SOME PROBLEMS ON FREE GROUPS

by

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STATEMENT

Of Chapter 1, only the section on "n-transformations" is original. The rest of the thesis is my work, except that occasionally a result or proof that is not mine is included for the convenience of the reader. At each such place I have indicated that the work is not my own.

M J Dunwoody

PREFACE

The Department of Mathematics in the Institute of Advanced Studies of the Australian National University was created in 1961. I feel particularly honoured to have been awarded the first Research Scholarship in the department. The work for this thesis was done during my stay in Canberra from September, 1961 to the present time.

I thank Professor B. H. Neumann, F.A.A., F.R.S. and Professor Hanna Neumann for suggesting some of the problems I have worked on and for taking an interest in my work.

I am greatly indebted to my supervisor, Dr M. F. Newman. He was always accessible and infused in me some of his own enthusiasm for mathematics.

I would also like to thank Dr Tekla Taylor for some interesting discussions.

Dr L. G. Kovács read part of the thesis and Mrs F. Munns inserted symbols on the stencils.

Most of the results of Chapters 2 and 3 have been published in my two papers [3] and [4]. Some of the material of Chapter 5 has been submitted for publication in Archiv der Mathematik.

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LIST OF NOTATIONS

Notations not introduced in the text

Many of the notations used are standard in modern publications on group theory. The following list is of those notations that I have used, which do not fall into this category, but which are not introduced later on.

Dom f, Dom φ	the domain of the function f or the mapping φ
Im f, Im φ	the image (range) of the function f or the mapping φ
$\lvert S \rvert$	the cardinal of the set S

Let G be a group, let $s, t, \ldots \in G$, and let S, T be subsets of G. Let θ be a homomorphism of G.

$\mathrm{sgp}\{s, t, \ldots\}$	the subgroup of G generated by $s, t, \ldots$
$\mathrm{sgp}\{S, T, \ldots\}$	the subgroup of G generated by $S, T, \ldots$
e	the identity element
E	$\mathrm{sgp}\{e\}$
s^t	$t^{-1}st$
s^{-t}	$t^{-1}s^{-1}t$
$[s,t]$	$s^{-1}s^t$
$[S,T]$	$\mathrm{sgp}\{[s,t] \mid s \in S, t \in T\}$
$Z(G)$	the centre of G

$\mathfrak{A}(G)$	the automorphism group of G
$\lvert G\rvert$	the order of G : a finite or infinite cardinal
$\operatorname{Ker}\theta$	the kernel of θ

<u>Notations introduced in the text</u>

INTRODUCTION

In this thesis several different problems concerning free groups are tackled. If there is a central theme, it is provided not by the problems tackled so much as by the method of solution. This is with the exception of the last half of Chapter 5, which is something of a digression.

Let θ be a homomorphism of the group H onto the group G, then θ maps every set of generators of H onto a set of generators of G. On the other hand, simple examples show that a set of generators of G need not be the image under θ of a set of generators of H. A set of generators of G which does have this property is said to have $P(\theta)$. Most of the problems tackled in this thesis are reduced to a problem of whether of not a certain set of generators of a group has $P(\theta)$ for particular homomorphisms θ. If $\mathrm{Ker}\,\theta$ is finite, then Gaschütz [6] has shown that a set of generators of G has $P(\theta)$ provided only that the set has at least as many elements as a minimal set of generators of H. No such simple criterion exists if $\mathrm{Ker}\,\theta$ is infinite. However a necessary condition can be found if the factor group of H by its derived group is free abelian of rank n, where n is the minimal number of generators of H. This result is applied to give most of the main results of the thesis.

Let π be a fixed homomorphism of a free group of rank n onto a group G, then all the sets of n generators of G having $P(\pi\beta)$, for some automorphism β of G, form a T-system. T-systems were introduced by B. H. Neumann and H. Neumann [14]: they are important in the study of characteristic subgroups of free groups. The method I have outlined above provides a new way of distinguishing between the T-systems of a group. This is described in Chapter 2.

In Chapter 3, groups G and H are constructed such that G is a homomorphic image of the n-generator group H and such that G has a set of n generators which does not have $P(\theta)$ for any homomorphism θ of H onto G. This provides a negative answer to a question raised by the Neumanns ([14], Problem 7.32). A related question from [14] concerning the hypercharacteristic subgroups of free groups is also answered negatively.

Let R and S be normal subgroups of the finitely generated free group F. If F/R and F/S are isomorphic and the derived group of F/R is finite, then the problem of whether $F/R\cap[F,F]$ and $F/S\cap[F,F]$ are isomorphic can be reduced to the problem of whether or not a particular set of generators of F/R has $P(\theta)$ for any homomorphism θ of $F/R\cap[F,F]$ onto F/R. This latter problem, and hence the original problem, is almost completely solved in Chapter 4. Some results on the T-systems of various soluble groups are also proved in Chapter 4.

Chapter 5 concerns the properties of the group $F/v(R)$, where F is a free group, and $v(R)$ is a word subgroup of the normal subgroup R of F. The problem of whether $F/v(S)$ is isomorphic to $F/v(R)$ can again be reduced to the sort of problem discussed above, if suitable restrictions are placed on F/R and v. The results obtained on this topic are given in the first half of the chapter.

The residual finiteness of a group G cannot in general be deduced from the residual finiteness of a normal subgroup N and its factor group G/N. However Baumslag [1] has shown that $F/v(R)$ is residually finite if F/R and $R/v(R)$ are residually finite. A new proof of this result is given in Chapter 5. It is shown that a subgroup topology of F/R (in the sense of [8]) can be converted to a subgroup topology of $F/v(R)$ in a simple way. Baumslag's result follows fairly easily.

CHAPTER 1

Introduction

This chapter provides a background of notation and results that are used in this thesis. Most of the definitions and results come from the Neumanns' paper [14]. However, I claim some originality for the treatment of "n-transformations". Some confusion about these mappings appears to exist in the above work, particularly in §4; Satz 4.6 is in fact trivially true (see Lemma 1.2).

n-vectors

An ordered set of n elements of a group G is called an n-vector of G; n-vectors will be denoted by small letters with a double underline, e.g. $\underline{\underline{g}}$, $\underline{\underline{h}}$. If $\underline{\underline{g}}$ is an n-vector, then the ith component of $\underline{\underline{g}}$ is denoted by g_i : thus

$$\underline{\underline{g}} = (g_1, g_2, \ldots, g_n) .$$

The set of all n-vectors of G is denoted by (n,G). If θ is a homomorphism and N a normal subgroup of G, then

$$\underline{\underline{g}}\theta = (g_1\theta, g_2\theta, \ldots, g_n\theta)$$

and

$$\underline{\underline{g}}N = (g_1N, g_2N, \ldots, g_nN)$$

are n-vectors of $G\theta$, G/N respectively. If $a \in G$, then

$$\underline{\underline{g}}^a = (g_1^a, g_2^a, \ldots, g_n^a) .$$

If $\operatorname{sgp}\{\underline{\underline{g}}\} = G$, then $\underline{\underline{g}}$ is called a generating n-vector. The set of all generating n-vectors of G is denoted by $[n,G]$.

Free groups

Throughout, F_n will denote the free group of rank n with generating n-vector $\underline{\underline{x}}$; i.e.,

$$F_n = \operatorname{gp}\{x_1, x_2, \ldots, x_n\} .$$

If $\underline{\underline{g}}$ is an n-vector of a group G , there is a unique homomorphism $\varphi_{\underline{\underline{g}}}$ of F_n into G such that $\underline{\underline{x}}\varphi_{\underline{\underline{g}}} = \underline{\underline{g}}$ ([9], p.93); Ker $\varphi_{\underline{\underline{g}}}$ is called the relation group of $\underline{\underline{g}}$ and is denoted by $R(\underline{\underline{g}})$. Clearly $F_n/R(\underline{\underline{g}}) \cong \operatorname{sgp}\{\underline{\underline{g}}\}$.

Word mappings

The mappings j_i , $i = 1, 2, \ldots$, are defined as follows:-

$$\operatorname{Dom} j_i = \{\underline{\underline{g}} \mid \underline{\underline{g}} \in (k,G) \text{ for some group } G ,\ k \geqq i\}$$

$$\underline{\underline{g}}j_i = g_i .$$

Word mappings are defined as follows:-

(a) $j_1, j_2, \ldots$ are word mappings

(b) if ω , υ are word mappings, then so is $\omega\upsilon^{-1}$, where

$$\operatorname{Dom} \omega\upsilon^{-1} = \operatorname{Dom} \omega \cap \operatorname{Dom} \upsilon$$

$$\underline{\underline{g}}\,\omega\upsilon^{-1} = (\underline{\underline{g}}\omega)(\underline{\underline{g}}\upsilon)^{-1} .$$

Thus $j_1 j_2 j_1^{-1} j_2^{-1}$ is a word mapping, where

$$\underline{\underline{g}} j_1 j_2 j_1^{-1} j_2^{-1} = g_1 g_2 g_1^{-1} g_2^{-1} , \quad \underline{\underline{g}} \in (k,G) , \quad k \geqq 2 .$$

<u>Lemma 1.1</u>. Let ω be a word mapping and θ a homomorphism of G. Then $\underline{\underline{g}}\omega\theta = \underline{\underline{g}}\theta\omega$ for every $\underline{\underline{g}} \in (n,G) \cap \mathrm{Dom}\,\omega$. (Note that on the left-hand side of this equation θ acts on a single element, while on the right it acts on an n-vector.)

<u>Proof</u>. This is a simple application of the homomorphism property of θ.

<u>n-transformations</u>

Let $\omega_1, \omega_2, \ldots, \omega_n$ be word mappings such that

$(n,F_n) \subseteqq \mathrm{Dom}\,\omega_i$, $i = 1, 2, \ldots, n$, then define α to be the mapping

$$\mathrm{Dom}\,\alpha = \{\underline{\underline{g}} \mid \underline{\underline{g}} \in (n,G) \text{ , for some group } G\}$$

$$\underline{\underline{g}}\alpha = (\underline{\underline{g}}\omega_1, \underline{\underline{g}}\omega_2, \ldots, \underline{\underline{g}}\omega_n) .$$

If $\underline{\underline{x}}\alpha$ is a generating n-vector of F_n , then α is called an n-transformation.

<u>Lemma 1.2</u>. Let α be an n-transformation and θ a homomorphism of G , then

$$\underline{\underline{g}}\alpha\theta = \underline{\underline{g}}\theta\alpha , \quad \underline{\underline{g}} \in (n,G) .$$

<u>Proof</u>. The lemma follows immediately from Lemma 1.1.

Lemma 1.3. An n-transformation is uniquely determined by the way it acts on the generating n-vector $\underline{\underline{x}}$ of F_n.

Proof. Let α, α' be n-transformations such that

$$\underline{\underline{x}}\alpha = \underline{\underline{x}}\alpha' ,$$

then for every n-vector $\underline{\underline{g}}$ of an arbitrary group G,

$$\underline{\underline{g}}\alpha = \underline{\underline{x}}\varphi_{\underline{\underline{g}}}\alpha = \underline{\underline{x}}\alpha\varphi_{\underline{\underline{g}}} = \underline{\underline{x}}\alpha'\varphi_{\underline{\underline{g}}} = \underline{\underline{x}}\varphi_{\underline{\underline{g}}}\alpha' = \underline{\underline{g}}\alpha' .$$

Thus $\alpha = \alpha'$, and the lemma is proved.

The product $\alpha\alpha'$ of two n-transformations α, α' is defined by

$$\underline{\underline{g}}(\alpha\alpha') = (\underline{\underline{g}}\alpha)\alpha' , \quad \underline{\underline{g}} \in (n,G) .$$

Theorem 1.4. The n-transformations with the above multiplication form a group isomorphic to $\mathfrak{A}(F_n)$.

Proof. Firstly it will be shown that $\alpha\alpha'$ is an n-transformation if α and α' are n-transformations. By definition $\underline{\underline{x}}\alpha$ is a generating n-vector of F_n, it is therefore also a set of free generators of F_n (see [9], p.109). Thus there is an automorphism γ of F_n such that $\underline{\underline{x}}\gamma = \underline{\underline{x}}\alpha$. Now, by Lemma 1.2,

$$((\underline{\underline{x}}\alpha)\alpha')\gamma^{-1} = ((\underline{\underline{x}}\alpha)\gamma^{-1})\alpha'$$

$$= \underline{\underline{x}}\alpha' ,$$

which generates F_n. Therefore $(\underline{\underline{x}}\alpha)\alpha'$ generates F_n. Clearly there is an n-transformation α^* such that $\underline{\underline{x}}\alpha^* = (\underline{\underline{x}}\alpha)\alpha'$, and, by applying $\varphi_{\underline{\underline{g}}}$ to both sides and repeated use of Lemma 1.2, it follows that $\underline{\underline{g}}\alpha^* = (\underline{\underline{g}}\alpha)\alpha'$ for every n-vector $\underline{\underline{g}}$. Therefore $\alpha\alpha'$ is an n-transformation.

It will be shown that there is a 1-1 mapping ρ of $\mathfrak{A}(F_n)$ onto the class of all n-transformations, and

$$(\gamma_1\gamma_2)\rho = \gamma_1\rho\gamma_2\rho$$

for every $\gamma_1, \gamma_2 \in \mathfrak{A}(F_n)$. This ensures that the n-transformations form a group, and in fact suffices to prove the theorem.

It has been seen that to every n-transformation α there is an automorphism γ of F_n such that $\underline{\underline{x}}\alpha = \underline{\underline{x}}\gamma$. Conversely, to every automorphism γ of F_n there is an n-transformation α such that $\underline{\underline{x}}\alpha = \underline{\underline{x}}\gamma$. But by Lemma 1.3, this α is unique. Therefore a 1-1 mapping ρ of $\mathfrak{A}(F_n)$ onto the set of all n-transformations can be defined such that

$$\gamma^{-1}\rho = \alpha .$$

Let $\gamma_1, \gamma_2 \in \mathfrak{A}(F_n)$, then

$$\underline{\underline{x}}(\gamma_1\gamma_2)\rho = \underline{\underline{x}}(\gamma_1\gamma_2)^{-1} = \underline{\underline{x}}\gamma_2^{-1}\gamma_1^{-1} = (\underline{\underline{x}}\gamma_2\rho)\gamma_1^{-1} .$$

But by Lemma 1.2,

$$(\underline{\underline{x}}\gamma_2\rho)\gamma_1^{-1} = (\underline{\underline{x}}\gamma_1^{-1})\gamma_2\rho = (\underline{\underline{x}}\gamma_1\rho)\gamma_2\rho = \underline{\underline{x}}(\gamma_1\rho\gamma_2\rho) ,$$

and since $(\gamma_1\gamma_2)\rho$ and $\gamma_1\rho\gamma_2\rho$ are both n-transformations, it follows from Lemma 1.3 that $(\gamma_1\gamma_2)\rho = \gamma_1\rho\gamma_2\rho$.

