ON CENTRE-EXTENDED-BY-METABELIAN GROUPS

A thesis submitted for the degree of Doctor of Philosophy in the Australian National University by

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The work in this thesis, except where a statement to the contrary appears, is my own. To this end it should be noted that the work contained in Chapter Two is based on a lecture given by Professor G. Higman of Oxford University.

Alexander G.R. Stewart
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Introduction

This thesis is concerned with varieties of centre-extended-by metabelian groups (i.e. groups with central second derived group) of exponent $p$, for some prime $p$, and of nilpotency class at most $p-1$. A complete classification of such varieties and the inclusion relations between them is given (cf. Chapter Five).

Let $C_c$, for all $c$ in $\{1, \ldots, p-1\}$, be the variety of all centre-extended-by-metabelian groups of exponent $p$ and class at most $c$. The distinct subvarieties of $C_{p-1}$ are given by $C_0, C_1, C_2$ and $X_{i,j}$ for all $j$ in $\{3, \ldots, p-1\}$ and, for each $j$, for all $i$ in $\{1, \ldots, j+1\}$ (except $(i,j) = (p, p-1)$). For all $c$ in $\{3, \ldots, p-1\}$, $C_c = X_{3,c}$. The variety $M_c$ of all metabelian groups of exponent $p$ and class at most $p-1$ is $X_{3,c}$ for all $c$ in $\{3, \ldots, p-1\}$. The subvarieties of $C_{p-1}$ are such that $C_0 < C_1 < C_2 < X_{3,3}$ and $X_{i,j} \cap X_{i',j'} = X_{k,r}$, where $k$ is the lesser of $i, i'$ and $r$ is the lesser of $j, j'$, while $X_{i,j} \cup X_{i',j'} = X_{k^*, r^*}$, where $k^*$ is the greater of $i, i'$ and $r^*$ is the greater of $j, j'$.

Therefore a diagram of the lattice of subvarieties of $C_c$ can be constructed recursively as follows. (As an illustration of this construction the diagram of the lattice of subvarieties of $X_{6,6}$ is given later). For $c$ at most 3 the diagram of the lattice of subvarieties of $C_c$ is given by the relevant subdiagram of
While for $c$ greater than 3 the diagram of the lattice of subvarieties of $C_c$ is obtained from the diagram of the lattice of subvarieties of $C_{c-1}$ by adding the following layer to that diagram in the appropriate places.

Here a circle labelled by $\bigcirc$ indicates that $\bigcirc$ is a subvariety of $C_{c-1}$ and is not part of the layer that is being added but is merely an indication of where the additional layer must be placed.
The following diagram is a diagram of the lattice of subvarieties of $\mathbb{C}_6$. 

The results in this chapter have arisen from an attempt to understand how the methods introduced by Higman for the study of varieties of groups of exponent $p$ (see [6]) could be used to obtain results about other familiar varieties, such as the solvable length and corresponding varieties.

One of the central parts of the proof of our result is a description of the lattice of subvarieties between $\mathbb{C}_{p-1}$ and all other subvarieties $\mathbb{C}_n$. This description is presented in Chapter Three. It is necessary to the presentation here of Higman's methods that a study of the lattice of subvarieties be given in that chapter.

Clearly the lattice of subvarieties is a partial ordering of subvarieties of the variety of subvarieties of $\mathbb{C}_{p-1}$. A study of this lattice is essential part of the description of the lattice. The given in Chapter Three. It is given in detail in Chapter Three. Moreover the results of Chapter Three were in the results of Brieskorn and Venanzio [2] and Willard.
A consequence of the main result of this thesis is a solution of Problem 13 of [9] (cf. Chapter Six).

The results in this thesis have arisen from an attempt to understand how the methods introduced by Higman for the study of varieties of groups of exponent \( p \) and class at most \( p-1 \) (cf. [6]) could be used to obtain results about other familiar varieties; for instance the soluble of length 3 and exponent \( p \) varieties.

One of the crucial parts of the proof of the main result is the description of the lattice of varieties between \( C_C^c \) and \( C_{c-1}^C \) for all \( c \) in \( \{1, \ldots, p-1\} \). This description is obtained in Chapter Four. It is here that Higman's methods are used and Chapter Two is taken up with an exposition of the relevant parts of Higman's theory.

Clearly the lattice of subvarieties of \( M_{p-1}^c \) is a sublattice of the lattice of subvarieties of \( C_{p-1}^p \). A description of this sublattice is an essential part of the description of the lattice and is given in Chapter Three. It is given first as its derivation makes use of Higman's methods in a simple and familiar setting. Moreover the results of Chapter Three confirm the results of Brisley and Weichsel (cf. [2] and [11]).
Chapter One

Definitions and Notation

Some definitions and a few well known results are collected here to establish the notation and for ease of reference.

1.1 Commutators

For brevity the set \( \{1, \ldots, c\} \) of all positive integers not greater than \( c \), will be denoted by \( \mathbb{I}_c \).

Where varietal and group theoretical terms and notation are used without explanation these terms and notation will have the meanings ascribed to them in [9].

Let \( G \) be a group. Then, as noted in the introduction, \( \gamma(G) \) will denote the \( c \) th term of the lower central series of \( G \). The second derived group of \( G \), that is the subgroup \([\gamma_2(G), \gamma_2(G)]\), will be denoted by \( G^{(2)} \).

If \( x_1, \ldots, x_c \) and \( y_1, \ldots, y_s \) are elements of \( G \), then \([x_1, x_2, \ldots, x_c; y_1, y_2, \ldots, y_s]\) will be used to denote the commutator \([[[x_1, x_2, \ldots, x_c], [y_1, y_2, \ldots, y_s]]]\) whenever \( c \) and \( s \) are integers greater than 1.
The abbreviation \([a,ib]\) is defined recursively for all non-negative integers \(i\) and all elements \(a,b\) of \(G\) by
\[
[a,ib] = \begin{cases} 
    a & \text{if } i = 0 \\
    [a, (i-1)b], b & \text{if } i > 0
\end{cases}
\]

In this thesis most commutator calculations are carried out modulo some term of the lower central series. Therefore the following special forms of the well known commutator identities are used frequently. The symbol \(=\) between products of commutators of the same weight means that the products are equal modulo commutators of higher weight. The number in square brackets associated with a statement indicates the relevant number in [5]. Let \(a,b,c,x_1,\ldots,x_z\) be elements of a group \(G\). Then

1.1.1 \([ab,c,x_1,\ldots,x_z] = [a,c,x_1,\ldots,x_z][b,c,x_1,\ldots,x_z]\)

[from 10.2.1.2]

1.1.2 \([a,bc,x_1,\ldots,x_z] = [a,b,x_1,\ldots,x_z][a,c,x_1,\ldots,x_z]\)

[from 10.2.1.3].

It is easily checked that

\([a^{-1},b] = [a,b,a^{-1}^{-1},b^{-1}]^{-1}[a,b]^{-1}\) and that

\([a,b^{-1}^{-1}] = [a,b,b^{-1}^{-1}]^{-1}[a,b]^{-1}\). Using these two results, 1.1.1 and 1.1.2, it can be shown that

1.1.3 \([a,b,x_1,\ldots,x_z] = [b,a,x_1,\ldots,x_z]^{-1}\) and
1.1.4 \([x_1, x_2, \ldots, x^a_1, \ldots, x_z]^a = [x_1, x_2, \ldots, x^a_1, \ldots, x_z]^a\]

for all integers \(a\).

Under the special conditions the Jacobi-Witt identity [10.2.1.4] becomes simply

1.1.5 \([a, b, c, x_1, \ldots, x_z][b, c, a, x_1, \ldots, x_z][c, a, b, x_1, \ldots, x_z] = 1\).

In a metabelian group this last identity holds with equality. It holds with equality in a centre-extended-by-metabelian group if \(z\) is at least 1.

1.2 Terminology from Category Theory

Categorical terms which are used without explanation will have the meanings ascribed to them in [8]. The only radical departure from the normal categorical conventions will be that the composition \(\alpha \beta\) of two maps will be defined as first apply \(\alpha\) then \(\beta\). Here maps means either functors or morphisms.

Only a restricted class of categories will be used. Let \(K\) be a field. Then \(K\) will be the category whose objects are the finite dimensional vector spaces over \(K\) and whose morphisms are the linear maps between them.

The term functor will be used only in the following restricted sense.
Definition 1.2.1 A functor $f$ on the category $\mathcal{K}$ is a mapping of $\mathcal{K}$ into itself that has the following properties:

1. If $\psi$, mapping $U$ to $U'$, is a morphism in $\mathcal{K}$, then the image $f^\psi$ of $\psi$ maps the image $U^f$ of $U$ into the image $U'^f$ of $U'$.

2. If $e$ is the identity map of $U$, then $e^f$ is the identity map of $U^f$.

3. If $\alpha; U \to U'$ and $\beta; U' \to U''$ are in $\mathcal{K}$, then
   $$(\alpha \beta)^f = \alpha^f \beta^f.$$  

The following examples of functors on $\mathcal{K}$ are given because they are useful in the later work.

1.2.2 The identity functor $i$ maps each object and each morphism onto itself.

1.2.3 A zero functor is one that maps each object in $\mathcal{K}$ to a zero dimensional vector space over $K$ and each morphism from $U$ to $U'$ in $\mathcal{K}$ onto the unique map from the image of $U$ to the image of $U'$.

Definition 1.2.4 The functor $t$ is a subfunctor of the functor $f$ if $U^t$ is a subspace of $U^f$ for all $U$ in $\mathcal{K}$ and if $\psi^t$ is the restriction of $\psi^f$ to $U^t$ whenever $\psi$ is a morphism from $U$ to an object in $\mathcal{K}$.

Note. This definition relies on the fact that the objects in $\mathcal{K}$ are set based. A subfunctor in this sense is not a subobject in the category of functors from $\mathcal{K}$ to $\mathcal{K}$. 
A functor has precisely one zero subfunctor.

**Definition 1.2.5** A functor is called **irreducible** if its only subfunctors are itself and its zero subfunctor.

**Definition 1.2.6** A functor \( f \) is a **quotient functor** of the functor \( f \) if for each \( U \) in \( K \) there is a \( K \)-epimorphism \( \varphi_U \) mapping \( U^f \) to \( U^t \) such that
\[
\psi^f \varphi_U = \varphi_U^t \psi^t \quad \text{for all linear maps from } U \text{ to an object } U' \text{ in } K.
\]

That is: if for each \( U \) in \( K \) there is a \( K \)-epimorphism \( \varphi_U \) from \( U^f \) to \( U^t \) for which the diagram
\[
\begin{array}{ccc}
U^f & \xrightarrow{\varphi_U} & U^t \\
\downarrow{\psi^f} & & \downarrow{\psi^t} \\
U'^f & \xrightarrow{\varphi_U'} & U'^t
\end{array}
\]

commutes for all \( \psi : U \to U' \) in \( K \).

The direct sum \( U \oplus U' \) of two objects \( U, U' \) in \( K \) will be considered as the vector space over \( K \) consisting of all ordered pairs \( (u, u') \) with \( u \) in \( U \), \( u' \) in \( U' \) and
\[
(u_1, u'_1) + (u_2, u'_2) = (u_1 + u_2, u'_1 + u'_2).
\]
The direct sum \( \alpha \oplus \beta \) of two linear maps \( \alpha \), from \( U \) to \( U^* \), and \( \beta \), from \( U' \) to \( U'^* \), is the linear map from \( U \oplus U' \) to \( U^* \oplus U'^* \) that maps \( (u, u') \) onto \( (u \alpha, u' \beta) \) for all \( u \) in \( U \) and \( u' \) in \( U' \).
Definition 1.2.7 The direct sum $t \oplus f$ of two functors $t$ and $f$ is defined by:

$$U^t \oplus f = U^t \oplus U^f \quad \text{and} \quad \psi \oplus f = \psi \oplus \psi^f \quad \text{for all } U,$$

where $\psi \in K$.

Let $U$ and $U'$ be objects in $K$. Then the tensor product $U \otimes U'$ will be regarded as the space whose underlying set is the set of all finite formal sums $\sum_{i} u_i \otimes u'_i$ with $u_i$ in $U$ and $u'_i$ in $U'$ for all $i$, where

$$u \otimes (u'_1 + u'_2) = u \otimes u'_1 + u \otimes u'_2 = (u'_1 + u'_2) \otimes u' = u'_1 \otimes u' + u'_2 \otimes u'$$

and $k(\otimes u') = ku \otimes u' = u \otimes ku'$ for all $k$ in $K$. The operation of $K$ on this set is defined by

$$k(\sum_{i} u_i \otimes u'_i) = \sum_{i} k(u_i) \otimes u'_i.$$

If $U$ has a basis $b_1, \ldots, b_n$ and $U'$ has a basis $b'_1, \ldots, b'_m$, then a basis for $U \otimes U'$ is given by the set of all elements of the form $b_{i_1} \otimes b'_{i_2}$ with $i_1$ in $I_n$ and $i_2$ in $I_m$.

If $\alpha$, mapping $U$ to $U^*$, and $\beta$, mapping $U'$ to $U'^*$ are linear maps, then $\alpha \otimes \beta$ is the linear mapping from $U \otimes U'$ to $U^* \otimes U'^*$ defined by

$$(b_{i_1} \otimes b'_{i_2})(\alpha \otimes \beta) = b_{i_1} \alpha \otimes b'_{i_2} \beta$$

on the basis elements of $U \otimes U'$ and extended by linearity to the whole space.

(The justification for considering the tensor product in this way can be found in Section 12 of [3])

Definition 1.2.8 The tensor product $t \otimes f$ of two functors is...
defined by
\[
U^t \otimes f = U^t \otimes U^f \quad \text{and} \quad \psi \otimes f = \psi \otimes \psi^f \quad \text{for all } U, \psi \in K.
\]

Only subfunctors and quotient functors of a particular type of functor will be considered in what follows. These are now defined.

**Definition 1.2.9** A functor \([c]\) on \(K\) is defined recursively for all positive integers by \([1] = \mathcal{I}\), the identity functor on \(K\), and for \(c > 1\) \([c] = [1] \otimes [c-1]\).

Because of the associativity of the tensor product, \(U[c]\) has the set of all \(b_{i_1} \otimes \ldots \otimes b_{i_c}\) for all \(i_1, i_2, \ldots, i_c\) in \(I_n\) as a basis if \(b_1, \ldots, b_n\) is a basis for \(U\).

The set of subfunctors of a functor \(f\) form a lattice in a natural way under the operations \(\cap\), \(\cup\) defined by:

If \(s\) and \(t\) are subfunctors of \(f\), then
\[
U^s \cup U^t = U^s + U^t \quad \text{and} \quad U^s \cap U^t = U^s \cap U^t \quad \text{for all } U \in K.
\]
Chapter Two

The Higman Theory

Throughout this chapter K will be used to denote the field GF(p) of integers modulo p. Furthermore c will always denote an integer less than p.

Let \( B_{=p, c} \) be the variety of all groups of exponent p and class at most c. In his address [6] Professor Graham Higman used the theory of representations of GL(n,K) on \( U[c] \), where U is an n-dimensional vector space over K, to obtain some interesting information about the lattice of varieties between \( B_{=p, c} \) and \( B_{=p, c-1} \).

The methods Higman used to obtain his results can be used to obtain corresponding results for the varieties \( Y_{=c} = B_{=p, c} \cap Y \), where Y is some variety of groups. In the later chapters of this thesis the particular cases \( Y = M \), the variety of all metabelian groups, and \( Y = G \), the variety of all centre-extended-by-metabelian groups, will be considered. However in this chapter a general \( Y \) will be taken.

To obtain information about the lattice of varieties between \( Y_{=c} \) and \( Y_{=c-1} \) a relationship will be established between this lattice and a known lattice, the lattice of all left ideals of \( KS_{c} \), where \( S_{c} \) is the symmetric group on \( c \) symbols.
This relationship will be obtained in several stages. The lattice of varieties between $Y \mathcal{C}_c$ and $Y \mathcal{C}_c^{-1}$ will be shown to be dual isomorphic to the lattice of subfunctors of a functor $L^c_\mathcal{C}$ on $K$. This latter lattice will then be shown to be a quotient lattice of the lattice of subfunctors of $[c]$ by defining an epimorphism $\varphi^c_Y$ from the lattice of subfunctors of $[c]$ onto it. Finally an isomorphism between the lattice of subfunctors of $[c]$ and the lattice of left ideals of $KS^c_c$ will be established. (It is this last stage that uses the representation theory of $GL(n,K)$.)

Thus the lattice of varieties between $Y \mathcal{C}_c$ and $Y \mathcal{C}_c^{-1}$ is dual isomorphic to a quotient lattice of the lattice of left ideals of $KS^c_c$.

To be able to describe the lattice of varieties a description of the lattice of left ideals of $KS^c_c$ is needed. One such description is given in the third section of this chapter. However the study of the effect of the epimorphism $\varphi^c_Y$ on the isomorphic image of this latter lattice in the two particular cases mentioned above is delayed until the later chapters as the methods used there differ from those used by Higman.
2.1 The Functor $L^c$ 

This section begins with the definition of the functor $L^c$. 

Let $U$ be an $n$-dimensional vector space over $K$. If $G_n$ is a free group of rank $n$ of the variety $Y_n$, then the factor group $G_n / \gamma_2(G_n)$ can be considered as a vector space over $K$. As such it is isomorphic to $U$. Let $\nu_U$ denote an isomorphism from $G_n / \gamma_2(G_n)$ to $U$. The $c$-th term $\gamma_c(G_n)$ of the lower central series of $G_n$ is an abelian group of exponent $p$ and so can also be considered as a vector space over $K$. (Here and throughout no notational distinction will be made between $\gamma_c(G_n)$ qua subgroup of $G_n$ and qua vector space as the context will make the meaning clear.) The first step in the definition of $L^c$ is to define $U^c$ as $\gamma_c(G_n)$. 

Now let $\psi$ be a linear mapping from $U$ to an $m$-dimensional vector space $U'$ over $K$. Then $U'$ is isomorphic to $G_m / \gamma_2(G_m)$ for some free group of rank $m$ of $Y_m$ under an isomorphism $\nu_U$, from the factor group,
considered as a vector space over $K$, onto $U'$. Let $G_n$ be freely generated by $g_1, \ldots, g_n$. Then

$$g_i \gamma_2 (G_n) \nu_p \psi U' = g_i' \gamma_2 (G_m)$$

for some representative $g_i'$ of the image coset in $G_m / \gamma_2 (G_m)$. There is a homomorphism $\xi$ from $G_n$ to $G_m$ such that $g_i \xi = g_i'$. The restriction of $\xi$ to $\gamma_c (G_n)$ is a homomorphism from $\gamma_c (G_n)$ into $\gamma_c (G_m)$ and this restriction, considered as a linear map, is

$$\psi \xi$$

This definition is independent of the choice of coset representative in the definition of $\xi$. For if $\xi'$ is another homomorphism from $G_n$ to $G_m$ obtained from $\psi$ in the same manner as $\xi$, then $g_i \xi'$ is equal to $g_i \xi \cdot z_i$ for some $z_i$ in $\gamma_2 (G_m)$. Hence

$$[g_i, \ldots, g_i]_c \xi' = [g_i \xi z_i, \ldots, g_i \xi z_i]_c$$

$$= [g_i \xi, \ldots, g_i \xi]_c \Pi c_s$$

by 1.1.1 and 1.1.2

$$= [g_i \xi, \ldots, g_i \xi]_c$$

since for each $s$, the factor $c_s$ is a commutator of weight $c$ with at least one entry from $\gamma_2 (G_m)$. Hence each factor is trivial. Thus $\xi$ and $\xi'$ have the same action on $\gamma_c (G_n)$ since it is generated by all commutators of the form $[g_i, \ldots, g_i]$ with $1, \ldots, c$ in $I_n$. 
The definition of $L^Y_c$ depends on the choices of the free groups, the choices of the isomorphisms $V_i$ and the choices of the free generators of the free groups. However the functors defined by the various choices are all equivalent and equivalent functors have isomorphic lattices of subfunctors. Therefore an arbitrary but fixed set of free groups, isomorphisms and free generators of the free groups will be used throughout this chapter.

The next theorem is the first stage in the development of the relationship mentioned in the introduction to this chapter.

**Theorem 2.1.1** The lattice of varieties between $Y^c$ and $Y^{c-1}$ is dual isomorphic to the lattice of subfunctors of $L_c^Y$.

**Proof** Throughout this proof use will be made of the fact that every variety between $Y^c$ and $Y^{c-1}$ defines a verbal subgroup contained in the $c$th term of the lower central series in each free group of $Y^c$ and is uniquely determined by those in the finite rank free groups. (c.f. 17.41 of [9]).

The proof is split into four parts. These are

**2.1.2** The definition of a mapping from the set of all subfunctors of $L_c^Y$ to the set of all varieties between $Y^c$ and $Y^{c-1}$. 
2.1.3 The proof that this mapping is one to one.
2.1.4 The proof that this mapping is onto.
2.1.5 The proof that this one to one correspondence takes unions to intersections and intersections to unions.

2.1.2 Let \( t \) be a subfunctor of \( L_c \). Then for each \( U \) in \( K_c \),

\[ U^t \text{ is a subspace of } U^c \] and so may be regarded as a subgroup of \( \gamma_c(G_n) \) for the relevant free group of \( Y_c \).

To show that \( U^t \) determines a verbal subgroup of \( G_n \) it must be shown that \( U^t \) is fully invariant.

Let \( \delta \) be an endomorphism of \( G_n \). Then \( \delta \) defines a

\[ a \text{ linear map } \psi \text{ from } U \text{ to } U \text{ for which } U^c \delta = U^c \psi. \]

Therefore \( U^t \psi^t = U^t \psi^c = U^t \delta \) and so \( U^t \delta \) is contained in \( U^t \) since \( U^t \psi^t \) is. Thus \( U^t \) is invariant under all endomorphism of \( G_n \) and is a verbal subgroup of \( G_n \).

