LAWS IN TORSION FREE NILPOTENT VARIETIES

with particular reference to the laws of free nilpotent groups

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Unless otherwise stated this thesis is my own work.

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Introduction

The substantive part of this thesis can be conveniently divided into two sections, the first consisting of Chapter 2 and the Appendix, the second consisting of Chapters 3, 4, 5 and 6.

In a paper [6] presented at the International Conference on the Theory of Groups held at the Australian National University in 1965 Professor Graham Higman developed a valuable tool for investigating the lattice of varieties between $\underline{\underline{N}}_c \wedge \underline{\underline{B}}_p$ and $\underline{\underline{N}}_{c-1} \wedge \underline{\underline{B}}_p$ where p is a prime, greater than c. A.G.R. Stewart provided an expanded explanation of this technique, first in his M.Sc. thesis and subsequently, and more fully, in his Ph.D. thesis [12]. In [12] Stewart applied the technique to find the lattice of subvarieties of the variety of centre-extended-by-metabelian groups, nilpotent of class c with prime exponent $p > c$.

Meanwhile L.G. Kovács and M.F. Newman found that the technique could be used for torsion-free nilpotent varieties, a torsion free variety being one whose free group of countable rank is torsion free. Chapter 2 of this thesis puts some of their results on record. My approach to the problem differs from that used by Kovács and Newman. The relationship between the two approaches is explained in the appendix which also outlines a third approach which I think would be best.

Chapters 3, 4 and 5 are aimed at finding out as much as possible about the laws of the free group of rank n in $\underline{\underline{N}}_c$, denoted $F_n(\underline{\underline{N}}_c)$.

The results of Chapter 2 are useful in this context, as Higman observed in [6].

Until recently not a great deal was known about the laws of $F_n(\underline{\underline{N}}_c)$ when $n < c$. T.C. Chau, in his Ph.D. thesis [3] found bases for the laws of these groups when $n < c \leq 6$. Independent proofs have been provided by Levin [8] and by Kovács, Newman and Pentony [7] that $F_{c-1}(\underline{\underline{N}}_c)$ generates $F_n(\underline{\underline{N}}_c)$ but that $F_{c-2}(\underline{\underline{N}}_c)$ does not. Levin showed in addition that, for $n < c - 1$, $F_n(\underline{\underline{N}}_c)$ generates a proper subvariety of the variety generated by $F_{n+1}(\underline{\underline{N}}_c)$.

Perhaps the most interesting of Chau's results was that $F_2(\underline{\underline{N}}_6)$ obeys a law of weight 5, that is a law which does not apply to $\underline{\underline{N}}_5$. In fact he showed that the law in question is the law which distinguishes $F_3(\underline{\underline{N}}_5)$ from $(\underline{\underline{N}}_5)$ which means that

$$\mathrm{Var}(F_2(\underline{\underline{N}}_6)) \wedge \underline{\underline{N}}_5 = \mathrm{Var}(F_3(\underline{\underline{N}}_5))$$

This means that the laws of $F_n(\underline{\underline{N}}_c)$ cannot be obtained by Higman's method alone, since it would only find the laws of weight c, that is the laws of $\mathrm{Var}F_n(\underline{\underline{N}}_c) \vee \underline{\underline{N}}_{c-1}$. On the other hand it leads to the conjecture that, whenever $2 \leq n < c$

$$\mathrm{Var}(F_n(\underline{\underline{N}}_c)) \wedge \underline{\underline{N}}_{c-1} = \mathrm{Var}(F_{n+1}(\underline{\underline{N}}_c)) \qquad (1)$$

which, if true would, together with Higman's results tell us a great deal about the laws of $F_n(\underline{\underline{N}}_c)$.

In Chapter 3 I obtain a set of laws of $F_n(\underline{\underline{N}}_c)$ whose weight is less than c. These laws are subsequently used to prove that (1) is true whenever n is greater than both $\frac{1}{2}(c-2)$ and 8 but is not true in general.

Chapter 4 develops a clumsy but easily used tool for commutator calculations, namely a basis for $F_n(\underline{\underline{N}}_c)$ many of whose elements are left normed commutators. With the use of this basis I provide a counterexample for (1).

Chapter 5 uses the basis developed in Chapter 4 together with the laws found in Chapter 3 to show that (1) is true whenever n is greater than both $\frac{1}{2}(c-2)$ and 8.

In Chapter 6 I state two conjectures which indicate the way in which I think the results obtained in Chapters 3, 4 and 5 should fit together.

I have preceeded each chapter with an introductory section, numbered n.0 where n is the number of the chapter, which gives a general idea of the main results of the chapter and how they are obtained. Towards this end they contain heuristic arguments which are certainly not intended as proofs. On some occasions, where it seemed useful, I have included similar arguments in the body of the chapter.

Notation Used in this Thesis

The following list is, obviously, incomplete and there are probably some unfortunate omissions. I have attempted to include all symbols which might cause confusion.

Throughout this list the following conventions apply.

n, m and c are integers, usually positive

q is a rational number

T and V are sets

R is a ring

N and M are R-modules

$\underline{\underline{V}}$ is a variety

D is a diagram

σ is a permutation

General Notation

Z	= the ring of integers
Z^+	= the positive integers
Z_n	= the first n positive integers
Q	= the field of rational numbers
ϕ	= the empty set

$T \setminus V$	$= \{\tau : \tau \in T \text{ and } \tau \notin V\}$
$\|T\|$	= the cardinality of T, where T is a set
$f\|_T$	= the restriction of f to the subset T of its domain
$\|q\|$	= the absolute value of q, where $q \in Q$
$n\|m$	: n divides m where $n,m \in Z$
$[q]$	= the integral part of q where $q \in Q$
S_n	= the symmetric group on n symbols

Module Notation

$GL(n,K)$	= the group of invertible $n \times n$ matrices with entries in the field K
$Hom_R(M,N)$	= the additive group of R-homomorphisms from the R-module M to the R-module N
$End_R(M)$	$= Hom_R(M,M)$
$M \dot{+} N$	= the external direct sum of M and N
$M \oplus N$	= the internal direct sum of M and N
$M \otimes_R N$	= the tensor product of the right R-module M with the left R-module N
$U^{[c]}$	= the c-fold tensor power of the bimodule U

Notation Defined in the Text

(Section and page numbers refer to the text)

1. Notation and Some Definitions

1.0. Unless otherwise stated the notation in this thesis is that used by Curtis and Reiner [4].

A notable exception is that I will normally write mappings, including permutations, on the right of their arguments whereas Curtis and Reiner write them on the left. Consequently I tend to be dealing with ~~left~~ right modules and ideals whereas [4] deals mainly with ~~right~~ left modules and ideals.

The term algebra will denote a set with some algebraic structure on it, not necessarily a ring of vectors as in [4]. The terms group algebra and Lie algebra have their usual meaning.

I will use Z to denote the set of integers, Z^+ to denote the set of positive integers and Z_n to denote the set of integers lying between 1 and n inclusive.

Unless otherwise stated the varietal notation is that of Hanna Neumann's book [11], one exception being that I have used double underlining in place of German script; thus $\underline{\underline{N}}_c$ is the variety of groups nilpotent of class at most c.

If G is a group $\gamma_c(G)$ is the c-th term in the lower central series of G.

I will use MacLane and Birkhoff's [9] notation for describing mappings, that is:

$\psi : A \to B$ means A is the domain and B the range of ψ.

$\psi : a \mapsto b$ means $a\psi = b$.

In order to facilitate commutator calculations I have adapted and adopted the notation used by Martin Ward in his Ph.D. thesis and subsequently published, [13]. I have attempted to include enough definitions and lemmas to make my treatment of this notation independent of the original. The rest of this chapter is included for this purpose. 14/

1.1. The Algebra of Expressions.

$\underset{\sim}{X}$ is some set with a bijection $\underset{\sim}{x} : Z \to X$. $\underset{\sim}{A}$ is the free algebra freely generated on $\underset{\sim}{X}$ by the operations of multiplication, commutation (both binary), inversion (unary) and identity (nullary), the only law being that multiplication is associative.

I will denote multiplication by juxtaposition of the operands; commutation by $[a,b]$, where a is the first operand and b the second; inversion by a^{-1} where a is the operand; and identity by $\underset{\sim}{1}$.

$\underset{\sim}{A}$ is the <u>algebra of expressions</u> and its elements are <u>expressions</u>.

If i is an integer $\underset{\sim}{x}_i$ is the image of i under $\underset{\sim}{x}$.

An elementary property of an algebra such as $\underset{\sim}{A}$ is that there exists a unique function, from $\underset{\sim}{A}$ to the positive integers (which I will call the <u>height</u> function and denote by ht) which has the following properties

(i) $\quad ht(\underset{\sim}{1}) = ht(\underset{\sim}{x}_i) = 1, \quad \forall i \in Z,$

(ii) $\quad ht(a^{-1}) = ht(a) + 1, \quad \forall a \in \underset{\sim}{A},$

(iii) $\quad ht(ab) = ht([a,b]) = ht(a) + ht(b) + 1, \quad \forall\, a,b \in \underset{\sim}{A}$

The notation $\prod_{i=1}^{n} a_i$ is defined recursively in the usual way, that is:

$$\prod_{i=1}^{1} a_i = a_1$$

$$\prod_{i=1}^{n} a_i = (\prod_{i=1}^{n-1} a_i)a_n, \quad \forall n > 1.$$

Exponentiation in $\underset{\sim}{A}$ is defined in the following rather artificial way:

$$a^0 = \underset{\sim}{1}, \quad a^1 = a, \quad a^{-1} \text{ is the inverse of } a,$$

$$\left.\begin{array}{l} a^n = a^{n-1}a \\ a^{-n} = a^{-n+1}a^{-1} \end{array}\right\} \quad \forall n > 1$$

It follows that $a^n a^m = a^{n+m}$ if $nm > 0$.

The purpose of $\underset{\sim}{A}$ is to provide an algebra in which commutator calculations can be broken down into small steps without the expression "collapsing". This is why it is equipped with only one law. As a result of this some care needs to be exercised when working in $\underset{\sim}{A}$ since, for example,

$$aa^{-1} \neq \underset{\sim}{1}, \quad a\underset{\sim}{1} \neq a, \quad [a,b] \neq a^{-1}b^{-1}ab.$$

<u>1.2. The Free Groups</u>: X is a set disjoint from $\underset{\sim}{X}$ but also having a bijection, this time $x : Z \to X$.

F is the free group freely generated by X and, for each variety $\underline{\underline{V}}$, $F(\underline{\underline{V}})$ is the $\underline{\underline{V}}$-free group freely generated by X.

For each integer i I will write x_i for the image of i under x.

It is worth noting that in order to evaluate the product of two elements of X it is necessary to know in which group one is working. This should always be clear from the context.

<u>1.3. The Homomorphisms</u> $\underset{\sim}{\rho}$: If commutation is defined on the groups $F(\underline{\underline{V}})$ and F in the usual way, that is

$$[a,b] = a^{-1}b^{-1}ab$$

then they become algebras of the same type as $\underset{\sim}{A}$. Since the only law in $\underset{\sim}{A}$ is also a law in any group it follows that the mapping

$$\underset{\sim}{x}_i \to x_i, \quad \forall i \in Z,$$

can be extended into a homomorphism from $\underset{\sim}{A}$ to F which I will denote by $\underset{\sim}{\rho}$ and, if $\underline{\underline{V}}$ is a variety, to a homomorphism from $\underset{\sim}{A}$ to $F(\underline{\underline{V}})$ which I will denote by $\underset{\sim}{\rho}_{\underline{\underline{V}}}$. I will denote $\underset{\sim}{\rho}_{\underline{\underline{N}}_c}$ by $\underset{\sim}{\rho}_c$.

The congruence relations $\sim$ and $\sim_{\underline{\underline{V}}}$ are defined on $\underset{\sim}{A}$ as follows:

$$a \sim b \text{ if } a\underset{\sim}{\rho} = b\underset{\sim}{\rho}$$

and $a \sim_{\underline{\underline{V}}} b$ if $a\underset{\sim}{\rho}_{\underline{\underline{V}}} = b\underset{\sim}{\rho}_{\underline{\underline{V}}}$.

I will denote $\sim_{\underline{\underline{N}}_c}$ by $\sim_c$.

<u>1.4. Definition:</u> $Z_n = \{i : i \in Z, 1 \leq i \leq n\}$.

$$\underset{\sim}{X}_n = \{\underset{\sim}{x}_i : i \in Z_n\}, \quad X_n = \{x_i : i \in Z_n\}.$$

$\underset{\sim}{A}_n$ is the subalgebra of $\underset{\sim}{A}$ generated by $\underset{\sim}{X}_n$.

F_n and $F_n(\underline{\underline{V}})$ are the subgroups of F and $F(\underline{\underline{V}})$, respectively, generated by X_n.

<u>1.5. Definition:</u> The functions weight (denoted wt), and weight in $\underset{\sim}{x}_i$ (denoted wt_i) from $\underset{\sim}{A}$ to $Z^+ \cup \{\infty\}$ are defined recursively on height as follows:

$wt(\underset{\sim}{1}) = \infty$; $wt_i(\underset{\sim}{1}) = \infty$,

$wt(\underset{\sim}{x}_j) = 1, \quad \forall j \in Z$; $wt_i(\underset{\sim}{x}_i) = 1$,

$wt_i(\underset{\sim}{x}_j) = 0, \quad \forall j \in Z \setminus \{i\}$,

$wt(a^{-1}) = wt(a), \quad \forall a \in \underset{\sim}{A}$; $wt_i(a^{-1}) = wt_i(a), \quad \forall a \in \underset{\sim}{A}$,

$wt(ab) = \min\{wt(a), wt(b)\}, \forall a,b \in \underset{\sim}{A}$; $wt_i(ab) = \min\{wt_i(a), wt_i(b)\}, \forall a,b \in \underset{\sim}{A}$,

$wt([a,b]) = wt(a) + wt(b), \forall a,b \in \underset{\sim}{A}$; $wt_i([a,b]) = wt_i(a) + wt_i(b), \forall a,b \in \underset{\sim}{A}$, (Addition and the partial ordering $\leq$ are extended to $Z^+ \cup \{\infty\}$ in the usual way. "min" in front of a set means the smallest element of the set.)

For each $a \in \underset{\sim}{A}$ the repetition pattern of a, denoted rep(a), is the function from Z to $Z^+ \cup \{\infty\}$ given by

$$i\,\text{rep}(a) = wt_i(a).$$

<u>Note</u>: The following results can be obtained by an obvious induction on height:

(i) If a is an expression and i an integer then

$$wt_i(a) = \infty \text{ if and only if } wt(a) = \infty .$$

(ii) If a is an expression then

$$\sum_{i \in Z} wt_i(a) \leq wt(a).$$

It follows from these two observations that repetition patterns are either infinite everywhere or finite everywhere and that in the latter case only a finite number of values of rep(a) are non zero.

<u>1.6. Definition</u>: An expression a is a <u>commutator</u> if

(i) $a \in \underset{\sim}{X}$

or (ii) $a = [a_1, a_2]$ where a_1 and a_2 are commutators.

An expression a is a <u>product of commutators of length</u> ℓ if it is of the form:

$$a = \prod_{i=1}^{\ell} a_i^{\varepsilon_i}$$

where each a_i is a commutator and each $\varepsilon_i \in \{-1,0,1\}$.
(The possibility of ε_i being 0 means that $\underset{\sim}{1}$ can appear in a.)

An expression a is a <u>homogeneous product of commutators</u> if it is of the form:

$$a = \prod_{i=1}^{\ell} a_i^{\varepsilon_i},$$

where each a_i is a commutator, each $\varepsilon_i \in \{1,-1\}$ and $\text{rep}(a) = \text{rep}(a_i)$ $\forall i \in Z_\ell$.

The set of <u>left normed</u> commutators is defined recursively on height as follows:

(i) $\underset{\sim}{x}_i$ is a left normed commutator $\forall i \in Z$.

(ii) If a is a left normed commutator so is $[a,\underset{\sim}{x}_i]$, $\forall i \in Z$.

2. The Higman Theory.

2.0. In a paper presented to the 1965 International Conference on the Theory of Groups, [6], Graham Higman established a relationship between the lattice of varieties of groups of prime exponent p lying between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$ and the lattice of right ideals of KS_c where K is the field with p elements.

L.G. Kovács and M.F. Newman have extended this approach to deal with the torsion free varieties between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$. In this chapter I will do likewise but will approach the problem from a different angle.

I originally intended to include at this point a discussion of the relationship between my approach and those of Higman and of Kovács and Newman. I found, however, that, in order to do this in a manner that I would consider adequate, it would be necessary for me to introduce a number of concepts and results which are not necessary for my own treatment and would therefore be dropped almost as soon as they were introduced. I have therefore relegated the discussion to an appendix and deal here only with the contents of this chapter.

The lattice of varieties between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$ is, of course, dual isomorphic to the lattice of those verbal subgroups of $F_c(\underline{\underline{N}}_c)$ which are contained in the bottom term of its lower central series, that is in $\gamma_c(F_c(\underline{\underline{N}}_c))$. Under this isomorphism the torsion free varieties correspond to those subgroups which give rise to torsion free factor groups of $F_c(\underline{\underline{N}}_c)$. I will refer to such subgroups as being "isolated".

The main aim of this chapter is to determine those minimal isolated fully invariant subgroups of $F_c(\underline{\underline{N}}_c)$ which lie in $\gamma_c(F_c(\underline{\underline{N}}_c))$. Suppose U is a free Z-module of rank c and basis $u_1, u_2, \ldots, u_c$. Then the set of endomorphisms of $F_c(\underline{\underline{N}}_c)$ can be mapped to the endomorphism ring of U under the mapping

$$(\theta : x_i \mapsto \prod_{j=1}^{m_i} x_{\lambda_{i,j}}^{\delta_{i,j}}) \mapsto (\varphi : u_i \mapsto \sum_{j=1}^{m_i} \delta_{i,j} u_{\lambda_{i,j}}).$$

Thus we can regard U and, more importantly, its c-fold tensor power, $U^{[c]}$, as $\mathrm{End}(F_c(\underline{\underline{N}}_c))$-modules and it is fairly clear that the $\mathrm{End}(F_c(\underline{\underline{N}}_c))$-submodules of $U^{[c]}$ are identical with its $\mathrm{End}_Z(U)$-submodules.

Now there is a natural homomorphism from $U^{[c]}$ to $\gamma_c(F_c(\underline{\underline{N}}_c))$ given by

$$u_{\lambda_1} \otimes u_{\lambda_2} \otimes \ldots \otimes u_{\lambda_c} \mapsto [x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_c}]$$

and, in fact this is an $\mathrm{End}(F_c(\underline{\underline{N}}_c))$-homomorphism. It follows that if we can express $U^{[c]}$ as a direct sum of minimal, isolated $\mathrm{End}_Z(U)$-submodules we can express $\gamma_c(F_c(\underline{\underline{N}}_c))$ as a direct sum of product/
minimal isolated verbal subgroups of $F_c(\underline{\underline{N}}_c)$.

In Section 2.1 I show that the lattice of isolated $\mathrm{End}_Z(U)$-submodules of $U^{[c]}$ is isomorphic to the lattice of $\mathrm{End}(Q \otimes_Z U)$-submodules of

$(Q \otimes_Z U)^{[c]}$. But this is just the lattice of representations of $GL(c,Q)$ in the c-fold tensor power. At the end of Section 2.1 I quote a number of results from Curtis and Reiner [4] which establish the relationship between this lattice and the lattice of right ideals of QS_c.

In Section 2.2 I simply quote, from Boener [1], a special decomposition of QS_c.

Section 2.3 then deals with the relation between $U^{[c]}$ and $\gamma_c(F_c(\underline{\underline{N}}_c))$ in rather more detail than I have done above.

For reasons which will become apparent in Section 2.3 I will deal with $\gamma_c(F_n(\underline{\underline{N}}_c))$ and Z-modules of rank n for an arbitrary positive integer n.

2.1. The Isolated Submodules of $U^{[c]}$.

2.1.0. (i) Notation and convention: In this section I will be using the notation and definitions of Curtis and Reiner [4] extensively.

Modules over commutative rings such as Q and Z are of necessity bimodules and I will treat them as such, usually without explanation. For example if L is a Z-module $Q \otimes_Z L$ can be and will be regarded as both a left and a right Q-module.

Curtis and Reiner use the notation $L \otimes_R M$ for the tensor product of L by M over R but denote the elements of this product by $\ell \otimes m$, dropping the subscript R. I will follow this convention in the belief that it is always possible to see, from the context, which ring the product is taken over. There are, however, some occasions on which it is necessary to exercise a little care in interpreting this notation.

I will denote the ring of R-endomorphisms of a right R-module L by $\mathrm{End}_R(L)$.

If G is a group End(G) is the algebra ~~of~~ formed by the endomorphisms of G under the operations of pointwise multiplication and composition. If G is abelian End(G) is, of course, a ring. In order to have a name for animals such as End(G) I will use the term <u>ringoid</u> to describe algebras with two binary operations.

(ii) <u>Definition</u>: A submodule L of an R-module U is <u>isolated</u> if the quotient module U/L considered as an abelian group is torsion free.

<u>2.1.1. Lemma</u>: (i) If L and M are torsion free Z-modules with submodules L_1 and M_1 respectively then $L_1 \otimes_Z M_1$ can be imbedded in the natural way in $L \otimes_Z M$.

(ii) If R is a commutative ring with identity and L is an R-module then

$$L \cong R \otimes_R L \cong L \otimes_R R$$

under the mappings $x \mapsto 1 \otimes x \mapsto x \otimes 1, \ \forall x \in L$.

(iii) If L is a torsion free Z-module then there is an imbedding of L in $Q \otimes_Z L$ given by

$$x \mapsto 1 \otimes x, \quad \forall x \in L.$$

(iv) If L is a R-module and $x \in Q \otimes_Z L$ then there exist an element, y of L and an element, q of Q such that

$$x = q \otimes y.$$

Proof: The first two results are well known and I simply give references for their proof.

(i) This result is due to Dieudonné. A proof can be found in Fuchs, [5], Theorem 64.4, page 254.

(ii) The first part of this isomorphism is proved in Curtis and Reiner, [4], Theorem (12.14), page 67 and, with the obvious modifications, this proof can be converted to a proof that

$$L \cong L \otimes_R R.$$

(iii) Suppose ψ is the isomorphism from L to $Z \otimes_Z L$ given by

$$x \mapsto 1 \otimes x$$

and φ is the natural mapping from $Z \otimes_Z L$ to $Q \otimes_Z L$. Then $\psi\varphi$ is the required injection.

(iv) Suppose $x \in Q \otimes_Z L$.

Let

$$x = \sum_{i=1}^{n} q_i \otimes y_i$$

and choose $m \in Z$ such that $mq_i \in Z$ for all $i \in Z_n$.

Then

$$x = \sum_{i=1}^{n} (\frac{m}{m} q_i \otimes y_i)$$

$$= \sum_{i=1}^{n} (\frac{1}{m} \otimes mq_i y_i)$$

$$= \frac{1}{m} \otimes \sum_{i=1}^{n} (mq_i y_i)$$

which completes the proof.

Note. The above results will be used repeatedly and, in general, without specific reference. In particular, if L and M are torsion free Z-modules with submodules L_1 and M_1 respectively then $L_1 \otimes_Z M_1$ will be regarded as a subset of $L \otimes_Z M$.

2.1.2: Suppose U is a free Z-module of rank n. Then $End_Z(U)$ is isomorphic to the ring of $n \times n$ matrices with entries from Z. The additive groups of these two rings are clearly Z-modules and can therefore be tensored with Q over Z. It is fairly obvious that these tensor products will be isomorphic to the additive group of $n \times n$ matrices with entries in Q and that, by defining multiplication on the tensor product in the obvious way, this can be extended to a ring isomorphism, in short that

$$Q \otimes_Z \mathrm{End}_Z(U) \cong \mathrm{End}_Q(Q \otimes_Z U).$$

This result is formally proved below.

<u>Lemma</u>: If U is a free, finite dimensional Z-module and an operation of multiplication is defined on $Q \otimes_Z \mathrm{End}_Z(U)$ by

$$(q_1 \otimes \psi_1)(q_2 \otimes \psi_2) = q_1q_2 \otimes \psi_1\psi_2$$

then $Q \otimes_Z \mathrm{End}_Z(U)$ becomes a ring and as such is isomorphic to $\mathrm{End}_Q(Q \otimes_Z U)$ under the mapping σ where

$$(p \otimes \psi)\sigma : (q \otimes x) \mapsto pq \otimes x\psi,$$

(where $p,q \in Q$, $x \in U$ and $\psi \in \mathrm{End}_Z(U)$).

<u>Proof</u>: The lemma will be proved in three parts

(i) σ maps elements of $Q \otimes_Z \mathrm{End}_Z(U)$ to Q-endomorphisms of $Q \otimes_Z U$.

(ii) σ preserves addition and multiplication.

(iii) σ is a bijection.

Once these have been established it will follow that the ringoid $Q \otimes_Z \mathrm{End}_Z(U)$ is isomorphic to $\mathrm{End}_Q(Q \otimes_Z U)$ and is hence a ring.

(i) Suppose that $\alpha,\beta \in Q \otimes_Z U$, that $x \in Q \otimes_Z \mathrm{End}_Z(U)$ and that $q \in Q$. It is obvious that

$$(q\alpha)(x\sigma) = q(\alpha(x\sigma)).$$

Let $\alpha = \frac{n_1}{m_1} \otimes a$, $\beta = \frac{n_2}{m_2} \otimes b$ and $x = p \otimes \psi$, where $p \in Q$; $n_1,n_2,m_1,m_2 \in Z$; $a,b \in U$ and $\psi \in \mathrm{End}_Z(U)$.