Let K_n denote the group of all n-transformations.

Lemma 1.5. Every n-transformation α acts as a permutation on $[n,G]$.

Proof. Since α has an inverse α^{-1} , it suffices to prove that $[n,G]\alpha \subseteq [n,G]$. Let $\underline{\underline{g}} \in [n,G]$, then

$$G = \mathrm{sgp}\,\{(\underline{\underline{g}}\alpha)\alpha^{-1}\} \subseteq \mathrm{sgp}\{\underline{\underline{g}}\alpha\} \subseteq \mathrm{sgp}\{\underline{\underline{g}}\} = G .$$

Therefore $\underline{\underline{g}}\alpha \in [n,G]$ and the result follows.

The group of permutations of $[n,G]$ obtained by restricting the n-transformations to $[n,G]$ is denoted by A .

T-systems

If β is an automorphism of G , then the mapping

$$\underline{\underline{g}} \to \underline{\underline{g}}\beta , \quad \underline{\underline{g}} \in [n,G] ,$$

also denoted by β , is easily seen to be a permutation of $[n,G]$. These permutations form a group B isomorphic to $\mathfrak{A}(G)$.

Since $\alpha\beta = \beta\alpha$ if $\alpha \in A$ and $\beta \in B$, it follows that AB is a group of permutations of $[n,G]$. The sets of transitivity for A, B and AB are called A-classes, B-classes and T-systems respectively; if $\underline{\underline{g}} \in [n,G]$, $\underline{\underline{g}}A$, $\underline{\underline{g}}B$ and $\underline{\underline{g}}AB$ denote the respective A-class, B-class and T-system containing $\underline{\underline{g}}$.

<u>Theorem 1.6</u>. Let $\underline{\underline{g}}$, $\underline{\underline{h}} \in [n,G]$, then $\underline{\underline{g}}$, $\underline{\underline{h}}$ belong to the same T-system if and only if

$$R(\underline{\underline{g}}) = R(\underline{\underline{h}})\gamma$$

for some automorphism γ of F_n .

<u>Proof</u>. Let $\underline{\underline{g}} = \underline{\underline{h}}\alpha\beta$, for some $\alpha \in K_n$ and $\beta \in B$.

Let γ be the automorphism of F_n such that $\underline{\underline{x}}\gamma^{-1} = \underline{\underline{x}}\alpha$,

then $\gamma^{-1}\varphi_{\underline{\underline{h}}} = \varphi_{\underline{\underline{h}}\alpha}$; for

$$\underline{\underline{x}}\gamma^{-1}\varphi_{\underline{\underline{h}}} = \underline{\underline{x}}\alpha\varphi_{\underline{\underline{h}}} = \underline{\underline{x}}\varphi_{\underline{\underline{h}}}\alpha = \underline{\underline{h}}\alpha .$$

Therefore

$$R(\underline{\underline{h}}\alpha\beta) = R(\underline{\underline{h}}\alpha) = \mathrm{Ker}\,(\gamma^{-1}\varphi_{\underline{\underline{h}}}) = (\mathrm{Ker}\,\varphi_{\underline{\underline{h}}})\gamma = R(\underline{\underline{h}})\gamma .$$

Conversely, let $R(\underline{\underline{g}}) = R(\underline{\underline{h}})\gamma$. Let α be the n-transformation such that

$$\underline{\underline{x}}\gamma^{-1} = \underline{\underline{x}}\alpha ,$$

then as above $\gamma^{-1}\varphi_{\underline{\underline{h}}} = \varphi_{\underline{\underline{h}}\alpha}$ and $R(\underline{\underline{h}}\alpha) = R(\underline{\underline{h}})\gamma$. Therefore $R(\underline{\underline{g}}) = R(\underline{\underline{h}}\alpha)$ and there is an automorphism β of G such that $\underline{\underline{g}} = \underline{\underline{h}}\alpha\beta$.

<u>Corollary 1.7</u>. The subgroup $I(\underline{\underline{g}}) = \bigcap_{\underline{\underline{h}} \in \underline{\underline{g}}AB} R(\underline{\underline{h}})$ is the

largest characteristic subgroup of F_n contained in $R(\underline{\underline{g}})$.

Proof. By Theorem 1.6,

$$I(\underline{\underline{g}}) = \bigcap_{\gamma \in \mathfrak{A}(F_n)} R(\underline{\underline{g}})\gamma ,$$

and this is clearly a characteristic subgroup of F_n . If I' is characteristic in F_n and is contained in $R(\underline{\underline{g}})$, then

$$I' = I'\gamma \leq R(\underline{\underline{g}})\gamma$$

for every automorphism γ of F_n. Therefore $I' \leq I(\underline{\underline{g}})$, and the result is proved.

Hypercharacteristic subgroups

If G, H are groups, then the set of all normal subgroups N of H such that $H/N \cong G$ is denoted by $\Sigma(H,G)$: i.e.,

$$\Sigma(H,G) = \{N \mid N \triangleleft H, H/N \cong G\} .$$

A normal subgroup K of a group G is said to be hypercharacteristic in G if $K \leq N$ for every $N \in \Sigma(G,G/K)$.

Lemma 1.8. If K is hypercharacteristic in G , then K is characteristic in G .

Proof. Let β be an automorphism of G , then $G\beta = G$ and

$$G/K\beta = G\beta/K\beta \cong G/K ,$$

that is $K\beta \in \Sigma(G,G/K)$. This means that

$$K\beta \geq K .$$

Similarly $K\beta^{-1} \geq K$, so that $K\beta = K$, and the lemma is proved.

<u>Lemma 1.9</u>. The intersection

$$I = \bigcap_{N \in \Sigma(H,G)} N$$

is a hypercharacteristic subgroup of $\overset{H}{\cancel{G}}$.

<u>Proof</u>. Let $M \in \Sigma(H,H/I)$. Let μ be an isomorphism of H/I onto H/M . If $N \in \Sigma(H,G)$, let $N\mu$ denote the normal subgroup containing M such that $N\mu/M = (N/I)\mu$. Now

$$M = \bigcap_{N \in \Sigma(H,G)} N\mu \text{ .}$$

But $N\mu \in \Sigma(H,G)$, since

$$H/N \cong \frac{H/I}{N/I} \cong \frac{(H/I)\mu}{(N/I)\mu} = \frac{H/M}{N\mu/M} \cong H/N\mu \text{ .}$$

Therefore $M \geqq I$ and the lemma is proved.

In particular, $\bigcap_{N \in \Sigma(F_n,G)} N$ is a hypercharacteristic subgroup of F_n . But $\Sigma(F_n,G) = \{R(\underline{\underline{g}}) \mid \underline{\underline{g}} \in [n,G]\}$; so that

$$U_n(G) = \bigcap_{\underline{\underline{g}} \in [n,G]} R(\underline{\underline{g}})$$

is a hypercharacteristic subgroup of F_n .

One might suppose that $U_n(G)$ was the largest hypercharacteristic subgroup of F_n contained in $R(\underline{\underline{g}})$. This is not the case; in Chapter 3 an example is constructed, for a particular group G , of a hypercharacteristic subgroup of F_2 contained in $R(\underline{\underline{g}})$ but not contained in $U_2(G)$.

Word subgroups

Let W be a set of word mappings. The function v which takes every group G to a subgroup $v(G)$, where

$$v(G) = \operatorname{sgp}\{\underline{\underline{g}}\omega \mid \omega \in W,\ \underline{\underline{g}} \in (k,G) \cap \operatorname{Dom}\omega,\ k \geqq 1\}$$

is called a word subgroup function and $v(G)$ a word subgroup of G .

Lemma 1.10. If θ is a homomorphism of G and v a word subgroup function, then

$$v(G\theta) = v(G)\theta ;$$

hence $v(G)$ is a fully invariant subgroup of G .

Proof. The lemma follows immediately from Lemma 1.1.

Lemma 1.11. A word subgroup of a group G is hypercharacteristic in G .

Proof. Let $v(G)$ be a word subgroup of G and suppose $G/N \cong G/v(G)$. Let θ be the natural homomorphism of G onto G/N , then $v(G\theta) = v(G)\theta$. But $v(G\theta) = E$, since $G\theta \cong G/v(G)$, and so $v(G) \leqq \operatorname{Ker}\theta = N$, proving the result.

Corollary 1.12. Every fully invariant subgroup of a free group F is hypercharacteristic in F .

Proof. The fully invariant subgroups of F are also word subgroups of F ([12], p.512); the corollary follows immediately.

Two word subgroup functions occur sufficiently often for them to be given special symbols. The word subgroup function associated with the set of word mappings $\{j_1^{-1}j_2^{-1}j_1j_2\}$ is denoted by δ ; thus

$$\delta(G) = [G,G]$$

the derived group of the group G . The word subgroup function associated with the set of word mappings $\{j_1^{-1}j_2^{-1}j_1j_2,\ j_1^k\}$ is denoted by ν_k ; in particular, $\nu_0 = \delta$.

CHAPTER 2

Introduction

In [14] the Neumanns give representatives of the 19 B-classes of generating 2-vectors of the alternating group $\underline{\underline{A}}_5$. They then show that there are two T-systems of generating 2-vectors by considering the action on the B-classes of a set of generators of the group K_2 of 2-transformations. Such a computation is feasible only if the number of B-classes is fairly small. This means that one is usually restricted to the generating 2-vectors of a group of fairly low order: by way of example, $\underline{\underline{A}}_6$ has 53 B-classes of generating 2-vectors, while $\underline{\underline{A}}_5$ has 1668 B-classes of generating 3-vectors.

In other cases some other method is required for distinguishing between the T-systems of a particular group. One such method is given by Higman's criterion:

> If $\underline{\underline{g}}$, $\underline{\underline{h}}$ are generating 2-vectors of the group G then $\underline{\underline{g}}$, $\underline{\underline{h}}$ belong to the same T-system only if the commutators $[g_1,g_2]$, $[h_1,h_2]$ have the same order.

B. H. Neumann [13] constructed a group with two generating 2-vectors $\underline{\underline{g}}$ and $\underline{\underline{h}}$ such that $[g_1,g_2]$ has order 2, while $[h_1,h_2]$ has order 4. Therefore $\underline{\underline{g}}$ and $\underline{\underline{h}}$ belong to different T-systems. Clearly this method can only be applied to generating 2-vectors. Also, if G is a metabelian group, it can be shown that the order of $[g_1, g_2]$ is the same for every generating 2-vector $\underline{\underline{g}}$. Thus if there were a metabelian group with more than one T-system of generating 2-vectors, this method would fail to distinguish them.

In this chapter a different method is described. If G is an n-generator group such that $G/\mathfrak{V}_k(G) \cong F_n/\mathfrak{V}_k(F_n)$, then $[n,G]$ can be partitioned into disjoint sets called T_k-systems. Each T_k-system is the union of T-systems. The T_k-systems of G can be determined comparatively easily, although some information about $\mathfrak{A}(G)$ is required.

To illustrate the method, the T_p-systems of generating n-vectors of a certain group S are determined (S actually depends on the integer $n \geq 2$, and two primes p and q). It is found that if $p > 3$, S has more than one T_p-system. Since S is always metabelian, taking $n = 2$ provides an example of a 2-generator metabelian group with more than one T-system.

$\underline{T_k\text{-systems}}$

Let A_n be the free abelian group of rank n with generating n-vector $\underline{\underline{a}}$. Let Λ_0 be the group $\{1, -1\}$ under multiplication.

Lemma 2.1. Let β be the automorphism of A_n such that

$$a_i\beta = a_1^{\beta_{i1}} \cdots a_n^{\beta_{in}}, \quad i = 1, 2, \ldots, n,$$

then the mapping τ_0 of $\mathfrak{A}(A_n)$ such that

$$\beta\tau_0 = \mathrm{Det}\,(\beta_{ij})$$

is a homomorphism into Λ_0.

Proof. If the $n \times n$ matrices (β_{ij}), (γ_{ij}) are associated with automorphisms β, γ in the above manner, then it is easily verified that $(\beta_{ij})(\gamma_{ij})$ is associated with $\beta\gamma$. By the multiplicative property of determinants,

$$\beta\tau_0\gamma\tau_0 = (\beta\gamma)\tau_0 . \qquad 2.1.1$$

However if ι is the identity automorphism

$$\iota\tau_0 = 1 . \qquad 2.1.2$$

Thus $\beta\tau_0\beta^{-1}\tau_0 = 1$. Since $\beta\tau_0$ and $\beta^{-1}\tau_0$ are both integers, it follows that τ_0 is a mapping into Λ_0. That τ_0 is a homomorphism follows immediately from 2.1.1 and 2.1.2.

Let

$$\Lambda_k = \{ i \mid (i,k) = 1,\ 1 \leqq i \leqq k \}, \quad k = 2, 3, \ldots,$$

then Λ_k is a group under multiplication mod k. Let

$V_k = \nu_k(A_n)$, for $k \geqq 0$.

<u>Lemma 2.2</u>. Let β be the automorphism of A_n/V_k, $k \geqq 2$, such that

$$a_i V_k \beta = (a_1 V_k)^{\beta_{i1}} \dots (a_n V_k)^{\beta_{in}}, \quad i = 1, 2, \dots, n,$$

then the mapping τ_k of $\mathfrak{A}(A_n/V_k)$, such that

$$\beta\tau_k \equiv \mathrm{Det}\,(\beta_{ij}) \pmod k$$

and

$$1 \leqq \beta\tau_k \leqq k,$$

is a homomorphism of $\mathfrak{A}(A_n/V_k)$ into Λ_k.

The proof is similar to that of Lemma 2.1.

<u>Lemma 2.3</u>. The group A_n/V_k, $k = 0, 1, 2, \dots$, has just one B-class of generating n-vectors.

<u>Proof</u>. If $\underline{\underline{a}}'V_k$ is an arbitrary generating n-vector of A_n/V_k, $R(\underline{\underline{a}}'V_k) = \nu_k(F_n)$; so that $R(\underline{\underline{a}}'V_k) = R(\underline{\underline{a}}V_k)$, and there is an automorphism β of A_n/V_k such that $\underline{\underline{a}}V_k\beta = \underline{\underline{a}}'V_k$.

A group G is said to be a (k,n)-group, for $n = 1, 2, \dots$ $k = 0, 2, 3, \dots$, if

(a) G can be generated by n elements,

and

(b) there is a homomorphism of G onto A_n/V_k.

Let θ be a homomorphism of G onto A_n/V_k, let $\underline{\underline{g}}$ be a generating n-vector of G, then $\underline{\underline{g}}\theta$ is a generating n-vector of A_n/V_k, and by Lemma 2.3, there is an automorphism γ of A_n/V_k such that

$$\underline{\underline{a}}V_k\gamma = \underline{\underline{g}}\theta .$$

Let D_k be the function taking $[n,G]$ into Λ_k, such that

$$D_k(\underline{\underline{g}}) = \gamma\tau_k .$$

In general, D_k will depend on the particular choice of θ, which is called the specified homomorphism.

