RAY-MODE DUALITY FOR SH WAVES AND THE EFFECT OF
EARTH STRUCTURE ON FREE OSCILLATION EIGENFREQUENCIES

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by

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STATEMENT

This thesis includes results of research undertaken while I was a full-time research scholar at the Research School of Earth Sciences, Australian National University during the period March 1975 to September 1978.

Except where mentioned in the acknowledgements, the work described in this thesis is my own. This thesis has never been submitted to another university or similar institution.

Chengsung Wang

Chengsung Wang
Canberra
September, 1978
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I would like to thank my wife, Ying-ying, for her encouragement and care which was necessary to complete this research.

During the period of three and a half years in which this research was carried out I was in receipt of an Australian National University postgraduate scholarship.
PUBLICATIONS DURING THE COURSE OF THIS THESIS


ABSTRACT

A major aim in geophysics is to extend the power of available data to constrain Earth structure. Brune (1964) used body wave data to derive information about free oscillations of higher overtone number than so far observed by direct methods. His technique, however, was not designed to cope with the presence of discontinuities in the upper mantle. As a result higher mode data derived by him and Brune and Gilbert (1974) are unable to detect the solotone effect, a persistent oscillatory component in the eigenfrequency spacing of high overtone data, which, as McNabb et al. (1976) have shown, is a consequence of the existence of internal discontinuities. This thesis is mainly concerned with an examination of the constraining power of high overtone eigenfrequencies, and with an extension of Brune's method to include the cases of internal discontinuities.

At the beginning of the present work, a computational study was carried out to show that the solotone effect in torsional oscillations could be derived from the summations of multiple reflections of SH waves from internal discontinuities recorded at small epicentral distances, using Brune's equation. Subsequently, a technique was developed to verify this result mathematically by deriving the McNabb et al. result from these summations. This was accomplished firstly for the case of a single discontinuity, and the analysis was then generalized to include an arbitrary number of discontinuities. The results of the above study were used to examine the relationship between the solotone effect and the depths and magnitudes of discontinuities.

Finally, the overtone behaviour of radial oscillations were investigated computationally. The differences between the solotone effects produced by discontinuities and by transition layers were examined, in order to determine the constraining power of these oscillations on the fine
structure at the core-mantle and inner-outer core boundaries as well as in the upper mantle.
# TABLE OF CONTENTS

## CHAPTER 1  GENERAL INTRODUCTION

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Aims</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Historical Background</td>
<td>3</td>
</tr>
<tr>
<td>1.3</td>
<td>Thesis Outline</td>
<td>5</td>
</tr>
<tr>
<td>1.4</td>
<td>Identification of Mode Numbers</td>
<td>6</td>
</tr>
</tbody>
</table>

## CHAPTER 2  ASYMPTOTIC OVERTONE STRUCTURE IN EIGENFREQUENCIES OF TORSIONAL NORMAL MODES OF THE EARTH: A MODEL STUDY

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>8</td>
</tr>
<tr>
<td>2.2</td>
<td>Asymptotic Formula, Asymptotic Behaviour of Overtone Eigenfrequencies, and the Solotone Effect</td>
<td>10</td>
</tr>
<tr>
<td>2.3</td>
<td>Brune's Phase Correlation Method</td>
<td>14</td>
</tr>
<tr>
<td>2.4</td>
<td>Asymptotic Behaviour of Eigenfrequencies Computed from SH Wave Seismograms by Brune's Phase Correlation Method</td>
<td>16</td>
</tr>
<tr>
<td>2.5</td>
<td>Discussion</td>
<td>27</td>
</tr>
</tbody>
</table>

## CHAPTER 3  RAY-MODE DUALITY FOR SH WAVES IN EARTH MODELS WITH CRUST AND MANTLE DISCONTINUITIES: I. THE CASE OF ONE DISCONTINUITY

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>30</td>
</tr>
<tr>
<td>3.2</td>
<td>Preliminaries</td>
<td>31</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Application of Brune's Phase Correlation Method for Small Epicentral Distances</td>
<td>31</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Crust and Mantle Models with a Single Discontinuity</td>
<td>33</td>
</tr>
<tr>
<td>3.2.3</td>
<td>Combinatorial Lemmas for the Summation of Multiple Reflections from Discontinuities</td>
<td>35</td>
</tr>
<tr>
<td>3.3</td>
<td>Analysis of Rays</td>
<td>36</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Ray Decomposition</td>
<td>38</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Summation of Equivalent Rays</td>
<td>40</td>
</tr>
<tr>
<td>3.4</td>
<td>Formulation of $f_1(t)$ and $f_2(t)$ and their Fourier Transforms</td>
<td>45</td>
</tr>
<tr>
<td>3.5</td>
<td>Derivation of $F_1(\omega)/F_2(\omega)$</td>
<td>46</td>
</tr>
<tr>
<td>3.6</td>
<td>Derivation of Eq. (3.1) from Brune's Formulation</td>
<td>48</td>
</tr>
</tbody>
</table>

### APPENDIX Expression of a ray in terms of fundamental ray components | 49 |

## CHAPTER 4  A STRATEGY FOR THE SUMMATION OF SEISMIC SIGNALS MULTIPLY-REFLECTED AT INTERNAL DISCONTINUITIES

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>51</td>
</tr>
<tr>
<td>4.2</td>
<td>Earth Models and Terminology</td>
<td>52</td>
</tr>
<tr>
<td>4.3</td>
<td>Ray Decomposition</td>
<td>56</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Properties of Rays</td>
<td>56</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Equivalent Rays</td>
<td>57</td>
</tr>
<tr>
<td>4.4</td>
<td>Summation of Equivalent Rays</td>
<td>59</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Numbers of Non-Equivalent Junctions</td>
<td>60</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Construction of Equivalent Rays</td>
<td>65</td>
</tr>
<tr>
<td>4.5</td>
<td>Summation of Multiple Reflections of Signals</td>
<td>71</td>
</tr>
<tr>
<td>4.6</td>
<td>Re-interpretation and Extension of a Filter for Removing Reverberations in Water Layer</td>
<td>72</td>
</tr>
</tbody>
</table>

### APPENDIX Uniqueness of a ray decomposition of Eq. (4.2) | 77 |
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Model B497 and its variations B497C, B497D and B497DC</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>Asymptotic overtone structures ($\eta \gamma_{l,n}$ curves) of the model Earth eigenfrequencies computed using Eq. (2.1) for models B497C, B497 and B497D.</td>
<td>12</td>
</tr>
<tr>
<td>2.3</td>
<td>Dependence of the amplitude of the oscillation in the solotone effect on the magnitude of a shear velocity discontinuity at a fixed depth of 531 km.</td>
<td>13</td>
</tr>
<tr>
<td>2.4</td>
<td>Dependence of the frequency of the oscillation in the solotone effect on the depth of a small shear velocity discontinuity of a fixed magnitude of 0.3 km/s</td>
<td>15</td>
</tr>
<tr>
<td>2.5</td>
<td>The ray paths and the corresponding model Earth response functions used for models B497DC and B497D.</td>
<td>18</td>
</tr>
<tr>
<td>2.6</td>
<td>Interpolation method used to evaluate, from the synthetic SH-wave pulses shown in Fig. 2.5, the eigenfrequencies $\eta \omega_{l}$ for which $l$ is an integer.</td>
<td>20</td>
</tr>
<tr>
<td>2.7</td>
<td>The phase difference functions $g_{l}(\omega)$ and the asymptotic overtone structures ($\eta \gamma_{l,n}$ curves), based on eigenfrequencies computed using Eqs. (2.3) and (2.4), for models B497C and B497DC.</td>
<td>22</td>
</tr>
<tr>
<td>2.8</td>
<td>The phase difference functions $g_{l}(\omega)$ and the asymptotic overtone structures ($\eta \gamma_{l,n}$ curves), based on eigenfrequencies computed using Eqs. (2.3) and (2.4), for model B497D.</td>
<td>26</td>
</tr>
<tr>
<td>3.1</td>
<td>Crust and mantle models with only one discontinuity between the Earth's surface and the core-mantle boundary.</td>
<td>34</td>
</tr>
<tr>
<td>3.2</td>
<td>The six fundamental ray components for Earth models containing only one discontinuity.</td>
<td>37</td>
</tr>
<tr>
<td>3.3</td>
<td>The two rays which have the same ordered set (1,1,0).</td>
<td>41</td>
</tr>
<tr>
<td>3.4</td>
<td>The construction of one of the equivalent rays determined by the ordered set (2,3,1).</td>
<td>44</td>
</tr>
</tbody>
</table>
Crust and mantle models with \( N \) discontinuities between the Earth's surface and the core-mantle boundary.

The twelve fundamental ray components for models containing two discontinuities between the Earth's surface and the core-mantle boundary.

The two rays which have the same set of ray segments and the same set of reflections at and transmissions across interfaces, yet belong to the different ordered sets \((1; 1,1;0,0,0)\) and \((2;0,0;0,1,0)\).

The two ways of constructing the same ray with an ordered set \((1;0,1;0,1,0)\) from a ray with an ordered set \((1;0,1;0,0,0)\).

The three types of fundamental reflections which are associated with \( S_i \) and \( S_j \) \((1 \leq i < j \leq N)\) and have a longer length than \( t_i^0 \).

The fundamental reflections which are associated with \( S_0 \) (the Earth's surface) and \( S_j \) \((j=1,2,...,N)\) and have a longer length than \( t_0^0 \).

The triangular pyramids of the numbers \( m_i^{j+1} \) and the corresponding numbers \( k_i^{j+1} \).

The construction of the equivalent rays determined by the ordered set \((1;2,1;1,2,1)\).

Diagram showing that the wave reverberations in the water layer associated with a reflection from an interface below the water bottom can occur at both sides of the reflection.

Representations of terms in Expression (i), Expression (ii), \( i+j\mu_{L+j} \) and \( i+j\nu_{L+j} \) of Eq. (5.D4), and representations of the products of the terms in Expressions (i) and (ii) and of the terms in \( l+j\mu_{L+j} \) and in \( l+j\nu_{L+j} \).

The correspondence of an ordered pair \([U,V]\) of different sets which produces a term on the left-hand side of Eq. (5.D4) to an ordered pair which produces a term on the right-hand side.

\( N \) curves for Earth models with a discontinuity of almost the same magnitude.
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2</td>
<td>Diagram illustrating the meaning of $h(n)$ for an Earth model with a single discontinuity</td>
<td>121</td>
</tr>
<tr>
<td>6.3</td>
<td>The diagram showing the departure of overtone eigenfrequencies computed by MODE from $2\pi n/T$ for $l=1$ and $l=2$ (for an Earth model with a homogeneous crust and mantle).</td>
<td>122</td>
</tr>
<tr>
<td>6.4</td>
<td>Comparison with $\Gamma_1$ values derived from eigenfrequencies computed by MODE and those evaluated from Eq. (6.9) (for Earth models shown in Figs. 6.1A and 6.1C).</td>
<td>126</td>
</tr>
<tr>
<td>6.5</td>
<td>Comparison between $\Gamma_1$ values derived from eigenfrequencies computed by MODE and those evaluated from Eq. (6.14) (for an Earth model shown).</td>
<td>129</td>
</tr>
<tr>
<td>6.6</td>
<td>Comparison between $\Gamma_1$ values derived from eigenfrequencies computed by MODE and those evaluated from Eq. (6.17) (for an Earth model shown).</td>
<td>131</td>
</tr>
<tr>
<td>6.7</td>
<td>Comparison between $\Gamma_1$ values derived from eigenfrequencies computed by MODE and those derived from solutions of Eq. (5.26) calculated by a computer with the BISECTION algorithm (for an Earth model shown).</td>
<td>133</td>
</tr>
<tr>
<td>7.1</td>
<td>Comparison between amplitudes of the solotone oscillations corresponding to a discontinuity and to a transition layer, respectively, at 600 km depth.</td>
<td>135</td>
</tr>
<tr>
<td>7.2</td>
<td>Diagram showing how a discontinuity in a parameter is replaced by a linear transition layer.</td>
<td>139</td>
</tr>
<tr>
<td>7.3</td>
<td>Models 1066A, HOMO and some other models ACxIy and HCxIy derived from them.</td>
<td>140</td>
</tr>
<tr>
<td>7.4</td>
<td>$\overline{V}_0 - n$ curves constructed from ACxIy with varying thickness $y$.</td>
<td>143</td>
</tr>
<tr>
<td>7.5</td>
<td>$\overline{V}_0 - n$ curves constructed from HCxIy with varying thickness $y$.</td>
<td>144</td>
</tr>
<tr>
<td>7.6</td>
<td>$\overline{V}_0 - n$ curves constructed from ACxIy with varying thickness $x$.</td>
<td>146</td>
</tr>
<tr>
<td>7.7</td>
<td>$\overline{V}_0 - n$ curves constructed from HCxIy with varying thickness $x$.</td>
<td>147</td>
</tr>
<tr>
<td>7.8</td>
<td>The dependence of the magnitude of the C/M transition layer on the thickness of the layer for different Earth models.</td>
<td>148</td>
</tr>
<tr>
<td>7.9</td>
<td>Comparison of the effect of the continuous upper mantle of 1066A and that of the discontinuous one of 1066B on the $\overline{V}_0 - n$ curve.</td>
<td>151</td>
</tr>
</tbody>
</table>
Figure

7.10 Comparison of $\bar{n}_0 - n$ curves for models 1066A and 1066B

7.11 Observed error ranges and comparison of observed $\bar{n}_0$ values and those constructed from model 1066A and two perturbed models of 1066A.

7.12 Comparison of $\bar{n}_0$ and $[\bar{n}_0(AC800I0) + \bar{n}_0(AC0I800) - \bar{n}_0(AC800I800)]$ values.
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>P values and arrival times for the rays shown in Fig. 2.5A.</td>
<td>19</td>
</tr>
<tr>
<td>2.2</td>
<td>Values of $\eta^{(m)}_L$ derived from the data of Brune and Gilbert (1974), compared with values from models 1066A and 1066B.</td>
<td>29</td>
</tr>
</tbody>
</table>
1.1 AIMS

Virtually all the direct information about the structure of the Earth's interior has been derived from measurements of the times, amplitudes and periods of seismic waves. Studies of body and surface waves have provided a great deal of information about the distributions of compressional and shear wave velocities ($\alpha$ and $\beta$) in the Earth's interior. However, the diversity in the calculated velocity distributions has been a major problem. In addition, the determination of density from velocity distributions was impossible without an appeal to plausible assumptions concerning chemical and phase homogeneity in layers with smooth velocity profiles.

Since the differential equations for the free oscillations of the Earth explicitly involve $\rho$, $\alpha$, and $\beta$ (or $\rho$, $k$ and $\mu$, where $k$ and $\mu$ are the bulk and rigidity moduli), important additional direct constraints on the solution of these problems have been provided by the observed eigenfrequencies (or eigenperiods) of the Earth. The first indisputable free oscillation data were those presented at an international meeting in Helsinki in 1960 (Bullen, 1975, p. 301). Using improved techniques of data acquisition, the set of free oscillation data has been greatly enlarged by Pekeris (1966), Derr (1969), Dziewonski and Landisman (1970), Mendiguren (1973), Dziewonski and Gilbert (1972, 1973), Brune and Gilbert (1974) and Gilbert and Dziewonski (1975). The chief contribution of these data has been in the refinement of existing Earth models and thus in substantially reducing the diversity of Earth models. However, the existing set of free oscillation data (consisting mainly of eigenperiods of low overtone numbers) is incapable of resolving fine structure in the
Earth. For example, using the same refinement technique and the same set of free oscillation data, Gilbert and Dziewonski (1975) obtained two different Earth models, 1066A and 1066B, from different starting models. The differences between these models are mainly concerned with limited structural features such as discontinuities and rapid transition regions in the upper mantle, such that 1066A has a 'continuous' upper mantle, and 1066B a 'discontinuous' upper mantle.

Free oscillations are defined such that their wavelengths decrease with increasing overtone number. As a consequence, it is not possible to distinguish between continuous and discontinuous models (such as 1066A and 1066B) unless free oscillation data of sufficiently large overtone number are available. It is therefore clear that, in order to obtain better resolution of the Earth's fine structure, free oscillation data of high overtone number are required. These data are related to body waves (e.g., Sato et al., 1963), but have not yet been observed directly due to attenuation effects.

Applying the constructive interference condition for body waves of the same apparent phase velocity, Brune (1964) devised a phase correlation method (reviewed in §2.3) which makes possible the evaluation of normal modes of high overtone number directly from body wave pulses in seismograms. As pointed out by Ben-Menaham (1964), Brune's method represents an application of a ray-mode duality to the representation of the Earth's displacement field. His method is, however, strictly applicable only to continuous Earth models, and requires modification in order to be applied to discontinuous ones.

As Cleary and Anderssen (1978) have pointed out, further development of background theory for free oscillation data inversion is needed. In particular, the relation between free oscillation eigenfrequencies and Earth structure requires more explicit formulation along the lines initiated by Anderssen and Cleary (1974).
In accordance with the above considerations, this thesis is chiefly concerned with

1. an examination of Brune's strategy for supplementing available observed eigenfrequencies (especially of high overtone number). This is done by showing that the equation derived by McNabb, Anderssen and Lapwood (1976) for the asymptotic behaviour of overtone eigenfrequencies can be derived by an adaptation of Brune's method to almost vertical SH waves multiply-reflected in discontinuous Earth models;

2. the development of background theory which describes explicitly the dependence of free oscillation overtone eigenfrequencies on Earth structure. To this end, the effects of discontinuities and transition regions in the Earth's interior on the free oscillation eigenfrequencies have been investigated theoretically and by model studies.

1.2 HISTORICAL BACKGROUND

For spherically symmetric, non-rotating, elastic and isotropic (SNREI) Earth models, the free oscillation eigenfrequencies $\frac{n}{\ell}$ with overtone number $n$ and angular order number $\ell$ are defined (e.g., Alterman, Jarosch and Pekeris, 1959) by the differential eigenvalue problem

$$\frac{\mu}{r^2} \frac{d}{dr} \left( r^2 \frac{du}{dr} \right) + \frac{du}{dr} \frac{d}{dr} \left( \frac{du}{dr} - \frac{U}{r} \right) + \left( \frac{\sigma^2}{r^2} - \frac{\ell (\ell + 1)}{r^2} \right) \mu U = 0 \quad (1.1)$$

$\sigma = n \sigma_{\ell}$, $U = U(r)$, $\mu = \mu(r)$, $\rho = \rho(r)$, $R_c \leq r \leq R_o$

with boundary conditions

$$\mu \frac{dU}{dr} - \frac{U}{r} = 0, \quad r = R_c \quad \text{and} \quad r = R_o,$$

where $\mu$ is rigidity, $\rho$ is density, and $R_c$ and $R_o$ are the radii of the core and the Earth's surface respectively.
In addition to the numerous investigations of the interdependence of free oscillation eigenfrequencies and Earth structure along the lines initiated by Backus and Gilbert (1967, 1968, 1970) (c.f. Gilbert and Dziewonski, 1975), a number of recent studies have concentrated on the properties of the asymptotic overtone eigenfrequencies. Anderssen, Osborne and Cleary (1974) found that the spacings between torsional eigenfrequencies $\sigma_n$ of successive overtone numbers $n$ and fixed angular order number $l$ contain information about the structure of SNREI Earth models. Subsequently, the effect of discontinuities in the Earth's interior on eigenfrequency spacings has been studied by various authors (Anderssen and Cleary, 1974; Lapwood, 1975; McNabb, Anderssen and Lapwood, 1976; Anderssen, Cleary and Dziewonski, 1975; Gilbert, 1975). The central conclusion of these studies is that, for fixed small $l$ and sufficiently large $n$, discontinuities in the Earth's interior cause a persistent oscillatory component, called the solotone effect, in the spacings.

In particular, McNabb, Anderssen and Lapwood (1976) derived an equation for the asymptotic behaviour of the eigenfrequencies of a Sturm-Liouville system (reduced from Eq. (1.1)) with discontinuous coefficients. For SNREI Earth models with $N$ discontinuities between the Earth's surface and the core-mantle boundary, their equation can be written as

$$\sin (a\sigma) = \sum_{j=1}^{2N-1} B_j \sin(b_j \sigma)$$  \hspace{1cm} (1.2)

where $\sigma$ is the shear wave radial travel time from the Earth's surface to the core-mantle boundary, and $B_j$ and $b_j$ are constants which depend on the parameters defining the discontinuities. The constants $B_j$ and $b_j$ have not been determined explicitly except for the cases where $N=1$ (Anderssen, 1977) and $N=2$ (Lapwood and Sato, 1977). The solotone effect can be interpreted in terms of Eq. (1.2).
1.3 THESIS OUTLINE

The dependence of the solotone effect on parameters defining the discontinuities in the Earth interior is investigated and a physical explanation of the solotone effect is sought in Chapter 2. It is found, for the case of torsional free oscillations, that some properties of the solotone effect vary systematically with the depths and the magnitudes of discontinuities, and that the solotone effect can be interpreted in terms of SH waves multiply-reflected from discontinuities.

For an Earth model with a single discontinuity between the surface and the core-mantle boundary, the basic equation (Eq. (1.2)) for the asymptotic behaviour of the torsional overtone eigenfrequencies is derived in Chapter 3, by summing almost vertical SH multiple reflections in the model by a ray analysis strategy and then adapting Brune's method to the summations. In Chapters 4 and 5, the ray analysis strategy is extended to Earth models with N discontinuities, and Eq. (1.2) is derived (with constants determined) using this extended strategy. This provides a theoretical proof that the solotone effect arises from multiple reflections between the Earth's surface, the core-mantle boundary and discontinuities between them, and establishes a basis for a ray-mode duality for discontinuous models.

The properties of the solotone effect summarised in Chapter 2 are derived in Chapter 6 from Eq. (1.2) for Earth models with (i) a single discontinuity which is either small or near the surface, or (ii) more than one small discontinuity. This connects the theoretical and model studies in the earlier chapters.

Finally, the effect on the radial free oscillations of the transition regions at the core-mantle and inner-outer core boundaries, and in the mantle, has been investigated by model studies. It has been established that the distribution of eigenfrequencies of low overtone
number \( n \) reflects the overall spherically symmetric structure of the Earth, while that of eigenfrequencies of high \( n \) tends to reflect vertically limited Earth structures such as discontinuities and very rapid transition regions. The details of this study are given in Chapter 7.

The results of the thesis are discussed in Chapter 8.

1.4 IDENTIFICATION OF MODE NUMBERS

The normal modes of the Earth can be grouped into two classes: spheroidal modes \( n^m_S \) and torsional modes \( n^m_T \). Since the deformation of the Earth's surface can be expressed uniquely in terms of the spherical harmonic functions \( [P^m \cos \theta \cos \lambda] \) (where \( \theta \) is the latitude and \( \lambda \) the longitude; \( l = 0,1,\ldots \) and \( m = 0,1,\ldots l \)), the integers \( l \) (called the angular order number or simply order number) and \( m \) define the pattern of surface displacement with regard to the source of disturbance. The remaining integer \( n \) (called the overtone number) defines the overtone pattern, i.e., the pattern of displacements as a function of depth. If Earth models are assumed to be non-rotating and spherically symmetric, the set \( (n, l, m) \) is degenerate and can be replaced by \( (n, l) \), since the deformation of the Earth's surface is now uniquely describable in terms of the zonal harmonics \( [P^l \cos \theta] \) \( (l = 0,1,\ldots) \). Under such conditions, there are \( l(l=0,1,\ldots) \) lines of nodal latitude in the surface pattern for spheroidal modes \( n^m_S \) and \( l-1(l=1,2,\ldots) \) lines for torsional modes \( n^m_T \). If, in addition, the Earth models are assumed to be solid, both \( n^S \) and \( n^T \) modes have \( n \) nodal surfaces in the Earth's interior (MacDonald and Ness, 1961). It is noted that the \( S_1 \) and \( T_1 \) modes do not exist since they involve changes in linear and angular momentum respectively.

The study of the Earth's free oscillations is complicated by the presence of a liquid outer core which consequently does not permit the transmission of shear waves. This complication does not affect the usual
interpretation (cf. MacDonald and Ness, 1961) of the notation $n_{T \ell}$ if attention is restricted to the torsional free oscillations in the mantle (in this case, $n$ is the number of nodal surfaces between the surface and the core-mantle boundary). However, $n$ in the notation $n_{S \ell}$ no longer denotes the number of nodal surface in the Earth's interior if it is assigned according to sequentially decreasing period for a given $\ell$. This leads to various difficulties (Anderssen, Cleary and Dziewonski, 1975). For example, if a new mode is discovered, it is often necessary to change the identification of some others. This makes the prediction of new modes and the study of the distribution of spheroidal overtone eigenfrequencies difficult.

For high eigenfrequencies of a small fixed angular order number $\ell$, Anderssen, Cleary and Dziewonski (1975) have classified the spheroidal modes into three sequences equivalent to the following three body wave types: PKIKP, (ScS)$_V$ or $J_V$ (shear waves in the inner core). For $K = PKIKP$, (ScS)$_V$ or $J_V$, the symbol $n_{S \ell} (K)$ was introduced by them to identify the normal modes corresponding to the body wave type $K$. As a consequence, the standard meaning of overtone number $n$ (cf. MacDonald and Ness, 1961) is retained. However, such an identification does not apply to normal modes with small $n$ and large $\ell$.

In this thesis, the classification of Anderssen, Cleary and Dziewonski (1975) is used for spheroidal normal modes, since we deal with normal modes of small $\ell$. Spheroidal normal mode eigenfrequencies corresponding to the different sequences are written as $\sigma_{\ell} (K)$ and torsional eigenfrequencies as $\sigma_{\ell} (ScS_H)$. However, in Chapters 2 through 7 the notation $\sigma_{\ell}$ is used when there is no ambiguity.
CHAPTER 2

ASYMPTOTIC OVERTONE STRUCTURE IN EIGENFREQUENCIES OF TORSIONAL NORMAL MODES OF THE EARTH: A MODEL STUDY

2.1 INTRODUCTION

Asymptotic behaviour of eigenfrequencies of normal modes of the Earth has been the subject of some investigation in the past few years (§1.2). The main conclusion of the investigation has been that discontinuities in the interior of a SNREI Earth model will produce a persistent oscillatory component (the solotone effect) in the asymptotic spacings between successive overtone eigenfrequencies. In order to understand this phenomenon, a model study is useful since the factors which may produce it can be studied separately. The present chapter shows that the oscillations in the solotone effect result from multiple reflections from discontinuities in the Earth's interior, and that the frequencies and amplitudes of the oscillations vary systematically with the depths and magnitudes of the discontinuities.

In this chapter the changes produced in the solotone effect by discontinuities at different depths and of different magnitudes are first examined for various Earth models. The asymptotic overtone structure in eigenfrequencies computed from synthetic SH-wave seismograms by Brune's (1964) phase correlation method is then constructed, to show that the solotone effect can be reproduced by summation of relevant multiple reflections from the discontinuities.

Earth models B497 (Gilbert, Dziewonski and Brune, 1973; Dziewonski and Gilbert, 1973, Appendix A1) and variations, including B497C, B497D and B497DC (Fig. 2.1), are used throughout the study. B497 is continuous except for a large discontinuity in density and wave velocity at 10 km depth; its construction was based on, among other
Fig. 2.1 Earth models used in this chapter. See the text for their descriptions.
constraints, certain eigenfrequencies (Brune and Gilbert, 1974) computed by Brune's phase correlation method. B497D was derived from B497 by the addition of a 0.77 km/s discontinuous increase in shear velocity at 600 km depth. The radial travel times for SH-waves from the Earth's surface to the core-mantle boundary for these two models were constrained to be the same. B497C and B497DC are the same, respectively, as B497 and B497D, except for the absence of the crustal layer.

2.2 ASYMPTOTIC FORMULA, ASYMPTOTIC BEHAVIOUR OF OVERTONE EIGENFREQUENCIES, AND THE SOLOTONE EFFECT

The free oscillation eigenfrequencies \( n_{\sigma} \) for overtone number \( n \) and angular order number \( l \) for SNREI Earth models are defined by Eq. (1.1). Using the variational method of Backus and Gilbert (1967), Gilbert wrote a computer program MODE to compute eigenfrequencies \( n_{\sigma} \) from Eq. (1.1). Throughout this thesis MODE is used for the investigation of the behaviour of the overtone structure in the eigenfrequencies.

Under the assumption that the coefficients in Eq. (1.1) contain no discontinuities, Anderssen and Cleary (1974) derived from Eq. (1.1), using the Sturm-Liouville transformation, the following asymptotic formula for \( n_{\sigma} \)

\[
\frac{2}{n_{\sigma}^2} = (n\pi)^{2} + A Y^{-2} + B Y^{-2} + O(n^{-2})
\]

(2.1)

for fixed \( l \) and suitably large \( n (n \geq 10 \) was sufficient for the models used in this study), where \( Y \) is the radial travel time for (ScS)_H waves between the Earth's surface and the core-mantle boundary and \( A \) and \( B \) are constants independent of \( n \). The formula

\[
n_{\gamma} = \pi(2n + 1)^{1/2}(n + 1, \sigma_{n}^2 - n_{\sigma}^2)^{-1/2}
\]

(2.2)

follows from Eq. (2.1), whenever \( n \) assumes values such that the leading term in Eq. (2.1) dominates the other non-constant terms and the second
and higher order effects can be neglected. With fixed $l$, $nY_l$ is expected to approach an asymptote for a continuous Earth model as $n$ increases (Anderssen, Osborne and Cleary, 1974; Anderssen and Cleary, 1974). The behaviour of $nY_l$ is quite different for models containing discontinuities; for these models $nY_l$ will oscillate about the asymptote in a persistent manner. This phenomenon is the solotone effect (McNabb, Anderssen and Lapwood, 1976).

Curves 1, 2 and 3 in Fig. 2.2 show the asymptotic overtone structures ($nY_l$ curves) in the eigenfrequencies of Eq. (1.1) for models B497C, B497 and B497D (Fig. 2.1) respectively. Curve 1 (B497C) shows that the $Y$ value quickly approaches its asymptote, the constant value of which is the radial travel time for SH waves from the surface of the model Earth to the core-mantle boundary. This is the characteristic asymptotic behaviour of the normal mode eigenfrequencies for a continuous Earth model.

It can be seen in Fig. 2.2 that the solotone effect is in the form of an oscillation about a value of $Y$ which depends on the model. By analogy with more common forms of oscillation we may describe the effect in terms of frequency (in units of cycles per step of increase in $n$) and amplitude (in units of seconds).

Curve 2 shows the effect produced by the 10 km discontinuity (crust-mantle boundary) of B497, and curve 3 the composite effect of the 10 and 600 km discontinuities of B497D. The 10 km discontinuity of B497 causes the low-frequency oscillation of curve 2, while the 600 km discontinuity of B497D causes the high-frequency oscillation of curve 3 about curve 2.

The amplitude of the oscillation in the solotone effect increases, for a fixed depth, with the magnitude of the discontinuity (Fig. 2.3), while the frequency of the oscillation is closely related
Fig. 2.2 Asymptotic overtone structures \((\gamma_n n)\) curves of the model Earth eigenfrequencies computed using Eq. (2.2). Curves 1, 2 and 3 correspond to models B497C, B497 and B497D (Fig. 2.1) respectively.
Fig. 2.3 Dependence of the amplitude of the oscillation in the solotone effect on the magnitude of a shear velocity discontinuity at a fixed depth of 531 km. Models used are shown above (the magnitudes of the discontinuities are 0.66, 1.16 and 1.66 km/s respectively).
to the radial travel times $\bar{x}_1$ and $\bar{x}_2$ from the Earth's surface and the core-mantle boundary, respectively, to the discontinuity. When the discontinuity is moved down from the Earth's surface towards the core-mantle boundary, the frequency of the solotone oscillation (Figs. 2.4A and 2.4B) increases as $\bar{x}_1$ increases until $\bar{x}_1 = \bar{x}_2'$, and then decreases as $\bar{x}_2$ decreases (Figs. 2.4G and 2.4H). When the absolute value of $(\bar{x}_1 - \bar{x}_2)$ becomes small, there is observed, in addition to the frequency mentioned above, a superposed frequency in the solotone effect (Figs. 2.4C-F) which is a function of both $\bar{x}_1$ and $\bar{x}_2$. A theoretical verification of these results can be found in Anderssen (1977) and Chapter 6.

