |F(x,t)|\,dt \\
&\le C|t_2 - t_1|\,.
\end{aligned}$$

Thus, $\mathscr{X}$ is Cauchy continuous with respect to t and therefore converges in C^0 to a unique limit $\mathscr{X}_T : \mathscr{M} \to \mathbb{R}^{n+1}$ as $t \to T$.

We next show that $\mathscr{X}_T$ is an immersion (cf. Hamilton 1982, Lemma 14.2): Using the evolution equation (2.5) for the metric, and the uniform parabolicity estimate (4.23), we have, for any $t_1 < t_2 \in [0,T)$ and any $(x,v) \in T\mathscr{M}$,

$$\begin{aligned}
\left|\log\frac{g_{(x,t_2)}(v,v)}{g_{(x,t_1)}(v,v)}\right| &= \left|\int_{t_1}^{t_2} \frac{\frac{\partial}{\partial t} g_{(x,\cdot)}(v,v)}{g_{(x,\cdot)}(v,v)} dt\right| \\
&\le \int_{t_1}^{t_2} \left|\frac{\frac{\partial}{\partial t} g_{(x,\cdot)}(v,v)}{g_{(x,\cdot)}(v,v)}\right| dt \\
&= 2\int_{t_1}^{t_2} \left|F\frac{\mathcal{W}_{(x,\cdot)}(v,v)}{g_{(x,\cdot)}(v,v)}\right| dt \\
&\le 2\widetilde{C}_0(t_2 - t_1)\,.
\end{aligned} \tag{4.26}$$

It follows that

$$e^{-2\widetilde{C}_0 t} g_{(x,0)}(v,v) \le g_{(x,t)}(v,v) \le e^{2\widetilde{C}_0 t} g_{(x,0)}(v,v)$$

for any $t \in [0,T)$ and any $(x,t) \in T\mathscr{M}$. It follows that $\mathscr{X}_T$ is a regular immersion. Smoothness of $\mathscr{X}_T$ follows from Proposition 4.27 and the evolution equations (CF) and (2.5).

Finally, using the smooth limit immersion $\mathscr{X}_T$ as an initial datum for (CF), the short-time existence theorem 3.7 allows us to extend $\mathscr{X}$ for a short time beyond T, contradicting the maximality assumption. □

4.4 Bounds for the maximal time

We now prove two estimates for the maximal time of existence of solutions of (CF).

Let us first prove a lower bound for the speed:

Lemma 4.30. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed and $\mathscr{X} : \mathscr{M} \times [0, T) \to \mathbb{R}$ a solution of* (CF) *satisfying $\vec{\kappa}(\mathscr{M} \times [0, T)) \subset \Gamma_0$ for some cone $\Gamma_0 \subset\subset \Gamma$. Then*

$$F(x,t) \geq \frac{F_{\min}}{\sqrt{1 - 2\underline{c}F_{\min}^2 t}}$$

for all $(x, t) \in \mathscr{M} \times [0, T)$, where $F_{\min} := \min_{\mathscr{M}\times\{0\}} F$ and $\underline{c} := \max\{F(\lambda)^2/\dot{F}^k(\lambda)\lambda_k^2 : \lambda \in \Gamma_0\}$.

Proof. Recalling that F satisfies the linearized flow (LF) and estimating $|\mathcal{W}|_F^2 \geq \underline{c}F^2$, we obtain

$$(\partial_t - \mathscr{L})F \geq \underline{c}F^3$$

The claim now follows from the maximum principle by comparing $\min_{\mathscr{M}\times\{t\}} F$ with the solution of the ordinary differential equation

$$\frac{du}{dt} = \underline{c}u^3 .$$

□

Proposition 4.31. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible speed function and $\Gamma \subset \Gamma_0$ a curvature cone. Suppose $\mathscr{X} : \mathscr{M} \times [0, T) \to \mathbb{R}^{n+1}$ is a solution of* (CF) *satisfying $\vec{\kappa}(\mathscr{M} \times [0, T)) \subset \Gamma_0$. Then the following hold:*

(i) $T \leq \frac{2r_+(\mathscr{M}_0)^2}{c_0}$*, where $c_0 := F(1, \ldots, 1)$ and $r_+(\mathscr{M}_0)$ is the circumradius of the initial hypersurface $\mathscr{M}_0 = \mathscr{X}(\mathscr{M}, 0)$.*

(ii) If $\vec{\kappa}(\mathscr{M}\times[0, T)) \subset \Gamma_0$ for some $\Gamma_0 \subset\subset \Gamma$, then $T \leq \frac{1}{2\underline{c}F_{\min}^2}$, where $F_{\min} := \min_{\mathscr{M}\times\{0\}} F$ and $\underline{c} := \max\{\dot{F}^k(\lambda)\lambda_k^2/F(\lambda)^2 : \lambda \in \Gamma_0\}$.

Proof. The first estimate follows from the avoidance principle (Theorem 4.32) by considering the evolution of all spheres which enclose the initial datum (recall Proposition 3.2). The second estimate follows immediately from the lower speed bound, Lemma 4.30. □

4.5 The avoidance principle

In this section we prove a comparison principle for solutions of (CF), which is well-known.

Theorem 4.32 (The avoidance principle)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed with a well-defined odd extension $F(z) := -F(-z)$ for $z \in -\Gamma := \{-z : z \in$*

$\Gamma\}$ *and let* $\mathscr{X}_i : \mathscr{M}_i \times [0,T) \to \mathbb{R}^{n+1}$, $i = 1,2$ *be two compact solutions of* (CF) *with* $\mathscr{X}_1(\mathscr{M}_1, 0) \cap \mathscr{X}_2(\mathscr{M}_2, 0) = \emptyset$. *Then the distance*

$$d_{\min}(t) := \min_{(x,y)\in\mathscr{M}_1\times\mathscr{M}_2} ||\mathscr{X}_1(x,t) - \mathscr{X}(y,t)||$$

is non-decreasing in t, *and, in particular,* $\mathscr{X}_1(\mathscr{M}_1, t) \cap \mathscr{X}_2(\mathscr{M}_2, t) = \emptyset$ *for* $t \in [0,T)$.

Proof. Define the extrinsic distance function $d : \mathscr{M}_1 \times \mathscr{M}_2 \times [0,T) \to \mathbb{R}$ by

$$d(x,y,t) = \|\mathscr{X}_1(x,t) - \mathscr{X}_2(y,t)\| \ .$$

We will show that the time derivative of d is non-negative at a spatial minimum. The claim then follows from the maximum principle. To simplify notation, we define

$$w(x,y,t) = \frac{\mathscr{X}_1(x,t) - \mathscr{X}_2(y,t)}{d(x,y,t)} \,,$$

and use scripts x and y to denote geometric quantities defined on $\mathscr{M}_1\times\mathscr{M}_2$ by pulling back those from $\mathscr{X}_1$ and $\mathscr{X}_2$ by the respective projections; for example, $F_x(\xi,\eta,\tau) := F(\xi,\tau)$.

With this notation in place, we find that d satisfies

$$\frac{\partial}{\partial t} d = \langle w, -F_x\nu_x + F_y\nu_y\rangle \,. \tag{4.27}$$

Suppose there is a spatial minimum of d at (x_0, y_0, t_0). Then, at this point, $\nabla^{\mathscr{M}_1\times\mathscr{M}_2} d = 0$ and $\mathrm{Hess}^{\mathscr{M}_1\times\mathscr{M}_2} d \geq 0$.

Let $\{x^i\}_{i=1}^n$ and $\{y^i\}_{i=1}^n$ be coördinates defined on neighbourhoods of x_0 and y_0 respectively, and write $\partial_i^x := \frac{\partial\mathscr{X}_1}{\partial x_i}$ and $\partial_i^y := \frac{\partial\mathscr{X}_2}{\partial y_j}$. Then

$$\partial_{x_i} d = \langle \partial_i^x, w\rangle \quad \text{and} \quad \partial_{y^i} d = -\langle \partial_i^y, w\rangle \,. \tag{4.28}$$

These vanish at the minimum (x_0, y_0, t_0); that is, $w(x_0, y_0, t_0)$ is orthogonal both to the tangent plane to $\mathscr{X}_1$ at (x_0, t_0) and the tangent plane to $\mathscr{X}_2$ at (y_0, t_0). Now, the assumption that F is odd implies the flow is invariant under change of orientation, since the sign of the Weingarten map changes with the orientation of the normal. So we may choose the orientations of $\mathscr{M}_1$ and $\mathscr{M}_2$ such that[2] $\nu_x = \nu_y = w$ at (x_0, y_0, t_0).

Next, using the vanishing of the gradients (4.28), we obtain, at the point (x_0, y_0, t_0), the identities

$$\nabla_i^{\mathscr{M}_1}\nabla_j^{\mathscr{M}_1} d = -\mathcal{W}_{ij}^x \langle \nu_x, w\rangle + \frac{1}{d}\, g_{ij}^x, \tag{4.29}$$

$$\nabla_i^{\mathscr{M}_2}\nabla_j^{\mathscr{M}_1} d = -\frac{1}{d}\left\langle \partial_j^x, \partial_i^y\right\rangle, \tag{4.30}$$

[2] See also the second Remark following the proof.

and

$$\nabla_i^{\mathscr{M}_2}\nabla_j^{\mathscr{M}_2} d = \mathcal{W}_{ij}^y \langle \nu_y, w\rangle + \frac{1}{d}\, g_{ij}^y. \tag{4.31}$$

Thus, for any vector $v \in \mathbb{R}^n$, we find

$$\begin{aligned} 0 &\leq v^i v^j \left(\nabla_i^{\mathscr{M}_1}\nabla_j^{\mathscr{M}_1} d + 2\,\nabla_i^{\mathscr{M}_2}\nabla_j^{\mathscr{M}_1} d + \nabla_i^{\mathscr{M}_2}\nabla_j^{\mathscr{M}_2} d\right) \\ &= -\mathcal{W}_{ij}^x v^i v^j \langle \nu_x, w\rangle + \frac{1}{d} g_{ij}^x v^i v^j + \mathcal{W}_{ij}^y v^i v^j \langle \nu_y, w\rangle + \frac{1}{d} g_{ij}^y v^i v^j - \frac{2}{d} v^i v^j \langle \partial_i^x, \partial_j^y\rangle \end{aligned}$$

at the point (x_0, y_0, t_0). We now choose the coördinates $\left\{x^i\right\}_{i=1}^n$ and $\left\{y^i\right\}_{i=1}^n$ to be orthonormal coördinates centred at x_0 and y_0 respectively. Since the tangent planes of the two hypersurfaces are parallel at x_0 and y_0, we may further assume that $\partial_i^x = \partial_i^y$ for all i at the point (x_0, y_0, t_0). Then, since $g_{ij}^x = g_{ij}^y = \delta_{ij}$ at (x_0, y_0, t_0), we obtain

$$\mathcal{W}_{ij}^x v^i v^j \leq \mathcal{W}_{ij}^y v^i v^j$$

at that point. It follows that $\mathcal{W}_{ij}^x \leq \mathcal{W}_{ij}^y$ at any spatial minimum of d. Since F is monotone, this implies $F_x \leq F_y$ at such a point. Thus, by (4.27), we obtain

$$\frac{\partial d}{\partial t} = -F_x + F_y \geq 0$$

at any spatial minimum of d. □

Remarks 4.6.

1. The homogeneity condition was not needed in the proof of Theorem 4.32.

2. The assumption that the speed is an odd function of the curvature can be relaxed if we make an additional topological assumption on the hypersurfaces to guarantee the correct orientation: If we require that $\mathscr{X}_1(\mathscr{M}_1, 0) = \partial\,\Omega_1$ and $\mathscr{X}_2(\mathscr{M}_2, 0) = \partial\,\Omega_2$ such that $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^{n+1}$ and the unit normal to $\mathscr{M}_i$ points out of Ω_i for $i = 1, 2$, then the above argument goes through unharmed, since this guarantees $\nu_x = w = \nu_y$ at the distance minimizing pair (x_0, y_0). This observation means that we can still compare compact solutions of (CF) with enclosing spheres, even if F has no odd extension.

The following example shows that the avoidance principle can be violated if the speed function is not odd and the topological assumption of the preceding remarks is not met.

Example 4.1. Observe that the function $F(z) := z_1 + z_2 + \frac{1}{2}\sqrt{z_1^2 + z_2^2}$ defines an admissible speed function on the cone $\Gamma = \mathbb{R}^2$. Under the corresponding flow, surfaces with opposite orientation can move closer together (and even cross): Consider the torus T obtained by rotating the circle $\{z \in \mathbb{R}^3 : (z_1 - 1)^2 + z_2^2 = r^2, z_3 = 0\}$, $r < 1$, about the z_2 axis. If we orient T with its inward pointing unit normal, we obtain $F_{\mathrm{T}}(p) = \left(\frac{1}{R} - \frac{1}{r} + \frac{1}{2}\sqrt{\frac{1}{R^2} + \frac{1}{r^2}}\right)$ at the point $p = (R, 0, 0)$, where $R := 1 - r$. On the other hand, the cylinder C obtained by rotating the line $\{z \in \mathbb{R}^3 : z_1 = R, z_3 = 0\}$ about the z_2 axis satisfies $F_{\mathrm{C}}(p) = -\frac{1}{2R}$ if we also orient using the inward pointing normal. Note that $F_{\mathrm{T}}(p) \geq \frac{1}{R} - \frac{1}{2r}$. Thus, we can achieve $F_{\mathrm{T}}(p) \geq -F_{\mathrm{C}}(p)$ if $\frac{1}{R} > \frac{1}{r}$; that is, if $r > \frac{1}{2}$. Since the normals of the two surfaces are pointing in opposite directions at the point p, this implies that the surfaces will begin to cross. Scaling down the cylinder by a small factor and capping off its ends sufficiently far away from the origin yields a configuration which contradicts the avoidance principle.

4.6 Preservation of embeddedness

In this section we prove that embedded initial data remain embedded under the evolution by (CF). This result is well-known.

Theorem 4.33. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed*[3] *with a well-defined odd extension $F(z) := -F(-z)$ for $z \in -\Gamma := \{-z : z \in \Gamma\}$, and $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ a solution of* (CF). *Suppose that $\mathscr{X}_0 := \mathscr{X}(\cdot, 0)$ is an embedding. Then $\mathscr{X}_t := \mathscr{X}(\cdot, t)$ is an embedding for all $t \in [0,T)$.*

Remark 4.7. In view of Theorem 4.33, we shall refer to a solution $\mathscr{X} : \mathscr{M}^n \times I \to \mathbb{R}^{n+1}$ of (CF) as an *embedded solution* if $\mathscr{X}_t := \mathscr{X}(\cdot, t)$ is an embedding for each $t \in I$.

Proof. Away from the diagonal $D := \{(x,x) : x \in \mathscr{M}\} \subset \mathscr{M} \times \mathscr{M}$, the proof is the same as that of the avoidance principle: At an off-diagonal local minimum of the distance function, the distance is non-decreasing. However, we must be more careful at points close to the diagonal $D := \{(x,x) : x \in \mathscr{M}\}$, on which d is neither smooth nor positive. We will show that a bound on $|\mathcal{W}|$ implies the existence of a neighbourhood E of D on which $d(x,y,t) = 0$ if and only if $x = y$, completing the proof.

We begin by restricting to a compact set: Fixing $\sigma \in (0,T)$, we will prove the claim for $t \in [0,\sigma]$. Set $C_\sigma := \max_{\mathscr{M} \times [0,\sigma]} |\mathcal{W}| < \infty$ and let d denote the extrinsic distance function, $d(x,y,t) := ||\mathscr{X}(x,t) - \mathscr{X}(y,t)||$. We will show that there is a neighbourhood of D, which depends on C_σ, for which $d(x,y,t) = 0$ if and only if $x = y$. To this end, consider points $x, y \in \mathscr{M}$ and a unit speed, length-minimizing geodesic $\gamma : [0,\beta] \to \mathscr{M}$ joining them (such that $\gamma(0) = x$ and $\gamma(\beta) = y$). Denote by s the parameter for γ, by $\gamma' = \gamma_* \partial_s$ its tangent vector, and by ${}^{\mathscr{X}}\!D$ the pullback connection of the Euclidean connection D on $\mathbb{R}^{n+1}$. Then the curvature bound implies

$$\left|\left|{}^{\mathscr{X}}\!D_s(\mathscr{X}_{t*}\gamma')\right|\right| = \left|\mathcal{W}(\gamma',\gamma')\right| \leq C_\sigma$$

for all $t \in [0,\sigma]$. Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\partial_s \left\langle \mathscr{X}_{t*}\gamma'(s), \mathscr{X}_{t*}\gamma'(0)\right\rangle\right| &= \left|\mathcal{W}(\gamma',\gamma')(s) \left\langle \nu(\gamma(s)), \mathscr{X}_{t*}\gamma'(0)\right\rangle\right| \\ &\leq C_\sigma\end{aligned}$$

[3] Note that, as in the proof of the avoidance principle, we will not actually require the speed be homogeneous.

for all $s \in [0, \beta]$ and all $t \in [0, \sigma]$. We now have

$$\begin{aligned}\left|\left\langle \mathscr{X}_{t*}\gamma'(s), \mathscr{X}_{t*}\gamma'(0)\right\rangle - 1\right| &= \left|\left\langle \mathscr{X}_{t*}\gamma'(s), \mathscr{X}_{t*}\gamma'(0)\right\rangle - \left\langle \mathscr{X}_{t*}\gamma'(0), \mathscr{X}_{t*}\gamma'(0)\right\rangle\right| \\ &= \left|\int_0^s \partial_\varepsilon \left\langle \mathscr{X}_{t*}\gamma'(\varepsilon), \mathscr{X}_{t*}\gamma'(0)\right\rangle d\varepsilon\right| \\ &\leq \int_0^s \left|\partial_\varepsilon \left\langle \mathscr{X}_{t*}\gamma'(\varepsilon), \mathscr{X}_{t*}\gamma'(0)\right\rangle\right| d\varepsilon \\ &\leq C_\sigma s \leq C_\sigma \beta\end{aligned}$$

for all $s \in [0, \beta]$ and all $t \in [0, \sigma]$. So, denoting by L the intrinsic distance function on $\mathscr{M}$, whenever $L(x, y) \leq (2C_\sigma)^{-1}$ we must have

$$\frac{1}{2} \leq \left\langle \mathscr{X}_{t*}\gamma'(s), \mathscr{X}_{t*}\gamma'(0)\right\rangle$$

for all $s \in [0, \beta]$ and all $t \in [0, \sigma]$. Utilising the Cauchy-Schwarz inequality once more, we find

$$\begin{aligned}d(x, y, t) = ||\mathscr{X}(x, t) - \mathscr{X}(y, t)|| &\geq \left|\left\langle \mathscr{X}(x, t) - \mathscr{X}(y, t), \mathscr{X}_{t*}\gamma'(0)\right\rangle\right| \\ &= \left|\int_0^\beta \partial_s \left\langle \mathscr{X}(\gamma(s), t), \mathscr{X}_{t*}\gamma'(0)\right\rangle ds\right| \\ &= \left|\int_0^\beta \left\langle \mathscr{X}_{t*}\gamma'(s), \mathscr{X}_{t*}\gamma'(0)\right\rangle ds\right| \\ &\geq \frac{\beta}{2} = \frac{L(x, y)}{2}\,.\end{aligned}$$

We have proved that, $d(x, y, t) = 0$ if and only if $x = y$ whenever $t \in [0, \sigma]$ and (x, y) lies in the strip $E := \left\{(x, y) \in \mathscr{M} \times \mathscr{M} : L(x, y) < \frac{1}{2C_\sigma}\right\}$. Thus, up to time σ, failure of embeddedness can only occur away from D. But this possibility may be ruled out as described above. Since σ was arbitrary, the claim follows. $\square$

Note that, by similar considerations as in Example 4.1, the conclusion of Theorem 4.33 fails without the oddness assumption for the flow speed.

5. A priori estimates for the curvature

5.1 Introduction

In §4.2 we saw that, so long as the second derivatives of the speed function which arise in the evolution equation for the curvature are not harmful, each solution of the flow (CF) admits a preserved cone of curvatures. Faith in the influence of diffusion leads to the belief that these cones should actually 'improve'. In the present section, we will see that this is indeed the case: Wherever the speed is becoming large, the curvature of the solution is pinched onto an optimal cone of curvatures. More precisely, we shall prove the following a priori estimate:

Theorem 5.1 (Curvature pinching, Andrews, Langford and McCoy (2014b; 2014a), Andrews and Langford (2014))**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a positive admissible speed function, and assume that one of the following auxiliary conditions is satisfied:*

1. *F is convex; or*
2. *$n = 2$; or*
3. *$\Gamma = \Gamma_+$ and F is concave.*

Then, given a curvature cone $\Gamma_0 \subset\subset \Gamma$, an initial volume scale $\alpha > 0$, an initial distance scale $R > 0$, and any $\varepsilon > 0$, there exists a constant $C_\varepsilon < \infty$ (depending only on n, F, Γ_0, α, R, and ε) such that, given any solution $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ of (CF) *with curvature satisfying $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \Gamma_0$, initial volume satisfying $\mu_0(\mathscr{M}) \leq \alpha$, and initial curvature satisfying $\min_{\mathscr{M}\times\{0\}} F \geq R^{-1}$ (alternatively, $\operatorname{diam}(\mathscr{X}_0(\mathscr{M})) \leq R$), the following estimate holds:*

$$\operatorname{dist}\left(\vec{\kappa}(x,t), \Lambda_m^+\right) \leq \varepsilon F(x,t) + C_\varepsilon$$

for all $(x,t) \in \mathscr{M} \times [0,T)$, where $m \in \{0,\ldots,n-1\}$ is the smallest integer such that

$$\Gamma_0 \subset\subset \Gamma_{m+1} := \begin{cases} \bigcap_{\sigma\in P_n}\{z \in \mathbb{R}^n : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} > 0\} & \text{if } m < n-1 \\ \Gamma & \text{if } m = n-1\,, \end{cases}$$

and, recalling that $c_m := F(\underbrace{0,\ldots,0}_{m-times},1,\ldots,1)$ is the value F takes on the unit cylinder

$\mathbb{R}^m \times S^{n-m}$,

$$\Lambda_m^+ := \begin{cases} \bigcap_{\sigma \in P_n} \{ z \in \overline{\Gamma}_+ : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} \geq c_m^{-1} F(z_1, \ldots, z_n) \} & \text{if } m < n-1 \\ \overline{\Gamma}_+ & \text{if } m = n-1 \,. \end{cases}$$

Remarks 5.1.
1. Recall from Lemma 4.9 that a hypersurface satisfying $\vec{\kappa} \in \Lambda_m \setminus \{0\}$ at all points is m-cylindrical ($\kappa_1 = \cdots = \kappa_m = 0$ and $\kappa_{m+1} = \cdots = \kappa_n > 0$) wherever it is not strictly m-convex ($\kappa_1 + \cdots + \kappa_m > 0$).

2. Given any speed function $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ satisfying the hypotheses of the theorem, any *preserved* cone $\Gamma_0 \subset\subset \Gamma$, and constants $\alpha > 0$, $0 < R < \infty$, and $\varepsilon > 0$, the estimate applies, with the same constant, $C(F, \Gamma_0, \alpha, R, \varepsilon)$, to all solutions arising from the class $\mathcal{C}(\Gamma_0, \alpha, R)$ of initial data $\mathscr{X}_0 : \mathscr{M}^n \to \mathbb{R}^{n+1}$ satisfying $\vec{\kappa}(\mathscr{M}) \subset \Gamma_0$, $\mu(\mathscr{M}) \leq \alpha$, and $\min_{\mathscr{M}} F \geq R^{-1}$ (alternatively, $\operatorname{diam}(\mathscr{X}_0(\mathscr{M})) \leq R$).

3. Recall that preserved cones always exist for surface flows and flows by convex speeds, but not in general for flows by concave speed functions (see Andrews, McCoy, and Zheng 2013, §5 for a counter example); although, as we have seen, if, in addition to concavity, F is inverse-concave, or if the curvature of the initial datum is sufficiently pinched, or if F vanishes on $\partial\Gamma_0$, then the flow will admit a preserved cone.

4. Theorem 5.1 is sharp in the sense that, whenever $\Lambda_m^+ \subset \overline{\Gamma}_0$, it is possible to construct a sequence of solutions of (CF) with curvature satisfying $\vec{\kappa} \in \overline{\Gamma}_0$ at all points which converges to a shrinking cylinder solution $\mathbb{R}^m \times S^{n-m}_{\sqrt{-c_m t/2}}$. (Note that the curvature of $\mathbb{R}^m \times S^{n-m}_{\sqrt{-c_m t/2}}$ lies in the boundary of Λ_m^+.)

Theorem 5.1 consists of two parts: The *convexity estimate*, Theorem 5.2, and the *cylindrical estimates*, Theorem 5.15. The convexity estimate, which corresponds to the $m = n-1$ case of Theorem 5.1, shows that positive solutions of the flow are becoming locally convex, in the sense that the scaling invariant ratio κ_1/F is becoming non-negative, at any point at which the curvature is becoming large. The cylindrical estimate says that it is possible to do better if the solution possesses better convexity properties; namely, if the solution is uniformly $(m+1)$-convex, $\kappa_1 + \cdots + \kappa_{m+1} \geq \beta F$, $\beta > 0$, then, at any point at which the curvature is becoming large, the solution is becoming m-cylindrical: $\kappa_1, \ldots, \kappa_m \approx 0$, $\kappa_{m+1} \approx \cdots \approx \kappa_n$, or strictly m-convex, $\kappa_1 + \cdots + \kappa_m > 0$ (note that uniform $(m+1)$-convexity is just enough to rule out the obstructions discussed in Remarks 5.1 4. Moreover, by Proposition 4.10, it is preserved under the flow).

As the reader is no doubt aware, pinching estimates of the flavour of Theorem 5.1 are now a standard means of studying singularities of geometric flow equations: Hamilton (1982) demonstrated that compact three-manifolds with positive Ricci curvature evolving by Ricci flow become isotropic at points where the scalar curvature is becoming large (see also Huisken 1985). This was the main step in the proof that three-manifolds with positive Ricci curvature converge to round spheres under the normalized Ricci flow. Soon afterwards, Huisken (1984) showed that compact hypersurfaces[1] with positive Weingarten curvature evolving by the mean curvature flow become umbilic wherever the mean curvature is becoming large. This was the main step in proving that convex hypersurfaces evolving by the normalized mean curvature flow converge to round spheres. Note that Huisken's estimate is contained in Theorem 5.1 (take $F = H$ and suppose $m = 0$). Subsequently, Hamilton (1995d) and Ivey (1993) proved a pinching estimate for solutions of the three-dimensional Ricci flow which implies that the curvature operator is becoming non-negative wherever its magnitude is large, which is a key diagnostic tool for Ricci flow with surgery (see, for example, Hamilton 1995a). The pinching phenomena for the Ricci flow were generalized by Böhm and Wilking (2008), who obtained conditions under which a general 'pinching set' will be preserved, and their methods were soon used by Brendle and Schoen (2009) to prove the 1/4-pinched differentiable sphere theorem (see also Nguyen (2008)). For the mean curvature flow, Huisken and Sinestrari (1999b; 1999a) proved that solutions with positive mean curvature are becoming convex wherever the mean curvature is becoming large, and, later, that 2-convex solutions are either becoming strictly convex or 1-cylindrical wherever the mean curvature is becoming large (Huisken and Sinestrari 2009, §5), both of which estimates are key components of Huisken and Sinestrari's construction of mean curvature flow with surgery (Huisken and Sinestrari 2009). The convexity and cylindrical estimates of Huisken and Sinestrari correspond to the $m = n - 1$ and $m = 1$ cases of Theorem 5.1 respectively.

It is interesting to note that, although the Hamilton and Hamilton–Ivey pinching estimates were obtained from clever maximum principle arguments, the Huisken and Huisken–Sinestrari pinching estimates could not be obtained so easily, requiring instead integral estimates and an iteration argument which makes use of the Michael–Simon Sobolev inequality. The latter methods play an important role in our proof of Theorem 5.1.

5.1.1 Outline of the proof of the pinching theorem

Theorem 5.1 combines the convexity estimate, Theorem 5.2, with the cylindrical estimates, Theorem 5.15. Each of these estimates is proved by analysing an appropriate 'pinching function' G for the 'pinching cone' in question (that is, a degree one homogeneous function whose zero set is the pinching cone), and seeking, for any $\varepsilon > 0$, an upper bound for the function $G_{\varepsilon,\sigma} := (G/F - \varepsilon)F^{\sigma}$ for some $\sigma > 0$. If this is possible, then we obtain

$$G/F \leq \varepsilon + CF^{-\sigma}\,, \tag{5.1}$$

which, exploiting homogeneity, is enough to obtain the result.

Recall that, given a degree zero homogeneous curvature function G, the curvature cones $\Gamma_C := \{z \in \Gamma : G(z) \leq C\}$ will be preserved by the flow so long as G satisfies the

[1] That is, of dimension at least two. The corresponding result in one space dimension is due to Gage and Hamilton (1986).

purely algebraic condition

$$0 \geq \mathscr{Q}(T,T) := (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs}$$

for any totally symmetric three-tensor T (at least wherever the auxiliary conditions $\dot{G}^{kl}T_{ikl} = 0$ and $G = 0$ hold). This was an easy consequence of the maximum principle; however, the upper bound for $G_{\varepsilon,\sigma}$ cannot be obtained so readily, since there is a reaction term appearing in its evolution which is not favourable (regardless of any algebraic condition we might impose[2] on G). Consequently, we need to work much harder to obtain the estimate (5.1). Namely, following ideas of Huisken, we seek to obtain a supremum bound by exploiting good integral estimates using Stampacchia's lemma and the Michael–Simon Sobolev inequality. The linchpin in the argument is the estimate

$$\frac{d}{dt}\,||(G_{\varepsilon,\sigma})_+||_{L^p(\mathscr{M}\times\{t\})} \leq K_{\varepsilon,\sigma,p} \tag{5.2}$$

for sufficiently large p, and small σ, where $(G_{\varepsilon,\sigma})_+$ is the positive part of $G_{\varepsilon,\sigma}$ and $K_{\varepsilon,\sigma,p}$ is a constant.

5.2 The convexity estimate

The purpose of this section is to prove the first half of Theorem 5.1:

Theorem 5.2 (Asymptotic convexity estimate, Andrews, Langford and McCoy (2014b; 2014a)). *Let $F : \Gamma^n \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a positive admissible speed function, and assume that one of the following auxiliary conditions is satisfied:*

1. *F is convex; or*
2. *$n = 2$.*

Then, given a curvature cone $\Gamma_0 \subset\subset \Gamma$, an initial volume scale $\alpha > 0$, an initial distance scale $R > 0$, and any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ (depending only on n, F, Γ_0, α, R, and ε) such that, given any solution $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ of (CF) *with curvature satisfying $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \Gamma_0$, initial volume satisfying $\mu_0(\mathscr{M}) \leq \alpha$, and initial curvature satisfying $\min_{\mathscr{M}\times\{0\}} F \geq R^{-1}$ (alternatively, $\mathrm{diam}(\mathscr{X}_0(\mathscr{M})) \leq R$), the following estimate holds:*

$$\mathrm{dist}\left(\vec{\kappa}(x,t), \overline{\Gamma}_+\right) \leq \varepsilon F(x,t) + C_\varepsilon$$

for all $(x,t) \in \mathscr{M} \times [0,T)$.

The first part of the proof concerns the construction of a suitable pinching function G for the pinching cone $\Lambda := \overline{\Gamma}_+$, and the derivation of its key properties. We will then show

[2] On the other hand, for certain special choices of the flow speed F, such estimates are indeed amenable to maximum principle arguments (see, for example, Schulze 2006; Alessandroni and Sinestrari 2010; Andrews and McCoy 2012)

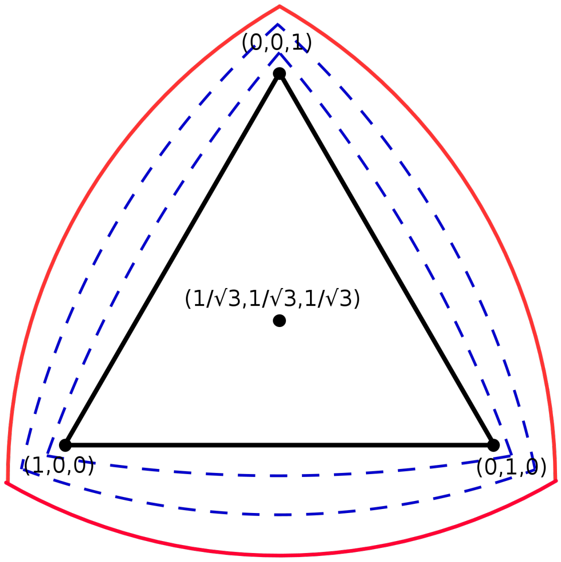

Figure 5.1: Intersection of the curvature space, $\mathbb{R}^3$, of a three-dimensional hypersurface with the unit sphere. The red curve is the boundary of some initial cone, Γ_0. The blue curves are the boundaries of the cones $\Gamma_\varepsilon := \cap_{i=1}^3\{z \in \Gamma_0 : z_i \geq -\varepsilon F(z)\}$ which 'pinch' onto $\overline{\Gamma}_+$ as $\varepsilon \to 0$.

that these properties are sufficient to obtain the essential integral estimate (5.2) for the positive part of the function $G_{\varepsilon,\sigma} := (G/F - \varepsilon)F^\sigma$. Finally, we adapt Huisken's iteration argument to obtain a supremum bound for $(G_{\varepsilon,\sigma})_+$, which quickly implies the desired estimate.

5.2.1 The pinching function

Our first task is to construct an appropriate pinching function $G : \Gamma_0 \to \mathbb{R}$. The construction is slightly different in each of the two cases considered.