Thus \( t \) determines verbal subgroups in the free groups of each finite rank in \( Y_c \), hence a variety between \( Y_c \) and \( Y_{c-1} \) and the mapping so defined is the one taken for the one in the proof.
2.1.3 If \( t_1 \) and \( t_2 \) are different subfunctors of \( L_c \), there is a \( U \) in \( K \) for which \( U^{t_1} \) and \( U^{t_2} \) are different. Thus \( U^{t_1} \) and \( U^{t_2} \) are different verbal subgroups of \( G_n \) and so the varieties determined by \( t_1 \) and \( t_2 \) must be different.

2.1.4 Let \( V \) be a variety between \( Y_c \) and \( Y'_{c-1} \). Then \( V \) determines verbal subgroups \( V(G_n) \) in each free group of \( Y=V_c \).

A mapping \( t \) of \( K \) into itself is defined as follows:-

Let \( U, \psi : U \rightarrow U' \) be in \( K \). Then \( U^t \) is defined to be \( V(G_n) \) and \( \psi^t \) as the restriction of \( \xi \) to \( V(G_n) \), where \( G_n \) and \( \xi \) are defined as they were in the definition of \( L_c \). To show that \( t \) is a subfunctor of \( L_c \) it is sufficient to show that \( U^t \psi^t = V(G_n) \xi \) is contained in \( U'^t = V(G_m) \). Each element in \( V(G_n) \) is a value of a law \( v(x_1, \ldots, x_s) \) of \( V \). But \( v(h_1, \ldots, h_s) \xi = v(h_1 \xi, \ldots, h_s \xi) \) which is in \( V(G_m) \) as required.

To complete the proof of 2.1.4 it suffices to note that the image of \( t \) under the mapping defined in 2.1.2 is obviously \( V \).
2.1.5 The image of the subfunctor $t^1 \cup t^2$ under the mapping defined in 2.1.2 is the variety determined by the verbal subgroups $U^1 \cup U^2$ of the free groups of finite rank of $Y^\subseteq_c$. Hence it must be the union of the two varieties determined by the sequences $U^1$ and $U^2$ of verbal subgroups of the $G_n$. Thus it is the union of the two images of the functors $t^1$ and $t^2$ under the mapping given in 2.1.2.

A similar argument holds for intersections and the theorem follows.

The next result is the second step in the development of the desired relationship.

Theorem 2.1.6 The lattice of subfunctors of $L_c$ is a quotient lattice of the lattice of subfunctors of $[c]$.

Proof To prove this an epimorphism $\phi^U_{Y^\subseteq} \psi$ from $U^\subseteq [c]$ onto $L_c^\subseteq$ will be defined for all $U$ in $K$ for which

$$\phi^U_{Y^\subseteq} \psi^{L_c} = \psi^C_{U'} \phi^U_{Y^\subseteq}$$

for all morphisms $\psi$ from $U$ to $U'$ in $K$. These epimorphisms will then be used to construct an epimorphism from the lattice of subfunctors of $[c]$ to the lattice of subfunctors of $L_c$. 
Let $U, G_n, U', G_m, \psi$ and $\xi$ be as they were in the definition of the functor $L^\Xi_c$. If $g_i \psi U = b_i$ for all $i$ in $I_n$, then the $b_i$ form a basis for $U$ and $U[c]$ has the set of all elements of the form $b_{i_1} \otimes \ldots \otimes b_{i_c}$ with $i_1, \ldots, i_c$ in $I_n$ as a basis. Let $\varphi^U_Y$ be the epimorphism from $U[c]$ to $U^\Xi_c$ obtained by a linear extension of the mapping defined on the basis of $U[c]$ by

$$b_{i_1} \otimes \ldots \otimes b_{i_c} \varphi^U_Y = [g_{i_1}, \ldots, g_{i_c}].$$

For each basis element of $U[c]$

$$b_{i_1} \otimes \ldots \otimes b_{i_c} \varphi^{U'} \psi = (b_{i_1} \psi \otimes \ldots \otimes b_{i_c} \psi) \varphi^{U'} = [g_{i_1} \xi, \ldots, g_{i_c} \xi]$$

while

$$b_{i_1} \otimes \ldots \otimes b_{i_c} \varphi^U \psi = [g_{i_1}, \ldots, g_{i_c}] \varphi^U \psi = [g_{i_1} \xi, \ldots, g_{i_c} \xi] = [g_{i_1} \xi, \ldots, g_{i_c} \xi].$$
Thus the two linear mappings defined by the two sides of 2.1.7 agree on the basis elements of $U^{[c]}$ hence on the whole of $U^{[c]}$.

An epimorphism $\varphi_Y$ from the lattice of subfunctors of $[c]$ onto the lattice of subfunctors of $L_c$ can be defined by:

Let $f$ be a subfunctor of $[c]$. Then $U^{\varphi_Y} = U^f \varphi_Y$ and $\psi$ is the restriction of $\psi$ to $U$ for all $U$ in $K$ and all linear maps $\psi$ out of $U$.

To show that $\varphi_Y$ has the desired properties it must first be shown that $f \varphi_Y$ is a subfunctor of $L_c$. To do this it is sufficient to show that the $U^\varphi_Y$ is contained in $U'$. But $U^\varphi_Y = U^f \varphi_Y$ and this is contained in $U' \varphi_Y = U'$ since $f$ is a subfunctor of $[c]$. 
Secondly it must be shown that it is a homomorphism between the two lattices. For all $U$ in $K$ and subfunctors $f_1$ and $f_2$ of $[c]$

\[
(f_1 \cup f_2) \varphi_Y = f_1 \varphi_Y U \cup f_2 \varphi_Y U
\]

Therefore \((f_1 \cup f_2) \varphi_Y = f_1 \varphi_Y U \cup f_2 \varphi_Y U\). by the definition of union of functors and because $\varphi_Y$ is a homomorphism.

A similar proof holds for intersections.

The fact that $\varphi_Y$ is an epimorphism follows easily from the definition of $\varphi_Y$ and the property 2.1.7 for if $t$ is a subfunctor of $L_c$, then the function $f$ for which $U^f$ is the complete inverse image of $U^t$ under $\varphi_Y$ in $U[c]$ and for which $\psi^f$ is the restriction of $\psi[c]$ to $U^f$ for all $U$ in $K$ and all $\psi$ out of $U$ gives a functor whose image under $\varphi_Y$ is $t$. The function $f$ is a functor because $U^f \psi^f = U^f \psi[c]$ and

\[
U^f [c] U' = U^f U L_c^f
\]
\[
L^c = U^t \psi 
\]
which is contained in \( U'^t \) and the result follows.

2.2 The Classical Theory

This section is the final stage in the development of the relationship between the lattice of varieties between \( Y \) and \( Y_{c-1} \) and the lattice of left ideals of \( KS_c \).

Theorem 2.2.1 The lattice of left ideals of \( KS_c \) is isomorphic to the lattice of subfunctors of \([c]\).

Proof Let \( x \) be in \( KS_c \). Then the first step in the proof will be to associate with \( x \) a linear mapping of \( U^{[c]} \) into itself for each \( U \) in \( K \). The image of \( U^{[c]} \) under this map determines a subspace of \( U^{[c]} \) and this subspace will be used to define a subfunctor \( c(x) \) of \([c]\). The mapping of \( KS_c \) to the set of subfunctors of \([c]\) given by \( x \) to \( c(x) \) will then be used to define a mapping from the left ideals of \( KS_c \) to the subfunctors of \([c]\) which will be shown to be a lattice isomorphism.

Therefore let \( b_1, \ldots, b_n \) be a basis for \( U \) and let
\[
x = \sum_{\sigma} x_\sigma. 
\]
A linear mapping, associated with \( x \), of \( \sigma \) in \( S_c \)
\( U^{[c]} \) into itself can be defined as a linear extension of the following mapping of the basis of \( U^{[c]} \):
for all $i_1, \ldots, i_c$ in $I_n$. No notational distinction will be made between $x$ qua element of $K_S$ and qua mapping of $U^c$ as the context will always make the meaning clear.

Although this definition appears to depend on the choice of basis for $U$ it can be easily checked that it is independent of the basis chosen.

A function $c(x)$ from $K$ into itself is defined by:

$$U^c(x) = U^c[x]$$ and $\psi^c(x)$ is the restriction of $[c]$ to $U^c(x)$ for all $U$ in $K$ and all linear mappings $\psi$ out of $U$. To show that this is a subfunctor of $[c]$ it is sufficient to prove that $U^c(x) \psi^c(x) = U^c[x] \psi^c[c]$ is contained in $U^c(x)$ for all $U$ in $K$ and for all linear maps $\psi$ from $U$ to $U'$ in $K$.

Therefore let bases $b_1, \ldots, b_n$ and $b'_1, \ldots, b'_m$ be chosen for $U$ and $U'$ respectively so that $b_i \psi = b'_i$ for $i$ in $I_r$ and $b_j \psi$ is trivial for all $j$ greater than $r$. Then

$$b_{i_1} \otimes \cdots \otimes b_{i_c} x \psi^c[c] = \sum_{\sigma \in S_c} x \mathop{\otimes}^{i_{\sigma}} b_{i_{\sigma}} \otimes \cdots \otimes b_{i_{c\sigma}} \psi^c[c]$$
if all the \( i_1, \ldots, i_c \) are at most \( r \). While

\[
\psi_{[c]} x = b_{i_1} \psi_{[c]} \psi_{[c]} x, \quad \text{provided } i_j \in I_r,
\]

If one of the \( i_j \) is greater than \( r \), then both \( x_{\psi_{[c]}} \)
and \( \psi_{[c]} x \) take \( b_{i_1} \psi_{[c]} \psi_{[c]} x \) to the trivial element of \( U_{[c]} \).

Therefore \( U_{[c]} x_{\psi_{[c]}} = U_{[c]} \psi_{[c]} x \) (2.2.2) and this is

contained in \( U'_{[c]} x = U'_{[c]} c(x) \) as required.

A relationship between the left ideals of \( KS_c \) and

the subfunctors of \([c]\) can be established by associating

the left ideal \( KS_e \) with the subfunctor \( c(e) \) where \( e \) is a
generating element of the left ideal. This relation is a

mapping since, if \( KS_e = KS e' \), then there are elements \( x \)

and \( y \) in \( KS_c \) for which \( e = xe' \) and \( e' = ye \). Hence \( U^c(e) \)

\[
= U_{[c]} e = U_{[c]} xe' \text{ which is contained in } U_{[c]} e' = U^c(e').
\]
The reverse inclusion, and hence the equality of \( U^c(e) \) and \( U^c(e') \), follows in a similar fashion from the fact that \( e' = ye \).

The proof that this mapping is a lattice isomorphism will be carried out in the following steps:

1. **The mapping is one to one.**
2. **The mapping is onto.**
3. **The mapping preserves the operations of union and intersection.**

Let \( e_1 \) and \( e_2 \) be two idempotents in \( KS_c \) and let it be supposed that \( c(e_1) = c(e_2) \). To obtain the desired result it must be shown that \( KS_c e_1 = KS_c e_2 \). This equality will be established by obtaining elements \( x \) and \( x' \) in \( KS_c \) for which \( e_1 = xe_2 \) and \( e_2 = xe_1 \).

The elements \( x \) and \( x' \) will be obtained in the following manner. If \( U \) is an \( n \)-dimensional space in \( K \), then \( U^c \) can be considered as a \( GL(n, K) \) module, where \( GL(n, K) \) is the group of all invertible mappings of \( U \) into itself. The algebra \( KS^c \) of all elements of \( KS_c \) considered as mappings of \( U^c \) is a homomorphic image of \( KS_c \) and is equal to \( \text{Hom}_{GL(n,K)}(U^c, U^c) \) the algebra of all \( GL(n,K) \)-endomorphisms of \( U^c \). (This result is the crux of the proof of 2.2.3 and it is proved later).
Therefore if two GL(n, K)-endomorphisms \( \eta_1 \) and \( \eta_2 \) of \( U^{[c]} \) can be found for which \( \varepsilon_1 = \eta_1 \varepsilon_2 \) and \( \varepsilon_2 = \eta_2 \varepsilon_1 \), then \( \varepsilon_1 \) and \( \varepsilon_2 \) would be in the same left ideal of \( KS^*_c \). However if the dimension of \( U \) is at least \( c \), then \( KS^*_c \) is isomorphic to \( KS^*_c \) under the natural mapping between them and so \( \eta_1 \) and \( \eta_2 \) would suffice for \( x \) and \( x' \).

The \( n \) dimensional space \( U \) can be turned into a GL(n, K) module by defining \( t\mu = t\mu^{[c]} \) for all \( t \) in \( U^{[c]} \) and all \( \mu \) in GL(n, K). (That it is in fact a module under this definition is easily checked.)

It follows from 2.2.2 that \( KS^*_c \) is a subalgebra of \( \text{Hom}_{GL(n, K)}(U^{[c]}, U^{[c]}) \). The reverse inclusion is more complex and relies on several results from [3]. It can be shown, using the methods of remark 2 on page 440 of [3], that \( KS^*_c \) is a symmetric algebra. Using the dual bases \( \sigma \) and \( \sigma^{-1} \) to define the pairing \( \tau \) a nucleus \( N \) of \( U^{[c]} \) can be defined. However, using the semi-simplicity of \( KS^*_c \) (cf. [3]) and the natural homomorphism from \( KS^*_c \) onto \( KS^*_c \), it can be shown that \( KS^*_c \) is semi-simple. Therefore Theorem 66.7 of [3] implies that \( U^{[c]} \) is regular with respect to the pairing \( \tau \). Hence it follows from Theorem 66.9 of [3] that for every \( \eta \) in \( \text{Hom}_{GL(n, K)}(U^{[c]}, U^{[c]}) \) there is a \( y \) in \( N \) for which \( ty = t\eta \) for all \( t \) in \( U^{[c]} \).
Therefore $K S^*$ contains $\text{Hom}_{GL(n, K)}(U[c], U[c])$ and the desired equality follows.

Thus the search for elements $x$ and $x'$ has been reduced to a search for two $GL(n, K)$ endomorphisms of $U[c]$ with the properties of $\eta_1$ and $\eta_2$.

Since $e_1$ and $e_2$ are idempotents in $K S^*$, it follows that

$$U[c] = U[c]e_1 \oplus U[c](1-e_1)$$

$$= U[c]e_2 \oplus U[c](1-e_2).$$

Now let $\eta^*$ be a $GL(n, K)$-homomorphism from $U[c](1-e_1)$ into $U[c](1-e_2)$.

Then a $GL(n, K)$-endomorphism $\eta_1$ of $U[c]$ can be defined by:

If $t = t_1 + t_2$ is in $U[c]$, where $t_1$ is in $U[c] e_1$ and $t_2$ is in $U[c](1-e_1)$, then $t\eta_1 = t_1 + t_2 \eta^*$.

However $t\eta_1 e_2 = (t_1 + t_2 \eta^*)e_2$

$$= t_1 e_2 + t_2 \eta^* e_2.$$ But $t_1$ is in $U[c] e_1$ which is equal to $U[c] e_2$, since $c(e_1) = c(e_2)$, and so

$t_1 e_2 = t_1$. However $t_2 \eta^*$ is in $U[c](1-e_2)$, hence equals $t'(1-e_2)$ for some $t'$ in $U[c]$ and so $t_2 \eta^* e_2 = t'(1-e_2)e_2 = t'e_2 - t'e_2^2 = 0.$
Therefore $t \eta_1 e_2 = t_1 = t e_1$. This holds for all $t$ in $U^{[c]}$ and so it follows that $e_1 = \eta_1 e_2$. The symmetry of the situation then implies that there is an $\eta_2$ with the desired properties and 2.2.3 follows if it can be shown that when $n$ is at least $c$, $KS^*_c$ is isomorphic to $KS^*_c$ under the natural mapping between them.

To establish that this mapping is in fact an isomorphism it suffices to show that $y$ is a non trivial mapping of $U^{[c]}$ whenever $y$ is a non trivial element of $KS^*_c$. If $U$ is of dimension at least $c$ and has a basis $b_1, \ldots, b_n$, then

$b_1 \otimes \ldots \otimes b_c$ is a basis element for $U^{[c]}$. But

$$b_1 \otimes \ldots \otimes b_c y = \sum_{\sigma \in S^*_c} y_\sigma b_{\sigma^{-1}} \otimes \ldots \otimes b_{\sigma^{-1}}.$$  

However $b_{\sigma^{-1}} \otimes \ldots \otimes b_{\sigma^{-1}}$ is different from $b_{\rho^{-1}} \otimes \ldots \otimes b_{\rho^{-1}}$ whenever $\sigma$ and $\rho$ are different. Therefore it follows from the independence of the basis elements of $U^{[c]}$ that $b_1 \otimes \ldots \otimes b_c y$ is trivial if and only if $y_\sigma$ is trivial for all $\sigma$ in $S^*_c$ and the result follows.
2.2.4 Let $f$ be a subfunctor of $[c]$. Then the existence of an idempotent $e^*$ in $KS_c$ for which $f = c(e^*)$ must be established.

Let $T^f$ be the set of all elements $x$ in $KS_c$ for which $U^f x$ is contained in $U^f$ for all $U$ in $K$. Since $U^{[c]} x y$ is contained in $U^f$, it follows that $yx$ is contained in $T^f$ whenever $x$ is.

Furthermore, if $x_1$ and $x_2$ are in $KS_c$, it follows from the fact that $U^{[c]}(x_1 + x_2)$ is contained in $U^{[c]} x_1 U^{[c]} x_2$ that $x_1 + x_2$ is in $T^f$ whenever both $x_1$ and $x_2$ are. Thus $T^f$ is a left ideal in $KS_c$ and it follows from the semi-simplicity of $KS_c$ that there is an idempotent $e^*$ in $KS_c$ for which $T^f = KS_c e^*$.

To complete the proof it must be shown that $U^{c}(e^*) = U^f$ for all $U$ in $K$. It follows from the definition of $e^*$ that $U^{c}(e^*)$ is contained in $U^f$. To obtain the equality it should be noted that $U^f$ and $U^{c}(e^*)$ are both $GL(n,K)$-submodules of $U^{[c]}$. For if $f'$ is a subfunctor of $[c]$, and $t$ is in $U^{f'}$, then $t \mu = t \mu^{[c]} = t \mu^{f'}$ which is in $U^{f'}$ for all $\mu$ in $GL(n,K)$ as required.
However it follows from 67.8 of [3] that $U[c]$ is a completely reducible $GL(n,K)$-module and that all its irreducible submodules are of the form $U[c]e$ where $e$ is a primitive idempotent in $KS_c$.

Because of the complete reducibility of $U[c], U^f$ can be written in the form

$$U^f = U^c(e^*) \otimes W,$$

where $W$ is a $GL(n,K)$ submodule of $U[c]$. Let it be supposed that $W$ is non trivial. Then it contains an irreducible submodule of $U[c]$. Let one such be $U[c]_e$.

---

(1) The results in [3] are proved for left modules and right ideals but the transition to right modules and left ideals does not effect the proofs at all. More important however is the fact that in [3] the results are proved over a field of characteristic zero. However, apart from Lemma 67.3, the results required for the proof of 67.8 only need the fact that $KS_c$ is semi-simple. A slight modification of the proof of Lemma 67.3 is required before its results can be proved for the characteristic $p$ case.
Then $U^{[c]} e$ is in $U^f$. If it can be shown that this implies that $U'^{[c]} e$ is in $U'^f$ for all other $U'$ in $\mathbb{K}$, then it would follow that $e$ is in $T^f = \mathbb{K} S_e^*$. This in turn implies that $U^{[c]} e$ is a non-trivial submodule of $U^c(\epsilon^*)$ and $W$. But this is a contradiction. Hence $W$ must be trivial.

Therefore to complete the proof of 2.2.4 it suffices to show that if $U^{[c]} e$ is contained in $U^f$, then $U'^{[c]} e$ is contained in $U'^f$ for all other $U'$ in $\mathbb{K}$. Let $U$ have a basis $b_1, \ldots, b_n$ and let $b'_1, \ldots, b'_m$ be a basis for an $m$-dimensional vector space $U'$ in $\mathbb{K}$. Then let $\psi$ be the linear mapping of $U$ into $U'$ for which $b_i \psi = b'_i$ for all $i$ not greater than the lesser of $m$ and $n$ and $b_j \psi$ is trivial for all other $j$ in $I_n$.

Now it follows from 2.2.2 that $U^{[c]} \psi^{[c]} = U^{[c]} \psi^{[c]} e$.

Since the left hand side is contained in $U'^f$ it suffices to show that the right hand side is equal to $U'^{[c]} e$.

If $m$ is at most $n$, then $U^{[c]} \psi^{[c]} = U'^{[c]}$ and the result follows. However, if $m$ is greater than $n$, then $U^{[c]} \psi^{[c]} e$ is isomorphic to $U^{[c]} e$ and so is non trivial.
But \( U^{[c]} \psi [c] \) is contained in \( U'[c] \) and hence is in the intersection of \( U'[c] \) and \( U^f \). But it follows from Theorem 67.8 of [3] that \( U'[c] \) is either trivial or an irreducible submodule of \( U[c] \), hence it is an irreducible submodule of \( U[c] \). However it intersects the other \( \text{GL}(n,K) \)-submodule \( U^f \) non-trivially and so \( U^f \cap U'[c] = U'[c] \) and 2.2.4 follows.

2.2.5 Let \( e^1 \) and \( e_2 \) be two idempotents in \( KS_c \). Furthermore let \( KS_{e^1} \cap KS_{e_2} = KS_{e_3} \) and \( KS_{e^1} \cup KS_{e_2} = KS_{e_4} \). Then it has to be shown that \( U = U \cup U_2 \) and \( U = U_1 \cap U_2 \) for all \( U \) in \( K \).

