SEMIGROUPS OF DIFFERENTIABLE FUNCTIONS

by

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STATEMENT

Unless it is stated in the text to the contrary, the material contained in this thesis is the product of my own research.

G. R. Wood.

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ABSTRACT

For over thirty years it has been known that the semigroup, under composition, of all mappings of an arbitrary set into itself has the property that every automorphism is inner. K.D. Magill, Jr, in the past ten years has shown that this property is held by many semigroups of functions and relations, in particular for the semigroup of all differentiable functions from the reals into the reals. The only new result given in the first chapter shows that the semigroup of Borel measurable functions on any T_1 topological space has the Magill property, namely the property that every automorphism is inner.

Even more recently S. Yamamuro has written a number of papers directed towards generalising the above result of Magill to semigroups of differentiable mappings defined on certain classes of locally convex spaces. The object of this thesis has been to continue that study. That the semigroup of Fréchet differentiable functions on an *FM*-space has the Magill property is the essential content of chapter two.

Showing that a semigroup of differentiable functions has the Magill property is closely related to showing that the algebraic structure of the semigroup characterises both topologically and algebraically the locally convex space on which the functions are defined. To show that the Magill property holds for subsemigroups of those considered above then becomes of interest. For this reason we consider in chapter three the semigroups of many times Fréchet differentiable mappings on *FM*-spaces and show that they too possess the Magill property, by using the results of chapter two as the first step in an inductive argument. Many times continuously Fréchet

differentiable functions are likewise treated.

An alternative proof of the result for many times continuously differentiable maps on a finite dimensional Banach space is given in chapter four. In this case the problem is equivalent to showing the differentiability with respect to the parameter of a one-parameter group of differentiable mappings, and so the classical theorem of Bochner and Montgomery may be applied. Further attention is also given in chapter four to the characterisation problem mentioned above. Using the notion of $\mathcal{S}$-category due to Bonic and Frampton we are able to give two theorems in this direction. Under certain conditions it is also shown in chapter four that if every automorphism of the group of invertible elements (units) in a semigroup is inner, then the same property will hold for the semigroup.

Admissibility of a class of spaces, a concept introduced by Magill, is extended in the final chapter to provide a framework in which to view the results. G.W. Mackey has shown that the group of continuous, linear, invertible mappings on a Hilbert space does characterise the space, but we are able to show that there exists an automorphism of this group which is not inner. The main theorem of the chapter then shows that for a large number of semigroups which contain this group, automorphisms which fix the group are inner. Certain 'd-automorphisms' of semigroups of differentiable functions are then shown to be inner.

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CHAPTER ONE

INTRODUCTION AND PRELIMINARIES

1.0 Introduction

The "elementary" algebraic properties of certain semigroups of mappings have been studied recently by a number of mathematicians. This thesis will survey and continue a portion of this work. We will be concerned largely with the nature of automorphisms of semigroups of differentiable functions defined on locally convex spaces. At all times the semigroup operation will be that of function composition. Unless it is stated to the contrary all topological spaces considered will be Hausdorff, and every vector space will be over the field of real numbers, R . We begin with a list of the basic notation and terminology which will be used throughout.

1.1 Notation and Terminology

The following lists are not intended to be exhaustive, but do contain the most frequently used items. Generally our usage agrees with Magill and Yamamuro.

Notation

N	:	the set of natural numbers
R	:	the set of real numbers with the usual topology. Elements of R will be denoted by Greek letters.
C	:	the set of complex numbers with the usual topology.
TVS	:	the set of all real, Hausdorff, topological vector spaces.
LCS	:	the set of all real, Hausdorff, locally convex spaces.
E, F, G	:	elements of *TVS* . When denoting a Banach space, Montel

space, etc., it will be made clear in the text. Elements of E will be denoted by Roman letters.

$\overline{E}$: the conjugate space of E with the topology of uniform convergence on bounded sets.

E_w : E with the weak topology.

$L(E, F)$: the space of all continuous linear mappings of E into F with the topology of uniform convergence on bounded sets. $L(E, E)$ is abbreviated to $L(E)$.

$U(E)$: the group of continuous linear invertible elements in $L(E)$ which have continuous inverses.

M : the set of all scalar mappings of a vector space into itself.

$C^0(E)$: the set of all continuous mappings of E into itself.

$\mathcal{D}_G(E)$: the set of all Gâteaux differentiable mappings of E into itself.

$\mathcal{D}_H(E)$: the set of all Hadamard differentiable mappings of E into itself.

$\mathcal{D}_F(E)$: the set of all Fréchet differentiable mappings of E into itself.

$\mathcal{D}_H^k(E)$: the set of all k times Hadamard differentiable mappings of E into itself.

$\mathcal{D}_F^k(E)$: the set of all k times Fréchet differentiable mappings of E into itself.

$\mathcal{J}C_F(E)$: the set of all Fréchet differentiable selfmaps of E with jointly continuous derivative.

$C_H^k(E)$: the set of all k times continuously Hadamard differentiable selfmaps of E .

$C_F^k(E)$: the set of all k times continuously Fréchet differentiable selfmaps of E .

If the space E , or the type of differentiation is clear, the notation will be simplified, for example to $\mathcal{D}_H$, or $C^k(E)$.

$CC(E)$: the set of all completely continuous selfmaps of E .

$\mathcal{B}(E)$: the set of all bounded and continuous selfmaps of E .

X, Y : sets, or topological spaces.

f, g, h : functions.

$A(X)$: a family of selfmaps of X equipped with some algebraic structure.

$S(X)$: a semigroup of selfmaps of X .

$H(X)$: the group of units in $S(X)$.

$Z(S)$: the centre of the semigroup S .

$M(X)$: the set of all Borel measurable selfmaps of the topological space X .

1 : the identity mapping.

c_a : the constant selfmap of X , whose single value is $a \in X$.

$I(X, Y)$: the set of all constant mappings from X to Y .

$T(E)$: the set of all translation mappings on E : all maps of the form $1 + c_a$, $a \in E$.

(c_0) : the space of all real sequences $\{\varepsilon_n\}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$.

l^1 : the Banach space of sequences $\{z_n\} \subset \mathbb{C}$ such that $\sum_{n=1}^{\infty} |z_n| < \infty$.

int V : the interior of the set V .

cl V : the closure of the set V .

ϕ : an automorphism, generally of a semigroup.

Terminology

For $y \in E$, $\overline{a} \in \overline{E}$, $\langle y, \overline{a} \rangle$ denotes the value of $\overline{a}$ at y .

For $x \in F$, $\overline{a} \in \overline{E}$, the map $x \otimes \overline{a}$ from E to F is given by $(x \otimes \overline{a})(y) = \langle y, \overline{a} \rangle x$, for $y \in E$.

When $a \in E$, and $\overline{a} \in \overline{E}$, $a \otimes \overline{a} \in L(E)$. Hence $(a \otimes \overline{a}) \otimes \overline{a} \in L\big(E, L(E)\big)$ which we write as $a \otimes^2 \overline{a}$. More generally, for $m \in \mathbb{N}$, we have the map $a \otimes^m \overline{a}$ in $L\big(\underbrace{E, \ldots, L(E)}_{m} \ldots\big)$, which we abbreviate to $L\big(E^m, E\big)$.

When $h \in \mathcal{D}_F^m(E)$, the mth Fréchet derivative of h at $x \in E$ is denoted by $h^{(m)}(x)$, and is an element of $L\big(E^m, E\big)$. If $a \in E$, $h^{(m)}(x)$ after m evaluations at a is an element of E denoted by $h^{(m)}(x)(a)^m$.

For $f : E \to F$, $a, x \in E$, and f differentiable in some sense at a , we define the remainder, $r[f, a, x]$, to be $f(a+x) - f(a) - f'(a)(x)$.

The sequence $\{a_1, a_2, \ldots\}$ will frequently be abbreviated to $\{a_n\}$. By $f^{-1}[X]$ we mean the inverse image of the set X under f . If $f \in \mathcal{D}_H(E)$ we define $df = \{f'(x) : x \in E\}$.

A discussion of properties of topological vector spaces which are used without reference may be found in either [17] or [48].

1.2 Preliminary definitions

We shall be concerned with three forms of differentiation: Gâteaux, Hadamard, and Fréchet. In order to present their definitions in a unified manner we adopt the terminology in [2, p. 86] and consider differentiability with respect to a system of sets. Gil de

Lamadrid (1955), Sebastião e Silva (1956-1957), and Miroslav Sova (1964) all arrived independently at the following method of differentiation.

Let $E, F \in TVS$, and $f : E \to F$. Then,

(1) f is *Gâteaux* differentiable at $a \in E$ if there exists a $u \in L(E, F)$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} r[f, a, \varepsilon x] = 0 \text{ , for each } x \in E \text{ ,}$$

where $r[f, a, \varepsilon x] = f(a+\varepsilon x) - f(a) - u(\varepsilon x)$.

(2) f is *Hadamard* differentiable at $a \in E$ if there exists a $u \in L(E, F)$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} r[f, a, \varepsilon x] = 0 \text{ ,}$$

uniformly for x in any sequentially compact subset of E .

(3) f is *Fréchet* differentiable at $a \in E$ if there exists a $u \in L(E, F)$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} r[f, a, \varepsilon x] = 0 \text{ ,}$$

uniformly for x in any bounded subset of E .

The Gâteaux, Hadamard, and Fréchet methods of differentiation may thus be considered as differentiation with respect to the system of all finite sets, all sequentially compact sets, and all bounded sets, respectively. For each type of differentiation, the continuous linear mapping u is uniquely determined, and is called the Gâteaux, Hadamard, and Fréchet derivative of f at a , respectively. We shall call it $f'(a)$. Note that for $c_a \in I(E)$, $c_a'(x) = 0$ for every x in E , and if $v \in L(E)$, $v'(x) = v$, for every x in E .

If $f : E \to E$ is differentiable at every point of E we say f

is differentiable on E and write $f \in \mathcal{D}_G(E)$, $f \in \mathcal{D}_H(E)$, or $f \in \mathcal{D}_F(E)$. Evidently $\mathcal{D}_F(E) \subseteq \mathcal{D}_H(E) \subseteq \mathcal{D}_G(E)$ and when $E = \mathbb{R}$ all reduce to the usual definition of the derivative. When the derivative $f' : E \to L(E, F)$ is continuous we say f is continuously differentiable. If $f : E \to F$, $E, F \in LCS$, is continuously Gâteaux differentiable it is known that f is Fréchet differentiable. For such continuously differentiable mappings we thus need not name the type of differentiation being considered.

When the mapping $f' : E \to L(E)$ between topological vector spaces is differentiable we say f is twice differentiable. It is then clear that inductively we have the families $\mathcal{D}^k_G(E)$, $\mathcal{D}^k_H(E)$, $\mathcal{D}^k_F(E)$, $C^k_G(E)$, $C^k_H(E)$, $C^k_F(E)$. We shall use f'' rather than $f^{(2)}$ for the second derivative of f .

1.3 Background to the differential calculus

Averbukh and Smolyanov in [1] and [2] have investigated the properties of such families of differentiable functions, showing in certain cases that they do form a semigroup with respect to the operation of function composition. In particular they show that the chain rule holds in $\mathcal{D}_H(E)$ and $\mathcal{D}_F(E)$, $E \in TVS$: if f and g are in $\mathcal{D}_F(E)$, for example, then $f \circ g \in \mathcal{D}_F(E)$ and

$$(f \circ g)'(x) = f'\big(g(x)\big) \circ g'(x) \text{ , for } x \in E \text{ .}$$

The composition of Gâteaux differentiable functions is however not necessarily Gâteaux differentiable, [2, p. 77]. Moreover they show that the Hadamard differentiation is the weakest for which the first order chain rule holds, [2, p. 74].

The higher order chain rules, [1, p. 234, Theorem 2.5] hold in $\mathcal{D}_H^k(E)$ and $\mathcal{D}_F^k(E)$, $E \in TVS$. For $C^k(E)$ the chain rule is obeyed provided E is a Fréchet space, [21]. In the locally convex space setting a discussion of the chain rule in $C^k(E)$ can be found in [46] or [21]. Although it concerns us only slightly, the definitions given do have a shortcoming: no longer do we have differentiability at a point implying continuity at that point. Averbukh and Smolyanov give an example of a $C^\infty(E)$ function which is not continuous in [2, p. 107].

If $E \in TVS$, E is termed sequential if for any set A in E every limit point of A is the limit of a sequence of points in A. Certainly every metrizable TVS is sequential, while Lloyd [22], has shown that every sequential LCS is bornological. In [2, p. 105] it is shown that any form of differentiability with respect to a system of bounded sets that contain all convergent sequences implies continuity if and only if the first space is sequential. In particular this will hold for both Hadamard and Fréchet differentiation.

It appears that for any definition of the derivative in locally convex spaces we can have either the second order chain rule or the property that differentiability implies continuity. An indication for this is the following: the usual proof of the first property relies upon the differentiability of the canonical map from $L(E) \times L(E)$ into $L(E)$, while if the second property is to hold then this map will be continuous. A result of Blair [4], Maissen [40], and Keller [14], shows this to be so if and only if E is normed.

For our purposes it is the chain rule which is of prime importance, and since the above definitions afford this property in the Hadamard and Fréchet cases, while the latter reduces to the usual form in the normed case, we adopt them here.

Let $S(X)$ denote a semigroup, under composition, of selfmaps of a set X. An automorphism ϕ of $S(X)$ is a one-to-one mapping of $S(X)$ onto itself which is multiplicative. That is,

$$\phi(fg) = \phi(f)\phi(g) \text{ , for every } f, g \in S(X) \text{ .}$$

We say the automorphism ϕ is inner if there is a bijection h of E such that $h, h^{-1} \in S(X)$, and

$$\phi(f) = hfh^{-1} \text{ , for every } f \in S(X) \text{ .}$$

1.4 Preliminary results

We now present two preliminary lemmas which will be fundamental to our work. Let X be a set, and $I(X)$ be the semigroup of constant selfmaps of X. With $S(X)$ as before, evidently

$$fc_a = c_{f(a)} \text{ and } c_a f = c_a \text{ , for } f \in S(X) \text{ .}$$

J. Schreier in 1937, [51], appears to have been the first to prove the following lemma:

LEMMA 1.1. *Let X be a set, $S(X)$ a semigroup of selfmaps of X such that $I(X) \subset S(X)$, and ϕ an automorphism of $S(X)$. Then there exists a bijection h of X such that*

$$\phi(f) = hfh^{-1} \text{ , for every } f \in S(X) \text{ .} \tag{1}$$

Proof. For any $x \in X$ we show $\phi(c_x)$ is again a constant mapping. Take arbitrary $y, z \in X$. Since ϕ is onto there is an $f \in S(X)$ such that $\phi(f) = c_y$. Then

$$\begin{aligned}\phi(c_x)(y) &= \phi(c_x)c_y(z)\\ &= \phi(c_x)\phi(f)(z)\\ &= \phi(c_x f)(z)\\ &= \phi(c_x)(z)\ .\end{aligned}$$

Now define $h(x) = y$, where $\phi(c_x) = c_y$. To show h is one-to-one, assume $h(x) = h(y)$. Then

$$\phi(c_x) = c_{h(x)} = c_{h(y)} = \phi(c_y)\ ,$$

whence it follows that $c_x = c_y$, or $x = y$. To show h is onto, we take an arbitrary $y \in X$. Then we can find an $f \in S(X)$ such that $\phi(f) = c_y$. As before we may show $f = \phi^{-1}(c_y)$ is a constant map, so there is an $x \in X$ such that $f = c_x$. Hence $h(x) = y$.

To complete the lemma we take $x, y \in X$, and $f \in S(X)$. Then

$$\begin{aligned}\phi(f)(x) &= \phi(f)c_x(y)\\ &= \phi\left(fc_{h^{-1}(x)}\right)(y)\\ &= \phi\left(c_{fh^{-1}(x)}\right)(y)\\ &= c_{hfh^{-1}(x)}(y)\\ &= hfh^{-1}(x)\ ,\end{aligned}$$

as required. //

If we impose additional restrictions we have the following lemma which is due essentially to Magill [26]. A demonstration of a similar result in this convex space setting appears in [66].

LEMMA 1.2. *Let* $E \in LCS$, ϕ *be an automorphism of the semi-group* $S(E)$, $C^\infty \cap C^0(E) \subset S(E) \subset C^0(E)$, *and* h *be as in Lemma* 1.1. *Then* $\langle h(x), \overline{a}\rangle$ *is continuous in* x, *for every* $\overline{a} \in \overline{E}$.

Proof. To show continuity at $a \in E$ we take $\varepsilon > 0$ and find an open neighbourhood U of a such that $|\langle h(x)-h(a), \overline{a}\rangle| < \varepsilon$, when $x \in U$. Let $\beta \in C^{\infty}(\mathrm{R})$ be such that

$$\beta(\xi) = \begin{cases} 0 & \text{if } |\xi| \geq \varepsilon , \\ 1 & \text{if } \xi = 0 , \end{cases}$$

and for b some non-zero element of E consider the mapping $g : E \to E$ defined by

$$g(x) = \beta\big(\langle x-h(a), \overline{a}\rangle\big)b + h(a) .$$

Evidently $g \in C^{\infty} \cap C^{0}(E) \subset S(E)$ so there is an $f \in S(E)$ such that $\phi(f) = g$. If $f(a) = a$ we have

$$h(a) = hf(a) = gh(a) = b + h(a) , \text{ or } b = 0 .$$

Thus $f(a) \neq a$. Since f is continuous we may find an open neighbourhood U of a such that if $x \in U$, $f(x) \neq a$. But h is one-to-one so $gh(x) = hf(x) \neq h(a)$, for $x \in U$. By the definition of β we have

$$|\langle h(x)-h(a), \overline{a}\rangle| < \varepsilon , \text{ for } x \in U . \qquad //$$

We add three frequently used facts:

(i) *ϕ uniquely determines the bijection h of Lemma* 1.1.

Suppose there exist bijections g, h such that (1) holds. For $x \in X$, $gc_x g^{-1} = hc_x h^{-1}$, so $g(x) = h(x)$, or $g \equiv h$.

(ii) *Any statement about h holds also for h^{-1}.*

Since $\phi^{-1}(f) = h^{-1}fh$, and ϕ^{-1} is an automorphism.

(iii) *We can assume $h(0) = 0$, for $X \in TVS$.*

Suppose $h(0) = a \neq 0$. Consider the automorphism ϕ_0, given by

$$\phi_0(f) = (1+c_a)^{-1}\phi(f)(1+c_a)$$

$$= \left[(1+c_a)^{-1}h\right]\phi(f)\left[(1+c_a)^{-1}h\right]^{-1}$$

$$= h_0 f h_0^{-1} .$$

Then $h_0(0) = (1+c_a)^{-1}h(0) = (1-c_a)(a) = 0$. Since the bijection $(1+c_a) \in C^\infty \cap C^0$, any property we show of h_0 will then hold for h .

Our starting point is a theorem of Magill which appeared in 1967, [26], in which he considers the semigroup of all differentiable functions from R to R :

THEOREM 1.1. *Every automorphism of* $\mathcal{D}(\mathrm{R})$ *is inner.*

Proof. By applying the above lemmas we reach the point where there exists a homeomorphism h of R such that

$$\phi(f) = hfh^{-1} \text{ , for every } f \in \mathcal{D}(\mathrm{R}) .$$

Such a homeomorphism is strictly monotone and so by a result in [45, p. 211, Theorem 4] has finite derivative almost everywhere. If x is such a point and y an arbitrary real number, since

$$\varepsilon^{-1}[h(y+\varepsilon)-h(y)] = \varepsilon^{-1}\left[h(1+c_{y-x})(x+\varepsilon)-h(1+c_{y-x})(x)\right]$$

$$= \varepsilon^{-1}\left[\phi(1+c_{y-x})h(x+\varepsilon)-\phi(1+c_{y-x})h(x)\right]$$

which converges to $\left[\phi(1+c_{y-x})h\right]'(x)$ as ε converges to zero, we have $h \in \mathcal{D}(\mathrm{R})$. //

In the light of his result of 1937 [51], our Lemma 1.1, Schreier had suggested the truth of Theorem 1.1 together with parallel results for the semigroups of all continuous maps and all measurable maps from the reals into the reals. Lemma 1.2 clearly settles the former case. However, the result of the following section appears to be new.

1.5 The semigroup of Borel measurable functions

In this section the inverse image of a set U under the map f will be denoted by $f^{-1}[U]$. By $M(X)$ we shall mean the family of Borel measurable selfmaps of a topological space X . That is, if $f \in M(X)$ and U is open in X , $f^{-1}[U]$ is Borel measurable.

If $f, g \in M(X)$, and U is open in X , $f^{-1}[U]$ is Borel measurable, and by [49, p. 13, Theorem 1.12 (b)],

$$g^{-1}[f^{-1}[U]] = (fg)^{-1}[U]$$

is Borel measurable, so $M(X)$ forms a semigroup. Then we have the following:

THEOREM 1.2. *Let* X *be a* T_1 *topological space. Then every automorphism* ϕ *of* $M(X)$ *is inner.*

Proof.