2.3 BRUNE'S PHASE CORRELATION METHOD

Brune (1964) has presented a method which permits the evaluation of overtone eigenfrequencies from pairs of body wave pulses of equal apparent phase velocity $d\Delta/dT = 1/p$ (where $p$ is the ray parameter), e.g. $[(\text{ScS})_\Delta, (2\text{ScS})_{2\Delta}]$ or $[(\text{S})_\Delta, (\text{SS})_{2\Delta}]$, where the first and second phases of the pairs are recorded at distances $\Delta$ and $2\Delta$ respectively. The method is particularly well suited to SH waves since there are no P/SV conversions at interfaces to complicate the result.

The pair $[(\text{ScS})_\Delta, (2\text{ScS})_{2\Delta}]$ is taken as an example and this method is reviewed briefly. Since $(\text{ScS})$ and $(2\text{ScS})_2$ have the same phase velocity, they can be represented approximately by the superimposition of normal modes having that phase velocity. The component frequencies satisfying the following constructive condition then correspond to overtone normal modes

$$\phi_2(\omega) + 2\pi n = \phi_1(\omega) + (T - \Delta \cdot dT/d\Delta) \cdot \omega$$

or

$$n = [(T - \Delta \cdot dT/d\Delta) \cdot \omega - g(\omega)] / 2\pi; \quad g(\omega) = \phi_2(\omega) - \phi_1(\omega)$$
Fig. 2.4 Dependence of the frequency of the oscillation in the solotone effect on a small shear velocity discontinuity of a fixed magnitude of 0.3 km/s. Models used are of constant density 4.5 g/cm$^3$ and have shear velocity functions shown at the right (the depths of the discontinuities are 300, 600, 1000, 1291, 1586, 1900, 2286 and 2586 km respectively).
where $\phi_1(\omega)$ and $\phi_2(\omega)$ are Fourier transform phases of $(ScS)_\Delta$ and $(2ScS)_{2\Delta}'$ respectively, and $T$ is the difference between the arrival times of $(2ScS)_{2\Delta}$ and $(ScS)_\Delta$. To avoid ambiguity $\sigma$ is used for the eigenfrequencies of Eq. (1.1) and $\omega$ for angular frequencies of waves recorded at the receiver. If $T$ is chosen correctly such that $g(\omega)$ is restricted to the range $\pm \pi$, $n$ corresponds to an overtone number of the Earth's free oscillations.

2.4 ASYMPTOTIC BEHAVIOUR OF EIGENFREQUENCIES COMPUTED FROM SH WAVE SEISMOGRAMS BY BRUNE'S PHASE CORRELATION METHOD

The advantage of Brune's method as formulated above is the removal of the source effect, but a disadvantage is the restriction of the application only to body wave pulses having the same apparent phase velocity. It will be shown that the solotone effect arises from the superposition of multiple reflections from discontinuities in the Earth's interior, and these reflections have apparent phase velocities which are not identical to those of the primary phases. The derivation of $\gamma$ values from Brune's (1964) and Brune and Gilbert's (1974) calculations of eigenfrequencies from SH data showed no solotone effect (Anderssen, Osborne and Cleary, 1974; Anderssen and Cleary, 1974), because the time window used by them would have excluded any such reflections from the analysis. The evaluation of eigenfrequencies from body-wave pulses including the multiple reflections from the discontinuities is therefore not strictly covered by the Brune formulation. This problem is not examined here, but it is assumed that, for small epicentral distances, changes in apparent phase velocity do not perturb the solution seriously.

The determination of $n\gamma$ values from eigenfrequencies of normal modes obtained from body-wave seismograms using Brune's (1964) method is, as noted previously, independent of the source function. Only the model Earth response functions are therefore used for synthetic seismogram
calculations. The output of the synthetic seismogram program can then be considered to be a sequence of δ-functions; this considerably simplifies the discussion of the solotone effect in terms of multiply-reflected body waves.

Figs. 2.5A and 2.5B represent some of the ray paths and the corresponding response functions for Earth models B497DC and B497D. They and their equivalents are considered to be the largest contributors to the solotone effect. The study is restricted to Δ ≤ 10° and Eq. (2.3a) is used as an approximation formula (Table 2.1 shows the p values and arrival times for the rays at Δ = 5° and 10° for B497DC). In Eq. (2.3a) T is taken as the difference between \((2S_cS)_{2\Delta}\) and \((S_cS)_{\Delta}\) arrival times, \(dT/d\Delta = p\) for \((S_cS)_{\Delta}\) and \((2S_cS)_{2\Delta}\), and \(\phi_1(\omega), \phi_2(\omega)\) are the Fourier transform phases of the summations of relevant pulses (Fig. 2.5) recorded at Δ and 2Δ with respect to the \((S_cS)_{\Delta}\) and \((2S_cS)_{2\Delta}\) arrival times. For a given epicentral distance Δ, the frequencies \(n\omega\) are determined by varying \(\omega\) until \(n\) is an integer (the corresponding \(g(\omega)\) are denoted by \(g(n,\omega)\)). The relation (Brune and Gilbert, 1974)

\[ \ell + 1/2 = n\omega \cdot dT/d\Delta \tag{2.4} \]

is used to give the angular order number \(\ell\) for the given Δ and each \(n\omega\) are designated by '.' in Fig. 2.6). The frequencies \(n\omega\) for which \(\ell\) is an integer are determined by interpolation (Fig. 2.6) from \(n\omega\) values
Fig. 2.5 The ray paths and the corresponding model Earth response functions used in this chapter for models B497DC(A) and B497D(B). These response functions (at $\Delta = 5$) are the output of the computer program written by Wiggins and Helmberger (1974) using the Cagniard-de Hoop algorithm.
### Table 2.1  
*p* values and arrival times for the rays shown in Fig. 2.5A

#### $\Delta = 5^\circ$

<table>
<thead>
<tr>
<th>ray</th>
<th>ScS</th>
<th>a</th>
<th>b</th>
<th>2ScS</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>p value</td>
<td>0.8904</td>
<td>0.8170</td>
<td>0.4674</td>
<td>0.8904</td>
<td>0.8521</td>
<td>0.6131</td>
</tr>
<tr>
<td>arrival time</td>
<td>939.127</td>
<td>1194.021</td>
<td>1619.886</td>
<td>1878.255</td>
<td>2133.141</td>
<td>2558.683</td>
</tr>
</tbody>
</table>

#### $\Delta = 10^\circ$

<table>
<thead>
<tr>
<th>ray</th>
<th>ScS</th>
<th>a</th>
<th>b</th>
<th>2ScS</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>p value</td>
<td>1.7640</td>
<td>1.6218</td>
<td>0.9323</td>
<td>1.7640</td>
<td>1.6900</td>
<td>1.2206</td>
</tr>
<tr>
<td>arrival time</td>
<td>945.774</td>
<td>1200.126</td>
<td>1623.387</td>
<td>1891.548</td>
<td>2145.869</td>
<td>2567.858</td>
</tr>
</tbody>
</table>
Fig. 2.6 Interpolation method used in this chapter to evaluate, from the synthetic SH-wave pulses shown in Fig. 2.5, the eigenfrequencies $\omega_n^l$ for which $l$ is an integer. • represents the $\omega_n^l$ values for which $n$ is an integer (using Eq. (2.3a) in the text) and $l$ (using Eq. (2.4)) is not. × represents the interpolated $\omega_n^l$ values for which both $n$ and $l$ are integers.
for different $\Delta$; ten distances ($\Delta = 1^\circ, 2^\circ, \ldots 10^\circ$) are used for this purpose. Eq. (2.2) is then used to construct the asymptotic structures for these eigenfrequencies.

Since $(T - \Delta dT/d\Delta)$ is almost the same for small $\Delta$ (i.e. for small $\ell$), from Eqs. (2.2) and (2.3a) it is clear that if $g(\omega) = 0$ for all $\omega$, the distribution of overtone eigenfrequencies is even, and there is no oscillation in the asymptotic structure $\gamma_L$. The solotone effect can therefore be interpreted in terms of the behaviour of $g(\omega)$.

The Fourier transform pairs for a delta function with amplitude $\tilde{A}_1$ at time $\tau_1$ are

$$\tilde{A}_1 \delta(t - \tau_1) \leftrightarrow \tilde{A}_1 \exp(-i\omega \tau_1). \quad (2.5)$$

If only the pair of pulses $[(ScS)_1, (2ScS)_2]$ is considered, the phase difference $g(\omega)$ of (2.3a) will be zero (Fig 2.7A); hence the overtone structure will be smooth (Fig. 2.7B). This corresponds to the continuous model B497C, and is equivalent to curve 1 of Fig. 2.2. When arrival pulses from crust/mantle discontinuities are included in the calculations, the phase functions become much more complex.

In the case of discontinuous models, where multiply-reflected arrivals from crust/mantle discontinuities must be considered, we can generalize Eq. (2.5) and write

$$\sum_{j=1}^{\infty} A_j \delta(t - \tau_j) \leftrightarrow \sum_{j=1}^{\infty} A_j \exp(-i\omega \tau_j) \quad (2.6)$$

where $A_j$ is the amplitude of the pulse arriving at time $\tau_j$. It can also be seen from Fig. (2.5) and Table 2.1 that the time separations from ScS and 2ScS for the multiply-reflected paths are almost of equal magnitude for small $\Delta$. We will use this simple model to demonstrate the behaviour of $g(\omega)$ for a discontinuous Earth model.
Fig. 2.7 The phase difference functions $g(n, \omega)$ of Eq. (2.3a) and the asymptotic overtone structures ($n, \gamma$ curves), based on eigenfrequencies computed using Eqs. (2.3a) and (2.4), for models B497C (A and B) and B497DC (C-F). See Fig. 2.1 for the models. Only the $[(ScS)_\Delta, (2ScS)_{2\Delta}]$ pairs are used for B497C. See Fig. 5A for pulses used for B497DC.
For the case of B497DC (with only one discontinuity at 600 km depth, cf. Fig. 2.1), since we have limited the number of rays to the most simple paths (Fig. 2.5A), the series in Eq. (2.6) is finite. This, of course, assumes that Eq. (2.6) converges rapidly. The frequency domain representations associated with the summations of relevant multiple reflections at $\Delta$ and $2\Delta$ (Fig. 2.5A), respectively, are

$$F_1 = A_0 [1 + 2\bar{k}_1 \exp(-i\omega t_1) + \bar{k}_2 \exp(-i\omega t_2)] \quad (2.7)$$

$$F_2 = B_0 [1 + 3\bar{k}_1 \exp(-i\omega t_1) + 2\bar{k}_2 \exp(-i\omega t_2)] \quad (2.7a)$$

where $t_2 = t_0 - t_1$ ($t_0$ is twice the radial travel time for SH waves from the Earth's surface to the core-mantle boundary, and $t_1$ is twice the radial travel time from the surface to the 600 km discontinuity); $A_0$ and $B_0$ are the amplitudes of $ScS$ (at $\Delta$) and $2ScS$ (at $2\Delta$) respectively; and $\bar{k}_1$ ($\bar{k}_2$) is the ratio of the amplitude of the first multiple (single) reflection from the top (bottom) of the discontinuity to $A_0$ at $\Delta$ (or to $B_0$ at $2\Delta$).

Since $\bar{k}_1$ and $\bar{k}_2$ are an order of magnitude smaller than 1 for the model, we can write Eqs. (2.7) and (2.7a) as

$$F_1 = A_0 [1 - k_1 \exp(-i\omega t_1)]^2 [1 - k_2 \exp(-i\omega t_2)] \quad (2.7b)$$

$$F_2 = B_0 [1 - k_1 \exp(-i\omega t_1)]^3 [1 - k_2 \exp(-i\omega t_2)]^2. \quad (2.7c)$$

Dividing $F_1$ by $F_2$, we get

$$F_1/F_2 = \frac{A_0}{B_0} [1 - k_1 \exp(-i\omega t_1)][1 - k_2 \exp(-i\omega t_2)]. \quad (2.8)$$

We compute the phase difference $\delta(\omega)$ of Eq. (2.3a) by taking the ratio of the imaginary and real parts of $F_1/F_2$. 
\[ g(\omega) = -\arctan \left( \frac{\bar{k}_1 \sin \omega \bar{t}_1 + \bar{k}_2 \sin \omega \bar{t}_2 - \bar{k}_1 \bar{k}_2 \sin \omega \bar{t}_0}{1 - \bar{k}_1 \cos \omega \bar{t}_1 - \bar{k}_2 \cos \omega \bar{t}_2 + \bar{k}_1 \bar{k}_2 \cos \omega \bar{t}_0} \right). \] (2.9)

We can simplify Eq. (2.9) and write

\[ g(\omega) = -\arctan \left[ \bar{k}_1 \sin \omega \bar{t}_1 + \bar{k}_2 \sin \omega \bar{t}_2 \right]. \] (2.10)

For a large discontinuity such that we cannot evaluate the phase difference \( g(\omega) \) by summation of a small number of paths (e.g. the crust-mantle discontinuity for models B497 and B497D), Eqs. (2.8) and (2.9) still hold (Chapter 3). Eq. (2.9) together with Brune's phase correlation equation, (2.3a), corresponds to McNabb, Anderssen and Lapwood's (1976) equation, (1.2), for evaluating the asymptotic torsional overtone eigenfrequencies of the Earth models with only one discontinuity (Chapter 3):

\[ \sin \frac{\bar{t}_0}{2} \omega = \bar{k} \sin \frac{(\bar{t}_2 - \bar{t}_1)}{2} \omega, \] (2.11)

where \( \bar{k} \) is a function of \( \bar{k}_1 \) and \( \bar{k}_2 \).

Eqs. (2.9) and (2.10) demonstrate that \( g(\omega) \) will oscillate with a frequency related to \( \bar{t}_1 \) and \( \bar{t}_2 \) and will have an amplitude determined by \( \bar{k}_1 \) and \( \bar{k}_2 \). The addition of more discontinuities to the Earth model will result in a complex superposition of frequencies to \( g(\omega) \). This effect will also be demonstrated experimentally here. See Chapter 5 for the general formulation and a rigorous verification of the correspondence between McNabb, Anderssen and Lapwood's (1976) result and the representation of multiple reflections from discontinuities in the Earth's interior.

The effect on \( g(\omega) \) of arrival pulses at \( \Delta = 5^\circ \) from the 600 km discontinuity reflections (Fig. 2.5A) is shown in Fig. 2.7C-E for comparison with the overtone structure (Fig. 2.7F) for model B497DC. Fig. 2.7C shows the effect when only pulses (a) and (c) and their equivalents (together with ScS and 2ScS) at \( \Delta \) and \( 2\Delta \), respectively, are included in the
calculations. Fig. 2.7D shows the effect when only pulses (b) and (d) are included. These figures show that the multiple reflections from the top and the bottom of the discontinuity have the same effect. Fig. 2.7E shows the superposition of the effect from these two types of pulses. Fig. 2.7F shows $\gamma_n^l$ computed from Eq. (1.1) superimposed on that derived from eigenfrequencies evaluated by Brune's phase correlation equation, (2.3a), with the phase difference $g_{\gamma_n}^{(\omega)}$ shown in Fig. 2.7E. It should be recalled that ten distances ($\Delta = 1^\circ$ through $10^\circ$) are used for this purpose, and Fig. 2.7E is only an example for $\Delta = 5^\circ$. The close agreement between the results from the two techniques is clearly demonstrated. This result suggests that the approximations we have made have not seriously affected the body-wave solutions. It demonstrates very well that the higher frequency oscillation in the solotone effect (curve 3 in Fig. 2.2) results from the multiple reflections from the 600 km discontinuity of model B497D.

Figs. 2.8A and 2.8B show the phase difference function $g_{\gamma_n}^{(\omega)}$ (at $\Delta = 5^\circ$) and $\gamma^l_n$ for model B497D when only the pulses (a1) - (a10) and (d1) - (d10) (Fig. 2.5B) and their equivalents (together with ScS and 2ScS) at $\Delta$ and $2\Delta$, respectively, are included in the calculations. Since the crust-mantle discontinuity of model B497D is quite large, and only a small number of paths are considered in the calculations of eigenfrequencies (those reflected from bottom of the 10 km discontinuity are neglected), the amplitude of the oscillation in the solotone effect (solid line in Fig. 2.8B) is smaller than that derived from eigenfrequencies computed from (1.1) (dotted line in Fig. 2.8B). Yet it demonstrates very well the low-frequency oscillation of curve 2 of Fig. 2.2. It can be seen here that $\gamma^l_n$ computed from body waves is tending to the solution obtained by Eq. (1.1). Since the effect of the crustal layer is not apparent at lower numbers, we have not attempted to improve this fit. In the case
Fig. 2.8 The phase difference functions $g(n \omega)$ of Eq. (2.3a) and the asymptotic overtone structures ($n \omega$: $n$ curves), based on eigenfrequencies computed using Eqs. (2.3a) and (2.4), for model B497D. See the text for description.
where $\gamma_L^n$ is determined from body waves, the apparent frequency of the oscillations in $\gamma_L^n$ remains fixed, but the amplitude of these oscillations is a function of the number of ray paths which were considered.

Figs. 2.8C and 2.8D demonstrate the superposition of the solutone effect from the two discontinuities of model B497D - using the paths shown in Fig. 2.5B. Although there is some discrepancy in these results, they are the result of a failure to include a sufficient number of multiple-reflections from the crust-mantle discontinuity; the more interesting test is that of the existence of a discontinuity in the mantle, and it has been shown that a few simple ray paths should be sufficient for that application.

2.5 DISCUSSION

Observations of normal modes of the Earth's free oscillations have provided a useful constraint in the construction of gross Earth models, and in recent years we have witnessed a rapid obsolescence of earlier models as more normal mode data come to hand. To date, however, the data have not provided significant information about upper mantle structure. The use of Brune's method permitted the derivation of frequencies of torsional oscillation modes of higher order number than have so far been directly observable. The data sample lengths used by Brune (1964) and Brune and Gilbert (1974) in their frequency analysis of body waves of ScS and SS type were too short, however, to include arrivals reflected from upper mantle discontinuities (Brune, private communication). This study has indicated that these reflected arrivals are responsible for the solutone effect. Thus, although the Brune and Gilbert data were included in the data set of Gilbert and Dziewonski (1975), from which the alternative models 1066A and 1066B (respectively continuous and discontinuous in the upper mantle) were derived, those
data are biased towards a continuous model. This is made clear in Table 2, in which the y values derived from the Brune and Gilbert data (Anderssen, Osborne and Cleary, 1974) are compared with the corresponding values from models 1066A and 1066B. Apart from the presence of a baseline change (which reflects the local nature of Brune and Gilbert's observations, the y values from model 1066B scatter about the mean significantly more than those from 1066A.

One of the major aims of the present study has been to ascertain what modifications of the Brune method are required to obtain upper mantle data in normal mode form. It is apparent, first of all, that the formulation of Brune is appropriate only for continuous models, and that some modification is therefore necessary. At the same time, the results indicate that, for small epicentral distances, deviations in phase velocity are not significant to the computation of the solotone effect. The study shows also that the window used in the Fourier analysis of body-wave phases of ScS type must have a length of at least 700 s if the relevant reflections from the supposed 600 km discontinuity are to be included. Such a procedure might be complicated by the necessity of avoiding the inclusion of other phases in the window, and for this reason other refinements to the technique might be necessary before the method becomes practicable.

The results also support the conclusion of Anderssen and Cleary (1974) that determination of torsional oscillations of high (beyond 10) overtone number n could yield useful information about upper mantle structure. It is possible that high-gain, long-period digital seismographs (see e.g. Berger et al. 1976) will eventually provide such data. It should be pointed out, however, that lateral variations in the depths of upper mantle discontinuities might be sufficient to smooth out the solotone effect. Consequently, the more localized information potentially available from Brune's method might prove to be of more value.
Table 2.2  Values of \( n \gamma_L^{(m)} \)\(^1\) \(= \pi(2nm + m^2)^{1/2}(n \sigma_L - m \sigma_L)^{-1}\) derived from the data of Brune and Gilbert\(^*\), compared with values from models 1066A and 1066B

<table>
<thead>
<tr>
<th>n</th>
<th>( \gamma_L^{(m)} )</th>
<th>( \gamma_L^{(m)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Brune and Gilbert</td>
<td>1066A</td>
</tr>
<tr>
<td>8</td>
<td>469.30</td>
<td>467.92</td>
</tr>
<tr>
<td>9</td>
<td>470.14</td>
<td>467.49</td>
</tr>
<tr>
<td>10</td>
<td>470.44</td>
<td>468.09</td>
</tr>
<tr>
<td>11</td>
<td>470.52</td>
<td>467.84</td>
</tr>
</tbody>
</table>

\(^*\) The data used are periods calculated for \( T_{14'} 12_{14'} T_{16'} 14_{16'} 10_{18'} 16_{18'} 11_{20'} 17_{20'} \) (cf. Anderssen et al., 1974).
CHAPTER 3
RAY-MODE DUALITY FOR SH WAVES IN EARTH MODELS
WITH CRUST AND MANTLE DISCONTINUITIES:
I. THE CASE OF ONE DISCONTINUITY

3.1 INTRODUCTION

Ben-Menahem (1964) constructed a theoretical derivation of Brune's (1964) formulation (reviewed in §2.3) for the evaluation of overtone eigenfrequencies of the Earth from body wave pulses, and pointed out that Brune's method represents a ray-mode duality of the Earth's displacement field. However, he did not explicitly examine its ramifications with respect to the possible existence of internal discontinuities.

As pointed out in Chapter 1, McNabb, Anderssen and Lapwood (1976) derived an equation, (1.2), for the asymptotic behaviour of torsional overtone eigenfrequencies of discontinuous SNREI Earth models and the solotone effect can be interpreted in terms of it. For Earth models with a single discontinuity between the surface and the core-mantle boundary, their equation can be written as

$$\sin(a \omega) = B_1 \sin(b_1 \omega),$$

where $a$ is the shear wave radial travel time from the Earth's surface to the core-mantle boundary; $b_1$ is the difference between the shear wave travel times from the model Earth's surface and the core-mantle boundary, respectively, to the discontinuity between them; and $B_1$ is the reflection coefficient at the discontinuity for rays incident normally from the bottom of the discontinuity (cf. Anderssen, 1977).

By synthetic SH seismograms and Brune's method, it has been shown in Chapter 2, for discontinuous Earth models, that the solotone effect arises from multiple reflections from discontinuities in the Earth's interior, and can be reconstructed from these multiple reflections. That
these two different approaches (the technique for summing SH multiple reflections from discontinuities and the analysis of a Sturm-Liouville system) yield the same result is consistent with a ray-mode duality for the representation of the displacement field in a SNREI Earth, and indicates that it is possible to derive the solotone effect and associated information about the Earth's structure directly from body wave pulses in seismograms. As a step towards the final goal of using the properties of the solotone effect as constraints in Earth modelling, the aim of this chapter is to verify that Eq. (3.1) can also be derived by applying Brune's (1964) phase correlation method to the summation of approximately vertically incident SH multiple reflections between the model Earth's surface, the core-mantle boundary and the discontinuity between them. This establishes a basis for a ray-mode duality for SNREI Earth models with a single discontinuity. A general derivation of Eq. (1.2) for Earth models with N discontinuities between the Earth surface and the core-mantle boundary is given in Chapter 5.

3.2 PRELIMINARIES

3.2.1 Application of Brune's Phase Correlation Method for Small Epicentral Distances

It has been shown in Chapter 2 that, for discontinuous Earth models, the solotone effect arises from multiple reflections from discontinuities between the Earth's surface and the core-mantle boundary. At least these multiple reflections should be included in the evaluation of the Earth's normal modes. But they have phase velocities which differ from those of the primary phases, e.g., $(\text{ScS})_\Delta$ and $(2\text{ScS})_{2\Delta}$. Therefore the evaluation of normal modes is not strictly covered by Eq. (2.3a) when discontinuities are present.

Until this method is extended to include body wave pulses of different phase velocities, it is necessary to limit attention to the
seismic wave pulses recorded at small epicentral distances (i.e., $\Delta < 5^\circ$). At such distances, all the rays corresponding to $kScS$ (a pulse which travels $k$ times the ray path of $ScS$) and the above-mentioned multiple reflections from discontinuities can be treated as vertically incident, for their phase velocities are all very large, so that $dT/d\Delta \approx 0$. Then since the second term on the right-hand side of Eq. (2.3a) includes a factor which is the product of $\Delta$ and $dT/d\Delta$, it can be neglected if only such wave pulses are taken into account and there is no need to consider the different phase velocities corresponding to different rays. Consequently, instead of using pairs of single body wave pulses, we can equally well use pairs $[h_1(t), h_2(t)]$ of wave trains of appropriate body wave pulses, corresponding to almost vertically incident rays, recorded at distances $\Delta$ and $2\Delta$ respectively. Thus, Eq. (2.3a) becomes

$$T(\omega) - 2m(\omega) = g(\omega); \quad g(\omega) = \phi_2(\omega) - \phi_1(\omega)$$

(3.2)

where $T$ is the difference between the starting times of $h_1(t)$ and $h_2(t)$, and $\phi_1(\omega)$ and $\phi_2(\omega)$ are the Fourier transform phases of $h_1(t)$ and $h_2(t)$, respectively.

Since $g(\omega)$ of Eq. (3.2) is independent of the source functions, $h_1(t)$ and $h_2(t)$ can be represented by series, $f_1(t)$ and $f_2(t)$, of $\delta$-functions (with each $\delta$-function representing a ray) to simplify our discussion. It is seen in Eq. (3.2) that if $g(\omega) = 0$ for any angular frequency $\omega$, the distribution of overtone eigenfrequencies is even. $g(\omega)$ is thus the term which produce the perturbation of the eigenfrequency distribution. If we let $F_1(\omega)$ and $F_2(\omega)$ be Fourier transforms of $f_1(t)$ and $f_2(t)$, then $g(\omega)$ can be computed by taking the ratio of the imaginary and real parts of $F_1(\omega)/F_2(\omega)$; i.e.

$$g(\omega) = -\text{arc tan}[\text{Im}(F_1/F_2)/\text{Re}(F_1/F_2)].$$

(3.3)
For Earth models with only one discontinuity between the surface and the core-mantle boundary, it will be shown that Eq. (3.1) can be derived from Eq. (3.2) if, starting from the onsets of \((\hat{S}\hat{S}\hat{S})_\Delta\) and \(((\hat{S}+1)\hat{S}\hat{S})_{2\Delta}\) pulses (where \(k \geq 1\)) in seismograms recorded at distances \(\Delta\) and \(2\Delta\) respectively, \(h_1(t)\) and \(h_2(t)\) are taken as the sums of \(m (m \geq 1)\) successive multiple reflections between the Earth's surface and the core-mantle boundary and all the associated multiple reflections from the discontinuity between them.

### 3.2.2 Crust and Mantle Models with a Single Discontinuity

Since we restrict attention to almost vertical rays between the Earth's surface and the core-mantle boundary, Earth models with plane boundaries and interfaces are sufficient for our purposes. It is assumed that the models are transversely isotropic and that the attenuation of waves in continuous media and the effect of transition zones between interfaces can be neglected. Under such conditions, only the reflections and transmissions of rays at the Earth's surface, the discontinuity and the core-mantle boundary have to be considered. Therefore, only the values of the densities and shear wave velocities just above or below the interfaces are important to the displacement amplitudes of the rays.

The Earth models used in this study, with only one discontinuity between the surface and the core-mantle boundary, are shown in Fig. 3.1, where \(D_i\) and \(S_i (i=0, 1, 2)\) represent the discontinuity and the ith interface respectively, \(\rho_i^-\) and \(\beta_i^- (\rho_i^+ \text{ and } \beta_i^+)\) are density and shear wave velocity just above (below) \(S_i\), and

\[
R_i^{(u)} = \text{reflection coefficient at } S_i \text{ for the vertical up-coming rays}
\]

\[
= \frac{\rho_i^+ \beta_i^+ - \rho_i^- \beta_i^-}{\rho_i^+ \beta_i^+ + \rho_i^- \beta_i^-}
\]
Fig. 3.1 Crust and mantle models with only one discontinuity between the Earth's surface and the core-mantle boundary. See the text (§3.2.2) for further description.
reflection coefficient at for the vertical down-going rays
\[ R_{1}^{(d)} = \frac{\rho_{11}^{\beta^-} - \rho_{11}^{\beta^+}}{\rho_{11}^{\beta^-} + \rho_{11}^{\beta^+}} = -R_{1}^{(u)} \; ; \]

transmission coefficient at S\(_{1}\) for the vertical up-coming rays
\[ T_{1}^{(u)} = \frac{2\rho_{11}^{\beta^+}}{\rho_{11}^{\beta^+} + \rho_{11}^{\beta^-}} \; ; \]

transmission coefficient at S\(_{1}\) for the vertical down-going rays
\[ T_{1}^{(d)} = \frac{2\rho_{11}^{\beta^-}}{\rho_{11}^{\beta^-} + \rho_{11}^{\beta^+}} \; . \]

We assume \( R_{0}^{(u)} = 1 \) and \( R_{2}^{(d)} = 1 \).

For convenience, the following constants are defined:

\[
\begin{align*}
    k_{1} & = R_{1}^{(u)} = -R_{1}^{(d)} \\
    k_{0} & = 1 \\
    k_{2} & = -1.
\end{align*}
\] (3.4)

3.2.3 Combinatorial Lemmas for the Summation of Multiple Reflections from Discontinuities

The following two lemmas are necessary in the formulation for summing multiple reflections from the Earth's surface, the core-mantle boundary and the discontinuity between them.

**LEMMA 3.2.3-A** (Bose-Eisteine statistics, e.g., Sears, 1950, P.320; or Hron, 1972, Appendix A)

The number of possible ways of distributing \( k (k=0, 1, 2, \ldots) \) identical balls into \( m (m \geq 1) \) distinct pockets (some of which can be empty)

is \( m+k-1 \binom{m+k-1}{k} \) where

\[
m+k-1 \binom{m+k-1}{k} = (m+k-1)! / [(m-1)! k!].
\]
For complex number $z$ with modulus less than 1 ($|z|<1$) and a positive integer $q$ we have

$$\sum_{k=0}^{q} \frac{q+k-1}{k!} z^k = \frac{1}{(1-z)^q} \quad (3.5)$$

### 3.3 ANALYSIS OF RAYS

In the literature two independent strategies for ray analysis have been given; namely by Hron (1971, 1972), in setting up the criteria for selection of phases in synthetic (theoretical) seismograms for layered media, and by Backus (1959), in deriving a filter to eliminate wave reverberations in a water layer. Their approaches, however, are not convenient for the formulation of the required series $f_1(t)$ and $f_2(t)$ defined in §3.2.1. To facilitate the discussion in this chapter, a different strategy of ray-analysis using fundamental ray components, defined below for Earth models shown in Fig. 3.1, is adopted.

#### DEFINITION 3.3

A fundamental ray component $r_{ij}^k (i\neq j; i=0,1,2; j=0,1,2)$ is a subset of a ray, which starts at interface $S_i$, travels without reflection to interface $S_j$, and is reflected there, and then returns without further reflection to $S_i$; it includes two reflections, at $S_i$ (its starting point) and $S_j$, and transmissions across any interfaces between $S_i$ and $S_j$ in travelling from $S_i$ to $S_j$ and back to $S_i$.