Flows by convex speeds

Recall the curvature function defined by (4.8) in the proof of Proposition 4.8 which was used to show that the curvature inequalities $\kappa_1 \geq -\varepsilon F$, $\varepsilon > 0$, are preserved. Our pinching function is a slight modification of this function: First, let $G_1 : \Gamma \to \mathbb{R}$ be defined as in (4.8); next, let $G_2 : \overline{\Gamma}_0 \setminus \{0\} \to \mathbb{R}$ be given by

$$G_2(\lambda) := MF(\lambda) + \mathrm{tr}(\lambda) - ||\lambda|| \,, \tag{5.3}$$

where $\mathrm{tr}(\lambda) := \sum_{i=1}^n \lambda_i$ and $M := 2\max\{\big(||\lambda|| - \mathrm{tr}(\lambda)\big)/F(\lambda) : \lambda \in \Gamma_0\}$; finally, let $K : [0,\infty) \times (0,\infty) \to \mathbb{R}$ be given by

$$K(x,y) := \frac{x^2}{y} \,. \tag{5.4}$$

Then our pinching function $G : \overline{\Gamma}_0 \setminus \{0\} \to \mathbb{R}$ is defined by

$$G(\lambda) := K(G_1(\lambda), G_2(\lambda)) \,. \tag{5.5}$$

Note, in particular, that, just as for G_1, G is non-negative, homogeneous of degree one, and vanishes if and only if $\lambda \in \overline{\Gamma}_+ \cap \Gamma_0$. The gain in making the modification is a small amount of convexity which allows us to obtain the following uniform estimate:

Lemma 5.3 (Cf. Andrews (1994a), Lemma 7.10)**.** *Let G be the curvature function defined by* (5.5)*. Then, for every $\varepsilon > 0$, there exists a constant $\gamma > 0$ (which depends only on F, Γ_0, and ε) such that*

$$\mathcal{Q}_{G,F}\big|_W(T,T) := \left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right)\Big|_W T_{kpq}T_{lrs} \le -\gamma \frac{||T||^2}{F(W)} \tag{5.6}$$

for all diagonal $W \in \mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda \in \Gamma_\varepsilon := \{z \in \Gamma : G(z) \ge \varepsilon F(z)\}$, and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. First, observe that

$$\begin{aligned}\left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right) = \dot{K}^1\left(\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_1^{pq,rs}\right) + \dot{K}^2\left(\dot{G}_2^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_2^{pq,rs}\right)\\ - \dot{F}^{kl}\ddot{K}^{\alpha\beta}\dot{G}^p_\alpha\dot{G}^q_\beta \,.\end{aligned}$$

Noting that $\dot{K}^1(x,y) > 0$, $\dot{K}^2(x,y) < 0$, and $\ddot{K}(x,y) \ge 0$ whenever x and y are positive, $\dot{G}_2 \ge M\dot{F}$, and recalling from the proof of Proposition 4.8 that

$$\left(\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_1^{pq,rs}\right)T_{kpq}T_{lrs} \le 0 \,,$$

we see that

$$\begin{aligned}\left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right)T_{kpq}T_{lrs} &\le \dot{K}^2\left(\dot{G}_2^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_2^{pq,rs}\right)T_{kpq}T_{lrs}\\ &\le \dot{K}^2\dot{F}^{kl}\ddot{N}^{pq,rs}T_{kpq}T_{lrs} \le 0 \,,\end{aligned}$$

where $N(z) := ||z||$. Since N is strictly convex in non-radial directions (recall Example 2.1), equality occurs only if T is radial; that is, if for each k we have $T_{kpq} = \mu_k W_{pq}$ for some constant μ_k. Since W is diagonal, it follows that T is also diagonal: $T_{klm} \ne 0$ only if $k = l = m$. Since $W \ne 0$, we must have $\lambda_n > 0$. But, since $T_{klm} = \mu_k W_{lm} = \mu_k \lambda_l \delta_{lm}$, we have, for any k,

$$T_{kkk} = \frac{\lambda_k}{\lambda_n} T_{knn} \,.$$

But T_{knn} vanishes unless $k = n$. Thus T has at most one non-zero component: T_{nnn}. It follows that W has at most one non-zero eigenvalue: If instead we had $\lambda_q \ne 0$ for some $q < n$, then we could obtain the contradiction $T_{nnn} = \frac{\lambda_n}{\lambda_q} T_{qnn} = 0$. But this implies that $G(W) = 0$, which contradicts $W \in K$. We conclude that $\mathcal{Q}_{G,F}$ can only vanish if

T vanishes. In particular, $\mathscr{Q}_{G,F}\big|_{\lambda}(T,T)F(\lambda)/\,||T||^2$ attains a negative maximum on the compact set $\{(\lambda,T)\in\overline{\Gamma}_0\times\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathbb{R}^n : ||\lambda|| = ||T|| = 1\}$. The estimate (5.6) now follows from homogeneity. □

We shall also require the following estimate for the zeroth order term which arises in the Simons-type identity (4.2):

Lemma 5.4. *Let G be the curvature function defined by* (5.5)*. For every $\varepsilon > 0$, there exists a constant $\delta > 0$ (which depends only on F, Γ_0, and ε) such that*

$$\mathscr{Z}_{G,F}\big|_W(W) := \left(\dot{F}^{pq}\dot{G}^{rs} - \dot{G}^{pq}\dot{F}^{rs}\right)\Big|_W W_{pq}W^2_{rs} \leq -\delta F(W)\,||W||^2 \tag{5.7}$$

for all diagonal $W\in\mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda\in\Gamma_\varepsilon := \{z\in\Gamma_0 : G(z)\geq\varepsilon F(z)\}$.

Proof. First, observe that

$$\begin{aligned}\mathscr{Z}_{G,F}\big|_W(W) &= \dot{F}^{pq}\dot{G}^{kl}\left(W_{pq}W^2_{kl} - W_{kl}W^2_{pq}\right)\\ &= \dot{K}^i\dot{F}^{pq}\dot{G}^{kl}_i\left(W_{pq}W^2_{kl} - W_{kl}W^2_{pq}\right)\\ &= \dot{K}^i\,\mathscr{Z}_{G_i,F}\big|_W(W)\,.\end{aligned}$$

Since $\dot{K}^2\leq 0$ and $\dot{G}_2\geq 0$, it follows that

$$\mathscr{Z}_{G,F}\big|_W(W)\leq\dot{K}^1\,\mathscr{Z}_{G_1,F}\big|_W(W)\,.$$

Since $\dot{K}^1 = 2G_1/G_2$ is bounded below by the constant $2\min_{\Gamma_\varepsilon}(G_1/G_2) > 0$ (which depends only on F, Γ_0 and ε) on the set Γ_ε, it suffices to prove the estimate for $\mathscr{Z}_{G_1,F}\big|_W(W)$.

So consider

$$\mathscr{Z}_{G_1,F}\big|_W(W) = (F\dot{G}^{kl}_1 - G_1\dot{F}^{kl})W^2_{kl}\leq F\dot{G}^k_1\lambda^2_k\,.$$

Since $\dot{G}^k_1$ is homogeneous of degree zero, and strictly negative on Γ_ε, we obtain

$$\mathscr{Z}_{G,F}\big|_W(W)\leq -\delta F\,||\lambda||^2\,,$$

where $-\delta := \max\{\dot{G}^k(\lambda) : \lambda\in\Gamma_\varepsilon,\ 1\leq k\leq n\} < 0$. □

Surface flows

We next consider the case that $n = 2$ and $F : \Gamma\subset\mathbb{R}^2\to\mathbb{R}$ is any positive admissible flow speed. Note that, by positivity and homogeneity of F, $\Gamma\subset\{z\in\mathbb{R}^2 : \max\{z_1,z_2\} > 0\}$. Thus, there exists a constant $a > 0$ (which depends on Γ_0) such that $\Gamma_0\subset\{z\in\mathbb{R}^2 : \min\{z_1,z_2\}\geq -\frac{a}{2}\max\{z_1,z_2\}\}$. Define $\varphi : [-a/2,\infty)\to\mathbb{R}$ by

$$\varphi(r) := \frac{-r}{a+r}\,.$$

Then our pinching function $G : \Gamma_0 \setminus \{0\} \to \mathbb{R}$ is defined by

$$G(\lambda) := F(\lambda)\varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right), \tag{5.8}$$

where $\lambda_{\min} := \min\{\lambda_1, \lambda_2\}$ and $\lambda_{\max} := \max\{\lambda_1, \lambda_2\}$. Observe that G is homogeneous of degree one and $G(\lambda) \leq 0$ if and only if $\lambda_{\min} \geq 0$. Furthermore, although G is not smooth at the positive ray, we shall only require smoothness outside of the positive cone.

We will need the following estimate, which corresponds to the estimate of Lemma 5.3:

Lemma 5.5. *Let G be the curvature function defined by (5.8) and set $Z := G/F$. Then, for every $\varepsilon > 0$, there exists $\gamma > 0$ (which depends only on F, Γ_0, and ε) such that*

$$\begin{aligned}\mathcal{Q}_{G,F}(T,T) := {}& \left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right) T_{kpq}T_{lrs} \\ \leq {}& F\ddot{F}^{kl,rs}\dot{Z}^{pq}T_{kpq}T_{lrs} + C|(DZ) * T| - 2\dot{F}^{kl}\dot{Z}^{pq}T_{kpq}\dot{F}^{rs}T_{lrs} \\ & - \gamma\frac{||T||^2}{F}\end{aligned} \tag{5.9}$$

at any diagonal $W \in \mathrm{Sym}(n)$ with eigenvalue pair $\lambda \in \Gamma_\varepsilon := \{z \in \Gamma : G(z) \geq \varepsilon F(z)\} \cap \{z \in \mathbb{R}^2 : z_2 > z_1\}$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, where $C < \infty$ is a constant (which depends only on F and Γ_0) and

$$(DZ) * T := (\dot{Z}^1 T_{111} + \dot{Z}^2 T_{122})T_{122} + (\dot{Z}^1 T_{211} + \dot{Z}^2 T_{222})T_{211}\,.$$

Proof. Writing $G = ZF$, we have

$$\mathcal{Q}_{G,F}(T,T) = F\mathcal{Q}_{Z,F}(T,T) - 2\dot{F}^{kl}\dot{Z}^{pq}T_{kpq}\dot{F}^{rs}T_{lrs}\,.$$

Since Z is degree zero homogeneous, Lemma 4.14 implies

$$\begin{aligned}\mathcal{Q}_{Z,F}\big|_W(T,T) = {}& \ddot{F}^{kl,rs}\dot{Z}^{pq}T_{kpq}T_{lrs} + \frac{2F\dot{Z}^1}{\lambda_2(\lambda_2 - \lambda_1)}\left[(T_{112})^2 + (T_{212})^2\right] \\ & - \frac{\dot{F}^1}{\dot{Z}^1}\frac{\ddot{Z}^{11}}{\dot{Z}^1}\left(\dot{Z}^1T_{111} + \dot{Z}^2T_{122}\right)^2 - \frac{\dot{F}^2}{\dot{Z}^2}\frac{\ddot{Z}^{22}}{\dot{Z}^2}\left(\dot{Z}^1T_{211} + \dot{Z}^2T_{222}\right)^2 \\ & + \left(2\frac{\dot{F}^1}{\lambda_2} - \frac{\dot{F}^2 - \dot{F}^1}{\lambda_2 - \lambda_1}\right)\left(\dot{Z}^1T_{111} + \dot{Z}^2T_{122}\right)T_{122} \\ & + \left(2\frac{\dot{F}^2}{\lambda_1} - \frac{\dot{F}^2 - \dot{F}^1}{\lambda_2 - \lambda_1}\right)\left(\dot{Z}^1T_{211} + \dot{Z}^2T_{222}\right)T_{211}\,.\end{aligned}$$

Consider together the terms

$$\begin{aligned}Q_1(\lambda, T) := {}& \frac{2F\dot{Z}^1}{\lambda_2(\lambda_2 - \lambda_1)}\left[(T_{112})^2 + (T_{212})^2\right] \\ & - \frac{\dot{F}^1}{\dot{Z}^1}\frac{\ddot{Z}^{11}}{\dot{Z}^1}\left(\dot{Z}^1T_{111} + \dot{Z}^2T_{122}\right)^2 - \frac{\dot{F}^2}{\dot{Z}^2}\frac{\ddot{Z}^{22}}{\dot{Z}^2}\left(\dot{Z}^1T_{211} + \dot{Z}^2T_{222}\right)^2\,.\end{aligned}$$

We will show that there exists $\gamma > 0$ (depending only on F, Γ_0, and ε) such that

$$Q_1(\lambda, T) \leq -\gamma \frac{||T||^2}{F^2}\,.$$

By homogeneity, it suffices to show that $\widetilde{Q}_1(\lambda, T) := F(\lambda)^2 Q_1(\lambda, T)/\,||T||^2$ has a negative upper bound on the compact set $K := \Gamma_\varepsilon \cap \{||\lambda|| = 1\} \times \{||T|| = 1\}$. First note that

$$\dot{Z}^1 = \varphi'\left(\frac{\lambda_1}{\lambda_2}\right)\frac{1}{\lambda_2}\,,$$

and

$$\dot{Z}^2 = -\varphi'\left(\frac{\lambda_1}{\lambda_2}\right)\frac{\lambda_1}{(\lambda_2)^2}\,.$$

Since

$$\varphi'(r) = \frac{-a}{(r+a)^2} < 0\,,$$

we have $\dot{Z}^1(\lambda) < 0$ and $\dot{Z}^2(\lambda) \neq 0$ for all $\lambda \in \Gamma_0 \setminus \overline{\Gamma}_+$. Next, we compute

$$\ddot{Z}^{11} = \varphi''\left(\frac{\lambda_1}{\lambda_2}\right)\frac{1}{(\lambda_2)^2}\,.$$

Since

$$\varphi''(r) = \frac{2a}{(r+a)^3} > 0\,,$$

we have $\ddot{Z}^{11}(\lambda) > 0$ for all $\lambda \in \Gamma_0 \setminus \overline{\Gamma}_+$. By homogeneity of Z (using the second and third of the identities (4.14)), this implies $\ddot{Z}^{ij}(\lambda) > 0$ on $\Gamma_0 \setminus \overline{\Gamma}_+$ for each i, j. It follows that $\widetilde{Q}_1 \leq 0$ on K. Suppose that $\widetilde{Q}(\lambda, T)$ vanishes for some $(\lambda, T) \in K$. Then $0 = T_{112} = T_{122} = (\dot{Z}^1 T_{111} + \dot{Z}^2 T_{122}) = (\dot{Z}^1 T_{211} + \dot{Z}^2 T_{222})$. But this implies that $T = 0$. Thus $\widetilde{Q}_1$ cannot vanish on K. By compactness of K we obtain the desired bound.

Finally, setting $C := \max\{C_1, C_2\}$, where

$$C_1 := \max\left\{F(\lambda)\left(2\frac{\dot{F}^1(\lambda)}{\lambda_2} - \frac{\dot{F}^2(\lambda) - \dot{F}^1(\lambda)}{\lambda_2 - \lambda_1}\right) : \lambda \in \Gamma_0\right\}\,,$$

and

$$C_1 := \max\left\{F(\lambda)\left(2\frac{\dot{F}^2(\lambda)}{\lambda_1} - \frac{\dot{F}^2(\lambda) - \dot{F}^1(\lambda)}{\lambda_2 - \lambda_1}\right) : \lambda \in \Gamma_\varepsilon\right\}\,,$$

we obtain

$$\frac{C}{F}|DZ * T| \geq \left(2\frac{\dot{F}^1}{\lambda_2} - \frac{\dot{F}^2 - \dot{F}^1}{\lambda_2 - \lambda_1}\right)\left(\dot{Z}^1 T_{111} + \dot{Z}^2 T_{122}\right) T_{122}$$
$$+ \left(2\frac{\dot{F}^2}{\lambda_1} - \frac{\dot{F}^2 - \dot{F}^1}{\lambda_2 - \lambda_1}\right)\left(\dot{Z}^1 T_{211} + \dot{Z}^2 T_{222}\right) T_{211}\,.$$

□

We also obtain the desired estimate for the zero order term:

Lemma 5.6. *Let G be the curvature function defined by* (5.8)*. Then, for every $\varepsilon > 0$, there exists a constant $\delta > 0$ (which depends only on F, Γ_0, and ε) such that*

$$\mathscr{Z}_{G,F}\big|_W(W) := \left(\dot{F}^{pq}\dot{G}^{rs} - \dot{G}^{pq}\dot{F}^{rs}\right)\Big|_W W_{pq}W_{rs}^2 \leq -\delta F(W)\,||W||^2 \tag{5.10}$$

for all diagonal $W \in \mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda \in \Gamma_\varepsilon := \{z \in \Gamma_0 : G(z) \geq \varepsilon F(z)\}$.

Proof. The proof is similar to the proof of Lemma 5.4. □

Evolution of the pinching function

Recall that our goal is to obtain, for any $\varepsilon > 0$, an upper bound for the function

$$G_{\varepsilon,\sigma} := \left(\frac{G}{F} - \varepsilon\right) F^\sigma$$

for some $\sigma > 0$. The first step is to understand the evolution of $G_{\varepsilon,\sigma}$.

Lemma 5.7. *In case F is convex, let G be defined by* (5.5)*. In case $n = 2$, let G be defined by* (5.8)*. Then the function $G_{\varepsilon,\sigma}$ satisfies the following evolution equation (away from umbilic points in case $n = 2$):*

$$\begin{aligned}(\partial_t - \mathscr{L})G_{\varepsilon,\sigma} = {} & F^{\sigma-1}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\,\nabla_k \mathcal{W}_{pq}\nabla_l \mathcal{W}_{rs} \\ & + \frac{2(1-\sigma)}{F}\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}|\nabla F|_F^2 \\ & + \sigma G_{\varepsilon,\sigma}|\mathcal{W}|_F^2\,.\end{aligned} \tag{5.11}$$

Proof. We first compute

$$\nabla G_{\varepsilon,\sigma} = F^{\sigma-1}\left(\nabla G - \frac{G}{F}\nabla F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\nabla F\,.$$

It follows that

$$\begin{aligned}\mathscr{L}G_{\varepsilon,\sigma} = {} & F^{\sigma-1}\left(\mathscr{L}G - \frac{G}{F}\mathscr{L}F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathscr{L}F - 2\frac{\sigma-1}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F \\ & + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,.\end{aligned} \tag{5.12}$$

Therefore,

$$(\partial_t - \mathscr{L})G_{\varepsilon,\sigma} = F^{\sigma-1}\left((\partial_t - \mathscr{L})G - \frac{G}{F}(\partial_t - \mathscr{L})F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}(\partial_t - \mathscr{L})F$$
$$+ 2\frac{1-\sigma}{F}\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2 .$$

The claim now follows by applying (4.6).

□

5.2.2 The integral estimates

Note that the final term of the evolution equation (5.11) is an obstruction to any application of the maximum principle when $\sigma > 0$. Instead, we will extract, by iteration, a supremum bound from the following integral estimate:

Proposition 5.8 (L^p-estimate)**.** *In case F is convex, let G be defined by* (5.5)*. In case $n = 2$, let G be defined by* (5.8)*. Then there exist constants $\ell > 0$ and $L < \infty$ (which depend only on F, Γ_0, and ε) such that*

$$\frac{d}{dt}\left|\left|\big(G_{\varepsilon,\sigma}(\cdot,t)\big)_+\right|\right|_{L^p(\mathscr{M},\,\mu(t))} \leq 0$$

for all $p > L$ and all $\sigma \in \left(0, \ell p^{-\frac{1}{2}}\right)$, where $(G_{\varepsilon,\sigma})_+ := \max\{G_{\varepsilon,\sigma}, 0\}$ is the positive part of $G_{\varepsilon,\sigma}$.

First observe that the evolution equations for $G_{\varepsilon,\sigma}$ (5.11) and the induced measure μ (2.16) yield the following evolution equation:

$$\begin{aligned}\frac{d}{dt}\int (G_{\varepsilon,\sigma})_+^p\, d\mu &= p\int (G_{\varepsilon,\sigma})_+^{p-1}\mathscr{L}G_{\varepsilon,\sigma}\, d\mu + p\int (G_{\varepsilon,\sigma})_+^{p-1}F^{\sigma-1}\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W})\, d\mu \\ &\quad + 2(1-\sigma)p\int (G_{\varepsilon,\sigma})_+^{p-1}\frac{\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F}{F}\, d\mu \\ &\quad - \sigma(1-\sigma)p\int (G_{\varepsilon,\sigma})_+^{p}\frac{|\nabla F|_F^2}{F^2}\, d\mu + \sigma p\int (G_{\varepsilon,\sigma})_+^{p}|\mathcal{W}|_F^2\, d\mu \\ &\quad - \int (G_{\varepsilon,\sigma})_+^{p} HF\, d\mu\,. \end{aligned} \tag{5.13}$$

We will use Young's inequality and the estimates of Lemmata 5.3 and 5.5 to control all but the two zero order terms:

Lemma 5.9. *There exist positive constants A_1, A_2, B_1, B_2, B_3 C_1 which depend only on*

F, Γ_0, and ε such that

$$\frac{d}{dt}\int (G_{\varepsilon,\sigma})_+^p d\mu \leq -\left(A_1 p(p-1) - A_2 p^{\frac{3}{2}}\right)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\, d\mu$$
$$-\left(B_1 p - B_2\sigma p - B_3 p^{\frac{1}{2}}\right)\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu$$
$$+ C_1(\sigma p + 1)\int (G_{\varepsilon,\sigma})_+^p |\mathcal{W}|^2 d\mu \tag{5.14}$$

for all $p \geq 2$ and $\sigma \in (0,1]$.

Remarks 5.2. 1. In fact, except for B_1, the constants are independent of ε.

2. In the case that F is convex, B_2 can be taken to be zero.

Proof of Lemma 5.9. The proof of the estimate is slightly different in each of the two cases at hand. We consider first the case that F is convex:

Case I: F is convex

Recall equation (5.13). We first integrate the diffusion term by parts:

$$\begin{aligned} p\int (G_{\varepsilon,\sigma})_+^{p-1}\mathscr{L} G_{\varepsilon,\sigma}\, d\mu = & -p(p-1)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\, d\mu \\ & - p\int (G_{\varepsilon,\sigma})_+^{p-1}\ddot{F}^{pq,rs}\,\nabla_p G_{\varepsilon,\sigma}\,\nabla_q \mathcal{W}_{rs}\, d\mu \\ \leq & -c_0 p(p-1)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\ & - p\int (G_{\varepsilon,\sigma})_+^{p-1}\ddot{F}^{pq,rs}\,\nabla_p G_{\varepsilon,\sigma}\,\nabla_q \mathcal{W}_{rs}\, d\mu\,, \end{aligned}$$

where $c_0 = \min\{\dot{F}^i(\lambda) : \lambda \in \Gamma_0,\ 1 \leq i \leq n\} > 0$. This yields a useful gradient term, but spits out an additional bad term due to the non-divergence form of the diffusion operator; however, due to homogeneity, the latter is easily estimated (wherever $G_{\varepsilon,\sigma} > 0$) using Young's inequality:

$$-p\int (G_{\varepsilon,\sigma})_+^p F\ddot{F}^{pq,rs}\frac{\nabla_p G_{\varepsilon,\sigma}}{G_{\varepsilon,\sigma}}\frac{\nabla_q \mathcal{W}_{rs}}{F}\, d\mu \leq c_1 p^{\frac{3}{2}}\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\, d\mu$$
$$+ c_1 p^{\frac{1}{2}}\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|_F^2}{F^2}\, d\mu\,,$$

where $c_1 = \max\left\{F(\lambda)\ddot{F}^{pq,rs}(\lambda) : \lambda \in \Gamma_0,\ 1 \leq p,q,r,s \leq n\right\}$. Comparing $|\cdot|_F$ with $|\cdot|$, we arrive at

$$p\int (G_{\varepsilon,\sigma})_+^{p-1}\mathscr{L} G_{\varepsilon,\sigma}\, d\mu \leq (c_1 c_2 p^{\frac{3}{2}} - c_0 p(p-1))\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\, d\mu$$
$$+ c_1 c_2 p^{\frac{1}{2}}\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,, \tag{5.15}$$

where $c_2 := \max\{\dot{F}^i(\lambda) : \lambda \in \Gamma_0,\ 1 \leq i \leq n\}$.

A further useful term is obtained from the second term of (5.13) via Lemma 5.3:

$$\begin{aligned} p\int (G_{\varepsilon,\sigma})_+^{p-1}F^{\sigma-1}\mathscr{Q}_{G,F}(\nabla\mathcal{W},\nabla\mathcal{W})\,d\mu \leq &-\gamma p\int (G_{\varepsilon,\sigma})_+^{p-1}F^{\sigma-1}\frac{|\nabla\mathcal{W}|^2}{F}d\mu \\ \leq &-c_3^{-1}\gamma p\int (G_{\varepsilon,\sigma})_+^{p}\frac{|\nabla\mathcal{W}|^2}{F^2}d\mu\,, \end{aligned} \tag{5.16}$$

where $c_3 = \max\{G(\lambda)/F(\lambda) : \lambda \in \Gamma_0\}$ and $\gamma > 0$ is the constant from Lemma 5.3 (which depends only on F, Γ_0, and ε).

The inner product term is estimated (wherever $G_{\varepsilon,\sigma} > 0$) using Young's inequality:

$$\begin{aligned} 2(1-\sigma)p\int (G_{\varepsilon,\sigma})_+^{p}\left\langle\frac{\nabla G_{\varepsilon,\sigma}}{G_{\varepsilon,\sigma}}\,,\,\frac{\nabla F}{F}\right\rangle_F d\mu \leq\ &(1-\sigma)p^{\frac{3}{2}}\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu \\ &+(1-\sigma)p^{\frac{1}{2}}\int (G_{\varepsilon,\sigma})_+^{p}\frac{|\nabla F|_F^2}{F^2}\,d\mu \\ \leq\ &c_2p^{\frac{3}{2}}\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu \\ &+c_4p^{\frac{1}{2}}\int (G_{\varepsilon,\sigma})_+^{p}\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,, \end{aligned} \tag{5.17}$$

where[3] $c_4 = c_2^3$.

Assuming $\sigma \in (0,1)$, we may discard the third to last term.

The final term is easily estimated using homogeneity of the integrand[4]:

$$-\int (G_{\varepsilon,\sigma})_+^{p}HF\,d\mu \leq c_5\int (G_{\varepsilon,\sigma})_+^{p}|\mathcal{W}|^2\,d\mu\,, \tag{5.18}$$

where $c_5 = \max\{-F(\lambda)\sum_{i=1}^n\lambda_i/\,||\lambda||^2 : \lambda \in \Gamma_0\}$. Combining the estimates (5.15)–(5.18) yields the claim.

Case II: $n = 2$

We again begin by integrating the diffusion term by parts:

$$\begin{aligned} p\int (G_{\varepsilon,\sigma})_+^{p-1}\mathscr{L}G_{\varepsilon,\sigma}\,d\mu =\ &-p(p-1)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu \\ &-p\int (G_{\varepsilon,\sigma})_+^{p-1}\ddot{F}^{pq,rs}\,\nabla_p G_{\varepsilon,\sigma}\,\nabla_q\mathcal{W}_{rs}\,d\mu \\ =\ &-p(p-1)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu \\ &-p\int (G_{\varepsilon,\sigma})_+^{p-1}F^{\sigma}\ddot{F}^{pq,rs}\,\nabla_p\left(\frac{G}{F}\right)\nabla_q\mathcal{W}_{rs}\,d\mu \\ &-\sigma p\int (G_{\varepsilon,\sigma})_+^{p}\ddot{F}^{pq,rs}\frac{\nabla_p F}{F}\,\nabla_q\mathcal{W}_{rs}\,d\mu\,. \end{aligned}$$

[3]Note that $|\nabla F|_F^2 = \dot{F}^k\dot{F}^p\dot{F}^q\,\nabla_k\mathcal{W}_{pp}\,\nabla_l\mathcal{W}_{qq}$.

[4]Note that we are allowing the possibility that $H < 0$ at some points of the hypersurface.

Applying Lemma 5.5 to the second term of (5.13) yields

$$
\begin{aligned}
p \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1} \mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W})\, d\mu& \\
\leq\; & - \gamma p \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma} \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu \\
& + p \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma} \ddot{F}^{pq,rs} \nabla_p \left(\frac{G}{F}\right) \nabla_q \mathcal{W}_{rs}\, d\mu \\
& - 2p \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma} \left\langle \nabla \left(\frac{G}{F}\right), \frac{\nabla F}{F} \right\rangle_F d\mu \\
& + Cp \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1} \left| \nabla \frac{G}{F} * \nabla \mathcal{W} \right| d\mu\,,
\end{aligned}
$$

where $\gamma > 0$ and $C < \infty$ are the constants from Lemma 5.5 and

$$
\begin{aligned}
\nabla \left(\frac{G}{F}\right) * \nabla \mathcal{W} :=\; & \nabla_1 \left(\frac{G}{F}\right) \nabla_1 \mathcal{W}_{22} + \nabla_2 \left(\frac{G}{F}\right) \nabla_2 \mathcal{W}_{11} \\
=\; & \big(F^{-\sigma} \nabla_1 G_{\varepsilon,\sigma} - \sigma F^{-1-\sigma} G_{\varepsilon,\sigma} \nabla_1 F\big) \nabla_1 \mathcal{W}_{22} \\
& + \big(F^{-\sigma} \nabla_2 G_{\varepsilon,\sigma} - \sigma F^{-1-\sigma} G_{\varepsilon,\sigma} \nabla_2 F\big) \nabla_2 \mathcal{W}_{11}\,.
\end{aligned}
\tag{5.19}
$$

Estimating $G_{\varepsilon,\sigma} \leq c_3 F^{\sigma}$ and $\langle \cdot\,, \cdot \rangle_F \leq c_2 |\cdot|^2$, we obtain

$$
\begin{aligned}
p \int (G_{\varepsilon,\sigma})_+^{p-1} \mathscr{L} G_{\varepsilon,\sigma}\, d\mu + p \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1} \mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W})\, d\mu& \\
\leq\; & - c_2 p(p-1) \int (G_{\varepsilon,\sigma})_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\
& - c_3 \gamma p \int (G_{\varepsilon,\sigma})_+^{p} \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu \\
& - \sigma p \int (G_{\varepsilon,\sigma})_+^{p} \ddot{F}^{pq,rs} \frac{\nabla_p F}{F} \nabla_q \mathcal{W}_{rs}\, d\mu \\
& - 2p \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1} \left\langle \nabla \left(\frac{G}{F}\right), \nabla F \right\rangle_F d\mu \\
& + Cp \int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1} \left| \nabla \frac{G}{F} * \nabla \mathcal{W} \right| d\mu\,.
\end{aligned}
$$

This provides the two good terms we need. The remaining terms can now be controlled by utilizing the homogeneity of F in conjunction with Young's inequality: First, we estimate

$$
-\sigma p \int (G_{\varepsilon,\sigma})_+^{p} F \ddot{F}^{pq,rs} \frac{\nabla_p F}{F^2} \nabla_q \mathcal{W}_{rs}\, d\mu \leq c_1 c_2 \sigma p \int (G_{\varepsilon,\sigma})_+^{p} \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,.
$$

Next, substituting $\nabla_k G_{\varepsilon,\sigma} = F^\sigma \nabla_k(G/F) + \frac{\sigma}{F} G_{\varepsilon,\sigma} \nabla_k F$, we estimate

$$\begin{aligned}
-2p\int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1}\left\langle \nabla\left(\frac{G}{F}\right), \nabla F\right\rangle_F d\mu &= p\int (G_{\varepsilon,\sigma})_+^p \left\langle \frac{\nabla G_{\varepsilon,\sigma}}{G_{\varepsilon,\sigma}}, \frac{\nabla F}{F}\right\rangle_F d\mu \\
&\quad - \sigma p \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla F|_F^2}{F^2}\, d\mu \\
&\le c_2 p^{\frac{3}{2}} \int (G_{\varepsilon,\sigma})_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\
&\quad + c_4 p^{\frac{1}{2}} \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,.
\end{aligned}$$

Using (5.19), we similarly estimate the term

$$\begin{aligned}
Cp\int (G_{\varepsilon,\sigma})_+^{p-1} F^{\sigma-1}\left|\nabla \frac{G}{F} * \nabla \mathcal{W}\right| d\mu &\le \frac{C}{2} p^{\frac{3}{2}} \int (G_{\varepsilon,\sigma})_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\
&\quad + \frac{C}{2}\left(p^{\frac{1}{2}} + c_2 \sigma p\right) \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,.
\end{aligned}$$

The remaining terms are estimated as in the case that F is convex. □

To show that the remaining bad term can be compensated by the two good terms, we need an estimate which involves both zero order and gradient terms. This is achieved by integrating the Simons-type identity (4.2), applying Lemmata 5.4 and 5.6, and controlling the error terms.

Lemma 5.10 (Poincaré-type inequality)**.** *There exist positive constants A_3, A_4, A_5, B_4, B_5, which depend only on F, Γ_0, and ε, such that*

$$\begin{aligned}
\int (G_{\varepsilon,\sigma})_+^p |\mathcal{W}|^2 &\le \left(A_3 p^{\frac{3}{2}} + A_4 p^{\frac{1}{2}} + A_5\right) \int (G_{\varepsilon,\sigma})_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\
&\quad + \left(B_4 p^{\frac{1}{2}} + B_5\right) \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu
\end{aligned} \tag{5.20}$$

for all $p \ge 2$ and $\sigma \in (0,1]$.