1. *There exists a bijection* h *of* X *such that* $\phi(f) = hfh^{-1}$, *for every* $f \in M(X)$.

Since the constant maps are Borel measurable, this is an application of Lemma 1.1.

2. h^{-1} *is Borel measurable.*

Take some fixed $a \in X$ and choose $b \neq h^{-1}(a)$. Define $\chi_U : X \to X$, for U open in X , as follows:

$$\chi_U(x) = \begin{cases} b & \text{when } x \in U , \\ h^{-1}(a) & \text{when } x \notin U . \end{cases}$$

Then clearly $\chi_U \in M(X)$, so $h\chi_U h^{-1} \in M(X)$, for every such set U .

Since X is T_1 , $X\backslash\{a\}$ is open, so

$$\left(h\chi_U h^{-1}\right)^{-1}[X\setminus\{a\}] = h\left[\chi_U^{-1}[h^{-1}[X\setminus\{a\}]]\right]$$

$$= h\left[\chi_U^{-1}[X\setminus\{h^{-1}(a)\}]\right] \text{, (since } h \text{ is a bijection)}$$

$$= (h^{-1})^{-1}[U] \text{ is Borel measurable.}$$

Thus h^{-1} is Borel measurable. In a similar way we have h Borel measurable and the theorem follows. //

1.6 Historical remarks

We now give a brief historical account of relevant investigations into the algebraic properties of families of functions. Although Schreier in 1937 [51], was the first to show that the semigroup of all selfmaps of an arbitrary set has the property that every automorphism is inner, Mal'cev [41] and Ljapin [18] each proved the result independently at a later date.

Between 1940 and 1948 there was considerable interest in families of continuous linear mappings. Since this topic is the subject matter of chapter five, we only sketch the results here for the sake of completeness. In 1940 Eidelheit [11] showed that every automorphism of the ring $L(E)$, E Banach, is inner. Further contributions to the ring case were made by Mackey in 1942 [24] and 1946 [23]. In the former paper he also considers the group $U(E)$, E a normed linear space, of continuous linear maps which have continuous inverses. Rickart [47] in 1948 was able to improve a result in Eidelheit's paper of 1940 to show that every automorphism of the semigroup $L(E)$ is inner, where E is a Banach space with dimension greater than one.

Almost two decades later Magill revived the subject and in a series of papers, [25] through to [39], has shown that the property that every automorphism is inner is held by many semigroups of functions and relations on topological spaces. Following Yamamuro in [66] we say a semigroup has the Magill property if every automorphism is inner. Nadler and Hofer, in for example [42], [43] and [12], [13] have published papers directed along related lines. Since 1967, however, Yamamuro has contributed a number of papers to the field, most directed towards generalising the result of Magill (Theorem 1.1) to semigroups of differentiable functions defined on certain classes of locally convex spaces.

In [62], Yamamuro has noted that no automorphism of the semigroup of constant selfmaps of a set is inner, and has shown that the same result is true of the semigroup of all completely continuous selfmaps of an infinite dimensional Banach space, $CC(E)$. However, for such a space E he has constructed both inner and outer automorphisms of the semigroup $1 + CC(E) = \{1+f : f \in CC(E)\}$. That $\mathcal{D}(\mathbb{C})$ does not possess the Magill property was shown recently by Warren, [56]. To date, no semigroup of all $\mathcal{D}^k$ or C^k mappings of a real, Hausdorff, locally convex space has been found which does not have the Magill property. In [62], the Magill property was also shown to hold for $\mathcal{B}(E)$, the semigroup of all continuous and bounded selfmaps of a Banach space, E.

In the event that the family $A(E)$ of continuous selfmaps of a Banach space E forms a near-ring, Yamamuro in [61] has proved that if $I(E), L(E) \subset A(E)$, then every near-ring automorphism of $A(E)$ is inner. Unfortunately not every semigroup automorphism is a near-ring automorphism.

Since the theory of Fréchet differentiation in a Banach space E has been fully investigated, [8, Chapter 8], it was natural to attempt to show that $\mathcal{D}_F(E)$ possessed the Magill property. By imposing restrictions on the automorphism, ϕ , the following pair of results were obtained by Yamamuro in [62] and [65]:

(i) Define ϕ to be a d-automorphism of $\mathcal{D}_F(E)$ when

$$d\phi(f) = \{\phi(f)'(x) : x \in E\} = \{\phi(f'(x)) : x \in E\} = \phi(df)$$

for each $f \in \mathcal{D}_F(E)$. Then every d-automorphism of $\mathcal{D}_F(E)$ is inner.

(ii) If ϕ is such that given $\varepsilon > 0$ and $\{\alpha_n\} \in (c_0)$ there is a $\delta > 0$ such that $\|x\| < \delta$ implies

$$\sup_{n\geq 1} \left\| \alpha_n^{-1}\phi(\alpha_n)(x)-x \right\| \leq \varepsilon\|x\| ,$$

ϕ is said to be uniform. Then an automorphism of $\mathcal{D}_F(E)$ is inner if and only if it is uniform.

1.7 Summary of chapter content

We are in a position now to give an outline of the remaining content of this thesis. In Chapter two we consider the semigroup $\mathcal{D}_F(E)$, $E \in LCS$, and prove a general theorem which implies that the Magill property does hold when E is a Fréchet Montel space. Similarly, the subsemigroups $\mathcal{D}_F^k(E)$ and $C^k(E)$, $k \in \mathbb{N}$, are treated in Chapter three. By arranging the problem in such a way that a classical theorem of Bochner and Montgomery concerning differentiability is applicable, we give a short proof in Chapter four to show that $C^k(E)$ has the Magill property, for E a finite dimensional Banach space.

There is an alternative interpretation of the problem. If the semigroups $S(E_i) \subset \mathcal{D}_H(E_i)$, $E_i \in LCS$, $i = 1, 2$, are known to have the Magill property, and we consider instead the situation in which an isomorphism exists between $S(E_1)$ and $S(E_2)$, then with only a notational change in our proofs we may find a bijection h from E_1 onto E_2 such that h is *a fortiori* Hadamard differentiable. Since

$$(h^{-1})'(h(x))h'(x) = h'(x)(h^{-1})'(h(x)) = 1 ,$$

for every $x \in E_1$, and $h'(x) \in L(E_1, E_2)$, E_1 and E_2 are linearly homeomorphic. In the remainder of Chapter four we give results showing that under certain conditions a semigroup of self-maps will characterise the space on which the maps are defined.

In Chapter five we extend the concept of the admissibility of a class of spaces, first introduced by Magill in [33]. A theorem is presented which extends both the near-ring result and the *d*-automorphism result mentioned earlier.

CHAPTER TWO

SEMIGROUPS OF FRÉCHET DIFFERENTIABLE MAPPINGS

2.0 Introduction

In order to phrase the results of both this and the following chapter in as general a form as possible we introduce the following notion: for $E \in LCS$, a map $f : E \to E$ is said to be weakly-$\mathcal{D}_F(E)$ if the map $f : E \to E_\omega$ is Fréchet differentiable, where E_ω denotes the space E endowed with the weak topology, $\sigma(E, \overline{E})$. Notice that this definition is more general than the definition for weak differentiability given in [59] and [66], where the domain of f was also given the weak topology. In the obvious way we also define weakly-$\mathcal{D}^k_F(E)$, weakly-$C^k(E)$, etc.

A space $E \in LCS$ will be said to have the property S if the dual of every separable subspace of E contains a countable total subset. Since the dual of a separable Fréchet space is weakly sequentially separable [17, p. 259, (5)] every Fréchet space has the property S .

The results in this chapter are essentially due to Yamamuro and appear in [60]. In that paper the results were phrased in Banach spaces with the property that weak sequential convergence implied strong convergence. The observations that Theorem 2.1 may be set in sequential LCS's with the property S , and Corollary 2.1 in FM-spaces, together with the generalisations in steps 3, 5, and 6 to arbitrary real numbers, necessary for Chapter three, are the author's sole claims to originality in the present chapter.

2.1 The main theorem

We shall prove the following theorem:

THEOREM 2.1. *Let E be a sequential locally convex space with the property* S. *Then if ϕ is an automorphism of $\mathcal{D}_F(E)$ there is a bijection h of E such that h and h^{-1} are weakly-$\mathcal{D}_F(E)$ and*

$$\phi(f) = hfh^{-1} \text{ , for every } f \in \mathcal{D}_F(E) \text{ .} \tag{1}$$

Proof.

1. *There exists a bijection h of E such that* (1) *holds.*

Since $c_a \in C^\infty(E) \subset \mathcal{D}_F(E)$, this is an application of Lemma 1.1.

From this point the method of proof must differ from that employed by Magill in Theorem 1.1. For locally convex spaces of dimension greater than one it is now known that there exist homeomorphisms which are not even Gâteaux differentiable at a single point. The following example of such a function is due to Dr S. Swierczkowski and Professor Jan Mycielski.

Suppose $E \in LCS$ has dimension greater than one. Then we may find non-zero $a \in E$, $\overline{a} \in \overline{E}$ such that $\langle a, \overline{a} \rangle = 0$. Let $\alpha : \mathrm{R} \to \mathrm{R}$ be a continuous function which is nowhere differentiable. Then the mapping

$$f(x) = x + \alpha(\langle x, \overline{a} \rangle)a$$

is a homeomorphism of E which is not Gâteaux differentiable at any point.

In brief, the proof runs as follows. With $\{\varepsilon_n\} \in (c_0)$, a calculation similar to that in Rolle's theorem enables us to show that the sequence $\left\{\varepsilon_n^{-1}h(\varepsilon_n a)\right\}$ does not converge weakly to zero, $0 \neq a \in E$. By exploiting the interplay between h and h^{-1} we are able to show the set $\left\{\varepsilon_n^{-1}h^{-1}(\varepsilon_n a)\right\}$ is bounded. From this we deduce the existence of a convergent subsequence, $\left\{\varepsilon_{n_k}^{-1}h^{-1}\left(\varepsilon_{n_k} a\right)\right\}$ and use well known properties of the Dini derivatives to show the limit, $\lim\limits_{\varepsilon\to 0} \varepsilon^{-1}h^{-1}(\varepsilon a)$, exists. The Fréchet differentiability of $h(a \otimes \overline{a})$ readily follows and the proof is completed by deducing the weak differentiability of h . We divide the remainder of the proof into nine steps.

2. *Let* λ *be a real-valued function of a real variable. If*

(i) $\lambda(0) = 0$,

(ii) λ *is continuous,*

(iii) there exists a sequence $\{\varepsilon_n\} \in (c_0)$ *such that*

$$\varepsilon_n^{-1}[\lambda(\xi\pm\varepsilon_n\eta)-\lambda(\xi)] \to 0 \ \textit{for any} \ \xi, \eta \in R ,$$

then $\lambda \equiv 0$.

For arbitrary η , consider the function

$$\mu(\xi) = \lambda(\xi\eta) - \xi\lambda(\eta) ,$$

which is continuous and for which $\mu(0) = \mu(1) = 0$. Then we can find an $\xi_0 \in (0, 1)$ at which μ takes a relative maximum or minimum value. Supposing, without loss of generality, it is the former, then for large values of n , $\mu(\xi_0\pm\varepsilon_n) \le \mu(\xi_0)$. Hence,

$$\lambda(\xi_0\eta\pm\varepsilon_n\eta) - (\xi_0\pm\varepsilon_n)\lambda(\eta) \le \lambda(\xi_0\eta) - \xi_0\lambda(\eta) ,$$

or,

$$\varepsilon_n^{-1}[\lambda(\xi_0\eta+\varepsilon_n\eta)-\lambda(\xi_0\eta)] \le \lambda(\eta) \le -\varepsilon_n^{-1}[\lambda(\xi_0\eta-\varepsilon_n\eta)-\lambda(\xi_0\eta)] .$$

Thus $\lambda \equiv 0$.

In steps 3, 5 and 6 we will prove results valid for any real number ξ , even though it will not be until the next chapter that the cases other than for $\xi = 0$ will be needed.

3. *For any non-zero $a \in E$, any $\xi \in \mathrm{R}$, and $\{\varepsilon_n\} \in (c_0)$, the sequence $\left\{\varepsilon_n^{-1}\left[h(\xi a+\varepsilon_n a)-h(\xi a)\right]\right\}$ does not converge weakly to zero.*

We first show the result for $\xi = 0$. Assume we can find a non-zero $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that $\lim_{n\to\infty} \left\langle \varepsilon_n^{-1} h(\varepsilon_n a), \bar{x}\right\rangle = 0$, for every $\bar{x} \in \bar{E}$. We shall show that for $\xi, \eta \in \mathrm{R}$,

$$\lim_{n\to\infty} \varepsilon_n^{-1}\langle h(\xi a+\varepsilon_n \eta a)-h(\xi a), \bar{x}\rangle = 0 \text{, for any } \bar{x} \in \bar{E} . \qquad (2)$$

With $\eta \in \mathcal{D}_F(E)$ the linear mapping $x \to \eta x$, we define $f \in \mathcal{D}_F(E)$ by $f = \xi c_a + \eta$. Then $f(\varepsilon_n a) = \xi a + \varepsilon_n \eta a$, and $f(0) = \xi a$, so

$$\begin{aligned}
&\varepsilon_n^{-1}\left[h(\xi a+\varepsilon_n \eta a)-h(\xi a)\right] \\
&\quad = \varepsilon_n^{-1}\left[hf(\varepsilon_n a)-hf(0)\right] \\
&\quad = \varepsilon_n^{-1}\left[\phi(f)h(\varepsilon_n a)-\phi(f)h(0)\right] \\
&\quad = \varepsilon_n^{-1}\left[\phi(f)'(0)\left(h(\varepsilon_n a)\right)+r\left(\phi(f), 0, h(\varepsilon_n a)\right)\right] \\
&\quad = \phi(f)'(0)\left[\varepsilon_n^{-1} h(\varepsilon_n a)\right] + \varepsilon_n^{-1} r\left(\phi(f), 0, \varepsilon_n\left[\varepsilon_n^{-1} h(\varepsilon_n a)\right]\right) .
\end{aligned}$$

Since $\phi(f)'(0) \in L(E)$ it follows that

$$\lim_{n\to\infty} \left\langle \phi(f)'(0)\left[\varepsilon_n^{-1} h(\varepsilon_n a)\right], \bar{x}\right\rangle = 0 ,$$

for every $\bar{x} \in \bar{E}$. Every weakly convergent sequence is bounded, so $\left\{\varepsilon_n^{-1} h(\varepsilon_n a)\right\}$ is bounded. Fréchet differentiability of $\phi(f)$ at zero then gives that

$$\lim_{n\to\infty} \varepsilon_n^{-1} r\left(\phi(f), 0, h(\varepsilon_n a)\right) = 0 ,$$

which completes the proof of (2).

The function $\lambda_{\overline{x}} : \mathrm{R} \to \mathrm{R}$, defined by $\lambda_{\overline{x}}(\xi) = \langle h(\xi a), \overline{x} \rangle$, for $\xi \in \mathrm{R}$, $\overline{x} \in \overline{E}$, is continuous by Lemma 1.2 so evidently satisfies the three conditions of step 2. Thus $\lambda_{\overline{x}}$ is identically zero. Since $\overline{x} \in \overline{E}$ is arbitrary, $h(\xi a) = 0$ for all $\xi \in \mathrm{R}$. But h is one-to-one, so we reach a contradiction.

Now suppose we can find an $\xi \in \mathrm{R}$ and a sequence $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} \left\langle \varepsilon_n^{-1}\left[h(\xi a+\varepsilon_n a)-h(\xi a)\right], \overline{x} \right\rangle = 0 \text{ , for every } \overline{x} \in \overline{E} .$$

Using a calculation similar to that above we may show

$$\begin{aligned} \varepsilon_n^{-1}h(\varepsilon_n a) &= \varepsilon_n^{-1}\left[h(\varepsilon_n a)-h(0)\right] \\ &= \phi(1-c_{\xi a})'(h(\xi a))\left[\varepsilon_n^{-1}(h(\xi a+\varepsilon_n a)-h(\xi a))\right] \\ &\quad + \varepsilon_n^{-1}r\left[\phi(1-c_{\xi a}), h(\xi a), \varepsilon_n\left(\varepsilon_n^{-1}(h(\xi a+\varepsilon_n a)-h(\xi a))\right)\right] . \end{aligned}$$

Again,

$$\lim_{n\to\infty} \left\langle \phi(1-c_{\xi a})'(h(\xi a))\left(\varepsilon_n^{-1}(h(\xi a+\varepsilon_n a)-h(\xi a))\right), \overline{x} \right\rangle = 0$$

for every $\overline{x} \in \overline{E}$. Further, the set $\left\{\varepsilon_n^{-1}\left[h(\xi a+\varepsilon_n a)-h(\xi a)\right]\right\}$ is bounded, so the limit of the second term is zero. Hence the sequence $\left\{\varepsilon_n^{-1}h(\varepsilon_n a)\right\}$ converges weakly to zero, a contradiction, so the statement follows.

4. *For any* $a \in E$ *and any* $\{\varepsilon_n\} \in (c_0)$ *, the sequence* $\left\{\varepsilon_n^{-1}h^{-1}(\varepsilon_n a)\right\}$ *is bounded.*

Assume there is a non-zero $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that the sequence $\left\{\varepsilon_n^{-1}h^{-1}(\varepsilon_n a)\right\}$ is unbounded. For some $\overline{a} \in \overline{E}$, and taking a subsequence of $\{\varepsilon_n\}$ if necessary, we have,

$$\lim_{n\to\infty} \left\langle \varepsilon_n^{-1} h^{-1}(\varepsilon_n a),\ \overline{a} \right\rangle = +\infty .$$

Since $(a \otimes \overline{a}) \in \mathcal{D}_F(E)$, $\phi(a \otimes \overline{a}) \in \mathcal{D}_F(E)$, and

$$\phi(a \otimes \overline{a})'(0)(a) = \lim_{n\to\infty} \varepsilon_n^{-1}\phi(a \otimes \overline{a})(\varepsilon_n a)$$

$$= \lim_{n\to\infty} \left[\varepsilon_n^{-1}\left\langle h^{-1}(\varepsilon_n a),\ \overline{a}\right\rangle\right]\left[\left\langle h^{-1}(\varepsilon_n a),\ \overline{a}\right\rangle\right]^{-1} h\left[\left\langle h^{-1}(\varepsilon_n a),\ \overline{a}\right\rangle a\right] .$$

Thus if $\delta_n = \left\langle h^{-1}(\varepsilon_n a),\ \overline{a}\right\rangle$, $\{\delta_n\} \in (c_0)$, so $\lim_{n\to\infty} \delta_n^{-1} h(\delta_n a) = 0$. This contradicts step 3.

5. *Given* $\xi \in \mathrm{R}$ *and non-zero* $a \in E$, *there exists an* $\overline{x}_\xi \in \overline{E}$ *such that* $\phi\left[(a \otimes \overline{x}_\xi)\left(1 - c_{h^{-1}(\xi a)}\right)\right]'(\xi a)(a) \neq 0$.

Suppose there exists an $\xi \in \mathrm{R}$ such that for all $\overline{x} \in \overline{E}$,

$$\phi\left[(a \otimes \overline{x})\left(1 - c_{h^{-1}(\xi a)}\right)\right]'(\xi a)(a) = 0 .$$

Take a sequence $\{\delta_n\} \in (c_0)$ such that $\delta_n \neq 0$, any n , and let M be the set of all $\overline{x} \in \overline{E}$ such that the sequence $\left\{\left\langle h^{-1}(\xi a + \delta_n a) - h^{-1}(\xi a),\ \overline{x}\right\rangle\right\}$ contains infinite non-zero members. If $\overline{x} \notin M$ then the sequence $\left\{\delta_n^{-1}\left\langle h^{-1}(\xi a+\delta_n a) - h^{-1}(\xi a),\ \overline{x}\right\rangle\right\}$ converges to zero. If $\overline{x} \in M$ we have

$$0 = \lim_{n\to\infty} \delta_n^{-1}\phi\left[(a \otimes \overline{x})\left(1 - c_{h^{-1}(\xi a)}\right)\right](\xi a + \varepsilon_n a)$$

$$= \lim_{n\to\infty} \delta_n^{-1}\left\langle h^{-1}(\xi a+\delta_n a) - h^{-1}(\xi a),\ \overline{x}\right\rangle \tau_n^{-1} h(\tau_n a) \qquad (3)$$

where $\tau_n = \left\langle h^{-1}(\xi a+\delta_n a) - h^{-1}(\xi a),\ \overline{x}\right\rangle$. Suppose the sequence $\left\{\delta_n^{-1}\left\langle h^{-1}(\xi a+\delta_n a) - h^{-1}(\xi a),\ \overline{x}\right\rangle\right\}$ does not converge to zero. Then there is a subsequence $\left\{\delta_{n_k}\right\}$ and a $\gamma \in R$, $\gamma > 0$, such that

$$\left|\delta_{n_k}^{-1}\left\langle h^{-1}\left(\xi a+\delta_{n_k} a\right)-h^{-1}(\xi a),\ \overline{x}\right\rangle\right| \geq \gamma\ ,\ \text{every}\ \ k\ .$$

Then by (3) the sequence $\left\{\tau_{n_k}^{-1} h\left(\tau_{n_k} a\right)\right\}$ converges to zero, which contradicts step 3. So for any $\overline{x} \in \overline{E}$, the sequence

$$\left\{\delta_n^{-1}\left\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\ \overline{x}\right\rangle\right\}$$

converges to zero, again contradicting step 3.