Then

$$(\alpha+\beta)x\sigma = (\frac{n_1m_2}{m_1m_2} \otimes a + \frac{n_2m_1}{m_1m_2} \otimes b)(p \otimes \psi)\sigma ,$$

$$= \frac{p}{m_1m_2} \otimes (n_1m_2a\psi + n_2m_1b\psi),$$

$$= \frac{pn_1m_2}{m_1m_2} \otimes a\psi + \frac{pn_2m_1}{m_1m_2} \otimes b\psi,$$

$$= \alpha(x\sigma) + \beta(x\sigma),$$

which completes this part of the proof.

(ii) It follows immediately from the definitions of σ and multiplication in $Q \otimes_Z \mathrm{End}_Z(U)$ that σ preserves multiplication. The proof that σ also preserves addition is a straightforward adaptation of the proof of (i) above.

(iii) Suppose $u_1,u_2,\ldots,u_n$ is a Z-basis for U. Then $1 \otimes u_1$, $1 \otimes u_2,\ldots,1 \otimes u_n$ is a Q-basis for $Q \otimes_Z U$. (Clearly ~~they~~ it ~~are~~ is a Q-generating set for $Q \otimes_Z U$ and any non trivial Q-linear expression in $1 \otimes u_1, \ldots, 1 \otimes u_n$, is a scalar multiple of a Z-linear expression and, by Lemma 2.1.1.(iii) we know that $1 \otimes u_1, \ldots, 1 \otimes u_n$ are Z-linearly independent.)

Suppose $\psi \in \mathrm{End}_Q(Q \otimes_Z U)$. For each (i,j) in Z_n^2 define $\psi_{i,j} \in Q$ by

$$(1 \otimes u_i)\psi = \sum_{j=1}^{n} \psi_{i,j} \otimes u_j, \quad \forall i \in Z_n.$$

Choose $m \in Z$ such that $m\psi_{i,j} \in Z$, $\forall (i,j) \in Z_n^2$.

Then

$$(1 \otimes u_i)\psi = \frac{1}{m} \otimes \sum_{j=1}^{n} m\psi_{i,j}u_j, \quad \forall i \in Z_n$$

$$= \frac{1}{m} \otimes u_i\psi', \quad \forall i \in Z_n$$

where ψ' is the Z-endomorphism of U given by

$$\psi' : u_i \mapsto \sum_{j=1}^{n} m\psi_{i,j}u_j \quad \forall i \in Z_n.$$

Hence for each ψ in $\mathrm{End}_Q(Q \otimes_Z U)$ there exist an integer, m, and an element ψ' of $\mathrm{End}_Z(U)$ such that

$$\psi = (\frac{1}{m} \otimes \psi')\sigma.$$

Thus σ maps $Q \otimes_Z \mathrm{End}_Z(U)$ onto $\mathrm{End}_Q(Q \otimes_Z U)$. Suppose now that x_1 and $x_2 \in Q \otimes_Z \mathrm{End}_Z(U)$ and

$$x_1\sigma = x_2\sigma.$$

Let $x_1 = \frac{n_1}{m_1} \otimes \psi_1 \quad x_2 = \frac{n_2}{m_2} \otimes \psi_2$.

Then $(1 \otimes u)x_1\sigma = (1 \otimes u)x_2\sigma, \quad \forall u \in U,$

thus

$$\frac{n_1}{m_1} \otimes u\psi_1 = \frac{n_2}{m_2} \otimes u\psi_2, \quad \forall u \in U,$$

$$1 \otimes n_1 m_2(u\psi_1) = 1 \otimes n_2 m_1(u\psi_2), \quad \forall u \in U,$$

$$n_1 m_2 \psi_1 = n_2 m_1 \psi_2,$$

$$\frac{n_1}{m_1} \otimes \psi_1 = \frac{n_2}{m_2} \otimes \psi_2,$$

and

$$x_1 = x_2,$$

which means that ~~s~~ σ is one to one.

2.1.3. (i) **Definition.** If R is a commutative ring, c a positive integer and U an R-module then the c-fold tensor product of U over R, $(U^{[c]})$ is defined inductively as follows:

$$U^{[1]} = U$$

$$U^{[c]} = U^{[c-1]} \otimes_R U \text{ if } c > 1.$$

I will denote the elements of $U^{[c]}$ by

$$\sum_{i=1}^{n} (u_{1,i} \otimes u_{2,i} \otimes \ldots \otimes u_{c,i}).$$

(ii) Note: The notation $U^{[c]}$ is ambiguous in that it gives no indication of the ring R. For example any Q-module is also a Z-module so if U is such a module $U^{[c]}$ might be a tensor power over Q or Z.

However, for the purposes of this exercise the problem can be overcome by adopting the following convention:

Whenever U is a free Z-module of finite rank $U^{[c]}$ is the c-fold tensor power of U over Z and $(Q \otimes_Z U)^{[c]}$ is the c-fold tensor power of $Q \otimes_Z U$ over Q.

(iii) Lemma: If R is a commutative ring, c a positive integer and U an R-module then the c-fold tensor power of U over R is a right $End_R(U)$-module under the action

$$\left(\sum_{i=1}^{n} u_{1,i} \otimes u_{2,i} \otimes \ldots \otimes u_{c,i}\right)\psi = \sum_{i=1}^{n} (u_{1,i}\psi \otimes u_{2,i}\psi \otimes \ldots \otimes u_{c,i}\psi)$$

for each $\psi \in End_R(U)$

Proof: This result is a simple, inductive extension of Curtis and Reiner [4], Theorem (12.10) page 63, using the endomorphism $\psi \otimes \psi \otimes \ldots \otimes \psi$ (c times).

2.1.4. Lemma: If U and V are torsion free Z-modules then there is a Q-isomorphism from $(Q \otimes_Z U) \otimes_Z V$ to $(Q \otimes_Z U) \otimes_Q (Q \otimes_Z V)$ given by:

$$\sum_{i=1}^{n} ((q_i \otimes u_i) \otimes v_i) \mapsto \sum_{i=1}^{n} ((q_i \otimes u_i) \otimes (1 \otimes v_i))$$

<u>Proof</u>: By 2.1.1. (ii)

$$(Q \otimes_Z U) \cong ((Q \otimes_Z U) \otimes_Q Q)$$

under the mapping

$$q \otimes u \mapsto (q \otimes u) \otimes 1$$

and, by the associativity of the tensor product,

$$(((Q \otimes_Z U) \otimes_Q Q) \otimes_Z V) \cong ((Q \otimes_Z U) \otimes_Q (Q \otimes_Z V))$$

under the mapping

$$(((q \otimes u) \otimes q) \otimes v) \mapsto ((q \otimes u) \otimes (q \otimes v))$$

and the lemma is proved.

<u>2.1.5. Lemma</u>: If U is a free Z-module then

$$Q \otimes_Z U^{[c]} \cong (Q \otimes_Z U)^{[c]}$$

under the isomorphism generated by

$$(q \otimes (u_1 \otimes u_2 \otimes \ldots \otimes u_c)) \mapsto (q \otimes u_1) \otimes (1 \otimes u_2) \otimes \ldots \otimes (1 \otimes u_c) .$$

Proof: By induction on c.

If $c = 1$ the result is trivial.

Suppose $d > 1$ and the statement is true for all $c < d$.

Then

$$Q \otimes_Z U^{[d]} = Q \otimes_Z (U^{[d-1]} \otimes_Z U)$$

$$\cong (Q \otimes_Z U^{[d-1]}) \otimes_Z U \quad \text{(Associativity of the tensor product.)}$$

$$\cong (Q \otimes_Z U^{[d-1]}) \otimes_Q (Q \otimes_Z U) \quad \text{(from 2.1.4)}$$

$$\cong (Q \otimes_Z U)^{[d-1]} \otimes_Q (Q \otimes_Z U) \quad \text{(by the inductive hypothesis)}$$

$$= (Q \otimes_Z U)^{[d]}$$

The form of the isomorphism can be readily checked by composing these three isomorphisms.

2.1.6. Lemma: If U is a finite dimensional Z-module then there is a lattice isomorphism, from the lattice of $Q \otimes_Z \mathrm{End}_Z(U)$-submodules of $Q \otimes_Z (U^{[c]})$ to the lattice of isolated $\mathrm{End}_Z(U)$-submodules of $U^{[c]}$.

The isomorphism is given by:

$$M \mapsto \{y : y \in U^{[c]} \text{ and } \exists q \in Q \setminus \{0\} \text{ such that } q \otimes y \in M\}$$

Proof: Let μ be the above mapping. It is sufficient to prove that, for any submodule M of $Q \otimes_Z U$:

(i) $M\mu$ is (a) isolated and (b) a submodule.

(ii) μ is (a) onto and (b) 1 - 1.

(iii) μ and μ^{-1} preserve set inclusion.

The last of these is trivial.

(i) Suppose M is a submodule of $Q \otimes_Z U^{[c]}$

and $x \in M\mu$

then $\exists y \in M$ and $q \in Q \setminus \{0\}$ such that

$$y = q \otimes x.$$

(a) Now suppose $n \in Z \setminus \{0\}$, $x_1 \in U^{[c]}$ and $x = nx_1$.

Then $M \ni q \otimes x = q \otimes nx_1 = nq \otimes x_1$

and it follows that $x_1 \in M\mu$. Hence $M\mu$ is isolated.

(b) Suppose $\psi \in End_Z(U)$.

Then $q \otimes x \in M$ so that

$$(q \otimes x)(1 \otimes \psi) \in M,$$

$$q \otimes x\psi \in M,$$

and $x\psi \in M\mu$.

Thus $M\mu$ is a submodule.

(ii) (a) Suppose L is an isolated $End_Z(U)$-submodule of $U^{[c]}$.

Then $Q \otimes_Z L$ is clearly a $Q \otimes_Z End_Z(U)$-submodule of $Q \otimes_Z U^{[c]}$.

Now suppose $x \in (Q \otimes_Z L)\mu$.

Then there exist $n_1, m_1, n_2, m_2 \varepsilon Z$ and $y \varepsilon L$ such that

$$\frac{n_1}{m_1} \otimes x = \frac{n_2}{m_2} \otimes y,$$

so that $1 \otimes n_1 m_2 x = 1 \otimes m_1 n_2 y$,

$$n_1 m_2 x \varepsilon L,$$

and

$$x \varepsilon L.$$

Thus μ is onto.

(b) Suppose M and L are $Q \otimes_Z \mathrm{End}_Z(U)$-submodules of $Q \otimes_Z U^{[c]}$, and

$$M\mu = L\mu.$$

Suppose $x \varepsilon M$. Then $\exists q \varepsilon Q \setminus \{0\}$, $y \varepsilon M\mu = L\mu$ such that

$$x = q \otimes y$$

and there exists $p \varepsilon Q \setminus \{0\}$ such that $p \otimes y \varepsilon L$ so that $(p \otimes y)(p^{-1} q \otimes j) \varepsilon L$, where j is the identity map on U, $q \otimes y \varepsilon L$, and $x \varepsilon L$. So $M \subseteq L$. Similarly $L \subseteq M$ and it follows that μ is 1 - 1.

2.1.7. Lemma 2.1.6 gives a lattice isomorphism between the lattice of isolated $End_Z(U)$-submodules of $U^{[c]}$ and the lattice of $End_Q(Q \otimes_Z U)$-submodules of $(Q \otimes_Z U)^{[c]}$.

I will later show that this second lattice is identical to the lattice of $GL(n,Q)$-submodules of $U^{[c]}$. To do this I will need some results from Curtis and Reiner [4].

(i) (§67, page 449 of [4]). If V is a finite dimensional Q-space, c a positive integer and S_c the symmetric group on Z_c then $V^{[c]}$ is a right S_c-module under the operation.

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_c)\sigma = (v_{1\sigma^{-1}} \otimes v_{2\sigma^{-1}} \otimes \ldots \otimes v_{c\sigma^{-1}}) \quad \forall \sigma \in S_c.$$

This action can be extended to QS_c in the obvious way.

Note: (a) This operation involves interchanging the places of the v's and does not correspond to an endomorphism of V, in fact it commutes with all of these.

(b) Curtis and Reiner write permutations on the left and multiply them accordingly. Since I adopt the opposite convention the above results have been paraphrased.

(ii) (Theorem 67.8 page 452 of [4].) Let V be a finite dimensional vector space over Q and let G be its group of Q-automorphisms. Then $V^{[c]}$ is a completely reducible G-module and its irreducible submodules are obtained as follows. Let e be a primitive idempotent in the group algebra QS_c. Then $V^{[c]}e$ is either zero or an irreducible

G-submodule of $V^{[c]}$. All irreducible G-submodules of $V^{[c]}$ are obtained in this way. Moreover, two irreducible G-modules $V^{[c]}e$ and $V^{[c]}e'$ are isomorphic if and only if QS_c and QS_ce' are isomorphic left QS_c-modules.

Note: If c is greater than or equal to the dimension of V then the lattice of G-submodules of $V^{[c]}$ is in fact isomorphic to the lattice of left ideals of QS_c.

2.1.8 Theorem: Let V be a free Z-module of finite rank. Then $U^{[c]}$ is a direct sum of minimal isolated $End_Z(U)$-modules and its minimal isolated submodules can be found as follows. Let e be a primitive idempotent in QS_c and suppose that n is a non zero integer such that $ne \in ZS_c$. Then the isolated closure of $U^{[c]}ne$ is either zero or a minimal isolated submodule of $U^{[c]}$. All such submodules are obtained in this way; moreover two modules $U^{[c]}ne$ and $U^{[c]}n'e'$ are isomorphic if and only if QS_ce and QS_ce' are isomorphic left QS_c-modules.

Proof: Let G be the group of Q-automorphisms of $(Q \otimes_Z U)$. Then the irreducible G-submodules of $(Q \otimes_Z U)^{[c]}$ are of the form $(Q \otimes_Z U)^{[c]}e$ where e is a primitive idempotent of QS_c. Since the action of QS_c on $(Q \otimes_Z U)^{[c]}e$ commutes with the action of G on $(Q \otimes_Z U)^{[c]}$ it follows that $(Q \otimes_Z U)^{[c]}e$ is an $End_Q(Q \otimes_Z U)$-submodule. Since any

$End_Q(Q \otimes_Z U)$-submodule of $(Q \otimes_Z U)^{[c]}$ is clearly a G-submodule it follows that $(Q \otimes_Z U)^{[c]}e$ is an irreducible $End_Q(Q \otimes_Z U)$-submodule of $(Q \otimes_Z U)^{[c]}$.

This required result then follows from Lemma 2.1.6 and the fact that if L_1 and L_2 are torsion free Z-modules, L_1 is isomorphic to L_2 if and only if $Q \otimes_Z L_1$ is isomorphic to $Q \otimes_Z L_2$ (from 2.1.1 (iii) and (iv)).

2.2 A Decomposition of QS_c.

The results in this section are well known. In particular they are proved in Boerner [1] and Curtis and Reiner [4] and I will give appropriate references in lieu of proof.

The definitions, and in particular the definition of a diagram, used in this section are derived from A.G.R. Stewart's Ph.D. Thesis, [12].

<u>2.2.1 (i) Definition.</u> <u>A diagram</u> D <u>of length</u> n is an injection of Z_n into Z_n^2 with the property that if $(i,j) \in Z_nD$, $1 \leq i_1 \leq i$ and $1 \leq j_1 \leq j$ then $(i_1,j_1) \in Z_nD$.

There is a useful convention for drawing diagrams which is most conveniently explained by an example. Suppose D is the diagram of length 6 given by:

$$1D = (1,2),\ 2D = (3,1),\ 3D = (1,3),$$

$$4D = (2,1),\ 5D = (1,1),\ 6D = (2,2)$$

then D is drawn

5	1	3
4	6	
2		

(ii) <u>Definition</u>: If $n \varepsilon Z$ then <u>a partition of</u> n <u>into</u> m <u>parts</u> is a monotonic decreasing function λ from Z_m to Z_n such that

$$\sum_{i=1}^{m} i\lambda = n.$$

For notational convenience I will write λ_i for $i\lambda$.

(iii) <u>Definition</u>: If D is a diagram of length n and λ is a partition of n into m parts such that

$$\lambda_i = \max\{j : (i,j) \varepsilon Z_n D\}, \quad \forall i \varepsilon Z_m ,$$

then D and λ are said to be <u>associated</u>.

<u>Note</u>: Clearly m above must be $\max\{i : (i,1) \varepsilon Z_n D\}$. Intuitively the partition corresponds to the shape of the diagram, or the array of empty squares. Since there are n squares to be filled it is obvious that each partition is associated with n! diagrams while each diagram is associated with one partition.

(iv) Definition: If D is a diagram of length n then the group of column permutations of D, denoted $C(D)$ is the subgroup of S_n given by

$$C(D) = \{\sigma : (iD = (j,k) \Rightarrow i^{\sigma}D = (\ell,k) \text{ for some } \ell \in Z_n)\}$$

and the group of row permutations, denoted $R(D)$, is the subgroup of S_n given by

$$R(D) = \{\rho : (iD = (j,k) \Rightarrow i\rho D = (j,\ell) \text{ for some } \ell \in Z_n)\}$$

Intuitively $C(D)$ permutes the entries of D within each column and $R(D)$ permutes them within each row.

(v) Definition: If D is a diagram of length n then $\mathcal{E}(D)$ is the element of ZS_n given by

$$\mathcal{E}(D) = \sum_{\substack{\sigma \in C(D) \\ \rho \in R(D)}} \mathcal{E}(\sigma)\sigma\rho \quad \text{where} \quad \mathcal{E}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

(vi) Definition: A standard diagram is a diagram, D, such that

if $i_1D = (j_1,k_1)$, $i_2D = (j_2,k_2)$, $j_1 \le j_2$ and $k_1 \le k_2$, then $i_1 \le i_2$.

I will denote the set of standard diagrams associated with a partition λ, by Δ_λ

(vii) I will use the usual notation for writing permutations in cyclic form, that is, $(x_1\ x_2\ x_3\ \ldots\ x_n)$ will denote the permutation

$$x_i \mapsto x_{i+1} \quad \forall\ i \in Z_{n-1}$$

$$x_n \mapsto x_1$$

<u>2.2.2 Theorem</u>: (i) (Theorem 28.15, p.197 of [4].) Let D be a diagram of length c. Then $\mathcal{E}(D)$ is a scalar multiple of a primitive idempotent of QS_c and $\mathcal{E}(D)QS_c$ is a minimal right ideal of QS_c.

Let D and D' be two diagrams of length c associated with the partitions λ_1 and λ_2 say. Then

$$\mathcal{E}(D)QS_c \cong \mathcal{E}(D')QS_c \quad \text{if and only if} \quad \lambda_1 = \lambda_2.$$

In fact if L is a minimal right ideal of QS_c then there exists a unique partition λ of c such that $L \cong \mathcal{E}(D)QS_c$ for any diagram D associated with λ. Thus each isomorphism class of minimal right ideals is associated with a partition of c and vice versa. If λ is a partition of c I will use QS_c^λ to denote the minimal two sided ideal of QS_c generated by those minimal left ideals associated with λ. right

(ii) (Theorem 4.5 p.114 of [1].)

$$QS_c^\lambda = \sum_{D \in \Delta_\lambda} \mathcal{E}(D)QS_c$$

and the sum is direct.

<u>2.2.3 Corollary</u>: Let D be a diagram of length c. Then $QS_c\mathcal{E}(D)$ is a minimal left ideal of QS_c. If D and D' are two diagrams of length c then $QS_c\mathcal{E}(D)$ is isomorphic to $QS_c\mathcal{E}(D')$ if and only if D and D' are associated with the same partition.

Furthermore if λ is a partition of c then, using the notation of 2.2.2,

$$QS_c^\lambda = \sum_{D\varepsilon\Delta_\lambda} QS_c\mathcal{E}(D)$$

and the sum is direct.

<u>Proof</u>: From 2.2.2(ii), and the fact that a semi-simple ring is the direct sum of its simple components, we know that

$$QS_c = \sum_\lambda \sum_{D\varepsilon\Delta_\lambda} \mathcal{E}(D)QS_c$$

where λ ranges over all partitions of c, and that the sum is direct. It follows that if D and D' are two different standard diagrams then

$$\mathcal{E}(D)\mathcal{E}(D') = 0,$$

and that

$$\sum_\lambda \sum_{D\varepsilon\Delta_\lambda} \mathcal{E}(D) \varepsilon Q$$

This means that $QS_c = \sum_\lambda \sum_{D\varepsilon\Delta_\lambda} QS_c\mathcal{E}(D)$

and the sum is direct.

But QS_c is the direct sum of its simple components and it is obvious that $\mathcal{E}(D) \in QS_c^{\lambda}$ if and only if D is associated with λ.

Hence

$$QS_c^{\lambda} = \sum_{D \in \Delta_\lambda} QS_c\mathcal{E}(D)$$

and the sum is direct. The remainder of the corollary follows immediately.

<u>2.2.4 Corollary</u>: Let V be a Q-space of dimension n, let λ be a partition of c into m parts and λ let D be a diagram associated with λ. Then $V^{[c]}\mathcal{E}(D)$ is trivial if and only if $m > n$. It follows that if $n \geq c$ then $V^{[c]}$ is a faithful QS_c-module.

<u>Proof</u>: Let $v_1, v_2, \ldots, v_n$ be a basis for V.

(i) Suppose $m > n$. Let $\mu_1, \mu_2, \ldots, \mu_m$ be the entries in the first column of D. That is set

$$\mu_i = ~~(1,i)~~ (i,1)D^{-1} \text{ for each } i \in Z_m .$$

Now suppose a is a basis element of $V^{[c]}$, that is

$$a = v_{\lambda_1} \otimes v_{\lambda_2} \otimes \ldots \otimes v_{\lambda_c} \quad \text{where } \lambda_i \in Z_n \ \forall i \in Z_c .$$

Since $m > n$ there must exist $j,k \in Z_m$ such that $j \neq k$ and $\lambda_{\mu_i} = \lambda_{\mu_j}$.

Thus $(\mu_i\ \mu_j) \in C(D)$ and $a(\mu_i\ \mu_j) = a$.

Let T be a right transversal of

$\langle 1, (\mu_i\ \mu_j)\rangle$ in $C(D)$, then

$$a\mathcal{E}(D) = a(1-(\mu_i\ \mu_j)) \sum_{\substack{\tau\varepsilon T \\ \rho\varepsilon R}} \mathcal{E}(\tau)\tau\rho$$

$$= 0\ .$$

Since a was an arbitary basis element of $V^{[c]}$ it follows that

$$V^{[c]}\mathcal{E}(D) = 0.$$

(ii) Suppose $m \leq n$. Suppose $iD = (\mu_i, \upsilon_i)$ for each i in Z_c and set

$$a = v_{\mu_1} \otimes v_{\mu_2} \otimes \ldots \otimes v_{\mu_c}\ ,$$

that is the index of the ith factor of a is the row of D in which the entry i occurs.

Now suppose $\sigma \in C(D)$ and $\rho \in R(D)$ are such that

$$a\sigma\rho = a \qquad (1)$$

Then $\mu_{i\rho^{-1}\sigma^{-1}} = \mu_i\ \forall\ i \in Z_c$.

But, clearly, $\mu_{i\rho} = \mu_i\ \forall\ i \in Z_c$ since $\rho \in R(D)$, so $\mu_i = \mu_{i\rho} = \mu_{i\rho\rho^{-1}\sigma^{-1}} = \mu_{i\sigma^{-1}}$ $\forall\, i \in Z_c$

But $iD = (\mu_i, \upsilon_i)$ $\forall\, i \in Z_c$, which means $i\sigma^{-1}D = (\mu_{i\sigma^{-1}}, \nu_i)$, since $\sigma \in C(D)$

$= (\mu_i, \nu_i)$.

Hence $\sigma = 1$ in any solution of (1), and it is clear that (1) is true if $\sigma = 1$ and $\rho \in R(D)$.

Thus $$a\mathscr{E}(D) = |R(D)|a + \sum_{\substack{\sigma \in C(D) \setminus \{1\} \\ \rho \in R(D)}} a\sigma\rho,$$

and each of the terms in the second part of the right hand side is a basis element of $V^{[c]}$ different from a.

It follows that

$$V^{[c]}\mathscr{E}(D) \ni a\mathscr{E}(D) \neq 0$$

and the corollary is proved.

<u>2.3 The Isolated Fully Invariant subgroups of</u> $F_n(\underline{\underline{N}}_c)$ <u>in</u> $\gamma_c(F_n(\underline{\underline{N}}_c))$.

<u>2.3.1 Lemma</u>: If U and V are free Z-modules, freely generated by $\{u_i : i \in Z_n\}$ and $\{v_i : i \in Z_m\}$ respectively then $U \otimes_Z V$ is a free Z-module, freely generated by $\{u_i \otimes v_j : i \in Z_n, j \in Z_m\}$.

<u>Proof</u>: Every free Z-module of rank n is a free abelian group of rank n. It follows that U is isomorphic to a direct sum of n copies of Z. Similarly V is isomorphic to a direct sum of m copies of Z.

Curtis and Reiner ([4], Theorem (12.12), p.64) show that

$$(M_1 \oplus M_2) \otimes_R N \cong M_1 \otimes_R N \dot{+} M_2 \otimes_R N$$

for any ring R and the proof given is readily converted to show that

$$M \otimes_R (N_1 \oplus N_2) \cong M \otimes_R N_1 \dot{+} M \otimes_R N_2 .$$

An obvious induction then shows that $U \otimes_Z V$ is isomorphic to the direct sum of nm copies of $Z \otimes_Z Z$ which is simply Z. Thus $U \otimes_Z V$ has rank nm and since $\{u_i \otimes v_j : i \in Z_n, j \in Z_m\}$ is clearly a generating set of $U \otimes V$ it must be a free generating set.