It follows from a similar argument to that used at the beginning of the proof for noting that \( U \) is in \( U[e^1] \) for some \( y \) in \( KS_{e^1} \), hence \( U[e^1] = U[e^1] ye' \) and so \( U[e^1] \) is contained in \( U[e'] \) for all \( U \) in \( K \). Since \( KS_{e^1} \) and \( KS_{e_2} \) are both contained in \( KS_{e_4} \) it follows that \( U \) and \( U \) are both contained in \( U[e_4] \). Hence \( U \) contains \( U \cup U' \).
Since $K_S^c$ is semi-simple it is a completely reducible left ideal. Therefore there are idempotents $e_1'$ and $e_2'$ in $K_S^c$ for which $K_S^c e_1' = K_S^c e_1' \oplus K_S^c e_3$ and $K_S^c e_2' = K_S^c e_2' \oplus K_S^c e_3$. Hence $K_S^c e_4' = K_S^c e_1' \oplus K_S^c e_2' \oplus K_S^c e_3$ and so $e_4'$ can be expressed in the form $e_4' = y_1 e_1' + y_2 e_2' + y_3 e_3$

where $y_1, y_2, y_3$ are in $K_S^c$. Therefore

$$c(e_4') = u[c](y_1 e_1' + y_2 e_2' + y_3 e_3) \text{ for all } u \text{ in } K.$$

However the right hand side is contained in $u[c] e_1' \cup u[c] e_2' \cup u[c] e_3$ which is contained in $u[c] e_1' \cup u[c] e_2'$

and the desired result follows.

It follows from a similar argument to that used at the beginning of the proof for unions that $U$ is contained in

$$c(e_3) \cap c(e_1) \cap c(e_2) \text{ for all } U \text{ in } K.$$

Now let $e_5'$ be a primitive idempotent in $K_S^c$ for which $u[c] e_5' \cap u[c] e_1' \cap u[c] e_2'$ is a non-trivial subspace of $U$. 


Then it follows from the same argument used in a similar situation in 2.2.4 that $U'$ is contained in 

$$c(\varepsilon_5) \cap c(\varepsilon_2)$$

for all other $U'$ in $K$. Now the set $I_1$, of all elements $x$ in $KS_c$ for which $U^c x$ is contained in $U^c \varepsilon_1$

for all $U$ in $K$, is a left ideal in $KS_c$ that has both $\varepsilon_1$ and $\varepsilon_5$ as members. If $I_1 = KS_c \varepsilon^*$, then it follows from the fact that $\varepsilon_1$ is in $KS_c \varepsilon^*$ that $U^c \varepsilon^*$ and $U^c \varepsilon_1$ are equal for all $U$ in $K$. Hence it follows from 2.2.3 that $KS_c \varepsilon^* = KS_c \varepsilon_1$. Therefore $\varepsilon_5$ is a member of $KS_c \varepsilon_1$. A similar argument implies that $\varepsilon_5$ is also a member of

$$KS_c \varepsilon_2$$

hence of $KS_c \varepsilon_3$. Thus $U$ is contained in $U^{c \varepsilon_3}$

$$c(\varepsilon_3) \cap c(\varepsilon_2)$$

for all $U$ in $K$ and $U^c = U^c \varepsilon_1 \cap U^c \varepsilon_2$ follows from the complete reducibility of $U^c$ for all $U$ in $K$.

This completes the proof of Theorem 2.2.1.
2.3 A Special Decomposition of $KS_c$

Let $P$ be a set of elements from $KS_c$ for which $KS_c = \sum_\theta KS_c \varepsilon$ is a decomposition of $KS_c$ into a direct sum of minimal left ideals. Then it follows from the results of the last section that $[c] = \sum_{\varepsilon \in P} c(\varepsilon)$. The results of Section 2.1 then imply that $L_\varepsilon = \bigcup_{\varepsilon \in P} c(\varepsilon) \phi_\varepsilon$. Therefore information about the $c(\varepsilon) \phi_\varepsilon$ is useful for the determination of the lattice of subfunctors of $L_\varepsilon$ hence of the lattice of varieties between $\underline{Y}_c$ and $\underline{Y}_{c-1}$. The object of this section is to obtain a set $P$ for which the $c(\varepsilon) \phi_\varepsilon$ can be determined explicitly and simply.

To obtain this set a special decomposition of $KS_c$ into the direct sum of minimal left ideals is obtained. Before this decomposition can be described several definitions and related introductory remarks are needed.

The results in this section can be considered as classical. They are included here to establish the notation and to provide a basis for the remaining chapters of this thesis. The treatment here is essentially that used in [1] where the result is proved for the group algebra over the field of complex numbers. In order to obtain the result in the present situation several results from [3] are used.
An idea essential to the obtaining of the desired result is that of a diagram, often called a Young diagram.

Definition 2.3.1 A diagram $D$ is a one to one mapping from the set $I_c$ into the cartesian product set $I_c \times I_c$ with the following property. If $(i,j)$ is in the image of $D$, then so is $(i',j')$ for all $i'$ not more than $i$ and all $j'$ not more than $j$.

Let $a$ be the maximum value of $i$ for which there is a $j$ such the $(i,j)$ is in the image of $D$. For all $i$ in $I_a$ let $\lambda_i$ be the maximum value of $j$ for which $(i,j)$ is in the image of $D$. Then it follows from Definition 2.3.1 that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_a$ and that

$$\sum_{i=1}^{a} \lambda_i = c.$$ Therefore $\lambda = (\lambda_1, \ldots, \lambda_a)$ is a partition of $c$, called the partition associated with $D$.

It is convenient to draw diagrams. For example. If $c$ is 6, the diagram $D$ with $1D = (1,1)$, $2D = (1,2)$, $3D = (1,3)$, $4D = (2,1)$, $5D = (2,2)$ and $6D = (3,1)$ will be drawn.

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It is a diagram associated with the partition $(3,2,1)$ of 6.
If $iD = (j,k)$, then $i$ is variously called the entry in the $(j,k)$ th position of $D$, an entry in the $j$ th row of $D$ or an entry in the $k$ th column of $D$.

To each diagram $D$ there corresponds two subgroups of $S_c$.

**Definition 2.3.2** The group $C(D)$ of column permutations of a diagram $D$ is the group of all $\sigma$ in $S_c$ for which $i\sigma$ is an entry in the same column of $D$ as $i$ is for all $i$ in $I_c$.

**Definition 2.3.3** The group $R(D)$ of row permutations of a diagram $D$ is the group of all permutations $\rho$ in $S_c$ for which $i\rho$ is an entry in the same row as $i$ for all $i$ in $I_c$.

For the example given above

$R(D) = \{(1), (12), (13), (23), (123), (132), \text{and all products of these with (45)}\}$

$C(D) = \{(1), (14), (16), (46), (146), (164), \text{and all products of these with (25)}\}$

Here and throughout $(1)$ is the identity element in $S_c$.

Obviously $R(D) \cap C(D) = (1)$ for all diagrams $D$.

**Definition 2.3.4** Let $D$ be a diagram. Then $\varepsilon(D)$ is the element of $K S_c$ given by

\[
\varepsilon(D) = \sum_{\sigma \text{ in } C(D)} \varepsilon_{\sigma} \sigma \rho, \text{ where } \varepsilon_{\sigma} \text{ is in } K, \\
\rho \text{ in } R(D)
\]

$\varepsilon_{\sigma} = 1$ if $\sigma$ is an even permutation and $\varepsilon_{\sigma} = -1$ if $\sigma$ is an odd permutation.
The following results can be obtained from those in Section 28 of [3]. For all diagrams D, $\text{KS}_c^e(D)$ is a minimal left ideal in $\text{KS}_c$. Moreover two left ideals obtained in this way from two different diagrams are isomorphic if, and only if, the partitions associated with these diagrams are the same. The set of $\text{KS}_c^e(D)$, where D runs through a set of diagrams, obtained by taking exactly one diagram associated with each partition of $c$, is a full set of non-isomorphic minimal left ideals of $\text{KS}_c$. Thus each minimal left ideal of $\text{KS}_c$ is isomorphic to one of the $\text{KS}_c^e(D)$. If M is a minimal left ideal of $\text{KS}_c$ that is isomorphic to $\text{KS}_c^e(D)$, where the partition associated with D is $\lambda$, then M will be called a minimal left ideal associated with the partition $\lambda$ of $c$.

(1) There are two changes. First the field is no longer the rationals. However $\text{KS}_c$ is still semi-simple and this is all that is needed. Secondly permutations are written on the right here and multiplied accordingly.
The \( e(D) \) are not idempotents but are scalar multiples of idempotents. If \( D \) is a diagram and \( \lambda \) is the partition of \( c \) associated with it, then \( k_\lambda^{-1} e(D) \) is an idempotent where \( k_\lambda \) is the dimension of \( KS_c e(D) \) taken modulo \( p \). (\( k_\lambda \) is not 0 since it follows from the proof of Lemma 28.13 of [3] that \( k_\lambda f \) in [3]) divides \( c! \) which is not trivial modulo \( p \).)

It is useful to note the following relationship. Since \( k_\lambda^{-1} e(D) \) is an idempotent,

\[
k_\lambda^{-1} e(D) = k_\lambda^{-1} e(D) \cdot k_\lambda^{-1} e(D) = k_\lambda^{-2} e(D)^2.
\]

Hence \( e(D)^2 = k_\lambda e(D) \). 2.3.5

A suitable subset of the \( e(D) \) will suffice for \( P \). To justify this statement some notation is required.

**Definition 2.3.6** A diagram \( D \) is a standard diagram if \( iD = (j, k), i'D = (j', k') \) and \( j \) at most \( j' \), \( k \) at most \( k' \) implies that \( i \) is at most \( i' \).

The set of all standard diagrams associated with a partition \( \lambda \) of \( c \) will be denoted by \( S_\lambda \).

It is evident that all standard diagrams have 1 in the \( (1,1) \) th position.

**Theorem 2.3.7** A decomposition of \( KS_c \) into a direct sum of minimal left ideals is given by

\[
KS_c = \sum D KS_c e(D), \quad \text{where the sum ranges over all standard diagrams } D.
\]
Proof The result is trivial for $c = 1$. Therefore in what follows $c$ will always be assumed to be greater than 1 and so $p$ will always be greater than 2.

Because $KS_c$ is semi-simple it follows from Theorem 25.15 of [3] that it can be written in the form

$$KS_c = \sum_{\lambda} B_{\lambda}$$

where the sum is over all partitions $\lambda$ of $c$ and, for each $\lambda$, $B_{\lambda}$ is the sum of all minimal left ideals of $KS_c$ associated with $\lambda$.

Therefore, for each partition $\lambda$ of $c$, $B_{\lambda}$ contains the sum

$$\sum_{D \in \mathcal{S}_{\lambda}} KS_c e(D).$$

This latter sum will be shown to be direct and to have dimension at least that of $B_{\lambda}$. Hence $B_{\lambda}$ will have to equal

$$\sum_{D \in \mathcal{S}_{\lambda}} KS_c e(D)$$

and the result will follow.

The proof that the sum is direct requires the definition of an order on $\mathcal{S}_{\lambda}$ and a couple of preliminary lemmas.

**Definition 2.3.9** An order on the standard diagrams corresponding to a partition $\lambda$ of $c$ is defined as follows: Let $D$ and $D^*$ be two standard diagrams corresponding to $\lambda$. Then $D$ is less than $D^*$ if the entries in the first $(i-1)$ rows and in the first $(j-1)$ spaces of the $i$th row of $D$ are the same as those in the corresponding positions of $D^*$ but the entry in the $(i,j)$ th position of $D$ is less than the corresponding entry in $D^*$.

This order is obviously a linear order on $\mathcal{S}_{\lambda}$. 
Lemma 2.3.10 If $D$ and $D^*$ are standard diagrams corresponding to a partition $\lambda$ of $c$ with $D$ less than $D^*$, then $\varepsilon(D) \cdot \varepsilon(D^*) = 0$.

Proof To prove this lemma it is sufficient to prove that there are two integers in $I_c$ which are in the same row in $D$ and in the same column of $D^*$.

If $k, \ell$ are two such integers and $\tau = (k\ell)$, then $\tau$ is in $R(D)$ and in $C(D^*)$.

However

$$\varepsilon(D)\tau = \sum_{\rho \in R(D)} \varepsilon_\sigma \rho \tau.$$ Since $\tau$ is in $R(D)$

$$\sigma \in C(D)$$

it follows that as $\rho$ runs through $R(D)$ so does $\rho \tau$. Thus $\varepsilon(D)\tau$ is the sum of the summands of $\varepsilon(D)$, in a different order, and so is equal to $\varepsilon(D)$.

Furthermore

$$\tau \varepsilon(D^*) = \sum_{\rho \in R(D^*)} \varepsilon_\sigma (\tau \sigma) \rho.$$ But it follows from the definition of the $\varepsilon_\sigma$, that $\varepsilon_\sigma = \varepsilon^{-1}$ and that $\varepsilon_\tau \varepsilon = \varepsilon_{\tau \sigma}$. Hence $\varepsilon_\sigma = \varepsilon_\tau \varepsilon_{\tau \sigma}$. Therefore

$$\tau \varepsilon(D^*) = \varepsilon_\tau \sum_{\rho \in R(D^*) \sigma \in C(D^*)} \varepsilon_{\tau \sigma} (\tau \sigma) \rho$$

$$= \varepsilon_\tau \varepsilon(D^*).$$

Thus $\varepsilon(D) \cdot \varepsilon(D^*) = \varepsilon(D)\tau \cdot \varepsilon(D^*)$ since $\tau \cdot \tau = (1)$

$$= \varepsilon_\tau \varepsilon(D) \cdot \varepsilon(D^*).$$

However $\varepsilon_\tau = -1$ and so the lemma follows from the fact that $K$ is $\text{GF}(p)$ for some odd prime $p$.

Therefore Lemma 2.3.10 has been reduced to the following lemma.
Lemma 2.3.11 If $D, D^*$ are in $S_\lambda$ and $D$ is less than $D^*$, then there are two integers in $I_c$ which are entries in the same row of $D$ and entries in the same column of $D^*$.

Proof Let the entries in the first $(i-1)$ rows and the first $(j-1)$ positions of the $i$th row of $D$ be the same as the corresponding entries in $D^*$. Furthermore let the entry $\ell$ in the $(i,j)$th position of $D$ be less than the corresponding entry in $D^*$.

If $\ell$ is the entry in the $(r,s)$th position of $D^*$, then $r$ must be at least $i$ from the definition of $\ell$. If $r$ were at least $i$ and $s$ were at least $j$, then it would follow from the definition of standard diagram that $\ell$ would be at least the entry in the $(i,j)$th position of $D^*$. This would contradict the definition of $\ell$ and so $s$ must be less than $j$. Therefore it follows from the fact that the entries in first $(j-1)$ positions of the $i$th row of $D$ are equal to those in the corresponding positions of $D^*$ that $r$ must be greater than $i$. If $k$ is the entry at the $(i,s)$th position of $D$ it is also the entry at the corresponding position of $D^*$. Therefore $k, \ell$ are entries in the same row of $D$ and in the same column of $D^*$ and the lemma follows.

Lemma 2.3.12 The sum $\sum_{D \in S_\lambda} KS_c \varepsilon(D)$ is direct.

Proof To prove that this sum is direct it is sufficient to show that if there are elements $x_D$ in $KS_c$ such that $\sum_{D \in S_\lambda} x_D \varepsilon(D) = 0$, then $x_D \varepsilon(D) = 0$ for all $D$ in $S_\lambda$.

Suppose that $x_D \varepsilon(D') = 0$ for all $D'$ in $S_\lambda$ greater than some $D^*$ in $S_\lambda$. 

Then \( \sum x_D e(D) = 0 \) where the sum now ranges over all \( D \) not greater than \( D^\alpha \) in \( S_L \). However multiplication of this latter equation on the right by \( e(D^\alpha) \) gives

\[
0 = \sum x_D e(D)e(D^\alpha) \\
= x_D e(D)e(D^\alpha) \quad \text{by Lemma 2.3.10} \\
= k_\lambda x_D e(D^\alpha) \quad \text{by 2.3.5}
\]

Therefore \( x_D e(D^\alpha) = 0 \) and an inductive argument completes the proof.

The second part of the proof of the Theorem relies heavily on results from [3].

The following results are proved in Sections 26 and 27 and summarized on page 185 of [3]. If \( M_\lambda \) is a minimal left ideal of \( KS_c \) associated with the partition \( \lambda \) of \( c \), then \( M_\lambda \) is a finite dimensional vector space over \( K \). Moreover if \( F^\lambda = \text{Hom}_{KS_c}(M_\lambda, M_\lambda) \), then \( F^\lambda \) is a finite dimensional division algebra over \( K \) and \( B_\lambda \) is isomorphic to \( \text{Hom}_{F^\lambda}(M_\lambda, M_\lambda) \). Let \( f^\lambda_\lambda \) be the dimension of \( F^\lambda \) over \( K \). Then \( B_\lambda \) is isomorphic, as an algebra, to \( F^\lambda_{f^\lambda_\lambda} \), the algebra of all \( \lambda \) by \( \lambda \) matrices with coefficients in \( F^\lambda \).

Let \( [X:Y] \) denote the dimension of \( X \) over \( Y \). Then

\[
[B_\lambda :K] = [F^\lambda_{f^\lambda_\lambda} :K] = (f^\lambda_\lambda)^2[F^\lambda :K].
\]
Hence the equating of the dimensions of both sides of 2.3.8 gives

\[ c' = \sum_{\lambda} (f^\lambda)^2 [F^\lambda : K]. \]

However it follows from Wedderburn's Theorem (26.8 of [3]) that \( f^\lambda \) is the number of minimal left ideals in a direct decomposition of \( B_\lambda \). Therefore, by Lemma 2.3.12 \( f^\lambda \) is at least \( \lambda \), the number of standard diagrams corresponding to \( \lambda \). Hence \( c' \) is at least \( \sum_{\lambda} f^2_\lambda \) \( [F^\lambda : K] \). But Theorem IV.4.3 of [1] asserts that \( c' = \sum_{\lambda} f^2_\lambda \). Since \( [F^\lambda : K] \) must be a positive integer it follows that \( [F^\lambda : K] = 1 \) and that \( f^\lambda = \lambda \) for all \( \lambda \). Thus \( B_\lambda = \sum_{D \in \mathbb{S}_\lambda} \mathbb{K} \varepsilon(D) \) and the theorem follows.
Chapter Three

The Lattice of Subvarieties of $\mathcal{M} = p-1$

This lattice has been described by Weichsel [11] and Brisley [2]. The description given in this chapter is obtained using the results developed in the last chapter and the methods employed here differ from those used in [2] and [11]. This work is included for two reasons. Firstly it gives an introduction, in a simpler and more familiar context, to the methods that will be used in obtaining a description of the lattice of subvarieties of $\mathcal{C} = p-1$. Moreover some of the results obtained form an essential part of the development of this latter lattice.

Throughout this chapter many of the symbols used will have the meanings ascribed to them in the last chapter, in particular $K$, $\varphi_Y$, $\varphi_Y$, $\varepsilon(D)$ and $c(x)$ for $x$ in $K \mathcal{S}$. Also $U$ will always be used to denote an $n$-dimensional vector space over $K$ (where $n$ may vary), $G_n$ the free group of rank $n$ of $\mathcal{Y}_c$ associated with it by the isomorphism $\gamma_U$ from the factor group $G_n/\gamma_2(G_n)$ to $U$ in the definition of

$$\mathcal{Y}_c, g_1, g_2, \ldots, g_n$$

the set of free generators of $G_n$, used in

$$\mathcal{Y}_c, b_1, \ldots, b_n$$

the basis of $U$ defined by $b_i = g_i \gamma_2(G_n) \gamma_U$. 

Firstly a subset of the set of all standard diagrams will be defined. It will then be shown that if $D$ is a standard diagram that is not contained in this subset, $c(g(D))b_1$ is a free functor (here $g$ is used as the proof is quite general).
The lattice of subvarieties of $M_{=p-1}$ is simply the
properly ascending chain $E = M_0 < M_1 < \ldots < M_{=p-1}$ of
varieties $M_c$ for $c$ in $\{0\} \cup I_{p-1}$. This result is obtained by
first showing that for all but one of the standard diagrams
associated with the partitions of $c$ the functor $c(e(D))_{\varphi_M}$
is a zero functor, then proving that, for this other standard
diagram $S, c(e(S))_{\varphi_M}$ is non-trivial. Thus

$$L_c^M$$

is a non-trivial image under $\varphi_M$ of an irreducible
subfunctor of $[c]$. Since $\varphi_M$ is a lattice homomorphism it
follows that $L_c^M$ is an irreducible functor and so $M_c = M_{=c-1}$ properly
contains $M_c = M_{=c-1}$ and there are no varieties strictly between
$M_c$ and $M_{=c-1}$. From this information it is a simple matter to
deduce that the lattice of subvarieties of $M_{=p-1}$ is what it is
claimed to be.

Since $[1]$ is an irreducible functor it is sufficient to
show that $[1]_{\varphi_M} = L_1^M$ is non trivial. This follows from the
fact that $M_{=1} = A_{=p}$. Hence for the remainder of this chapter
it can, and will, be assumed that $c$ is at least 2.

3.1 The Trivial Images

Firstly a subset of the set of all standard diagrams will
be defined. It will then be shown that if $D$ is a standard diagram
that is not contained in this subset, $c(e(D))_{\varphi_Y}$ is a zero functor.
(Here $Y$ is used as the proof is quite general)
Definition 3.1.1  A standard diagram is called a **special diagram** if 2 is the entry in its (2,1) position.

**Theorem 3.1.2**  Let $D$ be a standard diagram that is not a special diagram. Then $c(e(D))_{\phi_{\Xi}}$ is a zero functor.