Proof. Recall equation (5.12)

$$\begin{aligned}
\mathscr{L} G_{\varepsilon,\sigma} = F^{\sigma-1}&\left(\mathscr{L} G - \frac{G}{F}\mathscr{L} F\right) + \frac{\sigma}{F} G_{\varepsilon,\sigma} \mathscr{L} F - 2\frac{1-\sigma}{F}\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F \\
&- \frac{\sigma(1-\sigma)}{F^2} G_{\varepsilon,\sigma} |\nabla F|_F^2\,.
\end{aligned} \tag{5.21}$$

Applying the identity (4.2) yields

$$\begin{aligned}
\mathscr{L} G_{\varepsilon,\sigma} = &- F^{\sigma-1}\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + F^{\sigma-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k \nabla_l F \\
&- F^{\sigma-1}\mathscr{Z}_{G,F}(\mathcal{W}) + \frac{\sigma}{F} G_{\varepsilon,\sigma}\mathscr{L} F - 2\frac{(1-\sigma)}{F}\langle \nabla F, \nabla G_{\varepsilon,\sigma}\rangle_F \\
&+ \frac{\sigma(1-\sigma)}{F^2} G_{\varepsilon,\sigma} |\nabla F|_F^2\,.
\end{aligned} \tag{5.22}$$

We now estimate $\mathscr{Z}_{G,F}(\mathcal{W})$ using Lemmata 5.4 and 5.6. This yields

$$\begin{aligned}
\delta|\mathcal{W}|^2 &\leq F^{-1}\mathscr{Z}_{G,F}(\mathcal{W})\\
&= -F^{-\sigma}\mathscr{L}G_{\varepsilon,\sigma} - F^{-1}\mathscr{Q}_{G,F}(\nabla\mathcal{W},\nabla\mathcal{W}) + F^{-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k\nabla_l F\\
&\quad + \sigma F^{-1-\sigma}G_{\varepsilon,\sigma}\mathscr{L}F - 2(1-\sigma)F^{-1-\sigma}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F\\
&\quad + \sigma(1-\sigma)F^{-2-\sigma}G_{\varepsilon,\sigma}|\nabla F|_F^2\,.
\end{aligned}$$

Multiplying by $(G_{\varepsilon,\sigma})_+^p$ and integrating over $\mathscr{M}$, we obtain

$$\begin{aligned}
\delta\int(G_{\varepsilon,\sigma})_+^p|\mathcal{W}|^2\,d\mu \leq &-\int(G_{\varepsilon,\sigma})_+^pF^{-\sigma}\mathscr{L}G_{\varepsilon,\sigma}\,d\mu\\
&-\int(G_{\varepsilon,\sigma})_+^pF^{-1}\mathscr{Q}_{G,F}(\nabla\mathcal{W},\nabla\mathcal{W})\,d\mu\\
&+\int(G_{\varepsilon,\sigma})_+^pF^{-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k\nabla_l F\,d\mu\\
&+\sigma\int(G_{\varepsilon,\sigma})_+^{p+1}F^{-1-\sigma}G_{\varepsilon,\sigma}\mathscr{L}F\,d\mu\\
&-2(1-\sigma)\int(G_{\varepsilon,\sigma})_+^pF^{-1-\sigma}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F\,d\mu\\
&+\sigma(1-\sigma)\int(G_{\varepsilon,\sigma})_+^pF^{-2-\sigma}G_{\varepsilon,\sigma}|\nabla F|_F^2\,d\mu\,.
\end{aligned}\tag{5.23}$$

We will estimate each of the terms on the right similarly as in the proof of Lemma 5.9:

Integrating the first term by parts yields

$$\begin{aligned}
-\int(G_{\varepsilon,\sigma})_+^pF^{-\sigma}\mathscr{L}G_{\varepsilon,\sigma}\,d\mu = &\;p\int(G_{\varepsilon,\sigma})_+^{p-1}F^{-\sigma}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu\\
&-\sigma\int(G_{\varepsilon,\sigma})_+^pF^{-\sigma-1}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F\,d\mu\\
&+\int(G_{\varepsilon,\sigma})_+^pF^{-\sigma}\ddot{F}^{kl,rs}\nabla_k\mathcal{W}_{rs}\nabla_l G_{\varepsilon,\sigma}\,d\mu\,.
\end{aligned}$$

Estimating $F\ddot{F} \leq c_1$, $\dot{F}^i \leq c_2$, $G/F \leq c_3$, and $\sigma \leq 1$, and applying Young's inequality to the second and third terms, we obtain

$$\begin{aligned}
-\int(G_{\varepsilon,\sigma})_+^pF^{-\sigma}\mathscr{L}G_{\varepsilon,\sigma}\,d\mu \leq &\big(c_2c_3(p+1) + c_1c_3\big)\int(G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu\\
&+\big(c_2^3c_3 + c_1c_3\big)\int(G_{\varepsilon,\sigma})_+^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,.
\end{aligned}$$

The second term is easily estimated by bounding

$$\mathscr{Q}_{G,F}(\nabla\mathcal{W},\nabla\mathcal{W}) \geq -c_4|\nabla\mathcal{W}|^2/F\,,$$

where $-c_4 := \min\{F(\lambda)(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})(\lambda) : \lambda \in \Gamma_0, 1 \le k,l,p,q,r,s \le n\}$, yielding

$$-\int (G_{\varepsilon,\sigma})_+^p F^{-1} \mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W})\, d\mu \le c_4 \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,.$$

The third and fourth terms of (5.23) are estimated similarly to the first: For the third, we obtain

$$\begin{aligned}
\int (G_{\varepsilon,\sigma})_+^p F^{-2}(F\dot{G}^{kl} - G\dot{F}^{kl}) \nabla_k \nabla_l F\, d\mu &= \int (G_{\varepsilon,\sigma})_+^p \dot{Z}^{kl} \nabla_k \nabla_l F\, d\mu \\
&= -\int p(G_{\varepsilon,\sigma})_+^{p-1} \dot{Z}^{kl} \nabla_k G_{\varepsilon,\sigma} \nabla_l F\, d\mu \\
&\quad - \int (G_{\varepsilon,\sigma})_+^p \ddot{Z}^{kl,rs} \nabla_k \mathcal{W}_{rs} \nabla_l F\, d\mu \\
&\le c_5 p^{\frac{3}{2}} \int (G_{\varepsilon,\sigma})_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\
&\quad + \big(c_5 c_2^3 p^{\frac{1}{2}} + c_6\big) \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,,
\end{aligned}$$

where $Z := G/F$, $c_5 := \max\{F(\lambda)\dot{Z}^k(\lambda) : \lambda \in \Gamma_0, 1 \le k \le n\}$, and $c_6 := \max\{F(\lambda)^2 \dot{F}^{kl}(\lambda)\ddot{Z}^{pq,rs}(\lambda) : \lambda \in \Gamma_0, 1 \le k,l,p,q,r,s \le n\}$. For the fourth term, we obtain

$$\begin{aligned}
\sigma \int (G_{\varepsilon,\sigma})_+^{p+1} F^{-1-\sigma} \mathscr{L} F\, d\mu &= -\sigma(p+1) \int (G_{\varepsilon,\sigma})_+^p F^{-\sigma-1} \dot{F}^{kl} \nabla_k G_{\varepsilon,\sigma} \nabla_l F\, d\mu \\
&\quad + \sigma(1+\sigma) \int (G_{\varepsilon,\sigma})_+^{p+1} F^{-\sigma-2} |\nabla F|_F^2\, d\mu \\
&\quad - \sigma \int (G_{\varepsilon,\sigma})_+^{p+1} F^{-1-\sigma} \ddot{F}^{kl,rs} \nabla_k \mathcal{W}_{rs} \nabla_l F\, d\mu \\
&\le c_2^2 c_3 \sigma(p+1) p^{\frac{1}{2}} \int (G_{\varepsilon,\sigma})_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2\, d\mu \\
&\quad + c_3\big(c_2^2(p+1)p^{-\frac{1}{2}} + c_2^3 + c_7\big) \int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,,
\end{aligned}$$

where $c_7 := \max\{F(\lambda)\dot{F}^{kl}(\lambda)\ddot{F}^{pq,rs}(\lambda) : \lambda \in \Gamma_0,\ 1 \le k,l,p,q,r,s \le n\}\,.$

The penultimate term is easily estimated using Young's inequality:

$$\begin{aligned}
-2\frac{(1-\sigma)}{F} \langle \nabla F, \nabla G_{\varepsilon,\sigma} \rangle_F &\le (1-\sigma) G_{\varepsilon,\sigma} \left(\frac{|\nabla F|_F^2}{F^2} + \frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{G_{\varepsilon,\sigma}^2} \right) \\
&\le G_{\varepsilon,\sigma} \left(c_2^3 \frac{|\nabla \mathcal{W}|^2}{F^2} + c_2 \frac{|\nabla G_{\varepsilon,\sigma}|^2}{G_{\varepsilon,\sigma}^2} \right)\,.
\end{aligned}$$

The final term is estimated by applying $|\nabla F|_F^2 \le c_2^3 |\nabla \mathcal{W}|^2$. □

Combining Lemmata 5.9 and 5.10 yields

$$\begin{aligned}\frac{d}{dt}\int (G_{\varepsilon,\sigma})_+^p d\mu \leq & -\left(\alpha_0 p^2 - \alpha_1 \sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right)\int (G_{\varepsilon,\sigma})_+^{p-2} |G_{\varepsilon,\sigma}|^2 \, d\mu \\ & -\left(\beta_0 p - \beta_1 \sigma p^{\frac{3}{2}} - \beta_2 \sigma p - \beta_3 p^{\frac{1}{2}}\right)\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2} \, d\mu\end{aligned}$$

for some positive constants α_i and β_i, which depend only on F, Γ_0, and ε.

It is clear that constants $L < \infty$ and $\ell > 0$ (depending only on F, Γ_0, and ε) may be chosen such that

$$\left(\alpha_0 p^2 - \alpha_1 \sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right) \geq 0$$

and

$$\left(\beta_0 p - \beta_1 \sigma p^{\frac{3}{2}} - \beta_2 \sigma p - \beta_3 p^{\frac{1}{2}}\right) \geq 0$$

for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$. This completes the proof of Proposition 5.8.

5.2.3 The supremum estimate

We now extract an L^∞-bound for $(G_{\varepsilon,\sigma})_+$ from the L^p-estimate of the previous section. We will make use of the following lemma:

Lemma 5.11 (Stampacchia (1966)). *Let $\varphi : [k_0, \infty) \to \mathbb{R}$ be a non-negative, non-increasing function satisfying*

$$\varphi(h) \leq \frac{C}{(h-k)^\alpha}\varphi(k)^\beta, \qquad h > k > k_0\,, \tag{5.24}$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$\varphi(k_0 + d) = 0\,,$$

where $d^\alpha = C\varphi(k_0)^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}$.

Proof. We reproduce the proof from Stampacchia (1966): Consider the sequence defined by

$$k_r = k_0 + d - \frac{d}{2^r}, \quad r = 0, 1, 2, \ldots .$$

By assumption, we have

$$\varphi(k_{r+1}) \leq C\frac{2^{(r+1)\alpha}}{d^\alpha}\varphi(k_r)^\beta \tag{5.25}$$

for all $r = 0, 1, \ldots$. We will now prove by induction that

$$\varphi(k_r) \leq \varphi(k_0) 2^{-r\mu} \tag{5.26}$$

for all $r \in \mathbb{N}$, where $\mu := \frac{\alpha}{\beta-1} > 0$. Clearly (5.26) holds trivially for $r = 0$. Supposing (5.26) holds up to some integer r, we find by (5.25) and the definition of d that

$$\begin{aligned}\varphi(k_{r+1}) &\leq C\frac{2^{(r+1)\alpha}}{d^\alpha}\varphi(k_0)^\beta 2^{-r\mu\beta}\\ &= \varphi(k_0)2^{-(r+1)\mu\beta}\end{aligned}$$

which completes the proof of (5.26). Now, by the monotonicity assumption, we have

$$0 \leq \varphi(k_0 + d) \leq \varphi(k_r) \qquad \text{for all} \quad r = 0, 1, \ldots .$$

But, by (5.26), $\varphi(k_r) \to 0$ as $r \to \infty$. □

Now, given any $k \geq k_0$, where $k_0 := \sup_{\sigma\in(0,1)} \sup_{\mathscr{M}} G_{\varepsilon,\sigma}(\cdot, 0)$, set

$$v_k(x,t) := \big(G_{\varepsilon,\sigma}(x,t) - k\big)_+^{\frac{p}{2}} \quad \text{and} \quad A_k(t) := \{x \in \mathscr{M} : v_k(x,t) > 0\}\,.$$

We will show that $|A_k| := \int_0^T \int_{A_k(t)} d\mu(\cdot,t)\,dt$, $k \geq k_1$, satisfies the conditions of Stampacchia's Lemma for some $k_1 \geq k_0$. This provides us with a constant d for which $|A_{k_1+d}|$ vanishes. Theorem 5.2 follows. Since $|A_k|$ is clearly non-negative and non-increasing with respect to k, it remains to demonstrate that an inequality of the form (5.24) holds.

We begin by noting that

Lemma 5.12. *There are constants $L_1 \geq L$, $0 < \ell_1 \leq \ell$, and c_1, $c_2 > 0$ (depending only on F, Γ_0, and ε) such that, for all $p \geq L_1$ and $\sigma \leq \ell_1 p^{-\frac{1}{2}}$, the following estimate holds:*

$$\frac{d}{dt}\int v_k^2\,d\mu + \frac{1}{c_1}\int |\nabla v_k|^2\,d\mu \leq c_2(\sigma p + 1)\int_{A_k} G_{\varepsilon,\sigma}^p F^2\,d\mu\,. \tag{5.27}$$

Proof. Observe that

$$\frac{d}{dt}\int v_k^2\,d\mu = \int p(G_{\varepsilon,\sigma} - k)_+^{p-1}\,\partial_t G_{\varepsilon,\sigma}\,d\mu - \int v_k^2 HF\,d\mu$$

and

$$|\nabla v_k|^2 = \frac{p^2}{4}(G_{\varepsilon,\sigma} - k)_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,.$$

Thus, proceeding as in Lemma (5.9), we obtain

$$\frac{d}{dt}\int v_k^2 d\mu \leq -\frac{4}{p^2}\left(A_1 p(p-1) - A_2 p^{\frac{3}{2}}\right)\int |\nabla v_k|^2 \, d\mu$$
$$- \left(B_1 p - B_2 \sigma p - B_3 p^{\frac{1}{2}}\right)\int (G_{\varepsilon,\sigma} - k)_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu$$
$$+ C_1(\sigma p + 1)\int (G_{\varepsilon,\sigma} - k)_+^p F^2 d\mu$$

for some positive constants A_1, A_2, B_1, B_2, B_3, and C_1 (which depend only on F, Γ_0, and ε). The claim follows. □

Now set $\sigma' = \sigma + \frac{n}{p}$. Then

$$\int_{A_k} F^n \, d\mu \leq \int_{A_k} F^n \frac{G_{\varepsilon,\sigma}^p}{k^p}\, d\mu = k^{-p}\int_{A_k} G_{\varepsilon,\sigma'}^p \, d\mu \leq k^{-p}\int \left(G_{\varepsilon,\sigma'}\right)_+^p \, d\mu \,. \tag{5.28}$$

If $p \geq \max\left\{L_1, \frac{4n^2}{\ell_1^2}\right\}$ and $\sigma \leq \frac{\ell_1}{2}p^{-\frac{1}{2}}$, then $p \geq L_1$ and $\sigma' \leq \ell_1 p^{-\frac{1}{2}}$, so that, by Proposition 5.8,

$$\int_{A_k} F^n \, d\mu \leq k^{-p}\int \left(G_{\varepsilon,\sigma'}(\cdot,0)\right)_+^p \, d\mu_0 \leq \mu_0(\mathscr{M})\left(\frac{k_0}{k}\right)^p . \tag{5.29}$$

Choosing k sufficiently large, the right hand side of this inequality can be made arbitrarily small. We will use this fact in conjunction with the Michael–Simon Sobolev inequality to exploit the good gradient term in (5.27).

Theorem 5.13 (Michael–Simon Sobolev inequality (Michael and Simon 1973))**.** *Let $\mathscr{M}^n$, $n \geq 2$, be a smooth, immersed submanifold of $\mathbb{R}^{n+k}$ of dimension n, and let u be a non-negative, smooth function with compact support. Then there exists a constant $c_S > 0$, which depends only on n, such that the following estimate holds:*

$$\left(\int u^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq c_S \int \left(|\nabla u| + |\vec{H}|u\right) d\mu \,,$$

where μ is the induced measure of $\mathscr{M}$, $\vec{H}$ its mean curvature vector, and $|\cdot|$ its induced norm.

An application of Hölder's inequality to the Michael–Simon Sobolev inequality yields the following Gagliardo–Nirenberg type inequality:

Corollary 5.14 (Cf. Huisken (1984). See also Baker (2010))**.** *Let u be as above. Then*

$$\left(\int u^{p^*} \, d\mu\right)^{\frac{1}{p^*}} \leq c_S \frac{p(n-1)}{n-p}\left(\int |\nabla u|^p \, d\mu\right)^{\frac{1}{p}} + c_S\left(\int |\vec{H}|^n \, d\mu\right)^{\frac{1}{n}}\left(\int u^{p^*} \, d\mu\right)^{\frac{1}{p^*}}$$

for any $1 < p < n$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Proof. Applying the Michael–Simon Sobolev inequality to the function $v := u^\beta$, where $\beta > 0$ is to be chosen, we obtain

$$\left(\int u^{\beta\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}} \leq c_S\beta\int u^{\beta-1}|\nabla u|\,d\mu + c_S\int|\vec{H}|u^\beta\,d\mu\,.$$

Applying the Hölder inequality to both of the integrals on the right yields

$$\begin{aligned}\left(\int u^{\beta\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}} \leq{}& c_S\beta\left(\int u^{(\beta-1)\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int|\nabla u|^p\,d\mu\right)^{\frac{1}{p}}\\ &+ c_S\left(\int|\vec{H}|^n\,d\mu\right)^{\frac{1}{n}}\left(\int u^{\beta\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}}\end{aligned}$$

for any $1 \leq p < n$. Setting $\beta := p(n-1)/(n-p)$, so that $p^* = \beta\frac{n}{n-1} = (\beta-1)\frac{p}{p-1}$, the claim follows. □

We want to take advantage of the good gradient term in (5.27), so we need the corollary with $p = 2$. Setting $q := n/(n-2)$ if $n > 2$, or any positive number if $n = 2$, and squaring both sides, we obtain

$$\left(\int u^{2q}\,d\mu\right)^{\frac{1}{q}} \leq 2(c_S\beta)^2\int|\nabla u|^2\,d\mu + 2c_S^2\left(\int|H|^n\,d\mu\right)^{\frac{2}{n}}\left(\int u^{2q}\,d\mu\right)^{\frac{1}{q}},$$

where $\beta = p(n-1)/(n-p)$.

Finally, setting $c_3 := \max\{|\sum_{i=1}^n\lambda_i|/F(\lambda) : \lambda \in \Gamma_0\}$, we obtain the desired inequality:

$$\begin{aligned}\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}} &\leq 2(c_S\beta)^2\int|\nabla v_k|^2\,d\mu + c_S^2c_3\left(\int F^n\,d\mu\right)^{\frac{2}{n}}\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}}\\ &\leq c_4\left(\int|\nabla v_k|^2\,d\mu + \left(\int F^n\,d\mu\right)^{\frac{2}{n}}\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}}\right),\end{aligned} \tag{5.30}$$

where $c_4 := \max\{2c_S^2\beta^2, c_S^2c_3\}$.

It follows from (5.29) and (5.30) that, for any $p \geq \max\{L_1, 4n^2/\ell_1^2\}$ and $\sigma \leq \frac{\ell_1}{2}p^{-\frac{1}{2}}$, we have

$$\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}} \leq 2c_4\int|\nabla v_k|^2\,d\mu$$

for all $k \geq k_1$, where k_1 is chosen such that $k_1^p \geq 2c_4\mu_0(\mathscr{M})k_0^p$ (for example, $k_1 := (2c_4\alpha)^{\frac{1}{p}}k_0$).

Therefore, from (5.27), we have, for all $k \geq k_1$,

$$\frac{d}{dt}\int v_k^2\,d\mu + \frac{1}{2c_1c_4}\left(\int v^{2q}\,d\mu\right)^{\frac{1}{q}} \leq c_2(\sigma p+1)\int_{A_k}F^2G_{\varepsilon,\sigma}^p\,d\mu\,.$$

Integrating this over time, noting that $A_k(0) = \emptyset$, and assuming, without loss of generality, that $c_1 c_4 \geq \frac{1}{2}$, we find

$$\sup_{[0,T)} \left(\int_{A_k} v_k^2 \, d\mu \right) + \int_0^T \left(\int v^{2q} \, d\mu \right)^{\frac{1}{q}} dt \leq c_5(\sigma p + 1) \int_0^T \int_{A_k} F^2 G^p_{\varepsilon,\sigma} \, d\mu \, dt \,, \tag{5.31}$$

where $c_5 := 4c_1c_2c_4$. We now exploit the interpolation inequality for L^p spaces:

$$||f||_{q_0} \leq ||f||_r^{1-\theta} \, ||f||_q^\theta \,, \tag{5.32}$$

where $\theta \in (0,1)$ and $\frac{1}{q_0} = \frac{\theta}{q} + \frac{1-\theta}{r}$. Setting $r = 1$ and $\theta = \frac{1}{q_0}$, we may assume that $1 < q_0 < q$. Then, applying (5.32), we find

$$\int_{A_k} v_k^{2q_0} \, d\mu \leq \left(\int_{A_k} v_k^2 \, d\mu \right)^{q_0 - 1} \left(\int_{A_k} v^{2q} \, d\mu \right)^{\frac{1}{q}} .$$

Applying the Hölder inequality yields

$$\left(\int_0^T \int_{A_k} v_k^{2q_0} \, d\mu \, dt \right)^{\frac{1}{q_0}} \leq \left(\sup_{[0,T)} \int_{A_k} v_k^2 \, d\mu \right)^{\frac{q_0-1}{q_0}} \left(\int_0^T \left(\int_{A_k} v^{2q} \, d\mu \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q_0}} .$$

Using Young's inequality, $ab \leq \left(1 - \frac{1}{q_0}\right) a^{\frac{q_0}{q_0-1}} + \frac{1}{q_0} b^{q_0}$, on the right hand side, we obtain

$$\begin{aligned}
\left(\int_0^T \int_{A_k} v_k^{2q_0} \, d\mu \, dt \right)^{\frac{1}{q_0}} &\leq \left(1 - \frac{1}{q_0} \right) \sup_{[0,T)} \int_{A_k} v_k^2 \, d\mu + \frac{1}{q_0} \int_0^T \left(\int_{A_k} v^{2q} \, d\mu \right)^{\frac{1}{q}} dt \\
&\leq \sup_{[0,T)} \int_{A_k} v_k^2 \, d\mu + \int_0^T \left(\int_{A_k} v^{2q} \, d\mu \right)^{\frac{1}{q}} dt \,.
\end{aligned}$$

Recalling (5.31), we arrive at

$$\left(\int_0^T \int_{A_k} v_k^{2q_0} \, d\mu \, dt \right)^{\frac{1}{q_0}} \leq c_5(\sigma p + 1) \int_0^T \int_{A_k} F^2 G^p_{\varepsilon,\sigma} \, d\mu \, dt \,. \tag{5.33}$$

We now use the Hölder inequality to estimate

$$\int_0^T \int_{A_k} F^2 G^p_{\varepsilon,\sigma} \, d\mu \, dt \leq |A_k|^{1-\frac{1}{r}} \left(\int_0^T \int_{A_k} F^{2r} G^{pr}_{\varepsilon,\sigma} \, d\mu \,, dt \right)^{\frac{1}{r}} \leq c_6 |A_k|^{1-\frac{1}{r}} \tag{5.34}$$

and

$$\int_0^T \int_{A_k} v_k^2 \, d\mu \, dt \leq |A_k|^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A_k} v_k^{2q_0} \, d\mu \, dt \right)^{\frac{1}{q_0}} \tag{5.35}$$

whenever $\sigma \leq \frac{l_1}{4} p^{-\frac{1}{2}}$, and $2r > L_2 := \max\{L_1, \frac{4n^2}{l_1^2}, \frac{64}{l_1^2}\}$, where $c_6 := k_0^2 \left(T \mu_0(\mathscr{M})\right)^{\frac{1}{r}}$.

Finally, for $h > k \geq k_1$, we may estimate

$$|A_h| := \int_0^T \int_{A_h} d\mu\, dt = \int_0^T \int_{A_h} \frac{(G_{\varepsilon,\sigma} - k)_+^p}{(G_{\varepsilon,\sigma} - k)_+^p}\, d\mu\, dt \leq \int_0^T \int_{A_h} \frac{(G_{\varepsilon,\sigma} - k)_+^p}{(h-k)^p}\, d\mu\, dt\,.$$

Since $A_h(t) \subset A_k(t)$ for all $t \in [0, T)$ and $h \geq k$, and $v_k^2 := (G_{\varepsilon,\sigma} - k)_+^p$, we obtain

$$(h-k)^p |A_h| \leq \int_0^T \int_{A_k} v_k^2\, d\mu\, dt\,. \tag{5.36}$$

Putting together estimates (5.33), (5.34), (5.35) and (5.36), we arrive at

$$|A_h| \leq \frac{c_5 c_6 (\sigma p + 1)}{(h-k)^p} |A_k|^\gamma$$

for all $h > k \geq k_1$, where $\gamma := 2 - \frac{1}{q_0} - \frac{1}{r}$. Now fix $p := 2L_2$ and choose $\sigma < \frac{\ell_1}{4} p^{-\frac{1}{2}}$ sufficiently small that $\sigma p < 1$. Then, choosing $r > \max\{\frac{q_0}{q_0 - 1}, L_2\}$, so that $\gamma > 1$, we may apply Stampacchia's Lemma. We conclude that $|A_k| = 0$ for all $k > k_1 + d$, where $d^p = c_5 c_6 2^{\frac{\gamma p}{\gamma - 1} + 1} |A_{k_1}|^{\gamma - 1}$. We note that d is finite, since T is finite and

$$\int_{A_{k_1}} d\mu \leq \int_{A_{k_1}} \frac{(G_{\varepsilon,\sigma})_+^p}{k_1^p}\, d\mu \leq k_1^{-p} \int (G_{\varepsilon,\sigma})_+^p\, d\mu \leq k_1^{-p} \int (G_{\varepsilon,\sigma}(\cdot, 0))_+^p\, d\mu_0\,,$$

where the final estimate follows from Proposition 5.8. It follows that

$$\begin{aligned} G &\leq \varepsilon F + (k_1 + d) F^{1-\sigma} \\ &\leq 2\varepsilon F + C_\varepsilon \end{aligned} \tag{5.37}$$

for some constant C_ε, which depends only on ε, $k_1 + d$, and σ (hence only on n, F, Γ_0, $\mu_0(\mathscr{M})$, T, and ε).

Now, to see that an analogous estimate holds for the function $D := \operatorname{dist}(\vec{\kappa}, \Gamma_+)$, define, for any $\eta > 0$, $\Gamma_\eta := \{z \in \Gamma_0 : D(z) \geq \eta F(z)\}$ and set $A_\eta := \max_{\Gamma_\eta} \frac{D}{G}$ (which is finite by homogeneity of D and G, and positivity of G on Γ_η). Then, from (5.37), we obtain, for any $\delta > 0$,

$$\begin{aligned} D &\leq A_\delta G \\ &\leq A_\delta \left(\frac{\delta}{A_\delta} F + C_{\delta/(2A_\delta)} \right) = \delta F + A_\delta C_{\delta/(2A_\delta)} \end{aligned}$$

whenever $D > \delta F$.

Since, by Proposition 4.31, T may be bounded by a constant depending only on F, Γ_0, and R, this completes the proof of Theorem 5.2.

5.3 The cylindrical estimates

We now prove the second part of Theorem 5.1, the cylindrical estimates:

Theorem 5.15 (Cylindrical estimates, Andrews and Langford (2014)). *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a positive admissible speed function, and assume that one of the following auxiliary conditions is satisfied:*

1. *F is convex; or*

2. *$n = 2$; or*

3. *$\Gamma = \Gamma_+$ and F is concave.*

Then, given $m \in \{0, \ldots, n-2\}$, a curvature cone $\Gamma_0 \subset\subset \Gamma$ satisfying

$$\Gamma_0 \subset\subset \Gamma_{m+1} := \bigcap_{\sigma \in P_n} \{z \in \mathbb{R}^n : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} > 0\},$$

an initial volume scale $\alpha > 0$, an initial distance scale $R > 0$, and any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,m} < \infty$ (depending only on n, F, Γ_0, α, m, and ε) such that, given any solution $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ of (CF) *with curvature satisfying $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \Gamma_0$, initial volume satisfying $\mu_0(\mathscr{M}) \leq \alpha$, and initial curvature satisfying $\min_{\mathscr{M}\times\{0\}} F \geq R^{-1}$ (alternatively, $\mathrm{diam}(\mathscr{X}_0(\mathscr{M})) \leq R$), the following estimate holds:*

$$\mathrm{dist}\left(\vec{\kappa}(x,t), \Lambda_m^+\right) \leq \varepsilon F(x,t) + C_\varepsilon$$

for all $(x,t) \in \mathscr{M} \times [0,T)$, where, recalling that $c_m := F(\underbrace{0,\ldots,0}_{m-times},1,\ldots,1)$ is the value F takes on the unit cylinder $\mathbb{R}^m \times S^{n-m}$,

$$\Lambda_m^+ := \bigcap_{\sigma \in P_n} \{z \in \overline{\Gamma}_+ : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} \geq c_m^{-1} F(z_1, \ldots, z_n)\}.$$

5.3.1 The pinching functions

As for the convexity estimate, we begin by constructing appropriate pinching functions G_m for the pinching cones Λ_m. Our construction of G_m will be independent of the choice of m; so let us fix $m \in \{0, \ldots, n-2\}$ and assume that $\Gamma_0 \subset\subset \Gamma_m$.

Flows by convex speeds

Recall the function defined by (4.10) in Proposition 4.10, and used to show that the curvature inequalities $\kappa_1 + \cdots + \kappa_{m+1} - c_m^{-1} F \geq -\varepsilon F$ are preserved. We modify this function just as in the proof of the convexity estimate; more precisely, we set

$$G := K(G_1, G_2), \tag{5.38}$$

where K and G_2 are defined, respectively, as in (5.4) and (5.3), and G_1 is defined by (4.10) (for our fixed m).

The purpose of the modification is the following estimate:

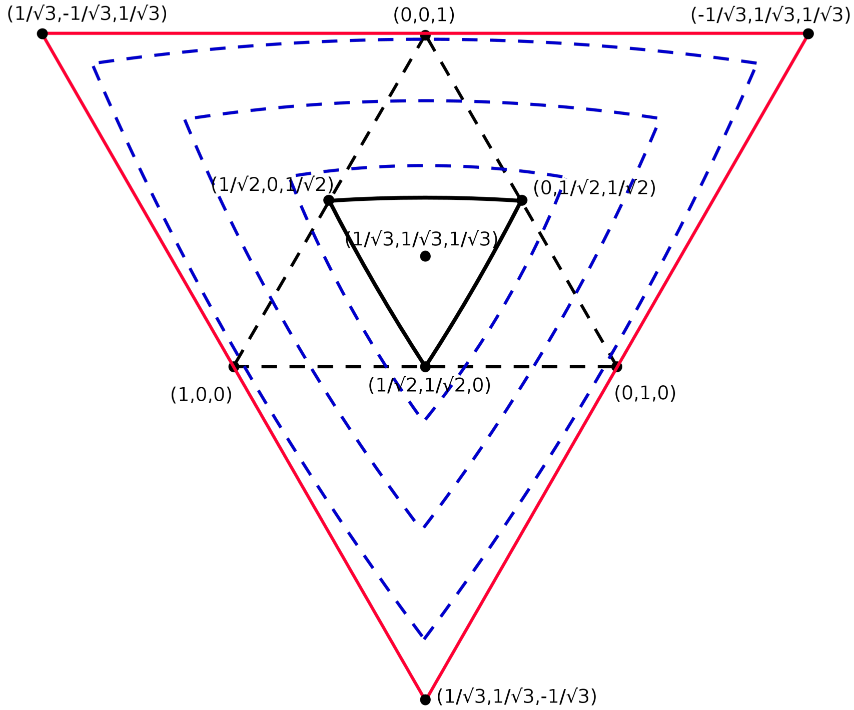

Figure 5.2: Intersection of the curvature space, $\mathbb{R}^3$, of a three-dimensional hypersurface with the unit sphere. The red curve is the boundary of the initial cone, Γ_1. The blue curves are the boundaries of the cones $\Gamma_\varepsilon := \cap_{\sigma\in P_3}\{z \in \Gamma_0 : z_{\sigma(1)} + z_{\sigma(2)} - c_2^{-1}F(z) \geq -\varepsilon F(z)\}$, which 'pinch' onto $\overline{\Lambda}_1$ as $\varepsilon \to 0$. Note that the initial condition is just enough to rule out the 'cylindrical points' $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Lemma 5.16. *Let G be the curvature function defined by* (5.38)*. Then, for every $\varepsilon > 0$, there exists a constant $\gamma > 0$ (which depends only on F, Γ_0, m and ε) such that*

$$\mathscr{Q}_{G,F}\big|_W(T,T) := \left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right)\Big|_W T_{kpq}T_{lrs} \leq -\gamma\frac{||T||^2}{F(W)} \tag{5.39}$$

for all diagonal $W \in \mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda \in \Gamma_\varepsilon := \{z \in \Gamma : G(z) \geq \varepsilon F(z)\}$, and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. Since we proved in Proposition 4.10 that

$$\left(\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_1^{pq,rs}\right)T_{kpq}T_{lrs} \leq 0\,,$$

the proof is exactly the same as the proof of Lemma 5.3 $\square$

The estimate for the zero order term $\mathscr{Z}$ is slightly different for the cylindrical estimates than for the convexity estimate due to the fact that the support of G contains points lying inside and points lying outside of the positive cone. We use the convexity estimate to control the points lying outside:

Lemma 5.17. *Let G be defined as in* (5.38)*. Then, for every $\delta > 0$, $\varepsilon > 0$, and $C > 0$, there exist constants $\gamma_1 > 0$, $\gamma_2 > 0$ (depending only on F, Γ_0, n, and ε) and $\gamma_3 > 0$*

(depending only on F, Γ_0, and n) such that

$$\mathscr{Z}_{G,F}\big|_W(W) := (F\dot{G}^{kl} - G\dot{F}^{kl})\big|_W W_{kl}^2 \geq \gamma_1 F^2(G - \delta\gamma_2 F)\big|_W - \gamma_3 CF^2\big|_W$$

for all diagonal matrices W with eigenvalue n-tuple $\lambda \in \Gamma_{\varepsilon,\delta,C} := \Gamma_\varepsilon \cap \Gamma_{\delta,C}$, where $\Gamma_\varepsilon := \{z \in \Gamma_0 : G(z) \geq \varepsilon F(z)\}$ and $\Gamma_{\delta,C} := \{z \in \Gamma_0 : z_i \geq -\delta F(z) - C$ for each $i\}$.