<u>2.3.2 Corollary:</u> If U is a free Z-module, freely generated by $\{u_i : i \in Z_n\}$ then $U^{[c]}$ is a free Z-module freely generated by $\{u_{\lambda_1} \otimes u_{\lambda_2} \otimes \ldots \otimes u_{\lambda_c} : \lambda_i \in Z_n \ \forall\, i \in Z_c\}$.

<u>2.3.3 Lemma:</u> Let U be a free Z-module, freely generated by $\{u_i : i \in Z_n\}$, let K be the Z-homomorphism: $U^{[c]} \to \gamma_c(F_n(\underset{=}{N}_c))$ given by

$$u_{\lambda_1} \otimes u_{\lambda_2} \otimes \ldots \otimes u_{\lambda_c} \mapsto [x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_c}]$$

and let L be an $End_Z(U)$-submodule of $U^{[c]}$.

Then LK is a fully invariant subgroup of $F_n(\underset{=}{N}_c)$ in $\gamma_c(F_n(\underset{=}{N}_c))$. All such subgroups are obtained in this way. Any isolated fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$ in $\gamma_c(F_n(\underline{\underline{N}}_c))$ can be written as LK where L is an isolated ~~sub~~ $End_Z(U)$-module of $U^{[c]}$.

Proof: We first note that the definition of K is possible by virtue of ~~2.2.2~~ 2.32. Moreover, since any element of $\gamma_c(F_n(\underline{\underline{N}}_c))$ can be written as a product of left normed commutators, K is an epimorphism.

Define a mapping E from $End(F_n(\underline{\underline{N}}_c))$ to $End_Z(U)$ as follows:

Let ψ be an endomorphism of $F_n(\underline{\underline{N}}_c)$. For each $i \in Z_n$ choose

$n_i, \lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,n_i}, \mathcal{E}_{i,1}, \ldots, \mathcal{E}_{i,n_i}$ such that

$$x_i\psi = \prod_{j=1}^{n_i} x_{\lambda_{i,j}}^{\mathcal{E}_{i,j}}$$

and set

$$u_i\psi E = \sum_{j=1}^{n_i} \mathcal{E}_{i,j}\, u_{\lambda_{i,j}}$$

(This is possible and is sufficient to define ψE because U is freely generated by the u_i.)

Now suppose $\psi \in End(F_n(\underline{\underline{N}}_c))$. Choose $n_i, \lambda_{i,j}, \mathcal{E}_{i,j}$ as above for each $i \in Z_n$, $j \in Z_{n_i}$. Then $((u_{\mu_1} \otimes u_{\mu_2} \otimes \ldots \otimes u_{\mu_c})\psi E)K$

$$= ((\sum_{j=1}^{n_{\mu_1}} \mathcal{E}_{\mu_1,j} u_{\lambda_{\mu_1,j}}) \otimes (\sum_{j=1}^{n_{\mu_2}} \mathcal{E}_{\mu_2,j} u_{\lambda_{\mu_2,j}}) \otimes \ldots \otimes (\sum_{j=1}^{n_{\mu_c}} \mathcal{E}_{\mu_c,j} u_{\lambda_{\mu_c,j}}))K$$

$$= \sum_{j_1=1}^{n_{\mu_1}} \sum_{j_2=1}^{n_{\mu_2}} \cdots \sum_{j_c=1}^{n_{\mu_c}} (\mathcal{E}_{\mu_1,j_1} \mathcal{E}_{\mu_2,j_2} \cdots \mathcal{E}_{\mu_c,j_c} (u_{\lambda_{\mu_1,j_1}} \otimes \cdots \otimes u_{\lambda_{\mu_c,j_c}}))K$$

$$= \prod_{j_1=1}^{n_{\mu_1}} \prod_{j_2=1}^{n_{\mu_2}} \cdots \prod_{j_c=1}^{n_{\mu_c}} [x_{\lambda_{\mu_1,j_1}}, x_{\lambda_{\mu_2,j_2}}, \ldots, x_{\lambda_{\mu_c,j_c}}]^{\mathcal{E}_{\mu_1,j_1}\mathcal{E}_{\mu_2,j_2}\cdots\mathcal{E}_{\mu_c,j_c}}$$

$$= [\prod_{j=1}^{n_{\mu_1}} x_{\lambda_{\mu_1,j}}^{\mathcal{E}_{\mu_1,j}}, \prod_{j=1}^{n_{\mu_2}} x_{\lambda_{\mu_2,j}}^{\mathcal{E}_{\mu_2,j}}, \ldots, \prod_{j=1}^{n_{\mu_c}} x_{\lambda_{\mu_c,j}}^{\mathcal{E}_{\mu_c,j}}]$$

$$= [x_{\mu_1}\psi, x_{\mu_2}\psi, \ldots, x_{\mu_c}\psi]$$

$$= (u_{\mu_1} \otimes u_{\mu_2} \otimes \ldots \otimes u_{\mu_c})K\psi.$$

Hence $(\psi E)K = K\psi$ for all $\psi \in End(F_n(\underline{N}_c))$.

Now define a mapping D from $End_Z(U)$ to $End(F_n(\underline{N}_c))$ as follows:

Suppose $\psi \in End_Z(U)$ then there exist $\alpha_{i,j} \in Z$ for all $i,j \in Z_n$ such that

$$u_i\psi = \sum_{i=1}^{n} \alpha_{i,j} u_j,$$

and we define ψD by

$$x_i(\psi D) = \prod_{i=1}^{n} x_j^{\alpha_{i,j}}.$$

A similar proof to that given above shows that

$$a\psi K = aK(\psi D), \quad \forall a \in U^{[c]} \quad \text{and} \quad \forall \psi \in End_Z(U).$$

Let L be an $End_Z(U)$-submodule of $U^{[c]}$ and let ψ be an endomorphism of $F(\underline{\underline{N}}_c)$.

Then $LK\psi = L(\psi E)K$

$\subseteq LK.$

Thus LK is fully invariant in $F_n(\underline{\underline{N}}_c)$ and it is clearly a subgroup of $\gamma_c(F_n(\underline{\underline{N}}_c))$.

Suppose M is a fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$ contained in $\gamma_c(F_n(\underline{\underline{N}}_c))$ and that ψ is a Z-endomorphism of U.

Then $MK^{-1}\psi = MK^{-1}\psi KK^{-1}$

$= MK^{-1}K(\psi D)K^{-1}$

$\subseteq MK^{-1}.$

Suppose a and $b \in MK^{-1}$. Then there exist α and $\beta \in M$ such that $aK = \alpha$ and $bK = \beta$. Since M is a subgroup $\alpha\beta^{-1} \in M$ and it follows that $\alpha - \beta \in MK^{-1}$. Thus MK^{-1} is a submodule of $U^{[c]}$.

Since $M = MK^{-1}K$ it follows that every fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$ contained in $\gamma_c(F_n(\underline{\underline{N}}_c))$ can be written as LK where L is some $End_Z(U)$-submodule of $\gamma_c(F_n(\underline{\underline{N}}_c))$.

Finally suppose that M is an isolated fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$ in $\gamma_c(F_n(\underline{\underline{N}}_c))$. Suppose there exist $a \in U^{[c]}$ and $n \in Z\setminus\{0\}$ such that

$$na \in MK^{-1}.$$

Let $aK = \alpha \in \gamma_c F_n(\underset{=}{N}_c)$.

Then $(na)K = n\alpha \in M$.

It follows that $\alpha \in M$ and hence that $a \in MK^{-1}$.

So MK^{-1} is isolated.

<u>2.3.4 Definition:</u> (i) $\Gamma_c(\underset{\sim}{A}_n)$ is the subalgebra of $\underset{\sim}{A}_n$ generated by the left normed commutators of weight c, their inverses and the operations of multiplication and identity.

(ii) An operation: $\Gamma_c(\underset{\sim}{A}_n) \times ZS_c \to \Gamma_c(\underset{\sim}{A}_n)$ is defined as follows:

Choose some ordering of S_c, in fact set $S_c = \{\sigma_i : i \in Z_{c!}\}$.

Each element of ZS_c can then be written uniquely in the form

$$\sum_{i=1}^{c!} \alpha_i \sigma_i \quad \text{where the } \alpha_i \in Z.$$

Define $\underset{\sim}{1}\alpha = \underset{\sim}{1}$, $\forall \alpha \in ZS_c$,

$$[\underset{\sim}{x}_{\lambda_1}, \underset{\sim}{x}_{\lambda_2}, \ldots, \underset{\sim}{x}_{\lambda_c}] \sum_{i=1}^{c!} \alpha_i \sigma_i = \prod_{i=1}^{c!} [\underset{\sim}{x}_{\lambda_{1\sigma_i^{-1}}}, \underset{\sim}{x}_{\lambda_{2\sigma_i^{-1}}}, \ldots, \underset{\sim}{x}_{\lambda_{c\sigma_i^{-1}}}]^{\alpha_i},$$

$$a^{-1}\alpha = a(-\alpha), \ \forall a \in \underset{\sim}{A}, \ \alpha \in ZS_c,$$

and

$$(ab)\alpha = a\alpha(b\alpha), \ \forall a,b \in \underset{\sim}{A}, \ \alpha \in ZS_c.$$

<u>Note</u>: This operation is not independent of the order chosen for S_c and is not a module action. However

<u>2.3.5 Theorem</u>: $\gamma_c(F_n(\underset{=}{N}_c))$ can be written as a direct product of minimal isolated fully invariant subgroups of $F_n(\underset{=}{N}_c)$.

If e is a primitive idempotent of QS_c and m a nonzero integer such that $me \in ZS_c$ then the isolated subgroup of $F_n(\underset{=}{N}_c)$ generated by $\Gamma_c(\underset{\sim}{A}_n)(me)\rho_c$ is either zero or a minimal isolated fully invariant subgroup of $F_n(\underset{=}{N}_c)$ contained in $\gamma_c(F_n(\underset{=}{N}_c))$. All such subgroups are obtained in this way.

<u>Proof</u>: Let U be a free Z-module freely generated by $u_1, u_2, \ldots, u_n$ and let P be the mapping from $\Gamma_c(\underset{\sim}{A}_n)$ to $U^{[c]}$ given by

$$[x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_c}]P = u_{\lambda_1} \otimes u_{\lambda_2} \otimes \ldots \otimes u_{\lambda_c},$$

$$a^{-1}P = -aP,$$

and $$abP = aP + bP.$$

Clearly $a\alpha P = aP\alpha$, $\alpha \in ZS_c$, and if K is defined as in Lemma 2.3.3,

$$PK = \rho_c\Big|_{\Gamma_c(\underset{\sim}{A}_n)}$$

Moreover P is a surjection.

It follows, from Lemma 2.1.8, that if e is a primitive idempotent of $\mathbb{Q}S_c$ then the isolated Z-submodule of $U^{[c]}$ generated by $\Gamma_c(\underset{\sim}{A}_n)(me)P$ is either zero or a minimal isolated $\mathrm{End}_Z(U)$-submodule of $U^{[c]}$.

Denote this submodule by L. LK is clearly a fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$ and is contained in $\gamma_c F_n(\underline{\underline{N}}_c)$. The isolated subgroup generated by LK is simply

$$M = \{a : a \in \gamma_c F_n(\underline{\underline{N}}_c) \text{ and } \exists$$
$$k \in Z \setminus \{0\} \text{ such that } a^k \in LK\}.$$

It is easily seen that M is fully invariant in $F_n(\underline{\underline{N}}_c)$.

Suppose M is not a minimal isolated fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$. Suppose, in fact, that M' is a non trivial isolated subgroup of $F_n(\underline{\underline{N}}_c)$ such that $M' \subset M$.

Clearly $M' \cap LK$ cannot be trivial since some power of every element of M must lie in LK. Hence $M'K^{-1} \cap L$ cannot be trivial. But L and $M'K^{-1}$ are both isolated $\mathrm{End}_Z(U)$-submodules of $U^{[c]}$ and it follows that $L \cap M'K^{-1}$ must be another. But L is minimal so we have a contradiction. Hence M must be either zero or a minimal isolated fully invariant subgroup of $F_n(\underline{\underline{N}}_c)$.

But

$$\Gamma_c(\underset{\sim}{A}_n)(me)\underset{\sim}{\rho}_c \subseteq LK \subseteq M$$

and it follows that the isolated subgroup generated by $\Gamma_c(\underset{\sim}{A}_n)(me)\underset{\sim}{\rho}_c$ is either zero or a minimal isolated fully invariant subgroup of $F_n(\underset{=}{N}_c)$.

For each diagram D of length c let M(D) be the isolated subgroup of $F_n(\underset{=}{N}_c)$ generated by $\Gamma_c \underset{\sim}{A}_n \mathcal{S}(D) \underset{\sim}{\rho}_c$. Then each M(D) is either a minimal isolated fully invariant subgroup or zero. Furthermore $\gamma_c(F_n(\underset{=}{N}_c))$ is spanned by

$$\{M(D) : D \text{ is a standard diagram of length } c\},$$

and it follows that a subset of standard diagrams must provide a direct product decomposition of $\gamma_c(F_n(\underset{=}{N}_c))$.

<u>2.3.6 Theorem</u>: If c is an integer greater than 2 then the maximal torsion free varieties between $\underset{=}{N}_c$ and $\underset{=}{N}_{c-1}$ are as follows.

Let e be a primitive idempotent of QS_c and m a non-zero integer such that $me \in ZS_c$. Let U be the isolated subgroup of $\gamma_c(F_c(\underset{=}{N}_c))$ generated by $\Gamma_c(\underset{\sim}{A}_c)(me)\underset{\sim}{\rho}_c$. Then $\mathrm{Var}(F_c(\underset{=}{N}_c)/U)$ is a maximal torsion free variety between $\underset{=}{N}_c$ and $\underset{=}{N}_{c-1}$. Its laws are generated by $[x_1,x_2,\ldots,x_{c+1}]$ and $\Gamma_c(\underset{\sim}{A}_c)(me)\underset{\sim}{\rho}_c$. All such varieties are obtained in this way.

<u>Proof</u>: This theorem follows immediately from 2.3.5 and the fact that a variety that is nilpotent of class $c \geqslant 2$ is generated by its c-generator groups, and by its free group of rank c (Neumann [11], 35.12 page 100).

2.3.7: The remainder of this thesis deals mainly with varieties of the form $\mathrm{Var}(F_n(\mathfrak{N}_c))$ where $n < c$. These do not always lie between $\mathfrak{N}_c$ and $\mathfrak{N}_{c-1}$. However $(\mathrm{Var}(F_n(\mathfrak{N}_c))) \vee \mathfrak{N}_{c-1}$ certainly does and is therefore susceptible the machinery we have developed above. The problem is to find which of the minimal isolated fully invariant subgroups of $F_c(\mathfrak{N}_c)$ in $\gamma_c(F_c(\mathfrak{N}_c))$ intersect $F_n(\mathfrak{N}_c)$ trivially. An equivalent problem is to find which of the idempotents, e, of QS_c satisfy the conditions

$$\Gamma_c(\mathbf{A}_n)(me)\rho_c = 1$$

and

$$\Gamma_c\mathbf{A}_c(me)\rho_c \neq 1,$$

where m is, as usual, a non-zero integer such that $me \in ZS_c$.

If e is such an idempotent it is obvious that $\Gamma_c(\mathbf{A}_c)(me)\rho$ is not in the kernel of the natural homomorphism from F_c to $F_c(\mathfrak{N}_c)$ but is in the kernel of every homomorphism from F_c to $F_n(\mathfrak{N}_c)$, in other words that it is a closed set of laws distinguishing $\mathrm{Var}(F_n(\mathfrak{N}_c))$ from $\mathfrak{N}_c$.

2.3.8 Lemma: Let e be a scalar multiple of a primitive idempotent of QS_c, associated with a partition of c into m parts and suppose k is a non-zero integer such that $ke \in ZS_c$ and that $n \in Z_c$. Then $\Gamma_c(\mathbf{A}_n)(ke)\rho_c$ is trivial if and only if either $\Gamma_c(\mathbf{A}_c)(ke)\rho_c$ is trivial or $m < n$.

Proof: Let U be a free Z-module with basis $u_1, u_2, \ldots, u_c$. Let U_n be the submodule of U generated by $u_1, u_2, \ldots, u_n$. Let P and K be the natural homomorphisms from $\Gamma_c(\underset{\sim}{A}_c)$ to $U^{[c]}$ and from $U^{[c]}$ to $\gamma_c(F_n(\underset{=}{N}_c))$ respectively. (P is defined in 2.3.5, page 38, K is defined in 2.3.3.)

Then $$\Gamma_c(\underset{\sim}{A}_n)(ke)\underset{\sim}{\rho}_c = \Gamma_c(\underset{\sim}{A}_n)(ke)PK$$
$$= \Gamma_c(\underset{\sim}{A}_n)P(ke)K$$
$$= U_n^{[c]}(ke)K\ .$$

Suppose $\Gamma_c(\underset{\sim}{A}_n)(ke)\underset{\sim}{\rho}_c$ is trivial. Then $U_n^{[c]}ke$ must be in the kernel of K. Since K maps $U^{[c]}$ to a torsion free Z-module its kernel must be isolated, and, since K is an $\mathrm{End}_Z(U)$-homomorphism its kernel must be an $\mathrm{End}_Z(U)$-submodule of $U^{[c]}$. But the isolated $\mathrm{End}_Z(U)$-submodule generated by $U^{[c]}ke$ is minimal and it follows that either $U^{[c]}(ke)K = 1$ or $U^{[c]}(ke) \cap \mathrm{Ker}(K)$ is trivial. But $U_n^{[c]}(ke)$ must be in this intersection so either

$$\Gamma_c(\underset{\sim}{A}_c)(ke)\underset{\sim}{\rho}_c = U^{[c]}(ke)K = 1$$

or

$$(Q \otimes_Z U_n)^{[c]}e = U_n^{[c]}(ke) = 0.$$

By Lemma 2.2.4 the second possibility yeilds $n > m$ and we have proved the necessity of the conditions stated. Their sufficiency is obvious.

<u>2.3.9 Theorem</u>: Let n and c be positive integers, $n \leq c$. Then the laws of $(\mathrm{Var}F_n(\underset{=}{N}_c)) \vee \underset{=}{N}_{c-1}$ are generated by

$$[x_1, x_2, \ldots, x_{c+1}]$$

and

$$\bigcup_{i=n+1}^{c} \bigcup_{D \in \Delta_i} \Gamma_c(\underset{\sim}{A}_n)\mathcal{E}(D)\underset{\sim}{\rho}$$

where Δ_i is the set of standard diagrams associated with partitions of c into i parts.

<u>Proof</u>: Denote $(\mathrm{Var}F_n(\underset{=}{N}_c)) \vee \underset{=}{N}_{c-1}$ by $\underset{=}{V}$. We know that $\underset{=}{N}_{c-1} \subseteq \underset{=}{V} \subseteq \underset{=}{N}_c$ and that $\underset{=}{V}$ is torsion free. It follows from Theorem 2.3.6 that the laws of $\underset{=}{V}$ are generated by $[x_1, x_2, \ldots, x_{c+1}]$ and the isolated subgroup of F_c generated by

$$\bigcup_{e \in I} \Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho} ,$$

where I is a set of elements of ZS_c which are scalar multiples of primitive idempotents of QS_c.

Let θ be the natural homomorphism from F_c to $F_c(\underset{=}{N}_c)$, so that $\underset{\sim}{\rho}\theta\big|_{\underset{\sim}{A}_c} = \underset{\sim}{\rho}_c\big|_{\underset{\sim}{A}_c}$, clearly any homomorphism from F_c to $F_n(\underset{=}{N}_c)$ can be factored through θ. It follows that, if $e \in I$, $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}$ is a set of laws in $F_n(\underset{=}{N}_c)$ if and only if $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}_c$ lies in the kernel of all homomorphisms from $F_c(\underset{=}{N}_c)$ to $F_n(\underset{=}{N}_c)$. But $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}_c$ is invariant under all endomorphisms of $F_c(\underset{=}{N}_c)$ so this last condition is equivalent to the condition that $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}_c \cap F_n(\underline{\underline{N}}_c) = 1$.

Now $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}_c \cap F_n(\underline{\underline{N}}_c) = \Gamma_c(\underset{\sim}{A}_n)e\underset{\sim}{\rho}_c$ so, using Lemma 2.3.8 we find that, if $e \in I$, $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}$ is a set of laws in $F_n(\underline{\underline{N}}_c)$ if and only if either e is a scalar multiple of a primitive idempotent of $\underset{\sim}{Q}S_c$ associated with a partition of c into more than n parts or $\Gamma_c(\underset{\sim}{A}_c)e\underset{\sim}{\rho}$ is a set of laws of $F_c(\underline{\underline{N}}_c)$.

The laws of $F_c(\underline{\underline{N}}_c)$ are generated by $[x_1,x_2,\ldots,x_{c+1}]$ so we need only consider the first alternative.

Since $\underline{\underline{V}}$ is torsion free the isolated closure of a set of laws of $\underline{\underline{V}}$ is a set of laws of $\underline{\underline{V}}$ and the required result follows from Lemma 2.2.3.

2.4 The Torsion Free Varieties between $\underline{\underline{N}}_4$ and $\underline{\underline{N}}_3$.

The theory developed in this Chapter can be applied to find the torsion free varieties between $\underline{\underline{N}}_4$ and $\underline{\underline{N}}_3$. This requires a considerable amount of tedious calculation which is omitted here. The partitions of 4 are $\lambda_1 = (1,1,1,1)$, $\lambda_2 = (2,1,1)$, $\lambda_3 = (2,2)$ $\lambda_4 = (3,1)$ $\lambda_5 = (4)$. It is fairly obvious $\Gamma_4(\underset{\sim}{A}_4)\mathcal{E}(D)\underset{\sim}{\rho}_4$ is trivial whenever D is a standard diagram associated with λ_1 or λ_5. In fact it is also trivial if D is associated with λ_3. The laws generated by idempotents associated with λ_2 are, by virtue of Theorem 2.3.9, the laws which distinguish $F_2(\underline{\underline{N}}_4) \vee \underline{\underline{N}}_3$ from $\underline{\underline{N}}_4$. $\mathrm{Var}F_2(\underline{\underline{N}}_4)$ is in fact the variety of metabelian groups of class 4 and its laws are generated by

$$[[x_1,x_2],[x_3,x_4]] \quad \text{and} \quad [x_1,x_2,x_3,x_4,x_5].$$

The first of these laws is a consequence of the second and $\Gamma_4(\mathbf{A}_4)\mathcal{E}(D_1)\boldsymbol{\rho}$, where

$$D_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} .$$

It follows that there are no torsion free varieties between the variety of metabelian groups of class 4 and $\mathfrak{N}_4$. It also follows that if D is any other primitive idempotent of QS_4 associated with λ_2, then the isolated closure of $\Gamma_4(\mathbf{A}_4)\mathcal{E}(D)\boldsymbol{\rho}_4$ must be either 1 or the isolated closure of

$$\Gamma_4(\mathbf{A}_4)\mathcal{E}(D_1)\boldsymbol{\rho}_4 .$$

This leaves only λ_4 to be considered. Put

$$D_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Then the isolated closure of $\Gamma_4(\mathbf{A}_4)\mathcal{E}(D_2)\boldsymbol{\rho}_4$ is a fully invariant subgroup of rank 45. It is equal to the fully invariant subgroup generated by $[x_2,x_1,x_1,x_2]$. All the other standard diagrams associated with D_2 give rise to trivial subgroups.

Thus there are precisely two torsion free varieties between $\mathfrak{N}_4$ and $\mathfrak{N}_3$, the variety of metabelian groups of class 4 and the variety whose laws are generated by

$$[x_2,x_1,x_1,x_2]$$

and

$$[x_1,x_2,x_3,x_4,x_5].$$

3. Some Laws of Low Weight in $F_n(\underset{=}{N}_c)$

3.0. Introduction: In this chapter I deal with some of the laws of weight w of $F_n(\underset{=}{N}_c)$ where $w \leq c$. It relies heavily on the notation developed in Chapter 1.

In Chapter 2 I showed that if D is a diagram corresponding to a partition of c into more than n parts then $\Gamma_c(\underset{\sim}{A}_c)\mathcal{E}(D)\underset{\sim}{\rho}$ is a set of laws of $F_n(\underset{=}{N}_c)$. A slightly different approach to this result gives some insight into the way in which it can be extended.

Suppose then that D is a diagram of length c associated with a partition of c into m parts. Let I be the set of entries in the first column of D and let G be the symmetric group on I regarded as a subgroup of S_c. Clearly G is a subgroup of $C(D)$, in fact a direct factor.