**Proof**  To prove this it is sufficient to show that

$U^c e(D)_{\phi_{\Xi}} \text{ is trivial for all } U \in \Xi$. This follows if it can be shown that

$$b_{i_1} \cdots b_{i_c} e(D)_{\phi_{\Xi}} = 1 \text{ for all } i_1, \ldots, i_c \text{ in } I_n.$$  

However it was remarked in Section 2.3 that a standard diagram has 2 in either the (2,1) or the (1,2) position. Since $D$ is not a special diagram it follows that 2 must be the entry in the (1,2) position. Therefore (12) is in $R(D)$ and $\{(1), (12)\}$ is a subgroup of $R(D)$. Hence there is a transversal $T$ of $\{(1), (12)\}$ in $R(D)$ with respect to which every element in $R(D)$ can be uniquely expressed in the form $\tau(1)$ or $\tau(12)$ with $\tau$ in $T$. But

$$e(D) = \sum_{\sigma \in C(D)} e_{\sigma} \sum_{\varphi \in R(D)} e_{\sigma\varphi} = \sum_{\sigma \in C(D)} \left\{ e_{\sigma} \sum_{\tau \in T} \tau \left\{ (1) + (12) \right\} \right\}.$$
Therefore

\[ b_i \ldots b_i c \in (D)\varphi_Y^U \]

\[ = \prod_{\sigma \in C(D)} \left\{ b_i (\sigma)^{-1} \ldots b_i (\sigma)^{-1} \right\} \{ (1) + (12) \} \varphi_Y^U \]

since \( \varphi_Y^U \) and \( x \), for all \( x \) in \( K \), are linear maps.

But \( b_i \ldots b_i c \in (1) + (12) \) \( \varphi_Y^U \)

\[ = \left\{ b_i \ldots b_i j \in C(D) \right\} \{ (1) + (12) \} \varphi_Y^U \]

\[ = [g_{j_1}, g_{j_2}, g_{j_3}, \ldots, g_{j_c}] [g_{j_2}, g_{j_1}, g_{j_3}, \ldots, g_{j_c}] \]

\[ = 1 \text{ by 1.1.3, since } \gamma_{c+1}(G_n) \text{ is trivial, and the result follows.} \]

Corollary 3.1.3 If \( D \) is a standard diagram associated with the partition \( (c) \) of \( c \), then \( c(e(D))\varphi_Y^U \) is a zero functor.

From the definition of \( D \) it follows that \( D \) has entries only in the first row. Therefore \( 2 \) is in the \((1,2)\) position and the result follows.

For the remainder of this chapter only the variety \( M \) will be considered so that \( G_n \) will always be a group in \( M_{=c} \).

The further reduction of the set of standard diagrams for which \( e(D)\varphi_M^U \) is non trivial requires the following well known identity (c.f. 34.51 of [9]) for metabelian groups.
Let \( X \) be in \( Y \) (\( G \)) and let \( y_1, \ldots, y_t \) be in \( G \).

Then
\[
[x, y_1, \ldots, y_t] = [x, y_1, \ldots, y_t] \tag{3.1.4}
\]

where \( \pi \) is in \( S_t \).

**Theorem 3.1.5** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a) \) be a partition of \( c \) with \( \lambda_1 \) at most \( c-2 \). Then if \( D \) is a special diagram associated with \( \lambda \), \( c(e(D))_{\varphi M} \) is a zero functor.

**Proof.** Once again it must be shown that \( U^{[c]} e(D)\varphi M \) is trivial for all \( U \) in \( K \). Since \( U^{[c]} e(D)\varphi M \) is generated by the elements \( b_{i_1} \cdots b_{i_c} e(D)\varphi M \) for all \( i_1, \ldots, i_c \) in \( I_n \), it is sufficient to show that each of these is trivial.

But \( b_{i_1} \cdots b_{i_c} e(D)\varphi M \)
\[
= b_{i_1} \cdots b_{i_c} \sum_{\sigma \text{ in } C(D)} e_{\sigma \varphi M} U_{\rho \text{ in } R(D)}
\]
\[
= \prod_{\rho \text{ in } R(D)} \left\{ b_{i_1} \cdots b_{i_c} \rho \sum_{\sigma \text{ in } C(D)} e_{\rho^{-1} \sigma \varphi M} U \right\} \tag{3.1.6}
\]

In order to simplify the proof that this is trivial some notation and results associated with that notation are required.
Definition 3.1.7 Let \( \rho \) be in \( S_c \) and let \( D \) be a diagram. Then
\( D_\rho \) is the diagram whose entry in the \((j,k)\)th position is the image under \( \rho \) of the entry in the \((j,k)\)th position of \( D \).

It follows from this definition that \( D \) and \( D_\rho \) must both be associated with the same partition of \( c \). More important for Theorem 3.1.5 is the following lemma.

Lemma 3.1.8 Let \( \rho \) be in \( S_c \) and let \( D \) be a diagram. Then
\[
\sum_{\sigma \in C(D_\rho)} \epsilon_\sigma \rho = \sum_{\sigma \in C(D)} \epsilon_\sigma \rho^{-1} \rho.
\]

Proof This result is proved by showing that \( \sigma \) is in \( C(D) \) if, and only if, \( \rho^{-1} \rho \) is in \( C(D_\rho) \) since it follows from 2.3.10 that
\[
\epsilon_\rho^{-1} \rho = \epsilon_\rho^{-1} \epsilon_\rho = \epsilon_\rho \epsilon_\rho^{-1} \epsilon_\rho = \epsilon_\rho.
\]

Now let \( \sigma \) be in \( C(D) \). Then \( i \) and \( i \sigma \) are entries in the same column of \( D \). Therefore \( i \rho \) and \( i \sigma \rho \) are entries in the same column of \( D_\rho \). But \( i \rho \cdot \rho^{-1} \sigma \rho = i \sigma \rho \) and, as this holds for all \( i \) in \( I_c \), \( \rho^{-1} \sigma \rho \) is in \( C(D_\rho) \).

Conversely if \( \sigma^* \) is in \( C(D_\rho) \), then, since \( D = (D_\rho) \rho^{-1} \), it follows from an argument similar to that used in the preceding paragraph that \( \rho \sigma^* \rho^{-1} \) is in \( C(D) \) and the result follows.

Therefore from 3.1.6 it follows that
\[
b_{\rho}^{1} \ldots b_{\rho}^{i_c} b_{\rho}^{i_c} \epsilon(D_\rho)_{\phi_M} \equiv
\prod_{\rho \in R(D)} \left\{ b_{\rho}^{1} \rho^{-1} \ldots b_{\rho}^{i_c} \rho^{-1} \sum_{\sigma \in C(D_\rho)} \epsilon_\sigma \rho^{-1} \rho \phi_M \right\}.
\]

Hence it is sufficient to prove that
\[ \sum_{j_1, \ldots, j_c \in I_n} \sigma_{\sigma \rho \theta}^U = 1 \text{ for all } j_1, \ldots, j_c \text{ in } C(D_\rho) \text{ and all } \rho \text{ in } R(D). \]

From the definitions of $D$ and $\rho$ it follows that $D_\rho$ has 1 as an entry in the first row and 2 as an entry in the second row.

Since $D_\rho$ is associated with $\lambda$ it falls into one of the following three categories.

3.1.9 There are elements $i, j$ in \{3, \ldots, c\} which are entries in the same column of $D_\rho$

Since 1, 2 are entries in the first two rows of $D_\rho$ and between them can be entries in at most two columns of $D_\rho$ it follows that such a pair $i, j$ can be found whenever $a$, the number of parts in the partition, is greater than 3 or $\lambda_2$ is greater than 2.

If $\lambda_2 = 2$ and $a = 3$, then either

(i) 1 is not the entry in the (1,1) position of $D_\rho$, in which case the entries in the (1,1) and (3,1) positions of $D_\rho$ are both different from 1 and 2 and so suffice for $i, j$.

(ii) 1 is the entry in the (1,1) position of $D_\rho$ and 2 is the entry in the (2,1) position of $D_\rho$. Then the entries in the (1,2) and (2,2) positions of $D_\rho$ suffice for $i$ and $j$.

(iii) 1 and 2 are the entries in the (1,1) and (2,2) position of $D_\rho$ respectively. Then the entries in the (2,1) and (3,1) positions suffice for $i$ and $j$.

If $\lambda = (c-2,2)$ or $(c-2,1,1)$ and 1 is the only entry in its column of $D_\rho$, then it easily follows that $D_\rho$ is a diagram in this category.
Finally if \( \lambda = (c-2,2) \) and 1, 2 are both entries in the same column of \( D_0 \), then \( D_0 \) falls into this category.

Therefore there are only the following situations left.

3.1.10 \( \lambda = (c-2,2) \) and 1 is an entry in one of the first two columns of \( D_0 \) while 2 is an entry in the other.

3.1.11 \( \lambda = (c-2,1,1) \) and 1, 2 are the entries in the \((1,1)\) and \((2,1)\) positions respectively.

These three situations are dealt with separately in the next three lemmas.

Lemma 3.1.12 Let \( D^* \) be a diagram and let there be two integers \( i \) and \( j \) in \( \{3,\ldots,c\} \) that are entries in the same column of \( D^* \).

Then

\[
\sum_{\sigma \in C(D^*)} e^{\sigma \sigma_M} = 1 \quad \text{for all } j_1, \ldots, j_c \text{ in } I_n.
\]

Proof Since \( i \) and \( j \) are in the same column of \( D^* \), \((ij)\) is in \( C(D^*) \). But \( \{(1),(ij)\} \) is a subgroup of \( C(D^*) \) so there is a transversal \( T \) of \( \{(1),(ij)\} \) in \( C(D^*) \) for which every element of \( C(D^*) \) can be written uniquely as either \( \tau(1) \) or \( \tau(ij) \) for some \( \tau \) in \( T \). Since \( e_{\tau \tau'} = e_{\tau} e_{\tau'} \) it follows that

\[
\sum_{\sigma \in C(D^*)} e^{\sigma \sigma} = \sum_{\tau \in T} \{ (1) - (ij) \}
\]

Therefore for all \( k_1, \ldots, k_c \) in \( I_n \)

\[
\left\{ (1) - (ij) \right\}^U \varphi_M
\]

\[
= \left\{ b_{k_1} \ldots b_{k_{i-1}} b_{k_i} b_{k_{i+1}} \ldots b_{k_{j-1}} b_{k_j} b_{k_{j+1}} \ldots b_{k_c} \right\}^U \varphi_M
\]

\[
- b_{k_1} \ldots b_{k_{i-1}} b_{k_j} b_{k_{i+1}} \ldots b_{k_{j-1}} b_{k_i} b_{k_{j+1}} \ldots b_{k_c} \right\}^U \varphi_M
\]
$$= [e_{k_1}, \ldots, e_{k_{i-1}}, e_{k_i}, e_{k_{i+1}}, \ldots, e_{k_{j-1}}, e_{k_j}, e_{k_{j+1}}, \ldots, e_{k_c}]$$

$$[e_{k_1}, \ldots, e_{k_{i-1}}, e_{k_i}, e_{k_{i+1}}, \ldots, e_{k_{j-1}}, e_{k_j}, e_{k_{j+1}}, \ldots, e_{k_c}]^{-1}$$

$$= 1$$ by 3.1.4 since both $i$ and $j$ are greater than 2.

Therefore

$$b_1 \Theta \cdots \Theta b_c \sum_{\sigma \in \text{C}(D^*)} \epsilon_{\sigma} \varphi_{M}^{U} = \prod_{\tau \in T} \left\{ b_{j_1\tau^{-1}} \Theta \cdots \Theta b_{j_c\tau^{-1}} \right\} \left\{ (1) - (ij) \right\} \varphi_{M}^{U}$$

$$= 1$$ and the Lemma follows.

**Lemma 3.1.13** Let $D^k$ be a diagram and let $i$ and $j$ be integers in $\{3, \ldots, c\}$. If 1 and $i$ are entries in the same column of $D^k$ and 2 and $j$ are entries in the same column of $D^k$, then

$$b_1 \Theta \cdots \Theta b_c \sum_{\sigma \in \text{C}(D^*)} \epsilon_{\sigma} \varphi_{M}^{U} = 1$$ for all $j_1, \ldots, j_c$ in $I_n$. 

**Proof** The set $\left\{ (1), (1i), (2j), (1i)(2j) \right\}$ is a subgroup of $\text{C}(D^k)$. Therefore it can be shown that there is a transversal $T$ of this subgroup in $\text{C}(D^k)$ for which

$$\sum_{\sigma \in \text{C}(D^k)} \epsilon_{\sigma} \varphi_{M}^{U} = \sum_{\tau \in T} \left\{ (1) - (1i) - (2j) + (1i)(2j) \right\} \varphi_{M}^{U}$$

Therefore $b_{k_1} \Theta \cdots \Theta b_{k_c} \left\{ (1) - (1i) - (2j) + (1i)(2j) \right\} \varphi_{M}^{U}$

$$= \left\{ b_{k_1}, b_{k_2}, \ldots, b_{k_i}, \ldots, b_{k_j}, \ldots, b_{k_{j+1}}, \ldots, b_{k_c} \right\} \left\{ b_{k_1}, b_{k_2}, \ldots, b_{k_i}, \ldots, b_{k_j}, \ldots, b_{k_c} \right\}$$
where the last two listed entries in each tensor are in
the $i$th and $j$th places.

$$
= [g_{k_1}, g_{k_2}, g_{k_1}, g_{k}, \ldots] [g_{k_1}, g_{k_2}, g_{k_1}, g_{k}, \ldots]^{-1}
$$

$$
[g_{k_1}, g_{k_2}, g_{k_1}, g_{k}, \ldots]^{-1}[g_{k_1}, g_{k_2}, g_{k_1}, g_{k}, \ldots]
$$

moving the entries in the $i$th and $j$th places to the
third and fourth respectively by 3.1.4

$$
= [g_{k_1}, g_{k_1}, g_{k_2}, g_{k}, \ldots][g_{k_1}, g_{k_1}, g_{k_2}, g_{k}, \ldots]
$$

by applying 1.1.5 to the pairs in each row
then 3.1.4 to the second factor.

$$
= 1 \text{ by 1.1.3}
$$

This holds for all $k_1, \ldots, k_c$ in $I_n$. Therefore

$$
b_{j_1} \otimes \cdots \otimes b_{j_c} \sum_{\sigma \in C(D^\times)} \epsilon_{\sigma} \left( \sigma \varphi_M \right)^U
$$

$$
= \prod_{\tau \in T} \left\{ b_{j_1 \tau} \otimes \cdots \otimes b_{j_c \tau} \right\} \left\{ (1)-(li)-(2j)+(li)(2j) \right\} \left( \sigma \varphi_M \right)^U
$$

$$
= 1.
$$

Finally

Lemma 3.1.14 If $1, 2, i$ are in the same column of a diagram $D^\times$, then

$$
b_{j_1} \otimes \cdots \otimes b_{j_c} \sum_{\sigma \in C(D^\times)} \epsilon_{\sigma} \left( \sigma \varphi_M \right)^U = 1 \text{ for all } j_1, \ldots, j_c
$$

in $I_n$. 

Proof: Clearly the alternating group on 1, 2, i
= \{(1), (12i), (1i2)\} is a subgroup of C(D^*). Therefore, as before, there is a transversal T of this subgroup in C(D^*) so that

\[ \sum_{\sigma \in C(D^*)} \epsilon(\sigma) = \sum_{\tau \in T} \epsilon(\tau) \{ (1) + (12i) + (1i2) \} \]

However, \( b_{k_1} \ldots b_{k_c} \{ (1) + (12i) + (1i2) \} \phi_M^U \)

\[ = \left\{ b_{k_1} \ldots b_{k_1} \ldots b_{k_1} \ldots + b_{k_1} \ldots b_{k_1} \ldots b_{k_2} \ldots \right\} \phi_M^U \]

\[ = [g_{k_1}, g_{k_2}, g_{k_1}, \ldots] [g_{k_1}, g_{k_1}, g_{k_2}, \ldots] [g_{k_2}, g_{k_1}, g_{k_1}, \ldots] \]

applying 3.1.4 to bring the element in the i-th place in each commutator to the third place.

= 1 by 1.1.5

The lemma follows by the usual argument and the theorem in complete.

3.2 The Non-trivial Image

In the last section it was shown that \( c(\epsilon(D))\phi_M^U \) is a zero functor for all standard diagrams except the special diagram associated with the partition (c-1,1) of \( c \). There is only one such diagram namely \( S \) with \( 1S = (1,1), 2S = (2,1) \) and \( iS = (1,i-1) \) for \( i \) in \( \{3, \ldots, c\} \). In this section \( c(\epsilon(S))\phi_M^U \) will be shown to be a non-trivial functor.

To do this it is sufficient to show that for at least one \( U \) in \( K \) there is a basis element of \( U^{[c]} \) which has a non-trivial image under \( \epsilon(S)\phi_M^U \)
Now \( \varepsilon(S) = \sum_{\rho} \{(1) - (12)\} \rho \)
\( \rho \) in \( R(S) \)

\[ = \sum_{\rho} \rho \{(1) - (1\rho2)\} \text{ since } 2\rho = 2. \]
\( \rho \) in \( R(S) \)

Then if \( U \) has dimension at least 2, \( b_1 \otimes b_2 \otimes b_1 \otimes \ldots \otimes b_1 \) is a basis element of \( U[c] \). Furthermore

\[ b_1 \otimes b_2 \otimes b_1 \otimes \ldots \otimes b_1 \varepsilon(S) \varphi_M^U \]

\[ = \prod_{\rho} \left\{ b_1 \otimes b_2 \otimes b_1 \otimes \ldots \otimes b_1 - b_1 \otimes b_2 \otimes b_1 \otimes \ldots \otimes b_1 \right\} \varphi_M^U \]
\( \rho \) in \( R(S) \)

where \( b_2 \) is in the \( l_\rho \) th place in the second tensor of each factor.

\[ = \prod_{\rho} [g_1, g_2, g_1, \ldots, g_1] \prod_{\rho} [g_2, g_1, \ldots, g_1]^{-1} \]
\( \rho \) in \( R(S) \)
\( l_\rho = 1 \)

since \( [g_1, g_1, \ldots] = 1 \) and \( l_\rho \) is not 2.

\[ = [g_2, g_1, \ldots, g_1]^{-(c-2)!} \text{ by 1.1.3} \]

Since \( c \) is less than \( p \), \((c-2)! \) \( c \) is not divisible by \( p \) and so to prove that \( b_1 \otimes b_2 \otimes b_1 \otimes \ldots \otimes b_1 \varepsilon(S) \varphi_M^U \) is not trivial it is sufficient to prove that \( [g_2, (c-1)g_1] \) is not trivial in \( G_n \) for \( n \) greater than 1.

The following example supplies this information

**Example 3.2.1** There is a group \( M \) with the following properties:

- \( M \) is (a) metabolism
- \( (b) \) generated by two elements \( m_1 \) and \( m_2 \)
(c) of exponent $p$ and class $(p-1)$
(d) such that $[m_2, (c-1)m_1]$ is not contained
in $\gamma_{c+1}(M)$.

It is well known, and can easily be verified by routine
calculations, which are omitted here, that the factor group
of a wreath product of two $p$-cycles by its centre is a
group with these properties.
Chapter Four

The Lattice of Varieties Between $C_c$ and $C_{c-1}$

In this chapter only subvarieties of $C_c$, the variety of all centre-extended-by-metabelian groups, will be considered. The symbols $K, \varphi_{C_c}, \varphi_C, \alpha(D)$ and $\alpha(x)$ for $x$ in $K_{S_c}$ will have the meanings ascribed to them in Chapter Two. Throughout this chapter $U$ will always represent an $n$-dimensional vector space over $K$ (where, as before, $n$ may vary), $G_n$ the free group of rank $n$ of $\mathbb{C}$ associated with $U$ in the definition of $L^C_c$, $\mathcal{V}_U$ the isomorphism from $G_n/\gamma_2(G_n)$ to $U$, $g_1, \ldots, g_n$ the set of free generators of $G_n$ used in the definition of $L^C_c$ and $b_1, \ldots, b_n$ the basis for $U$ defined by $b_i = g_i \gamma_2(G_n) \mathcal{V}_U$.

Since $C_i = M_i$ for $i$ less than 4 it will be assumed that $c$ is at least 4 throughout this chapter.

The results of this chapter may be summarized as follows. There are two varieties strictly between $C_c$ and $C_{c-1}$. One of these contains $M_c$ while the other does not. The two varieties intersect in $C_{c-1}$ and join in $C_c$.

These results are obtained by showing that the functor $L^C_c$ is the union of two distinct irreducible subfunctors.

By Theorem 3.1.2 it is sufficient to consider the special diagrams associated with the partitions of $c$ in order to obtain the structure of the lattice of subfunctors of $L^C_c$. 
It follows from the results of the last chapter that there is a \( U \) in \( K \) for which \( U^{[c]} \in (S) \varphi_{c} \), where \( S \) is the special diagram associated with the partition \((c-1,1)\) of \( c \), is not contained in the second derived group of \( G_{n} \). Hence it is not contained in \( \left[ G_{n}^{(2)} \right]_{c} \). Thus \( c(\varepsilon(S))\varphi_{c} \in L_{c} \). As the image of \( U^{[c]} \) under this functor is not contained in the second derived group of \( G_{n} \), it follows that the variety corresponding to it cannot contain \( M_{c} \).

This chapter is therefore concerned with the images, under \( \varphi_{c} \), of the \( \varepsilon(D) \) where \( D \) is a special diagram with less than \( c-1 \) entries in the first row.

In Section 4.1 it is shown that if \( D \) is a special diagram with less than \( c-2 \) entries in the first row, then \( \varepsilon(D)\varphi_{c} \) is a zero functor. The special diagrams, \( A \), associated with the partition \((c-2,1,1)\) are studied in Section 4.2. There it is shown that if \( c \) is odd, \( c(\varepsilon(A))\varphi_{c} \) is trivial for all such \( A \), while if \( c \) is even, \( c(\varepsilon(A))\varphi_{c} \) is non-trivial and equal to \( c(\varepsilon(A^{*}))\varphi_{c} \) for all other special diagrams \( A^{*} \) associated with \((c-2,1,1)\). Section 4.3 contains the considerations of the special diagrams \( B \) associated with the partition \((c-2,2)\) of \( c \). It is shown that the subfunctors of \( L_{c}^{c} \) derived from these are all trivial when \( c \) is even but that when \( c \) is odd, \( c(\varepsilon(B))\varphi_{c} \) is
non-trivial and equal to \( c(\varepsilon(B^*)\phi_C) \) for every other special diagram \( B^* \) associated with \((c-2,2)\). It follows from the work in these last two sections that \( U[c]_\varepsilon(A)\phi_C \) and \( U[c]_\varepsilon(B)\phi_C \) are contained in \( G_n^{(2)} \). Therefore the varieties corresponding to \( c(\varepsilon(A))\phi_C \) and \( c(\varepsilon(B))\phi_C \) are either trivial or contain \( M^c \), and the summarized results follow from the irreducibility of the \( c(\varepsilon(D)) \).