A fundamental ray component is not necessarily composed of adjacent segments of a ray. Fig. 3.2 shows the six fundamental ray components for models containing only one discontinuity, where '.' denotes reflections and 'x' transmissions.
Fig. 3.2 The six fundamental ray components for Earth models containing only one discontinuity. The definition of a fundamental ray component is in §3.3 and the Earth models are described in §3.2.2 and shown in Fig. 3.1.
3.3.1 Ray Decomposition

For Earth models with only one discontinuity (Fig. 3.1), we are interested only in the almost vertically incident rays corresponding to the multiple reflections between the Earth's surface, the core-mantle boundary and the discontinuity between them, which are recorded at small epicentral distances. For simplification, we treat these rays as vertically incident and all the ray segments as straight lines.

In this chapter, a 'ray' is defined as the stationary time path between sources and receiver which has a realizable sequence of reflections and transmissions at specified boundaries (e.g., $S_0$, $S_1$ and $S_2$ in Fig. 3.1). For convenience, the points at which these transmissions and reflections occur (together with source and receiver) will be referred to as ray junctions. To facilitate the discussion and manipulation below, we assume that (i) both source and receiver are at the Earth's surface, (ii) every ray starts with a reflection $R_0^{(u)}$ (although this is an artifact, it does not affect the result since $R_0^{(u)}$ was assumed to be unity), and (iii) every ray includes at least one kScS path ($k \geq 1$). We make this last assumption because of the definition of our required series, $f_1(t)$ and $f_2(t)$ (§3.2.1), of wave pulses.

Since the travel times and amplitudes of rays are the only two properties we are interested in, our strategy for ray decomposition is to decompose the ray into fundamental ray components (as shown in Fig. 3.2) which have in total the same length and the same set of reflections and transmissions as the original ray.

If a given ray possesses $m^0_2$ pairs of transmissions $T^{(d)}_1$ and $T^{(u)}_1$, $m^1_1$ reflections $R^{(d)}_1$ and $m^2_1$ reflections $R^{(u)}_1$, then we can express it as a linear combination of the six fundamental ray components shown in Fig. 3.2, i.e.

$$
\mathbf{R} = \mathbf{Z}_{2}^{0} \bar{R}_{0}^{0} + \mathbf{Z}_{2}^{2} \bar{R}_{0}^{2} + \mathbf{Z}_{2}^{0} \bar{R}_{1}^{0} + \mathbf{Z}_{2}^{1} \bar{R}_{1}^{1} + \mathbf{Z}_{2}^{1} \bar{R}_{1}^{1} + \mathbf{Z}_{2}^{2} \bar{R}_{1}^{2}
$$

(3.6)
with the constraints (see Appendix)

\[
\begin{align*}
Z_0^0 + Z_2^0 &= m_2^0 \\
Z_0^1 + Z_1^0 &= m_1^0 \\
Z_2^1 + Z_2^0 &= m_2^1.
\end{align*}
\] (3.7)

Since every ray consists of at least one kScS path (k≥l), we assume \(m_2^0≠0\).

Let \(t_{i}^{j}(i≠j)\) be the shear wave travel time along \(r_{i}^{j}\), and \(i_{A}^{j}(i≠j)\) be the product of coefficients of reflections and transmissions which define \(r_{i}^{j}\). Then, because \(r_{i}^{j}\) and \(r_{i}^{j}\) have the same length and the same set of reflections at and transmissions across the interfaces, it is clear that \(t_{i}^{j} = t_{i}^{j}\) and \(i_{A}^{j} = i_{A}^{j}\). Since the travel time and amplitude are the only properties of interest, \(r_{i}^{j}\) and \(r_{i}^{j}\) are equivalent and there is no need to consider them as separate entities. Therefore, we discuss the properties of rays in terms of \(m_2^0, m_1^0\) and \(m_2^1\).

For simplification, we denote \(r_{i}^{j}\) and \(r_{i}^{j}\) as \(r_{i}^{j}\) (i<j) and refer to them as fundamental reflections between \(S_i\) and \(S_j\). It is clear that \(m_{j}^{i}\) (i<j) are the numbers of fundamental reflections \(r_{i}^{j}\).

Using fundamental reflections, we rewrite Eq. (3.6) as

\[
t = m_2^0 t_2^0 + m_1^0 t_1^0 + m_2^1 t_2^1.
\] (3.8)

To discuss the properties of rays in terms of the properties of the fundamental reflections, we define the travel time and amplitude of the fundamental reflections \(r_{i}^{j}\) as

\[
\begin{align*}
t_{i}^{j} &\equiv t_{i}^{j} = t_{i}^{j} \\
i_{A}^{j} &\equiv i_{A}^{j} = j_{A}^{j} \ (i<j).
\end{align*}
\] (3.9)

The shear wave travel time for the ray of Eq. (3.8) is obviously

\[
t = m_2^0 t_2^0 + m_1^0 t_1^0 + m_2^1 t_2^1.
\] (3.10)
If we define the amplitude of a ray as the ratio of the amplitude of a signal recorded at the receiver to the amplitude of the signal in the ray direction at the source, then it is twice the product of $^iA_j$ for all the fundamental reflections the ray possesses (twice because the boundary condition of a free surface requires the Earth's surface displacement at the receiver to be the sum of displacements of the incident and reflected waves, and $R_{0}^{(u)} = 1$; Nuttli, 1961), i.e.,

$$\hat{A} = 2^0A_2^m_2^0A_1^m_1^0A_2^m_2^1$$  \hspace{1cm} (3.11)

(We neglect the energy dissipation and the effect of transition zones when rays pass through a continuous medium). Therefore the travel time and amplitude of a given ray are uniquely determined by the ordered set $(m_2^0, m_1^0, m_2^1)$ in terms of Eqs. (3.10) and (3.11).

### 3.3.2 Summation of Equivalent Rays

Although the travel time and amplitude of a given ray are uniquely and completely described by the ordered set $(m_2^0, m_1^0, m_2^1)$ (Eqs. (3.10) and (3.11)), an ordered set $(m_2^0, m_1^0, m_2^1)$ does not uniquely determine a ray (Fig. 3.3 shows the two rays which have the same ordered set $(1, 1, 0)$). Instead, a set of equivalent rays whose definition follows will share it.

**DEFINITION 3.3.2**

Two rays having the same source and receiver are equivalent if their respective decompositions are defined by the same ordered set $(m_2^0, m_1^0, m_2^1)$. 
Fig. 3.3 The two rays which have the same ordered set (1,1,0). They have the same length and the same set of reflections at and transmissions across the interfaces.
Since all the equivalent rays have the same expression of travel
time, Eq. (3.10), and amplitude, Eq. (3.11), we have therefore proved

THEOREM 3.3.2-A

All equivalent rays have the same travel time and amplitude.

To facilitate our summation of multiple reflections, we need
to calculate the amplitude of the set of equivalent rays determined by the
ordered set \((m_2^0, m_1^0, m_2^1)\), which is the sum of all the amplitudes of the
equivalent rays in the set. Since all the equivalent rays have the same
amplitude, it is therefore the product of the number \(M(m_2^0, m_1^0, m_2^1)\) of
equivalent rays in the set and the amplitude of any one of the equivalent
rays.

To get an explicit expression for \(M(m_2^0, m_1^0, m_2^1)\) we have to
construct all the equivalent ray paths for a given ordered set \((m_2^0, m_1^0, m_2^1)\).
Since the ordered set \((m_2^0, m_1^0, m_2^1)\) is the immediate result of our ray
decomposition strategy and any ray which consists of \(m_2^0 r_2^0, m_1^0 r_1^0\) and
\(m_2^1 r_2^1\) (where \(r_2^0, r_1^0\) and \(r_2^1\) are fundamental reflections) is an equivalent
ray for the ordered set \((m_2^0, m_1^0, m_2^1)\), our strategy for finding
\(M(m_2^0, m_1^0, m_2^1)\) is to put these fundamental reflections together in all
the possible ways which lead to rays with both source and receiver at
the Earth's surface. Under such a strategy, one may adopt \(r_j^0\) or \(r_j^1\)
or both (where \(r_j^0\) and \(r_j^1\) are fundamental ray components) in constructing
a ray so long as their total number is \(m_j^i\); and there may exist several
possible procedures which specify detailed steps to construct the
equivalent rays.

We present one such procedure here:

(i) Initially, put \(m_2^0 (m_2 = 1, 2, \ldots \infty)\) fundamental ray components
\(r_2^0\) between interfaces \(S_0\) and \(S_2\). There is only one way to do
this. Fig. 3.4A shows the situation for $m_2^0 = 2$ (with $m_1^0 = 0$ and $m_2^1 = 0$ at this stage, which corresponds to $2ScS$.

(ii) Next, introduce $m_1^0 (m_1^0 = 0, 1, \ldots \infty)$ fundamental ray components $r_{1}^0$ between interfaces $S_0$ and $S_1$. We distribute them into the $(m_2^0 + 1)$ junctions at $S_0$ (designated by '.' in Fig. 3.4A) of the ray set up step (i). According to LEMMA 3.2.3-A we have $\binom{m_2^0 + m_1^0}{m_1^0}$ equivalent ways to distribute these $m_1^0$ fundamental ray components. Each of the equivalent ways represents one equivalent ray. Fig. 3.4B shows one of the ten $(S^3_\binom{3}{3})$ possible equivalent rays for the situation when $m_2^0 = 2$ and $m_1^0 = 3$ (with $m_2^1 = 0$ at this stage).

(iii) Finally, put $m_2^1$ fundamental ray components $r_{1}^2$ between $S_1$ and $S_2$. We distribute them into the junctions at $S_2$ (designated by 'x' in Figs. 3.4A and 3.4B) of the ray set up in step (ii). (The number of junctions at $S_2$ is not affected by step (ii), therefore they are the same junctions of the ray in Fig. 3.4A). The number of the available junctions is $m_2^0$. According to LEMMA 3.2.3-A, we have $\binom{m_2^0 + m_1^1}{m_1^1} - \binom{m_2^1}{m_2^1}$ equivalent ways to put these $m_2^1$ fundamental ray components $r_{1}^2$. This means that, from each ray in (ii), we can construct $\binom{m_2^0 + m_1^1}{m_1^1} - \binom{m_2^1}{m_2^1}$ equivalent rays. The total number of possible equivalent rays for the ordered set $(m_2^0, m_1^0, m_2^1)$ is, therefore, the product of $\binom{m_2^0 + m_1^0}{m_1^0}$ and $\binom{m_2^0 + m_1^1}{m_1^1} - \binom{m_2^1}{m_2^1}$. Fig. 3.4C shows one of the twenty $(S^3_\binom{3}{3} \cdot \binom{3}{3})$ possible equivalent rays for the situation when $m_2^0 = 2$, $m_1^0 = 3$ and $m_2^1 = 1$. This step is independent of step (ii), therefore the order of steps (ii) and (iii) is reversible.
Fig. 3.4 The construction of one of the equivalent rays determined by the ordered set (2,3,1). This demonstrates the procedure described in §3.3.2 for obtaining an explicit expression for the number, \( M(m_2^0, m_1^0, m_1^1) \), of all the equivalent rays determined by the ordered set \( (m_2^0, m_1^0, m_1^1) \). '•' denotes (in Fig. A) the positions into which the fundamental ray components \( r_{10}^0 \) can be distributed and 'x' (in Figs. A and B) those into which the fundamental ray components \( r_{10}^2 \) can be distributed.
Since the ordered set \((m^0_2, m^0_1, m^1_2)\) uniquely determines a set of equivalent rays, the number of which, \(M(m^0_2, m^0_1, m^1_2)\), is independent of the procedures we use to construct it. In fact we have proved that

\[
M(m^0_2, m^0_1, m^1_2) = (m^0_2 + m^0_1 C m^1_1) (m^0_2 + m^1_1 - C m^1_1).
\]

If we define the travel time of the set of equivalent rays determined by the ordered set \((m^0_2, m^0_1, m^1_2)\) as the travel time of any equivalent ray in the set, then obviously it is expressed by Eq. (3.10).

Now we have proved

**THEOREM 3.3.2-B**

The total amplitude of the set of all equivalent rays determined by a given ordered set \((m^0_2, m^0_1, m^1_2)\) can be written as

\[
S(m^0_2, m^0_1, m^1_2) = 2(m^0_2 + m^0_1 C m^1_1) (m^0_2 + m^1_1 - C m^1_1) A_2^0 m^0_2 A_1^0 m^1_1 A^1_2 m^1_2
\]

while the travel time of the set can be written as Eq. (3.10).

### 3.4 FORMULATION OF \(f_1(t)\) AND \(f_2(t)\) AND THEIR FOURIER TRANSFORMS

To facilitate our formulation of \(f_1(t)\) and \(f_2(t)\), we group all the rays into different sets according to the numbers, \(m^0_2, m^0_1\) and \(m^1_2\), of fundamental reflections between \(S_0\) (the Earth's surface), \(S_1\) (the discontinuity) and \(S_2\) (the core-mantle boundary) (Fig. 3.1). Then each set is a set of equivalent rays determined by an ordered set \((m^0_2, m^0_1, m^1_2)\). Now, to formulate \(f_1(t)\) and \(f_2(t)\), we need only to sum these sets of equivalent rays.

By their definition in §3.2.1 and Eq. (3.13) (THEOREM 3.3.2-B), \(f_1(t)\) and \(f_2(t)\) can be written as
\[ f_1(t) = \sum_{m_2=0}^{k+m-1} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} S(m_2^0, m_1^0, m_2^1) \delta(t - [(m_2^0 - k) t_2^0 + m_1^0 t_2^0 + m_2^1 t_2^1]) \]  
\( (3.14) \)

\[ f_2(t) = \sum_{m_2=k+1}^{k+m} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} S(m_2^0, m_1^0, m_2^1) \delta(t - [(m_2^0 - k+1) t_2^0 + m_1^0 t_2^0 + m_2^1 t_2^1]) \]  
\( (3.14a) \)

The reason why we write the coefficients before \( t_2^0 \) as \((m_2^0 - k)\) and \((m_2^0 - k-1)\) is that \( f_1(t) \) and \( f_2(t) \) start from \( \hat{\kappa} \text{ScS} \) and \((k+1)\text{ScS} \) pulses in seismograms respectively. Their Fourier transforms are

\[ F_1(\omega) = 2^0 A_2 \sum_{m_2=0}^{k+m-1} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \left( m_2^0 + m_1^0 \right) \left( m_2^0 + m_1^0 - 1 \right) \left( m_2^1 \right) \delta(\omega - (m_2^0 - k)) \omega_1^{m_1^0} \omega_2^{m_2^1} \]  
\( (3.15) \)

\[ F_2(\omega) = 2^0 A_2 \sum_{m_2=k+1}^{k+m} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \left( m_2^0 + m_1^0 \right) \left( m_2^0 + m_1^0 - 1 \right) \left( m_2^1 \right) \delta(\omega - (m_2^0 - k+1)) \omega_1^{m_1^0} \omega_2^{m_2^1} \]  
\( (3.15a) \)

where \( \omega_j (i<j; i=0,1; j=1,2) = \frac{1}{\lambda_j} \exp(-i\omega t_j) \).

### 3.5 DERIVATION OF \( F_1(\omega)/F_2(\omega) \)

Before we can use Eq. (3.2) and (3.3) to derive Eq. (3.1), we have to obtain an explicit expression for \( F_1(\omega)/F_2(\omega) \) from Eqs. (3.15) and (3.15a). Thus, we have to simplify \( F_1(\omega) \) and \( F_2(\omega) \) first. As will be shown in this section, Eqs. (3.15) and (3.15a), and therefore the expression for \( F_1(\omega)/F_2(\omega) \), can be greatly simplified by the use of Eq. (3.5) (LEMMA 3.2.3-B).

Since \( F_1(\omega) \) and \( F_2(\omega) \) have the same mathematical form, we derive \( F_1(\omega) \) first. We need only a little modification of \( F_1(\omega) \) to get \( F_2(\omega) \). Since the expression to be summed in Eq. (3.15) can be expressed in the form \( p_1(m_2^0, m_1^0) p_2(m_2^0, m_2^1) p_3(m_2^0) \), Eq. (3.15) can be reduced to
\[
F_1(\omega) = 2^0A^k \sum_{m=0}^{k+m-1} \left[ \sum_{m_2=0}^{\infty} \left( m_2^0 + m_2^1 C_{1/m_2}^0 W_1^m \right) \right] \left( m_2^0 + m_2^1 - 1 C_{1/m_2}^1 W_2^m \right) W_2^{m_2(k-m)} 
\]

(3.16)

Because \( |0W_1| = |R^{(d)}_1| < 1 \) and \( |1W_2| = |R^{(u)}_1| < 1 \), using LEMMA 3.2.3-B (Eq. (3.5)) we simplify Eq. (3.16) as

\[
F_1(\omega) = 2^0A^k \sum_{m_2=0}^{k+m-1} \frac{0W_2^{(m_2-k)}}{(1-0W_1^m)(m_2^0 + 1)(1-1W_2^m)m_2^0} 
\]

(3.17)

\( F_1(\omega) \) cannot be further simplified. We now derive an expression for \( F_2(\omega) \). Comparing Eqs. (3.15) and (3.15a), it is clear that \( F_2(\omega) \) can be obtained by adding 1 to summation limits of Eq. (3.17) and multiplying Eq. (3.17)

\[
F_2^0 = 0A_2^0W_2^0 
\]

by \( 0A_2^0W_2^0 \):

\[
F_2(\omega) = 2^0A^\hat{k} \sum_{m_2=0}^{\hat{k}+m-1} \frac{0W_2^{(m_2-k-1)}}{(1-0W_1^m)(m_2^0 + 1)(1-1W_2^m)m_2^0} 
\]

(3.17a)

Now we divide Eq. (3.17) by Eq. (3.17a) to get \( F_1(\omega)/F_2(\omega) \):

\[
\frac{F_1(\omega)}{F_2(\omega)} = \frac{1}{0A_2^0} \frac{1}{(1-0W_1^m)(1-1W_2^m)} 
\]

\[
= \frac{1}{0A_2^0} \left[ 1+k \exp(-i\omega t_0^\dagger) \right] \left[ 1-k \exp(-i\omega t_1^\dagger) \right] 
\]

(3.18)

where \( k_1 = 0A_1^0 (= R^{(d)}_1) = 1A_2^0 (= R^{(u)}_1) \). It is interesting that Eq. (3.18) is dependent of \( \hat{k} \), the number of the successive multiple reflections between the Earth's surface and the core-mantle boundary which we summed to get \( F_1(\omega) \) and \( F_2(\omega) \), and \( \hat{k} \), which indicates that the series of wave pulses used for \( F_1(\omega) \) was started from \( \hat{k}\text{ScS} \), and that for \( F_2(\omega) \) from \( (\hat{k}+1)\text{ScS} \).
3.6 DERIVATION OF EQ. (3.1) FROM BRUNE'S FORMULATION

Now we can derive Eq. (3.1) from Eqs. (3.2), (3.3) and (3.18).

Substituting Eq. (3.18) into Eq. (3.3), we get

\[ g(\omega) = \arctan \frac{k \sin(\omega t_0^0) - k \sin(\omega t_1^1) - k^2 \sin(\omega t_0^0)}{1 + k \cos(\omega t_0^0) - k \cos(\omega t_1^1) - k^2 \cos(\omega t_0^0)}. \]  

(3.19)

Substituting Eq. (3.19) into Eq. (3.2) with \( T = t_2^0 \), taking the tangent of both sides, multiplying by \( \cos(\omega t_2^0) \) and the denominator of the right-hand side, we get after rearrangement

\[
\sin(\omega t_2^0) = k \left[ \sin(\omega t_0^0) \cos(\omega t_0^0) - \cos(\omega t_0^0) \sin(\omega t_0^0) \right] \\
- k \left[ \sin(\omega t_1^1) \cos(\omega t_0^0) - \cos(\omega t_1^1) \sin(\omega t_0^0) \right] \\
= k \sin[\omega(t_0^0 - t_0^0)] - k \sin[\omega(t_0^1 - t_0^0)].
\]

(3.20)

With \( t_2^0 = t_1^0 + t_2^0 \), Eq. (3.20) can be reduced to

\[
2\sin(\omega t_2^0) \cos(\omega t_2^0) = 2k \cos(\omega t_2^0) \sin(\omega t_2^0 - t_2^0/2).
\]

(3.21)

which is simply

\[
\sin(\omega t_2^0) = k \sin(\omega t_2^0 - t_2^0/2).
\]

(3.22)

if \( \cos(\omega t_2^0) \neq 0 \). Eq. (3.22) is the same formula as Eq. (3.1) if we let

\[ k_1 = B_1, \quad t_2^0/2 = a, \quad \text{and} \quad (t_2^0 - t_2^0)/2 = b_1. \]
APPENDIX EXPRESSION OF A RAY IN TERMS OF FUNDAMENTAL RAY COMPONENTS

Here we prove that if a given ray possesses \( m_2 \) pairs of transmissions \( T_{(d)} \) and \( T_{(u)} \), \( m_1 \) reflections \( R_{(d)} \) and \( m_1 \) reflections \( R_{(u)} \), then the following linear combination of fundamental ray components

\[
\hat{r} = \ell_0^0 r_0^0 + \ell_0^2 r_0^2 + \ell_1^0 r_1^0 + \ell_1^1 r_1^1 + \ell_2^0 r_1^0 + \ell_2^1 r_1^1 + \ell^2 + \ell^2
\]

(a)

with the constraints

\[
\begin{align*}
\ell_0^0 + \ell_0^2 &= m_0^0 \\
\ell_1^0 + \ell_1^1 &= m_0^1 \\
\ell_2^0 + \ell_2^1 &= m_1^1
\end{align*}
\]

(b)

has (i) the same length and (ii) the same set of transmissions and reflections as the given ray.

To prove property (i), we only have to prove that (a) and the given ray have the same number \( n_0^0 \) of pairs of ray segments between \( S_0 \) and \( S_1 \) and the same number \( n_1^0 \) of pairs between \( S_1 \) and \( S_2 \) (Fig. 3.1) (these ray segments must be of even number since the ray starts and finishes at \( S_0 \)). For (a), these two numbers can be expressed in terms

\[
\ell^x_y (x=0,1,2; y=0,1,2):
\]

\[
\begin{align*}
n_0 &= \ell_0^0 + \ell_1^1 + \ell_2^0 \\
n_1 &= \ell_1^0 + \ell_2^1 + \ell_0^0 \\
n_2 &= \ell_2^0 + \ell_1^1 + \ell_0^0
\end{align*}
\]

(c)

Substituting (b) into (c) we arrive at

\[
\begin{align*}
n_0 &= m_0^0 + m_0^1 \\
n_1 &= m_1^1 + m_0^0 \\
n_2 &= m_1^1 + m_0^0
\end{align*}
\]

(d)
which are clearly the same numbers for the given ray, since an $R_{1}^{(u)}$ represents a pair of ray segments between $S_{0}$ and $S_{1}$, and an $R_{1}^{(d)}$ represents a pair of ray segments between $S_{1}$ and $S_{2}$, while a pair of $T_{1}^{(d)}$ and $T_{1}^{(u)}$ represents a pair of ray segments between $S_{0}$ and $S_{1}$ and a pair between $S_{1}$ and $S_{2}$.

Since all the rays which have the same number of pairs of ray segments between $S_{0}$ and $S_{1}$ and the same number of pairs between $S_{1}$ and $S_{2}$ have the same number of reflections $R_{0}^{(u)}$ and the same number of reflections $R_{2}^{(d)}$, to prove property (ii) we only have to prove that (a) has the same number of pairs of $T_{1}^{(d)}$ and $T_{1}^{(u)}$, the same number of reflections $R_{1}^{(d)}$ and the same number of reflections $R_{1}^{(u)}$ as the given ray. For (a), we denote these numbers as $m_{0}^{0}$, $m_{1}^{0}$ and $m_{2}^{0}$. Since only $r_{2}^{0}$ or $r_{0}^{2}$ has a pair of $T_{1}^{(d)}$ and $T_{1}^{(u)}$, only $r_{1}^{0}$ or $r_{0}^{1}$ has a reflection $R_{1}^{(d)}$, and only $r_{2}^{1}$ or $r_{1}^{2}$ has a reflection $R_{1}^{(u)}$, $m_{y}^{x}$ for $x<y$, $x=0,1; y=1,2$ can be expressed in terms of $k_{y}^{x}$:

\[
\begin{align*}
-m_{2}^{0} &= k_{0}^{0} + k_{2}^{2} \\
-m_{1}^{0} &= k_{0}^{0} + k_{1}^{1} \\
-m_{2}^{1} &= k_{1}^{1} + k_{2}^{2}
\end{align*}
\]

(e)

It is clear that property (ii) has been proved by (b) and (e).
CHAPTER 4
A STRATEGY FOR THE SUMMATION OF SEISMIC SIGNALS
MULTIPLY-REFLECTED AT INTERNAL DISCONTINUITIES

4.1 INTRODUCTION

In seismological research it is often desirable to sum multiple reflections of waves or signals between discontinuities in a medium. Examples are the derivation of a ray-mode duality for seismic waves in the Earth and the derivation of mathematical filters for removing multiply-reflected waves between discontinuities from the records. In such cases a suitable strategy for ray analysis is required such that the summations of multiple reflections can be formulated in terms of a set of independent integral parameters and simplified by summing over these parameters.

As mentioned earlier, Backus (1959) and Hron (1971, 1972) have presented two independent strategies for ray analysis. Their approaches, however, are not convenient for this purpose, since they each adopted a set of integral parameters which are not independent of one another. In deriving the ray-mode duality for SH waves in Earth models with a single discontinuity between the surface and the core-mantle boundary, a novel strategy for the summation of almost vertical multiple reflections of waves between discontinuities was described in Chapter 3. The present chapter aims to extend this strategy to the general case of a medium with N discontinuities between two boundaries.

Since the main purpose of this chapter is to provide a basis for a more generalized analysis of the ray-mode duality for almost vertically incident SH waves in discontinuous Earth models, we deal here only with these waves. The application of the strategy, however, can be extended to different situations if the constants used are suitable modified. As an example the mathematical filter devised by Backus (1959) to eliminate wave reverberations in water layer is re-interpreted and extended to a general case.
4.2 EARTH MODELS AND TERMINOLOGY

Since we restrict attention to almost vertical rays between the Earth's surface and the core-mantle boundary, Earth models with plane boundaries and interfaces are sufficient for our purposes. As in Chapter 3 we assume transversely isotropic models (Fig. 4.1) which have plane interfaces. We denote the discontinuities by \( D_i \) \( (i = 1, 2, \ldots N) \) and the interfaces by \( S_j \) \( (j = 0, 1, \ldots N+1) \). Except for the Earth's surface \( (S_0) \) and the core-mantle boundary \( (S_{N+1}) \), the interfaces \( S_i \) correspond to the discontinuities \( D_i \). These interfaces define \( N+1 \) layers in which the density and elastic properties are continuous and functions of depth only. To simplify the discussion, we neglect the attenuation of waves in continuous media and the effect of transition layers between interfaces.

As far as the properties (travel times and amplitudes) of rays are concerned, we need only to consider the reflections and transmissions of rays at the interfaces under such conditions. In Fig. 4.1, \( R^{(u)}_i \) and \( T^{(u)}_i \) represent coefficients of reflection at and transmission across \( S_i \) for up-coming rays, and \( R^{(d)}_i \) and \( T^{(d)}_i \) the coefficients for down-going rays.

In this chapter, a 'ray' is defined as the stationary time path between source and receiver which has a realizable sequence of reflections and transmissions at specified boundaries \( (e.g., S_0, S_1, \ldots \text{ and } S_{N+1} \text{ in Fig. 4.1}) \). The points at which these transmissions and reflections occur (together with source and receiver) will be referred to as ray junctions (or simply junctions). As in Chapter 3, we assume that (i) both source and receiver are at the Earth's surface, (ii) every ray starts with a reflection \( R^{(u)}_0 \) (which does not affect our results, since \( R^{(u)}_0 = 1 \)), and (iii) every ray includes at least one \( k\text{ScS} \) path \( (k \geq 1) \).

In Chapter 3, we defined fundamental ray components and fundamental reflections in the strategy for ray analysis for Earth models.
Fig. 4.1 Earth models used in this chapter. See the text in §4.2 for detailed descriptions.
with a single discontinuity between the surface and the core-mantle boundary. For convenience we extend the definitions to Earth models with N discontinuities.

DEFINITION 4.2-A

A fundamental ray component $r_{ij}^{-i}$ ($i \neq j; i = 0, 1, \ldots N+1; j = 0, 1, \ldots N+1$) is a subset of a ray, which starts at interface $S_i$, travels without reflection to interface $S_j$, and is reflected there, and then returns without further reflection to $S_i$; it includes two reflections, at $S_i$ (its starting point) and $S_j$, and transmissions across any interfaces between $S_i$ and $S_j$ in travelling from $S_i$ to $S_j$ and back to $S_i$.

Fig. 4.2 shows the twelve fundamental ray components for models containing two discontinuities, where '.' denotes reflections and 'x' transmissions.

Let $t_{ij}^{-i}$ ($i \neq j$) be the shear wave travel time along $r_{ij}^{-i}$, and $A_{ij}^{-i}$ ($i \neq j$) be the product of coefficients of reflections and transmissions which define $r_{ij}^{-i}$. Then, because $r_{ij}^{-i}$ and $r_{ij}^{+i}$ have the same length and the same set of reflections at and transmissions across the interfaces, it is clear that $t_{ij}^{-i} = t_{ji}^{-j}$ and $A_{ij}^{-i} = A_{ji}^{-j}$. Since the travel time and amplitude are the only properties of interest, $r_{ij}^{-i}$ and $r_{ij}^{+i}$ are equivalent and there is no need to consider them as separate entities.

DEFINITION 4.2-B

A fundamental ray component $r_{ij}^{-i}$ or $r_{ij}^{+i}$ ($i<j$) is called a fundamental reflection $r_{ij}^{-i}$.

For convenience we define the travel time and amplitude of a fundamental reflection $r_{ij}^{-i}$ as...
Fig. 4.2 The twelve fundamental ray components for models containing two discontinuities between the Earth's surface ($S_0$) and the core-mantle boundary ($S_3$). Each fundamental ray component includes two reflections (designated by '.') and all the transmissions (designated by '×') across interfaces on its way.
\[
\begin{align*}
\tau_{ij} &= \tau_i^j = \tau_{ij}^j, \quad (i < j), \quad (4.1) \\
A_{ij} &= j_{A_i} = j_{A_i} = R_i^{(u)} (T_{i+1}^{(u)} T_{i+1}^{(u)}) \ldots (T_{j-1}^{(d)} T_{j-1}^{(d)}) R_j^{(d)}. \quad (4.1a)
\end{align*}
\]

A fundamental reflection is not necessarily composed of adjacent segments of a ray.

4.3 RAY DECOMPOSITION

4.3.1 Properties of Rays

In order to express rays and their properties in terms of a set of independent integral variables, we need to decompose the rays into suitable components. Our strategy for ray decomposition is to decompose the ray into fundamental reflections under the constraint that a fundamental reflection \(r^i_j\) (or \(r^j_i\), \(i < j\)) can be decomposed only when it is an integral part of the ray to be composed (for a reason to be given later).