First we note that $\Gamma_c(\underset{\sim}{A}_c)\mathcal{E}(D)\underset{\sim}{\rho}$ will be a set of laws in $F_n(\underset{=}{N}_c)$ if and only if $[\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_c]\mathcal{E}(D)\underset{\sim}{\rho}$ is a law in $F_n(\underset{=}{N}_c)$ and this second condition is equivalent to the condition that

$$[\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_c]\mathcal{E}(D)\theta \sim_c \underset{\sim}{1}$$

for any homomorphism $\theta : \underset{\sim}{A}_c \rightarrow \underset{\sim}{A}_n$. In fact it is possible to restrict θ by requiring the image of each of the generators $\underset{\sim}{x}_1 : i \in Z_c$ to be a product of generators and their inverses. The function ht (height) can be defined on such homomorphisms by setting

$$\mathrm{ht}(\theta) = \sum_{i=1}^{c} \mathrm{ht}(\underset{\sim}{x}_i\theta).$$

If $ht(\theta) = c$ it is obvious that θ must map generators to generators and it is obvious that, if $m > n$, I must contain two different integers i and j such that $x_i\theta = x_j\theta$. Since the permutation $(i\ j)$ is odd and in $G \leq C(D)$ it follows that

$$\prod_{\sigma \varepsilon C(D)} ([x_1\theta, x_2\theta, \ldots, x_c\theta]\sigma)^{\mathcal{E}(\sigma)} \sim_c 1$$

If, on the other hand, $ht(\theta) > c$ there must exist an $i \varepsilon Z_c$ such that $x_i\theta = ab$ for some $a,b \varepsilon A_n$ or $x_i\theta = x_j^{-1}$ for some $j \varepsilon Z_c$. I will here deal only with the first possibility, the second is similar. Define two homomorphisms, $\theta_1, \theta_2 : A_c \to A_n$ by

$$x_i\theta_1 = a, \quad x_i\theta_2 = b,$$
$$x_j\theta_1 = x_j\theta_2 = x_j\theta, \quad \forall j \varepsilon Z_c \setminus \{i\}.$$

Clearly $ht(\theta_1)$ and $ht(\theta_2)$ are both less than $ht(\theta)$ and

$$\prod_{\sigma \varepsilon C(D)} ([x_1, x_2, \ldots, x_c]\sigma)^{\mathcal{E}(\sigma)}\theta$$

$$\sim_c \prod_{\sigma \varepsilon C(D)} ([x_1, x_2, \ldots, x_c]\sigma)^{\mathcal{E}(\sigma)}\theta_1 \prod_{\sigma \varepsilon C(D)} ([x_1, x_2, \ldots, x_c]\sigma)^{\mathcal{E}(\sigma)}\theta_2 .$$

Thus we have a basis on which we could build an inductive proof that

$$\prod_{\sigma \varepsilon C(D)} ([x_1, x_2, \ldots, x_c]\sigma^{\mathcal{E}\sigma}\theta \sim_c 1$$

for any homomorphism θ from A_c to A_n and it is not difficult to get from here to the result we set out to prove.

Now suppose that c and w are two positive integers such that $c \geq w > c/2$ and suppose that $n \in Z_c$. Let I be a subset of Z_w with more than n elements. Finally let G be the group of permutations of I, regarded as a subgroup of S_w.

As before it is obvious that, if θ is a homomorphism: $\underset{\sim}{A}_w \to \underset{\sim}{A}_n$ of height w then

$$\prod_{\sigma \in G} ([\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_w]\sigma)^{\mathcal{E}(\sigma)} \theta \underset{c}{\sim} \underset{\sim}{1}.$$

(Note that the assumption $w > c/2$ ensures that expressions of weight w commute modulo $\underset{c}{\sim}$.)

However when we come to consider homomorphisms of height greater than w the situation becomes more complicated. Suppose θ is such a homomorphism and that $\underset{\sim}{x}_i \theta = ab$ for some $i \in Z_w$. Suppose $I' = I \setminus \{i\}$, G' is the group of permutations of I' and that T is a transversal of G′ in G. Then

$$[\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_w]\theta$$

$$= [\underset{\sim}{x}_1\theta, \underset{\sim}{x}_2\theta, \ldots, \underset{\sim}{x}_{i-1}\theta, ab, \underset{\sim}{x}_{i+1}\theta, \ldots, \underset{\sim}{x}_w\theta]$$

$$\underset{c}{\sim} [\underset{\sim}{x}_1\theta, \underset{\sim}{x}_2\theta, \ldots, \underset{\sim}{x}_{i-1}\theta, a, \underset{\sim}{x}_{i+1}\theta, \ldots \underset{\sim}{x}_w\theta]$$

$$[\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{i-1}, b, \underset{\sim}{x}_{i+1}, \ldots, \underset{\sim}{x}_w]$$

$$[\underset{\sim}{x}_1\theta, \underset{\sim}{x}_2\theta, \ldots \underset{\sim}{x}_{i-1}\theta, a, b, \underset{\sim}{x}_{i+1}\theta, \ldots, \underset{\sim}{x}_w\theta]\alpha,$$

where α is an expression of weight greater than $w + 1$. Thus, if we define θ_1 and θ_2 as before we have

$$\prod_{\sigma\varepsilon G} ([\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots \underset{\sim}{x}_w]\sigma)^{\mathcal{E}(\sigma)}\theta$$

$$\underset{c}{\sim} \prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} ([\underset{\sim}{x}_{1\tau^{-1}},\underset{\sim}{x}_{2\tau^{-1}},\ldots,\underset{\sim}{x}_{(i\tau-1)\tau^{-1}},ab,\underset{\sim}{x}_{(i\tau+1)\tau^{-1}},\ldots,\underset{\sim}{x}_{w\tau^{-1}}]\theta\sigma)^{\mathcal{E}(\sigma\tau)}$$

$$\underset{c}{\sim} \prod_{\sigma\varepsilon G} ([\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_w]\sigma)^{\mathcal{E}(\sigma)}\theta_1 \prod_{\sigma\varepsilon G} ([\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_w]\sigma)^{\mathcal{E}(\sigma)}\theta_2$$

$$\prod_{\tau\varepsilon T}\Big(\prod_{\sigma\varepsilon G'} ([\underset{\sim}{x}_{1\tau^{-1}},\underset{\sim}{x}_{2\tau^{-1}},\ldots,\underset{\sim}{x}_{(i\tau-1)\tau^{-1}},a,$$

$$b,\underset{\sim}{x}_{(i\tau+1)\tau^{-1}},\ldots,\underset{\sim}{x}_{w\tau^{-1}}]\sigma^{\mathcal{E}(\sigma\tau)})\theta\Big)$$

$$\prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} \alpha_{\tau\sigma}.$$

Where the $\alpha_{\tau\sigma}$ are expressions of weight greater than $w + 1$.

The first two products can be dealt with by an induction on the height of θ. The next term can be rewritten modulo $\underset{c}{\sim}$ as

$$\prod_{\tau\varepsilon T}\Big(\prod_{\sigma\varepsilon G'} ([\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_w,\underset{\sim}{x}_{w+1}]\sigma)^{\mathcal{E}(\sigma)}\theta_\tau\Big)^{\mathcal{E}(\tau)}$$

where $\theta_\tau : \underset{\sim}{x}_j \mapsto \begin{cases} \underset{\sim}{x}_{j\tau^{-1}}\theta & \text{if } 1 \le j < i \\ a & \text{if } j = i \\ b & \text{if } j = i + 1 \\ \underset{\sim}{x}_{j\tau^{-1}+1}\theta & \text{if } i + 2 \le j \le w + 1. \end{cases}$

The inside product is then similar to the expression we started with except that it has weight $w + 1$ and G' is the group of permutations of I' which has, at least, $|I| - 1$ elements. It thus seems reasonable to hope that if we start by requiring that $|I| \geq c - w + 1$ we can develop an inductive proof on $c - w$ which, combined with an induction on the height of θ, would show that

$$\prod_{\sigma \in G} [\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_w]\sigma^{\mathcal{E}(\sigma)}\theta \sim_c 1$$

and in fact the main Lemma in this chapter does roughly that. In fact it shows that, if $I_1, I_2, \ldots, I_p$ are disjoint subsets of Z_w, each with more than n elements, and

$$\sum_{i=1}^{p} (|I_i| - n) \geq c - w + 1,$$

and G is the subgroup of S_w generated by the groups of permutations of the I_i's, then

$$\prod_{\sigma \in G} ([\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_w]\sigma)^{\mathcal{E}(\sigma)}\theta \underset{c}{\sim} \underset{\sim}{1}$$

for any homomorphism, θ, from $\underset{\sim}{A}_w$ to $\underset{\sim}{A}_n$.

This means that if D is a diagram of length w associated with a partition λ of w into m parts, where $m > n$, such that

$$\sum_{i=n+1}^{m} \lambda_i \geq c - w + 1$$

then $\Gamma_w(\underset{\sim}{A}_w)\mathcal{E}(D)\underset{\sim}{\rho}$ is a set of laws in $F_n(\underline{\underline{N}}_c)$.

It follows from this and Theorem 2.3.9 that

$$(\mathrm{Var}F_n(\underline{\underline{N}}_c) \wedge N_w)^\vee \ \underline{\underline{N}}_{w-1} \subseteq \mathrm{Var}(F_{n+c-w}(\underline{\underline{N}}_w))^\vee \ \underline{\underline{N}}_{w-1}.$$

3.1. Definition: If r_1 and r_2 are two functions: $Z \to Z \cup \{\infty\}$ then $r_1 \leq r_2$ if $ir_1 \leq ir_2 \ \forall \ i \in Z$.

3.2. Lemma: If a and b are products of commutators with repetition patterns r_1 and r_2 respectively then there exist expressions α and β such that:

(i) $[a,b] \sim \alpha\beta$

(ii) α is a product of commutators, $\mathrm{rep}(\alpha) = r_1 + r_2$, $\mathrm{wt}(\alpha) = \mathrm{wt}(a) + \mathrm{wt}(b)$

and (iii) $\mathrm{rep}\beta \geq r_1 + r_2$, $\mathrm{wt}(\beta) > \mathrm{wt}(a) + \mathrm{wt}(b)$.

Proof: This lemma can be proved by a straight forward but tedious induction on the length of a and b using the following well known results:

$$[a_1a_2,b] \sim [a_1,b][a_1,b,a_2][a_2,b],$$

$$[a,b_1b_2] \sim [a,b_2][a,b_1][a,b_1,b_2],$$

$$[a_1^{-1},b] \sim [a_1,b]^{-1}[b,a,a^{-1}],$$

and $$[a,b_1^{-1}] \sim [a,b_1]^{-1}[b,a,b^{-1}].$$

<u>3.3. Lemma</u>: If a is a product of commutators with repetition pattern r then there is an expression α such that:

(i) $a^{-1} \sim \alpha$

and (ii) α is a product of commutators such that

$$\text{rep}(\alpha) = r, \quad \text{wt}(\alpha) = \text{wt}(a).$$

<u>Proof</u>: By induction on the length of a using the fact that $(a_1 a_2)^{-1} \sim a_2^{-1} a_1^{-1}$.

<u>3.4. Lemma</u>: If $a \in \underset{\sim}{A}$ there exist expressions α and β such that

(i) $a \sim \alpha\beta$,

(ii) α is a product of commutators such that

$$\text{rep}(\alpha) = \text{rep}(a), \text{wt}(\alpha) = \text{wt}(a)$$

and (iii) $\text{rep}(\beta) \geq \text{rep}(a), \text{wt}(\beta) > \text{wt}(a)$.

<u>Proof</u>: By induction on the height of a, using 3.2 and 3.3.

<u>3.5. Corollary</u>: For any expression a and integer c there is an expression α such that:

(i) $a \underset{c}{\sim} \alpha$

(ii) α is a product of commutators

and (iii) $\text{rep}(\alpha) = \text{rep}(a)$ and $\text{wt}(\alpha) = \text{wt}(a)$.

Proof: By induction on $c - wt(a)$ using 3.4.

3.6. Definition: (i) If K is a subset of Z, $S(K)$ is the group of permutations of K, considered as operating on the right. $S(K)$ will be regarded as a subgroup of $S(Z)$.

(ii) For each $\sigma \in S(Z)$, σ^* is the automorphism of $\underset{\sim}{A}$ defined by

$$\underset{\sim}{x}_i\sigma^* = \underset{\sim}{x}_{i\sigma}.$$

(iii) If $\sigma \in S(Z)$ then $\mathcal{E}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$

3.7. Lemma: If a is a commutator in $\underset{\sim}{A}_n$, $i \in Z_n$, $wt_i(a) = 1$ and φ is the endomorphism of $\underset{\sim}{A}$ defined by

$$\underset{\sim}{x}_j \mapsto \begin{cases} \underset{\sim}{x}_i\underset{\sim}{x}_{n+1} & \text{if } j = i, \\ \underset{\sim}{x}_j & \text{otherwise,} \end{cases}$$

then

$$a \sim a.a(i\ n{+}1)^*.b$$

where

(i) $b \in \underset{\sim}{A}_{n+1}$,

(ii) $wt_j(b) \geq wt_j(a)$, $\forall\, j \in Z_n$,

and (iii) $wt_{n+1}(b) \geq wt_i(a)$.

Proof: By induction on weight.

If $wt(a) = 1$ the result is trivial.

Suppose that w is an integer greater than 1 and the lemma is true for all commutators of weight less than w. Let a be a commutator of weight w with weight 1 in $\underset{\sim}{x}_i$.

Then $a = [a_1, a_2]$ and

either (i) $wt_i(a_1) = 1$ and $wt_i(a_2) = 0$,

or (ii) $wt_i(a_1) = 0$ and $wt_i(a_2) = 1$.

Suppose (i) is true.

Then $a\varphi = [a_1\varphi, a_2]$

and, by the inductive hypothesis, there exists b_1 such that;

$$wt_j(b_1) \geq wt_j(a_1), \quad \forall j \in Z_n,$$

$$wt_{n+1}(b_1) \geq wt_i(a_1),$$

and

$$a_1\varphi \sim a_1 \circ a_1(i\ n+1)^* . b_1 \ .$$

Hence

$$\begin{aligned} a\varphi &\sim [a_1(a_1(i\ n+1)^*)b_1, a_2] \\ &\sim [a_1 \circ a_1(i\ n+1)^*, a_2][a_1 \circ a_1(i\ n+1)^*, a_2, b_1][b_1, a_2] \\ &\sim [a_1, a_2][a_1, a_2, a_1(i\ n+1)^*][a_1(i\ n+1)^*, a_2] \\ &\qquad [a_1 \circ a_1(i\ n+1)^*, a_2, b_1][b_1, a_2] \\ &\sim [a \circ a(i\ n+1)^*[a_1, a_2, a_1(i\ n+1)^*] \\ &\qquad [a_1, a_2, a_1(i\ n+1)^*, a(i\ n+1)^*][a_1 \circ a_1(i\ n+1)^*, a_2, b_1] \\ &\qquad [b_1, a_2] \end{aligned}$$

$$\sim a \circ a(i\ n+1)^* \circ b \quad \text{(say)}$$

where b clearly satisfies the conditions above. The proof for case (ii) is similar.

Thus the lemma is true for all commutators of finite weight. Since the lemma requires that $wt_i(a) = 1$ it follows that commutators of weight ∞ do not occur.

<u>3.8. Corollary</u>: If a is a homogeneous product of commutators in $\underset{\sim}{A}_n$, $i \in Z_n$, $wt_i(a) = 1$ and φ is the endomorphism of $\underset{\sim}{A}$ generated by

$$\underset{\sim}{x}_j \mapsto \begin{cases} \underset{\sim}{x}_i \underset{\sim}{x}_{n+1} & \text{if } j = i \\ \underset{\sim}{x}_j & \text{otherwise,} \end{cases}$$

then

$$a\varphi \sim a \circ a(i\ n+1)^* \circ b,$$

where

(i) $b \in \underset{\sim}{A}_{n+1}$,

(ii) $wt_j(b) \geq wt_j(a)$, $\forall j \in Z_n$,

and (iii) $wt_{n+1}(b) \geq wt_i(a)$.

<u>3.9. Lemma</u>: If a is a homogeneous product of commutators in $\underset{\sim}{A}_n$, $i \in Z_n$, $wt_i(a) = 1$ and φ is the endomorphism of $\underset{\sim}{A}$ generated by

$$\underset{\sim}{x}_j \to \begin{cases} \underset{\sim}{x}_i^{-1} & \text{if } j = i \\ \underset{\sim}{x}_j & \text{otherwise,} \end{cases}$$

then

$$a\varphi \sim a^{-1}b,$$

where

(i) $b \in \underset{\sim}{A}_n$,

(ii) $wt_j(b) \geq wt_j(a)$ for all $j \in Z_n$

and (iii) $wt_i(b) \geq wt_i(a) + 1$.

Proof: This Lemma can be proved in much the same way as Lemma 3.7 and Corollary 3.8. using the fact that, if α and β are expressions,

$$[\alpha^{-1},\beta] \sim [\alpha,\beta]^{-1} [\beta,\alpha,\alpha^{-1}]$$

and

$$[\alpha,\beta^{-1}] \sim [\alpha,\beta]^{-1} [\beta,\alpha,\beta^{-1}]$$

3.10. Lemma: Let c, n and m be positive integers, α an element of $\underset{\sim}{A}_m$ and $I_1,I_2,\ldots,I_p$ be distinct disjoint subsets of Z_m. Let θ be a homomorphism from $\underset{\sim}{A}_m$ to $\underset{\sim}{A}_n$ which maps generators to products of generators and their inverses. That is, for each $i \in Z_m$,

$$\underset{\sim}{x}_i\theta = \prod_{j=1}^{h_i} \underset{\sim}{x}_{\lambda_{i,j}}^{\varepsilon_{i,j}},$$

where the h_i's and $\lambda_{i,j}$'s are positive integers and the $\varepsilon_{i,j}$ are integers.

Put

$$G = \prod_{i=1}^{p} S(I_i).$$

Then, if (i) $\sum_{i=1}^{m} wt_i(\alpha) = w \geq [\frac{c}{2}] + 1$,

(ii) $|I_i| > n$ for all $i \in Z_p$,

(iii) $\sum_{i=1}^{p} (|I_i| - n) \geq c - w + 1$,

and

(iv) $wt_i(\alpha) = 1$ for all $i \in \bigcup_{j=1}^{p} I_i$,

we have

$$(\prod_{\sigma \in G} (\alpha\sigma^*)^{\varepsilon(\sigma)})\theta \underset{c}{\sim} 1.$$

Proof: The proof uses a double induction, firstly on $c - w$.

If $c - w < 0$ the result is trivial since $wt(a) \geq \sum_{i=1}^{m} wt_i(a) = w > c$ in this case.

Suppose then that W is an integer such that $[\frac{c}{2}] + 1 \leq W < c$, that a is an expression in A_m and that conditions (i) to (iv) remain true if we substitute W for w and a for α. Suppose further that the lemma is true whenever $w > W$. (I will refer to this assumption as the first inductive hypothesis.)

From Lemma 3.4 we know that

$$a \underset{c}{\sim} \prod_{i=1}^{q} a_i^{e_i}$$

where each a_i is a commutator, $W \leq wt(a_i) \leq c$, $rep(a_i) \geq rep(a)$ and each $e_i \varepsilon Z$. Hence the a_i have weight at least $[\frac{c}{2}] + 1$ so $wt[a_i, a_j] > c$ and $a_i a_j \underset{c}{\sim} a_j a_i$.

It follows that

$$a \underset{c}{\sim} \prod_{f \varepsilon \Gamma} a_f$$

where a_f is 1 or a homogeneous product of commutators with repetition pattern f and

$$\Gamma = \{f : f \text{ is a mapping from } Z \text{ to } Z^+ \cup \{0\} \text{ such that } if \geq wt_i(a) \ \forall a \ \varepsilon \ Z; \ \sum_{i \varepsilon Z} if \leq c \text{ and } if = 0 \ \forall \ i \ \varepsilon \ Z \setminus Z_m\}.$$

Now suppose $f \ \varepsilon \ \Gamma \setminus \{rep(a)\}$.

Put

$$J = \{j : j \ \varepsilon \ Z_n \text{ and } jf > wt_j(a)\},$$

$$I'_i = I_i \setminus J \text{ for each } i \ \varepsilon \ Z_p,$$

$$P = \{i : |I'_i| > n\}$$

and

$$G' = \prod_{i \varepsilon P} S(I'_i).$$

Let T be a left transversal of G' in G. Suppose $\tau \varepsilon T$.

Then

$$\sum_{i=1}^{m} wt_i(a_f\tau^*) = W', \text{ say,}$$

$$= \sum_{i=1}^{m} wt_{i\tau^{-1}}(a_f)$$

$$= \sum_{i=1}^{m} wt_i(a_f)$$

$$\geq W + |J| > W.$$

But

$$\sum_{i\varepsilon P} (|I_i'| - n) = \sum_{i\varepsilon P} (|I_i| - |I_i \cap J| - n)$$

$$\geq \sum_{i=1}^{p} (|I_i| - |I_i \cap J| - n),$$

(since for $i \varepsilon Z_p \setminus P$, $|I_i| - |I_i \cap J| \leq n$)

$$= \sum_{i=1}^{p} (|I_i| - n) - |J|$$

$$\geq c - (W + |J|) + 1$$

$$\geq c - W' + 1.$$

Hence, by the first inductive hypothesis,

$$(\prod_{\sigma\varepsilon G} (a_f\sigma^*)^{\mathcal{E}(\sigma)})\theta \underset{c}{\sim} \prod_{\tau\varepsilon T} ((\prod_{\sigma\varepsilon G'} ((a_f\tau^*)\sigma^*)^{\mathcal{E}(\sigma)})\theta)^{\mathcal{E}(\tau)}$$

$$\underset{c}{\sim} \underset{\sim}{1}.$$

It follows that

$$(\prod_{\sigma\varepsilon G} (a\sigma^*)^{\mathcal{E}(\sigma)})\theta \underset{c}{\sim} \prod_{\sigma\varepsilon G} (a_r\sigma^*)^{\mathcal{E}(\sigma)}\theta,$$

where $r = rep(a)$.

If $a_r = \underset{\sim}{1}$ the result is trivial so we can assume that a_r is a homogeneous product of commutators with repetition pattern r.

This brings us to the second induction which is over

$$\sum_{i=1}^{p} \sum_{j\varepsilon I_i} (ht(\underset{\sim}{x}_j\theta) - 1) = h,$$

say. Suppose $h = 0$; then either

(i) $\underset{\sim}{x}_j\theta = \underset{\sim}{1}$ for some $j \varepsilon \bigcup_{i=1}^{p} I_i$,

or

(ii) for each $j \varepsilon \bigcup_{i=1}^{p} I_i$ there exists $k \varepsilon Z_n$ such that $\underset{\sim}{x}_j\theta = \underset{\sim}{x}_k$.

If (i) is true it is easily seen, from the fact that $a\sigma^*$ is a homogeneous product of commutators of weight 1 in $\underset{\sim}{x}_j$, that $((a\sigma^*)^{\mathcal{E}(\sigma)})\theta \sim \underset{\sim}{1}$ for all $\sigma \varepsilon G$ and the induction on $c - w$ is complete in this case.

If (ii) is true then, since $|I_1| > n$, and $a\theta \in \underset{\sim}{A}_n$ there must exist two distinct integers $q,k \in I_1$ such that $\underset{\sim}{x}_q\theta = \underset{\sim}{x}_k\theta$, which means $(q\ k)*\theta = \theta$. But $(q\ k) \in G$. Let T be a left transversal of $<1,(q\ k)>$ in G. Then

$$(\prod_{\sigma\in G} (a_r\sigma*)^{\mathcal{E}(\sigma)})\theta \underset{c}{\sim} \prod_{\tau\in T} (a_r\tau*\theta(a_r\tau*(q\ k)*\theta)^{-1})^{\mathcal{E}(\tau)}$$

$$\underset{c}{\sim} \underset{\sim}{1}.$$

Thus if $h = 0$ the induction on $c - w$ can be completed and the lemma is true.

Suppose now that H is a positive integer and that we have completed the induction on $c - w$ for all θ such that

$$0 \leq \sum_{i=1}^{p} \sum_{j\in I_i} (ht(\underset{\sim}{x}_j\theta) - 1) < H,$$

that is that the lemma is true for all such θ. (I will refer to this assumption as the second inductive hypothesis.)

Let ψ be a homomorphism from $\underset{\sim}{A}_m$ to $\underset{\sim}{A}_n$ which maps generators to products of generators and their inverses and suppose

$$\sum_{i=1}^{p} \sum_{j\in I_i} (ht(\underset{\sim}{x}_j\psi) - 1) = H.$$

I will complete the induction on $c - w$ for ψ and it will follow that this induction can always be completed.

There must exist integers i and j such that $i \in Z_p$, $j \in I_i$ and, either (i) $\underset{\sim}{x}_j\psi = uv$ for some $u,v \in \underset{\sim}{A}_n$

or (ii) $\underset{\sim}{x}_j\psi = \underset{\sim}{x}_k^{-1}$ for some $k \in Z_n$,

suppose $\underset{\sim}{x}_j\psi = uv$.

Define $\varphi \in \mathrm{Hom}(\underset{\sim}{A}_m, \underset{\sim}{A}_{m+1}) : \underset{\sim}{x}_j \mapsto \underset{\sim}{x}_j\underset{\sim}{x}_{m+1}, \underset{\sim}{x}_k \mapsto \underset{\sim}{x}_k, \ \forall\, k \in Z_m \setminus \{j\}$,

and

$$\psi_1 \in \mathrm{Hom}(\underset{\sim}{A}_{m+1}, \underset{\sim}{A}_n) : \underset{\sim}{x}_j \mapsto u,\ \underset{\sim}{x}_{m+1} \mapsto v,\ \underset{\sim}{x}_k \mapsto \underset{\sim}{x}_k\psi, \quad \forall k \in Z_m \setminus \{j\}.$$

Clearly $\psi = \varphi\psi_1$.

Let

$$G' = S(I_i \setminus \{j\}) \prod_{q \in Z_p \setminus \{i\}} S(I_q).$$

φ obviously commutes with σ^* whenever $\sigma \in G'$.

Let T be a left transversal of G' in G.