Note that if $\overline{x}$ satisfies

$$\phi\left[(a \otimes \overline{x})\left(1-c_{h^{-1}(\xi a)}\right)\right]'(\xi a)(a) \neq 0\ ,$$

so too does $(-\overline{x})$. Since

$$\phi\left[(a \otimes (-\overline{x}))\left(1-c_{h^{-1}(\xi a)}\right)\right]'(\xi a)$$

$$= \left[\phi(-1)\phi\left[(a \otimes \overline{x})\left(1-c_{h^{-1}(\xi a)}\right)\right]\right]'(\xi a)$$

$$= \phi(-1)'\left[\phi\left((a \otimes \overline{x})\left(1-c_{h^{-1}(\xi a)}\right)\right)(\xi a)\right]\phi\left((a \otimes \overline{x})\left(1-c_{h^{-1}(\xi a)}\right)\right)'(\xi a)$$

$$= \phi(-1)'(0)\phi\left((a \otimes \overline{x})\left(1-c_{h^{-1}(\xi a)}\right)\right)'(\xi a)\ .$$

But $\phi(-1)'(0)$ is a linear bijection, because

$$\phi(-1)'(0)\phi(-1)'(0) = \phi(1)'(0) = 1\ .$$

6. *For any* $a \in E$ *and* $\{\varepsilon_n\} \in (c_0)$ *there is a subsequence* $\left\{\varepsilon_{n_k}\right\}$ *such that the sequence* $\left\{\varepsilon_{n_k}^{-1} h\left(\varepsilon_{n_k} a\right)\right\}$ *is convergent.*

We can assume $a \neq 0$. For any $\xi \in R$ and the associated $\overline{x}_\xi \in \overline{E}$ it is evident from the equation

$$0 \neq \phi\left[\left(a \otimes \overline{x}_\xi\right)\left(1-c_{h^{-1}(\xi a)}\right)\right]'(\xi a)(a)$$

$$= \lim_{\delta\to 0} \delta^{-1}h\left(\left\langle h^{-1}(\xi a+\delta a)-h^{-1}(\xi a),\ \overline{x}_\xi\right\rangle a\right)$$

that the function $\left\langle h^{-1}(\xi a+\delta a)-h^{-1}(\xi a),\ \overline{x}_\xi\right\rangle$ takes non-zero values in every zero-neighbourhood. If not $\phi\left[\left(a \otimes \overline{x}_\xi\right)\left(1-c_{h^{-1}(\xi a)}\right)\right]'(\xi a)(a)$ has to be zero. Since $\langle h^{-1}(x),\ \overline{a}\rangle$ is continuous in x, there is a sequence $\{\delta_n\} \in (c_0)$ such that

$$\left\langle h^{-1}\left(\xi a+\delta_n a\right)-h^{-1}(\xi a),\ \overline{x}_\xi\right\rangle = \varepsilon_n \quad\text{or}\quad -\varepsilon_n .$$

So by taking a subsequence of $\{\varepsilon_n\}$ and replacing $\overline{x}_\xi$ by $-\overline{x}_\xi$ if necessary, we can assume that

$$\left\langle h^{-1}\left(\xi a+\delta_n a\right)-h^{-1}(\xi a),\ \overline{x}_\xi\right\rangle = \varepsilon_n \text{ , for every } n .$$

At the moment we need this only for $\xi = 0$. In this case we have that

$$0 \neq \phi\left(a \otimes \overline{x}_0\right)'(0)(a)$$

$$= \lim_{n\to\infty} \left(\delta_n^{-1}\left\langle h^{-1}\left(\delta_n a\right),\ \overline{x}_0\right\rangle\right)\varepsilon_n^{-1}h\left(\varepsilon_n a\right) .$$

By step 4, the sequence $\left\{\delta_n^{-1}\left\langle h^{-1}\left(\delta_n a\right),\ \overline{x}_0\right\rangle\right\}$ is bounded, so there is a subsequence $\left\{\delta_{n_k}\right\}$ of $\{\delta_n\}$ such that the limit $\lim_{k\to\infty} \delta_{n_k}^{-1}\left\langle h^{-1}\left(\delta_{n_k} a\right),\ \overline{x}_0\right\rangle = \alpha$, exists. Certainly α is non-zero, since the sequence $\left\{\varepsilon_{n_k}^{-1}h\left(\varepsilon_{n_k} a\right)\right\}$ is bounded, and $\phi\left(a \otimes \overline{x}_0\right)'(0)(a) \neq 0$, so the following limit exists:

$$\lim_{k\to\infty} \varepsilon_{n_k}^{-1}h\left(\varepsilon_{n_k} a\right) = \alpha^{-1}\phi\left(a \otimes \overline{x}_0\right)'(0)(a) .$$

7. *The limit* $\lim_{\varepsilon\to 0} \varepsilon^{-1}h(\varepsilon a)$ *exists for any* $a \in E$.

With $a \in E$ non-zero and arbitrary $\overline{a} \in \overline{E}$ we will show that the function $\lambda : \mathrm{R} \to \mathrm{R}$ defined by $\lambda(\xi) = \langle h(\xi a), \overline{a}\rangle$ is differentiable almost everywhere. By [50, p. 270] we must show that all of the Dini derivatives of λ at arbitrary $\alpha \in \mathrm{R}$ are finite. Suppose the upper right hand Dini derivative is infinite at $\alpha \in \mathrm{R}$. That is,

$$\overline{\lim_{\varepsilon\to 0^+}} \ \varepsilon^{-1}|\lambda(\alpha+\varepsilon)-\lambda(\alpha)| = +\infty .$$

But for $\{\varepsilon_n\} \in (c_0)$ we have

$$\varepsilon_n^{-1}[h(\alpha a+\varepsilon_n a)-h(\alpha a)]$$

$$= \varepsilon_n^{-1}[\phi(1+c_{\alpha a})h(\varepsilon_n a)-\phi(1+c_{\alpha a})(h(0))]$$

$$= \phi(1+c_{\alpha a})'(0)\left(\varepsilon_n^{-1}h(\varepsilon_n a)\right) + \varepsilon_n^{-1}r\left[\phi(1+c_{\alpha a}),\ 0,\ \varepsilon_n\left[\varepsilon_n^{-1}h(\varepsilon_n a)\right]\right] .$$

Since $\phi(1+c_{\alpha a})'(0) \in L(E)$ and the set $\left\{\varepsilon_n^{-1}h(\varepsilon_n a)\right\}$ is bounded by step 4, the sequence formed by the first term is bounded. As $\phi(1+c_{\alpha a}) \in \mathcal{D}_F(E)$ the same holds for the second term. So the set $\left\{\varepsilon_n^{-1}[h(\alpha a+\varepsilon_n a)-h(\alpha a)]\right\}$ is bounded and hence so too is the set $\left\{\varepsilon_n^{-1}|\lambda(\alpha+\varepsilon_n)-\lambda(\alpha)|\right\}$ whence the upper right derivate cannot be infinite. In a similar way the other Dini derivatives are shown to be finite.

We now consider the existence of the limit $\lim_{\varepsilon\to 0} \varepsilon^{-1}h(\varepsilon a)$. In the light of the result of step 6, it will be sufficient to show that if $\{\varepsilon_n\}, \{\delta_n\} \in (c_0)$ and $\lim_{n\to\infty} \varepsilon_n^{-1}h(\varepsilon_n a) = a_1$, $\lim_{n\to\infty} \delta_n^{-1}h(\delta_n a) = a_2$, then $a_1 = a_2$.

From step 4 it follows that $h(\xi a)$ is continuous with respect to ξ at $\xi = 0$. Using the translation map and the semigroup property we may transfer the continuity to an arbitrary point, showing it to be a continuous mapping of the separable space R into E . The following argument is carried out in the smallest closed linear subspace, F , containing the set $\{h(\xi a) : \xi \in \mathrm{R}\}$ and so is separable.

Since E has the property S , $\overline{F}$ contains a countable total subset. That is, we can find enumerable $\overline{a}_i \in \overline{F}$ such that when $\langle x, \overline{a}_i \rangle = 0$ for every $i = 1, 2, 3, \ldots$ then $x = 0$.

Consider the following functions of $\xi \in \mathrm{R}$,

$$\lambda_i(\xi) = \langle h(\xi a), \overline{a}_i \rangle , \quad i = 1, 2, 3, \ldots .$$

That each λ_i is differentiable almost everywhere implies there exists an $\alpha \in \mathrm{R}$ at which all the λ_i are differentiable. That is, the limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}[\lambda_i(\alpha+\varepsilon)-\lambda_i(\alpha)] ,$$

exists for every i . On the other hand we have,

$$\lim_{n\to\infty} \varepsilon_n^{-1}[h(\alpha a+\varepsilon_n a)-h(\alpha a)] = \phi(1+c_{\alpha a})'(0)(a_1) ,$$

while

$$\lim_{n\to\infty} \delta_n^{-1}[h(\alpha a+\delta_n a)-h(\alpha a)] = \phi(1+c_{\alpha a})'(0)(a_2) .$$

Thus

$$\langle \phi(1+c_{\alpha a})'(0)(a_1), \overline{a}_i \rangle = \langle \phi(1+c_{\alpha a})'(0)(a_2), \overline{a}_i \rangle$$

for every i , which implies

$$\phi(1+c_{\alpha a})'(0)(a_1) = \phi(1+c_{\alpha a})'(0)(a_2) .$$

But $(1-c_{\alpha a})(1+c_{\alpha a}) = 1$ so $\phi(1-c_{\alpha a})'(h(\alpha a))\phi(1+c_{\alpha a})'(0) = 1$, which means that $\phi(1+c_{\alpha a})'(0)$ is injective. Hence $a_1 = a_2$. We denote the limit, $\lim_{\varepsilon\to 0} \varepsilon^{-1}h(\varepsilon a)$, by $h^*(0)(a)$.

8. *For any* $a \in E$, $\bar{a} \in \bar{E}$ *such that* $\langle a, \bar{a}\rangle = 1$, $h(a \otimes \bar{a}) \in \mathcal{D}_F(E)$.

For brevity we let $h(a \otimes \bar{a}) = h_1$. We show h_1 to be Gâteaux differentiable at zero and that the Gâteaux derivative is a Fréchet derivative. Since $\varepsilon^{-1}[h_1(\varepsilon x)-h_1(0)] = \varepsilon^{-1}h(\varepsilon\langle x, \bar{a}\rangle a)$ it follows from step 7 that the limit as $\varepsilon \to 0$ exists and equals $\langle x, \bar{a}\rangle h^*(0)(a)$. This is certainly continuous and linear in x . For B a bounded set in E we must show that the remainder divided by ε ,

$$\varepsilon^{-1}[h_1(\varepsilon x)-h_1(0)-\langle x, \bar{a}\rangle h^*(0)(a)]$$

converges uniformly to zero on B . Clearly we need only consider $x \in B$ for which $\langle x, \bar{a}\rangle \neq 0$. Suppose the result false. Then we can find a zero-neighbourhood U , a sequence $\{x_n\} \subset B$ and $\{\varepsilon_n\} \in (c_0)$ such that

$$\langle x_n, \bar{a}\rangle[(\varepsilon_n\langle x_n, \bar{a}\rangle)^{-1}h(\varepsilon_n\langle x_n, \bar{a}\rangle a)-h^*(0)(a)] \notin U ,$$

for every n . But since the sequence $\{\langle x_n, \bar{a}\rangle\}$ is bounded, $\{\varepsilon_n\langle x_n, \bar{a}\rangle\} \in (c_0)$, so by step 7 we have a contradiction.

We now show, for $x \in E$,

$$h_1'(x) = [\phi(1+\langle x, \bar{a}\rangle c_a)h_1]'(0) .$$

With some calculation the expression,

$$\varepsilon^{-1}[h_1(x+\varepsilon y)-h_1(x)-[\phi(1+\langle x, \bar{a}\rangle c_a)h_1]'(0)(y)] ,$$

may be shown to equal

$$\varepsilon^{-1}[\phi(1+\langle x, \bar{a}\rangle c_a)h_1(\varepsilon y)-\phi(1+\langle x, \bar{a}\rangle c_a)h_1(0)-[\phi(1+\langle x, \bar{a}\rangle c_a)h_1]'(0)(y)] .$$

That $\langle a, \bar{a}\rangle = 1$ is used here, since only then do the maps $(a \otimes \bar{a})$ and $(1+\langle x, \bar{a}\rangle c_a)$ commute. But since h_1 is Fréchet differentiable at zero and $\phi(1+\langle x, \bar{a}\rangle c_a) \in \mathcal{D}_F(E)$ the expression converges to zero with epsilon, uniformly for $y \in B$. Hence $h_1 \in \mathcal{D}_F(E)$.

9. *For any* $a \in E$, $\bar{a} \in \bar{E}$ *such that* $\langle a, \bar{a}\rangle = 1$, $(a \otimes \bar{a})h \in \mathcal{D}_F(E)$.

$$\phi^{-1}[h(a \otimes \bar{a})] = h^{-1}[h(a \otimes \bar{a})]h = (a \otimes \bar{a})h \in \mathcal{D}_F(E) .$$

10. h *is weakly-*$\mathcal{D}_F(E)$.

Since a Fréchet differentiable function is also Gâteaux differentiable and the derivatives coincide we have

$$[(a \otimes \bar{a})h]'(0)(x) = \lim_{\varepsilon\to 0} \varepsilon^{-1}[(a \otimes \bar{a})h(\varepsilon x)]$$

$$= (a \otimes \bar{a})h^*(0)(x) .$$

Since for each $\bar{a} \neq 0$ we can find an $a \neq 0$ such that $\langle a, \bar{a}\rangle = 1$, it follows that $h^*(0)$ is linear. Moreover any net $\{x_\alpha\}$ convergent to zero in E is mapped by $[(a \otimes \bar{a})h]'(0)$ to a net convergent to zero in E . Thus $\{h^*(0)(x_\alpha)\}$ converges weakly to zero, so $h^*(0) \in L(E, E_w)$. We show h weakly Fréchet differentiable at zero. Since $(a \otimes \bar{a})h \in \mathcal{D}_F(E)$ we have

$$\varepsilon^{-1}[(a \otimes \bar{a})h(\varepsilon x)-((a \otimes \bar{a})h)'(0)(x)] = \langle \varepsilon^{-1}[h(\varepsilon x)-h^*(0)(x)], \bar{a}\rangle a$$

uniformly convergent to zero for x in any bounded set in E . But this is true for all $a, \bar{a}$ such that $\langle a, \bar{a}\rangle = 1$, so h is weakly-$\mathcal{D}_F(E)$. As before in step 8 we may move this point of weak differentiability to any other point and the theorem is proved. //

The Montel spaces form a class of arbitrary dimensional locally convex spaces which are barrelled and have the property that every bounded set is relatively compact. Note that when such a space is also normed it is necessarily finite dimensional. A Montel space which is also a Fréchet space is termed a Fréchet Montel space, [17, p. 369]. We prove,

COROLLARY 2.1. *If E is a Fréchet Montel space, $\mathcal{D}_F(E)$ has the Magill property.*

Proof. By the theorem we have a weakly-$\mathcal{D}_F(E)$ bijection h associated with an automorphism ϕ of $\mathcal{D}_F(E)$ such that

$$\phi(f) = hfh^{-1} \text{ , for every } f \in \mathcal{D}_F(E) \text{ .}$$

Since E is bornological $L(E, E_w) = L(E, E)$ so the weak Fréchet derivative at zero, $h^*(0)$, is an element of $L(E)$. Now suppose $\varepsilon^{-1}[h(\varepsilon x)-h^*(0)(x)]$ does not converge to zero in E, uniformly for x in any bounded set. Then there is a zero-neighbourhood U in E, a bounded set B, a sequence $\{\varepsilon_n\} \in (c_0)$ and a sequence $\{x_n\} \subset B$ such that

$$\varepsilon_n^{-1}\left[h(\varepsilon_n x_n)-h^*(0)(\varepsilon_n x_n)\right] \notin U \text{ , for every } n \text{ .}$$

But every weakly convergent sequence in a Montel space is strongly convergent to the same limit, which contradicts the theorem. The strong differentiability may be moved to any other point to complete the corollary. //

Note that the properties of the Fréchet Montel space E used were that

(i) E has the property S ;

(ii) E is bornological;

(iii) weak sequential convergence is equivalent to strong convergence.

In particular the result holds for all finite dimensional Banach spaces and for the infinite dimensional Banach space l^1. See [10, p. 296].

If we replace $\mathcal{D}_F$ by $\mathcal{D}_H$ the arguments of the theorem remain valid provided weak sequential convergence implies strong convergence (see step 3). Hence $\mathcal{D}_H(l^1)$ also has the Magill property.

CHAPTER THREE

SEMIGROUPS OF MANY TIMES FRÉCHET DIFFERENTIABLE MAPPINGS

3.0 Introduction

It was pointed out in Chapter one that to show every automorphism of a semigroup of differentiable functions is inner is tantamount to showing that the algebraic and topological properties of the underlying *TVS* are wholly determined by the algebraic structure of that semigroup of functions. In view of this it becomes of interest to find smaller semigroups of functions on *FM*-spaces which still retain the Magill property. Consequently we turn our attention now to the semigroups $\mathcal{D}_F^k(E)$, $k \in \mathrm{N}$, $E \in LCS$.

In order to obtain a theorem for such semigroups parallel to that in the previous chapter we must restrict ourselves to Fréchet spaces. With a little additional effort we shall find corresponding results for the semigroups $C^k(E)$, $k \in \mathrm{N}$. The contents of this chapter have been submitted for publication in [59]. Before proceeding to the main theorem (Theorem 3.1) we pause to obtain a certain property of the mth Fréchet derivative.

3.1 Preliminary results

Let $E_1, \ldots, E_m$, $F \in LCS$. By $LJ(E_1 \times \ldots \times E_m, F)$ we shall mean all jointly continuous m-linear maps from $E_1 \times \ldots \times E_m$ into F , while $LS(E_1 \times \ldots \times E_m, F)$ will refer to the corresponding family of separately continuous maps. It is readily shown that the inclusions,

$$LJ(E_1 \times \ldots \times E_m, F) \subseteq L(E_1, \ldots, L(E_m, F) \ldots)$$

$$\subseteq LS(E_1 \times \ldots \times E_m, F)$$

are valid at all times. For Theorem 3.1 we require to show that elements of L are jointly sequentially continuous elements of LS .

When $m = 2$ and E_1, E_2 are Fréchet spaces, Köthe has shown in [17, p. 172, (3)] that $LS = LJ$. This may be generalised to m-linear maps in a straightforward manner, so that the desired result follows in the special case when $E_1, \ldots, E_m$ are Fréchet. Since it is hoped that the main result of this chapter may be capable of being generalised to a larger class of *LCS*'s we obtain the more general result.

In the following natural way we may associate with every element $\tilde{u}$ of the space $L(E_1, \ldots, L(E_m, F) \ldots)$, $E_1, \ldots, E_m, F \in LCS$, a map u in $LS(E_1 \times \ldots \times E_m, F)$: we define

$$u(x_1, \ldots, x_m) = \Big[\ldots [[\tilde{u}(x_1)](x_2)] \ldots (x_m)\Big] ,$$

for $x_i \in E_i$, $i = 1, \ldots, m$. For simplicity we consider only the case where $m = 2$.

RESULT 3.1. *Let* B_1, B_2 *be bounded sets in* E_1, E_2 *respectively, and* $\tilde{u}_n \to \tilde{u}_0$ *in* $L(E_1, L(E_2, F))$. *Then* $(u_n - u_0)(x_1, x_2) \to 0$ *in* F , *uniformly for* $x_1 \in B_1$, $x_2 \in B_2$.