Under such a strategy, any given ray has a unique ray decomposition (see APPENDIX for proof)

\[
\begin{align*}
\hat{\tau} &= m_0 r_0^0 + (m_0 r_0^0 + m_1 r_1^1) + \ldots (m_0 r_0^0 + m_1 r_1^1 + \ldots m_N r_N^N), \quad (4.2)
\end{align*}
\]

where \(m_{N+1}^i = 1, 2, \ldots \infty\) and other \(m_j^i = 0, 1, 2, \ldots \infty\). The shear wave travel time for the ray of Eq. (4.2) is obviously

\[
\begin{align*}
\hat{\tau} &= m_0 t_0^0 + (m_0 t_0^0 + m_1 t_1^1 + \ldots \ m_1 t_1^1 + \ldots m_N t_N^N), \quad (4.3)
\end{align*}
\]

If we define the amplitude of a ray as the ratio of the amplitude of a signal recorded at the receiver to the amplitude of the signal in the ray direction at the source, then it is twice the product of \(i_{A_j}\) for all the fundamental reflections the ray possesses. (cf. Nuttli, 1961); i.e.,
\[ \hat{A} = 2 \left( a_n^0 m_{N+1}^0 \right) \left( a_n^0 m_N^1 \right) \cdots \left( a_1^0 \right) \left( a_2^1 \right) \cdots \left( a_{N+1}^N \right). \] (4.4)

Therefore the travel time and amplitude of a given ray are uniquely determined by the ordered set \((m_{N+1}^0, m_N^0, m_{N+1}^1, \ldots, m_1^0, m_2^1, \ldots, m_{N+1}^N)\) in terms of Eqs. (4.3) and (4.4). For brevity we denote the ordered set \((m_{N+1}^0, m_N^0, m_{N+1}^1, \ldots, m_1^0, m_2^1, \ldots, m_{N+1}^N)\) by \((m_{jN})\), where the subscript indicates that the ordered set applies to the case of \(N\) discontinuities.

The reason why we need the constraint that a fundamental reflection can be decomposed only when it is an integral part of the ray to be decomposed is that otherwise some rays do not have the unique decomposition of Eq. (4.2). Fig. 4.3 shows two rays which have the same set of ray segments and the same set of reflections at and transmissions across interfaces. Under this constraint these two rays are represented by the ordered sets \((1, 1, 1, 0, 0, 0)\) (Fig. 4.3A) and \((2, 0, 0, 0, 1, 0)\) (Fig. 4.3B) respectively. If this constraint is removed, the ordered set for either ray is not unique, since, after the reorganization of ray segments as well as reflections at and transmissions across interfaces, the ray in Fig. 4.3A can be converted to that in Fig. 4.3B, and vice versa. Since the ray segments which constitute a fundamental reflection and are separated by other fundamental reflections will be joined together after the removal of these separating fundamental reflections, this constraint does not conflict with our previous statement that the fundamental reflection is not necessarily composed of continuous ray segments.

### 4.3.2 Equivalent Rays

An ordered set \((m_{jN})\) does not uniquely determine a ray (Fig. 3.3 shows two rays which have the same ordered set \((1, 1, 0)\) for Earth models with only one discontinuity). Instead, it uniquely determines a set of equivalent rays defined below.
Fig. 4.3 The two rays which have the same set of ray segments and the same set of reflections at and transmissions across interfaces, yet belong to the different ordered sets \((1;1,1;0,0,0)\) (Fig. A) and \((2;0,0;0,1,0)\) (Fig. B).
DEFINITION 4.3.2-A

Two rays having the same source and receiver are equivalent if their respective decompositions are defined by the same ordered set \((m^i_j)_N\).

Since all the equivalent rays have the same expressions for travel time, Eq. (4.3), and amplitude, Eq. (4.4), we have therefore proved

THEOREM 4.3.2-A

All equivalent rays have the same travel time and amplitude.

4.4 SUMMATION OF EQUIVALENT RAYS

From the above discussion it is clear that if we can express the summation of all the equivalent rays determined by the ordered set \((m^i_j)_N\) in terms of \(m^i_j\), \(t^i_j\) and \(\Lambda_j\), the summation of rays of interest simply becomes the summation of sets of equivalent rays.

Obviously, the summation of all the equivalent rays in the set determined by the ordered set \((m^i_j)_N\) has a travel time expressed by Eq. (4.3) and an amplitude which is the product of the amplitude (Eq. (4.4)) of any one of the equivalent rays and the number, \(M[(m^i_j)_N]\), of equivalent rays in the set. The construction of \(M[(m^i_j)_N]\) is therefore our main task in summing the equivalent rays.

Since the ordered set \((m^i_j)_N\) is the immediate result of our ray decomposition strategy, our strategy for finding \(M[(m^i_j)_N]\) for a given ordered set is to put together all the fundamental reflections corresponding to this ordered set in all the possible ways which lead to rays with both source and receiver at the Earth's surface. To ensure that we construct only the rays corresponding to the given ordered set,
we need the constraint that a fundamental reflection can be inserted in
a ray at only one ray junction. Under such a strategy, one may adopt
$r_j^i$ or $r_i^j$ or both as appropriate in constructing a ray so long as their
total number is $m_j^i$, and there may exist several possible procedures
which specify detailed steps to construct the equivalent rays.

We want to define here one such procedure in which all the
fundamental reflections $r_j^i$ will be put together in the order of numbers
$m_j^i$ in the ordered set [i.e., $m_{N+1}^0$, $r_{N+1}^0$ are introduced first, then $m_{N}^0$, $r_{N}^0$
and $m_{N+1}^1$, $r_{N+1}^1$, etc.; remembering that $(m_j^i)_N = (m_{N+1}^0$, $m_N^0$, $m_{N+1}^1$, ...
$m_{1}^0$, $m_{2}^1$, ... $m_{N+1}^N)$] such that there is a single ray construction
corresponding to each equivalent ray. This requires the exclusion of
the possibility that an equivalent ray can be constructed in more than
one way (Fig. 4.4 shows two ways for constructing the same ray). This
requirement can be met if the fundamental reflections are distributed
only into the non-equivalent ray junctions defined below:

**DEFINITION 4.4-A**

Two ray junctions are non-equivalent for a fundamental
reflection $r_j^i$ if the insertion of an $r_j^i$ ($r_j^i$ or $r_j^i$ as appropriate; $i<j$)
in one ray junction and of an $r_j^i$ in the other produce different rays.

4.4.1 Numbers of Non-Equivalent Junctions

For $1 \leq i < j \leq N$, all the fundamental reflections which are associated
with $S_i$ and $S_j$ and have a longer length than $r_j^i$ can be classified into the
three types shown in Fig. 4.5, according to the distribution of
reflections at and transmissions across $S_i$ and $S_j$. Since an $r_j^i$ has a
reflection at its starting point, it can be inserted in a ray only in
such a way that, after the insertion of the $r_j^i$, the ray changes direction
at the starting point of the newly introduced $r_j^i$. For fundamental
Fig. 4.4 The two ways of constructing the same ray with an ordered set (1;0,1;0,1,0) from a ray with an ordered set (1;0,1;0,0,0). One way is to put an $r^1$ at junction 1 (Fig. A), and the other is to put an $r^2$ at junction 2 (Fig. B).
Fig. 4.5 The three types (A, B and C) of fundamental reflections which are associated with $S_i$ and $S_j$ ($1 \leq i, j \leq N$) and have a longer length than $r_{ij}^{-1}$. '.' on interfaces denotes a reflection and 'x' a transmission. The ray junctions designated by brackets are the chosen non-equivalent junctions for $r_{ij}^{-1}$. 
reflections of type A shown in Fig. 4.5, we can only insert an \( r^j_i \) at junction 2 (or 4') or insert an \( r^i_j \) at junction 4 (or 2'). Junctions 2 and 4 (or 2' and 4') are non-equivalent for \( r^i_j \), therefore for each fundamental reflection of type A, we have two non-equivalent junctions at which to put \( r^j_i \). For fundamental reflections of type B, we can put an \( r^j_i \) at junction 6 or put an \( r^j_i \) at junction 7. But junctions 6 and 7 are equivalent for \( r^j_i \). Therefore we can choose only one non-equivalent junction from them (we choose the first one, i.e., junction 6). Similarly, we can only choose one non-equivalent junction (junction 9) from junctions 9 and 10 for each fundamental reflection of type C. Thus, the number of total available non-equivalent junctions at \( S_i \) and \( S_j \) into which the \( m^i_j \) fundamental reflections \( r^i_j \) can be distributed is

\[
1^i_j = 2 \sum_{x=0}^{i-1} m^x + \sum_{y=j+1}^{N+1} m^y + \sum_{y=j+1}^{N+1} m^y (1 \leq i < j \leq N). \tag{4.5}
\]

Since all the fundamental reflections which are associated with \( S_i \) (i = 1, 2, ...N) and \( S_N+1 \) (the core-mantle boundary) and have a length longer than \( r^i_{N+1} \) belong to type C in Fig. 4.5, following the discussion in the last paragraph we can take all the non-equivalent junctions for \( r^i_{N+1} \) as the ray junctions at \( S_{N+1} \). Their number can be expressed as

\[
1^{i}_{N+1} = \sum_{x=0}^{i-1} m^x_{N+1} (i = 1, 2, ...N) \tag{4.6}
\]

All the fundamental reflections which reach \( S_0 \) (the Earth's surface) and transmit across \( S_j \) (j = 1, 2, ...N) can be described by Fig. 4.6. It is clear that we can choose the non-equivalent junctions as the ray junctions (including both source and receiver) at \( S_0 \), and their number is
Fig. 4.6 The fundamental reflections which are associated with $S_0$ (the Earth's surface) and $S_j$ ($j=1,2,...N$) and have a longer length than $r_j^0$ (the first two ray segments belong to one fundamental reflection and the last two to another).
\[ I_j^0 = \sum_{y=j+1}^{N+1} m_y^0 + 1 \quad (j = 1, 2, \ldots, N). \]  

(4.7)

Since we initiate the building of a ray by putting \( m_{N+1}^0 \) fundamental reflections \( r_{N+1}^0 \) between source and receiver, and there is only one way to do so, we therefore define

\[ I_{N+1}^0 = 1. \]  

(4.8)

\( I_j^i \) can be written as \( I_{j+i}^i \) (\( J = 1, 2, \ldots, N+1; \ i = 0, 1, \ldots, N-J+1 \)), and an important property of \( I_{j+i}^i \) is that \( I_{j+i}^i \) are functions of only those \( m_{j+i}^i \) with \( j > J \).

4.4.2 Construction of Equivalent Rays

To simplify the notation in our discussion, we let

\[ k_{j+i}^i = \left( I_{j+i}^i + m_{j+i}^i - 1 \right) C m_{j+i}^i . \]  

(4.9)

Then according to LEMMA 3.2.3-A and Eq. (4.9) we have the following theorem:

THEOREM 4.4.2-A

The numbers of all possible ways to put \( m_{j+i}^i \) (\( J = 1, 2, \ldots, N+1; \ i = 0, 1, \ldots, N-J+1 \)) fundamental reflections \( r_{j+i}^i \) into the \( I_{j+i}^i \) non-equivalent junctions are \( k_{j+i}^i \).

Now we can define our procedure for constructing all the equivalent rays determined by the ordered set \( (m_{N+1}^0, m_N^0, m_{N+1}^1; \ldots, m_1^0, m_2^1, \ldots, m_{N+1}^N) \) by the following algorithm:

(a) The fundamental reflections are introduced in the order of numbers \( m_j^i \) in the ordered set, as stated above. All the numbers \( m_j^i \) in the ordered set can be expressed in the form \( m_{j+i}^i \) (\( J = 1, 2, \ldots, N+1; \ i = 0, 1, \ldots, N-J+1 \);
i = 0, 1, ...N-J+1) and classified into N+1 classes according to
the number of J (each class is referred to as the Jth class, which
has N-J+2 elements). If all the numbers \( m_i^j \) are arranged as a
triangular pyramid (Figs. 4.7A-a and 4.7B-a) with each class of
\( m_i^j \) represented by a row of the pyramid, then the order of \( m_i^j \) in
the ordered set is the order of the elements of the triangular
pyramid, counting from the left-most element to the right-most
element of each row and from the top to the bottom row of the
pyramid;

(b) All the fundamental reflections between \( S_0 \) and \( S_j \) (j= 1, 2, ...N)
are taken as \( r_i^0 \) and distributed into ray junctions at \( S_0 \) of the
ray being set up, while all the fundamental reflections between \( S_i \)
(\( i = 1, 2, \ldots N \)) and \( S_{N+1} \) are taken as \( r_i^{N+1} \) and distributed into
the ray junctions at \( S_{N+1} \);

(c) The fundamental reflections between \( S_i \) and \( S_j \) (i<j; \( i = 1, 2, \ldots N-1; \)
\( j = 2, 3, \ldots N \)) are distributed into the non-equivalent junctions
shown in Fig. 4.5 (designated by brackets) and taken as \( r_i^j \) or \( r_j^i \)
as appropriate.

MODELS WITH TWO DISCONTINUITIES

Following this procedure, we first built up the number
\[ M(m_0^0, m_1^0, m_3^0; m_1^1, m_2^1, m_3^1) \]
for models with only two discontinuities
between the Earth's surface and the core-mantle boundary, and then extend
the result to the general case of N discontinuities. We construct all
the equivalent rays corresponding to the ordered set \( (m_3^0; m_2^0, m_3^1; m_1^0, m_2^1, m_3^1) \)
in the following steps:

Step (1):
Initially put \( m_3^0 \) (\( m_3^0 = 1, 2, \ldots \infty \)) fundamental ray components \( r_3^0 \)
between \( S_0 \) (the Earth's surface) and \( S_3 \) (the core-mantle boundary). There
A. TWO DISCONTINUITIES

(a)

3rd class $\rightarrow$ $m_3^0$
2nd class $\rightarrow$ $m_2^0, m_2^1$ $\rightarrow$ $\kappa_2^0, \kappa_3^1$
1st class $\rightarrow$ $m_1^0, m_1^1, m_1^2$ $\rightarrow$ $\kappa_1^0, \kappa_2^1, \kappa_3^2$

(b)

B. N DISCONTINUITIES

(a)

$(N+1)^{th}$ class $\rightarrow$ $m_{N+1}^0$
$N^{th}$ class $\rightarrow$ $m_N^0, m_N^1, m_N^{N+1}$ $\rightarrow$ $\kappa_N^0, \kappa_N^1$
$(N-1)^{th}$ class $\rightarrow$ $m_{N-1}^0, m_{N-1}^1, m_{N-1}^{N+1}$ $\rightarrow$ $\kappa_{N-1}^0, \kappa_N^1, \kappa_{N+1}^2$

(b)

1st class $\rightarrow$ $m_1^0, m_2^1, \ldots, m_{N-1}^N, m_N^{N+1}$ $\rightarrow$ $\kappa_1^0, \kappa_2^1, \ldots, \kappa_N^{N-1}, \kappa_N^{N+1}$

$$[\kappa_i^j = (i_j^i + m_j^i - 1)C_m^i]$$

Fig. 4.7 The triangular pyramids of the numbers $m_{J+i}^i$ (Figs. A-a and B-a) and the corresponding numbers $\kappa_{J+i}^i$ (Figs. A-b and B-b) of ways of distributing $m_{J+i}^i$ into $i_{J+i}$ non-equivalent junctions of the ray being built up. The product of all the elements of the pyramid of $\kappa_{J+i}^i$ is the number $M[(m_j^i)_{N-j}]$. Fig. A is for a special case of two discontinuities and Fig. B for a general case of $N$ discontinuities. See the text for the definitions of $i_{J+i}^i, \kappa_{J+i}^i$ and $M[(m_j^i)_{N-j}]$. 
is only one way to do this. Fig. 8A shows the situation for \( m_3^0 = 1 \) (with other \( m_j^i = 0 \) at this stage) which corresponds to ScS.

**Step (2):**

Distribute \( m_2^0 \) fundamental ray components \( r_2^0 \) into the \( I_2^0 \) (= \( m_3^0 + 1 \)) ray junctions at \( S_0 \) (designated by 'A' in Fig. 4.8A) of the ray set up in step (1) (according to THEOREM 4.4.2-A, we have \( \kappa_2^0 \) ways to do so), and then distribute \( m_3^1 \) fundamental ray components \( r_1^3 \) into the \( I_3^1 \) (= \( m_3^0 \)) ray junctions at \( S_3 \) (designated by '*' in Fig. 4.8A; we have \( \kappa_3^1 \) ways to do so). The introductions of \( m_2^0 r_2^0 \) and \( m_3^1 r_1^3 \) into the ray set up in step (1) are independent, therefore their order is reversible. Since each way of ray construction represents an equivalent ray, we have \( \kappa_2^0 \cdot \kappa_3^1 \) equivalent rays at this stage. Fig. 4.8B shows one of the three \( \binom{3}{2} \cdot \binom{1}{1} \) equivalent rays for the situation when \( m_3^0 = 1 \), \( m_2 = 2 \) and \( m_3^1 = 1 \) (with other \( m_j^i = 0 \) at this stage).

**Step (3):**

For each of the equivalent rays set up in Step (2), we distribute \( m_1^0 r_1^0 \) into the \( I_1^0 \) (= \( m_3^0 + m_2^0 + 1 \)) ray junctions at \( S_0 \) (designated by '•') in Fig. 4.8B), \( m_2^1 r_2^1 \) (each \( r_2^1 \) can be taken at \( r_2^1 \) or \( r_1^2 \) as appropriate) into the \( I_2^1 \) (= \( 2m_3^0 + m_2^0 + m_3^1 \)) non-equivalent junctions at \( S_1 \) and \( S_2 \) (designated by 'm' in Fig. 4.8B), and \( m_3^2 r_2^3 \) into the \( I_3^2 \) (= \( m_3^0 + m_3^1 \)) ray junctions at \( S_3 \). These three kinds of fundamental reflections can be introduced into the ray set up in Step (2) independently; therefore they can be put in simultaneously.

According to THEOREM 4.4.2-A, we can construct \( \kappa_1^0 \cdot \kappa_2^1 \cdot \kappa_3^1 \) equivalent rays from each of the equivalent rays set up in Step (2). The total number of equivalent rays determined by the ordered set \( (m_3^0; m_2^0; m_3^1; m_1^0; m_2^1; m_3^2) \) is therefore

\[
M(m_3^0; m_2^0; m_3^1; m_1^0; m_2^1; m_3^2) = \kappa_1^0 \cdot \kappa_2^1 \cdot \kappa_3^1 \cdot \kappa_2^0 \cdot \kappa_3^1
\]  

(4.10)
Fig. 4.8 The construction of the equivalent rays determined by the ordered set (1;2,1;1,2,1). This demonstrates the procedure for obtaining an explicit expression for the number, \( M(m_0^0 m_0^1 m_1^0 m_1^1 m_2^0 m_2^1 m_3^0 m_3^1 m_1 m_2 m_3) \), of all the equivalent rays determined by the ordered set \((m_0^0 m_0^1 m_1^0 m_1^1 m_2^0 m_2^1 m_3^0 m_3^1 m_1 m_2 m_3)\). 'A', '*', '.', '•' and 'x' denote the non-equivalent junctions for \(r_2^0, r_3^1, r_1^0, r_2^1\) and \(r_3^2\) respectively.
Fig. 4.8C shows one of the $360 \left(4^2 \cdot 6^2 \cdot 2^2 \cdot 1^2 \cdot 1^1 \right)$ equivalent rays for the situation when $m_0^0 = 1$, $m_0^2 = 2$, $m_0^3 = 1$, $m_1^0 = 1$, $m_2^1 = 2$ and $m_3^2 = 1$.

Fig. 4.8C results from Fig. 4.8B by putting one $r_1^0$ at junction 1, one $r_1^2$ at junction 2, one $r_2^1$ at junction 3 and one $r_2^3$ at junction 4.

Corresponding to the triangular pyramid of $m_j^i$ shown in Fig. 4.7A-a, we build a triangular pyramid (Fig. 4.7A-b) of the numbers of possible ways to put $m_j^i$ fundamental reflections $r_j^i$ in a ray. Since the element in the first row of the pyramid in Fig. 4.7A-b is unity, and the product of the elements in each other row of the pyramid in Fig. 4.7A-b can be expresses as

$$\prod_{i=0}^{2-J+1} \kappa_{J+i}$$

where $J (J = 1, 2)$ indicates the $(2-J+2)$th row of the pyramid and $\prod$ denotes series multiplication, Eq. (4.10) can be rewritten as

$$M(m_0^0, m_0^1, m_0^2, m_1^0, m_2^1, m_3^2) = \prod_{J=1}^{2} \prod_{i=0}^{2-J+1} \kappa_{J+i}$$

(4.11)

where $\kappa_{J+i}$ are defined by Eq. (4.9).

MODELS WITH N DISCONTINUITIES

Now we extend the result to a general case of $N$ discontinuities between the Earth's surface and the core-mantle boundary. Following the procedure stated and demonstrated above, we need $N+1$ steps to construct all the equivalent rays determined by the ordered set $(m_j^i)_N$, with each step distributing the fundamental reflections corresponding to the numbers $m_j^i$ in a row of the triangular pyramid shown in Fig. 4.7B-a. Corresponding to the pyramid of $m_j^i$ shown in Fig. 4.7B-a, the pyramid of the numbers of possible ways to put the $m_j^i$ fundamental reflections $r_j^i$ in a ray is shown in Fig. 4.7B-b. The element of the first row of the pyramid of Fig. 4.7B-b is unity, and the product of the elements in each other row of the pyramid in Fig. 4.7B-b can be expressed as
where \( J(J = 1, 2, \ldots N) \) indicates the \((N-J+2)\)th row of the pyramid. Since each way of constructing the equivalent rays corresponding to the ordered set \( (m_j^i)_N \) represents an equivalent ray, the total number \( M \) of the equivalent rays determined by the ordered set can be written as

\[
M[(m_j^i)_N] = \prod_{J=1}^{N} \prod_{i=0}^{N-J+1} \kappa_i^{j+i}.
\] (4.12)

Since the ordered set \( (m_j^i)_N \) uniquely determines a set of equivalent rays, the total number of the equivalent rays in the set is independent of the procedures we use to construct it, and is therefore proved to be defined by Eq. (4.12). Now we have proved (by Eqs. (4.4) and (4.12))

**THEOREM 4.4.2-B**

The total amplitudes of the summation of all equivalent rays determined by a given ordered set \( (m_j^i)_N \) can be written as

\[
S[(m_j^i)_N] = 2 \prod_{J=1}^{N} \prod_{i=0}^{N-J+1} (\kappa_i^{j+i} A_{N+1}^{m_j^i+i}) 0_{N+1}^{m_j^i}. \] (4.13)

**4.5 SUMMATION OF MULTIPLE REFLECTIONS OF SIGNALS**

To facilitate the summation of the required rays, we first group the rays into different sets according to the numbers \( m_j^i \) of fundamental reflections between discontinuities. Then, each set is a set of equivalent rays determined by an ordered set \( (m_j^i)_N \). Now, to get the summation \( f(t) \) of the multiple reflections, we need only to sum these sets of equivalent rays.

Since the travel time of any ray in the set of equivalent rays determined by the ordered set \( (m_j^i)_N \) is the same, it is clear that \( f(t) \) can be formulated in the following way:
\[ f(t) = \sum_{i} \sum_{m_j} \hat{f}(t-\hat{c}+c) \]

(4.14)

where \( \hat{c} \) is expressed in Eq. (4.3) and \( \hat{s} \) in Eq. (4.13); \( c \) is a constant related to the starting time of the series of multiple reflections on records; \( \hat{f} \) is a time function related to the travel time \( \hat{c} \) of any ray in the set of equivalent rays (\( \hat{f} \) can be represented by a \( \delta \)-function if the source function is not of interest); and the summation is over the related \( m_j \). If the source or receiver are not at the Earth's surface, some appropriate modification of Eq. (4.14) is required.

Examples of immediate application of the developed strategy for ray analysis can be seen in Chapter 5 for deriving a ray-mode duality for SH waves in discontinuous Earth's models. As an illustration of another application, in the next section the mathematical filter of Backus (1959) for removing wave reverberations in water layers from the record is re-interpreted and extended to a general case.

4.6 RE-INTERPRETATION AND EXTENSION OF A FILTER FOR REMOVING REVERBERATIONS IN WATER LAYER

Let \( \tau \) be the two-way travel time for vertically incident P-waves in the water layer, and \( b \) the reflection coefficient at bottom of the water layer for down-going waves. If the shot and receiver are assumed to be on the water surface, the series of (vertical) wave reverberations in the water layer can be written (with the reflection coefficient at the water surface assumed to be -1) as

\[ 1 - b\delta(t - \tau) + b^2\delta(t - 2\tau) - b^3\delta(t - 3\tau) + ... \]

(4.15)

and its Fourier transform is

\[ 1 - bz + b^2z^2 - b^3z^3 + ... = 1/(1 + bz) \]

(4.16)

where \( z = \exp(-i\omega\tau) \) and \( \omega \) is the wave frequency.
According to Backus (1959) and Robinson (1967, p.136) the water layer effect on signals which arrive at the water surface only once from an interface below the bottom of the water layer can be regarded as a linear filter. These signals pass through the water layer twice, once downward and once upward. Therefore the water layer effect can be considered as the transfer function

\[ K(\omega) = \frac{1}{(1 + bz)^2}, \quad (4.17) \]

and the filter for removing the wave reverberations in the water layer is

\[ H(\omega) = (1 + bz)^2. \quad (4.18) \]

The physical significance of Eqs. (4.17) and (4.18) is not immediately obvious. Therefore, a re-interpretation of the reverberation filter in terms of equivalent rays may be useful.

In this section we derive the mathematical filter for elimination of the wave reverberations (in water layer) associated with a wave which arrives at the water surface \( j \) times from the deep interfaces (i.e., those below the bottom of water layer). Eq. (4.18) will be interpreted as a simple case of the derived filter.

To facilitate our mathematical manipulation, we assume that the source and receiver are at the water surface. We are interested only in rays corresponding to a wave which arrives the water surface \( j \) times from the deep interfaces and the associated wave reverberations in the water layer. Any of these rays can be written in the form

\[ \hat{r} = \hat{r}_j + mr, \quad (4.19) \]

where \( \hat{r}_j \) is the ray path arriving the water surface \( j \) times from deep interfaces, \( r \) is a fundamental reflection (defined in §4.2), or wave reverberation, in the water layer, \( m \) is the number of fundamental reflections \( r \). The travel time and amplitude of the ray in Eq. (4.19) are
obviously

\[ \hat{t} = \hat{t}_j + m\tau \quad (4.20) \]
\[ \hat{A} = \hat{A}_j \cdot (-b)^m, \quad (4.20a) \]

where \( \tau \) is the travel time of \( r \), \( b \) is the reflection coefficient at bottom of the water layer for down-going vertically incident P waves and \( \hat{A}_j \) and \( \hat{t}_j \) are the amplitude and travel time of \( \hat{r}_j \).

We group all the rays of interest into different sets according to the number \( m \). Then each set is a set of equivalent rays determined by the ordered pair \((j, m)\). Since the \( m \) fundamental reflections \( r \) can be distributed into the \((j + 1)\) non-equivalent junctions for \( r \) (Fig. 4.6 and Eq. (4.7)), according to Eqs. (4.9) and (4.12) the total amplitude (Eq. (4.13)) of the set of equivalent rays determined by the ordered pair \((j, m)\) can be written as

\[ S(j, m) = \hat{A}_j \left[ \binom{j+m}{m} (-b)^m \right] \quad (4.21) \]

If we start the series of wave pulses from \( \hat{t}_j \), we can represent the summation \( f(t) \) (Eq. (4.14)) of the above-mentioned arriving wave pulses (with \( c = \hat{t}_j \)) as

\[ f(t) = \sum_{m=0}^{\infty} S(j,m) \delta (t - m\tau). \quad (4.22) \]

The Fourier transform of \( f(t) \) is

\[ F(\omega) = \sum_{m=0}^{\infty} S(j,m) \exp (-i\omega m\tau) \quad (4.23) \]

(where 'i' represents a complex number) which is simply (by Eq. (3.5))

\[ F(\omega) = \hat{A}_j / (1 + bz)^{j+1} \quad (4.24) \]

where \( z = \exp(-i\omega \tau) \). Therefore, the filter for removing wave reverberation is

\[ (1 + bz)^j + 1. \quad (4.25) \]
Equation (4.18) is the same as Eq. (4.25) when \( j = 1 \). It is clear from the above discussion that the simple forms of filters represented by Eqs. (4.18) and (4.25) result from the fact that the rays associated with wave reverberations in the water layer can be grouped into different sets of equivalent rays, and these sets, in turn, can be summed in the form of Eq. (3.5). Since in Eq. (4.21) the number \( {j+m \choose m} \) of equivalent rays in the set determined by the ordered pair \((j,m)\) is the number of ways of distributing the \( m \) wave reverberations in the water layer into the \((j+1)\) non-equivalent junctions (junctions at \( S_0 \), Fig. 4.6), the index of the filter in Eq. (4.25) arises from the fact that the wave reverberations in the water layer can occur at any of the \((j+1)\) non-equivalent junctions. Fig. 4.9 shows that the term is squared in Eq. (4.18) because wave reverberations in the water layer can occur at the two non-equivalent junctions for \( r \) (designated by ' . '), each at one side of the deep reflection.
Fig. 4.9 Diagram showing that the wave reverberations in the water layer associated with a reflection from an interface below the water bottom can occur at both sides of the reflection.
APPENDIX  UNIQUENESS OF RAY DECOMPOSITION OF EQ. (4.2)

We want to prove here that any given ray has a unique ray decomposition of Eq. (4.2) under our strategy for ray decomposition.

Since a ray is assumed to include at least one ScS path (i.e., \( r_{N+1}^0 \)), it reaches the core-mantle boundary (\( S_{N+1} \)) at least once. All the fundamental reflections can be written in the form \( r_{J+1}^i \) (\( J = 1, 2, \ldots N+1; i = 0, 1, \ldots N-J+1 \)). The segments of the fundamental reflections of larger \( J \) can be separated by the fundamental reflections of smaller \( J \), but the converse is not true. Therefore, if we decompose all the \( r_{1+i}^i \) first, then all the \( r_{2+i}^i \) ... and then all the \( r_{N+i}^i \), every fundamental reflection decomposed is an integral part of the ray just before the decomposition of the fundamental reflection, since after all the fundamental reflections \( r_{J+1}^i \) of a given \( J \) have been removed from the ray, all the fundamental reflections \( r_{J+1+i}^i \) are integral parts of the remanent ray. The remanent ray, after the removal of all \( r_{N+i}^i \), belongs to multiple ScS, which obviously can be decomposed into \( m_{N+1}^0 r_{N+1}^0 \) with \( m_{N+1}^0 \geq 1 \).

Since every fundamental reflection \( r_{J+1}^i \) includes a pair of successive reflections \( R_i^{(u)} \) and \( R_i^{(d)} \), decomposing an \( r_{J+1}^i \) (no matter whether \( r_{J+1}^i \) or \( r_{J+1}^{i+j} \)) means removing this pair of reflections (together with the associated ray segments) from the ray. For a given ray, the number of pairs of successive \( R_i^{(u)} \) and \( R_i^{(d)} \) is unique, no matter how these reflections are paired. Therefore, every \( m_{1+i}^i \) is unique. After removing all the \( r_{1+i}^i \) from the ray, the number of pairs of successive \( R_i^{(u)} \) and \( R_{2+i}^{(d)} \) is unique, therefore every \( m_{2+i}^i \) is unique. If we continue this process of argument until \( m_{N+1}^0 \) is proved to be unique, then we prove that every \( m_{J+1}^i \) is unique under the above-mentioned procedure for decomposing the ray.
The above procedure for decomposing the ray does not lose generality. Our strategy for ray decomposition includes a constraint that a fundamental reflection can be decomposed only when it is an integral part of the ray to be decomposed. Even if we decompose the fundamental reflections randomly under this constraint, the fundamental reflections $r^i_{J+i}$ ($J$ is given) whose ray segments are separated by the fundamental reflections $r^i_{J'+i}$ ($J'<J$) can become integral parts of the ray only when these fundamental reflections $r^i_{J'+i}$ have been removed.
CHAPTER 5
RAY-MODE DUALITY FOR SH WAVES IN EARTH MODELS WITH CRUST AND MANTLE DISCONTINUITIES: II. THE CASE OF N DISCONTINUITIES

5.1 INTRODUCTION

It has been shown in Chapter 3 that McNabb, Anderssen and Lapwood's (1976) equation (Eq. (1.2)) for the asymptotic behaviour of torsional overtone eigenfrequencies of discontinuous SNREI Earth models can be derived, for models with a single discontinuity between the surface and the core-mantle boundary, from an adaptation of Brune's (1964) method to summations of almost vertical SH waves multiply-reflected in the Earth's interior. The strategy of ray analysis was extended in Chapter 4 to a general case of Earth models with N discontinuities. By the developed ray-analysis strategy, this chapter aims to derive Eq. (1.2) for Earth models with N discontinuities, with all the constants defined explicitly in terms of Earth model parameters: the shear wave radial travel times between the Earth's surface, the core-mantle boundary and the discontinuities between them, and the reflection coefficients for vertically-incident SH waves at the discontinuities. To make our discussion clear, throughout this paper examples or demonstrations of steps in the derivation of Eq. (1.2) from summation of rays, for Earth models with only two discontinuities, are given where necessary.