4.1 Special Diagrams with less than \( c-2 \) entries in the first row

The proof that \( c(\varepsilon(D))\phi_C \) is trivial for all such special diagrams is very similar to the proof of Theorem 3.1.5.

From 3.1.4 it follows that if \( x,y_1,\ldots,y_t, z \) are in a group \( H \), then

\[
[x,y_1,\ldots,y_t,z] = [x,y_{l\pi},\ldots,y_{t\pi},m,z]
\]

where \( \pi \) is in \( S_t \) and \( m \) in \( H(2) \)

\[
= [x,y_{1\pi},\ldots,y_{t\pi},z][m,z][x,y_{1\pi},\ldots,y_{t\pi},m,z]
\]

Therefore if \( H \) is a centre-extended-by-metabelian group

\[
[x,y_1,\ldots,y_t,z] = [x,y_{l\pi},\ldots,y_{t\pi},z]
\]

for all \( \pi \) in \( S_t \). . . . 4.1.1

The following lemmas correspond to Lemmas 3.1.12 3.1.13 and 3.1.14 respectively.

\textbf{Lemma 4.1.2} Let \( D \) be a diagram, not necessarily standard.
If there are two distinct integers $i, j$ in $\{3, \ldots, c-1\}$ that appear as entries in the same column of $D$, then

$$b_{r_1} \cdots b_{r_c} \sum_{\sigma \in C(D)} e_{C(D)}^U = 1 \text{ for all }$$

$r_1, \ldots, r_c$ in $I$ and all $U$ in $K$.

**Proof** Using the same methods as in Lemma 3.1.12 it can be shown that there is a transversal $T$ of $\{(1), (ij)\}$ in $C(D)$ for which

$$b_{r_1} \cdots b_{r_c} \sum_{\sigma \in C(D)} e_{C(D)}^U$$

$$= \prod_{T \text{ in } T} \left\{ b_{r_{1T}}^{-1} \cdots b_{r_{cT}}^{-1} \left\{ (1) - (ij) \right\} C(D) \right\}^U$$

However

$$b_{s_1} \cdots b_{s_c} \left\{ (1) - (ij) \right\} C(D)$$

$$= [g_{s_1}, g_{s_2}, \ldots, g_{s_i}, g_{s_j}, \ldots, g_{s_c}]$$

$$[g_{s_1}, g_{s_2}, \ldots, g_{s_j}, \ldots, g_{s_i}, \ldots, g_{s_c}]^{-1}$$

where the third and fourth listed entries in each commutator are in the $i$th and $j$th places respectively. $= 1$ by 4.1.1 and the lemma follows.

**Lemma 4.1.3** Let $D$ be a diagram not necessarily standard. If there are two distinct integers $i, j$ in $\{3, \ldots, c-1\}$ for which $i, j$ are entries in the same column of $D$ and $2, j$ are entries in the same column of $D$, then
\[ b \sum_{r_1}^{c} \prod_{r_c}^{c} \epsilon^{U} \sigma_{\in \mathbb{C}(D)} = 1 \text{ for all } U \text{ in } \mathbb{K} \text{ and all } r_1, \ldots, r_c \text{ in } I_n. \]

**Proof** By the methods of Lemma 3.1.12 this can be reduced to showing that

\[ b \prod_{s_1}^{c} \left\{ (1)-(1i)-(2j)+(1i)(2j) \right\}^{U} \varphi_{C} = 1 \]

for all \( s_1, \ldots, s_c \) in \( I_n \).

But the left-hand side is equal to

\[ \left[ g_{s_1}, g_{s_2}, g_{s_1}, g_{s_j}, \ldots, g_{s_c} \right]^{-1} \left[ g_{s_i}, g_{s_2}, g_{s_1}, g_{s_j}, \ldots, g_{s_c} \right] \]

\[ \left[ g_{s_1}, g_{s_j}, g_{s_i}, g_{s_2}, \ldots, g_{s_c} \right]^{-1} \left[ g_{s_i}, g_{s_j}, g_{s_1}, g_{s_2}, \ldots, g_{s_c} \right] \]

by using 4.1.1 to move the entries in the \( i \)th and \( j \)th places in each commutator to the 3rd and 4th places respectively and to organize the other entries in the \( \{3, \ldots, c-1\} \) places into ascending order of subscript in the \( \{5, \ldots, c-1\} \) places.

\[ = \left[ g_{s_1}, g_{s_2}, g_{s_1}, g_{s_j}, \ldots, g_{s_c} \right]^{-1} \left[ g_{s_1}, g_{s_2}, g_{s_1}, g_{s_j}, \ldots, g_{s_c} \right] \]

by applying 1.1.5 to the pairs in each line of the previous expression and since \( \gamma_{c+1}(G_n) \) is trivial.

\[ = 1 \text{ by 4.1.1 and the lemma follows.} \]

**Lemma 4.1.4** Let \( D \) be a diagram, not necessarily standard. If there is an integer \( i \) in \( \{3, \ldots, c-1\} \) so that \( 1, 2, i \) are entries in the same column of \( D \), then
Proof By the arguments used at the beginning of Lemma 3.1.14 the proof of this lemma can be reduced to the proof of

\[ \sum_{s_1, \ldots, s_c} \left\{ (1) + (12i) + (1i2) \right\} \varphi_{c,i}^U = 1 \]

for all \( s_1, \ldots, s_c \) in \( I_n \).

But the left hand side of this expression is equal to

\[ [g_{s_1}, \ldots, g_{s_c}, \ldots, g_s] [g_{s_2}, \ldots, g_{s_c}, \ldots, g_s] \]

by application of 4.1.1 to interchange the \( i \)th and 3rd entries in each factor.

\[ = 1 \] by 1.1.5 since \( \gamma_{c+1}(G_n) \) is trivial.

These lemmas are now used to prove the following theorem.

**Theorem 4.1.5** Let \( D^* \) be a special diagram associated with the partition \( \lambda = (\lambda_1, \ldots, \lambda_c) \) of \( c \). If \( \lambda_1 \) is at most \( c-3 \), then \( c(\varepsilon(D^*), \varphi_C) \) is a zero functor.

**Proof** By an argument similar to that employed in the proof of Theorem 3.1.5 it is sufficient to prove that

\[ \sum_{k_1, \ldots, k_c} \varepsilon \sigma \varphi_C^U = 1 \] for all \( U \) in \( K \), all \( \varepsilon, \sigma \) in \( C(D^\rho) \), all \( k_1, \ldots, k_c \) and all \( \rho \) in \( R(D^\rho) \).

Now let \( D \) be the diagram \( D^\rho \). Then 1 is an entry in the first row of \( D \) while 2 is an entry in the second row.
If \( \lambda_2 \) is at least 3, then clearly two distinct integers \( i, j \) in \( \{3, \ldots, c-1\} \) can be found so that \( D \) satisfies the conditions of either Lemma 4.1.2 or 4.1.3, and the theorem follows in this case.

If \( a \), the number of parts in the partition and hence the number of entries in the first column, is at least four, then there are two subcases to consider. Either at most two of \( l, 2, c \) are entries in the first column of \( D \), in which case \( D \) satisfies the conditions of Lemma 4.1.2. Or \( l, 2, c \) are all entries in the first column of \( D \), in which case it follows from the fact that there are at least four entries in the first column of \( D \) that \( D \) satisfies the conditions of Lemma 4.1.4. Thus the theorem follows in this case also.

Since \( \lambda_1 \) is not more than \( c-3 \), it follows that \( \sum_{i=2}^{a} \lambda_i \) is at least three. If \( \lambda_2 \) were 1, then \( \lambda_i \) would be 1 for all \( i \) in \( \{2, \ldots, a\} \) and so \( a \) would be at least 4. On the other hand if \( a \) were 2, then \( \lambda_2 \) would be at least 3. Therefore the only case not already considered is the one where \( a = 3 \) and \( \lambda_2 = 2 \).

There are several subcases to be considered.

If \( l, 2, c \) are all entries in the first column of \( D \), then there are two distinct entries in the second column that are in \( \{3, \ldots, c-1\} \) and so \( D \) satisfies the conditions of Lemma 4.1.2.

If \( l, 2 \) are both entries in the first column of \( D \) while \( c \) is not, then the third entry \( i \) in the first column satisfies the conditions of Lemma 4.1.4.
If \(1, c\) are entries in the first column but \(2\) is not, it follows from the fact that the second column of \(D\) has at least two entries and the fact that \(2\) must be an entry in the second column that \(D\) satisfies the conditions of Lemma 4.1.3.

If \(2\) and \(c\) are both entries in the first column of \(D\) but \(1\) is not, then it follows that \(D\) satisfies either the conditions of Lemma 4.1.2, when \(1\) is not an entry in the second column of \(D\), or the conditions of Lemma 4.1.3, when \(1\) is an entry in the second column of \(D\).

Finally if there are less than two of \(1, 2, c\) in the first column of \(D\), then \(D\) satisfies the conditions of Lemma 4.1.2.

Thus in all cases
\[
b_1 \otimes \cdots \otimes b_c \sum_{\sigma \in \text{Sym}(1)} \epsilon\sigma \varphi_c = 1 \text{ and the theorem follows.}
\]

4.2 The Special Diagrams Associated with \((c-2, 1, 1)\)

The special diagrams associated with \((c-2, 1, 1)\) are uniquely determined by their entry in the \((3, 1)\) position. Therefore let \(A_i\) denote the diagram of this type with \(i\) the entry in the \((3, 1)\) position.

Throughout this section let \(U\) be an arbitrary but fixed vector space over \(K\) and \(b_1 \otimes \cdots \otimes b_c\) be a basis element of \(U\).

Then the first step in this section is the determination of the image of \(b_1 \otimes \cdots \otimes b_c\) under \(e(A_j)\varphi^U_c\) for all \(j\) in \(\{3, \ldots, c\}\).

For convenience this is split into two substeps

4.2.1 The image when \(j = c\)
4.2.2 The image when \(j\) is less than \(c\).
4.2.1 The diagram $A_c$ can be drawn as

\[
\begin{array}{cccc}
1 & 2 & \ldots & c-1 \\
2 & & & \\
c & & & \\
\end{array}
\]

Now $b_1 \ldots b_c \in (A_c)_{\varphi_C}^U$

\[
= \prod_{\rho \in R(A_c)} \left\{ b_{1 \rho}^{-1} \ldots b_{c \rho}^{-1} \sum_{\sigma \in C(A_{c \rho})} \varepsilon^\sigma \varphi_C^U \right\}
\]

For all $\rho \in R(A_c)$ $2\rho = 2$ and $c_\rho = c$.

There are two types of $\rho$ in $R(A_c)$

4.2.1 (i) $1_\rho = 1$ and 4.2.1 (ii) $1_\rho$ is not 1.

4.2.1 (i) For these

\[
\sum_{\sigma \in C(A_{c \rho})} \varepsilon^\sigma = (1)+(12c)+(1c2)-(12)-(2c)-(1c)
\]

Therefore if $s_j = r_{j \rho}^{-1}$,

\[
b_{s_1} \ldots b_{s_c} \sum_{\sigma \in C(A_{c \rho})} \varepsilon^\sigma \varphi_C^U
\]

\[
= [g_{s_1}, g_{s_2}, \ldots, g_{s_c}] [g_{s_c}, g_{s_1}, \ldots, g_{s_2}] [g_{s_2}, g_{s_3}, \ldots, g_{s_1}]
\]

\[
[g_{s_2}, g_{s_1}, \ldots, g_{s_c}]^{-1} [g_{s_1}, g_{s_c}, \ldots, g_{s_2}]^{-1} [g_{s_c}, g_{s_2}, \ldots, g_{s_1}]^{-1}
\]

where the entries in the $\{3, \ldots, c-1\}$ positions of each commutator are $g_{s_3}, \ldots, g_{s_{c-1}}$. 

by applying 1.1.3 to the second row and because
1p = 1, 2p = 2 and cp = c.
The entries in the \{3,...,c-1\} positions of each commutator
are therefore the elements \( g_r \), \( g_r \), ..., \( g_r \) in some order and
so by 4.1.1 may be taken to be in this order.

4.2.1 (ii) Here let \( 1p = j \). Then

\[
\sum_{\sigma \in C(A_{c \rho})} \varepsilon \sigma = (1)+(j2c)+(jc2)-(j2)-(2c)-(jc)
\]

As before let \( s_k = r_{k \rho} - 1 \). Then

\[
b_{r_{1 \rho} s_1} \cdots b_{r_{c \rho} s_c} \sum_{\sigma \in C(A_{c \rho})} \varepsilon \sigma \psi_C \]

\[
= b_{s_1} \cdots b_{s_c} \end{array}
\]

\[
= [g_{s_1}, g_{s_2}, g_{s_c}, \ldots, g_{s_1}] [g_{s_1}, g_{s_j}, g_{s_c}, \ldots, g_{s_j}]^{-1}
\]

\[
[g_{s_1}, g_{s_j}, g_{s_c}, \ldots, g_{s_j}] [g_{s_1}, g_{s_c}, g_{s_j}, \ldots, g_{s_c}]^{-1}
\]

\[
[g_{s_1}, g_{s_c}, g_{s_2}, \ldots, g_{s_2}] [g_{s_1}, g_{s_j}, g_{s_c}, \ldots, g_{s_j}]^{-1}
\]

using 4.1.1 to interchange the entries in the 3rd
and jth places in each commutator

\[
= [g_{s_j}, g_{s_2}, g_{s_c}, \ldots, g_{s_j}] [g_{s_j}, g_{s_c}, g_{s_2}, \ldots, g_{s_j}]
\]

\[
[g_{s_2}, g_{s_c}, g_{s_1}, \ldots, g_{s_j}]
\]

by applying 1.1.5 to the pairs in each row
But $s_j = r_{j-1} = r_1$, $s_2 = r_2$ and $s_c = r_c$ so

$$b_{r_1 \rho}^{-1} \otimes \cdots \otimes b_{r_c \rho}^{-1} \sum_{\sigma \in C(A_{\rho})} \epsilon_{\sigma \varphi_{C}}$$

$$= [g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_2}, g_{r_1}, \ldots, g_{r_c}]$$

where again the entries in the positions of each factor may be taken to be $g_{r_3}, \ldots, g_{r_c}$ in that order.

Therefore

$$b_{r_1} \otimes \cdots \otimes b_{r_c} \in (A_{\rho})_{\varphi_{C}}^{U}$$

$$= [g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_2}, g_{r_1}, \ldots, g_{r_c}] (c-3)! (c-1)$$

since there are $(c-3)!$ members of $R(A_{\rho})$ of type 4.2.1 (i) and $(c-3)! (c-3)$ of type 4.2.1 (ii).

As $p$ does not divide $(c-3)! (c-1)$ it follows that

$U^{[c]} (A_{\rho})_{\varphi_{C}}^{U}$ is generated by the set of all products of the form

$$[g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_2}, g_{r_1}, \ldots, g_{r_c}]$$

with $r_1, r_2, \ldots, r_c$ in $I_n$.

4.2.2 Let $\rho$ be in $R(A_{\rho})$. If $\rho$ is not equal to $c$, it follows from the fact that $2\rho = 2$ and $i_{\rho} = 1$ that $A_{i_{\rho}}$ satisfies the conditions of either Lemma 4.1.2 or 4.1.4 and so

$$b_{r_1 \rho}^{-1} \otimes \cdots \otimes b_{r_c \rho}^{-1} \sum_{\sigma \in C(A_{i_{\rho}})} \epsilon_{\sigma \varphi_{C}} = 1$$

for all $r_1, \ldots, r_c$ in $I_n$. 
Therefore to determine 
\[ b_{r_1} \cdots b_{r_c} e^{A_i}_{\varphi_C^U} \] it is sufficient to consider the 
\[ b_{r_1 \rho} \cdots b_{r_{c \rho}} \sum_{\sigma} \epsilon_\sigma \varphi_C^U \] where \( \rho \) is in \( R(A_i) \) 
and is such that \( l_{\rho} = c \).

For this type of \( \rho \) it follows that 
\[ \sum_{\sigma \in C(A_i \rho)} \epsilon_\sigma = (l)+(c_2i)+(ci2)-(2i)-(c2)-(ci). \]

Once more let \( s_k = r_{k \rho} \). Then 
\[ b_{s_1} \cdots b_{s_c} \sum_{\sigma} \epsilon_\sigma \varphi_C^U \] 
\[ = [g_{s_1}, g_{s_2}, g_{s_c}, \ldots, g_{s_i}] [g_{s_1}, g_{s_i}, g_{s_2}, \ldots, g_{s_c}]^{-1} \]
\[ [g_{s_1}, g_{s_i}, g_{s_c}, \ldots, g_{s_2}] [g_{s_1}, g_{s_c}, g_{s_i}, \ldots, g_{s_2}]^{-1} \]
\[ [g_{s_1}, g_{s_c}, g_{s_2}, \ldots, g_{s_i}] [g_{s_1}, g_{s_2}, g_{s_c}, \ldots, g_{s_i}]^{-1} \]
by applying 4.1.1 to interchange the entries in the 
3rd and i th position of each commutator.

\[ = [g_{s_i}, g_{s_2}, \ldots, g_{s_c}] [g_{s_c}, g_{s_i}, \ldots, g_{s_2}] [g_{s_2}, g_{s_i}, \ldots, g_{s_c}] \]
by applying 1.1.5 to the pairs in each row.

But \( s_2 = r_1 = r_1, \) \( s_2 = r_2 \) and \( s_i = r_i \).
Therefore it follows that

$$b \prod_{i=1}^{r} g_{r_i} \in (A_i)^{U}$$

$$= [g_{r_1}, g_{r_2}, \ldots, g_{r_i}]^{-1} [g_{r_1}, g_{r_2}, \ldots, g_{r_i}]^{-1} [g_{r_2}, g_{r_1}, \ldots, g_{r_i}]^{-1} \gamma_{c+1}^{c-3}!$$

by a re-arrangement of the order of the factors and the inverting of each factor using 1.1.3. Both of these operations can be carried out without affecting the result as \( \gamma_{c+1}^{c-3}! \) is trivial.

Because \( p \) does not divide \( (c-3)! \) it follows that

$$U[c] \in (A_i)^{U}$$

is generated by the set of all products of the form

$$[g_{r_1}, g_{r_2}, \ldots, g_{r_i}] [g_{r_1}, g_{r_2}, \ldots, g_{r_i}] [g_{r_2}, g_{r_1}, \ldots, g_{r_i}]$$

with \( r_1, \ldots, r_c \) in \( I_n \). Clearly this generating set is the same for all \( i \) in \( \{3, \ldots, c-1\} \) and coincides with the generating set for \( U[c] \in (A_i)^{U} \).

Theorem 4.2.3 Let \( A \) be a special diagram associated with the partition \( (c-2,1,1) \). If \( c \) is odd, \( c(e(A))^U \) is a zero functor. However, if \( c \) is even, \( c(e(A))^U \) is non trivial and is equal to \( c(e(A^*))^U \) for all other special diagrams \( A^* \) associated with \( (c-2,1,1) \).

Proof The equality in the last statement of the theorem follows directly from the fact that, for each \( U \) in \( K_n, U[c] \in (A)^U \) and \( U[c] \in (A^*)^U \) are subgroups of \( \gamma_{c}^{G_n} \) that have the same generating set.
Thus to complete the theorem it must be proved that
\[
[g_r_1, g_r_2, \ldots, g_r_c][g_r_c, g_r_1, \ldots, g_r_2][g_r_2, g_r_c, \ldots, g_r_1]
\]
is trivial for all \( U \) in \( K \) and all \( r_1, \ldots, r_c \) when \( c \) is odd but
that there is a \( U \) and \( r_1, \ldots, r_c \) in \( I_n \) for which it is non
trivial when \( c \) is even.

First the odd case is considered. This is proved using
several Lemmas in which commutator identities for centre-
extended-by-metabelian groups are established. The first is
for general \( c \) greater than 3.

Lemma 4.2.4 Let \( H \) be a group in \( C^c \), where \( c \) is at least 4.
Then for all \( i \) in \( \{2, \ldots, c-2\} \) and all \( h_1, \ldots, h_c \) in \( H \)
\[
[h_1, h_2, \ldots, h_i; h_c, h_{c-1}, \ldots, h_{i+1}]
\]
\[
= [h_1, h_2, \ldots, h_{c-2}; h_c, h_{c-1}]^{(-1)^{(c-i)}}.
\]

Proof The proof is by reverse induction on \( i \). The result is
obviously true for \( i = c-2 \). Therefore let it be assumed that
\( t \) is less than \( c-2 \) but at least 2 and that the result holds for
all \( i \) greater than \( t \).

Let \( x = [h_1, \ldots, h_t], y = [h_c, h_{c-1}, \ldots, h_{t+2}] \)
and \( z = h_{c-2} \). Then
\[
[h_1, \ldots, h_t ; h_c, h_{c-1}, \ldots, h_{t+2}, h_{t+1}]
\]
\[
= [y, z, x]^{-1}
\]
\[
= [z, x, y][x, y, z] \text{ by } 1.1.5 \text{ since } \gamma_{c+1}(H) \text{ is trivial.}
\]
\[
= [x, z, y]^{-1} \text{ by } 1.1.3 \text{ since } [x, y, z] \text{ is in } [H(2), H] = 1
\]
\[
= [h_1, \ldots, h_t, h_{t+1} ; h_c, h_{c-1}, \ldots, h_{t+2}]^{-1}
\]
by the inductive hypothesis.

and Lemma 4.2.4 follows.

An easy consequence of this Lemma is the following useful result

**Corollary 4.2.5** Let $H$ be in $\subseteq_c$. Then

$$[u, v, h_1, \ldots, h_{c-4}; x, y] = [x, y, h_1, \ldots, h_{c-4}; u, v](-1)^{(c-1)}$$

for all $u, v, x, y, h_1, \ldots, h_{c-4}$ in $H$.

