ON VARIETIES OF METABELIAN GROUPS OF PRIME-POWER EXPONENT

by

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STATEMENT

The results presented in this thesis are my own except where otherwise stated.

M.S. Brooks

M.S. Brooks.

PREFACE

The work for this thesis was carried out during my tenure of an Australian National University research scholarship. I much appreciate the generous financial assistance that this provided; not only has it supported me over the past three years but it also paid my return fare from Great Britain.

It is with pleasure that I express gratitude to my co-supervisors, Dr. L.G. Kovács and Dr. M.F. Newman for their ready availability and ever helpful advice whenever I have needed guidance.

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CONTENTS

INTRODUCTION

The work reported in this thesis is a contribution to the young, but growing, theory of metabelian varieties (i.e. varieties of metabelian groups). The basic (but in its full generality entirely hopeless) problem in this theory is to describe all metabelian varieties and the lattice lat($\underline{\underline{AA}}$) they form, and indeed most of the results obtained so far concern aspects of this problem.

Probably the most general, and certainly the most well-known, of these results is due to D.E. Cohen [3], who has shown that lat($\underline{\underline{AA}}$) has minimum condition. Other authors, such as Warren Brisley [1], R.A. Bryce [2], P.J. Cossey [4], L.G. Kovács and M.F. Newman (unpublished), and P.M. Weichsel [9], have given descriptions of various sublattices of lat($\underline{\underline{AA}}$). These sublattices are all distributive, whereas lat($\underline{\underline{AA}}$) itself is not, as has been shown by R.A. Bryce [2].

It follows from Cohen's result that every variety $\underline{\underline{V}}$ in lat($\underline{\underline{AA}}$) can be expressed as the irredundant join of finitely many join-irreducible varieties. Owing to non-distributivity not every $\underline{\underline{V}}$ has a unique expression of this kind, nevertheless a classification of the join-irreducible subvarieties of $\underline{\underline{AA}}$ would clearly provide a great deal of information about lat($\underline{\underline{AA}}$). In this direction L.G. Kovács and M.F. Newman, in work as yet unpublished, have classified the join-irreducibles of infinite exponent, and have shown further that for any $\underline{\underline{V}} \in \mathrm{lat}(\underline{\underline{A}}\underline{\underline{A}})$ the infinite exponent components in the expressions for $\underline{\underline{V}}$ as an irredundant join of join-irreducibles are unique. The join-irreducibles of finite, composite exponent have been considered by R.A. Bryce, who has obtained a reduction theorem relating to their classification. Although this theorem, which is also unpublished, does not actually lead to a classification, it does indicate that any such classification must necessarily be extremely complicated. The remaining case is that of the prime-power exponent join-irreducibles, and it is to certain aspects of the problem of classifying them that this thesis is devoted.

The principal result, which is expressed in the first part of Theorem 2.1.2, is a complete classification of the non-nilpotent join-irreducibles in $\mathrm{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$, where p is an arbitrary prime. It is shown that these non-nilpotent join-irreducibles form an ascending chain, so that any non-nilpotent variety $\underline{\underline{V}} \in \mathrm{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$ can be written $\underline{\underline{V}} = \underline{\underline{I}} \vee \underline{\underline{L}}$ where $\underline{\underline{I}}$ is a non-nilpotent join-irreducible, and $\underline{\underline{L}}$ is nilpotent. The second part of Theorem 2.1.2 says that this $\underline{\underline{I}}$ is unique (compare the result of L.G. Kovács and M.F. Newman mentioned above), but in Chapter 3 it is shown that at least $\mathrm{lat}(\underline{\underline{A}}_3\underline{\underline{A}}_9)$ is non-distributive, and, in particular, that the nilpotent component $\underline{\underline{L}}$ of $\underline{\underline{V}}$ is not always unique, even when "minimised". (See Remark 2.1.3). In addition to these results, a conjecture (item 2.9.5) is made regarding the non-nilpotent join-irreducibles in $\mathrm{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^{\beta+1}})$ which, if true, would reduce the classification problem of the join-irreducibles in $\mathrm{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^{\beta+1}})$ to that of the nilpotent join-irreducibles in the same lattice. This conjecture, which is similar to the reduction theorem of Bryce in the composite exponent situation, is proved for the case $\beta = 1$. Unfortunately, the classification problem for the nilpotent join-irreducibles appears very difficult.

The proof of Theorem 2.1.2 consists almost entirely of commutator calculations. In fact, such an extensive use is made of commutator calculus that it has been worthwhile to develop a new form of it which is tailor-made for the metabelian situation. This is described in Chapter 1 and is used there to provide a basis for the derived group of $F_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$. Although this result is only needed for the case $m = p$, $n = p^2$, it is given for general m,n as this does not make the proof any more difficult.

NOTATION AND TERMINOLOGY

Notation and terminology generally follows that in

Hanna Neumann. Varieties of Groups. Berlin, Heidelberg and New York. Springer 1967.

References to this book are frequent, and are indicated by the letters HN, usually followed by the relevant item number. Any notation or terminology neither explained below nor in the body of the thesis has exactly the meaning attached to it in HN. Note, however, that German letters are here represented by double-underlined Roman letters.

Logic and Sets:

$\Rightarrow$	logical implication
//	"end of proof" or, sometimes "no proof"
$\downarrow\downarrow$	signifies that a proof appears later. If the proof appears in a different section then the symbol is followed by the relevant section number.
$\emptyset$	the empty set
ω	the least infinite ordinal
$\aleph_0$	the smallest infinite cardinal
I	the set of non-negative integers
I^+	the set of positive integers

Groups:

The trivial element of every group is denoted by 1.

For the definitions below let H be a group; $H_1, H_2, \ldots$ subgroups of H; $h_1, h_2, \ldots$ elements of H with $\underline{\underline{h}} = \{h_1, h_2, \ldots\}$; $r_2, r_3, \ldots \in I$; and $k \in I^+ \backslash \{1\}$.

$H_1 \leq H$	H_1 is a subgroup of H
$gp(\underline{\underline{h}})$	the subgroup of H generated by $\underline{\underline{h}}$
$\langle \underline{\underline{h}} \rangle$	the fully invariant closure of $\underline{\underline{h}}$ in H
$h_1^{h_2}$	$h_2^{-1} h_1 h_2$
$[h_1, h_2]$	$h_1^{-1} h_1^{h_2}$
$[h_1, \ldots, h_k]$	defined recursively: $[h_1, \ldots, h_k] = [[h_1, \ldots, h_{k-1}], h_k]$
$[h_1, r_2 h_2]$	defined recursively: $[h_1, 0h_2] = h_1$, $[h_1, r_2 h_2] = [[h_1, (r_2 - 1)h_2], h_2]$
$[h_1, r_2 h_2, \ldots, r_k h_k]$	again defined recursively in the obvious manner
$[H_1, H_2]$	$gp(\{[h_1, h_2] \mid h_1 \in H_1,\ h_2 \in H_2\})$
$[H_1, r_2 H_2]$	defined recursively: similarly to above
$H_{(c)}$	$[H, (c-1)H]$ defined for all $c \in I^+$

The exponent of H is the smallest positive integer e such that $h^e = 1$ for all $h \in H$. If no such integer exists H is said to have infinite exponent.

Miscellaneous:

GF(p)	the field of integers modulo the prime p
suppδ	Let S be any set. The support of a function $\delta : S \to I$, denoted by suppδ, is defined by supp$\delta = \{s \in S \mid \delta(s) \neq 0\}$
[q]	the integer part of the non-negative rational number q, i.e. $[q] \in I$, $q - 1 < [q] \leq q$.
lat($\underline{\underline{V}}$)	the lattice of subvarieties of the variety $\underline{\underline{V}}$

The exponent of a variety $\underline{\underline{V}}$ is the least positive integer e such that $\underline{\underline{V}} \subseteq \underline{\underline{B}}_e$ or is infinite if no such e exists.

CHAPTER 1

THE DERIVED GROUP OF $F_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$

In this chapter the structure of the derived group $F'_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ of $F_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is investigated. Since $\underline{\underline{A}}_1$ is the variety of trivial groups, the variety $\underline{\underline{A}}_m\underline{\underline{A}}_n$ is abelian if m or n is 1, so that in these cases $F'_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is trivial. On the other hand when n = 0 the structure of $F'_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ becomes more complicated than can be handled by the methods presented here. For all other cases it is shown that $F'_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is free abelian of exponent m, and, more importantly, an explicit basis for it is exhibited. The description of this basis and a formal statement of results, is given in section 1.2, after the requisite notation has been introduced in 1.1. The proof of these results, modulo three principal lemmas, is given in 1.3, while the proofs of the three lemmas occupy sections 1.4 through 1.6. Finally, in 1.7 an alternative basis for $F'_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is described which, although easily obtainable from the original, is of a rather different nature.

1.1 A Commutator Calculus for Metabelian Groups

This section deals with the conventions, notation and terminology that will be adopted with regard to what is perhaps the most intensively exploited method of proof in this thesis, namely commutator calculus.

An inconvenience inherent in commutator calculus in general is that the word "commutator" is usually considered as having, simultaneously, two distinct meanings; on the one hand it is the name given to certain ELEMENTS of the group under consideration, while on the other it is the name given to certain purely FORMAL EXPRESSIONS to which the attributes such as weight can be ascribed. Although in most cases this presents no real difficulties, for the purposes of this thesis it does, and consequently I shall use the non-standard notation and terminology defined below. Part of the intuitive content of the definitions is that the word "commutator" will be reserved for the second of the meanings mentioned above, and "commutator-element" will be used for the first. Further, the two will be distinguished notationally by using parentheses in writing commutators, and brackets in writing commutator-elements.

The groups to which commutator calculus will be applied will almost always be metabelian and accordingly the definitions below are made with metabelian groups in mind, even though most of them are formulated in terms of arbitrary groups.

1.1.1 <u>Definition</u>: Let H be any group and let $k \in I^{+}\setminus\{1\}$. A <u>commutator</u> of <u>weight</u> k in H is an ordered k-tuplet $\tilde{c} = (h_1,\ldots,h_k)$ with $h_1,\ldots,h_k \in H$. For $1 \leq i \leq k$ the element h_i is referred to as the <u>i-th entry</u> of $\tilde{c}$.

The set of all commutators in H is denoted by $\tilde{C}(H)$

$\tilde{C}(H)$ (i.e. $\tilde{C} = \bigcup_{k=2}^{\infty} H^k$), and the weight of a commutator $\tilde{c} \in \tilde{C}(H)$ is denoted by $wt(\tilde{c})$.

1.1.2 <u>Definition</u>: Let H be any group. The <u>value</u> of a commutator $(h_1,\ldots,h_k)$ in H is defined as the element $[h_1,\ldots,h_k]$ of H. Any element of H that is the value of some commutator in H is called a <u>commutator-element</u>.

1.1.3 <u>Definition</u>: Let $\tilde{c}$ be a commutator in a group H. The <u>degree function</u> of $\tilde{c}$, denoted by $\delta_{\tilde{c}}$, is defined as follows: For any $h \in H$ define $\chi_h : H \to I$ by $\chi_h(h) = 1$ and $\chi_h(h') = 0$ for all $h' \neq h$. Then for $\tilde{c} = (h_1,\ldots,h_k)$ the degree function $\delta_{\tilde{c}} : H \to I$ is defined as $\sum_{i=1}^{k} \chi_{h_i}$.

1.1.4 Remarks: Let H be any group; $\tilde{c}$ a commutator in H; and $h \in H$. Then it follows immediately from Definition 1.1.1 and 1.1.3 that:-

(i) the set of entries of $\tilde{c}$ is precisely $\mathrm{supp}\delta_{\tilde{c}}$;

(ii) $\mathrm{supp}\delta_{\tilde{c}}$ is finite but non-empty;

(iii) $\delta_{\tilde{c}}(h)$ is the number of times h occurs as an entry in $\tilde{c}$;

(iv) $wt(\tilde{c}) = \sum_{h \in H} \delta_{\tilde{c}}(h)$.

1.1.5 Definition: Let H be any group. A pair of commutators in H are called similar if, and only if, they have the same first entry, the same second entry and the same degree function.

For any group H it is clear that similarity defines an equivalence relation on $\tilde{C}(H)$ and hence that $\tilde{C}(H)$ is the union of pair-wise non-intersecting "similarity classes". These similarity classes are the subject of the next definition:

1.1.6 Definition: Let H be any group. Denote by (h_1,h_2,δ) the (non-empty) similarity class containing commutators in H with degree function δ and first and second entries h_1 and h_2 respectively. Then (h_1,h_2,δ) is called the pseudo-commutator in H with first entry h_1, second entry h_2, and degree function δ. Third, fourth and further entries are not defined as such, but nevertheless any $h \in \mathrm{supp}\delta$ is called an entry of (h_1,h_2,δ).

The set of all pseudo-commutators in H is denoted by $\tilde{P}(H)$.

It follows from 1.1.4(iv) that similar commutators have the same weight. Thus:-

1.1.7 Definition: The weight of pseudo-commutator $\tilde{p}$ is defined to be the common weight of its members, and is denoted by $wt(\tilde{p})$.

1.1.8 Remark: Let H be any group, and let (h_1,h_2,δ) be a pseudo-commutator in H. Then $wt((h_1,h_2,\delta)) = \sum_{h \in H} \delta(h)$.

For metabelian groups the concept of pseudo-commutators is particularly useful. This is on account of the following well-known result. (See, for example, HN34.51).

1.1.9 Lemma: Let H be a metabelian group, and let $h_1,\ldots,h_k \in H$, $k \geq 2$. Then for any permutation π of $\{3,\ldots,k\}$

$$[h_1,h_2,h_3,\ldots,h_k] = [h_1,h_2,h_{3\pi},\ldots,h_{k\pi}]. \quad //$$

1.1.10 Corollary: In a metabelian group similar commutators have identical values. //

The above corollary makes possible the following definition, which provides the key to a simplified notation for elements of the derived group of a metabelian group.

1.1.11 Definition: Let H be a metabelian group. The value of a pseudo-commutator (h_1,h_2,δ) in H is defined to be the common value of its members, and is denoted by $[h_1,h_2,\delta]$.

A disadvantage of the (h_1,h_2,δ)-notation for pseudo-commutators is that it is generic rather than explicit. To overcome this, the degree function δ will, when necessary, be "listed" in the form $\{\delta(h)h | h \in \text{supp}\delta\}$. For example, the pseudo-commutator containing $(h_1,h_2,h_2,h_1,h_3,h_1,h_1)$ may be denoted by $(h_1,h_2,\{4h_1,2h_2,1h_3\})$. The notation will also be carried over to values of pseudo-commutators in the obvious manner.

1.2 Statement of the Main Theorem

For the remainder of this chapter let n denote an arbitrary but fixed integer greater than 1, and let $G(m) = F_\infty(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ where $m \in I^+$, $m \neq 1$. Further, let $\underline{\underline{g}}(m) = \{g_{mi} \mid i \in I^+\}$ denote a free generating set for G(m), where it is to be understood that $\underline{\underline{g}}(m)$ is well ordered by its indexing set, i.e. $g_{mi} \leq g_{mj}$ if, and only if, $i \leq j$.

1.2.1 <u>Definition</u>: A pseudo-commutator (a,b,δ) in G(m) will be called <u>basic</u> if, and only if,

(1) $\text{supp}\delta \subseteq \underline{\underline{g}}(m)$

(2) $b = \text{minsupp}\delta$ (i.e. b is the least element in $\text{supp}\delta$)

(3) $a \neq b$

(4) <u>either</u> (i) $\delta(a) \leq n$ and

$$\forall g_{mi} \in \underline{\underline{g}}(m)(g_{mi} \neq a \Rightarrow (g_{mi}) < n)$$

<u>or</u> (ii) $\delta(b) = n$, $a = \text{maxsupp}\delta$ and

$$\forall g_{mi} \in \underline{\underline{g}}(m)(g_{mi} \neq b \Rightarrow (g_{mi}) < n)$$

The set of basic pseudo-commutators in G(m) will be denoted by $\tilde{B}(m)$.

The main result of this chapter can now be stated as follows:

1.2.2 <u>Theorem</u>: The derived group $G'(m)$ of $G(m)$ is free abelian of exponent m. Further, the valuation mapping $\phi(m) : \tilde{B}(m) \to G(m)$ is one-to-one, and $\tilde{B}(m)\phi(m)$ is a basis for $G'(m)$. ↓↓(1.3)

It should perhaps be remarked that, in terms of basic commutators*, as defined in HN31.51, the basis $\tilde{B}(m)\phi(m)$ for $G'(m)$ consists of images under α (where $\alpha : X_\infty \to G(m)$ is the epimorphism induced by the natural map from $\underline{\underline{x}}$ to $\underline{\underline{g}}(m)$) of left-normed basic commutators in which no letter occurs more than (n-1) times, except that, in specific cases, one of the first two entries may occur n times. However, we shall not use basic commutator methods for the proof of 1.2.2, or, indeed, anywhere in this thesis.

1.3 Skeletal Proof of 1.2.2

The bulk of the proof of 1.2.2 will be carried out in finitely generated subgroups of G(0). For any integer r greater than 1 let $\underline{\underline{g}}_r(0) = \{g_{01}, \ldots, g_{0r}\}$, $(\underline{\underline{g}}_r(0) \subseteq \underline{\underline{g}}(0))$, and

*Caution: In this paragraph only, the word "commutator" is being used in the standard sense, and not as defined in 1.1.1.

let $G_r(0) = gp(\underline{g}_r(0))$. Let $\tilde{B}_r(0)$ denote the set of basic pseudo-commutators in $G_r(0)$; i.e. $\tilde{B}_r(0) = \tilde{B}(0) \cap \tilde{P}(G_r(0))$.

In this section it is shown how 1.2.2 is deduced from the following three lemmas:

1.3.1 Lemma: For all $r \geq 2$ the derived group $G_r'(0)$ of $G_r(0)$ is free abelian of exponent 0 and rank $(r-1)(n^r-1)$. ↓↓(1.4)

1.3.2 Lemma: For all $r \geq 2$ $|\tilde{B}_r(0)| = (r-1)(n^r-1)$. ↓↓(1.5)

1.3.3 Lemma: For all $r \geq 2$ $G_r'(0) = gp(\tilde{B}_r(0)\phi(0))$. ↓↓(1.6)

Actually, the rank of $G_r'(0)$ and the cardinality of $\tilde{B}_r(0)$ are not important in themselves; only their equality is required, and this is used to prove:

1.3.4 Lemma: For any integer $r \geq 2$ the valuation mapping $\phi(0)|_{\tilde{B}_r(0)} : \tilde{B}_r(0) \to G_r(0)$ is one-to-one, and $\tilde{B}_r(0)\phi(0)$ is a basis for $G_r'(0)$.

Proof: From 1.3.2 $|\tilde{B}_r(0)\phi(0)| \leq (r-1)(n^r-1)$, and equality holds only if $\phi(0)|_{\tilde{B}_r(0)}$ is one-to-one. On the other hand, since from 1.3.3 $\tilde{B}_r(0)\phi(0)$ is a generating set for $G_r'(0)$, it follows from 1.3.1 that $|\tilde{B}_r(0)\phi(0)| \geq (r-1)(n^r-1)$, and equality holds here only if $\tilde{B}_r(0)\phi(0)$ is a basis for $G'_r(0)$. //

<u>Proof of 1.2.2</u>: We deal first with the case $m = 0$.

Firstly, the mapping $\phi(0) : \tilde{B}(0) \to G(0)$ is one-to-one because any two distinct basic pseudo-commutators belonging to $\tilde{B}(0)$ are also members of $\tilde{B}_r(0)$ for sufficiently large r, and therefore have distinct values, since $\phi(0)|\tilde{B}_r(0)$ is one-to-one (from 1.3.4).

Secondly, $\tilde{B}(0)\phi(0)$ generates $G'(0)$ because any element w in $G'(0)$ is also a member of $G'_r(0)$ for large enough r, and $G'_r(0) = gp(\tilde{B}_r(0)\phi(0) \leq gp(\tilde{B}(0)\phi(0))$. (We have used 1.3.3). To verify that $\tilde{B}(0)\phi(0)$ is in fact a basis for $G'(0)$, it remains to show that no non-trivial relation exists among its members. Now if any such non-trivial relation did exist, say involving the values of basic pseudo-commutators $\tilde{p}_1,\ldots,\tilde{p}_k$, then, choosing r so that $\tilde{p}_1,\ldots,\tilde{p}_k \in \tilde{B}_r(0)$, it would also provide an example of a non-trivial relation among the members of $\tilde{B}_r(0)\phi(0)$. But this would mean that $\tilde{B}_r(0)\phi(0)$ could not be a basis for $G'_r(0)$, contradicting 1.3.4.

Finally, we must show that $G'(0)$ is free abelian of exponent 0, but since we have already exhibited a basis for $G'(0)$, it suffices to show that $G'(0)$ is torsion-free. For this simply note that $G'_r(0)$ is torsion-free for every $r \geq 2$ by 1.3.1, and $G'(0) = \bigcup_{r=2}^{\infty} G'_r(0)$.

To complete the proof of 1.2.2 we must deal with the case $m > 1$, and this we shall now do, essentially by showing that the restriction of the natural epimorphism $\theta : G(0) \to G(m)$ has the necessary properties.

For the remainder of this proof, let m denote an arbitrary but fixed integer greater than 1. Since $\underline{\underline{A}}_m\underline{\underline{A}}_n$ is a subvariety of $\underline{\underline{A}}_0\underline{\underline{A}}_n$, the natural mapping $\overline{\theta} : \underline{\underline{g}}(0) \to \underline{\underline{g}}(m)$, given by $g_{0i}\overline{\theta} = g_{mi}$ for all $i \in I^+$, extends to an epimorphism $\theta : G(0) \to G(m)$ with kernel $A_m(A_n(G(0)))$. From HN12.31, $A(G(0)\theta) = A(G(0)\theta$, so $G'(m) = G'(0)\theta$, and hence $G'(m)$ will be shown to be free abelian of exponent m if we can show that

1.3.5... $$\ker(\theta|_{G'(0)}) = B_m(G'(0))$$

To prove this, let F denote an absolutely free group of rank $\aleph_0$, so that $G(0) \cong F/A(A_n(F))$. In the same notation

$$G'(0) = A(G(0)) \cong A(F/A(A_n(F))) = A(F).A(A_n(F))/A(A_n(F)) = A(F)/A(A_n(F))$$

and hence $G(0)/G'(0) \cong F/A(F) \cong F_\infty(\underline{\underline{A}})$. Hence $G(0)/G'(0)$ is free abelian (of exponent 0) and it follows that $A_n(G(0))/G'(0)$, being a subgroup of a free abelian group, is also free abelian. ([5]p.143). Now $A_n(G(0))$ is abelian, (since $A(A_n(G(0))) = \{1\}$), and it follows that $G'(0)$ is a direct factor of $A_n(G(0))$. ([5]p.144). Hence denoting by C any complement of $G'(0)$ in $A_n(G(0))$, we have

$$\ker\theta = A_m(A_n(G(0))) = B_m(A_n(G(0))) = B_m(G'(0)\times C) = B_n(G'(0))\times B_m(C).$$

But this proves 1.3.5 for $\ker(\theta|_{G'(0)}) = \ker\theta \cap G'(0)$.

We show next that

1.3.6... If $\underline{\underline{b}}$ is a basis for $G'(0)$ then $\theta|_{\underline{\underline{b}}}$ is one-to-one and $\underline{\underline{b}}\theta$ is a basis for $G'(m)$.

Let $\underline{\underline{b}} = \{b_i | i \in I^+\}$ and suppose we have a relation of the kind

$$(b_{i_1}\theta)^{e_1}(b_{i_2}\theta)^{e_2}\dots(b_{i_k}\theta)^{e_k} = 1$$

where $e_1,\dots,e_k$ are integers, and the $b_{i_1},\dots,b_{i_k} \in \underline{\underline{b}}$ are pair-wise distinct. Then

$$b_{i_1}^{e_1}b_{i_2}^{e_2}\dots b_{i_k}^{e_k} \in \ker(\theta|_{G'(0)})$$

and since from 1.3.5 $\{b_i^n | i \in I^+\}$ is a basis for $\ker(\theta|_{G'(0)})$ it follows that $n|e_j$ for each $j \in \{1,\dots,k\}$. From this we conclude firstly that $\theta|_{\underline{\underline{b}}}$ is one-to-one, because if $i \neq j$ then the relation $(b_i\theta)(b_j\theta)^{-1} = 1$ cannot hold in $G'(m)$, and secondly that $\underline{\underline{b}}\theta$ is an independent set in $G'(m)$, as the only relations that can hold among the members of $\underline{\underline{b}}\theta$ are the trivial ones. (We are using the fact that $G'(m)$ has exponent m). This completes the proof of 1.3.6 because $G'(m) = G'(0)\theta = gp(\underline{\underline{b}})\theta = gp(\underline{\underline{b}}\theta)$, that is, $\underline{\underline{b}}\theta$ generates $G'(m)$.