By Corollary 3.8 we know that for each $\tau \in T$ there must exist an expression $b_\tau \in \underset{\sim}{A}_{n+1}$ such that

$$a_r\tau^*\varphi \sim a_r\tau^*(a_r\tau^*(j\ n{+}1)^*)b_\tau,$$

$$wt_k(b_\tau) \geq wt_k(a_r), \quad \forall k \in Z_m$$

and

$$wt_{m+1}(b_\tau) \geq wt_{j\tau^{-1}}(a_r) = 1.$$

Thus

$$\prod_{\sigma\varepsilon G} (a_r\sigma^*)^{\mathcal{E}(\sigma)}\psi \underset{c}{\sim} \prod_{\tau\varepsilon T}((\prod_{\sigma\varepsilon G'} (a_r\tau^*\varphi\sigma^*)^{\mathcal{E}(\sigma\tau)})\psi_1)$$

$$\underset{c}{\sim} (\prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} (a_r\tau^*\sigma^*)^{\mathcal{E}(\sigma\tau)}\psi_1)(\prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} (a_r\tau^*(j\ m+1)^*\sigma^{*\,\mathcal{E}(\sigma\tau)}\psi_1))$$

$$(\prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} (b_\tau\sigma^*)^{\mathcal{E}(\sigma\tau)}\psi_1)$$

$$\underset{c}{\sim} (\prod_{\sigma\varepsilon G} (a_r\sigma^*)^{\mathcal{E}(\sigma)}\psi_1)(\prod_{\sigma\varepsilon G} (a_r\sigma^*)^{\mathcal{E}(\sigma)}(j\ m+1)^*\psi_1)$$

$$(\prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} (b_\tau\sigma^*)^{\mathcal{E}(\sigma\tau)}\psi_1).$$

Now

$$\sum_{k\varepsilon Z_p}\sum_{q\varepsilon I_k} (ht(\underset{\sim}{x}_i\psi_1) - 1) = H - ht(ab) + ht(a)$$

$$< H$$

and

$$\sum_{k\varepsilon Z_p}\sum_{q\varepsilon I_k} (ht(\underset{\sim}{x}_i(j\ m+1)^*\psi_1) - 1)$$

$$= H - ht(ab) + ht(b)$$

$$< H$$

Hence, by the second inductive hypothesis

$$\Big(\prod_{\sigma\varepsilon G} (a_r\sigma^*)^{\mathcal{E}(\sigma)}\Big)\psi$$

$$\underset{c}{\sim} \prod_{\tau\varepsilon T} \Big(\prod_{\sigma\varepsilon G'} (b_\tau\sigma^*)^{\mathcal{E}(\sigma)}\psi_1\Big)^{\mathcal{E}(\tau)}.$$

For each $k \varepsilon Z_p$ let $I_k' = I \setminus \{j\}$.

Let $P = \{k : k \varepsilon Z_p$ and $|I_k'| > n\}$.

Clearly P is either Z_p or $Z_p \setminus \{i\}$, depending on the number of elements in I_i.

Now

$$\sum_{k\varepsilon P} (|I_k'| - n) = \sum_{k\varepsilon Z_p} (|I_k| - n) - 1$$

$$\geq c - (W+1) + 1.$$

It follows from the first inductive hypothesis that, if $wt_q(b_\tau) = 1$ for all

$$q \ \varepsilon \Big(\bigcup_{k=1}^{p} I_k\Big) \setminus \{j\}$$

then

$$\prod_{\sigma\varepsilon G'} (b_\tau\sigma^*)^{\mathcal{E}(\sigma)} \underset{c}{\sim} \underset{\sim}{1}.$$

If $wt_q(b_\tau) > 1$ for some $q \ \varepsilon \ (\bigcup_{k=1}^{p} I_k) \setminus \{j\}$ the same result follows from a proof almost identical to that used to eliminate the expressions $a_f : f \neq r$ (see page 59), and the induction is complete for case (i) (see page 63).

Now suppose case (ii) applies, that is that

$$\underset{\sim}{x}_j\psi = \underset{\sim}{x}_k^{-1} \text{ for some } k \text{ in } Z_n.$$

Define $\varphi \in \mathrm{Hom}(\underset{\sim}{A}_m,\underset{\sim}{A}_m) : \underset{\sim}{x}_j \mapsto \underset{\sim}{x}_j^{-1}, \underset{\sim}{x}_q \mapsto \underset{\sim}{x}_q, \quad \forall q \in Z_m \setminus \{j\}$,
and

$$\psi_1 \in \mathrm{Hom}(\underset{\sim}{A}_m,\underset{\sim}{A}_n) : \underset{\sim}{x}_j \mapsto x_k, \underset{\sim}{x}_q \mapsto \underset{\sim}{x}_q\psi \quad q \in Z_m \setminus \{j\}.$$

Define G' and T as on page 63. Clearly $\psi = \varphi\psi_1$ and

$$\prod_{\sigma\varepsilon G} (a_r\sigma^*)^{\mathscr{E}(\sigma)}\psi \underset{c}{\sim} \prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} (a_r\tau^*\varphi\sigma^*\psi_1)^{\mathscr{E}(\sigma\tau)}$$

$$\underset{c}{\sim} \prod_{\sigma\varepsilon G} (a_r\sigma^*)^{-\mathscr{E}(\sigma)}\psi_1 \prod_{\tau\varepsilon T}\prod_{\sigma\varepsilon G'} b_\tau\sigma^{*\mathscr{E}(\sigma)}\psi_1,$$

where, by Lemma 3.9, the b_τ are expressions such that
$wt_q(b_\tau) \geq wt_q(a_r), \quad \forall q \in Z_m$ and $wt_j(b_\tau) \geq wt_{j\tau^{-1}}(a_r) + 1$. As in case (i) we can eliminate the first product by the second inductive hypothesis and the second product by the first inductive hypothesis.

<u>3.11. Theorem:</u> Let c, n, w be positive integers such that $w \leq c$, let λ be a partition of c into k parts such that

$$\sum_{i=n+1}^{k} \lambda_i \geq c - w + 1$$

and let e be a scalar multiple of a primitive idempotent of QS_c such that e is associated with λ and

$$e \in ZS_c .$$

Then $\Gamma_w(\underset{\sim}{A}_w)e\underset{\sim}{\rho}$ is a set of laws in $F_n(\underline{\underline{N}}_c)$.

<u>Proof</u>: Suppose ψ is a homomorphism from F_w to $F_n(\underline{\underline{N}}_c)$.

Let

$$x_i\psi = \prod_{j=1}^{n_i} x_{\mu_{i,j}}^{\delta_{i,j}}$$

and define the homomorphism $\theta : \underset{\sim}{A}_w \to \underset{\sim}{A}_n$ by

$$\underset{\sim}{x}_i\theta = \prod_{j=1}^{n_i} \underset{\sim}{x}_{\mu_{i,j}}^{\delta_{i,j}} .$$

Clearly $\underset{\sim}{\rho}\psi = \theta\underset{\sim}{\rho}_c$, so

$$\Gamma_w(\underset{\sim}{A}_w)e\underset{\sim}{\rho}\psi = \Gamma_w(\underset{\sim}{A}_w)e\theta\underset{\sim}{\rho}_c .$$

Now suppose D is a diagram associated with λ. D has λ_{n+1} columns of length greater than n.

Set I_i equal to the set of entries in the ith column of D for $1 \leq i \leq \lambda_{n+1} = p$, say. More precisely $I_i = \{h : hD = (j,i)\}$ $\forall i \in Z_p$. Clearly $I_i \cap I_j$ is empty if $i \neq j$ and

$$\sum_{i=1}^{p} (|I_i| - n) = |\{h : hD = (j,i) \text{ for some } j > n\}|$$

$$= \sum_{j=n+1}^{p} |\{h : hD = (j,i)\}|$$

$$= \sum_{j=n+1}^{p} \max\{i : \exists\, h \in Z_w \text{ such that } hD = (j,i)\}$$

$$= \sum_{j=n+1}^{p} \lambda_j \geq c - w + 1.$$

Now

$$\sum_{i=1}^{p} (|I_i| - n) \leq w - n < w$$

so

$$c - w + 1 < w,$$

$$w > \frac{c+1}{2} ,$$

and

$$w \geq [\tfrac{c}{2}] + 1.$$

Suppose α is a commutator in $\underset{\sim}{A}_w$ of weight w. Let

$$a = [\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_w],$$

then $\alpha = a\xi$ for some endomorphisms ξ of $\underset{\sim}{A}_w$, and $wt_j(a) = 1$ for all $j \in Z_w$. Put $G = \prod_{i=1}^{p} S(I_i)$ and let T be a left transversal of G in $C(D)$.

Then

$$\alpha\mathcal{E}(D)\theta_{\mathcal{R}_c} = a\xi\mathcal{E}(D)\theta_{\mathcal{R}_c}$$

$$= a\mathcal{E}(D)\xi\theta_{\mathcal{R}_c}$$

$$= \prod_{\substack{\sigma\varepsilon C(D)\\ \rho\varepsilon R(D)}} (a(\mathcal{E}(\sigma)\sigma\rho))\xi\theta_{\mathcal{R}_c}$$

$$= \prod_{\substack{\sigma\varepsilon C(D)\\ \rho\varepsilon R(D)}} [\underset{\sim}{x}_{1\rho^{-1}\sigma^{-1}},\underset{\sim}{x}_{2\rho^{-1}\sigma^{-1}},\ldots,\underset{\sim}{x}_{c\rho^{-1}\sigma^{-1}}]^{\mathcal{E}(\sigma)}\xi\theta_{\mathcal{R}_c}$$

$$= \prod_{\substack{\sigma\varepsilon C(D)\\ \rho\varepsilon R(D)}} (a(\rho^{-1}\sigma^{-1})*)^{\mathcal{E}(\sigma)}\xi\theta_{\mathcal{R}_c}$$

$$= \prod_{\sigma\varepsilon G} ((\prod_{\substack{\tau\varepsilon T\\ \rho\varepsilon R(D)}} (a\rho*\tau*)^{\mathcal{E}(\tau)})\sigma*)^{\mathcal{E}(\sigma)}\xi\theta_{\mathcal{R}_c}$$

$$= 1,$$

since

$$\prod_{\substack{\tau\varepsilon T\\ \rho\varepsilon R(D)}} (a\rho*\tau*)^{\mathcal{E}(\tau)}$$

is clearly a homogeneous product of commutators with the same weight and repetition pattern as a, $\xi\theta$ is a homomorphism from $\underset{\sim}{A}_w$ to $\underset{\sim}{A}_n$ and all other conditions of Lemma 3.10 are satisfied.

It follows immediately that if α' is any element of $\Gamma_c(\underset{\sim}{A}_c)$ then $\alpha'\mathcal{E}(D)\theta_{\mathcal{R}_c} = 1$ and so

$$\Gamma_w(\underset{\sim}{A}_w)e_{\mathcal{R}}\psi = 1 \quad \text{for all} \quad \psi \varepsilon \mathit{Hom}(F_w,F_n(\underset{=}{N}_c)),$$

that is the elements of $\Gamma_w(\underset{\sim}{A}_w)e_{\mathcal{R}}$ are laws in $F_n(\underset{=}{N}_c)$.

<u>3.12. Theorem</u>: Let c, n, w be positive integers such that $w \leq c$. Then

$$(\mathrm{Var}(F_n(\underline{\underline{N}}_c)) \wedge \underline{\underline{N}}_w) \vee \underline{\underline{N}}_{w-1} \subseteq \mathrm{Var}(F_{n+c-w}(\underline{\underline{N}}_w)) \vee \underline{\underline{N}}_{w-1}.$$

<u>Proof</u>: From Theorem 2.3.9 we know that the only laws that distinguish the variety on the right hand side from $\underline{\underline{N}}_w$ are associated with partitions of w into more than $n + c - w$ parts. Suppose λ is such a partition. Clearly

$$\sum_{i=n+1}^{n+c-w+1} \lambda_i \geq c - w + 1$$

and the required result follows from Theorem 3.11.

4. Basic Left Normed Commutators.

4.0. In Chapter 3 I proved that certain laws hold in $F_n(\underset{=}{N}_c)$. In the next chapter I will show that, with certain conditions on n and c, these laws generate all the laws of $F_n(\underset{=}{N}_c)$.

To do this I will need to be able to identify elements of $\underset{\sim}{A}_n$ which are not in the kernel of $\underset{\sim}{\rho}_c$. Normally this is done with basic commutators.

The trouble with basic commutators is that they behave badly under homomorphisms from $\underset{\sim}{A}_n$ to $\underset{\sim}{A}_{n-1}$. That is, if ψ is such a homomorphism and a is a basic commutator in $\underset{\sim}{A}_n$ then $a\psi$ is unlikely to be one and the process of "collecting" it into a basic form is complicated and difficult to conceptualize.

It is known that if a is an element of $\underset{\sim}{A}$ then there is a product of left normed commutators, a' say, such that:

$$a \sim a'.$$

It can in fact be shown that, if a has weight c, a' can be a homogeneous product of commutators each with the same repetition pattern as a.

Now suppose a is a left normed commutator of weight c,

$$a = [\underset{\sim}{x}_{\lambda_1}, \underset{\sim}{x}_{\lambda_2}, \ldots, \underset{\sim}{x}_{\lambda_c}] \text{ say,}$$

and suppose

$$1 < i \leq c.$$

Then

$$a = [[[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-1}}],\underset{\sim}{x}_{\lambda_i}],\underset{\sim}{x}_{\lambda_{i+1}},\ldots,\underset{\sim}{x}_{\lambda_c}]$$

$$\underset{c}{\sim} [[\underset{\sim}{x}_{\lambda_i},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-1}}]],\underset{\sim}{x}_{\lambda_{i+1}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{-1}$$

$$\underset{c}{\sim} [([\underset{\sim}{x}_{\lambda_i},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-2}}],\underset{\sim}{x}_{\lambda_{i-1}}]$$

$$[\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_{i-1}},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-2}}]]^{-1}),\underset{\sim}{x}_{\lambda_{i+1}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{-1}$$

$$\underset{c}{\sim} [\underset{\sim}{x}_{\lambda_i},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-2}}],\underset{\sim}{x}_{\lambda_{i-1}},\underset{\sim}{x}_{\lambda_{i+1}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{-1}$$

$$[[\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_{i-1}}],[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-2}}],\underset{\sim}{x}_{\lambda_{i+1}},\ldots,\underset{\sim}{x}_{\lambda_c}]$$

$$\underset{c}{\sim} [\underset{\sim}{x}_{\lambda_i},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-3}}],\underset{\sim}{x}_{\lambda_{i-2}},\underset{\sim}{x}_{\lambda_{i-1}},\underset{\sim}{x}_{\lambda_{i+1}},\underset{\sim}{x}_{\lambda_{i+2}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{-1}$$

$$[\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_{i-2}},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-3}}],\underset{\sim}{x}_{\lambda_{i-1}},\underset{\sim}{x}_{\lambda_{i+1}},\underset{\sim}{x}_{\lambda_{i+2}},\ldots,\underset{\sim}{x}_{\lambda_c}]$$

$$[\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_{i-1}},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-3}}],\underset{\sim}{x}_{\lambda_{i-2}},\underset{\sim}{x}_{\lambda_{i+1}},\underset{\sim}{x}_{\lambda_{i+2}},\ldots,\underset{\sim}{x}_{\lambda_c}]$$

$$[\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_{i-1}},\underset{\sim}{x}_{\lambda_{i-2}},[\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-3}}],\underset{\sim}{x}_{\lambda_{i+1}},\underset{\sim}{x}_{\lambda_{i+2}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{-1}$$

$$\underset{c}{\sim} [\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_2},\ldots,\underset{\sim}{x}_{\lambda_{i-1}},\underset{\sim}{x}_{\lambda_{i+1}},\underset{\sim}{x}_{\lambda_{i+2}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{-1}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$[\underset{\sim}{x}_{\lambda_i},\underset{\sim}{x}_{\lambda_{i-1}},\underset{\sim}{x}_{\lambda_{i-2}},\ldots,\underset{\sim}{x}_{\lambda_1},\underset{\sim}{x}_{\lambda_{i+1}},\underset{\sim}{x}_{\lambda_{i+2}},\ldots,\underset{\sim}{x}_{\lambda_c}]^{\delta}$$

where $\varepsilon = \begin{cases} +1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even.} \end{cases}$

Thus if a is a left normed commutator of weight c, and $wt_j(a) = 1$, it is possible to find a homogeneous product of commutators, a', with the same repetition pattern as a, each commutator having first entry $\underset{\sim}{x}_j$, such that:

$$a \underset{c}{\sim} a'.$$

Now the number of commutators with this property, (i.e. first entry $\underset{\sim}{x}_j$, repetition pattern $rep(a)$) is given by

$$\frac{(c-1)!}{\prod_{k \varepsilon Z} (krep(a))!}$$

But a straightforward application of Witt's formula shows that this is simply the number of basic commutators with repetition pattern a. (See for example Magnus, Karras, and Solitar [10], Theorem 5.11, page 330.)

It follows that if B is a set of basic commutators of weight c in $\underset{\sim}{A}_n$ and B' is the set obtained from B by removing the basic commutators with repetition pattern $rep(a)$ and inserting, in their place, the left normed commutators with that repetition pattern and first entry j then $B'\rho_c$ generates $\gamma_c F_n(\underset{=}{N}_c)$ and $|(B'\rho_c)|$ is the rank of $\gamma_c F_n(\underset{=}{N}_c)$ regarded as a free Z-module. Hence $B'\rho_c$ is a basis for $\gamma_c F_n(\underset{=}{N}_c)$.

This displacement of basic commutators by left normed commutators only works for those repetition patterns which take the value 1 at some point.

The rest of this chapter is devoted largely to providing a formal proof of the above remarks. This formal proof has the added bonus that it provides an explicit formula for the process of moving a generator to the front of a left normed commutator. To do this it is necessary to define some rather artificial looking functions.

4.1. Definition: If T is a finite subset of Z^+, the permutation $[T] : Z^+ \to Z^+$ is defined as follows:

$[T]|_T$ is the monotonic decreasing bijection: $T \to Z_{|T|}$

$[T]|_{Z^+ \setminus T}$ is the monotonic increasing bijection: $Z^+ \setminus T \to Z^+ \setminus Z_{|T|}$.

4.2. Lemma: If T is a subset of Z_n, then

$$i[T] = i \quad \text{for all} \quad i > n$$

and $\quad i[T] \in Z_n \quad$ for all $\quad i \in Z_n$.

Proof: Since $[T]|_{Z^+ \setminus T}$ is a monotonic decreasing bijection, the $n - |T|$ smallest elements of $Z^+ \setminus T$ must be mapped to the $n - |T|$ smallest elements of $Z^+ \setminus [Z_{|T|}]$. But these constitute the sets $Z_n \setminus T$ and $Z_n \setminus Z_{|T|}$ respectively.

Hence $[T]$ maps $Z^+ \setminus Z_n$ onto itself and, being a monotonic increasing bijection on this set, must be the identity mapping.

It follows that $[T]$ maps Z_n onto itself which completes the proof.

4.3 <u>Lemma:</u> If $T \subseteq Z_n$ then

$$i[T \cup \{n+1\}] = \begin{cases} i[T] + 1 & \text{if } 1 \le i \le n, \\ 1 & \text{if } i = n + 1, \\ i[T] & \text{if } i > n + 1. \end{cases}$$

<u>Proof:</u> Put $T_1 = T \cup \{n+1\}$. Clearly $n + 1$ is the largest element of T_1 and $T \subseteq Z_{n+1}$. It follows from 4.1 that $(n+1)[T_1] = 1$ and from 4.2 that $i[T_1] = i = i[T]$ for all $i > n + 1$.
Now $[T]|_T$ is the monotonic decreasing bijection: $T \to Z_{|T|}$ and $[T_1]|_T$ is the monotonic decreasing bijection: $T \to Z_{|T|+1} \setminus \{1\}$.
Hence $i[T_1] = i[T] + 1$ for all $i \varepsilon T$. From 4.1 and 4.2 it follows that:

$[T]|_{Z_n \setminus T}$ is the monotonic increasing bijection: $Z_n \setminus T \to Z_n \setminus Z_{|T|}$

and

$[T_1]|_{Z_n \setminus T}$ is the monotonic increasing bijection: $Z_n \setminus T \to Z_{n+1} \setminus Z_{|T|+1}$

Hence $i[T_1] = i[T] + 1$ if $i \varepsilon Z_n \setminus T$.

4.4 Corollary: If $T \subseteq Z_n$ then,

$$i[T \cup \{n+1\}]^{-1} = \begin{cases} n+1 & \text{if } i = 1, \\ (i-1)[T]^{-1} & \text{if } 2 \leq i \leq n+1, \\ i[T]^{-1} & \text{if } i > n+1. \end{cases}$$

4.5 The following commutator identities are collected here for convenience.

Lemma: (i) If $a,b,d \in \underset{\sim}{A}$ and $wt(a) + wt(b) + wt(d) = c$ then

$$[a,b,d][d,a,b][b,d,a] \underset{c}{\sim} 1.$$

(ii) If $a,b,d \in \underset{\sim}{A}$ and $wt(a) + wt(b) + wt(d) = c$ then

$$[a,[b,d]] \underset{c}{\sim} [a,b,d][a,d,b]^{-1}.$$

(iii) If $a \in \underset{\sim}{A}$ and $a \underset{c}{\sim} b$ then

$$[a,d] \underset{c+1}{\sim} [b,d] \quad \text{for all } d \in \underset{\sim}{A}.$$

Proof: (i) is simply the Jacobi-Witt identity.

(ii)
$$\begin{aligned} [a,[b,d]] &\sim [b,d,a]^{-1} \\ &\underset{c}{\sim} [a,b,d][d,a,b] \quad \text{by (i)} \\ &\underset{c}{\sim} [a,b,d][a,d,b]^{-1}. \end{aligned}$$

(iii) If $a \underset{c}{\sim} b$ then there is an expression e of weight $c + 1$ such that:

$$a \sim be.$$

Hence $[a,d] \sim [be,d]$

$$\sim [b,d][b,e,d][e,d].$$

$$\underset{c+1}{\sim} [b,d],$$

since $wt[b,e,d] > wt[e,d] \geq c + 2.$

4.6 Lemma: If n is a positive integer and $a,b_1,b_2,\ldots,b_n$ are elements of $\underset{\sim}{A}$ and $w = wt(a) + \sum_{i=1}^{n} wt(b_i)$ then

$$[a,[b_1,b_2,\ldots,b_n]] \underset{w}{\sim} \prod_{T \subseteq Z_n \setminus \{1\}} [a,b_{1[T]^{-1}},b_{2[T]^{-1}},\ldots,b_{n[T]^{-1}}]^{(-1)^{|T|}}$$

(Note: For this product to make sense it is strictly necessary to have an order defined on the set of finite subsets of Z^+. In fact the order chosen makes no difference to the result since we are effectively working in $\gamma_{n+1}F(\underset{=}{N}_{n+1})$ which is abelian. However in order to avoid confusion choose some ordering on the set of finite subsets of Z^+. Then, if R is a subset of Z^+, $\prod_{T \subseteq R} \alpha_T$ will denote the product of the α_T taken in that order.)

Proof: By induction on n. If $n = 1$ then the right hand side of the statement becomes:

$$\prod_{T \subseteq \phi} [a, b_{1[T]^{-1}}]^{(-1)^{|T|}} = [a, b_{1[\phi]^{-1}}]^{(-1)^0} = [a, b_1]$$

which is the required result.

Suppose now that $m > 1$ and that the Lemma is true for all n such that $1 \leq n < m$. Then

$$[a,[b_1,\ldots,b_{m-1},b_m]]$$

$$= \quad [a,[[b_1,\ldots,b_{m-1}],b_m]]$$

$$\underset{w}{\sim} \quad [a,[b_1,\ldots,b_{m-1}],b_m][[a,b_m],[b_1,b_2,\ldots,b_{m-1}]]^{-1} \quad \text{by 4.5 (ii),}$$

$$\underset{w}{\sim} \quad [(\prod_{T \subseteq Z_{m-1} \setminus \{1\}} [a,b_{1[T]^{-1}},b_{2[T]^{-1}},\ldots,b_{(m-1)[T]^{-1}}]^{(-1)^{|T|}}),b_m]$$

$$\circ \prod_{T \subseteq Z_{m-1} \setminus \{1\}} [[a,b_m],b_{1[T]^{-1}},b_{2[T]^{-1}},\ldots,b_{(m-1)[T]^{-1}}]^{(-1)^{|T|+1}}$$

by the inductive hypothesis,

$$\underset{w}{\sim} \prod_{T \subseteq Z_{m-1} \setminus \{1\}} [a,b_{1[T]^{-1}},b_{2[T]^{-1}},\ldots,b_{(m-1)[T]^{-1}},b_{m[T]^{-1}}]$$

$$\circ \prod_{T \subseteq Z_{m-1} \setminus \{1\}} [a,b_{1[T\cup\{m\}]^{-1}},b_{2[T\cup\{m\}]^{-1}},\ldots,b_{(m-1)[T\cup\{m\}]^{-1}},b_{m[T\cup\{m\}]^{-1}}]$$

by 4.4,

$$\underset{w}{\sim} \prod_{T \subseteq Z_{m} \setminus \{1\}} [a,b_{1[T]^{-1}},b_{2[T]^{-1}},\ldots,b_{(m-1)[T]^{-1}},b_{m[T]^{-1}}]$$

which completes the proof.

<u>4.7 Corollary:</u> If a is a homogeneous product of commutators of weight c then there is a homogeneous product of left normed commutators, b say,

with the same repetition pattern as a and such that:

$$a \underset{c}{\sim} b \, .$$

<u>Proof</u>: By induction on c. If $c \leq 2$ the result is trivial. Suppose $w > 2$ and that the Corollary is true for all $c < w$.

Let a be a homogeneous product of commutators of weight w.

Suppose $a = \prod_{i=1}^{n} a_i^{\varepsilon_i}$ where each a_i is a commutator. Then

$$a_n = [a_{n,1}, a_{n,2}]$$

$$\underset{w}{\sim} [\prod_{i=1}^{m_1} b_{1,i}^{\varepsilon_{1,i}}, \prod_{i=1}^{m_2} b_{2,i}^{\varepsilon_{2,i}}]$$

where $b_{j,i}$ is a left normed commutator with the same repetition pattern as $a_{n,j}$

(this follows from the inductive hypothesis).