<u>Lemma 2.4</u>. If θ is an epimorphism of the group H onto the group G, and $\operatorname{Ker}\theta$ is characteristic in H, then to every automorphism β of H there is an induced automorphism β^θ of G such that

$$\beta\theta = \theta\beta^\theta .$$

The mapping $\beta \to \beta^\theta$ is a homomorphism of $\mathfrak{A}(H)$ into $\mathfrak{A}(G)$.

<u>Proof</u>. Let β^θ be defined by

$$h\theta\beta^\theta = h\beta^{\theta}$$

for every $h \in H$. Then β^θ is a mapping of G, for if $h\theta = h'\theta$, where $h, h' \in H$, then $h^{-1}h' \in \operatorname{Ker}\theta$, so $(h^{-1}h')\beta \in \operatorname{Ker}\theta$, since $\operatorname{Ker}\theta$ is characteristic; that is

$(h^{-1}h')\beta\theta = e$, and so $h\theta\beta^\theta = h'\theta\beta^\theta$. The reverse argument shows that β^θ is 1-1. The proof that β^θ is a homomorphism of G onto itself is trivial.

Let $\beta \in \mathcal{A}(H)$, then $\beta\theta = \theta\beta^\theta$, so that

$$\theta(\beta^\theta)^{-1} = \beta^{-1}\theta .$$

But $\beta^{-1}\theta = \theta(\beta^{-1})^\theta$, so that $(\beta^\theta)^{-1} = (\beta^{-1})^\theta$. If $\beta, \gamma \in \mathcal{A}(H)$, then $\beta\gamma\theta = \theta(\beta\gamma)^\theta$. But

$$(\beta\gamma)\theta = \beta\theta\gamma^\theta = \theta\beta^\theta\gamma^\theta ,$$

so that $(\beta\gamma)^\theta = \beta^\theta\gamma^\theta$, and the homomorphism property has been proved.

If G is a (k,n)-group and θ is the specified homomorphism of G onto A_n/V_k , then $\operatorname{Ker}\theta$ is characteristic in G ; for $\operatorname{Ker}\theta = \gamma_k(G)$, which is in fact fully invariant in G . Let B_k be the homomorphism of $\mathcal{A}(G)$ into Λ_k such that

$$B_k(\beta) = \beta^\theta\tau_k \qquad 2.4.1$$

for every $\beta \in \mathcal{A}(G)$.

<u>Lemma 2.5</u>. If $\underline{\underline{g}} \in [n,G]$, and $\beta \in \mathcal{A}(G)$, then

$$D_k(\underline{\underline{g}}\beta) = D_k(\underline{\underline{g}})B_k(\beta) .$$

<u>Proof</u>. If $\underline{\underline{a}}V_k\gamma = \underline{\underline{g}}\theta$, then

$$\underline{\underline{g}}\beta\theta = \underline{\underline{g}}\theta\beta^{\theta} = \underline{\underline{a}}V_k\gamma\beta^{\theta} ,$$

so that

$$D_k(\underline{\underline{g}}\beta) = (\gamma\beta^{\theta})\tau_k = \gamma\tau_k\beta^{\theta}\tau_k = D_k(\underline{\underline{g}})B_k(\beta)$$

Lemma 2.6. The function B_k is independent of the specified homomorphism.

Proof. Let θ, θ' be two homomorphisms of G onto A_n/V_k , and let $\underline{\underline{g}}$ be a generating n-vector of G . Then $R(\underline{\underline{g}}\theta) = R(\underline{\underline{g}}\theta')$, so that there is an automorphism μ of A_n/V_k such that

$$\theta' = \theta\mu . \qquad 2.6.1$$

If β is an automorphism of G , then

$$\beta\theta' = \beta\theta\mu = \theta\beta^{\theta}\mu = \theta'\mu^{-1}\beta^{\theta}\mu ,$$

so that

$$\beta^{\theta'} = \mu^{-1}\beta^{\theta}\mu .$$

But $(\mu^{-1}\beta^{\theta}\mu)\tau_k = \beta^{\theta}\tau_k$, since Λ_k is abelian. This completes the proof of Lemma 2.6.

Lemma 2.7. Every (0,n)-group G is a (k,n)-group for $k = 2, 3, \ldots$, and if B_k is the homomorphism of $\mathfrak{A}(G)$ into Λ_k as above, then

$$\operatorname{Im} B_k \subseteq \{1, k-1\} .$$

Proof. That G is a (k,n)-group for $k = 2, 3, \ldots$ follows immediately from the definition of a (k,n)-group.

Let π be the natural homomorphism of A_n onto A_n/V_k. Let γ be the automorphism of A_n such that

$$a_i\gamma = a_i^{\gamma_{i1}} \ldots a_n^{\gamma_{in}}, \quad i = 1, 2, \ldots, n,$$

then

$$a_iV_k\gamma^\pi = a_i\pi\gamma^\pi = a_i\gamma\pi = (a_iV_k)^{\gamma_{i1}} \ldots (a_nV_k)^{\gamma_{in}}.$$

But by Lemma 2.1,

$$\mathrm{Det}\,(\gamma_{ij}) = \pm 1.$$

Therefore

$$\gamma^\pi\tau_k = 1 \text{ or } k-1.$$

Let θ be a homomorphism of G onto A_n, then by Lemma 2.6, there is no loss of generality in taking $\theta\pi$ as the specified homomorphism of G onto A_n/V_k. If β is an automorphism of G, then

$$B_k(\beta) = \beta^{(\theta\pi)}\tau_k = (\beta^\theta)^\pi\tau_k = 1 \text{ or } k-1,$$

and the lemma is proved.

Theorem 2.8. There is a homomorphism C_k of the group K_n of n-transformations into Λ_k such that

$$D_k(\underline{\underline{g}}\alpha) = D_k(\underline{\underline{g}})C_k(\alpha) \qquad 2.8.1$$

for every generating n-vector $\underline{\underline{g}}$ of the (k,n)-group G, and every $\alpha \in K_n$.

The function C_k is independent of the specified homomorphism, and

$$\operatorname{Im} C_k = \{1, k-1\} .$$

<u>Proof</u>. Let ρ be the isomorphism of $\mathfrak{A}(F_n)$ onto K_n given in Theorem 1.4, and let B_k be the homomorphism of $\mathfrak{A}(F_n)$ into Λ_k defined by 2.4.1. Let C_k be the mapping of K_n into Λ_k defined by

$$C_k(\alpha) = B_k(\alpha\rho^{-1})^{-1} , \quad \alpha \in K_n . \qquad 2.8.2$$

Then C_k is a homomorphism; for B_k and ρ^{-1} are homomorphisms, and the mapping taking every element of Λ_k into its inverse is a homomorphism, since Λ_k is abelian. Let θ be the specified homomorphism of G onto A_n/V_k. If $\varphi_{\underline{\underline{g}}}\theta$ is taken as the specified homomorphism of F_n onto A_n/V_k, then

$$D_k(\underline{\underline{x}}) = D_k(\underline{\underline{g}})$$

and

$$D_k(\underline{\underline{x}}\alpha) = D_k(\underline{\underline{x}}\alpha\varphi_{\underline{\underline{g}}}) = D_k(\underline{\underline{x}}\varphi_{\underline{\underline{g}}}\alpha) = D_k(\underline{\underline{g}}\alpha) .$$

Therefore, in order to prove 2.8.1, it suffices to show that

$$D_k(\underline{\underline{x}}\alpha) = D_k(\underline{\underline{x}})C_k(\alpha) .$$

But $\underline{\underline{x}}\alpha = \underline{\underline{x}}(\alpha\rho^{-1})^{-1}$, and by Lemma 2.5,

$$D_k(\underline{\underline{x}}(\alpha\rho^{-1})^{-1}) = D_k(\underline{\underline{x}})B_k((\alpha\rho^{-1})^{-1}) = D_k(\underline{\underline{x}})C_k(\alpha) .$$

By Lemma 2.6 and 2.8.2, C_k is independent of the specified homomorphism. Since F_n is a $(0,n)$-group, it follows from Lemma 2.7 that

$$\mathrm{Im}\, B_k \subseteq \{1, k-1\} .$$

That $\mathrm{Im}\, C_k \subseteq \{1, k-1\}$ follows immediately from 2.8.2. If α_1 is the n-transformation for which

$$\underline{\underline{x}}\alpha_1 = (x_1^{-1}, x_2, x_3, \ldots, x_n) ,$$

then $C_k(\alpha_1) = k-1$. Thus $\{1, k-1\} \subseteq \mathrm{Im}\, C_k$, and the theorem is proved.

If G is a (k,n)-group, the image of a T-system of generating n-vectors of G under D_k is called a D-class; thus if $\underline{\underline{g}}$ is a fixed element of $[n,G]$,

$$\{D_k(\underline{\underline{g}}\alpha\beta) \mid \alpha \in A, \beta \in B\}$$

is a D-class.

<u>Lemma 2.9</u>. The D-classes are disjoint. In fact they are sets of transitivity of Λ_k under a subgroup of its right

regular representation.

Proof. The lemma follows immediately from Lemma 2.5 and Theorem 2.8

The set of all generating n-vectors which map onto a particular D-class under D_k is called a T_k-system of G.

Theorem 2.10. Each T_k-system is a union of T-systems. The T_k-systems of a (k,n)-group G are independent of the specified homomorphism.

Proof. The first part of the theorem follows immediately from Lemma 2.9.

Let θ, θ' be two homomorphisms of G onto A_n/V_k, and let D_k, D'_k be the associated mappings of $[n,G]$ into Λ_k. Now $\theta' = \theta\mu$ (see 2.6.1) for some automorphism μ of A_n/V_k. If $\underline{\underline{g}} \in [n,G]$ and

$$\underline{\underline{a}}V_k\gamma = \underline{\underline{g}}\,\theta ,$$

then

$$\underline{\underline{a}}V_k\gamma\mu = \underline{\underline{g}}\,\theta' ,$$

so that

$$D'_k(\underline{\underline{g}}) = \gamma\tau_k\mu\tau_k = D_k(\underline{\underline{g}})\mu\tau_k . \qquad 2.10.1$$

But $\mu\tau_k$ is independent of $\underline{\underline{g}}$. It follows that if

$\underline{\underline{g}}$, $\underline{\underline{h}} \in [n,G]$, then $D_k(\underline{\underline{g}}) = D_k(\underline{\underline{h}})$ if and only if $D'_k(\underline{\underline{g}}) = D'_k(\underline{\underline{h}})$.

The theorem follows immediately.

<u>An example</u>

Let p, q $(\neq 1)$ be primes such that p divides $q-1$. There is then an integer r such that

$$r^p \equiv 1 \pmod q ,$$

$$r \not\equiv 1 \pmod q .$$

Let P be the elementary p-group of order p^n with generating n-vector $\underline{\underline{u}}$. Let Q be the elementary q-group of order q^n with generating n-vector $\underline{\underline{v}}$. If μ_i is the automorphism of Q such that

$$\underline{\underline{v}}\mu_i = (v_1, \ldots, v_{i-1}, v_i^r, v_{i+1}, \ldots, v_n) , \quad 1 \leqq i \leqq n ,$$

then $\mu_i^p = \iota$, the identity automorphism, and $\mu_i\mu_j = \mu_j\mu_i$. It follows that the splitting extension S of Q by P can be formed, in which u_i induces the automorphism μ_i on Q ; i.e.,

$$S = gp\{\underline{\underline{u}}, \underline{\underline{v}} \mid u_i^p = v_i^q = [u_i,u_j] = [v_i,v_j] = [u_i,v_j] = e,$$

$$u_i^{-1}v_iu_i = v_i^r , \quad i, j = 1, 2, \ldots, n, i \neq j\} .$$

Let β be an automorphism of S . If

$$Q_i = sgp\{v_i\} , \quad i = 1, 2, \ldots, n ,$$

then the centralizer of Q_i has order $q^n p^{n-1}$, and the Q_i are the only subgroups with this property. Therefore

$$\underline{\underline{v}}\beta = (v_{1\pi}^{\rho_1}, v_{2\pi}^{\rho_2}, \ldots, v_{n\pi}^{\rho_n}) ,$$

where π is a permutation of $\{1, 2, \ldots, n\}$, and $q \nmid \rho_i$, $i = 1, 2, \ldots, n$. Since $[u_i\beta, v_j\beta] = e$, if $i \neq j$,

$$\underline{\underline{u}}\beta \equiv (u_{1\pi}^{\nu_1}, u_{2\pi}^{\nu_2}, \ldots, u_{n\pi}^{\nu_n}) \pmod{Q} , \qquad 2.10.2$$

where $p \nmid \nu_i$, $i = 1, 2, \ldots, n$. Also

$$(u_i\beta)^{-1} v_i\beta\, u_i\beta = (v_i\beta)^r ,$$

i.e.,

$$u_{i\pi}^{-\nu_i} v_{i\pi}^{\rho_i} u_{i\pi}^{\nu_i} = v_{i\pi}^{\rho_i r} , \quad i = 1, 2, \ldots, n .$$

It follows that

$$\rho_i r^{\nu_i} \equiv \rho_i r \pmod{q} .$$

But $q \nmid \rho_i$, therefore

$$r^{\nu_i} \equiv r \pmod{q} ,$$

so that

$$\nu_i \equiv 1 \pmod{p} , \quad i = 1, 2, \ldots, n . \qquad 2.10.3$$

If $n \geq 2$, S is a (p,n)-group. For A_n/V_p is a homomorphic image of S, and S can be generated by n elements, for instance

$$\underline{\underline{s}} = (u_1v_2,\ u_2v_3,\ \ldots,\ u_{n-1}v_n,\ u_nv_1) \qquad 2.10.4$$

is a generating n-vector of S , since

$$\text{sgp}\{u_iv_j\} = \text{sgp}\{u_i,\ v_j\}$$

if $i \neq j$. Let θ be the homomorphism of S onto A_n/V_p such that $\underline{\underline{s}}\theta = \underline{\underline{a}}V_p$. Then, by taking θ as the specified homomorphism, it can be seen from 2.10.2 and 2.10.3 that

$$B_p(\beta) = 1$$

if π is an even permutation, and

$$B_p(\beta) = p-1$$

if π is an odd permutation. Now

$$\underline{\underline{s}}_m = ((u_1v_2)^m,\ u_2v_3,\ \ldots,\ u_{n-1}v_n,\ u_nv_1)\ , \quad 1 \leq m \leq p\ ,$$

is a generating n-vector of S , and

$$D_p(\underline{\underline{s}}_m) = m\ .$$

Thus $\operatorname{Im} D_p = \Lambda_p$. But it has been shown above and in Theorem 2.8 that $\operatorname{Im} B_p = \operatorname{Im} C_p = \{1,\ p-1\}$. Therefore the D-classes of S are the sets

$$\{m,\ p-m\}\ , \quad 1 \leq m \leq p/2\ .$$

It follows that S has $[p/2]$ T_p-systems of generating n-vectors. Therefore, by Theorem 2.10, S has at least $[p/2]$ T-systems of generating n-vectors.