Remark 5.1. Note that, by the convexity estimate (Theorem 5.2), for every $\delta > 0$ there exists $C_\delta > 0$ (depending only on n, F, Γ_0, α, R and δ) such that the set Γ_{δ,C_δ} is preserved.

Proof of Lemma 5.17. First note that, just as in the proof of Lemma 5.4, it suffices to prove the estimate for $\mathscr{Z}_{G_1,F}$. So let W be a diagonal matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Let $l \in \{0, \ldots, m\}$ be the number of non-positive eigenvalues. Then, recalling equation (4.11),

$$\begin{aligned}
\mathscr{Z}_{G_1,F}(W) &= \sum_{p>q} \big(\dot{G}_1^p \dot{F}^q - \dot{G}_1^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q) \\
&= \sum_{p>q} \big(Q^p \dot{F}^q - Q^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q) \\
&= \left(\sum_{p>q>l} + \sum_{p>l\geq q} + \sum_{l\geq p>q}\right)\big(Q^p \dot{F}^q - Q^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q)\,,
\end{aligned}$$

where $Q^a := \sum_{\sigma\in O_a} \varphi'(r_\sigma)$, $O_a := \{\sigma \in H_m : a \in \sigma(\{1, \ldots, m+1\})\}$ (see Proposition 4.10).

Recalling (see Lemma 4.11) that $Q^p\dot{F}^q - Q^q\dot{F}^p \geq \dot{F}^p(Q^p - Q^q) \geq 0$ whenever $\lambda_p \geq \lambda_q$, we may discard the final sum and part of the first to obtain

$$\begin{aligned}
\mathscr{Z}_{G_1,F}(W) \geq{}& \sum_{p=m+2}^{n}\sum_{q=l+1}^{m+1} \big(Q^p \dot{F}^q - Q^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q) \\
&+ \sum_{p=l+1}^{n}\sum_{q=1}^{l} \big(Q^p \dot{F}^q - Q^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q) \\
={}& \sum_{p=m+2}^{n}\sum_{q=l+1}^{m+1} \big(Q^p \dot{F}^q - Q^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q) - F^2\sum_{i=1}^{l}\lambda_i \\
&+ F^2\sum_{i=1}^{l}\lambda_i + \sum_{p=l+1}^{n}\sum_{q=1}^{l} \big(Q^p \dot{F}^q - Q^q \dot{F}^p\big)\lambda_p\lambda_q(\lambda_p - \lambda_q)\,.
\end{aligned}$$

So consider the function

$$Z_1(z) := \sum_{p=m+2}^{n}\sum_{q=l+1}^{m+1} \Big(\big(Q^p(z)\dot{F}^q(z) - Q^q(z)\dot{F}^p(z)\big)\Big)z_p z_q(z_p - z_q) - F(z)^2\sum_{i=1}^{l} z_i\,.$$

Observe that $Z_1 \geq 0$. We claim that $Z_1 > 0$ on the cone $\Gamma^>_{\varepsilon,l} := \{z \in \Gamma_\varepsilon : z_1 \leq \cdots \leq z_l \leq 0 < z_{l+1} \leq \cdots \leq z_n\}$: Suppose, to the contrary, that $Z_1(z) = 0$ for some $z \in \Gamma^>_{\varepsilon,l}$. Then

$z_1 = \cdots = z_l = 0$ and $\left(Q^p(z)\dot{F}^q(z) - Q^q(z)\dot{F}^p(z)\right) z_p z_q(z_p - z_q) = 0$ for all $p > m+1 \geq q > l$. But, by Lemma 4.11, the latter implies that, for all $p > m+1 \geq q > l$, either $z_p = z_q$, or $r_\sigma(\lambda) \geq 0$ for all $\sigma \in O_{q,p}$. Note that the latter case cannot occur: Since $p > m+1 \geq q$, there is a permutation $\sigma \in O_{q,p}$ such that $0 \leq r_\sigma(z) = (z_1 + \cdots + z_{m+1} - c_m^{-1}F(z))/F(z)$, which implies $G(z) = 0$, contradicting $z \in \Gamma_\varepsilon$. On the other hand, if $z_p = z_q$ for all $p > m+1 \geq q > l$, then, by convexity of Λ_m (Lemma 4.9), $\lambda \in \Lambda_m$, so that we again obtain the contradiction $G(z) = 0$. Thus, $Z_1 > 0$ on $\Gamma^>_{\varepsilon,l}$. Since Z_1 is homogeneous of degree three, it follows that

$$Z_1(\lambda) \geq c_1 F(\lambda)^2 G(\lambda)\,,$$

where $c_1 := \min_l \min_{\Gamma^>_{\varepsilon,l}} \frac{Z_1}{F^2 G} > 0$.

Now consider

$$Z_2 := F^2 \sum_{i=1}^{l} \lambda_i + \sum_{p=l+1}^{n} \sum_{q=1}^{l} \left(Q^p \dot{F}^q - Q^q \dot{F}^p\right) \lambda_p \lambda_q (\lambda_p - \lambda_q)\,.$$

Note that, by homogeneity, $c_2 := \sup\{Q^p(z)\dot{F}^q(z) - Q^q(z)\dot{F}^p(z) : z \in \Gamma_0,\ 1 \leq p,\ q \leq n\} < \infty$. Thus, Z_2 is easily controlled using the 'convexity estimate' $\lambda_1 \geq -\delta F - C$:

$$\begin{aligned}
Z_2 \geq & \ -lF^2(\delta F + C) + (n-l)c_2 \lambda_n \sum_{q=1}^{l} \lambda_q(\lambda_n - \lambda_q) \\
\geq & \ -nF^2(\delta F + C) + 2nc_2 c_3^2 F^2 \sum_{q=1}^{l} \lambda_q \\
\geq & \ -nF^2(\delta F + C) - 2nc_2 c_3^2 F^2(\delta F + C) \\
\geq & \ -n(1 + 2c_2 c_3^2) F^2 (\delta F + C)\,,
\end{aligned}$$

where $c_3 := \max\{|\lambda_i|/F(\lambda) : \lambda \in \Gamma_0, 1 \leq i \leq n\}$.

The claim follows. □

Surface flows

In the setting of surface flows, we are left only with the choice $m = 0$. In this case, our initial cone satisfies $\Gamma_0 \subset\subset \Gamma_+$, and our pinching set is the positive ray. We define our pinching function by

$$G(\lambda) := F(\lambda)\frac{(\lambda_2 - \lambda_1)^2}{(\lambda_2 + \lambda_1)^2} = F(\lambda)\frac{2\,||\lambda||^2 - \mathrm{tr}(\lambda)}{(\mathrm{tr}(\lambda))^2}\,. \tag{5.40}$$

Lemma 5.18. *Let $G : \Gamma_0 \to \mathbb{R}$ be the curvature function defined by* (5.40) *and set $Z := G/F$. Then, for every $\varepsilon > 0$, there exists $\gamma > 0$ (which depends only on F, Γ_0, and ε)*

such that

$$\begin{aligned}\mathscr{Q}_{G,F}(T,T) &:= \left(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs}\right)T_{kpq}T_{lrs}\\ &\le F\ddot F^{kl,rs}\dot Z^{pq}T_{kpq}T_{lrs} + C|(DZ)*T| - 2\dot F^{kl}\dot Z^{pq}T_{kpq}\dot F^{rs}T_{lrs}\\ &\quad -\gamma\frac{\|T\|^2}{F}\end{aligned} \tag{5.41}$$

at any diagonal $W\in\mathrm{Sym}(n)$ with eigenvalue pair $\lambda\in\Gamma_\varepsilon := \{z\in\Gamma_0 : G(z)\ge \varepsilon F(z)\}$ and all totally symmetric $T\in\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathbb{R}^n$, where $C<\infty$ is a constant (which depends only on F and Γ_0) and

$$(DZ)*T := (\dot Z^1T_{111}+\dot Z^2T_{122})T_{122} + (\dot Z^1T_{211}+\dot Z^2T_{222})T_{211}\,.$$

Proof. The proof is similar to that of Lemma 5.5. □

Since in the case of surface flows our solution is already convex, the estimate for the zero order term is simplified:

Lemma 5.19. *Let G be defined as in* (5.40)*. Then, for every $\varepsilon>0$, there exists $\gamma>0$ (depending only on F, Γ_0, and ε) such that*

$$\mathscr{Z}_{G,F}\big|_W(W) := (F\dot G^{kl} - G\dot F^{kl})\big|_W W^2_{kl} \ge \gamma F^2 G$$

for all diagonal matrices W with eigenvalue pair $\lambda\in\Gamma_\varepsilon := \{z\in\Gamma_0 : G(z)\ge\varepsilon F(z)\}$.

Proof. Let W be any diagonal 2×2 matrix with positive eigenvalues $\lambda_1\le\lambda_2$. First note that

$$F\dot G(W^2) - G\dot F(W^2) = F^2\dot Z(W^2)\,,$$

where Z is defined by

$$Z(\lambda) = \frac{G}{F} = \frac{(\lambda_2-\lambda_1)^2}{(\lambda_1+\lambda_2)^2}\,.$$

Since

$$\dot Z^1(\lambda) = -\frac{4\lambda_2(\lambda_2-\lambda_1)}{(\lambda_1+\lambda_2)^3}\quad\text{and}\quad \dot Z^2(\lambda) = \frac{4\lambda_1(\lambda_2-\lambda_1)}{(\lambda_1+\lambda_2)^3}\,,$$

we find

$$F^2\dot Z(W^2) = \frac{4F(\lambda)^2}{(\lambda_1+\lambda_2)^3}\lambda_1\lambda_2(\lambda_2-\lambda_1)^2 > 0\,.$$

The claim now follows from homogeneity; precisely, we have

$$F^2\dot Z(W^2)\ge\gamma_\varepsilon F^2 G\,,$$

where

$$\gamma_\varepsilon := \min\left\{\frac{4\lambda_1\lambda_2}{F(\lambda)(\lambda_1+\lambda_2)} : \lambda \in \Gamma_\varepsilon\right\}.$$

□

Flows by concave speeds

Note that the only case we consider for flows by concave speeds is the $m = 0$ case. Our pinching function in this case is constructed as in the $m = 0$ case for flows by convex speeds, but with G_1 given by

$$G(\lambda) := G_1(\lambda) := F(\lambda)\sum_{i=1}^{n}\varphi\left(\frac{c_0^{-1}F(\lambda)-\lambda_i}{F(\lambda)}\right) \tag{5.42}$$

as in the proof of Proposition 4.13. Proceeding as in Lemma 5.16 yields the following estimate:

Lemma 5.20. *Let G be the curvature function defined by* (5.42)*. Then, for every $\varepsilon > 0$, there exists a constant $\gamma > 0$ (which depends only on F, Γ_0, m and ε) such that*

$$\mathscr{Q}_{G,F}\big|_W(T,T) := \left(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\right)\Big|_W T_{kpq}T_{lrs} \leq -\gamma\frac{||T||^2}{F(W)} \tag{5.43}$$

for all diagonal $W \in \mathrm{Sym}(n)$ with eigenvalue n-tuple $\lambda \in \Gamma_\varepsilon := \{z \in \Gamma : G(z) \geq \varepsilon F(z)\}$, and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proceeding as in Lemma 5.17 (and using the fact that $\Gamma_0 \subset\subset \Gamma_+$) yields the following estimate for the zero order term:

Lemma 5.21. *Let G be defined as in* (5.42)*. Then, for $\varepsilon > 0$ there exists $\gamma > 0$ (depending only on F, Γ_0, and ε) such that*

$$\mathscr{Z}_{G,F}\big|_W(W) := (F\dot{G}^{kl} - G\dot{F}^{kl})\big|_W W_{kl}^2 \geq \gamma F(W)^2 G(W)$$

for all diagonal matrices W with eigenvalue n-tuple $\lambda \in \Gamma_\varepsilon := \{z \in \Gamma_0 : G(z) \geq \varepsilon F(z)\}$.

5.3.2 The integral estimate

Now consider, for any $\varepsilon > 0$ and $\sigma > 0$, the function

$$G_{\varepsilon,\sigma} := \left(\frac{G}{F} - \varepsilon\right)F^\sigma.$$

We will proceed much as in the proof of the convexity estimate to obtain the following integral estimate:

Proposition 5.22 (L^p-estimate)**.** *In case F is convex, let G be defined by* (5.38)*. In case $n = 2$, let G be defined by* (5.40)*. In case F is concave, let G be defined by* (5.42)*. Then there exist constants $\ell > 0$, $L < \infty$ (which depend only on F, Γ_0, n, and ε) and $K > 0$ (which depends only on F, Γ_0, n, α, ε, σ, and p) such that*

$$\frac{d}{dt}\left|\left|\big(G_{\varepsilon,\sigma}(\cdot,t)\big)_+\right|\right|_{L^p(\mathscr{M},\mu(t))} \leq K$$

for all $p > L$ and all $\sigma \in \left(0, \ell p^{-\frac{1}{2}}\right)$, where $(G_{\varepsilon,\sigma})_+ := \max\{G_{\varepsilon,\sigma}, 0\}$ is the positive part of $G_{\varepsilon,\sigma}$.

Exactly as in the proof of the convexity estimate (using Lemmata 5.16 and 5.18) we obtain

Lemma 5.23. *There exist positive constants A_1, A_2, B_1, B_2, B_3 C_1 which depend only on F, Γ_0, m, and ε such that*

$$\begin{aligned}\frac{d}{dt}\int (G_{\varepsilon,\sigma})_+^p d\mu \leq &-\left(A_1 p(p-1) - A_2 p^{\frac{3}{2}}\right)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\, d\mu\\ &-\left(B_1 p - B_2\sigma p - B_3 p^{\frac{1}{2}}\right)\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\\ &+ C_1(\sigma p + 1)\int (G_{\varepsilon,\sigma})_+^p |\mathcal{W}|^2 d\mu \end{aligned} \tag{5.44}$$

for all $p \geq 2$ and $\sigma \in (0, 1]$.

To estimate the final term, we make use of Lemmata 5.17 and 5.19:

Lemma 5.24 (Cf. Huisken and Sinestrari (2009), §5)**.** *There exist positive constants A_3, A_4, A_5, B_3, B_4, C_2 which depend only on F, Γ_0, m, and ε such that:*

$$\begin{aligned}\int (G_{\varepsilon,\sigma})_+^p \frac{Z(\mathcal{W})}{F}\, d\mu \leq &\left(A_3 p^{\frac{3}{2}} + A_4 p^{\frac{1}{2}} + A_5\right)\int (G_{\varepsilon,\sigma})_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\, d\mu\\ &+\left(B_3 p^{\frac{1}{2}} + B_4\right)\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,.\end{aligned}$$

Proof. As in Lemma 5.10, contraction of the commutation formula for $\nabla^2 \mathcal{W}$ with $\dot F$ and $\dot G$ yields the identity

$$\begin{aligned}\mathscr{L}G_{\varepsilon,\sigma} = &-F^{\sigma-1}\mathscr{Q}_{G,F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + F^{\sigma-1}\mathscr{Z}_{G,F}(\mathcal{W}) + F^{\sigma-2}(F\dot G^{kl} - G\dot F^{kl})\nabla_k\nabla_l F\\ &+\frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{(1-\sigma)}{F}\langle \nabla F, \nabla G_{\varepsilon,\sigma}\rangle_F + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,.\end{aligned}$$

The claim is now proved by integrating against $(G_{\varepsilon,\sigma})_+^p F^{-\sigma}$ and estimating the error terms using integration by parts, Young's inequality, and homogeneity (via Lemma 4.25) similarly as in Lemma 5.10. □

Now recall Lemmata 5.17 and 5.19. In case F is convex, we set $\delta = \varepsilon/(2\gamma_2)$ and apply the convexity estimate (Theorem 5.2) to obtain

$$\frac{\varepsilon}{2}\gamma_1 F^2 \leq \frac{\mathscr{Z}_{G,F}(\mathcal{W})}{F} + \gamma_3 C_{\varepsilon/(2\gamma_2)} F \tag{5.45}$$

whenever $G \geq \varepsilon F$. In case $n = 2$ or F is concave, (5.45) holds wherever $G \geq \varepsilon F$ with $\gamma_3 = 0$.

We now use Young's inequality to estimate (Cf. Huisken and Sinestrari 2009, §5)

$$F = F^{-\sigma p} F^{1+\sigma p} \leq F^{-\sigma p}\left(\frac{b^q}{q} F^{q(1+\sigma p)} + \frac{b^{-q'}}{q'}\right)$$

for any $b > 0$ and $q > 0$, where q' is the Hölder conjugate of q: $\frac{1}{q} + \frac{1}{q'} = 1$. Choosing $q = \frac{2+\sigma p}{1+\sigma p}$, so that $q' = 2 + \sigma p$, we obtain

$$\begin{aligned} F &\leq b^{\frac{2+\sigma p}{1+\sigma p}} \frac{1+\sigma p}{2+\sigma p} F^2 + \frac{b^{-(2+\sigma p)}}{2+\sigma p} F^{-\sigma p} \\ &\leq b^{\frac{2+\sigma p}{1+\sigma p}} F^2 + b^{-(2+\sigma p)} F^{-\sigma p} . \end{aligned}$$

Now choose $b := \left(\frac{\varepsilon\gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}}\right)^{\frac{1+\sigma p}{2+\sigma p}}$, so that

$$\gamma_3 C_{\varepsilon/(2\gamma_2)} F \leq \frac{\varepsilon\gamma_1}{4} F^2 + K F^{-\sigma p} ,$$

where

$$K := \gamma_3 C_{\varepsilon/(2\gamma_2)} \left(\frac{\varepsilon\gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}}\right)^{-(1+\sigma p)} .$$

Returning to equation (5.45), we find

$$\frac{\varepsilon\gamma_1}{4} F^2 \leq K F^{-\sigma p} + \frac{\mathscr{Z}_{G,F}(\mathcal{W})}{F} .$$

Bounding $G_{\varepsilon,\sigma} \leq c_1 F^\sigma$, where $c_1 := \max\{G(\lambda)/F(\lambda) : \lambda \in \Gamma_0\}$, and $|\mathcal{W}|^2 \leq c_2 F^2$, where $c_2 := \max\{||\lambda||^2 / F(\lambda)^2 : \lambda \in \Gamma_0\}$, we obtain

$$(G_{\varepsilon,\sigma})_+^p |\mathcal{W}|^2 \leq \widetilde{K} + c_3 (G_{\varepsilon,\sigma})_+^p \frac{\mathscr{Z}_{G,F}(\mathcal{W})}{F} ,$$

for some constants $\widetilde{K} > 0$ (depending only on n, F, Γ_0, ε, σ and p), and $c_3 > 0$ (depending only on n, F, Γ_0, and ε).

Combining Lemma 5.24, Proposition[5] 2.3, and inequality (5.44) now yields

$$\frac{d}{dt}\int (G_{\varepsilon,\sigma})_+^p d\mu \le \widetilde{K}\mu_0(\mathscr{M}) - \left(\alpha_0 p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right)\int (G_{\varepsilon,\sigma})_+^{p-2}|G_{\varepsilon,\sigma}|^2\, d\mu$$
$$- \left(\beta_0 p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}}\right)\int (G_{\varepsilon,\sigma})_+^p \frac{|\nabla \mathcal{W}|^2}{F^2}\, d\mu\,.$$

for some positive constants α_i and β_i (which depend only on n, F, Γ_0, and ε), and K (which depends only on n, F, Γ_0, ε, σ and p).

It is clear that $L < \infty$ and $\ell > 0$ may be chosen such that

$$\left(\alpha_0 p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right) \ge 0$$

and

$$\left(\beta_0 p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}}\right) \ge 0$$

for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$. This proves Proposition 5.22.

The proof of Theorem 5.15 is now completed by proceeding with Huisken's iteration argument.

5.3.3 The supremum estimate

The argument has already been laid out in §5.2. The main difference which appears is in the estimate (5.29), where Proposition 5.22 instead yields

$$\int_{A_k} F^n\, d\mu \le k^{-p}\left(KT + \int \left(G_{\varepsilon,\sigma'}(\cdot,0)\right)_+^p\, d\mu_0\right) \le \frac{KT + \mu_0(\mathscr{M})k_0^p}{k^p}\,.$$

The rest of the proof goes through with only minor changes.

5.4 An infinitesimal description of singularities

We now apply scaling techniques and the curvature estimates of the preceding sections to analyse the structure of singularities of the flow (CF). We will see that the convexity estimate (through the splitting theorems, Theorems 4.21 and 4.23) forces an infinitesimal separation of variables at a singularity. As a consequence, we deduce that a certain sequence of rescalings of the flow about a singularity converges to a product of flat directions with a strictly convex solution of a lower dimensional flow. Moreover, if the singularity is occurring at a sufficiently fast rate, then (through Andrews' Harnack inequality, Theorem B.1) the strictly convex part must move by translation. In the special case of flows of convex hypersurfaces, we find that the only rescaling limits are shrinking spheres.

[5] Note that, since Γ_0 is convex, we have $H > 0$.

Theorem 5.25 (Infinitesimal description of singularities)**.** *Let $n \geq 2$ be an integer and $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ an admissible flow speed, and suppose that one of the following auxiliary conditions hold:*

1. *F is convex; or*
2. *$n = 2$; or*
3. *$\Gamma = \Gamma_+$ and F is concave.*

Let $\Gamma_0 \subset\subset \Gamma$ be a curvature cone and $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ a compact solution of (CF) *satisfying $\vec{\kappa}(\mathscr{M} \times [0,T)) \subset \Gamma_0$, and consider a sequence $\{\mathscr{X}_k\}_{k\in\mathbb{N}}$ of rescaled solutions defined by*

$$\begin{aligned} \mathscr{X}_k : \mathscr{M} \times [\alpha_k, T_k] &\to \mathbb{R}^{n+1} \\ (x,t) &\mapsto \lambda_k \left(\mathscr{X}\left(x_k, t_k + \frac{t}{\lambda_k^2} \right) - \mathscr{X}(x_k,t_k) \right), \end{aligned} \tag{5.46}$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ is a sequence of positive numbers, $\{(x_k,t_k)\}_{k\in\mathbb{N}}$ is a sequence of points, and for each k we have set $\alpha_k := -\lambda_k^2 t_k$ and $T_k := T - t_k - \frac{1}{k}$. Then the following statements hold:

- ***Type-I singularities**: If $\limsup_{t\nearrow T} \left(\sqrt{T-t}\max_{\mathscr{M}\times\{t\}} |\mathcal{W}|\right) < \infty$, choose the sequences $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{(x_k,t_k)\}_{k\in\mathbb{N}}$ such that*

  $$\lambda_k^2 := \max_{\mathscr{M}\times[0,T-\frac{1}{k}]} |\mathcal{W}|^2 = |\mathcal{W}(x_k,t_k)|^2 .$$

  Then the sequence (5.46) *converges locally smoothly along a sub-sequence to a maximal, ancient limit solution $\mathscr{X}_\infty : \left(\mathbb{R}^k \times \Sigma^{n-k}\right) \times (-\infty, T_\infty) \to \mathbb{R}^{n+1}$, $0 \leq k \leq n-1$, $T_\infty < \infty$, where the product is isometric and $\mathscr{X}_\infty\big|_{\{0\}\times\Sigma^{n-k}\times(-\infty,T_\infty)}$ is strictly convex, maps into an $(n-k+1)$-dimensional subspace, and solves the flow* (CF) *with speed given by the restriction of F to Γ_+^{n-k}. If, moreover, $\Gamma_0 \subset\subset \Gamma_m := \bigcap_{\sigma\in P_n}\{z \in \mathbb{R}^n : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} > 0\}$ for some $m \in \{0,\ldots,n-2\}$, then either $k < m$ or $k = m$, $\Sigma^{n-m} \cong S^{n-m}$, and $\mathscr{X}_\infty$ is a shrinking cylinder.*

- ***Type-II singularities**: If $\limsup_{t\nearrow T} \left(\sqrt{T-t}\max_{\mathscr{M}\times\{t\}} |\mathcal{W}|\right) = \infty$, choose the sequences $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{(x_k,t_k)\}_{k\in\mathbb{N}}$ such that*

  $$\lambda_k^2 := \max_{(x,t)\in\mathscr{M}\times[0,T-\frac{1}{k}]} |\mathcal{W}(x,t)|^2 \left(T - \frac{1}{k} - t\right) = |\mathcal{W}(x_k,t_k)|^2 \left(T - \frac{1}{k} - t_k\right) .$$

  Then the sequence (5.46) *converges locally smoothly along a sub-sequence to an eternal limit solution $\mathscr{X}_\infty : \left(\mathbb{R}^k \times \Sigma^{n-k}\right) \times (-\infty, \infty) \to \mathbb{R}^{n+1}$, $0 \leq k \leq n-1$, where the product is isometric and $\mathscr{X}\big|_{\{0\}\times\Sigma^{n-k}\times(-\infty,\infty)}$ is strictly convex, maps into an $(n-k+1)$-dimensional subspace, and solves the flow* (CF) *with speed given by the restriction of F to Γ_+^{n-k}. If, moreover, $\Gamma_0 \subset\subset \Gamma_m$ for some $m \in \{0,\ldots,n-2\}$, then $k < m$. Finally, in the case that F is convex, $\mathscr{X}_\infty$ moves by translation.*

Remarks 5.3. 1. In particular, if $\Gamma = \Gamma_+$ (so that the solutions are convex) the only possible rescaling limit is the shrinking sphere.

2. As remarked in Remark 4.5, the existence of preserved cones in the first two cases is automatic. For concave speeds, additional assumptions are necessary (inverse-concavity, strong enough initial curvature pinching, or vanishing of F on $\partial\Gamma_0$ are sufficient).

3. When the speed is given by the mean curvature, it is further known that the limit flows of type-I singularities are necessarily either shrinking cylinders or products of an $(n-1)$-plane with one of the (non-embedded) shrinking Abresch–Langer solutions of the curve shortening flow (see Abresch and Langer (1986), Huisken (1990; 1993), and Stone (1994)); however, this fact relies on Huisken's monotonicity formula (Huisken 1990, Theorem 3.1), which, as yet, has no replacement for flows other than the mean curvature flow. On the other hand, for $(m+1)$-convex flows, our result yields new information even for the mean curvature flow (except in the cases $m = 0$, 1, which follow from the work of Huisken (1984) and Huisken and Sinestrari (2009)), owing to the new cylindrical estimates (Theorem 5.15).

We shall first prove that the respective sequences converge, sub-sequentially, to an ancient, respectively eternal, limit solution. This follows from the following lemma by applying the compactness theorem (Theorem C.4):

Lemma 5.26 (Cf. Huisken and Sinestrari (1999b))**.**

(i) For each $k \in \mathbb{N}$, $\mathscr{X}_k(x_k, 0) = 0$.

(ii) For each $k \in \mathbb{N}$, $|\mathcal{W}_k(x_k, 0)| = 1$, *where* $\mathcal{W}_k$ *is the Weingarten curvature of* $\mathscr{X}_k$.

(iii) As $k \to \infty$, *we have*

$$t_k \to T\,, \quad \lambda_k \to \infty\,, \quad \alpha_k \to -\infty\,, \quad \text{and} \quad T_k \to T_\infty\,,$$

where $0 < T_\infty < \infty$ *if the singularity is of type-I and* $T_\infty = \infty$ *if it is of type-II.*

(iv) In the type-I case, we have $\max_{\mathscr{M}\times[\alpha_k,T_k]} |\mathcal{W}_k| \leq 1$ *for all* $k \in \mathbb{N}$. *In the type-II case, we have the following estimate: For any* $\varepsilon > 0$ *and any* $\overline{T} > 0$ *there exists* $k_0 \in \mathbb{N}$ *such that*

$$\max_{M\times[\alpha_k,\overline{T}]} |\mathcal{W}_k|^2 \leq 1 + \varepsilon \tag{5.47}$$

for all $k \geq k_0$.

Proof. The proof is essentially that of Huisken and Sinestrari (1999b, §4), who considered the case that $\mathscr{X}$ is a solution of the mean curvature flow.

Parts (i) and (ii) are immediate from the definitions and the scaling behaviour of $\mathcal{W}_k$.

Next consider part (iii): First note that, if the singularity is of type-I, then there is some constant $C > 0$ such that

$$T_k := |\mathcal{W}(x_k, t_k)|^2 \left(T - \frac{1}{k} - t_k\right) \leq |\mathcal{W}(x_k, t_k)|^2 \,(T - t_k) < C\,.$$

Thus, T_k is bounded. Now,

$$T_{k+1} - T_k = \lambda_{k+1}^2 (T - t_{k+1}) - \lambda_k^2 (T - t_k) \geq \lambda_k^2 (t_{k+1} - t_k)\,.$$

Moreover, for type-I singularities, $t_{k+1} \geq t_k$; this is because $\max_{\mathscr{M}\times[0,T-\frac{1}{k+1}]} |\mathcal{W}|$ cannot occur at an earlier time than $\max_{\mathscr{M}\times[0,T-\frac{1}{k}]} |\mathcal{W}|$. So T_k is non-decreasing, and must therefore approach some finite limit T_∞.

If instead the singularity is of type-II, then, for all $R > 0$, there exist $t_R \in [0, T)$ and $x_R \in M$ such that

$$|\mathcal{W}(x_R, t_R)|^2 (T - t_R) > 2R\,.$$

On the other hand, there is some sufficiently large $k_R \in \mathbb{N}$ such that

$$t_R < T - \frac{1}{k}\,, \qquad |\mathcal{W}(x_R, t_R)|^2 \left(T - \frac{1}{k} - t_R\right) > R$$

for all $k > k_R$. Therefore, by definition,

$$T_k = \max_{(x,t)\in\mathscr{M}\times[0,T-1/k]} |\mathcal{W}(x,t)|^2 \left(T - \frac{1}{k} - t\right) \geq |\mathcal{W}(x_R, t_R)|^2 \left(T - \frac{1}{k} - t_R\right) > R$$

for all $k > k_R$. Since R was arbitrary, we find $T_k \to \infty$ as $k \to \infty$.

Since $\left(T - \frac{1}{k} - t_k\right)$ is bounded, it follows from the definition of T_k that $\lambda_k \to \infty$ as $k \to \infty$. Therefore, since $|\mathcal{W}|$ remains bounded whilst $t < T$, we must have $t_k \to T$. It follows that $\alpha_k \to -\infty$.

Finally, we consider part (iv): Since the statement for type-I singularities is trivially satisfied, we consider the type-II case: Note first that

$$|\mathcal{W}_k(x, \tau)|^2 = \lambda_k^{-2} |\mathcal{W}(x, \lambda_k^{-2}\tau + t_k)|^2\,.$$

By the definition of λ_k and the choice of (x_k, t_k), we also have

$$|\mathcal{W}(x, \lambda_k^{-2}\tau + t_k)|^2 \left(T - \frac{1}{k} - (\lambda_k^{-2}\tau + t_k)\right) \leq \lambda_k^2 \left(T - \frac{1}{k} - t_k\right)\,.$$

Therefore,

$$|\mathcal{W}_k(x, \tau)|^2 \leq \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t_k - \lambda_k^{-2}\tau} = \frac{T_k}{T_k - \tau} = 1 + \frac{\tau}{T_k - \tau}\,.$$

Since $T_k \to \infty$ and $\tau \leq \overline{T}$, the claim follows. □

We now complete the proof of Theorem 5.25.

Proof of Theorem 5.25. Parts (i) and (iv) of the lemma allow us to apply Theorem C.4 to extract, for each k, sub-limits $(\mathscr{X}_\infty, \mathscr{M}_\infty \times [-\alpha_k, T_k], (x_\infty, 0))$ of the blow-up sequence. Taking a diagonal sub-sequence and applying part (iii), we conclude that $(\mathscr{X}_k, \mathscr{M} \times [-\alpha_k, T_k], (x_k, 0))$ has a sub-sequence which converges locally smoothly (in space-time) to a limit solution $(\mathscr{X}_\infty, \mathscr{M}_\infty \times (-\infty, T_\infty), (x_\infty, 0))$, where $0 < T < \infty$ if the sequence is of type-I and $T = \infty$ if the sequence is of type-II.

Applying Theorem 5.1 we deduce that the limit solutions $\mathscr{X}_\infty$ are weakly convex and, if the underlying flow is $(m+1)$-convex, (strictly) m-convex (unless $\mathscr{X}_\infty$ is a shrinking m-cylinder).