Hence

$$a \underset{w}{\sim} \prod_{i=1}^{m_1} \prod_{k=1}^{m_2} ([b_{1,i}, b_{2,k}])^{\varepsilon_{1,i}+\varepsilon_{2,k}} \, .$$

Now $b_{2,k}$ is a left normed commutator so that 4.6 can be applied to find a product of left normed commutators, $b_{n,i,k}$ such that:

$$\begin{aligned} \text{rep}(b_{n,i,k}) &= \text{rep}(b_{1,i}) + \text{rep}(b_{2,k}) \\ &= \text{rep}(a_{n,1}) + \text{rep}(a_{n,2}) \\ &= \text{rep}(a_n) \\ &= \text{rep}(a) \end{aligned}$$

and

$$[b_{1,i}, b_{2,k}] \not\sim_c b_{n,i,k} \text{ .}$$

It follows that a_n can be written as a product of left normed commutators and the required result follows easily by induction on n.

<u>4.8 Corollary</u>: Let $a = [x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_c}]$ and suppose that $\lambda_n = i$. Then

$$a \sim_c \prod_{T \subseteq Z_{n-1} \setminus \{1\}} [x_i, x_{\lambda_{1[T]^{-1}}}, x_{\lambda_{2[T]^{-1}}}, \ldots, x_{\lambda_{(n-1)[T]^{-1}}}, x_{\lambda_{n+1}}, \ldots, x_{\lambda_c}]^{-1^{(|T|+1)}}$$

<u>4.9 Definition</u>: (i) The mapping L_n from $\{\theta : Z_n \to Z\}$ to the set of left normed commutators of weight n is defined for each positive integer n as follows:

If $n = 1$, $\theta L_n = x_{1\theta}$.

If $n > 1$, $\theta L_n = [\theta|_{Z_{n-1}} L_{n-1}, x_{n\theta}]$.

If φ is a function from M to Z where $Z_n \subseteq M$ I will write φL_n for $\varphi|_{Z_n} L_n$.

Clearly every left normed commutator of weight n can be written as θL_n for some $\theta : Z_n \to Z$ and $\theta_1 L_n = \theta_2 L_n$ if and only if $\theta_1|_{Z_n} = \theta_2|_{Z_n}$.

(ii) If a is a left normed commutator of weight n and $a = \theta L_n$ then, for each $i \in Z_n$, the <u>ith entry of a</u> is defined as $x_{i\theta}$.

(iii) The set of (*)-basic commutators is defined as follows. If r is a finite repetition pattern, that is a function: $Z \to Z^+$ with only a finite number of non-zero points, the (*)-basic commutators with repetition pattern r are:

(i) the basic commutators with repetition pattern r if $ir \neq 1$ for all $i \in Z$

(ii) the left normed commutators with repetition pattern r and first entry $\underset{\sim}{x}_j$ if $j = \min\{i : ir = 1\}$.

4.10 Theorem: If B is the set of (*)-basic commutators of weight c then $B\rho_c$ is a basis for the Z-module $\gamma_c F_n(\underset{=}{N}_c)$.

Proof: $\gamma_c F_n(\underset{=}{N}_c)$ is a free Z-module with rank equal to the number of basic commutators of weight c. It is generated by the images under ρ_c of the basic commutators of weight c. It follows from 4.8 that it is generated by $B\rho_c$.

Witt's formula for the number of basic commutators with weight c and repetition pattern r is:

$$\frac{1}{c} \sum_{d \in D} \frac{c!\, \mu(d)}{\prod_{i \in Z} (\frac{ir}{d})!}$$

where

$$D = \{d : d \text{ divides } ir, \ \forall i \in Z\}$$

and μ is the Moebius function.

(See, for example (Magnus, Karrass, Solitar [10], Theorem 5.11, page 330). Clearly if $ir = 1$ for some i, $D = \{1\}$ and Witt's formula reduces to

$$\frac{(n-1)!}{\prod_{j\in Z^+} (jr)!}$$

But this is just the number of left normed commutators with first entry $\underset{\sim}{x}_i$ and repetition pattern r. (The number of ways of arranging copies of the generators other than $\underset{\sim}{x}_i$ in order with the requisite repetition pattern.)

It follows that the number of elements in $B_{\underset{\sim}{\ell}_c}$ is equal to the rank of $\gamma_c F_n(\underset{=}{N}_c)$ and that $B_{\underset{\sim}{\ell}_c}$ is a basis.

<u>4.11 Definition</u>: (i) A (*)-basic commutator is <u>type (1)</u> if it is basic, <u>type (2)</u> if it is not.

(ii) The (*)-basic commutators are ordered as follows:

The type (1) commutators have their usual ordering.

The type (2) commutators of a given weight and first entry have an arbitrary ordering.

The type (2) commutators of weight w are greater than the type (1) commutators of weight w and less than the type (1) commutators of weight $w + 1$.

If a_1 and a_2 are type (2) commutators of the same weight, w say, and if

$$a_1 = \theta_1 L_w,\ a_2 = \theta_2 L_w \text{ and } 1\theta_1 < 1\theta_2,$$

then

$$a_1 < a_2 .$$

(iii) An expression in $\underset{\sim}{A}_n$ is a <u>(c,n, *)-basic expression</u> if it is of the form:

$$\prod_{i=1}^{m} b_i^{\delta_i}$$

where m is the number of (*)-basic commutators of weight at most c in $\underset{\sim}{A}_n$, $b : i \mapsto b_i$ is the monotonic increasing bijection from Z^+ to the set of (*)-basic commutators in $\underset{\sim}{A}_n$ and the δ_i are integers (possibly zero, in which case $b_i^{\delta_i} = \underset{\sim}{1}$).

<u>4.12 Theorem</u>: For each element, a, of $\underset{\sim}{A}_n$ and for each integer c, there is a unique (c,n,*)-basic expression b such that $a \underset{c}{\sim} b$.

<u>Proof</u>: By induction on c.

If $c = 1$ then the set of (*)-basic commutators of weight at most c is just $\underset{\sim}{X}_n$ and $\underset{\sim}{A}_n \underset{\sim}{\rho}_c$ is the free abelian group freely generated by X_n and the result follows. Suppose $w > 1$ and the Theorem is true for all $c < w$. Let b_1 be the unique (w-1,n,*)-basic expression for which

$$a \underset{w-1}{\sim} b_1 .$$

Then

$$a \underset{w}{\sim} b_1 a_1,$$

where a_1 is an expression of weight at least w, since it is in the kernel of ρ_{w-1}.

It follows from 4.10 that

$$a_1\rho_w = \prod_{i=1}^{m} (\beta_i \rho_w)^{\delta_i}$$

where m is the number of (*)-basic commutators of weight w in A_n, $\beta : i \mapsto \beta_i$ is the monotonic bijection from Z_m to the set of these commutators and the δ_i are integers.

It follows that

$$a \underset{w}{\sim} b_1 \prod_{i=1}^{m} \beta_i^{\delta_i}$$

which is (w,n,*)-basic.

Now suppose

$$b_1 \prod_{i=1}^{m} \beta_i^{\delta_i} \underset{w}{\sim} b_2 \prod_{i=1}^{m} \beta_i^{\varepsilon_i}$$

Then

$$b_1 \prod_{i=1}^{m} \beta_i^{\delta_i} \underset{w-1}{\sim} b_2 \prod_{i=1}^{m} \beta_i^{\varepsilon_i}$$

so

$$b_1 \underset{w-1}{\sim} b_2 \quad \text{and} \quad b_1 = b_2$$

by the inductive hypothesis.

Thus

$$\left(\prod_{i=1}^{m} \beta_i^{\delta_i}\right)\rho_w = \left(\prod_{i=1}^{m} \beta_i^{\mathcal{E}_i}\right)\rho_w$$

and, by 4.10, it follows that

$$\delta_i = \mathcal{E}_i \quad \text{for all} \quad i \in Z_m$$

which proves the expression is unique.

4.13 An Application. The result I will prove in this section is now well known, independent proofs having been published by Levin [8] and by Kovács, Newman and Pentony [7]. This, third, proof is included here to illustrate the use of *-basic commutators.

Let c be an integer greater than two and let G be the group of permutations of $Z_c \setminus \{1\}$.

Let

$$a = \prod_{\sigma \in G} ([x_1, x_2, \ldots, x_c]\sigma^*)^{\mathcal{E}(\sigma)} .$$

It is obvious from the antisymmetric nature of a that $a\rho$ is a law in $F_{c-2}(N_c)$. Since the factors of a are all different (*)-basic commutators it is equally obvious that $a\rho_c$ is non-trivial. Thus $a\rho$ is a law distinguishing $Var(F_{c-2}(N_c))$ from N_c.

4.14. The following lemma will be useful for strengthening the result of Chapter 3.

<u>Lemma</u>: Let c be an integer, greater than 4, let λ be a partition of c with $\lambda_1 = 3$ and $\lambda_2 = 2$ and let D be a diagram associated with λ such that $1D = (1,3)$, $2D = (2,1)$, $3D = (1,1)$ and $4D = (1,2)$, then $\Gamma_c(\underset{\sim}{A}_c)\mathcal{E}(D)\underset{\sim}{\rho}_c$ is nontrivial.

<u>Proof</u>: First note that the top part of D can be drawn as follows

3	4	1
2	?	
?	?	

Let $a = [\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_c]$. I will show that $a\mathcal{E}(D)\underset{\sim}{\rho}_c \neq 1$.

Put $m = (c-1)!$ and let $b_1, b_2, \ldots b_m$ be the *-basic commutators with the same repetition pattern as a. For convenience let $a = b_1$.

For each permutation τ of Z_c define the integers $\delta_{1,\tau}, \delta_{2,\tau}, \ldots, \delta_{m,\tau}$ by

$$a\tau^* = \prod_{i=1}^{m} b_i^{\delta_{i,\tau}}.$$

I will show that

$$\sum_{\sigma\varepsilon C(D)} \sum_{\rho\varepsilon R(D)} \mathcal{E}(\sigma)\delta_{1,\rho\sigma} = 6,$$

which means that

$$\prod_{\substack{\sigma\varepsilon C(D)\\ \rho\varepsilon R(D)}} (a\rho^*\sigma^*)\underset{\sim}{\rho}_c^{\mathcal{E}(\sigma)} \neq 1$$

which in turn means

$$\prod_{\substack{\sigma\varepsilon C(D)\\ \rho\varepsilon R(D)}} (a\sigma\rho)^{\mathcal{E}(\sigma)} \mathcal{R}_c \neq 1 .$$

Note that if $\rho \varepsilon R(D)$, $\sigma \varepsilon C(D)$ and $m \varepsilon Z_c$ are such that $m\rho\sigma = 1$ then $m\rho = 1$, which means $m \varepsilon \{1,3,4\}$.

Thus, for all $\sigma \varepsilon C(D)$ and all $\rho \varepsilon R(D)$, the collecting process given in Corollary 4.8 for moving $\underset{\sim}{x}_1$ to the front of $a\rho^*\sigma^*$ will only affect, at most, the first four positions. This means that $\delta_{1,\rho\sigma} \neq 0$ only if $i\rho\sigma = i$ for all i greater than 4. Since $\sigma \varepsilon C(D)$, i and $i\rho$ must be in the same column, but $\rho \varepsilon R(D)$ so i and $i\rho$ are in the same row. Since D is one to one it follows that $i\rho = i$ and hence that $i\sigma = i$.

We have proved that, if $\sigma \varepsilon C(D)$, $\rho \varepsilon R(D)$ and $\delta_{1,\rho\sigma} \neq 0$ then $i\rho = i\sigma = i$. This in turn means that

$$\sum_{\sigma\varepsilon C(D)} \sum_{\rho\varepsilon R(D)} \mathcal{E}(\sigma)\delta_{1,\rho\sigma} = \sum_{\sigma\varepsilon C(D_1)} \sum_{\rho\varepsilon R(D_1)} \mathcal{E}(\sigma)\delta_{1,\rho\sigma}$$

where

D_1 =

3	4	1
2		

.

The sum on the right hand side can be readily calculated and is, in fact, 6, as promised.

<u>4.15 Theorem</u>: Let c,w and n be positive integers such that $c > w$, $2w = 2n + c + 2$ and $w \geq 7$.

Then

$$((\mathrm{Var}F_n(\underset{=}{N}_c)) \wedge \underset{=}{N}_w) \vee \underset{=}{N}_{w-1} \subset \mathrm{Var}F_{n+c-w}(\underset{=}{N}_w) \vee \underset{=}{N}_{w-1}$$

where the inclusion is proper.

In fact if m is an integer, greater than or equal to $\frac{w-1}{2}$, then there is a law of weight w in $F_n(\underset{=}{N}_c)$ which is not a law in $F_m(\underset{=}{N}_w)$.

<u>Proof</u>: I will first prove the second statement and then show it implies the first.

Let m be the smallest integer which is not less than $\frac{w-1}{2}$.

Define a partition λ of w as follows:

$$\lambda_1 = 3, \quad \lambda_i = 2 \text{ for all } i \text{ such that } 1 < i < m$$

$$\lambda_m = \begin{cases} 2 & \text{if } w \text{ is odd,} \\ 1 & \text{if } w \text{ is even.} \end{cases}$$

It is easily checked that λ is in fact a partition of w into m parts.

Let D be a diagram associated with λ such that $1D = (1,3)$, $2D = (2,1)$, $3D = (1,1)$ and $4D = (1,2)$.

Let

$$\alpha = \Gamma_w(\underset{\sim}{A}_w)\mathcal{E}(D)\varrho.$$

By Lemma 4.14 α is not a set of laws in $\underset{=}{N}_w$ and hence, by Lemma 2.3.8 is not a set of laws in $F_m(\underset{=}{N}_w)$.

On the other hand

$$\sum_{i=n+1}^{m} \lambda_i = 2(m-1-n) + \lambda_m = k, \text{ say.}$$

If w is odd, $k = 2(m-n) = w - 1 - 2n = c - w + 1$, and, if w is even, $k = 2(\frac{w}{2} - n - 1) + 1 = c - w + 1$ so, by Theorem 3.11 α is a set of laws in $F_n(\underset{=}{N}_c)$ and the second part of the statement is proved.

By virtue of 3.12 it now suffices to prove that $n + c - w \geq \frac{w-1}{2}$.

By hypothesis $w = 2n + c - w + 2 \geq 2n + 3$ so

$$n \leq \frac{w-3}{2}.$$

But $n + c - w = w - n - 2 \geq w - \frac{w-3}{2} - 2 = \frac{w-1}{2}$ and the proof is complete.

<u>4.16 Corollary</u>: Let c be a positive integer, greater than 7, and let w and n be defined as follows

$$w = c - 1 \text{ and } n = \frac{c}{2} - 2 \text{ if } c \text{ is even,}$$
$$w = c - 2 \text{ and } n = \frac{c-1}{2} - 3 \text{ if } c \text{ is odd.}$$

Then $(\mathrm{Var}(F_n(\underset{=}{N}_c)) \wedge \underset{=}{N}_w) \vee \underset{=}{N}_{w-1}$ is a proper subvariety of

$$\mathrm{Var}(F_{n+c-w}(\underset{=}{N}_w)) \vee \underset{=}{N}_{w-1}.$$

Note: This Corollary disproves the conjecture

$$\mathrm{Var}(F_n(\underline{\underline{N}}_c)) \wedge \underline{\underline{N}}_{c-1} = \mathrm{Var}(F_{n+1}(\underline{\underline{N}}_{c-1})).$$

5. The Laws of $F_n(\underset{=}{N}_c)$ for Large n.

5.0. In this chapter I will show that, provided n is sufficiently large the laws of $F_n(\underset{=}{N}_c)$ obtained in chapter 3 are the only laws of $F_n(\underset{=}{N}_c)$.

I will need results from all the previous chapters to do this.

5.1 Lemma: If α is an expression of weight w and i is an integer such that $wt_i(\alpha) \geq 1$ and if ψ is the endomorphism of $\underset{\sim}{A}$ given by

$$\underset{\sim}{x}_j \mapsto \begin{cases} \underset{\sim}{x}_j & \text{if } j \neq i, \\ [\underset{\sim}{x}_h, \underset{\sim}{x}_k] & \text{if } j = i, \end{cases}$$

then $wt(\alpha\psi) \geq wt(\alpha) + wt_i(\alpha)$.

Proof: The proof is by induction on height. If $ht(\alpha) = 1$ the result is trivial.

Suppose that a is an expression with height greater than one and that the Lemma is true for all α such that $ht(\alpha) < ht(a)$ and that $wt_i(a) = 1$.

There are three possibilities.

(i) $a = a_1^{-1}$ where $wt(a) = wt(a_1)$, $wt_i(a) = wt_i(a_1)$,

(ii) $a = a_1 a_2$ where $wt(a) = \min\{wt(a_1), wt(a_2)\}$,

$wt_i(a) = \min\{wt_i(a_1), wt_i(a_2)\}$,

(iii) $a = [a_1, a_2]$ where $wt(a) = wt(a_1) + wt(a_2)$,

$wt_i(a) = wt_i(a_1) + wt_i(a_2)$.

Suppose, for example that (ii) holds. Then

$$\begin{aligned} wt(a\psi) &= \min\{wt(a_1\psi), wt(a_2\psi)\} \\ &\geq \min\{(wt(a_1) + wt_i(a_1)), (wt(a_2) + wt_i(a_2))\} \\ &\geq \min\{wt(a_1), wt(a_2)\} + \min\{wt_i(a_1), wt_i(a_2)\} \\ &= wt(a) + wt_i(a) \quad \text{as required.} \end{aligned}$$

The other two cases are even more straightforward.

<u>5.2 Lemma</u>: If a is a non-trivial basic product of commutators in $\underset{\sim}{A}_n$, each factor having weight at most c and if $a\underset{\sim}{\rho}_c$ is in the kernel of every homomorphism from $F_n(\underline{\underline{N}}_c)$ to $F_{n-1}(\underline{\underline{N}}_c)$ then

$$wt_i(a) \geq 1, \quad \text{for all } i \in Z_n.$$

<u>Proof</u>: Suppose there exists $i \in Z_n$ such that $wt_i(a) = 0$. Let ψ be the homomorphism from $\underset{\sim}{A}_n$ to $\underset{\sim}{A}_{n-1}$ given by:

$$\underset{\sim}{x}_j \mapsto \begin{cases} \underset{\sim}{x}_j & \text{if } 1 \leq j < i, \\ \underset{\sim}{1} & \text{if } j = i, \\ \underset{\sim}{x}_{j-1} & \text{if } n \geq j > i. \end{cases}$$

Clearly, if α is a commutator,

$$\alpha\psi \underset{c}{\sim} \underset{\sim}{1} \quad \text{if } wt_i(\alpha) \geq 1$$

and $$\alpha\psi = \alpha \quad \text{if } wt_i(\alpha) = 0.$$

(This can be formally proved by induction on $wt(\alpha)$, using the fact that $[\alpha_1, \underset{\sim}{1}] \sim [\underset{\sim}{1}, \alpha_2] \sim \underset{\sim}{1}$ for any $\alpha_1, \alpha_2 \in \underset{\sim}{A}$.)

Thus, if,

$$a = \prod_{k=1}^{n} b_k^{\varepsilon_k}$$

then

$$a\psi \underset{c}{\sim} \prod_{k=1}^{n} b_k^{\delta_k} = a_1 \quad \text{(say)},$$

where

$$\delta_k = \begin{cases} 0 & \text{if } wt_i(b_k) \geq 1, \\ \varepsilon_k & \text{if } wt_i(b_k) = 0 . \end{cases}$$

Clearly a_1 is a basic expression in $\underset{\sim}{A}_{n-1}$ and, since

$$0 = wt_i(a) = \min\{wt_i(b_k) : \varepsilon_k \neq 0\} ,$$

it follows that at least one of the δ_k is non-zero, so a_1 is not in the kernel of ρ_c.

Let ψ' be the homomorphism $F_n(\underset{=}{N}_c) \to F_{n-1}(\underset{=}{N}_c)$ induced by ψ. By hypothesis $a\rho_c \in Ker(\psi')$ but

$$a\rho_c\psi' = a\psi\rho_c \neq 1$$

a contradiction which concludes the proof.

<u>5.3 Lemma</u>: If a is a left normed commutator of weight w and i is an integer such that

$$a = \theta L_w, \quad wt_i(a) = 1, \quad \text{and} \quad 1\theta \neq i,$$

and if ψ is the endomorphism of $\underset{\sim}{A}$ generated by

$$\underset{\sim}{x}_h \mapsto \begin{cases} \underset{\sim}{x}_h & \text{if } h \neq i \\ [\underset{\sim}{x}_j, \underset{\sim}{x}_k] & \text{if } h = i \end{cases}$$

where j, k are some elements of Z, then

$$a\lambda \sim_{w+1} \theta_1 L_{w+1}(\theta_2 L_{w+1})^{-1},$$

where
$$\theta_1 : h \mapsto \begin{cases} h\theta & \text{if } 1 \leq h < i\theta^{-1} \\ j & \text{if } h = i\theta^{-1}, \\ k & \text{if } h = i\theta^{-1} + 1, \\ (h-1)\theta & \text{if } i\theta^{-1} + 2 \leq h, \end{cases}$$

and
$$\theta_2 = (i\theta_1 \; i\theta_1^{-1} + 1)\theta,$$

that is θ_2 is defined in the same way as θ except that the positions of j and k are reversed.

<u>Proof</u>: By induction on $w - i\theta^{-1}$. If $w = i\theta^{-1}$, then

$$a\psi = [\theta L_{w-1}, [x_j, x_k]],$$

$$\underset{w+1}{\sim} [\theta L_{w-1}, x_j, x_k][\theta L_{w-1}, x_k, x_j]^{-1} \text{ by 4.5 (ii)},$$

$$\underset{w+1}{\sim} \theta_1 L_{w+1}(\theta_2 L_{w+1})^{-1}$$

Suppose now that c is an integer, greater than zero, and that the Lemma is true whenever $0 \le w - i\theta^{-1} < c$. Suppose that α is a left normed commutator, $\alpha = \varphi L_w$, $wt_i(\alpha) = 1$ and that $w - i\varphi^{-1} = c$.

Define φ_1, φ_2 in the same way as θ_1 and θ_2. Then we have

$$w\psi = [\varphi L_{w-1}\psi, x_{w\varphi}],$$

$$\underset{w+1}{\sim} [\varphi_1 L_w (\varphi_2 L_w)^{-1}, x_{w\varphi}], \text{ by the inductive hypothesis,}$$

~~and Lemma 4.5 (ii)~~

$$\underset{w+1}{\sim} [\varphi_1 L_w, x_{(w+1)\varphi_1}][\varphi_2 L_w, x_{(w+1)\varphi_2}]^{-1},$$

$$= \varphi_1 L_{w+1}(\varphi_2 L_{w+1})^{-1}.$$

<u>5.4 Lemma</u>: Let n, c be positive integers with $n > \frac{c}{2}$ and $n > 9$ and let a be a $(c,n,*)$-basic expression such that

$$a\rho_c\psi = 1 \quad \text{for all} \quad \psi : F_n(\underline{N}_c) \to F_{n-1}(\underline{N}_c).$$

Then

$$wt(a) \geq c.$$

<u>Proof</u>: By Lemma 5.2 we know that

$$wt_i(a) \geq 1 \text{ for all } i \in Z_n$$

and it follows that

$$wt(a) > \frac{c}{2}.$$

Let $a = \prod_{i=1}^{m} b_i^{\delta_i}$ where $b_1, \ldots, b_m$ are (*)-basic commutators and suppose $wt(a) = w < c$. Let $\lambda = \min\{\zeta : \delta_\zeta \neq 0\}$ then clearly $wt(b_\lambda) = w < c$ and $wt_i(b_\lambda) \geq 1$ for all $i \in Z_n$. For each $k \in Z^+$ put $I_k = \{i : wt_i(b_\lambda) = k\}$. Then

$$
\begin{aligned}
w &= wt(b_\lambda) \\
&= \sum_{i=1}^{n} wt_i(b_\lambda) \\
&= \sum_{k=1}^{w} k|I_k| \\
&\geq |I_1| + 2(n - |I_1|) \\
&= 2n - |I_1| \\
&> c - |I_1| \\
&\geq (w+1) - |I_1|.
\end{aligned}
$$

Hence $|I_1| \geq 2$.

Suppose $\xi = \min\{I_1\}$ then, by definition b_λ is a left normed commutator with first entry $\underset{\sim}{x}_\xi$.

Let $b_\lambda = \theta_\lambda L_w$.

Now b_λ can be regarded as a string of symbols each chosen from $\underset{\sim}{X}_n$, the first being $\underset{\sim}{x}_\xi$ with $\underset{\sim}{x}_i$ being repeated $wt_i(b_\lambda)$ times. It will be necessary to consider the relative positions of these symbols and towards this end I introduce the following terminology.

Two integers, i and j are adjacent in b_λ if there are integers, $k,\ell \leq w$ such that $|\ell-k| = 1$ and $k\theta_\lambda = i$ and $\ell\theta_\lambda = j$.

Intuitively i and j are adjacent in b_λ if one of the sequences $\ldots, \underset{\sim}{x}_i, \underset{\sim}{x}_j, \ldots$ and $\ldots, \underset{\sim}{x}_j, \underset{\sim}{x}_i, \ldots$ occur in b_λ.

I will prove that there are integers r,s,t in Z_n such that $r \in I_1$ ξ,r,s,t are all distinct and no two of r,s and t are adjacent.

This part of the proof is messy being based on consideration of each of the following four possible cases;

(i) $|I_1| = 2$,

(ii) $|I_1| = 3$ and the two elements of $I_1\setminus\{\xi\}$ are adjacent,

(iii) $|I_1| = 3$ and the two elements of $I_1\setminus\{\xi\}$ are not adjacent,

(iv) $|I_1| \geq 4$.