Before we can proceed further, we must relate the basic pseudo-commutators in $G(m)$ to those in $G(0)$. To do this, for any $\tilde{p} \in \tilde{B}(0)$, say $\tilde{p} = (g_{0i_1}, g_{0i_2}, \{d_1 g_{0i_1}, \ldots, d_k g_{0i_k}\})$ where $i_1, \ldots, i_k$ are pair-wise distinct positive integers, let

$$\tilde{p}\theta^* = (g_{0i_1}\theta, g_{0i_2}\theta, \{d_1 g_{0i_1}\theta, \ldots, d_k g_{0i_k}\theta\}) =$$

$$(g_{mi_1}, g_{mi_2}, \{d_1 g_{mi_1}, \ldots, d_k g_{mi_k}\}).$$

Reference to 1.2.1 shows that $\tilde{p}\theta^* \in \tilde{B}(m)$ and, in fact, that $\theta^* : \tilde{B}(0) \to \tilde{B}(m)$ is onto.

The definition of θ^* shows further that $\tilde{p}\theta^*\phi(m) = \tilde{p}\phi(0)\theta$ for every $\tilde{p} \in \tilde{B}(0)$, or in other words that the diagram

1.3.7...

$$\begin{array}{ccc} \tilde{B}(0) & \xrightarrow{\theta^*} & \tilde{B}(m) \\ \downarrow \phi(0) & & \downarrow \phi(m) \\ \tilde{B}(0)\phi(0) & \xrightarrow{\theta|\tilde{B}(0)\phi(0)} & \tilde{B}(m)\phi(m) \end{array}$$

is commutative.

This fact, together with 1.3.6, we now use to complete the outstanding parts of the proof 1.1.2, i.e. to prove that $\phi(m)$ is one-to-one and that $\tilde{B}(m)\phi(m)$ is a basis for $G'(m)$. Since, as we have already remarked, $\theta^* : \tilde{B}(0) \to \tilde{B}(m)$ is onto, and $\phi(m) : \tilde{B}(m) \to \tilde{B}(m)\phi(m)$ is onto by definition, it follows from 1.3.7 that $\tilde{B}(0)\phi(0)\theta|_{\tilde{B}(0)\phi(0)} = \tilde{B}(m)\phi(m)$. Taking $\underline{b}$ to be $\tilde{B}(0)\phi(0)$ in 1.3.6 therefore shows that $\tilde{B}(m)\phi(m)$ is a basis for $G'(m)$. Similarly, 1.3.6 shows that $\theta|_{\tilde{B}(0)\phi(0)}$ is one-to-one, and hence, using that $\phi(0)$ is one-to-one and θ^* is onto, 1.3.7 shows that $\phi(m)$ is also one-to-one. //

1.4 The Proof of 1.3.1

We will need the following simple observation:

1.4.1 <u>Lemma</u>: If R is a free abelian group of rank r (and exponent 0), T a subgroup of R such that $R/T \cong Q_1 \times Q_2$ where Q_1 is free abelian of rank q and Q_2 is finite, then T is free abelian of rank r - q.

Proof: The freeness of T is immediate, since every subgroup of a free abelian group is free abelian. Let the rank of T be t. Denoting the torsion-free-rank of an abelian group X by $r_0(X)$, [5] p.140 gives $r_0(R) = r_0(R/T)+r_0(T)$. (See, for example, [5] p.140, but note that the author means "torsion-free-rank" when he says "rank"). But $r_0(R) = r$, $r_0(T) = t$ and $r_0(R/T) = r_0(Q_1) + r_0(Q_2) = q + 0 = q$. //

<u>Proof of 1.3.1:</u>

Let F_r be an absolutely free group of rank r, and within it consider the verbal subgroups $A_n(F_r)$, $A(F_r)$ and $A(A_n(F_r))$; clearly these are arranged as in Fig. 1. We claim:

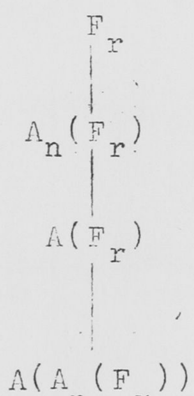


Fig. 1. Verbal Subgroups of F_r.

(i) $F_r/A_n(F_r)$ is finite, and has order n^r

(ii) $F_r/A(F_r)$ is free abelian of rank r

(iii) $A_n(F_r)/A(F_r)$ is free abelian of rank r

(iv) $A_n(F_r)/A(A_n(F_r))$ is free abelian of rank $(r-1)n^r + 1$

(v) $A(F_r)/A(A_n(F_r))$ is free abelian of rank $(r-1)(n^r-1)$

For the proofs we have:

(i) $F_r/A_n(F_r) \cong F_r(\underline{\underline{A}}_n)$ and so $F_r/A_n(F_r)$ is free abelian of exponent n and rank r.

(ii) Similarly $F_r/A(F_r) \cong F_r(\underline{\underline{A}})$

(iii) Use (i), (ii) and 1.4.1

(iv) From Schreier's formula and (i), $A_n(F_r)$ is (absolutely) free of rank $(r-1)n^r+1$, and hence

$$A_n(F_r)/A(A_n(F_r)) \cong F_{(r-1)n^r+1}/A(F_{(r-1)n^r+1}) \cong F_{(r-1)n^r+1}(\underline{\underline{A}})$$

(v) Use (iii), (iv) and 1.4.1.

But (v) is the required conclusion, for $G_r(0) \cong F_r/A(A_n(F_r))$ and hence

$$G_r'(0) = A(G_r(0)) \cong A(F_r/A(A_n(F_r))) = A(F_r)/A(A_n(F_r)). \quad //$$

1.5 The Proof of 1.3.2

Clearly, a basic pseudo-commutator (a,b,δ) in $G(0)$ is a member of $\tilde{B}_r(0)$ if, and only if, $\text{supp}\delta \subseteq \underline{\underline{g}}_r(0)$. Thus we merely have to count the pseudo-commutators that satisfy the conditions (2)-(4) of 1.2.1 (with $m = 0$ in (4)) and a strengthened version of condition (1), namely

$$(1)^* \quad \text{supp}\delta \subseteq \underline{\underline{g}}_r(0)$$

We count those $(a,b,\delta) \in \tilde{B}_r(0)$ with a given set of entries, say $\text{supp}\delta = \underline{\underline{a}} = \{a_1,\ldots,a_s\}$, where in view of conditions (1)* and (3) $\underline{\underline{a}} \subseteq \underline{\underline{g}}_r(0)$ and $2 \leq s \leq r$, and we may assume without loss of generality that $a_1 = \min\underline{\underline{a}}$ and

$a_s = \max \underline{\underline{a}}$. Since any pseudo-commutator (a,b,δ) with $\mathrm{supp}\delta = \underline{\underline{a}}$ automatically satisfies the condition (1)*, conditions (2) and (3) reduce this task to that of counting those members of the set

$$\tilde{S}(\underline{\underline{a}}) = \{(a_i, a_1, \{d_1 a_1, \ldots, d_s a_s\}) \mid 2 \leq i \leq s;\ d_1, \ldots, d_s \in I^+\}$$

that satisfy condition (4). Now for $(a_i, a_1, \{d_1 a_1, \ldots, d_s a_s\}) \in \tilde{S}(\underline{\underline{a}})$ to satisfy condition (4)(i) i can be chosen in (s-1) ways; d_i in n ways; and the remaining members of $\{d_1, \ldots, d_s\}$ in (n-1) ways each. Alternatively, for $(a_i, a_1, \{d_1 a_1, \ldots, d_s a_s\}) \in \tilde{S}(\underline{\underline{a}})$ to satisfy condition (4)(ii) i must be s; d_1 must be n; and $d_2, \ldots, d_s$ can be chosen in (n-1) ways each. Since (4)(i) and 4(ii) are mutually exclusive conditions this gives a total of $(s-1)n(n-1)^{s-1} + (n-1)^{s-1}$ basic pseudo-commutators in $\tilde{S}(\underline{\underline{a}})$. That is

$$|\tilde{B}_r(0) \cap \tilde{S}(\underline{\underline{a}})| = (s-1)n(n-1)^{s-1} + (n-1)^{s-1}$$

$$= ns(n-1)^{s-1} - (n-1)^s$$

Now let $D = \{\underline{\underline{a}} \mid \underline{\underline{a}} \subseteq \underline{\underline{g}}_r(0),\ |\underline{\underline{a}}| \geq 2\}$. Then it follows immediately from the various definitions that

(i) $\tilde{p} \in \tilde{B}_r(0) \Longrightarrow \exists \underline{\underline{a}} \in D : \tilde{p} \in \tilde{S}(\underline{\underline{a}})$,

(ii) $\forall \underline{\underline{a}}_1, \underline{\underline{a}}_2 \in D \quad \underline{\underline{a}}_1 \neq \underline{\underline{a}}_2 \Longrightarrow \tilde{S}(\underline{\underline{a}}_1) \cap \tilde{S}(\underline{\underline{a}}_2) = \emptyset$

(iii) $|\{\underline{\underline{a}} | \underline{\underline{a}} \in D; |\underline{\underline{a}}| = s\}| = \binom{r}{s}$.

Hence
$$|\tilde{B}_r(0)| = \sum_{\underline{\underline{a}} \in D} |\tilde{B}_r(0) \cap \tilde{S}(\underline{\underline{a}})|$$

$$= \sum_{s=2}^{r} \binom{r}{s} \{ns(n-1)^{s-1} - (n-1)^s\}$$

$$= \sum_{s=2}^{r} \frac{r}{s}\binom{r-1}{s-1} ns(n-1)^{s-1} - \sum_{s=2}^{r} \binom{r}{s}(n-1)^s$$

$$= rn((1+(n-1))^{r-1}-1) - ((1+(n-1))^r - r(n-1)-1)$$

$$= (r-1)(n^r-1). \quad //$$

1.6 The Proof of 1.3.3

The proof of 1.3.3 consists entirely of calculations with commutator-elements, and will make much use of the following well-known identities:

1.6.1 Remarks: Let T be any metabelian group, $t_1, t_2, \ldots \in T$. Then

(1) T' is abelian and hence $[t_1,t_2,\ldots] = 1$ whenever $t_1,t_2 \varepsilon T'$ or $t_i \varepsilon T'$ for $i \geq 3$.

(2) If $d_1,d_2,\ldots \varepsilon T'$ then $[\Pi d_i,t_1,t_2\ldots] = \Pi[d_i,t_1,t_2\ldots]$.

(3) $[t_1,t_2][t_2,t_1] = 1$. Using (2) this generalises to: If $(t_1,t_2,\delta) \varepsilon \tilde{P}(T)$ then $[t_1,t_2,\delta][t_2,t_1,\delta] = 1$.

(4) $[t_1t_2,t_3] = [t_1,t_3]^{t_2}[t_2,t_3] = [t_1,t_3][t_2,t_3][t_1,t_3,t_2]$
$[t_1,t_2t_3] = [t_1,t_2]^{t_3}[t_1,t_2] = [t_1,t_2][t_1,t_3][t_1,t_2,t_3]$

(5) $[t_1,t_2,t_3][t_2,t_3,t_1][t_3,t_1,t_2] = 1$. Using (2) this generalises to: If $(t_1,t_2,\delta) \varepsilon \tilde{P}(T)$ then for any $t_3 \varepsilon \operatorname{supp}\delta$ $[t_1,t_2,\delta][t_2,t_3,\delta][t_3,t_1,\delta] = 1$.

In the sequel the indentities 1.6.1(1)-(5) will frequently be used without explicit mention. Another useful identity is the following:

1.6.2 <u>Lemma:</u> Let T be a metabelian group; $t,u \varepsilon T$; and $k \varepsilon I^+$. Then $[t,u^k] = \prod_{i=1}^{k} [t,iu]^{\binom{k}{i}}$.

Proof: We use induction on k. The case $k = 1$ is immediate, and the inductive step is

$$[t,u^k] = [t,uu^{k-1}] = [t,u][t,u^{k-1}][t,u,u^{k-1}]$$

$$= [t,u]\left(\prod_{i=1}^{k-1}[t,iu]^{\binom{k-1}{i}}\right)\left(\prod_{i=1}^{k-1}[t,u,iu]^{\binom{k-1}{i}}\right)$$

$$= [t,u]\left(\prod_{i=2}^{k\ k-1}[t,iu]^{\binom{k-1}{i}+\binom{k-1}{i-1}}\right)[t,ku]$$

$$= \prod_{i=1}^{k}[t,iu]^{\binom{k}{i}} \quad . \quad //$$

Note that with the help of 1.6.1(2) and (3) this result becomes applicable in more general situations. For example

$$[t_1^k,t_2,t_3] = \prod_{i=1}^{k}[t_1,t_2,(i-1)t_1,t_3]^{\binom{k}{i}} .$$

Of course, to prove 1.3.3 we need to know more about $G_r(0)$ than just that it satisfies the metabelian law. The further information that is needed is contained in:

1.6.3 <u>Lemma</u>: For any $m,n \in I$

$$A_m(A_n) = \langle [[w,x],[y,z]],[w,y,z^n],[x^n,y^n],[x,y]^m,x^{mn}\rangle .$$

Proof: Denoting the right-hand side above by W, we have immediately that $A_m(A_n) \geq W$. To prove the reverse inclusion let H be any group for which $W(H) = \{1\}$. Then the laws $[[w,x],[y,z]]$ and $[x,y]^m$ ensure that $A_m(A(H)) = \{1\}$, and the laws $[x^n,y^n]$ and $(x^n)^m$ ensure that $A_m(B_n(H)) = \{1\}$. Further, $A(H)$ and $B_n(H)$ commute element-wise, $[x,y,z^n]$ being a law in H. Hence $A_m(A_n(H)) = A_m(A(H).B_n(H)) = \{1\}$. We have thus shown that for any group H; $W(H) = \{1\} \Longrightarrow A_m(A_n(H)) = \{1\}$, and this means that $W \geq A_m(A_n)$. //

Actually, 1.6.3 will not be needed in its entirety until the next chapter; here we simply use the laws $[x,y,z^n]$ and $[x^n,y^n]$ to deduce some further identities (Lemmas 1.6.4-1.6.7) that hold in groups belonging to $\underline{\underline{AA}}_n$. Of course, $G_r(0) \varepsilon \underline{\underline{AA}}_n$ for all $r \varepsilon I^+$, and in fact all of these further identities will be needed for the proof of 1.3.3.

1.6.4 Lemma: Let $T \varepsilon \underline{\underline{AA}}_n$; $t,u \varepsilon T$. Then

$$[t,u^{-1}] = \prod_{i=1}^{n} [t,iu]^{-\binom{n-1}{i-1}}$$

Proof: We have

$$[t,u^{n-1}] = [t,u^{-1}u^n] = [t,u^{-1}][t,u^n][t,u^{-1},u^n],$$

and hence, since $[t,u^{-1},u^n] = 1$ (by 1.6.3),

$$[t,u^{-1}] = [t,u^{n-1}][t,u^n]^{-1}.$$

Using 1.6.2, we conclude that

$$[t,u^{-1}] = \Big(\prod_{i=1}^{n-1}[t,iu]^{\binom{n-1}{i}}\Big)\Big(\prod_{i=1}^{n}[t,iu]^{\binom{n}{i}}\Big)^{-1}$$

$$= \prod_{i=1}^{n}[t,iu]^{-\binom{n-1}{i-1}} . \quad //$$

1.6.5 Lemma: Let $T \in \underline{\underline{AA}}_n$; $t,u,v \in T$; $k \in I$. Then there exist integers $e_0(k),\ldots,e_{n-1}(k)$ such that

$$[t,u,kv] = \prod_{i=0}^{n-1}[t,u,iv]^{e_i(k)} .$$

Proof: The proof is by induction on k. For $0 \leq k \leq n - 1$ there is nothing to prove. For $k = n$, we have from 1.6.2 and 1.6.3 that $1 = [t,u,v^n] = \prod_{i=1}^{n}[t,u,iv]^{\binom{n}{i}}$ and hence $[t,u,nv] = \prod_{i=1}^{n-1}[t,u,iv]^{-\binom{n}{i}}$. The inductive step for $k \geq n$, is

$$[t,u,(k+1)v] = [t,u,kv,v]$$

$$= [\prod_{i=1}^{n-1}[t,u,iv]^{e_i(k)},v]$$

$$= \left(\prod_{i=2}^{n-1} [t,u,iv]^{e_{i-1}(k)}\right)[t,u,nv]^{e_{n-1}(k)}$$

$$= \left(\prod_{i=2}^{n-1} [t,u,iv]^{e_{i-1}(k)}\right)\left(\prod_{i=1}^{n-1} [t,u,iv]^{-\binom{n}{i}}\right)^{e_{n-1}(k)}$$

$$= \prod_{i=1}^{n-1} [t,u,iv]^{e_i(k+1)}$$

where $e_1(k+1) = -ne_{n-1}(k)$ and $e_i(k+1) = e_{i-1}(k) - \binom{n}{i}e_{n-1}(k)$ for $2 \leq i \leq n-1$. //

1.6.6 <u>Lemma</u>: Let $T \in \underline{\underline{AA}}_n$; $(t,u,\delta) \in \tilde{P}(T)$ with $\delta = \{nt,nu\}$. Then $[t,u,\delta] = \prod_{\substack{i=1 \\ i+j<2n}}^{n} \prod_{j=1}^{n} [t,u,\delta_{ij}]^{-\binom{n}{i}\binom{n}{j}}$, where $\delta_{ij} = \{it,ju\}$.

Proof: We have, from 1.6.2 and 1.6.3,

$$1 = [t^n,u^n] = \prod_{j=1}^{n} [t^n,ju]^{\binom{n}{j}} = \prod_{j=1}^{n} \left(\prod_{i=1}^{n} [t,u,\{it,ju\}]^{\binom{n}{i}}\right)^{\binom{n}{j}},$$

and the result follows. //

1.6.7 <u>Lemma</u>: Let $T \in \underline{\underline{AA}}_n$; $t,u,v \in T$. Then

$$[t,nu,v] = [v,nu,t].\prod_{i=1}^{n-1} ([v,iu,t][t,iu,v]^{-1})^{\binom{n}{i}}.$$

Proof: From 1.6.3, 1.6.1(5) and (3), and 1.6.2, we have

$$1 = [t,v,u^n] = [t,u^n,v][v,u^n,t]^{-1}$$

$$= \left(\prod_{i=1}^{n} [t,iu,v]^{\binom{n}{i}}\right)\left(\prod_{i=1}^{n} [v,iu,t]^{\binom{n}{i}}\right)^{-1},$$

and the result follows. //

We are now ready to prove 1.3.3. Throughout the proof we shall abbreviate $G_r(0)$, $\tilde{B}_r(0)$, $\phi(0)$ and $\underline{\underline{g}}_r(0) = \{g_{01},\ldots,g_{0r}\}$ to G_r, $\tilde{B}_r$, ϕ and $\underline{\underline{g}}_r = \{g_1,\ldots,g_r\}$ respectively; no ambiguity should result from this.

<u>Proof of 1.3.3</u>: The proof is broken into five steps. Defining subsets $\tilde{S}_1,\ldots,\tilde{S}_6$ of $\tilde{P}(G_r)$ by

$$\tilde{S}_1 = \{\tilde{p} \in \tilde{P}(G_r) | wt(\tilde{p}) = 2\}$$

$$\tilde{S}_2 = \{(a,b,\delta) \in \tilde{P}(G_r) | supp\delta \subseteq \underline{\underline{g}}_r \cup \underline{\underline{g}}_r^{-1}\} \text{ where } \underline{\underline{g}}_r^{-1} = \{g_1^{-1},\ldots,g_r^{-1}\}$$

$$\tilde{S}_3 = \{(a,b,\delta) \in \tilde{P}(G_r) | supp\delta \subseteq \underline{\underline{g}}_r\}$$

$$\tilde{S}_4 = \{(a,b,\delta) \in \tilde{S}_3 | b = minsupp\delta;\ a \neq b\}$$

$$\tilde{S}_5 = \{(a,b,\delta) \in \tilde{S}_4 | \delta(a) \leq n;\ \delta(b) \leq n;\ \delta(a) + \delta(b) < 2n;$$
$$\delta(c) < n \text{ for } a \neq c \neq b\}$$

$$\tilde{S}_6 = \{(a,b,\delta) \in \tilde{S}_5 | \delta(b) = n \Rightarrow a = maxsupp\delta\} = \tilde{B}_r$$

we show in the i-th step that $\tilde{S}_i\phi \subseteq gp(\tilde{S}_{i+1}\phi)$. We then have that

$$G_r' = gp([a,b]\ a,b \in G_r) = gp(\tilde{S}_1\phi) \subseteq gp(\tilde{S}_2\phi) \subseteq \cdots$$
$$\cdots \subseteq gp(\tilde{S}_6\phi) = gp(\tilde{B}_r\phi) \subseteq G_r'$$

and hence $G_r' = gp(\tilde{B}_r\phi)$ as the lemma claims.

Step 1: For any $a,b \in G_r$ we can write $a = a_1a_2\ldots a_{\ell(a)}$ and $b = b_1b_2\ldots b_{\ell(b)}$ where $a_i, b_j \in \underline{\underline{g}}_r \cup \underline{\underline{g}}_r^{-1}$ for each $i \in \{1,\ldots,\ell(a)\}$, $j \in \{1,\ldots,\ell(b)\}$. Then $[a,b] = [a_1\ldots a_{\ell(a)}, b_1\ldots b_{\ell(b)}]$ can be "expanded" using 1.6.1(4) and (2) to give an expression of the form $[a,b] = \prod_{k=1}^{s} [c_k, d_k, \delta_k]$ where for each $k \in \{1,\ldots,s\}$

$\mathrm{supp}\delta_k \subseteq \{a_1,\ldots,a_{\ell(a)}, b_1,\ldots,b_{\ell(b)}\}$. Thus $[a,b] \in gp(\tilde{S}_2\phi)$ and hence $\tilde{S}_1\phi \subseteq gp(\tilde{S}_2\phi)$.

Step 2: Let $(a,b,\delta) \in \tilde{S}_2$ with $\sum_{i=1}^{r} \delta(g_i^{-1}) = s$. If $s = 0$ then already $(a,b,\delta) \in \tilde{S}_3$, so certainly $[a,b,\delta] \in gp(\tilde{S}_3\phi)$. For $s > 0$, assume inductively that if $(a',b',\delta') \in \tilde{S}_2$ with $\sum_{i=1}^{r} \delta'(g_i^{-1}) < s$ then $[a',b',\delta'] \in gp(\tilde{S}_3\phi)$. Choosing $k \in \{1,\ldots,r\}$ such that $\delta(g_k^{-1}) > 0$, 1.6.4 shows

$$[a,b,\delta] = \prod_{j=1}^{n} [a,b,\delta_j]^{-\binom{n-1}{j-1}} \quad \text{where } \delta_j = \delta - \chi_{g_k - I^+} j\chi_{g_k},$$

$j = 1,\ldots,n$. But for each $j \in \{1,\ldots,n\}$ $\sum_{i=1}^{r} \delta_j(g_i^{-1}) = s - 1$, and it follows that $[a,b,\delta] \in gp(\tilde{S}_3\phi)$. Hence $\tilde{S}_2\phi \subseteq gp(\tilde{S}_3\phi)$.

Step 3: Let $\tilde{p} = (a,b,\delta) \in \tilde{S}_3$. Then using 1.6.1(5) and (3), for any $c \in \text{supp}\delta$ $\tilde{p}\phi = [a,c,\delta][b,c,\delta]^{-1}$. In particular, putting $c = \text{minsupp}\delta$, this shows that $\tilde{p}\phi \in gp(\tilde{S}_4\phi)$. (The cases $a = c$ and/or $b = c$ do not upset this, since, of course, $[c,c,\delta] = 1$). Hence $\tilde{S}_3\phi \subseteq gp(\tilde{S}_4\phi)$.

Step 4: Let $\tilde{p} \in \tilde{S}_4$, say $\tilde{p} = (a_1,a_2,\{d_1a_1,\ldots,d_sa_s\})$ where $\{a_1,\ldots,a_s\} \subseteq \underline{g}_r$ and $d_1,\ldots,d_s \in I^+$, for some s, $2 \leq s \leq r$. Then writing

$$\tilde{p}\phi = [a_1,a_2,(d_1-1)a_1,(d_2-1)a_2,d_3a_3,\ldots,d_sa_s],$$

we can use 1.6.5 to give

$$\text{1.6.8}\ldots \quad \tilde{p}\phi = \prod_{i_1=0}^{n-1} \prod_{i_2=0}^{n-1} \cdots \prod_{i_s=0}^{n-1} [a_1,a_2,i_1a_1,\ldots,i_sa_s]^{e_{i_1i_2\ldots i_s}}$$

where, in the notation of 1.6.5,

$$e_{i_1i_2\ldots i_s} = e_{i_1}(d_1-1).e_{i_2}(d_2-1).e_{i_3}(d_3)\ldots e_{i_s}(d_s).$$

Of the pseudo-commutators

$$(a_1,a_2,\{(i_1+1)a_1,(i_2+1)a_2,i_3a_3,\ldots,i_sa_s\})$$

whose values occur as factors of the product on the right-hand side of 1.6.8 the only ones which are not members of $\tilde{S}_5$ are those in which $i_1 = i_2 = n - 1$. However, for these 1.6.6 gives

1.6.9... $[a_1,a_2,\{na_1,na_2,i_3a_3,\ldots,i_sa_s\}]$

$$= \prod_{\substack{i=1 \\ i+j<2n}}^{n} \prod_{j=1}^{n} [a_1,a_2,\{ia_1,ja_2,i_3a_3,\ldots,i_sa_s\}]^{-\binom{n}{i}\binom{n}{j}}$$

and here every $[a_1,a_2,\{ia_1,ja_2,i_3a_3,\ldots,i_sa_s\}] \varepsilon \tilde{S}_5\phi$. Hence, between them, 1.6.8 and 1.6.9 show that $\tilde{p}\phi \varepsilon gp(\tilde{S}_5\phi)$, and so $\tilde{S}_4\phi \subseteq gp(\tilde{S}_5\phi)$.