CHAPTER 3

Introduction

In [14] the Neumanns posed the following problem:

A. Let G and H be n-generator groups and let G be a homomorphic image of H. If $\underline{\underline{g}}$ is a generating n-vector of G, does there exist a generating n-vector $\underline{\underline{h}}$ of H, and a homomorphism θ of H onto G such that $\underline{\underline{h}}\theta = \underline{\underline{g}}$?

Gaschütz [6] showed that the answer is yes if the kernel of a homomorphism of H onto G is finite. In this chapter it is shown that the answer is no in some other cases. If G is a finite (k,n)-group with trivial centre, then a group H can be constructed such that every homomorphism of H onto G maps all the generating n-vectors of H into a particular T_k-system of generating n-vectors of G. Examples of such groups are given.

In Theorem 3.8 it is shown that the answer to A is no for a pair of groups to which the above method cannot be applied.

Problem A was originally raised in connection with another problem ([14], Problem 7.3.1):

B. Let G, H be n-generator groups and let G be a homomorphic image of H, is then $U_n(H) \subseteqq U_n(G)$?

A positive answer to A would imply a positive answer to B.

For if to every $\underline{\underline{g}} \in [n,G]$ there is an $\underline{\underline{h}} \in [n,H]$ which can be mapped homomorphically onto $\underline{\underline{g}}$, that is, there exists $\underline{\underline{h}} \in [n,H]$ such that $R(\underline{\underline{h}}) \leq R(\underline{\underline{g}})$, then

$$U_n(H) = \bigcap_{\underline{\underline{h}} \in [n,H]} R(\underline{\underline{h}}) \leq \bigcap_{\underline{\underline{g}} \in [n,G]} R(\underline{\underline{g}}) = U_n(G) .$$

Problem B is equivalent to the following problem:

C. If R is a normal subgroup of F_n, does $U_n(F_n/R)$ contain every hypercharacteristic subgroup of F_n contained in R ?

For suppose the answer to B is yes and S is a hypercharacteristic subgroup of F_n contained in R, then F_n/R is a homomorphic image of F_n/S and so $U_n(F_n/S) \leq U_n(F_n/R)$. But $S = U_n(F_n/S)$, since S is hypercharacteristic in F_n, and so the answer to C is yes. Conversely, if the answer to C is yes and G, H are as in B, then there exist normal subgroups I, J of F_n such that $F_n/I \cong G$, $F_n/J \cong H$ and $J \leq I$. But then $U_n(H) \leq J \leq I$, and so $U_n(H) \leq U_n(F_n/I)$. But $U_n(F_n/I) = U_n(G)$, and so the answer to B is yes.

In the last section of this chapter groups Q and M are constructed such that M is a homomorphic image of Q but $U_2(Q) \not\leq U_2(M)$. I have been unable to show that the answer to B and C is no for $n > 2$.

<u>The inverse images of generating n-vectors under homomorphisms</u>

Let θ be a homomorphism of the group H onto the group G, then a generating n-vector $\underline{g}$ of G is said to have property $P(\theta)$, if there exists a generating n-vector $\underline{h}$ of H such that $\underline{h}\theta = \underline{g}$.

The following theorem is due to Gaschütz ([6], Satz 1), and is stated here for convenience.

<u>Theorem 3.1.</u> If θ is a homomorphism of the n-generator group H onto the group G, and $\operatorname{Ker}\theta$ is finite, then every generating n-vector $\underline{g}$ of G has $P(\theta)$.

By means of a simple counter-example, Gaschütz showed that the theorem is not true if the finiteness condition on $\operatorname{Ker}\theta$ is removed. It is perhaps worth noting in this connection that if θ is a homomorphism of F_n onto G, then $\underline{g}$ has $P(\theta)$ if and only if it belongs to the same A-class as $\underline{x}\theta$. For suppose $\underline{y}\theta = \underline{g}$, where $\underline{y}$ is a generating n-vector of $\cancel{H}$ F_n, then $\underline{x}\alpha = \underline{y}$ for some n-transformation α, and

$$\underline{g} = \underline{y}\theta = \underline{x}\alpha\theta = \underline{x}\theta\alpha :$$

the reverse argument proves the converse.

Let H be a group, then a generating n-vector $\underline{g}$ of the group G is said to have property $P(H)$ if $\underline{g}$ has $P(\theta)$

for some homomorphism θ of H onto G .

Every generating n-vector $\underline{\underline{g}}$ of G has $P(F_n)$, because $\underline{\underline{x}}\varphi_{\underline{\underline{g}}} = \underline{\underline{g}}$.

<u>Lemma 3.2</u>. A generating n-vector $\underline{\underline{g}}$ of G has $P(H)$ if and only if there exists a generating n-vector $\underline{\underline{h}}$ of H such that $R(\underline{\underline{h}}) \leqq R(\underline{\underline{g}})$.

<u>Proof</u>. This is immediate from the definition of $P(H)$.

<u>Lemma 3.3</u>. If $\underline{\underline{g}}$ has $P(H)$, then so has $\underline{\underline{g}}\alpha\beta$, where α is an n-transformation and β an automorphism of G .

<u>Proof</u>. If $\underline{\underline{h}}$ is a generating n-vector of H , and θ a homomorphism of H onto G such that $\underline{\underline{h}}\theta = \underline{\underline{g}}$, then $\underline{\underline{g}}\alpha\beta = \underline{\underline{h}}\theta\alpha\beta = \underline{\underline{h}}\alpha\theta\beta$, by Lemma 1.2. It follows that $\underline{\underline{g}}\alpha\beta$ has $P(\theta\beta)$, proving the lemma.

From Lemma 3.3, it can be seen that the set of all generating n-vectors of G having $P(H)$ is a union of T-systems. It will be shown that this set can be a proper subset of $[n,G]$.

<u>Theorem 3.4</u>. Let H be a $(0,n)$-group, G a (k,n)-group and θ a homomorphism of H onto G , then there exists a $\lambda \in \Lambda_k$ such that $D_k(\underline{\underline{h}}\theta) = \lambda$ or $k-\lambda$, for every generating n-vector $\underline{\underline{h}}$ of H .

Proof. Let φ be the specified homomorphism of G onto A_n/V_k, and φ' a homomorphism of H onto A_n. Now

$$R(\underline{\underline{h}}\theta\varphi) = \nu_k(F_n) \geqq \nu_0(F_n) = R(\underline{\underline{h}}\varphi') .$$

Therefore by Lemma 3.2, there is a homomorphism θ' of A_n onto A_n/V_k such that $\underline{\underline{h}}\theta\varphi = \underline{\underline{h}}\varphi'\theta'$, and so the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] & A_n/V_k \\ H \arrow[u, "\theta"] \arrow[r, "\varphi'"'] & A_n \arrow[u, "\theta'"'] \end{tikzcd}$$

is commutative. Now A_n is a (k,n)-group, and taking θ' as the specified homomorphism of A_n onto A_n/V_k,

$D_k(\underline{\underline{h}}\varphi') = D_k(\underline{\underline{h}}\theta)$. However, by Lemma 2.3,

$$\underline{\underline{h}}\varphi' = \underline{\underline{a}}\gamma$$

for some automorphism γ of A_n. Thus,

$$D_k(\underline{\underline{h}}\varphi') = D_k(\underline{\underline{a}})B_k(\gamma) ,$$

by Lemma 2.5. But by Lemma 2.7, $B_k(\gamma) = 1$ or $k-1$, and the theorem follows by putting $\lambda = D_k(\underline{\underline{a}})$.

Lemma 3.5. If G and H are groups and $\Sigma(H,G)$ consists of the single element N, then a homomorphism θ of H onto G maps a T-system of H into a T-system of G, and the mapping is independent of the particular homomorphism chosen.

Proof. Note that N is hypercharacteristic in H and hence, by Lemma 1.8, characteristic in H. Let β be an automorphism of H, then by Lemma 2.4, there is an automorphism β^θ of G such that $\theta\beta^\theta = \beta\theta$. Let α be an n-transformation and let $\underline{\underline{h}}$ be a generating n-vector of H, then

$$\underline{\underline{h}}\alpha\beta\theta = \underline{\underline{h}}\alpha\theta\beta^\theta = \underline{\underline{h}}\theta\alpha\beta^\theta ,$$

by Lemma 1.2. Clearly, $\underline{\underline{h}}AB$ is mapped into $\underline{\underline{h}}\theta AB$ and the first part of the lemma is proved. If φ is another homomorphism of H onto G, then $\mathrm{Ker}\varphi = \mathrm{Ker}\,\theta = N$, so that there is an automorphism μ of G such that $\varphi = \theta\mu$. But then

$$\underline{\underline{h}}\varphi = \underline{\underline{h}}\theta\mu \in \underline{\underline{h}}\theta B ,$$

and the lemma is proved.

Theorem 3.6. If H is a $(0,n)$-group and G is a (k,n)-group and $\Sigma(H,G)$ has just one element, then every homomorphism of H onto G maps $[n,H]$ into a particular T_k-system of generating n-vectors of G.

Proof. Let θ be a homomorphism of H onto G, then by Theorem 3.4, $[n,H]$ is mapped into a T_k-system of G. But by Lemma 3.5, this T_k-system, which is a union of T-systems, is independent of θ.

It follows that if G, H are as in Theorem 3.6 and G has more than one T_k-system, then the generating n-vectors of G having $P(H)$ will form a proper subset of $[n,G]$.

Example 3.7. There are groups G, H such that

(a) G is a (p,n)-group, where $n \geq 2$, p is a prime, $p > 3$, and G has more than one T_p-system,

(b) H is a $(0,n)$-group,

(c) $\Sigma(H,G)$ consists of just one element.

Details. Let G be the group S constructed at the end of Chapter 2; S satisfies condition (a).

Let $S_i = \mathrm{sgp}\{u_i, v_i\}$, for $i = 1, 2, \ldots, n$, then S is the direct product of the subgroups S_i . If $z = u_i^{\mu} v_i^{\nu}$, then $v_i^z = v_i^{r^{\mu}}$, so that z commutes with v_i only if $z = v_i^{\nu}$. But v_i^{ν} commutes with u_i only if $v_i^{\nu} = e$. Thus $Z(S_i) = E$. But

$$Z(S) = Z(S_1) \times Z(S_2) \times \ldots \times Z(S_n) ,$$

so that S has trivial centre.

Let $R = R(\underline{\underline{s}})$, where $\underline{\underline{s}}$ is the generating n-vector of S given by 2.10.4. Let

$$H = F_n / R \cap \delta(F_n) ,$$

so that H is a $(0,n)$-group. Let θ be a homomorphism of H onto S, then $Z(H)\theta = E$, since $Z(H)\theta \leq Z(H\theta) = Z(S)$. Now $[R,F_n] \leq R$, since R is normal in F_n; also $[R,F_n] \leq \delta(F_n)$. It follows that

$$R/R\cap\delta(F_n) \leq Z(H) \leq \operatorname{Ker}\theta .$$

But $H/(R/R\cap\delta(F_n)) \cong F_n/R \cong S$. Since $H/\operatorname{Ker}\theta \cong S$ and S is finite, it follows that $\operatorname{Ker}\theta = R/R\cap\delta(F_n)$. Thus, condition (c) is satisfied.

There are examples of groups G and H such that the set of generating n-vectors of G having $P(H)$ is not a union of T_k-systems.

Theorem 3.8. Let $\underline{\underline{A}}_5$ be the alternating group of permutations on five symbols. Let W be the free product of a cyclic group of order 2 and a cyclic group of order 3. Then a generating 2-vector $\underline{\underline{h}}'$ of $\underline{\underline{A}}_5$ has $P(W)$ if and only if $\underline{\underline{h}}'$ belongs to the same T-system as $\underline{\underline{h}} = ((12)(34), (135))$.

Proof. Let

$$W = gp\{w_1, w_2 \mid w_1^2 = w_2^3 = e\},$$

then $R(\underline{\underline{h}}) \geq R(\underline{\underline{w}})$, and so, by Lemma 3.2, $\underline{\underline{h}}$ has $P(W)$. By Lemma 3.3, $\underline{\underline{h}}'$ has $P(W)$ if $\underline{\underline{h}}'$ belongs to the same T-system as $\underline{\underline{h}}$.

Conversely, suppose $\underline{\underline{h}}' = \underline{\underline{w}}'\theta$ for some generating 2-vector $\underline{\underline{w}}'$ of W , and some homomorphism θ of W onto $\underline{\underline{A}}_5$. By Gruško's Theorem (see [10], §39), $\underline{\underline{w}}' = \underline{\underline{w}}\alpha$ for some n-transformation α , so that $\underline{\underline{h}}' = \underline{\underline{w}}\alpha\theta = \underline{\underline{w}}\theta\alpha$. But of the representatives of the 19 B-classes of $\underline{\underline{A}}_5$ given in [14], §10, only $\underline{\underline{g}}_{19}$ (= $\underline{\underline{h}}$) has a relation group containing $R(\underline{\underline{w}})$. It follows that $\underline{\underline{w}}\theta = \underline{\underline{h}}\beta$ for some automorphism β of $\underline{\underline{A}}_5$, and $\underline{\underline{h}}' = \underline{\underline{h}}\beta\alpha$, proving the theorem.

The hypercharacteristic subgroups of F_2

Theorem 3.9. There exists a normal subgroup R of F_2 such that $U_2(F_2/R)$ does not contain every hypercharacteristic subgroup of F_2 contained in R .