Proof. Given a zero-neighbourhood U in F , let $W_{B_2,U}$ be the zero-neighbourhood in $L(E_2, F)$ given by $\{t \in L(E_2, F) : t(B_2) \subset U\}$. Then there is an $n_0 \in \mathrm{N}$ such that $n \geq n_0$ implies $(\tilde{u}_n - \tilde{u}_0)(B_1) \subset W_{B_2,U}$, or

$$(u_n - u_0)(B_1, B_2) = [(\tilde{u}_n - \tilde{u}_0)(B_1)](B_2) \subset U \; . \quad //$$

RESULT 3.2. *If* $\tilde{u} \in L(E_1, L(E_2, F))$ *then* $u : E_1 \times E_2 \to F$ *is sequentially continuous.*

Proof. Let $x_n \to x_0$ in E_1, $y_n \to y_0$ in E_2. Then

$$u(x_n, y_n) - u(x_0, y_0) = u(x_0, y_n - y_0) + u(x_n - x_0, y_0)$$
$$+ u(x_n - x_0, y_n - y_0) \; .$$

Every convergent sequence is bounded so the expression converges to zero as $n \to \infty$, by the method of the previous result. //

RESULT 3.3. *The evaluation mapping from*

$$L(E_1, L(E_2, F)) \times (E_1 \times E_2)$$

into F *is sequentially continuous.*

Proof. Let $\tilde{u}_n \to \tilde{u}_0$ in $L(E_1, L(E_2, F))$, $x_n \to x_0$ in E_1, and $y_n \to y_0$ in E_2. Then

$$u_n(x_n, y_n) - u_0(x_0, y_0) = (u_n - u_0)(x_n, y_n) + [u_0(x_n, y_n) - u_0(x_0, y_0)] \; .$$

Convergence to zero of the first term follows from Result 3.1 and of the second term by Result 3.2. //

For arbitrary values of m an expansion corresponding to that used in the proof of Result 3.2 may be readily obtained by induction.

COROLLARY 3.1. *Let* $f \in C^m(E)$, $E \in LCS$, $m \in \mathrm{N}$, *and* $x_n \to x_0$, $y_n^i \to y^i$, $i = 1, \ldots, m$. *Then*

$$f^{(m)}(x_n)(y_n^1) \ldots (y_n^m) \to f^{(m)}(x_0)(y^1) \ldots (y^m) \; , \text{ as } n \to \infty \; .$$

3.2 The main theorem

We proceed to the proof of the following:

THEOREM 3.1. *Let* E *be a Fréchet space. Then if* ϕ *is an automorphism of* $\mathcal{D}_F^k(E)$ *there is a bijection* h *of* E *such that* h *and* h^{-1} *are weakly-*$\mathcal{D}_F^k(E)$ *and*

$$\phi(f) = hfh^{-1} \text{ , for every } f \in \mathcal{D}_F^k(E) \text{ .} \tag{1}$$

Proof.

1. *There exists a bijection* h *of* E *such that* (1) *holds.*

This is again an application of Lemma 1.1.

It was pointed out in Chapter two that the elegant method of Magill used to show h once differentiable is no longer applicable when the space has dimension greater than one. A further difficulty is encountered in the present situation. Even in the case where $E = \mathrm{R}$, the derivative of the map h is everywhere finite, and

$$(h^{-1})'(h(x))h'(x) = 1 \text{ , for } x \in \mathrm{R} \text{ ,}$$

so that $h'(x) \neq 0$ for any x . Hence h' is certainly not a bijection, with the result that the method cannot be used in advancing to derivatives of higher order.

2. *For any* $\overline{a} \in \overline{E}$ *the function* $\langle h(x), \overline{a} \rangle$ *of* E *into* R *is continuous with respect to* $x \in E$.

Since differentiability implies continuity in a Fréchet space, this is a consequence of Lemma 1.2.

3. $h(a \otimes \overline{a}) \in \mathcal{D}_F^k(E)$ *for every* $a \in E$, $\overline{a} \in \overline{E}$ *such that* $\langle a, \overline{a} \rangle = 1$.

As before we let $h(a \otimes \overline{a}) = h_1$, and also $h^{-1}(a \otimes \overline{a}) = h_2$. The proof is by induction. By noting that the constant map c_a , and the one-dimensional map $(a \otimes \overline{a})$, belong to $\mathcal{D}_F^k(E)$, the case $k = 1$

follows as in Chapter two, steps 2 to 8. Now we assume $h_1 \in \mathcal{D}_F^m(E)$, $1 \leq m < k$, and show $h_1 \in \mathcal{D}_F^{m+1}(E)$.

This reduces in essence to showing that the limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\left[h_1^{(m)}(\varepsilon a)(a)^m - h_1^{(m)}(0)(a)^m\right]$$

exists. For this we are led to consider the differentiability of the real-valued functions of a real variable,

$$\lambda_{\bar{x}}(\xi) = \left\langle h_1^{(m)}(\xi a)(a)^m , \bar{x}\right\rangle , \text{ for each } \bar{x} \in \bar{E} .$$

When m is odd we are able to show that for $\{\varepsilon_n\} \in (c_0)$ the sequence $\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\alpha a+\varepsilon_n a)(a)^m - h_1^{(m)}(\alpha a-\varepsilon_n a)(a)^m\right]\right\}$ is bounded, for $\alpha \in R$. Using a longstanding result of Khintchine [16] it is then shown relatively readily that the $\lambda_{\bar{x}}$ are differentiable almost everywhere. Yet in the even case pursuing a similar path with the sequence

$$\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\alpha a+\varepsilon_n a)(a)^m + h_1^{(m)}(\alpha a-\varepsilon_n a)(a)^m - 2h_1^{(m)}(\alpha a)(a)^m\right]\right\}$$

and using a result of Zygmund [68] yields only the finiteness of the Dini derivatives of each $\lambda_{\bar{x}}$ on a dense set in R . With more effort differentiability almost everywhere does follow and the method of the odd case takes over. Regretfully the calculations are necessarily lengthy since we are constantly dealing with expansions of higher order derivatives of composition functions. Firstly, we show

3.1. $h_1^{(m)}(\xi a)(a)^m$ *is continuous with respect to* $\xi \in R$

If $\{\varepsilon_n\} \in (c_0)$, $h_1^{(m)}(\xi a)(a)^m$ is equal to the limit

$$\lim_{n\to\infty} \varepsilon_n^{-1}\left[h_1^{(m-1)}(\xi a+\varepsilon_n a)(a)^{m-1} - h_1^{(m-1)}(\xi a)(a)^{m-1}\right] .$$

By a result of Banach, [3, p. 397] such a limit function is continuous on a dense set. Suppose that α is such a point of continuity and ξ is an arbitrary real number. Then if $\{\varepsilon_n\} \in (c_0)$,

$$h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m = h_1^{(m)}[(1+c_{\xi a-\alpha a})(\alpha a+\varepsilon_n a)](a)^m$$

$$= [\phi(1+c_{\xi a-\alpha a})h_1]^{(m)}(\alpha a+\varepsilon_n a)(a)^m .$$

That $\langle a, \bar{a}\rangle = 1$ is used here, again to ensure the commutativity of the maps $(a \otimes \bar{a})$ and $(1-c_{\xi a-\alpha a})$. Using the expression given in [1, p. 234] for the expansion of a higher order derivative of a composition function, it is evident that the last term converges to

$$[\phi(1+c_{\xi a-\alpha a})h_1]^{(m)}(\alpha a)(a)^m = h_1^{(m)}(\xi a)(a)^m$$

as ε_n tends to zero. Hence $h_1^{(m)}(\xi a)(a)^m$ is a continuous function of ξ .

As in the previous chapter we have the following pair of results:

3.2. *Given* $\xi \in \mathrm{R}$, $a \neq 0$ *in* E , *and* $\{\varepsilon_n\} \in (c_0)$, *the sequence* $\left\{\varepsilon_n^{-1}[h(\xi a+\varepsilon_n a)-h(\xi a)]\right\}$ *does not converge weakly to zero.*

3.3. *Given* $\xi \in \mathrm{R}$ *and non-zero* $a \in E$, *there exists an* $\bar{x}_\xi \in \bar{E}$ *such that* $\phi\left[(a \otimes \bar{x}_\xi)\left(1-c_{h^{-1}(\xi a)}\right)\right]'(\xi a)(a) \neq 0$.

For fixed $\xi \in \mathrm{R}$ we let

$$S(\bar{x}_\xi) = \left\{\eta \in \mathrm{R} : \phi\left[(a \otimes \bar{x}_\xi)\left(1-c_{h^{-1}(\eta a)}\right)\right]'(\eta a)(a) \neq 0\right\} .$$

We show $S(\bar{x}_\xi)$ is open. Suppose $\eta \in S(\bar{x}_\xi)$, then

$$\langle h_2'(\eta a)(a), \overline{x}_\xi \rangle h_1'(0)(a) = h_1'(0)\left(a \otimes \overline{x}_\xi\right)h_2'(\eta a)(a)$$

$$= \left[h_1\left(a \otimes \overline{x}_\xi\right)\left\{1-c_{h^{-1}(\eta a)}\right\}h_2\right]'(\eta a)(a)$$

$$= \phi\left[\left(a \otimes \overline{x}_\xi\right)\left\{1-c_{h^{-1}(\eta a)}\right\}\right]'(\eta a)(a) ,$$

since $(a \otimes \overline{a})\left(a \otimes \overline{x}_\xi\right) = \left(a \otimes \overline{x}_\xi\right)$.

So $\langle h_2'(\eta a)(a), \overline{x}_\xi \rangle \neq 0$. By 3.1 this is a continuous function of η, showing $S\left(\overline{x}_\xi\right)$ to be open.

We are now in a position to show $h_1 \in \mathcal{D}_F^{m+1}(E)$, but must deal with the odd and even cases separately.

3.4. *Case where* m *odd*.

We show for arbitrary $\overline{x} \in \overline{E}$ that the continuous map $\lambda_{\overline{x}}$, defined at the beginning of step three, has the property that

$$\limsup_{\varepsilon \to 0}\left|\varepsilon^{-1}\left[\lambda_{\overline{x}}(\xi+\varepsilon)-\lambda_{\overline{x}}(\xi-\varepsilon)\right]\right| < \infty$$

for every $\xi \in R$.

3.4.1. *For any sequence* $\{\varepsilon_n\} \in \left(c_0\right)$ *, the set*

$$\left\{\varepsilon_n^{-1}\left[h_1^{(m)}\left(\varepsilon_n a\right)(a)^m - h_1^{(m)}\left(-\varepsilon_n a\right)(a)^m\right]\right\}$$

is bounded.

With $\overline{x}_0$ the functional associated with $\xi = 0$, as in 3.3, and arbitrary $\{\delta_n\} \in \left(c_0\right)$, consider the expression

$$[\phi(a \otimes \bar{x}_0)(a \otimes \bar{a})]^{(m+1)}(0)(a)^{m+1}$$

$$= \lim_{n\to\infty} \delta_n^{-1}\left[(h_1(a \otimes \bar{x}_0)h_2)^{(m)}(\delta_n a)(a)^m - (h_1(a \otimes \bar{x}_0)h_2)^{(m)}(0)(a)^m\right]$$

$$= \lim_{n\to\infty} \delta_n^{-1}\Big[(h_1(a \otimes \bar{x}_0))^{(m)}(h_2(\delta_n a))(h_2'(\delta_n a)(a))^m$$

$$- (h_1(a \otimes \bar{x}_0))^{(m)}(0)(h_2'(0)(a))^m$$

$$+ \sum_{1\le q<m} \sum \sigma_m\Big[(h_1(a \otimes \bar{x}_0))^{(q)}(h_2(\delta_n a))\left[h_2^{(i_1)}(\delta_n a)(a)^{i_1}\right]$$

$$\cdots \left[h_2^{(i_q)}(\delta_n a)(a)^{i_q}\right]$$

$$-(h_1(a \otimes \bar{x}_0))^{(q)}(0)\left[h_2^{(i_1)}(0)(a)^{i_1}\right] \cdots \left[h_2^{(i_q)}(0)(a)^{i_q}\right]\Big]\Big] .$$

The second summation is over all q-tuples of positive integers $i_1, \ldots, i_q$, such that $i_1 + \ldots + i_q = m$, and σ_m is an integer coefficient. We may now write down a similar expansion for $[\phi(-a \otimes \bar{x}_0)a \otimes \bar{a}]^{(m+1)}(0)(a)^{m+1}$, and show

$$[\phi(a \otimes \bar{x}_0)a \otimes \bar{a}]^{(m+1)}(0)(a)^{m+1} + [\phi(-a \otimes \bar{x}_0)a \otimes \bar{a}]^{(m+1)}(0)(a)^{m+1}$$

$$= \lim_{n\to\infty} \delta_n^{-1}\Big[\Big[(h_1(a \otimes \bar{x}_0))^{(m)}\left(h^{-1}(\delta_n a)\right) -$$

$$(h_1(a \otimes \bar{x}_0))^{(m)}\left(-h^{-1}(\delta_n a)\right)\Big](h_2'(\delta_n a)(a))^m$$

$$+ \sum_{1\le q<m} \sum \sigma_m\Big[(h_1(a \otimes \bar{x}_0))^{(q)}(h_2(\delta_n a))\left[h_2^{(i_1)}(\delta_n a)(a)^{i_1}\right]$$

$$\cdots \left[h_2^{(i_q)}(\delta_n a)(a)^{i_q}\right]$$

$$+(-1)^q(h_1(a \otimes \bar{x}_0))^{(q)}(-h_2(\delta_n a))\left[h_2^{(i_1)}(\delta_n a)(a)^{i_1}\right]$$

$$\cdots \left[h_2^{(i_q)}(\delta_n a)(a)^{i_q}\right]$$

$$-\left(1+(-1)^q\right)\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}(0)\left[h_2^{\left(i_1\right)}(0)(a)^{i_1}\right]$$

$$\ldots\left[h_2^{\left(i_q\right)}(0)(a)^{i_q}\right]\Big]\Big] . \quad (2)$$

We wish to show that the sequences formed by the terms within the double summation are bounded. If q is odd the term becomes,

$$\delta_n^{-1}\left[\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}\left(h_2\left(\delta_n a\right)\right)-\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}\left(-h_2\left(\delta_n a\right)\right)\right]$$

$$\left[h_2^{\left(i_1\right)}\left(\delta_n a\right)(a)^{i_1}\right] \ldots\left[h_2^{\left(i_q\right)}\left(\delta_n a\right)(a)^{i_q}\right]$$

$$=\left\langle\delta_n^{-1} h^{-1}\left(\delta_n a\right), \bar{x}_0\right\rangle\left\langle h^{-1}\left(\delta_n a\right), \bar{x}_0\right\rangle^{-1}$$

$$\left[\left[h_1^{(q)}\left\{\left\langle h^{-1}\left(\delta_n a\right), \bar{x}_0\right\rangle a\right\}-h_1^{(q)}(0)\right]-\left[h_1^{(q)}\left\{-\left\langle h^{-1}\left(\delta_n a\right), \bar{x}_0\right\rangle a\right\}-h_1^{(q)}(0)\right]\right]$$

$$\left\{\left\langle h_2^{\left(i_1\right)}\left(\delta_n a\right)(a)^{i_1}, \bar{x}_0\right\rangle a\right\} \ldots\left\{\left\langle h_2^{\left(i_q\right)}\left(\delta_n a\right)(a)^{i_q}, \bar{x}_0\right\rangle a\right\},$$

since if $\left\langle h^{-1}\left(\delta_n a\right), \bar{x}_0\right\rangle=0$, the expression vanishes. By noting that $\left\{\delta_n^{-1} h^{-1}\left(\delta_n a\right)\right\}$ converges, $\left\{\left\langle h^{-1}\left(\delta_n a\right), \bar{x}_0\right\rangle\right\} \in\left(c_0\right)$, $h_1^{(q)}$ is Fréchet differentiable, and $h_1^{(q)}(\xi a)(a)^q$ is continuous in ξ, it is evident that the sequence converges.

If q is even, by adding and subtracting a suitable term we have,

$$\delta_n^{-1}\left[\left[\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}\left(h_2\left(\delta_n a\right)\right)+\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}\left(-h_2\left(\delta_n a\right)\right)\right.\right.$$

$$\left.-2\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}(0)\right]\left[h_2^{\left(i_1\right)}\left(\delta_n a\right)(a)^{i_1}\right] \ldots\left[h_2^{\left(i_q\right)}\left(\delta_n a\right)(a)^{i_q}\right]$$

$$+2\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}(0)\left[h_2^{\left(i_1\right)}\left(\delta_n a\right)(a)^{i_1}\right] \ldots\left[h_2^{\left(i_q\right)}\left(\delta_n a\right)(a)^{i_q}\right]$$

$$\left.-2\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}(0)\left[h_2^{\left(i_1\right)}(0)(a)^{i_1}\right] \ldots\left[h_2^{\left(i_q\right)}(0)(a)^{i_q}\right]\right] .$$

Convergence of the first term follows in a manner similar to the case where q is odd, while the second and third terms may be rearranged as,

$$2\Big[\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)$$

$$\ldots\Big(h_2^{(i_{q-1})}(\delta_n a)(a)^{i_{q-1}}\Big)\Big(\delta_n^{-1}\Big[h_2^{(i_q)}(\delta_n a)(a)^{i_q}-h_2^{(i_q)}(0)(a)^{i_q}\Big]\Big)$$

$$+\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)$$

$$\ldots\Big(\delta_n^{-1}\Big[h_2^{(i_{q-1})}(\delta_n a)(a)^{i_{q-1}}-h_2^{(i_{q-1})}(0)(a)^{i_{q-1}}\Big]\Big)\Big(h_2^{(i_q)}(0)(a)^{i_q}\Big)$$

$$\vdots$$

$$+\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)\Big(\delta_n^{-1}\Big[h_2^{(i_1)}(\delta_n a)(a)^{i_1}-h_2^{(i_1)}(0)(a)^{i_1}\Big]\Big)$$

$$\ldots\Big(h_2^{(i_q)}(0)(a)^{i_q}\Big)\Big] .$$

Since $i_j < m$, $j = 1, \ldots, q$, $h_2^{(i_j)}$ is Fréchet differentiable and so *a fortiori* Gâteaux differentiable. Moreover, by Result 3.2, $[h_1(a\otimes\bar{x}_0)]^{(q)}(0) : E^q \to E$ is continuous, so convergence with n follows. This technique for showing convergence will be used frequently, but elsewhere we shall refrain from presenting these detailed calculations.

The first term in (2) is

$$\big(\langle h_2'(\delta_n a)(a), \bar{x}_0\rangle\big)^m\Big(\delta_n^{-1}\langle h_2(\delta_n a), \bar{x}_0\rangle\Big)\Big\langle h^{-1}(\delta_n a), \bar{x}_0\Big\rangle^{-1}$$

$$\Big[h_1^{(m)}\Big(\Big\langle h^{-1}(\delta_n a), \bar{x}_0\Big\rangle a\Big)-h_1^{(m)}\Big(-\Big\langle h^{-1}(\delta_n a), \bar{x}_0\Big\rangle a\Big)\Big](a)^m .$$

Recall that $\langle h_2'(0)(a), \bar{x}_0\rangle \neq 0$ and that given $\{\varepsilon_n\} \in (c_0)$ there is a sequence $\{\delta_k\} \in (c_0)$ and a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$\left\langle h^{-1}(\delta_k a),\ \bar{x}_0\right\rangle = \varepsilon_{n_k}\ , \text{ for every } k\ .$$

Hence we can conclude that given $\{\varepsilon_n\} \in (c_0)$ there is a subsequence $\left\{\varepsilon_{n_k}\right\}$ such that the set

$$\left\{\varepsilon_{n_k}^{-1}\left[h_1^{(m)}\left(\varepsilon_{n_k} a\right)(a)^m - h_1^{(m)}\left(-\varepsilon_{n_k} a\right)(a)^m\right]\right\}$$

is bounded. Immediately we have 3.4.1. This property is now transferred to an arbitrary $\xi \in R$.

3.4.2. *For any sequence* $\{\varepsilon_n\} \in (c_0)$, *and* $\xi \in R$, *the set* $\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m - h_1^{(m)}(\xi a-\varepsilon_n a)(a)^m\right]\right\}$ *is bounded.*

Consider

$$\begin{aligned}
&\varepsilon_n^{-1}\left[h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m - h_1^{(m)}(\xi a-\varepsilon_n a)(a)^m\right] \\
&\quad = \varepsilon_n^{-1}\left[\left[\phi(1+c_{\xi a})h_1\right]^{(m)}(\varepsilon_n a)(a)^m - \left[\phi(1+c_{\xi a})h_1\right]^{(m)}(-\varepsilon_n a)(a)^m\right] \\
&\quad = \varepsilon_n^{-1}\Bigg[\sum_{1\le q\le m} \sum \sigma_m(\phi(1+c_{\xi a}))^{(q)}(h(\varepsilon_n a))\left\{h_1^{(i_1)}(\varepsilon_n a)(a)^{i_1}\right\} \\
&\qquad\qquad \cdots \left\{h_1^{(i_q)}(\varepsilon_n a)(a)^{i_q}\right\} \\
&\qquad\qquad - \phi(1+c_{\xi a})^{(q)}(h(-\varepsilon_n a))\left\{h_1^{(i_1)}(-\varepsilon_n a)(a)^{i_1}\right\} \\
&\qquad\qquad \cdots \left\{h_1^{(i_q)}(-\varepsilon_n a)(a)^{i_q}\right\}\Bigg]\ .
\end{aligned}$$

In order to show the boundedness of the sequence formed by the term appearing when $q = 1$ we rewrite it as

$$\begin{aligned}
&\varepsilon_n^{-1}\left[\phi(1+c_{\xi a})'(h(\varepsilon_n a)) - \phi(1+c_{\xi a})'(0)\right]\left\{h_1^{(m)}(\varepsilon_n a)(a)^m\right\} \\
&\quad - \varepsilon_n^{-1}\left[\phi(1+c_{\xi a})'(h(-\varepsilon_n a)) - \phi(1+c_{\xi a})'(0)\right]\left\{h_1^{(m)}(-\varepsilon_n a)(a)^m\right\} \\
&\quad + \phi(1+c_{\xi a})'(0)\left[\varepsilon_n^{-1}\left\{h_1^{(m)}(\varepsilon_n a)(a)^m - h_1^{(m)}(-\varepsilon_n a)(a)^m\right\}\right]\ .
\end{aligned}$$

We may assume $k \geq 2$, so boundedness follows for the first pair of sequences since $\phi\left(1+c_{\xi a}\right)''(0) \in L(E, L(E))$. From 3.4.1 it follows that the third sequence is bounded. Decomposing the term formed for $1 < q \leq m < k$ in a similar way shows that it too forms a bounded sequence.