Step 5: Let $\tilde{p} \varepsilon \tilde{S}_5$, say $\tilde{p} = (a_1,a_2,\delta)$ where $supp\delta = \{a_1,\ldots,a_s\}(\subseteq \underline{\underline{g}}_r)$ for some s; $2 \leq s \leq r$. If $\tilde{p} \notin \tilde{S}_6$ then necessarily $\delta(a_2) = n$ and $a_1 \neq maxsupp\delta$. In this case, assuming $maxsupp\delta = a_s$ (there is no loss of generality in this assumption) we obtain from 1.6.7 that

1.6.10... $\tilde{p}\phi = [a_1,a_2,\delta] = [a_s,a_2,\delta] \prod_{i=1}^{n-1} ([a_s,a_2,\delta_i][a_1,a_2,\delta_i]^{-1})^{\binom{n}{i}}$

where $\delta_i = \delta - (n-i)\chi_{a_2}$ for $i = 1,\ldots,n - 1$. But each of the pseudo-commutators $(a_s,a_2,\delta),(a_s,a_2,\delta_i),(a_1,a_2,\delta_i)$, $i = 1,\ldots,n-1$, whose value occurs as factors of the product on the right-hand side of 1.6.10 is a member of $\tilde{S}_6$, so $\tilde{p}\phi \in gp(\tilde{S}_6\phi)$. Hence $\tilde{S}_5\phi \subseteq gp(\tilde{S}_6\phi)$ and the proof of 1.3.3 is complete. //

1.7 An Alternative Basis for G'(m)

We shall need only one preliminary lemma, which is, as it were, the "reverse" of 1.6.2:

1.7.1 Lemma: Let T be a metabelian group; $t,u \in T$; and $k \in I^+$. Then $[t,ku] = \prod_{i=1}^{k} [t,u^i]^{(-1)^{k-i}\binom{k}{i}}$.

Proof: The proof is by induction on k, and is analagous to that of 1.6.2. We therefore omit the details. //

1.7.2 Definition: The mapping $\xi(m) : \tilde{B}(m) \to \tilde{P}(G(m))$ is defined by the following rule: For any $(a_1,a_2,\delta) \in \tilde{B}(m)$ with $\mathrm{supp}\delta = \{a_1,\ldots,a_s\}$ $(\subseteq \underline{g}(m))$, $s \geq 2$, say, let

$$(a_1,a_2,\delta)\xi(m) = (a_1^{\delta(a_1)},a_2^{\delta(a_2)},\{1a_1^{\delta(a_1)},\ldots,1a_s^{\delta(a_s)}\}).$$

We shall denote the set $\tilde{B}(m)\xi(m)$ by $\tilde{D}(m)$.

Note that $\xi(m)$ is clearly one-to-one.

The promised alternative basis for $G'(m)$ is given by the following:

1.7.3 <u>Theorem</u>: The valuation mapping $\psi(m) : \tilde{D}(m) \to G(m)$ is one-to-one, and $\tilde{D}(m)\psi(m)$ is a basis for $G'(m)$.

Proof: It is clear that we need only prove the analogues of 1.3.2 and 1.3.3. To be precise, for any $r \geq 2$ let $\tilde{D}_r(0) = \tilde{B}_r(0)\xi(0)$. Then the theorem is proved once we have verified the following two statements:

1.7.4... For any $r \geq 2$ $|\tilde{D}_r(0)| = (r-1)(n^r-1)$

1.7.5... For any $r \geq 2$ $G'_r(0) = gp(\tilde{D}_r(0)\psi(0))$.

Now 1.7.4 is immediate from 1.3.2, since $\xi(0)|_{\tilde{B}_r(0)}$ is one-to-one. To verify 1.7.5 it is sufficient, in view of 1.3.3, to show that $\tilde{B}_r(0)\phi(0) \subseteq gp(\tilde{D}_r(0)\psi(0))$. But this is almost immediate, for if $(a_1, a_2, \{d_1 a_1, \ldots, d_s a_s\}) \in \tilde{B}_r(0)$ (where, as usual, for some s, $2 \leq s \leq r$, $\{a_1, \ldots, a_s\} \subseteq \underline{\underline{g}}_r(0)$ and $d_1, \ldots, d_s \in I^+$) then 1.7.1 gives

$$[a_1, a_2, \{d_1 a_1, \ldots, d_s a_s\}]$$

$$= \prod_{i_1=1}^{d_1} \cdots \prod_{i_s=1}^{d_s} [a_1^{i_1}, a_2^{i_2}, \{1a_1^{i_1}, \ldots, 1a_s^{i_s}\}]^{\prod_{j=1}^{s} (-1)^{d_j - i_j} \binom{d_j}{i_j}}$$

and, since $1 \leq i_j \leq d_j$, for each $j \in \{1, \ldots, s\}$, each

$[a_1^{i_1}, a_2^{i_2}, \{1a_1^{i_1}, \ldots, 1a_s^{i_s}\}] \in \tilde{D}_r(0)\psi(0)$. //

CHAPTER 2

THE SUBVARIETIES OF $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$

For the whole of this chapter let p denote a prime number, arbitrarily chosen, but fixed throughout.

The main result is stated in 2.1, and concerns the structure of lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$). The proof of this result, modulo seven principal lemmas, is given in 2.2, while the seven lemmas are proved in sections 2.3 through 2.7. The powerful result of D.E. Cohen [3], that lat($\underline{\underline{A}}\underline{\underline{A}}$) has minimum condition, is not used in any of these proofs, and in fact, as is shown in 2.8, the minimum condition for lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$) may be independently deduced from the main result presented here.

In section 2.9, the last in this chapter, an interesting relationship between lat($\underline{\underline{A}}_p\underline{\underline{A}}_p$) and lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$) is discussed.

2.1 Statement of the Main Theorem

2.1.1 **Definition**: For all $\alpha \in I^+$ the varieties $\underline{\underline{C}}_\alpha$ and $\underline{\underline{I}}_\alpha$ are defined as follows:

$$\underline{\underline{C}}_\alpha = \underline{\underline{N}}_\alpha\underline{\underline{A}}_p \wedge \underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$$

$$\underline{\underline{I}}_\alpha = \begin{cases} \underline{\underline{C}}_\alpha \wedge \underline{\underline{B}}_{p^2} & 1 \leq \alpha \leq p-1 \\ \underline{\underline{C}}_\alpha & \alpha \geq p \end{cases}$$

2.1.2 Theorem: The varieties $\underline{\underline{I}}_1, \underline{\underline{I}}_2, \ldots$ form a properly ascending infinite chain of (proper) subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. This chain, with $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ itself adjoined, makes up a complete list of non-nilpotent join-irreducible subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. Moreover, to every non-nilpotent proper subvariety $\underline{\underline{V}}$ of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ there exists a nilpotent variety $\underline{\underline{L}}$ and a unique $\underline{\underline{I}}_\alpha$ such that $\underline{\underline{V}} = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}$. ↓↓(2.2)

2.1.3 Remark: Let $\underline{\underline{V}}$ be an arbitrary, but fixed non-nilpotent subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. By Theorem 2.1.2 we have

2.1.4... $$\underline{\underline{V}} = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}$$

where $\underline{\underline{I}}_\alpha$ is uniquely determined by $\underline{\underline{V}}$, and $\underline{\underline{L}}$ is nilpotent. Clearly $\underline{\underline{L}}$ is not uniquely determined by $\underline{\underline{V}}$; for example, it can always be enlarged by adjoining a nilpotent subvariety of $\underline{\underline{I}}_\alpha$ of sufficiently high class. Nevertheless, since by Lyndon [7] lat($\underline{\underline{L}}$) has minimum condition, there does exist an $\underline{\underline{L}}$ which is minimal with respect to satisfying 2.1.4, and the question naturally arises as to whether such a minimal $\underline{\underline{L}}$ is unique. This question is taken up in Chapter 3 where it is shown by way of an example of non-distributivity in lat($\underline{\underline{A}}_3\underline{\underline{A}}_9$) that, in general, the answer is negative.

2.2 Skeletal Proof of 2.1.2

This section comprises a series of lemmas which culminate in the proof of 2.1.2. In the interests of simplicity of presentation the proofs of seven of the most fundamental of the lemmas are postponed until later sections, but apart from these the argument is complete.

Many of the lemmas describe properties of $F_\infty(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$, and these are built up from the foundations laid in Chapter 1. This group $F_\infty(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$ is denoted throughout the chapter by G. Theorem 1.2.2 tells us that G' is free abelian of exponent p, and the basis for G' that it exhibits enables us to express elements of G' in a canonic fashion. In the present context, however, the notation may be simplified somewhat, and so, for the sake of clarity, the basis for G' is redescribed here.

Let $\underline{\underline{g}} = \{g_i \mid i \in I^+\}$ be a free generating set for G, ordered by the rule: $g_i \leq g_j$ if, and only if, $i \leq j$. Basic pseudo-commutators are defined as follows:

2.2.1 Definition: A pseudo-commutator (a,b,δ) in G is called basic if, and only if,

(1) $\text{supp}\delta \subseteq \underline{\underline{g}}$

(2) $b = \text{minsupp}\delta$

(3) $a \neq b$

(4) either (i) $\delta(a) \leq p^2$ and $\forall g_i \in \underline{\underline{g}} \, (g_i \neq a \Rightarrow \delta(g_i) < p^2)$

or (ii) $\delta(b) = p^2$, $a = \mathrm{maxsupp}\delta$

and $\forall g_i \in \underline{\underline{g}} \, (g_i \neq b \Rightarrow \delta(g_i) < p^2)$.

Denoting the set of basic pseudo-commutators in G by $\tilde{B}$, the basis for G' given by 1.2.2 may now be expressed by:

2.2.2 Theorem: The valuation mapping $\phi : \tilde{B} \to G$ is one-to-one, and $\tilde{B}\phi$ is a basis for G'. //

The notion of expressing elements of G' canonically in terms of $\tilde{B}\phi$ is formalised as follows:

2.2.3 Definition: If $w \in G'$, $w \neq 1$, then w is said to be expressed in normal form when written $w = b_1^{e_1} \ldots b_s^{e_s}$, where $b_1, \ldots, b_s$ are pair-wise distinct members of $\tilde{B}\phi$ and $e_1, \ldots, e_s$ are integers satisfying $e_j \not\equiv 0 \pmod p$ for each $j \in \{1, \ldots s\}$.

Clearly, an expression of an element of G' in normal form is unique up to the arrangement of the product and congruence modulo p of the indices. That is, if $w \in G'$ is expressed in normal form both by $w = b_1^{e_1} \ldots b_s^{e_s}$ and by

$w = c_1^{f_1} \cdots c_t^{f_t}$, then $s = t$ and, for some permutation π of $\{1,\ldots,s\}$, $b_i = c_{i\pi}$ and $e_i \equiv f_{i\pi} \pmod p$ for each $i \in \{1,\ldots,s\}$.

In addition to basic pseudo-commutators, "special" pseudo-commutators, and the accompanying attribute of "p-complexity", will be needed. These are defined as follows:

2.2.4 <u>Definition</u>: A pseudo-commutator (a,b,δ) in G is called <u>special</u> if, and only if,

(1) $\mathrm{supp}\,\delta \subseteq \underline{\underline{g}}$

(2) $b = g_1$

(3) $a = g_2$

(4) $\delta(a) = \delta(b) = 1$.

The <u>p-complexity</u> of a special pseudo-commutator $\tilde{q} = (a,b,\delta)$ in G is defined as $(1 + \sum_{i=3}^{\infty} [\delta(g_i)/p])$ and is denoted by $\mathrm{comp}(\tilde{q})$.

The definition of normal form makes possible the definition of "weight" for elements of G'. In addition, since basic pseudo-commutators may also be special, "special" elements (of G') and the "p-complexity" of "special" elements can be defined. This is all done as follows:

2.2.5 Definition: Let w be a non-trivial element of G', expressed in normal form by $w = b_1^{e_1}\ldots b_s^{e_s}$. Then the weight of w, denoted by $wt(w)$, is defined as $\min(wt(b_j\phi^{-1}) \mid j \in \{1,\ldots,s\})$. Further, if $b_j\phi^{-1}$ is special for each $j \in \{1,\ldots,s\}$, then w is itself called special, and its p-complexity, denoted by $comp(w)$ is defined as $\min(comp(b_j\phi^{-1}) \mid j \in \{1,\ldots,s\})$. The trivial element is also considered to be special, but both its weight and its p-complexity are taken as greater than that of every non-trivial element; say $wt(1) = comp(1) = \omega$.

Note that for $w_1,w_2 \in G'$ $wt(w_1w_2) \geq \min(wt(w_1),wt(w_2))$ and that this inequality can be strict. Also if w_1 and w_2 are both special then so is w_1w_2, and $comp(w_1w_2) \geq \min(comp(w_1),comp(w_2))$, where again the inequality can be strict.

Since for certain considerations special elements are particularly convenient, it is useful to have a method of obtaining special elements from non-special ones. What is meant by this, and how it is done, is explained by the following definition and lemma, but for simplicity "non-special" is generalised to "arbitrary":

2.2.6 Definition: Let $\tau : G \to G$ and $\kappa_i : G \to G$, $i \in I^+$, be the endomorphisms of G induced respectively by the maps

$$\bar{\tau} : \underline{\underline{g}} \rightarrow \underline{\underline{g}};\ g_j\bar{\tau} = g_{j+2} \text{ for all } j \in I^+,$$

and $$\bar{\kappa}_i : \underline{\underline{g}} \rightarrow G;\ g_j\bar{\kappa}_i = \begin{cases} g_j[g_2, g_1], & j = i \\ g_j & \text{otherwise} \end{cases}$$

Then for all $w \in G'$, and all $i \in I^+$, define $w^{(i)}$ by

$$w^{(i)} = (w\tau\kappa_{i+2})(w\tau)^{-1}$$

2.2.7 Lemma: For all $w \in G'$, and all $i \in I^+$, $w^{(i)}$ is special. Moreover, if w is non-trivial then so is $w^{(i)}$ for at least one value of i. ↓↓(2.4)

This completes the prepatory remarks about elements of G. Of course, the information about G required to prove 2.1.2 concerns the verbal subgroups of G, and in this connection the following notation will be used: The lattice of fully invariant subgroups (equivalently; verbal subgroups) of G is denoted by lat(G), and if $U \in$ lat(G) then id(U) denotes the ideal in lat(G) generated by U; i.e. $\text{id}(U) = \{V \in \text{lat}(G) \mid V \leq U\}$. Also, an economy in writing will often be achieved by setting $\text{id}^{\#}(U) = \text{id}(U) \setminus \{\{1\}\}$.

The lattice dual-isomorphism $\mu : \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}) \rightarrow \text{lat}(G)$, defined by $\underline{\underline{V}}\mu = V(G)$ for all $\underline{\underline{V}} \in \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$, or more particularly its inverse, will be employed to interpret statements about lat(G) as statements about lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$), and those properties of μ which are described in, or follow immediately from, sections 3 and 4 of HN will often be used without explicit mention.

Throughout this chapter the A_p-subgroup of G is denoted by M. Thus $M = A_p(G) = \underline{\underline{A}}_p\mu$ and hence M is the unique maximal verbal subgroup of G. The first major step towards the proof of 2.1.2 is the following:

2.2.8 Lemma: For all $W \in id^{\#}(G')$ there exist $c, d \in I^+$, $d \neq 1$, such that $W = M_{(c)} \cap W.G_{(d)}$. ↓↓

To see how far this gets us, note firstly that for all $\alpha \in I^+$ $M_{(\alpha+1)} = N_\alpha(A_p(G)) = \underline{\underline{C}}_\alpha\mu$. Secondly, note that if some $W \in id^{\#}(G')$ can be written $W = M_{(1)} \cap W.G_{(d)}$ for some $d \geq 2$ then $W \geq G_{(d)}$ and hence $W\mu^{-1}$ is nilpotent. Noting finally that $G' = A(G) = A_{p^3}(G)$, we have

2.2.9 Corollary: Let $\underline{\underline{W}}$ be a non-nilpotent proper subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$, $\underline{\underline{W}}$ of exponent p^3. Then there exists $\alpha \in I^+$ and a nilpotent variety $\underline{\underline{L}}$ such that $\underline{\underline{W}} = \underline{\underline{C}}_\alpha \vee \underline{\underline{L}}$. //

The proof of 2.2.8 depends on the following five lemmas:

2.2.10 <u>Lemma</u>: If for a non-trivial element $w \in G'$ the integers c and d are defined by

$$c = \min(\text{comp}(w^{(i)}) \mid i \in I^+)$$

and $$d = \max(0, \text{wt}(w) - cp)$$

then $$w \in [M_{(c)}, dG].$$ ↓↓(2.4)

2.2.11 <u>Lemma</u>: Let w be a non-trivial special element of G', with $\text{comp}(w) = c$. Then there exists $e \in I^+$ such that $\langle w \rangle \geq [M_{(c)}, eG]$. ↓↓(2.5)

2.2.12 <u>Lemma</u>: If $w \in G_{(k)}$, where $k \in I^+ \setminus \{1\}$, then $\text{wt}(w) \geq k$. ↓↓(2.3)

2.2.13 <u>Lemma</u>: For all $c, e \in I$, $c \geq 2$, $M_{(c)} \not\geq [M_{(c-1)}, eG]$. ↓↓(2.7)

2.2.14 <u>Lemma</u>: For all $c \in I^+$ $[M_{(c)}, pG] \geq M_{(c+1)}$. ↓↓(2.4)

In consequence of the first two of these lemmas we have:

2.2.15 <u>Lemma</u>: Let $w \in G'$, $w \neq 1$. Then there exist $e \in I^+$ such that $M_{(c)} \geq \langle w \rangle \geq [M_{(c)}, eG]$, where $c = \min(\text{comp}(w^{(i)}) \mid i \in I^+)$.

Proof: It is immediate from the definition (2.2.6) that $w^{(i)} \in \langle w \rangle$ for all $i \in I^+$. In particular, choosing an integer $i_w \in I^+$ such that

$$\mathrm{comp}(w^{(i_w)}) = \min(\mathrm{comp}(w^{(i)}) \mid i \in I^+) = c,$$

it follows that $\langle w \rangle \geq \langle w^{(i_w)} \rangle$ and hence, from 2.2.11, that there exists $e \in I$ such that $\langle w \rangle \geq [M_{(c)}, eG]$. On the other hand, 2.2.10 specifies an integer $d \in I$ such that $w \in [M_{(c)}, dG]$, and from this we have, a fortiori, that $w \in M_{(c)}$. Hence $M_{(c)} \geq \langle w \rangle$ and the lemma is proved. //

The above lemma easily generalises to give the following:

2.2.16 <u>Lemma</u>: Let $W \in id^{\#}(G')$. Then there exist integers $c, e \in I^+$ such that $M_{(c)} \geq W \geq [M_{(c)}, eG]$.

Proof: Let $\{w_\lambda \mid \lambda \in \Lambda\}$ be the complete set of non-trivial elements of W. From 2.2.15 we have that for each $\lambda \in \Lambda$ there exist $c_\lambda, e_\lambda \in I^+$ such that $M_{(c_\lambda)} \geq \langle w_\lambda \rangle \geq [M_{(c_\lambda)}, e_\lambda G]$, and since $W = \bigcup_{\lambda \in \Lambda} \langle w_\lambda \rangle$ it follows that $\bigcup_{\lambda \in \Lambda} M_{(c_\lambda)} \geq W \geq \bigcup_{\lambda \in \Lambda} [M_{(c_\lambda)}, e_\lambda G]$.
Now choose $\bar\lambda \in \Lambda$ such that $c_{\bar\lambda} = \min(c_\lambda \mid \lambda \in \Lambda)$ and write $c = c_{\bar\lambda}$ and $e = e_{\bar\lambda}$. Then, since $M_{(c)} \geq M_{(c+1)} \geq \ldots$, we have

$$M_{(c)} = \bigcup_{\lambda \in \Lambda} M_{(c_\lambda)} \geq W \geq \bigcup_{\lambda \in \Lambda} [M_{(c_\lambda)}, e_\lambda G] \geq [M_{(c)}, eG]. \quad //$$

The proof of 2.2.8 comes easily from 2.2.16 and one further lemma, 2.2.18 below. The proof of the latter uses the following observation, which is very similar to 2.2.12:

2.2.17 <u>Lemma</u>: If $w \in M_{(k)} \cap G'$, $k \in I^+$, then $\min(\text{comp}(w^{(i)}) | i \in I^+) \geq k$.

Proof: If $w = 1$ the lemma is immediate, so assume $w \neq 1$. Then from 2.2.15 there exists $e \in I^+$ such that $\langle w \rangle \geq [M_{(k')}, eG]$, where $k' = \min(\text{comp}(w^{(i)}) | i \in I^+)$. From this it follows that $M_{(k)} \geq [M_{(k')}, eG]$, but unless $k' \geq k$ this contradicts 2.2.13. //

2.2.18 <u>Lemma</u>: For all $c, e \in I^+$, $[M_{(c)}, eG] \geq M_{(c)} \cap G_{(cp+e)}$.

Proof: It is sufficient to show that every non-trivial element of $M_{(c)} \cap G_{(cp+e)}$ is a member of $[M_{(c)}, eG]$. So let w be any such element. Then from 2.2.12 and 2.2.17 there exist $a_1, a_2 \in I$ such that $\text{wt}(w) = cp + e + a_1$ and $\min(\text{comp}(w^{(i)}) | i \in I^+) = c + a_2$. Hence by 2.2.10, $w \in [M_{(c+a_2)}, dG]$ where $d = \max(0, cp+e+a_1-(c+a_2)p) = \max(0, e+a_1-a_2p)$. Now it follows from 2.2.14 that $[M_{(c+a_2)}, dG] \leq [M_{(c)}, (d+a_2p)G]$ and thus $w \in [M_{(c)}, d'G]$ where

$$d' = d + a_2p = \max(0, e+a_1-a_2p) + a_2p$$

$$= \max(a_2p, e+a_1)$$

$$\geq e + a_1 \geq e.$$

This shows that $w \in [M_{(c)}, eG]$, as required. //

<u>Proof of 2.2.8</u>: From 2.2.16 and 2.2.18 it follows that for all $W \in id\#(G')$ there exists $c,e \in I^+$ such that $M_{(c)} \geq W \geq M_{(c)} \cap G_{(cp+e)}$. Setting $d = cp + e$ (note that $d \geq 2$) this gives

$$W = W(M_{(c)} \cap G_{(d)}) = M_{(c)} \cap W.G_{(d)},$$

the latter equality holding by reason of the modularity of lat(G). //

The second step towards the proof of 2.1.2 is the following:

2.2.19 <u>Lemma</u>: For all $c,d \in I^+$, $c \neq 1$,
$M_{(c)} \not\geq M_{(c-1)} \cap G_{(d)}$.

Proof: Assume to the contrary that for some $c,d \in I^+$, $c \geq 2$, $M_{(c)} \geq M_{(c-1)} \cap G_{(d)}$. Then, since clearly $M_{(c-1)} \cap G_{(d)} \geq [M_{(c-1)}, dG]$, it follows that $M_{(c)} \geq [M_{(c-1)}, dG]$, and this contradicts 2.2.13. //

2.2.20 Corollary (i): The variety $\underline{\underline{C}}_1$ is non-nilpotent.

Proof: If $\underline{\underline{C}}_1$ were nilpotent then we would have that $M_{(2)} \geq G_{(d)}$ for some $d \in I^+$. But this is impossible, since $G_{(d)} = M_{(1)} \cap G_{(d)}$ and $M_{(2)} \not\geq M_{(1)} \cap G_{(d)}$. //

2.2.21 Corollary (ii): Let $\alpha, \beta \in I^+$ with $\alpha < \beta$, and let $\underline{\underline{L}}$ be a nilpotent subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. Then $\underline{\underline{C}}_\alpha \vee \underline{\underline{L}} \not\geq \underline{\underline{C}}_\beta$.