Suppose $|I_1| = 2$, set r equal to the element $I_1 \setminus \{\xi\}$. Then since $wt_r(a) = 1$ there can be only two elements of Z_n which are adjacent to r. Let the set of these elements be R. Let s be any element in $Z_n \setminus (\{\xi, r\} \cup R)$.

Now $2n-1 > c - 1 \geq w \geq 2 + 2|I_2| + 3(\sum_{i=3}^{w} |I_i|)$

$$= 2 + 2|I_2| + 3(n - 2 - |I_2|)$$

$$= 3n - |I_2| - 4,$$

so $|I_2| > n - 3$, but clearly $|I_2| \leq n - 2$, so $|I_2| = n - 2$. It follows that $wt_s(a) = 2$ and that there are at most four integers adjacent to s in a. Denote the set of these integers by S. It follows that

$$|\{\xi, r, s\} \cup R \cup S| \leq 9 < n,$$

so that there exists $t \in Z_n \setminus (\{\xi, r, s\} \cup R \cup S)$ and the result is proved in this case.

Now suppose case (ii) applies. Take one of the elements of $I_1 \setminus \xi$ as r. Let the set of integers adjacent to r in b_λ be R. Clearly $|R| = 2$.

This time we have

$$2n - 1 > 3 + 2|I_2| + 3(n - 3 - |I_2|)$$

so $$|I_2| \geq n - 5.$$

Now at most one element of $\{\xi,r\} \cup R$ is in I_2 so we can choose s from $I_2 \setminus (\{\xi,r\} \cup R)$. Denote the set of integers adjacent to s in b by S. Then $|S| \leq 4$ so that

$$|\{\xi,r,s\} \cup R \cup S| \leq 9 < n$$

and we can find a t which satisfies the required conditions.

If either case (iii) or case (iv) applies there are at least two non-adjacent elements in $I_1 \setminus \{\xi\}$. Call one r and the other s. If R and S are defined as before.

$$|\{\xi,r,s\} \cup R \cup S| \leq 7 < n$$

and we can find a t to satisfy the required conditions.

Returning now to the main proof let $r,s,t \in Z_n \setminus \{\xi\}$ be three distinct integers no two of which are adjacent in b_λ and such that $r \in I_1$. Let $\underset{\sim}{A}^*_{n-1}$ be the subalgebra of $\underset{\sim}{A}$ generated by $\underset{\sim}{X}_n \setminus \{\underset{\sim}{x}_r\}$. Let ψ be the homomorphism from $\underset{\sim}{A}_n$ to $\underset{\sim}{A}^*_{n-1}$ given by

$$\underset{\sim}{x}_i \mapsto \begin{cases} \underset{\sim}{x}_i & \text{if } i \neq r \\ [\underset{\sim}{x}_s, \underset{\sim}{x}_t] & \text{if } i = r. \end{cases}$$

Let $F^*_{n-1}(\underset{=}{N}_c)$ be the subgroup of $F_n(\underset{=}{N}_c)$ generated by $X_n \setminus \{x_r\}$, and let ψ' be the homomorphism from $F_n(\underset{=}{N}_c)$ to $F^*_{n-1}(\underset{=}{N}_c)$ induced by ψ and let ϕ be the isomorphism from $F^*_{n-1}(\underset{=}{N}_c)$ to $F_{n-1}(\underset{=}{N}_c)$

generated by

$$x_i \mapsto \begin{cases} x_i & \text{if } i < r \\ x_{i-1} & \text{if } i > r \end{cases} .$$

Then, by hypothesis, $a\rho_{\sim c}$ is in the kernel of $\psi'\varphi$, which means that

$$1 = a\rho_{\sim c}\psi'\varphi = a\psi\rho_{\sim c}\varphi .$$

But φ is an isomorphism, so

$$1 = a\psi\rho_{\sim c}, \quad a\psi \underset{c}{\sim} \underset{\sim}{1}$$

and hence $a\psi \underset{w+1}{\sim} \underset{\sim}{1}$.

Now, by Lemma 5.1.

$$wt(b_i\psi) \geq wt(b_i) + wt_r(b_i) \quad \forall\, i \in Z_m$$

$$\geq wt(b_i) + 1$$

so $b_i\psi \underset{w+1}{\sim} \underset{\sim}{1}$ unless $wt(b_i) \leq w$ and $wt_r(b_i) = w + 1 - wt(b_i)$.
It follows that

$$\underset{\sim}{1} \underset{w+1}{\sim} a\psi = \prod_{i=1}^{m} (b_i^{\varepsilon_i})\psi$$

$$= \prod_{i=1}^{m} (b_i\psi)^{\varepsilon_i}$$

$$\underset{w+1}{\sim} \prod_{i=\lambda}^{m'} (b_i\psi)^{\varepsilon_i}$$

where $m' = \max\{i : wt(b_i) = w\}$ and $\mathcal{E}_i = 0$ if $wt_r(b_i) \neq 1$.

Now, for $\lambda \leq i \leq m'$, b_i is a left normed commutator of weight w,

$$b_i = \theta_i L_w \text{ say.}$$

By Lemma 5.3.

$$b_i \psi \underset{w+1}{\sim} (\theta_{i,1} L_{w+1})(\theta_{i,2} L_{w+1})^{-1}$$

where
$$\theta_{i,1} : \ell \mapsto \begin{cases} \ell\theta_i & \text{if } 1 \leq \ell < r\theta_i^{-1} \\ s & \text{if } \ell = r\theta_i^{-1} \\ t & \text{if } \ell = r\theta_i^{-1} + 1 \\ (\ell-1)\theta_i & \text{if } r\theta_i^{-1} + 2 \leq \ell \leq w+1 \end{cases}$$

and

$$\theta_{i,2} : \ell \mapsto \begin{cases} \ell\theta_i & \text{if } 1 \leq \ell < r\theta_i^{-1} \\ t & \text{if } \ell = r\theta_i^{-1} \\ s & \text{if } \ell = r\theta_i^{-1} + 1 \\ (\ell-1)\theta_i & \text{if } r\theta_i^{-1} + 2 \leq \ell \leq w+1 \end{cases}$$

Let $\beta_{i,j}$ be the $(w + 1, n, *)$-basic expressions which are given by

$$\theta_{i,j} L_{w+1} \underset{w+1}{\sim} \beta_{i,j}$$

and let

$$\beta_{i,j} = \underset{\sim}{1}\,\underset{\sim}{1} \cdots \underset{\sim}{1} \prod_{k=1}^{m''} \beta_k^{\delta_{i,j,k}}$$

where $\beta_1, \ldots, \beta_{m''}$ are the (*)-basic commutators of weight $w + 1$ in $\underset{\sim}{A}_n$.

(The product of $\underset{\sim}{1}$'s at the front corresponds to the commutators of weight less than $w + 1$ which will have zero exponents.)

Suppose $\theta_{\lambda,1}L_{w+1} = \beta_\mu$. We will now find those ordered pairs $(i,j) \in Z_{n'} \times Z_2$ which satisfy the condition $\mathcal{E}_i\delta_{i,j,\mu} \neq 0$.

The process of collecting left normed commutators described in 4.8. clearly preserves the repetition pattern. It follows that if $\delta_{i,j,\mu} \neq 0$, then $\text{rep}(\theta_{i,j}L_{w+1}) = \text{rep}(\theta_{\lambda,1}L_{w+1})$. Hence $1 = \text{wt}_\xi(\theta_{i,j}L_{w+1}) = \text{wt}_\xi(\theta_i L_w)$ $= \text{wt}_\xi(b_i)$ by the definition of $\theta_{i,j}$ and it follows that b_i must have weight 1 in $\underset{\sim}{x}_\xi$.

If $\text{wt}_\ell(b_i) = 1$ for some $\ell < \xi$ then $i \leq \lambda$ by the definition of the ordering of (*)-basic commutators. Hence $\xi = \min\{\ell : \text{wt}_\ell(b_i) = 1\}$ and so $\xi = 1\theta_i = 1\theta_{i,j}$. It follows that $\theta_{i,j}L_{w+1}$ is (*)-basic and hence that $\theta_{i,j} = \theta_{\lambda,1}$.

As we observed before, the fact that $\mathcal{E}_i$ is nonzero implies that $\text{wt}_r(b_i) = 1$.

Suppose that $j = 1$ and that $r\theta_\lambda^{-1} < r\theta_i^{-1}$. Then $r\theta_i^{-1} + 1 \geq r\theta_\lambda^{-1} + 2$, so $(r\theta_i^{-1})\theta_\lambda = (r\theta_i^{-1} + 1)\theta_{\lambda,1} = (r\theta_i^{-1} + 1)\theta_{i,1} = t$ and, since r and t are not adjacent in b_λ, it follows that $r\theta_i^{-1} - 1 > r\theta_\lambda^{-1}$, so that $(r\theta_i^{-1} - 1)\theta_\lambda = (r\theta_i^{-1})\theta_{\lambda,1} = (r\theta_i^{-1})\theta_{i,1} = s$.

Hence s and t are adjacent in b_λ, a contradiction.

Similarly, if we assume $r\theta_\lambda^{-1} < r\theta_i^{-1}$ and $j = 2$, we obtain

$$(r\theta_i)\theta_\lambda = s \quad \text{and} \quad (r\theta_i - 1)\theta_\lambda = t,$$

and if we assume $r\theta_\lambda^{-1} > r\theta_i^{-1}$, we obtain

$$(r\theta_i^{-1}\theta_\lambda) = \begin{cases} s & \text{if } j = 1, \\ t & \text{if } j = 2, \end{cases}$$

and

$$(r\theta_i^{-1} + 1)\theta_\lambda = \begin{cases} t & \text{if } j = 1 \\ s & \text{if } j = 2, \end{cases}$$

all of which are contradictions.

Finally if we suppose that $r\theta_i^{-1} = r\theta_\lambda^{-1}$ we find that $j = 2$ and $\theta_i = \theta_\lambda$, that is, $i = \lambda$.

Hence

$$0 = \sum_{i=1}^{m'} \sum_{j=1}^{2} \mathcal{E}_i \delta_{i,j,\mu}$$

$$= \mathcal{E}_\lambda .$$

But we chose λ so that $\mathcal{E}_\lambda \neq 0$ and we have a contradiction which proves that $wt(a) = c$.

5.5 Theorem: Let n and c be positive integers such that $n > \frac{c}{2} - 1$ and $n > 8$. Then

$$F_{n+1}(\underset{=}{N}_{c-1}) \in Var(F_n(\underset{=}{N}_c)).$$

Proof: Let K be the set of elements of $F_{n+1}(\underline{\underline{N}}_c)$ which are in the kernel of every homomorphism from $F_{n+1}(\underline{\underline{N}}_c)$ to $F_n(\underline{\underline{N}}_c)$.

Suppose ψ is a homomorphism from $F_{n+1}(\underline{\underline{N}}_c)$ to $F_n(\underline{\underline{N}}_c)$. ψ induces a homomorphism from $F_{n+1}(\underline{\underline{N}}_c)/K$ to $F_n(\underline{\underline{N}}_c)$. The intersection of the kernels of all homomorphisms from $F_{n+1}(\underline{\underline{N}}_c)/K$ to $F_n(\underline{\underline{N}}_c)$ induced in this way is clearly trivial. It follows that $F_{n+1}(\underline{\underline{N}}_c)/K \in \operatorname{Var} F_n(\underline{\underline{N}}_c)$.

But, by Lemma 5.4, $K \subseteq \gamma_c(F_{n+1}(\underline{\underline{N}}_c))$. It follows that the natural epimorphism from $F_{n+1}(\underline{\underline{N}}_c)$ to $F_{n+1}(\underline{\underline{N}}_{c-1})$ induces an epimorphism from $F_{n+1}(\underline{\underline{N}}_c)/K$ to $F_{n+1}(\underline{\underline{N}}_{c-1})$, so $F_{n+1}(\underline{\underline{N}}_{c-1}) \in \operatorname{Var}(F_{n+1}(\underline{\underline{N}}_c)/K) \subseteq \operatorname{Var}(F_n(\underline{\underline{N}}_c))$.

5.6 Corollary: If n and c are positive integers such that $n > \frac{c}{2} - 1$ and $n > 8$ and w is a positive integer less than c then

$$F_{n+c-w}(\underline{\underline{N}}_w) \in \operatorname{Var}(F_n(\underline{\underline{N}}_c)).$$

5.7 Theorem: For each pair of positive integers i and j such that $i > j$ define

$$U_{i,j} = \bigcup_{w \in Z_j} \bigcup_{\lambda \in \Lambda_{w,i+j-w}} \bigcup_{D \in \Delta_\lambda} \Gamma_w(\underset{\sim}{A}_i)\mathcal{E}(D)\underset{\sim}{\rho},$$

where $\Lambda_{w,m}$ is the set of partitions of w into more than m parts and Δ_λ is the set of standard diagrams associated with λ.

Let n and c be positive integers such that $n > \frac{c}{2} - 1$ and $n > 8$. Then the set of laws of $F_n(\underline{\underline{N}}_c)$ is the fully invariant isolated subgroup of F generated by $[x_1, x_2, \ldots, x_{c+1}]$ and $U_{n,c}$.

Proof: Let G be the fully invariant subgroup of F generated by $[x_1, x_2, \ldots, x_{c+1}]$ and $U_{n,c}$. We know, from Theorem 3.11 that every element of G is a law of $F_n(\underline{\underline{N}}_c)$ so it is only necessary to show that every law of $F_n(\underline{\underline{N}}_c)$ is in G.

Define the weight function, wt, on F by

$$wt(a) = \begin{cases} \infty & \text{if } a \text{ is a law in } \underline{\underline{N}}_w \text{ for all } w \in Z, \\ \max\{w : a \text{ is a law in } \underline{\underline{N}}_{w-1}\} & \text{otherwise.} \end{cases}$$

It is obvious that every law of $F_n(\underline{\underline{N}}_c)$ that has weight at least $c + 1$ is in G.

Suppose that $1 < w < c + 1$ and that every law of $F_n(\underline{\underline{N}}_c)$ that has weight at least $w + 1$ is in G. Suppose that u is a law of $F_n(\underline{\underline{N}}_c)$ and that $wt(u) = w$.

By Corollary 5.6, u is a law in $F_{n+c-w}(\underline{\underline{N}}_w)$ and, by ~~Lemma~~ Theorem 2.3.9 there exists a element v of F, which is a consequence of $[x_1, x_2, \ldots, x_{w+1}]$ and so has weight at least $w + 1$, and an element u_1 of G such that

$$u = u_1 v.$$

Since u and u_1 are laws in $F_n(\underline{\underline{N}}_c)$ it follows that v is and hence by the inductive hypothesis that $v \in G$ which means $u \in G$.

<u>5.8 Corollary</u>: Let c, w and n be positive integers such that $n > \frac{c}{2} - 1$, $n > 8$ and $c > w$.

Then $(\mathrm{Var}(F_n(\underline{\underline{N}}_c)) \wedge \underline{\underline{N}}_w) = \mathrm{Var}(F_{n+c-w}(\underline{\underline{N}}_w))$.

6. Two Conjectures Regarding the Laws of $F_n(\underset{=}{N}_c)$.

6.0 The following conjectures arise out of the results of Chapters 3, 4 and 5. They provide, I think, a convenient way of summarising the results of those chapters.

6.1 Conjecture: If c and n are positive integers such that $c > n \geq 2$ then

$$(\mathrm{Var}F_n(\underset{=}{N}_c)) \wedge \underset{=}{N}_{c-1} = \mathrm{Var}(F_{n+1}(\underset{=}{N}_{c-1})) \qquad (6.1.1)$$

if and only if either (i) $n > \frac{1}{2}(c-3)$

or (ii) $c = 7$ and $n = 2$.

Evidence: Corollary 5.8 shows that (6.1.1) holds if $n > 8$ and $n > \frac{1}{2}(c-2)$. Corollary 4.16 shows that if c is even and $n = \frac{1}{2}(c-4)$ then (6.1.1) is not true. Arguments similar to that in Section 4.14 but using different partitions should be capable of eliminating smaller values of n. This would mean that, in the case where c is even there is only one outstanding possibility, $n = \frac{1}{2}(c-2)$. This gap is surprisingly small considering that the main lemma of Chapter 5 has a little less finesse than the average stampeding elephant.

If c is odd then the gap is rather larger.

The main reason for the conjecture is a belief that the laws obtained in Theorem 3.11 generate all the laws of $F_n(\underset{=}{N}_c)$ for the relevant values of n.

Suppose then that $c > w > n > \frac{1}{2}(c-3)$, and that λ is a partition of n into k parts where $k > n$ and that

$$\sum_{i=n+1}^{k} \lambda_i \geq c - w + 1 \qquad (6.1.2)$$

Then suppose further that $\lambda_{n+1} \geq 2$, that is $\lambda_i \geq 2$ for $i \leq n + 1$. Then

$$\sum_{i=n+1}^{k} \lambda_i = w - \sum_{i=1}^{n} \lambda_i \leq w - 2n < w - c + 3 \leq 1,$$

which is a contradiction since we assumed $\lambda_{n+1} \geq 2$. Thus the only partitions of w satisfying (6.1.2) are those whose $n + 1$th part is one. Suppose λ is such a partition then

$$\sum_{i=n+2}^{k} \lambda_i = \sum_{i=n+1}^{k} \lambda_i - 1 \geq (c-1)-w + 1.$$

Thus the laws corresponding to λ will be laws in $F_{n+1}(\underset{=}{N}_{c-1})$.

It follows that, if we assume that all the laws of $F_n(\underset{=}{N}_c)$ are generated by those given in Theorem 3.11 then

$$\mathrm{Var}(F_{n+1}(\underset{=}{N}_{c-1})) \subseteq \mathrm{Var}(F_n(\underset{=}{N}_c)) \wedge \underset{=}{N}_{c-1}$$

when $n > \frac{1}{2}(c-3)$. Under the same assumption the reverse inclusion is easily obtained, and the sufficiency of condition (i) is plausible.

Now suppose that $n \leqslant \frac{1}{2}(c-3)$. Define a function λ from Z_{n+1} to Z by

$1\lambda = (c-1) - 2n$

$i\lambda = 2$ for all i such that $2 \leq i \leq n + 1$.

Clearly 1λ is greater than 2 and λ is a partition of $c - 1$ into $n + 1$ parts. By Theorem 3.11 the laws associated with λ are laws in $F_n(\underset{=}{N}_c)$. By Theorem 2.3.9 they are laws in $F_{n+1}(\underset{=}{N}_{c-1})$ only if they are laws in $\underset{=}{N}_{c-1}$. In section A.4.2 of the appendix I indicate why I believe this second possiblity only eventuates when $c = 7$ and $n = 2$. This is the justification for my belief that one of conditions (i) and (ii) must hold whenever (6.1.1) does.

<u>6.2 Conjecture</u>: The laws given in Theorem 3.11 do not, in general, generate all the laws of $F_n(\underset{=}{N}_c)$.

<u>Evidence</u>: My main reason for believing this is a feeling that the proof of Lemma 3.10 can be made even more horrible. Briefly this lemma uses the fact that there are only n expressions of height one in $\underset{\sim}{A}_n$ which means that any homomorphism: $\underset{\sim}{A}_w \to \underset{\sim}{A}_n$ must either map two generators to the same expression or map a generator to an expression of height greater than one. There are approximately n^2 expressions of weight 2 in $\underset{\sim}{A}_n$ and it is possible that, when $n^2 < w$, this could become important.

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Appendix

Categories Multiplicities and Things.

A.0. Chapter 2 stems largely from a paper by Higman [6]. The major changes are that where Higman deals with varieties of groups of prime exponent I have dealt with torsion free varieties and that while Higman uses categoric concepts I have avoided them.

The categoric approach has the advantage that it avoids the computational detail of Section 2.3 and gives a much more understandable picture of what is happening. At the same time the categoric approach needs a fair amount of preliminary work to translate the varietal problem to a categoric one. This appendix is included by reason of the first of these two considerations and is an appendix by reason of the second.

To some extent this appendix duplicates work contained in Stewart's Ph.D. Thesis [12]. However Stewart's treatment of some aspects of the relationship between functors and varieties is rather terse and I have found it useful for my own understanding to expand his account considerably.

Higman confines his remarks to nilpotent varieties. In Section A.2 I show that the relationship between functors and varieties is more general. I doubt that this generalisation is useful but think it is interesting.

In Section A.3. I give an indication of the way in which A.2 can be applied to nilpotent varieties and, in particular, to its application to nilpotent varieties of prime exponent. The details of this application are well covered by Stewart and I have given only a brief discussion which attempts mainly to show the relationship between Higman's paper, Stewart's thesis and my thesis.

In Section A.4 I discuss three ways of applying Higman's technique to torsion free nilpotent varieties, my own, Kovács and Newman's and the way I think it should be done. I also indicate the reason behind one of the conjectures of Chapter 6.

A.1.1. (i) Definition. A concrete category $\underline{\underline{K}}$ is a class of elements P called the objects of $\underline{\underline{K}}$, together with two functions α and $Hom_{\underline{\underline{K}}}$.

This first function α assigns to each object of $\underline{\underline{K}}$ a set called the underlying set of that object. The second function $Hom_{\underline{\underline{K}}}$ assigns to each pair of objects, (A,B) say, a subset $Hom_{\underline{\underline{K}}}(A,B)$ of the functions from $A\alpha$ to $B\alpha$, in a way that satisfies the following conditions.

(a) If A is any object of $\underline{\underline{K}}$ the identity function on $A\alpha$ denoted 1_A, is in $Hom_{\underline{\underline{K}}}(A,A)$.

(b) If A,B and C are any three objects of $\underline{\underline{K}}$ and $\theta \in Hom_{\underline{\underline{K}}}(A,B)$ and $\varphi \in Hom_{\underline{\underline{K}}}(B,C)$ then $\theta\varphi \in Hom_{\underline{\underline{K}}}(A,C)$.

The elements of $Hom_{\underline{\underline{K}}}(A,B)$ are called the $\underline{\underline{K}}$-morphisms (or simply the morphisms) from A to B.

(ii) Convention: Since the only categories I use are concrete I will henceforth use the term category to mean concrete category. I will normally identify objects with their underlying sets, dispensing with the mapping α.

If $\underline{\underline{K}}$ is a category I will write $A \in \underline{\underline{K}}$ to mean A is an object of $\underline{\underline{K}}$.

A subset B of an object A of a category $\underline{\underline{K}}$ is a subobject (in $\underline{\underline{K}}$) of A if B is an object of $\underline{\underline{K}}$ and the inclusion mapping of B in A is in $\mathrm{Hom}_{\underline{\underline{K}}}(B,A)$.

A.1.2 Examples. The following examples of categories are included here partly to elucidate the definition but mainly because they will be used later in this appendix.

(i) Every variety of groups can be regarded as a category whose morphisms are the group homomorphisms.

(ii) For each variety $\underline{\underline{V}}$ the class of $\underline{\underline{V}}$-free groups of finite rank together with their group homomorphisms form a category which I will denote by $\underline{\underline{F}}_0(\underline{\underline{V}})$.

(iii) For each commutative ring R the class of R-modules together with their R-homomorphisms form a category which I will denote by $\underline{\underline{R}}$.

(iv) For each commutative ring R the class of free R-modules of finite rank together with their R-homomorphisms form a category which I will denote by $\underline{\underline{R}}_0$.

A.1.3 Definition. A functor, f say, from a category $\underline{\underline{U}}$ to a category $\underline{\underline{V}}$ is a mapping which assigns to each object A of $\underline{\underline{U}}$ an object f(A) of $\underline{\underline{V}}$ and to each morphism ψ in $\mathrm{Hom}_{\underline{\underline{U}}}(A,B)$ a morphism $f(\psi)$ in $\mathrm{Hom}_{\underline{\underline{V}}}(f(A),f(B))$, in such a way that

(i) $f(1_A) = 1_{f(A)}$ for all $A \varepsilon \underline{\underline{U}}$ and

(ii) whenever $\theta \varepsilon \mathrm{Hom}_{\underline{\underline{U}}}(A,B)$ and $\varphi \varepsilon \mathrm{Hom}_{\underline{\underline{U}}}(B,C)$ then $f(\theta\varphi) = f(\theta)f(\varphi)$.

A.1.4 Examples. The following functors will be useful later.

(i) For each category $\underline{\underline{K}}$ there is an identity functor which fixes the objects and morphisms of $\underline{\underline{K}}$. I will denote this functor by $e_{\underline{\underline{K}}}$.

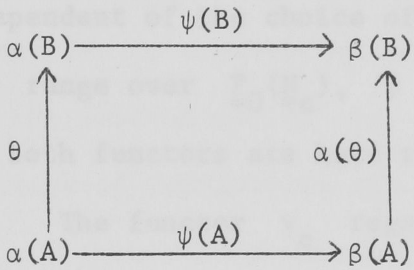
(ii) There is a functor from $\underline{\underline{N}}_c$ to $\underline{\underline{Z}}$ which maps each group in $\underline{\underline{N}}_c$ to the cth term of its lower central series and each morphism $\theta \varepsilon \mathrm{Hom}_{\underline{\underline{N}}_c}(A,B)$ to $\theta|_{\gamma_c(A)}$.

(iii) The functor ab. from $\underline{\underline{O}}$, the category of all groups, to $\underline{\underline{Z}}$, which maps each group to its commutator factor group and each group homomorphism to the homomorphism it induces between the appropriate factor groups.

(iv) For each commutative ring R there is a functor $[c]_R$ from $\underline{\underline{R}}$ to $\underline{\underline{R}}$ which maps each R-module to its c-fold tensor power over R and maps each R-morphism to the R-morphism it induces in the appropriate tensor power. Since the ring R will usually be obvious from the context I will drop the subscript on [c].

(v) For each integral domain R and each positive integer c there is a functor L_c from $\underline{\underline{R}}_0$ to $\underline{\underline{R}}_0$ which maps each free R-module U to the cth component of the free Lie Algebra freely generated by U.