As was pointed out in Chapter two, step seven, no loss of generality is suffered if at this stage we assume E to be separable. A result in [17, p. 259] then gives that $\overline{E}$ is weakly sequentially separable, which means that every element of $\overline{E}$ is the weak limit of a subsequence of a fixed sequence, $\{\overline{a}_i\}$ of elements of $\overline{E}$. Notice that such a set is also total. We now show

3.4.3. *For some* $\alpha \in R$ *the limit*

$$\lim_{\varepsilon \to 0} \left\langle \varepsilon^{-1}\left[h_1^{(m)}(\alpha a+\varepsilon a)(a)^m - h_1^{(m)}(\alpha a)(a)^m\right], \overline{a}_i \right\rangle$$

exists, for every $i = 1, 2, \ldots$.

Recall that each $\lambda_i = \lambda_{\overline{a}_i}$ is continuous, while from 3.4.2 it follows that

$$\limsup_{\varepsilon \to 0} \left| \varepsilon^{-1}\left[\lambda_i(\xi+\varepsilon) - \lambda_i(\xi-\varepsilon)\right] \right| < \infty$$

for every $\xi \in R$. An early result of Khintchine [16, p. 217] shows that this is sufficient for each λ_i to be differentiable almost everywhere. We deduce the existence of an $\alpha \in R$ at which each of the functions λ_i is differentiable. Coupled with the following, this enables us to show that the limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\left[h_1^{(m)}(\varepsilon a)(a)^m - h_1^{(m)}(0)(a)^m\right]$$

exists.

3.4.4. *Given* $\{\varepsilon_n\} \in (c_0)$ *there is a subsequence* $\left\{\varepsilon_{n_k}\right\}$ *such* *that the limit,* $\lim_{k\to\infty} \varepsilon_{n_k}^{-1}\left[h_1^{(m)}\left(\varepsilon_{n_k} a\right)(a)^m - h_1^{(m)}(0)(a)^m\right]$ *, exists.*

Although the inductive assumption was that $h_1 \in \mathcal{D}_F^m(E)$ we may also assume $h_2 \in \mathcal{D}_F^m(E)$, since any property true of h can also be shown for h^{-1} . In fact we have $h_2^{(m)}(\xi a)(a)^m$ continuous in ξ . With $\overline{x}_0$ as before, and $\xi \in \mathrm{R}$, we examine the expression

$$\left[\phi\left\{(a \otimes \overline{a})\left(a \otimes \overline{x}_0\right)\left(1-c_{h^{-1}(\xi a)}\right)\right\}(a \otimes \overline{a})\right]^{(m+1)}(\xi a)(a)^{m+1} .$$

For $\{\delta_n\} \in (c_0)$, this is the limit of the sequence with nth term

$$\delta_n^{-1}\left[\left[h_1\left(a \otimes \overline{x}_0\right)\left\{1-c_{h^{-1}(\xi a)}\right\}h_2\right]^{(m)}\left(\xi a+\delta_n a\right)(a)^m - \left[h_1\left(a \otimes \overline{x}_0\right)\left\{1-c_{h^{-1}(\xi a)}\right\}h_2\right]^{(m)}(\xi a)(a)^m\right] ,$$

which with some computation may be shown to equal,

$$\delta_n^{-1}\left[h_1^{(m)}\left\{\left\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\ \overline{x}_0\right\rangle a\right\}-h_1^{(m)}(0)\right]\left(\langle h_2'(\xi a)(a),\ \overline{x}_0\rangle a\right)^m$$

$$+\ \delta_n^{-1}\left[h_1^{(m)}\left\{\left\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\ \overline{x}_0\right\rangle a\right\}\left(\langle h_2'(\xi a+\delta_n a)(a),\ \overline{x}_0\rangle a\right)^m - h_1^{(m)}\left\{\left\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\ \overline{x}_0\right\rangle a\right\}\left(\langle h_2'(\xi a)(a),\ \overline{x}_0\rangle a\right)^m\right]$$

$$+\ \delta_n^{-1}\left[\sum_{1<q<m}\sum \sigma_m\left\{\left[h_1\left(a \otimes \overline{x}_0\right)\left\{1-c_{h^{-1}(\xi a)}\right\}\right]^{(q)}\left\{h^{-1}(\xi a+\delta_n a)\right\}\left\{h_2^{(i_1)}(\xi a+\delta_n a)(a)^{i_1}\right\}\cdots\left\{h_2^{(i_q)}(\xi a+\delta_n a)(a)^{i_q}\right\}\right.\right.$$

$$\left.\left.-\left[h_1\left(a \otimes \overline{x}_0\right)\left\{1-c_{h^{-1}(\xi a)}\right\}\right]^{(q)}\left(h^{-1}(\xi a)\right)\left\{h_2^{(i_1)}(\xi a)(a)^{i_1}\right\}\cdots\left\{h_2^{(i_q)}(\xi a)(a)^{i_q}\right\}\right\}\right]$$

$$+\delta_n^{-1}\Big[h_1'\Big(\Big\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\ \bar{x}_0\Big\rangle a\Big)-h_1'(0)\Big]\Big(\Big\langle h_2^{(m)}(\xi a)(a)^m,\ \bar{x}_0\Big\rangle a\Big)$$

$$+\Big\langle \delta_n^{-1}\Big\{h_2^{(m)}(\xi a+\delta_n a)(a)^m-h_2^{(m)}(\xi a)(a)^m\Big\},\ \bar{x}_0\Big\rangle$$

$$h_1'\Big(\Big\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\ \bar{x}_0\Big\rangle a\Big)(a)\ .$$

Firstly, assume $m > 1$. When suitable terms are added to and subtracted from the general term of the second sequence, the scalar coefficients taken out, and the continuity result of step 3.1 applied to the remainder, the second sequence may be shown to converge. By fixing q in the third sequence, again adding and subtracting suitable terms, and observing that $i_j < m$, $j = 1, \ldots, q$ so that $h_1^{(i_j)}$ is Fréchet differentiable, we are able to use the fact that differentiability implies continuity when the first space is sequential to show that the third sequence converges. Thus the three central terms form convergent sequences, as does the final term for all ξ in a set of full measure A , by the results of 3.4.3. Choosing $\xi \in S(\bar{x}_0) \cap A$ we are then able to find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ for which the limit

$$\lim_{k\to\infty} \varepsilon_{n_k}^{-1}\Big[h_1^{(m)}\big(\varepsilon_{n_k} a\big)-h_1^{(m)}(0)\Big](a)^m$$

exists, since $\langle h_2'(\xi a)(a),\ \bar{x}_0\rangle \neq 0$. Note that when $m = 1$ only the final pair of sequences remain and the proof goes through as before.

3.4.5. *The limit,* $\lim_{\varepsilon\to 0} \varepsilon^{-1}\Big[h_1^{(m)}(\varepsilon a)(a)^m-h_1^{(m)}(0)(a)^m\Big]$ *exists.*

In view of the conclusion of 3.4.4 we must show that if $\{\delta_n\}, \{\varepsilon_n\} \in (c_0)$ and

$$\lim_{n\to\infty} \varepsilon_n^{-1}\Big[h_1^{(m)}(\varepsilon_n a)(a)^m-h_1^{(m)}(0)(a)^m\Big] = a_1\ ,$$

$$\lim_{n\to\infty} \delta_n^{-1}\left[h_1^{(m)}(\delta_n a)(a)^m - h_1^{(m)}(0)(a)^m\right] = a_2 ,$$

then $a_1 = a_2$. Now with α as in 3.4.3,

$$\varepsilon_n^{-1}\left[h_1^{(m)}(\alpha a+\varepsilon_n a)(a)^m - h_1^{(m)}(\alpha a)(a)^m\right]$$

$$= \phi(1+c_{\alpha a})'(0)\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\varepsilon_n a)(a)^m - h_1^{(m)}(0)(a)^m\right]\right\}$$

$$+ \varepsilon_n^{-1}[\phi(1+c_{\alpha a})'(h_1(\varepsilon_n a)) - \phi(1+c_{\alpha a})'(0)]\left[h_1^{(m)}(\varepsilon_n a)(a)^m\right]$$

$$+ \sum_{2\le q\le m} \sum \sigma_m \varepsilon_n^{-1}\left[\phi(1+c_{\alpha a})^{(q)}(h_1(\varepsilon_n a))\left[h_1^{(i_1)}(\varepsilon_n a)(a)^{i_1}\right] \cdots \left[h_1^{(i_q)}(\varepsilon_n a)(a)^{i_q}\right]\right.$$

$$\left. - \phi(1+c_{\alpha a})^{(q)}(0)\left[h_1^{(i_1)}(0)(a)^{i_1}\right] \cdots \left[h_1^{(i_q)}(0)(a)^{i_q}\right]\right] .$$

All but the first term on the right hand side converge to a value independent of the sequence $\{\varepsilon_n\} \in (c_0)$. Hence, by 3.4.3,

$$\langle \phi(1+c_{\alpha a})'(0)(a_1), \bar{a}_i \rangle = \langle \phi(1+c_{\alpha a})'(0)(a_2), \bar{a}_i \rangle ,$$

for every $i = 1, 2, \ldots$. Since $\{\bar{a}_i\}$ is total and $\phi(1+c_{\alpha a})'(0)$ is one-to-one, we have $a_1 = a_2$.

3.5. *Case where* m *even.*

Due to the fact that

$$h_1^{(m)}(-\varepsilon a)(a)^m = \begin{cases} (h_1.-1)^{(m)}(\varepsilon a)(a)^m , & \text{for } m \text{ even,} \\ -(h_1.-1)^{(m)}(\varepsilon a)(a)^m , & \text{for } m \text{ odd,} \end{cases}$$

we are led in the even case to examine a second order difference quotient.

3.5.1. *For* $\{\varepsilon_n\} \in (c_0)$, *the set*

$$\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\varepsilon_n a)(a)^m + h_1^{(m)}(-\varepsilon_n a)(a)^m - 2h_1^{(m)}(0)(a)^m\right]\right\}$$

is bounded.

As in the odd case we calculate $\left[\phi\left(a \otimes \bar{x}_0\right) a \otimes \bar{a}\right]^{(m+1)}(0)(a)^{m+1}$ and $\left[\phi\left(-a \otimes \bar{x}_0\right) a \otimes \bar{a}\right]^{(m+1)}(0)(a)^{m+1}$ and find that their sum is, for $\{\delta_n\} \in (c_0)$,

$$\lim_{n\to\infty} \left(\langle h_2'(\delta_n a)(a), \bar{x}_0\rangle\right)^m \left\{\delta_n^{-1}\left\langle h^{-1}(\delta_n a), \bar{x}_0\right\rangle\right\}\left\langle h^{-1}(\delta_n a), \bar{x}_0\right\rangle^{-1}$$

$$\left[h_1^{(m)}\left(\left\langle h^{-1}(\delta_n a), \bar{x}_0\right\rangle a\right) + h_1^{(m)}\left(-\left\langle h^{-1}(\delta_n a), \bar{x}_0\right\rangle a\right) - 2h_1^{(m)}(0)\right](a)^m$$

$$+ 2\delta_n^{-1}\left[\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(m)}(0)\left(h_2'(\delta_n a)(a)\right)^m - \left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(m)}(0)\left(h_2'(0)(a)\right)^m\right]$$

$$+ \sum_{1\le q<m} \sum \sigma_m \delta_n^{-1}\left[\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}\left(h^{-1}(\delta_n a)\right)\left(h_2^{(i_1)}(\delta_n a)(a)^{i_1}\right) \cdots \left(h_2^{(i_q)}(\delta_n a)(a)^{i_q}\right)\right.$$

$$+ (-1)^q\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}\left(-h^{-1}(\delta_n a)\right)\left(h_2^{(i_1)}(\delta_n a)(a)^{i_1}\right) \cdots \left(h_2^{(i_q)}(\delta_n a)(a)^{i_q}\right)$$

$$\left. - \left(1+(-1)^q\right)\left(h_1\left(a \otimes \bar{x}_0\right)\right)^{(q)}(0)\left(h_2^{(i_1)}(0)(a)^{i_1}\right) \cdots \left(h_2^{(i_q)}(0)(a)^{i_q}\right)\right] .$$

Since m is even, h_2' is Fréchet differentiable so the middle sequence converges. The term within the double summation is identical to that in 3.4.1 where it was shown to give rise to a bounded sequence. Again, $\langle h_2'(0)(a), \bar{x}_0\rangle \neq 0$, allowing us to conclude that given $\{\varepsilon_n\} \in (c_0)$ there is a subsequence $\left\{\varepsilon_{n_k}\right\}$ such that the set

$$\left\{\varepsilon_{n_k}^{-1}\left[h_1^{(m)}\left(\varepsilon_{n_k} a\right)(a)^m + h_1^{(m)}\left(-\varepsilon_{n_k} a\right)(a)^m - 2h_1^{(m)}(0)(a)^m\right]\right\}$$

is bounded. Then 3.5.1 follows.

3.5.2. *For* $\{\varepsilon_n\} \in (c_0)$ *, and* $\xi \in \mathrm{R}$ *, the set*

$$\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m+h_1^{(m)}(\xi a-\varepsilon_n a)(a)^m-2h_1^{(m)}(\xi a)(a)^m\right]\right\}$$

is bounded.

Using the translation map, $(1+c_{\xi a})$, and the technique of 3.4.2 we can show the above expression is

$$\varepsilon_n^{-1}\Big[\phi(1+c_{\xi a})'(h(\varepsilon_n a))\left[h_1^{(m)}(\varepsilon_n a)(a)^m\right] +$$

$$\phi(1+c_{\xi a})'(h(-\varepsilon_n a))\left[h_1^{(m)}(-\varepsilon_n a)(a)^m\right] - 2\phi(1+c_{\xi a})'(0)\left[h_1^{(m)}(0)(a)^m\right]\Big]$$

$$+ \varepsilon_n^{-1}\Big[\sum_{2\le q\le m} \sum \sigma_m\Big[\phi(1+c_{\xi a})^{(q)}(h(\varepsilon_n a))\left[h_1^{(i_1)}(\varepsilon_n a)(a)^{i_1}\right]$$

$$\dots \left[h_1^{(i_q)}(\varepsilon_n a)(a)^{i_q}\right]$$

$$+ \phi(1+c_{\xi a})^{(q)}(h(-\varepsilon_n a))\left[h_1^{(i_1)}(-\varepsilon_n a)(a)^{i_1}\right] \dots \left[h_1^{(i_q)}(-\varepsilon_n a)(a)^{i_q}\right]$$

$$- 2\phi(1+c_{\xi a})^{(q)}(0)\left[h_1^{(i_1)}(0)(a)^{i_1}\right] \dots \left[h_1^{(i_q)}(0)(a)^{i_q}\right]\Big]\Big].$$

Boundedness of the sequence formed by the first term in this expression follows since we may rewrite the term as

$$\phi(1+c_{\xi a})'(0)\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\varepsilon_n a)(a)^m+h_1^{(m)}(-\varepsilon_n a)(a)^m-2h_1^{(m)}(0)(a)^m\right]\right\}$$

$$+ \varepsilon_n^{-1}\left[\phi(1+c_{\xi a})'(h(\varepsilon_n a))-\phi(1+c_{\xi a})'(0)\right]\left[h_1^{(m)}(\varepsilon_n a)(a)^m\right]$$

$$+ \varepsilon_n^{-1}\left[\phi(1+c_{\xi a})'(h(-\varepsilon_n a))-\phi(1+c_{\xi a})'(0)\right]\left[h_1^{(m)}(-\varepsilon_n a)(a)^m\right] .$$

We use 3.5.1 and the fact that $\phi(1+c_{\xi a})''(0) \in L(E, L(E))$.

Twofold application of the procedure of 3.4.2 shows boundedness for the sequence of terms within the double summation.

In the terminology of Zygmund, [68], the continuous functions λ_i , $i = 1, 2, \dots$ have the property Λ on R . That is,

$$\lambda_i(\xi+\varepsilon) + \lambda_i(\xi-\varepsilon) - 2\lambda_i(\xi) = O(\varepsilon) \ , \quad \xi \in \mathrm{R} .$$

As indicated in [68, p. 55] this is insufficient to ensure the differentiability of λ_i at even a single point. However, it does mean that the set of points at which all four Dini derivatives of λ_i are finite is everywhere dense.

3.5.3. *Given* $\{\varepsilon_n\} \in (c_0)$ *, the set*

$$\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\varepsilon_n a)(a)^m - h_1^{(m)}(0)(a)^m\right]\right\}$$

is bounded.

The calculations of 3.4.4 suffice to show that if $\{\delta_n\} \in (c_0)$, and the set $\left\{\left\langle \delta_n^{-1}\left[h_2^{(m)}(\xi a+\delta_n a)(a)^m - h_2^{(m)}(\xi a)(a)^m\right], \bar{x}_0\right\rangle\right\}$ is bounded in R , then so too is the set

$$\left\{\left\langle h^{-1}(\xi a+\delta_n a) - h^{-1}(\xi a), \bar{x}_0\right\rangle^{-1} \left[h_1^{(m)}\left(\left\langle h^{-1}(\xi a+\delta_n a) - h^{-1}(\xi a), \bar{x}_0\right\rangle a\right) - h_1^{(m)}(0)\right](a)^m\right\}$$

in E . Choosing ξ to be in the dense set in which all four Dini derivatives of $\lambda_{\bar{x}_0}$ are finite, as well as in the open set $S(\bar{x}_0)$, we deduce the existence of a subsequence $\left\{\varepsilon_{n_k}\right\}$ of $\{\varepsilon_n\}$ for which the set $\left\{\varepsilon_{n_k}^{-1}\left[h_1^{(m)}\left(\varepsilon_{n_k} a\right)(a)^m - h_1^{(m)}(0)(a)^m\right]\right\}$ is bounded. Immediately we have 3.5.3.

3.5.4. *Given* $\{\varepsilon_n\} \in (c_0)$ *,* $\xi \in$ R *, the set*

$$\left\{\varepsilon_n^{-1}\left[h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m - h_1^{(m)}(\xi a)(a)^m\right]\right\}$$

is bounded.

Since the above expression is equal to

$$\varepsilon_n^{-1}\left[\phi(1+c_{\xi a})'(h(\varepsilon_n a))\left\{h_1^{(m)}(\varepsilon_n a)(a)^m\right\}-\phi(1+c_{\xi a})'(0)\left\{h_1^{(m)}(0)(a)^m\right\}\right]$$

$$+\sum_{2\le q\le m}\sum\sigma_m\varepsilon_n^{-1}\left[\phi(1+c_{\xi a})^{(q)}(h(\varepsilon_n a))\left\{h_1^{(i_1)}(\varepsilon_n a)(a)^{i_1}\right\}\right.$$

$$\ldots\left\{h_1^{(i_q)}(\varepsilon_n a)(a)^{i_q}\right\}$$

$$\left.-\phi(1+c_{\xi a})^{(q)}(0)\left\{h_1^{(i_1)}(0)(a)^{i_1}\right\}\ldots\left\{h_1^{(i_q)}(0)(a)^{i_q}\right\}\right],$$

the result follows from 3.5.3, and standard methods.

3.5.5. *For any* $i = 1, 2, \ldots$, λ_i *is differentiable almost everywhere.*

If $\{\varepsilon_n\} \in (c_0)$ we have that the set

$$\left\{\left\langle\varepsilon_n^{-1}\left[h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m-h_1^{(m)}(\xi a)(a)^m\right], \bar{a}_i\right\rangle\right\}$$

is bounded, any $\xi \in R$, any $i = 1, 2, \ldots$. Thus all four Dini derivatives of λ_i are finite at every point in R, so by [50, p. 270], λ_i is differentiable almost everywhere. Following the argument of the odd case from here leads to the existence, in the even case also, of the limit

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left[h_1^{(m)}(\varepsilon a)(a)^m-h_1^{(m)}(0)(a)^m\right].$$

We call this $\left\{h_1^{(m)}\right\}^*(0)(a)^{m+1}$.