Proof: Assume the contrary. Then for some $\alpha, \beta, d \in I^+$, and some $W \in \mathrm{lat}(G)$, where $\alpha < \beta$ and $W \geq G_{(d)}$, we have $M_{(\alpha+1)} \cap W \leq M_{(\beta+1)}$. Setting $c = \alpha + 2$ and $a = \beta - \alpha$ (so $a \geq 1$ and $c \geq 3$) we conclude that

$$M_{(c-1)} \cap G_{(d)} \leq M_{(c-1)} \cap W \leq M_{(c+a-1)} \leq M_{(c)}$$

which contradicts 2.2.19. //

The next step in the argument is Lemma 2.2.22 below. In this lemma, and frequently thereafter, the notation G^{p^2} is used as a shorthand for the verbal subgroup $B_{p^2}(G)$. Although this notation conflicts with that for cartesian powers, the meaning will always be clear from the context.

2.2.22 Lemma: For each $c \in \{2, \ldots, p\}$, $M_{(c)} = M_{(c)} \cdot G^{p^2} \cap G'$. ↓↓

2.2.23 <u>Corollary</u>: For each $\alpha \in \{1,\ldots,p-1\}$,
$\underline{\underline{C}}_\alpha = \underline{\underline{I}}_\alpha \vee \underline{\underline{A}}_{p^3}$. //

The proof of 2.2.22 depends on the following lemma:

2.2.24 <u>Lemma</u>: $M_{(p)} \geq G^{p^2} \cap G'$. ↓↓(2.6)

<u>Proof of 2.2.22</u>: If $c \geq 2$ then $M_{(c)} \leq G'$, so for $c \in \{2,\ldots p\}$ we have from 2.2.24 that

$$G' \geq M_{(c)} \geq M_{(p)} \geq G^{p^2} \cap G'$$

Hence, using modularity, we have

$$M_{(c)} = M_{(c)}.(G^{p^2} \cap G') = M_{(c)}.G^{p^2} \cap G'. \quad //$$

The corollary to Lemma 2.2.8 considered the non-nilpotent subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ which have exponent p^3. The corollary to the following lemma concerns those having exponent p^2.

2.2.25 <u>Lemma</u>: Let $V = G^{p^2}.W$, $W \in id(G')$. Then there exist $c,d \in I^+$, $c \leq p$, $d > 1$, such that $V = G^{p^2}.M_{(c)} \cap V.G_{(d)}$.↓↓

2.2.26 <u>Corollary</u>: Let $\underline{\underline{V}}$ be a non-nilpotent (proper) subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$, $\underline{\underline{V}}$ of exponent p^2. Then there exists $\alpha \in \{1,\ldots,p-1\}$ and a nilpotent variety $\underline{\underline{L}}$ such that $\underline{\underline{V}} = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}$.

Proof: If $\underline{\underline{V}} \in \text{lat}(\underline{\underline{A}}_p \underline{\underline{A}}_{p^2})$ has exponent p^2 then $\underline{\underline{V}}\mu = V = G^{p^2}.W$ for some $W \in \text{id}(G')$, and from 2.2.25 $V = G^{p^2}.M_{(c)} \cap V.G_{(d)}$ for some $c,d \in I^+$, $c \leq p$, $d \nmid 1$. Now if $c = 1$ then $V \geq G_{(d)}$, making $\underline{\underline{V}} = V\mu^{-1}$ nilpotent. Hence if $\underline{\underline{V}}$ is non-nilpotent then

$$\underline{\underline{V}} = (G^{p^2}.M_{(c)})\mu^{-1} \vee (V.G_{(d)})\mu^{-1} = (\underline{\underline{B}}_{p^2} \wedge \underline{\underline{C}}_\alpha) \vee \underline{\underline{L}}$$

where $\underline{\underline{L}}$ is nilpotent and $\alpha = c-1 \in \{1,\ldots,p-1\}$. The conclusion follows. //

The following three lemmas lead up to the proof of 2.2.25:

2.2.27 Lemma: Let $a \in G$; $b \in G'$; and $r \in I^+$. Then

$$(ab)^r = a^r \prod_{i=1}^{r} [b,(i-1)a]^{\binom{r}{i}}.$$

Proof: Routine induction on r. //

2.2.28 Lemma: $[g_2,g_1,(p^2-1)g_3] \in G^{p^2}$.

Proof: From 2.2.27 we have

$$(g_3[g_2,g_1])^{p^2} = g_3^{p^2} \prod_{i=1}^{p^2} [g_2,g_1,(i-1)g_3]^{\binom{p^2}{i}}$$

but for each $i \in \{1,\ldots,p^2-1\}\binom{p^2}{i} \equiv 0 (\text{mod } p)$, and since G' has exponent p this leads to

$$(g_3^{-1})^{p^2}(g_3[g_2,g_1])^{p^2} = [g_2,g_1,(p^2-1)g_3].$$

The conclusion follows: //

2.2.29 <u>Lemma</u>: Let $W \in id(G')$, $W \geq G^{p^2} \cap G'$. Then there exist $c,d \in I^+$, $c \leq p$, $d \neq 1$, such that $W = M_{(c)} \cap W.G_{(d)}$.

Proof: In view of 2.2.8 it is only required to prove that $c \leq p$. To do this, note from 2.2.28 that $[g_2,g_1,(p^2-1)g_3] \in M_{(c)}$ and since $[g_2,g_1,(p^2-1)g_3]$ is special with p-complexity p it follows from 2.2.11 that $[M_{(p)},eG] \leq M_{(c)}$ for some $e \in I^+$. But from 2.2.13 this is impossible unless $c \leq p$. //

<u>Proof of 2.2.25</u>: Since $G^{p^2}.W = G^{p^2}.W.(G^{p^2} \cap G')$, we may assume without loss of generality that $W \geq (G^{p^2} \cap G')$. Hence, using 2.2.29, 2.2.22 and modularity, there exist $c \in \{1,\ldots,p\}$, $d \in I^+\setminus\{1\}$ such that

$$V = G^{p^2}.W = G^{p^2}.(M_{(c)} \cap W.G_{(d)})$$

$$= G^{p^2}(G^{p^2}.M_{(c)} \cap G' \cap W.G_{(d)})$$

$$= G^{p^2}(G^{p^2}.M_{(c)} \cap W.G_{(d)})$$

$$= G^{p^2}.M_{(c)} \cap G^{p^2}.W.G_{(d)}$$

$$= G^{p^2}.M_{(c)} \cap V.G_{(d)}. \quad //$$

Sufficient material is now available to prove the following two lemmas, and from these Theorem 2.1.2 will be deduced.

2.2.30 <u>Lemma</u>: Let $\underline{\underline{V}}$ be a non-nilpotent proper subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. Then there exists $\alpha \in I^+$ and a nilpotent variety $\underline{\underline{L}}$ such that $\underline{\underline{V}} = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}$.

Proof: The exponent of $\underline{\underline{V}}$ is either p^2 or p^3, for the exponent must divide p^3 and cannot be p since by Meier-Wunderli [8] any metabelian variety of prime exponent is nilpotent. If the exponent is p^2 then 2.2.26 applies leaving nothing to prove. If, on the other hand, $\underline{\underline{V}}$ has exponent p^3 then from 2.2.9 there exists $\alpha \in I^+$ and a nilpotent variety $\underline{\underline{L}}$ such that $\underline{\underline{V}} = \underline{\underline{C}}_\alpha \vee \underline{\underline{L}}$. Now either $\alpha \geq p$, so $\underline{\underline{C}}_\alpha = \underline{\underline{I}}_\alpha$ and we are finished, or $\alpha \in \{1,\ldots,p-1\}$ in which case $\underline{\underline{C}}_\alpha = \underline{\underline{I}}_\alpha \vee \underline{\underline{A}}_{p^3}$ by 2.2.23, and thus $\underline{\underline{V}} = \underline{\underline{I}}_\alpha \vee (\underline{\underline{A}}_{p^3} \vee \underline{\underline{L}})$. Since $\underline{\underline{A}}_{p^3} \vee \underline{\underline{L}}$ is nilpotent, this completes the proof. //

2.2.31 <u>Lemma</u>: The varieties $\underline{\underline{I}}_\alpha$, $\alpha \in I^+$, are non-nilpotent, and if $\alpha < \beta$ then $\underline{\underline{I}}_\alpha \vee \underline{\underline{L}} \ngeq \underline{\underline{I}}_\beta$ for any nilpotent subvariety $\underline{\underline{L}}$ of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$.

Proof: By 2.2.23 $\mathbf{C}_1 = \mathbf{I}_1 \vee \mathbf{A}_{p^3}$, and by 2.2.20 $\mathbf{C}_1$ is non-nilpotent. It follows that $\mathbf{I}_1$, and hence $\mathbf{I}_\alpha$ for all $\alpha \in I^+$, is non-nilpotent.

To prove the second part, note that for all $\alpha \in I^+$ $\mathbf{C}_\alpha$ has exponent p^3, so that $\mathbf{C}_\alpha \supseteq \mathbf{A}_{p^3}$ and hence, trivially, $\mathbf{C}_\alpha = \mathbf{C}_\alpha \vee \mathbf{A}_{p^3}$. Combining this with 2.2.23 it follows that $\mathbf{C}_\alpha = \mathbf{I}_\alpha \vee \mathbf{A}_{p^3}$ for <u>all</u> $\alpha \in I^+$. Now let $\mathbf{L} \in \operatorname{lat}(\mathbf{A}_p\mathbf{A}_{p^2})$, $\mathbf{L}$ nilpotent, and let $\alpha, \beta \in I^+$ with $\alpha < \beta$. Suppose, contrary to lemma, that $\mathbf{I}_\alpha \vee \mathbf{L} \supseteq \mathbf{I}_\beta$. Then it follows that $(\mathbf{I}_\alpha \vee \mathbf{L}) \vee \mathbf{A}_{p^3} \supseteq \mathbf{I}_\beta \vee \mathbf{A}_{p^3}$, i.e. that $\mathbf{C}_\alpha \vee \mathbf{L} \supseteq \mathbf{C}_\beta$, and this contradicts 2.2.21. //

<u>Proof of 2.1.2</u>: That each member of the infinite ascending chain of (proper) subvarieties $\mathbf{I}_1 \subseteq \mathbf{I}_2 \subseteq \ldots$ is non-nilpotent is given by 2.2.31, and from the same source it is clear that the chain ascends properly. (Put $\mathbf{L} = \mathbf{E}$ and $\beta = \alpha + 1$). Jumping now to the last part of the theorem, in view of 2.2.30 it is only required to show that if $\alpha, \beta \in I^+$, $\alpha \neq \beta$, and $\mathbf{L}_1, \mathbf{L}_2$ are nilpotent subvarieties of $\mathbf{A}_p\mathbf{A}_{p^2}$, then $\mathbf{I}_\alpha \vee \mathbf{L}_1 \neq \mathbf{I}_\beta \vee \mathbf{L}_2$. But this follows from 2.2.31, for we may assume without loss of generality that $\alpha < \beta$, so that $\mathbf{I}_\alpha \vee \mathbf{L}_1 \not\supseteq \mathbf{I}_\beta$ and therefore, in particular, $\mathbf{I}_\alpha \vee \mathbf{L}_1 \neq \mathbf{I}_\beta \vee \mathbf{L}_2$.

Now let $\Omega = \{\mathbf{I}_\alpha \mid \alpha \in I^+\} \cup \{\mathbf{A}_p\mathbf{A}_{p^2}\}$ and let Ω^* denote the set of non-nilpotent join-irreducible subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$. From 2.2.30 it is clear that $\Omega^* \subseteq \Omega$, so that the proof will be complete once it has been shown that every member of Ω is join-irreducible.

Firstly, $\underline{\underline{I}}_1$ is join-irreducible because by 2.2.30 it has no non-nilpotent proper subvarieties. Secondly, $\underline{\underline{I}}_\beta$, $\beta \in I^+\backslash\{1\}$ is join-irreducible because of the following consideration:

Suppose to the contrary that $\underline{\underline{I}}_\beta = \underline{\underline{V}}_1 \vee \underline{\underline{V}}_2$ where each of $\underline{\underline{V}}_1$ and $\underline{\underline{V}}_2$ is a proper subvariety of $\underline{\underline{I}}_\beta$. Then at least one of $\underline{\underline{V}}_1, \underline{\underline{V}}_2$ must be non-nilpotent, say $\underline{\underline{V}}_1$, so using 2.2.30 we can write $\underline{\underline{V}}_1 = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}_1$, where $\underline{\underline{L}}_1$ is nilpotent and $1 \leq \alpha < \beta$. (The latter because $\underline{\underline{V}}_1$ is a <u>proper</u> subvariety of $\underline{\underline{I}}_\beta$). Regarding $\underline{\underline{V}}_2$, either it is nilpotent, say $\underline{\underline{V}}_2 = \underline{\underline{L}}_2$, or non-nilpotent, say $\underline{\underline{V}}_2 = \underline{\underline{I}}_\gamma \vee \underline{\underline{L}}_2$ where $\underline{\underline{L}}_2$ is nilpotent and without loss of generality we may assume that $1 \leq \gamma \leq \alpha$. Setting $\underline{\underline{L}} = \underline{\underline{L}}_1 \vee \underline{\underline{L}}_2$, both cases give $\underline{\underline{I}}_\beta = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}$, which is impossible.

Finally we must show that $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ is join-irreducible. But if it were not, then, as before, we would have that $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} = \underline{\underline{I}}_\alpha \vee \underline{\underline{L}}$ for some $\alpha \in I^+$ and nilpotent variety $\underline{\underline{L}}$, and this is impossible, for it implies that $\underline{\underline{I}}_\alpha \vee \underline{\underline{L}} = \underline{\underline{I}}_{\alpha+1} \vee \underline{\underline{L}}$. //

<u>2.3 The Proof of 2.2.12:</u>

The fact that the p-group G has derived group of exponent p leads to several simplifications in calculations involving commutator elements of G. Essentially, these simplifications result from the four identities listed in the following lemma:

2.3.1 Lemma: Let $u,v,w \in G$. Then

(i) $[u,pv] = [u,v^p]$

(ii) $[u,v,p^2w] = 1$

(iii) $[u,v,\{p^2u,p^2v\}] = 1$

(iv) $[u,p^2w,v] = [v,p^2w,u]$.

Proof: (i) By 1.6.2:

$$[u,v^p] = \prod_{i=1}^{p} [u,iv]^{\binom{p}{i}}$$

But for $i \in \{1,\ldots,p-1\}$ $\binom{p}{i} \equiv 0 \pmod p$ and the conclusion follows.

(ii) By 1.6.3, $[x,y,z^{p^2}]$ is a law in G, and hence using 1.6.2 we have

$$1 = [u,v,w^{p^2}] = \prod_{i=1}^{p^2} [u,v,iw]^{\binom{p^2}{i}}$$

But for $i \in \{1,\ldots,p-1\}$ $\binom{p^2}{i} \equiv 0 \pmod p$ and the conclusion follows.

(iii) By 1.6.6:

$$[u,v,\{p^2u,p^2v\}] = \prod_{\substack{i=1 \\ i+j<2p^2}}^{p^2} \prod_{j=1}^{p^2} [u,v,iu,jv]^{-\binom{p^2}{i}\binom{p^2}{j}}$$

and the conclusion follows as before.

(iv) By 1.6.7:

$$[u,p^2w,v] = [v,p^2w,u] \prod_{i=1}^{p^2-1} ([v,iw,u][u,iw,v]^{-1})^{\binom{p^2}{i}}$$

and the conclusion again follows similarly. //

The next lemma is more directly relevant to the aim of this section, but before moving on to this lemma it is perhaps helpful to remark on a convention used in its proof (and in the proofs of future lemmas too). When an arbitrary finite subset of $\underline{g}$ is denoted by $\{a_1,\ldots,a_s\}$ it is <u>not</u> assumed that $a_1<\ldots< a_s$, although of course it <u>is</u> assumed that $a_i \neq a_j$ if $i \neq j$. However, note that a phrase such as "Let $(a_1,a_2,\delta) \in \tilde{B}$ with $\mathrm{supp}\delta = \{a_1,\ldots,a_s\}$" tacitly involves the assumption that $\min\{a_1,\dots,a_s\} = a_2$, and that $\max\{a_1,\dots,a_s\} = a_1$ if $\delta(a_2) = p^2$.

2.3.2 Lemma: Let (a_1,a_2,δ) be a pseudo-commutator in G with supp$\delta \subseteq \underline{\underline{g}}$ and non-trivial value. Then $wt([a_1,a_2,\delta]) = wt((a_1,a_2,\delta))$.

Proof: Let supp$\delta = \{a_1,\dots,a_s\}$ where $s \geq 2$ since $a_1 \neq a_2$. From 2.3.1(ii) and (iii) and the assumption that $[a_1,a_2,\delta] \neq 1$ it follows that $\delta(a_1) \leq p$; $\delta(a_2) \leq p$; $\delta(a_j) < p^2$ for $j \in \{3,\dots,s\}$; and $\delta(a_1)$ and $\delta(a_2)$ cannot both be p^2.

There are now two cases to consider.

(i) Suppose $\min\{a_1,\dots,a_s\} = a_i$, where $a_1 \neq a_i \neq a_2$. By 1.6.1(5) and (3) $[a_1,a_2,\delta] = [a_1,a_i,\delta][a_2,a_i,\delta]^{-1}$ and it follows from the restrictions on the values of the $\delta(a_j)$ $j = 1,\dots,s$ that the pseudo-commutator (a_1,a_i,δ) is basic unless $\delta(a_2) = p^2$, in which case $[a_1,a_i,\delta] = 1$ (by 2.3.1(ii)). A similar statement holds for (a_2,a_i,δ), so we conclude that the expression in normal form $[a_1,a_2,\delta]$ involves only the values of basic pseudo-commutators with degree function δ. Thus $wt([a_1,a_2,\delta]) = \sum_{j=1}^{s} \delta(a_j) = wt((a_1,a_2,\delta))$.

(ii) The alternative case occurs when $\min\{a_1,\dots,a_s\}$ is a_1 or a_2. In fact we may assume it is a_2 for clearly $wt((a_1,a_2,\delta)) = wt((a_2,a_1,\delta))$ and $wt([a_1,a_2,\delta])$ $= wt([a_2,a_1,\delta]^{-1}) = wt([a_2,a_1,\delta])$. Further, if $\delta(a_2) = p^2$ then we may assume that $\max\{a_1,\dots,a_s\} = a_1$. For if the max is

a_j then $[a_1,a_2,\delta] = [a_j,a_2,\delta]$ (by 2.3.1(iv)),

and thus

$$\mathrm{wt}([a_1,a_2,\delta]) = \mathrm{wt}([a_j,a_2,\delta]).$$

At this stage we are in fact assuming that (a_1,a_2,δ) is basic, so there is now nothing to prove. //

2.3.3 Corollary: Let (a_1,a_2,δ) be a pseudo-commutator in G with supp$\delta \subseteq \underline{\underline{g}}$ and non-trivial value. Then for all $a \in \underline{\underline{g}}$

$$\mathrm{wt}([[a_1,a_2,\delta],a]) \geq \mathrm{wt}([a_1,a_2,\delta]) + 1. \quad //$$

The above corollary generalises considerably:

2.3.4 Lemma: Let $w \in G'$, $v \in G$, with $w \neq 1 \neq v$. Then $\mathrm{wt}([w,v]) \geq \mathrm{wt}(w) + 1$.

Proof: Since G has finite exponent $v = g_{i_1}g_{i_2}\ldots g_{i_s}$ for some $i_1,\ldots,i_s \in I^+$ (not necessarily all distinct). Thus $[w,v] = [w,g_{i_1}\ldots g_{i_s}]$ and we may now proceed by induction on s. To deal with the preliminary case, s = 1, first express w in normal form by $w = b_1^{e_1}\ldots b_t^{e_t}$ say, and note that for each $j \in \{1,\ldots,t\}$ $\omega > \mathrm{wt}(b_j) \geq \mathrm{wt}(w)$. Then

$$\begin{aligned}
wt([w,g_{i_1}]) &= wt([b_1^{e_1}\ldots b_t^{e_t},g_{i_1}]) \\
&= wt([b_1,g_{i_1}]^{e_1}\ldots[b_t,g_{i_1}]^{e_t}) \\
&\geq \min(wt([b_j,g_{i_1}])\mid j \in \{1,\ldots,t\}) \\
&\geq \min(wt(b_j) + 1\mid j \in \{1,\ldots,t\}) \quad \text{(by 2.3.3)} \\
&\geq \min(wt(b_j)\mid j \in \{1,\ldots,t\}) + 1 \\
&\geq wt(w) + 1.
\end{aligned}$$

The inductive step is as follows:

$$\begin{aligned}
wt([w,g_{i_1}\ldots g_{i_s}]) &= wt([w,g_{i_1}\ldots g_{i_{s-1}}][w,g_{i_1}\ldots g_{i_{s-1}},g_{i_s}][w,g_{i_s}]) \\
&\geq \min(wt([w,g_{i_1}\ldots g_{i_{s-1}}]),wt([w,g_{i_1}\ldots g_{i_{s-1}},g_{i_s}]),wt([w,g_{i_s}])) \\
&\geq \min(wt(w)+1,\ wt(w)+2,\ wt(w)+1) \quad \text{(inductive hypothesis and case } s = 1) \\
&\geq wt(w) + 1. \quad //
\end{aligned}$$

Proof of 2.2.12: Since $G_{(c+1)} = [G_{(c)},G]$ for all $c \in I^+$, Lemma 2.2.12 easily follows from 2.3.4 by induction on c. //

2.4 The Proofs of 2.2.7, 2.2.10 and 2.2.14

We deal with Lemma 2.2.14 first, as it is needed for the proof of 2.2.10. However, rather than proving 2.2.14 directly, we first prove a stronger result, Lemma 2.4.2 below, and subsequently deduce 2.2.14 as a corollary. The reason for this indirect approach is that Lemma 2.4.2 will be needed in section 2.5.

We begin with a definition:

2.4.1 Definition: For all $c \in I^+$, $e \in I$ the verbal subgroups $U(c,e)$ and $V(c,e)$ of G are defined as follows:

$$U(c,e) = \{[y_1^p, \ldots, y_c^p, z_1, \ldots, z_e]\}(G)$$

$$V(c,e) = \{[x_1, x_2, y_2^p, \ldots, y_c^p, z_1, \ldots, z_e]\}(G)$$

The following examples should remove any uncertainty as to the intended meaning of the notation used in the definition:

$U(1,0) = \{y_1^p\}(G) = B_p(G)$; $\qquad V(1,0) = \{[x_1, x_2]\}(G) = A(G)$;

$U(2,2) = \{[y_1^p, y_2^p, z_1, z_2]\}(G)$; $\qquad V(2,2) = \{[x_1, x_2, y_2^p, z_1, z_2]\}(G)$.

Similar notations will be used frequently in the sequel, but no further comments on interpretation should be necessary.

2.4.2 <u>Lemma</u>: For all $c \in I^+$ and $e \in I$,

$[M_{(c)}, eG] = U(c,e).V(c,e)$. ↓↓

The proof of 2.4.2 uses the following two lemmas:

2.4.3 <u>Lemma</u>: Let $m \in M$. Then there exist $v \in G$ and $v' \in G'$ such that $m = v^p v'$.