Proof. Let

$$C = gp\{c_1, \ldots, c_5 \mid c_i^{11} = [c_i, c_j]^{11} = [[c_i, c_j], c_k] = e,$$
$$[c_i, c_j] = e \ (i-j \not\equiv \pm 1 \ (\mathrm{mod}\ 5)),$$
$$1 \leq i, j, k \leq 5\} .$$

Let

$$D' = gp\{d_1, d \mid [d_1, d] = d_1^{11} = d^{11} = e\} ,$$

then D' has an automorphism ρ of order 11 such that

$$d_1\rho = d_1 d$$

$$d\rho = d .$$

Let D be the splitting extension of D' by a cyclic group of order 11 generated by an element d_2 which induces ρ on D'. Then $d = [d_1, d_2]$, and D has the presentation

$$D = \mathrm{gp}\{d_1, \ldots, d_5 \mid d_1^{11} = d_2^{11} = d_3 = d_4 = d_5 = e,$$

$$[[d_1,d_2],d_1] = [[d_1,d_2],d_2] = e,$$

$$[d_1,d_2]^{11} = e\} .$$

Thus $R(\underline{\underline{d}}) \geqq R(\underline{\underline{c}})$, and there is a homomorphism θ of C onto D such that $\underline{\underline{c}}\theta = \underline{\underline{d}}$. It follows that $[c_1, c_2] \neq e$, since $[c_1, c_2]\theta = [d_1, d_2] \neq e$. The 5-vector

$$\underline{\underline{c}}' = (c_2^3,\ c_3^3,\ c_4^3,\ c_5^3,\ c_1^3)$$

is a generating 5-vector of C, and $R(\underline{\underline{c}}') = R(\underline{\underline{c}})$, so that there is an automorphism ν_1 such that $\underline{\underline{c}}\nu_1 = \underline{\underline{c}}'$. Similarly there is an automorphism ν_2 such that

$$\underline{\underline{c}}\nu_2 = (c_1,\ c_2^3,\ c_3^9,\ c_4^5,\ c_5^4) .$$

It is easily verified that $\nu_1^5 = \nu_2^5 = \iota$, the identity automorphism; while if $\pi = \nu_1^{-1}\nu_2^{-1}\nu_1\nu_2$, then

$$\underline{\underline{c}}\pi = (c_1^3,\ c_2^3,\ c_3^3,\ c_4^3,\ c_5^3)$$

and $\pi^5 = \nu_1^{-1}\pi^{-1}\nu_1\pi = \nu_2^{-1}\pi^{-1}\nu_2\pi = \iota$. Let

$$K = \mathrm{gp}\{k_1,\ k_2 \mid k_1^5 = k_2^5 = [k_1,k_2]^5 = [[k_1,k_2],k_i] = e,$$
$$i = 1,\ 2\},$$

then, by the above, there is a homomorphism φ of K into $\mathfrak{A}(C)$ such that $\underline{\underline{k}}\varphi = (\nu_1,\ \nu_2)$. Let M be the splitting extension of C by K in which k_1, k_2 induce on C the respective automorphisms ν_1, ν_2 : i.e.,

$$M = \mathrm{gp}\{c_1,\ \ldots,\ c_5,\ k_1,\ k_2 \mid \text{relns. of } C,\ \text{relns. of } K,$$
$$\underline{\underline{c}}^{k_1} = (c_2^3,\ c_3^3,\ c_4^3,\ c_5^3,\ c_1^3),$$
$$\underline{\underline{c}}^{k_2} = (c_1,\ c_2^3,\ c_3^9,\ c_4^5,\ c_5^4)\}\ .$$

Now $\underline{\underline{m}} = (k_1,\ c_1k_2)$ is a generating 2-vector of M ; for $[c_1,k_2] = e$, so that $\mathrm{sgp}\{c_1k_2\} = \mathrm{sgp}\{c_1,\ k_2\}$; but $C \leq \mathrm{sgp}\{c_1,\ k_1\}$, and the statement is proved. It follows that M is a (5,2)-group. The homomorphism τ such that

$$\underline{\underline{m}}\tau = \underline{\underline{a}}V_5$$

is taken as the specified homomorphism. If $\underline{\underline{g}} \in [2,M]$ and

$$\underline{\underline{g}} \equiv (k_1^{\sigma_1}k_2^{\sigma_2},\ k_1^{\rho_1}k_2^{\rho_2}) \pmod{\delta(M)}\ ,$$

then

$$D_5(\underline{\underline{g}}) \equiv \sigma_1\rho_2 - \sigma_2\rho_1 \quad (\bmod 5) .$$

If g, g', h are elements of a group G and $Z(G) \geq \delta(G)$, then $[gg',h] = [g,h][g',h]$ and $[h,gg'] = [h,g][h,g']$. Now $\delta(K) \leq Z(K)$, and $K \cong M/C$. Therefore

$$[g_1,g_2] \equiv [k_1^{\sigma_1}k_2^{\sigma_2}, k_1^{\rho_1}k_2^{\rho_2}] \quad (\bmod C)$$

$$= [k_1,k_2]^{\sigma_1\rho_2}[k_2,k_1]^{\sigma_2\rho_1}$$

$$= [k_1,k_2]^{\sigma_1\rho_2-\sigma_2\rho_1}$$

$$= [k_1,k_2]^{D_5(\underline{\underline{g}})} .$$

Let $w_1 = [x_1,x_2]^{-1}x_1^5[x_1,x_2]x_1^{-15}$. Let $\underline{\underline{g}} \in [2,M]$ and let $D_5(\underline{\underline{g}}) = 1$. It will be shown that $w_1\varphi_{\underline{\underline{g}}} \in \delta(C)$. Now

$$[x_1,x_2]\varphi_{\underline{\underline{g}}} = [g_1,g_2] = c[k_1,k_2]$$

for some $c \in C$. Therefore

$$w_1\varphi_{\underline{\underline{g}}} = [k_1,k_2]^{-1}g_1^5[g_1^5,c][k_1,k_2]g_1^{-15}$$

Also $g_1^5 \in C$, since K has exponent 5, and $[k_1,k_2]$ induces the automorphism π on C. But π induces on $C/\delta(C)$ the automorphism which maps every element into its cube. Therefore

$$w_1\varphi_{\underline{\underline{g}}} \equiv (g_1^5)\pi g_1^{-15} \equiv e \pmod{\delta(C)} .$$

Let $w_2 = [x_2,x_1]x_2^5[x_1,x_2]x_2^{-20}$. A similar argument to the above shows that $w_2\varphi_{\underline{\underline{g}}} \in \delta(C)$ if $D_5(\underline{\underline{g}}) = 4$. Note that $[k_1,k_2]^4$ induces on $C/\delta(C)$ the automorphism which maps every element into its fourth power. Since K has exponent 5 , $w_1\varphi_{\underline{\underline{g}}}$ and $w_2\varphi_{\underline{\underline{g}}}$ both belong to C regardless of the value of $D_5(\underline{\underline{g}})$. If $D_5(\underline{\underline{g}}) = 1$, $w_1\varphi_{\underline{\underline{g}}} \in \delta(C) \leq Z(C)$, so that

$$[w_1\varphi_{\underline{\underline{g}}}, w_2\varphi_{\underline{\underline{g}}}] = e :$$

similarly, the above equation holds if $D_5(\underline{\underline{g}}) = 4$. Thus

$$w = [w_1,w_2] \in R(\underline{\underline{g}})$$

if $D_5(\underline{\underline{g}}) = 1$ or 4 .

Let $R = R(\underline{\underline{m}})$, let $Q = F_2/R\cap\delta(F_2)$, and let μ be the homomorphism of Q onto $\overset{M}{\cancel{L}}$ such that

$$\underline{\underline{x}}(R\cap\delta(F_2))\mu = \underline{\underline{m}} .$$

Now Q is a $(0,\overset{2}{\cancel{n}})$-group and $D_5(\underline{\underline{m}}) = 1$, so that by Theorem 3.4, $D_5(\underline{\underline{q}}\mu) = 1$ or 4 for every generating 2-vector $\underline{\underline{q}}$ of Q . Thus

$$w\varphi_{\underline{\underline{q}}}\mu = w\varphi_{\underline{\underline{q}}\mu} = e .$$

Also $\delta(Q) = \delta(F_2)/R\cap\delta(F_2) \cong R\delta(F_2)/\overset{R}{\cancel{\delta(F_2)}} \cong \delta(L)$. Therefore, since $\delta(\overset{M}{\cancel{L}})$ is finite, μ maps $\delta(Q)$ isomorphically onto $\delta(\overset{M}{\cancel{L}})$. But $w\varphi_{\underline{\underline{q}}} \in \delta(Q)$ and $w\varphi_{\underline{\underline{q}}}\mu = e$. Therefore $w\varphi_{\underline{\underline{q}}} = e$: that is, $w \in R(\underline{\underline{q}})$ for every generating $\overset{2}{\cancel{n}}$-vector $\underline{\underline{q}}$ of Q , and so $w \in U_2(Q)$.

Let $\underline{\underline{m}}' = (c_2k_1,\ c_1k_2^2)$, then $\underline{\underline{m}}' \in [2,M]$ and

$$\begin{aligned} w_1\varphi_{\underline{\underline{m}}'} &\equiv [k_1,k_2]^{-2}(c_2k_1)^5[k_1,k_2]^2(c_2k_1)^{-15} \pmod{\delta(C)} \\ &= (c_2k_1)^{45}(c_2k_1)^{-15} \\ &= (c_2k_1)^{30} . \end{aligned}$$

But

$$\begin{aligned} (c_2k_1)^5 &= k_1^5k_1^{-5}c_2k_1^5k_1^{-4}c_2k_1^4k_1^{-3}c_2k_1^3k_1^{-2}c_2k_1^2k_1^{-1}c_2k_1 \\ &= c_2c_1^4c_5^5c_4^9c_3^3 , \end{aligned}$$

so that

$$w_1\varphi_{\underline{\underline{m}}'} \equiv c_1^2c_2^6c_3^7c_4^{10}c_5^8 \pmod{\delta(C)} .$$

Similarly,

$$\begin{aligned} w_2\varphi_{\underline{\underline{m}}'} &\equiv [k_1,k_2]^{-2}(c_1k_2^2)^5[k_1,k_2]^2(c_1k_2^2)^{-20} \pmod{\delta(C)} \\ &= (c_1k_2^2)^{25} \\ &= c_1^3 . \end{aligned}$$

Therefore

$$\begin{aligned} w\varphi_{\underline{\underline{m}}'} &= [c_1^2 c_2^6 c_3^7 c_4^{10} c_5^8, c_1^3] \\ &= [c_2^6, c_1^3]\,[c_5^8, c_1^3] \\ &= [c_1, c_2]^4\,[c_5, c_1]^2 \\ &\neq e \, . \end{aligned}$$

Thus $w \notin R(\underline{\underline{m}}')$ and so $w \notin U_2(M)$. Since $w \in U_2(Q)$, it follows that $U_2(Q) \nleq U_2(M) = U_2(F_2/R)$. But $U_2(Q) \leq R \cap \delta(F_2)$, and so the theorem has been proved.

<u>Corollary 3.10</u>. The union of two hypercharacteristic subgroups of F_2 is not necessarily hypercharacteristic in F_2 .

<u>Proof</u>. The group $M \cong F_2/R$ constructed above is finite. Therefore ([14], Satz 7.6) $F_2/U_2(M)$ is finite. Suppose H is a hypercharacteristic subgroup of F_2 such that

$$R \geqq H \geqq U_2(M) ,$$

then by Theorem 3.1, every element of $\Sigma(F_2,M)$ contains an element of $\Sigma(F_2,F_2/H)$. Therefore

$$U_2(M) \geqq U_2(F_2/H) .$$

But, since H is hypercharacteristic in F_2 , every element of $\Sigma(F_2,F_2/H)$ contains H . Therefore

$$H = U_2(F_2/H) ,$$

and so $U_2(M) = H$. It has been shown that $U_2(M)$ is a maximal hypercharacteristic subgroup of F_2 contained in R . But $U_2(Q)U_2(M)$ is contained in R and properly contains $U_2(M)$. Therefore $U_2(Q)U_2(M)$ is not hypercharacteristic in F_2 .

CHAPTER 4

Introduction

The n-transformations play quite an important role in group theory: for instance Gruško's Theorem (see [10], §39) states that if $\underline{\underline{g}}$ is a generating n-vector of a free product, then there exists an n-transformation α such that every component of $\underline{\underline{g}}\alpha$ belongs to one of the free factors. It is therefore of some interest to investigate to what extent n-transformations are transitive on the generating n-vectors of an arbitrary group. In this connection the following problem is posed:

D. If G is an m-generator group and $\underline{\underline{g}} \in [n,G]$, where $n > m$, does there exist an n-transformation α such that each of the first $n-m$ components of $\underline{\underline{g}}\alpha$ is e?

This is equivalent to the following problem:

E. If $R \lhd F_n$ and F_n/R has m generators, does there exist a generating n-vector of F_n such that $n-m$ of its components belong to R?

A negative answer to D. would in turn provide a positive answer to the following question:

F. If G is an n-generator group, can G have more than one T-system of generating (n+1)-vectors?

Theorem 4.1 shows that the answer to D is yes if G is soluble and its derived group is finite. However I think it very unlikely that the answer to D is yes in general: I suspect that the finite direct products of simple groups would provide counter-examples.

Theorem 4.2 gives a complete description of the A-classes of finitely generated abelian groups. The theorem is closely related to some results of Liebeck [11].

The third section of the chapter is devoted to the following problem:

G. If $R, S \triangleleft F_n$ and $F_n/R \cong F_n/S$, under what condition

is $F_n/R\cap\delta(F_n) \cong F_n/S\cap\delta(F_n)$?

A sufficiency condition is found for the case when $\delta(F_n/R)$ is finite. This condition is shown to be necessary when $Z(F_n/R) = E$. The results for $F_n/R\cap\delta(F_n)$ can be fairly easily extended to the group $F_n/[F_n,R]$.

Finally, using some of the previous results of this chapter, a description is obtained of the T-systems of a rather restricted class of metabelian groups.

<u>The A-classes of soluble groups</u>

If π is a permutation of the set $\{1, 2, \ldots, n\}$, then α_π will denote the n-transformation such that

$$\underline{\underline{x}}\alpha_\pi = (x_{1\pi}, x_{2\pi}, \ldots, x_{n\pi}) .$$

If $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, then $\alpha_{-i}, \alpha_{i:j}$ will denote the n-transformations for which

$$\underline{\underline{x}}\alpha_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_i^{-1}, x_{i+1}, \ldots, x_n)$$

$$\underline{\underline{x}}\alpha_{i:j} = (x_1, x_2, \ldots, x_{j-1}, x_i x_j, x_{j+1}, \ldots, x_n) .$$

These n-transformations generate the group K_n of all n-transformations (see [9], p.111).

<u>Theorem 4.1</u>. Let G be a soluble group with a finite derived group. If G can be generated by $n-1$ elements, then G has just one A-class of generating n-vectors.

<u>Proof</u>. The group G possesses a finite normal subgroup G^* such that G/G^* is free abelian of rank $m < n$. The proof of the theorem is by induction on the length c of a principal series of G-admissible subgroups of G^*. If $c = 0$, then G is free abelian of rank m. Let $\underline{\underline{h}} \in [m,G]$, $\underline{\underline{g}} \in [n,G]$, and let A_n be the free abelian group of rank n with generating n-vector $\underline{\underline{a}}$. There is a homomorphism θ of A_n onto G such that

$$\underline{\underline{a}}\theta = \underline{\underline{g}} .$$

But since G is free abelian, there is a homomorphism φ of G into A_n such that

$$\underline{\underline{h}}\varphi = \underline{\underline{b}} ,$$

where $\underline{\underline{b}}$ is an arbitrary m-vector of A_n . Choose $\underline{\underline{b}}$ so that

$$b_i\theta = h_i , \quad i = 1, 2, \ldots, m ,$$

then $\varphi\theta$ is the identity automorphism. Also, if $g \in G$,

$$g = g(g\theta\varphi)^{-1} g\theta\varphi .$$

However $g(g\theta\varphi)^{-1} \in \operatorname{Ker}\theta$ and $g\theta\varphi \in \operatorname{Im}\varphi$. Thus

$$G = \operatorname{Ker}\theta \times \operatorname{Im}\varphi ,$$

since $\operatorname{Im}\varphi \cap \operatorname{Ker}\theta = E$.