It follows directly from this last result and the fact that $p$ is at least 5 that $[u, v, h_1, \ldots, h_{c-4}; u, v] = 1$ whenever $c$ is even.

**Lemma 4.2.6** Let $H$ be a group in $\subseteq_c$ and let $u, v, w, x, y, h_1, \ldots, h_{c-5}$ be in $H$. Then

$$P = [u, v, x, h_1, \ldots, h_{c-5}; y, w][w, u, x, h_1, \ldots, h_{c-5}; y, v]$$

$$[v, w, x, h_1, \ldots, h_{c-5}; y, u]$$

$$= 1$$

whenever $c$ is odd.

(it follows from the restrictions placed on $c$ at the outset that $c$ must be at least 5.)

**Proof** It follows from 1.1.5 that

\[
1 = 
[u, v, x, \ldots; y, w][x, u, v, \ldots; y, w][v, x, u, \ldots; y, w] \\
[w, u, x, \ldots; y, v][x, w, u, \ldots; y, v][u, x, w, \ldots; y, v] \\
[v, w, x, \ldots; y, u][x, v, w, \ldots; y, u][w, x, v, \ldots; y, u] \\
= P [x, u, v, \ldots; y, w][u, x, w, \ldots; y, v] \\
[v, x, u, \ldots; y, w][x, v, w, \ldots; y, u] \\
[x, w, u, \ldots; y, v][w, x, v, \ldots; y, u]
\]

by re-arranging the factors.
= \prod [v, w, y, \ldots; x, u] [u, w, y, \ldots; v, x] [u, v, y, \ldots; x, w]

by applying 1.1.3 to the factors in the second column, then applying 4.2.5 to the factors in both columns and finally applying 1.1.6 to the pairs in each row.

= \prod [x, u, y, \ldots; v, w] [v, x, y, \ldots; u, w] [x, w, y, \ldots; u, v] by 4.2.5

= \prod [v, w, x, \ldots; y, u] [v, w, u, \ldots; x, y] [w, u, x, \ldots; y, v] [w, u, v, \ldots; x, y] [u, v, x, \ldots; y, w] [u, v, w, \ldots; x, y]

by application of 1.1.5, then 4.2.5 and finally 1.1.3 if necessary.

= \prod^2 as the factors in the second column form a trivial product by 1.1.5.

Since \( p \) must be at least 7 it follows that \( \prod = 1 \) and the lemma is proved.

It is now a simple matter to complete the odd case which follows from the next Lemma.

**Lemma 4.2.7** Let \( c \) be an odd integer greater than 4 and let \( H \) be a group in \( \mathbb{C}_c \). Then for all \( u, v, w, y, h_1, \ldots, h_{c-4} \) in \( H \)

\[ P^* = [u, v, h_1, \ldots, h_{c-4}; y, w] [w, u, h_1, \ldots, h_{c-4}; y, v] \]

\[ [v, w, h_1, \ldots, h_{c-4}; y, u] \]

= 1

**Proof** Let \( x = [u, v, h_1, \ldots, h_{c-4}] \). Then the first factor in \( P^* \) is

\[ [x, y, w] = [y, w, x]^{-1} [w, x, y]^{-1} \] by 1.1.5

\[ = [x; y, w] [x, w, y] \] by 1.1.3 ..... 4.2.8

Application of a similar manipulation to the other factors of \( P^* \) gives

\[ P^* = [u, v, h_1, \ldots, h_{c-4}; w, y] [u, v, h_1, \ldots, h_{c-4}; y, w] [w, u, h_1, \ldots, h_{c-4}; v, y] [w, u, h_1, \ldots, h_{c-4}; y, v] [v, w, h_1, \ldots, h_{c-4}; u, y] [v, w, h_1, \ldots, h_{c-4}; y, u] \]
= 1 since the factors in the first column, after an application of 4.1.1 to move the c-1 st entry to the 3 rd position in each commutator of that column, form a trivial product by 1.1.5 while the factors in the second column form a trivial product by Lemma 4.2.6.

Thus the odd case is complete.

Remark 4.2.9 The proof of the last lemma shows that

\[ U^C(A) U \] is contained in \( G_{n}^{(2)} \) for all \( U \) in \( K \).

Moreover Corollary 4.2.5 implies that if \( U \) is of dimension smaller than 3, then \( U^C(A) U \) is generated by trivial elements so is itself trivial regardless of whether \( c \) is even or odd.

To complete the theorem it is sufficient to establish that when \( c \) is even

\[
[g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_1}, g_{r_2}, \ldots, g_{r_c}] [g_{r_2}, g_{r_1}, \ldots, g_{r_c}]
\]

and

is non trivial for some \( U \) in \( K \) and some \( r_1, \ldots, r_c \) in \( I_n \).

If \( U \) is of dimension less than three, there is no \( r_1, \ldots, r_c \) in \( I_n \) for which the above expression is non trivial.

But if \( U \) is of dimension at least three and \( r_1 = 1, r_c = 3, r_i = 2 \) for all \( i \) in \( \{2, \ldots, c-1\} \), then the expression reduces to

\[
[g_{1}, (c-3)g_{2}; g_{2}, g_{2}]^{-2}
\] by applications of 4.2.5 and 1.1.3.
Therefore it is sufficient to construct a centre-extended-by metabelian group of exponent \( p \) and class \( c \), for every even integer \( c \) between 3 and \( p \), in which \([v,(c-3)u;v,w,u]\) is non trivial for some elements \( u,v,w \) in that group. The construction of this example is deferred until the end of Section 4.3.

**4.3 The Special Diagrams Associated with \((c-2,2)\)**

The results of this section can be summarized into the following theorem

**Theorem 4.3.1** Let \( B \) be a special diagram associated with the partition \((c-2,2)\) of \( c \). If \( c \) is even, then \( c(e(B))_{\varphi C} \) is a zero functor. However, if \( c \) is odd, \( c(e(B))_{\varphi C} \) is non trivial and equal to \( c(e(B^*))_{\varphi C} \) where \( B^* \) is another special diagram corresponding to \((c-2,2)\).

**Proof** The first step in this proof is the calculation of a set of generating elements for \( Ue(B)_{\varphi C} \) for all \( U \) in \( K \) and all special diagrams \( B \) associated with \((c-2,2)\).

This set of generating elements can be calculated by calculating the \( b_{r_1}b_{r_2}...b_{r_c}e(B)_{\varphi C}^{U} \) for all basis elements of \( U^{[c]} \). However, \( b_{r_1}b_{r_2}...b_{r_c}e(B)_{\varphi C}^{U} = \prod_{\rho \text{ in } R(B)} b_{r_{1\rho}}^{-1}b_{r_{c\rho}}^{-1} \sum_{\sigma \text{ in } C(B_\rho)} e_{\sigma}^{U} \). Therefore it is sufficient to calculate each factor in the above product.

For convenience \( s_j \) will sometimes be used to denote \( r_{j\rho}^{-1} \).
The special diagrams associated with $(c-2,2)$ all have 3 as the entry in their $(1,2)$ position so that they are uniquely determined by their entry in the $(2,2)$ position. Therefore $B_i$, with $i$ in $\{4, \ldots, c\}$, will be used to denote the special diagram of this type with $i$ as its entry in the $(2,2)$ position. The calculations are carried out in two main cases depending upon whether $i$ is less than $c$ or is $c$ and within each of these main cases in several subcases depending on the action of $\rho$ in $R(B)$.

4.3.2 Here $i$ is in $\{4, \ldots, c-1\}$. If $k$ is an integer in $I_c$ but not in $\{1, 2, i, c\}$, then the elements of $R(B_i)$ can be split into the following disjoint subsets.

(i) $3 \rho = k, 1_\rho$ is not $c$
(ii) $3 \rho = k, 1_\rho = c, 2_\rho = 2$
(iii) $3 \rho = k, 1_\rho = c, 2_\rho = i$
(iv) $3 \rho = 1, 1_\rho$ is not $c$
(v) $3 \rho = 1, 1_\rho = c, 2_\rho = i$
(vi) $3 \rho = 1, 1_\rho = c, 2_\rho = 2$
(vii) $3 \rho = c, 1_\rho = 1, 2_\rho = 2$
(viii) $3 \rho = c, 1_\rho = 1, 2_\rho = i$
(ix) $3 \rho = c, 1_\rho = k, 2_\rho = 2$
(x) $3 \rho = c, 1_\rho = k, 2_\rho = i$

Lemmas 4.1.2 and 4.1.3 imply the triviality of

$$ b_1 \otimes \cdots \otimes b_c \sum_{\sigma \in C(B_i^p)} e_{\sigma \rho}^{U} $$

for all $\rho$ in (i), (ii), (iv) and (x).

For all $\rho$ in $R(B_i)$ the diagram $B_i^\rho$ can be drawn as
\[
\begin{array}{cc}
1_0 & 3_0 \\
2_0 & 5_0 \\
\end{array}
\]

and so

\[
\sigma \in C(B_{1\rho}^1) = \left\{ (1) - (1\rho 2\rho) \right\} \left\{ (1) - (3\rho 1\rho) \right\}.
\]

Therefore

\[
\sum \varepsilon_{\sigma} \sigma = \left\{ (1) - (c_1) \right\} \left\{ (1) - (k_2) \right\}
\]

So in

\[
4.3.2 (iii)
\]

Here \( C(B_{1\rho}^1) = \left\{ (1) - (c) \right\} \left\{ (1) - (k) \right\} \)

Therefore

\[
b_{s_1} a_1 \ldots a_{s_c} = \sum_{\sigma \in C(B_{1\rho}^1)} \varepsilon_{\sigma} \sigma
\]

\[
= [g_{s_1}, g_{s_2}, g_{s_3}, \ldots, g_{s_{c-1}}, g_{s_c}] [g_{s_1}, g_{s_2}, g_{s_3}, \ldots, g_{s_{c-1}}, g_{s_c}]^{-1}
\]

applying 4.1.1 to move the \( k \)th entry and the \( i \)th entry in each commutator to the 3rd and \( (c-1) \)th position in each factor.

\[
= [g_{s_1}, g_{s_2}, \ldots, g_{s_{c-1}}, g_{s_c}] [g_{s_1}, g_{s_2}, \ldots, g_{s_{c-1}}, g_{s_c}]^{-1}
\]

by applying 1.1.5 to the pairs in each column.

\[
= [g_{s_1}, g_{s_2}, \ldots, g_{s_{c-1}}, g_{s_c}] \text{ by 4.2.8}
\]

\[
= [g_{r_1}, g_{r_2}, \ldots, g_{r_{c-1}}, g_{r_1}]^{(-1)^c}
\]

by 4.2.5 and 1.1.3

Using methods similar to these it can be shown that in the following cases

\[
b_{r_{1\rho}} a_{r_1} \ldots a_{r_{c\rho}} \sum_{\sigma \in C(B_{1\rho}^1)} \varepsilon_{\sigma} \sigma equals
\]

\[
4.3.2 (v) [g_{r_1}, g_{r_2}, \ldots, g_{r_{c-1}}, g_{r_1}]^{(-1)^c}.
\]
4.3.2 (vi) $[g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_1}] (-1)^c$.

4.3.2 (vii) $[g_{r_1}, g_{r_2}, \ldots; g_{r_1}, g_{r_3}]^2$

4.3.2 (viii) and (ix) $[g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_1}]$.

Where in all case the unnoted entries are the $g_{r_j}$, for $j$ not in $\{1,2,3,i\}$ in some fixed order.

Therefore $b_1 \cup b_2 \cup \ldots \cup b_c \in (B_i)_0$.

$$= [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_1}]^c + (-1)^c \beta$$

where $-\alpha = \text{The number of } \rho \text{ of type (viii)} + \text{the number of } \rho \text{ of type (ix)} + 2 \times \text{The number of } \rho \text{ of type (vii)}$

$$= (c-4)! + (c-4)! (c-4) + 2(c-4)!$$

$$= (c-4)! (c-1)$$

and $\beta = \text{The number of } \rho \text{ of type (iii)} + \text{the number of } \rho \text{ of type (vi)} + 2 \times \text{the number of } \rho \text{ of type (v)}$

$$= (c-4)! (c-4) + (c-4)! + 2(c-4)!$$

$$= -\alpha.$$

4.3.3 Here the diagram being considered is $B_c$.

The $\rho$ in $R(B_c)$ can be arranged into the following subsets.

(i) $1\rho = 1$, $2\rho = 2$
(ii) $1\rho = 1$, $2\rho = c$
(iii) $3\rho = 1$, $2\rho = 2$
(iv) $3\rho = 1$, $2\rho = c$
(v) $2\rho = 2$ but neither $1\rho$ nor $3\rho$ equals 1
(vi) $2\rho = c$ but neither $1\rho$ nor $3\rho$ equals 1.
Calculations similar to those employed in 4.3.2 give that

\[ b_{r_1 \ldots r_c} \sum_{\sigma \in C(B_{c \rho})} \epsilon \sigma \varphi_{C}^{U} \]

is equal to

(i) \[ [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^2 \]

(ii) \[ [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{(-1)^{c-1}} \]

(iii) \[ [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}] \]

(iv) \[ [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{(-1)^{c-1} \cdot 2} \]

(v) \[ [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}] \]

(vi) \[ [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{(-1)^{c-1}} \] for \( \rho \) in the relevant set.

Therefore \( b_{r_1 \ldots r_c} \sum_{\sigma \in C(B_{c \rho})} \epsilon \varphi_{C}^{U} \)

\[ = [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{\alpha^*} + (-1)^{c-1} \beta^* \]

where \( \alpha^* = \) (the number of \( \rho \) of type (i)) + 2 + the number of \( \rho \) of type (iii) + the number of \( \rho \) of type (v)

\[ = (c-3)! + (c-3)! + (c-3)! + (c-3)! \cdot (c-4) \]

\[ = (c-3)! (c-1) \]

and \( \beta^* = \) the number of \( \rho \) of type (ii) + 2( the number of \( \rho \) of type (iv) + the number of \( \rho \) of type (vi)

\[ = (c-3)! + 2(c-3)! + (c-3)! \cdot (c-4) \]

\[ = \alpha^*. \]
The results of 4.3.2 and 4.3.3 imply that $U^c(\varepsilon(B))_{\varphi_C}$ is generated either by all elements of the form

$$[g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{\alpha} + (-1)^{c+1} \alpha \text{ with } r_1, \ldots, r_c \text{ in } I_n,$$

or by all elements of the form

$$[g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{\alpha^*} + (-1)^{c-1} \alpha^* \text{ with } r_1, \ldots, r_c \text{ in } I_n.$$

When $c$ is even, these elements all have a trivial exponent and so $U^c(\varepsilon(B))_{\varphi_C}$ is trivial for all $U$ in $\mathbb{K}$. Therefore $\varepsilon(B))_{\varphi_C}$ is a zero functor whenever $c$ is an even integer.

However when $c$ is odd, $U^c(\varepsilon(B))_{\varphi_C}$ is generated by the

$$[g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{2\alpha} \text{ or } [g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]^{2\alpha^*}.$$

Since $p$ is greater than $c$, which was assumed to be at least 4, $2\alpha$ and $2\alpha^*$ are not divisible by $p$. Therefore $U^c(\varepsilon(B))_{\varphi_C}$ is generated by the set of all commutators of the form $[g_{r_1}, g_{r_2}, \ldots; g_{r_3}, g_{r_c}]$ with $r_1, \ldots, r_c$ in $I_n$. Since this holds for all $U$ in $\mathbb{K}$, it follows that $c(\varepsilon(B))_{\varphi_C} = c(\varepsilon(B^*))_{\varphi_C}$ for another special diagram $B^*$ associated with $(c-2,2)$. 

Hence to complete the theorem it is sufficient to establish the existence of a group in $\mathbb{C}_c$ with $c$ odd, which contains elements $u, v, w$ for which $[v, (c-3)u; w, u]$ is non-trivial.

This is established by the following example which suffices for the even case as well.

**Example 4.3.4** For each prime $p$ not less than 5 and each integer $c$ in $\{4, \ldots, p-1\}$, there is a group $H$ with the following properties:

1. $H$ is generated by three elements $u, v, w$.
2. such that $[v, (c-3)u; w, u]$ is non-trivial.
3. of class $c$.
4. centre-extended-by-metabelian
5. of exponent $p$.

**Details** Let $N$ be the nilpotent-of-class-2-and-of-exponent-$p$-group generated by the elements $v_0, v_1, \ldots, v_{c-2}, w_0, \ldots, w_{c-2}$ which satisfy the following relationships and their consequences but no others:

(a) $[v_i, v_j] = [w_i, w_j] = 1$ for all $i, j$ in $\{0, \ldots, c-2\}$
(b) $[v_i, w_j] = 1$ for all $i, j$ in $\{0, \ldots, c-2\}$ for which $i+j$ is greater than $c-2$.
(c) $[v_{i+1}, w_j][v_i, w_{j+1}][v_{i+1}, w_{j+1}] = 1$ for all $i, j$ in $\{0, \ldots, c-2\}$.

Let $u$ be the mapping of $N$ into itself defined on the generators of $N$ by

\[
\begin{align*}
v^u_1 &= v_1v_{i+1} ; \quad w^u_1 &= w_1w_{i+1} \quad \text{for } i \text{ in } \{0, \ldots, c-3\} \\
v^u_{c-2} &= v_{c-2} ; \quad w^u_{c-2} &= w_{c-2}.
\end{align*}
\]
To show that this can be extended to an automorphism of $N$ it must be shown that it preserves the relations on the generators. (In the following calculations if $v_{c-1}$ and $w_{c-1}$ appear they will be considered to be 1).

Firstly (a). Now

$$[v_i^u, v_j^u] = [v_i^1, v_{i+1}^1, v_j^1, v_{j+1}^1] = [v_i^1, v_j^1] [v_i^1, v_{j+1}^1] [v_{i+1}^1, v_j^1] [v_{i+1}^1, v_{j+1}^1]$$

$= 1$ and similarly $[w_i^u, w_j^u] = 1$ for all $i, j$ in $\{0, \ldots, c-2\}$.

(b) Since

$$[v_i^u, w_j^u] = [v_i^1, w_j^1] [v_{i+1}^1, w_j^1] [v_i^1, w_{j+1}^1] [v_{i+1}^1, w_{j+1}^1]$$

it follows that if $i + j$ is greater than $c-2$, then $[v_i^u, w_j^u] = 1$.

(c) Finally

$$[v_{i+1}^u, w_j^u] [v_i^u, w_{j+1}^u] [v_{i+1}^u, w_{j+1}^u] = [v_{i+1}^1, w_j^1] [v_{i+1}^1, w_{j+1}^1] [v_{i+1}^1, w_{j+1}^1] [v_{i+1}^1, w_{j+1}^1]$$

$$[v_i^1, w_{j+1}^1] [v_i^1, w_{j+2}^1] [v_{i+1}^1, w_{j+1}^1] [v_{i+1}^1, w_{j+2}^1]$$

$$[v_{i+1}^1, w_{j+1}^1] [v_{i+1}^1, w_{j+2}^1] [v_{i+1}^1, w_{j+1}^1] [v_{i+1}^1, w_{j+2}^1]$$

$= 1$ as the products of the factors in each of the columns are trivial by the relation (c).

Obviously the relations corresponding to $N$ being nilpotent of class 2 are preserved under $u$. Therefore it remains to check that $(v_i^u)^p = (w_i^u)^p$ for all $i$ in $\{0, \ldots, c-2\}$. Now
\[(v^u_i)^p = (v_i v_{i+1})^p = v_i^p v_{i+1}^p [v_i+1, v_i]^p \bmod (p-1) = 1 \text{ since } p \text{ is odd and } N \text{ is of exponent } p.\]

A similar argument holds for the \(w_i\) and so \(u\) can be extended to an automorphism of \(N\).

Let \(H\) be the group obtained by extending \(N\) by the cyclic group generated by an element which has the action of \(u\) on \(N\). (For simplicity this generating element will also be denoted by \(u\).)

It will now be shown that \(H\) satisfies the conditions (i),..., (v).

(i) It is clear that \(H\) is generated by \(u, v_o, \ldots, v_{c-2}, w_o, \ldots, w_{c-2}\). However,

\[
[v_i, u] = v_i^{-1} u v_i = v_{i+1} \text{ if } i \text{ is less than } c-2 = 1 \text{ if } i = c-2.
\]

A simple induction argument implies that

\[v_i = [v_o, iu] \text{ for all } i \text{ in } \{0, \ldots, c-2\}, \quad [v_o, (c-1)u] = 1.\]

Similarly \(w_i = [w_o, iu] \text{ for all } i \text{ in } I_{c-2} \text{ and } [w_o, (c-1)u] = 1.\)

Thus \(H\) is generated by \(u, v_o\) and \(w_o\).

(ii) From the above argument

\[[v_o, (c-3)u; w_o, u] = [v_{c-3}, w_1] \text{ and this is non-trivial by construction.}\]

(iii) Now \([v_i, w_j, w_k] = [v_i, w_j, v_k] = 1\) since \(N\) is nilpotent of class 2.
Moreover

\[ [v_i, w_j, u] \]

\[ = [v_i, w_j]^{-1} [v_i, w_j]^u \]

\[ = [v_i, w_j]^{-1} [v_i, w_j] [v_i, w_{j+1}] [v_i, w_{j+1}] [v_i, w_{j+1}] \]

\[ = 1 \text{ for all } i, j \text{ in } \{0, \ldots, c-2\} \]

Therefore it follows from the condition (a) on \( N \) that the only commutators of weight \( c+1 \) in \( H \) that need be considered are \([v_0, (c-1)w_0] \) and \([w_0, (c-1)u, v_0] \).

However these are both trivial by the results of (i).

Thus \( \gamma_{c+1}(H) = \{1\} \) and \( H \) is of class \( c \).

(iv) Clearly every element in \( H^{(2)} \) is in \( \gamma_2(N) \) and so is a product of elements of the form \([v_i, w_j]\) for \( i, j \) in \( \{0, \ldots, c-2\}\). However it has already been shown that these commute with the generators of \( H \). Thus \( [H^{(2)}, H] \) is trivial and \( H \) is centre-extended-by-metabelian.