Proof: Clearly $m \equiv v_1^p \ldots v_s^p \pmod{G'}$ for some $v_1, \ldots, v_s \in G$. But $v_1^p \ldots v_s^p \equiv (v_1 \ldots v_s)^p \pmod{G'}$. Thus writing $v = v_1 \ldots v_s$, we have $m = v^p v'$ for some $v' \in G'$ //

2.4.4 <u>Lemma</u>: Let $c \in I^+ \setminus \{1\}$; $t_1, \ldots, t_c \in G$; and $v_1', \ldots, v_c' \in G'$. Then

$$[t_1 v_1', \ldots, t_c v_c'] = [t_1, \ldots, t_c][v_2', t_1, t_3, \ldots, t_c]^{-1}[v_1', t_2, \ldots, t_c].$$

Proof: The proof is by induction on c. For c = 2

$$\begin{aligned}[t_1 v_1', t_2 v_2'] &= [t_1, t_2]^{v_1' v_2'}[t_1, v_2']^{v_1'}[v_1', t_2]^{v_2'}[v_1', v_2'] \\ &= [t_1, t_2][t_1, v_2'][v_1', t_2] \\ &= [t_1, t_2][v_2', t_1]^{-1}[v_1', t_2].\end{aligned}$$

For c > 2 the inductive step is as follows:

$$[t_1v_1',\ldots,t_cv_c'] = [[t_1v_1',\ldots,t_{c-1}v_{c-1}'],t_cv_c']$$

$$= [[t_1v_1',\ldots,t_{c-1}v_{c-1}'],t_c]^{v_c'}$$
$$.[[t_1v_1',\ldots,t_{c-1}v_{c-1}'],v_c']$$

$$= [[t_1v_1',\ldots,t_{c-1}v_{c-1}'],t_c]$$

$$= [[t_1,\ldots,t_{c-1}][v_2',t_1,t_3,\ldots,t_{c-1}]^{-1}$$
$$.[v_1',t_2,\ldots,t_{c-1}],t_c]$$

$$= [t_1,\ldots,t_c][v_2',t_1,t_3,\ldots,t_c]^{-1}[v_1',t_2,\ldots,t_c].//$$

Proof of 2.4.2: It is immediate that $[M_{(c)},eG] \geq U(c,e).V(c,e)$; only the reverse inclusion requires proof. Now by definition

$$[M_{(c)},eG] = gp([m_1,\ldots,m_c,w_1,\ldots,w_e] \mid m_1,\ldots,m_c \varepsilon M; w_1,\ldots,w_e \varepsilon G),$$

so that in view of 2.4.3 it is sufficient to prove

2.4.5... $\forall v_1,\ldots,v_c,w_1,\ldots,w_e \ \varepsilon\ G;\ \forall v_1',\ldots,v_c' \ \varepsilon\ G';$

$$[v_1^pv_1',\ldots,v_c^pv_c',w_1,\ldots,w_e] \ \varepsilon\ U(c,e).V(c,e).$$

In proving 2.4.5 the case c = 1 is a little exceptional, so we consider it separately first: If e = 0 the statement is trivial, for it merely asserts that $v_1^pv_1' \ \varepsilon\ G^pG'$ for all

$v_1 \varepsilon G$, $v_1' \varepsilon G'$. If, on the other hand, $e > 0$ then

$$\begin{aligned}[v_1^p v_1', w_1, \ldots, w_e] &= [[v_1^p v_1', w_1], w_2, \ldots, w_e] \\ &= [[v_1^p, w_1]^{v_1'}[v_1', w_1], w_2, \ldots, w_e] \\ &= [v_1^p, w_1, \ldots, w_e][v_1', w_1, \ldots, w_e]\end{aligned}$$

so 2.4.5 follows because clearly $[v_1^p, w_1, \ldots, w_e] \varepsilon U(1,e)$ and, since $V(1,e) = G_{(2+e)}$, $[v_1', w_1, \ldots, w_e] \varepsilon V(1,e)$.

For the proof of 2.4.2 it now remains to prove 2.4.5 for the case $c \geq 2$: Using 2.4.4 we have

$$\begin{aligned}&[v_1^p v_1', \ldots, v_c^p v_c', w_1, \ldots, w_e] \\ &= [v_1^p, \ldots, v_c^p, w_1, \ldots, w_e][v_2', v_1^p, v_3^p, \ldots, v_c^p, w_1, \ldots, w_e]^{-1} \\ &\qquad .[v_1', v_2^p, \ldots, v_c^p, w_1, \ldots, w_e]\end{aligned}$$

Further, any $v' \varepsilon G'$ can of course be written in the form

$v' = \prod_{i=1}^{k} [u_{1i}, u_{2i}]^{e_i}$, so that we deduce from the equation immediately above that

$$\begin{aligned}&[v_1^p v_1', \ldots, v_c^p v_c', w_1, \ldots, w_e] \\ &= [v_1^p, \ldots, v_c^p, w_1, \ldots, w_e] . \prod_{i=1}^{\ell} [u_{1i}, u_{2i}, v_{2i}^p, \ldots, v_{ci}^p, w_1, \ldots, w_e]^{f_i}\end{aligned}$$

for some integers $f_1, \ldots, f_\ell$ and for some $u_{1i}, u_{2i}, v_{2i}, \ldots v_{ci} \varepsilon G$, $i = 1, \ldots, \ell$. This finishes the proof, for

$[v_1^p,\ldots,v_c^p,w_1,\ldots,w_e] \in U(c,e)$ and

$[u_{1i},u_{2i},v_{2i}^p,\ldots,v_{ci},w_1,\ldots,w_e] \in V(c,e)$ for each

$i \in \{1,\ldots,\ell\}$. //

<u>Proof of 2.2.14</u>: In view of 2.4.2 it is sufficient to show that for all $c \in I^+$ $U(c,p) \geq U(c+1,0)$ and $V(c,p) \geq V(c+1,0)$. Only the first of these two inclusions is proved here, since the proof of the second follows a completely parallel course.

$$\begin{aligned} U(c,p) &= \{[y_1^p,\ldots,y_c^p,z_1,\ldots,z_p]\}(G) \\ &= gp([v_1^p,\ldots,v_c^p,w_1,\ldots,w_p] \mid v_1,\ldots,v_c,w_1,\ldots,w_p \in G) \\ &\geq gp([v_1^p,\ldots,v_c^p,pv_{c+1}] \mid v_1,\ldots,v_{c+1} \in G) \\ &= gp([v_1^p,\ldots,v_{c+1}^p] \mid v_1,\ldots,v_{c+1} \in G) \text{ (by 2.3.1(i))} \\ &= \{[y_1^p,\ldots,y_{c+1}^p]\}(G) = U(c+1,0). \quad // \end{aligned}$$

We come now to the proofs of 2.2.7 and 2.2.10.

It is clear from Definition 2.2.6 that for each (fixed) $i \in I^+$ the mapping of G' into itself given by $w \to w^{(i)}$ for all $w \in G'$ is an endomorphism of G'. The first objective, therefore, will be to describe the effect of these endomorphisms of G' on members of the basis $\tilde{B}\phi$. Such a description is a little too involved to give in a single statement, but all the necessary information is contained in items 2.4.5 through 2.4.8 below:

2.4.5 <u>Definition</u>: For any function $\delta : \underline{\underline{g}} \to I$, and any $i \in I^+$, define the function $\delta^{(i)}$ by the following rules:

$$\delta^{(i)}(g_1) = \delta^{(i)}(g_2) = 1$$

$$\delta^{(i)}(g_{i+2}) = \delta(g_i) - 1$$

$$\delta^{(i)}(g_j) = \delta(g_{j-2}) \text{ for all } j \in I^+\backslash\{1,2,i+2\}.$$

2.4.6 <u>Lemma</u>: Let $(g_{i_1}, g_{i_2}, \delta)$ be a pseudo-commutator in G with supp$\delta \subseteq \underline{\underline{g}}$. Then

(i) $[g_{i_1}, g_{i_2}, \delta]^{(i_1)} = [g_2, g_1, \delta^{(i_1)}]$

(ii) $[g_{i_1}, g_{i_2}, \delta]^{(i_2)} = [g_2, g_1, \delta^{(i_2)}]^{-1}$

(iii) $[g_{i_1}, g_{i_2}, \delta]^{(i)} = 1$ for all $i \in I^+\backslash\{i_1, i_2\}$.

Proof: Let supp$\delta = \{g_{i_1}, \ldots, g_{i_s}\}$ and set $d_j = \delta(g_{i_j})$, $j = 1, \ldots, s$. Then we can write

$$[g_{i_1},g_{i_2},\delta] = [g_{i_1},g_{i_2},(d_1-1)g_{i_1},(d_2-1)g_{i_2},d_3g_{i} ,\dots,d_sg_{i_s}]$$

and so

$$[g_{i_1},g_{i_2},\delta]\tau$$

$$= [g_{i_1+2},g_{i_2+2},(d_1-1)g_{i_1+2},(d_2-1)g_{i_2+2},d_3g_{i_3+2},\dots,d_sg_{i_s+2}].$$

Now by definition

$$[g_{i_1},g_{i_2},\delta]^{(i_1)} = [g_{i_1},g_{i_2},\delta]\tau\kappa_{i_1+2}([g_{i_1},g_{i_2},\delta])^{-1}$$

but

$$[g_{i_1},g_{i_2},\delta]\tau\kappa_{i_1+2}$$

$$= [g_{i_1+2}[g_2,g_1],g_{i_2+2},(d_1-1)g_{i_1+2}[g_2,g_1]$$

$$,(d_2-1)g_{i_2+2},d_3g_{i_3+2},\dots,d_sg_{i_s+2}]$$

$$= [g_{i_1+2},g_{i_2+2},(d_1-1)g_{i_1+2},(d_2-1)g_{i_2+2},d_3g_{i_3+2},\dots,d_sg_{i_s+2}]$$

$$.[g_2,g_1,g_{i_2+2},(d_1-1)g_{i_1+2},(d_2-1)g_{i_2+2},d_3g_{i_3+2},\dots,d_sg_{i_s+2}]$$

(by 2.4.4)

$$= [g_{i_1},g_{i_2},\delta]\tau[g_2,g_1,\delta^{(i_1)}]$$

and part (i) of the lemma follows. The proof of part (ii) is so similar that we omit it. Part (iii) is again proved along similar lines, except in this case the application of 2.4.4 gives

$$[g_{i_1},g_{i_2},\delta]\tau\kappa_{i+2} = [g_{i_1},g_{i_2},\delta]\tau \text{ for all } i \in I^+\backslash\{i_1,i_2\}.$$

But this, of course, is just what we need. //

2.4.7 Remark: It is clear from 2.4.5 that if (g_{i_1},g_{i_2},δ) is a pseudo-commutator in G with supp$\delta \subseteq g$ then the pseudo-commutators $(g_2,g_1,\delta^{(i_1)})$ and $(g_2,g_1,\delta^{(i_2)})$ are special.

2.4.8 Lemma: Let $(g_{i_1},g_{i_2},\delta) \in \tilde{B}$ and let $\{k,\ell\} = \{1,2\}$. Then for both k = 1 and k = 2

$$\text{(i)} \quad \delta(g_{i_\ell}) < p^2 \Rightarrow (g_2,g_1,\delta^{(i_k)}) \in \tilde{B}$$

$$\text{(ii)} \quad \delta(g_{i_\ell}) = p^2 \Rightarrow [g_2,g_1,\delta^{(i_k)}] = 1$$

Proof: From the definition $\delta^{(i_k)}$ and the fact that (g_{i_1},g_{i_2},δ) is basic it follows that $\delta^{(i_k)}(g_j) \leq p^2$ for all $j \in I^+$ unless $\delta(g_{i_\ell}) = p^2$, in which case $\delta^{(i_k)}(g_{i_\ell+2}) = p^2$. Part (i) of the lemma now follows immediately and for part (ii) simply observe that $[g_2,g_1,p^2g_{i_\ell+2}] = 1$ by 2.3.1(ii) //

We are now in possession of enough information to prove the first part of Lemma 2.2.7, viz:

2.4.9 Lemma: For all $w \in G'$ and all $i \in I^+$, $w^{(i)}$ is special.

Proof: For $w = 1$ there is nothing to prove, so let w be expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$, $t \geq 1$. Then for any $i \in I^+$ $w^{(i)} = (b_1^{(i)})^{e_1} \ldots (b_t^{(i)})^{e_t}$ and since a product of special elements is itself special it is sufficient to prove

2.4.10... If $(g_{i_1}, g_{i_2}, \delta) \in \tilde{B}$ then $[g_{i_1}, g_{i_2}, \delta]^{(i)}$ is special for all $i \in I^+$.

Now if $i_1 \neq i \neq i_2$ then 2.4.10 is immediate from 2.4.6(iii). Consider next the case $i = i_1$. From 2.4.6(i) $[g_{i_1}, g_{i_2}, \delta]^{(i_1)} = [g_2, g_1, \delta^{(i_1)}]$ and hence from 2.4.8(ii) $[g_{i_1}, g_{i_2}, \delta]^{(i_1)} = 1$ if $\delta(g_{i_2}) = p^2$. On the other hand if $\delta(g_{i_2}) < p^2$ then from 2.4.8(i) and 2.4.7 $(g_2, g_1, \delta^{(i_1)})$ is both basic and special and hence $[g_{i_1}, g_{i_2}, \delta]^{(i_1)}$ is special (but this time non-trivial). The proof for the case $i = i_2$ is similar, but starts with 2.4.6(ii). //

As will be shown presently, the second part of Lemma 2.2.7 follows from Lemma 2.4.11 below. However, I should point out that 2.4.11 is not really essential for this, since a proof of the result can also be obtained by putting together suitable parts of the various subsequent lemmas. But although such a proof might be more natural, the proof given here is tidier and more direct. Moreover, Lemma 2.4.11 is of interest on another score, for it may well also provide the starting point for a shorter proof of 2.2.10 than is given here. (Unfortunately my efforts in this direction have been unsuccessful).

2.4.11 Lemma: For all $w \in G'$ and all $v \in G$ $[w,v] \in \langle w^{(i)} | i \in I^+ \rangle$. ↓↓

The proof of 2.4.11 uses the following definition, lemma and corollary:

2.4.12 Definition: For each $v \in G$ and $i \in I^+$ let $\bar{\sigma}(v,i) : \underline{\underline{g}} \to G$ be the mapping defined by

$$g_1\bar{\sigma}(v,i) = v$$

$$g_2\bar{\sigma}(v,i) = g_i$$

$$g_j\bar{\sigma}(v,i) = g_{j-2} \text{ for all } j \in I^+\setminus\{1,2\}.$$

Then define $\sigma(v,i) : G \to G$ to be the endomorphism of G induced by mapping $\overline{\sigma}(v,i)$.

2.4.13 <u>Lemma</u>: For all $(g_{i_1}, g_{i_2}, \delta) \in \tilde{B}$, and all $v \in G$,

$$([g_{i_1}, g_{i_2}, \delta]^{(i_1)}\sigma(v,i_1))([g_{i_1}, g_{i_2}, \delta]^{(i_2)}\sigma(v,i_2)) = [[g_{i_1}, g_{i_2}, \delta], v].$$

Proof: One checks easily that

$$[g_2, g_1, \delta^{(i_1)}]\sigma(v,i_1) = [g_{i_1}, v, \delta+\chi_v]$$

and

$$[g_2, g_1, \delta^{(i_2)}]\sigma(v,i_2) = [g_{i_2}, v, \delta+\chi_v].$$

Hence we have

$$([g_{i_1}, g_{i_2}, \delta]^{(i_1)}\sigma(v,i_1))([g_{i_1}, g_{i_2}, \delta]^{(i_2)}\sigma(v,i_2))$$

$$= ([g_2, g_1, \delta^{(i_1)}]\sigma(v,i_1))([g_2, g_1, \delta^{(i_2)}]^{-1}\sigma(v,i_2)) \quad \text{(by 2.4.6)}$$

$$= [g_{i_1}, v, \delta+\chi_v][g_{i_2}, v, \delta+\chi_v]^{-1}$$

$$= [g_{i_1}, g_{i_2}, \delta+\chi_v] \quad \text{(by 1.6.1(3) and (5))}$$

$$= [[g_{i_1}, g_{i_2}, \delta], v]. \quad //$$

2.4.14 <u>Corollary</u>: Let $(g_{i_1}, g_{i_2}, \delta) \in \tilde{B}$; let $v \in G$; and let J be a finite subset of I^+ such that $\{i_1, i_2\} \subseteq J$. Then

$$\prod_{j\in J} ([g_{i_1}, g_{i_2}, \delta]^{(j)} \sigma(v,j)) = [[g_{i_1}, g_{i_2}, \delta], v]$$

Proof: The proof is immediate from 2.4.13 and 2.4.6(iii)//

<u>Proof of 2.4.11</u>: For $w = 1$ there is nothing to prove, so let w be expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$, $t \geq 1$. For each $k \in \{1, \ldots, t\}$ let $b_k\phi^{-1} = (g_{i_{1k}}, g_{i_{2k}}, \delta_k)$ and set $J = \bigcup_{k=1}^{t} \{g_{i_{1k}}, g_{i_{2k}}\}$. Then for any $v \in G$ we have

$$\prod_{j\in J} (w^{(j)}\sigma(v,j)) = \prod_{j\in J} ((\prod_{k=1}^{t} b_k^{e_k})^{(j)} \sigma(v,j))$$

$$= \prod_{j\in J} ((\prod_{k=1}^{t} (b_k^{(j)})^{e_k})\sigma(v,j))$$

$$= \prod_{j\in J} (\prod_{k=1}^{t} (b_k^{(j)}\sigma(v,j))^{e_k})$$

$$= \prod_{k=1}^{t} (\prod_{j\in J} (b_k^{(j)}\sigma(v,j)))^{e_k}$$

$$= \prod_{k=1}^{t} [b_k, v]^{e_k} \qquad \text{(by 2.4.14)}$$

$$= [\prod_{k=1}^{t} b_k^{e_k}, v] \qquad \text{(by 1.6.1(2))}$$

$$= [w, v]$$

Hence $[w,v] \in \langle w^{(j)} | j \in J \rangle$, and 2.4.11 follows. //

Proof of 2.2.7: In view of 2.4.9 and 2.4.11 it is now sufficient to show that if $w \in G$, $w \neq 1$ then there exists $v \in G$ such that $[w,v] \neq 1$.

Let w be expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$, where for $j = 1,\ldots,t$ $b_j\phi^{-1} = (a_j, b_j, \delta_j)$ say, and choose v such that $v \in \underline{\underline{g}} \setminus \bigcup_{j=1}^{t} \mathrm{supp}\delta_j$. Then

$$[w,v] = [\prod_{j=1}^{t} b_j^{e_j}, v] = \prod_{j=1}^{t} [b_j, v]^{e_j}$$

and hence

2.4.15... $$[w,v] = \prod_{j=1}^{t} [a_j, b_j, \delta_j + \chi_v]^{e_j}.$$

But the pseudo-commutators $(a_1, b_1, \delta_1 + \chi_v), \ldots, (a_t, b_t, \delta_t + \chi_v)$ are all basic (because of the choice of v) and are pairwise distinct (because $\delta_1, \ldots, \delta_k$ are pairwise distinct), so that $[w,v]$ is in fact expressed in normal form by 2.4.15. It follows that $[w,v] \neq 1$. //

The remainder of this section is concerned solely with proving Lemma 2.2.10. To simplify the language of the argument the following notation and terminology has been adopted:

2.4.16 <u>Notation</u>: For any $w \in G'$ denote $\min(\text{comp}(w^{(i)}) | i \in I^{+})$ by $\text{mic}(w)$.

2.4.17 <u>Definition</u>: Let w be a non-trivial element of G' and set $c = \text{mic}(w)$ and $d = \max(0, \text{wt}(w) - cp)$. Then w is said to be <u>well-behaved</u> if, and only if, $w \in [M_{(c)}, dG]$.

In terms of 2.4.17 Lemma 2.2.10 says precisely that every non-trivial element of G' is well-behaved. The following lemma indicates how the task of proving this statement is reduced:

2.4.18 <u>Lemma</u>: If $w = \prod_{i=1}^{k} w_i \neq 1$, where,

(1) $w_1, \ldots, w_k$ are well-behaved members of G'

(2) $\text{wt}(w) = \min(\text{wt}(w_i) | i \in \{1, \ldots, k\})$

(3) $\text{mic}(w) = \min(\text{mic}(w_i) | i \in \{1, \ldots, k\})$,

then w is well-behaved.

Proof: Set $c = \text{mic}(w)$, $d = \max(0,\text{wt}(w)-cp)$ and for each $i \in \{1,\ldots,k\}$ set $c_i = \text{mic}(w_i)$, $d_i = \max(0,\text{wt}(w_i)-c_i p)$.

For any $i \in \{1,\ldots,k\}$ we know from (1) that $w_i \in [M_{(c_i)}, d_i G]$ and from (3) that $c_i \geq c$. Further, from 2.2.14 it follows that

$$[M_{(c_i)}, d_i G] = [M_{(c+(c_i-c))}, d_i G] \leq [M_{(c)}, (d_i+(c_i-c)p)G]$$

and hence that $w_i \in [M_{(c)}, d'G]$, where

$$\begin{aligned} d' &= d_i + (c_i-c)p \\ &= \max(0,\text{wt}(w_i)-c_i p) + (c_i-c)p \\ &= \max((c_i-c)p,\text{wt}(w_i)-cp) \\ &\geq \max(0,\text{wt}(w_i)-cp) \\ &\geq \max(0,\text{wt}(w)-cp) \quad \text{(from (2))} \\ &= d. \end{aligned}$$

It follows that $w_i \in [M_{(c)}, dG]$ for each $i \in \{1,\ldots,k\}$ and consequently that $w \in [M_{(c)}, dG]$. That is, w is well-behaved. //

It is perhaps worth remarking that neither condition (2) nor (3) of 2.4.18 is automatically satisfied.

In order to make use of 2.4.18 we obviously need some well-behaved elements to start with. The following lemma provides some:

2.4.19 <u>Lemma</u>: Every element $w \in G'$ whose expression in normal form is of the kind $w = b^e$ $(b \in \tilde{B}\phi)$ is well-behaved. ↓↓

In addition to the description of the elements $b^{(i)}$ given by 2.4.5 through 2.4.8, the proof of 2.4.19 uses Lemmas 2.4.21 through 2.4.24 below. These four lemmas have in common the following hypothesis:

2.4.20 <u>Hypothesis</u>: Let $(g_{i_1}, g_{i_2}, \delta)$ be a pseudo-commutator in G with $\mathrm{supp}\delta = \{g_{i_1}, \ldots, g_{i_s}\}$ $(\subseteq g)$, where $s \geq 2$. For each $j \in \{1, \ldots, s\}$ write $\delta(g_{i_j}) = q_j p + r_j$, where $0 \leq r_j < p$.

2.4.21 <u>Lemma</u>: Let $(g_{i_1}, g_{i_2}, \delta)$ be as in 2.4.20. Then for both $k = 1$ and $k = 2$

$$\mathrm{comp}((g_2, g_1, \delta^{(i_k)})) = \begin{cases} 1 + \sum_{j=1}^{s} q_j & \text{if } r_k \neq 0 \\ \sum_{j=1}^{s} q_j & \text{if } r_k = 0 \end{cases}$$

Proof: From the definition of $\delta^{(i_k)}$ we have

$$\sum_{i=3}^{\infty} [\delta^{(i_k)}(g_i)/p] = \sum_{\substack{i=1 \\ i \neq i_k}}^{\infty} [\delta(g_i)/p] + [(\delta(g_{i_k})-1)/p]$$

$$= \sum_{\substack{j=1 \\ j \neq i_k}}^{s} [(q_j p + r_j)/p] + [(q_k p + r_k - 1)/p]$$

$$= \begin{cases} \sum_{j=1}^{s} q_j \text{ if } r_k \neq 0 \\ (\sum_{j=1}^{s} q_j) - 1 \text{ if } r_k = 0 \end{cases}$$

and the lemma follows. //

2.4.22 <u>Lemma</u>: In addition to 2.4.20 let $(g_{i_1}, g_{i_2}, \delta) \in \tilde{B}$. Then for both $k = 1$, $\ell = 2$ and $k = 2$, $\ell = 1$

$$\text{comp}([g_{i_1}, g_{i_2}, \delta]^{(i_k)}) = \begin{cases} \omega \text{ if } \delta(g_{i_\ell}) = p^2 \\ 1 + \sum_{j=1}^{s}{}' q_j \text{ if } \delta(g_{i_\ell}) < p^2 \text{ and } r_k \neq 0 \\ \sum_{j=1}^{s} q_j \text{ if } \delta(g_{i_\ell}) < p^2 \text{ and } r_k = 0 \end{cases}$$

Proof: The lemma is a straightforward deduction from 2.4.6(i) and (ii), 2.4.8 and 2.4.21. The details are therefore omitted. //

2.4.23 <u>Lemma</u>: In addition to 2.4.20 let $(g_{i_1}, g_{i_2}, \delta) \in \tilde{B}$. Then

$$\operatorname{mic}([g_{i_1}, g_{i_2}, \delta]) = \begin{cases} 1 + \sum_{j=1}^{r} q_j & \text{if } r_1 \neq 0 \neq r_2 \\ \sum_{j=1}^{s} q_j & \text{otherwise} \end{cases}$$

Proof: Use 2.4.6(iii), 2.4.22 and the fact that $\delta(g_{i_1})$ and $\delta(g_{i_2})$ cannot both be p^2. //

2.4.24 <u>Lemma</u>: Let $(g_{i_1}, g_{i_2}, \delta)$ be as in 2.4.20. Let integers c' and d' be defined as follows:

(i) If $r_1 \neq 0$ and $r_2 \neq 0$ then set $c' = 1 + \sum_{j=1}^{s} q_j$ and

$$d' = \left(\sum_{j=1}^{s} r_j\right) - 2$$

(ii) If $r_1 \neq 0$ and $r_2 = 0$ then set $c' = \sum_{j=1}^{s} q_j$ and

$$d' = \left(\sum_{j=1}^{s} r_j\right) - 2 + p$$

(iii) If $r_1 = 0$ and $r_2 \neq 0$ then set $c' = \sum_{j=1}^{s} q_j$ and

$$d' = (\sum_{j=1}^{s} r_j) - 2 + p$$

(iv) If $r_1 = 0$ and $r_2 = 0$ then set $c' = \sum_{j=1}^{s} q_j$ and

$$d' = \sum_{j=1}^{s} r_j$$

Then $[g_{i_1}, g_{i_2}, \delta] \in [M_{(c')}, d'G]$.