It is now clear that $\operatorname{Ker}\theta$ is free abelian of rank n-m and that if $\underline{\underline{d}}$ is a generating (n-m)-vector of $\operatorname{Ker}\theta$, then

$$\underline{\underline{f}} = (h_1\varphi, h_2\varphi, \ldots, h_m\varphi, d_1, \ldots, d_{n-m})$$

is a generating n-vector of G . But A_n has just one A-class of generating n-vectors (see [2], p.90), so that there is an n-transformation α such that $\underline{\underline{f}} = \underline{\underline{a}}^{\alpha}$. Therefore

$$\underline{\underline{f}}\theta = \underline{\underline{a}}\alpha\theta = \underline{\underline{a}}\theta\alpha = \underline{\underline{g}}^{\alpha} .$$

But $\underline{\underline{f}}\theta = (h_1, h_2, \ldots, h_m, e, e, \ldots, e)$, so that every element of $[n,G]$ belongs to the same A-class as $(h_1, h_2, \ldots, h_m, e, e, \ldots, e)$, and the theorem has been proved for $c = 0$.

Suppose now that

$$E = M_0 < M_1 < M_2 < \dots < M_c = G^* ,$$

where M_i/M_{i-1} is a minimal normal subgroup of G/M_{i-1}, for $i = 1, 2, \dots, c$. Let $\underline{\underline{h}} \in [n-1,G]$, $\underline{\underline{g}} \in [n,G]$, then by the induction hypothesis there exists $\underline{\underline{g}}' \in \underline{\underline{g}}A$ such that

$$\underline{\underline{g}}' \equiv (e, h_1, h_2, \dots, h_{n-1}) \pmod{M_1} \qquad 4.1.1$$

If $g_1' = e$, then $G = \mathrm{sgp}\{g_2', \dots, g_n'\}$ and, by operating on $\underline{\underline{g}}'$ by a product of the n-transformations $\alpha_{j:1}$ and their inverses a generating n-vector of G is obtained satisfying 4.1.1 and whose first component is non-trivial. It will be assumed, therefore, that $e \neq g_1' = m \in M_1$. Let $G' = \mathrm{sgp}\{g_2', \dots, g_n'\}$; if $g \in G$, then $g = \hat{m}\hat{g}$ for $\hat{m} \in M_1$, $\hat{g} \in G'$. Therefore, since M_1 is abelian, $m^g = m^{\hat{g}}$. Clearly $\underline{\underline{g}}' = (m, g_2', \dots, g_n')$ can be transformed into $(\hat{g}^{-1}m\hat{g}, g_2', \dots, g_n')$ by an n-transformation. Therefore, for every $g \in G$, there is an n-transformation taking $\underline{\underline{g}}'$ into $\underline{\underline{g}}'' = (m^g, g_2', \dots, g_n')$. But

$$\underline{\underline{g}}'\alpha_{1:2} = (m^g, m^g g_2', g_3', \dots, g_n')$$

and

$$\underline{\underline{g}}''\alpha_{1:2}^{-1} = (m^g, m^{-g}g_2', g_3', \ldots, g_n') ,$$

and since both these n-vectors satisfy 4.1.1, their first components can be transformed by g^{-1} by means of an n-transformation; i.e., $\underline{\underline{g}}''$ belongs to the same A-class as $(m, m^g g_2', g_3', \ldots, g_n')$ or $(m, m^{-g}g_2', g_3', \ldots, g_n')$. It follows that multiplying g_2' by a product of conjugates of m and its inverse can be achieved by an n-transformation. But since M_1 is a minimal normal subgroup of G , every element of M_1 is a product of conjugates of m and its inverse. Since $h_1 = m^*g_2'$ for some $m^* \in M_1$, $\underline{\underline{g}}$ belongs to the same A-class as $(m, h_1, g_3', \ldots, g_n')$. By extending this process, it is easily seen that $\underline{\underline{g}}$ belongs to the same A-class as $(m, h_1, h_2, \ldots, h_{n-1})$. However $\underline{\underline{h}}$ is a generating (n-1)-vector of G , so that a product of the $\alpha_{j:1}$'s and their inverses will transform the above n-vector into $(e, h_1, h_2, \ldots, h_{n-1})$. Thus every generating n-vector of G belongs to the same A-class as $(e, h_1, h_2, \ldots, h_{n-1})$, and the theorem is proved.

Theorem 4.2. If G is an abelian group and n is the minimal number of generators of G, then G is a (k,n)-group for some $k = 2, 3, \ldots$.

If G is a $(0,n)$-group, let $k^* = 0$, if not, let k^* be the largest k for which G is a (k,n)-group, then the sets

$$S_\lambda = \{\underline{\underline{g}} \mid \underline{\underline{g}} \in [n,G],\ D_{k^*}(\underline{\underline{g}}) = \lambda \text{ or } k^* - \lambda\},\quad \lambda \in \Lambda_{k^*},$$

are the A-classes of generating n-vectors of G.

Proof. The first part of the theorem follows immediately from the elementary divisor theorem (see [10], §20). From that theorem, it also follows that there is a generating n-vector $\underline{\underline{h}}$ of G such that $h_1^{k^*} = e$. The homomorphism θ such that $\underline{\underline{h}}\theta = \underline{\underline{a}}V_{k^*}$ is taken as the specified homomorphism. Let $\underline{\underline{g}} \in S_\lambda$. Now $G/\mathrm{sgp}\{h_1\}$ is a soluble group with a finite derived group and $n-1$ generators. Therefore, by Theorem 4.1, there exists $\underline{\underline{g}}' \in \underline{\underline{g}}A$ such that

$$\underline{\underline{g}}' \equiv (e, h_2, h_3, \ldots, h_n) \pmod{\mathrm{sgp}\{h_1\}}.$$

Now $G/\mathrm{sgp}\{h_1\}$ has $n-1$ generators, and $\mathrm{sgp}\{h_1\}$ is minimal with respect to this property. But $g_1' \in \mathrm{sgp}\{h_1\}$ and $G/\mathrm{sgp}\{g_1'\}$ has $n-1$ generators. Therefore $\mathrm{sgp}\{h_1\} = \mathrm{sgp}\{g_1'\}$. It follows that by applying a product of the $\alpha_{1:j}$'s and their inverses to $\underline{\underline{g}}'$, an n-vector $\underline{\underline{g}}'' \in \underline{\underline{g}}A$ is obtained such that

$$\underline{\underline{g}}'' = (h_1^{\mu}, h_2, h_3, \dots, h_n) .$$

Now $D_{k*}(\underline{\underline{g}}'') = \mu$. But, by Theorem 2.8, $D_{k*}(\underline{\underline{g}}'') = \lambda$ or $k^*-\lambda$. It follows that $\mu = \lambda$ or $k^*-\lambda$, and $\underline{\underline{g}}$ belongs to the same A-class as $(h_1^{\lambda}, h_2, h_3, \dots, h_n)$. Thus each S_{λ} is contained in an A-class. On the other hand, by Theorem 2.8, S_{λ} is a union of A-classes. Thus the theorem has been proved for a particular choice of the specified homomorphism. However, it follows from 2.10.1 that the S_{λ} will only be permuted if the specified homomorphism is changed, and so the theorem follows.

<u>The group $F_n/R\cap\delta(F_n)$</u>

First a useful lemma will be proved. This lemma is probably well known.

<u>Lemma 4.3</u>. If G is a group and K a subgroup such that $KZ(G) = G$ and $K\delta(G) = G$, then $K = G$.

<u>Proof</u>. Let $g \in G$, then $g = kz$ for $k \in K$, $z \in Z(G)$, and $g^{-1}Kg = z^{-1}Kz = K$. Thus every element of G transforms K into itself, that is $K \lhd G$. Now

$$G/K = KZ(G)/K \cong Z(G)/Z(G)\cap K .$$

But $Z(G)$ is abelian, so that G/K is abelian. Therefore

$K \geqq \delta(G)$, and $G = K\delta(G) = K$.

For the rest of this section G will denote an n-generator group with finite derived group, and

$$H = F_n/R \cap \delta(F_n) ,$$

where $R \in \Sigma(F_n, G)$.

Lemma 4.4. If $\underline{\underline{g}} \in [n,G]$, then $F_n/R(\underline{\underline{g}}) \cap \delta(F_n) \cong H$ if and only if $\underline{\underline{g}}$ has $P(H)$.

Proof. If $\underline{\underline{g}}$ has $P(H)$, then by Lemma 3.2, there exists $N \in \Sigma(F_n, H)$ such that $N \leqq R(\underline{\underline{g}})$. But H is a $(0,n)$-group, therefore $N \leqq \delta(F_n)$. Now

$$\delta(F_n)/N \cong \delta(H) = \delta(F_n)/R \cap \delta(F_n) \cong \delta(F_n)R/R \cong \delta(G) \tag{4.4.1}$$

and

$$\delta(F_n)/\delta(F_n) \cap R(\underline{\underline{g}}) \cong \delta(F_n)R(\underline{\underline{g}})/R(\underline{\underline{g}}) \cong \delta(G) .$$

But $\delta(G)$ is finite, so that N and $\delta(F_n) \cap R(\underline{\underline{g}})$ both have the same finite index in $\delta(F_n)$. Thus, since $N \leqq \delta(F_n) \cap R(\underline{\underline{g}})$, it follows that

$$N = \delta(F_n) \cap R(\underline{\underline{g}}) ,$$

and so $F_n/R(\underline{\underline{g}}) \cap \delta(F_n) \cong H$.

Conversely, if $H \cong F_n/R(\underline{\underline{g}}) \cap \delta(F_n)$, then

$R(\underline{g}) \cap \delta(F_n) \in \Sigma(F_n, H)$, and $\underline{g}$ has $P(H)$ by Lemma 3.2.

Lemma 4.5. If θ is a homomorphism of H onto G , then $\operatorname{Ker}\theta \leqq Z(H)$.

Proof. Let $\underline{h}$ be a generating n-vector of H such that $R(\underline{h}) = R \cap \delta(F_n)$. Then $R(\underline{h}\theta) \geq R(\underline{h})$, and, as in the proof of Lemma 4.4, $R(\underline{h}) = R(\underline{h}\theta) \cap \delta(F_n)$. Now

$$[\operatorname{Ker}\theta, H] \equiv [R(\underline{h}\theta), F_n] \pmod{R \cap \delta(F_n)}.$$

But

$$[R(\underline{h}\theta), F_n] \leqq R(\underline{h}\theta) \cap \delta(F_n) ,$$

since $R(\underline{h}\theta) \triangleleft F_n$. Therefore, since

$$R(\underline{h}\theta) \cap \delta(F_n) = R \cap \delta(F_n) , \qquad 4.5.1$$

$$[\operatorname{Ker}\theta, H] = E ,$$

and the lemma is proved.

Let θ be a homomorphism of H onto G , let π, $\hat{\pi}$ be the natural homomorphisms of G onto $G/\delta(G)$ and H onto $H/\delta(H)$ respectively. Then, since $H/\delta(H)$ is free abelian, there is a homomorphism $\hat{\theta}$ of $H/\delta(H)$ onto $G/\delta(G)$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\pi"] & G/\delta(G) \\ H \arrow[u, "\theta"] \arrow[r, "\hat{\pi}"'] & H/\delta(H) \arrow[u, "\hat{\theta}"'] \end{tikzcd}$$

is commutative, i.e., $\theta\pi = \hat{\pi}\hat{\theta}$.

<u>Lemma 4.6</u>. If $\mathbf{g} \in [n,G]$, then $\mathbf{g}$ has $P(\theta)$ if and only if $\mathbf{g}\pi$ has $P(\hat{\theta})$.

<u>Proof</u>. If $\mathbf{g}$ has $P(\theta)$, there exists $\mathbf{h} \in [n,H]$ such that $\mathbf{h}\theta = \mathbf{g}$. But then $\mathbf{g}\pi = \mathbf{h}\theta\pi = \mathbf{h}\hat{\pi}\hat{\theta}$, so that $\mathbf{g}\pi$ has $P(\hat{\theta})$.

Conversely, let $\cancel{[}\mathbf{m} \in [n,H/\delta(H)]$ be such that delete [

$$\mathbf{m}\theta = \mathbf{g}\pi .$$

Define sets S, S' as follows

$$S = \{\mathbf{s} \mid \mathbf{s} \in (n,G),\ \mathbf{s}\pi = \mathbf{g}\pi\} .$$

$$S' = \{\mathbf{t} \mid \mathbf{t} \in (n,H),\ \mathbf{t}\hat{\pi} = \mathbf{m}\} .$$

Now $|S| = |\mathrm{Ker}\,\pi|^n = |\delta(G)|^n$, and $|S'| = |\mathrm{Ker}\,\hat{\pi}|^n = |\delta(H)|^n$. But $\delta(G) \cong \delta(H)$ by 4.3.1, so that $|S| = |S'|$. If $\mathbf{t} \in S'$, then $\mathbf{t}\theta \in S$, for

$$\mathbf{t}\theta\pi = \mathbf{t}\hat{\pi}\hat{\theta} = \mathbf{m}\hat{\theta} = \mathbf{g}\pi .$$

Also if $\mathbf{t}$, $\mathbf{t}' \in S'$ and $\mathbf{t}\theta = \mathbf{t}'\theta$, then

$$t_i\theta = t'_i\theta ,\quad i = 1, 2, \ldots, n ,$$

so that

$$t'_i t_i^{-1} \in \mathrm{Ker}\ \theta .$$

But

$$t_i\hat{\pi} \;=\; t'_i\hat{\pi} \;=\; m_i \;, \quad i = 1, 2, \ldots, n \;,$$

so that

$$t'_i t_i^{-1} \in \operatorname{Ker}\hat{\pi} \;=\; \delta(H) \;, \quad i = 1, 2, \ldots, n \;.$$

However, by 4.5.1, $\delta(H) \cap \operatorname{Ker}\theta = E$, so that

$$t'_i \;=\; t_i \;, \quad i = 1, 2, \ldots, n \;,$$

that is $\underline{\underline{t}}' = \underline{\underline{t}}$. Thus θ acts as a 1-1 mapping of S' into S . But $|S'| = |S|$, and so the mapping is also onto. Therefore, since $\underline{\underline{g}} \in S$, there exists $\underline{\underline{h}} \in S'$, such that $\underline{\underline{h}}\theta = \underline{\underline{g}}$. It remains to show that $\underline{\underline{h}} \in [n,H]$. Let $K = \operatorname{sgp}\{\underline{\underline{h}}\}$, then since $\underline{\underline{h}}\theta \in [n,G]$,

$$K \operatorname{Ker}\theta \;=\; H \;,$$

so that, by Lemma 4.5, $KZ(H) = H$. Also $\underline{\underline{h}}\hat{\pi} \in [n, H/\delta(H)]$, so that $K\delta(H) = H$. Therefore, by Lemma 4.3, $K = H$, and the lemma is proved.