5.2 THE SUMMATIONS f_1(t) AND f_2(t)

Earth models used for this chapter were described in §4.2 and shown in Fig. 4.1. In Fig. 4.1, the Earth's surface (S_0) and the core-mantle boundary (S_{N+1}) are assumed to be perfect reflectors for SH waves, i.e., R_0^{(u)} = 1 and R_{N+1}^{(d)} = 1. To facilitate the discussion in this chapter, the following constants are defined:
\[
\begin{aligned}
&k_i (i = 1, 2, \ldots, N) \equiv R_i (u) = -R_i \\
&k_0 \equiv 1, \\
&k_{N+1} \equiv -1.
\end{aligned}
\] (5.1)

It is clear that
\[
T_i (d) T_i (u) = 1 - k_i^2. 
\] (5.2)

Since we restrict attention to almost vertical kScS (k≥1) rays and multiple reflections of SH waves from crust and mantle discontinuities, the pair \([h_1(t), h_2(t)]\) of wave trains of body wave pulses recorded at epicentral distances \(\Delta\) and \(2\Delta\) respectively will be adapted to Eq. (3.2) to derive Eq. (1.2).

As pointed out earlier, since \(g(\omega)\) of Eq. (3.2) is independent of the source functions, \(h_1(t)\) and \(h_2(t)\) can be represented by series, \(f_1(t)\) and \(f_2(t)\), of \(\delta\)-functions (with each \(\delta\)-function representing a ray) to simplify the discussion. This chapter aims to show that, for Earth models with \(N\) discontinuities between the Earth's surface and the core-mantle boundary, Eq. (1.2) can be derived from Eq. (3.2) if, starting from the onsets of \((k\text{ScS})_\Delta\) and \(((k+1)\text{ScS})_{2\Delta}\) pulses (where \(k\geq 1\)) in seismograms recorded at distances \(\Delta\) and \(2\Delta\) respectively, \(f_1(t)\) and \(f_2(t)\) are taken as the sums of \(m(m\geq 1)\) successive multiple reflections between the Earth's surface and the core-mantle boundary and all the associated multiple reflections from the discontinuities between them.

5.3 FORMULATION OF \(f_1(t)\), \(f_2(t)\) AND THEIR FOURIER TRANSFORMS

To facilitate the formulation of \(f_1(t)\) and \(f_2(t)\), we express the ray in the form of Eq. (4.2), and then group all the rays into different sets according to the numbers, \(m_i^j (i<j; i = 0, 1, \ldots, N; j = 1, 2, \ldots, N+1)\), of fundamental reflections between the Earth's surface, the discontinuities and the core-mantle boundary (Fig. 4.1). Then each set is a set of
equivalent rays (defined in §4.3) determined by an ordered set
\((0_{N+1}; m^0_N, m^1_{N+1}; \ldots; m^0_1, m^1_2, \ldots m^0_{N+1})\). Now, to formulate \(f_1(t)\) and \(f_2(t)\), we need only to sum these sets of equivalent rays.

By its definition in §5.2 and Eq. (4.13), \(f_1(t)\) can be written as

\[
f_1(t) = \sum_{k+m-1} \left( \sum_{m_N=0}^{m_{N+1}=0} \left( \sum_{m_1=0}^{m_{N+1}=0} \left( \sum_{m_1=0}^{m_{N+1}=0} \right) \right) \right)
\]

where each \(\Sigma\) denotes a summation, and the \(\Sigma\)'s in each bracket are over all \(m^i_{J+i}\) for a given \(J\). Since \(I^i_{J+i}\) (Eqs. (4.5) through (4.8)) and therefore \(\kappa^i_{J+i}\) (Eq. (4.9)) are functions of only those \(m^i_{J'+i}\) with \(J' > J\), the summations must be carried out over \(m^i_{J+i}\) of lower \(J\) first. The reason why we write the coefficients before \(t^0_{N+1}\) as \((m^0_{N+1}-k)\) is that \(f_1(t)\) starts from the \(\hat{ScS}\) pulse in a seismogram. From the fact that the Fourier transform of \(A_1\delta(t-t_1)\) is \(A_1 \exp(-i\omega t_1)\), the Fourier transform of \(f_1(t)\) can be written as

\[
F_1(\omega) = 2\exp(i\omega t^0_{N+1}) \sum_{k+m-1} \left( \sum_{m_N=0}^{m_{N+1}=0} \left( \sum_{m_1=0}^{m_{N+1}=0} \left( \sum_{m_1=0}^{m_{N+1}=0} \right) \right) \right)
\]

where \(X^x_y (x<y; x = 0, 1, \ldots N, y = 1, 2, \ldots N+1) = X_A \exp(-i\omega t^x_y)\) (the symbol 'i' in the argument of the exponential function denotes a complex number).
To simplify our discussion, using the expression in [ ] of Eq. (5.4) we define a function \( G(m^0_{N+1}) \) (where \( m^0_{N+1} = 1, 2, \ldots \)) with

\[
G(m^0_{N+1}) = \left( \sum_{m^0_{N+1}=0}^{\infty} \sum_{m^0_{N-1}=0}^{\infty} \sum_{m^0_{N-2}=0}^{\infty} \ldots \sum_{m^0_{1}=0}^{\infty} \sum_{m^0_{0}=0}^{\infty} \right) \ldots \left( \sum_{m^0_{N-1}=0}^{\infty} \sum_{m^0_{N-2}=0}^{\infty} \ldots \sum_{m^0_{0}=0}^{\infty} \right)
\]

where all \( \kappa_{J+i} \) are functions of \( m^0_{N+1} \). Then \( F_1(\omega) \) can be written as

\[
F_1(\omega) = 2 \exp\left(i\omega^{\hat{t}^0_{N+1}}\right) \sum_{m^0_{N+1}=k}^{k+m-1} \left[G(m^0_{N+1}) W^{m^0_{N+1}}_{N+1} \right]. \tag{5.6}
\]

Similarly, using its definition in §5.2 and following the same procedure, we can write the Fourier transform \( F_2(\omega) \) of \( f_2(t) \) as

\[
F_2(\omega) = 2 \exp[i\omega^{\hat{t}^0_{N+1}}] \sum_{m^0_{N+1}=k+1}^{k+m} \left[G(m^0_{N+1}) W^{m^0_{N+1}}_{N+1} \right]. \tag{5.6a}
\]

5.4 THE EXPRESSION \( F_1(\omega)/F_2(\omega) \)

In order to use Eqs. (3.2) and (3.3) to derive Eq. (1.2), we have to obtain an explicit expression for \( F_1(\omega)/F_2(\omega) \) from Eqs. (5.6) and (5.6a). Eqs. (5.6) and (5.6a) include a common factor \( G(m^0_{N+1}) \). Thus, to get \( F_1/F_2 \) we have to simplify \( G(m^0_{N+1}) \) first.

To facilitate the simplification of \( G(m^0_{N+1}) \), we adopt the following notations:

\[
i_{D_{l+1}} = 1 - iW_{l+1} \quad (i = 0, 1, \ldots N) \tag{5.7}
\]
\[ j_{D+j} = j_{D(J-1)+j} + j_{E(J-1)+j} + j_{W+j} \]

\[ (J = 2, 3, \ldots N; j = 0, 1, \ldots N-J+1), \quad (5.7a) \]

with

\[ s_{k+S} = \begin{cases} 1 & \text{when } K = 1, 2 \\ K-2 & \text{when } K = 3, 4, \ldots N, \\ \prod_{d=1}^{K-d+s-1} x_{s+1}^{d+x} \end{cases} \]

\[ (5.7b) \]

where \( s = 0, 1, \ldots (N-K+1) \). Then it can be shown (APPENDIX A) that Eq. (5.5) can be simplified to

\[ G_{m+1} = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]

under the conditions that \( \left| i_{w_{j+1}} \right| < 1 \) (this is satisfied since \( \left| i_{A_{j+1}} \right| < 1 \)) and

\[ \left| \left( j_{E_{j-1+i}}/j_{D_{j-1+i}} \right) + i^{l+i}_{j_{E_{j+1}}/j_{D_{j+1}}} \right| < 1 \]

where \( J = 2, 3, \ldots N; i = 0, 1, \ldots N-J+1 \). From Eq. (5.8) we know that \( F_1(\omega) \) and \( F_2(\omega) \) (Eqs. (5.6) and (5.6a)) are geometric series. Thus we have

\[ F_1 = \frac{1}{0_{N+1}} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]

\[ (5.10) \]

Eq. (5.10) is independent of \( m \) and \( k \). Since in the following section (§5.5) we want to derive Eq. (1.2) with the coefficients expressed in terms of \( k \) and \( t_j \), we express Eq. (5.10) in the following notations:

For integers \( J (1 \leq J \leq N) \) and \( j (0 \leq j \leq N-J+1) \), let: \( j_{C_{j+1}} = \{ j, 1+j, 2+j, \ldots J+j \} \); \((r,s)\) be an element of \( Z = \{(1,1), (1,2), \ldots (1,2z-1,1,2z)\} \) which is derived from a subset \( Z' = \{1, 2, \ldots 1,2z-1,1,2z\} \) of \( j_{C_{j+1}} \) with
\( j \leq i_1 < i_2 < \ldots < i_{2z-1} < i_{2z} \leq J+j; \) \( P(\mathcal{J}_{J+j}) \) be the set of all non-empty, even-number-elemented subsets of \( \mathcal{J}_{J+j} \); and

\[
j_{J+j}^1 = 1 + \sum_{Z' \in P(\mathcal{J}_{J+j})} \prod_{(r,s) \in Z'} k_k \exp(-i\omega t_s), \quad (5.11)
\]

which has \( 2^J \) terms since the set \( P(\mathcal{J}_{J+1}) \) has \( 2^J - 1 \) elements (this is clear from the fact that \( \mathcal{J}_{J+1} \) has \( J+1 \) elements and from the combinatorial formula (cf. Spiegel, 1968, p.4): \( J+1C_0 + J+1C_2 + J+1C_4 + \ldots = 2^J \)).

Then we can write Eq. (5.10) as (APPENDIX C)

\[
\frac{F_1}{F_2} = \frac{1}{\mathcal{U}_{N+1}} \bigcup_0 \bigcup_{N+1}.
\]

The conditions of (5.9) are satisfied if (APPENDIX E)

(a) When \( N = 2 \), \( |k_1| + |k_2| + |k_1k_2| < 1 \).

(b) When \( N > 2 \),

\[
\begin{align*}
&\left\{ \begin{array}{l}
0_k_N \quad 1_k_{N-1} < 2,
\end{array} \right. \\
&1_k_{N+1}^{2} \quad 2_k_N < 2,
\end{align*}
\]

where

\[
j_{J+1}^1 = 1 + \sum_{Z' \in P(\mathcal{J}_{J+1})} \prod_{(r,s) \in Z'} k_k k_s.
\]

5.5 DERIVATION OF EQ. (1.2) FROM BRUNE'S FORMULATION

In this section, we derive Eq. (1.2) from Eqs. (3.2), (3.3) and the expression of \( F_1/F_2 \).

The most straightforward way to derive Eq. (1.2) is to substitute Eq. (5.12) into Eq. (3.3) to get \( g(\omega) \), then substitute \( g(\omega) \) into Eq. (3.2)
and simplify Eq. (3.2). But, since Eq. (5.12) is quite complicated when it is expressed in terms of $k_x$ and $t_j$, it is not practical to do so. Instead, we study the role of each term of Eq. (5.12) in Eq. (3.2), and then organize the terms of Eq. (5.12) to simplify Eq. (3.2).

We express Eq. (5.12) in this form:

$$\frac{F_1}{F_2} = 1 + \sum_i \hat{k}_i \exp(-i\omega \hat{t}_i),$$

(5.14)

(where $\hat{k}_i$ and $\hat{t}_i$ are constants) with each term being a term in Eq. (5.12). Then substituting Eq. (5.14) into Eq. (3.3) we get

$$g(\omega) = \arctan \left[ \frac{\sum_i \hat{k}_i \sin(\omega \hat{t}_i)}{1 + \sum_i \hat{k}_i \cos(\omega \hat{t}_i)} \right].$$

(5.15)

Substituting Eq. (5.15) into Eq. (3.2), taking the tangent of both sides, multiplying by $\cos(\omega T)$ and the denominator of the right-hand side, we get after rearrangement of terms

$$\sin(\omega T) = \sum_i \hat{k}_i [\sin(\omega \hat{t}_i) \cos(\omega T) - \cos(\omega \hat{t}_i) \sin(\omega T)]$$

$$= \sum_i \hat{k}_i \sin[\omega (\hat{t}_i - T)].$$

(5.16)

The role of each term of Eq. (5.14) (and therefore Eq. (5.12)) in Eq. (3.2) is clear from Eq. (5.16). The term of unity produces the term on the left-hand side of Eq. (5.16), while each other term containing in the argument of exponential function a time factor $\hat{t}_i \neq T$ generates a term on the right-hand side of Eq. (5.16). Those terms containing the time factor $\hat{t}_i = T$ contribute nothing to Eq. (3.2) since the terms with $\hat{t}_i = T$ vanish from the summation of Eq. (5.16).

We now organize the terms of Eq. (5.12) in such a way that, when $\hat{k}_i$ and $\hat{t}_i$ are replaced by explicit expressions of $k_x$ and $t_j$, Eq. (5.16)
can be simplified to Eq. (1.2) by the use of formulas of sum, difference
and product of trigonometric functions. Substituting

\[
\begin{align*}
0_{U_N} &= 1_{U_N} + (0_{U_N} - 1_{U_N}) \\
1_{U_{N+1}} &= 1_{U_N} + (1_{U_{N+1}} - 1_{U_N})
\end{align*}
\]

into Eq. (5.12), we get

\[
\frac{F_1}{F_2} = \begin{cases} (i) \frac{1_{U_N}}{F} + \sum_{i} (0_{U_N} - 1_{U_N})(1_{U_{N+1}} - 1_{U_N}), \\ (ii) \end{cases}
\]

(5.17)

where

\[
\hat{F} = (0_{U_N} - 1_{U_N}) + (1_{U_{N+1}} - 1_{U_N}) + 1_{U_N}.
\]

From APPENDIX D (Eq. (D.1)), Expression (ii) of Eq. (5.17) can be written
as

\[
(0_{U_N} - 1_{U_N})(1_{U_{N+1}} - 1_{U_N}) = (0_{U_{N+1}} + 1_{U_N}) \exp(-i\omega t_{N+1}),
\]

(5.19)

where \(1_{U_N}^*\) is the conjugate of \(1_{U_N}\). Since the coefficient of exponential
function on the right-hand side of Eq. (5.19) is a constant of real
number and \(t_{N+1}^0 = T\), Expression (ii) of Eq. (5.17) contributes nothing
to Eq. (3.2) (or Eq. (5.16)). Thus, to derive Eq. (1.2) we need only
to replace the terms of Eq. (5.16) by those corresponding to the terms
of \(1_{U_N} \hat{F}\) (when it is expressed in terms of \(k_x\) and \(t_j^0\)), and then simplify
Eq. (5.16).

Now to facilitate our simplification of Eq. (5.16) in the form
of Eq. (5.14) we write \(1_{U_N} \hat{F}\) as

\[
1_{U_N} \hat{F} = 1 + (\hat{F} - 1) + (1_{U_N} - 1) \hat{F},
\]

(5.20)
where \((F - 1)\) and \(\hat{U}_N - 1\) \(\hat{F}\) contain no terms of unity. Corresponding to Eq. (5.20), Eq. (5.16) can be written as

\[
\sin(\omega T) = \text{SNF}(\hat{F} - 1) + \text{SNF}[\hat{U}_N - 1], \tag{5.21}
\]

with

\[
\begin{align*}
\text{SNF}(\hat{F} - 1) &= 2\cos(\omega T/2) \text{SN}(0)_{C_N} - \cos(\omega T/2) \text{SN}(1)_{C_N} - \sin(\omega T/2) \text{CN}(1)_{C_N}, \tag{5.22} \\
\text{SNF}[\hat{U}_N - 1] &= \cos(\omega T/2) \text{SN}(1)_{C_N} - \sin(\omega T/2) \text{CN}(1)_{C_N} + 2\text{CN}(1)_{C_N} \text{SN}(0)_{C_N}. \tag{5.23}
\end{align*}
\]

(See APPENDIX F for proof of Eqs. (5.22) and (5.23)) where \(\text{SNF}(\hat{F} - 1)\) includes all the terms (on the right-hand side of Eq. (5.16)) corresponding to the terms in \((\hat{F} - 1)\), \(\text{SNF}[\hat{U}_N - 1]\) includes those corresponding to the terms in \((\hat{U}_N - 1)\ \hat{F}\), and

\[
\text{SN}(i)_{C_N} (i = 0, 1) = \sum_{Z' \in P(i)_{C_N}} \left\{ \prod_{(r,s) \in Z} k_r k_s \sin(\omega \sum_{(r,s) \in Z} t_r - t_s/2) \right\} \tag{5.24}
\]

\[
\text{CN}(i)_{C_N} = \sum_{Z' \in P(i)_{C_N}} \left\{ \prod_{(r,s) \in Z} k_r k_s \cos(\omega \sum_{(r,s) \in Z} t_r - t_s/2) \right\} \tag{5.24a}
\]

(see §4 for definitions of \(Z, Z', \) and \(P(i)_{C_N}\)). Substituting Eqs. (5.22) and (5.23) into Eq. (5.21), and using the relation \(\sin(\omega T) = 2\sin(\omega T/2)\cos(\omega T/2)\), we get after rearrangement of terms

\[
2\sin(\omega T/2) \left[ \cos(\omega T/2) + \text{CN}(1)_{C_N} \right] = 2\left[ \cos(\omega T/2) + \text{CN}(1)_{C_N} \right] \text{SN}(0)_{C_N} \tag{5.25}
\]

which is simply

\[
\sin(\omega T/2) = \text{SN}(0)_{C_N} \tag{5.26}
\]

if \(\cos(\omega T/2) + \text{CN}(1)_{C_N} \neq 0\). The right-hand side of Eq. (5.26) has \(2^N - 1\) terms since \(P(0)_{C_N}\) includes \(2^N - 1\) non-empty elements. This completes our derivation of Eq. (1.2) from the summation of multiple ScsS pulses and
associated multiple reflections from discontinuities, since Eq. (5.26) is the same as Eq. (1.2) if we let \( \frac{T}{2} = a \), and the constants defining the terms of \( SN(\theta_C) \) by \( B_j \) and \( b_j \). See APPENDIX G for demonstration of the procedure stated in this paragraph for Earth models with two discontinuities between the surface and the core-mantle boundary.

Using the definition of Eq. (5.24), we can write Eq. (5.26) (or Eq. (1.2)) in simple algebraic form for the cases of Earth models with two and three internal discontinuities \( t_{ij} \) are two-way shear wave radial travel times between interfaces \( S_i \) and \( S_j \) (Fig. 4.1), and \( k_i \) are defined in Eq. (5.1)).

(i) When \( N = 2 \), \( ^0C_2 = \{0,1,2\} \) and \( ^0P_2 = \{z'\} = \{(0,1),(0,2),(1,2)\} \).

Thus
\[
\sin(\omega T_2^0) = k_{01}k_1\sin[\omega(t_{1-2}^0 T_1)] + k_{02}k_2\sin[\omega(t_{2-2}^0 T_2)] + k_{12}k_1k_2\sin[\omega(t_{2-2}^0 T_2)],
\]
with \( T = t_3^0 \) (\( S_3 \) is the core-mantle boundary).

(ii) When \( N = 3 \), \( ^0C_3 = \{0,1,2,3\} \) and \( ^0P_3 = \{z'\} = \{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3),(0,1,2,3)\} \). Thus,
\[
\sin(\omega T_3^0) = k_{01}k_1\sin[\omega(t_{1-2}^0 T_1)] + k_{02}k_2\sin[\omega(t_{2-2}^0 T_2)] + k_{03}k_3\sin[\omega(t_{3-2}^0 T_3)]
\]
\[
+ k_{12}k_1k_2\sin[\omega(t_{1-2}^0 T_1)] + k_{13}k_1k_3\sin[\omega(t_{1-3}^0 T_3)] + k_{23}k_2k_3\sin[\omega(t_{2-3}^0 T_3)]
\]
\[
+ k_{01}k_1k_2k_3\sin[\omega(t_{0}^0 T_1 + t_{2-3}^0 T_3)],
\]
with \( T = t_4^0 \) (\( S_4 \) is the core-mantle boundary).

5.6 DISCUSSION

For the case of SH waves, in Chapter 2 it was shown by a model study that the solotone effect arises from the multiple reflections of waves from the Earth’s surface, the core-mantle boundary, and the discontinuities between them. Because the solotone effect is the
immediate result of Eq. (5.26) or Eq. (1.2), the present study has shown theoretically that the above-mentioned multiple reflections are responsible for the solotone effect. Since there is no P-SV conversion at discontinuities for vertically incident waves, the derived formula (i.e., Eq. (5.26)) for the asymptotic behaviour of torsional overtone eigenfrequencies for discontinuous Earth models is also valid for the case of SV waves, and the technique used in this paper could be extended to study the solotone effect for PKIKP waves.

Although in this chapter we did not explicitly restrict attention to high frequency waves, Eq. (5.26) is a formula for high eigenfrequencies. There are two reasons for this: (i) The effect of transition zones was neglected: for waves with very short wavelengths (compared with the thickness of the transition zone) this effect is negligible; when the wavelength increases, the reflections and phase shifts of waves at transition zones should be taken into account (e.g., Wolf, 1937 and Richards, 1972). (ii) Earth models with plane boundaries and interfaces were used: for waves at very high frequencies, the coefficient matrices (Eqs. (15) and (16) of Gilbert, 1975) for spherically stratified and plane stratified media are the same in the case of vertically incident rays, because in this case both \( p \) (the ray parameter for the former medium) and \( q \) (the horizontal wave slowness or the reciprocal of phase velocity for the latter medium) are equal to zero; when the frequency of waves decreases, the effect of sphericity should be taken into account. Since the observation of eigenfrequencies is easier when the overtone numbers of the eigenfrequencies are smaller, the study of eigenfrequency behaviour in which the above-mentioned effects of transition zones and sphericity of Earth models are involved is desirable.
APPENDIX A SIMPLIFICATION OF EQ. (5.5)

Before we attempt to simplify Eq. (5.5), we examine the property of the following mathematical expression \( \hat{E} \):

\[
\hat{E} = \left( \sum_{i_1} \sum_{i_2} \ldots \sum_{i_x} \right) \left( \sum_{j_1} \sum_{j_2} \ldots \sum_{j_y} \right) e_1(i_1, i_2, \ldots, i_x; j_1, j_2, \ldots, j_y) e_2(i_1, i_2, \ldots, i_x), \tag{A1}
\]

where \( e_1 \) is a function of two sequences of integral variables while \( e_2 \) is a function of a single sequence, and the first two brackets group the summation symbols \( \sum \) over integral variables of different sequences into two sets. Since \( e_2 \) is not a function of the sequence \( [j_1, j_2, \ldots, j_y] \), Eq. (A1) can be written as

\[
\hat{E} = \left( \sum_{i_1} \sum_{i_2} \ldots \sum_{i_x} \right) \lambda(i_1, i_2, \ldots, i_x) e_2(i_1, i_2, \ldots, i_x), \tag{A2}
\]

where

\[
\lambda(i_1, i_2, \ldots, i_x) = \sum_{j_1} \sum_{j_2} \ldots \sum_{j_y} e_1(i_1, i_2, \ldots, i_x; j_1, j_2, \ldots, j_y). \tag{A3}
\]

Since \( \kappa_{i_{j+1}}^i \) in Eq. (5.5) are not functions of \( m_{j+1}^X (j' < j) \), to facilitate our simplification of Eq. (5.5), by repeatedly using the property of \( \hat{E} \) stated above we introduce a sequence \( Y_1, Y_2, \ldots, Y_N \) with each member \( Y_J (J = 1, 2, \ldots, N) \) of the sequence being defined as follows:

\[
Y_1 = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots \sum_{i_N=0}^{\infty} \left\{ \prod_{m_{1+i}^{i+1}} \left[ \kappa_{i_{j+1}}^i \right] \right\}, \tag{A4}
\]
\[ Y_J = \sum_{m_J=0}^{\infty} \sum_{m_{J+1}=0}^{N-J+1} \sum_{m_{N+1}=0}^{N+1} \{Y_{J-1}^{N-J+1} \prod_{i=0}^{\infty} \left[ K_{J+i}^{i} \left( J_{J+i}^{i} \right)^{m_{J+i}} \right] \} \]

(A4a)

when \( J \geq 2 \).

It is clear that Eq. (5.5) can be simply written as

\[ G(m_{N+1}) = Y_N \]  

(A5)

Thus, if we can simplify \( Y_1 \) first, and then \( Y_2, Y_3, \ldots \), the simplified form of \( G(m_{N+1}) \), as a member \( Y_N \) of the sequence, can be obtained. To do so we need only to prove that Eqs. (A4) and (A4a) can be simplified to the following relations:

\[ Y_1 = \prod_{i=0}^{N} \left( \frac{1}{D_{J+i}} \right)^{i}^{1+i} \]  

(A6)

\[ Y_J = \prod_{i=0}^{N-J+1} \left( \frac{1}{E_{J+i}} \right)^{i}^{J+i} \prod_{i=0}^{N-J} \left( \frac{1}{R_{J+i}} \right)^{i}^{J+i+1} \]  

(A6a)

(when \( J = 2, 3, \ldots N \)) under the conditions of (5.9).

Since each \( K_{J+i}^{i} \) in Eqs. (A4) and (A5) is independent of all the \( m_{J+x}^{x} \), for \( x \neq i \) (Eqs. (4.5) through (4.9)), the following formula is used in this proof:

\[ \prod_{n_1}^{\infty} \prod_{n_2}^{\infty} \prod_{n_k}^{\infty} B(n_1)B(n_2)\ldots B(n_k) = \prod_{n_1}^{\infty} \left[ \sum_{n_1}^{\infty} B(n_1) \right] \prod_{n_2}^{\infty} \left[ \sum_{n_2}^{\infty} B(n_2) \right] \ldots \prod_{n_k}^{\infty} \left[ \sum_{n_k}^{\infty} B(n_k) \right] \]  

(A7)

After applying Eq. (A7) to Eqs. (A4) and (A4a), Eqs. (A4) and (A4a) can be simplified by the use of the following formula (e.g., Knopp, 1956, P.151)

\[ \sum_{k=0}^{\infty} q^{k-1} \frac{k!}{k^{k}} = 1/(1-z)^{q} \]  

(A8)
where \( z \) is a complex number with modulus less than 1 (\(|z|<1\)) and \( q \) is a positive integer. Adopting our notations defined so far, we can rewrite Eq. (A8) as

\[
\sum_{i=mj+i=0}^{\infty} [\kappa_{J+i}(z)]^{i} = 1/(1-z)^{J+i}.
\] (A8a)

By the following induction, we prove now that Eqs. (A4) and (A4a) can be simplified to Eqs. (A6) and (A6a).

(i) When \( J=1 \), applying Formulas (A7) and (A8a) to Eq. (A4) we get Eq. (A6).

(ii) When \( J=2 \), from Eqs. (A4a) and (A6) we have

\[
\sum_{i=mj+i=0}^{\infty} \frac{\kappa_{2+i}(\frac{\pi}{2} \sqrt{1+2+i})}{i^{m_2+2+i}}
\] (A9)

By APPENDIX B, Eq. (A9) can be reduced to

\[
Y_2 = \left[ \prod_{i=0}^{N-1} \left( \frac{1}{D_{1+i}^{1+2+i}} \right) \right] \left[ \prod_{i=0}^{N-2} \left( \frac{1+1+i}{D_{2+i}^{1+2+i}} \right) \right].
\] (A10)

Applying Formulas (A7) and (A8a) to Eq. (A10) we get, under the conditions that \(|i^{\pi_2+i}/(1+1+iD_{1+i}^{1+2+i})|<1,

\[
Y_2 = \left[ \prod_{i=0}^{N-1} \left( \frac{1}{D_{2+i}^{1+2+i}} \right) \right] \left[ \prod_{i=0}^{N-2} \left( \frac{1+1+i}{D_{2+i}^{1+2+i}} \right) \right].
\] (A11)
Eq. (A11) is the same as Eq. (A6a) for the case of \( J = 2 \) since 
\[ iE_{1+i} = 1 \text{ and } iE_{2+i} = 1 \text{ (Eq. (5.7b))}. \]

(iii) Assuming Eq. (A6a) is true for \( J = j \) \((2 \leq j \leq N-1)\), we want to prove that Eq. (A6a) is also true for \( J = j+1 \). For \( J = j \),

Eq. (A6a) can be written as

\[
Y_j = \left[ \prod_{i=0}^{N-j} \frac{E_{j+i}^{1+i}}{D_{j+i}^{1+i}} \right] \left[ \prod_{i=0}^{N-j} \frac{D_{j+i}^{1+i}}{E_{j+i}^{1+i}} \right]. \tag{A12}
\]

Using the relations among \( I_{j+1}^{1+i} \) (APPENDIX B), we can rewrite Eq. (A12) as

\[
Y_j = \left[ \prod_{i=0}^{N-j-1} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right] \left[ \prod_{i=0}^{N-j-1} \frac{D_{j+1+i}^{1+i} E_{j+i}^{1+i}}{D_{j+i}^{1+i} E_{j+i}^{1+i}} \right]. \tag{A13}
\]

From Eq. (A4a) we have

\[
Y_{j+1} = \sum_{k=0}^{N-j} \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

\[
= \left\{ Y_j \prod_{i=0}^{N-j} \frac{E_{j+1+i}^{1+i} D_{j+1+i}^{1+i} m_{j+1+i}^{i}}{E_{j+i}^{1+i} D_{j+i}^{1+i}} \right\} \]

Substituting Eq. (A13) into Eq. (A14) and then summing Eq. (A14) over \( m_{j+1+i}^{i} \), we get, under the conditions of (5.9),
\[ V_{j+1} = \prod_{i=0}^{N-j} \left( \frac{1}{iD_{j+1+i}} \right)^i E_{j+i} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \]

\[ \prod_{i=0}^{N-j-1} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \] \hspace{1cm} (A15)

Since (see end of this appendix for proof)

\[ \frac{1}{iD_{j+1+i}} \left( \frac{1+i}{i+1} \right) E_{j+i} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \left( \frac{1+i}{i+1} \right) E_{j+1+i} \]

for \(2 \leq j \leq N-1\), Eq. (A15) is the same as Eq. (A6a) for \(j=j+1\).

To prove Eq. (A16), we need only to prove it for the case of \(i=0\), since for any value of \(i\) (\(0 \leq i \leq N-j-1\)), the procedure of proof is the same.