Proof: Writing w for $[g_{i_1}, g_{i_2}, \delta]$ we have

$$w = [g_{i_1}, g_{i_2}, \{(q_1p+r_1)g_{i_1}, \ldots, (q_sp+r_s)g_{i_s}\}]$$

Using 2.3.1(i) we can rewrite w in the following forms:

For case (i):-

$$w = [[g_{i_1}, g_{i_2}, q_1g_{i_1}^p, \ldots, q_sg_{i_s}^p], (r_1-1)g_{i_1}, (r_2-1)g_{i_2}, r_3g_{i_3}, \ldots, r_sg_{i_s}].$$

For case (ii):-

$$w = [[g_{i_1}, g_{i_2}, q_1g_{i_1}^p, (q_2-1)g_{i_2}^p, q_3g_{i_3}^p, \ldots, q_sg_{i_s}^p], (r_1-1)g_{i_1}, (p-1)g_{i_2}, r_3g_{i_3}, \ldots, r_sg_{i_s}].$$

For case (iii):-

$$w = [[g_{i_1}, g_{i_2}, (q_1-1)g_{i_1}^p, q_2 g_{i_2}^p, \ldots, q_s g_{i_s}^p], (p-1)g_{i_1},$$

$$(r_2-1)g_{i_2}, r_3 g_{i_3}, \ldots, r_s g_{i_s}].$$

For case (iv):-

$$w = [[g_{i_1}^p, g_{i_2}^p, (q_1-1)g_{i_1}^p, (q_2-1)g_{i_2}^p, q_3 g_{i_3}^p, \ldots, q_s g_{i_s}^p],$$

$$r_3 g_{i_3}, \ldots, r_s g_{i_s}].$$

From these expressions the lemma follows immediately. //

<u>Proof of 2.4.19</u>: Choose $b \in \tilde{B}\phi$ and an integer $e \not\equiv 0 (\mathrm{mod}\ p)$ arbitrarily, and set $w = b^e$. As usual, set $c = \mathrm{mic}(w)$ and $d = \max(0, \mathrm{wt}(w) - cp)$. Now it follows from the relevant definitions that $\mathrm{wt}(b^e) = \mathrm{wt}(b)$ and $\mathrm{comp}((b^e)^{(i)}) = \mathrm{comp}((b^{(i)})^e) = \mathrm{comp}(b^{(i)})$ for all $i \in I^+$. Thus c and d are independent of e, so that we may assume without loss of generality that $e = 1$, for if $b \in [M_{(c)}, dG]$ then certainly $b^e \in [M_{(c)}, dG]$. Consequently we have $w = b = [g_{i_1}, g_{i_2}, \delta]$ for some $(g_{i_1}, g_{i_2}, \delta) \in \tilde{B}$, and as in 2.4.20 we write $\mathrm{supp}\delta = \{g_{i_1}, \ldots, g_{i_s}\}$, $s \geq 2$, and $\delta(g_{i_j}) = q_j p + r_j$, $0 \leq r_j < p$, $j = 1, \ldots, s$. Note that in terms of this notation we have

$$wt(w) = p \sum_{j=1}^{s} q_j + \sum_{j=1}^{s} r_j, \text{ for}$$

$$wt(w) = wt(b) = wt(b\phi^{-1}) = wt((g_{i_1}, g_{i_2}, \delta))$$

$$= \sum_{i=1}^{\infty} \delta(g_i) = \sum_{j=1}^{s} \delta(g_{i_j}) = \sum_{j=1}^{s} (q_j p + r_j).$$

The proof of the lemma requires the consideration of three cases, delimited according to the values of r_1 and r_2:

Case 1: Assume that $r_1 \neq 0 \neq r_2$. From 2.4.23 $c = \sum_{j=1}^{s} q_j + 1$, and hence from 2.4.24 $w \in [M_{(c)}, d'G]$ where $d' = (\sum_{j=1}^{s} r_j) - 2$. It remains to show that $d' \geq d$. But

$$\begin{aligned} d' &= wt(w) - p\sum_{j=1}^{s} q_j - 2 \\ &= wt(w) - p(c-1) - 2 \\ &= (wt(w) - pc) + (p-2) \\ &\geq wt(w) - pc, \end{aligned}$$

amd since clearly $d' \geq 0$ we have

$$d' \geq \max(0, wt(w) - pc) = d.$$

Case 2: Assume that either $r_1 \neq 0 = r_2$ or $r_1 = 0 \neq r_2$. Then from 2.4.23 $c = \sum_{j=1}^{s} q_j$ and hence from 2.4.24 $w \in [M_{(c)}, d'G]$ where

$$
\begin{aligned}
d' &= \left(\sum_{j=1}^{s} r_j\right) - 2 + p \\
&= wt(w) - p\sum_{j=1}^{s} q_j - 2 + p \\
&= (wt(w) - pc) + (p-2) \\
&\geq wt(w) - pc.
\end{aligned}
$$

But again $d' \geq 0$, so that $d' \geq d$ and thus $w \in [M_{(c)}, dG]$.

Case 3: The only remaining possibility for the values of r_1 and r_2 is $r_1 = r_2 = 0$. For this case 2.4.23 and 2.4.24 give $w \in [M_{(c)}, d'G]$ where $c = \sum_{j=1}^{s} q_j$ and

$$
\begin{aligned}
d' = \sum_{j=1}^{s} r_j &= wt(w) - p\sum_{j=1}^{s} q_j \\
&= wt(w) - cp \\
&= d \text{ (since } wt(w) - cp = d' \geq 0).
\end{aligned}
$$

Thus, once again, $w \in [M_{(c)}, dG]$, and the lemma is proved. //

In order to make full use of 2.4.18 we need a larger initial set of well-behaved elements than is provided by

2.4.19. We need, in fact, the set of "elementary" elements of G'; where an "elementary" element is defined as follows:

2.4.25 Definition: Let w be a non-trivial element of G' expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$. Then w is called elementary, with degree function δ if, and only if, the basic pseudo-commutators $b_1\phi^{-1}, \ldots, b_t\phi^{-1}$ all have (the same) degree function δ.

The next step in the argument, therefore, is to prove the following:

2.4.26 Lemma: Every non-trivial elementary element of G' is well-behaved.

Proof: Let w be an arbitrary non-trivial element of G' expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$ say, where $b_j\phi^{-1} = (g_{i_j}, g_{i_0}, \delta)$, $j = 1, \ldots, t$ and $\operatorname{supp}\delta = \{g_{i_0}, \ldots, g_{i_s}\}$, $s \geq t$. As usual, write $\delta(g_{i_j}) = q_j p + r_j$ for each $j \in \{0, \ldots, s\}$. In addition, set $w_j = b_j^{e_j}$, $j = 1, \ldots, t$ since, where possible, we shall be using 2.4.18.

Observe that if $t = 1$ then w is well-behaved by 2.4.19, so we shall assume that $t > 1$. The assumption implies that $\delta(g_{i_j}) < p^2$ for all $j \in \{0, \ldots, s\}$ (as otherwise there is only one basic pseudo-commutator with degree function δ) and consequently

2.4.27... $(g_2, g_1, \delta^{(i)})$ is basic for every $i \in I^+$.

Another fact that we need is the following:

2.4.28... $wt(w) = wt(w_1) = \ldots = wt(w_t) = p \sum_{j=0}^{s} q_j + \sum_{j=0}^{s} r_j$.

The proof of 2.4.28 is quite straightforward and is therefore omitted.

From 2.4.28 we have in particular that $wt(w) = \min(wt(w_j) | j \in \{1,\ldots,t\})$. Since from 2.4.19 each w_j is well-behaved it now follows from 2.4.18 that if $mic(w) = \min(mic(w_j) | j \in \{1,\ldots,t\})$ then w is well-behaved. Consequently we now make the added assumption that

2.4.29... $mic(w) \neq \min(mic(w_j) | j \in \{1,\ldots,t\})$.

In order to show that w is well-behaved despite this assumption (as the lemma claims) it is necessary to first enumerate the situations for which the assumption is valid. Now from 2.4.6 we have

$$w^{(i_0)} = [g_2, g_1, \delta^{(i_0)}]^{-\sum_{j=1}^{t} e_j}$$

$$w^{(i_j)} = [g_2, g_1, \delta^{(i_j)}]^{e_j} \text{ for each } j \in \{1,\ldots,t\}$$

$$w^{(i)} = 1 \text{ for all } i \in I^+ \backslash \{i_0,\ldots,i_t\}$$

and so it follows from 2.4.27 that

2.4.30... $\mathrm{mic}(w) =$

$$= \begin{cases} \min(\mathrm{comp}((g_2, g_1, \delta^{(i_j)})) \mid j \in \{0,\ldots,t\}) & \text{if } \sum_{j=1}^{t} e_j \not\equiv 0 \pmod p \\ \min(\mathrm{comp}((g_2, g_1, \delta^{(i_j)})) \mid j \in \{1,\ldots,t\}) & \text{if } \sum_{j=1}^{t} e_j \equiv 0 \pmod p \end{cases}$$

On the other hand for $j \in \{1,\ldots,t\}$ 2.4.6 gives

$$w_j^{(i_0)} = [g_2, g_1, \delta^{(i_0)}]^{-e_j}$$

$$w_j^{(i_j)} = [g_2, g_1, \delta^{(i_j)}]^{e_j}$$

$$w_j^{(i)} = 1 \text{ for all } i \in I^+ \backslash \{i_0, i_j\}$$

and hence, using 2.4.7 we have

$$\mathrm{mic}(w_j) = \min(\mathrm{comp}((g_2, g_1, \delta^{(i_0)})), \mathrm{comp}((g_2, g_1, \delta^{(i_j)}))).$$

Thus

2.4.31... $\min(\mathrm{mic}(w_j) \mid j \in \{1,\ldots,t\}) =$

$$\min(\mathrm{comp}((g_2, g_1, \delta^{(i_j)})) \mid j \in \{0,\ldots,t\}).$$

If 2.4.30 and 2.4.31 are now compared then Lemma 2.4.21 shows that 2.4.29 is satisfied if, and only if,

2.4.32...
$$\begin{cases} \text{(i)} & \sum_{j=1}^{t} e_j \equiv 0 \pmod p \\ \text{(ii)} & r_j \not\equiv 0 \text{ for each } j \in \{1,\ldots,t\} \\ \text{(iii)} & r_0 = 0 \\ \text{(iv)} & \mathrm{mic}(w) = 1 + \sum_{j=0}^{s} q_j \end{cases}$$

Thus to complete the proof of the lemma we must show that under conditions 2.4.32 $w \in [M_{(c)}, dG]$, where $c = 1 + \sum_{j=0}^{s} q_j$ and $d = \max(0, wt(w) - cp)$. To do this first note that

$$w = \prod_{j=1}^{t} [g_{i_j}, g_{i_0}, \delta]^{e_j}$$

$$= \prod_{j=1}^{t} ([g_{i_j}, g_{i_1}, \delta][g_{i_1}, g_{i_0}, \delta])^{e_j} \quad \text{(by 1.6.1(5) and (3))}$$

$$= \left(\prod_{j=2}^{t} [g_{i_j}, g_{i_1}, \delta]^{e_j} \right) [g_{i_1}, g_{i_0}, \delta]^{\sum_{j=1}^{t} e_j}$$

$$= \prod_{j=2}^{t} [g_{i_j}, g_{i_1}, \delta]^{e_j}$$

Now from 2.4.32(ii) and 2.4.24 it follows that for $j = 2,\ldots,t$ $[g_{i_j}, g_{i_1}, \delta] \in [M_{(c')}, d'G]$ where $c' = 1 + \sum_{j=0}^{s} q_j$ and $d' = (\sum_{j=0}^{s} r_j) - 2$. Hence $w \in [M_{(c)}, d'G]$ and it only remains to show that $d' \geq d$. But

$$d' = (\sum_{j=0}^{s} r_j) - 2 = wt(w) - p\sum_{j=0}^{s} q_j - 2 \quad \text{(by 2.4.28)}$$

$$= wt(w) - p(c-1) - 2$$

$$= (wt(w)-pc) + (p-2)$$

$$\geq wt(w) - pc.$$

and since $r_1 \geq 1$, $r_2 \geq 1$ we also have $d' \geq 0$. Thus $d' \geq \max(0, wt(w)-pc) = d$ and the proof is complete. //

Of course, not every non-trivial element of G' is elementary, and we now consider the question of expressing an arbitrary element in terms of elementary ones.

Let w be a non-trivial element of G' expressed in normal form by $w = b_1^{e_1}\ldots b_t^{e_t}$. By rearranging the order of b_i's if necessary, this expression can be written in the form

$$w = b_{11}^{e_{11}}\ldots b_{1t(1)}^{e_{1t(1)}} b_{21}^{e_{21}}\ldots b_{2t(2)}^{e_{2t(2)}}\ldots b_{s1}^{e_{s1}}\ldots b_{st(s)}^{e_{st(s)}}$$

$$= w_1\ldots w_s \text{ say}$$

where, for $j = 1,\ldots,s$, $w_j = b_{j1}^{e_{j1}} \ldots b_{jt(j)}^{e_{jt(j)}}$ is elementary with degree function δ_j say, and $\delta_1,\ldots,\delta_s$ are pairwise distinct. Thus the equation $w = w_1 \ldots w_s$ expresses w as the product of its elementary parts. Note that by definition

$$wt(w) = \min(wt(b_{jk}\phi^{-1}) \mid j \in \{1,\ldots,s\},\ k \in \{1,\ldots,t(j)\})$$

$$= \min(\min(wt(b_{jk}\phi^{-1}) \mid k \in \{1,\ldots,t(j)\}) \mid j \in \{1,\ldots,s\})$$

so that we have

$$wt(w) = \min(wt(w_j) \mid j \in \{1,\ldots,s\}).$$

Moreover, as we shall now prove, we also have

$$mic(w) = \min(mic(w_j) \mid j \in \{1,\ldots,s|).$$

Let $i \in I^+$. Then $w^{(i)} = w_1^{(i)} \ldots w_s^{(i)}$, and in turn

$$w_j^{(i)} = (b_{j1}^{e_{j1}})^{(i)} \ldots (b_{jt(j)}^{e_{jt(j)}})^{(i)} = (b_{j1}^{(i)})^{e_{j1}} \ldots (b_{jt(j)}^{(i)})^{e_{jt(j)}}$$

for all $j = 1,\ldots,s$. Now from 2.4.6 and 2.4.8 it follows that for any $k \in \{1,\ldots,t(j)\}$ either $b_{jk}^{(i)} = 1$ or $b_{jk}^{(i)} = [g_2, g_1, \delta_j^{(i)}]^{\pm 1}$ where $(g_2, g_1, \delta_j^{(i)})$ is basic. Consequently, if $w_j^{(i)}$ is non-trivial then it is expressed in normal form by $w_j^{(i)} = [g_2, g_1, \delta_j^{(i)}]^{e(i,j)}$ for some integer $e(i,j) \not\equiv 0 (\mathrm{mod}\ p)$. Since $\delta_j^{(i)} \neq \delta_{j'}^{(i)}$, if $j \neq j'$ it now follows that by defining $J_i = \{j \in \{1,\ldots,s\} \mid w_j^{(i)} \neq 1\}$ we can express $w^{(i)}$ in normal form by

$$w^{(i)} = \prod_{j \varepsilon J_i} w_j^{(i)} = \prod_{j \varepsilon J_i} [g_2, g_1, \delta_j^{(i)}]^{e(i,j)}$$

(For the degenerate case of $J_i = \emptyset$ we have, of course, $w^{(i)} = 1$). Hence

$$\begin{aligned} \mathrm{comp}(w^{(i)}) &= \min(\mathrm{comp}((g_2, g_1, \delta_j^{(i)}))|j \;\varepsilon\; J_i) \\ &= \min(\mathrm{comp}(w_j^{(i)})|j \;\varepsilon\; J_i) \\ &= \min(\mathrm{comp}(w_j^{(i)})|j \;\varepsilon\; \{1,\ldots,s\}) \end{aligned}$$

(since $\mathrm{comp}(1) = \omega$)

Using this, we conclude finally that

$$\begin{aligned} \mathrm{mic}(w) &= \min(\mathrm{comp}(w^{(i)})|i \;\varepsilon\; I^+) \\ &= \min(\min(\mathrm{comp}(w_j^{(i)})|j \;\varepsilon\; \{1,\ldots,s\})|i \;\varepsilon\; I^+) \\ &= \min(\min(\mathrm{comp}(w_j^{(i)})|i \;\varepsilon\; I^+)|j \;\varepsilon\; \{1,\ldots,s\}) \\ &= \min(\mathrm{mic}(w_j)|j \;\varepsilon\; \{1,\ldots,s\}), \end{aligned}$$

which is precisely the claim we set out to prove.

To summarise, we have shown by the above remarks that:

2.4.33 <u>Lemma</u>: If a non-trivial element of $w \;\varepsilon\; G'$ is expressed as the product of its elementary parts by $w = w_1 \ldots w_2$ then

$$wt(w) = \min(wt(w_j) \mid j \in \{1,\ldots,s\})$$

$$\text{and } mic(w) = \min(mic(w_j) \mid j \in \{1,\ldots,s\}) \quad //$$

The above lemma provides the necessary connecting link between Lemmas 2.4.18 and 2.4.26, for taken together the three lemmas imply that every non-trivial element of G' is well-behaved. In other words, we have proved Lemma 2.2.10.

2.5 The Proof of 2.2.11

Many of the methods employed in this section have their origin in the Ph.D. thesis of R.A. Bryce [2]. In order to indicate the exact extent of this "borrowing" I have included at each relevant point in the section the item number of the analagous definition or lemma in [2]. It will be observed, however, that Bryce's results (in contrast to his methods) cannot be employed here, since they relate to bigroups rather than groups. Consequently, all the following lemmas require, and are given, proof, so that in this sense the entire section in independent of [2].

We begin by proving two results, Lemmas 2.5.4 and 2.5.6, which lead to a more convenient formulation of 2.2.11. The first of these results requires the following definitions:

2.5.1 Definition: Let w be a non-trivial element of G' expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$, and for each $i \varepsilon \{1,\ldots,t\}$ let $b_i\phi^{-1}$ have degree function δ_i. Then the set of entries of w, denoted by $E(w)$ is defined by $E(w) = \bigcup_{i=1}^{t} \mathrm{supp}\delta_i$. In addition, define $E(1)$ to be $\emptyset$, and for any $w_1,\ldots,w_m \varepsilon G'$ denote $\bigcup_{i=1}^{m} E(w_i)$ by $E(w_1,\ldots,w_m)$.

2.5.2 Definition: Let w be a non-trivial element of G' expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$. Then w is called homogeneous if, and only if,

$$E(b_1) = E(b_2) = \ldots = E(b_t) \; (= E(w)).$$

Clearly, any non-trivial element $w \varepsilon G'$ is the product of its homogeneous parts; i.e. $w = w_1 \ldots w_s$ where $w_1,\ldots,w_s$ are non-trivial homogeneous elements of G' with $E(w_i) \neq E(w_j)$ if $i \neq j$. In connection with this we have

2.5.3 Lemma: If w is a non-trivial element of G' then $\langle w \rangle \geq \langle w' \rangle$ for every homogeneous part w' of w.

Proof: The lemma is a special case of HN33.45. //

Now if w is a non-trivial special element of G' it is clear that the homogeneous parts of w are themselves special and that at least one of them has the same p-complexity as w. Thus from 2.5.3 we have immediately:

2.5.4 <u>Lemma</u>: Let w be a non-trivial special element of G', with comp(w) = c. Then there exists a non-trivial homogeneous special element of w' ε G', also having p-complexity c, such that $\langle w \rangle \geq \langle w' \rangle$. //

The second result concerns the subgroups U(c,e) and V(c,e) defined by 2.4.1, and is a consequence of 2.4.2 and the following lemma:

2.5.5 <u>Lemma</u>: For all $c \in I^+$, $e \in I$, $V(c,e) \geq U(c,e+1)$.

Proof: It is sufficient to show that

$$[v_1^p,\ldots,v_c^p,w_1,\ldots,w_{e+1}] \in V(c,e)$$

where the integers c and e, $c \in I^+$, $e \in I$, have been chosen arbitrarily, as have the elements $v_1,\ldots,v_c,w_1,\ldots,w_{e+1} \in G$. Now from the definition of V(c,e) it is immediate that

$$[w_{e+1}, v_1^p, v_2^p, \ldots, v_c^p, w_1, \ldots, w_e] \in V(c,e)$$

$$\text{and} \quad [w_{e+1}, v_2^p, v_1^p, \ldots, v_c^p, w_1, \ldots, w_e] \in V(c,e)$$

Hence

$$[w_{e+1}, v_2^p, v_1^p, \ldots, v_c^p, w_1, \ldots, w_e]$$
$$.[w_{e+1}, v_1^p, v_2^p, \ldots, v_c^p, w_1, \ldots, w_e]^{-1} \in V(c,e).$$

But by 1.6.1(3) and (5)

$$[w_{e+1}, v_2^p, v_1^p][w_{e+1}, v_1^p, v_2^p]^{-1} = [v_1^p, v_2^p, w_{e+1}]$$

and the result follows. //

2.5.6 <u>Lemma</u>: For all $c, e \in I^+$, $V(c,e-1) \geq [M_{(c)}, eG]$.

Proof: Trivially, $V(c,e-1) \geq V(c,e)$, so from 2.5.5 and 2.4.2 we have $V(c,e-1) \geq U(c,e).V(c,e) = [M_{(c)}, eG]$. //

From 2.5.4 and 2.5.6 it follows that Lemma 2.2.11 is equivalent to the following:

2.5.7. <u>Lemma</u>: Let w be a non-trivial homogeneous special element of G', with comp(w) = c. Then there exists $e \in I$ such that $\langle w \rangle \geq V(c,e)$. ↓↓

The proof of 2.5.7 is preceded by a sequence of preliminary lemmas, and it is the proofs of these that Bryce's methods are employed. I should perhaps remark that my original proof of 2.5.7, obtained before Bryce's work was available, was very much more complicated, so much so in fact, that I am not entirely convinced that it was valid.

2.5.8 <u>Lemma</u>: For all $u,v \in G$, $w \in G'$ and $I \in I^+$, $[w,(uv)^i] = [w,u^iv^i]$.

Proof: For some $c \in G'$ $(uv)^i = u^iv^ic$, so

$$[w,(uv)^i] = [w,u^iv^ic] = [w,u^iv^i]^c[w,c] = [w,u^iv^i]. \quad //$$

2.5.9 <u>Lemma</u>: (c.f. 4.2.5 in [2])
If $W \in id(G')$, and if for fixed elements $w_1,\ldots,w_m \in G'$ and all $v \in G$ $\prod_{i=1}^{m} [w_i,v^i] \in W$, then for all

$$v_1,\ldots,v_m \in G \; [w_m,v_m^m,v_{m-1}^{m-1},\ldots,v_1] \in W.$$

Proof: The proof is by induction on m. For m = 1 there is nothing to prove, so assume the assertion is true for $m = k - 1 \in I^+$ and now consider the case $m = k$.

Suppose, then, that for some $w_1,\ldots,w_k \in G'$

2.5.10... $$\prod_{i=1}^{k} [w_i, v^i] \in W \text{ for all } v \in G.$$

It follows immediately that for any v_k $G \prod_{i=1}^{k} [w_i, (v_k v)^i] \in W$

for all $v \in G$, and hence, by 2.5.8, that

$$\prod_{i=1}^{k} [w_i, v_k^i][w_i, v^i][w_i, v_k^i, v^i] \in W \text{ for all } v \in G.$$

Using 2.5.10 again, we conclude that

$$\prod_{i=1}^{k} [w_i, v_k^i, v^i] \in W \text{ for all } v \in G.$$

Since W is normal in G, 2.5.10 also implies that $\prod_{i=1}^{k} [w_i, v_k^i, v] \in W$ for all $v \in G$ (by 1.6.1(2)). Thus

2.5.11... $$\prod_{i=1}^{k} [w_i, v_k^i, v]^{-1} [w_i, v_k^i, v^i] \in W \text{ for all } v \in G.$$

Setting $w_i' = [w_{i+1}, v_m^{i+1}]$ for $i = 0,\ldots,k-1$, and using the identity $[w_i', v]^{-1} [w_i', v^{i+1}] = [w_i', v^i]^v$, we can rewrite 2.5.11 in the form

$$\prod_{i=1}^{k-1} [w_i', v^i]^v \in W \text{ for all } v \in G.$$

Since W is normal it follows that

$$\prod_{i=1}^{k-1} [w_i', v^i] \in W \text{ for all } v \in G.$$

By the inductive hypothesis this implies that

$$[w_{k-1}', v_{k-1}^{k-1}, v_{k-2}^{k-2}, \ldots, v_1] \in W \text{ for all } v_{k-1}, \ldots, v_1 \in G.$$

But $w_{k-1}' = [w_k, v_k^k]$ and v_k was chosen arbitrarily, so the induction is complete. //

2.5.12 <u>Definition</u>: (c.f. 4.2.6 in [2]).