Next, we apply the splitting theorems (Theorems 4.21, 4.23) to deduce that the limit splits as a product of k flat directions (with $k \leq m$ if the flow is $(m+1)$-convex) with a strictly convex solution of the corresponding $(n-k)$-dimensional flow. Finally, for flows by convex speeds, Proposition B.3 implies that $\mathscr{X}_\infty\big|_{\{0\}\times\Sigma^k_\infty\times(-\infty,\infty)}(\Sigma^k_\infty)$ moves by translation, since, by Lemma 5.26 (iv), the maximal value of $|\mathcal{W}_\infty|$ (and hence F) is attained at $(x_\infty, 0)$. $\square$

6. Non-collapsing

Towards the end of Section 4 we studied the extrinsic distance function d, and used its evolution under the flow to prove two useful geometric statements: that initially disjoint solutions remain disjoint, and initially embedded solutions remain embedded under (CF). In this section, we study two new extrinsic quantities, the *interior* and *exterior ball curvatures*. The interior ball curvature is defined at each point of a (compact) hypersurface as the curvature of the largest ball which is enclosed by the hypersurface and touches it at that point. The exterior ball curvature is defined similarly by considering enclosing regions (precise definitions are given in Definition 6.3). We will prove that, under certain concavity conditions on the flow speed, embedded solutions of (CF) preserve ratios of one or both of the ball curvatures to the speed, so long as the latter is positive[1]. Namely, we shall prove the following statements:

Theorem 6.1 (Non-collapsing, Andrews, Langford, and McCoy (2013) and Andrews and Langford (2013))**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be a positive admissible speed function and let $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ be an embedded solution of* (CF)*. Then the following statements hold:*

1. *If F is convex, or if $\Gamma = \Gamma_+$ and F is inverse-concave, then $\mathscr{X}$ is* exterior non-collapsing*; that is,*

$$\underline{k}(x,t) \geq k_0 F(x,t)$$

for all $(x,t) \in \mathscr{M} \times [0,T)$, where $\underline{k}(\cdot,t)$ is the exterior ball curvature of $\mathscr{X}_t$ and $k_0 := \inf_{\mathscr{M}\times\{0\}} \left(\frac{\underline{k}}{F}\right)$.

2. *If F is concave, then $\mathscr{X}$ is* interior non-collapsing*; that is,*

$$\overline{k}(x,t) \leq K_0 F(x,t)\,,$$

for all $(x,t) \in \mathscr{M} \times [0,T)$, where $\overline{k}(\cdot,t)$ is the interior ball curvature of $\mathscr{X}_t$ and $K_0 := \sup_{\mathscr{M}\times\{0\}} \left(\frac{\overline{k}}{F}\right)$.

Remarks 6.1. 1. Recall (Theorem 4.33) that embeddedness is preserved by (CF) if F has an odd extension. In particular, this is the case if its cone of definition lies in the positive mean half-space (which automatically holds if F is concave and positive).

[1] In fact, the speed may be replaced by any positive solution of the linearized flow.

2. Theorem 6.1 also holds, with a suitable modification, for flows in space-forms (see Andrews et al. (2014) and Andrews and Langford (2013)). We omit the proof of this fact here, since the rest of this thesis is concerned with flows in $\mathbb{R}^{n+1}$.

Theorem 6.1 constitutes a generalization of a result of Andrews (2012) and Sheng and Wang (2009) for the mean curvature flow to flows by a large class of non-linear speed functions.

We shall complete the section with some applications of Theorem 6.1, including a new proof that convex hypersurfaces shrink to round points under flows by concave, inverse-concave speeds.

6.1 The interior and exterior ball curvatures

Let us recall the following classical result:

Lemma 6.2 (Jordan–Brouwer Separation Theorem)**.** *Every smooth, connected, properly embedded hypersurface $\mathscr{X} : \mathscr{M}^n \hookrightarrow \mathbb{R}^{n+1}$ separates $\mathbb{R}^{n+1}$ into two regions; that is $\mathbb{R}^{n+1} \setminus \mathscr{X}(\mathscr{M})$ is open and has two connected components; moreover, if $\mathscr{X}(\mathscr{M})$ is compact, then one of the components is pre-compact, and the other is unbounded.*

We now define the interior and exterior ball curvatures:

Definition 6.3 (Interior and exterior ball curvatures)**.** *Let $\mathscr{X} : \mathscr{M}^n \to \mathbb{R}^{n+1}$, $n \geq 1$, be a smooth, connected, proper (but possibly non-compact) hypersurface embedding equipped with a smooth 'outer' unit normal field ν. We shall say that a smooth hypersurface embedding $\mathscr{Y} : \mathscr{N}^n \to \mathbb{R}^{n+1}$ with 'outer' normal μ* touches *$\mathscr{X}$ at $x \in \mathscr{M}$ if $\mathscr{X}(\mathscr{M})$ and $\mathscr{Y}(\mathscr{N})$ agree to first order at $\mathscr{X}(x)$; that is, there is a point $y \in \mathscr{N}$ such that $\mathscr{X}(x) = \mathscr{Y}(y)$ and $\nu(x) = \mu(y)$.*

Let Ω_{int} and $\Omega_{\text{ext}} = \mathbb{R}^{n+1} \setminus \overline{\Omega}_{\text{int}}$ be the open regions separated by the hypersurface $\mathscr{X}(\mathscr{M}^n)$ such that ν points out of Ω_{int}; that is, for each $x \in \mathscr{M}$, $\mathscr{X}(x) - s\nu(x) \in \Omega_{\text{int}}$ for all sufficiently small s. Then the interior ball curvature *of $\mathscr{X}$ is the function $\overline{k} : \mathscr{M} \to \mathbb{R}$ which at a point $x \in \mathscr{M}$ is equal to the boundary curvature of the largest smooth, connected region with totally umbilic boundary which is contained in Ω_{int} and touches $\mathscr{X}$ at x and the* exterior ball curvature *of $\mathscr{X}$ at x is the function $\underline{k} : \mathscr{M} \to \mathbb{R}$ which at each point $x \in \mathscr{M}$ gives the boundary curvature of the largest smooth, connected region with totally umbilic boundary which is contained in Ω_{ext} and touches $\mathscr{X}$ at x.*

Remarks 6.2.

1. The interior and exterior ball curvatures are well-defined since every complete umbilic hypersurface of $\mathbb{R}^{n+1}$ is either a round sphere or a hyperplane (and hence, given any two smooth regions Ω_1 and Ω_2 with touching, umbilic boundaries, either $\overline{\Omega}_1 \subset \overline{\Omega}_2$ or $\overline{\Omega}_2 \subset \overline{\Omega}_1$).

2. Note that the boundary curvature of a touching ball is positive, that of a touching half-space is zero, and that of a touching ball compliment is negative.

The proof of Theorem 6.1 is an application of the maximum principle. The following Proposition gives a useful analytic characterization of $\overline{k}$ and $\underline{k}$:

Proposition 6.4. *Consider the function* $k : \mathscr{M} \times \mathscr{M} \setminus \{(x,x) : x \in \mathscr{M}\} \to \mathbb{R}$ *defined by*

$$k(x,y) := \frac{2\left\langle \mathscr{X}(x) - \mathscr{X}(y)\,,\, \nu(x)\right\rangle}{||\mathscr{X}(x) - \mathscr{X}(y)||^2}\,.$$

Then the interior ball curvature of $\mathscr{X}$ *is given by*

$$\overline{k}(x) = \sup_{y\in\mathscr{M}\setminus\{x\}} k(x,y)$$

and the exterior ball curvature of $\mathscr{X}$ *is given by*

$$\underline{k}(x) = \inf_{y\in\mathscr{M}\setminus\{x\}} k(x,y)\,.$$

Proof. Observe that $\sup_{y\in\mathscr{M}\setminus\{x\}} k(x,y) = \kappa \in \mathbb{R}$ if and only if

$$2\left\langle \mathscr{X}(x) - \mathscr{X}(y), \nu(x)\right\rangle \leq \kappa\, ||\mathscr{X}(x) - \mathscr{X}(y)||^2 \quad \text{for all } y \in \mathscr{M} \setminus \{x\} \tag{6.1}$$

and there is no smaller number satisfying (6.1).

Suppose that (6.1) holds for some $\kappa \in \mathbb{R}$. If $\kappa > 0$, then (6.1) is equivalent to

$$\left|\left|\big(\mathscr{X}(x) - \kappa^{-1}\nu(x)\big) - \mathscr{X}(y)\right|\right| \geq \kappa^{-1} \qquad \text{for all } y \in \mathscr{M} \setminus \{x\}\,;$$

that is, the ball of boundary curvature κ centred at $\mathscr{X}(x) - \kappa^{-1}\nu(x)$ is contained in Ω_1. Therefore, $\sup_{y\in\mathscr{M}\setminus\{x\}} k(x,y) \leq \kappa$, with $\kappa > 0$, if and only if there is a ball of boundary curvature κ contained in Ω_1 whose boundary contains $\mathscr{X}(x)$.

If $\kappa < 0$, then (6.1) is equivalent to

$$\left|\left|\big(\mathscr{X}(x) - \kappa^{-1}\nu(x)\big) - \mathscr{X}(y)\right|\right| \leq -\kappa^{-1} \qquad \text{for all } y \in \mathscr{M} \setminus \{x\}\,;$$

that is, the complement of the ball of boundary curvature κ centred at $\mathscr{X}(x) - \kappa^{-1}\nu(x)$ is contained in Ω_1. Therefore, $\overline{k} \leq \kappa < 0$, if and only if there is a ball compliment of boundary curvature κ contained in Ω_1 and whose boundary contains $\mathscr{X}(x)$.

Finally, if $\kappa = 0$, then

$$2\left\langle \mathscr{X}(x) - \mathscr{X}(y), \nu(x)\right\rangle \leq 0 \quad \text{for all } y \in \mathscr{M} \setminus \{x\}\,,$$

which implies that the half-space $L := \{Y \in \mathbb{R}^{n+1} : \langle \mathscr{X}(x) - Y\nu(x)\rangle > 0\}$ is contained in Ω_1. Working backwards, we see that the converse also holds. Therefore $\overline{k}(x) \leq 0$ if and only if there is a half-space contained in Ω_1 whose boundary contains $\mathscr{X}(x)$.

The first claim now follows. The proof of the second claim is the same. $\square$

The function k is also closely related to the principal curvatures of the hypersurface:

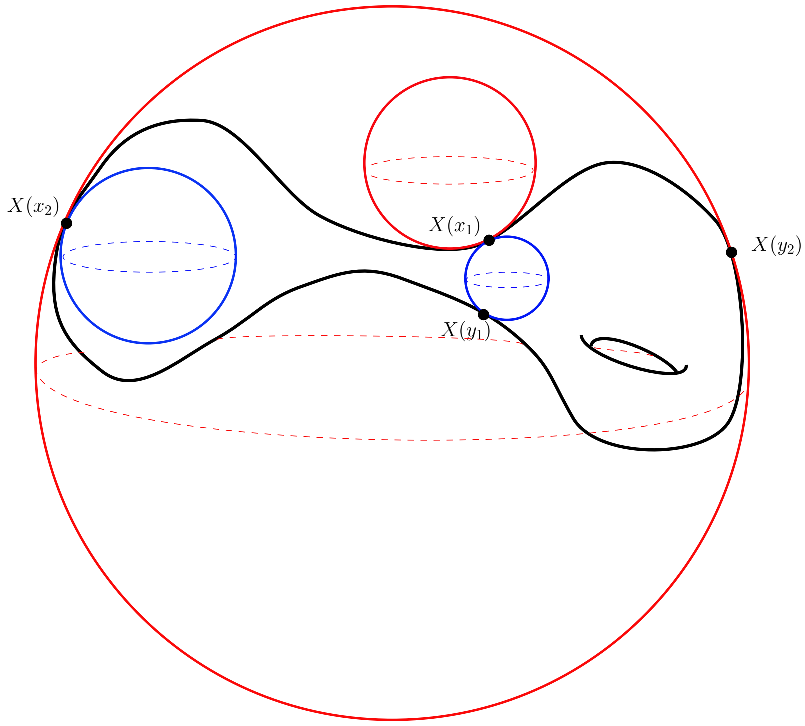

Figure 6.1: Exterior and interior ball curvatures. At x_1, we have $\overline{k}(x_1) = k(x_1, y_1)$ and $\underline{k}(x_1) = \kappa_1(x_1) < 0$. At x_2, we have $\overline{k}(x_2) = \kappa_n(x_2)$ and $\underline{k}(x_1) = k(x_2, y_2) > 0$.

Proposition 6.5. *Let $\mathscr{X} : \mathscr{M}^n \to \mathbb{R}^{n+1}$ be a proper embedding and $\gamma : (-s_0, s_0) \to \mathscr{M}$ any regular curve. Then*

$$\lim_{s\to 0} k(x, \gamma(s)) = \frac{\mathcal{W}_x(v, v)}{g_x(v, v)}\,,$$

where $\gamma(0) = x$ and $\gamma'(0) = v$. In particular, $\overline{k}(x) \geq \kappa_n(x)$ and $\underline{k}(x) \leq \kappa_1(x)$, and $\overline{k}$ and $\underline{k}$ are both bounded on any smooth, compact embedded hypersurface.

Proof. By definition, we have

$$k(x, \gamma(s)) = \frac{2\,\langle \mathscr{X}(x) - \mathscr{X}(\gamma(s))\,,\, \nu(x)\rangle}{||\mathscr{X}(x) - \mathscr{X}(\gamma(s))||^2}\,.$$

Since $\mathscr{X}$ is an embedding, the extrinsic distance is comparable to the intrinsic distance, so that the denominator is comparable to $s^2|\gamma'(0)|^2$ for small s; in fact, setting $d(s) := ||\mathscr{X}(x) - \mathscr{X}(\gamma(s))||$, we easily compute

$$d^2(s) = s^2|\gamma'(0)|^2 + \mathcal{O}(s^3)\,.$$

Next, observe that the numerator, $f(s) := 2\,\langle \mathscr{X}(x) - \mathscr{X}(\gamma(s))\,,\, \nu(x)\rangle$, expands as

$$f(s) = s^2\mathcal{W}(\gamma'(0), \gamma'(0)) + \mathcal{O}(s^3)\,.$$

Thus,

$$k(x,\gamma(s)) = \frac{s^2\mathcal{W}(\gamma'(0),\gamma'(0)) + \mathcal{O}(s^3)}{s^2|\gamma'(0)|^2 + \mathcal{O}(s^3)} = \frac{\mathcal{W}(\gamma'(0),\gamma'(0)) + \mathcal{O}(s)}{|\gamma'(0)|^2 + \mathcal{O}(s)}\,.$$

The claim follows. □

Next, we show that, for strictly convex hypersurfaces, the interior (resp. exterior) ball curvature and the largest (resp. smallest) principal curvature must agree at a point where the former is maximized (resp. minimized). This will be used in §6.6

Proposition 6.6. *Let $\mathscr{X} : \mathscr{M}^n \to \mathbb{R}^{n+1}$ be a strictly convex proper embedding.*

1. *Suppose that $\sup_{\mathscr{M}} \overline{k}$ is attained at a point $x_0 \in \mathscr{M}$, then $\overline{k}(x_0) = \kappa_n(x_0)$.*
2. *Suppose that $\inf_{\mathscr{M}} \underline{k}$ is attained at a point $x_0 \in \mathscr{M}$, then $\underline{k}(x_0) = \kappa_1(x_0)$.*

Proof. To prove the first claim, set $\kappa := \sup_{\mathscr{M}} \overline{k} = \sup_{\mathscr{M}\times\mathscr{M}\setminus D} k$ and suppose, to the contrary, that there exists a point $(x_0, y_0) \in \mathscr{M} \times \mathscr{M} \setminus D$ such that $\kappa_n(x_0) < k(x_0, y_0) = \overline{k}(x_0) = \kappa$. We claim that the tangent planes $\mathscr{X}_*T_{x_0}\mathscr{M}$ and $\mathscr{X}_*T_{y_0}\mathscr{M}$ must be parallel. To prove this, observe that, for any $v \in T_{x_0}\mathscr{M}$,

$$0 = \nabla_v\, k(x_0, y_0) = \frac{2\,\langle \mathscr{X}(x_0) - \mathscr{X}(y_0)\,,\,(\mathcal{W}_{x_0} - \kappa I)(v)\rangle}{||\mathscr{X}(x_0) - \mathscr{X}(y_0)||^2}\,.$$

Since $\mathcal{W}_{x_0} < \kappa I$, we find $\mathscr{X}(x_0) - \mathscr{X}(y_0) = ||\mathscr{X}(x_0) - \mathscr{X}(y_0)||\,\nu(x_0)$. The claim follows, since $\mathscr{X}_*T_{y_0}\mathscr{M}$ is tangent to the ball $B := B_{1/\kappa}(\mathscr{X}(x_0) - \nu(x_0)/\kappa)$.

Next, we claim that $\overline{k}$ is constant on $\mathscr{M}$; in fact, since $\mathscr{X}$ is a proper, convex embedding, Ω_{int} is a convex region, which therefore lies between the planes $\mathscr{X}_*T_{x_0}\mathscr{M}$ and $\mathscr{X}_*T_{y_0}\mathscr{M}$. But this implies that every ball contained in Ω_{int} has boundary curvature no less than κ, which, since $\kappa = \sup_{\mathscr{M}} \overline{k}$, implies the claim.

Finally, we claim that $\mathscr{X}(\mathscr{M}) = \partial\, B$, contradicting the assumption $\kappa_n(x_0) < \kappa$. So suppose, to the contrary, that there is a point $z \in \mathscr{X}(\mathscr{M}) \setminus \partial\, B$. Then, since $\overline{k}$ is constant, the largest interior ball touching $\mathscr{X}$ at $\mathscr{X}(z)$ must touch $\mathscr{X}_*T_{x_0}\mathscr{M}$, (at z', say). But this point must lie on $\mathscr{X}(\mathscr{M})$, since $\mathscr{X}(\mathscr{M})$ lies in between $T_{x_0}\mathscr{M}$ and Ω_{int}. It follows from convexity that the line joining $\mathscr{X}(z')$ and $\mathscr{X}(x_0)$ lies in $\mathscr{X}(\mathscr{M})$, which contradicts strict convexity of $\mathscr{X}$.

To prove the second claim, set $\kappa := \inf_{\mathscr{M}} \underline{k} = \inf_{\mathscr{M}\times\mathscr{M}\setminus D} k$ and suppose, to the contrary, that there exists a point $(x_0, y_0) \in \mathscr{M}\times\mathscr{M}\setminus D$ such that $\kappa_1(x_0) > k(x_0, y_0) = \kappa$. Note that $\kappa > 0$, since, otherwise (by convexity of Ω_{int}), the line joining $\mathscr{X}(x_0)$ and $\mathscr{X}(y_0)$ would be contained in $\mathscr{X}(\mathscr{M}) = \partial\,\Omega_{\mathrm{int}}$, contradicting strict convexity. In particular, this implies that $\mathscr{X}(\mathscr{M})$ is compact, since it lies inside the closure of the ball $B := B_{1/\kappa}(\mathscr{X}(x_0) - \nu(x_0)/\kappa)$. Next, we note that, for similar reasons as above, the tangent planes $\mathscr{X}_*T_{x_0}\mathscr{M}$ and $\mathscr{X}_*T_{y_0}\mathscr{M}$ must be parallel. But this implies that $\mathrm{diam}(\mathscr{X}(\mathscr{M})) = ||\mathscr{X}(x_0) - \mathscr{X}(y_0)|| = 2/\kappa$. Thus, every other enclosing ball must have diameter at least $\mathrm{diam}(\mathscr{X}(\mathscr{M}))$, and hence curvature at most κ. But, since $\kappa = \inf \underline{k}$, it follows that $\underline{k} \equiv \kappa$. But this implies that $\mathscr{X}(\mathscr{M}) = \partial\, B$, since the only closed ball of boundary curvature κ which contains $\mathscr{X}(x_0)$ and $\mathscr{X}(y_0)$ is $\overline{B}$. This contradicts the assumption $\kappa_1(x_0) > \kappa$, proving the claim.

$\square$

To prove Theorem 6.1, we will show, by using the smooth quantity k as an upper (lower) support function, that $\overline{k}$ ($\underline{k}$) is a subsolution (supersolution) of the linearized flow (LF). Unfortunately, k is defined on a non-compact set. Motivated by Proposition 6.5, we will show that, in the case that $\mathscr{M}$ is compact, k extends naturally to a continuous function on a suitable compactification of $\mathscr{M} \times \mathscr{M} \setminus \{(x,x) : x \in \mathscr{M}\}$: The diagonal, $D := \{(x,x) : x \in \mathscr{M}\}$, is a (compact) submanifold of $\mathscr{M} \times \mathscr{M}$ of dimension and codimension n. The normal space $N_{(x,x)}D$ of D at (x,x) is the n-dimensional subspace $\{(u,-u) :\ u \in T_x\mathscr{M}\}$ of $T_{(x,x)}(\mathscr{M} \times \mathscr{M}) \cong T_x\mathscr{M} \times T_x\mathscr{M}$. The tubular neighbourhood theorem ensures that there is some $r > 0$ such that the exponential map is a diffeomorphism on $\{((x,u),(x,-u)) \in T(\mathscr{M} \times \mathscr{M}) :\ 0 < |u| < r\}$. We 'blow-up' along D to define a manifold with boundary $\widehat{\mathscr{M}}$ which compactifies $(\mathscr{M} \times \mathscr{M}) \setminus D$ as follows: As a set, $\widehat{\mathscr{M}}$ is defined as the disjoint union of $(\mathscr{M} \times \mathscr{M}) \setminus D$ with the unit tangent bundle $S\mathscr{M} = \{(x,v) \in T\mathscr{M} : |v| = 1\}$. The manifold-with-boundary structure is defined by the atlas generated by all charts for $(\mathscr{M} \times \mathscr{M}) \setminus D$, together with the charts $\widehat{Y}$ from $S\mathscr{M} \times (0,r)$ defined by taking a chart Y for $S\mathscr{M}$, and setting $\widehat{Y}(z,s) := \big(\exp(sY(z)), \exp(-sY(z))\big)$. The extension of k to $\widehat{\mathscr{M}} \times [0,T)$ is then defined by setting

$$k(x,y) := \mathcal{W}_x(y,y)$$

for every $(x,y) \in S\mathscr{M} = \partial\widehat{\mathscr{M}}$.

Lemma 6.7. *The extension $k : \widehat{\mathscr{M}} \to \mathbb{R}$ is continuous.*

Proof. Continuity clearly holds away from $\partial\widehat{\mathscr{M}}$. The proof of continuity at $\partial\widehat{\mathscr{M}}$ is similar to the proof of Proposition 6.5. $\square$

6.2 Evolution of the ball curvatures under the flow

We now investigate how motion of the embedding affects the interior and exterior ball curvatures. So let $\mathscr{X} : \mathscr{M}^n \times (0,T) \to \mathbb{R}^{n+1}$ be a smooth family of smooth embeddings with 'outer' unit normal field ν. Then we can define an interior and exterior ball curvature of the embedding at each time; that is, we define the *interior ball curvature*, $\overline{k}$, of $\mathscr{X}$ at $(x,t) \in \mathscr{M} \times (0,T)$ as the boundary curvature of the largest region with totally umbilic boundary contained in Ω_t, and the *exterior ball curvature*, $\underline{k}$, of $\mathscr{X}$ at $(x,t) \in \mathscr{M} \times (0,T)$ as the boundary curvature of the largest region with totally umbilic boundary contained in $\mathbb{R}^{n+1} \setminus \overline{\Omega}_t$, where Ω_t is the open region in $\mathbb{R}^{n+1}$ with boundary $\mathscr{X}_t(\mathscr{M})$ that $\nu(\cdot,t)$ points out of. It then follows from Proposition 6.4 that

$$\overline{k}(x,t) = \sup_{y \in \mathscr{M} \setminus \{x\}} k(x,y,t)$$

and

$$\underline{k}(x,t) = \inf_{y \in \mathscr{M} \setminus \{x\}} k(x,y,t)\,,$$

where

$$k(x,y,t) := \frac{2\left\langle \mathscr{X}(x,t) - \mathscr{X}(y,t)\,,\, \nu(x,t)\right\rangle}{||\mathscr{X}(x,t) - \mathscr{X}(y,t)||^2}$$

if $(x,y,t) \in \operatorname{int}\left(\widehat{\mathscr{M}} \times \{t\}\right) = (\mathscr{M} \times \mathscr{M}) \setminus D \times \{t\}$ and

$$k(x,y,t) := \mathcal{W}_{(x,t)}(y,y)$$

if $(x,t,y) \in S_{(x,t)}\mathscr{M} = \partial\left(\widehat{\mathscr{M}} \times \{t\}\right)$.

Just as for the proof of the avoidance principle, we shall need to compute derivatives, up to second order, over the product $\mathscr{M} \times \mathscr{M}$. We note, however, that the computation here has an important difference: Previously, the two points 'x' and 'y' have appeared in a symmetric way, so that the choice of coefficients of the highest order term was necessarily determined by information at both points. However, in the present situation, x and y play different roles. Accordingly, we are able to make a choice of coefficients in the second derivatives which depends on x but not on y, thus removing any need to compare the curvature at different points. We therefore consider operators of the form

$$\widehat{\mathscr{L}} := \dot{F}_x^{kl}\, \nabla_{\partial_{x^k} + \Lambda_k{}^p\,\partial_{y^p}} \nabla_{\partial_{x^l} + \Lambda_l{}^p\,\partial_{y^p}}\ .$$

where Λ is an arbitrary matrix. We shall compute the relevant derivatives working in local normal coordinates $\{x^i\}$ near x_0 and $\{y^i\}$ near $y_0 \neq x_0$. As in the previous computations over $\mathscr{M} \times \mathscr{M}$, we define

$$d(x,y,t) := |\mathscr{X}(x,t) - \mathscr{X}(y,t)|\,; \quad w(x,y,t) := \frac{\mathscr{X}(x,t) - \mathscr{X}(y,t)}{d}\,,$$

and

$$\partial_i^x = \frac{\partial \mathscr{X}}{\partial x^i}\,; \quad \partial_i^y = \frac{\partial \mathscr{X}}{\partial y^i}\,,$$

and use sub- and super-scripts x and y to denote, respectively, pullback by the projections onto the first and second factors of the product $\mathscr{M} \times \mathscr{M}$; for example: $F_x(\xi,\eta,\tau) = F(\xi,\tau)$. With these notations in place, we find

$$\nabla_{\partial_{x^i} + \Lambda_i{}^p\,\partial_{y^p}} k = \frac{2}{d^2}\left(\left\langle \partial_i^x - \Lambda_i{}^p\,\partial_p^y\,,\, \nu_x - kdw\right\rangle + \left\langle dw\,,\, \mathcal{W}^x(\partial_{x^i})\right\rangle\right)$$

Since $\overline{k}$ and $\underline{k}$ are defined by taking extrema over the second factor, we only need to compute the derivatives of k at such an extremum. Observe that the vanishing of the

y-derivatives at an 'off-diagonal' extremum $y \in \mathscr{M} \setminus \{x\}$ of $k(x, \cdot, t)$ determines the tangent plane at y:

Lemma 6.8. *Suppose that the point (x_0, y_0) is an off-diagonal extremum of k at time t_0; That is, $y_0 \in \mathscr{M} \setminus \{x_0\}$ is an extremum of $k(x_0, \cdot, t_0)$. Then*

$$\nu_y = \nu_x - dkw$$

at (x_0, y_0, t_0).

Proof of Lemma 6.8. By Proposition 6.4, there is an interior ball B of radius $1/k$ touching at $\mathscr{X}(x_0, t_0)$ and $\mathscr{X}(y_0, t_0)$. The outward normals to B at these points agree with the outward normals to the hypersurface $\mathscr{X}(\mathscr{M}, t_0)$. In particular,

$$\begin{aligned}\nu(y_0, t_0) &= k(x_0, y_0, t_0)\left(\mathscr{X}(y_0, t_0) - \left(\mathscr{X}(x_0, t_0) - \frac{1}{k(x_0, y_0, t_0)}\nu(x_0, t_0)\right)\right) \\ &= (\nu_x - dkw)|_{(x_0, y_0, t_0)} .\end{aligned}$$

□

We now compute the second derivatives:

$$\begin{aligned}&\nabla_{\partial_{x^j} + \Lambda_j{}^p\, \partial_{y^p}} \nabla_{\partial_{x^i} + \Lambda_i{}^p\, \partial_{y^p}} k \\ &\quad = \frac{2}{d^2}\Big\{ \langle -\mathcal{W}^x{}_{ij}\nu_x + \Lambda_i{}^p\Lambda_j{}^q\mathcal{W}^y{}_{pq}\nu_y \,,\, \nu_x - kdw\rangle + \left\langle \partial^x_i - \Lambda_i{}^p\, \partial^y_p \,,\, \mathcal{W}^x{}_j{}^q\, \partial^x_q \right\rangle \\ &\qquad - \nabla_{\partial_{x^j} + \Lambda_j{}^q\, \partial_{y^q}} k \left\langle \partial^x_i - \Lambda_i{}^p\, \partial^y_p \,,\, dw\right\rangle - k\left\langle \partial^x_i - \Lambda_i{}^p\, \partial^y_p \,,\, \partial^x_j - \Lambda_j{}^q\, \partial^y_q \right\rangle \\ &\qquad + \left\langle \partial^x_j - \Lambda_j{}^q\, \partial^y_q \,,\, \mathcal{W}^x{}_i{}^p\, \partial^x_p \right\rangle + \left\langle dw \,,\, \nabla\, \mathcal{W}^x{}_{ij} - (\mathcal{W}^x)^2_{ij}\nu_x \right\rangle \\ &\qquad - \nabla_{\partial_{x^i} + \Lambda_i{}^p\, \partial_{y^p}} k \left\langle \partial^x_j - \Lambda_j{}^q\, \partial^y_q \,,\, dw\right\rangle \Big\} .\end{aligned}$$

The time derivative is

$$\partial_t\, k = \frac{2}{d^2}\left(\langle -F_x\nu_x + F_y\nu_y \,,\, \nu_x - kdw\rangle + \langle dw \,,\, \nabla\, F_x\rangle\right) .$$

At an off-diagonal extremum (x_0, y_0, t_0) we obtain

$$\begin{aligned}\left(\partial_t - \widehat{\mathscr{L}}\right) k = k\dot{F}_x(\mathcal{W}^2_x) + \frac{2}{d^2}\Big\{&(F_y - \dot{F}^{ij}_x\Lambda_i{}^p\Lambda_j{}^q\mathcal{W}^y{}_{pq})\,\langle \nu_y \,,\, \nu_x - kdw\rangle \\ &+ k\dot{F}^{ij}_x \left\langle \partial^x_i - \Lambda_i{}^p\, \partial^y_p \,,\, \partial^x_j - \Lambda_j{}^q\, \partial^y_q \right\rangle - 2\dot{F}^{ij}_x \left\langle \partial^x_j - \Lambda_j{}^q\, \partial^y_q \,,\, \mathcal{W}^x{}_i{}^p\, \partial^x_p \right\rangle \\ &+ 2\dot{F}^{ij}_x \left(\nabla_{\partial_{x^i} + \Lambda_i{}^p\, \partial_{y^p}} k\right) \left\langle \partial^x_j - \Lambda_j{}^q\, \partial^y_q \,,\, dw\right\rangle \Big\} .\end{aligned}$$

Since the tangent plane at y_0 is the reflection of the tangent plane at x_0, we may choose the orthonormal basis at y_0 to be the reflection of the one at x_0; that is, we may choose

$$\partial^y_i = \partial^x_i - 2\,\langle \partial^x_i \,,\, w\rangle\, w .$$

Moreover, we have $\partial_{y^i} k = 0$ at (x_0, y_0, t_0). Making use of these observations, we obtain

$$\left(\partial_t - \widehat{\mathscr{L}}\right) k = \dot{F}_x(\mathcal{W}_x^2)k + 2\dot{F}_x^{ij}\, \partial_i\, k(\mathcal{W}^x - kI)^{-1}{}_j{}^p(\partial_p\, k) + \frac{2}{d^2}\Big\{F_y - F_x + F_x^{ij}\Big[(k\delta_{ij} - \mathcal{W}^x{}_{ij}) - 2\Lambda_i{}^p(k\delta_{pj} - \mathcal{W}^x{}_{pj}) + \Lambda_i{}^p\Lambda_j{}^q(k\delta_{pq} - \mathcal{W}^y{}_{pq})\Big]\Big\} \tag{6.2}$$

at an off-diagonal extremum.

Observe that the first term on the right is exactly the reaction term appearing in the linearized flow. Moreover, the gradient term is negative at an off-diagonal y-maximum of k, where $k = \overline{k} > \kappa_n$, and positive at an off-diagonal y-minimum, where $k = \underline{k} < \kappa_1$ (note that equation (6.2) only holds at interior points). We will now show that, under certain conditions, the final term can also be controlled. The boundary case is more direct, since we are able to use κ_n (κ_1) as a support function for $\overline{k}$ ($\underline{k}$).

6.3 Flows by convex speed functions

We first consider flows by convex speed functions. We will prove that solutions are exterior non-collapsing. This is a direct consequence of the following statement:

Theorem 6.9. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be a convex admissible speed function. Then the exterior ball curvature $\underline{k}$ of any embedded solution of the flow* (CF) *is a viscosity supersolution of the linearized flow* (LF)*.*

Before proving Theorem 6.9, we note that it implies the desired non-collapsing estimate:

Corollary 6.10 (Exterior non-collapsing for flows by convex speed functions)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be a convex admissible speed function. Then every embedded solution $\mathscr{X} : \mathscr{M} \times [0, T) \to \mathbb{R}^{n+1}$ of CF is* exterior non-collapsing*; that is,*

$$\underline{k} \geq k_0 F\,,$$

where $\underline{k}$ is the exterior ball curvature of $\mathscr{X}$ and $k_0 := \inf_{\mathscr{M}\times\{0\}}\left(\frac{\underline{k}}{F}\right)$.

Remark 6.1. Of course, we may replace the speed function by any positive subsolution of (LF).

Proof of Corollary 6.10. The claim follows immediately from the maximum principle (see, for example, Da Lio (2004), or the direct argument of Andrews, Langford, and McCoy (2013)).