<u>Lemma 4.7</u>. If G is not a (k,n)-group for every $k = 2, 3, \ldots$, and θ is a homomorphism of H onto G , then every $\underline{\underline{g}} \in [n,G]$ has $P(\theta)$. If G is a (k,n)-group for some $k = 2, 3, \ldots$, and k^* is defined as in Theorem 4.2, then the set of elements of $[n,G]$ having $P(H)$ is a union of T_{k^*}-systems. In fact if $\underline{\underline{g}}'$ has $P(\theta)$ and $\underline{\underline{g}}$ belongs to the same T_{k^*}-system as $\underline{\underline{g}}'$, then there exists an auto-

morphism β of G such that $\underline{\underline{g}}$ has $P(\theta\beta)$.

Proof. Let $\underline{\underline{g}}, \underline{\underline{g}}' \in [n,G]$, and let $\underline{\underline{g}}'$ have $P(\theta)$.

If G is not a (k,n)-group for every $k \geq 2$, then by Theorem 4.2, $G/\delta(G)$ has $n-1$ generators. Therefore, by Theorem 4.1, there exists an n-transformation α such that $\underline{\underline{g}}'\pi\alpha = \underline{\underline{g}}\pi$.

Let $\underline{\underline{h}}' \in [n,H]$ be such that $\underline{\underline{h}}'\theta = \underline{\underline{g}}'$, then $\underline{\underline{h}}'\theta\pi\alpha = \underline{\underline{g}}\pi$.

But

$$\underline{\underline{h}}'\theta\pi\alpha = \underline{\underline{h}}'\hat{\pi}\hat{\theta}\alpha = \underline{\underline{h}}'\hat{\pi}\alpha\hat{\theta} , \qquad 4.7.1$$

so that $\underline{\underline{g}}\pi$ has $P(\hat{\theta})$, and the first part of the lemma follows from Lemma 4.6.

Let G be a (k,n)-group for some $k = 2, 3, \ldots$, and let $\underline{\underline{g}}$ belong to the same T_{k*}-system as $\underline{\underline{g}}'$. By Theorem 2.8, there exists an automorphism β of G such that

$$D_{k*}(\underline{\underline{g}}\beta^{-1}) = D_{k*}(\underline{\underline{g}}') \text{ or } k* - D_{k*}(\underline{\underline{g}}') .$$

It follows from Theorem 4.2 that there is an n-transformation α such that $\underline{\underline{g}}'\pi\alpha = \underline{\underline{g}}\beta^{-1}\pi$.

But from 4.7.1, it follows that $\underline{\underline{g}}'\pi\alpha$ has $P(\hat{\theta})$, so that by Lemma 4.6, $\underline{\underline{g}}\beta^{-1}$ has $P(\theta)$, and so $\underline{\underline{g}}$ has $P(\theta\beta)$, proving the lemma.

Let θ be a homomorphism of H onto G . If G has trivial centre, then $Z(H)\theta = Z(H\theta) = E$, and so $Z(H) \leqq \operatorname{Ker}\theta$. But by Lemma 4.5, $\operatorname{Ker}\theta \leqq Z(H)$. It follows that

$$\operatorname{Ker}\theta = Z(H), \qquad 4.7.2$$

i.e., $\Sigma(H,G)$ consists of just the one normal subgroup $Z(H)$. Hence, by Theorem 3.6, the set of generating n-vectors of G having $P(H)$ is contained in a T_k-system, if G is a (k,n)-group. Therefore, combining this result with ~~Theorem~~ Lemma 4.7 and Lemma 4.3, the following theorem is obtained:

Theorem 4.8. If G is an n-generator group with a finite derived group and $\underline{\underline{g}}, \underline{\underline{g}}' \in [n,G]$, then

$$F_n/R(\underline{\underline{g}})\cap\delta(F_n) \cong F_n/R(\underline{\underline{g}}')\cap\delta(F_n) \qquad 4.8.1$$

if G is not a (k,n)-group. If G is a (k,n)-group for some $k = 2, 3, \ldots$, and k^* is defined as in Theorem 4.2, then 4.8.1 is satisfied if $\underline{\underline{g}}$ and $\underline{\underline{g}}'$ belong to the same T_{k^*}-system. This condition is necessary if G has trivial centre.

The T-systems of some metabelian groups

A group is called metabelian if its derived group is abelian.

Lemma 4.9. If H is a metabelian $(0,2)$-group and $\underline{\underline{h}}, \underline{\underline{h}}'$ are generating 2-vectors of H such that

$$\underline{\underline{h}}\delta(H) = \underline{\underline{h}}'\delta(H),$$

then there is an automorphism γ of H such that

$$\underline{\underline{h}}\gamma = \underline{\underline{h}}' .$$

Proof. Let J denote the group-ring of $H/\delta(H)$ over the integers. Just for the proof of this lemma $\delta(H)$ will be regarded as an additive right J-module. If $a = [h_1, h_2]$, then every element of $\delta(H)$ is a sum of conjugates of a and $-a$, that is $\delta(H) = aJ$. In particular

$$a' = [h_1', h_2'] = aj$$

for some $j \in J$.

Let $u \in F_2$, then, since $\underline{\underline{h}}\delta(H) = \underline{\underline{h}}'\delta(H)$,

$$u\varphi_{\underline{\underline{h}}}\delta(H) = u\varphi_{\underline{\underline{h}}'}\delta(H) . \qquad 4.9.1$$

Put $u^* = u\varphi_{\underline{\underline{h}}}\delta(H)$, then

$$(u^{-1}[x_1, x_2]u)\varphi_{\underline{\underline{h}}} = au^* ,$$

$$(u^{-1}[x_1, x_2]u)\varphi_{\underline{\underline{h}}'} = a'u^*$$

by 4.9.1. It follows that if $w \in \delta(F_2)$, so that

$$w\varphi_{\underline{\underline{h}}} = aj^*$$

for some $j^* \in J$, then

$$w\varphi_{\underline{\underline{h}}'} = a'j^* .$$

But

$$a'j^* = ajj^* = aj^*j \; ,$$

since J is commutative. Therefore $aj^* = 0$ implies $a'j^* = 0$; that is $w\varphi_{\underline{\underline{h}}} = e$ implies $w\varphi_{\underline{\underline{h}}'} = e$, if $w \in \delta(F_2)$. Since H is a $(0,2)$-group, every element of $R(\underline{\underline{h}})$ belongs to $\delta(F_2)$. Therefore $R(\underline{\underline{h}}) \leqq R(\underline{\underline{h}}')$. Similarly $R(\underline{\underline{h}}') \leqq R(\underline{\underline{h}})$, and the lemma follows immediately.

<u>Theorem 4.10</u>. A metabelian $(0,2)$-group H has just one T-system of generating 2-vectors.

<u>Proof</u>. Let $\underline{\underline{h}}, \hat{\underline{\underline{h}}} \in [2,H]$. Now $H/\delta(H)$ has just one A-class of generating 2-vectors (see [2], p.90). Therefore there exists a 2-transformation α such that

$$\hat{\underline{\underline{h}}}\alpha \equiv \underline{\underline{h}} \pmod{\delta(H)} .$$

The theorem follows immediately from Lemma 4.9.

<u>Theorem 4.11</u>. Let G be a finite 2-generator metabelian group with trivial centre. If G is not a $(k,2)$-group for every $k \geqq 2$, then G has just one T-system of generating 2-vectors. If G is a $(k,2)$-group for some $k \geqq 2$, and k^* is the largest integer for which this is so, then every T_{k^*}-system is a T-system of generating 2-vectors of G .

<u>Proof</u>. Let $\underline{\underline{g}}, \underline{\underline{g}}'$ belong to the same T_{k^*}-system, if G is a $(k,2)$-group for some $k \geqq 2$; if not, let $\underline{\underline{g}}, \underline{\underline{g}}'$

be arbitrary elements of $[2,G]$. Let

$$H = F_2/R(\underline{\underline{g}}) \cap \delta(F_2) ,$$

and let θ be a homomorphism of H onto G such that $\underline{\underline{g}}$ has $P(\theta)$. Then by Lemma 4.7, there is an automorphism β of G such that $\underline{\underline{g}}'\beta$ has $P(\theta)$. Let $\underline{\underline{h}}, \hat{\underline{\underline{h}}} \in [2,H]$ be such that

$$\underline{\underline{h}}\theta = \underline{\underline{g}} ,$$

$$\hat{\underline{\underline{h}}}\theta = \underline{\underline{g}}'\beta .$$

By Theorem 4.10, there is a 2-transformation α and an automorphism γ of H , such that

$$\underline{\underline{h}}\alpha\gamma = \hat{\underline{\underline{h}}} .$$

Now by 4.7.2, $\operatorname{Ker}\theta = Z(H)$, which is a characteristic subgroup of H . Therefore by Lemmas 2.4 and 1.2,

$$\hat{\underline{\underline{h}}}\theta = \underline{\underline{h}}\alpha\gamma\theta = \underline{\underline{h}}\alpha\theta\gamma^{\theta} = \underline{\underline{h}}\theta\alpha\gamma^{\theta} ;$$

that is,

$$\underline{\underline{g}}'\beta = \underline{\underline{g}}\alpha\gamma^{\theta} ,$$

and the theorem is proved.

CHAPTER 5

Introduction

Let F be a free group and v a word subgroup function. If R is a normal subgroup of F, then so is $v(R)$. This chapter is devoted to an investigation of what properties of F/R are inherited by $F/v(R)$. Firstly, the following problem is discussed:

H. If $F/R \cong F/S$, is then $F/v(R) \cong F/v(S)$?

Gaschütz [5] showed that the answer is yes for a particular word subgroup function, if F is finitely generated and F/R is finite. Using a similar technique to that of Gaschütz, it is shown in Theorem 5.3 that the answer to H is yes if $R/v(R)$ is finite.

However the answer to H is no for some other word subgroup functions. In particular if F_n/R is a (k,n)-group with more than one T_k-system, then there exists a normal subgroup S of F_n such that $F_n/R \cong F_n/S$, but $F_n/\delta(R) \ncong F_n/\delta(S)$. This fact is a consequence of Theorem 5.4. I tried unsuccessfully to extend this result to the case when F_n/R has more than one T-system of generating n-vectors. In particular, if $\underline{\underline{g}}$, $\underline{\underline{h}}$ are representatives of the two T-systems of generating 2-vectors of $\underline{\underline{A}}_5$ (the alternating group on 5 symbols), it is an unsolved question whether $F_2/\delta(R(\underline{\underline{g}})) \cong F_2/\delta(R(\underline{\underline{h}}))$.

Recently Baumslag [1] proved that if F/R and $R/v(R)$ are residually finite (or of p-power order), then $F/v(R)$ has the same property. This result generalizes a theorem of Gruenberg ([7], Theorem 7.1). Baumslag's result is obtained here (Theorem 5.11) as a fairly immediate consequence of Theorem 5.10, which, I think, is of some independent interest. Theorem 5.10 could be proved using techniques similar to those used by Takahasi in [16], but I think a more interesting approach is provided if Schreier systems are used, as here.

<u>The isomorphism properties of $F/v(R)$</u>

Let v be a non-trivial word subgroup function as described at the end of Chapter 1.

The following theorem has recently been proved by Peter M. Neumann [15].

<u>Theorem 5.1</u>. If S and T are normal subgroups of a non-abelian free group, then $v(S) \leqq v(T)$ implies $S \leqq T$. Hence $v(S) = v(T)$ only if $S = T$.

A group G is called a Hopf group if G is not isomorphic to any of its proper factor groups.

Let G be an n-generator Hopf group, $n \geq 2$, and let

$$H = F_n/v(R) ,$$

where $R \in \Sigma(F_n, G)$. Let π be a homomorphism of H onto

G with kernel $R/v(R)$.

Lemma 5.2. Let $\underline{\underline{g}} \in [n,G]$, then $F_n/v(R(\underline{\underline{g}})) \cong H$ only if $\underline{\underline{g}}$ has $P(\pi\beta)$ for some automorphism β of G . This condition is sufficient if G is finite and $F_m/v(F_m)$ is a Hopf group for every $m \geqq 1$.

Proof. If $\underline{\underline{h}} \in [n,H]$, then there exists an isomorphism μ of H onto $F_n/R(\underline{\underline{h}})$ such that

$$\underline{\underline{h}}\mu = \underline{\underline{x}}R(\underline{\underline{h}}) .$$

Now $\varphi_{\underline{\underline{h}}\pi} = \varphi_{\underline{\underline{h}}\mu\mu^{-1}\pi}$, and $\varphi_{\underline{\underline{h}}\mu}$ is the natural homomorphism of F_n onto $F_n/R(\underline{\underline{h}})$. Therefore

$$\mathrm{Ker}\,\mu^{-1}\pi = R(\underline{\underline{h}}\pi)/R(\underline{\underline{h}}) .$$

But $k \in \mathrm{Ker}\,\pi$ if and only if $k\mu \in \mathrm{Ker}\,\mu^{-1}\pi$. Hence μ maps $R/v(R) = \mathrm{Ker}\,\pi$ isomorphically onto $R(\underline{\underline{h}}\pi)/R(\underline{\underline{h}})$.

If G is finite, R and $R(\underline{\underline{h}}\pi)$ are both free groups of rank $1 + |G|(n-1)$ (see [9], p.104). If G is infinite, R and $R(\underline{\underline{h}}\pi)$ are both free groups of countably infinite rank (see Theorem 5.7). In either case, therefore, $R \cong R(\underline{\underline{h}}\pi)$, and so

$$R(\underline{\underline{h}}\pi)/v(R(\underline{\underline{h}}\pi)) \cong R(\underline{\underline{h}}\pi)/R(\underline{\underline{h}}) . \qquad 5.2.1$$

Therefore, by Lemma 1.11,

$$R(\underline{\underline{h}}) \geqq v(R(\underline{\underline{h}}\pi)) \ . \qquad 5.2.2$$

If $F_n/v(R(\underline{\underline{g}})) \cong H$, choose $\underline{\underline{h}}' \in [n,H]$ such that $R(\underline{\underline{h}}') = v(R(\underline{\underline{g}}))$. But then by 5.2.2,

$$v(R(\underline{\underline{h}}'\pi)) \leqq R(\underline{\underline{h}}') = v(R(\underline{\underline{g}})) \ .$$

It follows from Theorem 5.1, that $R(\underline{\underline{h}}'\pi) \leqq R(\underline{\underline{g}})$. But G is a Hopf group, so that in fact $R(\underline{\underline{h}}'\pi) = R(\underline{\underline{g}})$, and there is an automorphism β of G such that

$$\underline{\underline{h}}'\pi\beta = \underline{\underline{g}} \ .$$

If G is finite, then $R(\underline{\underline{h}}\pi)$ has finite rank. If $F_m/v(F_m)$ is a Hopf group for every $m \geqq 1$, then clearly $R(\underline{\underline{h}}\pi)/v(R(\underline{\underline{h}}\pi))$ is a Hopf group. It therefore follows from 5.2.1 and 5.2.2 that

$$R(\underline{\underline{h}}) = v(R(\underline{\underline{h}}\pi)) \ .$$

Let $\underline{\underline{h}}^* \in [n,H]$ be such that $\underline{\underline{h}}^*\pi\beta = \underline{\underline{g}}$, where $\beta \in \mathfrak{A}(G)$, then

$$R(\underline{\underline{h}}^*) = v(R(\underline{\underline{h}}^*\pi)) = v(R(\underline{\underline{h}}^*\pi\beta)) = v(R(\underline{\underline{g}})) \ ,$$

and the lemma is proved.