If we deploy \(E_j\), \(E_{j+1}\) and \(E_j\) (defined in Eq. (5.7b)) in the following form, then the proof of Eq. (A16) becomes a straightforward matter.

\[ O_{E_j} = \left( D_2 \right)^{D_3} \ldots D_{j-1} \ldots \left( D_{j-2} \right)^{D_{j-1}} \ldots \left( D_{j-1} \right)^1 D_j \]

\[ 1_{E_{j+1}} = \left( D_3 \right)^{D_4} \ldots D_{j-1} \ldots \left( D_{j} \right)^1 D_j \]

\[ 1_{E_j} = \left( D_3 \right)^{D_4} \ldots D_{j-1} \ldots \left( D_{j-1} \right)^1 D_j \]

It is clear that the left-hand side of Eq. (A16) (for the case of \(i=0\)) can be reduced to

\[ \left( D_2 \right)^{D_3} \ldots D_{j-1} \ldots \left( D_{j-2} \right)^{D_{j-1}} \left( D_{j-1} \right)^1 D_j \]

which is \(O_{E_{j+1}}\) (defined in Eq. (5.7b)).
APPENDIX B PROPERTIES OF $I_{J+i}^i$

Here we list and prove the relations among $I_{J+i}^i$.

For $N \geq 2$ and $1 \leq J < N$, we have the following relations among $I_{J+i}^i$:

(i) $I_{J+1}^0 = I_{J+1}^0 + m_{J+1}^0$ \hspace{1cm} (B1)

(ii) $I_{N+1}^{N-J+1} = I_{N+1}^{N-J} + m_{N+1}^{N-J}$ \hspace{1cm} (B1a)

(iii) For $1 \leq i \leq N-J$

\[ I_{J+i}^i = (I_{J+i+1}^{i-1} + m_{J+i+1}^{i-1}) + (I_{J+i+1}^{i+1} + m_{J+i+1}^{i+1}) - I_{J+i+1}^{i-1} \] \hspace{1cm} (B1b)

[proof]

(i) $I_{J+1}^0 = (\sum_{y=J+1}^{N+1} m_y^0) + 1$ \hspace{1cm} (by Eq. (4.7))

\[ = [(\sum_{y=J+2}^{N+1} m_y^0) + 1] + m_{J+1}^0 \]

\[ = I_{J+1}^0 + m_{J+1}^0. \]

(ii) $I_{N+1}^{N-J+1} = I_{N+1}^{N-J} + m_{N+1}^{N-J}$ \hspace{1cm} (by Eq. (4.6))

\[ = (\sum_{x=0}^{N-J-1} m_x^{N+1}) + m_{N+1}^{N-J} \]

\[ = I_{N+1}^{N-J+1} + m_{N+1}^{N-J}. \]

(iii) 1. When $i=1$

\[ I_{J+1}^1 = 2(\sum_{y=J+2}^{N+1} m_y^0) + m_{J+1}^0 + \sum_{y=J+2}^{N+1} m_y^1 \] \hspace{1cm} (by Eq. (4.5))

\[ = [(\sum_{y=J+2}^{N+1} m_y^0) + 1] + m_{J+1}^0 + [\sum_{y=J+2}^{N+1} (m_y^0 + m_y^1)] + 1 \]

\[ = [I_{J+1}^0 + m_{J+1}^0] + [\sum_{y=J+2}^{N+1} (m_y^0 + m_y^1)] - 1 \] \hspace{1cm} (B2)

Similarly,

\[ I_{J+2}^1 = [I_{J+2}^0 + m_{J+2}^0] + [\sum_{y=J+3}^{N+1} (m_y^0 + m_y^1)] - 1 \] \hspace{1cm} (B3)

Subtracting Eq. (B3) from Eq. (B2) we have, after rearrangement,

\[ I_{J+1}^1 = [I_{J+1}^0 + m_{J+1}^0] + [I_{J+2}^1 + m_{J+2}^1] - I_{J+2}^0. \]
2. When \( i = N - J \)

\[
\begin{align*}
\sum_{N-J}^{N-J-1} x^N_N &= 2 \left( \sum_{x=0}^{N-J-2} x = 0 \right) + \left( \sum_{x=0}^{N-J-2} x = 0 \right) + \left( \sum_{x=0}^{N-J-2} x = 0 \right) \quad \text{(by Eq. (4.5))}
\end{align*}
\]

\[
\sum_{N-J}^{N-J-1} x^{N+1}_N + \sum_{N-J}^{N-J-1} x^{N+1}_N + \sum_{N-J}^{N-J-1} x^{N+1}_N + \sum_{N-J}^{N-J-1} x^{N+1}_N
\]

From (ii) we have

\[
\sum_{N-J}^{N-J-1, N-J-1} x = 0 \quad \text{(B5)}
\]

Subtracting Eq. (B5) from Eq. (B4) we get, after rearrangement,

\[
\sum_{N}^{N-J} x = 0 + \sum_{N}^{N-J} x = 0 \quad \text{(B5)}
\]

Subtracting Eq. (B5) from Eq. (B4) we get, after rearrangement,

\[
\sum_{N}^{N-J} x = 0 + \sum_{N}^{N-J} x = 0 \quad \text{(B5)}
\]

Similarly,

\[
\sum_{y=J+i+1}^{y=J+i+1} y = 0 + \sum_{y=J+i+1}^{y=J+i+1} y = 0
\]

Subtracting Eq. (B7) from Eq. (B6), we get after rearrangement

\[
\sum_{y=J+i+1}^{y=J+i+1} y = 0 + \sum_{y=J+i+1}^{y=J+i+1} y = 0
\]

Subtracting Eq. (B7) from Eq. (B6), we get after rearrangement

\[
\sum_{y=J+i+1}^{y=J+i+1} y = 0 + \sum_{y=J+i+1}^{y=J+i+1} y = 0
\]

Subtracting Eq. (B7) from Eq. (B6), we get after rearrangement

\[
\sum_{y=J+i+1}^{y=J+i+1} y = 0 + \sum_{y=J+i+1}^{y=J+i+1} y = 0
\]

Subtracting Eq. (B7) from Eq. (B6), we get after rearrangement

\[
\sum_{y=J+i+1}^{y=J+i+1} y = 0 + \sum_{y=J+i+1}^{y=J+i+1} y = 0
\]
APPENDIX C  VERIFICATION OF EQ. (5.12)

We prove here that in terms of Eq. (5.11), Eq. (5.10) can be express as Eq. (5.12).

To complete this proof, we need only to prove

$$\frac{j_{D_{j+1}}}{j_{E_{j+1}}} = j_{U_{j+1}}'$$  \hspace{1cm} (C1)

since \(0_{D_N}/0_{E_N}\) and \(1_{D_{N+1}}/1_{E_{N+1}}\) are special cases when \(J=N\), and \(j=0\) and \(j=1\) respectively. Because the procedures of proving Eq. (C1) for different values of \(j\) and a given value of \(J\) are the same, we need only to prove Eq. (C1) for a given value of \(j\).

(I) When \(J=1\)

$$\frac{j_{D_{1+j}}}{j_{E_{1+j}}} = 1 - j_{W_{1+j}}$$

$$ = 1 + k_j k_{1+j} \exp(-i\omega t_{1+j})$$ \hspace{1cm} (C2)

(\(It\ should\ be\ recalled\ that\ X_{W} = X_{A_y} \exp(-i\omega t)\) and \(X_{A_y}\) is defined in Eq. (4.10), which is a special case of Eq. (C1) for \(J=1\).

(II) Assuming Eq. (C1) is true for \(J=L\) \((1 \leq N - 1\)), i.e.,

$$\frac{j_{D_{L+j}}}{j_{E_{L+j}}} = j_{U_{L+j}}'$$ \hspace{1cm} (C3)

we have to prove that

$$\frac{j_{D_{L+1+j}}}{j_{E_{L+1+j}}} = j_{U_{L+1+j}}'$$ \hspace{1cm} (C4)

From Eqs. (5.7b) and Eqs. (A16) and (C3), we have

$$\frac{j_{D_{L+1+j}}}{j_{E_{L+1+j}}} = \frac{j_{D_{L+j}}}{j_{E_{L+j}}} \frac{1+j_{D_{L+1+j}}}{1+j_{E_{L+1+j}}} - \frac{j_{W_{L+1+j}}}{j_{E_{L+1+j}}} \frac{1+j_{E_{L+1+j}}}{1+j_{D_{L+1+j}}}$$

$$ = \{j_{U_{L+j}} + 1+j_{U_{L+1+j}} - j_{W_{L+1+j}}/1+j_{U_{L+j}}\}$$ \hspace{1cm} (C5)

Substituting \(j_{U_{L+j}} = 1+j_{U_{L+j}} + (j_{U_{L+j}} - 1+j_{U_{L+j}})\)

and \(1+j_{U_{L+1+j}} = 1+j_{U_{L+1+j}} + (1+j_{U_{L+1+j}} - 1+j_{U_{L+1+j}})\) \hspace{1cm} (C6a)

into Eq. (C5), we get

$$\frac{j_{D_{L+1+j}}}{j_{E_{L+1+j}}} = \{j_{U_{L+j}} + 1+j_{U_{L+1+j}} - 1+j_{U_{L+1+j}}\} + \{(j_{U_{L+j}} - 1+j_{U_{L+j}})$$
\[ (1^+jU_{L+1}^+ - 1^+jU_{L+j}^+) - jW_{L+1^+ j}^{1^+jU_{L+j}}. \] (C7)

Since (APPENDIX D)

\[ \left\{ (1^+jU_{L+j} - 1^+jU_{L+j}) \right\} \left\{ 1^+jU_{L+1^+ j} - 1^+jU_{L+j} \right\} - jW_{L+1^+ j}^{1^+jU_{L+j}} \]

\[ = k_jk_{L+1^+ j}^\text{exp}(-i\omega L^+_{L+1^+ j}) 1^+jU_{L+j}^+, \] (C8)

(Where 1^+jU_{L+j}^+ is a complex conjugate of 1^+jU_{L+j}), Eq. (C7) becomes

\[ \frac{j^D_{L+1}+j}{j^E_{L+1^+ j}} = 1^+jU_{L+j} + 1^+jU_{L+1^+ j} - 1^+jU_{L+j} \]

\[ + k_jk_{L+1^+ j}^\text{exp}(-i\omega L^+_{L+1^+ j}) 1^+jU_{L+j}^+. \] (C9)

which is the same as Eq. (C4) in the light of the following explanation.

Each term contained in the last expression in Eq. (C9) is of the form

\[ k_jk_{L+1^+ j}^\text{exp}(-i\omega L^+_{L+1^+ j}) \left[ k_{i_1}k_{i_2}^\text{exp}(i\omega t_{i_2}^+) \right] \left[ k_{i_3}k_{i_4}^\text{exp}(i\omega t_{i_4}^+) \right] \]

\[ \ldots [k_{2x-1}k_{2x}^\text{exp}(i\omega t_{2x-1}^+)], \] (C10)

where \( j \leq i_1 < i_2 < \ldots < i_{2x} < L^+L+j \) and \( x = 0, 1, 2, \ldots \). Since

\[ t_{L^+L+j}^j - [t_{i_1}^j - t_{i_2}^j + \ldots - t_{i_{2x}}^j] = t_{i_1}^j + t_{i_2}^j + \ldots + t_{L+j}^j. \] (C11)

(when \( x = 0 \), the only term left on the right-hand side is \( t_{L^+L+j}^j \)),

Expression (C10) can be rewritten as

\[ \left[ k_jk_{i_1}^j \text{exp}(-i\omega t_{i_1}^+ \right] \left[ k_{i_2}k_{i_3}^j \text{exp}(-i\omega t_{i_3}^+ \right] \left[ k_{i_4}k_{i_5}^j \text{exp}(-i\omega t_{i_5}^+ \right] \ldots \left[ k_{2x}k_{2x+1}^j \text{exp}(-i\omega t_{2x+1}^+ \right]. \] (C12)

It is clear now that all the terms contained in the last expression in Eq. (C9) correspond to the set of all the sets \( \{(j,i_1), (i_2,i_3), \ldots (i_{2x},L^+L+j)\} \) which are derived from the subsets \( \{j, i_1, i_2, \ldots \} \) of \( \{j, j+1, j+2, \ldots, J+j, J^+L+j\} \) with \( j \leq i_1 < i_2 < \ldots < i_{2x} < J+j \) (when \( x = 0 \), \( \{(j,i_1), (i_2, i_3), \ldots (i_{2x}, L+j)\} \)).

From the definition of \( XU_y \) (Eq. (5.11)), it is clear that, except for those discussed in the last paragraph, all the terms of \( j^D_{L+1^+ j} \) form the same set of terms in \( j^D_{L+j} + 1^+jU_{L+1^+ j} \)

\[ - 1^+jU_{L+j}. \]
We prove here the relation of Eq. (C8) Multiplying both sides of Eq. (C8) by $1 + j w_{L+1+j}$ and expressing $j w_{L+1+j}$ as

$-k_j (1-k_{j+1}^2) (1-k_{j+2}^2) \ldots (1-k_{L+1}^2) k_{L+1+j} \exp(-i \omega t_{L+1+j})$, we can write Eq. (C8) as

$$
\begin{align*}
& (j u_{L+j} - 1+j u_{L+1+j}) (1+j u_{L+1+j} - 1+j u_{L+j}) + k_j (1-k_{L+1+j}^2) (1-k_{L+2+j}^2) \\
& \quad \ldots (1-k_{L+j}^2) k_{L+1+j} \exp(-i \omega t_{L+1+j}) \\
= & (1+j u_{L+j} - 1+j u_{L+1+j}) k_j k_{L+1+j} \exp(-i \omega t_{L+1+j}) ,
\end{align*}
$$

(D1)

where $j$ and $L$ are given integers. Since

$$
\begin{align*}
t^j u_1 + u_2 + \ldots + u_{2x-2} = t^j u_1 + (u_2 + u_4 + \ldots + u_{2x-1}) ,
\end{align*}
$$

(D2)

where $u_1$ and $x$ are integers and $1+j \leq u_1 < u_2 < \ldots < u_{2x-1} \leq L+j$.

Each term of $(j u_{L+j} - 1+j u_{L+1+j}) (1+j u_{L+1+j} - 1+j u_{L+j})$ can be written in the form

$$
\begin{align*}
& k_j k_{u_1} k_{u_2} \ldots k_{u_{2x-1}} \exp(-i \omega t_{L+1+j}) \exp[i\omega (t_{u_2} + u_3 + \ldots + u_{2x-1})] \\
& \quad \ldots (t_{v_{2y-2}+1} + t_{v_{2y-1}+1}) ,
\end{align*}
$$

(D3)

where $v_1$ and $y$ are integers and $1+j \leq v_1 < v_2 < \ldots < v_{2y-1} \leq L+j$.

Thus, in order to express Eq. (D1) in terms of $k_i$ and $t_b^a$, we define the following notations:

Let $B = \{1+j, 2+j, \ldots, L+j\}$, $(x, s)$ be an element of

$X = \{(i_1, i_2), (i_3, i_4), \ldots, (i_{2x-1}, L+1+j)\}$ which is derived from a subset of $X' = \{i_1, i_2, i_3, i_4, \ldots, i_{2x-1}\}$ of $B$ with $1+j \leq i_1 < \ldots < i_{2x-1}$.
\[ i_2 < \ldots < i_{2x-1} \leq L+j; \text{ and } Q(B) \text{ be the set of all the odd-number-elemented subsets of } B. \]

If \( U \) and \( V \) are sets in the form of \( X \), and \( U', V' \) sets in the form of \( X' \), then, dividing Eq. (D1) by \( k_j k_{L+1+j} \exp(-i\omega_j L+1+j) \) and using the notations defined above, we can write Eq. (D1) as

\[ \begin{align*}
\text{(i)} & \quad \sum [\prod k_{r,s} \exp(i\omega_r L+1+j)] \\
\text{(ii)} & \quad \sum [\prod k_{r,s} \exp(-i\omega_r L+1+j)] \\
\text{(iii)} & \quad + (l-k_1^2)(l-k_2^2)\ldots(l-k_{L+j}^2) = 1+\sum_{L+j}^1 1+j_{L+j}. 
\end{align*} \]

To build up this proof, we need only to prove the relation of Eq. (D4). This relation can be proved by comparing the terms on both sides (all the terms are expressed in terms of \( k_i \) and \( t^a_i \), and each term in the product of Expressions (i) and (ii) corresponds to a pair of sets \( U \) and \( V \), where \( U \) produces a term in Expression (i) and \( V \) produces a term in Expression (ii) of Eq. (D4)). Before the comparison of terms, to make our discussion clear we represent graphically the terms in Expressions (i) and (ii) on the left-hand side and in \( 1+\sum_{L+j}^1 1+j_{L+j} \) on the right-hand side of Eq. (D4).

In Expression (i), the term corresponding to the set \( U=\{(u_1, u_2), \ldots, (u_{2x-1,L+1+j})\} \) can be written in the form

\[ \begin{align*}
(k_{u_1} k_{u_2} \ldots k_{u_{2x-1}}) \exp[i\omega(t_{u_2}^2+t_{u_4}^2+\ldots+t_{L+1+j}^2)] .
\end{align*} \]

As shown in Fig.5.1A-(a), we represent this expression by a series of disconnected links (except for the last link, each link including a thick solid line, a hurdle and two dots), with each link denoting a pair of values in \( U \). The thick solid line \( u_a^u u_b^u \) and the accompanying hurdles represent \( \exp(i\omega u_a^u u_b^u) \), and the dot
Figure 5.1 Representations of terms in Expression (i) (Figure A-(a)), Expression (ii) (Figure A-(b)), $1 + j_L^U L^+ j$ (Figure B-(a)) and $1 + j_L^U L^+ j$ (Figure B-(b)) of Equation (D4), and representations of the product (Figure A-(c)) of terms represented by Figures A-(a) and A-(b) (which produces a term of the product of Expressions (i) and (ii)) and the product (Figure B-(c)) of terms represented by Figures B-(a) and B-(b) (which produces a term on the right-hand side of Equation (D4)). See the text for detailed description.
at \( u_i \) represents \( k_{u_i} \). Notice that the last link must finish at point \((L+1+j)\), and that between points \( l+j \) and \( L+j \) (end-points included) there are an odd number of dots (dots must be situated at points representing integers).

In Expression (ii), the term corresponding to the set \( V=\{(v_1,v_2), (v_3,v_4), \ldots (v_{2y-1},L+1+j)\} \) can be written in the form

\[
(k_{v_1}k_{v_2}\ldots k_{v_{2y-1}})\exp\left(-i\omega(t_{v_1}^2 + t_{v_2}^3 + \ldots + t_{L+1+j}^{2y-1})\right),
\]

and represented by a series of reversed links shown in Fig. S1A-(b), with the dots at \( v_i \) representing \( k_{v_i} \) and the thick solid line \( v_a^\alpha v_b^\beta \) and the accompanying reversed hurdle representing \( \exp(-i\omega t_{v_i}^\beta) \). The product, shown in Fig. S1A-(c), of the two terms represented by Figs. S1A-(a) and S1A-(b) can be written as \( k_p \exp(i\omega t_p) \), \( k_p \) being the product of all the coefficients of Expressions (D5) and (D6) and

\[
t_p = (t_{u_1} + t_{u_3} + \ldots + t_{L+1+j}) - (t_{v_1} + t_{v_3} + \ldots + t_{L+1+j}).
\]

Notice that the corresponding parts of solid lines in Fig. S1A-(a) and Fig. S1A-(b) cancel each other. We situate the dots from Fig. S1A-(a) above the axis line and those from Fig. S1A-(b) below the axis line, though they all represents \( k_1 \).

Similarly, in \( 1^+j \) the term corresponding to the set \( \{(a_1,a_2), (a_3,a_4), \ldots (a_{2r-1},a_{2r})\} \) (with \( l+j \leq a_1 < a_2 < \ldots \leq a_{2r} \leq L+j \)) is represented by a series of links shown in Fig. S1B-(a), while in \( 1^+j L+1 \) the term corresponding to \( \{(b_1,b_2), (b_3,b_4), \ldots (b_{2s-1},b_{2s})\} \) (with \( l+j \leq b_1 < b_2 < \ldots \leq b_{2s} \leq L+j \)) is represented by a series of reversed links shown
in Fig. 5.1B-(b). Unlike Figs. 5.1A-(a) and 5.1A-(b), either Fig. 5.1B-(a) or Fig. 5.1B-(b) is of even-number of dots between 1+j and L+j (end-points included), and the last link can finish at any point of integer depending on the set of pairs of integers we choose. The product of the two terms represented by Figs. 5.1B-(a) and 5.1B-(b) is shown in Fig. 5.1B-(c), where a dot above the axis line represents a $k_i$ from Fig. 5.1B-(a), and a dot below the axis line a $k_i$ from Fig. 5.1B-(b).

We now compare the terms on both sides of Eq. (D4):

(A) Both sides contain a term of unity.

(B) Except for unity, on the right-hand side, all the terms containing no exponential part (i.e. consisting of product of $k_i$ only) are of this form $k_{c_1}^2 k_{c_2}^2 \ldots k_{c_{2z-1}}^2 k_{c_{2z}}^2$ (product of an even number of squares of $k_i$). The terms of this kind are produced by the products of two terms, one in $1+j_{L+j}$ and the other in $1+j_{L+j}$, corresponding to the same set of pairs of values. No term containing only product of an odd number of squares of $k_i$ is permitted to exist, since each term in $1+j_{L+j}$ or $1+j_{L+j}$ consists of an even number of $k_i$. On the left-hand side, the only terms containing no exponential part are also of the form $k_{c_1}^2 k_{c_2}^2 \ldots k_{c_{2z-1}}^2 k_{c_{2z}}^2$. They come from Expression (iii) and are one-to-one correspondent to those terms mentioned above. The product of Expressions (i) and (ii) produces some terms which are products of an odd number of squares of $k_i$ and are cancelled by terms of the same form but of different sign from Expression (iii).

(C) For the terms other than those mentioned above, we define here a term-to-term correspondence between the right-hand side and the left-hand side of Eq. (D4). On the left-hand side, any term of this kind can be represented by an
ordered pair \([U, V] = \{(u_1, u_2), (u_3, u_4), \ldots (u_{2x-1}, L+1+j), \}
\{(v_1, v_2), (v_3, v_4), \ldots (v_{2y-1}, L+1+j)\}\) of different sets, with

the first set producing a term of Expression (i) and the

second set producing a term of Expression (ii), and

\[1+j \leq u_1 < u_2 < \ldots < u_{2x-1} \leq L+j; \quad 1+j \leq v_1 < v_2 < \ldots < v_{2y-1} \leq L+j.\]

On the right-hand side of Eq. (D4), any term of this kind

can be represented by an ordered pair \([(a_1, a_2), (a_3, a_4), \ldots (a_{2r-1}, a_{2r})] \}

\([(b_1, b_2), (b_3, b_4), \ldots (b_{2s-1}, b_{2s})]\) of different sets, with the first set producing a term of

\[1+j \leq u_1 < u_2 < \ldots < u_{2r} < L+j; \quad 1+j \leq b_1 < b_2 < \ldots < b_{2s} < L+j.\]

Therefore, if we define in the following steps a

one-to-one correspondence between the ordered pairs

producing the terms on the left-hand side of Eq. (D4)

and the ordered pairs producing the terms on the right-hand side, the term-to-term correspondence between

the left-hand side and the right-hand side of Eq. (D4) is clear.

(I) We represent \(U\) and \(V\) in the pair \([U, V]\) in the

form of Figs. 5.1A-(a) and 5.1A-(b); then:

(A) When \(u_{2x-1} < v_{2y-1}\):

(a) if \(u_{2x-1} < v_{2y-1}\), \([U, V]\) corresponds to

\[\{(u_1, u_2), (u_3, u_4), \ldots (u_{2x-3}, u_{2x-2}), (u_{2x-1}, v_{2y-1})\}\]

which produces a term in \(1+j \leq u_{L+j} \rightarrow 1+j \leq u_{L+j}\) (Fig. 5.2A-(a));

(b) if \(u_{2x-1} > v_{2y-1}\), \([U, V]\) corresponds to

\[\{(u_1, u_2), (u_3, u_4), \ldots (u_{2x-3}, u_{2x-2}), (v_1, v_2), (v_3, v_4), \ldots (v_{2y-3}, v_{2y-2})\}\]

When \(u_{2x-1} = v_{2y-1}\), we search from the right to the left
Figure 5.2 The correspondence of an ordered pair \([U, V]\) of different sets (upper pair of each figure) which produces a term on the left-hand side of Equation (D4) to an ordered pair (lower pair of each figure) which produces a term on the right-hand side. The upper pair is in the form of Figures 5-A-(a) and 5-A-(b), and the lower is in the form of Figures 5-B-(a) and 5-B-(b). The pairs enclosed by broken lines show how the upper pair is changed to become a lower pair. Figures A-(a), A-(b), B-(a), B-(b), C-(a) and C-(b) illustrate steps (I)-A-(a), A-(b), B-(a), B-(b), C-(a) and C-(b) for defining the one-to-one correspondence of the upper ordered pairs to the lower ones mentioned above.
along the axes of $u_1$ and $v_1$ (two axes together) to find the first (i.e., the largest) number $\tilde{x}$ at which only one dot appears (only in the axis of $u_1$ or only in the axis of $v_1$). Then

B-(a). If $\tilde{x} = u_{2w_1}$, i.e., it is the second element of a pair

\[(u_{2w_1-1}, u_{2w_1}) \] in $U$, $[U, V] = \{(u_1, u_2), (u_3, u_4), \ldots, (u_{2x-1}, L_{1+j})\}, \{(v_1, v_2), (v_3, v_4), \ldots, (v_{2y-1}, L_{1+j})\} \] (where $v_{2y}$ is the number (in $U$) which is smaller than and closest to $u_{2w_1}$) corresponds to $\{(u_1, u_2), (u_3, u_4), \ldots, (u_{2w_1-1}, u_{2w_1+1}), (u_{2w_2}, u_{2w_2+1})\}$ (Fig. 5.2B-(a));

(b). If $\tilde{x} = v_{2w_2}$, i.e., it is the second element of a pair

\[(v_{2w_2-1}, v_{2w_2}) \] in $V$, $[U, V] = \{(u_1, u_2), (u_3, u_4), \ldots, (u_{2x-1}, L_{1+j})\}, \{(v_1, v_2), (v_3, v_4), \ldots, (v_{2y-1}, L_{1+j})\} \] (where $u_{2w_2}$ is the number (in $U$) which is smaller than and closest to $v_{2w_2}$) corresponds to $\{(u_1, u_2), (u_3, u_4), \ldots, (u_{2w_2-1}, u_{2w_2+1}), (v_{2w_2}, u_{2w_2+1})\}$ (Fig. 5.2B-(b));

C-(a). If $\tilde{x} = u_{2w_3-1}$, i.e., it is the first element of a pair

\[(u_{2w_3-1}, u_{2w_3}) \] in $U$, $[U, V] = \{(u_1, u_2), (u_3, u_4), \ldots, (u_{2x-1}, L_{1+j})\}, \{(v_1, v_2), (v_3, v_4), \ldots, (v_{2y-1}, L_{1+j})\} \] (where $v_{2w_3-1}$ is the number (in $V$) which is smaller than and closest to $u_{2w_3-1}$) corresponds
to \{(u_1, u_2), (u_3, u_4), \ldots, (u_{2w_3-3}, u_{2w_3-2}), (u_{2w_3}, u_{2w_3+1})\},
\{(u_{2x-2}, u_{2x-1})\}, \{(v_1, v_2), (v_3, v_4), \ldots, (v_{2w_3-1}, u_{2w_3-1})\},
\{(v_{2w_3}, v_{2w_3+1})\}, \ldots, \{(v_{2y-2}, v_{2y-1})\}\} (Fig. 5.2C-(a));

(b). If \(x = v_{2w_4-1}\), i.e., it is the first element of a pair \((v_{2w_4-1}, v_{2w_4})\) in \(V\), \([U, V] = \{(u_1, u_2),
(u_3, u_4), \ldots, (u_{2w_4-1}, u_{2w_4}), \ldots, (u_{2x-1}, L+1+j)\},
\{(v_1, v_2), (v_3, v_4), \ldots, (v_{2w_4-1}, v_{2w_4}), \ldots, (v_{2y-1},
L+1+j)\}\} (where \(u_{2w_4-1}\) is the number (in \(U\) which is smaller than and closest to \(v_{2w_4-1}\)) corresponds to \[\{(u_1, u_2), (u_3, u_4), \ldots, (u_{2w_4-1}, v_{2w_4-1}), (u_{2w_4},
\ldots, (u_{2x-2}, u_{2x-1})\}, \{(v_1, v_2), (v_3, v_4), \ldots,
(v_{2w_4-3}, v_{2w_4-2}), (v_{2w_4}, v_{2w_4+1}), \ldots, (v_{2y-2}, v_{2y-1})\}\]
(Fig. 5.2C-(b)).

(II) The correspondence of the ordered pairs \[\{(a_1, a_2),
(a_3, a_4), \ldots, (a_{2T-1}, a_{2T})\}, \{(b_1, b_2), (b_3, b_4), \ldots, (b_{2S-1},
b_{2S})\}\] to the ordered pairs \([U, V]\) can be defined similarly.
APPENDIX A PROOF OF RELATION (5.9)

We prove here that under Conditions (5.13), (5.13a) and (5.13b) Relation (5.9) holds.

(A) when \( N=2 \):

we prove that

\[
|{\mathcal{W}_2}/({\mathcal{D}_1}{\mathcal{D}_2})| < 1 \text{ and } |{\mathcal{W}_3}/({\mathcal{D}_2}{\mathcal{D}_3})| < 1
\]

({\mathcal{E}_1} and {\mathcal{E}_2} are unity) under Condition (5.13), i.e., under the condition

\[
|k_1| + |k_2| + |k_1k_2| < 1. \tag{E1}
\]

Multiplying \( |{\mathcal{D}_1}{\mathcal{D}_2}| \) and using the definition of \( {\mathcal{D}_1+i} \) (Eq. (5.7)), the inequality \( |{\mathcal{W}_2}/({\mathcal{D}_1}{\mathcal{D}_2})| < 1 \) can be written as

\[
|{\mathcal{W}_2}| < |(1-\mathcal{W}_1)(1-\mathcal{W}_2)|. \tag{E2}
\]

(It should be recalled that \( x_{-y} = -k_y(1-k_y^2)(1-k_y^2) \ldots (1-k_y^2) k_y \exp(-i\omega_{-y}) \), where \( x<y \). Then the inequality

(E2) can be rewritten as

\[
|k_1(1-k_1^2)k_2\exp(-i\omega_2)| < |1+k_0k_1\exp(-i\omega_1) + k_1k_2\exp(-i\omega_2) + k_0k_1^2k_2\exp(-i\omega_2)|, \tag{E3}
\]

which is satisfied if (substituting \( k_0=1 \))

\[
|k_2| (1-k_1^2) < 1 - |k_1| - |k_1k_2| - k_1^2 |k_2|, \tag{E4}
\]

since the right-hand side of (E4) is smaller than or equal to the right-hand side of (E3), and \( (1-k_1^2)>0 \). After the
rearrangement of (E4), we get (El). This proves that
\[ |0w_2/(1D_1 D_2)| < 1 \text{ under Condition (El)}. \]

Similarly, we can prove that \[ |1w_3/(1D_2 D_3)| < 1 \] under Condition (El), since \( k_3 = -1 \) (S is the core-mantle boundary for the case of two discontinuities).

(B) when \( N>2 \):

We prove that (5.7) holds for \( J=3, 4, \ldots, N \) and \( i=0, 1, \ldots, N-J+1 \), under Conditions (5.13a) and (5.13b).