□

We now prove Theorem 6.9.

Proof of Theorem 6.9. Consider, for an arbitrary point $(x_0, t_0) \in \mathscr{M} \times [0, T)$, an arbitrary lower support function ϕ for $\underline{k}$; that is, ϕ is C^2 on a neighbourhood of (x_0, t_0) in $\mathscr{M} \times [0, t_0]$, and lies below $\underline{k}$ with equality at (x_0, t_0). Then we need to prove the differential inequality for ϕ at (x_0, t_0).

Observe that, for all x close to x_0 and all $t \le t_0$ close to t_0, we have $k(x, y, t) \ge \underline{k}(x, t) \ge \phi(x, t)$ for each $y \ne x$ in $\mathscr{M}$ and $k(x, \xi, t) \ge \underline{k}(x, t) \ge \phi(x, t)$ for all $(x, t, \xi) \in S_{(x,t)}\mathscr{M}$. Furthermore, equality $\underline{k} = \phi$ holds when $(x, t) = (x_0, t_0)$.

Now, by the definition of $\underline{k}$, we may divide the proof into the following dichotomy: Either we have $k(x_0, y_0, t_0) = \underline{k}(x_0, t_0) < \kappa_1(x_0, t_0)$ for some $y_0 \ne x_0$, or we have $k(x_0, \xi_0, t_0) = \underline{k}(x_0, t_0) = \mathcal{W}_{(x_0,t_0)}(\xi_0, \xi_0) = \kappa_1(x_0, t_0)$ for some $(x_0, t_0, \xi_0) \in S_{(x_0,t_0)}\mathscr{M}$.

We consider the latter case first:

The boundary case

Suppose that the infimum is attained on the diagonal; that is, $\underline{k}(x_0, t_0) = \kappa_1(x_0, t_0)$. Since $\underline{k} \le \kappa_1$, ϕ is a lower support for κ_1 at (x_0, t_0). But recall (see 4.19 in Remark 4.6) that κ_1 satisfies

$$\begin{aligned}(\partial_t - \mathscr{L})\kappa_1 \ge{}& |\mathcal{W}|_F^2 \kappa_1 + \ddot{F}(\nabla_1 \mathcal{W}, \nabla_1 \mathcal{W}) \\ &+ 2 \sup_{\Lambda_{k1}=0} \dot{F}^k \left[2\Lambda_k{}^p \nabla_1 \mathcal{W}_{kp} - (\Lambda_k{}^p)^2(\kappa_p - \kappa_1)\right]\end{aligned}$$

in the viscosity sense. Thus,

$$\begin{aligned}(\partial_t - \mathscr{L})\phi \ge{}& |\mathcal{W}|_F^2 \phi + \ddot{F}(\nabla_1 \mathcal{W}, \nabla_1 \mathcal{W}) \\ &+ 2 \sup_{\Lambda_{k1}=0} \dot{F}^k \left[2\Lambda_k{}^p \nabla_1 \mathcal{W}_{kp} - (\Lambda_k{}^p)^2(\kappa_p - \phi)\right] .\end{aligned}$$

The claim now follows from convexity of F (take, say, $\Lambda = 0$).

The interior case

Next consider the case that the infimum is not attained on the diagonal. Then $\phi(x_0, t_0) = \underline{k}(x_0, t_0) = k(x_0, y_0, t_0) < \kappa_1(x_0, t_0)$ for some $y_0 \ne x_0$, and $k(x, y, t) \ge \underline{k}(x, t) \ge \phi(x, t)$ for all points x near x_0, times $t \le t_0$ near t_0, and all $y \in \mathscr{M} \setminus \{x\}$. This implies that $\frac{\partial \phi}{\partial t}(x_0, t_0) \ge \frac{\partial k}{\partial t}(x_0, y_0, t_0)$, that the gradient of $\phi - k$ on $\mathscr{M} \times \mathscr{M}$ vanishes at (x_0, y_0, t_0) and that the Hessian of $\phi - k$ on $\mathscr{M} \times \mathscr{M}$ is non-positive definite at (x_0, y_0, t_0). Thus, from (6.2), we obtain

$$\begin{aligned}0 \ge{}& (\partial_t - \widehat{\mathscr{L}})(k - \phi) \\ ={}& -(\partial_t - \mathscr{L})\phi + \dot{F}(\mathcal{W}^2)k + 2\dot{F}_x^{ij}\,\partial_i k(\mathcal{W}^x - kI)^{-1}{}_j{}^p(\partial_p k) + \frac{2}{d^2}\Big\{F_y - F_x \\ &+ F_x^{ij}\Big[(k\delta_{ij} - \mathcal{W}^x{}_{ij}) - 2\Lambda_i{}^p(k\delta_{pj} - \mathcal{W}^x{}_{pj}) + \Lambda_i{}^p\Lambda_j{}^q(k\delta_{pq} - \mathcal{W}^y{}_{pq})\Big]\Big\} .\end{aligned}$$

If we simply choose Λ to be the identity matrix, then

$$\begin{aligned}(\partial_t - \mathscr{L})\phi \geq \dot{F}(\mathcal{W}^2)k + 2\dot{F}_x^{ij}\,\partial_i\, k(\mathcal{W}^x - kI)^{-1}{}_j{}^p(\partial_p\, k) \\ + \frac{2}{d^2}\left\{F_y - F_x - F_x^{ij}\left[\mathcal{W}^y{}_{ij} - \mathcal{W}^x{}_{ij}\right]\right\}.\end{aligned}$$

Convexity of F implies

$$F(B) \geq F(A) + \dot{F}_A\,(B - A) = F(A) + \dot{F}_A(B) - \dot{F}_A(A) \tag{6.3}$$

for any A, $B \in \mathrm{Sym}(n)$ with eigenvalue n-tuples in Γ. So the term on the second line is non-positive. Moreover, since $\kappa_1 > \underline{k}$ when the supremum is not attained on the diagonal, the operator $(\mathcal{W}^x - kI)^{-1}$ is positive definite at (x_0, y_0, t_0), so that and the gradient term is non-negative. Putting this together, we arrive at

$$(\partial_t - \mathscr{L})\phi \geq \dot{F}(\mathcal{W}^2)\phi + 2\dot{F}_x^{ij}\,\partial_i\, k(\mathcal{W}^x - \phi I)^{-1}{}_j{}^p(\partial_p\, k) \geq \dot{F}(\mathcal{W}^2)\phi\,. \tag{6.4}$$

as required.

This completes the proof of Theorem 6.9. □

6.4 Flows by concave speed functions

Next, we consider flows by concave speed functions. We will prove that solutions are interior non-collapsing. This follows from the following statement:

Theorem 6.11. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be a concave admissible speed function. Then the interior ball curvature of any embedded solution of the flow* (CF) *is a viscosity subsolution of the linearized flow* (LF)*.*

Corollary 6.12 (Interior non-collapsing for flows by concave speed functions)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be a concave admissible speed function for the flow. Then every embedded solution $\mathscr{X} : \mathscr{M} \times [0, T) \to \mathbb{R}^{n+1}$ of CF is* interior non-collapsing*; that is,*

$$\overline{k} \leq K_0 F\,,$$

where $\overline{k}$ is the interior ball curvature of $\mathscr{X}$ and $K_0 : \sup_{\mathscr{M}\times\{0\}}\left(\frac{\overline{k}}{F}\right)$.

Remark 6.2. Of course, we may replace the speed function by any positive supersolution of (LF).

Proof of Theorem 6.11. The proof is similar to the proof of Theorem 6.9: Note that the viscosity inequalities are reversed and now $\overline{k} \geq \kappa_n$ (whereas before we had $\underline{k} \leq \kappa_1$) with strict inequality in the interior case. Thus, to prove the boundary case, we apply the subsolution property (4.20) and concavity of F. To prove the interior case, we replace the inequality (6.3) with the corresponding inequality for concave functions. □

6.5 Flows by inverse-concave speed functions

Finally, we consider flows by concave speed functions. We will show, by modifying the proof of Theorem 6.9, that solutions are exterior non-collapsing.

Theorem 6.13. *Let $F:\Gamma_+\subset\mathbb{R}^n\to\mathbb{R}$, $n\geq 1$, be an inverse-concave admissible speed function. Then the exterior ball curvature of any embedded solution of the flow* (CF) *is a viscosity supersolution of the linearized flow* (LF)*.*

Corollary 6.14 (Exterior non-collapsing for flows by inverse-concave speed functions)**.** *Let $F:\Gamma_+\subset\mathbb{R}^n\to\mathbb{R}$, $n\geq 1$, be an inverse-concave admissible speed function. Then every embedded solution $\mathscr{X}:\mathscr{M}\times[0,T)\to\mathbb{R}^{n+1}$ of CF is* exterior non-collapsing*; that is,*

$$\underline{k}\geq k_0 F\,,$$

where $\underline{k}$ is the exterior ball curvature of $\mathscr{X}$ and $0<k_0:=\inf_{\mathscr{M}\times\{0\}}\left(\frac{\underline{k}}{F}\right)$.

Remark 6.3. Of course, we may replace the speed function by any positive subsolution of (LF).

Proof of Theorem 6.13. We consider first the boundary case.

The boundary case

Let (x_0,t_0) be any point in $\mathscr{M}\times[0,T)$ such that $\underline{k}(x_0,t_0)=\kappa_1(x_0,t_0)$ and let ϕ be a lower support function for $\underline{k}$ at (x_0,t_0). Then, just as in the proof of Theorem 6.9, it suffices to show that

$$\ddot{F}(\nabla_1\mathcal{W},\nabla_1\mathcal{W})+2\sup_{\Lambda_{k1}=0}\dot{F}^k\left[2\Lambda_k{}^p\nabla_1\mathcal{W}_{kp}-(\Lambda_k{}^p)^2(\kappa_p-\phi)\right]\geq 0$$

at (x_0,t_0).

First, we make use of an observation of Brendle:

Lemma 6.15 (Brendle (2013b), Proposition 8)**.** *Let $\mathscr{X}:\mathscr{M}^n\to\mathbb{R}^{n+1}$ be a properly embedded hypersurface and suppose that $\underline{k}=\kappa_1$ at a point $x_0\in\mathscr{M}$. Then $\nabla_1\mathcal{W}_{11}$ vanishes at x_0, where e_1 is a principal direction of κ_1.*

Proof of Lemma 6.15. Consider the function $Z:\mathscr{M}\times\mathscr{M}\to\mathbb{R}$ defined by

$$Z(x,y):=2\left\langle\mathscr{X}(x)-\mathscr{X}(y)\,,\,\nu(x)\right\rangle-\kappa_1(x)\left|\left|\mathscr{X}(x)-\mathscr{X}(y)\right|\right|^2\,.$$

Note that, for each $x\in\mathscr{M}$, $Z(x,y)\geq 0$ for all $y\in\mathscr{M}$ with equality at $y=x$. Let $\gamma:\mathbb{R}\to\mathscr{M}$ be the geodesic defined by $\gamma(s):=\exp_{x_0}se_1$. We consider the Taylor expansion of $f(s):=Z(x_0,\gamma(s))$. First note that $f(0)=0$. Next, we compute

$$f'=-2\left\langle\mathscr{X}_*\gamma'\,,\,\nu(x_0)\right\rangle+2\kappa_1(x_0)\left\langle\mathscr{X}_*\gamma'\,,\,\mathscr{X}(x_0)-\mathscr{X}(\gamma)\right\rangle\,.$$

In particular, $f'(0) = 0$. Next, we compute

$$f'' = 2\left\langle \mathcal{W}(\gamma',\gamma')\nu(\gamma)\,,\,\nu(x_0)\right\rangle - 2\kappa_1(x_0)\left\langle \mathcal{W}(\gamma',\gamma')\nu(\gamma)\,,\,\mathscr{X}(x_0) - \mathscr{X}(\gamma)\right\rangle - 2\kappa_1(x_0)\,.$$

Thus, $f''(0) = 0$. Finally, we compute

$$\begin{aligned} f''' &= 2\left\langle \nabla_{\gamma'}\mathcal{W}(\gamma',\gamma')\nu(\gamma) + \mathcal{W}(\gamma',\gamma')\mathcal{W}(\gamma')\,,\,\nu(x_0)\right\rangle \\ &\quad - 2\kappa_1(x_0)\left\langle \nabla_{\gamma'}\mathcal{W}(\gamma',\gamma')\nu(\gamma) + \mathcal{W}(\gamma',\gamma')\mathcal{W}(\gamma')\,,\,\mathscr{X}(x_0) - \mathscr{X}(\gamma)\right\rangle\,. \end{aligned}$$

Thus, $f'''(0) = 2\nabla_1\mathcal{W}_{11}\big|_{x_0}$. We conclude that

$$f(s) = \frac{s^3}{3}\nabla_1\mathcal{W}_{11}\big|_{x_0} + \mathcal{O}(s^4)\,.$$

Since $f \geq 0$, it follows that $\nabla_1\mathcal{W}_{11}\big|_{x_0} = 0$. □

Applying Lemma 6.15 and the following proposition completes the proof in the boundary case:

Proposition 6.16. *Let $F : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be an inverse-concave admissible speed function. Then, for all diagonal $W \in GL(n)$ with positive eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and all symmetric $B \in GL(n)$ with $B_{11} = 0$, it holds that*

$$\ddot{F}^{pq,rs}B_{pq}B_{rs} + 2\sup_{\Lambda_{i1}=0}\dot{F}^k\left[2\Lambda_k{}^pB_{kp} - (\Lambda_k{}^p)^2(\lambda_p - \lambda_1)\right] \geq 0$$

at W.

Proof of Proposition 6.16. Note that it suffices to prove the claim in the case all λ_i are distinct. Set

$$Q := \ddot{F}^{pq,rs}B_{pq}B_{rs} + 2\sup_{\Lambda_{i1}=0}\dot{F}^k\left[2\Lambda_k{}^pB_{kp} - (\Lambda_k{}^p)^2(\lambda_p - \lambda_1)\right]\,.$$

Observe that the supremum occurs when $\Lambda_{kp} = (\lambda_p - \lambda_1)^{-1}B_{kp}$ for $p > 1$. With this choice, we obtain

$$Q = \ddot{F}^{pq,rs}B_{pq}B_{rs} + 2\dot{F}^{kl}R^{pq}B_{kp}B_{kq}\,,$$

where $R^{pq} := (\lambda_p - \lambda_1)^{-1}\delta^{pq}$ for $p, q \neq 1$ and zero otherwise. Therefore, it suffices to prove that

$$0 \leq \left(\ddot{F}^{pq,rs} + 2\dot{F}^{pr}R^{qs}\right)B_{pq}B_{rs}$$

for any symmetric B with $B_{11}=0$. Applying Theorem 2.5, we obtain

$$\begin{aligned}\left(\ddot{F}^{pq,rs}+2\dot{F}^{pr}R^{qs}\right)B_{pq}B_{rs} &= \ddot{F}^{pq}B_{pp}B_{qq}+\sum_{p\neq q}\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}B_{pq}^2+2\sum_{p=1,\,q=2}^{n}\frac{\dot{F}^p}{\lambda_q-\lambda_1}B_{pq}^2\\ &= \ddot{F}^{pq}B_{pp}B_{qq}+2\sum_{p>1,\,q>1}\frac{\dot{F}^p\delta^{pq}}{\lambda_p-\lambda_1}B_{pp}B_{qq}\\ &\quad+\sum_{p\neq q}\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}B_{pq}^2+2\sum_{p=2}^{n}\frac{\dot{F}^1}{\lambda_p-\lambda_1}B_{p1}^2\\ &\quad+2\sum_{\substack{p>1,\,q>1,\\ p\neq q}}\frac{\dot{F}^p}{\lambda_q-\lambda_1}B_{pq}^2\,.\end{aligned}$$

We first estimate

$$\begin{aligned}\ddot{F}^{pq}B_{pp}B_{qq}+2\sum_{p>1,q>1}\frac{\dot{F}^p\delta^{pq}}{\lambda_p-\lambda_1}B_{pp}B_{qq} &\geq \ddot{F}^{pq}B_{pp}B_{qq}+2\sum_{p=2,q=2}\frac{\dot{F}^p}{\lambda_p}\,\delta^{pq}B_{pp}B_{qq}\\ &=\left(\ddot{F}^{pq}+2\frac{\dot{F}^p}{\lambda_p}\,\delta^{pq}\right)B_{pp}B_{qq}\geq 0\,,\end{aligned}$$

where the final inequality follows from inverse-concavity of F (Lemma 2.11). The remaining terms are

$$\begin{aligned}&\sum_{p\neq q}\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}B_{pq}^2+2\sum_{p=2}^{n}\frac{\dot{F}^1}{\lambda_p-\lambda_1}B_{p1}^2+2\sum_{\substack{p>1,\,q>1,\\ p\neq q}}\frac{\dot{F}^p}{\lambda_q-\lambda_1}B_{pq}^2\\ &\qquad=\sum_{\substack{p>1,\,q>1,\\ p\neq q}}\left(\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}+2\frac{\dot{F}^p}{\lambda_q-\lambda_1}\right)B_{pq}^2+2\sum_{p=2}^{n}\left(\frac{\dot{F}^p-\dot{F}^1}{\lambda_p-\lambda_1}+\frac{\dot{F}^1}{\lambda_p-\lambda_1}\right)B_{p1}^2\\ &\qquad\geq\sum_{\substack{p>1,\,q>1,\\ p\neq q}}\left(\frac{\dot{F}^p-\dot{F}^q}{\lambda_p-\lambda_q}+\frac{\dot{F}^p}{\lambda_q}+\frac{\dot{F}^q}{\lambda_p}\right)B_{pq}^2+2\sum_{p=2}^{n}\left(\frac{\dot{F}^p}{\lambda_p-\lambda_1}\right)B_{p1}^2\,.\end{aligned}$$

The second term is clearly non-negative. Non-negativity of the first term follows from inverse-concavity of F (Lemma 2.11). This completes the proof. □

The interior case

Let (x_0,t_0) be any point in $\mathscr{M}\times[0,T)$ such that $\underline{k}(x_0,t_0)=k(x_0,y_0,t_0)>\kappa_1(x_0,t_0)$ for some $y_0\in\mathscr{M}\times\mathscr{M}\setminus D$ and let ϕ be a lower support function for $\underline{k}$ at (x_0,t_0). Then, just as in the proof of Theorem 6.9, it suffices to show that

$$F_y-F_x+F_x^{ij}\left[(k\delta_{ij}-\mathcal{W}^x{}_{ij})-2\Lambda_i{}^p(k\delta_{pj}-\mathcal{W}^x{}_{pj})+\Lambda_i{}^p\Lambda_j{}^q(k\delta_{pq}-\mathcal{W}^y{}_{pq})\right]\geq 0\,.$$

This follows from the following proposition.

Proposition 6.17. *Let $F:\Gamma_+\subset\mathbb{R}^n\to\mathbb{R}$, $n\geq 1$, be an inverse-concave admissible* *speed function. Then, for any $k \in \mathbb{R}$, any diagonal, positive definite $B \in GL(n)$, and any positive definite $A \in GL(n)$ with $k < \min_i\{\lambda_i(A)\}$, we have*

$$0 \le F(B) - F(A) + \dot{F}^{ij}(A) \sup_{\Lambda} \Big[(k\delta_{ij} - A_{ij}) - 2\Lambda_i{}^p(k\delta_{pj} - A_{pj}) \\ + \Lambda_i{}^p\Lambda_j{}^q(k\delta_{pq} - B_{pq})\Big] \,.$$

Proof of Proposition 6.17. Since the expression in the square brackets is quadratic in Λ, it is easy to see that the supremum is attained with the choice $\Lambda = (A - kI)(B - kI)^{-1}$, where I denotes the identity matrix. Thus, given any positive definite A, we need to show that

$$0 \le Q(B) := F(B) - F(A) \\ - \dot{F}^{ij}(A)\Big((A - kI)_{ij} - \big[(A - kI)\cdot(B - kI)^{-1}\cdot(A - kI)\big]_{ij}\Big) \,.$$

Since B is diagonal, and the expression $Q(B)$ is invariant under similarity transformations with respect to A, we may diagonalize A to obtain

$$Q(B) := F(b) - F(a) - \dot{F}^i(a)\left[(a_i - k) - \frac{(a_i - k)^2}{b_i - k}\right] ,$$

where we have set $a = \lambda(A)$ and $b = \lambda(B)$. We are led to consider the function q defined on Γ_+ by

$$q(z) := F(z) - F(a) - \dot{F}^i(a)\left[(a_i - k) - \frac{(a_i - k)^2}{z_i - k}\right] .$$

We compute

$$\dot{q}^i = \dot{F}^i - \dot{F}^i(a)\frac{(a_i - k)^2}{(z_i - k)^2} ,$$

and

$$\ddot{q}^{ij} = \ddot{F}^{ij} + 2\dot{F}^i(a)\frac{(a_i - k)^2}{(z_i - k)^3}\delta^{ij} = \ddot{F}^{ij} + 2\frac{\dot{F}^i\delta^{ij}}{z_i - k} - 2\frac{\dot{q}^i\delta^{ij}}{z_i - k} .$$

It follows that

$$\ddot{q}^{ij} + 2\frac{\dot{q}^i\delta^{ij}}{z_i - k} = \ddot{F}^{ij} + 2\frac{\dot{F}^i\delta^{ij}}{z_i - k} > \ddot{F}^{ij} + 2\frac{\dot{F}^i\delta^{ij}}{z_i} \ge 0 \,, \tag{6.5}$$

where the final inequality follows from inverse-concavity of F (Lemma 2.11).

It follows that q has a unique local minimum at the point $z = a$, where it vanishes. This completes the proof of Proposition 6.17. □

This completes the proof of Theorem 6.13. □

6.6 Some consequences of the non-collapsing estimates

We shall now derive some consequences of the non-collapsing estimates. The main result is our new proof of the convergence theorem for flows of convex hypersurfaces, Theorem 6.24; although it seems Proposition 6.18 and Lemma 6.21 should also prove useful for flows of non-convex hypersurfaces.

First, we observe that, since the ratio $\overline{k}/F$ is scaling invariant, interior collapsing solutions such as the Grim Reaper cannot arise as blow-up limits of solutions which are interior non-collapsing:

Proposition 6.18. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be a concave admissible speed. Let $\mathscr{X} : \mathscr{M}^n \to \mathbb{R}^{n+1}$ be a limit of rescalings of a compact, embedded solution of* (CF)*. Then $\mathscr{X}$ is interior non-collapsing, and, in particular, not a product of the Grim Reaper.*

Proof. The first claim follows from scaling invariance of $\overline{k}/F$. Now, recall that the Grim Reaper is the curve

$$\Gamma(x,t) := (x, \gamma(x) + t)$$

where $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}$ is defined by $\gamma(x) := -\log\cos x$. Recall (from the proof of Proposition 3.1) that the curvature of γ at x is $\kappa(x) = \cos x$. Thus, $\kappa(x) \to 0$ as $x \to \pm\pi/2$. On the other hand, since $\mathbb{R}^{n-1} \times \Gamma_t(-\pi/2, \pi/2)$ lies between the planes $x_n = \pm\pi/2$, the interior ball curvature is bounded by $2/\pi$. Thus, $\overline{k}/F \propto \overline{k}/\kappa \to \infty$ as $x_n \to \pm\pi/2$. The claim follows. ☐

For flows of convex hypersurfaces, non-collapsing quickly implies that the ratio of circumradius to inradius remains bounded:

Proposition 6.19. *Let $F : \Gamma^n_+ \to \mathbb{R}$, $n \geq 1$, be a positive admissible speed function and let $\mathscr{X} : \mathscr{M}^n \times [0, T) \to \mathbb{R}^{n+1}$ be a compact, embedded solution of* (CF)*. Define the circumradius $r_+(t) := \inf\{r : \Omega_t \subset B_r(p) \text{ for some } p \in \mathbb{R}^{n+1}\}$ and the inradius $r_-(t) := \sup\{r : B_r(p) \subset \Omega_t \text{ for some } p \in \mathbb{R}^{n+1}\}$ of $\mathscr{X}_t(\mathscr{M})$ (recall that Ω_t is the region enclosed by $\mathscr{X}_t(\mathscr{M})$). Then the following estimates hold:*

1. *If F is convex or inverse-concave, then*

$$r_+ \leq \frac{1}{\delta k_0 c_0} r_- , \tag{6.6}$$

where $c_0 := F(1, \ldots, 1)$, $\delta := \min_{\mathscr{M}\times\{0\}} \kappa_1/\kappa_n$ and $k_0 := \min_{\mathscr{M}\times\{0\}} \underline{k}/F$.

2. *If F is concave and there exists $\delta > 0$ such that the pinching estimate $\kappa_1 \geq \delta\kappa_n$ holds, then*

$$r_-(t) \geq \frac{\delta c_0}{K_0} r_+(t) , \tag{6.7}$$

where $K_0 := \max_{\mathscr{M}\times\{0\}} \overline{k}/F$.

Proof. To prove the first claim, we estimate $r_+(t) \leq 1/\max_{\mathscr{M}\times\{t\}} \underline{k} \leq 1/k_0 \max_{\mathscr{M}\times\{t\}} F$ using exterior non-collapsing. Next, we estimate $F \geq c_0\kappa_1$ using monotonicity of F, and $\kappa_1 \geq \delta\kappa_n$ using the fact that the cone $\Gamma_0 := \{\kappa_1/F \geq k_0\} \subset\subset \Gamma_+$ is preserved. Finally, it follows from Proposition 6.6 that, at a point of $\mathscr{M}$ where $\kappa_n(\cdot,t)$ is maximized, the ball tangent to $\mathscr{X}$ of radius $1/\kappa_n$ is enclosed by $\mathscr{X}_t(\mathscr{M})$, so that $r_-(t) \geq 1/\max_{\mathscr{M}\times\{t\}} \kappa_n$.

The proof of the second claim is similar: By interior non-collapsing, we have $r_-(t) \geq 1/\min_{\mathscr{M}\times\{t\}} \overline{k} \geq 1/K_0 \min_{\mathscr{M}\times\{t\}} F$. Monotonicity of F implies $F \leq c_0\kappa_n$, so that, from the pinching assumption, $F \leq c_0\delta^{-1}\kappa_1$. Finally, it follows from Proposition 6.6 that, at a point of $\mathscr{M}$ where $\kappa_1(\cdot,t)$ is minimized, the ball tangent to $\mathscr{X}$ of radius $1/\kappa_1$ encloses $\mathscr{X}_t(\mathscr{M})$, so that $r_+(t) \leq 1/\min_{\mathscr{M}\times\{t\}} \kappa_1$. □

Two-sided non-collapsing also quickly yields the following instantaneous Harnack estimate for flows of convex hypersurfaces by concave, inverse-concave speeds:

Proposition 6.20. *Let $F : \Gamma_+^n \to \mathbb{R}$ be an admissible flow speed and $\mathscr{X} : \mathscr{M} \times [0,T) \to \mathbb{R}^{n+1}$ a solution of* (CF)*. Suppose that F is concave and inverse-concave. Then, for all $t \in [0,T)$, it holds that*

$$\min_{\mathscr{M}\times\{t\}} F \geq \frac{k_0}{K_0} \max_{\mathscr{M}\times\{t\}} F\,,$$

where $k_0 := \min_{\mathscr{M}\times\{0\}} \underline{k}/F$ and $K_0 := \max_{\mathscr{M}\times\{0\}} \overline{k}/F$.

Proof. Observe that, since $\overline{k}$ is the curvature of an enclosed sphere, and $\underline{k}$ is the curvature of an enclosing sphere, we have $\max_{\mathscr{M}\times\{t\}} \underline{k} \leq \min_{\mathscr{M}\times\{t\}} \overline{k}$. The proof is now a simple application of the non-collapsing estimates. □

Next, we make use of the gradient term in the evolution of the ball curvatures in conjunction with the strong maximum principle:

Lemma 6.21. *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be a positive admissible speed function and let $\mathscr{X} : \mathscr{M}^n \times (-\alpha, 0] \to \mathbb{R}^{n+1}$ be an embedded (connected but possibly non-compact) solution of* (CF)*. Set $c_k := F(\underbrace{0,\dots,0}_{k\text{-times}},1,\dots,1)$ (if it is defined) for each $k \in \{0,\dots,n-1\}$.*

1. *Suppose that F is convex. Then, if $\underline{k}/F$ attains a non-negative global minimum, either $\underline{k} \equiv 0$ or $\mathscr{X}$ is a shrinking sphere: $\mathscr{X}(\mathscr{M},t) = S^n_{\sqrt{r_0 - c_0 t/2}}$.*

2. *Suppose that $\Gamma = \Gamma_+$ and F is inverse-concave. Then, if $\underline{k}/F$ attains a positive global minimum, $\mathscr{X}$ is a shrinking sphere.*

3. *Suppose that F is concave. Then, if $\overline{k}/F$ attains its global maximum, $\mathscr{X}$ is a shrinking cylinder: $\mathscr{X}(\mathscr{M},t) = \mathbb{R}^k \times S^{n-k}_{\sqrt{r_0 - c_k t/2}}$ for some $k \in \{0,\dots,n-1\}$.*

Proof. Consider first the case that F is convex. Set $k_0 := \min_{\mathscr{M}\times(-\alpha,0]} \underline{k}/F$. Since F is a solution of (LF), Theorem 6.9 implies that $\underline{k} - k_0 F$ is a non-negative viscosity supersolution of (LF). By the strong maximum principle, we conclude that $\underline{k} \equiv k_0 F$. In particular, $\underline{k}$ is smooth. Now consider, for any $t_0 \in (-\alpha, 0]$, the set $U_{t_0} := \{x \in \mathscr{M} : \underline{k}(x,t_0) < \kappa_1(x,t_0)\}$.

By construction, for every $x_0 \in U_{t_0}$, there is a point $y_0 \in \mathscr{M} \setminus \{x_0\}$ such that $\underline{k}(x_0, t_0) = k(x_0, y_0, t_0)$. Since $k \geq \underline{k} \equiv k_0 F$, we have $k(x, y, t) - k_0 F(x, t) \geq 0$, with equality at (x_0, y_0, t_0). Thus, recalling the inequality (6.4), we obtain

$$0 \geq (\partial_t - \widehat{\mathscr{L}})(k - k_0 F) \geq \sum_{i=1}^{n} \dot{F}^i \frac{(\partial_{x^i} k)^2}{\kappa_i - k} \geq 0$$

at (x_0, t_0). Since $x_0 \in U_{t_0}$ was arbitrary, it follows that $0 \equiv \nabla \underline{k} \equiv k_0 \nabla F$ on $U_{t_0} \times \{t_0\}$. Therefore, either $\underline{k} \equiv 0$ or $k_0 > 0$ and F is constant on $U_{t_0} \times \{t_0\}$. Consider the latter case: Since U_{t_0} is open and $\kappa_1 > \underline{k} > 0$ on U_{t_0}, it follows from a result of Ecker and Huisken (1989) that $\mathscr{X}_{t_0}(U_{t_0})$ is umbilic. Since a single, complete round sphere satisfies $\kappa_1 \equiv \underline{k}$, we are forced to conclude that $U_{t_0} \subsetneq \mathscr{M}$. In that case, either U_{t_0} is empty or it has a non-empty boundary in $\mathscr{M}$. The latter case is easily ruled out: Suppose that U_{t_0} is non-empty, so that there is a point $x_0 \in \partial U_{t_0}$. By continuity, (x_0, t_0) is also an umbilic point, so that $c_0^{-1} F(x_0, t_0) = \kappa_1(x_0, t_0) = \underline{k}(x_0, t_0) = k_0 F(x_0, t_0)$. We conclude $k_0 = c_0^{-1}$, which implies $\mathscr{X}_{t_0}(\mathscr{M})$ is a round sphere, again contradicting the assumption that U_{t_0} is non-empty. Thus U_{t_0} is empty; that is, $\kappa_1(\cdot, t_0) \equiv \underline{k}(\cdot, t_0) \equiv k_0 F(\cdot, t_0)$. In this case, Lemma 6.15 implies that $\nabla \kappa_1(\cdot, t_0) \equiv 0$, so that $\nabla F(\cdot, t_0) \equiv 0$. Since $\kappa_1 = \underline{k} > 0$, we conclude, from the Ecker–Huisken result, that $\mathscr{X}_{t_0}(\mathscr{M})$ is a round sphere. The claim follows since t_0 was arbitrary.

The proof of the statement for inverse-concave speeds is similar, since the result of Ecker and Huisken also applies to inverse-concave functions (see[2] Andrews, McCoy, and Zheng 2013, Lemma 11 and the remark following it).

Finally, consider the case that F is concave. Set $K_0 := \max_{\mathscr{M} \times (-\alpha, 0]} \overline{k}/F$. As above, by applying the strong maximum principle to the evolution of $\overline{k} - K_0 F$, we obtain $\overline{k} \equiv K_0 F$. Consider now the set $V_{t_0} := \{x \in \mathscr{M} : \overline{k}(x, t_0) > \kappa_n(x, t_0)\}$. By construction, for any $x_0 \in V_{t_0}$, there is a point $y_0 \in \mathscr{M} \setminus \{x_0\}$ such that $\overline{k}(x_0, t_0) = k(x_0, y_0, t_0)$. Since $k \leq \overline{k} \equiv K_0 F$, we have $k(x, y, t) - K_0 F(x, t) \leq 0$, with equality at (x_0, y_0, t_0). Thus,

$$0 \geq (\partial_t - \widehat{\mathscr{L}})(K_0 F - k) \geq \sum_{i=1}^{n} \dot{F}^i \frac{(\partial_{x^i} k)^2}{k - \kappa_i} \geq 0$$

at (x_0, t_0). Since $x_0 \in V_{t_0}$ was arbitrary, it follows that $0 \equiv \nabla \overline{k} \equiv K_0 \nabla F$ on $V_{t_0} \times \{t_0\}$. It now follows from the Ecker–Huisken result that $\mathscr{X}_{t_0}(V_{t_0})$ is isoparametric with at most two distinct principal curvatures, and hence a union of parts of a round, orthogonal hypercylinder $\mathscr{H}_k := \mathbb{R}^k \times S^{n-k}$ for some $0 \leq k \leq n - 1$. Since a complete cylinder $\mathbb{R}^k \times S^{n-k}$, $0 \leq k \leq n - 1$ satisfies $\kappa_n \equiv \overline{k}$, we are forced to conclude that $V_{t_0} \subsetneq \mathscr{M}$. In that case, either V_{t_0} is empty or it has a non-empty boundary in $\mathscr{M}$. As before, the latter case is easily ruled out: Suppose that V_{t_0} is non-empty, then, calculating at an interior point $x_0 \in V_{t_0}$, we find $c_k^{-1} F(x_0, t_0) = \kappa_n(x_0, t_0) < \overline{k}(x_0, t_0) = K_0 F(x_0, t_0)$ so that $K_0 > c_k^{-1}$. But, since $V_{t_0} \subsetneq \mathscr{M}$, the same calculation at a boundary point $x_0 \in \partial V_{t_0}$ yields

[2]The result is also easily obtained by applying the maximum principle to the elliptic counterparts of equations (4.19) and (4.20).