3.6. $h_1^{(m)}$ *is Fréchet differentiable.*

We begin by showing

$$h_1^{(m+1)}(0) = \left\{h_1^{(m)}\right\}^*(0)(a)^{m+1}\otimes^{m+1}\bar{a}$$

which is certainly an element of $L(\underbrace{E, \ldots, L(E}_{m+1})\ldots)$ which we abbreviate to $L(E^{m+1}, E)$. Thus $h_1^{(m+1)}(0)(x)$ will equal

$\langle x, \bar{a}\rangle \left[h_1^{(m)}\right]^*(0)(a)^{m+1} \otimes^m \bar{a}$.

We have to show that for each bounded set B in E

$$\varepsilon^{-1}\left[h_1^{(m)}(\varepsilon x)-h_1^{(m)}(0)\right] - \langle x, \bar{a}\rangle \left[h_1^{(m)}\right]^*(0)(a)^{m+1} \otimes^m \bar{a}$$

is uniformly convergent to zero for $x \in B$. Since

$$h_1^{(m)}(x) = h_1^{(m)}(\langle x, \bar{a}\rangle a)(a)^m \otimes^m \bar{a}$$

the expression is zero if $\langle x, \bar{a}\rangle = 0$. So we need consider only those x for which $\langle x, \bar{a}\rangle \neq 0$. Suppose the result is false. Then

$$\langle x, \bar{a}\rangle \left[(\varepsilon\langle x, \bar{a}\rangle)^{-1}\left[h_1^{(m)}(\varepsilon\langle x, \bar{a}\rangle a)(a)^m \otimes^m \bar{a} - h_1^{(m)}(0)(a)^m \otimes^m \bar{a}\right] - \left[h_1^{(m)}\right]^*(0)(a)^{m+1} \otimes^m \bar{a}\right]$$

does not converge to zero uniformly for $x \in B$. Hence we can find a zero-neighbourhood U in E , $\{\varepsilon_n\} \in (c_0)$, $\{x_n\} \subset B$, and bounded sequences $\left\{x_n^1\right\}, \ldots, \left\{x_n^m\right\}$, such that

$$\left\langle x_n^1, \bar{a}\right\rangle \ldots \left\langle x_n^m, \bar{a}\right\rangle \langle x_n, \bar{a}\rangle \left[(\varepsilon_n\langle x_n, \bar{a}\rangle)^{-1}\left(h_1^{(m)}(\varepsilon_n\langle x_n, \bar{a}\rangle a)(a)^m - h_1^{(m)}(0)(a)^m\right] - \left[h_1^{(m)}\right]^*(0)(a)^{m+1}\right] \notin U ,$$

for every n . But the sets $\left\{\left\langle x_n^i, \bar{a}\right\rangle\right\}$ are bounded, $i = 1, \ldots, m$, and $\{\varepsilon_n\langle x_n, \bar{a}\rangle\} \in (c_0)$ so from the definition of $\left[h_1^{(m)}\right]^*(0)(a)^{m+1}$ we have a contradiction. We now show

$$h_1^{(m+1)}(x) = \left[\phi\left(1+\langle x, \bar{a}\rangle c_a\right)h_1\right]^{(m+1)}(0) ,$$

which certainly exists in $L\left(E^{m+1}, E\right)$, since we have shown $h_1^{(m)}$ is Fréchet differentiable at zero. Given a bounded set B in E we must show

$$\varepsilon^{-1}\Big[h_1^{(m)}(x+\varepsilon y)-h_1^{(m)}(x)-\big[\phi\big(1+\langle x,\ \overline{a}\rangle c_a\big)h_1\big]^{(m+1)}(0)(\varepsilon y)\Big]$$

$$= \varepsilon^{-1}\Big[h_1^{(m)}(\langle x,\ \overline{a}\rangle a+\varepsilon\langle y,\ \overline{a}\rangle a)-h_1^{(m)}(\langle x,\ \overline{a}\rangle a) - \big[\phi\big(1+\langle x,\ \overline{a}\rangle c_a\big)h_1\big]^{(m+1)}(0)(\varepsilon y)\Big]$$

converges to zero uniformly for y in B. As before it is evident we need consider only those y in B for which $\langle y,\ \overline{a}\rangle \neq 0$. But the above expression is

$$\langle y,\ \overline{a}\rangle\Big[(\varepsilon\langle y,\ \overline{a}\rangle)^{-1}\big[\big(\phi\big(1+\langle x,\ \overline{a}\rangle c_a\big)h_1\big)^{(m)}(\varepsilon\langle y,\ \overline{a}\rangle a) - \big(\phi\big(1+\langle x,\ \overline{a}\rangle c_a\big)h_1\big)^{(m)}(0)\big]-\big(\phi\big(1+\langle x,\ \overline{a}\rangle c_a\big)h_1\big)^{(m+1)}(0)(a)\Big]$$

which converges uniformly to zero for y in B. Hence $h_1 \in \mathcal{D}_F^{m+1}(E)$, so by induction, $h_1 \in \mathcal{D}_F^k(E)$.

4. *$(a \otimes \overline{a})h \in \mathcal{D}_F^k(E)$, for all a, $\overline{a}$ such that $\langle a,\ \overline{a}\rangle = 1$.*

Since $h(a \otimes \overline{a}) \in \mathcal{D}_F^k(E)$ it follows that

$$\phi^{-1}[h(a \otimes \overline{a})] = h^{-1}[h(a \otimes \overline{a})]h = (a \otimes \overline{a})h \in \mathcal{D}_F^k(E) .$$

5. *h is weakly-$\mathcal{D}_F^k(E)$.*

The proof is by induction. The case $k = 1$ was treated in Chapter two. Now assume h is weakly-$\mathcal{D}_F^m(E)$, some m, $1 \leq m < k$. Unless otherwise stated $[(a \otimes \overline{a})h]^{(m)}$ and $[h(a \otimes \overline{a})]^{(m)}$ will refer in this section to strong mth Fréchet derivatives, while $h^{(m)}$ will denote the weak mth Fréchet derivative of h. Note that since $L(E)$ and $L(E,\ E_w)$ are equal as sets, strong differentiability implies weak differentiability and the derivatives coincide.

5.1. *$h^{(m)}$ is Gâteaux differentiable at zero.*

With $a, \bar{a}$ as before, $h_1^{(m+1)}(0)(a)$ exists and equals

$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left[h_1^{(m)}(\varepsilon a)-h_1^{(m)}(0)\right]$, an element of $L(E^m, E)$. But this is

$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left[h^{(m)}(\varepsilon a)-h^{(m)}(0)\right]$, an element of

$$L(\underbrace{E, \ldots, L}_{m}(E, E_w) \ldots) = L\left(E^m, E_w\right) ,$$

since the topology on $L(E, E_w)$ is weaker than the topology on $L(E)$. We denote this limit by $\left(h^{(m)}\right)^*(0)(a)$.

5.2. $\left(h^{(m)}\right)^*(0) \in L\left[E, L\left(E^m, E_w\right)\right]$.

It is readily shown that if $\{S_\alpha\}$ is a net in $L\left(E^m, E_w\right)$ then S_α converges to zero in $L\left(E^m, E_w\right)$ if and only if $(a \otimes \bar{a})S_\alpha$ converges to zero in $L\left(E^m, E_w\right)$, for every $a, \bar{a}$ such that $\langle a, \bar{a}\rangle = 1$. Now

$$[(a \otimes \bar{a})h]^{(m+1)}(0)(y) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[\left((a \otimes \bar{a})h\right)^{(m)}(\varepsilon y)-\left((a \otimes \bar{a})h\right)^{(m)}(0)\right]$$

in $L(E^m, E)$

$$= \lim_{\varepsilon\to 0} (a \otimes \bar{a})\varepsilon^{-1}\left[h^{(m)}(\varepsilon y)-h^{(m)}(0)\right]$$

in $L\left(E^m, E_w\right)$

$$= (a \otimes \bar{a})\left(h^{(m)}\right)^*(0)(y) ,$$

using the 'only if' of the above result. Since for non-zero $\bar{a}$ we can find non-zero a such that $\langle a, \bar{a}\rangle = 1$, it follows that $\left(h^{(m)}\right)^*(0)$ is linear. Any net convergent to zero is mapped by $[(a \otimes \bar{a})h]^{(m+1)}(0)$ into a net convergent to zero, so using the 'if' direction of the result gives $\left(h^{(m)}\right)^*(0) \in L\left(E^{m+1}, E_w\right)$.

5.3. $h^{(m)}$ *is weakly Fréchet differentiable.*

We show this property at zero. Let

$$r\left[h^{(m)}, 0, y\right] = h^{(m)}(y) - h^{(m)}(0) - \left(h^{(m)}\right)^*(0)(y) .$$

We require that $\varepsilon^{-1}r\left[h^{(m)}, 0, \varepsilon y\right]$ should converge to zero in $L\left(E^m, E_w\right)$ uniformly for y in any bounded subset of E . Suppose this is false. Then there exists a sequence $\{\varepsilon_n\} \in (c_0)$, bounded sequences $\{y_n\}$, $\left\{y_n^i\right\}$, $i = 1, \ldots, m$, and $\overline{a} \in \overline{E}$, such that

$$\left\langle \varepsilon_n^{-1} r\left[h^{(m)}, 0, \varepsilon_n y_n\right]\left(y_n^1\right) \ldots \left(y_n^m\right), \overline{a}\right\rangle$$

does not converge to zero with n . But $[(a \otimes \overline{a})h]^{(m)}$ is Fréchet differentiable at zero, so for any bounded sets B, B_i , $i = 1, \ldots, m$ in E ,

$$\varepsilon^{-1}r\left[\left((a \otimes \overline{a})h\right)^{(m)}, 0, \varepsilon y\right]\left(y^1\right) \ldots \left(y^m\right)$$

converges to zero in E , uniformly for $y \in B$, $y^i \in B_i$, $i = 1, \ldots, m$. That is,

$$\langle \varepsilon^{-1}r\left[h^{(m)}, 0, \varepsilon y\right]\left(y^1\right) \ldots \left(y^m\right), \overline{a}\rangle$$

converges to zero uniformly for $y \in B$, $y^i \in B_i$, $i = 1, \ldots, m$, a contradiction. We may use a method similar to that in 3.6 to move this point of weak differentiability to any other point, so completing the proof of the theorem. //

Corresponding to the corollary of Chapter two we have:

COROLLARY 3.2. *If* E *is a Fréchet Montel space,* $\mathcal{D}_F^k(E)$ *has the Magill property.*

Proof. We use induction. The case $k = 1$ follows as in the previous chapter. Assume $h \in \mathcal{D}_F^m(E)$, some m , $1 \le m < k$, and suppose $h^{(m)}$, the strong mth derivative, does not have Fréchet derivative at zero given by $h^{(m+1)}(0)$. This is the weak Fréchet derivative at zero of the strong mth derivative of h . Then there exists a neighbourhood U of zero, bounded sets $B, B_1, \ldots, B_m$, a sequence $\{\varepsilon_n\} \in (c_0)$ and sequences $\{y_n\} \subset B$, $\{y_n^i\} \subset B_i$, $i = 1, \ldots, m$ such that $\varepsilon_n^{-1} r\left[h^{(m)}, 0, \varepsilon_n y_n\right]\left(y_n^1\right) \ldots \left(y_n^m\right) \notin U$, for every n . Using once again the fact that every weakly convergent sequence in a Montel space is strongly convergent to the same limit, we contradict Theorem 3.1. //

3.3 The semigroup C^k

We now show that parallel results hold for the semigroups of many times continuously Fréchet differentiable maps.

THEOREM 3.2. *Let E be a Fréchet space. If ϕ is an automorphism of $C^k(E)$ there is a bijection h of E such that h and h^{-1} are weakly-$C^k(E)$ and*

$$\phi(f) = hfh^{-1} \text{, for every } f \in C^k(E) \text{ .} \tag{3}$$

Proof. As usual, there exists a bijection h such that (3) holds. Moreover, for a pair $a, \bar{a}$, $\langle a, \bar{a}\rangle = 1$, we have $h_1 = h(a \otimes \bar{a}) \in \mathcal{D}_F^k(E)$. We now show

1. $h_1 \in C^k(E)$.

Suppose $h_1^{(k)} : E \to L\left(E^k, E\right)$ is not continuous at $x \in E$. Then we can find a sequence $x_n \to x$ such that $h_1^{(k)}\left(x_n\right) \not\to h_1^{(k)}(x)$.

That is, there exist bounded sequences $\{x_n^1\}, \ldots, \{x_n^k\}$, each in E, and a zero-neighbourhood U in E such that

$$h_1^{(k)}(x_n)\{x_n^1\} \ldots \{x_n^k\} - h_1^{(k)}(x)\{x_n^1\} \ldots \{x_n^k\} \notin U ,$$

for each n, or

$$\left[\left\langle x_n^1, \overline{a}\right\rangle\right] \ldots \left[\left\langle x_n^k, \overline{a}\right\rangle\right]\left[h_1^{(k)}(\langle x_n, \overline{a}\rangle a)(a)^k - h_1^{(k)}(\langle x, \overline{a}\rangle a)(a)^k\right] \notin U ,$$

for every $n \in \mathbb{N}$. By step 3.1 of Theorem 3.1, $h_1^{(k)}(\xi a)(a)^k$ is continuous in ξ, and since the sequence $\left\{\left\langle x_n^i, \overline{a}\right\rangle\right\}$ is bounded, $i = 1, \ldots, k$, we reach a contradiction.

2. $(a \otimes \overline{a})h = \phi^{-1}[h(a \otimes \overline{a})] \in C^k(E)$.

As before we can show h is weakly-$\mathcal{D}_F^k(E)$. To complete the proof we show

3. h is weakly-$C^k(E)$.

Suppose $h^{(k)} : E \to L\left(E^k, E_w\right)$ is discontinuous at $x \in E$. Then there is a sequence $x_n \to x$, bounded sequences $\{x_n^1\}, \ldots, \{x_n^k\}$ in E, and an $\overline{a} \in \overline{E}$ such that

$$\left\langle h^{(k)}(x_n)\{x_n^1\} \ldots \{x_n^k\} - h^{(k)}(x)\{x_n^1\} \ldots \{x_n^k\}, \overline{a}\right\rangle$$

does not converge to zero. But $[(a \otimes \overline{a})h]^{(k)} : E \to L(E^k, E)$ is continuous, a contradiction. //

COROLLARY 3.3. *If* E *is an FM-space, every automorphism* ϕ *of* $C^k(E)$ *is inner.*

Proof. As before, we have $h \in \mathcal{D}_F^k(E)$. Suppose $h^{(k)}$ is discontinuous at $x \in E$. With $x_n \to x$, $\{x_n^1\}, \ldots, \{x_n^k\}$ as above we have the sequence

$$h^{(k)}(x_n)\left(x_n^1\right)\ldots\left(x_n^k\right) - h^{(k)}(x)\left(x_n^1\right)\ldots\left(x_n^k\right)$$

not convergent to zero. But by the theorem this sequence is weakly, and therefore strongly, convergent to zero since we are in Montel space. //

Note that a similar treatment of the semigroups $\mathcal{D}_F^{\infty}(E)$, and $C^{\infty}(E)$, of indefinitely Fréchet differentiable and indefinitely continuously Fréchet differentiable selfmaps of Fréchet Montel space respectively, reveals that each of their automorphisms is inner.

CHAPTER FOUR

A SPECIALISATION, CHARACTERISATION, AND REDUCTION

4.0 Introduction

This chapter is divided into four sections. In the first we give an alternative and far shorter proof that the semigroup $C^k(R^n)$ has the Magill property. Whereas in the previous chapter this was achieved by elementary methods we show here that the problem may be arranged in such a way that a classical theorem concerning differentiability is applicable. The second section is devoted to a reinterpretation of the automorphism problem in terms of the $\mathcal{S}$-categories of Bonic and Frampton [6], and a number of results in this direction are given. We include a section mentioning a number of unsolved problems, and conclude with a reduction of the semigroup automorphism problem to the group automorphism problem.

4.1 The semigroup $C^k(R^n)$, again

We present a different and brief proof of the following result, which is to appear in [58]:

THEOREM 4.1. *If E is a finite dimensional Banach space, every automorphism ϕ of $C^k(E)$ is inner.*

Proof.

1. As before we have a bijection h of E such that $\phi(f) = hfh^{-1}$, for every $f \in C^k(E)$. Since the weak and strong topologies in E coincide it follows from Lemma 1.2 that h is continuous. In order to show $h \in C^k(E)$ we rearrange the problem in the manner below.

Definition. A family $\{\psi(\xi) : \xi \in R\}$ of selfmaps of E is said to be a one-parameter group if

$$\psi(\xi+\eta) = \psi(\xi)\psi(\eta) \text{ , for any } \xi, \eta \in R .$$

Chernoff and Marsden in 1970, [7, p. 1044, Theorem 1], showed that if E is a metric linear space and $\psi(\xi)(x)$ is separately continuous on $R \times E$, then it is jointly continuous. In view of this the result of Bochner and Montgomery, 1945, [5, p. 691, Theorem 4], can be stated as follows: if E is finite dimensional, $\{\psi(\xi)\}$ a one-parameter group with $\psi(\xi)(x)$ separately continuous, and $\psi(\xi) \in C^k(E)$ for each $\xi \in R$, then $\psi : R \times E \to E$ is jointly k times continuously differentiable.

We define a one-parameter group of C^k selfmaps of E, $\{\psi(\xi)\}$, by $\psi(\xi) = \phi(e^{\xi})$, $\xi \in R$. Continuity with respect to the parameter follows readily from the continuity of $h(\xi a)$ with respect to ξ. We show that the k times continuous differentiability with respect to the parameter suffices to give h in $C^k(E)$, by proving:

2. *For* $\alpha \in R$, $\alpha > 0$, *and* $x \in E$, $\frac{d^k}{d\alpha^k} h(\alpha x)$ *exists and is continuous in* α.

Tedious differentiation shows that if $\alpha = e^{\xi}$, $y \in E$, and $m \in N$, then

$$\frac{d^m}{d\xi^m} h(e^{\xi}y) = \sum_{r=1}^{m} c_r^m e^{r\xi} \frac{d^r}{d\alpha^r} h(\alpha y) \tag{1}$$

providing we assume that these derivatives exist. The coefficients $c_r^m \in N$, $r = 1, \ldots, m$ are given inductively by $c_1^m = c_m^m = 1$ while $c_r^m = rc_r^{m-1} + c_{r-1}^{m-1}$ for $1 < r < m$. The result is obtained using the above and complete induction.

When $k = 1$, since $\psi(\xi)(x)$ is continuously differentiable with respect to ξ, we have the existence of

$$\frac{d}{d\xi} he^{\xi}h^{-1}(x) = e^{\xi} \frac{d}{d\alpha} h(\alpha y) ,$$

where $y = h^{-1}(x)$, $\alpha = e^{\xi}$. Hence for $\alpha > 0$, $\frac{d}{d\alpha} h(\alpha y)$ exists. Since $\frac{d}{d\xi} h(e^{\xi}y)$ is continuous in ξ , $\frac{d}{d\alpha} h(\alpha y) = e^{-\xi} \frac{d}{d\xi} h(e^{\xi}y)$ is also continuous in ξ . Theorems 8 and 9 of [15, p. 95] ensure the continuity in α .

Assuming now that the result holds for all natural numbers less than some $m \in \mathbb{N}$, $m \leq k$, we may use (1) and the existence and continuity in ξ of $\frac{d^m}{d\xi^m} h(e^{\xi}y)$ to give the existence and continuity in α of $\frac{d^m}{d\alpha^m} h(\alpha y)$ for all $\alpha > 0$ in an entirely parallel manner.

If we let $\{e_i\}$ be the standard basis for $E = \mathbb{R}^n$, $(\{e_i\}, \{\overline{e}_i\})$ be a biorthogonal pair, and $h_i = h(e_i \otimes \overline{e}_i)$, we may use an argument almost identical to that in the previous chapter to show that $h_i \in C^k(E)$, $i = 1, \ldots, n$, and thus that $h \in C^k(E)$. //

Note that when $k = 1$ this result is included in [64]. Here it is shown that for the semigroup of C^1 maps with bounded derivative on an arbitrary real Banach space every continuous automorphism is inner. When the space is finite dimensional every automorphism becomes continuous.