We prove that
\[ \left| \left(0E_{N-1}/0D_{N-1}\right) \left(1E_N/1D_N\right) \right| < 1 \] (E5)
under Condition (22a). Multiplying \[ \left| \left(0D_{N-1}/0F_{N-1}\right) \left(1D_N/1F_N\right) \right| \] and using Eq. (C1), we rewrite (E5) as
\[ \left| 0w_N \right| < \left| 0u_{N-1} \right| \] (E6)

Since \( 1 - \sum \bar{k}_1 \exp(-i\omega \bar{\tau}_1) \leq 1 + i \left| \sum \bar{k}_1 \exp(-i\omega \bar{\tau}_1) \right| \) (where \( \bar{k}_1 \) and \( \bar{\tau}_1 \) are constants of real numbers) and \( (0k_{N-1} 1k_N) \) is term-to-term correspondent to \( (0u_{N-1} 1u_N) \), each term of which is of the form \( \bar{k}_1 \exp(-i\omega \bar{\tau}_1) \), we have
\[ 1 - (0k_{N-1} 1k_N) \leq \left| 0u_{N-1} 1u_N \right| \] (E7)

From (E7) we know that inequality (E6) is satisfied if
\[ \left| 0w_N \right| < 1 - (0k_{N-1} 1k_N) \] (E8)

After rearrangement, (E8) can be rewritten as
Since (end of this appendix)

\[ i K_{J-1+i} \frac{1+i}{1+i} K_{J+1} + |W_{J+i}| = i K_{J+i} \frac{1+i}{1+i} K_{J-1+i}, \]  

(E10)

(E9) is Condition (5.13a). This proves that (E5) holds under Condition (5.13a).

Similarly, we can prove that \( \left( \frac{E_N - 1}{D_N} \right) \frac{2E_{N+1} - 1}{2D_{N+1}} \) \( W_{N+1} \) \( <1 \) under Condition (5.13b). Using the same procedure which we adopted to prove (E5), we know that (5.9) holds for \( J=3,4,\ldots N \) if

\[ i K_{J+i} \frac{1+i}{1+i} K_{J-1+i} < 2. \]  

(E11)

Since (see §5.4 for definition of \( ^i C_{J+i} \))

(a) when \( J+i \leq N \), \( ^i C_{J+i} \) is a subset of \( ^0 C_N \) and \( ^1 C_{J-1+i} \) is a subset of \( ^1 C_{N-1} \), therefore

\[ i K_{J+i} \frac{1+i}{1+i} K_{J-1+i} < 2 \]  

(from Condition (5.13a));

(b) when \( J+j = N+1 \), \( ^j C_{J+j} \) \( (j \geq 1) \) is a subset of \( ^1 C_{N+1} \)
and \( ^1 C_{J-1+j} \) is a subset of \( ^2 C_N \), therefore

\[ ^j K_{J+j} \frac{1+j}{1+j} K_{J-1+j} < 2 \]  

(from Condition (5.13b)),

(E11) always holds under Conditions (5.13a) and (5.13b). Thus, we have proved (5.9) for \( J=3,4,\ldots N \).

From Eqs. (C4) and (C5), we have

\[ i U_{J-1+i} \frac{1+i}{1+i} U_{J+i} - i W_{J+i} = i U_{J+i} \frac{1+i}{1+i} U_{J-1+i}. \]  

(E12)
To prove Eq. (E10), we need only to prove that the right-hand sides, and also the left-hand sides, of Eqs. (E10) and (E12) have a term-to-term correspondence. From the definitions of $w_{j+i}$ (Eq. (5.11)) and $k_{j+i}$ (Eq. (5.13c))

and from

$$|w_{j+i}| = |k_{j+i}| (1-k_{i+1}^2)(1-k_{i+2}^2)\ldots(1-k_{j+i-1}^2)$$

and

$$-i w_{j+i} = k_{j+i} (1-k_{i+1}^2)(1-k_{i+2}^2)\ldots(1-k_{j+i-1}^2),$$

these term-to-term correspondences are clear.
APPENDIX F PROOFS OF RELATIONS (5.22) AND (5.23)

For simplicity, we adopt the following notations:

Let \( \hat{x} = \{(0,i_1), (i_2, i_3), \ldots (i_{2x-2}, i_{2x-1})\} \) which is derived from a subset \( \hat{x}' = \{0, i_1, i_2, i_3, \ldots i_{2x-2}, i_{2x-1}\} \) of \( 0_{C_N} = \{0, 1, 2, \ldots N\} \) with \( 1 \leq i_1 < i_2 < \ldots < i_{2x-1} \leq N \). Let \( \hat{P}(0_{C_N}) \) be the set of all non-empty subsets \( \hat{x}' \) of \( 0_{C_N} \). Besides, if \( \hat{V} \) is a sum with each term of the form \( \hat{x} \exp(-i\omega \hat{x}) \), let \( \text{SNF}(\hat{V}) \) denote a sum of terms on the right-hand side of Eq. (5.16), which correspond to the terms of \( \hat{V} \).

We first prove Relation (5.22). From Eq. (5.18), \( \hat{F} - 1 \) can be written as

\[
\hat{F} - 1 = (0_{U_N} - 1_{U_N}) + (1_{U_{N+1}} - 1_{U_N}) + (1_{U_N} - 1).
\]

Corresponding to Eq. (F1) we write \( \text{SNF}(\hat{F} - 1) \) as

\[
\text{SNF}(\hat{F} - 1) = \text{SNF}(0_{U_N} - 1_{U_N}) + \text{SNF}(1_{U_{N+1}} - 1_{U_N}) + \text{SNF}(1_{U_N} - 1).
\]

Now, we discuss the \( \text{SNF} \)'s on the right-hand side of Eq. (F2) separately.

Using Eq. (5.16) and the notations defined above, we can write

\[
\text{SNF}(0_{U_N} - 1_{U_N}) = \sum_{\hat{x} \in \hat{P}(0_{C_N})} \left( \prod_{(r, s) \in X} \sin \left[ \frac{\omega}{2} \left( \sum t_s - T \right) \right] \right). \tag{F3}
\]

Since there is a term-to-term correspondence between \( 0_{U_N} - 1_{U_N} \) and \( 1_{U_{N+1}} - 1_{U_N} \), i.e.,

\[
k_0 k_1 k_2 \ldots k_{2x-1} \exp(-i\omega(t_{i_1} + t_{i_2} + \ldots t_{i_{2x-2}})) \leftrightarrow \]

\[
k_1 k_2 \ldots k_{2x-1} k_{N+1} \exp(-i\omega(t_{i_1} + t_{i_2} + \ldots t_{i_{2x-2}})),
\]

\[
k_{N+1} = -k_0 \text{ and } t_{i_2} + t_{i_3} + \ldots t_{i_{2x-1}} = T - (t_{i_1} + t_{i_2} + \ldots t_{i_{2x-2}})
\]

(it should be recalled that \( t^0_{N+1} = T \)), using Eq. (5.16) we can write

\[
\text{SNF}(1_{U_{N+1}} - 1_{U_N}) \]

as
Using the relation

$$\sin[\omega(t - T)] + \sin(\omega t) = 2\cos(\omega t/2)\sin(\omega(t - T/2)),$$

we can sum Eqs. (F3) and (F4) to get

$$\text{SNF}(U_{N+1} - U_N) + \text{SNF}(U_{N+1} - U_N) = 2\cos(\omega t/2)\sin[\omega(t - T + T/2)].$$

Using the notations defined in Eqs. (5.24) and (5.24a) and the formula for difference of trigonometric functions, we write (see definitions of $Z, Z', \psi_j$ and $F(\psi_j)$ in §5.4)

$$\text{SNF}(U_N - 1) = \sum_{\text{N}} \left[ \left( \prod_{\text{N}} k_{x,y} \right) \sin[\omega(\xi + t_{x,y} - T)] \right] = \cos(\omega t/2)\sin\psi + \sin(\omega t/2)\cos\psi.$$

Since

$$\hat{F}(C_N^0) + F(C_N^1) = F(C_N^1),$$

it is clear from Eqs. (F6), (F7) and (F8) that Eq. (F2) can be reduced to Eq. (5.22) after rearrangement of terms.

We now turn to prove Relation (5.23). Using Eq. (5.18) we write

$$(U_N - 1)^4 = (U_N - 1) + (U_N - 1)(U_N - 1) + (U_N - 1)(U_N - 1)(U_N - 1) + (U_N - 1)(U_N - 1)(U_N - 1)(U_N - 1)$$

and

$$\text{SNF}(U_N - 1)^4 = \text{SNF}(U_N - 1) + \text{SNF}(U_N - 1)(U_N - 1) + \text{SNF}(U_N - 1)(U_N - 1)(U_N - 1) + \text{SNF}(U_N - 1)(U_N - 1)(U_N - 1)(U_N - 1).$$
Since SNF\( (1^{\text{UN}} - 1) \) has been discussed (Eq. (F7)), we need only to discuss the other SNF's on the right-hand side of Eq. (F10). Let \( \hat{K}_1 \exp(-i\omega_1) \) be a term of \( (1^{\text{UN}} - 1) \) and \( \hat{K}_j \exp(-i\omega_j) \) a term of \( (0^{\text{UN}} - 1^{\text{UN}}) \). Then, corresponding to \( \hat{K}_1 \hat{K}_j \exp[-i\omega(\hat{t}_1 + \tau_j)] \) the term in SNF\( (1^{\text{UN}} - 1)(0^{\text{UN}} - 1^{\text{UN}}) \) can be written as

\[
\hat{K}_1 \hat{K}_j \sin[\omega(\hat{t}_1 + \tau_j - \tau)].
\]

Since there is a term-to-term correspondence between \( (0^{\text{UN}} - 1^{\text{UN}}) \) and \( (1^{\text{UN}+1} - 1^{\text{UN}}) \), using a similar procedure in which we get the terms of Eq. (F4) and corresponding to the above term in SNF\( (1^{\text{UN}} - 1)(0^{\text{UN}} - 1^{\text{UN}}) \)
we have in SNF\( (1^{\text{UN}} - 1)(1^{\text{UN}+1} - 1^{\text{UN}}) \) a term

\[
\hat{K}_1 \hat{K}_j \sin[\omega(\hat{t}_1 - \tau_j)].
\]

Using the formula for difference of trigonometric functions, we get

\[
\text{SNF}(1^{\text{UN}} - 1)(0^{\text{UN}} - 1^{\text{UN}}) + \text{SNF}(1^{\text{UN}} - 1)(1^{\text{UN}+1} - 1^{\text{UN}}) = 2CN(1^{\text{CN}}), \quad (F11)
\]

where \( CN(1^{\text{CN}}) \) is defined in Eq. (5.24a).

If we let \( \hat{K}_1 \exp(-i\omega_1) \) be one term, and \( \hat{K}_2 \exp(-i\omega_2) \) another term in \( (1^{\text{UN}} - 1) \), corresponding to

\[
\hat{K}_1 \hat{K}_2 \exp[-i\omega(\hat{t}_1 + \hat{t}_2)]
\]

we have a term in SNF\( (1^{\text{UN}} - 1)(1^{\text{UN}} - 1) \)

\[
\hat{K}_1 \hat{K}_2 \sin[\omega(\hat{t}_1 + \hat{t}_2 - \tau)]
\]

\[
\hat{K}_1 \hat{K}_2 \sin[\omega(\hat{t}_1 - \hat{t}_2)] \cos[\omega(\hat{t}_2 - \hat{t}_2)] + \cos[\omega(\hat{t}_1 - \hat{t}_2)] \sin[\omega(\hat{t}_2 - \hat{t}_2)].
\]

From Eq. (F12) we can write

\[
\text{SNF}(1^{\text{UN}} - 1)(1^{\text{UN}} - 1) = \text{SNF}(1^{\text{CN}}) \times \text{SNF}(1^{\text{CN}}) + \text{SNF}(1^{\text{UN}}) \times \text{SNF}(1^{\text{UN}})
\]

\[
= 2CN(1^{\text{CN}}), \quad (F13)
\]

From Eqs. (F7), (F8), (F11) and (F13), it is clear that Eq. (F10) can be reduced to Eq. (5.23).
APPENDIX G  DERIVATION OF EQ.(1.2) FOR MODELS WITH TWO DISCONTINUITIES

We demonstrate here the procedure stated in §5.5 for simplifying Eq. (3.2) (or Eq.(5.16)) to Eq.(1.2) for Earth’s models with two discontinuities. In this case,

\[ 0_u^2 = 1 + k_0 k_1 \exp(-i\omega t_0) + k_1 k_2 \exp(-i\omega t_1) + k_0 k_2 \exp(-i\omega t_2), \quad (G1) \]

\[ 1_u^3 = 1 + k_1 k_2 \exp(-i\omega t_2) + k_2 k_3 \exp(-i\omega t_3) + k_1 k_3 \exp(-i\omega t_4), \quad (G1a) \]

\[ 1_u^2 = 1 + k_1 k_2 \exp(-i\omega t_1). \quad (G1b) \]

(see Eq.(5.11) for definition of \( \frac{t}{u} \)). Thus, using Eq.(5.18) for \( N=2 \) and the relations \( k_0 = -k_3 = 1, \ t_0^3 = T, \ t_3 = T - t_0 \) and \( t_3^2 = T - t_0^2 \) we can write

\[ \text{SNF} \left( \frac{t}{u} - 1 \right) = k_1 \left[ \sin[\omega(t_1 - T)] - \sin[\omega t_0^0] \right] + k_2 \left[ \sin[\omega(t_2 - T)] - \sin[\omega t_1^0] \right] + k_1 k_2 \sin[\omega(t_2^1 - T)], \quad (G2) \]

where

\[ \text{SN} \left( \frac{t}{u}^2 \right) = k_1 \sin[\omega(t_1^0 - T)] + k_2 \sin[\omega(t_2^0 - T)] + k_1 k_2 \sin[\omega(t_2^1 - T)], \quad (G2a) \]

and

\[ \text{SNF} \left( \frac{t}{u}^3 \right) = k_1 k_2 \sin[\omega(t_2^1 - T)] + k_1 k_2 \left[ k_1 \sin[\omega(t_1^0 + t_2 - T)] - \sin[\omega(t_1^0 - t_2)] \right] + k_2 \left[ \sin[\omega(t_0^0 + t_2) - \sin[\omega(t_0^0 - t_2^1)] \right] + k_1 k_2 \sin[\omega(t_2^1 - T)] \]

\[ = k_1 k_2 \cos[\omega(T - t_1^0)] \sin[\omega(t_1^0 - T)] - \sin[\omega(T - t_2)] \cos[\omega(t_2^1 - T)] \]

\[ + 2 \cos[\omega(t_2^1 - T)] \sin(0 \alpha \omega), \quad (G3) \]

Substituting Eqs. (G2) and (G3) into Eq.(5.21) we get after rearrangement of terms

\[ 2 \sin[\omega(T - t_2)] \{ \cos[\omega(T - t_2)] + k_1 k_2 \cos[\omega(t_1^0 - T)] \} \]

\[ = 2 \{ \cos[\omega(T - t_2)] + k_1 k_2 \cos[\omega(t_1^0 - T)] \} \text{SN} \left( \frac{t}{u}^2 \right), \quad (G4) \]

which is simply

\[ \sin[\omega(T - t_2)] = \text{SN} \left( \frac{t}{u}^2 \right), \]

\[ = k_1 \sin[\omega(t_1^0 - T)] + k_2 \sin[\omega(t_2^0 - T)] + k_1 k_2 \sin[\omega(t_2^1 - T)] \quad (G5) \]

if \( \cos[\omega(T - t_2)] + k_1 k_2 \cos[\omega(t_1^0 - T)] \neq 0 \). Eq. (G5) is the same as Eq.(5.26) or Eq.(1.2) for the case of two discontinuities.
CHAPTER 6

THE EFFECT OF CRUST AND MANTLE DISCONTINUITIES ON THE ASYMPTOTIC SPACING BETWEEN SUCCESSIVE TORSIONAL OVERTONE EIGENFREQUENCIES OF THE EARTH

6.1 INTRODUCTION

As indicated in §1.2, McNabb, Anderssen and Lapwood (1976) derived an equation (Eq. (1.2), with constants determined explicitly only for the cases of one and two internal discontinuities) for the asymptotic behaviour of torsional overtone eigenfrequencies from a Sturm-Liouville system associated with torsional free oscillations of the discontinuous SNREI Earth. The same equation (Eq. (5.26), with constants determined for all cases) has been obtained in Chapters 3 and 5 from the summation of SH multiple reflections from discontinuities. As mentioned earlier, the solotone effect can be interpreted in terms of this equation.

Some properties of the solotone effect, which vary systematically with the depths and magnitudes of discontinuities, have been investigated in Chapter 2. Some of these properties have been quantitatively verified by Anderssen (1977) for small discontinuities about half-way (in terms of the shear wave radial travel time) between the Earth's surface and the core-mantle boundary. This chapter aims to summarise the properties of the solotone effect and to show that these properties can be mathematically derived from Eq. (5.26) for the asymptotic behaviour of torsional overtone eigenfrequencies when the SNREI Earth models contain between the surface and the core-mantle boundary (i) a single discontinuity which is either small or near the surface; or (ii) more than one small discontinuity.
6.2 PROPERTIES OF THE SOLOTONE EFFECT INVESTIGATED FROM MODEL STUDIES

As mentioned in §2.2, Gilbert wrote computer program MODE to compute eigenfrequencies of SNREI Earth models from Eq. (1.1). In Chapter 2 the properties of the solotone effect were studied by examining the behaviour of the $\gamma_n^{n\theta}$ curves (where $\gamma_n^{n\theta} = \pi (2n + 1)^{\frac{3}{2}} (n + 1)^{\frac{1}{2}} - n^{\frac{1}{2}} \sigma^2_\theta$; $\sigma_\theta$ is an eigenfrequency) derived from torsional eigenfrequencies computed by MODE for various Earth models, because of the advantage that the baseline of a $\gamma_n^{n\theta}$ curve is an estimate of the shear wave radial travel time between the Earth's surface and the core-mantle boundary (Anderssen and Cleary, 1974).

Since the solotone effect is virtually a persistent oscillatory component in the asymptotic spacings

$$\gamma_n^{n\theta} \approx n^{\theta}$$

between eigenfrequencies of successive overtone numbers $n$ and a fixed small angular order number $\theta$, any proper function of $n$ (e.g., $\gamma_n^{n\theta}$ mentioned above) which is associated with discontinuous SNREI Earth models and contains $\gamma_n^{n\theta}$ as a factor will contain an oscillatory component.

In order to simplify the mathematical manipulation, we will deal with the properties of the solotone effect exhibited in $\gamma_n^{n\theta}$ curves instead of in $\gamma_n^{n\theta}$ curves. Since the solotone effect is an oscillatory component in $\gamma_n^{n\theta}$, it can be described in terms of frequency (in units of cycles per step of increase in overtone number $n$) and amplitude (in units of radians per second).

In this study the properties of the solotone effect are re-examined in the $\gamma_n^{n\theta}$ curves derived from torsional eigenfrequencies computed by MODE. It is found that the properties exhibited in the $\gamma_n^{n\theta}$ curves and those exhibited in the $\gamma_n^{n\theta}$ curves of Chapter 2 are the same. These properties can be summarized as follows:
I. EARTH MODELS WITH A SINGLE DISCONTINUITY

(i) If we defined the magnitude of a discontinuity as the absolute value of the reflection coefficient for vertically incident SH waves at the discontinuity, then, for a discontinuity at a fixed depth, the amplitude of the solotone effect increases with the magnitude of the discontinuity (Fig. 2.3);

(ii) If $\tau_1$ and $\tau_2$ denote the shear radial travel times from the Earth's surface and the core-mantle boundary, respectively, to the discontinuity, then, as the depth of a discontinuity of a fixed magnitude increases, the amplitude of the solotone effect increases as $\tau_1$ increases until $\tau_1 = \tau_2$, and then decreases as $\tau_2$ decreases (Fig. 6.1);

(iii) As the depth of a discontinuity increases, the frequency of the solotone effect increases as $\tau_1$ increases until $\tau_1 = \tau_2$, and then decreases as $\tau_2$ decreases (Fig. 6.1);

(iv) There is an observed superposed frequency in the solotone effect if a discontinuity is not close to the Earth's surface or the core-mantle boundary (Figs. 6.1C through 6.1F).

II. EARTH MODELS WITH MORE THAN ONE DISCONTINUITY

For Earth models with only small discontinuities, the composite solotone effect caused by the discontinuities can be described as the superposition of the solotone effects caused by individual discontinuities. For models with large discontinuities the above statement is still true, except for some second-order interferences.

Except for I.(ii), these properties have already been summarized in Chapter 2 and by Anderssen (1977). Properties I.(i), I.(iii) and II have been quantitatively verified by Anderssen (1977) for the cases of one and more small discontinuities about half-way between the Earth's surface and the core-mantle boundary.
Fig. 6.1 \( n' \)-\( n \) curves for Earth models with a discontinuity of the same magnitude. These curves show how the properties of the solotone effect vary with the depth of a discontinuity. Models used are the same as those shown in Fig. 2.4. Each model contains a discontinuity of 0.3 km/sec in shear wave velocity (as shown at the right) and is of a constant density of 4.5 g/cm\(^3\).
6.3 THE FORM OF SOLUTIONS OF EQ. (5.26)

To derive mathematically the properties of the solotone effect summarized in the last section, we have to solve Eq. (5.26) for \( \sigma \) which is a function of \( n \), and then substitute \( \sigma \) into Eq. (6.1). We limit our discussion to the condition that

\[
\sum Z' P(0_{C_N}) (r,s) \sum k_k < 1
\]

(see §5.4 for definitions of \( Z, Z', 0_{C_N} \) and \( P(0_{C_N}) \), and Eq. (5.1) for definitions of \( k_k \)). Then, since the time factor in the argument of every sine function on the right-hand side of Eq. (5.26) is smaller than \( T/2 \), the solutions of Eq. (5.26) can be written in the form

\[
\sigma = 2[n + h(n)]\pi/T,
\]

where \( |h(n)| < 1 \). The expression \( 2h(n)\pi/T \) is the perturbation from evenly distributed overtone eigenfrequencies, which is contained in the right-hand side of Eq. (5.26). In Fig. (6.2) we illustrate the meaning of \( h(n) \) for an Earth model with a single discontinuity. Substituting Eq. (6.3) into Eq. (6.1) we get

\[
\tilde{\Gamma}_l = \frac{2\pi}{T} [1 + h(n+1) - h(n)].
\]

From Eq. (6.4) it is clear that \( h(n+1) - h(n) \) is responsible for the solotone effect. In Eqs. (6.3) and (6.4) the dependence of \( \sigma \) and \( \Gamma_l \) on angular order number \( l \) is neglected, since we are dealing with eigenfrequencies and \( \Gamma_l \) of fixed small \( l \) (say \( l \leq 10 \) and large \( n \), and the dependence of eigenfrequencies and \( \Gamma_l \) on a fixed \( l \) decreases as \( n \) increases. Fig. 6.3 shows, for an Earth model with a homogeneous mantle (including crust), the departure of overtone eigenfrequencies computed by MODE from \( 2\pi n/T \). From Fig. 6.3 it is clear that the dependence of \( \Gamma_l \) on \( l \) decreases more rapidly with \( n \) than the dependence of \( \sigma_l \) and the dependence of \( \Gamma_l \) (i.e., \( l = 1 \)) can be neglected when \( n \geq 10 \).
Fig. 6.2 Diagram illustrating the meaning of $h(r)$ for an Earth model with a single discontinuity.

Parameters used are: $T = 940$ sec, $T_1 = T_2 = 300$ sec, $k = 0.3$. 

$\sin(\frac{T}{2})$
Fig. 6.3 For an Earth model with a homogeneous mantle (including crust; shear wave velocity being 6.14 km/sec and density 4.5 g/cm\(^3\)) the diagram shows the departure of overtone eigenfrequencies computed by MODE from \(2\pi n/T\) for \(l=1\) and \(l=n\).
6.4 MATHEMATICAL DERIVATION OF PROPERTIES OF THE SOLOTONE EFFECT

In this section we derive the properties of the solotone effect summarized in §6.2 from Eqs. (5.26) and (6.4). This involves solving Eq. (5.26) for $\sigma$. Since the derivation of exact solutions of Eq. (5.26) is impossible or difficult, the derivation of properties of the solotone effect will be pursued only for Earth models containing (i) a single discontinuity which is either small or near the surface or (ii) more than one small discontinuity, based on approximate solutions of Eq. (5.26).

6.4.1 Earth Models with a Single Discontinuity

In this case, if we let $k_0 = 1$ Eq. (5.26) can be written as

$$\sin(\frac{T}{2}) = k_1 \sin(\frac{t_1 - t_2}{2})$$

where $T = t_2 = t_1 + t_1^0$, $t_1^0$ and $t_2^1$ are twice the shear wave radial travel times from the Earth's surface and the core-mantle boundary, respectively, to the discontinuity, and $k_1$ is the reflection coefficient for up-coming SH waves incident vertically at the discontinuity.

Eq. (6.5) contains a sine function of both $t_1^0$ and $t_2^1$. From the relation that $t_1^0 + t_2^1 = T$, Eq. (6.5) can also be written in forms such that only $t_1^0$ or only $t_2^1$ is involved in the equation. Since some of the properties of the solotone effect are described in terms of $t_1^0$ only when $t_1^0 \leq T/2$ and of $t_2^1$ only when $t_2^1 < T/2$, we write Eq. (6.5) in two other forms such that only $t_1^0$ is involved in one form and $t_2^1$ is involved in the other. Substituting the relation $(t_1^0 - t_2^1)/2 = t_1^0 - T/2$ into Eq. (6.5), applying the formula $\sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b)$ to the right-hand side, and then dividing both sides by $1 + k_1 \cos(t_1^0 \sigma)$, we get after rearrangement of terms
\[
\tan(\frac{T}{2}) = k_1 \sin(t_1^0) / \sqrt{1 + k_1 \cos(t_1^0)}.
\] (6.6)

Similarly, substituting the relation \((t_1^0 - t_2^0) / 2 = T/2 - t_2^0\) into Eq. (6.5) we get

\[
\tan(\frac{T}{2}) = -k_1 \sin(t_2^0) / \sqrt{1 - k_1 \cos(t_2^0)}.
\] (6.6a)

For convenience in the discussion, we use Eq. (6.6) when \(t_1^0 \leq T/2\) and Eq. (6.6a) when \(t_2^0 < T/2\).

### 6.4.1.1 A Small Discontinuity

In this case \(|k_1| \ll 1\), and Eqs. (6.6) and (6.6a) can be written as

\[
\tan(\frac{T}{2}) = k_1 \sin(t_1^0); \ t_1^0 \leq T/2
\] (6.7)

and

\[
\tan(\frac{T}{2}) = -k_1 \sin(t_2^0); \ t_2^0 < T/2.
\] (6.7a)

We assume \(|h(n)| \ll 1\) in this case (this is clear from Fig. 6.2).

Substituting Eq. (6.3) into Eqs. (6.7) and (6.7a), respectively, with \(\tan(h(n)\pi) = h(n)\pi\) we get

\[
h(n) \approx \frac{k_1}{\pi} \sin(\frac{2t_1^0}{T}\pi); \ t_1^0 \leq T/2
\] (6.8)

and

\[
h(n) \approx -\frac{k_1}{\pi} \sin(\frac{2t_2^1}{T}\pi); \ t_2^1 < T/2.
\] (6.8a)

Substituting Eqs. (6.8) and (6.8a) into Eq. (6.4), respectively, we get

\[
\Gamma \approx \frac{2\pi}{T} \left\{1 + \frac{2k_1}{\pi} \sin(\frac{t_1^0}{T}\pi) \cos(\frac{2t_0^0}{T}(n + \frac{1}{2})\pi)\right\}; \ t_1^0 \leq T/2
\] (6.9)
and

\[ n \Gamma_\ell = \frac{2\pi}{T} \left( 1 - \frac{2k_1}{\pi} \sin \left( \frac{T}{2} \right) \cos \left( \frac{2t_2}{T} (n + \frac{1}{2}) \pi \right) \right) \] \tag{6.9a}

The coefficient \( k_1 \) in Eqs. (6.9) and (6.9a) reflects Property I.(i) of the soliton effect, while the sine functions reflect Property I.(ii) and cosine functions Property I.(iii). Figs. 6.4A and 6.4B show, for Earth models shown in Figs. 6.1A and 6.1C respectively, the comparison of the asymptotic spacings \( \Gamma_\ell \) evaluated from eigenfrequencies computed by MODE (solid lines) with those evaluated from eigenfrequencies calculated from Eq. (6.9) (designated by '+').

Property I.(iv) arises from the fact that \( \Gamma_\ell \) values are evaluated only at discrete points along \( n \)-coordinate (i.e., only at points where \( n \) is an integer). When the discontinuity is near the Earth's surface or the core-mantle boundary such that \( 2t_1^0/T \) or \( 2t_2^1/T \) is about the order of 0.1 or smaller, this kind of frequency does not appear (e.g., Figs. 6.1A and 6.1H) because there are enough points on the \( n \)-coordinate to produce gradual change of the cosine functions of Eqs. (6.9) and (6.9a) with \( n \).

Substituting the relation \( t_1^0 = T/2 + (t_1^0 - t_2^1)/2 \) into Eq. (6.9) or substituting \( t_2^1 = T/2 - (t_1^0 - t_2^1)/2 \) into Eq. (6.9a), we get

\[ n \Gamma_\ell = \frac{2\pi}{T} \left( 1 - \frac{2k_1}{\pi} \sum_{n=1}^{N} \sin \left( \frac{T}{2} (n + \frac{1}{2}) \pi \right) \right) \] \tag{6.10}

From Eq. (6.10) it is clear that when the discontinuity is about half-way (in terms of shear wave radial travel time) between the Earth's surface and the core-mantle boundary such that \( |(t_1^0 - t_2^1)/T| \) is of the order of 0.1 or smaller, there are enough points along the \( n \)-axis to produce the sine function in Eq. (6.10), and therefore the oscillations of the observed superposed frequency (Property I.(iv)) appear as the envelope of the rapid
Fig. 6.4 Comparison between $\gamma_n$ values derived from eigenfrequencies computed by MODE (solid lines) and those evaluated from Eq. (6.9) (designated by '+'). Figs. A and B are for models shown in Figs. 6.1A and 6.1C respectively.
solotone oscillations (Fig. 6.1D and 6.1E). Eq. (6.10) is equivalent to Eq. (15) of Anderssen (1977) if we let $\cos[(t_1^0 - t_2^1) \pi / 2] = 1$. Since the second term of Eq. (6.10) changes sign for almost each incremental step of $n$, the solotone oscillations (Property I.(iii)) of high frequency do not contain information about the depth of the discontinuity. Instead, this information is contained in the slowly-changing envelope.

For a discontinuity which is not close to the Earth's surface or the core-mantle boundary or half-way between them, the observed superposed oscillations (Property I.(iv)) may provide misleading information because they look like the solotone effect caused by another discontinuity near the Earth's surface or the core-mantle boundary (Figs. 6.1C and 6.1F). It is not difficult to show, from Eqs. (6.9) and (6.9a), that the superposed frequency can be estimated by

$$\frac{T - Kt_1^0}{T} \sin\left(2\frac{\pi}{T}n\right) = 0; \ (t_1^0 < \frac{T}{2}) \ (6.11)$$

or

$$\frac{T - Kt_2^1}{T} \sin\left(2\frac{\pi}{T}n\right) = 0; \ (t_2^1 < \frac{T}{2}) \ (6.12)$$

where $K$ is the integer closest to $T/t_1^0$ (when $t_1^0 < T/2$) or $T/t_2^1$ (when $t_2^1 < T/2$).