A.1.5 Definition. (i) Let U and V be categories and α and β functors from U to V. A natural transformation from α to β is a mapping, ψ say, which assigns to each object A of U a morphism $\psi(A) \in \mathrm{Hom}_V(\alpha(A), \beta(A))$ in such a way that the following diagram commutes for every pair of objects A and B in U and every $\theta \in \mathrm{Hom}_U(A,B)$.



(ii) Let U, V, α, β, ψ satisfy the conditions of (i). Then ψ is called a natural equivalence, and α and β are said to be equivalent, if $\psi(A)$ is an isomorphism for all objects A of U, that is if $\psi(A)$ has an inverse in $\mathrm{Hom}_V(\beta(A), \alpha(A))$ for all $A \in U$.

(iii) If ψ, α, β, U and V satisfy the conditions of (i) and, for each $A \in U$, $\alpha(A)$ is a subobject of $\beta(A)$ and $\psi(A)$ is the inclusion map of $\alpha(A)$ in $\beta(A)$ then α is said to be a subfunctor of β. The subfunctors of a given functor are considered to be partially ordered by inclusion.

(iv) If ψ, α, β, U and V satisfy the conditions of (i) and $\psi(A)$ is a epimorphism for all $A \in V$ then β is a quotient functor of α.

A.1.6 Examples. (i) Let G be a free-nilpotent-of-class-c group, freely generated by $g_1, g_2, \ldots, g_n$. Define a homomorphism $\psi(G)$ from $L_c(ab.(G))$ to $\gamma_c(G)$ by

$$\psi(G) : [g_{\lambda_1} \circ \gamma_2(G), g_{\lambda_2} \circ \gamma_2(G), \ldots, g_{\lambda_c} \circ \gamma_2(G)] \mapsto [g_{\lambda_1}, g_{\lambda_2}, \ldots, g_{\lambda_c}].$$

Magnus Karrass and Solitar [10] Theorem 5.12, page 337 show that this homomorphism is in fact an isomorphism. It is not difficult to see that it is independent of the choice of generating set for G and that, if we let G range over $\underline{\underline{F}}_0(\underline{\underline{N}}_c)$, ψ is a natural equivalence from $L_c ab.$ to γ_c where both functors are here regarded as functors from $\underline{\underline{N}}_c$ to $\underline{\underline{Z}}$.

(ii) The functor γ_c regarded as a functor from $\underline{\underline{N}}_c$ to $\underline{\underline{N}}_c$ is a subfunctor of $e_{\underline{\underline{N}}_c}$.

(iii) The functor L_c from $\underline{\underline{Z}}$ to $\underline{\underline{Z}}$ is a quotient functor of the functor $[c]$ from $\underline{\underline{Z}}$ to $\underline{\underline{Z}}$.

A.1.7 Definition. A functor α from $\underline{\underline{U}}$ to $\underline{\underline{V}}$ is an epifunctor if for each object A of $\underline{\underline{V}}$ there is an object B of $\underline{\underline{U}}$ such that $\alpha(B)$ is isomorphic to A in $\underline{\underline{V}}$ and for every pair of objects B and C of $\underline{\underline{U}}$ and every morphism θ in $\mathrm{Hom}_{\underline{\underline{V}}}(\alpha(B),\alpha(C))$ there is a morphism φ in $\mathrm{Hom}_{\underline{\underline{U}}}(B,C)$ such that $\alpha(\varphi) = \theta$.

A.1.8 Example. The functor ab. regarded as a functor from $\underline{\underline{F}}_0(\underline{\underline{N}}_c)$ to $\underline{\underline{Z}}_0$ is an epifunctor.

A.1.9 Lemma. Let $\underline{\underline{U}}$, $\underline{\underline{V}}$ and $\underline{\underline{W}}$ be categories and suppose that α is an epifunctor from $\underline{\underline{U}}$ to $\underline{\underline{V}}$ and that β is a functor from $\underline{\underline{V}}$ to $\underline{\underline{W}}$. Suppose further that the subfunctors of β form a lattice under the obvious partial ordering. Suppose $\underline{\underline{W}}$ has the property that whenever $A, B \in \underline{\underline{W}}$, θ is a $\underline{\underline{W}}$ isomorphism from A to B and C is a subobject of A then $C\theta$ is a subobject of B.

Then the subfunctors of $\beta\alpha$ form a lattice which is isomorphic to the lattice of subfunctors of β.

Proof. There is clearly an inclusion preserving mapping from the lattice of subfunctors of β to the set of subfunctors of $\beta\alpha$ given by $t \mapsto t\alpha$. I will show that this mapping has an inverse which is also inclusion preserving and the required result will follow.

In order to obtain the inverse mapping without invoking the axiom of choice a certain amount of manoeuvring is necessary.

For each quadruplet (B,A,θ,t) such that $B \in \underline{\underline{V}}$, $A \in \underline{\underline{U}}$, θ is an isomorphism in $\mathrm{Hom}_{\underline{\underline{V}}}(\alpha(A),B)$ and t is a subfunctor of $\beta\alpha$ define

$$f(B,A,\theta,t) = t(A)(\beta(\theta)) \subseteq \beta(B).$$

Suppose now that $B \in \underline{\underline{V}}$ and A_1, A_2 are objects of $\underline{\underline{U}}$ with isomorphisms θ_1 and θ_2 in $\mathrm{Hom}_{\underline{\underline{V}}}(\alpha(A_1),B)$ and $\mathrm{Hom}_{\underline{\underline{V}}}(\alpha(A_2),B)$ respectively. Let t be a subfunctor of $\beta\alpha$ and let φ be a morphism in $\mathrm{Hom}_{\underline{\underline{U}}}(A_1,A_2)$ such that $\alpha\varphi = \theta_1\theta_2^{-1}$. Let i and j be the inclusion maps of $t(A_1)$ and $t(A_2)$ in $\beta\alpha(A_1)$ and $\beta\alpha(A_2)$ respectively.

$$\begin{aligned} \text{Then}\quad f(B,A_1,\theta_1,t) &= t(A_1)\beta(\theta_1) \\ &= t(A_1)i(\beta\alpha(\varphi))\beta(\theta_2) \\ &= t(A_1)t(\varphi)j\beta(\theta_2) \\ &\subseteq t(A_2)\beta(\theta_2) \\ &= f(B,A_2,\theta_2,t). \end{aligned}$$

The reverse inclusion can be proved similarly.

Thus, for any object B of $\underline{\underline{V}}$ and any subfunctor t of $\beta\alpha$ there is only one set in the class of all values of $f(B,A,\theta,t)$ obtained by allowing A and θ to vary over all permissible objects and isomorphisms. Denote it by $f(B,t)$.

The restriction we placed on $\underline{\underline{W}}$ ensures that $f(B,t)$ is a subobject of $\beta(B)$. Now suppose B_1 and $B_2 \in \underline{\underline{V}}$ and $\psi \in \mathrm{Hom}_{\underline{\underline{V}}}(B_1,B_2)$. Let A_1 and $A_2 \in \underline{\underline{U}}$ and let θ_1 and θ_2 be $\underline{\underline{V}}$-isomorphisms from $\alpha(A_1)$ to B_1 and from $\alpha(A_2)$ to B_2 respectively. Let φ be a $\underline{\underline{U}}$-morphism from A_1 to A_2 such that $\alpha(\varphi) = \theta_1\psi\theta_2^{-1}$. Let t be a subfunctor of $\beta\alpha$.

$$\begin{aligned} \text{Then}\quad f(B_1,t)\beta(\psi) &= t(A_1)\beta(\theta_1\psi) \\ &= t(A_1)\beta\alpha(\varphi)\beta(\theta_2) \\ &\subseteq f(B_2,t). \end{aligned}$$

It follows that for any subfunctor t of $\beta\alpha$ we can define a subfunctor t^* of β by

$$t^*(B) = f(B,t), \quad \text{for all } B \in \underline{\underline{V}}.$$

Clearly

$$t^*\alpha = t \qquad \text{for any subfunctor } t \text{ of } \beta\alpha$$

and

$$(t\alpha)^* = t \qquad \text{for any subfunctor } t \text{ of } \beta$$

and the proof is complete.

A.2. Varieties and Functors.

A.2.0. In this section I will be dealing with the relationship between lattices of subfunctors and lattices of varieties. This section is derived from Higman's paper [6] but is a little more general.

I will need a little of the notation developed in Chapter 1, particularly the free groups $F(\underline{\underline{V}})$ and $F_n(\underline{\underline{V}})$ which remain distinct representatives of their isomorphism classes. I will, of course, also need the notation developed in the first part of this appendix.

A.2.1. Definition. (i) For each variety of groups, $\underline{\underline{V}}$ the functor $e_0(\underline{\underline{V}})$ is the restriction of the functor $e_{\underline{\underline{V}}}$ to the category $\underline{\underline{F}}_0(\underline{\underline{V}})$. It is regarded as a functor from $\underline{\underline{F}}_0(\underline{\underline{V}})$ to $\underline{\underline{V}}$.

(ii) For each verbal subgroup, w, of $F(\underline{\underline{V}})$ w_0 is the subfunctor of $e_0(\underline{\underline{V}})$ given by

$$w_0(G) = \{a\theta : a \in F(\underline{\underline{V}}),\ \theta \in \mathrm{Hom}(F(\underline{\underline{V}}),G)\}$$

(Hanna Neumann [11], 12.31, page 5 , shows that this is a functor when $\underline{\underline{V}}$ is the variety of all groups. The adaptation to relatively free groups is trivial, as is the fact that w_0 so defined is a subfunctor of $e_{0,\underline{\underline{V}}}$.)

(iii) For each subfunctor, t, of $e_0(\underline{\underline{V}})$ t* is the subgroup of $F(\underline{\underline{V}})$ given by

$$t^* = \bigcup_{n\varepsilon Z^+} t(F_n(\underline{\underline{V}})).$$

t^* is clearly fully invariant in $F(\underline{\underline{V}})$ and hence verbal.

A.2.2 Lemma. If t is a subfunctor of $e_0(\underline{\underline{V}})$ and w is a verbal subgroup of $F(\underline{\underline{V}})$ then

(i) $t(F_n(\underline{\underline{V}})) = t^* \cap F_n(\underline{\underline{V}})$,

and

(ii) $w_0(F_n(\underline{\underline{V}})) = w \cap F_n(\underline{\underline{V}})$.

Proof. Both parts can be proved along the same lines as 12.62 on page 7 of Hanna Neumann's book [11]. I will prove only (i) which has slightly less similarity to that result.

For each pair of integers $n > m \geq 1$ define $\pi_{n,m}$ to be the projection homomorphism from $F_n(\underline{\underline{V}})$ to $F_m(\underline{\underline{V}})$, that is,

$$\pi_{n,m} : x_i \mapsto \begin{cases} x_i & \text{if } i \varepsilon Z_m \\ 1 & \text{otherwise.} \end{cases}$$

Then $\pi_{n,m}$ acts as the identity mapping on $F_m(\underline{\underline{V}})$ and it follows that $t(\pi_{n,m})$ acts as the identity mapping on $(t(F_n(\underline{\underline{V}}))) \cap F_m(\underline{\underline{V}})$.

Thus $(t(F_n(\underline{\underline{V}}))) \cap F_m(\underline{\underline{V}}) \subseteq t(F_m(\underline{\underline{V}}))$.

The reverse inclusion can be obtained immediately by considering the action of t on the inclusion homomorphism from $F_m(\underline{\underline{V}})$ to $F_n(\underline{\underline{V}})$.

Thus $t(F_m(\underline{\underline{V}})) = F_m(\underline{\underline{V}}) \cap tF_n(\underline{\underline{V}})$ for all $n > m \geq 1$.

It follows that

$$t(F_m(\underline{\underline{V}})) = \bigcup_{n=m}^{\infty} (F_m(\underline{\underline{V}}) \cap t(F_n(\underline{\underline{V}}))) = F_m(\underline{\underline{V}}) \cap (\bigcup_{n\varepsilon Z^+} t(F_n(\underline{\underline{V}})))$$
$$= F_m(\underline{\underline{V}}) \cap t^*$$

<u>A.2.3 Lemma.</u> The mappings

(i) $t \mapsto t^*$ for each subfunctor t of $e_0(\underline{\underline{V}})$,

and (ii) $w \mapsto w_0$ for each verbal subgroup w of $F(\underline{\underline{V}})$,

are lattice isomorphisms between the lattice of subfunctors of $e_0(\underline{\underline{V}})$ and the lattice of verbal subgroups of $F(\underline{\underline{V}})$. In fact (i) is the inverse of (ii).

<u>Proof.</u> Suppose w is a verbal subgroup of $F(\underline{\underline{V}})$.

Then $(w_0)^* = \bigcup_{n\varepsilon Z^+} w_0(F_n(\underline{\underline{V}})) = \bigcup_{n\varepsilon Z^+} (F_n(\underline{\underline{V}}) \cap w) = w$.

Suppose now that t is a subfunctor of $e_0(\underline{\underline{V}})$ and that $G \varepsilon \underline{\underline{F}}_0(\underline{\underline{V}})$. Then G is a $\underline{\underline{V}}$-free group of finite rank. This means that there exists a positive integer n such that there is an isomorphism, θ say, in $\mathrm{Hom}(F_n(\underline{\underline{V}}),G)$. Thus $(t^*)_0(G) = (t^*)_0(F_n(\underline{\underline{V}}))(t^*)_0(\theta)$

$$= (t^* \cap F_n(\underline{\underline{V}}))\theta\Big|_{t^* \cap F_n(\underline{\underline{V}})}$$
$$= t(F_n(\underline{\underline{V}}))\, t(\theta)$$
$$= t(G).$$

Thus each mapping is the inverse of the other and, since both are clearly inclusion preserving, they must be lattice isomorphisms.

A.2.4 Theorem. Let $\underline{\underline{V}}$ be a variety of groups. The lattice of subfunctors of $e_0(\underline{\underline{V}})$ is dual isomorphic to the lattice of subvarieties of $\underline{\underline{V}}$ under the dual isomorphism

$$t \mapsto \mathrm{Var}(F(\underline{\underline{V}})/t^*)$$

Proof. This theorem follows trivially from the preceding lemma and the dual isomorphism between subvarieties of $\underline{\underline{V}}$ and verbal subgroups of $F(\underline{\underline{V}})$.

Note: (i) This theorem enables us to find the subvarieties of $\underline{\underline{V}}$ without having to work with groups of infinite rank. That is to say it puts into a categoric form the relationship that must exist in a sequence of verbal subgroups

$$w_1 \le F_1(\underline{\underline{V}}), w_2 \le F_2(\underline{\underline{V}}), \ldots, w_n \le F_n(\underline{\underline{V}}), \ldots$$

if it is to be the sequence of the n-variable laws of a subvariety of $\underline{\underline{V}}$.

(ii) It is worth noting that this isomorphism does not hold for the lattice of subfunctors of $e(\underline{\underline{V}})$. In fact $e(\underline{\underline{V}})$ can have two different functors whose restrictions to $F_0(\underline{\underline{V}})$ are equal. For instance suppose $\underline{\underline{V}}$ is torsion free and let t_1 and t_2 be the functors defined by

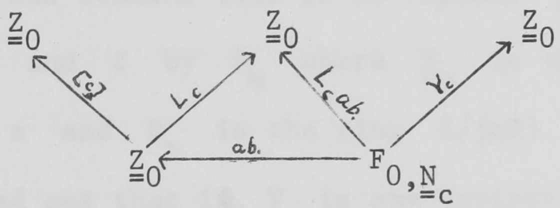
$$t_1 : G \mapsto \{ab : a \in \gamma_2(G),\ b \in G \text{ and } b^n = 1 \text{ for some } n \in Z\setminus\{0\}\}$$

$$t_2 : G \mapsto \{a : a \in \gamma_2(G)\}.$$

A.3 The Varieties Between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$.

From Theorem A.2.4 we know that the lattice of varieties between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$ is dual isomorphic to the lattice of those subfunctors of $e_0(\underline{\underline{N}}_c)$ which are subfunctors of γ_c, here regarded as a functor from $F_0(\underline{\underline{N}}_c)$ to $\underline{\underline{N}}_c$. Since γ_c in fact maps $F_0(\underline{\underline{N}}_c)$ to $\underline{\underline{Z}}_0$ it can be regarded as a functor from $F_0(\underline{\underline{N}}_c)$ to $\underline{\underline{Z}}$ and, so regarded, its subfunctor structure remains the same.

We will consider the following diagram.



I have already observed that there is a natural equivalence from $L_c ab.$ to γ_c (A 1.6.1 p. A.7).

It follows that the lattices of subfunctors of these two functors are isomorphic. Furthermore, since ab. is an epifunctor both lattices must be isomorphic to the lattice of subfunctors of L_c. Finally there is a natural transformation ψ from $[c]$ to L_c which is obtained as follows. Let U be a free Z-module of rank n, with basis $u_1, u_2, \ldots, u_n$. Then $\psi(U)$ is defined by

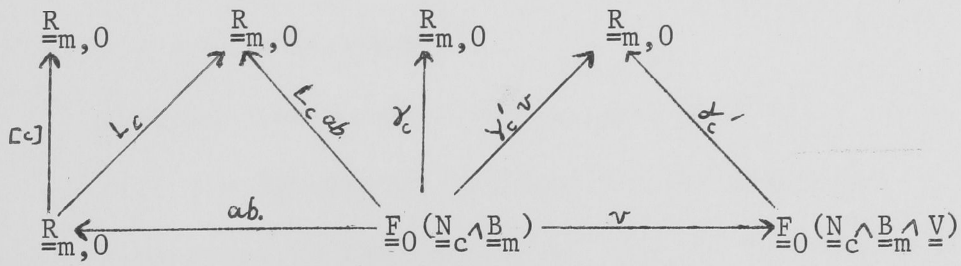
$\psi(U) : (u_{\lambda_1} \otimes u_{\lambda_2} \otimes \ldots \otimes u_{\lambda_c}) \mapsto [u_{\lambda_1}, u_{\lambda_2}, \ldots, u_{\lambda_c}]$ for any

$\lambda_1, \lambda_2, \ldots, \lambda_c \in Z_n$.

It follows that there is an inclusion preserving mapping from the lattice of subfunctors of $[c]$ to the lattice of subfunctors of L_c and thence to the lattice of subfunctors of γ_c.

We have thus established a relationship between the subfunctors of $[c]$ regarded as a functor from $\underline{\underline{Z}}_0$ to $\underline{\underline{Z}}_0$ and the varieties between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$.

All this remains true if we replace $\underline{\underline{N}}_c$ by $\underline{\underline{N}}_c \wedge \underline{\underline{B}}_m$, $\underline{\underline{N}}_{c-1}$ by $\underline{\underline{N}}_{c-1} \wedge \underline{\underline{B}}_m$ and Z by R_m where $\underline{\underline{B}}_m$ is the variety of groups of exponent m and R_m is the ring $Z/(mZ)$. In fact Stewart [12] has pointed out that if $\underline{\underline{V}}$ is any variety of groups, whose laws are V, we can modify the diagram on page A16 to obtain the following diagram



where $\underline{\underline{R}}_{m,0}$ is the category of free $\underline{\underline{R}}_m$-modules of finite rank where v is the functor $G \mapsto G/V(G)$, γ_c' is the functor which maps each group in $\underline{\underline{F}}_0(\underline{\underline{N}}_c \wedge \underline{\underline{B}}_m \wedge \underline{\underline{V}})$ to the cth term of its lower central series, the morphisms being mapped in the obvious way in each case.

There is a natural transformation φ from γ_c to $\gamma_c'v$, obtained by setting $\varphi(G)$ equal to the restriction to $\gamma_c(G)$ of the natural epimorphism from G to $G/V(G)$. It follows that there is an inclusion preserving mapping from the lattice of subfunctors of γ_c to the lattice of subfunctors of $\gamma_c'v$. Since v is an epifunctor the latter lattice is isomorphic to the lattice of subfunctors of γ_c' and we have found a relationship between the subfunctors of $[c]$ and the varieties between $\underline{\underline{N}}_c \wedge \underline{\underline{B}}_m \wedge \underline{\underline{V}}$ and $\underline{\underline{N}}_{c-1} \wedge \underline{\underline{B}}_m \wedge \underline{\underline{V}}$.

If m is a prime then R_m is a field and $\underline{\underline{R}}_{m,0}$ is the category of finite dimensional R_m-spaces. If $m > c$ then $[c]$ is completely reducible and its irreducible subfunctors can be obtained from the primitive idempotents of $R_m S_c$.

Stewart used this result to find the lattice of subvarieties of the variety of centre-extended-by-metabelian groups, of exponent p (a prime) and nilpotency class $c < p$.

A.4. Three Approaches to the Torsion Free Case.

A.4.1. The functor [c], regarded as a functor from $\underline{\underline{Z}}$ to $\underline{\underline{Z}}$ is not completely reducible and, in fact has no minimal subfunctors. Similarly there are no minimal verbal subgroups of $F_n(\underline{\underline{N}}_c)$ in $\gamma_c(F_n(\underline{\underline{N}}_c))$ and no maximal varieties between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$.

However if we restrict ourselves to torsion free varieties between $\underline{\underline{N}}_c$ and $\underline{\underline{N}}_{c-1}$, which means isolated subfunctors of [c], we can find a direct decomposition.

I have not been able to make the concept of isolation fit properly into a categorical context. I have therefore, in Chapter 2, taken advantage of the fact that a variety that is nilpotent of class at most c , where c is greater than 2, is generated by its free groups of rank c.

This means that instead of having to consider the behaviour of γ_c on $\underline{\underline{F}}_0(\underline{\underline{N}}_c)$ it is sufficient to consider it on $\{F_n(\underline{\underline{N}}_c) : n \in Z_c\}$. Having made this step the categoric notation is unecessary and can be replaced by module notation.

My approach in Chapter 2 was to show that if U is a free Z-module of rank n, the lattice of isolated $End_Z(U)$-submodules of $U^{[c]}$ was lattice isomorphic to the lattice of $End_Q(Q \otimes_Z U)$-submodules of $(Q \otimes_Z U)^{[c]}$ and to find the fully invariant subgroups of $F_n(\underline{\underline{N}}_c)$ in $\gamma_c(F_n(\underline{\underline{N}}_c))$ from there. In the process I effectively short circuited the functor L_c, which was probably not a good idea.

A.4.2. Kovács and Newman approached the problem from the other end. They formed the divisible completion of $F_n(\underline{\underline{N}}_c)$, (a group which is closed under root extraction in which $F_n(\underline{\underline{N}}_c)$ can be imbedded) and showed that there is a lattice isomorphism between the lattice of fully invariant subgroups of this group and the lattice of isolated subgroups of $F_n(\underline{\underline{N}}_c)$. This construction is more complicated than simply tensoring a $\underline{\underline{Z}}$-module with Q but has the following advantage.

Let the divisible completion of $F_n(\underline{\underline{N}}_c)$ be G. Then $\gamma_c(G)$ is a Q-space and as such is isomorphic to the component of degree c in the free Lie algebra of rank n over Q. Specifically let U be a Q-space of dimension n. Then $\gamma_c(G) \cong L_c(U)$ and, if we regard $\gamma_c(G)$ and $L_c(U)$ as GL(n,Q)-modules in the obvious way, then the isomorphism is a GL(n,Q)-isomorphism. Now the GL(n,Q)-submodules of $\gamma_c(G)$ are the fully invariant subgroups of G which are contained in $\gamma_c(G)$.

The minimal GL(n,Q)-submodules of $\gamma_c(G)$ can thus be associated with minimal irreducible GL(n,Q)-submodules of $U^{[c]}$ and hence with primitive idempotents of QS_c which in turn can be associated with partitions of n. Turning the process around we can associate with each partition of QS_c a set, possibly empty, of isomorphic irreducible GL(n,Q)-submodules of $\gamma_c(G)$. In Chapter 6 I showed that it would be interesting to know which partitions gave rise to non empty sets of irreducible submodules when n = c.

Brandt [2] has shown that in this case the character of the representation of $GL(c,Q)$ afforded by $L_c(U)$ is given by

$$M \mapsto \frac{1}{c} \sum_{d|c} \mu(d) s_d^{c/d}$$

where μ is the Moebius function and s_d is the trace of M^d. From this it is possible to obtain the multiplicities of the irreducible representations corresponding to the partitions of c. Thrall [13] and Brandt [2] have published these multiplicities for $c \leq 10$. (Thrall's table for $c = 10$ is incorrect, the corrected table is published in [2].) As we have seen we are interested in partitions which have non zero multiplicity.

The partitions of c into one and c parts both have zero multiplicity whenever $c > 2$. When $7 \leq c \leq 10$ these partitions are the only ones with zero multiplicity. The partition of 6 into 3 equal parts and the partition of 4 into two equal parts both have zero multiplicity. However the partition of 8 into four equal parts and that of 10 into 5 equal parts have multiplicities 1 and 2 respectively. It thus seems reasonable to expect that, when $c \geq 7$, all partitions other than the two extremes give rise to non trivial isolated fully invariant subgroups of $\gamma_c(F_c(\underline{\underline{N}}_c))$, which is the basis for Conjecture 6.1 on page 107.

A.4.3. The third possible approach to the problem of finding isolated Z-modules is to let U be a free Z-module of rank n and form $Q \otimes_Z L_c(U)$. It should then be possible to show, using the approach of Section 2.1, that this is isomorphic as an $End_Q(Q \otimes_Z U)$-module to $L_c(Q \otimes_Z U)$, and that the $End_Q(Q \otimes_Z U)$-submodule structure of $Q \otimes_Z L_c(U)$ is the same as the isolated $End_Z(U)$-submodule structure of $L_c(U)$.

If this can be done we would have an aproach that retained the simplicity of Chapter 2 without blocking the use of Brandt's formula.