4.2 The characterisation problem

In Chapters two and three we gave proofs which showed that certain semigroups of selfmaps $S(E)$ of a locally convex space E have the Magill property. These may be rewritten in a straightforward fashion to show that if ϕ is an isomorphism between $S(E)$ and

$S(F)$, then ϕ may be represented as conjugation by some invertible element of $S(E, F)$, the "S-type" maps from E into F . If all maps are Hadamard differentiable then as outlined in Chapter one, the semigroup $S(E)$ affords a topological and algebraic characterisation of the space E . This notion will be dealt with formally at the beginning of the next chapter. At present we devote ourselves to this more general problem of finding conditions under which a semigroup of selfmaps will characterise the underlying topological vector space. If a topological characterisation only is required the question has a ready answer, (Theorem 4.2) but for a topological and algebraic characterisation the problem is more difficult (Theorem 4.4).

We let $I(E)$ denote the set of all constant mappings from E into itself, and $T(E)$ the corresponding set of all translation mappings. We introduce the following idea, due essentially to Bonic and Frampton [6], but generalised by Lloyd in [20] to *TVS*'s :

Definition. An S-category is a category S whose objects comprise all open subsets of all topological vector spaces. For any pair of objects U and V , the morphisms $S(U, V)$ are functions from U into V with the usual composition as their product. We require that the following conditions be satisfied:

S1: $S(U, V) \subset C^0(U, V)$ for every U, V ; $I(E), T(E) \subset S(E, E)$ and $L(E, F) \subset S(E, F)$ for every $E, F \in TVS$;

S2: if $f \in S(U, V)$ and W is an open subset of V containing $f(U)$, then $f \in S(U, W)$;

S3: if $f \in C^0(U, V)$ and for each $x \in U$ there is an open set W with $x \in W \subset U$ such that $f|_W \in S(W, V)$, then $f \in S(U, V)$;

S4: if $f_1 \in S(U_1, V_1)$ and $f_2 \in S(U_2, V_2)$ then

$f_1 \times f_2 \in S(U_1 \times U_2, V_1 \times V_2)$.

Examples of S-categories in topological vector spaces are C^0, $\mathcal{D}_H^k \cap C^0$, and $\mathcal{D}_F^k \cap C^0$, $k \in \mathrm{N}$, and C^k , $k \in \mathrm{N}$ in sequential locally convex spaces. Since the composition of Gâteaux differentiable maps is not necessarily Gâteaux differentiable, $\mathcal{D}_G^k$ and C_G^k, $k \in \mathrm{N}$ are not S-categories. Let supp f denote the support of the real-valued function f .

Definition. Let $E \in TVS$ and S be an S-category. E is said to be S-smooth if given $a \in E$ and a neighbourhood V of a , there exists an $f \in S(E, \mathrm{R})$ such that $f(a) > 0$, $f(x) \geq 0$ for $x \in E$, and supp $f \subset V$.

For results on the smoothness of certain spaces, see [6] and [20]. For an S-category S we call the semigroup $S(E, E)$, $S(E)$. In the terminology of Magill [33], an isomorphism ϕ from $S(E)$ onto $S(F)$ is said to be induced by a homeomorphism h from E onto F if $\phi(f) = hfh^{-1}$, for every $f \in S(E)$. Clearly if every isomorphism is induced by a homeomorphism then the semigroup does topologically characterise the space. We now prove:

THEOREM 4.2. *Let E and F be S-smooth topological vector spaces. Then every isomorphism ϕ from $S(E)$ onto $S(F)$ is induced by a homeomorphism.*

Proof. With only a change in notation we may show there exists a bijection $h : E \to F$ such that $\phi(f) = hfh^{-1}$, for each $f \in S(E)$, as in the proof of Lemma 1.1. We show the continuity of h at arbitrary $a \in E$. Since F is S-smooth, given a neighbourhood V of $h(a)$ we may find a $\beta \in S(F, \mathrm{R})$ such that

$\beta(h(a)) = 1$, $\beta(x) \geq 0$ for $x \in V$, and supp $\beta \subset V$. Take some $b \in F$, $b \neq h(a)$, and define $g \in S(F)$ by

$$g(x) = \beta(x)(b-h(a)) + h(a) \text{ , for } x \in F \text{ .}$$

Following the method of Lemma 1.2 from here gives the continuity of h . //

COROLLARY 4.1. *Let* $E, F \in TVS$. *Then every isomorphism from* $C^0(E)$ *onto* $C^0(F)$ *is induced by a homeomorphism. Conversely, if* E *and* F *are homeomorphic,* $C^0(E)$ *and* $C^0(F)$ *are isomorphic.*

Proof. E and F are completely regular [17, p. 147] and so C^0-smooth. On the other hand, if h is a homeomorphism from E onto F then ϕ given by $\phi(f) = hfh^{-1}$ for each $f \in C^0(E)$ is an isomorphism from $C^0(E)$ onto $C^0(F)$. //

It follows that $C^0(E)$, $E \in TVS$, has the Magill property. Corollary 4.1 is not new. In [33] Magill defined a class of topological spaces, $\mathcal{E}$, to be S-admissible if every isomorphism from $C^0(E)$ onto $C^0(F)$, where $E, F \in \mathcal{E}$, is induced by a homeomorphism. In a later paper [25], he defines an S^*-space as follows:

Definition. A topological space X is an S^*-space if it is T_1 and for each closed subset F of X and each point p in $X \backslash F$ there exists a function f in $C^0(X)$ and a point y in X such that $f(x) = y$ for each x in F and $f(p) \neq y$.

The main theorem of this paper shows the class of S^*-spaces to be S-admissible. We are able to deduce that every TVS is an S^*-space, and hence Corollary 4.1, from Theorem 3 of the same paper, which states: every completely regular Hausdorff space containing at least two distinct points which are connected by an arc is an S^*-space.

It is of interest to note that in an earlier paper on the subject [32], Magill defined a further S-admissible class of spaces, S-spaces, in the following way:

Definition. Let X be a topological space and x be a point of X. An open set G containing x is an S-neighbourhood of x if it consists of x alone or if there exists a continuous function f mapping $\mathrm{cl}\, G$ into X such that $f(x) \neq x$, but $f(y) = y$ for each $y \in \mathrm{cl}\, G \backslash G$.

Definition. A topological space is an S-space if it is Hausdorff and every point has a basis of S-neighbourhoods.

In [25] it is pointed out that there are S^*-spaces which are not S-spaces, but it is not known whether every S-space is an S^*-space. However, we are able to show:

THEOREM 4.3. *Every real Hausdorff topological vector space* E *is an* S-*space.*

Proof. Let x be in E. Given a neighbourhood U of x we must find an S-neighbourhood of x, V, inside U. But E is regular so we may take an open set V containing x such that $\mathrm{cl}\, V \subset U$. Furthermore, E is completely regular, so there exist continuous functions from E into $\mathbb{R}$ which separate disjoint closed sets. Now $E \backslash V$ is closed and $x \notin E \backslash V$ so there is a continuous map $g : E \to [0, 1]$ such that $g(x) = 1$, $g(y) = 0$ for $y \in E \backslash V$. Choose non-zero $w \in E$ and define $f : \mathrm{cl}\, V \to E$, a continuous map, by $f(z) = z + g(z)w$, for $z \in \mathrm{cl}\, V$. Then $f(x) \neq x$, and $f(y) = y$, $y \in \mathrm{cl}\, V \backslash V$, so E is an S-space. //

Admissibility will be discussed further in Chapter five. We now turn to the more difficult characterisation problem mentioned previously.

Let $\mathcal{JC}_F(E)$ be the set of Fréchet differentiable selfmaps of E

with jointly continuous derivative. In sequential locally convex spaces JC_F forms an S-category. We prove:

THEOREM 4.4. *Let* S *be an* S*-category such that* $S(G) \subseteq JC_F(G)$ *for every Fréchet space* G *, and* E *and* F *be* S*-smooth Fréchet spaces. Then if* $S(E)$ *and* $S(F)$ *are isomorphic,* E *and* F *are linearly homeomorphic.*

Proof. The proof is in eight steps. Let ϕ be the isomorphism from $S(E)$ onto $S(F)$.

1. *There exists a homeomorphism* $h : E \to F$ *such that*

$$\phi(f) = hfh^{-1} \text{ , for every } f \in S(E) \text{ .}$$

This follows as in Theorem 4.2.

2. *The limit,* $\lim_{\varepsilon \to 0} \varepsilon^{-1}[h(x+\varepsilon y)-h(x)]$ *exists, for every* $x, y \in E$.

The proof when $x = 0$ is as in Chapter two, steps 2 to 7. Using the translation map this limit may be moved to non-zero x values. Let $h^*(x)(y)$ denote the above limit. Then,

3. $h^*(x)(y)$ *is continuous in* x *, for fixed* y .

For arbitrary $w, x, y, z \in E$, and $\{\varepsilon_n\} \in (c_0)$, we may use standard techniques to show

$$\begin{aligned} h^*(x+z)(y) &= \lim_{n\to\infty} \varepsilon_n^{-1}\left[h\left(x+z+\varepsilon_n y\right)-h(x+z)\right] \\ &= \lim_{n\to\infty} \left\{\phi\left(1+c_{x-w}\right)'\left(h(w+z)\right)\left[\varepsilon_n^{-1}\left(h\left(w+z+\varepsilon_n y\right)-h(w+z)\right)\right]\right. \\ &\quad \left. +\varepsilon_n^{-1} r\left[\phi\left(1+c_{x-w}\right),\, h(w+z),\, \varepsilon_n\left[\varepsilon_n^{-1}\left(h\left(w+z+\varepsilon_n y\right)-h(w+z)\right)\right]\right]\right\}. \end{aligned}$$

The sequence $\left\{\varepsilon_n^{-1}\left[h\left(w+z+\varepsilon_n y\right)-h(w+z)\right]\right\}$ is convergent, so forms a compact set. Since $\phi\left(1+c_{x-w}\right) \in \mathcal{D}_H(F)$ the second term has limit zero. Thus

$$h^*(x+z)(y) = \phi\left(1+c_{x-w}\right)'\left(h(w+z)\right)\left(h^*(w+z)(y)\right) .$$

By the result of Banach, [3], used earlier, $h^*(x)(y)$ for fixed y is continuous in x on a dense set in E. Suppose w is such a point, and let $z_n \to 0$. Then for arbitrary $x \in E$,

$$\begin{aligned}\lim_{n\to\infty} h^*\left(x+z_n\right)(y) &= \lim_{n\to\infty} \phi\left(1+c_{x-w}\right)'\left(h\left(w+z_n\right)\right)\left(h^*\left(w+z_n\right)(y)\right) \\ &= \phi\left(1+c_{x-w}\right)'\left(h(w)\right)\left(h^*(w)(y)\right)\end{aligned}$$

since h is continuous and $\phi\left(1+c_{x-w}\right) \in JC_F(F)$. But this is just $h^*(x)(y)$.

4. *$h^*(x)(y)$ is linear in y, for each x.*

It will suffice to show $h^*(x)$ is additive, for each $x \in E$. We shall prove the following:

RESULT 4.1. *Let $E, F \in LCS$, $f : E \to F$ be such that $\lim_{\varepsilon\to 0} \varepsilon^{-1}[f(x+\varepsilon y)-f(x)] = f^*(x)(y)$, exists for each $x, y \in E$. Then if $f^*(x)(y)$ is continuous in x for fixed y, $f^*(x)$ is additive, for each $x \in E$.*

Proof. We generalise the method given in the Banach space case in [55, p. 38-39]. Let the topology of F be determined by the set Q of (continuous) seminorms. Take $p \in Q$, $x, y_1, y_2 \in E$, and $\varepsilon > 0$. There exists a $\delta > 0$ such that if $|t| < \delta$, then

$$f^*(x)\left(y_1\right) = \tfrac{1}{t}\left[f\left(x+ty_1\right)-f(x)\right] + \alpha_1 ,$$

$$f^*(x)\left(y_2\right) = \tfrac{1}{t}\left[f\left(x+ty_2\right)-f(x)\right] + \alpha_2 ,$$

$$f^*(x)\left(y_1+y_2\right) = \tfrac{1}{t}\left[f\left(x+ty_1+ty_2\right)-f(x)\right] + \alpha_3 ,$$

where $p\left(\alpha_i\right) \le \frac{\varepsilon}{4}$, $i = 1, 2, 3$. Hence

$$p[f^*(x)(y_1+y_2)-f^*(x)(y_1)-f^*(x)(y_2)]$$

$$\leq \frac{1}{|t|} p[f(x+ty_1+ty_2)-f(x+ty_2)-f(x+ty_1)+f(x)] + \frac{3\varepsilon}{4} . \quad (1)$$

For fixed t , $|t| < \delta$, we can find an $\overline{a} \in \overline{F}$, by [17, p. 191, (8)], such that

$$\langle f(x+ty_2+ty_1)-f(x+ty_2)-f(x+ty_1)+f(x), \overline{a}\rangle$$

$$= p[f(x+ty_2+ty_1)-f(x+ty_2)-f(x+ty_1)+f(x)]$$

and

$$|\langle z, \overline{a}\rangle| \leq p(z) \text{ , for all } z \in F .$$

Using the Mean Value Theorem for functionals [19, Theorem 1.4], we can find τ_1, τ_2 , $0 < \tau_1, \tau_2 < t$, such that

$$\langle f(x+ty_2+ty_1)-f(x+ty_2), \overline{a}\rangle = t\langle f^*(x+ty_2+\tau_1 ty_1)(y_1), \overline{a}\rangle$$

and

$$\langle f(x+ty_1)-f(x), \overline{a}\rangle = t\langle f^*(x+\tau_2 ty_1)(y_1), \overline{a}\rangle .$$

Hence

$$p[f(x+ty_2+ty_1)-f(x+ty_2)-f(x+ty_1)+f(x)]$$

$$= t\langle f^*(x+ty_2+\tau_1 ty_1)(y_1)-f^*(x+\tau_2 ty_1)(y_1), \overline{a}\rangle$$

$$\leq |t| p[f^*(x+ty_2+\tau_1 ty_1)(y_1)-f^*(x+\tau_2 ty_1)(y_1)] . \quad (2)$$

But $f^*(x)(y)$ is continuous at x so for suitably small δ we have

$$p[f^*(x+ty_2+\tau_1 ty_1)(y_1)-f^*(x+\tau_2 ty_1)(y_1)] \leq \frac{\varepsilon}{4} . \quad (3)$$

Combining (1), (2), and (3) gives

$$p[f^*(x)(y_1+y_2)-f^*(x)(y_1)-f^*(x)(y_2)] \leq \varepsilon .$$

Since ε was arbitrary, the left hand side is zero. By a result in [57, p. 216, (vii)], Q is total and so since p was arbitrary, $f^*(x)$ is additive.

5. *h is Gâteaux differentiable.*

It suffices to show $h^*(x)(y)$ is continuous in y, for fixed x. We use the result of Banach, as before, and the linearity of $h^*(x)$.

6. *$h^*(x)(y)$ is jointly continuous.*

The proof is based on the following result [9, p. 256, Problem 11]: Let E be a Baire space and F, G metric spaces. Let $f : E \times F \to G$ be a map which is separately continuous. Then for each point $y \in F$ there is a dense set $A_y \subset E$ such that f is jointly continuous at (w, y) for every $w \in A_y$.

If (x, y) is an arbitrary point in $E \times E$ we can thus find a $(w, y) \in E \times E$ at which h^* is jointly continuous. For $x_n \to x$, $y_n \to y$ we have

$$\begin{aligned}\lim_{n\to\infty} h^*(x_n)(y_n) &= \lim_{n\to\infty} h^*\left[(x-w)+(w+x_n-x)\right](y_n)\\ &= \lim_{n\to\infty} \phi(1+c_{x-w})'\left[h(w+x_n-x)\right]\left[h^*(w+x_n-x)(y_n)\right]\\ &= \phi(1+c_{x-w})'[h(w)][h^*(w)(y)]\text{ , since } \phi(1+c_{x-w}) \in JC_F(E)\text{ ,}\\ &= h^*(x)(y)\text{ .}\end{aligned}$$

7. *h is Hadamard differentiable.*

This is a consequence of the following unpublished result of Yamamuro:

RESULT 4.2. *Let $E, F \in LCS$, and $f : E \to F$ be Gâteaux differentiable. If $f^*(x)(y)$ is jointly continuous at (a, y), for all $y \in E$, then f is Hadamard differentiable at $a \in E$.*

Proof. Suppose f is not Hadamard differentiable at $a \in E$. Then there is a continuous seminorm p, $\{\varepsilon_n\} \in (c_0)$, and $y_n \to y_0$ such that

$$p\left[\varepsilon_n^{-1}\big(f(a+\varepsilon_n y_n)-f(a)\big)-f'(a)(y_n)\right] \not\to 0 \quad \text{as} \quad n \to \infty .$$

A consequence of [19, Theorems 1.4 and 1.5] is the following Mean Value Theorem: Let $f : E \to F$, $E, F \in LCS$, be Gâteaux differentiable. Then for each $\varepsilon > 0$, $a, y \in E$, and continuous seminorm p , there exists a $\tau \in (0, 1)$ such that

$$p\left[\varepsilon^{-1}\big(f(a+\varepsilon y)-f(a)\big)-f'(a)(y)\right] \le p\left[f'(a+\tau\varepsilon y)(y)-f'(a)(y)\right] .$$

Thus for each $n \in \mathbb{N}$ we can find a $\tau_n \in (0, 1)$ such that

$$p\left[\varepsilon_n^{-1}\big(f(a+\varepsilon_n y_n)-f(a)\big)-f'(a)(y_n)\right] \le p\left[f'(a+\tau_n\varepsilon_n y_n)(y_n)-f'(a)(y_n)\right]$$

which by the assumption goes to zero as n tends to infinity, a contradiction.

8. *E and F are linearly homeomorphic.*

For $x \in E$, $h'(x)$ is now a linear homeomorphism from E onto F , as shown in Chapter one, Section seven.

An immediate consequence of [54, p. 178, Corollary 4] is that every reflexive Banach space is C^1-smooth. Thus for example, Theorem 4.4 then shows that the semigroup $C^1(E)$ characterises the topological and algebraic structure of a reflexive Banach space, E .

4.3 Comments and unsolved problems.

Since the Fréchet and Hadamard derivatives coincide on Fréchet Montel spaces it is of interest to note that with the exception of the semigroup $\mathcal{D}_F(l^1)$, all semigroups of differentiable functions shown to possess the Magill property involve the Hadamard differentiability. The importance of this form of differentiation has been growing in recent years following the work of Miroslav Sova [52] and [53], who first observed that it was equivalent to the quasi-differentiability of Dieudonné [8, p. 151, Problem 4]. Moreover, Lloyd in [20] has shown that every separable locally convex space is $\mathcal{D}_H$-smooth. Both the Magill property and smoothness being measures of the compatibility of the space with the functions defined on it, it remains an open question whether $E \in LCS$ being S-smooth implies that the semigroup $S(E)$ has the Magill property. The converse is false since l^1 is not $\mathcal{D}_F$-smooth [6, p. 882].

Beyond sequential spaces the proof of Lemma 1.2 breaks down for semigroups of differentiable functions, since no longer does differentiability imply continuity. Despite smoothness properties of the space this loss of topological link between the semigroup of functions and the space would appear to make it less likely that the Magill property would hold.

As mentioned in Chapter one, a great deal is known about Fréchet differentiation in a Banach space E, so doubtless the most interesting semigroup remaining is $\mathcal{D}_F(E)$. The semigroups $\mathcal{D}_H^k(E)$ and $C^k(E)$, E a Fréchet space, $k \in \mathbb{N}$, would seem possible contenders for the Magill property.

It would be of interest to know whether the Magill property is hereditary. For example, given that F is a subspace of E, and $S(E)$ a semigroup of selfmaps with the Magill property, can we show that $S(F)$ has the Magill property? The problem would seem difficult for two reasons. Firstly, the lack of relevant extension theorems for functions means that $S(E)$ and $S(F)$ are only tenuously related, and secondly, there is no general link between automorphisms of a semigroup and those of a subsemigroup, should this fortuitously be the relationship between $S(E)$ and $S(F)$. For similar reasons, products, quotients, and conjugates also defy this approach. However, we do have the result of the following section.

4.4 Reduction to the group of units

Every automorphism ϕ of a semigroup $S(E)$ of selfmaps of $E \in TVS$, when restricted to the group of invertible elements, $H(E)$, becomes a group automorphism. Let us suppose that $I, L, T \subset S$. It would be of interest to know when every group automorphism of $H(E)$ is inner, since $S(E)$ will then have the Magill property, as we show below.

As usual we have a bijection h, $h(0) = 0$, such that

$$\phi(f) = hfh^{-1}, \text{ for every } f \in S(E) .$$

Suppose further that there exists a $k \in H(E)$ such that

$$\phi(f) = kfk^{-1}, \text{ for every } f \in H(E) .$$

Then

$$k^{-1}hf = fk^{-1}h, \text{ for every } f \in H(E) . \tag{1}$$

For $\alpha \in \mathrm{R}$, the associated map α belongs to $H(E)$, so

$$k\alpha k^{-1}(0) = h\alpha h^{-1}(0) = 0 .$$

Thus if $k^{-1}(0) = b$,

$$k(\alpha b) = k(\beta b) = 0, \text{ for } \alpha, \beta \in \mathrm{R},$$

but since k is a bijection, $b = 0$.