(v) Because \( H \) is a \( p \)-group of class less than \( p \), it is regular (cf 12.4 of [5]) and therefore \( H \) has exponent \( p \) because its generators have order \( p \).
Chapter Five

The Lattice of Subvarieties of $C_{=p-1}$

5.1 Introduction

Throughout this chapter only subvarieties of $C_{=p-1}$ will be considered so that whenever the word variety is used it will mean subvariety of $C_{=p-1}$.

In Chapter Three it was shown that for all $c$ less than $p$ there are no subvarieties strictly between $M_{=c}$ and $M_{=c-1}$ and that $M_{=c-1}$ is a proper subvariety of $M_{=c}$. Then in Chapter Four it was shown that for all $c$ in $\{4, \ldots, p-1\}$ there are two distinct varieties strictly between $C_{=c}$ and $C_{=c-1}$, one of which contains $M_{=c}$ while the other does not, which join in $C_{=c}$ and intersect in $C_{=c-1}$. The variety strictly between $C_{=c}$ and $C_{=c-1}$ which contains $M_{=c}$ will be denoted by $X_{=c-1,c}$ while the other is denoted by $X_{=c,c-1}$. Therefore

$$X_{=c,c-1} \cup X_{=c-1,c} = C_{=c-1}$$

and

$$X_{=c,c-1} \cap X_{=c-1,c} = C_{=c}$$

or the lattice of varieties between $C_{=c}$ and $C_{=c-1}$ can be drawn as follows:

![Diagram of lattice of subvarieties]

$C_{=c}$

$X_{=c-1,c}$

$C_{=c-1}$

$X_{=c,c-1}$
Since $M_i = C_i$ for all $i$ in $\{0,\ldots,3\}$ it follows that the lattice of subvarieties of $C_3$ is given by:

\[ \begin{array}{c}
C_3 \\
\downarrow \\
C_2 \\
\downarrow \\
C_1 \\
\downarrow \\
C_0 
\end{array} \]

Let $j$ be in $\{3,\ldots,p-1\}$. Then a variety $X_{i,j}$ can be defined for all $i$ in $\{3,\ldots,j+1\}$, except for $i=c$ when $j = p-1$, by:

\[ X_{3,j} = M_j \] while for $i$ greater than $3$,

\[ X_{i,j} = X_{3,j} \cup X_{i,i-1}. \]

The following results will be obtained in Section 5.2. Firstly $C_c = X_c$ for all $c$ in $\{3,\ldots,p-1\}$ and for all such $c$ the two definitions of $X_{c-1,c}$ are compatible. The $X_{i,j}$ are such that $X_{i,j}$ differs from $X_{i',j'}$ whenever the ordered pair $(i,j)$ differs from the ordered pair $(i',j')$. Furthermore $X_{i,j} \cap X_{i',j'} = X_{k,r}$ where $k$ is the lesser of $i,i'$ and $r$ is the lesser of $j,j'$, while $X_{i,j} \cup X_{i',j'} = X_{k^*,r^*}$ where $k^*$ is the greater of $i,i'$ and $r^*$ is the greater of $j,j'$. 
In Section 5.3 it will be shown, under certain assumptions, that the $X_{i,j}$ for all appropriate $i$ and $j$ together with $C_0, C_1$ and $C_2$ are the only subvarieties of $C_\geq 1$.

The later sections of the chapter are taken up with the proofs that the assumptions made in Section 5.3 hold.

Thus for $c$ greater than 3 a diagram of the lattice of subvarieties of $X_{c,c}$ can be obtained from the diagram of the lattice of subvarieties of $X_{c-1,c-1}$ by adding the following layer to the latter diagram in the appropriate position.

![Diagram of lattice](image)

It is obvious from this diagram that the lattice is distributive with $X_{3,c}, X_{c,c-1}$ and $C_0, C_1, C_2$ as its join irreducible elements.

In this chapter much use is made of the fact that the lattice is modular, that is it obeys the following law:-
5.1.1 Let \( V, W, Z \) be subvarieties of \( \mathbb{C}^{p-1} \). If \( Z \) is a subvariety of \( W \), then
\[
W \cap (V \cup Z) = (W \cap V) \cup Z \quad \text{(cf. [10])}
\]

5.2 The Distinctness Results

Lemma 5.2.1 For all \( c \) in \( \{4, \ldots, p-1\} \), \( \mathbb{C}^c = \mathbb{C}^c_{c,c} \).

Proof Now \( \mathbb{C}^c_{c,c} \) is a subvariety of \( \mathbb{C}^c \) that contains \( \mathbb{C}^c_{c,c-1} \).

But \( \mathbb{C}^c_{3,c} \) is not contained in \( \mathbb{C}^c_{c,c-1} \), for if it were, \( \mathbb{C}^c_{3,c} \) would be in \( \mathbb{C}^c_{c,c-1} \cap \mathbb{C}^c_{c-1,c} = \mathbb{C}^c_{c-1} \) and so equal to \( \mathbb{C}^c_{3,c-1} \) contrary to the results of Chapter Three. Therefore \( \mathbb{C}^c_{c,c-1} \) is a proper subvariety of \( \mathbb{C}^c_{c,c} \). Since the only proper subvarieties of \( \mathbb{C}^c \) that contain \( \mathbb{C}^c_{c,c-1} \) properly are \( \mathbb{C}^c_{c,c-1} \) and \( \mathbb{C}^c_{c-1,c} \) and since \( \mathbb{C}^c_{c-1,c} \) does not contain \( \mathbb{C}^c_{c,c-1} \) the result follows.

The next lemma justifies the notation used for the variety between \( \mathbb{C}^c \) and \( \mathbb{C}^c_{c-1} \) that contains \( \mathbb{M}^c = \mathbb{C}^3_{3,c} \).

Lemma 5.2.2 For every \( c \) greater than 3, \( \mathbb{C}^c_{c-1,c} \), defined by
\[
\mathbb{C}^c_{3,c} \cup \mathbb{C}^c_{c-1,c-2},
\]
is the variety strictly between \( \mathbb{C}^c_{c,c} \) and \( \mathbb{C}^c_{c-1,c-1} \) that contains \( \mathbb{C}^c_{3,c} \).

Proof Obviously \( \mathbb{C}^c_{c-1,c} \) contains \( \mathbb{C}^c_{3,c} \).

The variety strictly between \( \mathbb{C}^c_{c,c} \) and \( \mathbb{C}^c_{c-1,c-1} \) that contains \( \mathbb{C}^c_{3,c} \) clearly contains \( \mathbb{C}^c_{c-1,c} \). That it is \( \mathbb{C}^c_{c-1,c} \) will follow if it can be shown that \( \mathbb{C}^c_{c-1,c-1} \) is a proper subvariety of \( \mathbb{C}^c_{c-1,c} \).
However, 

\[ X_{c-1, c} = X_{c-1, c-2} \cup X_{3, c} \]

\[ = X_{c-1, c-2} \cup (X_{3, c-1} \cup X_{3, c}) \]

\[ = (X_{c-1, c-2} \cup X_{3, c-1}) \cup X_{3, c} \] by the associativity of union

\[ = X_{c-1, c-1} \cup X_{3, c}. \]

Since \( X_{3, c} \) is different from \( X_{3, c-1} \) it follows that \( X_{3, c} \) is not contained in \( X_{c-1, c-1} \). Therefore \( X_{c-1, c} \) has \( X_{c-1, c-1} \) as a proper subvariety and the result follows.

Theorem 5.2.3 Let \( j, j' \) be in \{3, ..., p-1\} and let \( i, i' \) be in \{3, ..., j+1\}, \{3, ..., j'+1\} respectively. Then

\[ X_{i, j} \cup X_{i', j'} = X_{k^*, r^*} \] where \( k^* \) is the greater of \( i, i' \) and \( r^* \) is the greater of \( j, j' \). Moreover \( X_{i, j} \cap X_{i', j'} = X_{k, r} \) where \( k \) is the lesser of \( i, i' \) and \( r \) is the lesser of \( j, j' \).

Proof By definition

\[ X_{i, j} \cup X_{i', j'} = (X_{3, j} \cup X_{3, i-1}) \cup (X_{3, j'} \cup X_{3, i'-1}) \]

\[ = (X_{3, j} \cup X_{3, j'}) \cup (X_{3, i-1} \cup X_{3, i'-1}) \] by the associativity and commutativity of union. But

\[ X_{3, j} \cup X_{3, j'} = X_{3, r^*} \] by the results of Chapter Three. Now

\[ X_{i, i-1} \] contains \( X_{i-1, i-1} = C_{i-1} \) by definition.
However if $s$ is less than $i$, then $X_{s,s-1}$ is contained in $X_{i-1,i-1}$ as it is contained in $X_{s,s}$ which is a subvariety of $X_{i-1,i-1}$. Therefore $X_{s,s-1}$ is contained in $X_{i,i-1}$ whenever $s$ is less than $i$. Thus

$$X_{i,i-1} \cup X_{i',i'-1} = X_{i',i'-1}$$

and

$$X_{i,i} \cup X_{i',i'} = X_{i',i'} \cup X_{k^*,k^*-1} = X_{k^*,k^*-1}$$

as required.

Attention is now focused on the intersection of the two varieties. The following result is useful for the desired result and is extracted as a lemma.

**Lemma 5.2.4** For all possible $i,j,s$,

$$X_{i,j} \cap X_{j,s} = X_{j,s}$$

where $t$ is the lesser of $j,s$.

**Proof** Since $X_{j,s}$ contains $X_{j,s}$ whenever $j$ is at least $s$ it follows that $X_{i,j} \cap X_{j,s} = X_{j,s}$ whenever $s$ is at most $j$.

However if $j$ is less than $s$, then $X_{i,j} \cap X_{j,s}$ contains $X_{j,s}$. It cannot contain $X_{j,s+1}$ for if it did, then $X_{j,s+1}$ would be a subvariety of $X_{i,j}$ and hence of $X_{j+1,j}$, contrary to the definition of $X_{j+1,j}$. Moreover $X_{i,j} \cap X_{j,s}$ is a subvariety of $X_{j+1,j}$, and so by the results of Chapter Three it must be $X_{j,s}$ and the lemma follows.

Now to return to the proof of the theorem.

Without loss of generality it can be assumed that $i'$ is not more than $i$. Therefore
Theorem 5.2.5 For all $c$ greater than 3, $X_{i,c}$ is not equal to $X_{j,c}$ whenever $i$ and $j$ are different integers in $\{3, \ldots, c+1\}$.

Proof It has already been established that $X_{c,c}$ contains $X_{i,c}$ for all $i$ in $\{3, \ldots, c-1\}$. Therefore since $X_{c+1,c}$ properly contains $X_{c,c}$, $X_{c+1,c}$ differs from $X_{c,c}$ and the $X_{i,c}$.

Thus it is sufficient to prove the result for $i$ and $j$ in $\{3, \ldots, c\}$.

The proof now proceeds by induction on $c$. The results is obviously true for $c$ equal to 4. Therefore let $s$ be greater than 4 and let it be assumed that the result is true for $s-1$. 

Since $X_{3,c}$ is not equal to $X_{3,c-1}$ for all $c$ in $\{3, \ldots, p-1\}$ it follows that $X_{j,c}$ is different from $X_{k,c-1}$ for all $j$ in $\{3, \ldots, c+1\}$ and all $k$ in $\{3, \ldots, c\}$ as the former contains $X_{3,c}$ and so is not a subvariety of $X_{c,c-1}$ while the latter is a subvariety of $X_{c,c-1}$.

Therefore the next theorem is sufficient to establish required distinctness results.

Theorem 5.2.5 For all $c$ greater than 3, $X_{i,c}$ is not equal to $X_{j,c}$ whenever $i$ and $j$ are different integers in $\{3, \ldots, c+1\}$.

Proof It has already been established that $X_{c,c}$ contains $X_{i,c}$ for all $i$ in $\{3, \ldots, c-1\}$. Therefore since $X_{c+1,c}$ properly contains $X_{c,c}$, $X_{c+1,c}$ differs from $X_{c,c}$ and the $X_{i,c}$.

Thus it is sufficient to prove the result for $i$ and $j$ in $\{3, \ldots, c\}$.

The proof now proceeds by induction on $c$. The results is obviously true for $c$ equal to 4. Therefore let $s$ be greater than 4 and let it be assumed that the result is true for $s-1$. 

Since $X_{i,j}$ is a subvariety of $X_{i',j'}$ hence of $X_{i,j}$.

Theorem is complete.
If for some \( i, j \) in \( \{3, \ldots, s\} \) \( X_{i,s} = X_{j,s} \), then 
\[
X_{i,s-1} = X_{i,s} \cap X_{s-1,s-1} = X_{j,s} \cap X_{s-1,s-1} = X_{j,s-1}.
\]
Thus \( i \) and \( j \) are equal by the inductive hypothesis and the result follows.

These results establish that the lattice is at least what it is claimed to be.

5.3 The Non-existence Results

As was mentioned in the introduction some of the results in this section rely on the assumption of two results which will be established in the later sections.

These assumed results are

5.3.1 For all \( c \) greater than 4, the only variety strictly between 
\( X_{c,c} \) and \( X_{c-2,c} \) is \( X_{c-1,c} \) and

5.3.2 For all \( c \) greater than 4, the only varieties strictly between \( X_{1,c} \) and \( X_{i-1,c-1} \) for all \( i \) in \( \{4, \ldots, c\} \).

The aim of this section is to show that the only subvarieties 
for \( i \) in \( \{3, \ldots, p-1\} \) and, for each \( j \), for \( i \) in \( \{3, \ldots, j+1\} \), (except \( i = p \) when \( j = p-1 \),) \( G_o, G_1 \) and \( G_2 \).

The first few results are independent of the two assumptions.

The following lemma is a simple consequence of the centre-
extended-by-metabelian law and is sufficient to establish that the subvarieties of \( X_{4,4} \) are all of the desired type.

**Lemma 5.3.3** Let \( c \) be in \( \{4, \ldots, p-1\} \) and let \( \mathcal{V} \) be a variety such that 
\[
\mathcal{V} \cap M = X_{3,c-1}.
\]
Then \( \mathcal{V} \) is a subvariety of \( X_{c,c} \).


Proof Let $H$ be a group in $V$. Then the factor group $H/H^{(2)}$ is metabelian and so is in $V \cap M = X_{3,c-1}$. This implies that $\gamma_c(H)$ is contained in $H^{(2)}$ and so $\gamma_{c+1}(H) = [\gamma_c(H), H]$ is a subgroup of $[H^{(2)}, H]$ hence is trivial. Therefore $H$ is of class at most $c$ and the result follows.

The next result is an easy consequence of Lemma 5.3.3 and partially accounts for the lopsidedness of the diagram of the lattice.

**Lemma 5.3.4** If $V$ is a variety for which $V \cap X_{c,c} = X_{c,c-1}$ for some $c$ in $\{4, \ldots, p-1\}$, then $V = X_{c,c-1}$.

**Proof** Now $V \cap X_{3,c} = (V \cap X_{c,c}) \cap X_{3,c}$

$$= X_{c,c-1} \cap X_{3,c}$$

$$= X_{3,c-1}$$

Thus it follows from 5.3.4 that $V$ is a subvariety of $X_{c,c}$ and so must be $X_{c,c-1}$.

**Lemma 5.3.5** Let $c$ be in $\{4, \ldots, p-1\}$ and let $i$ be in $\{4, \ldots, c+1\}$. If $V$ is a proper subvariety of $X_{i,c}$ that contains $X_{i-1,c}$, then $V = X_{i-1,c}$.

**Proof** The result has already been established for $i = c$ or $c+1$. Therefore let $i$ be in $\{4, \ldots, c-1\}$.

The proof now proceeds by induction on $c$ and is vacuously true for $c = 4$. Therefore let $s$ be greater than 4 and assume that the result is true for $s-1$. 

Now
\[ V = V \cap X_{1,s} \]
\[ = V \cap (X_{i,s-1} \cup X_{i-1,s}) \]
\[ = (V \cap X_{i,s-1}) \cup X_{i-1,s} \] by 5.1.1

However \( V \cap X_{i,s-1} \) is contained in \( X_{i,s-1} \) and, since both \( X_{i,s-1} \) and \( X_{i-1,s-1} \) contain \( X_{i-1,s-1} \), contains \( X_{i-1,s-1} \). Thus by the inductive hypothesis it is either \( X_{i,s-1} \) or \( X_{i-1,s-1} \).

But if \( V \cap X_{i,s-1} \) were \( X_{i,s-1} \), then \( V \) would contain both \( X_{i,s-1} \) and \( X_{i-1,s} \) and so would contain \( X_{i,s} \) contrary to the assumptions on \( V \).

Therefore \( V \cap X_{i,s-1} = X_{i-1,s-1} \) and so
\[ V = X_{i-1,s-1} \cup X_{i-1,s} \]
\[ = X_{i-1,s} \] and the lemma follows.

Using the last result the following useful identity can be obtained.

The following theorem uses the assumed results and is sufficient to prove that the lattice is only what it is claimed to be.

**Theorem 5.3.7** Let \( c \) be an integer in \( \{4, \ldots, p-1\} \). If \( V \) is a subvariety of \( X_{c,c} \) that is not contained in \( X_{c-1,c-1} \), then \( V \) is either \( X_{i,c} \) for some \( i \) in \( \{3, \ldots, c\} \) or \( X_{c,c-1} \).
Proof The proof is by induction on \( c \). The result has already been established for \( c = 4 \). Therefore let \( s \) be greater than 4 and let it be supposed that the result is true for \( s-1 \).

Now it follows from Lemma 5.3.3 that \( V \cap M \) contains \( X_{3,s-1} \). Therefore \( V \cap X_{s-1,s-1} \) is one of the \( X_{i,s-1} \) for \( i \in \{3, \ldots, s-1\} \).

The proof of the inductive step is itself an inductive one and it will be called the inner induction.

The inner induction is an inductive proof on \( i \in \{1, \ldots, s-3\} \) that the only subvarieties of \( X_{s,s} \) which are not subvarieties of \( X_{s-1,s-1} \) and whose intersection with \( X_{s-1,s-1} \) contains \( X_{s-i,s-1} \) are the \( X_{j,s} \) for \( j \in \{s-i, \ldots, s\} \) and \( X_{s,s-1} \).

That this holds for \( i = 1 \) follows directly from the results of Chapter Four. Therefore let it be assumed that \( t \) is greater than 1 and that the inner inductive assumption holds for all \( i \) less than \( t \). Then let \( V \) be a subvariety of \( X_{s,s} \) that is not contained in \( X_{s-1,s-1} \) and such that \( X_{s-1,s-1} \cap V = X_{s-t,s-1} \).

Now \( W = V \cup X_{s-t+1,s-1} \) is a subvariety of \( X_{s,s} \) not contained in \( X_{s-1,s-1} \) whose intersection with \( X_{s-1,s-1} \) contains \( X_{s-t+1,s-1} \). Therefore it follows from the inner inductive assumption that one of the following possibilities must hold:-
5.3.8 \( W = X_{j,s} \) for some \( j \) in \( \{s-t+1, \ldots, s-1\} \).

5.3.9 \( W = X_{s,s} \)

or 5.3.10 \( W = X_{s,s-1} \).

5.3.8 In this case

\[
X_{j,s-1} = X_{s-1,s-1} \cap X_{j,s} \quad \text{by 5.2.3}
\]

\[
= X_{s-1,s-1} \cap (V \cup X_{s-t+1,s-1})
\]

\[
= (X_{s-1,s-1} \cap V) \cup X_{s-t+1,s-1} \quad \text{by 5.1.1}
\]

\[
= X_{s-t,s-1} \cup X_{s-t+1,s-1}
\]

and so \( j = s-t+1 \) by 5.2.5.

Therefore \( V \) is a variety that is contained in \( X_{s-t+1,s} \) and that properly contains \( X_{s-t,s-1} \). Therefore by 5.3.2 and the assumptions on \( V \), \( V = X_{s-t,s} \).

5.3.9 Here

\[
X_{s-1,s-1} = X_{s-1,s-1} \cap (V \cup X_{s-t+1,s-1})
\]

\[
= (X_{s-1,s-1} \cap V) \cup X_{s-t+1,s-1} \quad \text{by 5.1.1}
\]

\[
= X_{s-t,s-1} \cup X_{s-t+1,s-1}
\]

\[
= X_{s-t+1,s-1}.
\]
Therefore \( s-t+1 = s-1 \) and so \( t \) must equal 2 by 5.2.5.

Thus \( V \) contains \( X \equiv_{s-2,s-1} \).

However \( V \) cannot be \( X \equiv_{s-2,s} \) or \( X \equiv_{s-1,s} \) since if it were \( W \) would be \( X \equiv_{s-1,s} \). Therefore \( X \equiv_{s-2,s} \) is a proper subvariety of \( X \equiv_{s-2,s} U V \). Moreover \( X \equiv_{s-2,s} U V \) cannot be \( X \equiv_{s-1,s} \) as this would imply that \( W = X \equiv_{s-1,s} \). Therefore it follows from 5.3.1 that \( X \equiv_{s-2,s} U V = X \equiv_{s,s} \).

Thus
\[
X \equiv_{s,s-1} = X \equiv_{s,s-1} \cap (X \equiv_{s-2,s} U V)
\]

\[
= (X \equiv_{s,s-1} \cap X \equiv_{s-2,s}) U V \quad \text{by 5.1.1}
\]

\[
= X \equiv_{s-2,s-1} U V \quad \text{by 5.3.6}
\]

\[
= V. \quad \text{But this contradicts the assumption on} \ V \ \text{and so there is no} \ V \ \text{that satisfies the conditions of 5.3.9.}
\]

5.3.10 In this case
\[
X \equiv_{s-1,s-1} = X \equiv_{s-1,s-1} \cap (V U X \equiv_{s-t+1,s-1})
\]

\[
= X \equiv_{s-t+1,s-1} \quad \text{as before. Thus, once again,} \ t = 2.
\]

Now \( V \) cannot be contained in \( X \equiv_{s-1,s} \), for if it were it would be a subvariety of \( X \equiv_{s-1,s} \cap X \equiv_{s,s-1} = X \equiv_{s-1,s-1} \) contrary to the assumptions on \( V \). Thus \( V U X \equiv_{s-2,s} \) properly contains \( X \equiv_{s-2,s} \) and is different from \( X \equiv_{s-1,s} \).
Therefore it follows from 5.3.1 that $\forall \subseteq X = X_{s,s}$.