For each $W \in id(G')$ and $q, e \in I$ the subset $W_{q,e}$ of G' is defined by

$$W_{q,e} = \{u \in G' \mid [u, v_1^p, \ldots, v_q^p, w_1, \ldots, w_e] \in W \text{ for all } v_1, \ldots, v_q, w_1, \ldots, w_e \in G\}.$$

2.5.13 <u>Lemma</u>: If $W \in id(G')$ and $q, q', e, e' \in I$ then

(i) $(W_{q,e})_{q',e'} = W_{q+q',e+e'}$

(ii) $W_{q,e} \in id(G')$

Proof: Since (i) is immediate from the definition we need only prove (ii). Now $W_{q,e}$ is a subgroup by 1.6.1(2) so it only remains to show that $W_{q,e}$ is fully invariant.

Let $u \in W_{q,e}$ and let θ be an endomorphism of G, u and θ chosen arbitrarily. Now choose a set $\{a_1,\ldots,a_q,b_1,\ldots,b_e\} \subseteq \underline{\underline{g}}\backslash E(u)$. Then for any $v_1,\ldots,v_q,w_1,\ldots,w_e \in G$ there exists an endomorphism θ^* of G such that $u\theta^* = u\theta$, $a_i\theta^* = v_i$ $i = 1,\ldots,q$, and $b_i\theta^* = w_i$, $i = 1,\ldots,e$. Since $[u,a_1^p,\ldots,a_q^p,b_1,\ldots,b_e] \in W$, and W is fully invariant, application of the endomorphism θ^* shows that $[u\theta,v_1^p,\ldots,v_q^p,w_1,\ldots,w_e] \in W$. Hence $u\theta \in W_{q,e}$ and the lemma is proved. //

2.5.14 <u>Lemma:</u> If $w \in G'$; $W \in id(G')$; $i \in I^+$; and if for all $v \in G$ $[w,v^i] \in W$ then

(i) $g.c.d(i,p^3) = 1 \Longrightarrow [w,v] \in W$ for all $v \in G$

(ii) $g.c.d(i,p^3) = p \Longrightarrow [w,v^p] \in W$ for all $v \in G$.

Proof: (i) There exist integers a and b such that $ai + bp^3 = 1$ and since G has exponent p^3 it follows that

$$[w,v] = [w,v^{ai+bp^3}] = [w,(v^a)^i(v^b)^{p^3}] = [w,(v^a)^i] \in W.$$

(ii) In this case we have $a'i + b'p = p$ for some integers a', b' and the conclusion follows similarly. //

2.5.15 <u>Lemma</u>: (c.f. 5.3.1 in [2]).

Let $\prod_{i=1}^{m} [w_i, ia] \in W$, where $0 < m = qp + r$, $0 \leq q$, $r < p$, $w_1, \ldots, w_m \in G'$, $a \in \underline{\underline{g}} \backslash E(w_1, \ldots, w_m)$, and $W \in id(G')$. Then $w_m \in W_{q,m-q}$.

Proof: Using Lemma 1.7.1 we have

$$\prod_{i=1}^{m} [w_i, ia] = \prod_{i=1}^{m} [w_i', a^i], \text{ where } w_i' = \prod_{j=1}^{m} w_j^{(-1)^{j-1}\binom{j}{i}}, \quad i=1,\ldots,m$$

Note that $w_m' = w_m$. Now for any $v \in G$ there exists an endomorphism θ of G such that $w_i'\theta = w_i'$, $i = 1,\ldots,m$ and $a\theta = v$, so it follows that

$$\prod_{i=1}^{m} [w_i', v^i] \in W \text{ for all } v \in G.$$

Thus, by 2.5.9, $[w_m', v_m^m, v_{m-1}^{m-1}, \ldots, v] \in W$ and, since $w_m' = w_m$, the conclusion follows by employing 2.5.14. //

2.5.16 <u>Lemma</u>: (c.f., again, 5.3.1 in [2]).

Let $w = \prod_{i=1}^{p^2-1} [w_i, ia]$, where $w_1, \ldots, w_{p^2-1} \in G'$ and $a \in \underline{\underline{g}} \backslash E(w_1, \ldots, w_{p^2-1})$. Then for each $i \in \{1, \ldots, p^2-1\}$ there exists $e_i \in I$ such that $w_i \in \langle w \rangle_{q_i, e_i}$, where $q_i = [i/p]$.

Proof: If in the previous lemma we put $m = p^2 - 1 = (p-1)p + (p-1)$ and $W = \langle w \rangle$, the case $i = p^2 - 1$ follows immediately (with $e_{p^2-1} = (p-1)p$).

In particular this means that

$$[w_{p^2-1},(p^2-1)a] = [w_{p^2-1},(p-1)a^p,(p-1)a] \in \langle w \rangle_{0,(p-1)^2}$$

But trivially $w \in \langle w \rangle_{0,(p-1)^2}$, and therefore

$$\prod_{i=1}^{p^2-2} [w_i,ia] \in \langle w \rangle_{0,(p-1)^2}$$

If we now employ 2.5.15 again, but this time with $m = p^2 - 2$ and $W = \langle w \rangle_{0,(p-1)^2}$ (the latter is permissible by 2.5.13(ii)), we obtain the assertion of the lemma for the case $i = p^2 - 2$.

With another $p^2 - 3$ applications of this procedure, the lemma is proved. //

2.5.17 Lemma: (c.f. 5.3.2 in [2]).
Let $s \in I^+$ and let $\underline{D} = \{1,\ldots,p^2-1\}^s$, so that each $\underline{d} \in \underline{D}$ is an s-tuplet $\underline{d} = (d_1,\ldots,d_s)$ with $1 \leq d_i \leq p^2 - 1$ for $i = 1,\ldots,s$.

Let $w = \prod_{\underline{d} \in \underline{D}} [w_{\underline{d}},d_1a_1,\ldots,d_sa_s]$ where $w_{\underline{d}} \in G'$ for all

$\underline{d} \in \underline{D}$ and $\{a_1,\ldots,a_s\} \subseteq \underline{\underline{g}}\setminus E(w_{\underline{d}} \mid \underline{d} \in \underline{D})$. Then for each $\underline{d} \in \underline{D}$ there exists $e_{\underline{d}} \in I$ such that $w_{\underline{d}} \in \langle w\rangle_{q_{\underline{d}},e_{\underline{d}}}$, where $q_{\underline{d}} = \sum_{i=1}^{s} [d_i/p]$.

Proof: The proof is by induction on s. For s = 1 the lemma reduces to 2.5.16, and the inductive step is as follows:

For each $d_s \in \{1,\ldots,p-1\}$ set $\underline{D}_{d_s} = \{(d_1',\ldots,d_s') \in \underline{D} \mid d_s' = d_s\}$ and let

2.5.18... $$w_{d_s} = \prod_{\underline{d}\in\underline{D}_{d_s}} [w_{\underline{d}}, d_1a_1,\ldots,d_{s-1}a_{s-1}]$$

We then have $w = \prod_{d_s=1}^{p^2-1} [w_{d_s}, d_sa_s]$, and thus, by 2.5.16, for each $d_s \in \{1,\ldots,p^2-1\}$ there exists $e_{d_s} \in I$ such that

2.5.19... $$w_{d_s} \in \langle w\rangle_{q_{d_s},e_{d_s}} \quad \text{where } q_{d_s} = [d_s/p]$$

Further, from 2.5.18 and the inductive hypothesis we have that if $\underline{d} \in \underline{D}_{d_s}$ then there exists $e_{\underline{d}}' \in I$ such that

$$w_{\underline{d}} \in \langle w_{d_s}\rangle_{q_{\underline{d}}',e_{\underline{d}}'}, \text{where } q_{\underline{d}}' = \sum_{i=1}^{s-1} [d_i/p].$$

Thus, using 2.5.19, we have for any $\underline{d} = (d_1,\ldots,d_s) \in \underline{D}$

$$w_{\underline{d}} \in \langle w_{d_s} \rangle_{q'_{\underline{d}}, e'_{\underline{d}}} \leq (\langle w \rangle_{q_{d_s}, e_{d_s}})_{q'_{\underline{d}}, e'_{\underline{d}}} = \langle w \rangle_{q_{\underline{d}}, e_{\underline{d}}}$$

where $e_{\underline{d}} = e_{d_s} + e'_{\underline{d}}$. This completes the proof. //

<u>Proof of 2.5.7</u>: Let w be a non-trivial homogeneous special element of G' with comp(w) = c and $E(w) = \{g_1, g_2, a_1, \ldots, a_s\}$ and let w be expressed in normal form by

$$w = \prod_{i=1}^{t} [g_2, g_1, \delta_i]^{e_i}.$$

Setting $\delta_i(a_j) = d_{ij}$ for all $i \in \{1,\ldots,t\}$, $j \in \{1,\ldots,s\}$, we can rewrite this expression in the form

$$w = \prod_{i=1}^{t} [[g_2, g_1]^{e_i}, d_{i1}a_1, \ldots, d_{is}a_s]$$

and thus, in the notation of 2.5.17

$$w = \prod_{\underline{d} \in \underline{D}} [w_{\underline{d}}, d_1 a_1, \ldots, d_s a_s]$$

where for $\underline{d} = (d_1,\ldots,d_s)$ $w_{\underline{d}}$ is defined by

$$w_{\underline{d}} = \begin{cases} [g_2, g_1]^{e_i} & \text{if } d_{ij} = d_j \text{ for } j = 1,\ldots,s \\ 1 & \text{otherwise} \end{cases}$$

The assumption that comp(w) = c implies that for some $i^* \varepsilon \{1,\ldots,t\}$, $c = 1 + \sum_{j=1}^{s} [d_{i^*j}/p]$, and hence that there exists $\underline{d}^* \varepsilon \underline{D}$ such that $w_{\underline{d}^*} = [g_2,g_1]^{e_{i^*}}$ and, again in the notation of 2.5.17, $q_{\underline{d}^*} = c - 1$. Thus we conclude from 2.5.17 that there exists $e \varepsilon I$ (namely $e = e_{\underline{d}^*}$) such that $[g_2,g_1]^{e_{i^*}} \varepsilon \langle w\rangle_{c-1,e}$. It follows that $[g_2,g_1] \varepsilon \langle w\rangle_{c-1,e}$ and consequently that $[u_1,u_2] \varepsilon \langle w\rangle_{c-1,e}$ for all $u_1,u_2 \varepsilon G$. That is, for all $u_1,u_2,v_2,\ldots,v_c,w_1,\ldots,w_e \varepsilon G$

$$[u_1,u_2,v_2^p,\ldots,v_c^p,w_1,\ldots,w_e] \varepsilon \langle w\rangle,$$

and this says precisely that $V(c,e) \leq \langle w\rangle$. //

2.6 The Proof of 2.2.24

The following simple observation will be required:

2.6.1 Lemma: Let R be a reduced free group of rank $\aleph_0$ and let r be a member of some free generating set for R. Then for any integer e, $r^e \varepsilon R'$ only if $r^e = 1$.

Proof: Let $\underline{\underline{r}} = \{r_i | i \varepsilon I^+\}$ be a free generating set for R chosen in such a way that $r_1 = r$. Now if $r^e \varepsilon R'$ for some $e \varepsilon I^+$ then, denoting $gp(r_1)$ by R_1, we have $r^e \varepsilon A(R) \cap R_1$. But by HN13.42 $A(R) \cap R_1 = A(R_1)$, and since R_1 is abelian the conclusion follows. //

The proof of 2.2.24 depends on the characterisation of $G^{p^2} \cap G'$ given by Lemma 2.6.2 below. The idea for the proof of this lemma was suggested to me by L.G. Kovács.

2.6.2 <u>Lemma</u>: $G^{p^2} \cap G' = \langle g_2^{-p^2} g_1^{-p^2} (g_1 g_2)^{p^2} \rangle$.

Proof: Set $V = \langle g_2^{-p^2} g_1^{-p^2} (g_1 g_2)^{p^2} \rangle$. Since $(g_1 g_2)^{p^2} = g_1^{p^2} g_2^{p^2} c$ for some $c \in G'$, it is clear that $V \leq G^{p^2} \cap G'$. Hence, if we write $H = G/V$ and $H^{p^2} = B_{p^2}(H)$, then we shall have completed the proof when we have shown that $H^{p^2} \cap H' = \{1\}$.

So let $w \in H^{p^2}$; say $w = a_1^{p^2} a_2^{p^2} \ldots a_s^{p^2}$ for some $a_1, \ldots, a_s \in H$. Now from the definition of H it follows that for all $a,b \in H$ $(ab)^{p^2} = a^{p^2} b^{p^2} = b^{p^2} a^{p^2}$. (The second equality holds because $[x^{p^2}, y^{p^2}]$ is a law in G).
Thus, writing $a_i = h_{i1}^{e_{i1}} \ldots h_{i\ell(i)}^{e_{i\ell(i)}}$ for each $i \in \{1, \ldots, s\}$ where for all i,j $e_{ij} = \pm 1$ and h_{ij} is a member of some (fixed) free generating set $\underline{\underline{h}}$, we have

$$w = (h_{11}^{e_{11}} \ldots h_{1\ell(1)}^{e_{1\ell(1)}})^{p^2} \ldots (h_{s1}^{e_{s1}} \ldots h_{s\ell(s)}^{e_{s\ell(s)}})^{p^2}$$

$$= h_{11}^{e_{11}p^2} \ldots h_{s\ell(s)}^{e_{s\ell(s)}p^2}$$

$$= h_{i_1}^{\alpha_1 p^2} \ldots h_{i_k}^{\alpha_k p^2} \quad \text{say}$$

where $h_{i_1},\ldots,h_{i_k}$ are pair-wise distinct members of $\underline{\underline{h}}$ and $\alpha_1,\ldots,\alpha_k$ are integers.

Now assume additionally that $w \in H'$. Then if for $j \in \{1,\ldots,k\}$ the endomorphisms $\sigma_j : H \to H$ are defined by $h_{i_j}\sigma_j = h_{i_j}, h_i\sigma_j = 1$ for $i \neq i_j$, it follows that $h_{i_j}^{\alpha_j p^2} = w\sigma_j \in H'$ for each $j \in \{1,\ldots,k\}$. Hence, from 2.6.1, $h_{i_1}^{\alpha_1 p^2} = h_{i_2}^{\alpha_2 p^2} = \ldots = h_{i_k}^{\alpha_k p^2} = 1$, and thus $w = 1$. This completes the proof. //

Proof of 2.2.24: In view of 2.6.2 it is sufficient to show that $g_2^{-p^2} g_1^{-p^2}(g_1g_2)^{p^2} \in M_{(p)}$, or equivalently that $(g_1g_2)^{p^2} \equiv g_1^{p^2} g_2^{p^2} (\mathrm{mod}\ M_{(p)})$. To do this, first write $(g_1g_2)^p = g_1^p g_2^p d$, where $d \in G'$, and note that $g_1^p, g_2^p, d \in M$. Now $M/M_{(p)}$ is a p-group of class less than p and as such is regular. Thus

$$(g_1g_2)^{p^2} = ((g_1g_2)^p)^p = (g_1^p g_2^p d)^p \equiv (g_1^p)^p (g_2^p)^p d^p (\mathrm{mod}\ M_{(p)})$$

and the result follows since $d^p = 1$. (G' has exponent p). //

2.7 Proof of 2.2.13

Many of the ideas for this section were suggested to me by L.G. Kovács.

Let $c, e \in I^+$, $c \geq 2$, c and e otherwise arbitrary but fixed throughout. A wreath product of finite p-groups, denoted by G^*, is defined by $G^* = R\,\mathrm{wr}(S \times T)$, where

$$R = \mathrm{gp}(r \mid r^p = 1)$$

$$S = S_1 \times \ldots \times S_{c-2};\; S_i = \mathrm{gp}(s_i \mid s_i^{p^2} = 1),\; i \in \{1, \ldots, c-2\}$$

$$T = T_0 \times \ldots \times T_e;\quad T_j = \mathrm{gp}(t_j \mid t_j^p = 1),\; j \in \{0, \ldots, e\},$$

and of course $S = \{1\}$ if $c = 2$. The base group of G^* will be denoted by K, and is to be considered as consisting of all functions from $S \times T$ into R, with multiplication defined component-wise. Additionally, for each $i \in \{1, \ldots, c-2\}$, $j \in \{0, \ldots, e\}$, notation will be abused to the extent of considering S_i and T_j (and so also S and T) as subgroups of G^* via the standard embedding.

If we now define $M^* = A_p(G^*)$, then it is clear that since $G^* \in \underline{\underline{A}}_p \underline{\underline{A}}_{p^2}$ it is sufficient for the proof of 2.2.13 to show:

2.7.1 <u>Lemma</u>: $M^*_{(c)} \not\leq [M^*_{(c-1)}, {}_eG^*]$. ↓↓

To prove 2.7.1 two facts about G^* will be required.

These are 2.7.2 and 2.7.3 below, both of which follow from results of H. Liebeck [6].

2.7.2 <u>Lemma</u>: $M^*_{((c-2)(p-1)+2)} = \{1\}$.

Proof: Clearly $M^* \leq K.S^p = \overline{M}^*$ say. Now from the proof of HN22.14 it follows that $\overline{M}^* \cong R^T \text{wr } S^p$ where R^T denotes the direct product of $|T|$ copies of R. Thus, from [6] Theorem 5.1, $\overline{M}^*$ has nilpotency class $(c-2)(p-1) + 1$ and the conclusion follows. //

2.7.3 <u>Lemma</u>: Let $k \in K$ be defined by $k(1) = r$ and $k(v) = 1$ for all $v \in (S \times T) \setminus \{1\}$. Then

$$[k, t_0, (p-1)s_1^p, \ldots, (p-1)s_{c-2}^p, t_1, \ldots, t_e] \neq 1.$$

Proof: It follows from part (a) of the proof of Theorem 5.1 in [6] that

$$[k, (p^2-1)s_1, \ldots, (p^2-1)s_{c-2}, (p-1)t_0, \ldots, (p-1)t_e] \neq 1$$

and hence, a fortiori, that

$$[k, (p-1)ps_1, \ldots, (p-1)ps_{c-2}, t_0, \ldots, t_e] \neq 1.$$

By using 2.3.1(i) this is equivalent to

$$[k,(p-1)s_1^p,\ldots,(p-1)s_{c-2}^p,t_0,t_1,\ldots,t_e] \neq 1$$

and the conclusion follows since by [6] Corollary 5.7 an alteration of the order of entries occurring after k leaves the commutator-element unchanged. //

Proof of 2.7.1: With k defined as in 2.7.3 let $w = [k,t_0,s_1^p,\ldots,s_{c-2}^p,t_1,\ldots,t_e]$. Since clearly $w \in [M^*_{(c-1)},eG^*]$, 2.7.1 will be proved when it is shown that $w \notin M^*_{(c)}$. If we suppose to the contrary that $w \in M^*_{(c)}$, then it follows that

$$[w,(p-2)s_1^p,\ldots,(p-2)s_{c-2}^p] \in M^*_{(c+(c-2)(p-2))},$$

i.e. that

$$[k,t_0,(p-1)s_1^p,\ldots,(p-1)s_{c-2}^p,t_1,\ldots,t_e] \in M^*_{((c-2)(p-1)+2)}$$

But from 2.7.2 and 2.7.3 this is impossible. //

2.8 Two Consequences of the Main Theorem

Neither of the two theorems about $\mathrm{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$ proved in this section are original, but are included here as by-products of Theorem 2.1.2.

Firstly:

2.8.1 __Theorem__: $\mathrm{lat}(\underline{\underline{A}}_p \underline{\underline{A}}_{p^2})$ has minimum condition.

As already remarked, this is a special case of D.E. Cohen's result [3] that $\mathrm{lat}(\underline{\underline{A}}\underline{\underline{A}})$ has minimum condition. However, the proof of 2.8.1 given below is quite independent of Cohen and is of interest for two reasons:

(1) It makes no use of any kind of representation theory (in contrast to Cohen's proof).

(2) It is a measure of the strength of Theorem 2.1.2.

The proof of 2.8.1 uses the following consideration:

A lattice Λ is called __join-continuous__ if for every $x \in \Lambda$ and every chain $\{y_\gamma \mid \gamma \in \Gamma\} \subseteq \Lambda$, $x \vee (\bigwedge_{\gamma\in\Gamma} y_\gamma) = \bigwedge_{\gamma\in\Gamma} (x \vee y_\gamma)$. It is readily checked that $\mathrm{lat}(\underline{\underline{V}})$ is join-continuous for every variety $\underline{\underline{V}}$, so that the following unpublished theorem of L.G. Kovács is relevant:

2.8.2 __Theorem__: Let Λ be a complete modular and join-continuous lattice. Then Λ has minimum condition if

(i) every element of Λ is the join of finitely many join-irreducible elements

and (ii) the set of join-irreducible elements of Λ has minimum condition (with respect to the partial order it inherits from Λ). //

The converse of 2.8.2 is also true; the second part of that is trivial and, as is well-known, the first part follows by very elementary considerations.

Proof of 2.8.1: It will be shown that 2.8.2(i) and (ii) are satisfied when $\Lambda = \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$.

(i) Let $\underline{\underline{V}} \in \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_{p^2})$ be a minimal counter-example. Then by 2.1.2 $\underline{\underline{V}}$ is nilpotent, which is impossible since by Lyndon [7] $\text{lat}(\underline{\underline{L}})$ has minimum condition for every nilpotent variety $\underline{\underline{L}}$.

(ii) Suppose there exists a properly descending infinite chain of join-irreducible subvarieties $\underline{\underline{V}}_1 \supset \underline{\underline{V}}_2 \supset \ldots$ From the classification of non-nilpotent join-irreducible subvarieties given by 2.1.2 it is immediate that every properly descending chain of non-nilpotent join-irreducibles breaks off, so that $\underline{\underline{V}}_k$ is nilpotent for some $k \in I^+$. But this is impossible since $\text{lat}(\underline{\underline{V}}_k)$ has minimum condition (again by Lyndon). //

The other consequence of 2.1.2 to be noted here is the following, which is a special case of a result of L.G. Kovacs and M.F. Newman (unpublished).

2.8.3 <u>Theorem</u>: A subvariety of $\mathbf{A}_p\mathbf{A}_{p^2}$ is non-nilpotent if, and only if, it contains $\mathbf{A}_p\mathbf{A}_p$. ↓↓

2.8.4 <u>Corollary</u>: Every proper subvariety of $\mathbf{A}_p\mathbf{A}_p$ is nilpotent. //

From 2.1.2 the variety $\mathbf{I}_1$ is non-nilpotent and contained in all non-nilpotent subvarieties of $\mathbf{A}_p\mathbf{A}_{p^2}$. Thus for the proof of 2.8.3 we need only show:

2.8.5 <u>Lemma</u>: $\mathbf{I}_1 = \mathbf{A}_p\mathbf{A}_p$.