Theorem 5.3. Let F be a free group and let R, S be normal subgroups of F. If $F/R \cong F/S$ and $R/v(R)$ is finite, then $F/v(R) \cong F/v(S)$.

Proof. If $v(R) = R$, then $v(S) = S$, and the theorem is trivial. If $v(R) \neq R$, and $R/v(R)$ is finite, then R has finite rank, for if not, $v(R)$ would contain a free generator of R and hence all of R . It follows from Theorem 5.8 that F has finite rank and from Theorem 5.7 that F/R is finite. If $F \cong F_1$, then $F/R \cong F/S$ implies $R = S$, and the theorem is trivial. Finally, suppose $F = F_n$ for $n \geqq 2$, and that $S = R(\underline{\underline{g}})$, where $\underline{\underline{g}} \in [n, F_n/R]$. Then by Lemma 5.2, $F_n/v(S) \cong F_n/v(R)$ if $\underline{\underline{g}}$ has $P(\pi\beta)$ for some automorphism β of F_n/R . But by Theorem 3.1, $\underline{\underline{g}}$ has $P(\pi)$, and so the theorem is proved.

Theorem 5.4. Let v be a word subgroup function such that $v(A) = E$ for every abelian group A . Let G be a Hopf (k,n)-group for $k, n \geqq 2$, and let $\underline{\underline{g}}, \underline{\underline{g}}' \in [n,G]$, then

$$F_n/v(R(\underline{\underline{g}})) \cong F_n/v(R(\underline{\underline{g}}')) \qquad 5.4.1$$

only if $\underline{\underline{g}}$ and $\underline{\underline{g}}'$ belong to the same T_k-system.

Proof. Let $H = F_n/v(R(\underline{\underline{g}}))$, then H is a $(0,n)$-group, since

$$v(R(\underline{\underline{g}})) \leqq \delta(R(\underline{\underline{g}})) \leqq \delta(F_n) .$$

Let π be the homomorphism of H onto G such that

$$(\underline{\underline{x}}R(\underline{\underline{g}}))\pi = \underline{\underline{g}} ,$$

then $\operatorname{Ker} \pi = R(\underline{\underline{g}})/v(R(\underline{\underline{g}}))$. By Lemma 5.2, if 5.4.1 is satisfied, then $\underline{\underline{g}}'$ has $P(\pi\beta)$ for some automorphism β of G .

This implies that $\underline{\underline{g}}'\beta^{-1}$ has $P(\pi)$. But by Theorem 3.4,

$$D_k(\underline{\underline{g}}) = D_k(\underline{\underline{g}}'\beta^{-1}) \text{ or } k - D_k(\underline{\underline{g}}'\beta^{-1}) ,$$

and the theorem is proved.

Schreier systems

Let X be a set of free generators of a free group F . Let $f \in F$, $f \neq e$, then f can be uniquely represented as a reduced word in the elements of $X \cup X^{-1}$; say

$$f = f_1 f_2 \ldots f_m , \quad f_i \in X \cup X^{-1} , \quad i = 1, 2, \ldots, m .$$

The length of f is denoted by $\underline{f}$; i.e., $\underline{f} = m$. Also

$$f^{(i)} = f_1 f_2 \ldots f_i , \quad 0 < i \leq \underline{f} ,$$

$$f^{(0)} = e .$$

A set of elements T of F is said to be a Schreier system if

(a) $f \in T$ implies that $f^{(i)} \in T$ for $0 \leq i \leq \underline{f}$.

Schreier (see, for instance, [9], p.95) showed that if U is a subgroup of F , then there is a Schreier system that is a complete set of right coset representatives of U .

The following lemma is a slight generalization of this result.

Lemma 5.5. Let U be a subgroup of F. Let T be a Schreier system such that for every pair $t, t' \in T$

(b) $Ut = Ut'$ implies that $t = t'$.

Then there is a Schreier system M that is a complete set of right coset representatives of U and such that $T \subseteqq M$.

Proof. Consider the set W of all Schreier systems T' such that $T \subseteqq T'$ and which satisfy (b). If this set is ordered by set inclusion, then it is clear that the union of every simply ordered subset of W belongs to W. By Zorn's Lemma, W contains a maximal element M. Let $f = f_1 f_2 \ldots f_m \in F$, let k be the largest integer such that $Uf^{(k)} = Ud$ for some $d \in M$. If $k < \underline{f}$, then $M \cup \{df_{k+1}\}$ is a Schreier system satisfying (b), since

$$Udf_{k+1} = Uf^{(k+1)} \neq Ud'$$

for every $d' \in M$. This contradicts the maximality of M. Therefore $k = \underline{f}$, and M is a complete set of coset representatives of U.

Since $\{e\}$ is a Schreier system satisfying (b), it follows that the existence of a Schreier system of right coset representatives of U has been proved. Let T be such a Schreier system, and let φ be the function of F onto T

such that

$$Uf = U\varphi(f) , \quad f \in F .$$

Schreier's Theorem ([9], Theorem 7.2.1) is now stated for convenience in the following form.

Theorem 5.6. Every subgroup U of F is a free group. The set

$$\{tx\varphi(tx)^{-1} \mid t \in T, \ x \in X, \ tx \neq \varphi(tx)\}$$

is a set of distinct free generators of U .

The following two theorems are well known. However I could not find proofs of them in the literature.

Let $r(H)$ denote the rank (a finite or infinite cardinal) of a free group H .

Theorem 5.7. Let N be a non-trivial normal subgroup of F and let F/N have infinite order, then

$$r(N) \geqq |F/N| .$$

Proof. Let T be a Schreier system of right coset representatives of N ; thus $|T| = |F/N|$. Let $n \in N$, $n \neq e$, and let

$$n = n_1 n_2 \ldots n_s , \quad n_i \in XUX^{-1} , \quad i = 1, 2, \ldots, s ,$$

be the representation of n as a reduced word. Let T_i , $0 \leqq i \leqq s-1$, be defined as follows:

$$T_i = \{tn^{(i)} \mid t, \ tn^{(i)} \in T, \ tn^{(i+1)} \notin T\} .$$

If $t \in T$, then $tn \notin T$, since $Ntn = Nt$. It follows that for every $t \in T$, there exists an i such that $tn^{(i)} \in T_i$, and clearly

$$|T| = |T_0| + |T_1| + \dots + |T_{s-1}| .$$

Since $|T|$ is infinite, for some r, $0 \leqq r \leqq s-1$,

$$|T| = |T_r| .$$

If $n_{r+1} \in X$, then by Theorem 5.6,

$$V = \{dn_{r+1}\varphi(dn_{r+1})^{-1} \mid d \in T_r\}$$

is a subset of a set of free generators of N such that

$$|V| = |T| ;$$

while if $n_{r+1} \in X^{-1}$, V^{-1} has the same property. This completes the proof of the theorem.

<u>Theorem 5.8</u>. Let N be a non-trivial normal subgroup of F, then $r(N) \geqq r(F)$.

<u>Proof</u>. If $r(F)$, $r(N)$ and F/N are finite, then (see [9], p. 104)

$$r(N) = 1 + |F/N|(r(F)-1) ;$$

the result follows immediately for this case.

A set of generators of N together with a set of representatives of the cosets of N in F will form a set of generators of F. Therefore

$$r(F) \leq r(N) + |F/N| . \qquad 5.8.1$$

It follows that if $r(N)$ and $|F/N|$ are finite, then so is $r(F)$. It remains, therefore, to deal with the case when either $r(N)$ or F/N is infinite. Inequality 5.8.1 then becomes

$$r(F) \leq \mathrm{Max}\,(r(N), |F/N|) ,$$

and the theorem follows from Theorem 5.7

The residual properties of $F/v(R)$

Let v be a word subgroup function. Let F be a free group and X a set of free generators of F .

The following well known lemma about word subgroups of free groups will be required.

Lemma 5.9. If $X' \subseteqq X$ and $F' = \mathrm{sgp}\{X'\}$, then

$$v(F') = v(F) \cap F' ,$$

and $F'/v(F')$ is isomorphic to a subgroup of $F/v(F)$.

Proof. Let σ be the epimorphism of F onto F' such that

$$x'\sigma = x' , \quad x' \in X' ,$$

$$x\sigma = e , \quad x \in X , \quad x \notin X' .$$

If $f \in F'$, then $f = f\sigma$. But $f \in v(F)$ implies that $f\sigma \in v(F')$. Hence if $f \in F' \cap v(F)$, then $f = f\sigma \in v(F')$: that is $F' \cap v(F) \leq v(F')$. Trivially $v(F') \leq F' \cap v(F)$ and the first part of the lemma is proved. Finally,

$$F'/v(F') = F'/v(F) \cap F' \cong F'v(F)/v(F) .$$

Theorem 5.10. Let Σ be a set of subgroups of F, closed under finite intersections. If $I = \bigcap_{U \in \Sigma} U$, then

$$v(I) = \bigcap_{U \in \Sigma} v(U) .$$

Proof. Trivially $v(I) \leqq \bigcap_{U \in \Sigma} v(U)$.

Let $u \in I$, $u \notin v(I)$; it will be shown that $u \notin v(U)$ for some $U \in \Sigma$. Let T be a Schreier system of right coset representatives of I, and Y the set of free generators of I as given in Theorem 5.6, so that

$$u = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_r^{\varepsilon_r} , \quad y_i \in Y , \quad \varepsilon_i = \pm 1 , \quad 1 \leqq i \leqq r .$$

Let $y_i = t_i x_i \varphi(t_i x_i)^{-1}$, $t_i \in T$, $x_i \in X$. Let

$$A = \{t_1, \varphi(t_1 x_1), t_2, \varphi(t_2 x_2), \ldots, t_r, \varphi(t_r x_r)\} ,$$

so that $A \subseteqq T$. Let

$$B = \{a^{(i)} \mid a \in A, 0 \leqq i \leqq \underline{a}\} ,$$

so that $B \subseteqq T$. Also B is finite. Set

$$C = \{b b_1^{-1} \mid b, b_1 \in B , b \neq b_1\} .$$

Let $b, b_1 \in B$, then $b b_1^{-1} \in I$ implies that $Ib = Ib_1$, and so $b = b_1$. It follows that if $c \in C$, then $c \notin I$.

Thus there exists $U_c \in \Sigma$ such that $c \notin U_c$. Let

$$J = \bigcap_{c \in C} U_c ,$$

then, since C is finite, $J \in \Sigma$. If $Jb = Jb_1$, where $b, b_1 \in B$, then $Jbb_1^{-1} = J$, and so $bb_1^{-1} \in J$. If $b \neq b_1$, then $bb_1^{-1} \in C$. But C and J are disjoint. Therefore $Jb = Jb_1$ implies $b = b_1$. Thus B is a Schreier system satisfying condition (b) of Lemma 5.5 for subgroup J. Hence there is a Schreier system M that is a complete set of right coset representatives of J and such that $B \subseteqq M$.

Let φ' be the function of F onto M given by

$$J\varphi'(f) = Jf , \quad f \in F ,$$

then

$$J\varphi'(t_i x_i) = Jt_i x_i \geqq It_i x_i = I\varphi(t_i x_i) .$$

Hence $J\varphi'(t_i x_i) = J\varphi(t_i x_i)$. But $\varphi(t_i x_i) \in A \subseteqq M$. Therefore $\varphi'(t_i x_i) = \varphi(t_i x_i)$. Now

$$Y' = \{dx\varphi'(dx)^{-1} \mid d \in M, x \in X, dx \neq \varphi'(dx)\}$$

is a set of free generators for J. But $t_i \in A \subseteqq M$. Therefore

$$y_i = t_i x_i \varphi(t_i x_i)^{-1} = t_i x_i \varphi'(t_i x_i)^{-1} \in Y'$$

for $i = 1, 2, \ldots, r$. Let $K = \mathrm{sgp}\{y_1, y_2, \ldots, y_r\}$, then by Lemma 5.9,

$$v(K) = K \cap v(I) = K \cap v(J) .$$

But $u \in K$, $u \notin v(I)$. Therefore $u \notin v(J)$ and the theorem follows immediately.

A group G is said to have property P residually if, for every $g \in G$, $g \neq e$, there is a normal subgroup N such that $g \notin N$ and G/N has property P.

A group property P is called a root property if it satisfies the following conditions:-

1) if group G has P, then every subgroup of G has P;
2) if groups G and H have P, then the direct product $G \times H$ has P;
3) if $G \geq H > K$ is a series of subgroups, each normal in its predecessor, and G/H and H/K have P, then K contains a subgroup L, normal in G, such that G/L has P.

This definition was introduced by Gruenberg [7]. Solubility, finiteness and "having p-power order" are all root properties ([7], p.33).

<u>Theorem 5.11</u>. Let P be a root property. Let R be a normal subgroup of F such that F/R and $R/v(R)$ have P residually, then $F/v(R)$ has P residually.

Proof. Let $f \in F$, $f \notin v(R)$. It is required to find a normal subgroup N of F, such that $v(R) \leq N$, $f \notin N$ and F/N has P.

Let

$$\Sigma_P = \{S \mid R \leq S \lhd F, F/S \text{ has } P\},$$

then ([7], p.33) Σ_P is closed under finite intersections and

$$\bigcap_{S \in \Sigma_P} S = R .$$

Therefore, by Theorem 5.10,

$$v(R) = \bigcap_{S \in \Sigma_P} v(S) .$$

In particular, there exists $S' \in \Sigma_P$ such that $f \notin v(S')$. Since $S' \geq R$, $r(S') \leq r(R)$ by Theorem 5.8. Therefore, by Lemma 5.9, $S'/v(S')$ is isomorphic to a subgroup of $R/v(R)$. Since P satisfies 1), it follows that $S'/v(S')$ has P residually. Therefore there exists a normal subgroup $K/v(S')$ of $S'/v(S')$ such that S'/K has P, but $f \notin K$. Since P satisfies 3) and since F/S' has P, there exists a normal subgroup $N/v(S')$ of $F/v(S')$ such that F/N has P and $K \geq N$. Clearly $f \notin N$ and $N \geq v(R)$, so that N has the required properties.

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