Now $1+c_x \in H(E)$, so using (1) we have

$$k^{-1}h(1+c_x)(0) = (1+c_x)k^{-1}h(0),$$

or

$$k^{-1}h(x) = x,$$

so $k \equiv h$, since x was arbitrary. That is, $h \in H(E)$ and $S(E)$ has the Magill property. Note that if $h(0) = a \neq 0$, we may show (see p. 10, (iii)) that $h_0 \in H(E)$, as before, and hence $h = (1+c_a)h_0 \in H(E)$.

CHAPTER FIVE

ADMISSIBILITY AND FAMILIES OF CONTINUOUS LINEAR MAPPINGS

5.0 Admissibility

We begin this final chapter by introducing three notions which will allow us to view our results in a clearer light.

Let $\mathbf{E}$ be a class of locally convex spaces. For each pair $E, F \in \mathbf{E}$ let $A(E, F)$ be a family of mappings from E into F in which an operation of addition is defined pointwise, and such that the family $\{A(E) : E \in \mathbf{E}\}$, where $A(E) = A(E, E)$ is equipped with an operation of multiplication given by function composition, comprises only one of the following algebraic structures: semigroups, groups, near-rings, rings. Then $\mathbf{E}$ is said to be:

i) *C_A-admissible* if when $E, F \in \mathbf{E}$ are such that $A(E)$ is isomorphic to $A(F)$, then E is linearly homeomorphic to F. That is, $A(E)$ characterises the space E.

ii) *M_A-admissible* if for every $E \in \mathbf{E}$, every automorphism ϕ of $A(E)$ is inner. That is, there exists an invertible h in $A(E)$ such that $h^{-1} \in A(E)$ and $\phi(f) = hfh^{-1}$, for every $f \in A(E)$.

iii) *R_A-admissible* if every isomorphism from $A(E)$ onto $A(F)$, $E, F \in \mathbf{E}$, can be represented as conjugation by an invertible element of $A(E, F)$ whose inverse lies in $A(F, E)$. That is, following the terminology of Hofer in [12], every isomorphism is representable.

It is to be understood that all isomorphisms are with respect to the initially chosen algebraic structure. We note the following simple facts:

1) *Every R_A-admissible class is M_A-admissible.*

2) *If* $A(E, F) \subseteq \mathcal{D}_H(E, F)$ *, for every* $E, F \in \mathcal{E}$ *, then every* R_A*-admissible class is* C_A*-admissible.*

Since there exists an invertible element h in $A(E, F) \subseteq \mathcal{D}_H(E, F)$, $h'(x)$, $x \in E$, is a linear homeomorphism from E onto F .

It is not known whether every M_A-admissible class is R_A-admissible. Unless it is stated otherwise, the algebraic structure in the sequel is assumed to be a semigroup.

Although the result of Chapter two indicates only that the class of *FM*-spaces is $M_{\mathcal{D}_F}$-admissible it is evident that the proof also shows them to be $R_{\mathcal{D}_F}$-admissible. In Chapter three it is shown that they are $R_{\mathcal{D}_F^k}$, R_{C^k} , $R_{\mathcal{D}_F^\infty}$, and R_{C^∞}-admissible. The class of all *TVS*'s is shown in Corollary 4.1 to be R_{C^0}-admissible. Note that the *S*-admissibility of Magill [33] corresponds to our R_{C^0}-admissibility. For an *S*-category in which $S(E) \subset \mathcal{J}C_F(E)$ for each E , Theorem 4.4 can be rephrased to give that the class of such *S*-smooth Fréchet spaces is C_S-admissible. We also note here that the method of Chapter two together with that in [67] suffices to show that the class of sequential *LCS*'s with the separability property $\mathcal{S}$ is $C_{\mathcal{D}_F}$-admissible.

5.1 An automorphism which is not inner

In this section we firstly give an example of a class of spaces which is "*C*-admissible" but not "*M*-admissible", for some system of selfmaps. We shall be involved with the group $U(E)$ of all linear homeomorphisms (units) of a locally convex space E. By way of introduction we give a brief survey of existing results concerning the characterisation of topological and algebraic properties of a locally convex space by means of families of continuous linear selfmaps defined on the space.

We deal with the *ring* $L(E)$ initially. Eidelheit [11], in 1940 showed that the class of Banach spaces was R_L-admissible. The finite dimensional version of this result had been given by Nagumo in 1933, [44]. By dropping the completeness assumption, Mackey in 1942 [24], showed that the class of normed linear spaces was C_L-admissible. Four years later [23] he further generalised this result to show that the class of all *LCS*'s equipped with the Mackey topology is C_L-admissible. For a discussion of the Mackey topology, see [48, p. 62]. The same result holds if the Mackey topology is replaced by the weak topology. It is interesting to note at this stage that the result of Chapter two could be reinterpreted to give that the class of *LCS*'s equipped with both the weak topology and the separability property S, is $R_{\mathcal{D}_F \cap \mathcal{C}^0}$-admissible. However it is the Mackey topology (Banach space) case which would be of greater interest.

When $L(E)$ is regarded as a *semigroup*, Eidelheit showed in the paper of 1940 that every continuous automorphism of $L(E)$, E Banach with dimension ≥ 2, is inner. Rickart [47] in 1948 showed that the continuity was unnecessary. A different proof of this was given by Yamamuro in [63].

In the case of the *group* of continuous linear selfmaps of $E \in LCS$, $U(E)$, with continuous inverses, the situation is more intriguing. In his paper of 1942, [24], Mackey showed that for Banach spaces E and F, $U(E)$ is isomorphic to $U(F)$ if and only if either

i) E is linearly homeomorphic to F, or

ii) E and F are mutually conjugate.

Since a Hilbert space is self-conjugate it is evident that the class of Hilbert spaces is C_U-admissible. However, we are able to show

RESULT 5.1. *The class of Hilbert spaces is not* M_U*-admissible.*

Proof. We show that the automorphism ϕ of $U(H)$, H a Hilbert space, given by $\phi(u) = (u^{-1})^*$, $u \in U(H)$, is not inner. Here u^* is the adjoint of u, as discussed in [57, p. 98]. Note that if ϕ is inner then ϕ fixes a non-identity element of $U(H)$. The inner product of $x, y \in H$ will be denoted by $\langle x, y \rangle$. Firstly we show

1) $\phi(u) = u$ *if and only if* u *is an isometry.*

Suppose $u = (u^{-1})^*$. Then if $x \in H$,

$$\langle ux, ux \rangle = \langle x, u^*ux \rangle = \langle x, u^{-1}ux \rangle = \langle x, x \rangle .$$

That is, $\|ux\| = \|x\|$, for each $x \in H$, so u is an isometry.

Conversely, suppose u is an isometry. Since u is onto u is unitary, so $uu^* = 1 = u^*u$. Thus $u^* = u^{-1}$ or $\phi(u) = (u^{-1})^* = u$.

We complete the proof by showing

2) ϕ *cannot be conjugation by some isometry,* v.

Suppose v is an isometry and $vuv^{-1} = (u^{-1})^*$, for all $u \in U(H)$. Then

$$\|u^{-1}\| = \|(u^{-1})^*\| = \|vuv^{-1}\| \le \|v\|\|u\|\|v^{-1}\| = \|u\| ,$$

since $\|v\| = \|v^{-1}\| = 1$, v being an isometry. Again

$$\|u\| = \|u^*\| = \|vu^{-1}v^{-1}\| \le \|v\|\|u^{-1}\|\|v^{-1}\| = \|u^{-1}\| ,$$

so $\|u\| = \|u^{-1}\|$, for every $u \in U$, a contradiction. //

In the penultimate paragraph of [24], Mackey raises the question of the extent to which isomorphisms between the groups $U(E)$ and $U(F)$, $E, F \in LCS$, are representable. It is clear than when E and F are mutually conjugate yet not linearly homeomorphic, an isomorphism between $U(E)$ and $U(F)$ cannot be representable. However, even when E and F are both mutually conjugate and linearly homeomorphic we may use a different method of proof to generalise the above result to the following:

RESULT 5.2. *Given mutually conjugate, linearly homeomorphic Banach spaces, E and F , there exists an isomorphism from $U(E)$ onto $U(F)$ which is not representable.*

Proof. Suppose the isomorphism $\phi : U(E) \to U(\overline{E}) = U(F)$, given by $\phi(u) = \left(u^{-1}\right)^*$, is representable. Then for some linear homeomorphism v from E onto $\overline{E}$,

$$\left(u^{-1}\right)^* = vuv^{-1} \text{ , for every } u \in U(E) \text{ .} \tag{1}$$

Select $u = 1 + a \otimes \overline{a}$ in $U(E)$, where $\langle a, \overline{a} \rangle = 1$. Then

$$u^{-1} = 1 - \tfrac{1}{2}(a \otimes \overline{a}) \text{ , } u^* = 1 + \overline{a} \otimes a \text{ , } \left(u^{-1}\right)^* = 1 - \tfrac{1}{2}(\overline{a} \otimes a) \text{ ,}$$

where $\overline{a} \otimes a : \overline{E} \to \overline{E}$ is given by

$$\overline{a} \otimes a(\overline{x}) = \langle a, \overline{x} \rangle \overline{a} \text{ , for each } \overline{x} \in \overline{E} \text{ .}$$

From (1) we have,

$$v(1+a \otimes \overline{a})(x) = \left(1-\tfrac{1}{2}(\overline{a} \otimes a)\right)v(a) \text{ ,}$$

or

$$\langle x, \overline{a} \rangle v(a) = -\tfrac{1}{2}\langle a, v(x) \rangle \overline{a} \text{ , for all } x \in E \text{ .} \tag{2}$$

Since v is a function,

$$c\langle x, \overline{a} \rangle = \langle a, v(x) \rangle \text{ , for all } x \in E \text{ ,}$$

and some $c \in \mathrm{R}$.

Thus $\langle a, v(a) \rangle = c \langle a, \overline{a} \rangle = c$. But from (2), letting $x = a$

and evaluating at a we have, $\langle a, v(a)\rangle = -\frac{1}{2}\langle a, v(a)\rangle$, so $\langle a, v(a)\rangle = c = 0$.

Again from (2) we have $v(a) = 0$, a contradiction since $a \neq 0$. //

When E is a reflexive Banach space, the direct sum of E with its conjugate space $\overline{E}$, $E \oplus \overline{E}$, has conjugate $\overline{E} \oplus E$, indicating that Result 5.2 is in fact a strict generalisation of Result 5.1.

5.2 The main theorem

Although not every automorphism of $U(H)$ is inner, for H a Hilbert space, if we form the semigroup $U \cup I(H)$ by adjoining the constant mappings we have immediately from Lemma 1.1 that every automorphism is given as conjugation by some bijection. This leads us to prove the theorem below.

THEOREM 5.1. *Let E be a locally convex space of dimension greater than two, equipped with either the weak topology or the Mackey topology. Let S be a semigroup of selfmaps of E such that $I \subset S$, $U \subset S$, and ϕ be an automorphism of S . Then if $\phi(U) = U$, ϕ is inner and $h \in U$.*

Proof. The proof is in seven stages, as follows:

1. *There exists a bijection h such that $\phi(f) = hfh^{-1}$, for all $f \in S$.* This is an application of Lemma 1.1. Since ϕ fixes U , h fixes zero, so we need not assume this fact in the way that we have previously. By $Z(U)$ we shall mean the centre of the group U . The scalar mapping $x \to \xi x$, $0 \neq \xi \in R$, is denoted by ξ , and the set of all such maps, M .

2. $Z(U) = M$.

Evidently $M \subseteq Z(U)$, so it remains to show $Z(U) \subseteq M$. Suppose $u \in Z(U)$. Since $1 + a \otimes \overline{a} \in U$ if and only if $\langle a, \overline{a}\rangle \neq -1$, we have

$$u(1+a \otimes \bar{a})(x) = (1+a \otimes \bar{a})u(x) ,$$

or

$$\langle x, \bar{a} \rangle u(a) = \langle u(x), \bar{a} \rangle a,$$

for every $a, \bar{a}, x$ such that $\langle a, \bar{a} \rangle \neq -1$. If $\langle a, \bar{a} \rangle = -1$ we may use $-\bar{a}$ to show the above relation still holds. Since u is a function there exists an $\xi \in \mathrm{R}$, $\xi \neq 0$, such that

$$\xi \langle x, \bar{a} \rangle = \langle u(x), \bar{a} \rangle , \text{ for every } \bar{a}, x .$$

Thus $u(x) = \xi x$, or $u \in M$.

Now ϕ will be an automorphism of the group U , so preserves the centre of U . Thus, since $\phi(0) = 0$, there exists a real-valued function of the real variable ξ , such that

$$\phi(\xi)(x) = \lambda(\xi)(x) , \text{ for all } x \in E , \ \xi \in \mathrm{R} .$$

We need some properties of λ .

2.1. λ *is one-to-one and onto.*

This follows immediately from the corresponding properties of ϕ .

2.2. λ *fixes* $+1, -1$, *and* 0 .

Certainly $\lambda(1) = \phi(1) = 1$, while since

$$[\lambda(-1)]^2 = \phi(-1)\phi(-1) = \phi(1) = 1 ,$$

and $\phi(-1) \neq \phi(1)$, $\lambda(-1) = -1$. Moreover $\lambda(0) = \phi(0) = 0$.

3. *Given* $a, b \in E$, *there exists* $\mu, \rho \in \mathrm{R}$ *such that* $h^{-1}(a+b) = \mu h^{-1}(a) + \rho h^{-1}(b)$.

With $\langle x, \bar{x} \rangle \neq -1$, $\phi(1+x \otimes \bar{x})(y)$ is linear in y , so

$$h\left[h^{-1}(a+b)+\langle h^{-1}(a+b), \bar{x} \rangle x\right] = h\left[h^{-1}(a)+\langle h^{-1}(a), \bar{x} \rangle x\right] + h\left[h^{-1}(b)+\langle h^{-1}(b), \bar{x} \rangle x\right] \quad (3)$$

for arbitrary $a, b \in E$. When both $\langle h^{-1}(a), \bar{x} \rangle = 0$ and

$\langle h^{-1}(b), \bar{x}\rangle = 0$, we have from (3),

$$h^{-1}(a+b) + \langle h^{-1}(a+b), \bar{x}\rangle x = h^{-1}(a+b) ,$$

so $\langle h^{-1}(a+b), \bar{x}\rangle = 0$, since x may be chosen to be non-zero. This means $h^{-1}(a+b)$ lies in the subspace spanned by $h^{-1}(a)$ and $h^{-1}(b)$, so the result follows.

4. *When* $h^{-1}(a)$ *and* $h^{-1}(b)$ *are linearly independent, then* $\mu = \rho$.

Choose $\bar{x}$ such that $\langle h^{-1}(a), \bar{x}\rangle = 1 = \langle h^{-1}(b), \bar{x}\rangle$. Then with $x = -h^{-1}(a) - h^{-1}(b)$, equation (3) becomes

$$\begin{aligned} h\left[h^{-1}(a+b)-(\mu+\rho)\left(h^{-1}(a)+h^{-1}(b)\right)\right] &= h\left[-h^{-1}(b)\right] + h\left[-h^{-1}(a)\right] \\ &= -(a+b) , \text{ since } \lambda(-1) = -1 . \end{aligned}$$

Thus

$$\begin{aligned} h^{-1}(a+b) &= \frac{\mu+\rho}{2}\left(h^{-1}(a)+h^{-1}(b)\right) \\ &= \mu h^{-1}(a) + \rho h^{-1}(b) . \end{aligned}$$

Since $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent, $\frac{\mu+\rho}{2} = \mu$, or $\rho = \mu$. Note that μ cannot be zero.

5. *h preserves linearly independent sets of elements.*

Suppose $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent, and $\alpha a + \beta b = 0$, for some $\alpha, \beta \in \mathrm{R}$. Then

$$\begin{aligned} 0 &= h^{-1}(\alpha a+\beta b) \\ &= \mu\left[h^{-1}(\alpha a)+h^{-1}(\beta b)\right] , \text{ since } h^{-1}(\alpha a) \text{ and } h^{-1}(\beta b) \\ &\qquad \text{are linearly independent,} \\ &= \mu\left[\lambda^{-1}(\alpha)h^{-1}(a)+\lambda^{-1}(\beta)h^{-1}(b)\right] . \end{aligned}$$

Thus $\lambda^{-1}(\alpha) = \lambda^{-1}(\beta) = 0$, so $\alpha = \beta = 0$.

In a similar way we may show this result for any finite set of linearly independent elements.

6. *h is linear.*

In [24, p. 245, Lemma A], Mackey has generalised a theorem of projective geometry to show that if E and F are linear spaces with dimension greater than two, then any mapping of E onto F which affords a one-to-one correspondence between one-dimensional subspaces and preserves linear independence is necessarily linear.

7. *h is continuous.*

For this we need the conditions on the topology of E.

1) *E has the weak topology.*

Suppose the net $\{x_\alpha\}$ converges weakly to zero. For $\langle x, \overline{x}\rangle \neq -1$,

$$\begin{aligned}\phi(1+x\otimes\overline{x})(x_\alpha) &= h(1+x\otimes\overline{x})h^{-1}(x_\alpha)\\ &= h\left[h^{-1}(x_\alpha)+\left\langle h^{-1}(x_\alpha), \overline{x}\right\rangle x\right]\\ &= x_\alpha + \left\langle h^{-1}(x_\alpha), \overline{x}\right\rangle h(x)\end{aligned}$$

converges weakly, with α, to zero. Thus $\left\langle h^{-1}(x_\alpha), \overline{x}\right\rangle$ converges to zero, and since $\overline{x}$ may be arbitrarily chosen, h^{-1} is continuous with respect to the weak topology at zero, hence everywhere. We may show the same result for h.

2) *E has the Mackey topology.*

Since $L(E) \subset L(E_w)$ [48, p. 39, Proposition 13], we may use the method above and [48, p. 62, Proposition 14] to obtain the result. //

REMARKS.

1. Theorem 5.1 holds when E is a Fréchet space, $\dim E > 2$.

2. If A is a semigroup of continuous selfmaps of a Banach

space E , $\dim E > 2$, such that $I, L \subset A$ and ϕ is an automorphism of A , then the following statements are equivalent:

a) ϕ fixes L ,

b) ϕ is additive,

c) ϕ is additive on the one-dimensional maps.

That b) implies c) is immediate. The method of [61] may be used to show c) implies a), while b) follows from a) as in Theorem 5.1. Theorem 5.1 is thus a generalisation of the near-ring result [61], mentioned in Chapter one, Section six.

3. For a Hilbert space H , $\dim H > 2$, $U \cup I(H)$ has the Magill property (Theorem 5.1), while $U(H)$ has not (Result 5.1).

5.3 *d*-automorphisms

In the remainder, $E \in LCS$ is as in the previous section and S is a semigroup of selfmaps of E such that $I, U \subset S \subset \mathcal{D}_H$. We generalise the definition of a d-automorphism of S [62], as follows: ϕ is a d-automorphism if

$$d\phi(f) = \{\phi(f)'(x) : x \in E\} = \{\phi(f'(x)) : x \in E\} = \phi(df) ,$$

for every invertible $f \in S$ for which $f^{-1} \in S$. Then,

THEOREM 5.2. *Every d-automorphism of S is inner.*

Proof. By the previous theorem we need only show $\phi(U) = U$.

$U \subseteq \phi(U)$: Let $u \in U$. Then there exists an invertible $f \in S$ such that $\phi(f) = u$. Now

$$\{u\} = \{\phi(f)'(x) : x \in E\} = \{\phi(f'(x)) : x \in E\} ,$$

so $f'(x)$ is constant with respect to x . By the Mean Value Theorem for functionals [19], since $f(0) = 0$, $f \in L$. Further, $u^{-1} = \phi(f)^{-1} = \phi(f^{-1})$, so similarly,

$$\{u^{-1}\} = \{\phi(f^{-1})'(x) : x \in E\} = \{\phi[(f^{-1})'(x)] : x \in E\} ,$$

so $f^{-1} \in L$. Thus $f \in U$.

$\phi(U) \subseteq U$: If $f \in \phi(U)$, f is invertible and there exists a $u \in U$ such that $\phi(u) = f$. Now

$$df = \{\phi(u)'(x) : x \in E\} = \{\phi(u)\} ,$$

so $\phi(u) \in L$. Similarly,

$$df^{-1} = \{\phi(u^{-1})'(x) : x \in E\} = \{\phi(u^{-1})\},$$

so $\phi(u)^{-1} \in L$. Thus $f \in U$. //

Since this result does not require mappings in the semigroup to be continuous, it gives for instance that every d-automorphism of $\mathcal{D}^k_H(E)$, $\mathcal{D}^k_F(E)$, and $C^k(E)$, $k \in \mathbb{N}$, is inner, for E with the Mackey or weak topology.

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