6.4.1.2 A Discontinuity near the Earth's Surface

For a discontinuity near the Earth's surface and a discontinuity near the core-mantle boundary, the mathematical derivation of properties of the solotone effect is similar. But since no geophysical evidence has so far been observed for the existence of discontinuities near the core-mantle boundary, we treat here only the case of a discontinuity near the Earth's surface.
In this case $|t_1^0/T| << 1$. Substituting Eq. (6.3) into Eq. (6.6) we get

$$h(n) = \frac{1}{\pi} \text{arc tan} \left[ \frac{k_1 \sin(2t_1^0 n\pi/T)}{1 + k_1 \cos(2t_1^0 n\pi/T)} \right]. \quad (6.12)$$

Substituting Eq. (6.12) into Eq. (6.4), using the trigonometric relation

$$\text{arc tan}(x) - \text{arc tan}(y) = \text{arc tan} \left( \frac{x-y}{1+xy} \right) \quad (6.13)$$

and the relations $\text{arc tan}(0) = 0$, $\sin(0) = 0$ and $\cos(0) = 1$ when $0<<1$, we can write Eq. (6.4) as

$$n_1^T = \frac{2\pi}{T} \left\{ 1 + \frac{2k_1 t_1^0}{T} \frac{\cos[2(n + \frac{1}{2}) t_1^0 n\pi/T] + k_1}{1 + 2 k_1 \cos[2(n + \frac{1}{2}) t_1^0 n\pi/T] + k_1^2} \right\}. \quad (6.14)$$

Note that when $k_1<<1$, Eq. (6.14) can be reduced to Eq. (6.9). The coefficient of the second term of Eq. (6.14) reflects Properties I.(i) and I.(ii) of the solotone effect, while the cosine functions reflect Property I.(iii). Property I.(iv) does not exist in this case.

It is clear from Eq. (6.14) that the solotone oscillations caused by small discontinuities (i.e., $k_1<<1$) near the Earth's surface are negligible, and only those caused by large discontinuities are important in this case. The solotone effect caused by a large discontinuity near the Earth's surface can no longer be represented by a simple sinusoidal function of $n$, since the second and third terms of the denominator of Eq. (6.14) cannot be neglected. For the Earth model shown in Fig. 6.5A, Fig. 6.5B shows the shape of the $n_1^T$ curve derived from eigenfrequencies computed by MODE and Fig. 6.5C shows the comparison of part of these $n_1^T$ values (solid line) with those evaluated from Eq. (6.14) (designated by '+').
Fig. 6.5 For an Earth model shown in Fig. A, Fig. B shows the shape of the $\Gamma_n$ curve derived from eigenfrequencies computed by MODE, and Fig. C the comparison of part of these $\Gamma_n$ values (solid line) with those evaluated from Eq. (6.14) (designated by '++').
6.4.2 Earth Models with more than One Small Discontinuity

We derive here Property II (§6.2) of the solotone effect only for the case of \( N \) small discontinuities such that

\[
\sum_{i=1}^{N} |k_i| \ll 1. \tag{6.15}
\]

Substituting \( k_0 = 1 \) and neglecting the terms consisting of coefficient of second or higher order of \( k_i \), we can rewrite Eq. (5.26) as

\[
\sin \left( \frac{T}{2} \sigma \right) = \sum_{i=1}^{N} k_i \sin \left( \frac{t_i^0 - t_i}{N+1} \right). \tag{6.16}
\]

If we substitute Eq. (6.3) into Eq. (6.16) and follow the same procedure by which we derived Eqs. (6.9) and (6.9a), it is not difficult to show that

\[
\Gamma_1 = \frac{2\pi}{T} \left\{ 1 + \sum_{i=1}^{J} \frac{2k_i}{\pi} \sin \left( \frac{t_i^0}{T} \right) \cos \left[ \frac{2t_i^0}{T} (n + \frac{1}{2}) \pi \right] \right. \]
\[
\left. - \sum_{i=J+1}^{N} \frac{2k_i}{\pi} \sin \left( \frac{t_i^0}{N+1} \right) \cos \left[ \frac{2t_i^0}{T} (n + \frac{1}{2}) \pi \right] \right\}, \tag{6.17}
\]

where \( J \) is the number of discontinuities above the half-way (in terms of the shear wave radial travel time) between the Earth's surface and the core-mantle boundary.

From Eq. (6.17) it is clear that in the case of small discontinuities the composite solotone effect is the superposition of the effects corresponding to individual discontinuities. For a model shown in Fig. 6.6A which includes two discontinuities in shear wave velocity at depths of 400 km and 600 km respectively, Fig. 6.6B shows the composite solotone effect exhibited in the \( \Gamma_1 \) - \( n \) curve derived from torsional eigenfrequencies computed by MODE, and Fig. 6.6C shows the comparison
Fig. 6.6 For an Earth model shown in Fig. A which includes two discontinuities in shear wave velocity at 400 km and 600 km respectively, Fig. B shows the composite solotone effect exhibited in the $\eta_1^\Gamma - \eta_1$ curve derived from eigenfrequencies computed by MODE, and Fig. C the comparison of part of these $\eta_1^\Gamma$ values (solid line) with those evaluated from Eq. (6.17) (designated by '+').
of part of the $n_{11}$ values shown in Fig. 6.6B (solid line) with those evaluated from Eq. (6.17) (designated by '+').

For Earth models with large discontinuities, it is difficult to obtain approximate solutions of Eq. (5.26) for $\sigma$, so computational solutions were used. For the Earth model shown in Fig. 6.7A, we compare in Fig. 6.7B $n_{11}$ values (solid line) derived from eigenfrequencies computed by MODE with those (designated by '+') derived from solutions of Eq. (5.26) calculated by a computer with the BISECTION algorithm (e.g., Pennington, 1970, p. 274). The almost exact coincidence, except for small $n$'s, of these two sets of data is clear. This proves indirectly not only that the solotone effect is controlled by Eq. (5.26), but also that the variational method of Backus and Gilbert (1967) is highly accurate in calculating eigenfrequencies of the Earth models.
Fig. 6.7 For an Earth model shown in Fig. A, Fig. B shows the comparison of $\Gamma$ values (solid line) derived from eigenfrequencies $n_1$ computed by MODE with those (designated by '+') derived from solutions of Eq. (5.26) calculated by a computer with the BISECTION algorithm.
CHAPTER 7
EFFECT OF EARTH STRUCTURE ON RADIAL OSCILLATIONS

7.1 INTRODUCTION

So far, only the effect of discontinuities in the Earth's crust and mantle on torsional free oscillations has been discussed in this thesis. Transition regions in the Earth are important structural features, but their effect on the Earth's free oscillations has not yet been examined in sufficient detail. However, at least for torsional oscillations, it is clear that in this case the solotone effect is modified such that its amplitude decreases with increasing overtone number. This is a consequence of the fact that the amplitudes of waves reflected from transitions of finite width decrease as the frequency of the incident wave increases (cf. Wolf, 1937). This point is illustrated in Fig. 7.1.

Since (Anderssen, Cleary and Dziewonski, 1975) the differential equation for radial free oscillations can be written as a Sturm-Liouville system of the type examined by McNabb, Anderssen and Lapwood (1975), the above results should remain valid for radial oscillations. Anderssen, Cleary and Dziewonski (1975) showed that the spacings between observed radial (and, for $\ell = 1$, spheroidal) eigenfrequencies of successive overtone numbers contain an oscillatory component. In this chapter the effects of the core-mantle (C/M) and inner-outer core (I/O) boundaries, and of the crust and upper mantle structure, on radial oscillations are studied, in order to confirm that results for torsional free oscillations are applicable to radial oscillations, and to provide a general basis for further Earth modelling.

Anderssen, Cleary and Dziewonski (1975) also showed, in their figure 1, that the spacings between observed normal modes and between the
Fig. 7.1 Comparison between amplitudes of the solotone oscillations corresponding to a discontinuity (dashed line) and to a transition layer (solid line), respectively, at 600 km depth (where $\eta_1''$ is defined in Eq. (6.1)).
corresponding eigenfrequencies of model 1066A or 1066B (Gilbert and Dziewonski, 1975) have some minor differences. We have examined the sensitivity of these differences to changes in the structure of the core boundaries, and the possibility of obtaining further information of the Earth's structure from the observed radial normal modes.

We restrict attention to radial oscillations, since they correspond to vertical PKIKP rays with no P/SV conversion at interfaces. The results show that each of the structural regions considered produces an oscillation of a particular period in the spacings between eigenfrequencies of successive overtone numbers. The period is determined by the location of the region, whereas the amplitude of the oscillation is closely related to the velocity and density contrast across the region and decreases at a rate related to its width.

7.2 TERMINOLOGY

7.2.1 \( \tilde{\gamma} \) Curves

The effect of Earth structure on free oscillations is exhibited in the spacing of eigenfrequencies. Therefore, any proper parameter which depends on the overtone eigenfrequency spacing will reflect the effect of Earth structure. For fixed angular order number \( \ell \), one such parameter is

\[
\tilde{\gamma}_n^\ell (\kappa) = \pi / \left[ n + 1 \sigma_\ell (\kappa) - n \sigma_\ell (\kappa) \right],
\]

where \( \sigma_\ell (\kappa) \) are the eigenfrequencies of any of the \((\text{ScS})_H\), \((\text{ScS})_V\), PKIKP and \( J_V \) sequences (§1.4). This parameter was first used by Anderssen, Cleary and Dziewonski (1975) and has the advantage that, when \( \ell \) is small, the baseline for the \( \tilde{\gamma}_n^\ell (\kappa) - n \) curve yields a reliable estimate of the radial travel time for PKIKP waves (from oscillations of PKIKP type, such as radial oscillations) or for S waves between the surface and the core-mantle boundary (from oscillations of \( \text{SH} \) or \( \text{SV} \) type, such as torsional free oscillations). Since we are dealing here mainly with
free oscillations equivalent to PKIKP waves, for brevity we simply use $\tilde{\gamma}_l$ for $\gamma_l^{PKIKP}$ and $\sigma_l$ for $\sigma_l^{PKIKP}$ except where otherwise stated. It should be noted that the parameter $\tilde{\gamma}_l$ differs slightly from the parameter $\gamma_l$ defined by Eq. (2.2), which also yields a valid estimate of radial travel times.

In order to examine the effect of Earth structure on $\tilde{\gamma}_l$, its values are normally plotted as a function of $n$. This chapter is concerned mainly with the effects on $\tilde{\gamma}_l$ of discontinuities and transition layers. These effects occur as oscillations in the $\tilde{\gamma}_l$ curves, which are described in terms of amplitude (in unit of seconds) and frequency (in cycles per unit $n$) or period (in units of $n$).

7.2.2 Magnitudes of Discontinuities and Transition Layers

We define a transition layer as a spherically symmetric region of limited thickness in the Earth's interior, in which the density and the compressional and shear wave velocities are monotone functions of depth. It is clear that a discontinuity is the limiting case where a transition layer has zero thickness. In order to be consistent with normal terminology, however, we will restrict the case of the term 'transition layer' to regions of finite thickness.

The radial free oscillations are equivalent to vertically incident PKIKP waves. If the results for torsional eigenfrequencies are applicable to radial free oscillations, the amplitude of the oscillation in the $\tilde{\gamma}_0$ curve caused by a discontinuity is related to the reflection coefficient for these waves at the discontinuity, which in turn is related to the magnitude of the discontinuity which is defined as the absolute value of the reflection for vertically incident P waves, i.e.

$$|\rho^+\alpha^+ - \rho^-\alpha^-|/(\rho^+\alpha^+ + \rho^-\alpha^-)$$

(7.1)
where \((\rho^+, \alpha^+)\) and \((\rho^-, \alpha^-)\) are pairs of density and P wave velocity on opposite sides of the discontinuity. This can be generalized to the case of a transition layer. Since the reflection coefficient for these waves at a transition layer is closely related to the contrast in the acoustic impedance (product of density and velocity) at the two ends of the transition layer (cf. Wolf, 1937), we define the magnitude of a transition layer to be (7.1), where \((\rho^+, \alpha^+)\) and \((\rho^-, \alpha^-)\) denote the pairs of density and P wave velocity values at the two ends of the transition layer.

Let \(R(\omega, d)\) denote the absolute value of the reflection coefficient for waves of angular frequency \(\omega\) normally incident at a transition layer with thickness \(d\). Then \(R(\omega, d)\) approaches (7.1) as \(\omega\) approaches zero, and, generally speaking, \(R(\omega, d)\) decreases as \(\omega\) increases. The rate of decrease is closely associated with \(d\): the larger \(d\), the more rapid the decrease in \(R(\omega, d)\) (cf. Wolf, 1937). This implies that, for waves with wavelengths much greater than \(d\), the transition layer can be treated as a discontinuity, while the reflection of waves with wavelengths much smaller than \(d\) at the transition layer is negligible.

### 7.3 EFFECTS OF CORE BOUNDARIES

The effects of the I/O and C/M boundaries will be studied by systematic replacement of either or both of the I/O and C/M discontinuities by linear transition layers as shown in Fig. 7.2.

#### 7.3.1 Earth Models Used

Models 1066A (Fig. 7.3A; Gilbert and Dziewonski, 1975) and HOMO (with homogeneous mantle, outer core and inner core; Fig. 7.3B), and a set of models derived from them by systematic replacement of either or both of the I/O and C/M discontinuities by linear transition layers, are used in this section.
Fig. 7.2 Diagram showing how a discontinuity in a parameter is replaced by a linear transition layer.
Fig. 7.3 Models 1066A (Fig. A) and HOMO (Fig. B) and some other models ACxIy and HCxIy derived from them. See text for descriptions of ACxIy and HCxIy.
Model 1066A is continuous except at the crust-mantle, I/O and C/M boundaries, where the discontinuities in density and elastic wave velocities are large. It has been confirmed experimentally that, compared to the effects of the core boundaries, the effect of the crust-mantle discontinuity on the radial free oscillations is negligible for 1066A. It is therefore not necessary to take its existence into account. For convenience the Earth models derived from 1066A are referred to as ACxIy, where ACxIy contains a linear transition layer ACx of thickness x at the C/M boundary and a linear transition layer AIy of thickness y at the I/O boundary. Model ACOIO corresponds to 1066A. Some other examples of ACxIy are shown in Fig. 7.3A. Although it is known from the observations of short period reflections such as PcP and PKiKP that the changes in elastic properties at both core boundaries are quite abrupt, for the purposes of this investigation models with transition layers of several hundred km at the core boundaries have been included.

The simple model HOMO was constructed from model PEM (Dziewonski, Hales and Lapwood, 1975) by taking the average values of density, P and S wave velocities for the mantle (including the crust), the outer core and the inner core, in the sense that total mass and the P and S wave radial travel times are preserved. The models derived from HOMO are referred to as HCxIy, where HCxIy, like ACxIy, is a model with a C/M transition layer of thickness x and an I/O transition layer of thickness y.

7.3.2 Result

Our results show that the $\sqrt{n}$ curves for models 1066A and HOMO contain, for high overtone numbers (say, $n>20$), two distinct oscillations with periods of 5.5 and 2.4, which are produced by the I/O and C/M discontinuities respectively. If either of the discontinuities is replaced by a linear transition layer, the amplitude of the corresponding
oscillation decreases as the overtone number \( n \) increases. The rate of the decrease is closely related to the thickness of the transition layer: the thicker the transition layer, the more rapid the amplitude decrease. This is related to the above-mentioned property that, for a fixed value of \( d \), \( R(\omega,d) \) decreases as \( \omega \) increases. We now discuss separately the effects at the core boundaries.

7.3.2.1 Effect of a Discontinuity or Transition Layer at the I/O Boundary

Figs. 7.4 and 7.5 show the \( n_0 \) curves constructed from ACxIy and HCxIy, respectively, with varying thickness \( y \). The \( n_0 \) curves in the right-hand column of each figure contain, for \( n \geq 20 \), an oscillation with period 5.5. Except for small \( n \), these curves have a negligibly small C/M boundary effect, because the C/M discontinuity has been replaced by an 800 km transition layer. They are much simpler than those in the left-hand columns, where the discontinuity has been retained.

In both Figs. 7.4 and 7.5, when \( y \approx 0 \) the amplitude of the oscillation with period 5.5 in the \( n_0 \) curve decreases with increasing \( n \), and the rate of the decrease is closely associated with the value of \( y \). In addition, there is a close agreement in behaviour, as \( y \) increases, between the corresponding curves in Figs. 7.4 and 7.5. For \( y \geq 200 \) km, the decrease in the amplitude of the oscillation is restricted to the range \( 0 \leq n \leq 20 \), except for a slowly-decreasing ripple (with a period of about 8) in Fig. 7.4, which is produced by the transition zone in the upper mantle of 1066A.

It is therefore clear that the oscillation in Figs. 7.4A and 7.5A having a period of 5.5 arises from the I/O discontinuity. It follows that the oscillation having a period of 2.4 in the left-hand columns of Figs. 7.4 and 7.5 is produced by the C/M discontinuity, since in both 1066A and HOMO the only two dominant structural features which
Fig. 7.4 \( \bar{\eta}_n - n \) curves constructed from \( ACxIy \) with varying \( y \) and fixed \( x \) values of 0 km (left column) and 800 km (right column).
Fig. 7.5 $\gamma_n$-$n$ curves constructed from HCxIy with varying y and fixed x values of 0 km (left column) and 800 km (right column).
can cause large amplitude oscillations in the \( n_Y \) curves are the I/O
and C/M discontinuities.

7.3.2.2 Effect of a Discontinuity or Transition Layer at the C/M
Boundary

Figs. 7.6 and 7.7 show \( n_Y \) curves constructed from models ACxIy
and HCxIy, respectively, with varying x values. The effect of the I/O
boundary on the \( n_Y \) curves in the right columns of Figs. 7.6 and 7.7 is
greatly reduced when \( n \) is large, since it is replaced by a 300 km thick
transition layer. For \( n \geq 20 \), these \( n_Y \) curves contain an oscillation
with period 2.4. As the C/M transition layer is extended to 800 km,
the amplitude of this oscillation is significantly reduced. It can
therefore be identified as the effect of the C/M transition layer.

As \( x \) increases, the change in the form of the \( n_Y \) curves is
more complex for ACxIy than for HCxIy. The character of the change
common to both 1066A and HOMO is: when \( x = 0 \) (i.e., the C/M boundary
is a discontinuity), the oscillation with period 2.4 is persistent
(cf. Figs. 7.6A and 7.7A), whereas when \( x \neq 0 \) (i.e., the C/M boundary
is a transition layer), the amplitude of this oscillation decreases
with increasing \( n \) (cf. Figs. 7.6D through 7.6G and 7.7B through 7.7G).
The rate of decrease is closely related to the value of \( x \).

In addition, the character of the change for ACxIy has the
following two properties:

(i) The amplitude of the oscillation with period 2.4 increases with
increasing \( x \) (cf. the right-hand curves of Figs. 7.6B through
7.6D for the range 10 \( \leq x \leq 30 \)). This increase corresponds to an
increase in the magnitude of the C/M transition layer when \( x \) is
increased. Such an increase in magnitude will occur for most
existing Earth models. The dependence of this magnitude on \( x \)
is shown in Fig. 7.8 for models 1066A, 1066B (Gilbert and Dziewonski,
1975), B497 (Dziewonski and Gilbert, 1973, Appendix A1), PEM
Fig. 7.6 $n_0-n$ curves constructed from ACxIy with varying x and fixed y values of 0 km (left column) and 300 km (right column).
Fig. 7.7 $\bar{V}_0$-n curves constructed from HCxIy with varying x and fixed y values of 0 km (left column) and 300 km (right column).
Fig. 7.8 The dependence of the magnitude of the C/M transition layer on the thickness of the layer for different Earth models.
(Dziewonski, Hales and Lapwood, 1975) and C2 (Anderson and Hart, 1976).

(ii) The amplitude of the oscillation with period 2.4 increases with increasing \( n \); in particular, for \( n > 10 \) when \( x = 50 \) km, and for \( 10 < n < 45 \) when \( x = 100 \) km. Its connection with any special characteristic of the C/M boundary is unknown.

Because of (i) and (ii), the decrease (with \( n \)) of the amplitude of the oscillation with period 2.4 is much slower for ACxIy than for HCxIy.

7.4 EFFECT OF THE CRUST AND UPPER MANTLE STRUCTURE

From §7.3 it follows that a discontinuity or a transition layer will produce in the \( \vec{\gamma}_{n0} \) curve an oscillation with a particular frequency determined by its position. The oscillation produced by a discontinuity is persistent, whereas the amplitude of the oscillation produced by a transition layer decreases with \( n \) at a rate closely related to the thickness of the transition layer. In this section the effect of crust and upper mantle structure on the \( \vec{\gamma}_{n0} \) curve will be discussed based on this conclusion.

The upper mantle (extending from the base of the crust to a depth of about 1000 km) is the region where the density and P and S wave velocities are most complicated and where the existing Earth models differ most. These Earth models can be divided into two classes: those containing a discontinuous upper mantle, which generates a persistent
oscillation in the $n_0$ curve, and those containing a continuous upper mantle, which generates an oscillation with decreasing amplitude. To illustrate, the effect of the continuous upper mantle of 1066A and that of the discontinuous upper mantle of 1066B are compared in Fig. 7.9. The $n_0$ curves for models 1066A and 1066B (constructed from the same set of geophysical data) are also compared in Fig. 7.10.

The oscillation corresponding to the upper mantle structure has a period of 8 (approximately). Except for discontinuities, the effect of upper mantle structure on the $n_0$ curve is mainly restricted to $n \leq 20$ (Fig. 7.9). Therefore, for $n > 20$, the oscillation only corresponds to the upper mantle discontinuities and rapid transition layers. For example, the slowly-decreasing ripples marked by arrows in Figs. 7.4 and 7.6 arise from the transition region in the upper mantle. Since the discontinuities and rapid transition layers generate oscillations with similar periods, their individual properties cannot be separated. Thus the effects of the 421 km and 671 km discontinuities in model 1066B combine to form an oscillation with period 8 (Figs. 7.9 and 7.10).

For torsional oscillations, it has been shown in Chapter 6 that the amplitude of the solotone effect corresponding to a discontinuity in the upper mantle or crust is proportional to $k_1 \sin(t_1 \pi / \alpha)$ while the period of the solotone effect is $\alpha / t_1$, where $k_1$ is the reflection coefficient at the discontinuity for normally incident SH waves, and $t_1$ and $\alpha$ are the shear wave radial travel times from the Earth's surface to the discontinuity and the core-mantle boundary respectively. Numerical experimentation has indicated that this result can be extended to radial oscillations. Thus the oscillation generated by a discontinuity or transition layer in the upper mantle or crust will have a period $z_0 / z_1$ and an amplitude closely associated with the product of the magnitude of the discontinuity or transition layer and $\sin(z_1 \pi / z_0)$, where $z_1$ and $z_0$ are the P wave radial travel times from the Earth's surface to the
Fig. 7.9 Comparison of the effect of the continuous upper mantle of 1066A (solid line) and that of the discontinuous one of 1066B (dashed line) on the $\rho$-$\Delta H$ curves. The models used are derived in a way such that the radial P wave travel times from the core-mantle discontinuity to the Earth's center are kept the same as 1066A and 1066B respectively.
Fig. 7.10 Comparison of $\bar{v}_n$ curves for models 1066A (solid line) and 1066B (dashed line).
discontinuity or transition layer and the Earth's center respectively. Therefore, as a discontinuity or a transition layer in the upper mantle is moved toward the surface of a model Earth, the oscillation in the $nY_0$ curve corresponding to it becomes smaller in amplitude and longer in period.

Accordingly, an oscillation corresponding to a discontinuity or transition layer in the crust will have a very long period and a very small amplitude, since $z_1/z_0$ will be very small. In particular, for a discontinuity at a depth less than 40 km and with a magnitude less than 0.5, the corresponding oscillation will have a peak-to-peak amplitude less than 10s.

It follows from the above that although the dominant effects in $nY_0$ curves from radial oscillations of PKIKP type are produced by the C/M and I/O discontinuities (as indicated by Anderssen, Cleary and Dziewonski, 1975), it may be possible to separate out upper mantle effects on the basis of period.

7.5 COMPARISON BETWEEN OBSERVED AND SYNTHETIC DATA

Fig. 7.11B shows the differences between $nY_0$ values for model 1066A (Curve 1) and for observed free oscillation data (designated by 'x'). It can be seen that 1066A fits the observed $nY_0$ values well for $n \leq 2$ and that, in general, when $n$ increases, the fit becomes worse. This trend indicates that the differences in $nY_0$ values could be reduced by changing the parameter values concerning discontinuities and/or rapid transitions in the model.

Shown in Fig. 7.11A (Curves 2 and 3) are two perturbed models of 1066A which fit, except for $5Y_0$, the observed $nY_0$ values better than 1066A. Fig. 7.11 confirms that the radial normal mode data for small overtone numbers cannot distinguish an Earth model with discontinuous core boundaries from an Earth model with continuous ones. Although the
Fig. 7.11 Error ranges (designated by triangles) and comparison of observed $v_p$ values (designated by 'x') and those constructed from 1066A (curve 1) and two perturbed models of 1066A (curves 2 and 3). One perturbed model is derived from 1066A by shifting the C/M and I/O discontinuities toward the Earth's surface by 10 and 70 km respectively, and the other by replacing the discontinuities by 20 and 140 km thick transition layers in the way shown.
perturbed models are not realistic, the results show that the differences between $Y_0$ values derived from observed data and eigenfrequencies of 1066A are sensitive to the changes in parameter values defining the core boundaries, and indicate that if sufficiently refined normal mode data, including high overtone numbers, can be obtained, the Earth model can be improved significantly.

Fig. 7.11B also shows the error ranges (designated by triangles) of the observed $Y_0$ values, based on the observation errors listed by Gilbert and Dziewonski (1975). It is clear from these that, in addition to data of higher overtone number, a refinement of the present data is required for satisfactory constraint on the Earth's fine structure.

If lateral variations in the radius of the C/M boundary exist, as suggested, for example, by Hide and Horai (1968), then because of the averaging properties of free oscillations, the effect of these variations would be similar to that of a transition layer at the boundary. It follows from the above results that such variations would not be detectable by the present radial oscillation data.

7.6 CONCLUSIONS

The above results may be summarized as follows:

1. A $Y_0$ curve comprises a baseline component whose value is determined by the P wave radial travel time from the surface to the centre of the Earth, and an oscillatory component which reflects the structure in the Earth's interior, including gradual slopes, transition layers and discontinuities;

2. The effect of a discontinuity on the $Y_0$ curve is seen as a persistent oscillation, while the effect of a transition layer is seen as an oscillation whose amplitude decreases with $n$ at a rate closely associated with the thickness of the layer. The amplitude of the oscillation is closely related to the magnitude of the discontinuity or
transition layer, while the period is related to its position. A consequence of this is that the effects of gradual slopes on the curve are limited to small overtone numbers (say \( n \leq 20 \) only; (3) It is known from the observation of short period waves that the I/O and C/M boundaries are discontinuities or very rapid transition layers of large magnitudes, therefore their effects on the Earth's radial oscillations are persistent or almost persistent oscillations in the curve having periods of about 5.5 and 2.4 respectively. Except for very rapid transition layers or discontinuities, the effect of upper mantle structure is mostly restricted to \( n \leq 20 \). The oscillation corresponding to the structure of the upper mantle has a period of about 8. The effect of crustal structure on the radial oscillations is seen as an oscillation of very long period and very small amplitude (less than 10's from peak-to-peak); (4) As a consequence of the above, the oscillations of very small \( n \) (say, \( n \leq 5 \)) in the curve are a composite effect of all the Earth's interior structures (gradual slopes, transition layers and discontinuities), i.e., they reflect the overall spherically symmetric structure of the Earth. When \( n \) increases, the oscillations tend to reflect local structures such as very rapid transitions and discontinuities, i.e., their effects are proportionally greater. When \( n > 20 \), the oscillations are mostly the effects of these very rapid transition layers and discontinuities; (5) The exact manner in which the effects of all the structural features of the Earth on the radial free oscillations are composed still requires a detailed study. By neglecting the minor effects and the interference among the effects of major structural features, however, we can represent the overall composite effect of the Earth's structure as the sum of all the individual oscillations in the curve. Fig. 12 shows the comparison of (1066A) (solid line) and \([\tilde{\gamma}_0 (AC800I0) + \tilde{\gamma}_0 (AC0I800)]\) (designated by 'x') structures, where
Fig. 7.12 Comparison of $n_0$ (solid line) and $[n_0(AC800I0) + n_0(AC0I800) - n_0(AC800I800)]$ (designated by 'x') values.
\( \tilde{\gamma}_0(k) \) is the \( \tilde{\gamma}_0 \) value for model \( k \);

(6) A comparison between observed and synthetic \( \tilde{\gamma}_0 \) values confirms that the radial normal mode data for small overtone number cannot distinguish a continuous Earth model from a discontinuous one. Because of the large error ranges of the observed \( \tilde{\gamma}_0 \) values and the nature of free oscillations of low overtone number, information about the fine structure of the Earth and lateral variations in the radius of the C/M boundary cannot be detected by the present radial oscillation data.
CHAPTER 8
GENERAL DISCUSSION

A major aim in geophysics is to extend the power of available data to constrain Earth structure. This can be achieved in a number of ways:

(i) by improvising the quantity and quality of data;
(ii) by developing theory which permits a more sophisticated interpretation of existing data, and
(iii) by developing theory which indicated the type of additional data that should, if possible, be collected.

This thesis has been mainly concerned with this last aspect as well as the development of a technique which will facilitate the accomplishment of (i) in that it has centred around an examination of the constraining power of free oscillation data of high overtone number which are not yet available, and the refinement of a technique by which such data might be obtained indirectly.

In particular, the thesis has been directed towards extending Brune's method of deriving higher overtone eigenfrequency information from body wave data to include the effect of internal discontinuities in the Earth. The initial insight into this problem came from model studies which showed that the solotone effect for torsional oscillations could be derived by the summation of SH waves reflected at internal discontinuities. This highlighted the problem that Brune's method in its original form was not applicable to discontinuous models. In fact the window lengths used by Brune (1964) and Brune and Gilbert (1974) in obtaining data excluded reflections from such discontinuities, and consequently the resulting data were incapable of constraining the fine structure of the upper mantle in the studies by Gilbert, Dziewonski and Brune (1973) and Gilbert and Dziewonski (1975). An associated point, concerning the size of the errors associated with the neglect of discontinuity effects, has
recently been made in an independent study by Kennett and Nolet (1978),
where the solotone effect was examined by relating the high frequency
torsional mode dispersion to the intercept time as a function of
slowness derived from SH travel times, using an approach of Pekeris (1965)
as formalized by Woodhouse (1978).

A theoretical framework was then developed to establish the
validity of the results obtained from the model experiments. In the
course of this, it was shown that, for the case of one discontinuity in
the crust or mantle, the equation derived by McNabb, Anderssen and
Lapwood (1976) for the asymptotic behaviour of torsional overtone
eigenfrequencies can be obtained from an extension of Brune's equation
to summations of SH waves recorded at small distances and multiply-reflected
at the Earth's surface, the core-mantle boundary, and the internal
discontinuity. Following this, the study was generalized to include an
arbitrary number of discontinuities. A geophysical interpretation of
the constants in the McNabb et al. formulation was obtained, and the
ray-mode duality established for discontinuous Earth models.

The effect of the magnitudes and depths of the discontinuities
on the solotone oscillations was examined in some detail. Independent
work on the same problem has recently been published by Sato and Lapwood
(1977a,b) and Lapwood and Sato (1977) where, for homogeneous spherical
layers, the solotone effect was examined using eigenfrequencies evaluated
from an exact frequency equation, in terms of spherical Bessel functions,
and some asymptotic formulae.

The overtone behaviour of radial eigenfrequencies was investigated
computationally in a manner similar to that used for the torsionals. The
differences between the solotone effects produced by discontinuities and by
transition layers were examined, as a means of determining the constraining
power of these oscillations on the resolution of fine structure at the
core-mantle and inner-outer core boundaries, as well as in the upper mantle.
It was found that the solotone effect produced by a transition layer decayed in amplitude as the overtone number increased. The study confirmed that presently available data are unable to detect lateral variations in core radii of the type proposed, for example by Hide and Horai (1968).

Gilbert and Dziewonski (1975) have indicated that it is possible, using their stacking and stripping procedures, to obtain free oscillation data of higher overtone number than has been so far observed. It is clear that such data would be useful if it could be extended to sufficiently high overtone number with sufficient precision of measurement. In practice, however, such information might be obtained more readily by a careful application of an extended Brune technique at short epicentral distances.
REFERENCES