Proof: By definition $\mathbf{I}_1 = \mathbf{A}\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2} \wedge \mathbf{B}_{p^2}$, so it is immediate that $\mathbf{I}_1 \supseteq \mathbf{A}_p\mathbf{A}_p$. For the reverse inclusion use 1.6.3 to show that $A(A_p).A_p(A_{p^2}).B_{p^2} \geq A_p(A_p)$. //

2.9 An Alternative Description of the Varieties $\mathbf{I}_\alpha$

2.9.1 <u>Definition</u>: For each $\alpha \in I^+$ define a variety $\overline{\mathbf{I}}_\alpha$ as follows:

$$\overline{\mathbf{I}}_\alpha = \begin{cases} \mathbf{N}_\alpha \wedge \mathbf{A}_p\mathbf{A}_p \wedge \mathbf{B}_p & \alpha \in \{1,\ldots,p-1\} \\ \mathbf{N}_\alpha \wedge \mathbf{A}_p\mathbf{A}_p & \alpha \geq p. \end{cases}$$

2.9.2 Theorem: For all $\alpha \in I^+$

$\underline{\underline{I}}_\alpha = \overline{\underline{\underline{I}}}_\alpha \underline{\underline{A}}_p \wedge \underline{\underline{A}}_p \underline{\underline{A}}_{p^2}$. ↓↓

One lemma is required:

2.9.3. Lemma: For each $c \in \{2,\ldots,p\}$
$M_{(c)}.M^p = M_{(c)}.G^{p^2}$.

Proof: Since $M \geq G^p$ it is immediate that $M_{(c)}.M^p \geq M_{(c)}.G^{p^2}$. For the reverse inclusion it is clearly sufficient to show that $M^p \leq M_{(p)}.G^{p^2}$. Now an arbitrary element of M can be written in the form $w_1^p w_2^p \ldots w_s^p c$ with $w_1,\ldots,w_s \in G$ and $c \in G'$. Hence an arbitrary element $w \in M^p$ can be written

$$w = (w_{11}^p \ldots w_{1s(1)}^p c_1)^p (w_{21}^p \ldots w_{2s(2)}^p c_2)^p \ldots (w_{t1}^p \ldots w_{ts(t)}^p c_t)^p$$

where the intended meaning of the notation is clear. As in the proof of 2.2.24 (section 2.6) we now use the facts that $M/M_{(p)}$ is regular and G' has exponent p to deduce that

$$w = w_{11}^{p^2} \ldots w_{ts(t)}^{p^2} \pmod{M_{(p)}}.$$

But this shows that $w \in M_{(p)}.G^{p^2}$ and hence that $M^p \leq M_{(p)}G^{p^2}$ as required. //

<u>Proof of 2.9.2</u>: The case $\alpha \geq p$ is immediate, for then

$$\overline{\mathbf{I}}_\alpha \mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2} = (\mathbf{N}_\alpha \wedge \mathbf{A}_p\mathbf{A}_p)\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}$$

$$= \mathbf{N}_\alpha\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_p\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2} \quad \text{(by HN21.23)}$$

$$= \mathbf{N}_\alpha\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}$$

$$= \mathbf{C}_\alpha = \mathbf{I}_\alpha$$

Now let $\alpha \in \{1,\ldots,p-1\}$. Then it follows from 2.9.3 that

$$(\mathbf{N}_\alpha\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}) \wedge (\mathbf{B}_p\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}) = (\mathbf{N}_\alpha\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}) \wedge (\mathbf{B}_{p^2} \wedge \mathbf{A}_p\mathbf{A}_{p^2})$$

and hence that

$$\mathbf{C}_\alpha \wedge \mathbf{B}_p\mathbf{A}_p = \mathbf{C}_\alpha \wedge \mathbf{B}_{p^2} = \mathbf{I}_\alpha .$$

Thus

$$\overline{\mathbf{I}}_\alpha \mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2} = (\mathbf{N}_\alpha \wedge \mathbf{B}_p \wedge \mathbf{A}_p\mathbf{A}_p)\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}$$

$$= \mathbf{N}_\alpha\mathbf{A}_p \wedge \mathbf{B}_p\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_p\mathbf{A}_p \wedge \mathbf{A}_p\mathbf{A}_{p^2}$$

$$= \mathbf{C}_\alpha \wedge \mathbf{B}_p\mathbf{A}_p = \mathbf{I}_\alpha . \quad //$$

On page 108 the description of $\text{lat}(\mathbf{A}_p\mathbf{A}_p)$ obtained by M.F.Newman (oral communication) is reproduced, and from this it is immediate that $\overline{\mathbf{I}}_\alpha$ is join-irreducible for every $\alpha \in I^+$. It is this fact that makes Theorem 2.9.2 interesting, for one wonders whether a similar situation occurs in general for varieties $\underline{\underline{A}}_p\underline{\underline{A}}_p\beta$, $\beta \in I$. I suspect that this is true, and express the conjecture formally by means of the following definition:

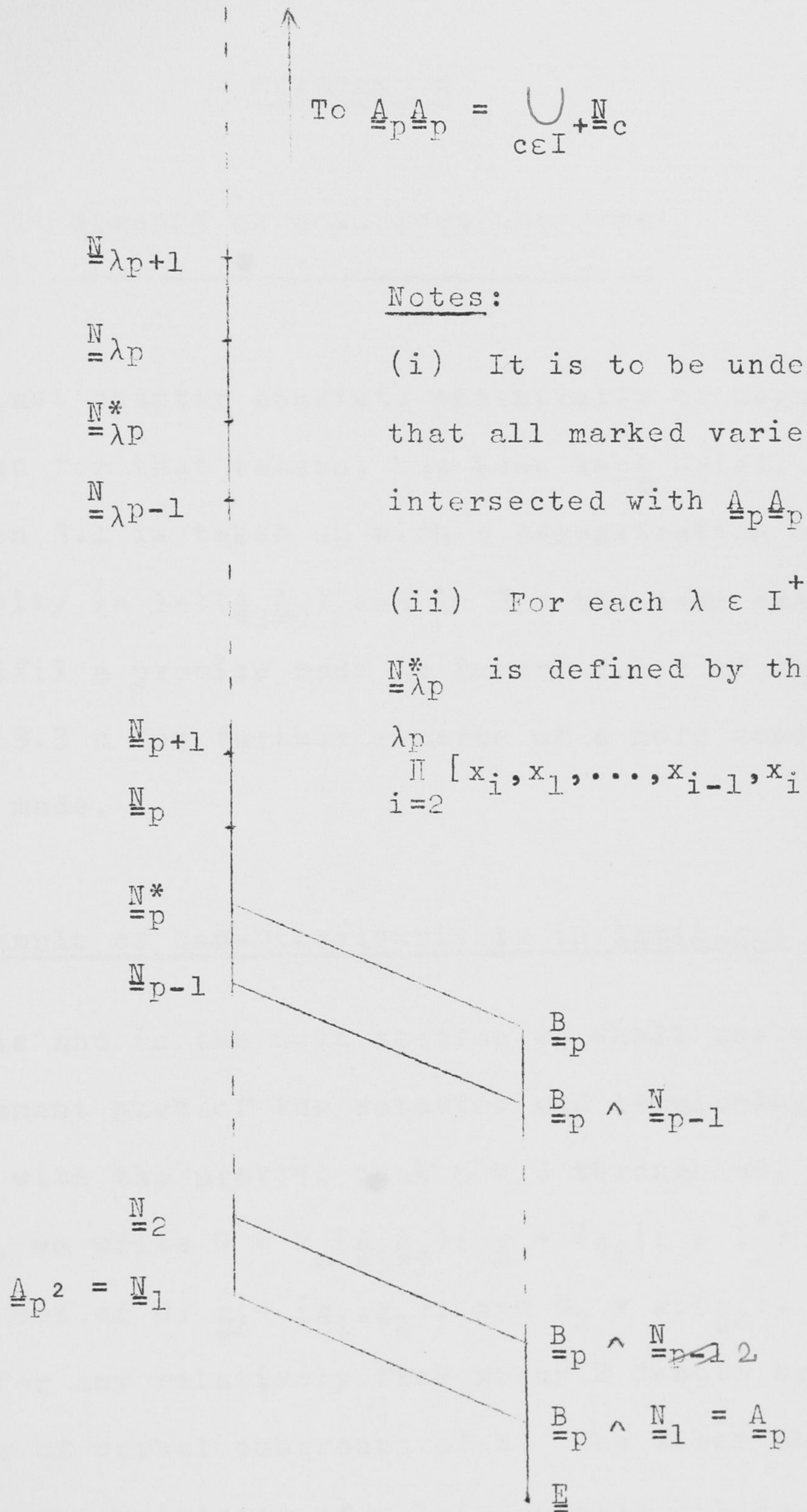
2.9.4 <u>Definition</u>: For all $\beta \in I$ let the mapping $\lambda_\beta : \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_p\beta) \to \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_p\beta+1)$ be defined by
$\underline{\underline{U}}\lambda_\beta = \underline{\underline{U}}\underline{\underline{A}}_p \wedge \underline{\underline{A}}_p\underline{\underline{A}}_p\beta+1$ for all $\underline{\underline{U}} \in \text{lat}(\underline{\underline{A}}_p\underline{\underline{A}}_p\beta)$.

2.9.5 <u>Conjecture</u>: For all $\beta \in I$, every non-nilpotent join irreducible subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p\beta+1$ is the image under λ_β of some join-irreducible (but possibly nilpotent) subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p\beta$.

From 2.8.4 it is immediate that the conjecture is true for $\beta = 0$, and from 2.1.2, 2.9.2 and the remarks preceeding 2.9.4 it follows that the conjecture is also true for $\beta = 1$. Further supporting evidence is provided by R.A. Bryce's study in [2] of "bivarieties" $\underline{\underline{A}}_p\alpha \circ \underline{\underline{A}}_p\beta$, but it must be admitted that this evidence is very indirect.

Finally, note that not every join-irreducible subvariety of $\underline{\underline{A}}_p\underline{\underline{A}}_p\beta$ leads via λ_β to a join-irreducible of $\underline{\underline{A}}_p\underline{\underline{A}}_p\beta$. For example, the subvariety $\underline{\underline{A}}_{p^2}$ of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ is join-irreducible, but, as is easily checked, $\underline{\underline{A}}_{p^2}\lambda_1 = \underline{\underline{A}}_p\underline{\underline{A}}_p \vee \underline{\underline{A}}_{p^3}$.

FIG. 2



Notes:

(i) It is to be understood that all marked varieties are intersected with A_pA_p

(ii) For each $\lambda \varepsilon I^+$ the variety $N^*_{\lambda p}$ is defined by the law:

$$\prod_{i=2}^{\lambda p} [x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\lambda p}]$$

THE SUBVARIETY LATTICE OF A_pA_p

CHAPTER 3

REMARKS ON NON-DISTRIBUTIVITY

This last chapter consists essentially of negative results, and for that reason, has been kept brief.

Section 3.1 is taken up with a demonstration of non-distributivity in lat($\underline{\underline{A}}_3\underline{\underline{A}}_9$) and in 3.2 the same example is used to fulfil a promise made in Remark 2.1.3 of Chapter 2. Finally in 3.3 a few further remarks of a more general nature are made.

3.1 An Example of Non-Distributivity in lat($\underline{\underline{A}}_3\underline{\underline{A}}_9$)

In this and in the next section we shall use without further comment much of the notation and terminology of Chapter 2, with the proviso that p = 3 throughout. Thus in particular, we write $G = F_\infty(\underline{\underline{A}}_3\underline{\underline{A}}_9)$; $\underline{\underline{g}} = \{g_i \mid i \in I^+\}$ a free generating set of G; $\underline{\underline{g}}_2 = \{g_1, g_2\}$; and $G_2 = gp(\underline{\underline{g}}_2)$. In addition, for any relatively free group H denote by lat(H) the lattice of verbal subgroups of H. The first objective is to prove the existence of a lattice epimorphism from lat(G) to lat(G_2):

Let $\xi_1 : G \to G_2$ be the natural projection endomorphism. If $W \varepsilon \text{lat}(G)$ then $W = V(G)$ for some closed set of words V, and hence by HN12.31.

3.1.1... $W\xi_1 = V(G)\xi_1 = V(G\xi_1) = V(G_2) \varepsilon \text{lat}(G_2)$

Thus ξ_1 induces an onto mapping $\Xi_1 : \text{lat}(G) \to \text{lat}(G_2)$ defined by

3.1.2... $W\Xi_1 = W\xi_1$ for all $W \varepsilon \text{lat}(G)$.

From HN13.42 $V(G_2) = V(G) \cap G_2$ for any closed set of words V so that from 3.1.1 and 3.1.2 we have

3.1.3... $W\Xi_1 = W \cap G_2$ for all $W \varepsilon \text{lat}(G)$.

From 3.1.2 it is clear that Ξ_1 is a join-homomorphism while from 3.1.3 it is equally clear that Ξ_1 is a meet-homomorphism, so that Ξ_1 is, in fact, a lattice epimorphism.

Now set $G^* = G_2/(G_2)_{(12)}$ and let $\xi_2 : G_2 \to G^*$ be the natural epimorphism. If $\Xi_2 : \text{lat}(G_2) \to \text{lat}(G^*)$ is now defined by analogy with 3.1.2 then by HN13.32 Ξ_2 is also a lattice epimorphism. Thus $\Xi = \Xi_1\Xi_2 : \text{lat}(G) \to \text{lat}(G^*)$ is a lattice epimorphism and it follows that the non-distributivity of lat(G), and hence of $\text{lat}(\underline{\underline{A}}_3\underline{\underline{A}}_9)$, will be established by demonstrating non-distributivity in lat(G*).

The example we shall provide occurs among the subgroups of $G^*_{(11)}$ which, of course, is the least non-trivial term of the lower central series of G^*. We need the following description of $G^*_{(11)}$:

Let $g_1\xi_2 = g_1^*$ and $g_2\xi_2 = g_2^*$, so that $\underline{\underline{g}}^* = \{g_1^*,g_2^*\}$ is a free generating set for G^*. If now for each $i \in \{2,\ldots,9\}$ we set $w_i = [g_2^*,ig_1^*,(10-i)g_2^*]$ then, we claim, $G^*_{(11)}$ is an elementary 3-group with basis $\{w_2,\ldots,w_9\}$. The first part is immediate, for G' is free abelian of exponent 3. For the second part note that

$$G^*_{(11)} = (G_2)_{(11)}\xi_2 = (G_{(11)} \cap G_2)\xi_2$$

and that it follows from Lemma 2.2.12 that $G_{(11)} \cap G_2$ has a basis consisting of the values of all basic pseudo-commutators in G with set of entries $\{g_1,g_2\}$ and weight not less than 11. Of these ξ_2 kills all those, and only those, of weight not less than 12 (again by 2.2.12) and what remains is precisely the set $\{w_2,\ldots,w_9\}$. Thus $w_2,\ldots,w_9$ generate $G^*_{(11)}$ and it is easy to see that any dependence among them would involve dependence among the basis for $G_{(11)} \cap G_2$.

The next task is to obtain a usable criterion by which to determine whether any given subgroup of $G^*_{(11)}$ is fully invariant in G^*:

Let α,β,γ be the automorphisms of G^* given by

$$g_1^*\alpha = g_1^*g_2^* \qquad g_1^*\beta = g_2^* \qquad g_1^*\gamma = g_1^{*-1}$$

$$g_2^*\alpha = g_2^* \qquad g_2^*\beta = g_1^* \qquad g_2^*\gamma = g_2^*$$

Let $M^* = A_p(G^*)$ and for any endomorphism η of G^* denote by η/M^* the endomorphism of G^*/M^* induced by η. We claim that $\{\alpha/M^*,\beta/M^*,\gamma/M^*\}$ forms a generating set for the automorphism group of G^*/M^*. To see this, note that G^*/M^* is just a two-dimensional vector space over GF(3) so that with a suitable interpretation we can write

$$\alpha/M^* = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad \beta/M^* = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \gamma/M^* = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

and it is readily checked that these three matrices generate $GL(2,3) \cong \mathrm{Aut}(G^*/M^*)$. To make use of this information we need the following two results which can be proved easily from the facts that $G^*_{(12)} = 1$ and $G^*_{(11)}$ has exponent 3.

(i) If η_1,η_2 are endomorphisms of G^* such that $\eta_1/M^* = \eta_2/M^*$ then $\eta_{1|G^*_{(11)}} = \eta_{2|G^*_{(11)}}$

(ii) If η is an endomorphism of G^* such that $\ker(\eta/M^*) \neq \{1\}$ then $\ker(\eta_{|G^*_{(11)}}) = G^*_{(11)}$.

Now suppose that S is a subgroup of $G^*_{(11)}$ that admits the automorphisms α,β,γ, and let η be an arbitrary endomorphism of G^*. Either $\ker(\eta|_{G^*_{(11)}}) = G^*_{(11)}$ in which case $G^*_{(11)}$ certainly admits η, or, by (ii), $\eta/M^* \in \mathrm{Aut}(G^*/M^*)$. In the latter case we have $\eta/M^* = \nu/M^*$ for some $\nu \in gp(\alpha,\beta,\gamma)$ and since S admits ν it follows from (i) that S admits η. We have thus shown that a subgroup S of $G^*_{(11)}$ is fully invariant in G^* if (and trivially only if) it admits α,β,γ.

The action of these automorphisms on $w_2,\ldots,w_9$ is easily calculated and has been tabulated on page 114. From these tables it is a purely routine matter to verify that the subgroups

$$D_1 = gp(w_2, w_3w_5w_7, w_4w_6w_8, w_9)$$

$$D_2 = gp(w_2w_4, w_3w_5w_7, w_4w_6w_8, w_7w_9)$$

$$U = gp(w_4, w_7)$$

each admit α,β,γ and hence are fully invariant in G^*, but that

3.1.4... $$\{1\} = (U \cap D_1).(U \cap D_2) \neq U \cap D_1.D_2 = U$$

which gives the required non-distributivity. A diagram of the full sublattice of $\mathrm{lat}(G^*)$ contained in $G^*_{(11)}$ is given by Fig. 3.

THE ACTION OF THE AUTOMORPHISMS α,β,γ			
w_i	$w_i\alpha$	$w_i\beta$	$w_i\gamma$
w_2	w_2	w_9^{-1}	w_2
w_3	$w_2^{-1}w_3$	w_8^{-1}	w_3^{-1}
w_4	w_4	w_7^{-1}	w_4
w_5	† $w_2 \quad w_4 \; w_5$	w_6^{-1}	w_5^{-1}
w_6	$w_2^{-1}w_3w_4 \; w_5^{-1}w_6$	w_5^{-1}	w_6
w_7	† $w_4^{-1} \quad w_7$	w_4^{-1}	w_7^{-1}
w_8	† $w_2 \quad w_4^{-1}w_5^{-1} \quad w_7w_8$	w_3^{-1}	w_8
w_9	$w_2^{-1}w_3w_4^{-1}w_5 \; w_6^{-1}w_7w_8^{-1}w_9$	w_2^{-1}	w_9^{-1}

† N.B. For display purposes only the elements in these products are not all juxtaposed.

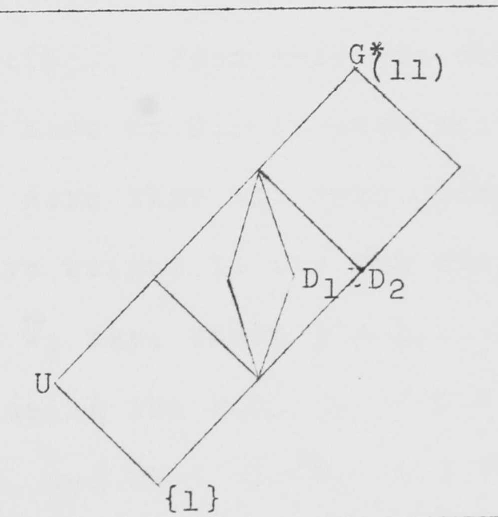


FIG. 3. A SUBLATTICE OF lat(G*)

3.2 A Non-Uniqeness Result

Continuing with the example in the last section we show next that $U = M^*_{(4)}$. Since $M^*_{(4)} = (M_{(4)} \cap G_2)\xi_2$, we will do this by showing that the image under ξ_2 of $M_{(4)} \cap G_2$ is generated by w_4 and w_7.

Note from 2.3.1(i) and 2.4.2 (with e = 0 in the latter) that $M_{(4)} \leq G_{(11)}$. Thus by 2.2.12 if $w \in M_{(4)} \cap G_2$ is expressed in normal form by $w = b_1^{e_1} \ldots b_t^{e_t}$ then $wt(b_i\phi^{-1}) = wt(b_i) \geq 11$ for each $i \in \{1,\ldots,t\}$. However, as we are only interested in the image of w under ξ_2 we may assume that $wt(b_i) = 11$ for each i. Using the notation of 2.4.16 we now claim further that $mic(b_i) \geq 4$ for each i. The justification for this is as follows: Because $w \in G_2$ the elements $b_1^{e_1},\ldots,b_t^{e_t}$ are the elementary parts of w and thus by 2.4.33 $mic(w) = min(mic(b_i) | i \in \{1,\ldots,t\})$, since clearly $mic(b_i^{e_i}) = mic(b_i)$. From this the claim follows, for since $w \in M_{(4)}$ we have by 2.2.17 that $mic(w) \geq 4$. To complete the argument note that the only elements $b_i \in \tilde{B}\phi \cap G_2$ which have weight 11 are the elements $[g_2, jg_1, (10 - j)g_2] = \overline{w}_j$ say, where j = 2,...,9, and of these it can be checked by using the methods of 2.4 that $mic(\overline{w}_4) = mic(\overline{w}_7) = 4$, and that $mic(\overline{w}_j) = 3$ for $4 \neq j \neq 7$. Since $\overline{w}_j\xi_2 = w_j$, we are home.

Now define $\overline{D}_1, \overline{D}_2 \in \text{lat}(G)$ by

$$\overline{D}_1 = \langle\{\overline{w}_2, \overline{w}_3\overline{w}_5\overline{w}_7, \overline{w}_4\overline{w}_6\overline{w}_7, \overline{w}_9\}\rangle . G_{(12)}$$

$$\overline{D}_2 = \langle\{\overline{w}_2\overline{w}_4, \overline{w}_3\overline{w}_5\overline{w}_7, \overline{w}_4\overline{w}_6\overline{w}_7, \overline{w}_7\overline{w}_9\}\rangle . G_{(12)}$$

Then, clearly, $\overline{D}_1\Xi = D_1$ and $\overline{D}_2\Xi = D_2$. As we also have $M_{(4)}\Xi = M^*_{(4)} = U$, it follows from 3.1.4 (and the fact that Ξ is a lattice homomorphism) that

$$(M_{(4)} \cap \overline{D}_1).(M_{(4)} \cap \overline{D}_2) \neq M_{(4)} \cap \overline{D}_1.\overline{D}_2$$

In terms of varieties this means

3.2.1... $(\underline{\underline{I}}_3 \vee \underline{\underline{L}}_1) \wedge (\underline{\underline{I}}_3 \vee \underline{\underline{L}}_2) \neq \underline{\underline{I}}_3 \vee (\underline{\underline{L}}_1 \wedge \underline{\underline{L}}_2)$

where $\underline{\underline{L}}_1$ and $\underline{\underline{L}}_2$ are nilpotent. If now $\underline{\underline{L}}'_1, \underline{\underline{L}}'_2$ and $\underline{\underline{V}}$ are defined by

$$\underline{\underline{L}}'_1 = \underline{\underline{L}}_1 \wedge (\underline{\underline{I}}_3 \vee \underline{\underline{L}}_2)$$

$$\underline{\underline{L}}'_2 = \underline{\underline{L}}_2 \wedge (\underline{\underline{I}}_3 \vee \underline{\underline{L}}_1)$$

$$\underline{\underline{V}} = (\underline{\underline{I}}_3 \vee \underline{\underline{L}}_1) \wedge (\underline{\underline{I}}_3 \vee \underline{\underline{L}}_2)$$

then, by using 3.2.1 and modularity, we have

(i) $\underline{\underline{V}} = \underline{\underline{I}}_3 \vee \underline{\underline{L}}'_1$

(ii) $\underline{\underline{V}} = \underline{\underline{I}}_3 \vee \underline{\underline{L}}'_2$

(iii) $\underline{\underline{V}} \neq \underline{\underline{I}}_3 \vee (\underline{\underline{L}}'_1 \wedge \underline{\underline{L}}'_2)$

This is just the situation we need to answer the question posed in 2.1.3, for if there existed a unique minimal (nilpotent) variety $\underline{\underline{L}}$ satisfying $\underline{\underline{V}} = \underline{\underline{I}}_3 \vee \underline{\underline{L}}$ then from (i) and (ii) we would have $\underline{\underline{L}}_1' \supseteq \underline{\underline{L}}$, $\underline{\underline{L}}_2' \supseteq \underline{\underline{L}}$ and hence $\underline{\underline{L}}_1' \wedge \underline{\underline{L}}_2' \supseteq \underline{\underline{L}}$. But that is impossible, for we would then have

$$\underline{\underline{V}} = \underline{\underline{I}}_3 \vee \underline{\underline{L}}_1' \supseteq \underline{\underline{I}}_3 \vee (\underline{\underline{L}}_1' \wedge \underline{\underline{L}}_2') \supseteq \underline{\underline{I}}_3 \vee \underline{\underline{L}} = \underline{\underline{V}}$$

which contradicts (iii).

3.3 Further Remarks

It is clear that the example we have seen of non-distributivity in lat(G*) not only demonstrates that lat($\underline{\underline{A}}_3\underline{\underline{A}}_9$) is non-distributive; it in fact demonstrates that lat($\underline{\underline{A}}_3\underline{\underline{A}}_9 \wedge \underline{\underline{N}}_{11}$) is non-distributive. Even this can be sharpened, for by a similar example it can be shown that lat($\underline{\underline{A}}_3\underline{\underline{A}}_9 \wedge \underline{\underline{N}}_9$) is non-distributive. (The "larger" example was chosen for inclusion here because it yields, in addition, the result of 3.2).

I have also shown, by an example similar to the second example mentioned above, that lat($\underline{\underline{A}}_5\underline{\underline{A}}_{25} \wedge \underline{\underline{N}}_{25}$) is non-distributive, and I am convinced that this example can be generalised to cover lat($\underline{\underline{A}}_p\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{p^2}$) for all odd primes p. However, a general example such as this involves some rather complicated identities in GF(p) which at present I am unable to handle.

With regard to $\mathrm{lat}(\underline{\underline{A}}_2\underline{\underline{A}}_4)$, it appears that $\mathrm{lat}(F_2(\underline{\underline{A}}_2\underline{\underline{A}}_4))$ is distributive; whether or not $\mathrm{lat}(F_r(\underline{\underline{A}}_2\underline{\underline{A}}_4))$ is non-distributive for some $r \in I^+$, I do not know.

Lastly, by way of contrast, it is worth remarking that M.F. Newman (unpublished) has shown that $\mathrm{lat}(\underline{\underline{A}}_{p^2}\underline{\underline{A}}_p)$ <u>is</u> distributive for all primes p.

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