$K_0 = c_k^{-1}$, a contradiction. We conclude that V_{t_0} is empty, in which case $\kappa_n \equiv \overline{k}$. In this case, a similar computation as in Lemma 6.15 implies that $\nabla \kappa_n \equiv 0$ on $\mathscr{M} \times \{t_0\}$. It follows that $\nabla F \equiv 0$ on $\mathscr{M} \times \{t_0\}$, which, by the Ecker–Huisken result, implies that $\mathscr{X}(\mathscr{M}, t_0)$ is a round, orthogonal hypercylinder. The claim follows since t_0 was arbitrary. □

By applying Lemma 6.21 to a limit of rescalings about a singularity, we are able to deduce that the collapsing ratios improve asymptotically along flows of convex hypersurfaces:

Corollary 6.22. *Let $F : \Gamma_+^n \to \mathbb{R}$, $n \geq 1$, be a positive admissible speed function and let $\mathscr{X} : \mathscr{M}^n \times [0, T) \to \mathbb{R}^{n+1}$ be a compact, embedded*[3] *solution of* (CF). *Set $c_0 := F(1, \ldots, 1)$. Suppose that F is concave and inverse-concave. Then we obtain the following estimates:*

1. *For every $\varepsilon > 0$, there exists $F_\varepsilon < \infty$ such that*

$$F(x,t) > F_\varepsilon \quad \Rightarrow \quad \underline{k}(x,t) \geq (1-\varepsilon) c_0^{-1} F(x,t)\,.$$

2. *For every $\varepsilon > 0$, there exists $F_\varepsilon < \infty$ such that*

$$F(x,t) > F_\varepsilon \quad \Rightarrow \quad \overline{k}(x,t) \leq (1+\varepsilon) c_0^{-1} F(x,t).$$

Proof. Suppose the first estimate is false. Then there exists $\varepsilon_0 > 0$ and a sequence $(x_i, t_i) \in \mathscr{M} \times [0, T)$ such that $t_i \nearrow T$, $F(x_i, t_i) \nearrow \infty$, and $\min_{\mathscr{M} \times [0, t_i]} \frac{\underline{k}}{F} = \frac{\underline{k}}{F}(x_i, t_i) \to (1 - \varepsilon_0) c_0^{-1}$. Note that, by Theorem 6.1, $\varepsilon_0 < 1$.

Set $\lambda_i := F(x_i, t_i)$ and consider the blow-up sequence

$$\mathscr{X}_i(x,t) := \lambda_i \left(\mathscr{X}\left(x, \lambda_i^{-2} t + t_i\right) - \mathscr{X}_i(x_i, t_i) \right) \,.$$

Note that, by Proposition 6.20, and since $\min F$ is monotone non-decreasing, there is a constant C such that, for all $t \in [-\lambda_i^{-2}, 0]$ and all $i \in \mathbb{N}$, the estimate $\max_{\mathscr{M} \times \{t\}} F_i \leq C$ holds. Since, furthermore, $\mathscr{X}_i(x_i, 0) = 0$ for each $i \in \mathbb{N}$, it follows from Theorem C.4 that the sequence $\mathscr{X}_i$ converges locally smoothly along a sub-sequence to a smooth limit flow $\mathscr{X}_\infty : \mathscr{M}_\infty \times (-\infty, 0] \to \mathbb{R}^{n+1}$. Moreover, since the ratio $\underline{k}/F$ is invariant under rescaling, we have

$$\frac{\underline{k}_i}{F_i}(x_i, 0) = \frac{\underline{k}}{F}(x_i, t_i) \geq k_0 > 0\,,$$

which implies that the image of each $\mathscr{X}_i$ is contained in a compact set. We conclude that the convergence is global and $\mathscr{M}_\infty \cong \mathscr{M}$.

Finally, we note that, by construction, $\inf_{\mathscr{M} \times (-\infty, 0]} \underline{k}/F$ is attained at time $t = 0$. We can now conclude from Lemma 6.21 that $\mathscr{X}_\infty$ is a shrinking sphere, which contradicts the assumption $\varepsilon_0 > 0$, proving the first claim.

The proof of the second estimate is similar, making use of the third statement of Lemma 6.21. □

[3]For $n \geq 2$, the embeddedness assumption is superfluous: Every smooth, convex, compact immersed submanifold of $\mathbb{R}^{n+1}$, $n \geq 2$, is embedded.

Remark 6.4. We expect that Lemma 6.21 should also prove useful for flows of non-convex hypersurfaces when combined with Theorem 5.1; however, at present, we are unable to obtain a smooth blow-up limit which attains $\inf \underline{k}/F$ (or $\sup \overline{k}/F$) in the non-compact case. This problem has recently been overcome by Haslhofer and Kleiner (2013a) and Haslhofer and Kleiner (2013b) and Brendle (2013a) in the case of the mean curvature flow: Haslhofer and Kleiner exploited two-sided non-collapsing to obtain local estimates for the derivatives of $\mathcal{W}$ depending only on the value of H at a single point, thereby allowing a local blow-up procedure. Brendle, on the other hand, proved asymptotically improving estimates for $\overline{k}$ and $\underline{k}$ directly using a weak version of Huisken's iteration argument and incorporating the Huisken–Sinestrari convexity and cylindrical estimates.

It now follows from Proposition 6.20 that, for flows by concave, inverse-concave speeds, the ratio of circumradius to inradius improves to unity as the maximal time is approached.

Corollary 6.23. *Let $F : \Gamma_+^n \to \mathbb{R}$, $n \geq 1$, be a positive admissible speed function and let $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact, embedded solution of* (CF)*. Suppose that F is concave and inverse-concave. Then, for all $\varepsilon > 0$, there exists $F_\varepsilon < \infty$ such that*

$$\max_{\mathscr{M}\times\{t\}} F > F_\varepsilon \quad \Rightarrow \quad r_+(t) \leq (1+\varepsilon) r_-(t)\,. \tag{6.8}$$

This quickly implies that flows by concave, inverse-concave speeds shrink convex hypersurfaces to round points:

Theorem 6.24 (Huisken (1984) and Andrews (2007))**.** *Let $F : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible speed function for* (CF) *and let $\mathscr{X}_0 : \mathscr{M}^n \to \mathbb{R}^{n+1}$ be a smooth, compact, convex embedding. Suppose that F is concave and inverse-concave. Then there exists a unique maximal solution $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ of the curvature flow* (CF) *with $\mathscr{X}(\cdot,0) = \mathscr{X}_0$ such that $T < \infty$, the maps $\mathscr{X}_t := \mathscr{X}(\cdot,t)$ converge in C^0 to a constant map $p \in \mathbb{R}^{n+1}$ as $t \to T$, and the rescaled embeddings $\widetilde{\mathscr{X}}_t := \widetilde{\mathscr{X}}(\cdot,t)$ defined by*

$$\widetilde{\mathscr{X}}(x,t) := \frac{\mathscr{X}(x,t) - p}{\sqrt{2(T-t)}}$$

converge in C^2 to a limit embedding with image equal to the sphere $S^n_{c_0^{-1}}$ of radius c_0^{-1} as $t \to T$, where $c_0 := F(1,\ldots,1)$.

Remark 6.5. Following Huisken (1984, §§9–10), the convergence statement can be improved to C^∞ using the curvature derivative estimates from Proposition 4.27; however, as the new ingredient in our proof is the use of the non-collapsing estimates, we have chosen to omit these details.

Proof of Theorem 6.24. Without loss of generality, we assume $c_0 = 1$. First note that, by the local and global existence Theorems 3.7 and 4.29, there exists $T < \infty$ and a unique smooth time-dependent immersion $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfying (CF) such that $\mathscr{X}(\cdot,0) = \mathscr{X}_0$ and $\max_{\mathscr{M}\times\{t\}} F \to \infty$ as $t \to T$. Moreover, by Theorem 4.33, $\mathscr{X}_t$ is an embedding for each $t \in [0,T)$.

Next, we show that the solution converges uniformly to a point $p \in \mathbb{R}^{n+1}$ in the Hausdorff metric: Observe that $|\mathscr{X}(x_1,t) - \mathscr{X}(x_2,t)| \leq 2r_+(t)$ for every x_1, $x_2 \in \mathscr{M}$ and every $t \in [0,T)$, where $r_+(t)$ denotes the circumradius of $\mathscr{X}_t(\mathscr{M})$. Since $F > 0$, $\mathscr{X}$ remains in the compact region $\overline{\Omega}_0$ enclosed by $\mathscr{X}_0(\mathscr{M})$. Thus, it suffices to show that $r_+ \to 0$ as $t \to T$. But this follows directly from non-collapsing: Since $\underline{k}(x,t)$ is the curvature of the smallest ball which encloses the hypersurface $\mathscr{X}(\mathscr{M},t)$, and touches it at $\mathscr{X}(x,t)$, we have

$$\frac{1}{r_+} \geq \max_{\mathscr{M}\times\{t\}} \underline{k} \geq k_0 \max_{\mathscr{M}\times\{t\}} F\,.$$

But $\max_{\mathscr{M}\times\{t\}} F \to \infty$.

We shall now prove Hausdorff convergence of the rescaled hypersurfaces $\widetilde{\mathscr{X}}(\mathscr{M},t)$ to the unit sphere: Note that, by Corollary 6.23, the ratio of circumradius to inradius of the solution approaches unity as $t \to T$: there exists, for every $\varepsilon > 0$, a time $t_\varepsilon \in [0,T)$ such that $r_+(t) \leq (1+\varepsilon)r_-(t)$ for all $t \in [t_\varepsilon, T)$. Now, by the avoidance principle, the remaining time of existence at each time t is no less than the lifespan of a shrinking sphere of initial radius $r_-(t)$, and no greater than the lifespan of a shrinking sphere of initial radius $r_+(t)$. This observation yields

$$r_-(t) \leq \sqrt{2(T-t)} \leq r_+(t) \leq (1+\varepsilon)r_-(t)\,. \tag{6.9}$$

for all $t \in [t_\varepsilon, T)$. It follows that the circum- and in-radii of the rescaled solution each approach unity as $t \to T$. We can also control the distance from the final point p to the centre p_t of any in-sphere of $\mathscr{X}(\mathscr{M},t)$: For any $t' \in [t,T)$, the final point p is enclosed by $\mathscr{X}(\mathscr{M},t')$, which is enclosed by the sphere of radius $\sqrt{r_+(t)^2 - 2(t'-t)}$ about p_t. Taking $t' \to T$ and applying (6.9) gives

$$|p - p_t| \leq \sqrt{r_+(t)^2 - 2(T-t)} \leq \sqrt{(1+\varepsilon)^2 \cdot 2(T-t) - 2(T-t)}\,,$$

Thus

$$\frac{|p-p_t|}{\sqrt{2(T-t)}} \leq \sqrt{(1+\varepsilon)^2 - 1}\,. \tag{6.10}$$

This yields the desired Hausdorff convergence of $\widetilde{\mathscr{X}}$ to the unit sphere.

Next, we obtain bounds for the curvature of the rescaled solution $\widetilde{\mathscr{X}}$: Using Corollary 6.22, Proposition 6.20 and the inequalities $r_-(t) \leq \sqrt{2(T-t)} \leq r_+(t)$ derived above, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{\sqrt{2(T-t)}} \leq \frac{1}{r_-(t)} \leq \min_{x\in\mathscr{M}} \overline{k}(x,t) &\leq (1+\varepsilon)\min_{x\in\mathscr{M}} F\\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\min_{x\in\mathscr{M}} \underline{k}(x,t) \leq \frac{1+\varepsilon}{1-\varepsilon}\min_{x\in\mathscr{M}} \kappa_1(x,t)\,,\end{aligned}$$

and

$$\frac{1}{\sqrt{2(T-t)}} \geq \frac{1}{r_+(t)} \geq \max_{x\in\mathscr{M}} \underline{k}(x,t) \geq (1-\varepsilon) \max_{x\in\mathscr{M}} F$$
$$\geq \frac{1-\varepsilon}{1+\varepsilon} \max_{x\in\mathscr{M}} \overline{k}(x,t) \geq \frac{1-\varepsilon}{1+\varepsilon} \max_{x\in\mathscr{M}} \kappa_n(x,t)$$

for t sufficiently close to T. It follows that, for any $\varepsilon > 0$, the rescaled hypersurfaces satisfy

$$\frac{1-\varepsilon}{1+\varepsilon} \leq \frac{1}{\sqrt{n}} |\widetilde{\mathscr{W}}|^2 \leq \frac{1+\varepsilon}{1-\varepsilon}$$

for t sufficiently close to T. This yields convergence of the second derivatives of $\widetilde{\mathscr{X}}$.

Convergence of the rescaled metric, and hence the first derivatives of $\widetilde{\mathscr{X}}$, now follows similarly as in (4.26). □

Remark 6.6. The proof of Theorem 6.24 was particularly facilitated by the use of two-sided non-collapsing. We note also that, via Proposition 6.19, one-sided non-collapsing also simplifies the arguments of Andrews (1994a) for the convergence of flows by convex speeds, or flows of sufficiently pinched initial data by concave speeds.

A. Fully non-linear scalar parabolic PDE

In this appendix, we collect results from the literature on scalar parabolic equations which are needed to obtain, in Section 3, local existence and, in Section 4, global regularity of solutions of (CF).

A.1 Local existence

The first result we require is a local existence theorem for fully non-linear scalar parabolic equations on closed Riemannian manifolds. This will be used to obtain short-time existence of solutions of (CF) (Theorem 3.7).

The following result will suffice; for a proof, we refer the reader to Baker (2010, Main Theorem 5).

Theorem A.1. *Let $(\mathscr{M}, g)$ be a smooth, closed, n-dimensional Riemannian manifold with Levi-Civita connection ∇. Consider the following initial value problem:*

$$\begin{cases} \partial_t\, u(x,t) = F\left(\nabla^2 u(x,t), \nabla\, u(x,t), u(x,t), x, t\right) \\ u(x,0) = u_0(x)\,, \end{cases} \tag{A.1}$$

where $u_0 \in C^\infty(\mathscr{M})$. Suppose that F is smooth, and uniformly elliptic at u_0; that is, there exist constants $0 < \lambda \le \Lambda < \infty$ such that

$$\lambda\,||\xi|| \le \dot{F}^{ij}(\nabla^2 u_0(x), \nabla\, u_0(x), u_0(x), x, 0)\xi_i\xi_j \le \Lambda\,||\xi||$$

for all $\xi \in \mathbb{R}^n$ and $x \in \mathscr{M}$, where $\dot{F}$ is the derivative of F with respect to its first variable. Then there exists $\beta > 0$, and $u \in C^\infty(\mathscr{M}^n \times [0,\beta))$ satisfying (A.1)*. Moreover, u is unique: For any smooth $u' : \mathscr{M} \times [0, \beta') \to \mathbb{R}$ satisfying* (A.1) *with $u'(\cdot, 0) \equiv u_0$, we have $u_{\mathscr{M}\times([0,\beta)\cap[0,\beta'))} \equiv u'_{\mathscr{M}\times([0,\beta)\cap[0,\beta'))}$.*

A.2 Global regularity

Next we require some regularity results for fully non-linear parabolic scalar PDE on domains $\Omega \subset \mathbb{R}^n$. These results are used to obtain long-time regularity results for solutions of (CF) (Theorem 4.29).

So consider fully non-linear equations

$$\partial_t u(x,t) = F\left(D^2u(x,t), Du(x,t), u(x,t), x, t\right) , \tag{A.2}$$

where $F \in C^2(\mathscr{O})$ for some open subset $\mathscr{O} \subset \mathrm{Sym}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. It will be useful to define the parabolic balls $Q_r(x,t) := B_r(x) \times (t - r^2, t]$, $Q_r := Q_r(0,0)$.

The following theorem of Andrews (2004, Theorem 6), which slightly generalizes well-known results of Evans (1982) and Krylov (1982), provides Hölder continuity up to second order of a solution of equation (A.2):

Theorem A.2. *Let Ω be a domain in $\mathbb{R}^n$, $T > 0$. Let $u \in C^4\left(\Omega \times (0,T]\right)$ be a solution of the fully non-linear equation*

$$\partial_t u(x,t) = F\left(D^2u(x,t), Du(x,t), u(x,t), x, t\right) ,$$

where $F : \mathrm{Sym}(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times (0,T]$ is C^2. Suppose that there exist $0 < \lambda \le \Lambda < \infty$ such that $\lambda I \le \dot F \le \Lambda I$, where $\dot F$ is the derivative of F with respect to the first argument. Suppose in addition that there exists $K < \infty$ such that $\ddot F^{pq,rs} M_{pq} M_{rs} \le K \dot F^{pq} \dot F^{rs} M_{pq} M_{rs}$ for any $M \in \mathrm{Sym}(n)$, where $\ddot F$ is the second derivative of F with respect to the first argument. Then for any $\tau \in (0,T)$ and $\Omega' \subset\subset \Omega$, there exist $\alpha \in (0,1)$, $C > 0$ such that

$$\sup_{(x,t)\neq(y,s)\in\Omega'\times[\tau,T]} \left(\frac{|\,\partial_t u(x,t) - \partial_t u(y,t)|}{|x-y|^\alpha + |s-t|^{\frac{\alpha}{2}}} + \frac{|D^2u(x,t) - D^2(y,t)|}{|x-y|^\alpha + |s-t|^{\frac{\alpha}{2}}} \right) + \sup_{x\in\Omega',\, s\neq t\in[\tau,T]} \frac{|Du(x,t) - Du(x,s)|}{|s-t|^{\frac{1+\alpha}{2}}} \le C\,.$$

The constant α depends on λ and Λ, and the constant C depends on λ, Λ, τ, $\sup_{\Omega\times(0,T]}\left(|D^2u| + |Du|\right)$, $\sup_{\Omega\times(0,T]} |\,\partial_t u|$, $d(\Omega', \partial\,\Omega)$, K, and bounds for the first and second derivatives of F.

The above Hölder estimate requires convexity of the level sets of F; in two space dimensions we require the following stronger result (Andrews 2004, Theorem 5), which dispenses with this additional assumption:

Theorem A.3. *Let Ω be a domain in $\mathbb{R}^2$ and $T > 0$. Let $u \in C^2\left(\Omega \times [0,T)\right)$ be a solution of the fully non-linear equation*

$$\partial_t u(x,t) = F\left(D^2u(x,t), Du(x,t), u(x,t), x, t\right) ,$$

where $F : \mathrm{Sym}(2) \times \mathbb{R}^2 \times \mathbb{R} \times \Omega \times (0,T]$ is Lipschitz in all arguments. Suppose that there exist $0 < \lambda \le \Lambda < \infty$ such that $\lambda I \le \dot F \le \Lambda I$, where $\dot F$ is the derivative of F with respect to the first argument. Then for any $\tau \in (0,T)$ and $\Omega' \subset\subset \Omega$, there exist $\alpha \in (0,1)$, $C > 0$

such that

$$\sup_{(x,t)\neq(y,s)\in\Omega'\times[\tau,T]}\left(\frac{|\,\partial_t\,u(x,t)-\partial_t(y,t)|}{|x-y|^\alpha+|s-t|^{\frac{\alpha}{2}}}+\frac{|D^2u(x,t)-D^2u(y,t)|}{|x-y|^\alpha+|s-t|^{\frac{\alpha}{2}}}\right)$$
$$+\sup_{x\in\Omega',\,s\neq t\in[\tau,T]}\frac{|Du(x,t)-Du(x,s)|}{|s-t|^{\frac{1+\alpha}{2}}}\leq C\,.$$

The constant α depends on λ and Λ, and the constant C depends on λ, Λ, τ, $\sup_{\Omega\times(0,T]}\left(|D^2u|+|Du|\right)$, $\sup_{\Omega\times(0,T]}|\,\partial_t\,u|$, $d(\Omega',\partial\,\Omega)$, and bounds for the first derivatives of F.

We now state the all important Schauder estimate (see, for example, Lieberman 1996, Theorem 4.9):

Theorem A.4. *Given constants $0<\lambda\leq\Lambda<\infty$, A, B, $C\geq 0$, and $\alpha\in(0,1)$, there exists $K<\infty$ depending only on n, λ, Λ, A, B, C, and α such that for any smooth solution $u:Q_1\to\mathbb{R}$ of an equation*

$$\partial_t\,u=a^{ij}u_{ij}+b^iu_i+cu+f \tag{A.3}$$

which is uniformly parabolic: $\lambda\,||\xi||^2\leq a^{ij}(x,t)\xi_i\xi_j\leq\Lambda\,||\xi||^2$ for all $(x,t)\in Q_1$ and all $\xi\in\mathbb{R}^n$; and has bounded, Hölder continuous coefficients: $\sup_{Q_1}|a^{ij}|\leq A$, $\sup_{Q_1}|b^i|\leq B$, $\sup_{Q_1}|c|\leq C$; $\sup_{(x,t)\neq(y,s)\in Q_1}\frac{|a^{ij}(x,t)-a^{ij}(y,s)|}{||x-y||^\alpha+|t-s|^{\frac{\alpha}{2}}}\leq A$, $\sup_{(x,t)\neq(y,s)\in Q_1}\frac{|b^i(x,t)-b^i(y,s)|}{||x-y||^\alpha+|t-s|^{\frac{\alpha}{2}}}\leq B$, $\sup_{(x,t)\neq(y,s)\in Q_1}\frac{|c(x,t)-c(y,s)|}{||x-y||^\alpha+|t-s|^{\frac{\alpha}{2}}}\leq C$, the following estimate holds:

$$\begin{aligned}&\sup_{Q_{1/2}}\left(||Du||+\left|\left|D^2u\right|\right|+||\partial_t\,u||\right)\\&+\sup_{(x,t)\neq(y,s)\in Q_{1/2}}\left(\frac{|\,\partial_t\,u(x,t)-\partial_t\,u(y,t)|}{|x-y|^\alpha+|s-t|^{\frac{\alpha}{2}}}+\frac{|D^2u(x,t)-D^2(y,t)|}{|x-y|^\alpha+|s-t|^{\frac{\alpha}{2}}}\right)\\&+\sup_{x\in B_{1/2}(0),s\neq t\in(-1/4,0]}\frac{|Du(x,t)-Du(x,s)|}{|x-y|^\alpha+|s-t|^{\frac{\alpha}{2}}}\\&\qquad\leq K\left(\sup_{Q_1}\left(|u|+|f|\right)+\sup_{(x,t)\neq(y,s)\in Q_1}\frac{|f(x,t)-f(y,t)|}{|x-y|^\alpha+|s-t|^{\frac{\alpha}{2}}}\right).\end{aligned}$$

B. The differential Harnack estimate

In this appendix we use the following differential Harnack estimate of Andrews to prove that strictly convex, eternal solutions of flows by convex or inverse-concave speeds necessarily move by translation.

Theorem B.1 (Differential Harnack estimate, Andrews (1994b))**.** *Let $F : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible speed function and $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ a (strictly convex) solution of* (CF)*. Suppose that F is inverse-concave. Then*

$$\partial_t F - \mathcal{W}^{-1}(\nabla F, \nabla F) + \frac{F}{2t} \geq 0\,.$$

Remark B.1. Recall that, in particular, every convex speed function defined on Γ_+ is inverse-concave.

We note that a similar estimate for the mean curvature flow was also proved by Hamilton (1995b) (see also Chow (1991) and the recent work of Ecker (2007; 2014) relating Harnack inequalities to entropy monotonicity), and used to show that convex, eternal solutions of the mean curvature flow, on which the maximum of the mean curvature is attained, necessarily move by translation. We utilize Theorem B.1 to prove a similar statement for the class of flows by inverse-concave admissible speeds. But first, we give an outline of Andrews' proof of Theorem B.1:

Proof of Theorem B.1. The proof of Theorem B.1 is a consequence of the following observation:

Lemma B.2 (Andrews (1994b), Lemma 5.1)**.** *Let $F : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible speed function and $\mathscr{X} : \mathscr{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ a (strictly convex) solution of* (CF)*. Define the speed as a function of the principal radii, $\vec{\rho} := (\kappa_1^{-1}, \ldots, \kappa_n^{-1})$, of $\mathscr{X}$ by setting $\chi(x,t) = \chi(\vec{\rho}(x,t))$, where $\chi : \Gamma_+ \to \mathbb{R}$ is defined by*

$$\chi(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) := -F(\lambda_1, \ldots, \lambda_n)\,.$$

Then, in the Gauß map parametrization, the 'Harnack quantity' $P := \partial_t \chi$ satisfies

$$\partial_t P = \dot{\chi}^{ij}\overline{\nabla}_i\overline{\nabla}_j P + \mathrm{tr}(\dot{\chi})P + \ddot{\chi}(Q,Q)\,, \tag{B.1}$$

where $\overline{\nabla}$ is the canonical (time-independent) connection on S^n, and $Q := \nabla_t \mathcal{W}^{-1}$.

Proof. The proof is a short computation which makes use of identities related to the support function and the Gauß map parametrization. See Andrews (1994b, Lemma 5.1). □

Now consider the function $R := 2t\,\partial_t\,\chi + \chi$. From Lemma B.2 and the evolution equation for χ (Andrews 1994b, Theorem 3.6),

$$\partial_t\,\chi = \dot{\chi}^{ij}\overline{\nabla}_i\overline{\nabla}_j\chi + \mathrm{tr}(\dot{\chi})\chi\,,$$

we obtain

$$\partial_t\,R = \dot{\chi}^{ij}\overline{\nabla}_i\overline{\nabla}_j R + \mathrm{tr}(\dot{\chi})R + 2P + 2t\ddot{\chi}(Q,Q)\,.$$

Applying Lemma 2.12 now yields

$$\partial_t\,R \leq \dot{\chi}^{ij}\overline{\nabla}_i\overline{\nabla}_j R + R\left(\mathrm{tr}(\dot{\chi}) + \frac{2P}{\chi}\right)\,.$$

Since R is initially non-positive, the maximum principle yields $P + \frac{\chi}{2t} \leq 0$ for all $t > 0$. The claim now follows since $P = -\,\partial_t\,F$, which is equal to $-\,\partial_t\,F + \mathcal{W}^{-1}(\nabla\,F, \nabla\,F)$ with respect to the flow's original parametrization (Andrews 1994b, Lemma 3.10). □

We now use Theorem B.1 to prove that strictly convex eternal solutions necessarily move by translation:

Proposition B.3 (Cf. Hamilton (1995b))**.** *Let $n \geq 1$ be an integer and $F : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$ an inverse-concave admissible speed. Let $\mathscr{X} : \Sigma^n \times (-\infty, \infty) \to \mathbb{R}^{n+1}$ be a strictly convex, eternal solution of* (CF) *such that $\sup_{\Sigma\times(-\infty,\infty)} F$ is attained. Then $\mathscr{X}$ moves by translation.*

Proof. Since F is inverse-concave and the flow is invariant under time-translations, Theorem B.1 implies that every solution $\mathscr{X} : \mathscr{M} \times [t_0, T) \to \mathbb{R}^{n+1}$ of (CF) satisfies

$$\partial_t\,F - \mathcal{W}^{-1}(\nabla\,F, \nabla\,F) + \frac{F}{2(t-t_0)} \geq 0 \tag{B.2}$$

for all $t > t_0$. Thus, fixing t and taking $t_0 \to -\infty$, we see that any strictly convex, eternal solution of (CF) satisfies

$$\partial_t\,F - \mathcal{W}^{-1}(\nabla\,F, \nabla\,F) \geq 0$$

for all $t \in \mathbb{R}$. Equivalently, $P \leq 0$ for all $t \in \mathbb{R}$ in the Gauß map parametrization.

Now, since χ is a concave function of $\mathcal{W}^{-1}$, equation (B.1) implies

$$\partial_t\,P \leq \dot{\chi}^{ij}\overline{\nabla}_i\overline{\nabla}_j P + \mathrm{tr}(\dot{\chi})P\,.$$

Thus, by the strong maximum principle, $P = 0$ at an interior space-time point only if equality holds identically. Since $P = 0$ at the point where $\sup F$ is attained, we deduce that P vanishes identically on $\Sigma \times (-\infty, \infty)$. In particular, from (B.1), we obtain

$$0 \equiv Q = \nabla_t \mathcal{W}^{-1} ,$$

since, by Lemma 2.12 and strict monotonicity of F, χ is strictly concave. Returning to the original parametrization (using, for example, Andrews 1994b, Lemma 3.10), we obtain

$$0 \equiv \nabla_t \mathcal{W} + \nabla_V \mathcal{W} ,$$

where we define $V := -\mathcal{W}^{-1}(\operatorname{grad} F)$. Substituting $\nabla_t \mathcal{W} = \nabla \operatorname{grad} F + F\mathcal{W}^2$, we obtain, for any $u \in T\Sigma$,

$$\begin{aligned} 0 =& \ \nabla_u \operatorname{grad} F + F\mathcal{W}^2(u) + \nabla_u \mathcal{W}(V) \\ =& \ \nabla_u(\operatorname{grad} F + \mathcal{W}(V)) + \mathcal{W}(F\mathcal{W}(u) - \nabla_u V) . \end{aligned}$$

It follows that $\nabla V - F\mathcal{W} = 0$.

Now define the vector field $T := \mathscr{X}_* V - F\nu$. Then, for any $u \in T\Sigma$,

$$^{\mathscr{X}}\!D_u T = \mathscr{X}_* \left(\nabla_u V - F\mathcal{W}(u)\right) - g\left(\mathcal{W}(V) + \operatorname{grad} F, u\right)\nu = 0$$

and

$$^{\mathscr{X}}\!D_t T = {}^{\mathscr{X}}\!D_t \mathscr{X}_* V - \partial_t F \nu - F\mathscr{X}_* \operatorname{grad} F .$$

Since $P \equiv 0$, the second equation becomes

$$\begin{aligned} ^{\mathscr{X}}\!D_t T &= {}^{\mathscr{X}}\!D_t \mathscr{X}_* V - g(\mathcal{W}^{-1}(\operatorname{grad} F), \operatorname{grad} F)\nu - F\mathscr{X}_* \operatorname{grad} F \\ &= {}^{\mathscr{X}}\!D_t \mathscr{X}_* V + g(V, \operatorname{grad} F)\nu - F\mathscr{X}_* \operatorname{grad} F . \end{aligned}$$

Since V is tangential, we have

$$\left\langle {}^{\mathscr{X}}\!D_t \mathscr{X}_* V \,,\, \nu \right\rangle = -\left\langle \mathscr{X}_* V \,,\, {}^{\mathscr{X}}\!D_t \nu \right\rangle = -g(V, \operatorname{grad} F) .$$

It follows that the normal component of ${}^{\mathscr{X}}\!D_t T$ is zero. Finally, the tangential part of ${}^{\mathscr{X}}\!D_t \mathscr{X}_* V$ is, by definition, $({}^{\mathscr{X}}\!D_t \mathscr{X}_* V)^\top =: \mathscr{X}_* \nabla_t V = -F\mathscr{X}_* \mathcal{W}(V) = F\mathscr{X}_* \operatorname{grad} F$; so the tangential component of ${}^{\mathscr{X}}\!D_t T$ also vanishes. That is, T is parallel.

Now set $\widetilde{\mathscr{X}}(x,t) := \mathscr{X}(\phi(x,t),t)$, where ϕ is the solution of $\frac{d\phi^i}{dt} = V^i$ with initial condition $\phi(x,0) = x$, so that

$$\frac{\partial \widetilde{\mathscr{X}}}{\partial t} = \frac{\partial \mathscr{X}}{\partial x^i}\frac{d\phi^i}{dt} + \frac{\partial \mathscr{X}}{\partial t} = T .$$

This completes the proof. $\square$

C. A compactness theorem

Given an admissible speed function, we consider the set of all solutions of the flow (CF). Under an appropriate topology, we will prove that subsets with uniform initial curvature control are compact, so long as the flow admits preserved cones and Hölder estimates for the curvature (by the results of Section 4, this is the case, for example, if $n = 2$; or if the speed function $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ is convex and $\Gamma_+ \subset \Gamma$; or if F is concave and either $\Gamma = \Gamma_+$ and F is inverse-concave, or $\Gamma \subset \Gamma_+$ is sufficiently 'pinched').

We will first introduce a topology on the space $C^\infty(\mathscr{M}, \mathbb{R}^N)$ of smooth maps $\mathscr{X} : \mathscr{M} \to \mathbb{R}^N$, and prove that, in the case of immersions, subsets with uniformly bounded extrinsic geometry are compact. The compactness theorem for solutions of (CF) is a simple extension of this result. We note that our topology is weaker than other topologies that have been considered in similar settings (cf. Langer (1985), Breuning (2011), and Baker (2010)), in that we assume no local area bound. As a result, our convergence result is local, in that each convergent sub-sequence only picks up a single connected component of the limit.