Hence $X_{s-1,s} = X_{s-1,s} \cap (\forall \cup X_{s-2,s})$

$= (X_{s-1,s} \cap \forall) \cup X_{s-2,s}$ by 5.1.1.

However $X_{s-1,s} \cap \forall$ is contained in $X_{s-1,s} \cap X_{s,s-1} = X_{s-1,s-1}$

and contains $X_{s-2,s-1}$. It cannot be $X_{s-1,s-1}$ and so by 5.3.5

it must be $X_{s-2,s-1}$. Therefore

$X_{s-1,s} = X_{s-2,s-1} \cup X_{s-2,s}$

$= X_{s-2,s}$ which contradicts Theorem 5.2.5.

Thus there are no varieties that satisfy the conditions

of 5.3.10.

Thus the inner induction, and hence Theorem 5.3.7, follows.
5.4 The Proof of 5.3.1

The Proof is achieved by proving the corresponding result for the lattice of verbal subgroups of $G$, where $G$ is a free group of $X^c,c$.

By 5.2.3 $X \equiv_{c-1,c} X \equiv_{c-1,c-1} \cup X \equiv_{3,c}$ and $X \equiv_{c-2,c} X \equiv_{c-2,c-2} \cup X \equiv_{3,c}$. However $X \equiv_{3,c}$ is the variety of all metabelian groups of exponent $p$ and class at most $c$ so corresponds to the verbal subgroup $G^{(2)}$. Moreover for $i$ in $\{3,\ldots,c\}$ $X \equiv_{i,i}$ consists of all groups of class at most $i$ in $X \equiv_{c,c}$ and so the verbal subgroup corresponding to $X \equiv_{i,i}$ is $\gamma_{i+1}^{(G)}$. Therefore the verbal subgroups of $G$ corresponding to $X \equiv_{c,c}, X \equiv_{c-1,c}$ and $X \equiv_{c-2,c}$ are $\gamma_{c+1}^{(G)} = 1, \gamma_{c}^{(G)} \cap G^{(2)}$ and $\gamma_{c-1}^{(G)} \cap G^{(2)}$ respectively.

It should be noted that, since there are no varieties strictly between $X \equiv_{i,c}$ and $X \equiv_{i-1,c}$ for all $i$ in $\{4,\ldots,c\}$, there are no verbal subgroups between $\gamma_{c-1}^{(G)} \cap G^{(2)}$ and $\gamma_{c}^{(G)} \cap G^{(2)}$. Moreover $\gamma_{c}^{(G)} \cap G^{(2)}$ is a minimal verbal subgroup of $G$.

Thus 5.3.1 is equivalent to the following Theorem.
Theorem 5.4.1  The only verbal subgroup of $G$ properly contained in $\gamma_{c-1}(G) \cap G(2)$ is $\gamma_c(G) \cap G(2)$.

Proof  Let $v$ be an element of $\gamma_{c-1}(G) \cap G(2)$ that is not contained in $\gamma_c(G) \cap G(2)$. Then the verbal subgroup $V$ of $G$ generated by $v$ and $\gamma_c(G) \cap G(2)$ is contained in $\gamma_{c-1}(G) \cap G(2)$ and properly contains $\gamma_c(G) \cap G(2)$ hence must be $\gamma_{c-1}(G) \cap G(2)$.

Now every element of $V$ can be written as a product of a finite number of images of $v$ under the endomorphisms of $G$ and an element of $\gamma_c(G) \cap G(2)$. (All images of $v$ under the endomorphisms of $G$ and all elements of $\gamma_c(G) \cap G(2)$ are in $G(2)$ and so are central in $G$.)

Let $\{g_1, g_2, \ldots\}$ be a set of free generators for $G$. Then it follows from the fact that $V = \gamma_{c-1}(G) \cap G(2)$ that there are endomorphisms $\eta_1, \eta_2, \ldots, \eta_r$ of $G$ and there is an element $x$ in $\gamma_c(G) \cap G(2)$ for which

$$[g_1, (c-4)g_2; g_3, g_2] = \prod_{i=1}^{r} \eta_i \cdot x.$$ 

Since $x$ is in $\gamma_c(G)$ and $\gamma_{c+1}(G)$ is trivial $x$ can be expressed as a product of the form $\prod_{j \in J} x_j$ where $J$ is some index set and the $x_j$ are commutators of the form $[g_{j_1}, \ldots, g_{j_c}]$. 
Let $J^*$ be the subset of $J$ consisting of all those $j$ for which at least one of the $j_1, \ldots, j_c$ is 1. Then

$$[g_1, (c-4)g_2; g_3, g_2] = \prod_{i=1}^{r} \prod_{j \in J^*} x_j \prod_{j \in J \setminus J^*} x_j$$

If $\delta_1$ is the endomorphism of $G$ for which

$g_1\delta_1 = 1$ and $g_i\delta_1 = g_i$ for all $i$ greater than 1, then

$$[g_1, (c-4)g_2; g_3, g_2] \delta_1 = \prod_{i=1}^{r} \prod_{j \in J^*} x_j \prod_{j \in J \setminus J^*} x_j \delta_1.$$ 

Thus $1 = \prod_{i=1}^{r} \prod_{j \in J \setminus J^*} x_j$ since the entries in $x_j$ are unaltered by $\delta_1$ for all $j$ in $J \setminus J^*$. Therefore

$$\prod_{j \in J \setminus J^*} x_j = \prod_{i=1}^{r} (\nu_j \delta_1)^{-1}$$

Therefore it follows that

$$[g_1, (c-4)g_2; g_3, g_2] = \prod_{i=1}^{r} (\nu_j \delta_1)^{-1} x_j$$

Now let $\theta$ be the endomorphism of $G$ for which

$g_1\theta = [g_1, g_2]$ and $g_i\theta = g_i$ for all $i$ greater than 1. Then

$$[g_1, (c-3)g_2; g_3, g_2] = [g_1, (c-4)g_2; g_3, g_2] \theta$$

$$= \prod_{i=1}^{r} (\nu_j \delta_1 \theta)^{-1} x_j \theta.$$
Since $g_1$ appears at least once as an entry in $x_j$ for each $j$ in $J^*$, $x_j^\theta$ is of weight at least $c+1$ and so is trivial.

Thus $[g_1, (c-3)g_2;g_3,g_2]$ is in the verbal closure of $v$.

However it follows from Example 4.3.4 that $[g_1, (c-3)g_2;g_3,g_2]$ is a non-trivial element of $G$. Thus the intersection of the verbal closure of $v$ and $\gamma_c(G)\cap G^{(2)}$ is a non-trivial verbal subgroup of $G$ that is contained in $\gamma_c(G)\cap G^{(2)}$. Hence it is $\gamma_c(G)\cap G^{(2)}$ and the theorem follows.

5.5 The Proof of 5.3.2

This result will also be established by proving the corresponding result for the lattice of verbal subgroups of $G$, where $G$ is a free group of $X$ generated by $\{g_1, g_2, \ldots\}$.

It follows from the results of Chapter Four that 5.3.2 need only be proved for $i$ less than $c$.

Now $X_{i,c-1} = X_{i,c} \cap X_{c-1,c-1}$. Therefore the verbal subgroups of $G$ corresponding to the varieties $X_{i-1,c-1} \cap X_{i,c}$ and $X_{i-1,c}$ are respectively

$$Y_1 = (\gamma_i(G) \cap G^{(2)}) \cup \gamma_c(G)$$

$$Y_2 = (\gamma_{i+1}(G) \cap G^{(2)}) \cup \gamma_c(G)$$

$$Y_3 = \gamma_i(G) \cap G^{(2)}$$

and $$Y_4 = \gamma_{i+1}(G) \cap G^{(2)}.$$
Therefore 5.3.2 is equivalent to the following theorem.

**Theorem 5.5.1** The only verbal subgroups of G strictly between $Y_1$ and $Y_4$ are $Y_2$ and $Y_3$.

**Proof** Let $v$ be an element of $Y_1$ that is not contained in $Y_2$. Then it will be shown that the verbal subgroup $V$ of $G$ generated by $v$ and $Y_4$ contains $Y_3$. Therefore it follows from the modularity of the lattice that $V$ is either $Y_3$ or $Y_1$ and the theorem follows from this.

Now the verbal subgroup $V^*$ of $G$ generated by $v$ and $Y_2$ is contained in $Y_1$ and properly contains $Y_2$. Therefore it follows that $V^* = Y_1$. Hence $[g_1,(i-3)g_2;g_3,g_2]$ is in $V^*$. Therefore there are endomorphisms $\eta_1, \eta_2, \ldots, \eta_r$ of $G$ and there is an element $z$ in $Y_2$ for which

$$[g_1,(i-3)g_2;g_3,g_2] = \prod_{j=1}^{r} \eta_j^z$$

Since $Y_2 = (\gamma_{i+1}(G) \circ G^{(2)}) \cup \gamma_c(G)$ it follows that $z$ is a product of the form $z_1z_2$ where $z_1$ is in $\gamma_c(G)$ and $z_2$ is in $\gamma_{i+1}(G) \circ G^{(2)}$. Therefore

$$[g_1,(i-3)g_2;g_3,g_2] \cdot z_1^{-1} = \prod_{j=1}^{r} \eta_j^z z_2$$

which is in $V$.

These results will be used to show that $[g_1,(i-3)g_2;g_3,g_2]$ is in $V$. But it follows from Example 4.3.4 that $[g_1,(i-3)g_2;g_3,g_2]$ is not in $\gamma_{i+1}(G)$ and so is not in $Y_4$. 


From this it follows that the intersection of $V$ and $Y_3$ is a verbal subgroup of $G$ contained in $Y_3$ that properly contains $Y_4$ and so by 5.3.5 must be $Y_3$. Therefore $V$ contains $Y_3$ as required.

Since $z_1^{-1}$ is in $\gamma_c(G)$ it can be expressed as a product of the form $\prod_{j \in J} x_j$ where $J$ is some index set and $x_j = [g_{j_1}, \ldots, g_{j_c}]$ for some integers $j_1, \ldots, j_c$.

Let $k$ be a positive integer and let $\delta_k$ be the endomorphism of $G$ for which $g_s \delta_k = g_s$ for all integers $s$ different from $k$ and $g_k \delta_k = 1$. Then $V$ contains $([g_1, (i-3)g_2; g_3, g_2]. z_1^{-1}) \delta_k$ which is the product of all those factors of $[g_1, (i-3)g_2; g_3, g_2] \prod_{j \in J} x_j$ that do not have $g_k$ as an entry.

Hence by an inductive type argument on the integers greater than 3 it follows that

$$w = \prod_{j \in J'} x_j \text{ is in } V$$

where $J'$ is that subset of $J$ which consists of all $j$ for which $j_1, \ldots, j_c$ are in $\{1, 2, 3\}$.

Therefore $w (w \delta_1)^{-1} (w \delta_2)^{-1} (w \delta_3)^{-1}$ is in $V$ or

$$[g_1, (i-3)g_2; g_3, g_2] \prod_{j \in J^*} x_j \text{ is in } V$$

where $J^*$ is that subset of $J'$ consisting of all $j$ for which $g_1, g_2$, and $g_3$ all appear at least once as an entry in $x_j$. 
Now let $\alpha$ be a primitive root of $p$ and let $\theta_k$, for each positive integer $k$, be the endomorphism of $G$ that fixes $g_s$ for all $s$ different from $k$ but takes $g_k^\alpha$ to $g_k$. Furthermore, for each positive integer $k$ let $J^*_k$ be the subset of $J^*$ consisting of all $j$ for which $x_j$ has $k$ entries equal to $g_1$. Then $V$ contains

\[
\left\{ [g_1,(i-3)g_2;g_3,g_2] \prod_{k=1}^{c-3} \left( \prod_{j \in J^*_k} x_j \right) \right\} \theta_1
\]

\[
= [g_1,(i-3)g_2;g_3,g_2] \alpha \prod_{k=1}^{c-3} \left( \prod_{j \in J^*_k} x_j \right) \alpha^k
\]

where $y$ is a product of commutators of the form

$[g_1,g_2,mg_1,(i-3)g_2;g_3,g_2]$ for some positive integer $m$.

Therefore $y$ is in $V$ and so it follows that

\[
\prod_{k=2}^{c-3} \left( \prod_{j \in J^*_k} x_j \right) \alpha^{(\alpha^{k-1} - 1)} \text{ is in } V. \quad \text{But } \alpha \text{ is not divisible by } p \text{ and so it follows that }
\]

\[
\prod_{k=2}^{c-3} \left( \prod_{j \in J^*_k} x_j \right) (\alpha^{k-1} - 1)
\]

is in $V$. A simple inductive argument then implies that

\[
\prod_{k=t}^{c-3} \left( \prod_{j \in J^*_k} x_j \right) \prod_{r=1}^{t-l} (\alpha^{k-r} - 1) \text{ is in } V \text{ for all }
\]

$t \in \{2,\ldots,c-3\}$ ...5.5.2.
Therefore in particular

$$\left( \prod_{j \in J^*_3} x_j \right)_{r=1}^{c-4} \prod_{r=1}^{(\alpha^{c-3-r} - 1)} \text{ is in } V.$$ 

Because $\alpha$ is a primitive root of $p$, $p$ does not divide $(\alpha^l - 1)$ for all $l$ less than $p-1$ and therefore $p$ does not divide $\prod_{r=1}^{c-4} (\alpha^{c-3-r} - 1)$. Hence $\prod_{j \in J^*_3} x_j$ is in $V$ and it follows from 5.5.2 that

$$\prod_{k=t}^{t-1} (\alpha^{k-r} - 1)$$

is in $V$ for all $t$ in $\{2, \ldots, c-4\}$. An obvious induction argument then implies that $\prod_{j \in J^*_k} x_j$ is in $V$ for all $k$ in $\{2, \ldots, c-3\}$. Therefore

$$[g_1, (i-3)g_2; g_3, g_2] \prod_{j \in J^*_1} x_j \text{ is in } V.$$ 

A similar argument with

$$[g_1, (i-3)g_2; g_3, g_2]$$

and the endomorphism $\theta_3$ implies that

$$[g_1, (i-3)g_2; g_3, g_2] \prod_{j \in J^*_1} x_j \text{ is in } V$$

where $J^*_{11}$ is the subset of $J^*_1$ consisting of all $j$ for which $g_3$ appears exactly once as an entry in $x_j$. Therefore, for all $j$ in $J^*_1$, $x_j$ has one entry equal to $g_1$, one entry equal to $g_3$, and $(c-2)$ entries equal to $g_2$. 


Therefore

\[ \left\{ [g_1, (i-3)g_2; g_3, g_2] \prod_{j \in J_{11}} x_j \right\} \theta_2 \]

\[ = [g_1, (i-3)g_2; g_3, g_2]^{(i-2)} \quad \text{is in } \mathcal{V}, \]

where \( y^* \) is a product of commutators in \( Y_4 \).

Thus

\[ \left( \prod_{j \in J_{11}} x_j \right)^{(\alpha^{c-2} - \alpha^{i-2})} \text{ is in } \mathcal{V}. \]

\[ \alpha^{c-2} - \alpha^{i-2} = \alpha^{i-2} (\alpha^{c-i} - 1) \]

which is not divisible by \( p \) since \( \alpha \) is a primitive root of \( p \) and \( i \) is less than \( c \). Hence

\[ \prod_{j \in J_{11}} x_j \text{ is in } \mathcal{V}. \]

Therefore

\[ [g_1, (i-3)g_2; g_3, g_2] \]

is in \( \mathcal{V} \) as required.
Chapter Six

An Application

In this chapter a solution to problem 13 of [9] will be given. For convenience this problem is restated here.

**Problem 13** Let \( U_{c+1} \) be a variety of class \( c+1 \) and let \( U_c \) be the subvariety of all groups of class \( c \) in \( U_{c+1} \). Is the condition that the centre of \( F_{c+1}(U_{c+1}) \) is exactly \( \gamma_{c+1}(F_{c+1}(U_{c+1})) \) necessary to ensure that if \( F_k(U_{c+1}) \) generates \( U_{c+1} \), then \( F_k(U_c) \) generates \( U_c \)?

In this chapter it is shown that some condition is necessary but it is not the one given in Problem 13.

The crux of this solution is the following theorem.

**Theorem 6.1.1** Let \( c \) be an integer in \( \{4, \ldots, p-1\} \). Furthermore, for each positive integer \( n \), let \( G_n \) be a free group of rank \( n \) in the variety \( X_{c,c} \). Then \( G_3 \) generates \( X_{c,c} \) while \( G_2 \) generates \( X_{c,c} \) if, and only if, \( c \) is odd.

**Proof** In the definition of \( L_c \) a 3-dimensional vector space \( U \) over \( GF(p) \) was associated with \( G_3 \). Let \( \varphi_C^U \) be defined as it was in Chapter Two. Then if \( D \) is the special diagram associated with the partition \( (c-1,1) \) of \( c \), it follows from the remarks at the beginning of Chapter Four that \( (U^{[c]} \in D) \varphi_C^U \) is a non-trivial verbal subgroup of \( G_3 \) that is contained in \( \gamma_c(G_3) \) but is not contained in \( G_3(2) \).
Furthermore, if \( D^k \) is a special diagram associated with either the partition \((c-2,2)\), when \( c \) is odd, or the partition \((c-2,1,1)\), when \( c \) is even, then it follows from Theorem 4.3.1, Theorem 4.2.3 and Example 4.3.4 that \((U^\mathcal{C}_g(D^k))_{\mathcal{C}^U} \) is a non-trivial verbal subgroup of \( G_3 \) that is contained in \( G_3^{(2)} \cap \gamma_c(G_3) \).

Therefore the variety generated by \( G_3 \) contains both \( X_{c-1,c} \) and \( X_{c,c-1} \) and so must be \( X_{c,c} \).

Let \( U^k \) be the vector space over \( GF(p) \) that was associated with \( G_2 \) in the definition of \( L_c \). Then it follows from an argument similar to that used above, that \((U^k[c]_g(D^k))_{\mathcal{C}^U} \) is a non-trivial verbal subgroup of \( G_2 \) that is contained in \( \gamma_c(G_2) \) but not in \( G_2^{(2)} \) when \( D \) is the special diagram associated with \((c-1,1)\).

Let \( c \) be odd and let \( D' \) be a special diagram associated with the partition \((c-2,2)\) of \( c \). Theorem 4.3.1 implies that 
\[(U^k[c]_g(D'))_{\mathcal{C}^U} \text{ contains } [g_1, (c-3)g_2; g_1, g_2] \text{ where } g_1, g_2 \text{ are the free generators of } G_2 \text{ given by the definition of } L_c^c.\]

If it could be shown that this element is non-trivial, it would follow that \((U^k[c]_g(D'))_{\mathcal{C}^U} \) is a non-trivial verbal subgroup of \( G_2 \) that is contained in \( G_2^{(2)} \cap \gamma_c(G_2) \). Therefore the variety generated by \( G_2 \) would contain both \( X_{c-1,c} \) and \( X_{c,c-1} \) and so be \( X_{c,c} \).
Therefore to complete the sufficiency part of the second statement of the theorem it is sufficient to prove that there is a centre-extended-by-metabelian group $H$ of exponent $p$ and class $c$ in which $[y, (c-3)u; y, u]$ is non-trivial for some elements $u$ and $y$. Consider the group $H$ of Example 4.3.4 and the elements $y = v_0 w_0$ and $u$ of $H$. Now $[v_0 w_0, (c-3)u; v_0 w_0]$

$$= [v_0, (c-3)u; v_0, u] [v_0, (c-3)u; w_0, u]$$

$$= [w_0, (c-3)u; v_0, u] [w_0, (c-3)u; w_0, u]$$

$$= [w_{c-3}, v_1] [v_{c-3}, w_1] \text{ (by condition (a) on } N\text{), } [v_{c-3}, w_1]^2 \text{ by condition (c) on } N \text{ since } c \text{ is odd. But } H \text{ is of exponent } p \text{ for some prime } p \text{ greater than } 2 \text{ and } [v_{c-3}, w_1] \text{ is non-trivial by construction.}$$

Therefore $[y, (c-3)u; y, u]$ is non-trivial as required.

Now let $c$ be even and let $D^k$ be a special diagram associated with the partition $(c-2, 1, 1)$ of $c$. It follows from Theorem 4.2.3 and Lemma 4.2.4 that $(U_k^c[D^k])_{c(D^k)}^c$ is trivial. Therefore the variety generated by $G_2$ is a subvariety of $X_{c, c-1}$, so is not $X_{c, c}$ and the Theorem is complete.

Let $c = 4$ and let $U_5 = X_{5, 5},$ then $U_4 = X_{4, 4}$. It follows from Theorem 6.1.1 that $F_2(X_{5, 5})$ generates $X_{5, 5}$ while $F_2(X_{4, 4})$ does not generate $X_{4, 4}$. Therefore some condition is needed.

However let $c = 5$ and let $U_6 = X_{6, 6},$ then $U_5 = X_{5, 5}$. By Theorem 6.1.1, $F_3(X_{6, 6})$ is the smallest rank free group that generates $X_{6, 6}$ but $F_3(X_{5, 5})$ generates $X_{5, 5}$. Thus if $F_k(U_6)$ generates $U_6$, then $F_k(U_5)$ generates $U_5$. Moreover it follows from Example 4.3.4 that $F_6(X_{6, 6})^2$ is not contained in $X_{6, 6}(F_6(X_{6, 6}))$. 
Therefore the centre of $F_6(X_{x_6,6})$, which contains $F_6(X_{x_6,6})^{(2)}$, is not $\gamma_6(F_6(X_{x_6,6}))$. Hence the condition given in Problem 13 is not a necessary condition for the truth of the statement.


References