The result relies on the well-known Cheeger–Gromov compactness theorem for Riemannian manifolds, which we now state:

Theorem C.1 (Compactness theorem for Riemannian manifolds, Gromov (1981) and Hamilton (1995a))**.** *Suppose that $\{(\mathscr{M}_k, g_k, O_k)\}_{k\in\mathbb{N}}$ is a sequence of pointed, complete Riemannian manifolds satisfying the following conditions:*

(i) Uniformly bounded geometry*: For each $m \in \mathbb{N}$ there exists $C_m > 0$ such that $|{}^k\nabla^m R_k| \leq C_m$ for every $k \in \mathbb{N}$, where ${}^k\nabla$, and R_k denote the Levi-Civita connection and Riemann tensor of g_k, and*

(ii) Injectivity radius bound*: There exists $\kappa > 0$ such that $\mathrm{Inj}_k(O_k) \geq \kappa$ for every $k \in \mathbb{N}$, where Inj_k denotes the injectivity radius of g_k.*

Then there exists a sub-sequence of $\{(\mathscr{M}_k, g_k, O_k)\}_{k\in\mathbb{N}}$ which converges in the Cheeger–Gromov topology to a pointed complete Riemannian manifold $(\mathscr{M}_\infty, g_\infty, O_\infty)$; that is, there exists an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of $\mathscr{M}_\infty$ such that $O_\infty \in U_1$ and a sequence of diffeomorphisms $\Phi_k : U_k \to V_k \subset \mathscr{M}_k$ such that $\Phi_k(O_\infty) = O_k$ and $\Phi_k^* g_k$ converges in[1] *$\Gamma^\infty(T^*\mathscr{M}_\infty \otimes T^*\mathscr{M}_\infty)$ to g_∞ on compact subsets of $\mathscr{M}_\infty$. Moreover, the limit $(\mathscr{M}_\infty, g_\infty, O_\infty)$ satisfies $|{}^\infty\nabla^m R_\infty| \leq C_m$ for every $m \in \mathbb{N}$ and $\mathrm{Inj}_\infty(O_\infty) \geq \kappa$.*

[1]See, for example, Andrews and Hopper (2011, Chapter 8) for a discussion of smooth convergence of sections of vector bundles.

C.1 Local-smooth convergence of immersions

As above, given a manifold $\mathscr{M}^n$ of dimension n, we denote by $C^\infty(\mathscr{M}^n, \mathbb{R}^N)$ the space of smooth maps $\mathscr{X} : \mathscr{M}^n \to \mathbb{R}^N$. We equip $C^\infty(\mathscr{M}^n, \mathbb{R}^N)$ with the topology induced by the following notion of convergence:

Definition C.2 (Local-smooth convergence of pointed maps). *For each $k \in \mathbb{N} \cup \{\infty\}$ let $\mathscr{M}_k$ be a smooth, complete manifold, x_k a point of $\mathscr{M}_k$, and $\mathscr{X}_k : \mathscr{M}_k \to \mathbb{R}^N$ a smooth map. We say that the sequence $\{(\mathscr{X}_k, \mathscr{M}_k, x_k)\}_{k\in\mathbb{N}}$ of pointed maps* converges locally smoothly *to $(\mathscr{X}_\infty, \mathscr{M}_\infty, x_\infty)$ if there exists an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of $\mathscr{M}_\infty$ with $x_\infty \in U_1$, and a sequence of smooth diffeomorphisms $\{\Phi_k : U_k \to V_k \subset \mathscr{M}_k\}_{k\in\mathbb{N}}$ such that $\Phi_k(x_\infty) = x_k$ for every $k \in \mathbb{N}$ and $\Phi_k^*\mathscr{X}_k$ converges to $\mathscr{X}_\infty$ in $C^\infty_{\mathrm{loc}}(\mathscr{M}_\infty, \mathbb{R}^N)$.*

Consider now the subspace $\mathrm{Imm}(\mathscr{M}^n) \subset C^\infty(\mathscr{M}, \mathbb{R}^{n+1})$ of smooth hypersurface immersions $\mathscr{X} : \mathscr{M}^n \to \mathbb{R}^{n+1}$. Our next result is a compactness theorem for subsets of $\mathrm{Imm}(\mathscr{M}^n)$. The result follows directly from the compactness theorem for Riemannian manifolds and the Arzelà–Ascoli theorem.

Theorem C.3 (Local compactness theorem for pointed submanifolds). *Let $\mathscr{X}_k : \mathscr{M}_k \to \mathbb{R}^N$ be a sequence of smooth immersions of smooth, complete manifolds $\mathscr{M}_k$ of dimension n, and $x_k \in \mathscr{M}_k$ a sequence of points. Suppose that the following hold:*

(i) Bounded extrinsic geometry*: For every $m \in \mathbb{N} \cup \{0\}$ there exists $C_m > 0$ such that $|{}^{\mathscr{X}_k}\nabla^m \mathcal{W}^{\mathscr{X}_k}|_{\mathscr{X}_k} \le C_m$ for every $k \in \mathbb{N}$, where ${}^{\mathscr{X}_k}\nabla$, $\mathcal{W}^{\mathscr{X}_k}$ and $|\cdot|_{\mathscr{X}_k}$ are respectively the Levi-Civita connection, Weingarten tensor, and norm induced by the immersion $\mathscr{X}_k$.*

(ii) Accumulation*: There exists $R > 0$ such that $\mathscr{X}_k(x_k) \in B_R(0)$ for every $k \in \mathbb{N}$.*

Then there is a sub-sequence of $\{(\mathscr{X}_k, \mathscr{M}_k, x_k)\}_{k\in\mathbb{N}}$ which converges locally smoothly to a pointed immersion $(\mathscr{X}_\infty, \mathscr{M}_\infty, x_\infty)$ which satisfies $|{}^{\mathscr{X}_\infty}\nabla^m \mathcal{W}^{\mathscr{X}_\infty}|_{\mathscr{X}_\infty} \le C_m$ for all $m \in \mathbb{N}$ and $\mathscr{X}_\infty(x_\infty) \in \overline{B}_R(0)$.

Remark C.1. We note that the statement of Theorem C.3 is a special case of a more general result of Cooper (2010).

Proof of Theorem C.3. Our first step is to extract a sub-sequence of $\{(\mathscr{M}_k, g_k, x_k)\}_{k\in\mathbb{N}}$ (where g_k denotes the metric induced on $\mathscr{M}_k$ by $\mathscr{X}_k$) which converges in the sense of Cheeger–Gromov. Note first that, by Klingenberg's Lemma (see, for example, Chavel 1993, Theorem III.2.4), the injectivity radii of the sequence are bounded by a constant depending only on the uniform curvature bound (see also the proof of Theorem 4.33). Note next that, via the Gauß equation, the extrinsic geometry bounds of condition (i) yield uniform bounds on the Riemann tensor of g_k and its covariant derivatives. Thus the conditions of the compactness theorem for pointed Riemannian manifolds are met, and we obtain a sub-sequence of $\{(\mathscr{M}_k, g_k, x_k)\}_{k\in\mathbb{N}}$ which converges to a pointed, complete

Riemannian manifold $(\mathscr{M}_\infty, g_\infty, x_\infty)$ in the sense of Cheeger–Gromov. That is (passing to the convergent sub-sequence) there exists an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of $\mathscr{M}_\infty$ with $x_\infty \in U_1$, and a sequence of diffeomorphisms $\{\Phi_k : U_k \to V_k \subset \mathscr{M}_k\}_{k\in\mathbb{N}}$ with $\Phi_k(x_\infty) = x_k$ such that $\Phi_k^* g_k$ converges smoothly to g_∞ on each compact set $K \subset \mathscr{M}$.

The next step is an application of the Arzelà–Ascoli theorem to extract a limit immersion (in the sense of Definition C.2). We claim that, for any integer, $m \geq 0$, and any compact set, $K \subset \mathscr{M}_\infty$, the m-th derivative of $\Phi_k^*\mathscr{X}_k$ is bounded on K with respect to g_∞, independent of k. To see this, first note that the Cheeger–Gromov convergence of the metrics yields the claim for $m = 1$ (the claim being trivial for $m = 0$). Since smooth convergence of the metrics $\Phi_k^* g_k$ implies smooth convergence of their induced Levi-Civita connections, the claim for the higher derivatives follows easily by induction, employing the Gauß equation and the extrinsic geometry bounds assumed by condition (i). It now follows from condition (ii) and the Arzelà–Ascoli theorem that, passing to a further sub-sequence, we have $C^\infty_{\mathrm{loc}}(\mathscr{M}_\infty, \mathbb{R}^{n+1})$-convergence of $\Phi_k^*\mathscr{X}_k$ to a limit immersion $\mathscr{X}_\infty : \mathscr{M}_\infty \to \mathbb{R}^{n+1}$ such that $|{}^{\mathscr{X}_\infty}\nabla^m \mathcal{W}^{\mathscr{X}_\infty}|_{\mathscr{X}_\infty} \leq C_m$ for all $m \in \mathbb{N}$ and $\mathscr{X}_\infty(x_\infty) \in \overline{B}_R(0)$. $\square$

C.2 Local-smooth convergence of curvature flows

We will now use Theorem C.3 to prove the compactness theorem for solutions of (CF).

Theorem C.4 (Local compactness theorem for curvature flows)**.** *Let $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ be an admissible flow speed which admits curvature derivative estimates (that is, the conclusion of Proposition 4.27 holds) and let $\Gamma_0 \subset\subset \Gamma$ be a curvature cone. Let $\{\mathscr{X}_k : \mathscr{M}_k \times (-\sigma, 0] \to \mathbb{R}^{n+1}\}_{k\in\mathbb{N}}$ be a sequence of solutions of the curvature flow* (CF) *satisfying $\vec{\kappa}^k(\mathscr{M} \times (-\sigma, 0]) \subset \Gamma_0$ for every k, and let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points $x_k \in \mathscr{M}_k$. Suppose that the following hold:*

(i) Curvature bound*: There exists $C_0 > 0$ such that $\sup_{\mathscr{M}_k\times(-\sigma,0]} |\mathcal{W}_k| \leq C_0$ for all $k \in \mathbb{N}$.*

(ii) Accumulation*: There exists $R > 0$ such that $\mathscr{X}_k(x_k, 0) \in B_R(0)$ for every $k \in \mathbb{N}$.*

Then there is a sub-sequence of $(\mathscr{X}_k, \mathscr{M} \times [-\sigma/2, 0], x_k)$ which converges, in the sense of definition C.2, to a complete, pointed time-dependent immersion $(\mathscr{X}_\infty, \mathscr{M}_\infty \times [-\sigma/2, 0], x_\infty)$ which solves (CF) *and satisfies $\vec{\kappa}(\mathscr{M}_\infty \times [-\sigma/2, 0]) \subset \Gamma_0$, $\sup_{\mathscr{M}_\infty\times[-\sigma/2,0]} |\mathcal{W}_\infty| \leq C_0$, and $\mathscr{X}_\infty(x_\infty, 0) \in \overline{B}_R(0)$.*

Remarks C.1.

1. In order to obtain the conclusion of Proposition 4.27 it suffices that the Weingarten tensor of the solution admits $C^{2,\alpha}$ estimates (see Remarks 4.5)
2. If the cone Γ_0 is preserved, the assumption $\vec{\kappa}^k \subset \Gamma_0$ need only be made at some initial time $t_0 \leq -\sigma/2$ (see Remarks 4.5).

Proof of Theorem C.4. By Proposition 4.27, the bound on $\mathcal{W}_k$ implies bounds on the derivatives of $\mathcal{W}_k$ to all orders (which depend only on n, F, C_0 and Γ_0) uniformly in time. Thus, in particular, we may apply Theorem C.3 to the sequence of immersions $\mathscr{X}_{k,0} := \mathscr{X}_k(\cdot, 0)$ to extract a smooth, complete manifold $\mathscr{M}_\infty$, a point $x_\infty \in \mathscr{M}_\infty$, an immersion $\mathscr{X}_{\infty,0} : \mathscr{M}_\infty \to \mathbb{R}^{n+1}$, an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of $\mathscr{M}_\infty$, and a sequence of smooth diffeomorphisms $\Phi_k : U_k \to V_k \subset \mathscr{M}_k$ such that $\Phi_k(x_\infty) = x_k$ for each k, and $\Phi_k^*\mathscr{X}_k\big|_{\mathscr{M}\times\{0\}}$ converges locally smoothly along a sub-sequence to $\mathscr{X}_{\infty,0}$. Let us now define the diffeomorphisms

$$\begin{aligned}\Psi_k : U_k \times [-\sigma/2, 0] &\to V_k \times [-\sigma/2, 0]\\ (x, t) &\mapsto (\Phi_k(x), t)\,.\end{aligned}$$

We claim that the sequence $\Psi_k^*\mathscr{X}_k$ converges subsequentially in $C^\infty_{\mathrm{loc}}(\mathscr{M}_\infty\times[-\sigma/2, 0], \mathbb{R}^{n+1})$ to a limit map $\mathscr{X}_\infty : \mathscr{M}_\infty \times [-\sigma/2, 0] \to \mathbb{R}^{n+1}$ (which necessarily solves (CF)); In fact, it follows (as in the proof of Theorem C.3) from the curvature derivative estimates mentioned above that $\Psi_k^*\mathscr{X}_k$ has uniform bounds for its spatial derivatives to all orders (independent of time). Bounds for the time derivatives and mixed spatial-temporal derivatives then follow from the evolution equation (CF). The claim now follows from the Arzelà–Ascoli theorem. $\square$

Bibliography

Abresch, U. and J. Langer (1986). "The normalized curve shortening flow and homothetic solutions". In: *J. Differential Geom.* 23.2, pp. 175–196. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214440025.

Alessandroni, Roberta and Carlo Sinestrari (2010). "Convexity estimates for a nonhomogeneous mean curvature flow". In: *Math. Z.* 266.1, pp. 65–82. ISSN: 0025-5874. DOI: 10.1007/s00209-009-0554-3. URL: http://dx.doi.org/10.1007/s00209-009-0554-3.

Andrews, Ben (1994a). "Contraction of convex hypersurfaces in Euclidean space". In: *Calc. Var. Partial Differential Equations* 2.2, pp. 151–171. ISSN: 0944-2669. DOI: 10.1007/BF01191340. URL: http://dx.doi.org/10.1007/BF01191340.

— (1994b). "Harnack inequalities for evolving hypersurfaces". In: *Math. Z.* 217.2, pp. 179–197. ISSN: 0025-5874. DOI: 10.1007/BF02571941. URL: http://dx.doi.org/10.1007/BF02571941.

— (2004). "Fully nonlinear parabolic equations in two space variables". arXiv:math/0402235 [math.AP].

— (2007). "Pinching estimates and motion of hypersurfaces by curvature functions". In: *J. Reine Angew. Math.* 608, pp. 17–33. ISSN: 0075-4102. DOI: 10.1515/CRELLE.2007.051. URL: http://dx.doi.org/10.1515/CRELLE.2007.051.

— (2010). "Moving surfaces by non-concave curvature functions". In: *Calc. Var. Partial Differential Equations* 39.3-4, pp. 649–657. ISSN: 0944-2669. DOI: 10.1007/s00526-010-0329-z. URL: http://dx.doi.org/10.1007/s00526-010-0329-z.

— (2012). "Noncollapsing in mean-convex mean curvature flow". In: *Geom. Topol.* 16.3, pp. 1413–1418. ISSN: 1465-3060. DOI: 10.2140/gt.2012.16.1413. URL: http://dx.doi.org/10.2140/gt.2012.16.1413.

Andrews, Ben and Paul Bryan (2011). "Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem". In: *J. Reine Angew. Math.* 653, pp. 179–187. ISSN: 0075-4102. DOI: 10.1515/CRELLE.2011.026. URL: http://dx.doi.org/10.1515/CRELLE.2011.026.

Andrews, Ben and Christopher Hopper (2011). *The Ricci flow in Riemannian geometry.* Vol. 2011. Lecture Notes in Mathematics. A complete proof of the differentiable 1/4-pinching sphere theorem. Springer, Heidelberg, pp. xviii+296. ISBN: 978-3-642-16285-5. DOI: 10.1007/978-3-642-16286-2. URL: http://dx.doi.org/10.1007/978-3-642-16286-2.

Andrews, Ben and Mat Langford (2013). "Two-sided non-collapsing curvature flows." To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).*

— (2014). "Cylindrical estimates for hypersurfaces moving by convex speed functions." To appear in *Analysis and PDE.*

Andrews, Ben, Mat Langford, and James McCoy (2013). "Non-collapsing in fully non-linear curvature flows". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 30.1, pp. 23–32. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2012.05.003. URL: http://dx.doi.org/10.1016/j.anihpc.2012.05.003.

Andrews, Ben, Mat Langford, and James McCoy (2014a). "Convexity estimates for fully non-linear surface flows." To appear in *J. Differential Geom.*

— (2014b). "Convexity estimates for hypersurfaces moving by convex curvature functions." To appear in *Analysis and PDE.*

Andrews, Ben and James McCoy (2012). "Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature". In: *Trans. Amer. Math. Soc.* 364.7, pp. 3427–3447. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2012-05375-X. URL: http://dx.doi.org/10.1090/S0002-9947-2012-05375-X.

Andrews, Ben, James McCoy, and Yu Zheng (2013). "Contracting convex hypersurfaces by curvature". In: *Calc. Var. Partial Differential Equations* 47.3-4, pp. 611–665. ISSN: 0944-2669. DOI: 10.1007/s00526-012-0530-3. URL: http://dx.doi.org/10.1007/s00526-012-0530-3.

Andrews, Ben et al. (2014). "Non-collapsing for hypersurface flows in the sphere and hyperbolic space". To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).*

Angenent, Sigurd B. (1992). "Shrinking doughnuts". In: *Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989).* Vol. 7. Progr. Nonlinear Differential Equations Appl. Birkhäuser Boston, Boston, MA, pp. 21–38.

Baker, Charles (2010). "The mean curvature flow of submanifolds of high codimension." PhD thesis. Australian National University.

Ball, J. M. (1984). "Differentiability properties of symmetric and isotropic functions". In: *Duke Math. J.* 51.3, pp. 699–728. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-84-05134-2. URL: http://dx.doi.org/10.1215/S0012-7094-84-05134-2.

Böhm, Christoph and Burkhard Wilking (2008). "Manifolds with positive curvature operators are space forms". In: *Ann. of Math. (2)* 167.3, pp. 1079–1097. ISSN: 0003-486X. DOI: 10.4007/annals.2008.167.1079. URL: http://dx.doi.org/10.4007/annals.2008.167.1079.

Brendle, Simon (2013a). "A sharp bound for the inscribed radius under mean curvature flow". arXiv:1309.1459 [math.DG].

— (2013b). "Embedded minimal tori in S^3 and the Lawson conjecture". In: *Acta Math.* 211.2, pp. 177–190. ISSN: 0001-5962. DOI: 10.1007/s11511-013-0101-2. URL: http://dx.doi.org/10.1007/s11511-013-0101-2.

Brendle, Simon and Gerhard Huisken (2013). "Mean curvature flow with surgery of mean convex surfaces in R^3". arXiv:1309.1461 [math.DG].

Brendle, Simon and Richard Schoen (2009). "Manifolds with 1/4-pinched curvature are space forms". In: *J. Amer. Math. Soc.* 22.1, pp. 287–307. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-08-00613-9. URL: http://dx.doi.org/10.1090/S0894-0347-08-00613-9.

Breuning, Patrick (2011). "Immersions with local Lipschitz representation." PhD thesis. Albert-Ludwigs-Universität, Freiburg.

Chavel, Isaac (1993). *Riemannian geometry—a modern introduction.* Vol. 108. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, pp. xii+386. ISBN: 0-521-43201-4; 0-521-48578-9.

Chow, Bennett (1985). "Deforming convex hypersurfaces by the nth root of the Gaussian curvature". In: *J. Differential Geom.* 22.1, pp. 117–138. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214439724.

— (1987). "Deforming convex hypersurfaces by the square root of the scalar curvature". In: *Invent. Math.* 87.1, pp. 63–82. ISSN: 0020-9910. DOI: 10.1007/BF01389153. URL: http://dx.doi.org/10.1007/BF01389153.

— (1991). "On Harnack's inequality and entropy for the Gaussian curvature flow". In: *Comm. Pure Appl. Math.* 44.4, pp. 469–483. ISSN: 0010-3640. DOI: 10.1002/cpa.3160440405. URL: http://dx.doi.org/10.1002/cpa.3160440405.

Clutterbuck, J. and O. C. Schnürer (2011). "Stability of mean convex cones under mean curvature flow". In: *Math. Z.* 267.3-4, pp. 535–547. ISSN: 0025-5874. DOI: 10.1007/s00209-009-0634-4. URL: http://dx.doi.org/10.1007/s00209-009-0634-4.

Cooper, Andrew A. (2010). "A compactness theorem for the second fundamental form." arXiv:1006.5697v4 [math.DG].

Crandall, Michael G., Hitoshi Ishii, and Pierre-Louis Lions (1992). "User's guide to viscosity solutions of second order partial differential equations". In: *Bull. Amer. Math. Soc. (N.S.)* 27.1, pp. 1–67. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-1992-00266-5. URL: http://dx.doi.org/10.1090/S0273-0979-1992-00266-5.

Da Lio, Francesca (2004). "Remarks on the strong maximum principle for viscosity solutions to fully nonlinear parabolic equations". In: *Commun. Pure Appl. Anal.* 3.3, pp. 395–415. ISSN: 1534-0392. DOI: 10.3934/cpaa.2004.3.395. URL: http://dx.doi.org/10.3934/cpaa.2004.3.395.

DeTurck, Dennis M. (1981/82). "Existence of metrics with prescribed Ricci curvature: local theory". In: *Invent. Math.* 65.1, pp. 179–207. ISSN: 0020-9910. DOI: 10.1007/BF01389010. URL: http://dx.doi.org/10.1007/BF01389010.

— (1983). "Deforming metrics in the direction of their Ricci tensors". In: *J. Differential Geom.* 18.1, pp. 157–162. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214509286.

Ecker, Klaus (2007). "A formula relating entropy monotonicity to Harnack inequalities". In: *Comm. Anal. Geom.* 15.5, pp. 1025–1061. ISSN: 1019-8385. URL: http://projecteuclid.org/euclid.cag/1210944228.

— (2014). "Entropy and differential Harnack type formulas for evolving domains". In: *Calc. Var. Partial Differential Equations* 50.1-2, pp. 455–480. ISSN: 0944-2669. DOI: 10.1007/s00526-013-0642-4. URL: http://dx.doi.org/10.1007/s00526-013-0642-4.

Ecker, Klaus and Gerhard Huisken (1989). "Immersed hypersurfaces with constant Weingarten curvature". In: *Math. Ann.* 283.2, pp. 329–332. ISSN: 0025-5831. DOI: 10.1007/BF01446438. URL: http://dx.doi.org/10.1007/BF01446438.

Evans, Lawrence C. (1982). "Classical solutions of fully nonlinear, convex, second-order elliptic equations". In: *Comm. Pure Appl. Math.* 35.3, pp. 333–363. ISSN: 0010-3640. DOI: 10.1002/cpa.3160350303. URL: http://dx.doi.org/10.1002/cpa.3160350303.

Evans, Lawrence Christopher (2010). "A strong maximum principle for parabolic systems in a convex set with arbitrary boundary". In: *Proc. Amer. Math. Soc.* 138.9, pp. 3179–3185. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-2010-10495-1. URL: http://dx.doi.org/10.1090/S0002-9939-2010-10495-1.

Fourès-Bruhat, Y. (1952). "Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires". In: *Acta Math.* 88, pp. 141–225. ISSN: 0001-5962.

Gage, M. and R. S. Hamilton (1986). "The heat equation shrinking convex plane curves". In: *J. Differential Geom.* 23.1, pp. 69–96. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214439902.

Gerhardt, Claus (1996). "Closed Weingarten hypersurfaces in Riemannian manifolds". In: *J. Differential Geom.* 43.3, pp. 612–641. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214458325.

— (2006). *Curvature problems.* Vol. 39. Series in Geometry and Topology. International Press, Somerville, MA, pp. x+323. ISBN: 978-1-57146-162-9; 1-57146-162-0.

Giga, Yoshikazu and Shun'ichi Goto (1992). "Geometric evolution of phase-boundaries". In: *On the evolution of phase boundaries (Minneapolis, MN, 1990–91)*. Vol. 43. IMA Vol. Math. Appl. Springer, New York, pp. 51–65. DOI: 10.1007/978-1-4613-9211-8_3. URL: http://dx.doi.org/10.1007/978-1-4613-9211-8_3.

Glaeser, Georges (1963). "Fonctions composées différentiables". In: *Ann. of Math. (2)* 77, pp. 193–209. ISSN: 0003-486X.

Grayson, Matthew A. (1987). "The heat equation shrinks embedded plane curves to round points". In: *J. Differential Geom.* 26.2, pp. 285–314. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214441371.

Gromov, Mikhael (1981). *Structures métriques pour les variétés riemanniennes*. Vol. 1. Textes Mathématiques [Mathematical Texts]. Edited by J. Lafontaine and P. Pansu. CEDIC, Paris, pp. iv+152. ISBN: 2-7124-0714-8.

Halldorsson, Hoeskuldur P. (2012). "Self-similar solutions to the curve shortening flow". In: *Trans. Amer. Math. Soc.* 364.10, pp. 5285–5309. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2012-05632-7. URL: http://dx.doi.org/10.1090/S0002-9947-2012-05632-7.

Hamilton, Richard S. (1982). "Three-manifolds with positive Ricci curvature". In: *J. Differential Geom.* 17.2, pp. 255–306. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214436922.

— (1986). "Four-manifolds with positive curvature operator". In: *J. Differential Geom.* 24.2, pp. 153–179. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214440433.

— (1995a). "A compactness property for solutions of the Ricci flow". In: *Amer. J. Math.* 117.3, pp. 545–572. ISSN: 0002-9327. DOI: 10.2307/2375080. URL: http://dx.doi.org/10.2307/2375080.

— (1995b). "Harnack estimate for the mean curvature flow". In: *J. Differential Geom.* 41.1, pp. 215–226. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214456010.

— (1995c). "Isoperimetric estimates for the curve shrinking flow in the plane". In: *Modern methods in complex analysis (Princeton, NJ, 1992)*. Vol. 137. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, pp. 201–222. DOI: 10.1016/1053-8127(94)00130-3. URL: http://dx.doi.org/10.1016/1053-8127(94)00130-3.

— (1995d). "The formation of singularities in the Ricci flow". In: *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*. Int. Press, Cambridge, MA, pp. 7–136.

Haslhofer, Robert and Bruce Kleiner (2013a). "Mean curvature flow of mean convex hypersurfaces". arXiv:1304.0926 [math.DG].

— (2013b). "On Brendle's estimate for the inscribed radius under mean curvature flow". arXiv:1309.3231 [math.DG].

Hirsch, Morris W. (1994). *Differential topology*. Vol. 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, pp. x+222. ISBN: 0-387-90148-5.

Huisken, Gerhard (1984). "Flow by mean curvature of convex surfaces into spheres". In: *J. Differential Geom.* 20.1, pp. 237–266. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214438998.

— (1985). "Ricci deformation of the metric on a Riemannian manifold". In: *J. Differential Geom.* 21.1, pp. 47–62. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214439463.

— (1987). "Deforming hypersurfaces of the sphere by their mean curvature". In: *Math. Z.* 195.2, pp. 205–219. ISSN: 0025-5874. DOI: 10.1007/BF01166458. URL: http://dx.doi.org/10.1007/BF01166458.

— (1990). “Asymptotic behavior for singularities of the mean curvature flow”. In: *J. Differential Geom.* 31.1, pp. 285–299. ISSN: 0022-040X. URL: `http://projecteuclid.org/euclid.jdg/1214444099`.

— (1993). “Local and global behaviour of hypersurfaces moving by mean curvature”. In: *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990).* Vol. 54. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, pp. 175–191.

— (1998). “A distance comparison principle for evolving curves”. In: *Asian J. Math.* 2.1, pp. 127–133. ISSN: 1093-6106.

Huisken, Gerhard and Alexander Polden (1999). “Geometric evolution equations for hypersurfaces”. In: *Calculus of variations and geometric evolution problems (Cetraro, 1996).* Vol. 1713. Lecture Notes in Math. Springer, Berlin, pp. 45–84. DOI: `10.1007/BFb0092669`. URL: `http://dx.doi.org/10.1007/BFb0092669`.

Huisken, Gerhard and Carlo Sinestrari (1999a). “Convexity estimates for mean curvature flow and singularities of mean convex surfaces”. In: *Acta Math.* 183.1, pp. 45–70. ISSN: 0001-5962. DOI: `10.1007/BF02392946`. URL: `http://dx.doi.org/10.1007/BF02392946`.

— (1999b). “Mean curvature flow singularities for mean convex surfaces”. In: *Calc. Var. Partial Differential Equations* 8.1, pp. 1–14. ISSN: 0944-2669. DOI: `10.1007/s005260050113`. URL: `http://dx.doi.org/10.1007/s005260050113`.

— (2009). “Mean curvature flow with surgeries of two-convex hypersurfaces”. In: *Invent. Math.* 175.1, pp. 137–221. ISSN: 0020-9910. DOI: `10.1007/s00222-008-0148-4`. URL: `http://dx.doi.org/10.1007/s00222-008-0148-4`.

Hungerbühler, N. and K. Smoczyk (2000). “Soliton solutions for the mean curvature flow”. In: *Differential Integral Equations* 13.10-12, pp. 1321–1345. ISSN: 0893-4983.

Ivey, Thomas (1993). “Ricci solitons on compact three-manifolds”. In: *Differential Geom. Appl.* 3.4, pp. 301–307. ISSN: 0926-2245. DOI: `10.1016/0926-2245(93)90008-O`. URL: `http://dx.doi.org/10.1016/0926-2245(93)90008-O`.

Krylov, N. V. (1982). “Boundedly inhomogeneous elliptic and parabolic equations”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 46.3, pp. 487–523, 670. ISSN: 0373-2436.

Langer, Joel (1985). “A compactness theorem for surfaces with L_p-bounded second fundamental form”. In: *Math. Ann.* 270.2, pp. 223–234. ISSN: 0025-5831. DOI: `10.1007/BF01456183`. URL: `http://dx.doi.org/10.1007/BF01456183`.

Lieberman, Gary M. (1996). *Second order parabolic differential equations.* World Scientific Publishing Co., Inc., River Edge, NJ, pp. xii+439. ISBN: 981-02-2883-X. DOI: `10.1142/3302`. URL: `http://dx.doi.org/10.1142/3302`.

McCoy, James Alexander (2011). “Self-similar solutions of fully nonlinear curvature flows”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 10.2, pp. 317–333. ISSN: 0391-173X.

Michael, J. H. and L. M. Simon (1973). “Sobolev and mean-value inequalities on generalized submanifolds of R^n”. In: *Comm. Pure Appl. Math.* 26, pp. 361–379. ISSN: 0010-3640.

Nguyen, Huy (2008). “Invariant curvature cones and the Ricci flow.” PhD thesis. Australian National University.

Schulze, Felix (2006). “Convexity estimates for flows by powers of the mean curvature”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 5.2, pp. 261–277. ISSN: 0391-173X.

Schwarz, Gerald W. (1975). “Smooth functions invariant under the action of a compact Lie group”. In: *Topology* 14, pp. 63–68. ISSN: 0040-9383.

Sheng, Weimin and Xu-Jia Wang (2009). “Singularity profile in the mean curvature flow”. In: *Methods Appl. Anal.* 16.2, pp. 139–155. ISSN: 1073-2772. DOI: `10.4310/MAA.2009.v16.n2.a1`. URL: `http://dx.doi.org/10.4310/MAA.2009.v16.n2.a1`.

Simons, James (1968). “Minimal varieties in riemannian manifolds”. In: *Ann. of Math. (2)* 88, pp. 62–105. ISSN: 0003-486X.

Smoczyk, Knut (1998). “Starshaped hypersurfaces and the mean curvature flow”. In: *Manuscripta Math.* 95.2, pp. 225–236. ISSN: 0025-2611. DOI: `10.1007/s002290050025`. URL: `http://dx.doi.org/10.1007/s002290050025`.

Stampacchia, Guido (1966). *Èquations elliptiques du second ordre à coefficients discontinus.* Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l’Université de Montréal, Montreal, Que., p. 326.

Stone, Andrew (1994). “A density function and the structure of singularities of the mean curvature flow”. In: *Calc. Var. Partial Differential Equations* 2.4, pp. 443–480. ISSN: 0944-2669. DOI: `10.1007/BF01192093`. URL: `http://dx.doi.org/10.1007/BF01192093`.

Tso, Kaising (1985). “Deforming a hypersurface by its Gauss-Kronecker curvature”. In: *Comm. Pure Appl. Math.* 38.6, pp. 867–882. ISSN: 0010-3640. DOI: `10.1002/cpa.3160380615`. URL: `http://dx.doi.org/10.1002/cpa.3160380615`.

Urbas, John I. E. (1991). “An expansion of convex hypersurfaces”. In: *J. Differential Geom.* 33.1, pp. 91–125. ISSN: 0022-040X. URL: `http://projecteuclid.org/euclid.jdg/1214446031`.

White, Brian (2000). “The size of the singular set in mean curvature flow of mean-convex sets”. In: *J. Amer. Math. Soc.* 13.3, 665–695 (electronic). ISSN: 0894-0347. DOI: `10.1090/S0894-0347-00-00338-6`. URL: `http://dx.doi.org/10.1090/S0894-0347-00-00338-6`.

— (2003). “The nature of singularities in mean curvature flow of mean-convex sets”. In: *J. Amer. Math. Soc.* 16.1, 123–138 (electronic). ISSN: 0894-0347. DOI: `10.1090/S0894-0347-02-00406-X`. URL: `http://dx.doi.org/10.1090/S0894-0347-02-00406-X`.